Fano schemes of lines on singular cubic hypersurfaces and their Picard schemes

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Abstract

To every cubic hypersurface X we associate the parameter space of lines contained in X; this is called the Fano scheme of lines on X and denoted $F(X)$. If X admits an isolated singular point of ADE-type, we prove that $F(X)$ admits hypersurface singularities of the same type transversally along the regular part of its singular locus. As was shown by H. Clemens and P. Griffiths, the Albanese variety $\mathrm{Alb}(F(X))$ of the Fano scheme of lines $F(X)$ on a smooth cubic threefold X is isomorphic to the intermediate Jacobian $IJ(X)$ of X. G. van der Geer and A. Kouvidakis generalised this result to nodal cubic threefolds, replacing the Albanese variety $\mathrm{Alb}(F(X))$ by the Picard scheme $\mathrm{Pic}^0(F(X))$.

We study more generally degenerations of the Picard scheme $\mathrm{Pic}^0(F(X))$ when the smooth cubic threefold X degenerates to a cubic threefold with unique singular point of type A_k and prove that these degenerations define points in Mumford's partial compactification $\mathcal{A}_5'$ of the moduli space $\mathcal{A}_5$ of principally polarised Abelian varieties of genus five.

Keywords: Cubic hypersurfaces, Fano scheme of lines, Picard scheme

Contents

Introduction

One of the most famous results in algebraic geometry asserts that any smooth cubic surface over the field of complex numbers, i.e. every hypersurface of degree three in $\mathbb{P}^3 = \mathbb{P}^3_{\mathbb{C}}$, contains precisely 27 lines. The first one to study the parameter space $F(Y)$ of lines on a hypersurface Y was G. Fano in his 1904 articles [Fan04a, Fan04b]. This parameter space is nowadays named after him and called the Fano scheme of lines.
The classical result on the 27 lines on a smooth cubic surface translates into the language of Fano schemes as the fact that the Fano scheme of lines $F(X)$ on a smooth cubic hypersurface $X \subset \mathbb{P}^3$ is zero-dimensional, reduced and of degree 27. Many of the classical geometers also computed the number of lines on cubic surfaces with an isolated double point; their results have been summarised in 1869 by A. Cayley [Cay69] and it can be deduced from his results that every cubic surface contains a line, but the number of lines decreases with the presence of singular points. For example, on a cubic hypersurface with an ordinary double point, one finds precisely 21 lines. On the other hand, the degree of the Fano scheme of lines on a cubic surface can still be seen to be 27, see [EH16] for a modern treatment, meaning that some of the lines on a singular cubic surface should be considered with multiplicity. For the Fano scheme of lines on a singular cubic surface this means that it can no longer be reduced. A modern survey of the geometrical considerations to compute the number of lines on singular cubic surfaces is given by I. Dolgachev in his book [Dol12]. Regardless of using classical or modern methods to compute the multiplicity of points of the Fano scheme of lines on a cubic surface, it is not explained *why* exactly these multiplicities occur besides the fact that they *do* occur. At least the classical geometers for example often derived the multiplicity of certain points of the Fano scheme of lines on a singular cubic surface only by knowing that the sum of the multiplicities should be 27 in the end.

Aside from the classical treatment of lines on cubic surfaces, the Fano scheme of lines did not receive much attention for some time after its introduction, but people began to study cubic threefolds, i.e. cubic hypersurfaces in $\mathbb{P}^4$, more intensively in the late 1960s. In 1973, H. Clemens and P. Griffiths [CG73] made use of the Fano scheme of lines again and showed that for a smooth cubic threefold X the intermediate Jacobian $IJ(X)$ of X is isomorphic to the Albanese variety of the Fano scheme of lines $F(X)$,

$$IJ(X) \cong \operatorname{Alb}(F(X)). \tag{I}$$

This result plays a central role in their proof of irrationality of all smooth cubic threefolds. They also considered the Fano scheme of lines on a cubic threefold with a single node. Afterwards, in 1977, A. B. Altman and S. L. Kleiman [AK77] published a general treatment of Fano schemes of lines on hypersurfaces of arbitrary degree and even over arbitrary algebraically closed fields and A. Collino and J. P. Murre reproved the result of H. Clemens and P. Griffiths using purely algebraic methods in their 1978 paper [CM78]. In the same year, W. Barth and A. van de Ven [BV78] showed that for a smooth cubic hypersurface $X \subset \mathbb{P}^n$ the Fano scheme of lines $F(X)$ always is non-empty, connected, and smooth of dimension $2n - 6$.

The natural question, how the varieties on both sides of (I) degenerate if the cubic threefold X degenerates to a singular one, has not been treated for more than thirty years after. In 2009, S. Casalaina-Martin and R. Laza studied cubic threefolds from a moduli-theoretic perspective and became interested in degenerations of the intermediate Jacobian of a cubic threefold. As one of their results in [CL09], they were able to show that the intermediate Jacobian of a cubic threefold degenerates to a point in Mumford's partial compactification $\mathcal{A}'_5$ of the moduli space $\mathcal{A}_5$ of principally polarised Abelian varieties of genus five, if the cubic threefold admits a unique singular point of type A_k ($k \leq 5$) or D_4. In particular, the intermediate Jacobian of any such cubic threefold is either given by a product of Jacobians of smooth curves or by a $\mathbb{C}^*$-extension of such a product and they were able to determine the respective extension data in many cases. The degeneration of the right-hand side of (I) has been studied in the simplest case, i.e. when the cubic threefold degenerates to a nodal one, by G. van der Geer and A. Kouvidakis in their 2010 article [vK10], replacing the Albanese variety of $F(X)$ by its Picard scheme. This can be done since the isomorphism (I) provides a principal polarisation on the Albanese variety of $F(X)$ and this principal polarisation establishes an identification of $\mathrm{Alb}(F(X))$ with its dual Abelian variety $\mathrm{Pic}^0(F(X))$. As was to be expected, their result is the same as the corresponding result on the intermediate Jacobian of such a cubic threefold in [CL09].
In 2015, S. Casalaina-Martin, S. Grushevsky, K. Hulek and R. Laza [CGHL15] generalised the results from [CL09] to cubic threefolds with arbitrary isolated ADE-singularities, and established a general framework to compute such degenerations along with the corresponding extension data. In particular they classified all degenerations of intermediate Jacobians of cubic threefolds of torus rank one and two. Using their general framework, many cases have been worked out in detail by K. Havasi in his 2016 thesis [Hav16].

Recently, in 2017, the Fano scheme of lines on a cubic fourfold with isolated ADE-singularities has been studied by R. Yamagishi [Yam17a] as an example of a singular symplectic variety in the sense of Beauville [Bea00].

This thesis aims to complete the study of degenerations of the varieties in (I) by contributing the study of degenerations of the right-hand side of (I), replacing the Albanese variety by the Picard scheme, just as G. van der Geer and A. Kouvidakis did in [vK10], but besides that by different methods. Our main result on the degenerations of the Picard scheme of the Fano scheme of lines $F(X)$ of a cubic threefold X is the following.

A Theorem (Degenerations of the Picard scheme, Theorem 3.34)**.** *Let X be a cubic threefold with a unique singular point of type A_k, let $\pi : \mathfrak{F} \to B$ be a smoothing (in the sense of Definition 2.1) of $\mathfrak{F}_0 = F(X)$ over a smooth curve B and denote by $\pi' : \mathfrak{F}' \to B$ the tail reduction of $\pi : \mathfrak{F} \to B$. Then, the degenerate Picard scheme, see Definition 2.17, of $\mathfrak{F}_0$ with respect to the family $\pi : \mathfrak{F} \to B$ is uniquely determined by $\mathrm{Pic}^0(\mathfrak{F}'_0)$ which has the form*

$$1 \longrightarrow K \longrightarrow \mathrm{Pic}^0(\mathfrak{F}'_0) \longrightarrow \mathrm{Pic}^0(\Sigma^\nu) \times \mathrm{Pic}^0(T) \longrightarrow 0,$$

where T is a smooth curve of genus $g(T) = \lceil \frac{k-1}{2} \rceil$ and Σ is a curve associated to X in a natural way. Moreover,

$$K = \begin{cases} \mathbb{C}^*, & \textit{if } k \textit{ is odd;} \\ 0, & \textit{if } k \textit{ is even.} \end{cases}$$

The curve T in the theorem above and therefore the degenerate Picard scheme of a cubic threefold X with a unique singular point of type A_k, depends on the initially chosen smoothing of $F(X)$, though its genus does not.

Our work is structured as follows. We begin the first chapter by collecting some well-known results about cubic hypersurfaces and their Fano schemes of lines. As very little is known about singularities of the Fano scheme of lines on a cubic hypersurface, we investigate the relation between the singularities of X and those of $F(X)$, and show that there *is* such a relation

(Theorem 1.29), a fact that is not obvious and in general false for hypersurfaces of degree different from three. This is done by developing a systematic approach for detecting types of hypersurface singularities, based on previously known results in singularity theory due to V. I. Arnold [Arn73, Arn74] and methods due to J. W. Bruce and C. T. C. Wall [BW79] used in their classification of cubic surfaces. We are able to give an algorithm that is capable of computing conditions on the coefficients of a polynomial P defining a hypersurface $Y = \{P = 0\}$ that are equivalent to $(Y, 0)$ being of a certain singularity type (Theorem 1.15). It follows from our local computations that the singular locus of $F(X)$ is isomorphic to a complete intersection Σ of type $(2, 3)$ that is associated to every singular cubic hypersurface in a natural way, and we show that for all $l \in \Sigma_{\text{reg}}$ the singularity $(F(X), l)$ is a hypersurface singularity whose type is independent of the point l . Using the systematic approach developed earlier, we compare the singularity types of X at its singular point p_0 and of $F(X)$ at a point $l \in \Sigma_{\text{reg}}$ and obtain the following theorem on the relation between singularities of X and $F(X)$.

B Theorem (Singularities of Fano schemes of lines on cubic hypersurfaces, Theorem 1.29)**.** *If $X \subset \mathbb{P}^n$ is a cubic hypersurface with unique singular point p_0 of type A_k, the types of (X, p_0) and $(F(X), l)$ are the same for all $l \in \Sigma_{\text{reg}}$.*

This generalises the analogous result by R. Yamagishi on singularities of the Fano scheme of lines on a cubic fourfold with isolated ADE-singularities, cf. [Yam17a]. His proof relies on methods from symplectic geometry which are only available in the particular case he treated.

In chapter two we recall the theory of semistable reduction for families of curves. In particular, we recall a constructive proof due to J. Harris and I. Morrison [HM98] from which a semistable reduction for families of curves can be computed explicitly (Theorem 2.3). As an example, we compute the degenerate Picard scheme for curves with unique singular point of type A_k (Corollary 2.19). Afterwards, we prove that the steps from the explicit proof of semistable reduction for families of curves can be generalised to families of varieties with curve singularities transversally along a smooth singular locus (Theorem 2.22). As it is unclear in which, if any, sense the result will be semistable, this procedure is called tail reduction.

C Theorem (Tail reduction for transverse curve singularities, Theorem 2.22)**.** *Let $\pi : \mathfrak{Z} \to B$ be a flat family of varieties over a smooth curve B such that $\mathfrak{Z}_b = \pi^{-1}(b)$ is smooth for $b \neq 0$ and such that $\mathfrak{Z}_0$ has smooth singular locus $S_0 \subset \mathfrak{Z}_0$ and a curve singularity of type A_k transversally along S_0. Then, there exists a branched cover $\varphi : B' \to B$ totally ramified over $0 \in B$ and a flat family of varieties $\pi' : \mathfrak{Z}' \to B'$ that differs from $\pi : \mathfrak{Z} \to B$ only in the central fibre and such that the central fibre $\mathfrak{Z}'_0 \subset \mathfrak{Z}'$ is reduced with smooth components intersecting transversally.*

By applying this to smoothings of Fano schemes of lines on cubic threefolds with a unique singular point of type A_1 or A_2, we obtain Theorem A for these special cases.

Chapter three contains the proof of our main result about the degenerations of the Picard scheme of $F(X)$ for a cubic threefold X. We begin by generalising a construction due to A. Collino and J. P. Murre [CM78] of a morphism $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$ from the symmetric square of a curve Σ naturally associated to a cubic threefold X with unique singular point of type A_1 to its Fano scheme of lines $F(X)$ by proving that the blowup of $F(X)$ along its singular locus results in $\mathrm{Hilb}^2(\Sigma)$.

D Theorem (Theorem 3.5)**.** *Let $X \subset \mathbb{P}^n$ be a cubic hypersurface with unique singular point of type A_k. Denote by Σ the associated complete intersection isomorphic to the singular locus of $F(X)$. Then there exists a natural map $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ that coincides with the blowup of $F(X)$ along Σ.*

We also provide a concrete geometric description of this map. As we need to compute a resolution of singularities of the Fano scheme of lines on a singular cubic threefold explicitly in the process of computing a tail reduction, we study resolutions of the Hilbert square of a singular curve. Just as the Fano scheme of lines on a singular cubic hypersurface, the Hilbert square of a singular curve in general has singular locus which is itself singular. We exhibit the local structure of the Hilbert square of a singular curve near the singular points of its singular locus to derive the following theorem which seems to be interesting on its own.

E Theorem (Theorem 3.23). *Let Y be a curve admitting a unique singular point y_0 of type A_k, $k \geq 3$, and denote by $\pi : \widetilde{\mathrm{Hilb}^2(Y)} \to \mathrm{Hilb}^2(Y)$ the blowup of $\mathrm{Hilb}^2(Y)$ along its singular locus. Then,*

$$(\widetilde{\mathrm{Hilb}^2(Y)}, p) \cong (\mathrm{Hilb}^2(\widetilde{Y}), q),$$

where p and q denote the singular points of the respective singular loci and $\widetilde{Y}$ is the blowup of Y at its singular point.

The statement of Theorem E was available in the literature, see [Yam17b], but the proof given there turns out to be incorrect.
Using this result we are then able to compute a resolution of the Fano scheme of lines on a singular cubic threefold by successive blowups along its singular locus explicitly (Theorem 3.34) and to deduce Theorem A.

Our final chapter four hints at generalisations of statements in this thesis and starting points for further research. We also give an intrinsic explanation of the multiplicities of the points of $F(X)$ when X is a singular cubic surface, i.e. explain *why* these multiplicities occur.

1 Singularities of the Fano scheme of lines on a cubic hypersurface

In this chapter we give the definitions of our main objects along with some known or easy to prove results that we will use throughout this thesis. We also give an algorithm that can be used to recognise singularity types from the defining equation of a hypersurface singularity. By considering local equations for cubic hypersurfaces in a specific normal form, we deduce that the Fano scheme of lines on this cubic has transversal singularities along the smooth part of its singular locus. After computing local equations for the Fano scheme of lines on a cubic, we show that our algorithm can be utilised to recognise the transversal singularity type of the Fano scheme of lines from its defining equations. By comparing the results for a cubic hypersurface with those for its Fano scheme of lines, we are then able to prove that the transversal singularity type of the Fano scheme of lines is the same as the singularity type of the cubic. The last section then explains some simplifications we have made when handing the problem to a computer. We always work over $\mathbb{C}$, the field of complex numbers.

1.1 Generalities on singular cubic hypersurfaces and their Fano scheme of lines

We begin by introducing the basic notions and objects that we will use in this thesis. Along these lines, we recall known facts and use this opportunity to develop our notation. Besides some introductory examples, we are always interested in cubic hypersurfaces. These are irreducible subschemes $X = \{f = 0\} \subset \mathbb{P}^n$ of the projective space defined by a single homogenous polynomial f of degree three. To every cubic hypersurface or, more generally, to any projective variety $Y \subset \mathbb{P}^n$, we can associate the Fano scheme of lines $F(Y)$ on this hypersurface which is the fine moduli space parameterising the one-dimensional linear subschemes $l \subset Y$ of the given variety Y. In other words, viewing $F(Y)$ as a set, it is

$$F(Y) = \{l \subset \mathbb{P}^n \mid l \text{ is a line contained in } Y\}.$$

The scheme structure on $F(Y)$ comes from viewing it as moduli space. This is done by means of Hilbert schemes and we now briefly recall their general definition. Let P be a polynomial in one variable and let $\mathfrak{h}_P$ be the contravariant functor from the category of schemes into the category of sets,

$$\mathfrak{H}_P : \mathfrak{Sch} \longrightarrow \mathfrak{Set},$$

that sends a scheme B to

$$\mathfrak{H}_P(B) = \left\{ \begin{array}{c} \text{flat families } Y \times B \supset Z \xrightarrow{b} B \text{ of subschemes of } Y, \\ \text{such that the fibres have Hilbert polynomial } P \end{array} \right\}$$

and a morphism $\alpha : B \to B'$ of schemes to

$$\mathfrak{H}_P(\alpha : B \to B') : (Y \times B' \supset Z' \xrightarrow{b'} B') \mapsto (Y \times B \supset Z = \alpha^* Z' \xrightarrow{\mathrm{pr}_1} B),$$

where by pr_1 we denote the projection onto the first factor.

By a well-known result due to A. Grothendieck, [Gro62], the functor $\mathfrak{H}_P$ can always be represented by a scheme denoted $\mathrm{Hilb}_P(Y)$ and which turns out to be a fine moduli space parameterising the subschemes of Y having Hilbert polynomial P.

1.1 Definition. *Let $n \in \mathbb{N}$ and $Y \subset \mathbb{P}^n$ be a scheme.*

(i) $\mathrm{Gr}(k, \mathbb{P}^n) := \mathrm{Hilb}_P(\mathbb{P}^n)$*, where $P(m) = \binom{m+k}{k}$, is called the Grassmannian of k-planes in $\mathbb{P}^n$. In particular,* $\mathrm{Gr}(1, \mathbb{P}^n)$ *is called the Grassmannian of lines in $\mathbb{P}^n$.*

(ii) $F(Y) := \mathrm{Hilb}_P(Y)$*, where $P(m) = m + 1$, is called the Fano scheme of lines on Y and a natural subscheme of* $\mathrm{Gr}(1, \mathbb{P}^n)$*, the Grassmannian of lines in $\mathbb{P}^n$.*

Some well-known examples of Fano schemes of lines include the following.

1.2 Examples. (i) Let $Y \subset \mathbb{P}^3$ be a smooth quadric surface. Then $F(Y)$ is isomorphic to a disjoint union of two smooth conics $Q_1, Q_2 \subset \mathbb{P}^2$. Each of these conics can be identified with the image of the second Veronese embedding $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, thus is isomorphic to $\mathbb{P}^1$. This reflects the fact that on a smooth quadric surface we find two different rulings, each of the rulings being given by a $\mathbb{P}^1$ of lines. For a detailed discussion of this example see [EH00, IV.3.2].

(ii) Let Y be the Fermat quartic in $\mathbb{P}^4$, i.e. the hypersurface

$$Y = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{P}^4.$$

Then $F(Y)$ has 40 irreducible components and each of these components is everywhere nonreduced, see [EH16, Example 6.69].

(iii) Let $X \subset \mathbb{P}^3$ be a smooth cubic surface. Then $F(X)$ consists of 27 reduced points, reflecting the well-known fact that every smooth cubic surface contains 27 lines.

We see that in general there is no reason to expect any relation between singularities of a hypersurface Y and those of its Fano scheme of lines $F(Y)$. The purpose of this chapter is to prove that if $Y = X$ is a cubic hypersurface with a unique singular point of type A_k for some k, then the singularities of $F(X)$ can be described in dependence of the singularities of X.
As a first result in this direction we state the following result due to W. Barth and A. Van de Ven. The proof given here is a more detailed version of their original proof.

1.3 Lemma ([BV78, Corollary 4])**.** *Assume that $n \geq 3$ and let $X = \{f = 0\} \subset \mathbb{P}^n$ be an irreducible cubic hypersurface. Let $l \subset X$ be a line not passing through any of the singular points of X. Then, l is a smooth point of the Fano scheme of lines on X. In particular, if X is smooth, $F(X)$ is also smooth.*

Proof. It is well-known, see for example [EH16, Theorem 6.21], that the tangent space to $F(X)$ at a point $l \in F(X)$ is

$$T_l F(X) = H^0(X, \mathcal{N}_{l/X}), \tag{1.1}$$

where $\mathcal{N}_{l/X}$ denotes the normal sheaf of l in X. As the expected dimension of $F(X)$ is $2(n-3)$, see [EH16, Proposition 6.1], we need to show that $H^0(X, \mathcal{N}_{l/X})$ has dimension $2(n-3)$ in order to prove the assertion. To do so, consider the normal sheaf sequence of l in X,

$$0 \longrightarrow \mathcal{N}_{l/X} \longrightarrow \mathcal{N}_{l/\mathbb{P}^n} \longrightarrow \mathcal{N}_{X/\mathbb{P}^n}\big|_l \longrightarrow 0. \tag{1.2}$$

A proof of exactness of this sequence is given in [EH16, Proposition 6.15]. Since l lies in the smooth locus of X, the normal sheaves are actually vector bundles, and since $l \cong \mathbb{P}^1$, they all split as a direct sum of line bundles $\mathcal{O}_l(k)$. Our exact sequence (1.2) thus can be viewed as

$$0 \longrightarrow \bigoplus_{i=1}^{n-2} \mathcal{O}_l(k_i) \longrightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_l(1) \longrightarrow \mathcal{O}_l(3) \longrightarrow 0 \tag{1.3}$$

where we used that X has degree three and that a line in $\mathbb{P}^n$ is given by $n-1$ linear equations. Since the first arrow in (1.3) is injective, $k_i \leq 1$ holds for all $i \in \{1, \dots, n-2\}$. It is simple to compute the first Chern class for the bundles in this sequence using that $c_1(A \oplus B) = c_1(A) + c_1(B)$ for vector bundles A, B and that $c_1(\mathcal{O}_{\mathbb{P}^n}(d)) = d$. Doing so gives

$$\sum_{i=1}^{n-2} k_i = n-4,$$

thus $k_i \geq -1$ for all $i \in \{1, \dots, n-2\}$. Since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0$ for $d > -2$, we can conclude that $H^1(\mathcal{N}_{l/X}) = \bigoplus_{i=1}^{n-2} H^1(\mathcal{O}_l(k_i)) = 0$. The cohomology sequence of (1.3) contains

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{N}_{l/X}) \arrow[r] \arrow[d, equal] & H^0(\mathcal{N}_{l/\mathbb{P}^n}) \arrow[r] \arrow[d, equal] & H^0(\mathcal{N}_{X/\mathbb{P}^n}\big|_l) \arrow[r] \arrow[d, equal] & H^1(\mathcal{N}_{l/X}) \arrow[d, equal] \\
& T_lF(X) & \mathbb{C}^{2(n-1)} & \mathbb{C}^4 & 0
\end{tikzcd}$$

and therefore, $\dim T_lF(X) = 2(n-1) - 4$. □

Thus, whenever X is an irreducible cubic hypersurface, $F(X)$ can be singular at most at points corresponding to lines passing through a singular point of X and it is well-known, see for example [AK77, Corollary 1.11], that the lines passing through a singular point of X indeed form the singular locus of $F(X)$. Our main interest is the Fano scheme of lines on a singular cubic hypersurface. We therefore make the following general assumption.

1.4 Assumption. *We assume that $n \geq 3$ and that $X = \{f = 0\} \subset \mathbb{P}^n$ is an irreducible cubic hypersurface in $\mathbb{P}^n$ having a unique double point $p_0 \in X$.*

We will now give a more precise and geometric description of the singular locus of $F(X)$. Take a linear change of coordinates that maps the point p_0 to the point $(1 : 0 : \cdots : 0)$. In these coordinates, as $p_0 \in X$ is a double point, the defining equation f for X can be written as

$$f(z_0 : \cdots : z_n) = z_0 f_2(z_1 : \cdots : z_n) + f_3(z_1 : \cdots : z_n), \tag{1.4}$$

where f_2 and f_3 are of degree two and three, respectively. The projection

$$\pi_0 : \mathbb{P}^n \dashrightarrow H = \{z_0 = 0\} \cong \mathbb{P}^{n-1}$$

from the point p_0 maps every point $p \in \mathbb{P}^n$ that is not the point p_0 to the intersection of the line $\langle p, p_0 \rangle$ with the hyperplane H. We may therefore think of H as parameterising lines in $\mathbb{P}^n$ passing through p_0. In fact, there is a natural morphism

$$\Phi : H \to \mathrm{Gr}(1, \mathbb{P}^n),\ p \mapsto \langle p, p_0 \rangle \tag{1.5}$$

that is an isomorphism onto its image, which is the Schubert variety of lines in $\mathbb{P}^n$ passing through p_0. The singular locus of $F(X)$ can therefore be identified with a subscheme $\Sigma \subset H$ via Φ. It is simple to check that

$$\Sigma = \{f_2 = f_3 = 0\} \subset H.$$

A detailed description of the structure of Σ is provided by the following lemma.

1.5 Lemma ([Wal99, Theorem 2.1 and Theorem 2.2]). *If all singularities of X are isolated, Σ is a complete intersection of type $(2,3)$ and*

$$\mathrm{Bl}_\Sigma H \cong \mathrm{Bl}_{p_0} X.$$

Moreover, the singularities of Σ are in bijection with the singularities of $\mathrm{Bl}_{p_0} X$ and singular points corresponding to each other under this bijection have the same singularity type.

Proof. A detailed proof can be found in [Hav16, Theorem 2.1.1 and Theorem 2.1.18]. □

Let $\Sigma_2 = \{f_2 = 0\}$ and $\Sigma_3 = \{f_3 = 0\}$. C. T. C. Wall also gives a precise description of how the singular points of $\Sigma = \Sigma_2 \cap \Sigma_3$ arise.

1.6 Lemma ([Wal99, Theorem 2.1]). *Every singular point of Σ is a singular point of Σ_2 but a smooth point of Σ_3.*

A simple observation following from Lemma 1.5 is that the singular locus Σ of $F(X)$ will itself be singular in general. Moreover, as Σ is a complete intersection, its dimension is $\dim(\Sigma) = n - 3 = \frac{1}{2}\dim(F(X))$ and $F(X)$ cannot be normal, if $n \leq 4$.
To complete our brief discussion on generalities on the Fano scheme of lines on a cubic hypersurface, we recall how local equations for it can be computed. A more detailed discussion than ours below can be found in [EH16, Section 6.1.1].
Let $l \subset X$ be a line passing through p_0. We can assume that $l = \{z_2 = \cdots = z_n = 0\}$ after a linear change of coordinates. A local neighbourhood U of l in $\mathrm{Gr}(1, \mathbb{P}^n)$ with local coordinates $x_2, \ldots, x_n, y_2, \ldots, y_n$ is then given by all lines passing through the points $(1 : 0 : x_2 : \cdots : x_n)$ and $(0 : 1 : y_2 : \cdots : y_n)$. Local equations for $F(X)$ are then obtained by taking a common parametrisation $\alpha : U \times \mathbb{P}^1 \to \mathbb{P}^n$ of the lines in U and considering the pullback of f under this parameterisation,

$$\begin{aligned}
&(\alpha^* f)((\lambda : \mu), (x_2, \ldots, x_n, y_2, \ldots, y_n)) \\
&\qquad = f(\lambda : \mu : \lambda x_2 + \mu y_2 : \cdots : \lambda x_n + \mu y_n) \\
&\qquad = F_{3,0}(x_2, \ldots, x_n, y_2, \ldots, y_n)\lambda^3 + F_{2,1}(x_2, \ldots, x_n, y_2, \ldots, y_n)\lambda^2\mu \\
&\qquad\quad + F_{1,2}(x_2, \ldots, x_n, y_2, \ldots, y_n)\lambda\mu^2 + F_{0,3}(x_2, \ldots, x_n, y_2, \ldots, y_n)\mu^3,
\end{aligned} \tag{1.6}$$

where $(\lambda : \mu)$ denote homogenous coordinates on the $\mathbb{P}^1$. Here, the polynomials $F_{3,0}, F_{2,1}, F_{1,2}, F_{0,3}$ are obtained by arranging the terms in (1.6) by their respective powers of λ and μ. A point $(x_2, \ldots, x_n, y_2, \ldots, y_n) \in U$ is a point of $F(X)$ if and only if every point of the respective line lies in X, that is, if (1.6) vanishes regardless of the choice of λ and μ. This vanishing for all λ, μ is equivalent to

$$\begin{aligned}
F_{3,0}(x_2, \ldots, x_n, y_2, \ldots, y_n) &= F_{2,1}(x_2, \ldots, x_n, y_2, \ldots, y_n) \\
= F_{1,2}(x_2, \ldots, x_n, y_2, \ldots, y_n) &= F_{0,3}(x_2, \ldots, x_n, y_2, \ldots, y_n) = 0,
\end{aligned} \tag{1.7}$$

which therefore are local equations for $F(X)$ in $U = U(l) \subset \mathrm{Gr}(1, \mathbb{P}^n)$. We can also conclude that the line corresponding to a point $(x_2, \ldots, x_n, y_2, \ldots, y_n) \in U$ passes through the point p_0, if $x_2 = \cdots = x_n = 0$, and consequently, that the image under Φ of the hyperplane H parameterising the lines through p_0 is, in these local coordinates, given by all points in U where $x_2 = \cdots = x_n = 0$. This shows that $\Sigma = F(X) \cap \{x_2 = \cdots = x_n = 0\}$. Note that Σ is a complete intersection by Lemma 1.5 and therefore cannot be a component of $F(X)$ of dimension greater than $2(n-3)$. In fact, as the expected dimension of $F(X)$ is $2(n-3)$ and the dimension of $\mathrm{Gr}(1, \mathbb{P}^n)$ is $2(n-1)$, the Fano scheme of lines on X is locally a complete intersection in $\mathrm{Gr}(1, \mathbb{P}^n)$. A. B. Altman and S. L. Kleiman [AK77] also derived this fact using different methods. It also follows from their work that $F(X)$ is reduced if X contains no triple point.

1.2 Recognising singularity types of singular points on hypersurfaces from equations

Usually, when one wants to detect the singularity type of an isolated hypersurface singularity from the defining equation, one computes the corank of the singularity, that is, the corank of the Hessian of the defining equation at the singular point, and then further invariants of the singularity such as the Milnor number. Respective algorithms have been implemented for example in SINGULAR [SIN], a computer algebra system for polynomial computations.
A similar problem that usually is not posed in the literature is: *given any polynomial P defining a hypersurface $Y = \{P = 0\} \subset \mathbb{C}^n$ (for some n) with isolated singular point $0 \in \mathbb{C}^n$, what are the conditions on the defining equation P that are equivalent to $(Y, 0)$ being of a certain type?*

In this section we show how to address this problem and how explicit conditions on the defining equation P of the isolated hypersurface singularity $(Y, 0)$ that imply, and in fact are equivalent to, that $(Y, 0)$ is of a certain type, can be obtained.
Our approach is based on results of V. Arnol'd [Arn73, Arn74], J. Bruce and C. T. C. Wall [BW79] together with some known facts from singularity theory. Although none of the results is new, combining them to obtain an explicit algorithm is an approach that, to our best knowledge, can be found nowhere in the literature though it might be known to experts.

1.2.1 Recognition Principle

The first result we are going to use is the so-called Recognition Principle. In order to formulate it, we need a series of definitions.

1.7 Definition (Semiquasihomogenity, [Arn74, Definition 2.1–2.5])**.** *Let $d \in \mathbb{N}$ and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$.*

(i) A monomial $m(z_1, \dots, z_n) = z_1^{k_1} \cdots z_n^{k_n} \in \mathbb{C}[z_1, \dots, z_n]$ is said to be of α-degree d, if with $\underline{k} = (k_1, \dots, k_n)$ we have that $\langle \alpha, \underline{k} \rangle = d$.

(ii) A polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ is called quasihomogenous of degree d with respect to the weight α, if every monomial in P has α-degree d. In other words, P is quasihomogenous of degree d with respect to the weight α if and only if $P(\lambda^{\alpha_1} z_1, \dots, \lambda^{\alpha_n} z_n) = \lambda^d P(z_1, \dots, z_n)$ holds for every $\lambda \in \mathbb{C}$.

(iii) A polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ is called semiquasihomogenous of degree d with respect to the weight α if it can be written as $P = P_0 + P_1$ such that

(a) P_0 is quasihomogenous of degree d with respect to the weight α;

(b) P_0 has at most an isolated singularity at the origin;

(c) every monomial in P_1 has α-degree strictly greater than d.

1.8 Example. The normal forms of the defining equations of hypersurface singularities of ADE-type are quasihomogenous of degree one with respect to the weights listed in Table 1.1. We call these weights the ADE-weights.

1.9 Lemma (Recognition Principle, [BW79, Lemma 1 and Corollary below])**.** *Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial.*

(i) If P is semiquasihomogenous of degree one with respect to an ADE-weight $\alpha(T)$, then its quasihomogenous part P_0 can be brought into the respective normal form of a singularity of type T after an analytic change of coordinates.

(ii) If P is as in (i), the hypersurface $Y = \{P = 0\}$ has an isolated singularity of type T at the origin.

Type T	Normal form	ADE-weight $\alpha(T)$
A_k, $k \geq 1$	$z_1^{k+1} + \sum_{i=2}^{n} z_i^2$	$\left(\frac{1}{k+1}, \frac{1}{2}, \dots, \frac{1}{2}\right)$
D_k, $k \geq 4$	$z_1^{k-1} + z_1 z_2^2 + \sum_{i=3}^{n} z_i^2$	$\left(\frac{1}{k-1}, \frac{k-2}{2(k-1)}, \frac{1}{2}, \dots, \frac{1}{2}\right)$
E_6	$z_1^3 + z_2^4 + \sum_{i=3}^{n} z_i^2$	$\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \dots, \frac{1}{2}\right)$
E_7	$z_1^3 + z_1 z_2^3 + \sum_{i=3}^{n} z_i^2$	$\left(\frac{1}{3}, \frac{2}{9}, \frac{1}{2}, \dots, \frac{1}{2}\right)$
E_8	$z_1^3 + z_2^5 + \sum_{i=3}^{n} z_i^2$	$\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{2}, \dots, \frac{1}{2}\right)$

Table 1.1: Normal forms of defining equations of hypersurface singularities of ADE-type and weights for quasihomogenity of degree one with respect to these weights.

1.10 Remark. Semiquasihomogenity is not preserved under coordinate changes, not even under linear ones, cf. [GLS07, page 124]. Therefore, before the Recognition Principle can be applied, one usually has to change coordinates such that a given polynomial P becomes semiquasihomogenous in the new coordinates.

Although we gave the Recognition Principle only for hypersurface singularities of ADE-type, it can be formulated for arbitrary isolated hypersurface singularities. This follows from the respective results in [Arn74], respectively from [GLS07, Corollary 2.18], but we will not need this more general result.

1.11 Example. Let X be the cubic surface given by the vanishing of

$$f(z_0 : z_1 : z_2 : z_3) = z_0 z_1 z_2 + z_1(a_3 z_1 z_3 + c_1 z_3^2) + z_2(b_2 z_2^2 + b_3 z_2 z_3 + c_2 z_3^2) + c_3 z_3^3.$$

X is singular at the point $p_0 = (1 : 0 : 0 : 0)$. Take the affine chart $U_0 = \{z_0 \neq 0\}$. It is clear that the singularity at the origin defined by $f = 0$ inside U_0 cannot be of type A_1 as the corank of the singularity is one. By using the weight $\alpha(A_2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\right)$ on $f(1, z_1, z_2, z_3)$, we have

terms of $\alpha(A_2)$-degree $= 1$	$z_1 z_2 + c_3 z_3^3$
terms of $\alpha(A_2)$-degree > 1	$z_1(a_3 z_1 z_3 + c_1 z_3^2) + z_2(b_2 z_2^2 + b_3 z_2 z_3 + c_2 z_3^2)$

and no terms of $\alpha(A_2)$-degree < 1. By the Recognition Principle, Lemma 1.9, the singularity is of type A_2, if $c_3 \neq 0$.

As we mentioned in Remark 1.10, before using the Recognition Principle, one may have to change coordinates first in order to obtain semiquasihomogenity. One way of finding such coordinates is provided by the Generalised Morse Lemma. We recall its statement and algorithmic proof for completeness.

1.12 Lemma (Generalised Morse Lemma, [GLS07, I, Theorem 2.47])**.** *If $P \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \dots, x_N\}$ defines an isolated hypersurface singularity of rank r and multiplicity two at the origin, then there exists a formal coordinate change φ such that*

$$(\varphi^* P)(x_1, \dots, x_N) = x_1^2 + \dots + x_r^2 + Q(x_{r+1}, \dots, x_N),$$

with $Q \in \mathfrak{m}^3$.

Proof. Since the hypersurface singularity defined by P has rank r, the quadratic part of P can be transformed to $x_1^2+\cdots+x_r^2$ by an analytic coordinate change, cf. the proof of [GLS07, Theorem 2.46]. We may therefore assume that P can be written as

$$P(x_1,\ldots,x_N)=x_1^2+\cdots+x_r^2+P_3(x_{r+1},\ldots,x_N)+\sum_{i=1}^{r}x_iQ_i(x_1,\ldots,x_N)$$

with $P_3\in\mathfrak{m}^3$ and $Q_i\in\mathfrak{m}^2$. Then the coordinate change $x_i\mapsto x_i-\frac{1}{2}Q_i$ for $i=1,\ldots,r$ and $x_i\mapsto x_i$ for $i=r+1,\ldots,N$ yields

$$P(x)=x_1^2+\cdots+x_r^2+P_3(x_{r+1},\ldots,x_N)+P_4(x_{r+1},\ldots,x_N)+\sum_{i=1}^{r}x_iR_i(x_1,\ldots,x_N),$$

with $P_4\in\mathfrak{m}^4$ and $R_i\in\mathfrak{m}^3$. By iterating this procedure, the assertion follows. □

1.13 Remark. If the quadratic part is given by $x_1x_2+x_4^2+\cdots+x_r^2$, we can either change coordinates to turn it into a sum of squares, or use in the proof the coordinate changes $x_1\mapsto x_1-Q_2$ and $x_2\mapsto x_2-Q_1$ for the first two coordinates instead of the ones above. This has the advantage that we can avoid dealing with complex coordinate changes when performing this algorithm on a computer.

1.14 Remark. The coordinate changes used in the proof of Lemma 1.12 are not uniquely determined. In fact, there are already several choices for normalising the quadratic part of the equation. However, after normalising the quadratic part, we are left with a polynomial of the form

$$P(x_1,\ldots,x_N)=x_1^2+\cdots+x_r^2+P_3(x_{r+1},\ldots,x_N)+\sum_{i=1}^{r}x_iQ_i(x_1,\ldots,x_N)$$

and by choosing each Q_i with as many terms as possible from $P-\sum_{j<i}x_jQ_j$, the coordinate changes $x_i\mapsto x_i-\frac{1}{2}Q_i$ used in the proof become uniquely determined. We call these coordinate changes the standard coordinate changes for the Generalised Morse Lemma with respect to the initial coordinate change used to normalise the quadratic part.

The following combination of the Generalised Morse Lemma and the Recognition Principle yields a powerful tool for the determination of which singularity types on a hypersurface appear in dependence on the coefficients of the defining equation.

1.15 Theorem. *Let $Y\subset\mathbb{C}^N$ be a hypersurface defined by a polynomial $P\in\mathbb{C}[x_1,\ldots,x_N]$ and assume that the origin is an isolated singular point of Y of corank one. Then, there are polynomials $C_1,\ldots,C_{k+1}$ in the coefficients of P and depending on the choice of an analytic coordinate change such that the conditions*

$$C_1=\cdots=C_k=0,\ C_{k+1}\neq 0$$

on the coefficients of P are equivalent to $(Y,0)$ being of type A_k. Moreover, each C_i is homogenous of degree $i-2$ and fixing the analytic coordinate change they depend on, there is an explicit algorithm computing them.

Proof. Let $k\in\mathbb{N}$. Using the coordinate changes from the proof of Lemma 1.12, P can be brought to the form

$$P(x)=x_1^2+\cdots+x_{N-1}^2+P_3(x_N)+\cdots+P_k(x_N)+P_{k+1}(x_N)+\sum_{i=1}^{N-1}x_iQ_i(x_1,\ldots,x_N)\quad(1.8)$$

with $P_i \in \mathfrak{m}^i$ and $Q_i \in \mathfrak{m}^k$. By taking the weight $\alpha(A_k) = \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{k+1}\right)$, we have

terms of $\alpha(A_k)$-degree < 1	$P_3(x_N) + \cdots + P_k(x_N)$
terms of $\alpha(A_k)$-degree $= 1$	$x_1^2 + \cdots + x_{N-1}^2 + P_{k+1}(x_N)$
terms of $\alpha(A_k)$-degree > 1	$\sum\limits_{i=1}^{N-1} x_i Q_i(x_1, \dots, x_N)$

and writing $P_i(x_N) = C_i x_N^i$ we can conclude by the Recognition Principle, Lemma 1.9, that P defines a singularity of type A_k, if $C_3 = \cdots = C_k = 0$ and $C_{k+1} \neq 0$. The C_i are polynomials in the coefficients of P and depend on the choice of coordinate changes from the proof of Lemma 1.12 used to bring P to the form (1.8). As these coordinate changes are given in explicit terms, the form (1.8) and therefore also the C_i can be computed explicitly for each choice of such coordinates changes. By construction, each C_i is homogenous of degree $i - 2$.
On the other hand, if $X = \{P = 0\}$ has a singularity of type A_k at the origin, P is determined by its $(k+1)$-jet by [GLS07, Corollary 2.24]. That is, performing the coordinate changes from the proof of the Generalised Morse Lemma, Lemma 1.12, we can restrict ourselves to the $(k+1)$-jet, that is, ignore the terms $x_i Q_i$ in the expression above. Now if one of $P_3, \dots, P_k$ would be nonzero, the Milnor number

$$\mu(P) = \dim_{\mathbb{C}} \left(\mathbb{C}[x_1, \dots, x_N]/j(P)\right), \quad j(P) = \left\langle \frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_N} \right\rangle$$

would be less than k contradicting the assumption that the singularity is of type A_k. Likewise, if $P_{k+1} = 0$, the Milnor number is $\mu(P) = \infty$, yielding a contradiction. □

1.16 Definition (Coefficient conditions)**.** *For a polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$ and $k \in \mathbb{N}$ we call the conditions*

$$C_3 = \cdots = C_k = 0, \; C_{k+1} \neq 0$$

from Theorem 1.15 obtained by using the standard coordinate changes for the proof of the Generalised Morse Lemma, cf. Remark 1.14, the coefficient conditions for a singularity of type A_k. They are dependent on the choice of a coordinate change used to normalise the quadratic part of P.

Besides the general situation treated here, we will usually be concerned with equations defining hypersurface singularities whose quadratic part has been normalised already. In such a situation, the coefficient conditions for a singularity of type A_k depend only on P as we defined them to be the conditions from Theorem 1.15 obtained with respect to the standard coordinate changes for the proof of the Generalised Morse Lemma as specified in Remark 1.14, thus fixed a choice of coordinate changes.

1.2.2 Application to cubic hypersurfaces

We will use Theorem 1.15 to compute the conditions on the coefficients of the defining equation f for X that are equivalent to X having a singularity of type A_k at the point p_0. As we are taking a similar approach for determining the singularities of $F(X)$ later, we first bring the defining equation f for X into a specific normal form. This normal form is made in a way that we not only change the coordinates of the point p_0 to $(1 : 0 : \cdots : 0)$ but also make a line l passing through p_0 to be given by $z_2 = \cdots = z_n = 0$. This has the advantage that we can immediately obtain a local normal form for $F(X)$ around l, i.e. we obtain normal forms for both X and $F(X)$ at once.

Let thus (p_0, l) be a pair consisting of the point $p_0 \in \mathbb{P}^n$ and a line $l \subset \mathbb{P}^n$ passing through p_0. As we aim to find the conditions for a singularity of type A_k, we assume that the corank of $\Sigma_2 = \{f_2 = 0\} \subset H \cong \mathbb{P}^{n-1}$ is either zero (for $k = 1$) or one (for $k \geq 2$). After changing coordinates, we may assume that $p_0 = (1 : 0 : \cdots : 0)$ and $l = \{z_2 = \cdots = z_n = 0\}$. Recall from (1.4) that f can be written as

$$f(z_0 : \cdots : z_n) = z_0 f_2(z_1 : \cdots : z_n) + f_3(z_1 : \cdots : z_n),$$

where f_2, f_3 are homogenous of degree $2, 3$, respectively. The condition that $l \subset X$ implies that there are no terms of the form $z_0 z_1^2$ or z_1^3 in f. We can sort the terms in f_2, f_3 by their powers of z_1, i.e. write

$$\begin{aligned} f_2(z_1 : \cdots : z_n) =& z_1 L_2(z_2 : \cdots : z_n) + Q_2(z_2 : \cdots : z_n), \\ f_3(z_1 : \cdots : z_n) =& z_1^2 L_3(z_2 : \cdots : z_n) + z_1 Q_3(z_2 : \cdots : z_n) + C_3(z_2 : \cdots : z_n), \end{aligned}$$

with L_2, L_3 linear, Q_2, Q_3 quadratic, and C_3 cubic polynomials. If (X, p_0) is of type A_1, the corank of f_2 equals zero and thus $L_2 \neq 0$. We may bring f_2 to the normal form of a non-degenerate quadric even by using linear coordinate changes. This is a special case of Lemma 1.12 and it is a well-known result from linear algebra that for a non-degenerate quadric one is able to do this with linear coordinate changes only, cf. the theory of quadratic forms over $\mathbb{C}$. Consequently, in appropriate coordinates,

$$f_2(z_1 : \cdots : z_n) = z_1 z_2 + z_3^2 + \cdots + z_n^2. \tag{A_1}$$

If the corank of the singularity equals one, there are two cases: In the first case, analogous to the previous case, we have that $L_2 \neq 0$. There are again coordinate changes that bring f_2 to the normal form of the defining equation of a quadric of corank one. That is, after applying a coordinate change, f_2 takes the same form as before but with one z_i^2 missing, where $i \in \{3, \ldots, n\}$. We may assume that $i = 3$ and therefore, that f_2 takes the form

$$f_2(z_1 : \cdots : z_n) = z_1 z_2 + z_4^2 + \cdots + z_n^2. \tag{$A_{k\geq 2},\ L_2 \neq 0$}$$

In the second case, $L_2 = 0$. Since f_2 does then not depend on z_1 and since the corank of f_2 has to be one, Q_2 defines a non-degenerate quadric and takes the form $Q_2(z_2 : \cdots : z_n) = z_2 z_3 + z_4^2 + \cdots + z_n^2$ after a change of coordinates. Thus, in this case,

$$f_2(z_1 : \cdots : z_n) = z_2 z_3 + z_4^2 + \cdots + z_n^2. \tag{$A_{k\geq 2},\ L_2 = 0$}$$

If the corank of Σ_2 is zero, Σ_2 is a smooth quadric and also the intersection $\Sigma = \Sigma_2 \cap \Sigma_3$ is smooth, cf. Lemma 1.6. On the other hand, if the corank of Σ_2 is one, Σ_2 has a singular point $q_0 \in \Sigma_2$. The intersection $\Sigma = \Sigma_2 \cap \Sigma_3$ is also singular at q_0, provided that $q_0 \in \Sigma$. According to Lemma 1.5 this is the case when $k \geq 3$. Therefore, there are in general two types of lines in the singular locus of $F(X)$: lines corresponding to a smooth point of Σ and lines corresponding to a singular point of Σ. Via the morphism Φ, see (1.5), if l_0 denotes the singular point of the singular locus of $F(X)$, it is identified with the line $l_0 = \langle p_0, q_0 \rangle$, i.e. the line passing through the singular points $q_0 \in \Sigma$ and $p_0 \in X$.
From the local normal forms for f_2 we obtained above, it follows easily that the case $(A_{k\geq 2},\ L_2 = 0)$ corresponds to $l = l_0$, that is, the case where the line we started with is the line connecting p_0 and q_0; and that the case $(A_{k\geq 2},\ L_2 \neq 0)$ corresponds to $l \neq l_0$, i.e. to a regular point of the singular locus Σ of $F(X)$.

One particular consequence of this discussion is that the local structure of $F(X)$ around every line $l \in \Sigma \backslash \{l_0\}$ is necessarily the same. Or, stated differently, that for any two lines $l, l' \in \Sigma \backslash \{l_0\}$ the germs $(F(X), l)$ and $(F(X), l')$ are isomorphic.

1.17 Theorem ([Eph78, Theorem 0.22]). *Let Z be any analytic space, $p \in Z$ be a point and denote by*

$$\mathrm{Isosing}(Z, p) = \{q \in Z \mid (Z, p) \cong (Z, q)\}$$

the isosingular locus of p. *Then, for every* $q \in \mathrm{Isosing}(Z,p)$ *there exists a neighbourhood* $U = U(q) \subset Z$ *of* q *and an analytic space* Y *such that*

$$Z \cap U \cong Y \times (U \cap \mathrm{Isosing}(Z,p)).$$

1.18 Definition (Transversal singularity)**.** *We say that a scheme* Z *has a transversal singularity along a locally closed scheme* $S \subset Z$ *if for every point* $s \in S$ *there exists a neighbourhood* $U = U(s) \subset Z$ *such that* $Z \cap U \cong Y \times (U \cap S)$, *where* Y *is an analytic space not depending on* s. *If* Y *has a unique singular point* y, *the transversal singularity type of* Z *along* S *is defined to be the type of* (Y,y).

From what we have shown, for every line $l \in \Sigma_{\mathrm{reg}} = \Sigma \setminus \{l_0\}$, the isosingular locus is $\mathrm{Isosing}(F(X),l) = \Sigma_{\mathrm{reg}}$ and by Theorem 1.17 we find a small neighbourhood $U = U(l)$ of l in $F(X)$ sucht hat $F(X) \cap U \cong Y \times (\Sigma_{\mathrm{reg}} \cap U)$. We will compute the transversal singularity type for $F(X)$ along Σ_{reg} in the next section and also show that it is the same as those of (X,p_0) in Theorem 1.29. As long as we are interested in computing the transversal singularity type of $F(X)$ along Σ_{reg} only, we may assume that we are in case ($A_{k\geq 2}$, $L_2 \neq 0$).
In this case, consider the polynomial f_3 and denote the coefficients of the terms $z_1 z_2 z_i$ by α_i for $i \in \{1,\dots,n\}$. The change of coordinates $z_0 \mapsto z_0 - \sum\limits_{i=1}^{n} \alpha_i z_i$ eliminates all of these terms from f, and f_3 then takes the general form

$$\begin{aligned} f_3(z_1 : \dots : z_n) = z_1 & \left(a_3 z_1 z_3 + z_1 \sum_{i=4}^{n} a_i z_i + z_3 \sum_{i=4}^{n} e_i z_i + \sum_{4 \leq i \leq j \leq n} A_{i,j} z_i z_j + c_1 z_3^2 \right) \\ & + z_2 \left(b_2 z_2^2 + z_2 \sum_{i=3}^{n} b_i z_i + z_3 \sum_{i=4}^{n} f_i z_i + \sum_{4 \leq i \leq j \leq n} B_{i,j} z_i z_j + c_2 z_3^2 \right) \\ & + c_3 z_3^3 + z_3^2 \sum_{i=4}^{n} c_i z_i + z_3 \sum_{4 \leq i \leq j \leq n} C_{i,j} z_i z_j + \sum_{4 \leq i \leq j \leq k \leq n} p_{i,j,k} z_i z_j z_k, \end{aligned} \tag{1.9}$$

with $a_i, b_i, c_i, e_i, f_i, A_{i,j}, B_{i,j} \in \mathbb{C}$ for all indices i,j appearing in (1.9). We can now apply the algorithm from Theorem 1.15 to compute the coefficient conditions for a singularity of type A_k.

1.19 Lemma. *With the notation as above,* (X,p_0) *is of type* A_2 *if and only if* $c_3 \neq 0$*; and of type* A_3 *if and only if* $c_3 = 0$ *and* $4c_1c_2 + c_4^2 + \dots c_n^2 \neq 0$.

Proof. The first assertion is a simple consequence of the Recognition Principle as there are no terms of $\alpha(A_2)$-degree strictly less than one in (1.9). For the second assertion, recall that we need to find the fourth powers of z_3 that occur after applying the algorithm from the proof of the Generalised Morse Lemma, Lemma 1.12, once. We may cover the ambient $\mathbb{P}^n$ with the standard affine charts $\{z_i \neq 0\}$ for $i \in \{0 \dots, n\}$. Then the only one out of these affine charts containing the point p_0 is the chart $\{z_0 \neq 0\}$. We therefore perform our calculations in this chart and write

$$f(1, z_1, \dots, z_n) = z_1 z_2 + z_4^2 + \dots + z_n^2 + g_3(z_3) + \sum_{\substack{i=1 \\ i \neq 3}}^{n} z_i g_i(z_1, \dots, z_n).$$

For $i \geq 4$, the coordinate change $z_i \mapsto z_i - \frac{1}{2} g_i$ produces a term $\frac{1}{4} c_i^2 z_3^4$ from the quadratic part of the equation and a term $\frac{1}{2} c_i^2 z_i^4$ from the cubic part of the equation. We therefore obtain a summand $\frac{1}{4} c_i^2 z_3^4$ for every $i \geq 4$. For $i = 1, 2$, the only fourth power of z_3^4 that arises comes from the product $(z_1 - g_2)(z_2 - g_1)$ and yields a term $c_1 c_2 z_3^4$. Summing up, the conditions to have a singularity of type A_3 are

$$c_1 c_2 + \frac{1}{4}(c_4^2 + \cdots + c_n^2) \neq 0,\ c_3 = 0,$$

as we asserted. □

1.20 Example (cf. [BW79])**.** If $X \subset \mathbb{P}^3$ is a cubic surface with defining equation f in the above normal form

$$f(z_0 : z_1 : z_2 : z_3) = z_0 z_1 z_2 + z_1 \left(a_3 z_1 z_3 + c_1 z_3^2\right) + z_2 \left(b_2 z_2^2 + b_3 z_2 z_3 + c_2 z_3^2\right) + c_3 z_3^3,$$

the method of Theorem 1.15 brings f in the chart $z_0 \neq 0$ to the form

$$\begin{aligned} z_1 z_2 + c_3 \cdot z_3^3 \\ -c_1 c_2 \cdot z_3^4 \\ +(a_3 c_2^2 + b_3 c_1^2) \cdot z_3^5 \\ -(4 a_3 b_3 c_1 c_2 + b_2 c_1^3) \cdot z_3^6 \\ +(4 a_3^2 b_3 c_2^2 + 6 a_3 b_2 c_1^2 c_2 + 4 a_3 b_3^2 c_1^2) \cdot z_3^7 \\ -(12 a_3^2 b_2 c_1 c_2^2 + 16 a_3^2 b_3^2 c_1 c_2 + 12 a_3 b_2 b_3 c_1^3) \cdot z_3^8 \\ \vdots \end{aligned}$$

A necessary condition for the singularity of X at p_0 to be isolated is that there exists an index $i \neq 3$ such that $c_i \neq 0$. This follows easily by computing the Jacobian of (1.9). Thus, the coefficients c_1 and c_2 cannot vanish simultaneously and we can deduce the respective coefficient conditions listed in Table 1.2. These conditions have also been obtained in [BW79] but with a

Type of (X, p_0)	Conditions on the coefficients
A_2	$c_3 \neq 0$
A_3	$c_3 = 0$ $c_1 c_2 \neq 0$
A_4	$c_1 = c_3 = 0$ $a_3, c_2 \neq 0$ **or** $c_2 = c_3 = 0$ $b_3, c_1 \neq 0$
A_5	$c_2 = c_3 = b_3 = 0$ $b_2, c_1 \neq 0$

Table 1.2: Singularities of cubic surfaces of type A_k for $k \geq 2$ and corresponding conditions on the coefficients of the defining equation.

less systematic approach. If one tries to find the conditions for a singularity of type A_6, one obtains that the coefficient b_2 needs to vanish. But this already implies that the coefficient of z_3^p vanishes for all p and provides a different proof of the fact that there exist no cubic surfaces with an isolated singular point of type A_k with Milnor number k greater than or equal to six.

It is in general hard to compute the coefficient conditions by hand. But since we have an explicit algorithm, the problem of computing the coefficient conditions can be handed to a computer.

1.3 Recognising transversal singularity types of the Fano scheme of lines on a cubic hypersurface

In this section we will explain how to find the transversal singularity type of $F(X)$ along Σ_{reg}. We begin by computing local equations for $F(X)$ in a neighbourhood of the fixed line $l = \{z_2 = \cdots = z_n = 0\}$ using ($A_{k\geq 2}$, $L_2 \neq 0$) and (1.9) which are stating that the defining equation f for X can be brought to the form

$$\begin{aligned} f(z_0 : \cdots : z_n) = {} & z_0(z_1 z_2 + z_4^2 + \cdots + z_n^2) \\ & + z_1 \left(a_3 z_1 z_3 + z_1 \sum_{i=4}^{n} a_i z_i + z_3 \sum_{i=4}^{n} e_i z_i + \sum_{4\leq i\leq j\leq n} A_{i,j} z_i z_j \right) \\ & + z_2 \left(b_2 z_2^2 + b_3 z_2 z_3 + z_2 \sum_{i=4}^{n} b_i z_i + z_3 \sum_{i=4}^{n} f_i z_i + \sum_{4\leq i\leq j\leq n} B_{i,j} z_i z_j \right) \\ & + c_3 z_3^3 + z_3^2 \sum_{i=4}^{n} c_i z_i + z_3 \sum_{4\leq i\leq j\leq n} C_{i,j} z_i z_j + \sum_{4\leq i\leq j\leq k\leq n} p_{i,j,k} z_i z_j z_k. \end{aligned} \tag{1.10}$$

If $\alpha : U \times \mathbb{P}^1 \to \mathbb{P}^n$ is a common parameterisation of the lines in U and $x_2, \dots, x_n, y_2, \dots, y_n$ denote local coordinates on U,

$$\begin{aligned} (\alpha^* f)(\lambda : \mu) = {} & \lambda(\mu(\lambda x_2 + \mu y_2) + (\lambda x_4 + \mu y_4)^2 + \cdots + (\lambda x_n + \mu y_n)^2) \\ & + \mu \Bigg(a_3 \mu(\lambda x_3 + \mu y_3) + \mu \sum_{i=4}^{n} a_i(\lambda x_i + \mu y_i) + (\lambda x_3 + \mu y_3) \sum_{i=4}^{n} e_i(\lambda x_i + \mu y_i) \\ & \quad + \sum_{4\leq i\leq j\leq n} A_{i,j}(\lambda x_i + \mu y_i)(\lambda x_j + \mu y_j) \Bigg) \\ & + (\lambda x_2 + \mu y_2) \Bigg(b_2(\lambda x_2 + \mu y_2)^2 + b_3(\lambda x_2 + \mu y_2)(\lambda x_3 + \mu y_3) \\ & \quad + (\lambda x_2 + \mu y_2) \sum_{i=4}^{n} b_i(\lambda x_i + \mu y_i) \\ & \quad + (\lambda x_3 + \mu y_3) \sum_{i=4}^{n} f_i(\lambda x_i + \mu y_i) \\ & \quad + \sum_{4\leq i\leq j\leq n} B_{i,j}(\lambda x_i + \mu y_i)(\lambda x_j + \mu y_j) \Bigg) \\ & + c_3(\lambda x_3 + \mu y_3)^3 + (\lambda x_3 + \mu y_3)^2 \sum_{i=4}^{n} c_i(\lambda x_i + \mu y_i) \\ & + (\lambda x_3 + \mu y_3) \sum_{4\leq i\leq j\leq n} C_{i,j}(\lambda x_i + \mu y_i)(\lambda x_j + \mu y_j) \\ & + \sum_{4\leq i\leq j\leq k\leq n} p_{i,j,k}(\lambda x_i + \mu y_i)(\lambda x_j + \mu y_j)(\lambda x_k + \mu y_k) \end{aligned}$$

$$= F_{3,0}(x_2,\dots,x_n,y_2,\dots,y_n)\lambda^3 + F_{2,1}(x_2,\dots,x_n,y_2,\dots,y_n)\lambda^2\mu$$
$$+ F_{1,2}(x_2,\dots,x_n,y_2,\dots,y_n)\lambda\mu^2 + F_{0,3}(x_2,\dots,x_n,y_2,\dots,y_n)\mu^3,$$

with

$$\begin{aligned} F_{3,0}(x_2,\dots,x_n,y_2,\dots,y_n) &= x_4^2+\dots+x_n^2+\mathfrak{h}_1, && \mathfrak{h}_1\in\mathfrak{m}^3,\\ F_{2,1}(x_2,\dots,x_n,y_2,\dots,y_n) &= x_2+\mathfrak{h}_2, && \mathfrak{h}_2\in\mathfrak{m}^2,\\ F_{1,2}(x_2,\dots,x_n,y_2,\dots,y_n) &= y_2+\mathfrak{h}_3, && \mathfrak{h}_3\in\mathfrak{m}^2,\\ F_{0,3}(x_2,\dots,x_n,y_2,\dots,y_n) &= a_3y_3+\dots+a_ny_n+\mathfrak{h}_4, && \mathfrak{h}_4\in\mathfrak{m}^2. \end{aligned} \tag{1.11}$$

By analysing the equations above, $\mathfrak{h}_1$ depends on the coordinates $x_2,\dots,x_n$ only, $\mathfrak{h}_4$ on the coordinates $y_2,\dots,y_n$, and $\mathfrak{h}_2(0,\dots,0,y_2,\dots,y_n)=0$. As we know that the singular locus of $F(X)\cap U$ is $F(X)\cap U\cap\Phi(H)=F(X)\cap U\cap\{x_2=\dots=x_n=0\}$, where Φ is the morphism defined in (1.5), it follows that

$$\Sigma\cap U\cong\{(y_2,\dots,y_n)\mid F_{1,2}(0,\dots,0,y_2,\dots y_n)=F_{0,3}(0,\dots,0,y_2,\dots y_n)=0\}.$$

We now deduce how to compute coefficient conditions for the transversal singularity type of $F(X)$ along Σ_{reg}.

1.21 Definition (cf. [FG02, page 109])**.** *Denote by $\mathbb{C}\{x_1,\dots,x_N\}$ the ring of convergent power series in N variables. Then, $P\in\mathbb{C}\{x_1,\dots,x_N\}$ is called x_1-regular of order a if there exists $\tilde{P}_0\in\mathbb{C}\{x_1\}$ such that $P(x_1,0\dots,0)=x_1^a\tilde{P}_0(x_1)$ and $\tilde{P}_0(0)\neq 0$.*

If $P\in\mathbb{C}\{x_1,\dots,x_N\}$, we denote by $\operatorname{jet}^p(P)$ the p-jet of P, that is, the sum of all terms of P of degree at most p.

1.22 Lemma. *Let $P\in\mathbb{C}\{x_1,\dots,x_N\}$, $P\neq 0$, x_1-regular of order* 1*. Then, for any natural number $p\geq 1$, there is an analytic coordinate change φ, such that* $\operatorname{jet}^p(\varphi^*P)=x_1$.

Proof. We may assume that the coefficient of x_1 is normalised to one and thus write

$$P(x_1,\dots,x_n)=x_1+x_1Q_1(x_2,\dots,x_N)+x_1^2Q_2(x_2,\dots,x_N)+\dots.$$

The case $p=1$ is trivial. Denote by $O_n(f)$ the polynomial consisting of all monomials of order n in P. For $p\geq 1$, the change of coordinates $x_1\mapsto x_1-O_2(P)$ eliminates all quadratic terms in P and produces additional terms of order at least three. Applying analogous coordinate changes successively yields the desired k-jet. ☐

Since this is possible for every p, we obtain that $\varphi^*f=x_1$ in the limit, i.e. after applying these coordinate changes successively. This possibly infinite succession of analytic coordinate changes results in a formal coordinate change and we thus have the following corollary.

1.23 Corollary. *Let $f\in\mathbb{C}\{x_1,\dots,x_N\}$, $f\neq 0$, x_1-regular of order* 1*. Then there exists a formal coordinate change φ such that $\varphi^*f=x_1$.*

If in equation (1.11) there is some $a_i\neq 0$, $F_{0,3}$ is y_3-regular of order 1 after possibly renaming the coordinates and there exists a formal coordinate change φ_1 such that $\varphi_1^*F_{0,3}=y_3$. If $a_3=\dots=a_n=0$, the polynomial $F_{0,3}$ has no linear term. Let φ_1 denote a formal coordinate change as in Corollary 1.23 and such that $\varphi_1^*F_{1,2}=y_2$. It is then simple to check from the local equations that

$$\varphi_1^*F_{0,3}=y_3^2\sum_{i=3}^n c_iy_i+y_3\sum_{i=4}^n e_iy_i+y_2\cdot P(y_2,\dots,y_n)$$

for some polynomial P. Using $\varphi^* F_{1,2} = y_2 = 0$ on this equation gives

$$\varphi_1^* F_{0,3} = y_3^2 \sum_{i=3}^{n} c_i y_i + y_3 \sum_{i=4}^{n} e_i y_i.$$

Thus, either $y_3 = 0$, or $y_3(c_3 y_3 + \cdots + c_n y_n) + e_4 y_4 + \cdots + e_n y_n = 0$. If some $e_i \neq 0$, we can assume after possibly renaming the coordinates, that $\varphi_1^* F_{0,3}$ is y_3-regular of order 1 and apply Corollary 1.23 to find a formal coordinate change φ_2 such that $(\varphi_2 \circ \varphi_1)^* F_{0,3} = y_3$. If $e_4 = \cdots = e_n = 0$, $\varphi_1^* F_{0,3} = y_3^2(c_3 y_3 + \cdots + c_n y_n)$, and thus either $y_3 = 0$ or $c_3 y_3 + \cdots + c_n y_n = 0$. We discussed already that for the singularity (X, p_0) to be isolated there needs to be some $c_i \neq 0$. Consequently, the polynomial $c_3 y_3 + \cdots + c_n y_n$ is y_3-regular of order 1 after possibly renaming the coordinates and there is a formal coordinate change φ_2 such that $(\varphi_2 \circ \varphi_1)^* F_{0,3} = y_3$. By applying Corollary 1.23 to the other equations in (1.11) also, we can conclude that there is a formal coordinate change φ such that the local equations (1.11) for $F(X)$ in this new coordinates take the form

$$\begin{aligned} \varphi^* F_{3,0}(x_2, \ldots, x_n, y_2, \ldots, y_n) &= x_4^2 + \cdots + x_n^2 + \mathfrak{h} \\ \varphi^* F_{2,1}(x_2, \ldots, x_n, y_2, \ldots, y_n) &= x_2, \\ \varphi^* F_{1,2}(x_2, \ldots, x_n, y_2, \ldots, y_n) &= y_2, \\ \varphi^* F_{0,3}(x_2, \ldots, x_n, y_2, \ldots, y_n) &= y_3, \end{aligned} \tag{1.12}$$

with a formal power series $\mathfrak{h}$. Note that each of the coordinate changes from the proof of Lemma 1.22 that we apply to bring the defining equations to the form (1.12) only affects the coordinates x_2, y_2 and y_3. Thus the same is true for the formal coordinate change φ and the image of Σ under these coordinate changes is $\{(x_2, \ldots, x_n, y_2, \ldots, y_n) \mid y_2 = y_3 = x_2 = \cdots = x_n = 0\}$.
If we now take any fixed point $(0, 0, y_4, \ldots, y_n) \in \Sigma \cap U$, we can read off equations for the transversal singularity along Σ_{reg} from (1.12) which are

$$x_2 = x_4^2 + \cdots + x_n^2 + \mathfrak{h} = 0,$$

that is, we obtain a hypersurface singularity of corank one inside $\mathbb{C}^{n-2}$ (with coordinates $x_3, \ldots, x_n$). We can then apply the coordinate changes from the proof of the Generalised Morse Lemma, Lemma 1.12, to deduce that $\mathfrak{f} = x_4^2 + \cdots + x_n^2 + \mathfrak{h}$ after a formal coordinate change takes the form

$$\mathfrak{f} = x_4^2 + \cdots + x_n^2 + C_3 x_3^3 + C_4 x_3^4 + \ldots, \tag{1.13}$$

with the C_i depending on the coefficients of f. If now $C_3 = \cdots = C_k = 0$ but $C_{k+1} \neq 0$, there exists, by Corollary 1.23, another formal coordinate change such that $\mathfrak{f}$ becomes $x_4^2 + \cdots + x_n^2 + C_{k+1} x_3^{k+1}$. In other words $\mathfrak{f}$ then is formally equivalent to the defining equation of a singularity of type A_k.

1.24 Theorem ([Art68, Corollary 1.6])**.** *Let Y_1, Y_2 be analytic spaces and $y_1 \in Y_1$, $y_2 \in Y_2$ be points. Then, Y_1 and Y_2 are formally isomorphic at y_1 and y_2 respectively, i.e. $\widehat{\mathcal{O}}_{Y_1,y_1} \cong \widehat{\mathcal{O}}_{Y_2,y_2}$ (where $\widehat{\cdot}$ denotes formal completion) if and only if $(Y_1, y_2) \cong (Y_2, y_2)$.*

Therefore, we can also compute coefficient conditions for the hypersurface singularity of $F(X)$ along Σ_{reg}. It could of course happen that $C_i = 0$ for all $i \geq 3$. In this case, the analytic space Y from Theorem 1.17 would have a non-isolated singularity at every point of Σ_{reg}. If there is at least one i such that $C_i \neq 0$, the argument above implies that the transversal type is of type A_k for some k.

The actual computation of the coefficient conditions again turns out to be very time consuming. We would therefore like to also give this computations to a computer. One obstruction to do so is the need for formal coordinate changes. If we are only interested in finding the coefficient conditions for a singularity of type A_k for, or up to, some fixed k, it suffices to work with a sufficiently high jet of the defining equations.

1.25 Lemma. *For computing the coefficient conditions $C_3, \dots, C_{k+1}$ from* (1.13) *for fixed k, it is sufficient to consider the formal coordinate changes used to transform the last three equations in* (1.11) *up to order $\lceil \frac{k+1}{3} \rceil$.*

Proof. We need to compute $\mathfrak{f}$ up to order $k+1$. Since in (1.11) the function $\mathfrak{h}_1$ lies in $\mathfrak{m}^3$, for obtaining the entire $(k+1)$-jet of $\mathfrak{h}$ it is sufficient to work with the $\lceil \frac{k+1}{3} \rceil$-jet of the last three equations. The coordinate changes performed to bring $\mathfrak{h}$ to the desired form $C_3 x_3^3 + C_4 x_4^4 + \dots$ are those from the proof of the Generalised Morse Lemma, Lemma 1.12, and only increase but never decrease the degree of terms in $\mathfrak{f}$. □

1.26 Example. Let us compute the coefficient conditions for a transversal singularity of type A_2. We have $\lceil \frac{k+1}{3} \rceil = 1$, since $k = 2$, that is, it suffices to work with the one-jet of the last three equations in (1.7). By working out the local equations, we obtain

$$
\begin{aligned}
F_{3,0}(x_2, \dots, x_n, y_2, \dots, y_n) &= x_4^2 + \dots + x_n^2 + x_2^2 \sum_{i=2}^{n} b_i x_i + x_2 \sum_{4 \le i \le j \le n} B_{i,j} x_i x_j + c_3 x_3^3 \\
&\quad + x_3^2 \sum_{i=4}^{n} c_i x_i + x_3 \sum_{4 \le i \le j \le n} C_{i,j} x_i x_j + \sum_{4 \le i \le j \le k \le n} p_{i,j,k} x_i x_j x_k, \\
\operatorname{jet}^1 F_{2,1}(x_2, \dots, x_n, y_2, \dots, y_n) &= x_2, \\
\operatorname{jet}^1 F_{1,2}(x_2, \dots, x_n, y_2, \dots, y_n) &= y_2, \\
\operatorname{jet}^1 F_{0,3}(x_2, \dots, x_n, y_2, \dots, y_n) &= a_3 y_3 + \dots + a_n y_n.
\end{aligned}
$$

We can compute the transversal singularity type at every point of Σ_{reg}, in particular at the point $y_2 = \dots = y_n = 0$. This eliminates the last two equations. We can moreover eliminate the variable x_2 using the second equation. The remaining equation then determines the transversal singularity type and is

$$
\begin{aligned}
F_{3,0}(0, x_3, \dots, x_n, 0, \dots, 0) &= x_4^2 + \dots + x_n^2 + c_3 x_3^3 + x_3^2 \sum_{i=4}^{n} c_i x_i + x_3 \sum_{4 \le i \le j \le n} C_{i,j} x_i x_j \\
&\quad + \sum_{4 \le i \le j \le k \le n} p_{i,j,k} x_i x_j x_k.
\end{aligned}
$$

We can now either conclude using the Recognition Principle, Lemma 1.9, that the coefficient condition for a singularity of type A_2 is $c_3 \neq 0$, or use the coordinate changes from Lemma 1.22 to bring this equation to the form $x_4^2 + \dots + x_n^2 + c_3 x_3^3$, since the equation is x_3-regular of order three. Comparing with Lemma 1.19, we see that $F(X)$ has a transversal singularity of type A_2 along Σ_{reg}, if and only if (X, p_0) is of type A_2.

For the remainder of this section only, let f_n be given by equation (1.10), that is, let f_n be the defining equation in the normal form from section 1.2.2 of a cubic hypersurface in $\mathbb{P}^n$ containing a double point. Then, f_n admits the following useful symmetry property:

(SP) Let $I = (i_0, \dots, i_n) \in \mathbb{N}_0^{n+1}$. If there exists in f_n a term of the form $C \cdot z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$, there also exist in f_n all terms of the form $C_\sigma \cdot z_0^{i_0} \cdots z_3^{i_3} z_{\sigma(4)}^{i_4} \cdots z_{\sigma(n)}^{i_n}$, where σ is any permutation of $\{4, \dots, n\}$. Here, both C and C_σ denote (non-zero multiples of) coefficients of f_n.

This property is immediately clear from the equation (1.10). All functions we derived from f_n, namely the coefficient conditions $C_{n,k} = C_k(f_n)$ and the local equations (1.7) for the Fano scheme, inherit the symmetry property (SP). Note that for the coefficient conditions C_i the coefficients C and C_σ are the same for all permutations σ of $\{4, \dots, n\}$. The following example illustrates the usefulness of (SP) for computing the coefficient conditions $C_{n,k}$.

1.27 Example. We have shown in Lemma 1.19 that the coefficient condition $C_{n,4}$ is

$$C_{n,4} = 4c_1c_2 + c_4^2 + \cdots + c_n^2.$$

We claim that using the symmetry property (SP), $C_{n,4}$ can be determined from $C_{5,4}$ already. Recall that $C_{n,k}$ is known to be homogenous of degree $k-2$ by Theorem 1.15. Therefore, every monomial in $C_{n,4}$ can be a product of at most two mutually distinct variables. As we have the symmetry property (SP), every such product appears for $n = 5$ already. Conversely, every product of at most two mutually distinct variables in $C_{n,4}$ is determined by such a product in $C_{5,4}$ using the symmetry property (SP). That is, since we find the term c_4^2 in $C_{5,4}$, the terms c_i^2 have to appear in $C_{n,4}$ for every $i \geq 4$; and $C_{n,4}$ can not contain any product of two distinct variables with index at least four, as if it would e.g. contain a term of the form c_ic_j for $i, j \geq 4$, $i \neq j$, the symmetry property (SP) implies that $C_{5,4}$ has to contain the term c_4c_5 which it does not.
Note that we needed n to be at least $5 = 3 + 2$ since only coordinates with index at least four are permuted in the definition of (SP).

The argument from Example 1.27 is not limited to the coefficient condition $C_{n,4}$ but easily generalises to the following corollary.

1.28 Corollary. *If f_n is given by equation* (1.10) *and $C_{n,k} = C_k(f_n)$ denotes the corresponding coefficient condition, then $C_{n,k}$ can be determined from $C_{k+1,k}$ using the symmetry property (SP).*

Proof. Just as in Example 1.27, one has to compute $C_{n,k}$ for the smallest n such that a product of $k-2$ mutually distinct coordinates with index at least four can appear and can then conclude using the symmetry property (SP). The respective smallest n is given by $n = 3+k-2 = k+1$. □

1.29 Theorem. *For $k \leq 5$, the singularity type of (X, p_0) is A_k if and only if the transversal singularity type of $F(X)$ along Σ_{reg} is A_k.*

Proof. We can compute the coefficient conditions for X and $F(X)$ to have a singularity of type A_k separately and compare them just as we did with Lemma 1.19 and Example 1.26. This is done by first bringing the defining equation f for X to the normal form (1.8). From this, the coefficient conditions for the cubic hypersurface X can be computed using the algorithm from the proof of Lemma 1.12. To compute the coefficient conditions for the Fano scheme of lines on X, we need to pass to local equations $F(X)$ first and bring them to the form (1.12) using the formal coordinate changes introduced above, cf. Lemma 1.22. This reduces us to another computation of the coefficient conditions of a hypersurface which can again be done by applying the algorithm from the proof of Lemma 1.12.

In order to prove the present theorem for fixed k, we handed the problem to a computer. This can be done, since Lemma 1.25 allows us to use algebraic coordinate changes instead of polynomial ones and since Corollary 1.28 reduces the computations for fixed k to computing finitely many equations.

We implemented the computation of local equations for $F(X)$, the algorithm from the proof of Lemma 1.12 and the computation of the normal form (1.12) in the computer algebra system SINGULAR [SIN] and also in Python [Pyt] and used this implementation to prove the theorem. The source code for the implementation in Python is provided in the appendix of this thesis together with some brief explanations on the code. □

1.30 Remark. The limitation to $k \leq 5$ in Theorem 1.29 is only due to lack of computational power.

1.31 Remark. The case of (X, p_0) of type A_1 follows immediately from our local computations which in particular show that the corank of (X, p_0) is the same as the corank of the transversal singularity of $F(X)$. Now the singularity of type A_1 is the unique singularity of corank zero, as is clear from the Generalised Morse Lemma, Lemma 1.12. This special case was proven before by H. Clemens and P. Griffiths [CG73, Theorem 7.8].

Instead of proving Theorem 1.29 in all dimensions for a fixed k, we can also check it for all k in a fixed dimension. It turns out that at least the case of of cubic surfaces and cubic threefolds can entirely be dealt with using this method.

1.32 Theorem. *If $X \subset \mathbb{P}^n$ is a cubic hypersurface and $n \leq 4$, Theorem 1.29 holds without any assumption on k.*

Proof. For fixed dimension $n \geq 3$, there is a maximal k such that there can be a cubic hypersurface $X \subset \mathbb{P}^n$ with isolated singular point of type A_k. For $n = 3$ this maximal k is known to be $k = 5$ by [BW79]; for $n = 4$ we show that the maximal k is $k = 11$, see Corollary 3.35. Consequently, to proof the assertion there are only finitely many cases to be checked and this can be done as in the proof of Theorem 1.29. □

1.4 Remarks on the implementation

In this section we briefly explain some further simplifications of the defining equations we have made to decrease the computation time. Consider the normal form (1.10) for the defining equation f for X. One step in computing the coefficient conditions on f to define a singularity of type A_k is to perform the Generalised Morse Lemma. In order to do so, let

$$
\begin{aligned}
g_1(z_1, \ldots, z_n) &= a_3 z_1 z_3 + z_1 \sum_{i=4}^{n} a_i z_i + z_3 \sum_{i=4}^{n} e_i z_i + \sum_{4 \leq i \leq j \leq n} A_{i,j} z_i z_j, \\
g_2(z_1, \ldots, z_n) &= b_2 z_2^2 + b_3 z_2 z_3 + z_2 \sum_{i=4}^{n} b_i z_i + z_3 \sum_{i=4}^{n} f_i z_i + \sum_{4 \leq i \leq j \leq n} B_{i,j} z_i z_j, \\
g_4(z_1, \ldots, z_n) &= c_4 z_3^2 + z_3 \sum_{i=4}^{n} C_{4,i} z_i + \sum_{4 \leq i \leq j \leq n} p_{4,i,j} z_i z_j, \\
&\vdots \\
g_n(z_1, \ldots, z_n) &= c_n z_3^2 + C_{n,n} z_3 z_n + p_{n,n,n} z_n^2, \\
f_3(z_3) &= c_3 z_3^3,
\end{aligned}
$$

and

$$
f(1, z_1, \ldots, z_n) = z_1 z_2 + z_4^2 + \cdots + z_n^2 + f_3(z_3) + \sum_{\substack{i=1 \\ i \neq 3}}^{n} z_i g_i(z_1, \ldots, z_n).
$$

1.33 Lemma. *The following terms have $\alpha(A_k)$-degree strictly greater than one and retain this property under the coordinate changes used in the proof of Lemma 1.12:*

- $p_{i,j,k} z_i z_j z_k$ *for* $i, j, k \geq 4$;
- $A_{i,j} z_1 z_i z_j$ *for* $i, j \geq 4$;
- $B_{i,j} z_2 z_i z_j$ *for* $i, j \geq 4$.

These terms therefore do not contribute to the conditions for a singularity of type A_k.

Proof. Recall that the weight $\frac{1}{k+1}$ is put on the variable z_3 and that the coordinate changes performed in the proof of Lemma 1.12 are

$$z_1 \mapsto z_1 - g_2, \quad z_2 \mapsto z_2 - g_1, \quad z_i \mapsto z_i - \frac{1}{2} g_i, \ i = 4, \dots, n.$$

It is clear that all of the terms in question have $\alpha(A_k)$-degree strictly greater than one and the only thing to prove is that this property is retained under these coordinate changes.
Terms of the form $p_{i,j,k} z_i z_j z_k$ for $i, j, k \geq 4$ give rise to terms in $g_4, \dots, g_n$. Consider a coordinate change $z_i \mapsto z_i - \frac{1}{2} g_i$. The defining equation in this new coordinates contains

$$\left(z_i - \frac{1}{2} p_{i,j,k} z_j z_k\right)^2 + \left(z_i - \frac{1}{2} p_{i,j,k} z_j z_k\right) g_i \left(z_1, \dots, z_i - \frac{1}{2} p_{i,j,k} z_j z_k, \dots, z_n\right).$$

In this, the term z_i^2 remains anyway, the term $-z_i p_{i,j,k} z_j z_k$ cancels out by construction and the term $\frac{1}{4} p_{i,j,k}^2 z_j^2 z_k^2$ clearly has $\alpha(A_k)$-degree two thus strictly greater than one. The remaining terms either have factor $-\frac{1}{2} p_{i,j,k} z_j z_k$ which already has $\alpha(A_k)$-degree one, thus multiplying it with anything that is not a constant increases the $\alpha(A_k)$-degree; or are terms coming from

$$z_i g_i \left(z_1, \dots, z_i - \frac{1}{2} p_{i,j,k} z_j z_k, \dots, z_n\right) = z_i g_i(z_1, \dots, z_n) + R(z_1, \dots, z_n).$$

The first term is the one that cancels out with the term mentioned earlier. In $R(z_1, \dots, z_n)$ every term again has a factor $\left(-\frac{1}{2} p_{i,j,k} z_j z_k\right)^p$ for some p, thus $\alpha(A_k)$-degree strictly greater than one as we have argued already.
The argument for the other terms $A_{i,j} z_1 z_i z_j$ and $B_{i,j} z_2 z_i z_j$ is analogous. □

Using Lemma 1.33 we can disregard some of the terms in the g_i. This results in the following simplification for $g_1, \dots, g_n$:

$$\begin{aligned}
g_1(z_1, \dots, z_n) &= a_3 z_1 z_3 + z_1 \sum_{i=4}^{n} a_i z_i + z_3 \sum_{i=4}^{n} e_i z_i, \\
g_2(z_1, \dots, z_n) &= b_2 z_2^2 + b_3 z_2 z_3 + z_2 \sum_{i=4}^{n} b_i z_i + z_3 \sum_{i=4}^{n} f_i z_i, \\
g_4(z_1, \dots, z_n) &= c_4 z_3^2 + z_3 \sum_{i=4}^{n} C_{4,i} z_i, \\
&\vdots \\
g_n(z_1, \dots, z_n) &= c_n z_3^2 + C_{n,n} z_3 z_n, \\
f_3(z_3) &= c_3 z_3^3.
\end{aligned}$$

There also is a major simplification possible when dealing with the local equations (1.11) for $F(X)$. The formal coordinate changes we use to linearise the last three equations in (1.11) only affect the coordinates x_2, y_2, y_3. The coordinates $y_4, \dots, y_n$ are therefore invariant under this formal coordinate change. As we can compute the transversal singularity type at any point of Σ_{reg}, in particular at the point where the local coordinates for Σ_{reg} all vanish, we can assume that $y_2 = \dots = y_n = 0$ just as we did in Example 1.26. But since $y_4, \dots, y_n$ remain unchanged under the formal coordinate change, we can assume $y_4 = \dots = y_n = 0$ from the start, i.e. before performing any coordinate change.

2 Tail reduction for transverse curve singularities

The probably most famous appearance of the Fano scheme of lines in the literature is in a paper by H. Clemens and P. Griffiths [CG73] from 1973 where it is shown that, for a smooth cubic threefold X, the intermediate Jacobian of X is isomorphic to the Albanese variety of $F(X)$. This result was generalised in 2010 by G. van der Geer and A. Kouvidakis [vK10] where they prove the same result for cubic threefolds with a unique singular point of type A_1 but replace the Albanese variety of $F(X)$ by its Picard scheme. A general framework for computing the intermediate Jacobian of a singular cubic threefold was then given in 2015 by S. Casalaina-Martin, S. Grushevsky, K. Hulek and R. Laza in [CGHL15] and many cases are worked out using their methods by K. Havasi in his 2016 thesis [Hav16].

The aim of the remainder of this thesis is to study the degenerate Picard scheme of $F(X)$ when X is a singular cubic threefold with a unique singular point of type A_k. Our approach is different from that given in [vK10] and based on the Semistable Reduction Theorem for curves. We therefore recall this theorem along with a proof due to J. Harris and I. Morrison [HM98] that gives us the opportunity to perform explicit calculations. Afterwards, we show how the Semistable Reduction Theorem can be used to compute the degenerate Picard scheme of a singular curve.

In order to apply these results to the Fano scheme of lines on a singular cubic threefold, we prove a generalisation of the methods in [HM98] to varieties admitting transverse curve singularities along a smooth singular locus of which the Fano scheme of lines on a cubic threefold with a unique singular point of type A_1 or A_2 is a particular example. We call this operation *tail reduction* as it is not clear in which, if any, sense the result is semistable. Finally, applying the tail reduction to $F(X)$ for X with a unique singular point of type A_1 or A_2 enables us to compute its degenerate Picard scheme.

2.1 Semistable reduction for curves

Throughout the remainder of this thesis, B always denotes a smooth curve, $0 \in B$ a point and $B^* = B \setminus \{0\}$ the complement of the point $0 \in B$.

2.1 Definition (Smoothing). *Let $\pi : \mathfrak{Y} \to B$ be a flat family of varieties. Then, $\pi : \mathfrak{Y} \to B$ is called smoothing of $\mathfrak{Y}_0$, if for all $b \in B^*$ the fibre $\mathfrak{Y}_b = \pi^{-1}(b)$ is smooth.*

Let $\pi : \mathfrak{C} \to B$ be a smoothing of the curve $\mathfrak{C}_0$ such that for all $b \in B^*$ we have that $g(\mathfrak{C}_b) \geq 2$, and assume that $\mathfrak{C}_0$ has a unique singular point of ADE-type, cf. Table 1.1. By a (semi-)stable reduction of $\pi : \mathfrak{C} \to B$ we mean the following Theorems 2.2 and 2.3.

2.2 Theorem (Stable Reduction Theorem for curves, cf. [HM98, Proposition 3.47]). *There exists a branched cover $\varphi : B' \to B$, totally ramified over $0 \in B$, and a flat family $\pi' : \mathfrak{C}' \to B'$*

of curves such that $\mathfrak{C}'_b \cong \mathfrak{C}_{\varphi(b)}$ *for all* $b \neq 0$ *and the central fibre* $\mathfrak{C}'_0$ *of the family* $\pi' : \mathfrak{C}' \to B'$ *is stable in the sense of Deligne-Mumford, [DM69, Definition 1.1], that is,* $\mathfrak{C}'_0$ *is reduced, connected, its only singularities are ordinary double points and every non-singular rational component of* $\mathfrak{C}'_0$ *meets other components of* $\mathfrak{C}'_0$ *in at least three points. Moreover, the central fibre* $\mathfrak{C}'_0$ *of* $\pi' : \mathfrak{C}' \to B$ *is uniquely determined by the family* $\pi : \mathfrak{C} \to B$*.*

The family $\pi : \mathfrak{C}' \to B'$ in Theorem 2.2 is called a stable reduction of the initial family $\pi : \mathfrak{C} \to B$. In general, the total space $\mathfrak{C}'$ of the stable reduction $\pi' : \mathfrak{C}' \to B'$ cannot assumed to be smooth but it is possible to obtain a similar statement with smooth total space, by weakening the assumptions on the central fibre $\mathfrak{C}'_0$. To be more precise, if a smooth total space $\mathfrak{C}'$ is needed, the central fibre $\mathfrak{C}'_0$ can in general only assumed to be semistable in the sense of Deligne-Mumford.

2.3 Theorem (Semistable Reduction Theorem for curves, cf. [HM98, Proposition 3.48])**.** *There exists a branched cover* $\varphi : B' \to B$*, totally ramified over* $0 \in B$*, and a flat family* $\pi' : \mathfrak{C}' \to B'$ *of curves such that* $\mathfrak{C}'_b \cong \mathfrak{C}_{\varphi(b)}$ *for all* $b \in B^*$ *and the central fibre* $\mathfrak{C}'_0$ *of the family* $\pi' : \mathfrak{C}' \to B'$ *is semistable in the sense of Deligne-Mumford, that is,* $\mathfrak{C}'_0$ *is reduced, connected, its only singularities are ordinary double points and every non-singular rational component of* $\mathfrak{C}'$ *meets other components of* $\mathfrak{C}'$ *in at least two points. Moreover, the total space* $\mathfrak{C}'$ *is smooth and the central fibre* $\mathfrak{C}'_0$ *of* $\pi' : \mathfrak{C}' \to B$ *is determined by the family* $\pi : \mathfrak{C} \to B$ *up to contractions of smooth rational components of* $\mathfrak{C}'$ *that meet the rest of* $\mathfrak{C}'$ *in fewer than three points.*

It is a general result, see [KKMS73, page 53], that for every flat family $\pi : \mathfrak{Y} \to B$ of varieties such that $\mathfrak{Y}_b$ is smooth for $b \in B^*$, there exists a finite base change $\varphi : B' \to B$ and a family $\pi' : \mathfrak{Y}' \to B'$ with smooth total space $\mathfrak{Y}'$ such that $\mathfrak{Y}'_b \cong \mathfrak{Y}_{\varphi(b)}$ for all $b \in B^*$ and with $\mathfrak{Y}'_0$ defining a simple normal crossing divisor in $\mathfrak{Y}'$, cf. Definition 2.4 below. This is what usually is called a semistable reduction of the family $\pi : \mathfrak{Y} \to B$ and Theorem 2.2 appears as a special case of this. The proof given for families of curves in [KKMS73, pages 98-108] is in most parts constructive, but fails to give a precise description of the central fibre $\mathfrak{Y}'_0$ of a stable reduction $\pi' : \mathfrak{Y}' \to B'$. The first entirely constructive proof is due to J. Harris and I. Morrison [HM98]. As we aim to generalise their method of proof to transversal curve singularities along a smooth singular locus on the central fibre, we give here complete proofs of Theorem 2.3 and Theorem 2.2 following the sketch in [HM98].

2.4 Definition (cf. [Kol07, Definition 1.44])**.** *Let* Y *be a smooth variety and* $D \subset Y$ *be a divisor.*

(i) We call D *a simple normal crossing divisor, if every irreducible component of* D *is smooth and if all intersections of irreducible components of* D *are transverse. In other words,* D *is a simple normal crossing divisor, if all of its components are smooth and if for every* $p \in D$ *there exists an analytic neighbourhood* $U = U(p)$ *of* p *with local coordinates* $x_1, \dots, x_N$ *such that* $D \cap U = \{x_1^{a_1} \cdot \dots \cdot x_r^{a_r} = 0\}$ *for some* $r \in \{1, \dots, N\}$ *and natural numbers* $a_1, \dots, a_r$*.*

(ii) We call D *a reduced simple normal crossing divisor, if* D *is a simple normal crossing divisor and all of its irreducible components are reduced. In other words,* D *is a reduced simple normal crossing divisor, if all of its components are smooth and if for every* $p \in D$ *there exists an analytic neighbourhood* $U = U(p)$ *of* p *with local coordinates* $x_1, \dots, x_N$ *such that* $D \cap U = \{x_1 \cdot \dots \cdot x_r = 0\}$ *for some* $r \in \{1, \dots, N\}$*.*

Proof of Theorems 2.2 and 2.3, cf. the sketch given in [HM98, pages 137-138].
Let $\pi : \mathfrak{C} \to B$ be a flat family of curves such that all fibres over points of B^* are smooth and of genus at least two.
The proof is organised as follows: We start by passing from the initial family $\pi : \mathfrak{C} \to B$ to a family $\widetilde{\pi} : \widetilde{\mathfrak{C}} \to B$ with smooth total space that differs from the original family only in the central fibre and such that $\widetilde{\mathfrak{C}}_0$, seen as a divisor in $\widetilde{\mathfrak{C}}$, is a simple normal crossing divisor.

Since $\widetilde{\mathfrak{C}}_0$ will in general admit nonreduced components, we perform a finite base change followed by normalisation to obtain a family such that the central fibre defines a reduced simple normal crossing divisor. In order to understand, how the components of the central fibre change during this procedure, we decompose this operation into several intermediate steps. The description of these intermediate steps and the analysis of their effect to the several components of $\widetilde{\mathfrak{C}}_0$ makes up the second step of the proof.
If we were only interested in proving the assertion, this step could be dealt with in a few lines. The advantage of the lengthy approach presented here is the complete analysis of the effect of these operations on the central fibre, making it possible to in the end compute the central fibre of a stable reduction explicitly.
The third and fourth step then establish Theorems 2.2 and 2.3 by first resolving the singularities of the total space, yielding Theorem 2.3, and then blowing down all components of the central fibre that intersect the rest of the central fibre in fewer than three points to obtain Theorem 2.2.

Step 1 (*Replacing the central fibre by one that has normal crossing singularities*)**.**

We begin with a strong embedded resolution of the singularities of $\mathfrak{C}_0$ inside $\mathfrak{C}$, that is, with a resolution of singularities of $\mathfrak{C}_0$ by successive blowups of $\mathfrak{C}$ until the total transform of $\mathfrak{C}_0$ defines a simple normal crossing divisor in the blown up ambient space $\mathfrak{C}$. This is always possible, see e.g. [Kol07, Theorem 1.47] for a proof, and can be done by successive blow ups of $\mathfrak{C}$ at the singular points of $\mathfrak{C}_0$ and its respective strict transforms. The total transform of $\mathfrak{C}_0$ under these blowups then consists of the strict transform of $\mathfrak{C}_0$, which is the normalisation of $\mathfrak{C}_0$, together with the exceptional divisors from the various blowups. We give a precise description of the configurations of exceptional divisors that can arise from embedded resolutions of curves with ADE-singularities and the multiplicities of the various components of the total transform in section 2.2. If we denote the blown up total space by $\widetilde{\mathfrak{C}}$, we obtained a family $\widetilde{\pi} : \widetilde{\mathfrak{C}} \to B$ that differs from the original family $\mathfrak{C} \to B$ only in the central fibre and such that $\widetilde{\mathfrak{C}}$ is smooth with $\widetilde{\mathfrak{C}}_0$ defining a simple normal crossing divisor in $\widetilde{\mathfrak{C}}$. Flatness of this family follows from [Har83, Proposition 9.7, page 257], since $\widetilde{\mathfrak{C}}$ is reduced, connected and dominates B.

To simplify our notation, we will from now on assume that our initial family $\pi : \mathfrak{C} \to B$ already had smooth total space $\mathfrak{C}$ and that $\mathfrak{C}_0$ defines a simple normal crossing divisor in $\mathfrak{C}$.

Step 2 (*From simple normal crossing to reduced simple normal crossing*)**.**

Write $\mathfrak{C}_0 = D = \sum a_j D_j$, seen as a divisor on $\mathfrak{C}$. Let l be the least common multiple of the multiplicities of the components of $\mathfrak{C}_0$, i.e. $l = \mathrm{lcm}(a_j \mid j)$. If $l = 1$, $\mathfrak{C}_0$ defines a reduced simple normal crossing divisor already and we can proceed with step four. We may therefore assume that $l \geq 2$. Consider the finite base change $\varphi : B' \to B$ given by $u \mapsto u^l = t$ and let $\psi : \widetilde{\mathfrak{C}} \to \varphi^*\mathfrak{C}$ be the normalisation of $\varphi^*\mathfrak{C} = \mathfrak{C} \times_B B'$. We claim that the central fibre $\widetilde{\mathfrak{C}}_0$ is reduced with smooth components intersecting transversally. This can be checked using local coordinates, see for example [ACG11, Chapter X, section 4]. By doing so it is, however, hard to keep track of what happens to the central fibre $\mathfrak{C}_0$ under the finite base change and normalisation. We will therefore decompose these operations into several intermediate steps for which it is easier to understand their effect on the various components of the central fibre.

Consider the prime factorisation $l = p_1 \cdot \dots \cdot p_r$ with the p_i not necessarily mutually distinct. We decompose the base change of order l into the composition of the base changes $\varphi_i : u \mapsto u^{p_i} = t$ and will, instead of normalising the pullback by φ take a partial normalisation of the pullback of the family by φ_i in each step.
We begin with $\widetilde{\mathfrak{C}}^0 = \mathfrak{C}$ and $B_0 = B$ and perform the following procedure successively for $i = 1, \dots, r$. Write $D^{i-1} = \sum a_j^{i-1} D_j^{i-1}$ and let

$$\varphi_i : B_i \to B_{i-1},\ u \mapsto u^{p_i} = t$$

be the finite base change of order p_i. Taking the fibre product $\mathfrak{C}^i = \widetilde{\mathfrak{C}}^{i-1} \times_{B_{i-1}} B_i = \varphi^*\widetilde{\mathfrak{C}}^{i-1}$ is the same as taking the branched covering of $\widetilde{\mathfrak{C}}^{i-1}$ of order p_i that is branched along the divisor

$D^{i-1} = \{t = 0\}$. Since $\widetilde{\mathfrak{C}}_0^{i-1}$ has normal crossing singularities only, $\widetilde{\mathfrak{C}}^{i-1}$ is locally around points of $\widetilde{\mathfrak{C}}_0^{i-1}$ given by $t = x^a y^b$ where a, b are from the a_j^{i-1}'s, or by $t = x^a$ where a is from the a_j^{i-1}'s. Here, x, y are local coordinates on $\widetilde{\mathfrak{C}}^{i-1}$ and t on B_{i-1}. The local equation of the resulting surface $\mathfrak{C}^i$ then is $u^{p_i} = x^a y^b$, where u is a local coordinate on B_i. The relation between the central fibre of $\mathfrak{C}^i \to B_i$ and $\widetilde{\mathfrak{C}}^{i-1} \to B_{i-1}$ is explained by

$$\varphi_i^* D^{i-1} = \{u^{p_i} = 0\} = p_i \cdot \{u = 0\}, \tag{2.1}$$

where $\{u = 0\}$ is the divisor corresponding to the central fibre $\mathfrak{C}_0^i \subset \mathfrak{C}^i$. As a divisor on $\mathfrak{C}^i$, the central fibre $\mathfrak{C}_0^i$ therefore equals $\frac{1}{p_i} \cdot \varphi_i^* D^{i-1}$. We can also see this in local coordinates: if $m \cdot D_j^{i-1} = \{z^m = 0\}$ is a component of D, where z is one of the local coordinates on $\widetilde{\mathfrak{C}}^{i-1}$, its variation with t is given by $\{z^m = t\}$ and the inverse image of this is $\{z^m = u^{p_i}\}$, giving a local equation for $\mathfrak{C}^i$. In particular, $\mathfrak{C}^i$ is singular along the inverse image $\varphi_i^*(mD_j^{i-1})$, if $m > 1$. Consider the map ψ_i that is in local coordinates defined by

$$\psi_i : v \mapsto v z^{\left\lfloor \frac{m}{p_i} \right\rfloor} = u.$$

Then, $\psi_i^*(\{z^m = u^{p_i}\}) = \{v^{p_i} = z^{m \mod p_i}\}$ is a local equation for $\widetilde{\mathfrak{C}}^i = \psi_i^*(\mathfrak{C}^i)$. In other words, taking the fibre product $\mathfrak{C}^i = \widetilde{\mathfrak{C}}^{i-1} \times_{B_{i-1}} B_i$ followed by replacing the coordinate u by the coordinate v introduced above, is the same as taking the branched cover of $\widetilde{\mathfrak{C}}^{i-1}$ of order p_i that is branched along the divisor $D^{i-1}_{\mod p_i} = \sum (a_j \mod p_i) D_j^{i-1}$ which we call the divisor D^{i-1} reduced modulo p_i.[1] In particular, the map ψ_i is defined globally on $\widetilde{\mathfrak{C}}^i$. Denote this branched cover by $\eta_i : \widetilde{\mathfrak{C}}^i \to \widetilde{\mathfrak{C}}^{i-1}$. We now describe the central fibre $\widetilde{\mathfrak{C}}_0^i = \{v = 0\}$ of $\widetilde{\mathfrak{C}}^i$. First, note that according to equation (2.1) above, $\widetilde{\mathfrak{C}}_0^i = \frac{1}{p_i} \eta_i^* D^{i-1}$. Let D_j^{i-1} be a component of D^{i-1}. Then, $\eta_i^* D_j^{i-1}$ is determined with respect to the following cases:

- if D_j^{i-1} is a component not intersecting any component of the branch locus $D^{i-1}_{\mod p_i}$, its inverse image is an unramified, p_i-sheeted cover of D_j^{i-1}, i.e. p_i distinct copies of D_j^{i-1}. Each of these copies of D_j^{i-1} has the same multiplicity as D_j^{i-1}.
- if D_j^{i-1} is a component intersecting the branch locus in r points, its inverse image is a p_i-sheeted cover of D_j^{i-1} that is branched at r points. This is a smooth curve whose genus can be computed using the Riemann-Hurwitz formula, see [Har83, IV, Corollary 2.4], which for this particular situation asserts that

$$2g(\eta_i^{-1}(D_j^{i-1})) - 2 = p_i(2g(D_j^{i-1}) - 2) + r(p_i - 1). \tag{2.2}$$

  The multiplicity of this curve equals the multiplicity of D_j^{i-1}.
- if D_j^{i-1} is contained in the branch locus, its inverse image is a p_i-fold copy of D_j^{i-1}. The multiplicity of $\eta_i^{-1} D_j^{i-1}$ therefore equals the multiplicity of D_j^{i-1} multiplied by p_i.

This finishes the description of the intermediate steps. It follows from the local description of such branched coverings, cf. the proof of Lemma 2.6, that these operations are independent of the order of the p_i's.

We claim that performing these steps for all primes from the prime factorisation of l yields the same result as pullback of order l followed by normalisation. Note that iteration of the procedure above indeed yields a normal total space $\widetilde{\mathfrak{C}}$ whose central fibre is reduced, connected with smooth components intersecting transversally. This can be seen as follows: For the reducedness, consider any component $a_j D_j$ of the central fibre of our initial family $\pi : \mathfrak{C} \to B$. Let $p_1, \ldots, p_s$ be the primes dividing a_j. Whenever we pass to the branched cover of order p_i as above where $i \in \{1, \ldots, s\}$, D_j is not a component of the branch divisor $D_{\mod p_i}$, thus will either split into

[1] We always work with a representative $a_j \mod p_i \in \{0, \ldots, p_i - 1\}$.

several copies of itself or become a curve of possibly different genus. In both cases the multiplicity of the components of the central fibre of the branched cover branched along $D_{\bmod p_i}$ that arise as the pullback of D_j is $\frac{a_j}{p_i}$. Consequently, after taking the respective branched covers of orders $p_1, \dots p_s$, the multiplicities of the components coming from D_i all are equal to $\frac{a_i}{p_1 \cdots p_s} = 1$. Since we have chosen l to be the least common multiple of all the a_i's, all components of the central fibre of $\widetilde{\mathfrak{C}}$ have multiplicity one and $\widetilde{\mathfrak{C}}_0$ is reduced.
It remains to prove normality of $\widetilde{\mathfrak{C}}$. Lemma 2.6 below gives a precise description of the singular locus of the covering space of a finite covering that is branched along a divisor. In particular, after performing the final step of this procedure, Lemma 2.6 tells us that the total space can be singular in points only, since the last branch divisor has reduced components intersecting in points only. Thus, since $\widetilde{\mathfrak{C}}$ is locally a subvariety of $\mathbb{C}^3$ defined by one equation and is smooth in codimension one, it is normal by Serre's criterion for normality, cf. [Liu02, Theorem 8.2.23]. To sum up, the procedure described above yields a normal total space $\widetilde{\mathfrak{C}}$ and a central fibre $\widetilde{\mathfrak{C}}_0$ as asserted, that is, reduced with smooth components intersecting transversally.

We now argue that the result is the same as when taking the base change φ of order l followed by passing to the normalisation $\mathfrak{C}'$ of $\varphi^*\mathfrak{C}$. That is, we want to show that $\widetilde{\mathfrak{C}} = \mathfrak{C}'$. This follows already from the uniqueness of semistable reduction that we will prove independently in step four. The remainder of this step gives an alternative and direct proof of this fact but is logically not necessary for the proof. Consider the following diagram

$$\begin{tikzcd}
\mathfrak{C}' \arrow[r, equal, "\sim"] \arrow[ddd, "\psi"'] & \widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}^r \arrow[dr, dotted] \arrow[dddl, "\mu"] & & & \\
& & \widetilde{\mathfrak{C}}^2 \arrow[d, "\psi_2"] \arrow[dr, "\eta_2"] \arrow[dd, bend right=60, "(\psi_1\times\mathrm{id})|_{\mathfrak{C}^2}\circ\psi_2"'] & & \\
& & \mathfrak{C}^2 = \widetilde{\mathfrak{C}}^1 \times_{B_1} B_2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "(\psi_1\times\mathrm{id})|_{\mathfrak{C}^2}"] & \widetilde{\mathfrak{C}}^1 \arrow[d, "\psi_1"] \arrow[dr, "\eta_1"] & \\
\varphi^*\mathfrak{C} \arrow[rr, dotted] \arrow[dr, "\mathrm{pr}_2"] & & (\varphi_1 \circ \varphi_2)^*\mathfrak{C} = \widetilde{\mathfrak{C}}^1 \times_{B_1} B_2 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathfrak{C}^1 = \widetilde{\mathfrak{C}}^0 \times_{B_0} B_1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \widetilde{\mathfrak{C}}^0 = \mathfrak{C} \arrow[d] \\
& B' = B_r \arrow[r, dotted] & B_2 \arrow[r, "\varphi_2"] & B_1 \arrow[r, "\varphi_1"] & B_0 = B
\end{tikzcd}$$

showing $\mu = (\psi_1 \times \mathrm{id}) |_{\mathfrak{C}^2} \circ \cdots \circ (\psi_{r-1} \times \mathrm{id}) |_{\mathfrak{C}^r} \circ \psi_r$. Then μ is a finite birational morphism, since every of the ψ_i is finite and birational. This gives two maps $\psi : \widehat{\mathfrak{C}} \to \varphi^*\mathfrak{C}$ and $\mu : \widetilde{\mathfrak{C}} \to \varphi^*\mathfrak{C}$, both finite regular morphisms from a normal variety to $\varphi^*\mathfrak{C}$ and one of them is the normalisation. It then follows from [Sha13a, Theorem 2.21] that $\mathfrak{C}' \cong \widetilde{\mathfrak{C}}$ as we asserted.
It remains to argue that $\pi' : \mathfrak{C}' \to B'$ is flat. This follows from [Har83, Proposition 9.2, page 254], stating that the finite base change preserves flatness, together with [Har83, Proposition 9.7, page 257], implying that in our particular situation passing to the normalisation of the total space retains flatness, see the argument in step one.
As we will later see, it is sufficient to perform the algorithm up to this point to be able to compute the degenerate Picard scheme of the family $\pi : \mathfrak{C} \to B$.

Step 3 (*Smoothing the total space*).

We now assume that we are in the situation obtained after step two, i.e. that we are given a family $\pi : \mathfrak{C} \to B$ such that the central fibre $\mathfrak{C}_0$ is reduced with smooth components intersecting transversally. The family $\pi : \mathfrak{C} \to B$ can thus be expressed in local coordinates x, y on $\mathfrak{C}$ and t on B by either $t^m = x$ or $t^m = xy$, where the first case appears locally around smooth points of the central fibre and the second case locally around the nodal points of the central fibre. The total space $\mathfrak{C}$ can be singular at most at the singular points of the central fibre $\mathfrak{C}_0$. Consider such a point, i.e. a point $p \in \mathfrak{C}$, where π is given by $t^l = xy$. If $l = 1$, the point p is a smooth point of $\mathfrak{C}$. If $l > 1$, p is a singular point of $\mathfrak{C}$ and its type is A_{l-1}. By taking a minimal resolution $\widetilde{\mathfrak{C}} \to \mathfrak{C}$ of the singularities of $\mathfrak{C}$, we replace each singular point p in $\mathfrak{C}$ of type A_{l-1} by a chain of $(l-1)$ rational curves, see for example [BHPV04, III, section 6 and 7]. The minimal resolution $\widetilde{\mathfrak{C}} \to \mathfrak{C}$ now gives a flat family $\widetilde{\pi} : \widetilde{\mathfrak{C}} \to B$ with smooth total space and central fibre $\widetilde{\mathfrak{C}}_0$ defining a reduced simple normal crossing divisor in $\widetilde{\mathfrak{C}}$. Flatness of this family follows again from [Har83, Proposition 9.7, page 257].

Step 4 (*Obtaining a (semi-)stable central fibre*). To obtain the semistable reduction, we have to blow down the smooth rational components of $\widetilde{\mathfrak{C}}_0$ that meet the rest of $\widetilde{\mathfrak{C}}_0$ only once. This yields the asserted family $\pi' : \mathfrak{C}' \to B'$ with semistable central fibre. To obtain a stable reduction, we also need to blow down all smooth rational components of $\mathfrak{C}'_0$ that meet the rest of $\mathfrak{C}'_0$ only twice. We refer to [HM98, Section 3.C] and [ACG11, Chapter X, section 4] for further details.

Step 5 (*Uniqueness*). It is sufficient to prove uniqueness for the central fibre of a stable reduction. The asserted uniqueness of the central fibre of a semistable reduction, that is, uniqueness up to contractions of smooth rational components of the central fibre that meet the rest of the central fibre in fewer than three points, is then an immediate consequence. For a proof of uniqueness of the central fibre of a stable reduction see [ACG11, Chapter X, section 5]. □

2.5 Remark. Our assumptions on the family $\pi : \mathfrak{C} \to B$ are more restrictive than necessary. See [HM98, Chapter 3] for a more general version of Theorem 2.2 and Theorem 2.3.

In the proof we used the the following Lemma 2.6 and Corollary 2.7.

2.6 Lemma. *Let $\eta : S \to T$ be a b-sheeted cover of a variety T that is branched along a divisor $B = \{f = 0\}$. Then each singular point of S lies over a singular point of T or a singular point of B.*

Proof. We may assume that S, T are affine, say $T = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]/\langle g\rangle)$, where $g = (g_1, \ldots, g_r)$ are defining equations for T, by working algebraically locally. The singular points of T are the points where the rank of the Jacobian of g, $\operatorname{Jac}(g)$, is not maximal. The b-sheeted covering of the ambient $\mathbb{C}^n$ branched along $B = \{f = 0\}$ is given by $\operatorname{Spec}(\mathbb{C}[t, x_1, \ldots, x_n]/\langle t^b - f\rangle) \to \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n])$. We have the following commutative diagram

$$\begin{tikzcd}
T = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]/\langle g\rangle) \arrow[r] & \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]) \\
S = \operatorname{Spec}(\mathbb{C}[t, x_1, \ldots, x_n]/(\langle g\rangle + \langle t^b - f\rangle)) \arrow[u, "\eta"] \arrow[r] & \operatorname{Spec}(\mathbb{C}[t, x_1, \ldots, x_n]/\langle t^b - f\rangle). \arrow[u]
\end{tikzcd}$$

The covering space S is singular in all points where the rank of the Jacobian matrix of its defining equations is not maximal. Since

$$\operatorname{Jac}(g, t^b - f) = \begin{pmatrix} \frac{\partial g_1}{\partial t} & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_r}{\partial t} & \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_n} \\ \frac{\partial (t^b - f)}{\partial t} & \frac{\partial (t^b - f)}{\partial x_1} & \cdots & \frac{\partial (t^b - f)}{\partial x_n} \end{pmatrix} = \left(\begin{array}{c|c} 0 & \\ \vdots & \operatorname{Jac}(g) \\ 0 & \\ \hline bt^{b-1} & \operatorname{Jac}(f) \end{array}\right),$$

we have that a point of S is a singular point if and only if

a) the rank of $\operatorname{Jac}(g)$ or of $(bt^{b-1}, \operatorname{Jac}(f))$ is not maximal, or,

b) $t = 0$, $\mathrm{Jac}(g)$ and $\mathrm{Jac}(f)$ have maximal rank, but $\mathrm{Jac}(f)$ is a linear combination of rows of $\mathrm{Jac}(g)$.

If $t = 0$ and $\mathrm{Jac}(f)$ is a linear combination of rows of $\mathrm{Jac}(g)$, there exist $\alpha_{i,j}$ such that

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{r} \alpha_{i,j} \frac{\partial g_j}{\partial x_i},\ i = 1, \ldots, n.$$

But this implies that on $T = \{g_1 = \cdots = g_r = 0\}$ the function f is constant and since $B \neq \emptyset$, f vanishes identically on all of T. This contradicts the assertion that $B = \{f = 0\}$ is a divisor in T, thus case b) cannot occur. Points in which $\mathrm{rank}(\mathrm{Jac}(g))$ is not maximal are the singular points of T; points in which $\mathrm{rank}(bt^{b-1}, \mathrm{grad}(f))$ is not maximal correspond to the singular points of B. Consequently, every singular point of S corresponds to either a singular point of T or one of B. □

2.7 Corollary. *Let $\eta : S \to T$ be a b-sheeted cover of a smooth variety T that is branched along a divisor $B \subset T$. Then there is a one-to-one correspondence between the singular points of B and those of S.*

As we mentioned in the proof of Theorem 2.3, for computing the degenerate Picard scheme of the family $\pi : \mathfrak{C} \to B$, it is sufficient to perform steps one and two from the proof of Theorem 2.3. This is, as we will prove in the following section, because performing these steps enables us to identify the, besides the normalisation $\mathfrak{C}_0^\nu$, only non-rational component T of the central fibre of a semistable reduction, the so-called tail of the semistable reductions

2.8 Definition (Tail reduction for families of curves). *The family obtained by performing the algorithm from the proof of Theorem 2.3 up to step two is called the tail reduction of the initial family $\pi : \mathfrak{C} \to B$.*

2.2 Computation of semistable reduction for curves with ADE-singularities

We compute semistable reductions for smoothings of curves with a unique singular point of type A_k explicitly. This enables us to describe the degenerate Picard scheme of such curves in section 2.3, and also turns out to be related to the degenerate Picard scheme of $F(X)$ when $X \subset \mathbb{P}^4$ is a singular cubic threefold with a unique singular point of type A_k, cf. section 2.4 for the case $k = 2$ and section 3.3 for the general case.

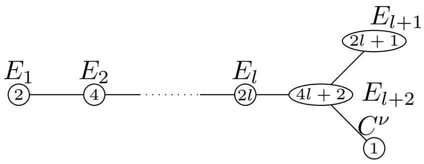
Let $\pi : \mathfrak{C} \to B$ be a family of curves defining a smoothing of $\mathfrak{C}_0$ and assume that $\mathfrak{C}_0$ has a unique singular point of type A_k. Following the algorithm from the proof of Theorem 2.3, we need to begin by taking successive blowups of $\mathfrak{C}$ at the singular point of $\mathfrak{C}_0$ until we have resolved the singularity of $\mathfrak{C}_0$. Table 2.1 displays, in dependence of the singularity type of the singular point of $C = \mathfrak{C}_0$, the singularities of the strict transform $\widetilde{C}$ and total transform $\widetilde{C} \cup E$ after a single blowup of the singular point of C. Here, E denotes the exceptional divisor of the blowup. For a discussion of multiplicities see Lemma 2.9 below.

C	A_1	A_2	$A_{k,k\geq 3}$	D_4	D_5	$D_{k,k\geq 6}$	E_6	E_7	E_8
$\widetilde{C}$	$2A_0$	A_0	A_{k-2}	$3A_0$	$2A_0$	$A_0 + A_{k-5}$	A_0	A_1	A_2
$\widetilde{C} \cup E$	$2A_1$	A_3	D_{k+1}	$3A_1$	$A_1 + A_3$	$A_1 + D_{k-2}$	A_5	D_6	E_7

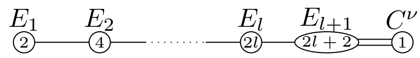
Table 2.1: Singularities of strict and total transform of a singular curve, [Dim87, Table 10.31]

It is simple to obtain this table using the local normal forms of ADE-singularities from Table 1.1 and by calculating the respective blowups locally. Table 2.1 already yields a precise description of the mutual intersections of the components of the total transform under a strong embedded resolution of a curve singularity of ADE-type. What cannot be deduced from Table 2.1 are the multiplicities of the components of the total transform under a strong embedded resolution.

2.9 Lemma. *Let C be a singular curve with a unique singular point of type A_k. Then, the central fibre of a strong embedded resolution of C has the following dual graph, depending on the parity of k,*



A_{2l}



A_{2l+1}

wherein E_i denotes the exceptional divisor from the i-th blowup and $l = \lceil \frac{k-1}{2} \rceil$. Each vertex of the dual graph corresponds to an irreducible component of the total transform of C, two vertices are joined by an edge for every irreducible component of their mutual intersection, and the circled numbers indicate the multiplicities of the respective component.

Proof. The curve C is locally around its singular point analytically isomorphic to the curve

$$\{x^2 - y^{k+1} = 0\} \subset \mathbb{C}^2,$$

where x, y denote the coordinates on $\mathbb{C}^2$. We compute successive blowups at the singular point of C and its strict transforms. Here, $x, y, u, v, \bar{x}, \bar{y}, \bar{u}, \bar{y}$ always denote local coordinates in the respective affine charts.
Blowing up the singular point yields the following total transforms in the respective charts:

$$\begin{aligned} &\text{chart 1:} \quad && u^2(1 - v^2u^{k-1}) = 0, \\ &\text{chart 2:} \quad && \bar{v}^2(\bar{u}^2 - \bar{v}^{k-1}) = 0. \end{aligned}$$

The first chart doesn't contain the singular point of the strict transform, therefore remains unchanged under succeeding blowups. Blowing up the second chart in the singular point of the strict transform, which now is of type A_{k-2}, gives

$$\begin{aligned} &\text{chart 2.1:} \quad && x^4y^2(1 - y^{k-1}x^{k-3}) = 0, \\ &\text{chart 2.2:} \quad && \bar{y}^4(\bar{x}^2 - \bar{y}^{k-3}) = 0. \end{aligned}$$

In chart 2.1 we see the transversal intersection of the exceptional divisor E_1 from the first blowup with the exceptional divisor E_2 from the second blowup. Moreover, the strict transform in this chart is smooth and the exceptional divisor E_1 does not contain the centre of the succeeding blowup, therefore remains unchanged under further blowups. In particular, it retains its multiplicity. We can moreover see that the situation for chart 2.2 is the same as for chart 2 of our initial blowup but with the multiplicity of the exceptional divisor raised by two. This enables us to conclude by performing a simple induction that the multiplicities of the exceptional divisors appearing until the strict transform is smooth, are the ones we asserted. If $k = 2l+1$ is odd, this proves the assertion already, since the components of the total transform under these blowups intersect each other transversally. If $k = 2l$ is even, the resulting charts after $l = \lceil \frac{k-1}{2} \rceil$ blowups are

$$\begin{aligned} &\text{chart } l.1: \quad && x^{2l}y^{2l-2}(1 - y^3x) = 0, \\ &\text{chart } l.2: \quad && \bar{y}^{2l}(\bar{x}^2 - \bar{y}) = 0. \end{aligned}$$

The total transform admits a singular point of type A_3 in the second of these charts and this point is the point of intersection of the strict transform $\bar{x}^2 - \bar{y} = 0$ with the exceptional divisor $E_l = \{\bar{y}^{2l} = 0\}$ from the last blowup. Blowing up the point of intersection affects only the second of these charts, as the first one does not contain the centre of blowup. We obtain

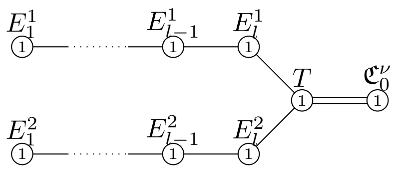
$$\begin{aligned} &\text{chart } (l+1).1: && v^{2l+1}(u^2 v - 1) = 0, \\ &\text{chart } (l+1).2: && \bar{v}^{2l}\bar{u}^{2l+1}(\bar{u} - \bar{v}) = 0. \end{aligned}$$

Therefore, the exceptional divisor $E_{l+1} =$ has multiplicity $2l+1$. The total transform admits a singular point of type D_4 in the second of these charts, we thus need to perform another blowup. Again, the first of these charts does not contain the centre of blowup. The resulting local equations are

$$\begin{aligned} &\text{chart } (l+2)\ .1: && x^{2l+1}y^{4l+2}(x-1) = 0, \\ &\text{chart } (l+2)\ .2: && \bar{y}^{2l}\bar{x}^{4l+2}(1-\bar{y}) = 0. \end{aligned}$$

The exceptional divisor E_{l+2} consequently appears with multiplicity $4l+2$. Moreover, the total transform admits singularities of type A_1 only, that is, it defines a simple normal crossing divisor in the blown-up total space. This completes the proof. □

2.10 Lemma. *Let $\mathfrak{C}_0$ admit a unique singular point of type A_k and assume that $k = 2l+1$ is odd. If $\pi' : \mathfrak{C}' \to B'$ denotes the tail reduction of the family $\pi : \mathfrak{C} \to B$, the dual graph of $\mathfrak{C}'_0$ is*



where T is a smooth curve of genus $g(T) = l$, $\mathfrak{C}_0^\nu$ denotes the normalisation of $\mathfrak{C}_0$ and every other component is a smooth rational curve.

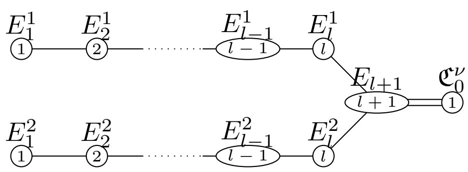
Proof. After passing to a strong embedded resolution, we can assume that the central fibre has dual graph as in Lemma 2.9. Following the algorithm from the proof of Theorem 2.3, we need to consider the least common multiple of the multiplicities of the components of the central fibre. Note that these multiplicities are $2, 4, 6, \ldots, 2l, 2(l+1), 1$, thus the least common multiple of this numbers is divisible by two,

$$2 \mid \operatorname{lcm}(1, 2, 4, 6, \ldots, 2(l+1)).$$

Let η_1 be the branched cover of order two branched along the divisor D^0 reduced modulo two, where D^0 denotes the divisor corresponding to the central fibre. It is clear that the branch divisor has only one component and that this component is $\mathfrak{C}_0^\nu$. Following the description of $\eta_1^*(D_j^0)$ from the proof of Theorem 2.3, we obtain that $g(\eta_1^*(E_{l+1})) = 0$ and that

$$\eta_1^*(2E_1 + 4E_2 + \cdots + 2lE_l) = (E_1^1 + \cdots + lE_l^1) + (E_1^2 + \cdots + lE_l^2)$$

with corresponding dual graph



wherein the multiplicities of the components $E_1^1, \dots, E_l^1, E_1^2, \dots, E_l^2$ are the natural numbers up to l. To see what happens to these components, note that the Riemann-Hurwitz formula (2.2) yields $g(\eta_i^*(D_j^{i-1})) = 0$ whenever the number r of intersection points with the branch locus is $r = 2$ and $g(D_j^{i-1}) = 0$. For every prime p and natural number $m \in \{1, \dots, l\}$, it is obvious that

$$m \mod p = 0 \Longrightarrow (m-1) \mod p \neq 0, \quad (m+1) \mod p \neq 0.$$

But this means that for each component $E \in \{E_1^1, E_1^2, \dots, E_l^1, E_l^2\}$ of the central fibre and any of the branched covers η of prime order from the proof of Theorem 2.3, E is either a component of the branch locus or intersects the branch locus in precisely two points. In both cases, the genus of E stays zero through all of the tail reduction process.

It remains to describe the effect of the respective branched coverings on the component E_{l+1}. This component always intersects the branch divisor in at least two points, since $\mathfrak{C}_0^\nu$ necessarily is a component of the branch locus for every prime. For a prime p there are two possibilities. Either $l+1 \mod p = 0$ and $l \mod p \neq 0$, or $l+1 \mod p \neq 0$. In the second case, E_{l+1} is a component of the branch divisor, therefore retains its genus when taking the pullback under the respective branched cover of order p from the proof of Theorem 2.3. In the first case, E_{l+1} is not a component of the branch divisor but intersects the branch divisor in precisely four points. Write $l+1 = p_1 \cdot \dots \cdot p_r$ with $p_1, \dots, p_r$ prime. If the number of intersection points with the branch divisor is four, the Riemann-Hurwitz formula (2.2) gives

$$g(\eta_i^{-1}(D_j^{i-1})) = p_i g(D_j^{i-1}) + p_i - 1, \tag{2.3}$$

where η_i is as in the proof of Theorem 2.3. In particular,

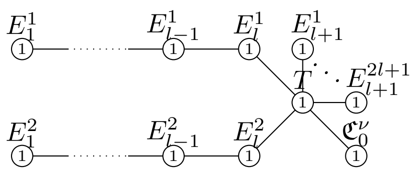
$$\begin{aligned} g((\eta_r \circ \eta_{r-1})^{-1}(D_j^{r-2})) &= p_r(p_{r-1} g(D_j^{r-2}) + p_{r-1} - 1) + p_r - 1 \\ &= p_r p_{r-1} g(D_j^{r-2}) + p_r p_{r-1} - 1, \end{aligned}$$

and, since $g(D_j^0) = 0$, by iterating this formula,

$$g(T) = g((\eta_r \circ \dots \circ \eta_1)^{-1}(E_{l+1})) = \prod_{i=1}^r p_i - 1 = l + 1 - 1 = l,$$

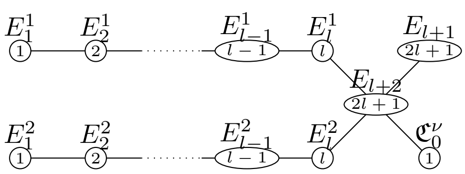
where $T = (\eta_r \circ \dots \circ \eta_1)^{-1}(E_{l+1})$. □

2.11 Lemma. *Let $\mathfrak{C}_0$ admit a unique singular point of type A_k and assume that $k = 2l$ is even. If $\pi : \mathfrak{C}' \to B'$ denotes the tail reduction of the family $\pi : \mathfrak{C} \to B$, the dual graph of $\mathfrak{C}_0'$ is*



where T is a smooth curve of genus $g(T) = l$, $\mathfrak{C}_0^\nu$ denotes the normalisation of $\mathfrak{C}_0$ and every other component is a smooth rational curve.

Proof. The argument for the components $E_1, \dots, E_l$ is the same as in the proof of Lemma 2.10. After taking the initial branched covering of order two, the dual graph of the central fibre is



Note that, since the multiplicity of the component E_{l+1} is odd, this component retains its multiplicity. Moreover, for every prime p, E_{l+1} cannot intersect the branch locus of the branched covering of order p in finitely many points as is clear from the dual graph given above, and therefore gives rise to $2l+1$ disjoint copies of E_{l+1} in the central fibre of the tail reduction.

It remains to discuss the component E_{l+2}. If p is any prime either $2l+1 \mod p = 0$ and $l \mod p \neq 0$ or $2l+1 \mod p \neq 0$ and $l \mod p = 0$. (In fact, if $l = \prod q_i$ with q_i prime, then $2l+1 = 2\prod q_i + 1$ and $l \mod p = 0$ implies that $2l+1 \mod p \neq 0$. Conversely, if $2l+1 \mod p = 0$, then $l \mod p \neq 0$ follows, by assuming the contrary, from the argument we just gave.) If $2l+1 \mod p \neq 0$ and $l \mod p = 0$, we have that E_{l+2} is a component of the branch divisor and therefore retains its multiplicity. In the other case, where $2l+1 \mod p = 0$ and $l \mod p \neq 0$, the intersection of E_{l+2} with the branch divisor consists of precisely three points. Just as in the proof of Lemma 2.10, we can compute the genus of $T = (\eta_1 \circ \cdots \circ \eta_r)^{-1}(E_{l+2})$, where $l = p_1 \cdots p_r$ and the η_i are as in the proof of Theorem 2.3, using an iterative formula coming from the Riemann-Hurwitz formula (2.2) with $r = 3$. The formula obtained this way is the same as (2.3) but with both genera multiplied by a factor of two. Since $g(D_j^0) = 0$, this doesn't affect the calculations we made in the proof of Lemma 2.10 and we likewise obtain $g(T) = l$. $\square$

2.12 Remark. It can be shown, using the same arguments as in [HM98, page 126], that the stable reduction of a family $\mathfrak{C} \to B$ as above with the central fibre having a unique singular point of type A_k has two components, one being the normalisation $\mathfrak{C}_0^\nu$ of $\mathfrak{C}_0$ and the other a smooth curve T of genus $g(T) = \lfloor \frac{k-1}{2} \rfloor$. The intersection of these two components is a single point, if k is even, and two distinct points, if k is odd. In other words, that the stable reduction is obtained by contracting the smooth rational components E_i^j in the notation of Lemma 2.10 and Lemma 2.11. A computation of the curves T can also be found in [Has00a].

2.13 Definition (Tail of a curve with respect to a degeneration)**.** *Let $\pi : \mathfrak{C} \to B$ be a smoothing of a curve $C = \mathfrak{C}_0$ with a unique singular point of type A_k and denote by $\pi' : \mathfrak{C}' \to B'$ the tail reduction of $\pi : \mathfrak{C} \to B$. Then the curve T in the notation of Lemma 2.10 and Lemma 2.11 is called the tail of $\mathfrak{C}_0$ with respect to the family $\pi : \mathfrak{C} \to B$.*

2.3 Degenerate Picard schemes of singular curves

In this section we utilise the algorithm from the proof of Theorem 2.3 to compute the degenerate Picard scheme of a curve with a unique singular point of type A_k. Let $\pi : \mathfrak{C} \to B$ be a smoothing of such a curve and denote by $\pi' : \mathfrak{C}' \to B'$ the tail reduction of $\pi : \mathfrak{C} \to B$. Thus, $\mathfrak{C}'_0$ is reduced with smooth components intersecting transversally. We denote the tail of $\mathfrak{C}_0$ with respect to the family $\pi : \mathfrak{C} \to B$ by T.

2.14 Lemma (Picard group of reducible varieties)**.** *Let $Y = Y_1 \cup Y_2$ be a compact, reducible variety such that $Y_1 \cap Y_2$ is connected, non-empty, Y_1, Y_2 are smooth along $Y_1 \cap Y_2$, and the intersection of Y_1, Y_2 along $Y_1 \cap Y_2$ is transverse. Then,*

$$\operatorname{Pic}(Y) \cong \operatorname{Pic}(Y_1) \times_{\operatorname{Pic}(Y_1 \cap Y_2)} \operatorname{Pic}(Y_2). \tag{2.4}$$

The same identity holds for the Picard scheme.

Proof. Consider the exact sequence

$$1 \to \mathcal{O}_Y^* \to \mathcal{O}_{Y_1}^* \times \mathcal{O}_{Y_2}^* \to \mathcal{O}_{Y_1 \cap Y_2}^* \to 1$$

in which the map $\mathcal{O}_{Y_1}^* \times \mathcal{O}_{Y_2}^* \to \mathcal{O}_{Y_1 \cap Y_2}^*$ is given by $(\lambda, \mu) \mapsto \lambda \mid_{Y_1 \cap Y_2} \cdot \mu \mid_{Y_1 \cap Y_2}^{-1}$. The associated long exact sequence in cohomology starts as follows

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\alpha} \mathbb{C}^* \longrightarrow \operatorname{Pic}(Y) \xrightarrow{\beta} \operatorname{Pic}(Y_1) \times \operatorname{Pic}(Y_2) \xrightarrow{\gamma} \operatorname{Pic}(Y_1 \cap Y_2),$$

where we used the assumptions on $Y_1 \cap Y_2$ to obtain $H^0(Y_1 \cap Y_2, \mathcal{O}^*_{Y_1\cap Y_2}) \cong \mathbb{C}^*$ and identified isomorphism classes of analytic line bundles with algebraic ones using Serre's GAGA theorems [Ser56]. The map α is surjective, so we may end the sequence on the left by $0 = \operatorname{coker}(\alpha)$. Furthermore, $\beta(L) = (L\mid_{Y_1}, L\mid_{Y_2})$ and $\gamma(L, M) = L\mid_{Y_1\cap Y_2} \otimes M\mid^{\vee}_{Y_1\cap Y_2}$ and we may end the sequence on the right by $\operatorname{img}(\beta) = \ker(\gamma) = \operatorname{Pic}(Y_1) \times_{\operatorname{Pic}(Y_1\cap Y_2)} \operatorname{Pic}(Y_2)$. Altogether this proves the assertion. □

2.15 Corollary. *If the intersection $Y_1 \cap Y_2$ in Lemma 2.14 has two connected components, there is an exact sequence*

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(Y_1) \times_{\operatorname{Pic}(Y_1\cap Y_2)} \operatorname{Pic}(Y_2) \longrightarrow 0. \tag{2.5}$$

Proof. The map $\alpha : H^0(\mathcal{O}^*_{Y_1} \times \mathcal{O}^*_{Y_2}) \to H^0(\mathcal{O}^*_{Y_1\cap Y_2})$ in the proof of Lemma 2.14 maps to the diagonal in $H^0(\mathcal{O}^*_{Y_1\cap Y_2}) \cong \mathbb{C}^* \times \mathbb{C}^*$, therefore is no longer surjective. The remainder of the proof carries over verbatim. □

2.16 Remark. If the intersection $Y_1 \cap Y_2$ in Lemma 2.14, respectively Corollary 2.15 consists of points only, $H^1(Y_1 \cap Y_2, \mathcal{O}^*_{Y_1\cap Y_2}) = 0$ and the map $\beta : \operatorname{Pic}(Y) \to \operatorname{Pic}(Y_1) \times \operatorname{Pic}(Y_2)$ from the proof of Lemma 2.14 is always surjective. Therefore, the fibre product in (2.4) and (2.5) then equals the ordinary product.

As all rational components of $\mathfrak{C}'_0$ intersect the rest of the $\mathfrak{C}'_0$ in precisely one point, it follows from Lemma 2.14 that $\operatorname{Pic}^0(\mathfrak{C}'_0) \cong \operatorname{Pic}^0(\mathfrak{C}^\nu_0 \cup T)$. If k is even, the intersection $\mathfrak{C}^\nu_0 \cap T$ is a single point and Lemma 2.14 identifies $\operatorname{Pic}^0(\mathfrak{C}'_0)$ with $\operatorname{Pic}^0(\mathfrak{C}^\nu_0) \times \operatorname{Pic}^0(T)$. If k is odd, the intersection $\mathfrak{C}^\nu_0 \cap T$ consists of two distinct points p_1, p_2 and by Corollary 2.15 we find an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \operatorname{Pic}^0(\mathfrak{C}'_0) \longrightarrow \operatorname{Pic}^0(\mathfrak{C}^\nu_0) \times \operatorname{Pic}^0(T) \longrightarrow 0. \tag{2.6}$$

Let $\mathcal{A}_g$ denote the moduli space of principally polarised Abelian varieties of dimension g and let further $\mathcal{A}^{\mathrm{Vor}}_g$ denote the second Voronoi compactification of $\mathcal{A}_g$. Every point of $\mathcal{A}^{\mathrm{Vor}}_g$ is completely determined by a set of degeneration data by results due to G. Faltings and C.-L. Chai [FC90, Chapter II] and V. Alexeev [Ale02]. Following the notation of [ABH02, section 2.1] we denote these degeneration data by (d0)–(d4). For k even, $\operatorname{Pic}^0(\mathfrak{C}'_0) \in \mathcal{A}_g$ and we do not have do deal with these degeneration data at all, since they are all trivial in this case. On the other hand, if k is odd, $\operatorname{Pic}^0(\mathfrak{C}'_0) \notin \mathcal{A}_g$ but the exact sequence (2.6) determines the degeneration data (d1) and, as (2.6) defines an extension of an element in $\mathcal{A}_{g-1}$ by a $\mathbb{C}^*$, in fact also the degeneration data (d0), (d2)–(d4) and therefore determines a unique point $\overline{\operatorname{Pic}^0(\mathfrak{C}'_0)} \in \mathcal{A}^{\mathrm{Vor}}_g$. The point $\overline{\operatorname{Pic}^0(\mathfrak{C}'_0)}$ corresponds to a semi-abelic variety containing $\operatorname{Pic}^0(\mathfrak{C}'_0)$ as an dense open subset.
Therefore, $\operatorname{Pic}^0(\mathfrak{C}'_0)$ determines a point of $\mathcal{A}^{\mathrm{Vor}}_g$ in both cases.
Since $\mathcal{A}^{\mathrm{Vor}}_g$ is a coarse moduli space, the family $\pi' : \mathfrak{C}' \to B'$ gives rise to a morphism

$$j' : B' \to \mathcal{A}^{\mathrm{Vor}}_g,\ b \mapsto \operatorname{Pic}^0(\mathfrak{C}'_b)$$

with $j'(0) = \overline{\operatorname{Pic}^0(\mathfrak{C}'_0)}$, the semi-abelic variety determined by $\operatorname{Pic}^0(\mathfrak{C}'_0)$. On the other hand, we have a morphism

$$j : B^* \to \mathcal{A}_g \subset \mathcal{A}^{\mathrm{Vor}}_g,\ b \mapsto \operatorname{Pic}^0(\mathfrak{C}_b)$$

that extends to a morphism $\bar{j} : B \to \mathcal{A}^{\mathrm{Vor}}_g$ by [Har83, I, Proposition 6.8]. Consider the diagram

$$\begin{tikzcd} B' \arrow[r, "j'"] \arrow[d, "\varphi"'] & \mathcal{A}^{\mathrm{Vor}}_g \\ B \arrow[ur, "\bar{j}"'] & \end{tikzcd}$$

with φ the branched cover of order l totally ramified over $0 \in B$ as in the proof of Theorem 2.3. Obviously, $(\bar{j} \circ \varphi) \mid_{(B')^*} = (j') \mid_{(B')^*}$, as the families $\mathfrak{C} \to B$ and $\mathfrak{C}' \to B'$ only differ in the central fibre. In other words, j' is a lift of $\bar{j}$ via φ. But this already implies $j'(0) = \bar{j}(0)$ by [Har83, I, Lemma 4.1].

2.17 Definition (Degenerate Picard scheme)**.** *With notations as above, we call the point $\bar{j}(0) \in \mathcal{A}_g^{\mathrm{Vor}}$ the degenerate Picard scheme of $\mathfrak{C}_0$ with respect to the family $\pi : \mathfrak{C} \to B$ and denote it by*

$$\bar{j}(0) = \lim_{b \to 0} \mathrm{Pic}^0(\mathfrak{C}_b) = \overline{\mathrm{Pic}^0(\mathfrak{C}'_0)}.$$

2.18 Remark. The degenerate Picard scheme $\lim_{b \to 0} \mathrm{Pic}^0(\mathfrak{C}_b)$ can indeed be understood as limit of points in $\mathcal{A}_g^{\mathrm{Vor}}$, justifying our notation. Moreover, it determines a point of Mumford's partial compactification $\mathcal{A}'_g$ of $\mathcal{A}_g$.

Our discussion above provides the following corollary.

2.19 Corollary (Degenerate Picard schemes of singular curves)**.** *Let $\pi : \mathfrak{C} \to B$ be a smoothing of a curve $\mathfrak{C}_0$ with a unique singular point of type A_k. Then, the degenerate Picard scheme of $\mathfrak{C}_0$ with respect to the family $\pi : \mathfrak{C} \to B$ is completely determined by the exact sequence*

$$1 \longrightarrow K \longrightarrow \mathrm{Pic}^0(\mathfrak{C}'_0) \longrightarrow \mathrm{Pic}^0(\mathfrak{C}_0^{\nu}) \times \mathrm{Pic}^0(T) \longrightarrow 0,$$

where T is a smooth curve of genus $g(T) = \lfloor \frac{k-1}{2} \rfloor$ and $\pi' : \mathfrak{C}' \to B'$ is the tail reduction of $\pi : \mathfrak{C} \to B$. Moreover,

$$K = \begin{cases} \mathbb{C}^*, & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

2.20 Corollary (Degenerate Picard schemes of the Hilbert square of a singular curve)**.** *Let Y be a curve with a unique singular point of type A_k and let $\mathfrak{Y} \to B$ be a smoothing of $Y = \mathfrak{Y}_0$. Then, as points of $\mathcal{A}_g^{\mathrm{Vor}}$,*

$$\lim_{b \to 0} \mathrm{Pic}^0(\mathrm{Hilb}^2(\mathfrak{Y}_b)) = \lim_{b \to 0} \mathrm{Pic}^0(\mathfrak{Y}_b).$$

Proof. The relative Hilbert scheme $\pi : \mathfrak{H} \to B$ with $\mathfrak{H}_b = \mathrm{Hilb}^2(\mathfrak{Y}_b)$ is flat over B by a result of A. Grothendieck [Gro62, IV]. Consider the maps

$$j_{\mathfrak{Y}} : B \setminus \{0\} \to \mathcal{A}_g, \; b \mapsto \mathrm{Pic}^0(\mathfrak{Y}_b),$$
$$j_{\mathfrak{H}} : B \setminus \{0\} \to \mathcal{A}_g, \; b \mapsto \mathrm{Pic}^0(\mathfrak{H}_b),$$

and denote by $\bar{j}_{\mathfrak{Y}}$, $\bar{j}_{\mathfrak{H}}$ their extensions to $\mathcal{A}_g^{\mathrm{Vor}}$. Since $\mathrm{Hilb}^2(\mathfrak{Y}_b) \cong \mathrm{Sym}^2(\mathfrak{Y}_b)$ for all $b \neq 0$ and $\mathrm{Pic}^0(\mathrm{Sym}^2(\mathfrak{Y}_b)) \cong \mathrm{Pic}^0(\mathfrak{Y}_b)$ for all $b \neq 0$ by Lemma 2.26, we have that $\bar{j}_{\mathfrak{Y}}(b) = \bar{j}_{\mathfrak{H}}(b)$ holds for $b \neq 0$. But then, $\bar{j}_{\mathfrak{Y}}(0) = \bar{j}_{\mathfrak{H}}(0)$ by [Har83, I, Lemma 4.1] proving the assertion. ☐

2.4 Tail reduction for transverse curve singularities

As we have shown in the previous section, computing the central fibre of a tail reduction of a family of curves enables us to compute the degenerate Picard scheme of its central fibre. In this section, we show that the algorithm from the proof of Theorem 2.3, which we used to compute the tail reduction, can be generalised to varieties with transverse curve singularities along a smooth singular locus. As a particular example of this, we consider the Fano scheme of lines on a cubic threefold X with a unique singular point p_0 of type A_1 or A_2. The singular locus Σ of $F(X)$ is then smooth, cf. Lemma 1.5, and $F(X)$ admits transversally along Σ a curve singularity of the same type as (X, p_0), cf. Theorem 1.29. We prove that the degenerate Picard scheme of $F(X)$ is the same as the degenerate Picard scheme of a curve C with unique singular point of the same type as (X, p_0).

2.21 Lemma (Strong embedded resolution for transverse curve singularities along a smooth singular locus)**.** *Let $Z_0 \subset Z$ be varieties. Let $S \subset Z$ be a variety such that its intersection S_0 with Z_0 is the singular locus of Z_0, i.e. $S_0 = S \cap Z_0 = (Z_0)_{\mathrm{sing}}$. If, for a fixed $k \in \mathbb{N}$, Z_0 has transverse curve singularities of type A_k along S_0, there exists a variety $\widetilde{Z}$ together with a map $b : \widetilde{Z} \to Z$ that can be factored as a sequence of blowups with centres lying above S and such that $b^{-1}(Z_0)$ is a simple normal crossing divisor in $\widetilde{Z}$. Moreover, all but one component of $b^{-1}Z_0$ are locally trivial algebraic fibre bundles over S with fibre $\mathbb{P}^1$ and the remaining component $\widehat{Z}_0$ of $b^{-1}(Z_0)$ is characterised by the property that $b\,|_{\widehat{Z}_0} : \widehat{Z}_0 \to Z_0$ is a resolution of singularities of Z_0.*

Proof. Consider the blowup $b_S : \mathrm{Bl}_S Z \to Z$ of the ambient variety Z along S. It restricts over Z_0 to $\mathrm{Bl}_{S_0} Z_0 \to Z_0$, the blowup of Z_0 along its singular locus, cf. [Har83, II, Corollary 7.15]. As Z_0 admits transversal singularities of type A_k along S_0, for every point $s \in S_0$ there exists an analytic neighbourhood $U = U(s) \subset Z_0$ such that $U \cong (S_0 \cap U) \times Y$, where $Y = \{x^{k+1} + y^2 = 0\} \subset \mathbb{C}^2$. Since S_0 is assumed to be smooth, we can assume that $S_0 \cap U \cong \Delta \subset \mathbb{C}^p$ for some p, where Δ is a domain, that is, open, connected and non-empty. We thus have an identification

$$U \cong (S_0 \cap U) \times Y \cong \Delta \times \{(x, y) \in \mathbb{C}^2 \mid x^{k+1} + y^2 = 0\} \subset \mathbb{C}^p \times \mathbb{C}^2. \tag{2.7}$$

Taking the blowup $\mathrm{Bl}_{S_0 \cap U} U \to U$ means blowing up the locus $x = y = 0$ in the above. Therefore, taking successive blowups along $S_0 \cap U$ and its respective strict transforms is the same as performing at all points of $S_0 \cap U$ at once an embedded resolution of a curve singularity of type A_k. The resulting total transform of U then has components of the form $(S_0 \cap Us) \times \mathbb{P}^1$ and $(S_0 \cap U) \times Y^\nu$, where Y^ν denotes the normalisation. Their mutual intersection and multiplicities are described by the respective dual graphs in Lemma 2.9 but with edges now being the products we just described and two vertices joined by an edge, if their intersection is isomorphic to $S_0 \cap U$. The local description we gave describes the total transform of Z_0 under the blowup of Z along S completely as the local blowups necessarily glue, cf. [Sha13b, Section 2.2]. We of course cannot expect that the components which locally over U are products of the form $S_0 \cap U$ times a curve also have this property globally and therefore only obtain total spaces of locally trivial analytic fibre bundles over S_0 instead of (global) products. As any locally trivial analytic $\mathbb{P}^1$ bundle over a smooth projective variety is algebraic, see [Har83, II, exercise 7.10], the assertion follows. $\square$

2.22 Theorem (Tail reduction for transverse curve singularities)**.** *Let $\pi : \mathfrak{Z} \to B$ be a flat family of varieties over a smooth curve B such that $\mathfrak{Z}_b = \pi^{-1}(b)$ is smooth for $b \neq 0$ and such that $\mathfrak{Z}_0$ has smooth singular locus $S_0 \subset \mathfrak{z}_0$ and a curve singularity of type A_k transversally along S_0. Then, there exists a branched cover $\varphi : B' \to B$ totally ramified over $0 \in B$ and a flat family of varieties $\pi' : \mathfrak{Z}' \to B'$ that differs from $\pi : \mathfrak{Z} \to B$ only in the central fibre and such that the central fibre $\mathfrak{Z}'_0 \subset \mathfrak{Z}'$ is reduced with smooth components intersecting transversally.*

Proof. Lemma 2.21 shows that step one from the proof of Theorem 2.3 can be generalised to varieties with a transversal curve singularity of type A_k along a smooth singular locus. We therefore can assume that $\mathfrak{Z}$ is smooth, $\mathfrak{Z}_0 \subset \mathfrak{Z}$ defines a simple normal crossing divisor, and that all but one component of $\mathfrak{Z}_0$ are total spaces of locally trivial algebraic $\mathbb{P}^1$-bundles over S. It is then left to show that the second step of the proof of Theorem 2.3 can likewise be generalised.

Let $D = \sum_j a_j D_j$ be the divisor corresponding to the central fibre $\mathfrak{Z}_0 \subset \mathfrak{Z}$. If $p \in \mathfrak{Z}_0$ is any point, it is clear from the local descriptions we gave in the proof of Lemma 2.21 and the smoothness of $\mathfrak{Z}$, that in an analytic neighbourhood $U = U(p) \subset \mathfrak{Z}$ of p in $\mathfrak{Z}$,

$$U \cong \Delta \times \{t = x^a y^b\} \subset \mathbb{C}^p \times \mathbb{C}^2 \times \mathbb{C},$$

where $S_0 \cap U \cong \Delta \subset \mathbb{C}^p$, x, y denote local coordinates on the factor $\mathbb{C}^2$ and t is a local coordinate on $B \cap \pi(U) \cong \Delta' \subset \mathbb{C}$. Let $z = (z_1, \ldots, z_p)$ denote coordinates on the factor $\mathbb{C}^p$ and let $c \in \Delta \subset \mathbb{C}^p$ be a point. Then, $\mathfrak{C} = \mathfrak{Z} \cap U \cap \{z = c\} \to \pi(B)$ is a flat family of curves and we write $\mathcal{D} = \sum_j a_j \mathcal{D}_j$ for the divisor corresponding to the central fibre $\mathfrak{C}_0 \subset \mathfrak{C}$. Note that on U,

$$D \cap U = \sum_j a_j(D_j \cap U) = \sum_j a_j(\Delta \times \mathcal{D}_j) \tag{2.8}$$

holds, as is again clear from the local description we gave in the proof of Lemma 2.21. Let $l = \mathrm{lcm}(a_j \mid j) = p_1 \cdot \dots \cdot p_r$ with $p_1, \dots, p_r$ prime. If H_i denotes the branched covering of $\mathfrak{Z}$ of order p_i branched along $D_{\bmod p_i}$, (2.8) implies that $\mathrm{H}_i \mid_U = \mathrm{id}_\Delta \times \eta_i$, where η_i is the branched cover of $\mathfrak{C}$ of order p_i branched along $\mathcal{D}_{\bmod p_i}$. Thus, the effect of applying H_i is to perform the operations from step two of the proof of Theorem 2.3 at all points of S_0 at once. The result after applying these operations successively for $i = 1, \dots, r$ as in the proof of Theorem 2.3, is a new family $\pi' : \mathfrak{Z}' \to B'$ whose central fibre has dual graph as in Lemma 2.10, if k is odd, and as in Lemma 2.11, if k is even, but with edges corresponding to total spaces of algebraic $\mathbb{P}^1$ bundles over S_0 and two vertices joined by an edge, if they intersect. Each intersection is a smooth curve isomorphic to S_0 by construction. □

Before applying Theorem 2.22 to a smoothing of the Fano scheme of lines on a cubic threefold $X \subset \mathbb{P}^4$ with a unique singular point p_0 of type A_1 or A_2, we describe a resolution of singularities of $F(X)$, cf. the description of the central fibre in Lemma 2.9.

Let $\Sigma = \Sigma_2 \cap \Sigma_3 \subset H \cong \mathbb{P}^3 \subset \mathbb{P}^4$ be the complete $(2,3)$-intersection associated to $X \subset \mathbb{P}^4$, parameterising the lines in X through p_0, as discussed in chapter one. As X admits a unique singular point of type A_1 or A_2, it follows from Lemma 1.5 that Σ is a smooth curve. We construct a morphism $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$ that will turn out to be the normalisation of $F(X)$. For cubic threefolds with a unique singular point of type A_1, this construction goes back to H. Clemens and P. Griffiths, [CG73, section 8], and has also been studied from a more algebraic perspective by A. Collino and J. P. Murre, [CM78].
For every point $\eta = P + Q \in \mathrm{Sym}^2(\Sigma)$ consider the plane $E_\eta = \langle s(P,Q), p_0 \rangle \cong \mathbb{P}^2$, where $s(P,Q)$ denotes the secant to Σ passing through the points P, Q. We interpret $s(P,Q)$ as the unique tangent to Σ at P, if $P = Q$. Since no plane can be contained in X, see Proposition 2.23 below, the intersection $X \cap E_\eta$ defines a plane cubic containing the lines $l_P = \langle P, p_0 \rangle$ and $l_Q = \langle Q, p_0 \rangle$, respectively the line $\langle P, p_0 \rangle$ twice, if $P = Q$. Every plane cubic containing two lines necessarily contains a third and if we denote this third line by l_η, we obtain a map $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$ by setting $\varphi(\eta) = l_\eta$.

2.23 Proposition ([Seg88, section 5 and section 38])**.** *Let $X \subset \mathbb{P}^4$ be a cubic threefold. If X contains a plane, then it contains at least four double points or a point of multiplicity at least three.*

The line l_η defines a singular point of $F(X)$ if and only if $p_0 \in l_\eta$. As the lines in X through p_0 are parameterised by Σ, this is the case if and only if $l_\eta = \langle R, p_0 \rangle$ for some $R \in \Sigma$, that is, if $R \in s(P,Q) \cap \Sigma$. But this means, according to Bézout's Theorem, cf. [Sha13a, 4, section 2.1], that $s(P,Q) \subset \Sigma_2$, as the intersection $s(P,Q) \cap \Sigma_2$ contains, counted with multiplicity, three points. The intersection $s(P,Q) \cap \Sigma_3$ consists of the points P, Q and a third point R, since if a line would be contained in Σ, the cubic threefold X would have an at least one-dimensional singular locus. Consequently, l_η defines a singular point of $F(X)$ if and only if $s(P,Q)$ is a line of a ruling of Σ_2.

We can also construct a rational inverse $\psi : F(X) \dashrightarrow \mathrm{Sym}^2(\Sigma)$ to φ geometrically. If $l \subset X$ is any line not passing through p_0, we can form the plane $E(l) = \langle l, p_0 \rangle \cong \mathbb{P}^2$. The intersection $E \cap X$ again defines a plane cubic and this cubic contains the line l by construction. The residual component of $E \cap X$ is a conic section and since p_0 is a singular point of this conic section, it decomposes as the union of two lines L, L'. Each of these lines is contained in X and passing through p_0. Therefore, L, L' determine points $P, P' \in \Sigma$ and we may define $\psi(l) = P + P' \in \mathrm{Sym}^2(\Sigma)$. It is simple to check that the constructions of φ and ψ reverse each other.

2.24 Lemma. *Let X be a cubic threefold with a unique singular point p_0 of type A_1 or A_2. Then, $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$, as constructed above, is the normalisation.*

Proof. A detailed treatment of the case of a cubic threefold X with unique singular point of type A_1 is given by G. van der Geer and A. Kouvidakis in [vK10], although they do not give an explicit proof of φ being a morphism. We prove finiteness of φ in Lemma 2.25 below and postpone the proof that φ is a morphism until chapter three where we generalise the construction of φ to cubic threefolds with a unique singular point of type A_k for arbitrary k. ☐

2.25 Lemma. *Let X be a cubic threefold with a unique singular point p_0 of type A_1 or A_2 and let $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$ be the normalisation as constructed above. If (X, p_0) is of type A_1, then $\varphi^{-1}(\Sigma) = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are disjoint curves both being isomorphic to Σ. If (X, p_0) is of type A_2, $\varphi^{-1}(\Sigma)$ is a single curve isomorphic to Σ.*

Proof. We begin with the case of a cubic threefold with a unique singular point of type A_1. Σ_2 is then a smooth quadric surface and admits two different rulings, cf. Example 1.2. For $i = 1, 2$ consider the maps

$$\iota_i : \Sigma \to \mathrm{Sym}^2(\Sigma),\ R \mapsto P + Q,$$

where P, Q are such that the unique line of the i-th ruling of Σ_2 passing through R intersects Σ_3 in R and the residual points P, Q. We claim that for $i = 1, 2$, the map ι_i is an embedding. Recall that $\Sigma \subset H$, where $H \cong \mathbb{P}^3$ is the hyperplane $\{z_0 = 0\} \subset \mathbb{P}^4$ and parameterises the lines in $\mathbb{P}^4$ passing through p_0 via the morphism Φ, see (1.5). The map $\varphi|_{\mathrm{img}(\iota_i)} \circ \iota_i$ is the isomorphism $\Phi : H \to \Phi(H)$ restricted to Σ,

$$\varphi|_{\mathrm{img}(\iota_i)} \circ \iota_i = \Phi|_\Sigma, \tag{2.9}$$

thus an isomorphism onto its image, the singular locus of $F(X)$. Therefore, $d\iota_i$ is injective and ι_i a closed immersion, that is, an embedding of Σ into $\mathrm{Sym}^2(\Sigma)$, and we write Σ_i for the image of Σ under ι_i. The curves Σ_1 and Σ_2 are disjoint as a point of intersection gives rise to a line being an element of both rulings of Σ_2 at the same time by going through the construction above. But such a line does not exist, cf. Example 1.2. It is clear that $\varphi^{-1}(\Sigma) = \Sigma_1 \cup \Sigma_2$ as points $P + Q \in \Sigma_1 \cup \Sigma_2 \subset \mathrm{Sym}^2(\Sigma)$ are the only points such that the respective secant $s(P, Q)$ defines an element of one of the rulings of Σ_2.

If the unique singular point of X is of type A_2, the same construction applies but with the difference that the two curves Σ_1, Σ_2 are no longer distinct. This is, because Σ_2 in this case is a quadric of corank one and admits a unique ruling giving rise to a single embedding $\iota : \Sigma \to \mathrm{Sym}^2(\Sigma)$. ☐

We need the following Proposition 2.26 – Lemma 2.28 to compute the degenerate Picard scheme of $F(X)$.

2.26 Proposition ([vK10, Lemma 3.1])**.** *If C is a smooth curve, then*

$$\mathrm{Pic}^0(\mathrm{Sym}^2(C)) \cong \mathrm{Pic}^0(C).$$

Proof. We recall the proof from [vK10, Lemma 3.1] for completeness. For every $p \in C$ define a divisor C_p on $\mathrm{Sym}^2(C)$ by

$$C_p = \{p + q \mid q \in C\},$$

and consider the inclusion $j_p : C \to \mathrm{Sym}^2(C),\ q \mapsto p+q$. The divisor C_p then is the image of C under the inclusion j_p. Now, given any divisor $D = \sum a_i p_i$ on C, we can associate the divisor $C_D = \sum a_i C_{p_i}$ on $\mathrm{Sym}^2(C)$ to it and this defines an inclusion

$$i : \mathrm{Pic}^0(C) \hookrightarrow \mathrm{Pic}^0(\mathrm{Sym}^2(C)),\ \mathcal{O}_C(D) \mapsto \mathcal{O}_{\mathrm{Sym}^2(C)}(C_D). \tag{2.10}$$

It is straightforward to check that this inclusion is a morphism of Abelian groups. We claim that each of the maps $j_p^* : \mathrm{Pic}^0(\mathrm{Sym}^2(C)) \to \mathrm{Pic}^0(C)$ is an inverse for i. First note that $j_p^* \circ i = \mathrm{id}_{\mathrm{Pic}^0(C)}$ showing that i is indeed injective. To prove surjectivity, recall that $\mathrm{Pic}^0(\mathrm{Sym}^2(C)) \cong H^1(\mathrm{Sym}^2(C), \mathcal{O}_{\mathrm{Sym}^2(C)})/H^1(\mathrm{Sym}^2(C), \mathbb{Z})$. As i is linear, it defines an isomorphism if we can show that $H^1(\mathrm{Sym}^2(C), \mathcal{O}_{\mathrm{Sym}^2(C)})$ and $H^1(C, \mathcal{O}_C)$ have the same dimension. The dimension of $H^1(C, \mathcal{O}_C)$ equals the genus $g = g(C)$ of C by Serre-dualtiy. On the other hand, the cohomology of the symmetric square of a smooth curve has been computed by I. G. Macdonald and it follows from [Mac62, Formula 11.1] that $\dim H^1(\mathrm{Sym}^2(C), \mathcal{O}_{\mathrm{Sym}^2(C)}) = \binom{g}{1} = g$ proving the assertion. □

2.27 Lemma. *Let C_1, C_2 be smooth compact curves. Then,*

$$\mathrm{Pic}^0(C_1 \times C_2) \cong \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2).$$

Proof. According to [Smi05, Theorem 3.3.12, page 40],

$$\mathrm{Pic}(C_1 \times C_2) \cong \mathrm{Pic}(C_1) \times \mathrm{Pic}(C_2) \times \mathrm{Hom}(\mathrm{Jac}(C_1), \mathrm{Jac}(C_2))$$

as Abelian groups. Now, since $\mathrm{Hom}(\mathrm{Jac}(C_1), \mathrm{Jac}(C_2)) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}_0$, see [BL04, Proposition 1.2.2], which is a discrete group, the assertion follows. □

2.28 Lemma. *Let $p : T \to M$ be a locally trivial algebraic fibre bundle over a smooth curve M with fibres isomorphic to a smooth curve F. Then,*

$$\mathrm{Pic}^0(T) \cong \mathrm{Pic}^0(M) \times \mathrm{Pic}^0(F).$$

Proof. Let U be a Zariski-open set in M such that $p^{-1}(U) \cong U \times F$. Since U is open, this induces a birational map $f : T \dashrightarrow M \times F$. Consider a resolution of indeterminancies of f, cf. [Sha13a, 4, Theorem 4.9], that is, a smooth surface S and birational morphisms $\alpha : S \to T$, $\beta : S \to M \times F$ such that the diagram

$$\begin{tikzcd} & S \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ T \arrow[rr, dashed, "f"] & & M \times F \end{tikzcd}$$

commutes. Since both of α, β factor as a composition of blowups of smooth points, there are integers m, n such that

$$\mathrm{Pic}(T) \oplus \mathbb{Z}^{\oplus m} \cong \mathrm{Pic}(S) \cong \mathrm{Pic}(M \times F) \oplus \mathbb{Z}^{\oplus n}.$$

Together with Lemma 2.27 this yields the assertion. □

2.29 Theorem. *Let X be a cubic threefold with a unique singular point of type A_1. Let $\pi : \mathfrak{F} \to B$ be a smoothing of $\mathfrak{F}_0 = F(X)$ and denote by $\pi' : \mathfrak{F}' \to B'$ the tail reduction of the family $\pi : \mathfrak{F} \to B$. Then, the degenerate Picard scheme of $\mathfrak{F}_0$ with respect to the family $\pi : \mathfrak{F} \to B$ is uniquely determined by $\mathrm{Pic}^0(\mathfrak{F}'_0)$ which has the form*

$$1 \longrightarrow \mathbb{C}^* \to \mathrm{Pic}^0(\mathfrak{F}'_0) \longrightarrow \mathrm{Pic}^0(\Sigma) \longrightarrow 0,$$

where Σ denotes the singular locus of $\mathfrak{F}_0$.

Proof. It follows from Theorem 2.22 and Lemma 2.24 that the dual graph of the tail reduction $\pi' : \mathfrak{F}' \to B'$ of the family $\pi : \mathfrak{F} \to B$ is given by

$$\overset{E_1}{\textcircled{1}} = \overset{\mathrm{Sym}^2(\Sigma)}{\textcircled{1}},$$

wherein Σ denotes the singular locus of $\mathfrak{F}_0$ and E_1 is the total space of a locally trivial algebraic fibre bundle over Σ with fibre $\mathbb{P}^1$. Using Corollary 2.15 there is an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}^0(\mathfrak{F}'_0) \longrightarrow \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \times_{\mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)\cap E_1)} \mathrm{Pic}^0(E_1) \longrightarrow 0.$$

By Lemma 2.28, $\mathrm{Pic}^0(E_1)$ can be identified with $\mathrm{Pic}^0(\Sigma)$. Moreover, the intersection $\mathrm{Sym}^2(\Sigma) \cap E_1 = \Sigma_1 \cup \Sigma_2$ by Lemma 2.25, where Σ_1, Σ_2 are disjoint curves isomorphic to Σ. We may thus identify the fibre product in the exact sequence above with

$$\mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \times_{\mathrm{Pic}^0(\Sigma_1 \cup \Sigma_2)} \mathrm{Pic}^0(\Sigma) = \{L \in \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \mid \ L\mid_{\Sigma_1} = L\mid_{\Sigma_2}\}.$$

Here, by $L\mid_{\Sigma_i}$ we mean $\iota_i^* L$, where $\iota_i : \Sigma \to \mathrm{Sym}^2(\Sigma)$ denotes the inclusion. By [vK10, Remark 6.2], $L\mid_{\Sigma_1} = L\mid_{\Sigma_2}$ holds for all $L \in \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma))$. Therefore, the fibre product in the above can be identified with $\mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma))$ and together with the identification $\mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \cong \mathrm{Pic}^0(\Sigma)$ from Proposition 2.26 this gives the exact sequence we asserted. The same arguments as in section 2.3 show that this determines the degenerate Picard scheme of $\mathfrak{F}_0$ with respect to the family $\pi : \mathfrak{F} \to B$. □

2.30 Remark. In [vK10], G. van der Geer and A. Kouvidakis proved the same result using different methods. In fact, they are even able to identify the degenerate Picard scheme $\lim\limits_{b \to 0} \mathrm{Pic}^0(\mathfrak{F}_b)$ with $\mathrm{Pic}^0(\mathfrak{F}_0)$.

2.31 Remark. In the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}^0(\mathfrak{F}'_0) \longrightarrow \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \longrightarrow 0$$

from Theorem 2.29, the fibre over any point $L \in \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma))$ consists of the group of isomorphisms $\mu : L\mid_{\Sigma_1} \to L\mid_{\Sigma_2}$. Every such isomorphism is a nowhere vanishing global section $\Sigma \to \mathrm{Hom}(L\mid_{\Sigma_1}, L\mid_{\Sigma_2})$ of the Hom-bundle $\mathrm{Hom}(L\mid_{\Sigma_1}, L\mid_{\Sigma_2}) \to \Sigma$. The bundle $\mathrm{Hom}(L\mid_{\Sigma_1}, L\mid_{\Sigma_2}) \to \Sigma$ thus is a line bundle with nowhere vanishing global section and therefore trivial.[2] If $s : \Sigma \to \Sigma \times \mathbb{C}$, $\sigma \mapsto (\sigma, s_\sigma)$ is any such section, the map $\sigma \mapsto s_\sigma$ is a regular map from a projective curve, hence constant. Thus, s_σ does not vary with σ and depends on s only. In particular, we find one global section for each complex number and the nowhere vanishing global sections correspond to sections s such that $s_\sigma \in \mathbb{C}^*$. This gives an alternative explanation of the $\mathbb{C}^*$ on the left in the exact sequence above.

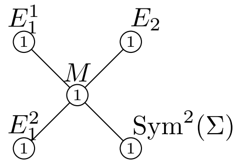
2.32 Theorem. *Let X be a cubic threefold with a unique singular point of type A_2. Let $\pi : \mathfrak{F} \to B$ be a smoothing of $\mathfrak{F}_0 = F(X)$ and $\pi' : \mathfrak{F}' \to B'$ its tail reduction. Then,*

$$\mathrm{Pic}^0(\mathfrak{F}'_0) \cong \mathrm{Pic}^0(\Sigma) \times E,$$

where E is an elliptic curve and Σ denotes the singular locus of $\mathfrak{F}_0$. Moreover, this completely determines the degenerate Picard scheme of $\mathfrak{F}_0$ with respect to the family $\pi : \mathfrak{F} \to B$.

[2] In lack of an adequate reference, we give here a sketch of proof. Let s be a nowhere vanishing section of a line bundle $L \to M$ over a smooth manifold M. Define $F : M \times \mathbb{C} \to L$ via $F(m, c) = cs(x)$. This gives the desired isomorphism $M \times \mathbb{C} \to L$. Conversely, if an isomorphism $F : M \times \mathbb{C} \to L$ is given, we obtain a nowhere vanishing global section s by defining $s(m) = F(m, c)$ where c is any non-zero element of $\mathbb{C}$.

Proof. It follows from Theorem 2.22 and Lemma 2.24 that the dual graph of the tail reduction $\pi' : \mathfrak{F}' \to B'$ of the family $\pi : \mathfrak{F} \to B$ is given by



where Σ denotes the singular locus of $\mathfrak{F}_0$, E_1^1, E_1^2, E_2 are total spaces of $\mathbb{P}^1$-bundles over Σ and M is the total space of a bundle over Σ whose fibres are elliptic curves. Let $E \in \{E_1^1, E_1^2, E_2\}$ and let $K(E)$ such that $\mathfrak{F}_0' = K(E) \cup E$. Then, using Lemma 2.14 and Lemma 2.28,

$$\begin{aligned}
\mathrm{Pic}^0(\mathfrak{F}_0') &= \mathrm{Pic}^0(E) \times_{\mathrm{Pic}^0(E \cap K(E))} \mathrm{Pic}^0(K(E)) \\
&= \mathrm{Pic}^0(\Sigma) \times_{\mathrm{Pic}^0(\Sigma)} \mathrm{Pic}^0(K(E)) \\
&= \mathrm{Pic}^0(\Sigma \cup_\Sigma K(E)) \\
&= \mathrm{Pic}^0(K(E)),
\end{aligned}$$

showing that the components E_1^1, E_1^2, E_2 of $\mathfrak{F}_0'$ do not contribute to the Picard scheme $\mathrm{Pic}^0(\mathfrak{F}_0')$. We may thus, using 2.14 and Lemma 2.26, compute

$$\begin{aligned}
\mathrm{Pic}^0(\mathfrak{F}_0') &= \mathrm{Pic}^0(M \cup \mathrm{Sym}^2(\Sigma)) \\
&= (\mathrm{Pic}^0(E) \times \mathrm{Pic}^0(\Sigma)) \times_{\mathrm{Pic}^0(\Sigma)} \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \\
&= \mathrm{Pic}^0(E) \times (\mathrm{Pic}^0(\Sigma) \times_{\mathrm{Pic}^0(\Sigma)} \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma)) \\
&= E \times \mathrm{Pic}^0(\Sigma).
\end{aligned}$$

The same arguments as in section 2.3 show that this determines the degenerate Picard scheme of $\mathfrak{F}_0$ with respect to the family $\pi : \mathfrak{F} \to B$. $\square$

3 Degenerations of the Picard scheme of the Fano scheme of lines on a cubic threefold

As we saw in the previous chapter, a tail reduction can be computed for varieties with curve singularities transversally along a smooth singular locus. Unfortunately, if X is a cubic threefold with a unique singular point of type A_k and $k \geq 3$, its Fano scheme of lines $F(X)$ has singular locus which is itself singular, see Lemma 1.5.

Our strategy to bypass this problem is the following: we begin by generalising the construction of a morphism $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$ from the previous chapter by proving that for cubic hypersurfaces with a unique singular point of ADE-type there always exists a morphism $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ and that this morphism is the blowup of $F(X)$ along Σ, a fact that seems to be interesting on its own. We proceed by showing that there is a desingularisation of $\mathrm{Hilb}^2(\Sigma)$ by successive blowups of the singular locus and compute such a resolution explicitly.

This explicit resolution of $F(X)$ then finally enables us to compute the degenerate Picard scheme of $F(X)$ in general.

3.1 The natural map $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$

We generalise the construction of a morphism $\varphi : \mathrm{Sym}^2(\Sigma) \to F(X)$ for X with unique singular point of type A_1 or A_2 from the previous chapter, cf. Lemma 2.24 and Lemma 2.25, to cubic threefolds with a unique singular point of type A_k for arbitrary $k \in \mathbb{N}$. It in fact turns out that for any singular cubic hypersurface $X \subset \mathbb{P}^n$ not containing a plane, we find a natural morphism $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ which is the blowup of $F(X)$ along its singular locus.

Throughout this section we assume that the unique singular point p_0 of X is of type A_k with $k \geq 2$ unless explicitly stated otherwise.

3.1.1 Geometric construction

We begin by defining the map $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ geometrically. Recall that after making p_0 the point with coordinates $(1 : 0 : 0 : 0 : 0)$, we denoted by $H \cong \mathbb{P}^3$ the hyperplane $\{z_0 = 0\}$ in the ambient $\mathbb{P}^4$ of X and by $\pi_0 : \mathbb{P}^4 \dashrightarrow H$ the projection from the point p_0, cf. chapter one. Consider the map

$$\langle \cdot \rangle : \mathrm{Hilb}^2(H) \to \mathrm{Gr}(1, H), \quad \eta \mapsto \langle \eta \rangle, \tag{3.1}$$

sending a length two subscheme of H to the line spanned by it. Denote by α the restriction of this map to the subscheme $\mathrm{Hilb}^2(\Sigma) \subset \mathrm{Hilb}^2(H)$, i.e. $\alpha = \langle \cdot \rangle \mid_{\mathrm{Hilb}^2(\Sigma)}$. Since for $\eta \in \mathrm{Hilb}^2(\Sigma)$, each point of the support of η corresponds to a line in X through p_0, if we form the plane

$$E_\eta = \langle \alpha(\eta), p_0 \rangle \cong \mathbb{P}^2,$$

it contains the lines $\langle x, p_0 \rangle$ for all points $x \in \operatorname{supp}(\eta)$. That is, E_η contains the cone over $\operatorname{supp}(\eta)$ with vertex p_0. These are two distinct lines, if the support of η consists of two points, and is the same line taken twice, if it consists of a single point only. The intersection $E_\eta \cap X$ is a plane cubic containing two lines (or a double line), and therefore also a third line $l_\eta \subset X$. This gives rise to a map

$$\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X), \quad \eta \mapsto l_\eta. \tag{3.2}$$

3.1 Remark. If $\pi_0 : \mathbb{P}^4 \dashrightarrow H$ denotes the projection from the point p_0 and $\eta \in \mathrm{Hilb}^2(\Sigma)$ is such that $p_0 \notin l_\eta$, then $\pi_0(l_\eta) = \alpha(\eta)$.

It is simple to check whether l_η for given $\eta \in \mathrm{Hilb}^2(\Sigma)$ defines a smooth or singular point of $F(X)$: as every line in X through p_0 intersects Σ in precisely one point, $l_\eta \in F(X)_{\mathrm{sing}}$ if and only if $l_\eta \cap \Sigma \neq \emptyset$. This condition can be rephrased as $\alpha(\eta) \subset \Sigma_2$ by Bézout's Theorem and is equivalent to saying that $\alpha(\eta)$ is a line of the ruling of Σ_2.

We can also construct an inverse $\psi : F(X) \dashrightarrow \mathrm{Hilb}^2(\Sigma)$ to φ geometrically as follows. If $l \in F(X)_{\mathrm{reg}}$ is any line in X not passing through p_0, we may form the plane

$$E(l) = \langle l, p_0 \rangle \cong \mathbb{P}^2.$$

The intersection $E(l) \cap X$ is a plane cubic and contains the line l. It therefore decomposes as $E(l) \cap X = l \cup C$ where C is a conic section with singular point p_0. But this means that C is the union of two (not necessarily distinct) lines passing through p_0. As each of them determines a point on Σ, the intersection $E(l) \cap X$ determines a length two subscheme of Σ. Thus,

$$\psi : F(X) \dashrightarrow \mathrm{Hilb}^2(\Sigma), \; l \mapsto E(l) \cap \Sigma \tag{3.3}$$

defines on $F(X)_{\mathrm{reg}}$ an inverse to φ by construction. The following lemma provides a detailed description of the locus $S = \varphi^{-1}(\Sigma) \subset \mathrm{Hilb}^2(\Sigma)$, that is, it provides a precise description of which length two subschemes of Σ give rise to lines passing through the point p_0 by means of the construction above.

3.2 Lemma. *The set*

$$S = \{\eta \in \mathrm{Hilb}^2(\Sigma) \mid \alpha(\eta) \subset \Sigma_2\} = \varphi^{-1}(\Sigma)$$

has two irreducible components S_1, S_2, where

$$\begin{aligned} S_1 =& \mathrm{Hilb}^2(\Sigma)_{\mathrm{sing}}, \\ S_2 =& \overline{\{\eta \in \mathrm{Hilb}^2(\Sigma) \mid q_0 \notin \operatorname{supp}(\eta), \; q_0 \in \alpha(\eta)\}}, \end{aligned}$$

and $q_0 \in \Sigma$ is the singular point. Moreover, $S_1 \cong \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ denotes the blowup of Σ at its singular point q_0; and $S_2 \cong \mathbb{P}^1$. The intersection $S_1 \cap S_2$ consists of two points if the singularity type of (Σ, q_0) is A_1 and of a single point otherwise.

Proof. As we are going to show independently in Lemma 3.21, the singular locus of $\mathrm{Hilb}^2(\Sigma)$ is

$$\mathrm{Hilb}^2(\Sigma)_{\mathrm{sing}} = \{\eta \in \mathrm{Hilb}^2(\Sigma) \mid q_0 \in \operatorname{supp}(\eta)\}$$

and isomorphic to $\widetilde{\Sigma}$, the blowup of Σ at its singular point q_0. We divide the proof into the following steps. First, we show that $\varphi(S_1 \cup S_2) \subset \Sigma$. For proving the inclusion $\varphi^{-1}(\Sigma) \subset S_1 \cup S_2$ we separately prove the inclusions $\varphi^{-1}(\Sigma_{\mathrm{reg}}) \subset S_1$ and $\varphi^{-1}(l_0) \subset S_2$. Here, l_0 denotes the singular point of the singular locus of $F(X)$.

In order to show that $\varphi(S_1 \cup S_2) \subset \Sigma$, we distinguish the cases $\eta \in S_1 \cap S_2$, $\eta \in S_1 \setminus S_2$ and $\eta \in S_2 \setminus S_1$. First, let $\eta \in S_1 \cap S_2$. Then η is a length two subscheme with $\operatorname{supp}(\eta) = \{q_0\}$, as follows from the construction below. Each length two subscheme of Σ supported at a single point corresponds to this point together with a tangent direction. It follows from the discussion below that η defines an element of the projectivised tangent cone to Σ at q_0. But since

$$\alpha(\eta) \subset \mathbb{P}(TC_{q_0}\Sigma) = \mathbb{P}(TC_{q_0}\Sigma_2) \cap \mathbb{P}(TC_{q_0}\Sigma_3)$$

and $\mathbb{P}(TC_{q_0}\Sigma_2) \cong \Sigma_2$, we conclude $\alpha(\eta) \subset \Sigma_2$. Let now $\eta \in S_2 \setminus S_1$, that is, $q_0 \notin \mathrm{supp}(\eta)$. The line $\alpha(\eta)$ is secant to Σ and passes through the singular point q_0 of Σ_2. The intersection multiplicity at this point is at least two, as q_0 is a double point of Σ_2. Furthermore, $\alpha(\eta)$ intersects Σ_2 in the residual point p of $\mathrm{supp}(\eta) \setminus \{q_0\}$ also. Since the intersection multiplicity of $\alpha(\eta)$ with Σ_2 is thus at least three, it has to be contained in the quadric Σ_2 by Bézout's Theorem.
For $\eta \in S_1 \setminus S_2$, the line $\alpha(\eta)$ is a line intersecting the quadric Σ_2 in the points of $\mathrm{supp}(\eta)$ and also in q_0, thus has to be contained in Σ_2 by the same argument.

It remains to show that $\varphi^{-1}(\Sigma) \subset S = S_1 \cup S_2$. To do so, we make the following construction of a rational map $\iota : \Sigma \dashrightarrow \mathrm{Hilb}^2(\Sigma)$: if $p \neq q_0$ is any point on Σ, there is a unique line $l = l(p)$ of the ruling of Σ_2 such that $p \in l$. The intersection $l \cap \Sigma_3$ defines a subscheme $\eta = \iota(p)$ of Σ of length two after removing the point p. If $\mathrm{supp}(\iota(p)) = \{q_0\}$, the line $l = l(p)$ intersects Σ_3 twice at q_0. Since Σ_3 is smooth at q_0, this means that l is a projectivised tangent line to Σ_3 at q_0. Moreover, the projectivised tangent cone to Σ_2 at q_0 is Σ_2 with the lines of the ruling being the tangent lines. Therefore, $\iota(p)$ corresponds to q_0 together with a line in the tangent cone to Σ at q_0. We used this in the above. From the explicit construction of ι and φ, it is simple to check that $(\varphi \circ \iota)(p) = \langle p, p_0 \rangle \in F(X)_{\mathrm{sing}} \setminus \{l_0\}$ and thus,

$$\varphi^{-1}(\langle p, p_0 \rangle) = \iota(p) \in S_1.$$

We are left to show that $\varphi^{-1}(l_0) \subset S_2$ and $S_2 \cong \mathbb{P}^1$. Let $\eta \in \mathrm{Hilb}^2(\Sigma)$ such that $q_0 \notin \mathrm{supp}(\eta)$ but $\alpha(\eta) \subset \Sigma_2$. Consider the projection $\pi_0' : H \dashrightarrow \mathbb{P}^2$ from the point q_0. Since Σ_2 is a cone over a smooth plane quadric and we are projecting from the vertex of the cone, the image of Σ_2 under this projection is a smooth plane quadric, the basis of the cone Σ_2. If $\mathrm{supp}(\eta) = \{p, q\}$ with p, q not necessarily distinct, they define a point on the smooth plane quadric since they lie on a line passing through q_0. On the other hand, every point p on this quadric determines a line of the ruling of Σ_2 (via $p \mapsto \langle p, q_0 \rangle$) and this line intersects Σ_3 in three points counted with multiplicity, one of them being q_0. If one of the other two points is also q_0, the line of the ruling is the geometric tangent to Σ at q_0 and in the intersection of S_1 with S_2. Otherwise, it is a point of S_2. By construction, $\varphi^{-1}(l_0) \subset S_2$, and $S_2 \cong \mathbb{P}^1$ follows since every smooth plane quadric is isomorphic to $\mathbb{P}^1$ via the second Veronese embedding. $\square$

3.1.2 Interpretation as blowup morphism

3.3 Remark. Consider the rational map $\pi_0 |_X : X \dashrightarrow H$ given by projecting from the singular point $p_0 = (1 : 0 : 0 : 0 : 0) \in X$ onto the hyperplane $H = \{z_0 = 0\} \cong \mathbb{P}^3$. This map is not only rational but birational as we have a rational inverse $\rho : H \dashrightarrow X$ defined by mapping each point $p \in H$ to the residual point of the intersection $\langle p, p_0 \rangle \cap X$ after removing the point p_0 with multiplicity two. To be more precise, we consider the intersection with multiplicity, i.e.

$$\langle p, p_0 \rangle \cap X = \sum_{x \in \langle p, p_0 \rangle \cap X} \mathrm{mult}_x(\langle p, p_0 \rangle \cap X) \cdot x \in \mathrm{Sym}^3(X)$$

and define

$$\rho(p) = \sum_{x \in \langle p, p_0 \rangle \cap X} \mathrm{mult}_x(\langle p, p_0 \rangle \cap X) \cdot x - 2p_0 \in X.$$

We then have a commutative diagram, cf. Lemma 1.5,

$$\begin{tikzcd} X \arrow[d, dashed, bend right, "\pi_0|_X"'] & \mathrm{Bl}_{p_0}X \arrow[l] \arrow[d, equal] \\ H \arrow[u, dashed, "\rho"] & \mathrm{Bl}_{\Sigma}H. \arrow[l] \end{tikzcd}$$

We are going to need a coordinate description for ρ. Let $x = (0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4$ be any point of H. The line L joining x and p_0 is

$$L = \{(\gamma : x_1 : x_2 : x_3 : x_4) \mid \gamma \in \mathbb{C}\} \cup \{p_0\} \subset \mathbb{P}^4.$$

A point on L is a point of X, if it is either the point p_0, or if

$$\gamma f_2(x_1 : x_2 : x_3 : x_4) + f_3(x_1 : x_2 : x_3 : x_4) = 0. \tag{3.4}$$

We easily see that the entire line is contained in X and passes through p_0, if $f_2(x) = f_3(x) = 0$. This reflects the fact that Σ parametrises the lines in X passing through p_0. If f_2 and f_3 do not vanish simultaneously at x, (3.4) can be solved for γ, yielding

$$\gamma = \gamma(x_1 : x_2 : x_3 : x_4) = -\frac{f_3(x_1 : x_2 : x_3 : x_4)}{f_2(x_1 : x_2 : x_3 : x_4)}, \quad f_2(x) \neq 0, \tag{3.5}$$

and the desired coordinate description for ρ therefore is

$$\rho(x_1 : x_2 : x_3 : x_4) = \begin{cases} (\gamma(x_1 : x_2 : x_3 : x_4) : x_1 : x_2 : x_3 : x_4), & f_2(x) \neq 0, \\ (1 : 0 : 0 : 0 : 0), & f_2(x) = 0, \ f_3(x) \neq 0. \end{cases} \tag{3.6}$$

Note that we identified $(\infty : x_1 : x_2 : x_3 : x_4)$ with $(1 : 0 : 0 : 0 : 0)$ in (3.6) which is justified by the fact that for $f_2(x) = 0$ and $f_3(x) \neq 0$,

$$\begin{aligned} L \cap X =& (\{(\gamma : x_1 : \cdots : x_4) \mid \gamma \in \mathbb{C}\} \cup \{p_0\}) \cap X \\ =& (\{(\gamma : x_1 : \cdots : x_4) \mid \gamma \in \mathbb{C}\} \cap X) \cup \{p_0\} \\ =& \{\gamma \in \mathbb{C} \mid \gamma f_2(x) + f_3(x) = 0\} \cup \{p_0\} \\ =& \{p_0\}. \end{aligned}$$

In particular, this shows in coordinates that ρ is continuous and a morphism away from Σ.

3.4 Remark. $\overline{\rho(\alpha(\eta) \setminus \Sigma)} = \varphi(\eta)$ holds for all $\eta \in \mathrm{Hilb}^2(\Sigma) \setminus (S_1 \cup S_2)$, analogous to Remark 3.1.

3.5 Theorem. *Let $X \subset \mathbb{P}^4$ be a cubic threefold with a unique singular point of type A_k. Denote by $\Sigma = \Sigma_2 \cap \Sigma_3$ the associated complete intersection isomorphic to the singular locus of $F(X)$. Then there exists a natural map $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ that coincides with the blowup of $F(X)$ along Σ.*

Proof. Consider the rational map

$$\mathrm{Hilb}^2((\Sigma_3)_{\mathrm{reg}}) \dashrightarrow (\Sigma_3)_{\mathrm{reg}}, \ \eta \mapsto \sum_{x \in \alpha(\eta) \cap \Sigma_3} \mathrm{mult}_x(\alpha(\eta) \cap \Sigma_3) \cdot x - \sum_{x \in \mathrm{supp}(\eta)} \mathrm{mult}_x(\eta) \cdot x$$

associating to every length two subscheme η of the regular locus of Σ_3 the residual point of intersection with Σ_3 of the line defined by η after removing the points of the support of η. The exceptional locus of this map is the set of all η such that the line defined by η is entirely contained in Σ_3. Recall from Lemma 1.6 that $\Sigma \subset (\Sigma_3)_{\mathrm{reg}}$. We may therefore restrict the map above to the subscheme $\mathrm{Hilb}^2(\Sigma)$ of $\mathrm{Hilb}^2((\Sigma_3)_{\mathrm{reg}})$ to obtain a rational map $\beta : \mathrm{Hilb}^2(\Sigma) \dashrightarrow \Sigma_3$. Since no line is contained in Σ, this map is everywhere defined, i.e. a morphism $\beta : \mathrm{Hilb}^2(\Sigma) \to \Sigma_3$. Let $\mathcal{U} = \{(l,p) \mid p \in l\} \subset \mathrm{Gr}(1,H) \times H$ be the universal line over $\mathrm{Gr}(1,H)$. We then have a morphism

$$\varepsilon : \mathrm{Hilb}^2(\Sigma) \to \mathcal{U},\ \eta \mapsto (\alpha(\eta), \beta(\eta)). \tag{3.7}$$

Note that a point $(l,p) \in \mathcal{U}$ is a point of $\mathrm{img}(\varepsilon)$, if

i) $p \in \Sigma_3$,

ii) for all $q \in l \cap \Sigma_3$, $q \neq p$ we have $q \in \Sigma_2$,

and that ε is an isomorphism onto its image, since on $\mathrm{img}(\varepsilon)$ we have an inverse morphism $\mathrm{img}(\varepsilon) \to \mathrm{Hilb}^2(\Sigma)$ given by $(p,l) \mapsto l \cap \Sigma_3 - p$. We aim to give a morphism $\sigma : \mathrm{img}(\varepsilon) \to F(X)$ such that $\varphi = \sigma \circ \varepsilon$ together with a coordinate description of σ.

In order to choose coordinates on a neighbourhood of $l = l(0,0) \in \mathrm{Gr}(1,H)$, we may pick two distinct points $x(0), y(0)$ on l and map them to the points $x(0) = (1:0:0:0) \in H$ and $y(0) = (0:1:0:0) \in H$ using a linear coordinate change on H. A neighbourhood of l in $\mathrm{Gr}(1,H)$ is then given by all lines $l = l(u,v)$ spanned by $x(u) = (1:0:u_3:u_4)$ and $y(v) = (0:1:y_3:y_4)$, cf. chapter one. To obtain a neighbourhood of (l,p) inside $\mathrm{img}(\varepsilon)$, we can assume without loss of generality that $p = y(v)$, that is, that one of the points we chose to generate l is the point p itself. Consequently, any point in a neighbourhood of (l,p) inside $\mathrm{img}(\varepsilon)$ can be written as $((0:1:v_3:v_4),(u_3,u_4,v_3,v_4)) \in H \times \mathrm{Gr}(1,H)$.

To define the map σ, we distinguish two cases. First, assume that $y(v) = (0:1:v_3:v_4) \in \Sigma_3 \setminus \Sigma_2$, that is, the point $((0:1:v_3:v_4),(u_3,u_4,v_3,v_4))$ is a point of $\varepsilon(\mathrm{Hilb}^2(\Sigma) \setminus (S_1 \cup S_2))$. As the line given by (u_3,u_4,v_3,v_4) is not contained in Σ_2, we may assume that $(1:0:u_3:u_4) \notin \Sigma_2$. To obtain the desired factorisation $\varphi = \sigma \circ \varepsilon$, the point $((0:1:v_3:v_4),(u_3,u_4,v_3,v_4))$ should be mapped to the line spanned by $\rho(x(u))$ and $\rho(y(v))$, cf. Remark 3.4. Following Remark 3.3 we compute

$$\begin{aligned}
\rho(x(u)) &= \rho(1:0:u_3:u_4) = (\gamma(x(u)):1:0:u_3:u_4) \in \mathbb{P}^4,\\
\rho(y(v)) &= \rho(0:1:v_3:v_4) = (0:0:1:v_3:v_4) \in \mathbb{P}^4,
\end{aligned}$$

where we used that $\gamma(y(v)) = 0$ for $y(v) \in \Sigma_3$. We therefore define

$$\sigma : \begin{cases} \varepsilon(\mathrm{Hilb}^2(\Sigma) \setminus (S_1 \cup S_2)) \to F(X), \\ ((0:1:v_3:v_4),(u_3,u_4,v_3,v_4)) \mapsto (\gamma(x(u)),u_3,u_4,0,v_3,v_4), \end{cases} \tag{3.8}$$

where the point on the right is to be understood as point in local coordinates on $\mathrm{Gr}(1,\mathbb{P}^4)$.
If $y(v) = (0:1:v_3:v_4) \in \Sigma_2 \cap \Sigma_3 = \Sigma$, it corresponds to a point of $\varepsilon(S_1 \cup S_2)$. For the point $x(u)$ we necessarily have $x(u) \in \Sigma_2$ but we can assume that $x(u) \notin \Sigma_3$. We then define

$$\sigma : \begin{cases} \varepsilon(S_1 \cup S_2) \to F(X), \\ ((0:1:v_3:v_4),(u_3,u_4,v_3,v_4)) \mapsto (0,0,0,0,v_3,v_4), \end{cases} \tag{3.9}$$

that is, we map the point $((0:1:v_3:v_4),(u_3,u_4,v_3,v_4))$ to the line spanned by $\rho(x(u)) = p_0$ and $y(v)$. By (3.6), respectively the continuity of ρ, σ is easily seen to be continuous. The desired factorisation $\varphi = \sigma \circ \varepsilon$ now holds by construction.

To complete the proof, it remains to show that $\mathrm{Hilb}^2(\Sigma)$ is isomorphic to $\widetilde{F(X)}$, the blowup of $F(X)$ along Σ. Let $\Phi : H \to \mathrm{Gr}(1,\mathbb{P}^4)$ be the map $p \mapsto \langle p, p_0 \rangle$ realising H as Schubert variety of all lines in $\mathbb{P}^4$ passing through p_0, see (1.5). The blowup $\mathrm{Bl}_{\Phi(H)}\mathrm{Gr}(1,\mathbb{P}^4)$ replaces each point $L \in \Phi(H)$ by a $\mathbb{P}^2$ parameterising the lines in $H \cong \mathbb{P}^3$ passing through p, where $\{p\} = L \cap H$. These lines are also parameterised by the image of the projection $\pi_p : H \cong \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from the point p in a natural way, cf. our discussion in chapter one. Therefore, the fibre $E_L = \mathrm{bl}^{-1}_{\Phi(H)}(L)$ of the exceptional bundle over $\Phi(H)$ is naturally isomorphic to the image of $\pi_{L\cap H}$,

$$E_L = \mathrm{bl}^{-1}_{\Phi(H)}(L) \cong \mathrm{img}(\pi_{L\cap H}). \tag{3.10}$$

Thus, if we are given a point $((0:1:v_3:v_4),(u_3,u_4,v_3,v_4)) \in \varepsilon(S_1\cup S_2)$, we can extend the map σ to a map $\bar{\sigma} : \mathrm{img}(\varepsilon) \to \widetilde{F(X)}$ by

$$\bar{\sigma} : \begin{cases} \varepsilon(S_1\cup S_2) \to \widetilde{F(X)}, \\ ((0:1:v_3:v_4),(u_3,u_4,v_3,v_4)) = (0,0,0,0,v_3,v_4) \times \pi_{y(v)}(x(u)). \end{cases} \tag{3.11}$$

Note that in our special choice of coordinates, $\pi_{y(v)}(x(u))$ is the projection onto the hyperplane $\{z_1 = 0\} \subset H$ and $\pi_{y(v)}(x(u)) = x(u)$, thus

$$\bar{\sigma}\mid_{\varepsilon(S_1\cup S_2)} ((0:1:v_3:v_4),(u_3,u_4,v_3,v_4)) = (0,0,0,0,v_3,v_4) \times (1:u_3:u_4) \tag{3.12}$$

and the assertion follows. □

3.6 Remark. The assumption of Theorem 3.5 that X is a cubic threefold is not necessary. The precise same arguments can be used to prove the theorem for arbitrary cubic hypersurfaces not containing a plane and with unique singular point of type A_k. We state this as Corollary 3.8.

3.7 Remark. It is interesting to compare Theorem 3.5 with earlier results of A. Beauville and R. Donagi [BD85] and B. Hassett [Has00b] stating that for some smooth cubic fourfolds $X \subset \mathbb{P}^5$ one is able to find a K3-surface S and an isomorphism $\mathrm{Hilb}^2(S) \cong F(X)$. In fact, a smooth complete $(2,3)$-intersection in $\mathbb{P}^4$ such as the curve Σ appearing in Theorem 3.5 is a K3-surface. Unfortunately, the construction of the surface S and the isomorphism $\mathrm{Hilb}^2(S) \cong F(X)$ in [BD85, Has00b] is purely Hodge-theoretic and it is not clear how this relates to the explicit geometric construction of the map $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ provided above.

3.8 Corollary. *Let $X \subset \mathbb{P}^n$ be a cubic hypersurface with a unique singular point of type A_k and not containing a plane. Denote by Σ the associated complete intersection isomorphic to the singular locus of $F(X)$. Then there exists a natural map $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ that coincides with the blowup of $F(X)$ along Σ.*

3.2 Hilbert square of singular curves

In order to perform the tail reduction procedure, we need to compute a resolution of $F(X)$ explicitly. As we have seen in the previous section, the blowup of the entire singular locus of $F(X)$ results in $\mathrm{Hilb}^2(\Sigma)$ where Σ is the associated complete intersection, cf. Theorem 3.5. Since we know already that successive blowups of the transversality locus, that is, of the singular locus of $\mathrm{Hilb}^2(\Sigma)$ without the singular point of the singular locus, resolve the transversal curve singularity, it is natural to ask if successive blowups of the entire singular locus of $\mathrm{Hilb}^2(\Sigma)$ and then of its strict transforms resolve the singularities of $\mathrm{Hilb}^2(\Sigma)$ and therefore of $F(X)$. This is proven in the current section. Since we have to work with local equations for $\mathrm{Hilb}^2(\Sigma)$ in order to compute the resolution explicitly, we begin by calculating local equations for $\mathrm{Sym}^2(\Sigma)$.

3.2.1 Symmetric square of singular curves

Let Y be a curve with a unique singular point y_0 of type A_k. Denote by $\pi : \widetilde{Y} \to Y$ the blowup of Y in the singular point y_0. The blown up curve $\widetilde{Y}$ is smooth, if $k = 1, 2$, or has a unique singular point $\widetilde{y}_0$ of type A_{k-2}, see Table 2.1 and also the local calculations from the proof of Lemma 2.9.

3.9 Lemma. *Let $\bar{\pi} : \mathrm{Bl}_{\mathrm{Sing}(Y\times Y)}(Y \times Y) \to Y \times Y$ be the blowup of the entire singular locus and let $p \in \bar{\pi}^{-1}(y_0, y_0)$. Then, there exists an isomorphism of analytic germs*

$$(\widetilde{Y} \times \widetilde{Y}, (\widetilde{y}_0, \widetilde{y}_0)) \cong (\mathrm{Bl}_{\mathrm{Sing}(Y\times Y)}(Y \times Y), p),$$

where $\widetilde{y}_0 \in \pi^{-1}(y_0)$.

Proof. Take a local analytic normal form $Y \cong \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}^2$ which is singular at the point $y_0 = (0, 0)$. It is simple to check using the Jacobian criterion that the product

$$Y \times Y = \{(x_1, x_2, y_1, y_2) \in \mathbb{C}^4 \mid x_1^{k+1} + x_2^2 = y_1^{k+1} + y_2^2 = 0\}$$

is singular along $(Y \times \{y_0\}) \cup (\{y_0\} \times Y)$. The subvariety $Y \times \{y_0\}$ is given by the ideal $I = \langle x_1^{k+1} + x_2^2, y_1, y_2\rangle$, the subvariety $\{y_0\} \times Y$ by the ideal $J = \langle x_1, x_2, y_1^{k+1} + y_2^2\rangle$. Consequently, $\mathrm{Sing}(Y \times Y)$ is given by the ideal

$$I \cap J = \langle x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_1^{k+1} + x_2^2, y_1^{k+1} + y_2^2\rangle = \langle f_0, \dots, f_5\rangle. \tag{3.13}$$

Consider the blowup of $\mathbb{C}^4$ along the subvariety $C = \{x_1y_1 = x_1y_2 = x_2y_1 = x_2y_2 = 0\}$ of $\mathbb{C}^2 \times \mathbb{C}^2$ that restricts to the singular locus of $Y \times Y$. Then,

$$\mathrm{Bl}_C\mathbb{C}^4 = \{(x_1, x_2, y_1, y_2) \times (a_0 : \dots : a_3) \in \mathbb{C}^4 \times \mathbb{P}^3 \mid a_if_j = a_jf_i,\ i, j \in \{0, \dots, 3\}\}.$$

Take the chart $U_0 = \{a_0 \neq 0\}$. In this chart, the total transform of $Y \times Y$ is given by equations

$$\begin{aligned} x_1^{k+1} + x_2^2 &= 0, \\ y_1^{k+1} + y_2^2 &= 0, \\ x_1y_2 - a_1x_1y_1 &= 0, \\ x_2y_1 - a_2x_1y_1 &= 0, \\ x_2y_2 - a_3x_1y_1 &= 0, \end{aligned}$$

and the exceptional divisor by $\{x_1y_1 = 0\}$. To obtain equations for the strict transform, we need to drop all factors corresponding to components of the exceptional divisor. Doing so in the third and fourth equation enables us to substitute the variables x_2, y_2 and results in local equations

$$\begin{aligned} x_1^2(x_1^{k-1} + a_2^2) &= 0, \\ y_1^2(y_1^{k-1} + a_1^2) &= 0, \\ a_1a_2 - a_3 &= 0. \end{aligned} \tag{3.14}$$

There are more factors corresponding to the exceptional divisor in the equations. By removing them we obtain the following local equations for the strict transform after substituting the variable a_3:

$$\begin{aligned} x_1^{k-1} + a_2^2 &= 0, \\ y_1^{k-1} + a_1^2 &= 0. \end{aligned}$$

These are local equations for $\widetilde{Y} \times \widetilde{Y}$ around the point $(\widetilde{y}_0, \widetilde{y}_0)$. □

3.10 Lemma. *The blowup $b : \widetilde{\mathbb{C}^2 \times \mathbb{C}^2} \to \mathbb{C}^2 \times \mathbb{C}^2$ of $\mathbb{C}^2 \times \mathbb{C}^2$ along the ideal defining the union $(\{y_0\} \times \mathbb{C}^2) \cup (\mathbb{C}^2 \times \{y_0\})$ is isomorphic to the self-product of the blowup $b_0 : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ of $\mathbb{C}^2$ in the point y_0. In other words, there is a commutative diagram*

$$\begin{tikzcd} & \widetilde{\mathbb{C}^2 \times \mathbb{C}^2} \arrow[d, "b"] \\ \widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2} \arrow[ur, "\cong"] \arrow[r, "b_0 \times b_0"] & \mathbb{C}^2 \times \mathbb{C}^2. \end{tikzcd}$$

In particular, the total transform of $Y \times Y$ via b is isomorphic to the self-product of the total transform of Y via b_0.

Proof. We begin by checking that all coordinate charts are isomorphic and that these isomorphisms identify the respective blowup maps with each other. Afterwards, we will show that the transition functions between the respective coordinate charts also get identified under these isomorphisms. Let us start by computing the coordinate charts for $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2} = \mathrm{Bl}_S(\mathbb{C}^2 \times \mathbb{C}^2) \subset \mathbb{C}^4 \times \mathbb{P}^3$, where S is such that $S \cap (Y \times Y) = (Y \times Y)_{\mathrm{sing}}$. Denote projective coordinates on the $\mathbb{P}^3$ by $(a_0 : a_1 : a_2 : a_3)$ and the respective affine coordinate charts by $U_i = \{a_i \neq 0\}$. We then have

$$\begin{aligned} (\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}) \cap U_0 =& \{y_2 - a_1 y_1 = x_2 - a_2 x_1 = a_1 a_2 - a_3 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^3, \\ (\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}) \cap U_1 =& \{y_1 - a_0 y_2 = x_2 - a_3 x_1 = a_3 a_0 - a_2 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^3, \\ (\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}) \cap U_2 =& \{y_2 - a_3 y_1 = x_1 - a_0 x_2 = a_1 - a_0 a_3 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^3, \\ (\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}) \cap U_3 =& \{y_1 - a_2 y_2 = x_1 - a_1 x_2 = a_1 a_2 - a_0 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^3. \end{aligned} \tag{3.15}$$

For $\widetilde{\mathbb{C}}^2 \times \widetilde{\mathbb{C}}^2 = \mathrm{Bl}_{y_0}(\mathbb{C}^2) \times \mathrm{Bl}_{y_0}(\mathbb{C}^2) \subset \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$ denote projective coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ by $(b_0 : b_1), (c_0 : c_1)$. The respective coordinate charts then are $U_{i,j} = \{b_i \neq 0\} \cap \{c_j \neq 0\}$ and

$$\begin{aligned} (\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}) \cap U_{0,0} =& \{y_2 - c_1 y_1 = x_2 - b_1 x_1 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^2, \\ (\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}) \cap U_{0,1} =& \{y_2 - c_1 y_1 = b_0 x_2 - x_1 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^2, \\ (\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}) \cap U_{1,0} =& \{c_0 y_2 - y_1 = x_2 - b_1 x_1 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^2, \\ (\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}) \cap U_{1,1} =& \{c_0 y_2 - y_1 = b_0 x_2 - x_1 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^2. \end{aligned} \tag{3.16}$$

We can regard the equations (3.15) for the local charts as equations in $\mathbb{C}^4 \times \mathbb{C}^2$ by taking the projection from $\mathbb{C}^4 \times \mathbb{C}^3$ given by forgetting about the coordinate a_3, respectively a_2, a_1, a_0, in the chart U_0, respectively U_1, U_2, U_3. Then, the identification of the local charts for $\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}$ with the local charts of $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}$ is immediately clear as the defining equations are the same up to relabelling coordinates. It is not hard to check that the restriction of the respective blowup maps are the same under this identification of the charts. For example, it is simple to see that $\mathrm{bl}_S|_{U_0} \colon U_0 \to \mathbb{C}^2 \times \mathbb{C}^2$ becomes the map $(\mathrm{bl}_{y_0} \times \mathrm{bl}_{y_0})|_{U_{0,0}} \colon U_{0,0} \to \mathbb{C}^2 \times \mathbb{C}^2$.
The transition functions between the local charts are derived from the standard transition function for $\mathbb{P}^3$ and $\mathbb{P}^1$, respectively, and it is not hard to check that they coincide under the identification of the local charts described above. We omit the explicit computation. □

3.11 Remark. If we consider the ambient spaces $\mathbb{C}^4 \times \mathbb{P}^3$ of $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}$ and $\mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$ of $\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}$, the proof of Lemma 3.10 also shows that we have a map $m : \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{C}^4 \times \mathbb{P}^3$ given by $\mathrm{id}_{\mathbb{C}^4} \times s$, where

$$s : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \quad ((b_0 : b_1), (c_0 : c_1)) \mapsto (b_0 c_0 : b_0 c_1 : b_1 c_0 : b_1 c_1) = (a_0 : a_1 : a_2 : a_3)$$

denotes the Segre-embedding which embeds the product $\mathbb{P}^1 \times \mathbb{P}^1$ as the quadric $\{a_1a_2 - a_0a_3 = 0\}$ inside $\mathbb{P}^3$. The variety $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}$ now is the image of $\widetilde{\mathbb{C}^2} \times \widetilde{\mathbb{C}^2}$ under the embedding m.

3.12 Remark. A more abstract proof of Lemma 3.10 can be found in [Ran16, Remark 2.2].

Lemma 3.10 provides a way to resolve the singularities of a product of curve singularities. We state this as the following corollary.

3.13 Corollary. *Let $Z \to Y \subset \mathbb{C}^2$ be the minimal embedded resolution of Y. Then there exists an embedded resolution $W \to Y \times Y$ of $Y \times Y$ which is given by successive blowups of the entire singular locus. Moreover, the strict transform of Y under these blowups is isomorphic to $Z \times Z$.*

The action of $\mathfrak{S}_2$ on the ambient $\mathbb{C}^2 \times \mathbb{C}^2$ lifts to an action on the blowup $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2}$ and this in turn extends to the ambient $\mathbb{C}^4 \times \mathbb{P}^3$ of the blown up $\mathbb{C}^4$. The action there is given by

$$(x_1, x_2, y_1, y_2) \times (a_0 : a_1 : a_2 : a_3) \mapsto (y_1, y_2, x_1, x_2) \times (a_0 : a_2 : a_1 : a_3). \tag{3.17}$$

That is, we have the usual action of $\mathfrak{S}_2$ by permutation of the factors on $\mathbb{C}^2 \times \mathbb{C}^2$, and are given an involution on the $\mathbb{P}^3$. The quotient of $\mathbb{C}^4 \times \mathbb{P}^3$ therefore is $\mathrm{Sym}^2(\mathbb{C}^2) \times \mathbb{P}^3/i$, where $i(a_0 : a_1 : a_2 : a_3) = (a_0 : a_2 : a_1 : a_3)$ denotes the involution as obtained in (3.17). If we consider the map $m = \mathrm{id}_{\mathbb{C}^4} \times s : \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{C}^4 \times \mathbb{P}^3$ from Remark 3.11, the action (3.17) of $\mathfrak{S}_2$ pulls back via m to an action on $\mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is the usual action of $\mathfrak{S}_2$ given by permutation of the factors, thus with quotient $\mathrm{Sym}^2(\mathbb{C}^2) \times \mathrm{Sym}^2(\mathbb{P}^1) \cong \mathrm{Sym}^2(\mathbb{C}^2) \times \mathbb{P}^2$. This gives rise to a commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "m=\mathrm{id}_{\mathbb{C}^4} \times s"] \arrow[d] & \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{P}^3 \arrow[d] \\ \mathrm{Sym}^2(\mathbb{C}^2) \times \mathbb{P}^2 \arrow[r] & \mathrm{Sym}^2(\mathbb{C}^2) \times \mathbb{P}^3/i. \end{tikzcd} \tag{3.18}$$

The centre of the blowup $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2} \to \mathbb{C}^2 \times \mathbb{C}^2$ is invariant under the action of $\mathfrak{S}_2$ as is clear from the coordinate description (3.13). Therefore, the blowup map $\widetilde{\mathbb{C}^2 \times \mathbb{C}^2} \subset \mathbb{C}^4 \times \mathbb{P}^3 \to \mathbb{C}^2 \times \mathbb{C}^2$ descends to a map $\widetilde{\mathrm{Sym}^2(\mathbb{C}^2)} \subset \mathrm{Sym}^2(\mathbb{C}^2) \times \mathbb{P}^2 \to \mathrm{Sym}^2(\mathbb{C}^2)$ being the blowup along $(I \cap J)/\mathfrak{S}_2$, in the notation of (3.13), and the ideal $(I \cap J)/\mathfrak{S}_2$ is the ideal defining the singular locus of $\mathrm{Sym}^2(Y)$. Lemma 3.9 and Lemma 3.10 now yield Corollaries 3.14 and 3.15 when passing to the quotient by the action of $\mathfrak{S}_2$.

3.14 Corollary. *Let $\bar{\pi} : \mathrm{Bl}_{\mathrm{Sing}(\mathrm{Sym}^2(Y))}\mathrm{Sym}^2(Y) \to \mathrm{Sym}^2(Y)$ be the blowup of the entire singular locus and let $p \in \bar{\pi}^{-1}(2y_0)$. Then there is an isomorphism of analytic germs*

$$(\mathrm{Sym}^2(\widetilde{Y}), 2\widetilde{y}_0) \cong (\mathrm{Bl}_{\mathrm{Sing}(\mathrm{Sym}^2(Y))}\mathrm{Sym}^2(Y), p),$$

where $\widetilde{y}_0 \in \pi^{-1}(y_0)$.

3.15 Corollary. *The blowup of $\mathrm{Sym}^2(Y)$ along its entire singular locus is isomorphic to $\mathrm{Sym}^2(\widetilde{Y})$. In particular, successive blowups of the entire singular locus provide a resolution $W \to \mathrm{Sym}^2(Y)$ of singularities of $\mathrm{Sym}^2(Y)$.*

If $W \to \mathrm{Sym}^2(Y)$ is the resolution from Corollary 3.15, it follows that $W \cong \mathrm{Sym}^2(Y^\nu)$. In fact, we have the following lemma saying that the resolution of $\mathrm{Sym}^2(Y)$ by successive blowups of the entire singular locus is the normalisation.

3.16 Lemma. *The normalisation of the symmetric square of Y is isomorphic to the symmetric square of the normalisation of Y,*

$$\mathrm{Sym}^2(Y)^\nu \cong \mathrm{Sym}^2(Y^\nu).$$

Proof. The following proof is due to M. A. van Opstall, [van06, Lemma 4.3], and included here for completeness. His assumption that the singular point of Y is of type A_1 is not necessary since not used in the proof.

Let $\nu_Y : Y^\nu \to Y$ be the normalisation of Y and $\nu_{Y\times Y} : (Y\times Y)^\nu \to Y\times Y$ be the normalisation of $Y \times Y$. Since $Y^\nu \times Y^\nu$ and $(Y \times Y)^\nu$ are both normal, there exists, by the universal property of normalisation, [Sha13a, 2, Theorem 2.1], a unique morphism φ such that the diagram

$$\begin{tikzcd} & (Y\times Y)^\nu \arrow[d, "\nu_{Y\times Y}"] \\ Y^\nu \times Y^\nu \arrow[ur, "\varphi"] \arrow[r, "\nu_Y\times\nu_Y"'] & Y\times Y \end{tikzcd}$$

commutes. By Zariski's Main Theorem , cf. [Har83, Chapter III, Corollary 11.4], the fibres of φ are connected. Since the diagram commutes, every positive dimensional fibre of φ would have to lie in a fibre of $\nu_Y \times \nu_Y$. But since the latter has zero dimensional fibres, there is no positive dimensional fibre of φ. Moreover, a zero-dimensional fibre can only be connected, if it is a point. Therefore, φ is a homeomorphism and it follows, again from Zariski's Main Theorem, that φ is an isomorphism. The action of $\mathfrak{S}_2$ on $Y^\nu \times Y^\nu$ induces an action of $\mathfrak{S}_2$ on $(Y \times Y)^\nu$ via φ. Repeating the argument for the quotients of the three spaces by the action of $\mathfrak{S}_2$ in the diagram above yields that $(Y \times Y)^\nu/\mathfrak{S}_2 \to (Y \times Y)/\mathfrak{S}_2$ is the normalisation and that $(Y \times Y)^\nu/\mathfrak{S}_2$ is isomorphic to $(Y^\nu \times Y^\nu)/\mathfrak{S}_2$. □

3.17 Corollary. *Let $W \to \mathrm{Sym}^2(Y)$ be the resolution obtained by successive blowups of the entire singular locus. Then, $W \cong \mathrm{Sym}^2(Y^\nu)$.*

Proof. We have seen that the blowup of the entire singular locus is a finite map between the strict transforms. Since the normalisation of $\mathrm{Sym}^2(Y)$ is smooth, it is the minimal resolution and we have a commutative diagram

$$\begin{tikzcd} W \arrow[d, "\text{contraction}"', "\text{of curves}"'] \arrow[dr, "\text{finite}"] & \\ \mathrm{Sym}^2(Y^\nu) \arrow[r, "\text{finite}"'] & \mathrm{Sym}^2(Y). \end{tikzcd}$$

By commutativity, if there were any curve contracted by the map $W \to \mathrm{Sym}^2(Y^\nu)$, the map $W \to \mathrm{Sym}^2(Y)$ could not be finite. Thus, the map $W \to \mathrm{Sym}^2(Y^\nu)$ has to be an isomorphism. □

The following lemma is needed for the proof of our main result, Theorem 3.34.

3.18 Lemma. *Let Y be a curve with a unique singular point y_0 of type A_{2k+1} and let $\nu : Y^\nu \to Y$ be the normalisation. Denote by y_1, y_2 the points of Y^ν such that $\nu(y_1) = \nu(y_2) = y_0 \in Y$, that is, the points lying over the singular locus of Y. Then, if $Y_{y_i} = \{p + y_i \mid p \in Y^\nu\}$, for every $L \in \mathrm{Pic}^0(\mathrm{Sym}^2(Y^\nu))$ the restrictions $L\,|_{Y_{y_1}}$ and $L\,|_{Y_{y_2}}$ are isomorphic.*

Proof. For $i = 1, 2$ denote by $j_{y_i} : Y^\nu \to \mathrm{Sym}^2(Y^\nu)$, $p \mapsto p + y_i$ the inclusion of Y^ν inside $\mathrm{Sym}^2(Y^\nu)$ with image Y_{y_i}. The assertion of the lemma is that for every $L \in \mathrm{Pic}^0(\mathrm{Sym}^2(Y^\nu))$, there is an isomorphism $j_{y_1}^* L \cong j_{y_2}^* L$. Since $\mathrm{Pic}^0(\mathrm{Sym}^2(Y^\nu))$ is generated by divisors of the form $Y_r - Y_s$ where $Y_r = \{p + r \mid p \in Y^\nu\}$ and $r, s \in Y^\nu$, cf. Lemma 2.26 or [vK10, Section 3], it suffices to show that

$$j_{y_1}^*(Y_r - Y_s) \sim j_{y_2}^*(Y_r - Y_s),$$

where $\sim$ denotes linear equivalence. But as $Y_{y_1} \cap Y_r = \{r + y_i\}$ we have $j_{y_i}^* Y_r = r$ and therefore,

$$j_{y_i}^*(Y_r - Y_s) = r - s,$$

and the assertion follows. □

3.2.2 Local equations for the symmetric square of a singular curve

We compute local equations for $\mathrm{Sym}^2(Y)$. Our calculations are similar to those in [Yam17b, section 2] but are more general and valid in any dimension. We only limit ourselves to the case of curves for the sake of brevity and readability.

Consider $\mathbb{C}^2 \times \mathbb{C}^2$ with local coordinates $((x_1, x_2), (y_1, y_2))$ and the action of $\mathfrak{S}_2$ given by permutation of the factors, that is, the action

$$(x_1, x_2, y_1, y_2) \sim (y_1, y_2, x_1, x_2).$$

The change of coordinates

$$u_i = \frac{x_i + y_i}{2}, \qquad v_i = \frac{x_i - y_i}{2},$$

turns the action of $\mathfrak{S}_2$ into an action of $\mathbb{Z}_2$ given by

$$(u_1, u_2, v_1, v_2) \sim (u_1, u_2, -v_1, -v_2).$$

The ring of invariants for this action is generated by

$$X_1 = u_1,\ X_2 = u_2,\ X_3 = v_1^2,\ X_4 = v_2^2,\ X_5 = v_1 v_2,$$

and there is a single relation between these generators which is $X_3X_4 - X_5^2 = 0$. Translation back into the coordinates (x_1, x_2, y_1, y_2) yields

$$X_1 = \frac{x_1 + y_1}{2},\ X_2 = \frac{x_2 + y_2}{2},\ X_3 = \frac{(x_1 - y_1)^2}{4},\ X_4 = \frac{(x_2 - y_2)^2}{4},\ X_5 = \frac{(x_1 - y_1)(x_2 - y_2)}{4}$$

with the same relation $X_3X_4 - X_5^2 = 0$ as before. In other words, we find that the coordinate ring of $\mathrm{Sym}^2(\mathbb{C}^2)$ is

$$\begin{aligned}\mathbb{C}[\mathrm{Sym}^2(\mathbb{C}^2)] &\cong \mathbb{C}[X_1, X_2, X_3, X_4, X_5]/\langle X_3X_4 - X_5^2\rangle \\ &= \mathbb{C}\left[\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \frac{(x_1 - y_1)^2}{4}, \frac{(x_2 - y_2)^2}{4}, \frac{(x_1 - y_1)(x_2 - y_2)}{4}\right] \\ &= \mathbb{C}[x_1, x_2, y_1, y_2]^{\mathfrak{S}_2}.\end{aligned}$$

Let $Y = \{x_1^{k+1} - x_2^2 = 0\} \subset \mathbb{C}^2$. Then, $Y \times Y$ is given by the ideal

$$\langle r_1, r_2\rangle = \langle x_1^{k+1} - x_2^2, y_1^{k+1} - y_2^2\rangle \subset \mathbb{C}[x_1, x_2, y_1, y_2].$$

To find equations for $\mathrm{Sym}^2(Y)$, we need to find the inverse image of this ideal under the map $f : \mathbb{C}[X_1, \dots, X_5] \to \mathbb{C}[x_1, x_2, y_1, y_2]$, i.e.

$$f^{-1}(\langle r_1, r_2\rangle) = \ker(f) + f^{-1}(\langle r_1, r_2\rangle \cap \mathrm{img}(f)).$$

In order to shorten our notation, we will often use the coordinates $X_1, \dots X_5$ in our calculations and view them as functions in x_1, x_2, y_1, y_2. When we write a term with a root such as $\sqrt{X_3}$, we mean the function obtained by formally cancelling root and square with each other. For example, the element $\sqrt{X_3}$ means the function

$$\sqrt{X_3} = \sqrt{\frac{(x_1 - y_1)^2}{4}} = \frac{x_1 - y_1}{2} \in \mathbb{C}[x_1, x_2, y_1, y_2].$$

We already computed $\ker(f) = \langle X_3X_4 - X_5^2\rangle$. The ideal $\langle r_1, r_2\rangle \cap \mathrm{img}(f)$ of $\mathrm{img}(f)$ consists of the $\mathfrak{S}_2$-invariant elements in the ideal generated by r_1, r_2 which are easily computed to be

$$\langle r_1, r_2\rangle \cap \mathrm{img}(f) = \langle r_1 + r_2, (r_1 - r_2)(x_1 - y_1), (r_1 - r_2)(x_2 - y_2)\rangle =: \langle g_1, g_2, g_3\rangle \subset \mathrm{img}(f).$$

We thus have to find for each g_i an element $f_i \in f^{-1}(g_i)$. We begin with $r_1 + r_2$ and $r_1 - r_2$. Since

$$r_1 + r_2 = x_1^{k+1} - x_2^2 + y_1^{k+1} - y_2^2$$

and

$$\begin{aligned} f\left((X_1 + \sqrt{X_3})^{k+1}\right) &= \left(\frac{x_1 + y_1}{2} + \sqrt{\frac{(x_1 - y_1)^2}{4}}\right)^{k+1} = x_1^{k+1}, \\ f\left((X_1 - \sqrt{X_3})^{k+1}\right) &= \left(\frac{x_1 + y_1}{2} - \sqrt{\frac{(x_1 - y_1)^2}{4}}\right)^{k+1} = y_1^{k+1}, \\ f\left(-(X_2^2 + X_4)\right) &= -\frac{(x_2 + y_2)^2}{4} - \frac{(x_2 - y_2)^2}{4} = -\frac{x_2^2 + y_2^2}{2}, \end{aligned}$$

we obtain

$$r_1 + r_2 = f\left(-2X_2^2 - 2X_4 + (X_1 + \sqrt{X_3})^{k+1} + (X_1 - \sqrt{X_3})^{k+1}\right). \tag{3.19}$$

Note the formula

$$2\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i = (X_1 + \sqrt{X_3})^{k+1} + (X_1 - \sqrt{X_3})^{k+1} \tag{3.20}$$

which allows us to rewrite (3.19) as

$$r_1 + r_2 = f\left(-2X_2^2 - 2X_4 + 2\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i\right). \tag{3.21}$$

For $r_1 - r_2 = x_1^{k+1} - x_2^2 - y_1^{k+1} + y_2^2$ notice that

$$f\left(\frac{1}{\sqrt{X_3}}(X_2 X_5)\right) = \frac{(x_2 + y_2)(x_1 - y_1)(x_2 - y_2)}{4(x_1 - y_1)} = \frac{x_2^2 - y_2^2}{4}$$

so that we can write

$$r_1 - r_2 = f\left((X_1 + \sqrt{X_3})^{k+1} - (X_1 - \sqrt{X_3})^{k+1} - 4\frac{1}{\sqrt{X_3}} X_2 X_5\right).$$

Using the formula

$$2\sqrt{X_3}\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i = (X_1 + \sqrt{X_3})^{k+1} - (X_1 - \sqrt{X_3})^{k+1} \tag{3.22}$$

we can write

$$r_1 - r_2 = f\left(2\sqrt{X_3}\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 4\frac{1}{\sqrt{X_3}} X_2 X_5\right).$$

Now, $x_1 - y_1 = f(2\sqrt{X_3})$ so that

$$(r_1 - r_2)(x_1 - y_1) = f\left(4X_3\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 8X_2 X_5\right). \tag{3.23}$$

Finally, $(x_2 - y_2) = f(2\sqrt{X_4}) = f\left(\frac{2}{\sqrt{X_3}}X_5\right)$ gives that

$$
\begin{aligned}
(r_1 - r_2)(x_2 - y_2) &= f\left(4X_5 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 8\frac{1}{X_3} X_2 X_5^2\right) \\
&= f\left(4X_5 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 8X_2X_4\right).
\end{aligned}
\tag{3.24}
$$

To sum up, we have that $f^{-1}(\langle r_1, r_2\rangle = f^{-1}(\langle r_1, r_2\rangle \cap \operatorname{img}(f)) + \ker(f)$ is generated by

$$
\begin{aligned}
g_1 &= -X_2^2 - X_4 + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i, \\
g_2 &= X_3 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2X_5, \\
g_3 &= X_5 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2X_4, \\
g_4 &= X_3X_4 - X_5^2,
\end{aligned}
\tag{3.25}
$$

which are thus local equations for $\operatorname{Sym}^2(Y)$. Note that the diagonal $\Delta = \{x_1 - y_1 = x_2 - y_2 = 0\}$ descends to $\operatorname{Sym}^2(\mathbb{C}^2)$ as $X_3 = X_4 = X_5 = 0$. After having computed local equations for the symmetric square of a curve with a unique singular point of type A_k, we turn our attention to the Hilbert square of such a curve.

3.19 Lemma. *Let Y be a curve with a unique singular point of type A_k. Then, the Hilbert-Chow morphism*

$$\operatorname{Hilb}^2(Y) \to \operatorname{Sym}^2(Y),\ \xi \mapsto \sum_{x \in \operatorname{supp}(\xi)} \operatorname{length}_x(\xi) \cdot x$$

is the blowup of the diagonal.

Proof. This is a special case of [ES14, Theorem 7.7] □

The following proposition describes the set of all length two subschemes of Y which are supported only at the singular point y_0 of Y, that is, the preimage $h^{-1}(2y_0)$ of the point $2y_0 \in \operatorname{Sym}^2(Y)$ under the Hilbert-Chow morphism $h : \operatorname{Hilb}^2(Y) \to \operatorname{Sym}^2(Y)$.

3.20 Proposition. *Every length two ideal*

$$I \subset \left(\frac{\mathbb{C}[x,y]}{\langle x^{k+1} + y^2\rangle}\right)_{\langle x,y\rangle}$$

in the localisation of $\frac{\mathbb{C}[x,y]}{\langle x^{k+1}+y^2\rangle}$ at the ideal $\langle x, y\rangle$ is of the form $I_a = \langle y + ax\rangle$, for some $a \in \mathbb{P}^1$ (regarded as $\mathbb{C} \cup \{\infty\}$), where $I_0 = \langle x^2, y\rangle$ and $I_\infty = \langle x, y^2\rangle$.

Proof. Let

$$R = \left(\frac{\mathbb{C}[x,y]}{\langle x^{k+1} + y^2\rangle}\right)_{\langle x,y\rangle} \cong \frac{\mathbb{C}[[x,y]]}{\langle x^{k+1} + y_2\rangle}$$

and $I = \langle f_1, \dots, f_r\rangle \subset R$ be an ideal of length two. Since any length m ideal contains x^m, y^m, we have $x^2, y^2 \in I$ and if $f \in \{f_1, \dots, f_r\}$, $f \neq 0$, is one of the generators for I, $f = ax + bxy + cy$ for

$a, b, c \in \mathbb{C}$. Let us write $f = yh(x) + ax$. If $h = 0$, $f = ax$ with $a \neq 0$ and $I = \langle x, x^2, y^2 \rangle = \langle x, y^2 \rangle$. If $h \neq 0$, then $f = x^m yu + ax$ for a unit $u \in R^\times$ and $m \geq 0$. In fact, if $c = 0$, $u = b \neq 0$ and $m = 1$, and if $c \neq 0$, $u = bx + c$ and $m = 0$. In both cases, if $a = 0$, we may replace the generator f for I by $x^m y$. Note however that $\langle x^2, x^m y, y^2 \rangle$ has length three, if $m = 1$ and there needs to be an element of the form $px + qy$ in I meaning that we can disregard (or replace by $px + qy$) the generator xy. For the generator $px + qy$ of I, we either have $q = 0$, $p \neq 0$ and $x \in I$ implying $I = \langle x, y^2 \rangle$, or $q \neq 0$, implying $I = \langle y + \frac{p}{q} x \rangle$. It remains to discuss the case $a \neq 0$. Recall that $f = x^m yu + ax$ for a unit u and note that

$$f = x^m yu + ax = \begin{cases} \left(c + \frac{bc}{a} y\right)\left(y + \frac{a}{c} x\right) + \langle x^2, y^2 \rangle, & c \neq 0, \\ (a + by)x, & c = 0. \end{cases}$$

In the former case, f can be replaced by $y + dx$ with $d = \frac{a}{c}$, in the latter by x. Finally, if I has two generators $y + px$, $y + qx$ with $p, q \neq 0$, then $x \in I$. Consequently, if $I \subset R$ has length two, it has at most two generators and then necessarily contains either x or y. If $I \subset R$ is generated by a single element, this element is of the form $y + ax$ for $a \in \mathbb{C}^*$. This proves the assertion. □

3.21 Lemma. *The singular locus of* $\mathrm{Hilb}^2(Y)$ *is isomorphic to the blowup of* Y *at its singular point* y_0.

Proof. By [Har83, II, Corollary 7.15], we find a commutative diagram

$$\begin{tikzcd} \mathrm{Bl}_{\mathrm{Sing}(\mathrm{Sym}^2(Y)) \cap \Delta} \mathrm{Sing}(\mathrm{Sym}^2(Y)) \arrow[r, hook] \arrow[d] & \mathrm{Hilb}^2(Y) \arrow[d, "h"] \\ \mathrm{Sing}(\mathrm{Sym}^2(Y)) \arrow[r, hook] & \mathrm{Sym}^2(Y). \end{tikzcd}$$

The singular locus of $\mathrm{Hilb}^2(Y)$ has to be contained in the preimage of the singular locus of $\mathrm{Sym}^2(Y)$. By Proposition 3.20, the preimage of the point $\{2y_0\} = \mathrm{Sing}(\mathrm{Sym}^2(Y)) \cap \Delta$ is a $\mathbb{P}^1$ and the preimage of the singular locus therefore is $\widetilde{Y} \cup \mathbb{P}^1$ where $\widetilde{Y}$ denotes the blowup of Y at its singular point. Consequently, the assertion follows if we can prove that the component isomorphic to $\mathbb{P}^1$ lies in the smooth locus of $\mathrm{Hilb}^2(Y)$ except for the point(s) of intersection with the component isomorphic to $\widetilde{Y}$.

Take a subscheme η entirely supported at the singular point of Y and corresponding to an ideal

$$I_a = \langle y + ax \rangle \subset \left(\frac{\mathbb{C}[x, y]}{\langle x^{k+1} + y^2 \rangle} \right)_{\langle x, y \rangle},$$

cf. Proposition 3.20. We argue, similarly to Lemma 1.3, that the tangent space

$$T_\eta \mathrm{Hilb}^2(Y) = H^0(\mathcal{N}_{\eta/Y})$$

is two-dimensional, showing that $\eta \in \mathrm{Hilb}^2(Y)$ is a smooth point. Using the normal sheaf sequence of η in Y, cf. (1.2), η is a smooth point of $\mathrm{Hilb}^2(Y)$, if $H^1(\mathcal{N}_{\eta/Y}) = 0$. Since η is zero-dimensional, this follows immediately from the fact that $\mathcal{N}_{\eta/Y}$ is coherent. □

We now compute local equations for $\mathrm{Hilb}^2(Y)$ by calculating the blowup of (3.25) along the diagonal. Recall that the diagonal is given by $X_3 = X_4 = X_5 = 0$ and consider the blowup of the ambient $\mathbb{C}^5$ along the diagonal, i.e. the variety

$$\mathrm{Bl}_\Delta \mathrm{Sym}^2(\mathbb{C}^2) = \left\{ (X_1, \ldots, X_5) \times (a_0 : a_1 : a_2) \in \mathbb{C}^5 \times \mathbb{P}^2 \;\middle|\; \begin{aligned} X_3 X_4 - X_5^2 &= 0 \\ a_0 X_4 - a_1 X_3 &= 0 \\ a_0 X_5 - a_2 X_3 &= 0 \\ a_1 X_5 - a_2 X_4 &= 0 \end{aligned} \right\} \subset \mathbb{C}^5 \times \mathbb{P}^2.$$

Take the chart $U_0 = \{a_0 \neq 0\}$. To calculate equations for the strict transform of $\mathrm{Sym}^2(Y)$ in this chart, we consider the defining equations for $\mathrm{Bl}_\Delta \mathrm{Sym}^2(Y) \cap U_0$ which are

$$\begin{aligned}
-X_2^2 - X_4 + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i &= 0, \\
X_3 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2X_5 &= 0, \\
X_5 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2X_4 &= 0, \\
X_3X_4 - X_5^2 &= 0, \\
X_4 - a_1X_3 &= 0, \\
X_5 - a_2X_3 &= 0.
\end{aligned}$$

Substitution of the variables X_4, X_5 using the last two equations gives

$$\begin{aligned}
-X_2^2 - a_1X_3 + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i &= 0, \\
X_3 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2a_2X_3 &= 0, \\
a_2X_3 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2a_1X_3 &= 0, \\
a_1X_3^2 - a_2^2X_3^2 &= 0.
\end{aligned}$$

Elimination of factors corresponding to the exceptional divisor $X_3 = 0$ then yields

$$\begin{aligned}
-X_2^2 - a_1X_3 + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i &= 0, \\
\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2a_2 &= 0, \\
a_2 \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2a_1 &= 0, \\
a_1 - a_2^2 &= 0.
\end{aligned}$$

Using the last equation, the third one is a multiple of the second and we can reduce the equations to

$$\begin{aligned}
-X_2^2 - X_3X_4^2 + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i} X_1^{k-2i+1} X_3^i &= 0, \\
\sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2i+1} X_1^{k-2i} X_3^i - 2X_2X_4 &= 0,
\end{aligned} \tag{3.26}$$

where we write X_4 for the variable a_2. Local equations for the other charts are computed similarly but are not relevant for our purposes as we are interested only in neighbourhoods of the singular point of the singular locus. Using the Jacobian criterion on the equations (3.26) it is simple to show that

$$\mathrm{Sing}(\mathrm{Hilb}^2(Y) \cap U_0) = \{X_2 - X_1X_4 = X_3 - X_1^2 = 2^{k-1}X_1^{k-1} - X_4^2 = 0\},$$

i.e. a curve with a singularity of type A_{k-2}. By working out the local equations in the other coordinate charts of the blowup one can check that the singular locus admits no further singular points. This provides a different proof of Lemma 3.21. In the same way one can deduce that over the special point of $\mathrm{Sym}^2(Y)$ there lies the smooth quadric $\{a_0a_1 - a_2^2 = 0\} \subset \mathrm{Proj}(\mathbb{C}[a_0, a_1, a_2]) \cong \mathbb{P}^2$ which is isomorphic to $\mathbb{P}^1$, reproving Proposition 3.20.

Now that we have local equations for $\mathrm{Hilb}^2(Y)$ we try to resolve its singularities by blowing up the entire singular locus. To simplify our calculations, consider the coordinate change $X_2 \mapsto X_2 + X_1X_4$, $X_3 \mapsto X_3 + X_1^2$. The defining equations for $\mathrm{Hilb}^2(Y)$ in these new coordinates then are

$$\begin{aligned} f_1 =& -X_2^2 - 2a_2X_1X_2 - a_2^2X_1^2 - a_2^2(X_3 + X_1^2) + \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor} \binom{k+1}{2i} X_1^{k-2i+1}(X_3 + X_1^2)^i, \\ f_2 =& \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor} \binom{k+1}{2i+1} X_1^{k-2i}(X_3 + X_1^2)^i - 2(X_2 + a_2X_1)a_2. \end{aligned}$$

In order to simplify these equations, we are going to use the formula

$$\sum_{i=j}^{\lfloor\frac{k+1}{2}\rfloor} \binom{k+1}{2i+1}\binom{i}{j} = 2^{k-2j}\binom{k-j}{j}, \tag{3.27}$$

as well as the formula

$$\sum_{i=j}^{\lfloor\frac{k+1}{2}\rfloor} \binom{k+1}{2i}\binom{i}{j} = (-1)^{k+1}2^{k-2j}\left(\binom{-j-1}{k-2j} + 2\binom{-j-1}{k-2j+1}\right). \tag{3.28}$$

3.22 Remark. Binomial identities such as (3.27) and (3.28) can be verified using the Wilf-Zeilberger method. An explanation and proof of this method can be found in [PWZ96]. The major advantage of the Wilf-Zeilberger method is that as long as one has come up with an identity that possibly could hold, it can be proven by a computer. Details on this are also explained in [PWZ96]. Formula (3.27) and other binomial identities in this thesis have been verified using the RISCErgoSum software [RIS] provided by the Research Institute for Scientific Computing in Linz. We thank Prof. Peter Paule and the RISC Linz for granting us access to the software. Formula (3.27) can also be found in [Yam17b, formula (2.2)].

Replace f_1 by $f_3 = f_1 - X_1 f_2$ to obtain

$$\begin{aligned}
f_1 - X_1 f_2 = &\ -X_2^2 - X_3X_4^2 + \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}\left(\binom{k+1}{2i} - \binom{k+1}{2i+1}\right)X_1^{k-2i+1}(X_3+X_1^2)^i \\
= &\ -X_2^2 - X_3X_4^2 + \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}\left(\binom{k+1}{2i} - \binom{k+1}{2i+1}\right)X_1^{k-2i+1}\sum_{j=0}^{i}\binom{i}{j}X_3^jX_1^{2i-2j} \\
= &\ -X_2^2 - X_3X_4^2 + \sum_{j=0}^{\lfloor\frac{k+1}{2}\rfloor}\sum_{i=j}^{\lfloor\frac{k+1}{2}\rfloor}\left(\binom{k+1}{2i} - \binom{k+1}{2i+1}\right)\binom{i}{j}X_1^{k-2j+1}X_3^j \\
= &\ -X_2^2 - X_3X_4^2 \\
&+ \sum_{j=0}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\left((-1)^{k+1}\left(\binom{-j-1}{k-2j} + 2\binom{-j-1}{k-2j+1}\right) - \binom{k-j}{j}\right)X_1^{k-2j+1}X_3^j \\
= &\ -X_2^2 + X_3(2^{k-1}X_1^{k-1} - X_4^2) \\
&+ \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\left((-1)^{k+1}\left(\binom{-j-1}{k-2j} + 2\binom{-j-1}{k-2j+1}\right) - \binom{k-j}{j}\right)X_1^{k-2j+1}X_3^j,
\end{aligned}$$

using (3.27) and (3.28). The equation f_2 can be simplified as follows

$$\begin{aligned}
f_2 = &\ \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}\binom{k+1}{2i+1}X_1^{k-2i}(X_3+X_1^2)^i - 2X_2X_4 - 2X_1X_4^2 \\
= &\ \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{k+1}{2i+1}X_1^{k-2j}X_3^j - 2X_2X_4 - 2X_1X_4^2 \\
= &\ \sum_{j=0}^{\lfloor\frac{k+1}{2}\rfloor}\sum_{i=j}^{\lfloor\frac{k+1}{2}\rfloor}\binom{i}{j}\binom{k+1}{2i+1}X_1^{k-2j}X_3^i - 2X_2X_4 - 2X_1X_4^2 \\
= &\ \sum_{j=0}^{\lfloor\frac{k+1}{2}\rfloor}2^{k-2j}\binom{k-j}{j}X_1^{k-2j}X_3^j - 2X_2X_4 - 2X_1X_4^2 \\
= &\ \sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor}2^{k-2j}\binom{k-j}{j}X_1^{k-2j}X_3^j - 2X_2X_4 + 2X_1(2^{k-1}X_1^{k-1} - X_4^2),
\end{aligned}$$

using (3.27). In order to further simplify f_3, define

$$F(j,k) = (-1)^{k+1}\left(\binom{-j-2}{k-2j-2} + 2\binom{-j-2}{k-2j-1}\right) - \binom{k-j-1}{j+1}.$$

It is simple to check that $F(j,k)$ satisfies, and is completely determined by, the following recursive relation

$$F(j,k+1) = \frac{(k-j)F(j,k)}{k-2j}, \quad F(j,2j+1) = 2. \tag{3.29}$$

Therefore, by examining this relation,

$$F(j,k) = \frac{2(k-j-1)!}{j!(k-2j-1)!} = 2\binom{k-j-1}{j}, \tag{3.30}$$

and we can write

$$\begin{aligned}
f_3 = & -X_2^2 + X_3(2^{k-1}X_1^{k-1} - X_4^2) + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j} F(j-1,k) X_1^{k-2j+1} X_3^j \\
= & -X_2^2 + X_3(2^{k-1}X_1^{k-1} - X_4^2) + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j+1} \binom{k-j}{j-1} X_1^{k-2j+1} X_3^j.
\end{aligned}$$

The identity (3.30) can also be proven using the methods described in Remark 3.22. To sum up, in the new coordinates the local equations for $\mathrm{Hilb}^2(Y)$ are

$$\begin{aligned}
f_3 = & -X_2^2 + X_3(2^{k-1}X_1^{k-1} - X_4^2) + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j+1} \binom{k-j}{j-1} X_1^{k-2j+1} X_3^j, \\
f_2 = & \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j} \binom{k-j}{j} X_1^{k-2j} X_3^j - 2X_2X_4 + 2X_1(2^{k-1}X_1^{k-1} - X_4^2).
\end{aligned} \tag{3.31}$$

The singular locus in the new coordinates now is $X_2 = X_3 = 2^{k-1}X_1^{k-1} - X_4^2 = 0$. Take the blowup of the ambient $\mathbb{C}^4$ along this locus. Defining equations for the blown up $\mathbb{C}^4$ then are

$$\{a_0X_3 - a_1X_2 = a_0t - a_2X_2 = a_1t - a_2X_3 = 0\} \subset \mathbb{C}^4 \times \mathbb{P}^2,$$

where we write $t = 2^{k-1}X_1^{k-1} - X_4^2$. In the chart $U_1 = \{a_1 \neq 0\}$, the total transform of $\mathrm{Hilb}^2(Y)$ is given by

$$\begin{aligned}
0 &= X_3^2 \left(-a_0^2 + a_2 + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j+1} \binom{k-j}{j-1} X_1^{k-2j+1} X_3^{j-2} \right), \\
0 &= X_3 \left(\sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j} \binom{k-j}{j} X_1^{k-2j} X_3^{j-1} - 2a_0X_4 + 2a_2X_1 \right), \\
0 &= 2^{k-1}X_1^{k-1} - X_4^2 - a_2X_3.
\end{aligned} \tag{3.32}$$

Local equations for the strict transform of $\mathrm{Hilb}^2(Y)$ are thus

$$\begin{aligned}
0 = & -a_0^2 + a_2 + \sum_{j=2}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j+1} \binom{k-j}{j-1} X_1^{k-2j+1} X_3^{j-2}, \\
0 = & \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} 2^{k-2j} \binom{k-j}{j} X_1^{k-2j} X_3^{j-1} - 2a_0X_4 + 2a_2X_1, \\
0 = & \, 2^{k-1}X_1^{k-1} - X_4^2 - a_2X_3.
\end{aligned} \tag{3.33}$$

We can eliminate the variable a_2 using the first equation. The corresponding calculations are

$$
\begin{aligned}
&\sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\binom{k-j}{j}X_1^{k-2j}X_3^{j-1} - 2a_0X_4 - 2X_1\left(-a_0^2 + \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-2}\right)\\
&= \sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\binom{k-j}{j}X_1^{k-2j}X_3^{j-1} - 2a_0X_4 + 2a_0^2X_1 - \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+2}\binom{k-j}{j-1}X_1^{k-2j+2}X_3^{j-2}\\
&= \sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\binom{k-j}{j}X_1^{k-2j}X_3^{j-1} - 2a_0X_4 + 2a_0^2X_1 - \sum_{l=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2l}\binom{k-l-1}{l}X_1^{k-2l}X_3^{l-1}\\
&= \sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\left(\binom{k-j}{j} - \binom{k-j-1}{j}\right)X_1^{k-2j}X_3^{j-1} - 2a_0X_4 + 2a_0^2X_1,\\
&= \sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\binom{k-j-1}{j-1}X_1^{k-2j}X_3^{j-1} - 2a_0X_4 + 2a_0^2X_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&2^{k-1}X_1^{k-1} - X_4^2 + X_3\left(-a_0^2 + \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-2}\right)\\
&= 2^{k-1}X_1^{k-1} - X_4^2 - a_0^2X_3 + X_3\sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-2}.
\end{aligned}
$$

Therefore, we obtained the following simplified equations

$$
\begin{aligned}
0 &= \sum_{j=1}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\binom{k-j-1}{j-1}X_1^{k-2j}X_3^{j-1} - 2a_0X_4 + 2a_0^2X_1,\\
0 &= 2^{k-1}X_1^{k-1} - X_4^2 - a_0^2X_3 + X_3\sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-2},
\end{aligned}
\tag{3.34}
$$

for the strict transform of $\mathrm{Hilb}^2(Y)$ under the blowup of the ambient $\mathrm{Hilb}^2(\mathbb{C}^2)$ along the singular locus of $\mathrm{Hilb}^2(Y)$. Consider the change of coordinates $X_4 \mapsto X_4 - 2a_0X_1$. Then, (3.34) becomes

$$
\begin{aligned}
\widetilde{f_1} &= \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j}\binom{k-j-1}{j-1}X_1^{k-2j}X_3^{j-1} - 2a_0X_4 + 2X_1(2^{k-3}X_1^{k-3} - a_0^2)\\
&= \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-2-2j}\binom{k-2-j}{j}X_1^{k-2-2j}X_3^{j} - 2a_0X_4 + 2X_1(2^{k-3}X_1^{k-3} - a_0^2),\\
\widetilde{f_2} &= 4X_1^2(2^{k-3}X_1^{k-3} - a_0^2) - X_4^2 - 4a_0X_1X_4 - a_0^2X_3 + \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-1}.
\end{aligned}
\tag{3.35}
$$

We see that the first equation of (3.35) is the same as the second defining equation for the Hilbert square of a curve with a singularity of type A_{k-2}, see (3.31).

Replace $\widetilde{f}_2$ by $\widetilde{f}_3 = \widetilde{f}_2 - 2X_1\widetilde{f}_1$ to obtain

$$
\begin{aligned}
\widetilde{f}_3 = {} & 4X_1^2(2^{k-3}X_1^{k-3} - a_0^2) - X_4^2 - 4a_0X_1X_4 - a_0^2X_3 + \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-1} \\
& - 2X_1\left(\sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-2-2j}\binom{k-2-j}{j}X_1^{k-2-2j}X_3^j - 2a_0X_4 + 2X_1(2^{k-3}X_1^{k-3} - a_0^2)\right) \\
= {} & - X_4^2 - a_0^2X_3 \\
& + \sum_{j=2}^{\lfloor\frac{k+1}{2}\rfloor} 2^{k-2j+1}\binom{k-j}{j-1}X_1^{k-2j+1}X_3^{j-1} - \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-1-2j}\binom{k-2-j}{j}X_1^{k-1-2j}X_3^j \\
= {} & - X_4^2 - a_0^2X_3 \\
& + \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-2j-1}\binom{k-j-1}{j}X_1^{k-2j-1}X_3^j - \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-1-2j}\binom{k-2-j}{j}X_1^{k-1-2j}X_3^j \\
= {} & - X_4^2 - a_0^2X_3 + \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-1-2j}\left(\binom{k-j-1}{j} - \binom{k-2-j}{j}\right)X_1^{k-1-2j}X_3^j \\
= {} & - X_4^2 - a_0^2X_3 + \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-1-2j}\binom{k-2-j}{j-1}X_1^{k-1-2j}X_3^j \\
= {} & - X_4^2 + X_3(2^{k-3}X_1^{k-3} - a_0^2) + \sum_{j=2}^{\lfloor\frac{k-1}{2}\rfloor} 2^{k-1-2j}\binom{k-2-j}{j-1}X_1^{k-1-2j}X_3^j.
\end{aligned}
$$

By comparing the equations $\widetilde{f}_1$ and $\widetilde{f}_3$ defining the strict transform in the chart U_0 with equations (3.31), we see that they are the same as (3.31) but with k replaced by $k-2$. This proves the following theorem.

3.23 Theorem. *Let Y be a curve admitting a unique singular point y_0 of type A_k and $k \geq 5$. Let $\pi : \widetilde{\mathrm{Hilb}^2(Y)} \to \mathrm{Hilb}^2(Y)$ be the blowup of $\mathrm{Hilb}^2(Y)$ along its singular locus. Then,*

$$(\widetilde{\mathrm{Hilb}^2(Y)}, p) \cong (\mathrm{Hilb}^2(\widetilde{Y}), q),$$

where p and q denote the singular points of the respective singular loci.

3.24 Remark. If Y is a curve with a unique singular point of type A_k, the singular locus of $\mathrm{Hilb}^2(Y)$ is a curve with a singular point of type A_{k-2}. For $k \leq 4$, the singular locus of the strict transform of $\mathrm{Hilb}^2(Y)$ under the blowup of the ambient space along the singular locus of $\mathrm{Hilb}^2(Y)$ is smooth. Therefore, in the notation of Theorem 3.23,

$$(\widetilde{\mathrm{Hilb}^2(Y)}, p) \cong (\mathrm{Hilb}^2(\widetilde{Y}), q)$$

holds for all points p, q of the respective singular loci.

3.25 Remark. R. Yamagishi treated this case in [Yam17b] but there are several errors in his computations. For example, he asserts and frequently uses the formula

$$\sum_{i=k}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2i}\binom{i}{k} = 2^{n-2k}\frac{n}{n-k}\binom{n-k+1}{k},$$

see [Yam17b, formula (2.1)]. This is incorrect, e.g. for $(n,k)=(4,2)$ the left-hand side is 5 whilst the right-hand side is 6. [1]

3.26 Remark. Note that the isomorphism in Theorem 3.23 does not come from a global isomorphism between $\mathrm{Hilb}^2(\widetilde{Y})$ and $\widetilde{\mathrm{Hilb}^2(Y)}$. In fact, let Y be a curve with unique singular point of type A_1. Then, $\widetilde{Y}$ is smooth, therefore $\mathrm{Hilb}^2(\widetilde{Y}) \cong \mathrm{Sym}^2(\widetilde{Y})$. On the other hand, as the singular locus of $\mathrm{Hilb}^2(Y)$ is isomorphic to $\widetilde{Y}$, it is smooth, and the singularities of $\mathrm{Hilb}^2(Y)$ can be resolved by a single blowup of the entire singular locus. Consider the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Hilb}^2(Y)} \arrow[r] \arrow[d] & \mathrm{Hilb}^2(Y) \arrow[d] \\
\mathrm{Hilb}^2(\widetilde{Y}) \arrow[r] & \mathrm{Sym}^2(Y)
\end{tikzcd}$$

where the horizontal maps are the blowups along the respective singular loci, the vertical map on the right is the Hilbert-Chow morphism and the vertical left on the left is given by the universal property of the normalisation $\mathrm{Hilb}^2(\widetilde{Y}) \to \mathrm{Sym}^2(Y)$. Then the horizontal maps are finite whilst the vertical map on the right contracts a rational curve in $\mathrm{Hilb}^2(Y)$. Therefore, the vertical map an the left also contracts a rational curve and $\mathrm{Hilb}^2(\widetilde{Y})$ cannot be globally isomorphic to $\widetilde{\mathrm{Hilb}^2(Y)}$.

3.27 Corollary. *Let Y be a curve with a unique singular point of type A_k. Then, the singularities of $\mathrm{Hilb}^2(Y)$ can be resolved by successive blowups of the singular locus of $\mathrm{Hilb}^2(Y)$. Under this succession of blowups, the fibre over the singular point of the singular locus of $\mathrm{Hilb}^2(Y)$ is a chain of $l = \left\lfloor \frac{k-1}{2} \right\rfloor$ reduced rational curves with dual graph*

$$\overset{E_1}{\bigcirc} - \overset{E_2}{\bigcirc} - \cdots - \overset{E_l}{\bigcirc}\ .$$

Moreover, the intersection matrix $(E_iE_j)_{i,j=1}^l$ is negative definite.

Proof. Every blowup gives a reduced rational curve that is contracted to the singular point of the singular locus of the preceding blowup, as long as the singular locus of the preceding blowup is still singular. Moreover, these curves intersect each other in at most one point and transverse at these points. This can be seen from the local equations above. It can also be seen as follows: let $W \to \mathrm{Hilb}^2(Y)$ be the resolution obtained by successive blowups of the entire singular locus. Then, $W \to \mathrm{Sym}^2(Y)$ is likewise a resolution and the universal property of normalisation gives a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "r"] \arrow[d, "c"'] & \mathrm{Hilb}^2(Y) \arrow[d, "h"] \\
\mathrm{Sym}^2(Y^\nu) \arrow[r] & \mathrm{Sym}^2(Y).
\end{tikzcd}$$

Since W and $\mathrm{Sym}^2(Y^\nu)$ are smooth surfaces, c factors as a composition of blowups of smooth points. Stated differently, c contracts a chain of smooth rational curves to a smooth point of $\mathrm{Sym}^2(Y^\nu)$. By commutativity of the diagram, one of these curves is the $\mathbb{P}^1$ contracted by the Hilbert-Chow morphism h, see Proposition 3.20 and the proof of Lemma 3.21, and the remaining curves are the fibre of r over the singular point of the singular locus of $\mathrm{Hilb}^2(Y)$. Their intersection matrix thus is negative definite by Grauert's criterion [Gra62, page 367]. As $l = \left\lfloor \frac{k-1}{2} \right\rfloor$ blowups of $\mathrm{Hilb}^2(Y)$ along its singular locus are needed to obtain a smooth

[1] In fact, the left-hand side clearly is always an integer but the right-hand side can become rational, see e.g. $(n,k)=(3,1)$ where the right-hand side results in $\frac{3}{2}$ (and the left-hand side equals one).

singular locus, the number of curves in the fibre of r over the singular point of the singular locus of $\mathrm{Hilb}^2(Y)$ is l, as asserted. Moreover, every such curve intersects the centre of a succeeding blowup in precisely one point, these curves thus have the asserted dual graph. □

3.28 Corollary. *Let Y be a curve with a unique singular point of type A_k and let $W \to \mathrm{Hilb}^2(Y)$ be the resolution of singularities of Y obtained by successive blowups of the singular locus. Then,* $\mathrm{Pic}^0(W) \cong \mathrm{Pic}^0(Y^\nu)$.

Proof. By Corollary 3.27 there is a sequence of blowups of smooth points $c : W \to \mathrm{Sym}^2(Y^\nu)$. Consequently, $\mathrm{Pic}(W) = \mathrm{Pic}(\mathrm{Sym}^2(Y^\nu)) \oplus \mathbb{Z}^l$, where l is the number of blowups. But this implies the assertion using that $\mathrm{Pic}^0(\mathrm{Sym}^2(Y^\nu)) \cong \mathrm{Pic}^0(Y^\nu)$ by Lemma 2.26. □

3.29 Remark. The results of this section can be generalised to varieties Y with a unique singular point of type A_k and dimension greater than one. To do so, one needs to add squares of new variables to the locally defining equation for Y. This changes the number of variables one has to deal with in the local computations for $\mathrm{Sym}^2(Y)$ and $\mathrm{Hilb}^2(Y)$ accordingly. The crucial step in proving our results was to find the correct formulas in the binomial coefficients which can still be applied regardless the number of variables.

3.3 Degenerations of the Picard scheme

We are now able to generalise our approach from chapter two, see Theorem 2.29 and Theorem 2.32, for computing the degenerate Picard scheme of a smoothing $\mathfrak{F} \to B$ of the Fano scheme $F(X) = \mathfrak{F}_0$ of lines on a singular cubic threefold X. Theorem 3.5 shows that the blowup of $F(X)$ along its singular locus Σ results in $\mathrm{Hilb}^2(\Sigma)$ and Theorem 3.23 shows that further blowups along the entire singular locus of $\mathrm{Hilb}^2(\Sigma)$ yield a resolution of $F(X)$. We begin by analysing the relative situation, i.e. by describing the total transform of the blowup of $\mathfrak{F}$ along the singular locus of $\mathfrak{F}_0$.

3.30 Proposition. *Let Y be a curve with unique singular point of type A_k, $k \geq 3$, and let $\pi : \mathfrak{H} \to B$ be a smoothing of $\mathfrak{H}_0 = \mathrm{Hilb}^2(Y)$ with regular total space $\mathfrak{H}$. If $r : \widetilde{\mathfrak{H}} \to \mathfrak{H}$ denotes the blowup of $\mathfrak{H}$ along $C = (\mathfrak{H}_0)_{\mathrm{sing}}$, then $r^{-1}(\mathfrak{H}_0) = W \cup E$, where W denotes the strict transform of $\mathfrak{H}_0$ and E is the total space of a $\mathbb{P}^1$-bundle over W_{sing}. Moreover, $W \cap E = W_{\mathrm{sing}} \cup P$, where P is the fibre of E over the unique point $c \in W_{\mathrm{sing}}$ with $r(c) = C_{\mathrm{sing}}$, hence isomorphic to $\mathbb{P}^1$.*

Proof. Denote by $E = r^{-1}((\mathfrak{H}_0)_{\mathrm{sing}})$ the exceptional divisor of the blowup. Our local calculations above have shown that the singular locus of $\mathrm{Hilb}^2(Y)$ is locally a complete intersection in the respective ambient space. As being locally a complete intersection is an intrinsic property of a scheme and not dependent on the ambient space, cf. [Har83, II, Remark 8.22.2], $(\mathfrak{H}_0)_{\mathrm{sing}} \subset \mathfrak{H}$ is locally given by two equations. Thus, for every point $p \in (\mathfrak{H}_0)_{\mathrm{sing}}$ the fibre $r^{-1}(p) \subset E$ is isomorphic to $\mathbb{P}^1$. The intersection $E \cap W$ can easily be computed from the local equations (3.33) describing the strict transform W of $\mathfrak{H}_0$ locally. In fact, by intersecting these equations with the local equation $X_3 = 0$ for E, one finds that the intersection $W \cap E$ has two components, one given by the singular locus of W and the other given (in the respective coordinates) by $X_1 = X_2 = X_3 = X_4 = a_0^2 - a_2 = 0$. By examining the other charts also, it is simple to check that $W \cap E = W_{\mathrm{sing}} \cup P$, where $P = \{a_0^2 - a_1 a_2 = 0\} \subset \mathrm{Proj}(\mathbb{C}[a_0, a_1, a_2])$, that is, $P \cong \mathbb{P}^1$, and that $P = r^{-1}(\eta_0)$ where η_0 is the singular point of $C = (\mathfrak{H}_0)_{\mathrm{sing}}$.

By Lemma 2.21, $r^{-1}(C_{\mathrm{reg}})$ defines a $\mathbb{P}^1$-bundle over the regular points of W_{sing} and therefore, using the above, if $U \subset \widetilde{\mathfrak{H}}$ is a neighbourhood of an arbitrary point $p \in W_{\mathrm{sing}}$,

$$E \cap U \cong (W_{\mathrm{sing}} \cap U) \times \mathbb{P}^1.$$

and this proves the assertion as every $\mathbb{P}^1$-fibration over a curve is a $\mathbb{P}^1$-bundle, cf. [Sch70, Satz 4.9]. □

3.31 Remark. The analogous result holds for a smoothing $\pi : \mathfrak{F} \to B$ of $\mathfrak{F}_0 = F(X)$, where X is a cubic threefold with a unique singular point of type A_k and $k \geq 3$. The only thing that needs to be checked is that the inverse image of the singular locus of $F(X)$ under the morphism $\varphi : \mathrm{Hilb}^2(\Sigma) \to F(X)$ has two components, one of them the singular locus of $\mathrm{Hilb}^2(\Sigma)$ and the other isomorphic to a $\mathbb{P}^1$. But this was proven in Lemma 3.2.

Let B be one-dimensional and $\pi : \mathfrak{F} \to B$ be a smoothing of $\mathfrak{F}_0 = F(X)$ where X is a cubic threefold with unique singular point of type A_k and $k \geq 3$. Then, by Proposition 3.30, taking the blowup of $\mathfrak{F}$ along the singular locus of $\mathfrak{F}_0$ results in a situation which is almost the same as described in Lemma 2.21 but with one fibre of the resulting $\mathbb{P}^1$-bundle being contained in the strict transform. The following corollary should be thought of as an analogue of Lemma 2.21 but taking into account the additional intersection of the resulting $\mathbb{P}^1$-bundles with the strict transform.

3.32 Corollary (Strong embedded resolution for Fano schemes of lines on cubic threefolds)**.** *Let $\pi : \mathfrak{F} \to B$ be a smoothing of $\mathfrak{F}_0 = F(X)$ where X is a cubic threefold with a unique singular point of type A_k and $k \geq 3$. Then there exists a sequence*

$$r : \widetilde{\mathfrak{F}} = \mathfrak{F}^s \xrightarrow{\quad r^s \quad} \cdots \xrightarrow{\quad r^2 \quad} \mathfrak{F}^1 \xrightarrow{\quad r^1 \quad} \mathfrak{F}^0 = \mathfrak{F}$$

of blowups with centres lying over the singular locus of $\mathfrak{F}_0$ such that $\widetilde{\mathfrak{F}}$ is smooth and

(i) $\widetilde{\mathfrak{F}}_0 = r^{-1}(\mathfrak{F}_0) \subset \widetilde{\mathfrak{F}}$ *is a simple normal crossing divisor;*

(ii) $\widetilde{\mathfrak{F}}_0 = W \cup E_1 \cup \cdots \cup E_s$ *where W denotes the strict transform of $\mathfrak{F}_0$ and each E_i is the total space of a $\mathbb{P}^1$-bundle over Σ^ν;*

(iii) if $C \subset W$ denotes the curve arising as inverse image of the singular locus $(\mathfrak{F}_0)_{\mathrm{sing}}$ inside W, i.e. $C = r^{-1}((\mathfrak{F}_0)_{\mathrm{sing}}) \cap W$, then for every point $p \in C$ there exists a neighbourhood U of p inside $\widetilde{\mathfrak{F}}$ such that the central fibre $\widetilde{\mathfrak{F}}_0$ of $\widetilde{\mathfrak{F}} \to B$ is locally a product

$$\widetilde{\mathfrak{F}}_0 \cap U \cong (C \cap U) \times K,$$

where K is a configuration of curves with dual graph as in the corresponding case of Lemma 2.9;

(iv) $E_i \cap W \cong \mathbb{P}^1$ *for* $i \in \left\{1, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor\right\}$.

Proof. After $l = \left\lfloor \frac{k-1}{2} \right\rfloor$ blowups as in Proposition 3.30, the singular locus C of the strict transform W of $\mathfrak{F}_0$ becomes a smooth curve isomorphic to Σ^ν and W has, depending on the parity of k, a singularity of type A_1 or A_2 transversally along its singular locus. If $E_1, \ldots, E_l$ denote (the strict transforms of) the exceptional divisors of these blowups, each is the total space of a $\mathbb{P}^1$-bundle over C and for every point $p \in C$ there exists a neighbourhood $U = U(p)$ such that

$$(W \cup E_1 \cup \cdots \cup E_l) \cap U \cong (C \cap U) \times K,$$

where K is a configuration of curves as for an embedded resolution of a curve with a unique singular point of type A_k after l blowups of its singular point, cf. Lemma 2.9 and Lemma 2.21. Following Proposition 3.30 there is a unique point $p \in C$ such that for $i \in \{1, \ldots, l\}$ the fibre $(E_i)_p$ of E_i over p is contained in W.
To obtain the normal crossing assertion, we only need to blow up once more along C, if k is odd, and thrice more, if k is even. This is clear by Lemma 2.21 as W admits a singularity of type A_1 or A_2 transversally along C and C is smooth. Note that after performing these blowups, the blown up total space $\widetilde{\mathfrak{F}}$ is locally given by $t - x^a y^b z^c = 0$, hence smooth. The asserted properties now all hold by construction, respectively have been proven in Lemma 2.21. $\square$

In order to compute the degeneration data for the Picard scheme of $\mathfrak{F}_0$, we would like to argue similarly to the proofs of Theorem 2.29 and Theorem 2.32 by computing a tail reduction of $\pi : \mathfrak{F} \to B$ explicitly using the algorithm provided by the proof of Theorem 2.22. The initial step in this algorithm is to modify the total space of a given family $\mathfrak{Z} \to B$ by a series of blowups $\widetilde{\mathfrak{Z}} \to \mathfrak{Z}$ such that $\widetilde{\mathfrak{Z}}$ is smooth and $\widetilde{\mathfrak{Z}}_0 \subset \widetilde{\mathfrak{Z}}$ defines a simple normal crossing divisor. If the central fibre $\mathfrak{Z}_0 \subset \mathfrak{Z}$ has curve singularities of type A_k transversally along a smooth singular locus, we have shown in Lemma 2.21 that this is possible by successive blowups of the singular locus of $\mathfrak{Z}_0$. However, Corollary 3.32 shows that the same is true for a smoothing $\pi : \mathfrak{F} \to B$ of the Fano scheme of lines $F(X) = \mathfrak{F}_0$ of a cubic threefold with a unique singular point of type A_k. Therefore, the algorithm from the proof of Theorem 2.22 generalises to $\pi : \mathfrak{F} \to B$ by taking in the initial step the embedded resolution of $\mathfrak{F}_0 \subset \mathfrak{F}$ from Corollary 3.32.

3.33 Definition. *Let $\pi : \mathfrak{F} \to B$ be a smoothing of $\mathfrak{F}_0 = F(X)$ where X is a cubic threefold with a unique singular point of type A_k. By the tail reduction of $\pi : \mathfrak{F} \to B$ we mean the family $\pi' : \mathfrak{F}' \to B'$ obtained by applying the algorithm from the proof of Theorem 2.22 to $\pi : \mathfrak{F} \to B$ but taking in the initial step the embedded resolution of $\mathfrak{F}_0 \subset \mathfrak{F}$ from Corollary 3.32.*

By computing a tail reduction of $\pi : \mathfrak{F} \to B$ explicitly, we are able to prove the following theorem which is our main result on the degenerate Picard scheme of the Fano scheme of lines on a cubic hypersurface with a unique singular point of type A_k.

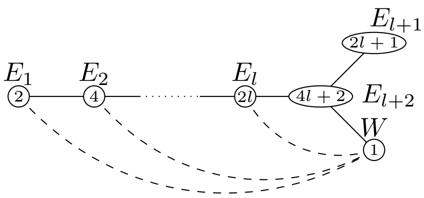
3.34 Theorem. *Let $\pi' : \mathfrak{F}' \to B'$ denote the tail reduction of $\pi : \mathfrak{F} \to B$. Then, the degenerate Picard scheme of $\mathfrak{F}_0$ with respect to the family $\pi : \mathfrak{F} \to B$ is uniquely determined by $\mathrm{Pic}^0(\mathfrak{F}'_0)$ which has the form*

$$1 \longrightarrow K \longrightarrow \mathrm{Pic}^0(\mathfrak{F}'_0) \longrightarrow \mathrm{Pic}^0(\Sigma^\nu) \times \mathrm{Pic}^0(T) \longrightarrow 0,$$

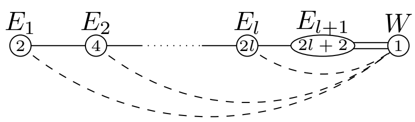
where T is a smooth curve of genus $g(T) = \lceil \frac{k-1}{2} \rceil$. Moreover,

$$K = \begin{cases} \mathbb{C}^*, & \textit{if } k \textit{ is odd;} \\ 0, & \textit{if } k \textit{ is even.} \end{cases}$$

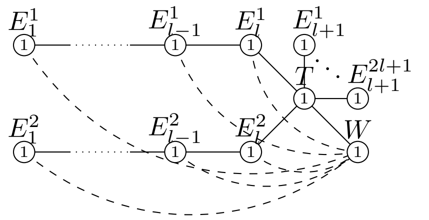
Proof. Following Corollary 3.32, taking successive blowups of $\mathfrak{F}$ along the singular locus of $\mathfrak{F}_0$ and its strict transforms provides a strong embedded resolution of $\mathfrak{F}_0 \subset \mathfrak{F}$. Denote by $\widetilde{\pi} : \widetilde{\mathfrak{F}} \to B$ the family obtained this way. Then the dual graph of $\widetilde{\mathfrak{F}}_0$ is given by



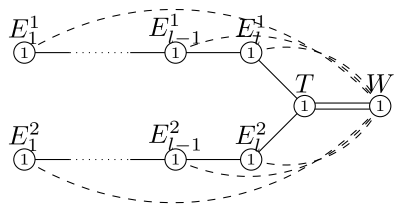
if k is even, and by



if k is odd. In these dual graphs, $l = \lceil \frac{k-1}{2} \rceil$, W denotes the strict transform of $\mathfrak{F}_0$ and each E_i is the total space of a $\mathbb{P}^1$ bundle over Σ^ν. Moreover, circled numbers indicate the multiplicity of the respective component and two vertices are joined by a solid edge for every irreducible component of their intersection being isomorphic to Σ^ν; and by a dashed edge for every irreducible component of their intersection being isomorphic to $\mathbb{P}^1$. The resulting dual graphs of the central fibre $\mathfrak{F}'_0$ of the tail reduction $\pi' : \mathfrak{F}' \to B'$ are then similar to those obtained in Lemma 2.10 and Lemma 2.11 except for the additional intersection with W which turn them into



if k is even, and



if k is odd. In both cases, let $E \in \{E_1^1, E_1^2 \dots, E_l^1, E_l^2\}$ and let $K(E)$ such that $\mathfrak{F}'_0 = K(E) \cup E$. As in the proof of Theorem 2.32, we can compute, using Lemma 2.14,

$$\begin{aligned} \mathrm{Pic}^0(\mathfrak{F}'_0) &= \mathrm{Pic}^0(K(E) \cup E) \\ &= \mathrm{Pic}^0(K(E)) \times_{\mathrm{Pic}^0(K(E)\cap E)} \mathrm{Pic}^0(E) \\ &= (*) \end{aligned}$$

and then, since $\mathrm{Pic}^0(E) \cong \mathrm{Pic}^0(\Sigma^\nu)$ by Lemma 2.28 and $\mathrm{Pic}^0(K(E) \cap E) \cong \mathrm{Pic}^0(\Sigma^\nu \cup_{pt.} \mathbb{P}^1) \cong \mathrm{Pic}^0(\Sigma^\nu)$ by Lemma 2.14,

$$\begin{aligned} (*) &= \mathrm{Pic}^0(K(E)) \times_{\mathrm{Pic}^0(\Sigma^\nu)} \mathrm{Pic}^0(\Sigma^\nu) \\ &= \mathrm{Pic}^0(K(E)), \end{aligned}$$

This shows that none of the components $E \in \{E_1^1, E_1^2 \dots, E_l^1, E_l^2\}$ contribute to the Picard scheme of $\mathfrak{F}'_0$ and consequently, that $\mathrm{Pic}^0(\mathfrak{F}'_0) \cong \mathrm{Pic}^0(W \cup T)$. Since $\mathrm{Pic}^0(W) \cong \mathrm{Pic}^0(\Sigma^\nu)$ by Corollary 3.28, we can conclude in the precise same way as in the proof of Theorem 2.32, if k is even. If k is odd, we can conclude the same way as in the proof of Theorem 2.29, if we can show that

$$\{L \in \mathrm{Pic}^0(W) \mid L\,|_{W_1} \cong L\,|_{W_2}\} \cong \mathrm{Pic}^0(W),$$

where $W_1, W_2 \cong \Sigma^\nu$ are the curves in W such that $r(W_1 \cup W_2) = \Sigma \subset F(X)$ and the map r is the resolution of singularities of $F(X)$ by successive blowups of the singular locus as in Corollary 3.32. This is because we used [vK10, Remark 6.2] in the proof of Theorem 2.29 and this result does not generalise to our case. Consider the commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "c"'] & \mathrm{Hilb}^2(\Sigma) \arrow[r] \arrow[d, "h"] & \mathfrak{F}_0 = F(X) \\ \mathrm{Sym}^2(\Sigma^\nu) \arrow[r] & \mathrm{Sym}^2(\Sigma). & \end{tikzcd}$$

from Corollary 3.27. The map c maps the curves W_1, W_2 isomorphically to curves $\Sigma_1, \Sigma_2 \subset \mathrm{Sym}^2(\Sigma^\nu)$ and $\Sigma_i = \{p + q_i \mid p \in \Sigma\}$, where q_1, q_2 are the points in Σ^ν that are mapped to the singular point q_0 of Σ under the normalisation $\Sigma^\nu \to \Sigma$. Moreover, the pullback by c is an isomorphism of the respective Picard schemes, cf. Corollary 3.28, and the result follows if we can show that for all $L \in \mathrm{Pic}^0(\mathrm{Sym}^2(\Sigma^\nu))$ the restrictions $L\,|_{\Sigma_1}$ and $L\,|_{\Sigma_2}$ are isomorphic. But this was proven in Lemma 3.18. $\square$

This result coincides with results about the degenerations of the intermediate Jacobian of a cubic threefold, which were first investigated for special cases by S. Casalaina-Martin and R. Laza, see [CL09, Table 1, page 22] and later computed in greater generality by S. Casalaina-Martin, S. Grushevsky, K. Hulek and R. Laza, cf. [CGHL15, Table 1, page 37].

3.35 Corollary. *If X is a cubic threefold with unique singular point of type A_k, then $k \leq 11$.*

Proof. Let $\pi : \mathfrak{F} \to B$ be a smoothing of $\mathfrak{F}_0 = F(X)$ coming from a smoothing $\mathfrak{X} \to B$ of $\mathfrak{X}_0 = X$. Then, $\mathrm{Pic}^0(\mathfrak{F}_b) \in \mathcal{A}_5$ for $b \neq 0$ and the degenerate Picard scheme of $\mathfrak{F}_0$ with respect to this family defines a point of $\mathcal{A}_5^{\mathrm{Vor}}$ as we defined it to be a limit of points in $\mathcal{A}_5^{\mathrm{Vor}}$. But if $k \geq 12$, Theorem 3.34 computes the degenerate Picard scheme of $\mathfrak{F}_0$ and yields a point in $\mathcal{A}_g^{\mathrm{Vor}}$ for $g \geq 6$ which yields a contradiction. $\square$

4 Concluding remarks and outlook

In this chapter, we remark on some further results we obtained but without including proofs. This is, because the methods used to prove these results are either the same as those already presented in this thesis or the proofs rely, at least partially, on lengthy calculations. We also want to pose some questions we left unanswered and explain which answers seem plausible to us. Moreover, we hint at methods that could possibly be used for answering these questions.

Finally, we think it is worth mentioning the implications of our results for the 27 lines on singular cubic surfaces.

4.1 Singular points of type D_k, E_6, E_7 or E_8 and the case of several isolated singular points

Through all of this thesis, we always limited ourselves to cubic hypersurfaces X with a unique singular point of type A_k. By using the local normal forms from Table 1.1, we can see that singularities of type D_k, $k \geq 4$, and E_6, E_7, E_8 have corank two whereas singularities of type A_k had corank one for $k \geq 2$ and corank zero for $k = 1$. As the Recognition Principle, Lemma 1.9, holds for all singularities of ADE-type, see [BW79], and also the algorithmic proof of the Generalised Morse Lemma 1.12 does not rely on the corank of the singularity, Theorem 1.15 generalises to the following theorem.

4.1 Theorem. *Let $Y \subset \mathbb{C}^N$ be a hypersurface defined by a polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$ and assume that the origin is an isolated singular point of Y of corank at most three. Then, there are polynomials $C_1, \dots, C_{k+1}$ in the coefficients of P and depending on the choice of an analytic coordinate change such that the conditions*

$$C_1 = \dots = C_k = 0, \ C_{k+1} \neq 0$$

on the coefficients of P are equivalent to $(Y, 0)$ being of type $T \in \{A_p, D_q, E_6, E_7, E_8 \mid p \geq 1, \ q \geq 4\}$. The number k of these conditions depends on the Milnor number of the singularity. Moreover, fixing the analytic coordinate change they depend on, there is an explicit algorithm computing them.

Sketch of proof. We already formulated the Recognition Principle, Lemma 1.9, for all singularities of ADE-type. Also the algorithmic proof of the Generalised Morse Lemma, Lemma 1.12, is not limited to polynomials defining a hypersurface singularity of corank one. Therefore if $P \in \mathbb{C}[x_1, \dots, x_N]$ is a polynomial in N variables and such that $\{P = 0\}$ defines a hypersurface singularity of corank two we find, by the Generalised Morse Lemma, for every fixed k a coordinate change such that

$$\varphi^* P(x_1, \dots, x_N) = x_1^2 + \dots + x_{N-2}^2 + P_3(x_{n-1}, x_N) + \dots + P_{k+1}(x_{n-1}, x_N) + \sum_{i=1}^{N-2} x_i Q_i(x_1, \dots, x_N)$$

with $P_i \in \mathfrak{m}^i$ and $Q_i \in \mathfrak{m}^k$. By applying the Recognition Principle to this equation, one obtains the desired polynomial conditions in the coefficients of P. □

If the cubic hypersurface $X \subset \mathbb{P}^n$ admits a unique singular point p_0 of ADE-type, the singular locus Σ of $F(X)$ is smooth or singular at isolated points only by Lemma 1.5 and Table 2.1. Just as we did for a singularity of type A_k, a suitable normal form for the defining equation f of X can be computed if (X, p_0) is of type D_k, $k \geq 4$, E_6, E_7 or E_8. Similarly to our calculations in chapter one, one finds that the type of $(F(X), l)$ for $l \in \Sigma_{\text{reg}}$ is independent of l and can be compared to the type of (X, p_0) by handing the problem to a computer. With our limited computational power we obtained the following generalisation of Theorem 1.29.

4.2 Theorem. *If $X \subset \mathbb{P}^n$ is a cubic hypersurface with unique singular point p_0 of ADE-type T and $l \in \Sigma_{\text{reg}} \subset F(X)$ is a regular point of the singular locus of the Fano scheme of lines on X, then the singularity types (X, p_0) and $(F(X), l)$ are the same for all $T \in \{A_1, \dots, A_5, D_4, \dots, D_7, E_6, E_7, E_8\}$.*

Sketch of proof. If $l \in \Sigma_{\text{reg}}$ is a line in X passing through the singular point p_0 of X and corresponding to a smooth point of the singular locus of $F(X)$, we have shown in section 1.2.2 how to obtain a simultaneous normal form for the defining equation f for X and the defining equations for $F(X)$ around l, if X has a unique singular point of type A_k. For other ADE-types, similar coordinate changes can be used to obtain such a simultaneous normal form. Then, computing the defining equations for $F(X)$, they are of a form similar to (1.11) and a formal coordinate change yields a defining equation for the hypersurface singularity $(F(X), l)$. The assertion then follows by comparing the respective coefficient conditions for (X, p_0) and $(F(X), l)$. □

The limitation in the theorem is again only due to lack of computational power, but covers all possible isolated ADE-singularities on cubic surfaces. For cubic threefolds, more cases can be covered by our methods, cf. Theorem 1.32, but since no classification of ADE-singularities on cubic threefolds is available in the literature, it is not clear if we were able to cover all possible cases.

For a cubic hypersurface $X \subset \mathbb{P}^n$ there can of course be more than a single singular point on it and it is therefore natural to ask about the singularities of $F(X)$ in such a case. Let $X \subset \mathbb{P}^n$ be such a cubic hypersurface and denote by $p_0, \dots, p_r$ its singular points such that all of the singularities (X, p_i) are of ADE-type. If we denote by Σ_i the set of lines passing through the singular point p_i, i.e.

$$\Sigma_i = \{l \cong \mathbb{P}^1 \subset X \mid p_i \in l\},$$

then it can be shown that

$$\Sigma = F(X)_{\text{sing}} = \bigcup_{i=0}^{r} \Sigma_i.$$

Each Σ_i can itself be singular and the singular locus Σ of $F(X)$ is singular at the singular points of the Σ_i as well the points of intersection $\Sigma_i \cap \Sigma_j$ for $i \neq j$. It follows from [Wal99, Theorem 2.1] that any three distinct components Σ_i, Σ_j, Σ_k of Σ cannot intersect in a point, i.e. $\Sigma_i \cap \Sigma_j \cap \Sigma_k = \emptyset$ whenever $i \neq j \neq k$. Local calculations similar to those in chapter one then provide the following.

4.3 Theorem. *For $i, j \in \{0, \dots, r\}$ and $i \neq j$, let $q_{i,j} \in \Sigma_i \cap \Sigma_j \subset F(X)$. Assume that (X, p_i) is of type T_i and that (X, p_j) is of type T_j. Then, $(F(X), q_{i,j})$ is of type $T_i \times T_j$.*

Sketch of proof. After computing a suitable normal form for the defining equation f for X, local equations for $F(X)$ around a point $q_{i,j}$ as in the assertion take the form

$$\begin{aligned} F_{3,0}(x_2,\dots,x_n,y_2,\dots,y_n) =& x_4^2+\dots+x_n^2+\mathfrak{h}_1, \quad && \mathfrak{h}_1 \in \mathfrak{m}^3,\\ F_{2,1}(x_2,\dots,x_n,y_2,\dots,y_n) =& x_2+\mathfrak{h}_2, && \mathfrak{h}_2 \in \mathfrak{m}^2,\\ F_{1,2}(x_2,\dots,x_n,y_2,\dots,y_n) =& y_2+\mathfrak{h}_3, && \mathfrak{h}_3 \in \mathfrak{m}^2,\\ F_{0,3}(x_2,\dots,x_n,y_2,\dots,y_n) =& y_4^2+\dots+y_n^2+\mathfrak{h}_4, && \mathfrak{h}_4 \in \mathfrak{m}^3. \end{aligned}$$

Using a formal coordinate change φ, one can then bring these equations to the form

$$\begin{aligned} \varphi^* F_{3,0}(x_2,\dots,x_n,y_2,\dots,y_n) =& x_4^2+\dots+x_n^2+\mathfrak{h},\\ \varphi^* F_{2,1}(x_2,\dots,x_n,y_2,\dots,y_n) =& x_2,\\ \varphi^* F_{1,2}(x_2,\dots,x_n,y_2,\dots,y_n) =& y_2,\\ \varphi^* F_{0,3}(x_2,\dots,x_n,y_2,\dots,y_n) =& y_4^2+\dots+y_n^2+\mathfrak{g}, \end{aligned}$$

where $\mathfrak{h} = \mathfrak{h}(x_2,\dots,x_n)$ depends on x-coordinates only and $\mathfrak{g} = \mathfrak{g}(y_2,\dots,y_n)$ depends on y-coordinates only. Therefore, $(F(X), q_{i,j})$ is formally equivalent to the product of two hypersurface singularities and the Recognition Principle can be applied separately to each of these hypersurface singularities. □

This generalises a result by R. Yamagishi, see [Yam17a, Theorem 2.4], on singularities of the Fano scheme of lines on a cubic fourfold.

We believe that also the computations for the degenerate Picard scheme of a cubic threefold can be generalised to other isolated ADE-singularities and combinations of isolated ADE-singularities on a cubic threefold.

4.2 27 lines on singular cubic surfaces

Let $X \subset \mathbb{P}^3$ be a cubic surface. If X is smooth, $F(X)$ consists of 27 reduced points, see Example 1.2. The classification of cubic surfaces due to J. W. Bruce and C. T. C. Wall, see [BW79], shows that every isolated singular point $p_0 \in X$ of ADE-type is of type A_1, A_2, A_3, A_4, A_5, D_4, D_5 or E_6. Table 4.1 shows the number of lines on a cubic surface X with a unique singular point of one of the types above. These numbers can for example be found in [Dol12, Table 9.1]. As the lines passing through p_0 are parameterised by Σ and Σ is zero-dimensional of degree six,

Type of (X,p_0)	A_1	A_2	A_3	A_4	A_5	D_4	D_5	E_6
$\#F(X)$	21	15	10	6	3	6	3	1

Table 4.1: Number of lines on singular cubic surfaces

we have, counted with multiplicity, six lines on X passing through p_0. It is simple, for example by using local equations for $F(X)$ as we did in chapter one, to count the number of points of

$F(X)$ with their respective multiplicity. We obtain, in dependence of the type of (X, p_0),

$$\begin{aligned}
A_1: \quad \sum_{l \in F(X)} \mathrm{mult}_l(F(X)) &= \sum_{l \in F(X)\setminus\Sigma} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma} \mathrm{mult}_l(F(X)) \\
&= 15 + 2 \cdot 6 = 27, \\
A_2: \quad \sum_{l \in F(X)} \mathrm{mult}_l(F(X)) &= \sum_{l \in F(X)\setminus\Sigma} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma} \mathrm{mult}_l(F(X)) \\
&= 9 + 3 \cdot 6 = 27, \\
A_3: \quad \sum_{l \in F(X)} \mathrm{mult}_l(F(X)) &= \sum_{l \in F(X)\setminus\Sigma} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma_{\mathrm{reg}}} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma_{\mathrm{sing}}} \mathrm{mult}_l(F(X)) \\
&= 5 + 4 \cdot 4 + 6 = 27, \\
A_4: \quad \sum_{l \in F(X)} \mathrm{mult}_l(F(X)) &= \sum_{l \in F(X)\setminus\Sigma} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma_{\mathrm{reg}}} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma_{\mathrm{sing}}} \mathrm{mult}_l(F(X)) \\
&= 2 + 3 \cdot 5 + 10 = 27, \\
A_5: \quad \sum_{l \in F(X)} \mathrm{mult}_l(F(X)) &= \sum_{l \in F(X)\setminus\Sigma} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma_{\mathrm{reg}}} \mathrm{mult}_l(F(X)) + \sum_{l \in \Sigma_{\mathrm{sing}}} \mathrm{mult}_l(F(X)) \\
&= 0 + 2 \cdot 6 + 15 = 27.
\end{aligned}$$

These multiplicities are explained as follows. By Theorem 1.32, $F(X)$ admits a hypersurface singularity of the same type as (X, p_0) at every point of Σ_{reg}. But a hypersurface singularity of type A_k inside $\mathbb{C}$ is just the point given by $\{x^{k+1} = 0\} \subset \mathbb{C}$, that is, a fat point of multiplicity $k+1$. The multiplicities of these lines appear in the literature, see for example [Dol12] or [Cay69], but are derived using different arguments. In order to understand how the multiplicity of the points $l_0 \in \Sigma_{\mathrm{sing}} \subset F(X)$ arises, we pose the following conjecture.

4.4 Conjecture. *Let $X \subset \mathbb{P}^n$ be a cubic hypersurface with a unique singular point of ADE-type T and let Γ be a variety of dimension $n-3$ with a unique singular point of the same ADE-type T. Then,*

$$(F(X), l_0) \cong (\mathrm{Hilb}^2(\Gamma), \gamma_0),$$

where l_0 and γ_0 denote the singular points of the respective singular loci.

We give a brief explanation on why one should expect this conjecture to hold true. First note that this conjecture has been verified for cubic fourfolds by R. Yamagishi, see [Yam17a, Theorem 2.5]. His proof relies on methods from symplectic geometry and also on normality of $F(X)$. By analysing his proof, one finds that the arguments from symplectic geometry he uses, are mainly used to avoid giving a precise description of the fibre $\pi^{-1}(\gamma_0)$ where $\pi : W \to \mathrm{Hilb}^2(\Gamma)$ is a resolution of singularities of $\mathrm{Hilb}^2(\Gamma)$. But such a precise description is part of our results in chapter three. Thus, as $F(X)$ is normal for all cubic hypersurfaces $X \subset \mathbb{P}^n$, if $n \geq 5$, the arguments from [Yam17a] could eventually be generalised to prove the conjecture for all $n \geq 5$. This would also generalise Theorem 1.32 to $n \geq 5$ and thus establish it in general as we proved it for $n \leq 4$. Further evidence to the conjecture comes from the multiplicities of $F(X)$ at the singular point l_0 of its singular locus, if $X \subset \mathbb{P}^3$ is a cubic surface. The multiplicities of the Hilbert square of a fat point of multiplicity $k+1$ are easy to compute and one obtains the one one listed in Table 4.2.

k	local equations inside $\mathrm{Hilb}^2(\mathbb{C}) \cong \mathbb{C}^2$	multiplicity at the origin
3	$a^3 + 2ab = a^2b + b^2 = 0$	6
4	$a^4 + 3a^2b + b^2 = a^3b + 2ab^2 = 0$	10
5	$a^5 + 4a^3b + 3ab^2 = a^4b + 3a^2b^2 + b^3 = 0$	15

Table 4.2: Multiplicity of the Hilbert square of $\{x^{k+1} = 0\} \subset \mathbb{C}$ at the singular point of its singular locus

Now these multiplicities are precisely those obtained by the classical geometers, for example by A. Cayley [Cay69], but which could not be explained intrinsically. A confirmation of Conjecture 4.4 at least for cubic surfaces provides such an intrinsic explanation of these multiplicities without ever using that the sum of the multiplicities should be 27 in the end.

Appendices

A. Source code for implementation in Python

In this section we give the source code we used to prove Theorem 1.29. We give brief explanations of what the code is doing. The input lines, marked by "In", can be copied into a Python console and executed, provided that all necessary packages have been installed. We also provide the respective output and added line-breaks for the purpose of presenting it in this thesis. The source code for our module "procs" loaded at the beginning is given below.
We begin by importing modules and libraries we are going to use.

In [1]:
```
import sympy as sp
import procs
import math
import time
sp.init_printing()
```

Now initialise all symbols we are going to use. These are the variables $z_1, \dots, z_n$, $x_2, \dots, x_n$, $y_2, \dots, y_n$ and also all coefficients of the defining equation f for the cubic hypersurface X. Note that there are symbols introduced which remain unused. This is to keep the code easier to read. Any number of variables can be specified below, we took, as an example, $n = 4$.

In [2]:
```
numberOfVars =4 #Can be adjusted
dz = dict(('z_%d'%k, sp.symbols('z_%d'%k)) for k in range(numberOfVars+1))
da = dict(('a_%d'%k, sp.symbols('a_%d'%k)) for k in range(numberOfVars+1))
db = dict(('b_%d'%k, sp.symbols('b_%d'%k)) for k in range(numberOfVars+1))
dc = dict(('c_%d'%k, sp.symbols('c_%d'%k)) for k in range(numberOfVars+1))
dd = dict(('d_%d'%k, sp.symbols('d_%d'%k)) for k in range(numberOfVars+1))
de = dict(('e_%d'%k, sp.symbols('e_%d'%k)) for k in range(numberOfVars+1))
df = dict(('f_%d'%k, sp.symbols('f_%d'%k)) for k in range(numberOfVars+1))
locals().update(dz)
locals().update(da)
locals().update(db)
locals().update(dc)
locals().update(dd)
locals().update(de)
locals().update(df)
```

We declare some polynomials which are used to build the defining equation f.

In [3]:
```
A,B,C,D,E,F,Q=0,0,0,0,0,0,0
lz=list(dz.values())
la=list(da.values())
lb=list(db.values())
lc=list(dc.values())
ld=list(dd.values())
le=list(de.values())
lf=list(df.values())

for ind in range(4,(numberOfVars+1)):
    Q+=lz[ind]**2
    A+=la[ind]*lz[ind]
    B+=lb[ind]*lz[ind]
    C+=lc[ind]*lz[ind]
    D+=ld[ind]*lz[ind]
    E+=le[ind]*lz[ind]
    F+=lf[ind]*lz[ind]
```

Now declare the function f.

In [4]:
```
f = z_0*(z_1*z_2+Q)+b_2*z_2**3+a_3*z_1**2*z_3+b_3*z_2**2*z_3+c_1*z_1*z_3**2+c_2*z_2*z_3**2+z_1**2*A\
    +z_2**2*B+z_1*z_3*E+z_2*z_3*F+z_3**2*C+c_3*z_3**3
```

We check the definition of f.

In [5]:
```
f
```

Out[5]: $a_3z_1^2z_3 + a_4z_1^2z_4 + b_2z_2^3 + b_3z_2^2z_3 + b_4z_2^2z_4 + c_1z_1z_3^2 + c_2z_2z_3^2 + c_3z_3^3 + c_4z_3^2z_4 + e_4z_1z_3z_4 + f_4z_2z_3z_4 + z_0\left(z_1z_2 + z_4^2\right)$

The following lines computes defining equations for the Fano scheme of lines on $X = \{f = 0\}$.

In [6]:
```
F=f.expand()
F1,F2,F3,F4=0,0,0,0

#Coordinates for the Grassmannian
dx = dict(('x_%d'%k, sp.symbols('x_%d'%k)) for k in range(numberOfVars+1))
dy = dict(('y_%d'%k, sp.symbols('y_%d'%k)) for k in range(numberOfVars+1))
locals().update(dx)
locals().update(dy)
lx=list(dx.values())
ly=list(dy.values())
#
L= sp.Symbol('L')
M= sp.Symbol('M')
F=F.subs(lz[0],L)
F=F.subs(lz[1],M)
#
for ind in range(2,numberOfVars+1):
    F=F.subs(lz[ind],L*lx[ind]+M*ly[ind])
#
F1 = procs.degree_k_terms(sp.poly(F,L),3,[L])
F2 = procs.degree_k_terms(sp.poly(F,L),2,[L])
F3 = procs.degree_k_terms(sp.poly(F,M),2,[M])
F4 = procs.degree_k_terms(sp.poly(F,M),3,[M])
#Remove the temporary variables L,M and turn polynomials into polynomials
#in x,y
F1 = sp.poly(F1.subs(L,1),lx+ly)
F2 = sp.poly(F2.subs({L:1,M:1}),lx+ly)
F3 = sp.poly(F3.subs({L:1,M:1}),lx+ly)
F4 = sp.poly(F4.subs(M,1),lx+ly)
```

As we explained in chapter one, we can set the variables $y_4, \ldots, y_n$ to zero for our purposes.

In [7]:
```
for var in ly[4:]:
    F1=F1.subs(var,0)
    F2=F2.subs(var,0)
    F3=F3.subs(var,0)
    F4=F4.subs(var,0)
#assure that these functions are still defined as functions of y_2,...,y_n
F1=sp.poly(F1,lx+ly)
F2=sp.poly(F2,lx+ly)
F3=sp.poly(F3,lx+ly)
F4=sp.poly(F4,lx+ly)
```

In order to perform our algorithm, we have to specify a bound. We have taken here $\mu = 5$ but this number may be adjusted to compute other cases.

In [8]:
```
mu = 5
bound = math.ceil((mu+1)/3)+1
```

We normalise the linear terms of F_2, F_3 and F_4, and begin by changing the 1-jet of F_3 to y_2 and normalise the coefficient of y_2, if necessary.

In [9]:
```
if procs.degree_k_terms(F3,1,lx+ly)!=ly[2]:
    #To make the linear term a multiple of y_2
    sub = procs.degree_k_terms(F3,1,lx+ly)
    F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
                .subs(ly[2],ly[2]-procs.expr_from_poly(sub-ly[2],lx+ly)).expand(),lx+ly)
    F2=sp.poly(procs.expr_from_poly(F2,lx+ly)
                .subs(ly[2],ly[2]-procs.expr_from_poly(sub-ly[2],lx+ly)).expand(),lx+ly)
    F3=sp.poly(procs.expr_from_poly(F3,lx+ly)
                .subs(ly[2],ly[2]-procs.expr_from_poly(sub-ly[2],lx+ly)).expand(),lx+ly)
    F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
                .subs(ly[2],ly[2]-procs.expr_from_poly(sub-ly[2],lx+ly)).expand(),lx+ly)

    #To change the factor of y_2 to one
    sub = procs.degree_k_terms(F3,1,lx+ly)
    F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
                .subs(ly[2],1/(sp.div(sub,ly[2])[0])*ly[2]),lx+ly)
```

```
    F2=sp.poly(procs.expr_from_poly(F2,lx+ly)
               .subs(ly[2],1/(sp.div(sub,ly[2])[0])*ly[2]),lx+ly)
    F3=sp.poly(procs.expr_from_poly(F3,lx+ly)
               .subs(ly[2],1/(sp.div(sub,ly[2])[0])*ly[2]),lx+ly)
    F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
               .subs(ly[2],1/(sp.div(sub,ly[2])[0])*ly[2]),lx+ly)
```

Now change the $\lceil\frac{\mu+1}{3}\rceil$-jet of F_3 to y_2.

```
In [10]: count =2
         while procs.jet(F3,bound,lx+ly)!=ly[2]:
             print('---------- step: '+str(count-1)+'/'+str(bound-1))
             start = time.time()
             #Calculate the terms in F3 of order count
             sub = sp.poly(procs.degree_k_terms(F3,count,lx+ly),lx+ly)

             F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
                        .subs(ly[2],ly[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
             F2=sp.poly(procs.expr_from_poly(F2,lx+ly)
                        .subs(ly[2],ly[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
             F3=sp.poly(procs.expr_from_poly(F3,lx+ly)
                        .subs(ly[2],ly[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
             F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
                        .subs(ly[2],ly[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
             #we can cut off terms that will never be used
             #note that we will be working with F1 in the end and need all terms
             #for this polynomial
             F1=sp.poly(procs.jet(F1,mu+1,lx+ly),lx+ly)
             F2=sp.poly(procs.jet(F2,bound,lx+ly),lx+ly)
             F3=sp.poly(procs.jet(F3,bound,lx+ly),lx+ly)
             F4=sp.poly(procs.jet(F4,bound,lx+ly),lx+ly)
             count+=1
             end = time.time()
             print('---------- done! ('+str(end-start)+'s)')

---------- step: 1/2
---------- done! (0.638685941696167s)
---------- step: 2/2
---------- done! (14.569494009017944s)
```

We can now eliminate the variable y_2.

```
In [11]: F1=sp.poly(F1.subs(ly[2],0),lx+ly)
         F2=sp.poly(F2.subs(ly[2],0),lx+ly)
         F4=sp.poly(F4.subs(ly[2],0),lx+ly)
```

We continue with F_2.

```
In [12]: count =2
         while procs.jet(F2,bound,lx+ly)!=lx[2]:
             print('---------- step: '+str(count-1)+'/'+str(bound-1))
             start = time.time()
             #Calculate the terms in F2 of order count
             sub = sp.poly(procs.degree_k_terms(F2,count,lx+ly),lx+ly)

             F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
                        .subs(lx[2],lx[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
             F2=sp.poly(procs.expr_from_poly(F2,lx+ly)
                        .subs(lx[2],lx[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
             F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
                        .subs(lx[2],lx[2]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)

             #we can cut off terms that will never be used
             F1=sp.poly(procs.jet(F1,mu+1,lx+ly),lx+ly)
             F2=sp.poly(procs.jet(F2,bound,lx+ly),lx+ly)
             F4=sp.poly(procs.jet(F4,bound,lx+ly),lx+ly)
             count+=1
             end = time.time()
             print('---------- done! ('+str(end-start)+'s)')

---------- step: 1/2
---------- done! (0.4918539524078369s)
---------- step: 2/2
---------- done! (13.766942977905273s)
```

```
In [13]: F1=sp.poly(F1.subs(lx[2],0),lx+ly)
         F4=sp.poly(F4.subs(lx[2],0),lx+ly)
```

Finally, we bring F_4 to the desired form.

In [14]:
```
if procs.degree_k_terms(F4,1,lx+ly)!=ly[3]:
    #normalisation is required
    sub = procs.degree_k_terms(F4,1,lx+ly)
    F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
               .subs(ly[3],ly[3]-procs.expr_from_poly(sub-ly[3],lx+ly)).expand(),lx+ly)
    F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
               .subs(ly[3],ly[3]-procs.expr_from_poly(sub-ly[3],lx+ly)).expand(),lx+ly)
    #
    sub = procs.degree_k_terms(F4,1,lx+ly)
    F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
               .subs(ly[3],1/(sp.div(sub,ly[3])[0])*ly[3]),lx+ly)
    F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
               .subs(ly[3],1/(sp.div(sub,ly[3])[0])*ly[3]),lx+ly)
```

In [15]:
```
count =2
while procs.jet(F4,bound,lx+ly)!=ly[3]:
    start = time.time()
    print('---------- step: '+str(count-1)+'/'+str(bound-1))
    #Calculate the terms in F4 of order count
    sub = sp.poly(procs.degree_k_terms(F4,count,lx+ly),lx+ly)
    F1=sp.poly(procs.expr_from_poly(F1,lx+ly)
               .subs(ly[3],ly[3]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
    F4=sp.poly(procs.expr_from_poly(F4,lx+ly)
               .subs(ly[3],ly[3]-procs.expr_from_poly(sub,lx+ly)).expand(),lx+ly)
    #we can cut off terms that will never be used
    F1=sp.poly(procs.jet(F1,mu,lx+ly),lx+ly)
    F4=sp.poly(procs.jet(F4,bound,lx+ly),lx+ly)
    count+=1
    end = time.time()
    print('---------- done! ('+str(end-start)+'s)')
```

```
---------- step: 1/2
---------- done! (0.30374884605407715s)
---------- step: 2/2
---------- done! (1.2940187454223633s)
```

In [16]:
```
F1=sp.poly(F1.subs(ly[3],0),lx+ly)
```

We reduced ourselves to computing the coefficient conditions for the following polynomial.

In [17]:
```
procs.expr_from_poly(F1,lx)
```

Out[17]:
$$a_3c_2^2x_3^5 + 2a_3c_2f_4x_3^4x_4 + a_3f_4^2x_3^3x_4^2 + a_4c_2^2x_3^4x_4 + 2a_4c_2f_4x_3^3x_4^2 + a_4f_4^2x_3^2x_4^3 + b_3c_1^2x_3^5 + 2b_3c_1e_4x_3^4x_4 + b_3e_4^2x_3^3x_4^2 + b_4c_1^2x_3^4x_4 + 2b_4c_1e_4x_3^3x_4^2 + b_4e_4^2x_3^2x_4^3 - c_1c_2x_3^4 - c_1f_4x_3^3x_4 - c_2e_4x_3^3x_4 + c_3x_3^3 + c_4x_3^2x_4 - e_4f_4x_3^2x_4^2 + x_4^2$$

To perform the algorithm from the proof of the Generalised Morse Lemma, we need to detect the non-squared variable in the above.

In [18]:
```
quadratic_part=procs.expr_from_poly(procs.degree_k_terms(F1,2,lx+ly),lx+ly)
for ind in range(3,len(lx)):
    if quadratic_part-lx[ind]**2!=0:
        var = lx[ind]
        break
residual_part = procs.expr_from_poly(F1,lx)-quadratic_part
```

The next loop performs the algorithm from the proof of the Generalised Morse Lemma to bring F_1 to the normal form of a singularity of type A_k.

In [19]:
```
count = 0
start=time.time()
while count != mu:
    residual_part = procs.expr_from_poly(F1,lx)-quadratic_part
    g=[]
    for ind in range(len(lx)):
        if lx[ind] == var:
            g.extend([0])
        else:
            g.extend([procs.expr_from_poly(procs.g_ind(sp.poly(residual_part,lx),ind,lx),lx)])
            residual_part-=lx[ind]*g[ind]
    for ind in range(len(lx)):
        if g[ind] != 0:
            F1=procs.jet(sp.poly(F1.subs(lx[ind],lx[ind]-sp.Rational(1/2)*g[ind]).expand(),lx),mu,lx)
            F1=sp.poly(F1,lx)
    count+=1
F1=sp.poly(F1,lx)
end=time.time()
print('successfully performed the Generalised Morse Lemma ('+str(end-start)+'s)')
```

```
successfully performed the Generalised Morse Lemma (8.086852073669434s)
```

We can now read off the conditions to admit a singularity of Milnor number $\leq \mu$.

In [20]: F1

Out[20]: $\text{Poly}\left(\left(a_3c_2^2 + b_3c_1^2 + \frac{c_1c_4f_4}{2} + \frac{c_2c_4e_4}{2}\right)x_3^5 + \left(-c_1c_2 - \frac{c_4^2}{4}\right)x_3^4 + c_3x_3^3 + x_4^2, x_0, x_1, x_2, x_3, x_4, \quad domain = \mathbb{Q}[a_3, b_3, c_1, c_2, c_3, c_4, e_4, f_4]\right)$

Now compute the coefficient conditions for f. To do so, we pass to the standard affine chart $z_0 \neq 0$ containing the singular point p_0.

In [21]:
```
lz_aff = lz[1:]
g = f.subs(z_0,1)
```

In [22]:
```
g=procs.gen_Morse2(sp.poly(g,lz_aff),mu,lz_aff)
```

```
the weight has to be put on z_3
1
2
3
4
5
Successfully performed the Generalised Morse Lemma (48.67490100860596s)
```

In [23]: g

Out[23]: $\text{Poly}\left(z_1z_2 + \left(a_3c_2^2 + b_3c_1^2 + \frac{c_1c_4f_4}{2} + \frac{c_2c_4e_4}{2}\right)z_3^5 + \left(-c_1c_2 - \frac{c_4^2}{4}\right)z_3^4 + c_3z_3^3 + z_4^2, z_1, z_2, z_3, z_4, \quad domain = \mathbb{Q}[a_3, b_3, c_1, c_2, c_3, c_4, e_4, f_4]\right)$

The source code for the module "procs" is the following.

```
"""

@author: Tobias Heckel
"""
import sympy as sp
import math
import time

def jet(F, mu, lz):
    #monomials wrt z of degree mu
    erg=sp.poly(0,lz)
    mons = F.monoms()
    for ind in range(len(mons)):
        if sum(mons[ind])<=mu:
            tmp=sp.poly(1,lz)
            for ind2 in range(len(lz)):
                tmp=tmp*lz[ind2]**mons[ind][ind2]
            erg+=F.as_dict()[mons[ind]]*tmp
    return erg.expand()

def degree_k_terms(F,mu,lz):
    #monomials wrt z of degree mu
    erg=sp.poly(0,lz)
    mons = F.monoms()
    for ind in range(len(mons)):
        if sum(mons[ind])==mu:
            tmp=sp.poly(1,lz)
            for ind2 in range(len(lz)):
                tmp=tmp*lz[ind2]**mons[ind][ind2]
            erg+=F.as_dict()[mons[ind]]*tmp
    return erg.expand()

def fano(F, lz):
    F=F.expand()
    F1,F2,F3,F4=0,0,0,0
    numberOfVars = len(lz)
    #Coordinates for the Grassmannian
    dx = dict(('x_%d'%k, sp.symbols('x_%d'%k)) for k in range(numberOfVars))
    dy = dict(('y_%d'%k, sp.symbols('y_%d'%k)) for k in range(numberOfVars))
    globals().update(dx)
    globals().update(dy)
    lx=list(dx.values())
    ly=list(dy.values())
    #
```

```
    L= sp.Symbol('L')
    M= sp.Symbol('M')
    F=F.subs(lz[0],L)
    F=F.subs(lz[1],M)
    #
    for ind in range(2,numberOfVars):
        F=F.subs(lz[ind],L*lx[ind]+M*ly[ind])
    #
    F1 = degree_k_terms(sp.poly(F,L),3,[L])
    F2 = degree_k_terms(sp.poly(F,L),2,[L])
    F3 = degree_k_terms(sp.poly(F,M),2,[M])
    F4 = degree_k_terms(sp.poly(F,M),3,[M])
    #Remove the temporary variables L,M and make polynomials into polynomials
    #in x,y
    F1 = sp.poly(F1.subs(L,1),lx+ly)
    F2 = sp.poly(F2.subs({L:1,M:1}),lx+ly)
    F3 = sp.poly(F3.subs({L:1,M:1}),lx+ly)
    F4 = sp.poly(F4.subs(M,1),lx+ly)
    return (F1,F2,F3,F4)

#returns the expression of a polynomial
def expr_from_poly(F,lz):
    erg=0
    if F==sp.poly(0,lz): return erg
    mons = F.monoms()
    mu=max([sum(r) for  r in mons])
    for ind in range(len(mons)):
        if sum(mons[ind])<=mu:
            tmp=1
            for ind2 in range(len(lz)):
                tmp=tmp*lz[ind2]**mons[ind][ind2]
            erg+=F.as_dict()[mons[ind]]*tmp
    return erg.expand()

#computation of the g_i polynomials from the proof
#of the Generalised Morse Lemma
def g_ind(pol,ind,lz):
    monoms = pol.monoms()
    erg=sp.poly(0,lz)
    for mon in monoms:
        if mon[ind]!=0:
            #get polynomial but lower exponent ind by one
            tmp=sp.poly(1,lz)
            for ind2 in range(len(mon)):
                if ind == ind2:
                    tmp=tmp*lz[ind2]**(mon[ind2]-1)
                else:
                    tmp=tmp*lz[ind2]**mon[ind2]
            erg+=pol.as_dict()[mon]*tmp
    return erg

#applies the Generalised Morse Lemma to poly
#up to the specified bound
#NOTE: this procedure assumes that the quadratic part is a sume of squares
def gen_Morse(poly, bound, lz):
    start = time.time()
    count = 0
    #compute the quadratic and residual part
    quadratic_part=expr_from_poly(degree_k_terms(poly,2,lz),lz)
    for ind in range(0,len(lz)):
        if degree_k_terms(sp.poly(quadratic_part,lz[ind]),2,[lz[ind]])==0:
            var = lz[ind]
            break
    residual_part = expr_from_poly(poly,lz)-quadratic_part
    print('the weight has to be put on '+str(var))

    while count != bound+1:
        residual_part = expr_from_poly(poly,lz)-quadratic_part
        g=[]
        for ind in range(len(lz)):#lx[ind]!=var needs to be assured!
            if lz[ind] == var:
                g.extend([0])
            else:
```

```
                g.extend([expr_from_poly(g_ind(sp.poly(residual_part,lz),ind,lz),lz)])
                residual_part-=lz[ind]*g[ind]
        #now comes the change of coordinates
        for ind in range(len(lz)):
            if g[ind] != 0:
                poly=jet(sp.poly(poly.subs(lz[ind],lz[ind]-sp.Rational(1/2)*g[ind]).expand(),lz),bound+1,lz)
                poly=sp.poly(poly,lz)
        count+=1
        print(count)
    end = time.time()
    print('Successfully performed the Generalised Morse Lemma ('+str(end-start)+'s)')
    return sp.poly(poly,lz)

#applies the Generalised Morse Lemma to poly
#up to the specified bound
#NOTE: this procedure assumes that the quadratic part equals
#z_1z_2+sum of squares of other variables
def gen_Morse2(poly, bound, lz):
    start = time.time()
    count = 0
    #compute the quadratic and residual part
    quadratic_part=expr_from_poly(degree_k_terms(poly,2,lz),lz)
    for ind in range(2,len(lz)):
        if degree_k_terms(sp.poly(quadratic_part,lz[ind]),2,[lz[ind]])==0:
            var = lz[ind]
            break
    residual_part = expr_from_poly(poly,lz)-quadratic_part
    print('the weight has to be put on '+str(var))

    while count != bound:
        residual_part = expr_from_poly(poly,lz)-quadratic_part
        g=[]
        for ind in range(len(lz)):#lz[ind]!=var needs to be assured!
            if lz[ind] == var:
                g.extend([0])
            else:
                g.extend([expr_from_poly(g_ind(sp.poly(residual_part,lz),ind,lz),lz)])
                residual_part-=lz[ind]*g[ind]
        #now comes the change of coordinates
        for ind in range(len(lz)):
            if g[ind] != 0:
                if ind == 0:
                     poly=jet(sp.poly(poly.subs(lz[ind],lz[ind]-g[ind+1]).expand(),lz),bound,lz)
                     poly=sp.poly(poly,lz)
                elif ind ==1:
                     poly=jet(sp.poly(poly.subs(lz[ind],lz[ind]-g[ind-1]).expand(),lz),bound,lz)
                     poly=sp.poly(poly,lz)
                else:
                    poly=jet(sp.poly(poly.subs(lz[ind],lz[ind]-sp.Rational(1/2)*g[ind]).expand(),lz),bound,lz)
                    poly=sp.poly(poly,lz)
        count+=1
        print(count)
    end = time.time()
    print('Successfully performed the Generalised Morse Lemma ('+str(end-start)+'s)')
    return sp.poly(poly,lz)
```

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