Integral points of bounded height via universal torsors

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Abstract

A conjecture of Manin's relates the number of rational points of bounded height on Fano varieties with their geometric properties. Analogously to this conjecture on rational points, we study the distribution of integral points of bounded height on three varieties: on a smooth Fano threefold of Picard number 2 and type 30 in the Mori–Mukai classification, on a quartic del Pezzo surface with an A_1- and an A_3-singularity, and on a toric threefold. We determine asymptotic formulas and interpret the leading term geometrically. For the proofs, we parametrize integral points using universal torsors, and use analytic techniques to count integral points on the torsor. This seems to be the first application of the torsor method to integral points. The asymptotic formula for our toric variety contradicts a result by Chambert-Loir and Tschinkel. We describe an obstruction that explains this contradiction, and study its relation with some constants that appear in asymptotic formulas for the number of integral points of bounded height.

Kurzfassung

Eine Vermutung von Manin stellt einen Bezug zwischen der Anzahl rationaler Punkte beschränkter Höhe auf Fano-Varietäten und geometrischen Eigenschaften her. Analog zu dieser Vermutung für rationale Punkte untersuchen wir die Verteilung ganzer Punkte beschränkter Höhe auf drei Varietäten: auf einer glatten dreidimensionalen Fano-Varietät von Picardrang 2 und Typ 30 in der Mori–Mukai-Klassifikation, auf einer quartischen del-Pezzo-Fläche mit A_1- und A_3-Singularität und auf einer dreidimensionalen torischen Varietät. Wir bestimmen asymptotische Formeln und interpretieren den führenden Term geometrisch. In den Beweisen parametrisieren wir die ganzen Punkte mit Hilfe universeller Torsore, und zählen ganze Punkte auf den universellen Torsoren mit analytischen Methoden. Dies scheint die erste Anwendung der Torsor-Methode zum Zählen ganzer Punkte zu sein. Die asymptotische Formel für die torische Varietät steht im Widerspruch zu einem Ergebnis von Chambert-Loir und Tschinkel. Wir beschreiben eine Obstruktion, die diesen Widerspruch erklärt und untersuchen ihren Zusammenhang mit einigen Konstanten, die ein Bestandteil asymptotischer Formeln für die Anzahl ganzer Punkte beschränkter Höhe sind.

Keywords: Integral points, universal torsors, Manin's conjecture
Schlagworte: Ganze Punkte, universelle Torsore, Manins Vermutung

Contents

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Introduction

A classical problem in number theory is the solubility of Diophantine equations: Given a system of polynomial equations

$$f_1(x_1, \dots, x_n) = \dots = f_s(x_1, \dots, x_n) = 0$$

with integral coefficients, does it have solutions over the integers? This kind of problem is related to algebraic geometry: Such a system of polynomials defines an algebraic variety, and solutions correspond to points on the variety. We can then ask whether solutions exist, and, if so, how many. If there are infinitely many, we have the following quantitative question: For some positive bound B, how many solutions in integers between $-B$ and B are there, and how does this number behave as B grows? In geometric terms, these solutions are those in a certain region of the variety: points of *height* at most B. Moreover, we can study the relation between the answer to these arithmetic questions and the geometry of the corresponding variety.

Manin's conjecture on rational points

One variant of this problem is Manin's conjecture [FMT89, BM90] about the number of rational points of bounded height on Fano varieties. Let X be a Fano variety over the field $\mathbb{Q}$ of rational numbers, by which we will mean a smooth, projective $\mathbb{Q}$-variety whose anticanonical bundle $\omega_X^\vee$ is ample. We equip X with an anticanonical height function. If $\omega_X^\vee$ is even very ample, we can construct one as follows: Choose an anticanonical embedding $f\colon X \to \mathbb{P}^n$. A rational point x on $\mathbb{P}^n$ can be represented as $x = (x_0 : \dots : x_n)$, where the x_i are coprime integers. This representation is then unique up to a sign. We define the height of such a point to be $H(x_0 : \dots : x_n) = \max\{|x_0|, \dots, |x_n|\}$, and the height of a rational point $x \in X(\mathbb{Q})$ to be the height $H(x) = H(f(x))$ of its image. We can then ask how the number

$$\#\{x \in X(\mathbb{Q}) \mid H(x) \leq B\}$$

of rational points of bounded height behaves asymptotically.

This number might be dominated by points on strict subvarieties – such *accumulating* subvarieties should be excluded from the analysis. For their complement V, Manin's conjecture predicts that the number of rational points of bounded height

$$N(B) = \#\{x \in V(\mathbb{Q}) \mid H(x) \leq B\}$$

is asymptotically

$$cB(\log B)^{\rho-1},$$

where $\rho = \operatorname{rk}\operatorname{Pic} X$ is the Picard number of X. Peyre [Pey95, Pey03] gave a conjectural interpretation of the leading constant c as a product of a constant α determined by the geometry of the effective cone, a cohomological constant β (cf. [BT98]) related to the Brauer group, and an adelic volume τ that can be regarded as a product of local densities.

Results of this kind include applications of the circle method for varieties of large dimension compared to their degree (e.g. [Bir62, Ski97, BHB17, FM17]), and varieties with a group action such as generalized flag varieties [FMT89], toric varieties [BT98], and other equivariant compactifications [CLT02, STBT04, STBT07]. Besides such general classes of varieties, there are results for e.g. some smooth del Pezzo surfaces [Bre02, BF04, BB11].

The class of varieties studied can be expanded to include singular varieties. In this case, the counting problem can be compared to that on a desingularization, which is smooth, but no longer Fano, although the anticanonical bundle is typically still big and nef. Peyre describes a framework for the interpretation of asymptotic formulas for rational points on more general smooth *almost Fano* varieties [Pey03]. Asymptotic formulas for singular or weak del Pezzo surfaces include [BB07, BBD07, BBP12] and many others.

Integral points of bounded height

Rational points on a complete variety X, and integral points on an arbitrary proper model $\mathcal{X}$ coincide as a consequence of the valuative criterion for properness. On non-complete varieties, this is no longer the case. A set-up analogous to Manin's conjecture in the case of integral points is the following: Let X be a smooth, projective variety, and D a divisor on X with strict normal crossings, such that the log-anticanonical bundle $\omega(D)^\vee$ is ample, or at least big. Let $U = X - D$, let $\mathcal{U}$ be an integral model of U, and let H be a log-anticanonical height function. Again, the number of integral points of bounded height on $\mathcal{U}$ might be dominated by points on a subvariety, and one should investigate the number of integral points of bounded height outside these accumulating subvarieties.

There are several results on this kind of problem: Varieties with a large number of variables compared to their degree can be studied using the circle method (e.g. [Bir62, Sch85]). In addition, there are numerous results of this kind on varieties with a group action, such as algebraic groups and homogeneous spaces (e.g. [DRS93, EM93, BR95, EMS96, Mau07, GOS09, WX16]) and partial equivariant compactifications [Mor99, CLT10b, CLT12, TBT13] (not all of them in the geometric setting described above). In these cases, the group structure is exploited for the proofs, for example using harmonic analysis. In [CLT10a], Chambert-Loir and Tschinkel describe a framework that allows the geometric interpretation of asymptotic formulas for the number of integral points of bounded height, similar to Peyre's in the case of rational points. In this case, asymptotic formulas take the form

$$cB(\log B)^{b-1},$$

where the constant c again consists of "local densities" (some of which are however supported on the boundary divisor D), cohomological constants, and combinatorial data associated with the effective cone and the divisor D, while the exponent $b-1$ depends on the Picard number of U and incidence properties of D.

Results

Our main results are similar asymptotic formulas, including on varieties without such kinds of group action.

- A Fano threefold obtained by blowing up $\mathbb{P}^3$ in a plane conic. This variety has Picard number 2 and is of type 30 in the Mori–Mukai-classification of Fano threefolds [MM82]. We take U to be the complement of a plane intersecting the conic in one or two rational points. (Chapter 3.)
- A quartic del Pezzo surface with an A_1- and an A_3-singularity defined by two explicit quadratic equations in $\mathbb{P}^4$. We take U to be the complement of either of the singular points and study integral points by considering the counting problem on a desingularization. (Chapter 4.)
- The toric variety obtained by consecutively blowing up $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in two intersecting lines. We count points on the complement U of the two exceptional divisors and a plane parallel to the two lines. (Chapter 5.)

The first two cases do not belong to the general classes of varieties for which results are known. The last one is a special case of the preprint [CLT10b]; however, our asymptotic formula contradicts parts of this result by Chambert-Loir and Tschinkel: Our exponent of $\log B$ is one less than the one in op. cit. This exemplifies a gap in their proof of which the authors were already aware and is explained by an obstruction preventing the existence of integral points on a region of the toric variety that should have dominated the asymptotic formula. We describe and analyze this obstruction and its implications on the shape of some constants arising in asymptotic formulas. The results take the following form:

Let X be one of the above varieties and U be one of the described open subvarieties. Let $\mathcal{U}$ be a certain integral model of U, and let H be a certain log-anticanonical height function. There exists an open, dense subvariety $V \subset X$ such that the number

$$N(B) = \{x \in \mathcal{U}(\mathbb{Z}) \cap V(\mathbb{Q}) \mid H(x) \leq B\}$$

of integral points of bounded height satisfies an asymptotic formula

$$N(B) = cB(\log B)^{b-1}(1 + o(1)).$$

Here, the constant c has the following shape: It is a sum over certain maximal faces A of the *Clemens complex*, an object encoding incidence properties of the divisor D. Such maximal faces correspond to maximal sets of divisor components that have a common intersection point. In the first two cases, the sum runs over all such maximal faces built out of a maximal number of divisor components, i.e., maximal dimensional faces. In the third case, the only maximal dimensional face has to be excluded as a consequence of an obstruction, and we instead have to take a face that is maximal with respect to inclusion, corresponding to a component that does not intersect any other component of D, but whose dimension is 1 less than the maximum. Each term term of this sum is then a constant $\alpha_{\underline{A}}$ that slightly generalizes a construction by Chambert-Loir and Tschinkel, times a product of local densities. For finite places, it is a local density of integral points on $\mathcal{U}$, and for the archimedean place, it is supported on the intersection of the divisor components belonging to A.

For the exponent, the number $b = \operatorname{rk}\operatorname{Pic}(U) + d + 1$ is determined by the Picard rank of U and the dimension d of these maximal faces, that is, in the first two cases, $d+1$ is the maximal number of divisor components having non-empty intersection, while in the last case, it is one less, since we had to exclude to only maximal dimensional face. Details on the factors and terms of this constant are given in Chapter 2.

Instead of methods exploiting a group action, which are no longer available, we use the *torsor method* in the proofs of these asymptotic formulas. Universal torsors have been defined and studied by Colliot-Thélène and Sansuc [CTS87]; their application to count rational points of bounded height goes back to Salberger [Sal98], who used them to reprove Manin's conjecture for toric varieties. The method allows the parametrization of rational points on the variety by integral points on a universal torsor. These integral points can be regarded as lattice points and counted using analytic techniques. The torsor method has since been successfully applied to count rational points on a number of varieties. Our results seem to be the first application of the method to integral points.

Outline

In Chapter 1, we provide a geometric setup and study integral points on a two examples to exemplify heuristics and expectations for their distribution. In Chapter 2, we recall the frameworks of Peyre and Chambert-Loir and Tschinkel for the interpretation of asymptotic formulas. We slightly generalize a construction in [CLT10b] to non-toric varieties to define a factor $\alpha_{\underline{A}}$ appearing in our asymptotic formulas. We study some of its properties and analyze its relation to an obstruction to the Zariski density of integral points on certain parts of varieties imposed by regular sections on certain subvarieties.

In the following three chapters, we determine asymptotic formulas for the number of integral points on subvarieties U on the three above-mentioned varieties, and interpret the formulas geometrically.

Chapter 1

Counting integral points of bounded height

1.1 The problem

On projective varieties, rational and integral points coincide as a consequence of the valuative criterion for properness, or, more elementarily, because it is always possible to multiply all coordinates of a rational point $(x_0 : \cdots : x_n)$ by the product of all denominators to get an integral point. For rational points, a counting problem on a non-complete variety U can be compared to the problem on a compactification X of U: If the rational points on the boundary $X - U$ were to contribute to an asymptotic formula for the compactification, the boundary would be accumulating and should be excluded when studying Manin's conjecture on X. For integral points, this does not hold: For the notion of integral points to make sense, we need an integral model $\mathcal{X}$ of X, that is, a flat and separated $\mathbb{Z}$-scheme $\mathcal{X}$ of finite type such that $\mathcal{X} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Q} \cong X$. There can be integral points on $\mathcal{X}$ that, as rational points, are on U, but lie on $\mathcal{X} - \mathcal{U}$ modulo a prime p, and thus are neither integral points on $\mathcal{U}$ nor on $\mathcal{X} - \mathcal{U}$.

The boundary $X - U$ carries a lot of information on the distribution of integral points. If $X - U$ is not a divisor, we can make it one by blowing up, and can then use embedded resolution of singularities to make it have strict normal crossings, all without changing the variety U. We consider the log-anticanonical bundle $\omega_X(D)^\vee$, and a height function H associated with it – if the bundle is very ample, we can take a log-anticanonical embedding $f\colon X \to \mathbb{P}^N$, and set $H(x) = H(f(x))$, using the standard height on $\mathbb{P}^N$. The number

$$\#\{x \in \mathcal{U}(\mathbb{Q}) \mid H(x) \le B\}$$

might again be dominated by points on accumulating subvarieties, which we thus want to exclude. If we simply were to make U smaller, we would affect the number even if we remove non-accumulating subvarieties (by also removing some points that are integral on neither the smaller subscheme nor on its complement). Removing integral points on a subvariety means only considering the set

$$\{x \in \mathcal{U}(\mathbb{Z}) \cap V(\mathbb{Q}) \mid H(x) \le B\}$$

of integral points of bounded height that, as rational points, are in the complement V of such accumulating subvarieties, or more formally, integral points whose generic point is in V. That we are counting points on the complement of such strict subvarieties means that we should only work on schemes whose set of integral points is Zariski dense – if it were not, integral points would all lie on a finite set of subvarieties that we should thus all exclude, and we would count the empty set. For rational points on Fano varieties, it would follow from a conjecture of Colliot-Thélène's (e.g. [CT03, p. 174]) that the set of rational points is Zariski dense as soon as it is non-empty; for integral points, this is wrong. (As an elementary example, consider $\mathcal{U} = \mathbb{G}_{\mathrm{m},\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$, the complement of two lines in $\mathbb{P}^2_{\mathbb{Z}}$. Both the anticanonical and log-anticanonical bundles are ample, but still every integral point lies on one of the two subvarieties $\{\pm 1\} \times \mathbb{A}^1_{\mathbb{Z}}$.)

In the next chapter, we will describe the machinery to associate height functions with arbitrary line bundles over arbitrary number fields. In this context, the question still makes sense as long as the log-anticanonical bundle is at least big, that is, as long as it is in the interior of the effective cone. In this case, there exists an open subvariety V such that the number of rational points of bounded height on V is finite for every B.

In total, this gives a set-up for counting integral points on a non-complete variety analogous to Manin's conjecture: Let X be a smooth, projective variety defined over a number field K. Let $D \subset X$ be a reduced, effective divisor with strict normal crossings, and assume that the log-anticanonical bundle $\omega_X(D)^\vee$ is at least big. Let $\mathcal{U}$ be an integral model of $U = X - D$. Let $H\colon X(K) \to \mathbb{R}_{>0}$ be a log-anticanonical height function. Consider a sufficiently small subset $V \subset X(K)$ that does not contain accumulating subvarieties. In general, V is expected to be the complement of a *thin subset*; in the examples considered here, a Zariski open subset will always suffice. If $\mathcal{U}(\mathbb{Z})$ is Zariski dense, how does the number of integral points of bounded height

$$N(B) = \{x \in \mathcal{U}(\mathfrak{o}_k) \cap V \mid H(x) \leq B\}$$

behave asymptotically?

Note that, with $D = 0$, this specializes to a variant of Manin's problem, relaxing the requirement that the anticanonical bundle be ample, and instead only requiring that it be big.

1.2 Example: $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta_{\mathbb{P}^1}$

As an example, we consider the variety $X = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ – where the first copy of $\mathbb{P}^1_{\mathbb{Q}}$ has the coordinate pair (x_0, x_1), and the second copy has the coordinate pair (y_0, y_1) – together with the diagonal divisor $\Delta_{\mathbb{P}^1_{\mathbb{Q}}} = V(x_0y_1 - x_1y_0)$. An integral model of X is $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$; an integral model of the open subvariety $U = X - \Delta_{\mathbb{P}^1}$ that we want to count integral points on is $\mathcal{U} = \mathcal{X} - \Delta_{\mathbb{P}^1_{\mathbb{Z}}}$. This variety is one of the easiest possible examples that is not a partial equivariant compactification of an algebraic group, although X is an equivariant compactification of the symmetric variety $U \cong \mathrm{SL}_2 / T$, where T is the torus of diagonal matrices. (This type of variety is studied e.g. in [EMS96, GOS09, WX16], although U is excluded in these results, since T is has non-trivial $\mathbb{Q}$-characters.) The anticanonical divisor of X is $\omega_X^\vee = \mathcal{O}_X(2,2)$, the diagonal $\Delta_{\mathbb{P}^1_\mathbb{Q}}$ has degree $(1,1)$, and thus the log-anticanonical bundle $\omega_X(\Delta_{\mathbb{P}^1_\mathbb{Q}})^\vee \cong \mathcal{O}_X(1,1)$ is very ample. The morphism

$$X \to \mathbb{P}^3_\mathbb{Q}, \quad ((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1).$$

is a log-anticanonical embedding and thus defines a log-anticanonical height function H by composing it with the standard height function $\mathbb{P}^3_\mathbb{Q}(\mathbb{Q}) \to \mathbb{R}_{>0}$.

We can represent a point on X in a way such that x_0, x_1, y_0, y_1 are integers with $\gcd(x_0, x_1) = \gcd(y_0, y_1) = 1$; this representation is unique up to the two choices of sign. As a consequence, we have a 4-to-1-correspondence between the sets $X(\mathbb{Q}) = \mathcal{X}(\mathbb{Z})$ of rational points on X, respectively integral points on $\mathcal{X}$, and the set

$$\{(x_0, x_1, y_0, y_1) \in \mathbb{Z}^4 \mid \gcd(x_0, x_1) = \gcd(y_0, y_1) = 1\}.$$

To use this expression for counting, we need to know what the height of a point represented in such a way is, and when such a point is integral on $\mathcal{U}$. Since x_0 and x_1, and y_0 and y_1 are coprime, so are the pairwise products: $\gcd(x_0y_0, x_0y_1, x_1y_0, x_1y_1) = 1$. We thus get the description

$$\begin{aligned} H(x_0, x_1, y_0, y_1) &= \max\{|x_0y_0|, |x_0y_1|, |x_1y_0|, |x_1y_1|\} \\ &= \max\{|x_0|, |x_1|\} \max\{|y_0|, |y_1|\}. \end{aligned}$$

of the above log-anticanonical height function. That a point in the above representation is an integral point on $\mathcal{U}$ means that it does not meet the divisor $\Delta_{\mathbb{P}^1_\mathbb{Z}}$ over any point of $\operatorname{Spec} \mathbb{Z}$. For the generic point this means $x_0y_1 - x_1y_0 \neq 0$; for closed points $p\mathbb{Z}$ this means $p \nmid x_0y_1 - x_1y_0$. A point is thus integral if and only if $x_0y_1 - x_1y_0$ is a unit of $\mathbb{Z}$.

Putting this together, we now have an explicit description of the counting function:

$$N(B) = \frac{1}{4} \# \left\{ (x_0, x_1, y_0, y_1) \in \mathbb{Z}^4 \,\middle|\, \substack{x_0y_1 - x_1y_0 \in \{\pm 1\}, \\ \gcd(x_0,x_1) = \gcd(y_0,y_1) = 1, \\ \max\{|x_0|,|x_1|\} \max\{|y_0|,|y_1|\} \leq B} \right\}.$$

Using the symmetry in the two possible values of the equation and cutting up the set into the subsets satisfying $\max\{|x_0|, |x_1|\} < \max\{|y_0|, |y_1|\}$ or not, we get

$$\begin{aligned} N(B) = \frac{1}{2}\Bigg(&\# \left\{ (x_0, x_1, y_0, y_1) \in \mathbb{Z}^4 \,\middle|\, \substack{x_0y_1 - x_1y_0 = 1, \\ \gcd(x_0,x_1) = \gcd(y_0,y_1) = 1, \\ \max\{|x_0|,|x_1|\} \max\{|y_0|,|y_1|\} \leq B, \\ \max\{|x_0|,|x_1|\} < \max\{|y_0|,|y_1|\}} \right\} \\ &+ \# \left\{ (x_0, x_1, y_0, y_1) \in \mathbb{Z}^4 \,\middle|\, \substack{x_0y_1 - x_1y_0 = 1, \\ \gcd(x_0,x_1) = \gcd(y_0,y_1) = 1, \\ \max\{|x_0|,|x_1|\} \max\{|y_0|,|y_1|\} \leq B, \\ \max\{|x_0|,|x_1|\} \geq \max\{|y_0|,|y_1|\}} \right\} \Bigg). \end{aligned} \tag{1.1}$$

We start by analyzing the first term in this expression, which we regard as a sum over x_0 and x_1

$$\sum_{\substack{x_0, x_1 \in \mathbb{Z}, \\ \gcd(x_0,x_1)=1, \\ |x_0|,|x_1| \leq \sqrt{B}}} \# \left\{ (y_0, y_1) \in \mathbb{Z}^2 \,\middle|\, \substack{x_0y_1 - x_1y_0 = 1, \\ \max\{|y_0|,|y_1|\} \leq B / \max\{|x_0|,|x_1|\}, \\ \max\{|x_0|,|x_1|\} < \max\{|y_0|,|y_1|\}} \right\}, \tag{1.2}$$

using the facts that the height condition and inequality imply $\max\{|x_0|, |x_1|\} \leq \sqrt{B}$ and that the equation implies the coprimality condition $\gcd(y_0, y_1) = 1$.

To treat such sums, we want to determine the number

$$\widetilde{N}(x_0, x_1, C) = \#\left\{(y_0, y_1) \in \mathbb{Z}^2 \;\middle|\; x_0 y_1 - x_1 y_0 = 1, \quad |y_0|, |y_1| \leq C\right\}.$$

The equation has a solution $(\hat{y}_0, \hat{y}_1)$ if and only if x_0 and x_1 are coprime; the other solutions then have the form $(\hat{y}_0 + kx_1, \hat{y}_1 + kx_0)$ for $k \in \mathbb{Z}$. If we assume that both x_0 and x_1 are non-zero, such a solution satisfies the inequalities if

$$\left|k + \tfrac{\hat{y}_0}{x_1}\right| \leq \tfrac{C}{|x_1|} \quad \text{and} \quad \left|k + \tfrac{\hat{y}_1}{x_0}\right| \leq \tfrac{C}{|x_0|},$$

that is, if k is in an intersection of two intervals. The number of integers in an interval is the length of the interval with an error of at most 1. Since the distance between the midpoints of the two intervals is

$$\left|\tfrac{\hat{y}_1}{x_0} - \tfrac{\hat{y}_0}{x_1}\right| = \left|\tfrac{x_0\hat{y}_1 - x_1\hat{y}_0}{x_0 x_1}\right| = \tfrac{1}{|x_0 x_1|} \leq 1,$$

the error we get when replacing the intersection of the two intervals by the smaller interval – which has length $2C/\max\{|x_0|, |x_1|\}$ – also is at most 1. We thus get

$$\widetilde{N}(x_0, x_1, C) = \frac{2C}{\max\{|x_0|, |x_1|\}} + O(1), \tag{1.3}$$

whenever x_0 and x_1 are coprime and both non-zero. If one of them, say x_0, is zero, the other one has to be 1 or -1, and the solutions are those $(y_0, y_1) \in \mathbb{Z}^2$ with $|y_0| \leq C$, $y_1 = x_1$, so (1.3) still holds.

Using this, we can now get rid of the condition

$$\max\{|x_0|, |x_1|\} < \max\{|y_0|, |y_1|\}$$

in (1.2). The error we introduce in doing so is

$$\sum_{\substack{(x_0,x_1)\in\mathbb{Z}^2,\\ \gcd(x_0,x_1)=1,\\ |x_0|,|x_1|\leq\sqrt{B}}} \#\left\{(y_0, y_1) \in \mathbb{Z}^2 \;\middle|\; \substack{x_0y_1 - x_1y_0 = 1,\\ \max\{|y_0|,|y_1|\}\leq\max\{|x_0|,|x_1|\}}\right\} \ll \sum_{\substack{x_0,x_1\in\mathbb{Z},\\ |x_0|,|x_1|\leq\sqrt{B}}} 1 \ll B$$

by (1.3). For the same reason, we can change the inequality $\max\{|x_0|, |x_1|\} \geq \max\{|y_0|, |y_1|\}$ in the second term of (1.1) to $>$, introducing another error $\ll B$. Now, by exchanging the roles of x and y, we get the same estimate for the second term and have simplified (1.1) to

$$\begin{aligned} N(B) &= \sum_{\substack{x_0,x_1\in\mathbb{Z},\\ \gcd(x_0,x_1)=1,\\ |x_0|,|x_1|\leq\sqrt{B}}} \#\left\{(y_0, y_1) \in \mathbb{Z}^2 \;\middle|\; \substack{x_0y_1 - x_1y_0 = 1,\\ \max\{|y_0|,|y_1|\}\leq B/\max\{|x_0|,|x_1|\},}\right\} + O(B) \\ &= \sum_{\substack{x_0,x_1\in\mathbb{Z},\\ \gcd(x_0,x_1)=1,\\ |x_0|,|x_1|\leq\sqrt{B}}} \frac{2B}{\max\{|x_0|, |x_1|\}^2} + O(B), \end{aligned}$$

since we have $\sum_{|x_0|,|x_1|\leq B} 1 \ll B$ for the sum over the error terms from another application of (1.3). We simplify this sum with a Möbius inversion and then perform a change of variables $x_i' = x_i/\alpha$:

$$\begin{aligned} N(B) &= \sum_{\substack{(x_0,x_1)\in\mathbb{Z}^2,\\ |x_0|,|x_1|\leq\sqrt{B}}} \sum_{\alpha|x_0,x_1} \mu(\alpha)\frac{2B}{\max\{|x_0|,|x_1|\}^2} + O(B) \\ &= \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2} \sum_{\substack{x_0',x_1'\in\mathbb{Z},\\ |x_0'|,|x_1'|\leq\sqrt{B}/\alpha}} \frac{2B}{\max\{|x_0'|,|x_1'|\}^2} + O(B). \end{aligned} \tag{1.4}$$

(Note that the sums are finite and thus absolutely convergent.) Using the symmetry in x_0 and x_1, the inner sum is

$$4 \sum_{\substack{x_0',x_1'\in\mathbb{Z},\\ |x_0'|\leq|x_1'|\leq\sqrt{B}/\alpha}} \frac{B}{|x_1'|^2} - 2 \sum_{\substack{x_0',x_1'\in\mathbb{Z},\\ |x_0'|=|x_1'|\leq\sqrt{B}/\alpha}} \frac{B}{|x_1'|^2}.$$

The second term is

$$8\sum_{x_1>0} \frac{B}{|x_1|^2} \ll B,$$

and the first one is

$$\begin{aligned} 8 \sum_{\substack{x_1'\in\mathbb{Z},\\ |x_1'|\leq\sqrt{B}/\alpha}} \frac{B}{|x_1'|} + O(B) &= 16B\left(\log\left(\frac{\sqrt{B}}{\alpha}\right) + O(1)\right) \\ &= 8B\log B + O(B(1+\log\alpha)). \end{aligned}$$

Plugging this back into (1.4), we arrive at

$$\begin{aligned} N(B) &= \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2}\big(8B\log B + O(B(1+\log\alpha)\big) + O(B) \\ &= 8B\log B \sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2} + O\left(B + \sum_{\alpha>0}\frac{B(1+\log\alpha)}{\alpha^2}\right). \end{aligned}$$

So, finally, we have an asymptotic formula

$$N(B) = \frac{8}{\zeta(2)} B\log B + O(B)$$

for the number of integral points of bounded height on $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta_{\mathbb{P}^1}$.

This example can be regarded as an application of the *torsor method*. The morphism $(\mathbb{A}^2 - (0,0)) \times (\mathbb{A}^2 - (0,0)) \to \mathbb{P}^1 \times \mathbb{P}^1$ is an example of a universal torsor. Integral points on $\mathbb{A}^4_{\mathbb{Z}}$ are simply 4-tuples (x_0, x_1, y_0, y_1) (formally, they are morphisms $\operatorname{Spec}\mathbb{Z} \to \mathbb{A}^4$, that is, given by homomorphisms $\mathbb{Z}[x_0, x_1, y_0, y_1] \to \mathbb{Z}$, which in turn are defined by the images of the generators). An integral point on $\mathbb{A}^4_{\mathbb{Z}}$ is contained in the open subscheme

$$\mathcal{T} = (\mathbb{A}^2_{\mathbb{Z}} - V(x_0, x_1)) \times (\mathbb{A}^2_{\mathbb{Z}} - V(y_0, y_1))$$

if the image of $\operatorname{Spec}\mathbb{Z}$ does not intersect the complement of $\mathcal{T}$. The image of the generic point (0) is in $V(x_0,x_1)$ if $x_0=x_1=0$, and the image of a closed point (p) of $\operatorname{Spec}\mathbb{Z}$ is in $V(x_0,x_1)$ if $p \mid x_0, x_1$, and hence the image does not intersect $V(x_0,x_1)$ if $\gcd(x_0,x_1)=1$. With the analogous criterion for $V(y_0,y_1)$, we can see that the set of integral points on $\mathcal{T}$ is

$$\mathcal{T}(\mathbb{Z}) = \{(x_0,x_1,y_0,y_1)\in\mathbb{Z}^4 \mid \gcd(x_0,x_1)=\gcd(y_0,y_1)=1\}.$$

Finally, all fibers of $\mathcal{T}\to\mathbb{P}^1_{\mathbb{Z}}\times\mathbb{P}^1_{\mathbb{Z}}$ are isomorphic to $\mathbb{G}_{\mathrm{m}}^2$, with $\mathbb{G}_{\mathrm{m}}(\mathbb{Z})=\{\pm1\}$, giving the 4-to-1-parametrization. This parametrization allowed us to regard integral points as lattice points satisfying an equation and gcd-conditions.

We will continue to use this example to illustrate constructions related to the geometric interpretation of asymptotic formulas. Since this interpretation involves a local-global-principle, it will be helpful to have a description of the *local solutions*, that is, of the set of $\mathbb{Z}_p$-points on $\mathcal{U}$. Completely analogous to $\mathbb{Z}$-points, the set of $\mathbb{Z}_p$-points is

$$\left\{((x_0:x_1),(y_0:y_1))\in\mathbb{P}^1\times\mathbb{P}^1 \;\middle|\; \substack{\gcd(x_0,x_1)=\gcd(y_0,y_1)=1,\\ x_0y_1-x_1y_0\in\mathbb{Z}_p^\times}\right\}.$$

This time, there is only one prime that can divide $\gcd(x_0,x_1)$, and thus the coprimality condition means precisely that $x_0\in\mathbb{Z}_p^\times$ or $x_1\in\mathbb{Z}_p^\times$ (and the same for y). We can also translate this to a condition of the absolute values: the p-adic integer x_0 is a unit if and only if its p-adic absolute value $|x|_p$ is 1, and the last condition in the set is equivalent to saying

$$|x_0y_1-x_1y_0|_p = 1. \tag{1.5}$$

We can also ask about the shape of this set in, say, $\mathbb{A}^2\cong V=\mathbb{P}^1\times\mathbb{P}^1-V(x_0y_0)$, using coordinates $(x,y)=(x_1/x_0,y_1/y_0)$. If $|x|_p<1$, we have $x_1\notin\mathbb{Z}_p^\times$, $x_0\in\mathbb{Z}_p^\times$. So, by the ultrametric triangle inequality, (1.5) holds precisely if $|y_1|_p=1$, that is, if $|y|_p\geq1$. Analogously, for $|x|_p>1$, the condition holds precisely if $|y|_p\leq1$. Finally, if $|x|_p=1$, we have $|x_0|_p=|x_1|_p=1$, so (1.5) always holds if $|y|_p\neq1$; if $|y|_p=1$, we also have $|y_0|=|y_1|=1$, so after dividing by $|x_0y_0|_p$, the condition reads $|x-y|_p=1$. In total, we get

$$\mathcal{U}(\mathbb{Z}_p)\cap V(\mathbb{Q}_p) = \left\{(x,y)\in\mathbb{Q}_p^2 \;\middle|\; \substack{|x|_p<1,\ |y|_p\geq1, \quad \text{or} \quad |x|_p>1,\ |y|_p\leq1, \quad \text{or}\\ |x|_p=1,\ |y|_p\neq1, \quad \text{or} \quad |x|_p=|y|_p=1,\ |x-y|_p=1}\right\}.$$

1.3 Distribution of integral points

Moreover, this example highlights a difference in the distribution of rational and integral points. In our geometric context, we would expect that the rational points are dense in a component of the set of real points as soon as there is at least one such rational point. (This holds true in cases where weak approximation holds, or, more generally, when the Brauer–Manin obstruction is the only one to weak approximation). Their distribution factors into asymptotic formulas as a volume of $X(\mathbb{R})$ with respect to a certain measure, that can be thought of as a *real density.* We cannot expect something similar to happen for integral points. Already for $\mathbb{A}^1$, we can see that $\mathbb{Z}$ is far from dense in $\mathbb{R}$, and the same holds for integral points in the example above. Note that, for this

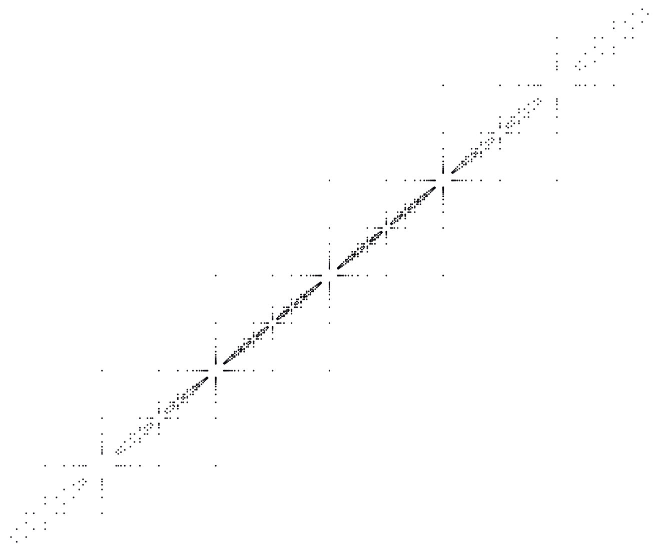

Figure 1.1: Integral points of height ≤ 15 on $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta_{\mathbb{P}^1}$.

reason, when studying strong approximation for integral points, $X(\mathbb{R})$ has to be omitted or replaced by its set $\pi_0(X(\mathbb{R}))$ of connected components.

Both examples demonstrate a different behaviour of integral points that appears in the geometric interpretation: They accumulate around the boundary. In the case of $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$, every neighbourhood of ∞ contains almost all integral points, since the absolute value of almost all integers is larger than a fixed number. In the previous example, again regarding the affine patch V defined by $x_0, y_0 \neq 0$, a point $(x, y) = (x_1/x_0, y_1/y_0)$ is close to the diagonal if $|x - y|$ is small, i.e. if $|x_1/x_0 - y_1/y_0|$ is small, with a similar picture on the affine patch $x_1, y_1 \neq 0$. Writing $\Delta_{\mathbb{P}^1}$ as a union of two suitable compact sets, one in each of these two affine patches, we can fit a set of the form

$$\left\{ \left| \frac{x_1}{x_0} - \frac{y_1}{y_0} \right|, \left| \frac{x_0}{x_1} - \frac{y_0}{y_1} \right| < \epsilon \right\}$$

into every analytic neighbourhood of $\Delta_{\mathbb{P}^1}$. Every point outside such a neighbourhood then satisfies

$$\epsilon \geq \left| \frac{x_1}{x_0} - \frac{y_1}{y_0} \right| = \left| \frac{x_1 y_0 - x_0 y_1}{x_0 y_0} \right| = \left| \frac{1}{x_0 y_0} \right|,$$

and the same holds for $|x_1 y_1|$, so there are only finitely many points. In particular, no real point $p \in U(\mathbb{R})$ can be approximated by integral points, in the sense that there cannot be a sequence of integral points $(p_n)_n$ with $p_n \neq p$ converging to it. On the other hand, every real point $(t, t) \in \Delta_{\mathbb{P}^1}(\mathbb{R})$ can be approximated by integral points: Take a sequence

$$\frac{x_1^{(n)}}{x_0^{(n)}} \to t,$$

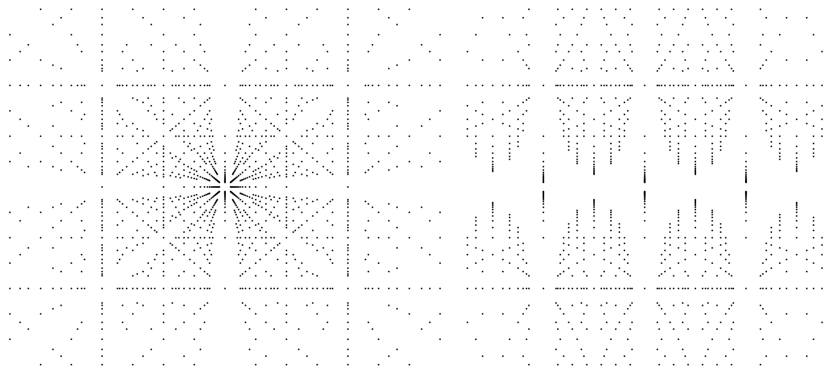

Figure 1.2: Integral points of height $\leq 2^9$ on $\mathbb{P}^2 - \{\mathrm{pt}\}$. To the right: the same integral points regarded as points on $Bl_{\{\mathrm{pt}\}}\mathbb{P}^2 - E$; the exceptional divisor E is a horizontal line.

with coprime numerator and denominator, and such that $x_1^{(n)}/x_0^{(n)} \neq t$ for all n; in particular, we then have $x_0^{(n)} \to \infty$. For every positive integer n, let $(y_0^{(n)}, y_1^{(n)})$ be an integral solution of $x_0^{(n)}y_1^{(n)} - x_1^{(n)}y_0^{(n)} = 1$. (By the coprimality assumption, such a solution exists.) Then we also have

$$\frac{y_1^{(n)}}{y_0^{(n)}} = \frac{x_0^{(n)}y_1^{(n)}}{x_0^{(n)}y_0^{(n)}} = \frac{x_1^{(n)}y_0^{(n)}+1}{x_0^{(n)}y_0^{(n)}} = \frac{x_1^{(n)}}{x_0^{(n)}} + \frac{1}{x_0^{(n)}y_0^{(n)}} \to t,$$

and get a sequence of points converging to (t, t).

1.4 Example: $\mathbb{P}^n - \{\mathrm{pt}\}$

In general, such a strong statement need not hold: There are well-behaved varieties U with infinitely many points away from the boundary, and whose integral points are even analytically dense – consider for example $\mathcal{U} = \mathbb{P}^n_{\mathbb{Z}} - P \cong Bl_P\mathbb{P}^n_{\mathbb{Z}} - E$, where $P = (1 : 0 \cdots : 0)$, the scheme $Bl_P\mathbb{P}^n_{\mathbb{Z}}$ is the blow up in P, E is the exceptional divisor, and $n \geq 2$. Integral points in this case are rational points with a modified coprimality condition:

$$\mathcal{U}(\mathbb{Z}) = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_0, \ldots, x_n \in \mathbb{Z},\ \gcd(x_1, \ldots, x_n) = 1\}.$$

To see that integral points are analytically dense, let us consider a box of the form

$$B = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid a_i \leq t_i \leq b_i\} \subset \mathbb{A}^n(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{R})$$

for positive real numbers a_i, b_i. Our aim is to show that it contains infinitely many integral points – since the situation is symmetric in the signs of the t_i, this will imply that every neighbourhood of every real point contains infinitely many integral points. The set of integral points in this box is

$$\{(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^n_{>0} \mid a_i \leq \frac{x_i}{x_0} \leq b_i,\ \gcd x_1, \ldots, x_n = 1\}.$$

We can estimate the number $N(B)$ of such points with bounded $x_0 \le B$. A Möbius inversion yields

$$\begin{aligned} N(B) &= \sum_{x_0 \le B} \sum_{\alpha > 0} \#\{(x_1, \dots, x_n) \in \mathbb{Z}^n_{>0} \mid x_0 a_i \le x_i \le x_0 b_i,\ \alpha \mid x_1, \dots, x_n\} \\ &= \sum_{x_0 \le B} \sum_{\alpha \le b x_0} \mu(\alpha) \#\{(x_1, \dots, x_n) \in \mathbb{Z}^n_{>0} \mid \frac{x_0 a_i}{\alpha} \le x_i \le \frac{x_0 b_i}{\alpha}\} \end{aligned}$$

with $b = \min_i\{b_i\}$, since the set is empty as soon as $\alpha > b x_0$. The number of integers x_i in each of those intervals is

$$\frac{x_0(b_i - a_i)}{\alpha} + O(1)$$

Multiplying these, we get an error term

$$O_{\underline{a}, \underline{b}, n}\left(\left(\frac{x_0}{\alpha}\right)^{n-1}\right),$$

since $x_0 b_i / \alpha \ge 1$, and thus all lower order terms are smaller. In total this gives

$$\begin{aligned} N(B) &= \sum_{x_0 \le B} \left(\sum_{\alpha \le b x_0} c_{\underline{a}, \underline{b}} \frac{\mu(\alpha)}{\alpha^n} x_0^n + O_{\underline{a}, \underline{b}, n}(x_0^{n-1}(1 + \log x_0)) \right) \\ &= c'_{\underline{a}, \underline{b}} B^{n+1} + O_{\underline{a}, \underline{b}, n}(B^n \log B), \end{aligned}$$

which tends to ∞. (The logarithmic factor in the error term is only necessary if $n = 2$.)

Still, if we consider a log-anticanonical height function on $(Bl_P \mathbb{P}^n, E)$, 100% of points of height $\le B$ are inside a neighbourhood of the boundary as $B \to \infty$. To verify this, let us first determine an asymptotic formula for the number of integral points of bounded log-anticanonical height. A computation on the homogeneous coordinate ring of the toric variety $Bl_P \mathbb{P}^n$ shows that a possible choice for a log-anticanonical height of an integral point is $\max_{1 \le i \le n}\{\left|x_i^{n+1}\right|, |x_0 x_i^n|\}$. A Möbius inversion as above then yields

$$N(B) = \sum_{\alpha > 0} \mu(\alpha) \sum_{0 \le x_0 \le B/\alpha^n} \#\{(x_1, \dots, x_n) \in \mathbb{Z}^n - \{0\} \mid \underset{\text{for all } i}{|x_i| \le \frac{B^{\frac{1}{n+1}}}{\alpha}, \frac{1}{\alpha}\left(\frac{B}{x_0}\right)^{\frac{1}{n}}}\}.$$

The first of the two conditions on every x_i is stronger if and only if $x_0 \le B^{1/(n+1)}$. In this region, there are at most

$$\frac{2B^{\frac{1}{n+1}}}{\alpha} + 1$$

possible values for each x_i, so we get a contribution of at most

$$\ll \sum_{\alpha > 0} \sum_{0 < x_0 \le B^{1/(n+1)}} \frac{B^{\frac{n}{n+1}}}{\alpha^n} \ll B$$

to the number of integral points, which will turn out to be an acceptable error. In the other region, for every $B^{1/(n+1)} \le x_0 \le B$ there are

$$\frac{2}{\alpha}\left(\frac{B}{x_0}\right)^{\frac{1}{n}} + O(1)$$

integers x_i in every interval. In total, we get

$$\frac{2^n B}{\alpha^n x_0} + O_n\left(\frac{B^{\frac{n-1}{n}}}{\alpha^{n-1} x_0^{\frac{n-1}{n}}}\right),$$

since $x_0 \leq B/\alpha^n$ implies that the terms of lower order are smaller. Summing the error term over x_0 and α, we see that the total arror is

$$\ll_n \sum_{\substack{\alpha>0,\\ |x_0|\leq B/\alpha^2}} \frac{B^{\frac{n-1}{n}}}{\alpha^{n-1} x_0^{\frac{n-1}{n}}} \ll_n \sum_{\alpha>0} \frac{B}{\alpha^{n-1+2/n}} \ll_n B,$$

and we arrive at

$$\begin{aligned}
N(B) &= \sum_{\alpha>0} \mu(\alpha) \sum_{B^{1/(n+1)}<x_0\leq B/\alpha^2} \frac{2^n B}{\alpha^n x_0} + O_n(B)\\
&= \sum_{\alpha>0} \mu(\alpha) \frac{2^n B}{\alpha^n}\left(\log B - 2\log\alpha - \frac{1}{n+1}\log B + O(1)\right) + O_n(B)\\
&= \sum_{\alpha>0} \mu(\alpha) \frac{n 2^n}{(n+1)\alpha^n} B\log B + O_n(B)\\
&= \frac{n}{n+1}\frac{2^n}{\zeta(n)} B\log B + O_n(B),
\end{aligned}$$

which can be checked to agree with [CLT10b].

Let us now determine the number of points outside a box-shaped neighbourhood V_0 of $(1 : 0 : \cdots : 0)$. Every such point satisfies $|x_i/x_0| > \epsilon$ for some i, so, using the height conditions, it needs to satisfy

$$x_0 < \frac{|x_i|}{\epsilon} \leq \frac{B^{\frac{1}{n+1}}}{\epsilon}.$$

Similarly to above, we can get an upper bound for the number $N(\epsilon; B)$ of integral points satisfying this inequality:

$$N(\epsilon; B) \ll \sum_{x_0<B^{1/(n+1)}/\epsilon} \#\{(x_1, \dots, x_n) \in \mathbb{Z}^n - \{0\} \mid |x_i| \leq B^{1/(n+1)}\} \ll \frac{B}{\epsilon},$$

which is asymptotically smaller than the total number $cB\log B$ of rational points. Hence the proportion of integral points lying inside V_0 tends to 1 for any neighbourhood V_0 of $(1 : 0 : \cdots : 0)$:

$$\frac{\#\{x \in \mathcal{U}(\mathbb{Z}) \cap V_0 \mid H(x) \leq B\}}{\#\{x \in \mathcal{U}(\mathbb{Z}) \mid H(x) \leq B\}} \to 1, \quad B \to \infty,$$

which agrees with the probability following from the equidistribution theorem in [CLT10b].

Chapter 2

Geometric framework

The aim of this chapter is to provide the necessary background for the geometric interpretation of asymptotic formulas for the number of integral points of bounded height on pairs (X, D). To this end, in Sections 2.2–2.3.2, we recall the frameworks by Peyre [Pey95, Pey03] in the context of rational points and by Chambert-Loir and Tschinkel [CLT10a] in the context of integral points. In the remainder of Section 2.3, we slightly generalize the construction of a divisor group and a cone by Chambert-Loir and Tschinkel [CLT10b] to non-toric varieties and study some of its properties. We will need these objects to geometrically interpret asymptotic formulas in the following chapters. In Section 2.4, we describe an obstruction to the Zariski density of integral points on parts of varieties that explains the phenomena on the toric variety in Chapter 5; moreover, we analyze its connections with constructions in the previous section and its relation to an obstruction described by Jahnel and Schindler [JS17].

2.1 Setting

Throughout this chapter, let K be a number field, $\mathfrak{o}_K$ its ring of integers, $\overline{K}$ an algebraic closure, K_v the completion at a place v, and k_v the residue field at a finite place v. We equip the completions with the absolute values $|\cdot|_v$ normalized such that

$$|x|_v = \left|N_{K_v/\mathbb{Q}_w}(x)\right|_w$$

at a place v lying above a place w of $\mathbb{Q}$, such that $|p|_p = 1/p$ on $\mathbb{Q}_p$, and with the usual absolute value on $\mathbb{R}$. Moreover, we equip each of the local fields with a Haar measure μ_v satisfying $\mu_v(\mathfrak{o}_{K_v}) = 1$ at finite places, the usual Lebesgue measure $\mathrm{d}\mu_v = \mathrm{d}x$ at real places, and $\mathrm{d}\mu_v = i\,\mathrm{d}z\,\mathrm{d}\overline{z} = 2\,\mathrm{d}x\,\mathrm{d}y$ at complex places.

We consider pairs (X, D) with assumptions on X that are similar to the ones in [Pey03] (replacing the anticanonical by the log-anticanonical divisor): Let X be a smooth, projective, geometrically integral K-variety, and D be a reduced, effective divisor with strict normal crossings. Let $U = X - D$, and let $\mathcal{U}$ be an integral model, by which we mean a flat and separated $\mathfrak{o}_K$-scheme of finite type together with an isomorphism between its generic fiber $\mathcal{U} \times_{\mathfrak{o}_K} K$ and U. We assume that

1. $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$,

2. the geometric Picard group $\mathrm{Pic}(X_{\overline{K}})$ is torsion free,

3. there is a finite number of effective divisors $D_1, \dots, D_n$ that generate the pseudo-effective cone $\overline{\mathrm{Eff}}_X = \overline{\{\sum a_i D_i \mid a_i \in \mathbb{R}_{\geq 0}\}} \subset \mathrm{Pic}(X)_{\mathbb{R}}$, and

4. the log-anticanonical bundle $\omega_X(D)^\vee$ is big, that is, it is in the interior of the (pseudo-)effective cone.

In particular, the anticanonical bundle $\omega_X^\vee$ is also big, and X is *almost Fano* in the sense of [Pey03].

For simplicity, we will assume some form of *splitness* of the pair (X, D): We assume that the canonical homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{K}})$ is an isomorphism and that all irreducible components of $D_{\overline{K}}$ are defined over K. This is weaker than the pair (X, D) being split in the sense of [Har17].

To fix further notation, for any open subvariety $V \subset X$ we let $E(V) = \mathcal{O}_X(V)^\times / K^\times$ be the finitely generated abelian group of invertible regular functions on V up to constants.

2.2 Metrics, heights, and Tamagawa measures

To fix notation, we begin this chapter by recalling several definitions needed for the geometric interpretation of asymptotic formulas.

2.2.1 Adelic metrics

We start with the definition of adelic metrics and methods to construct them, as found for example in [Pey03]. An *adelic metric* on a line bundle $\mathcal{L}$ on X is a collection of norms $\|\cdot\|_v : \mathcal{L}(x_v) \to \mathbb{R}_{\geq 0}$ on the lines $\mathcal{L}(x_v)$ for any completion K_v of K and any K_v-point $x_v \in X(K_v)$, satisfying the following conditions:

1. For every local section $s \in \Gamma(U, \mathcal{L})$, the map
$$U(K_v) \to \mathbb{R}_{\geq 0} \colon x_v \mapsto \|s(x_v)\|_v$$
is continuous with respect to the analytic topology.

2. For almost all finite places v, the norm is defined by an integral model $\mathcal{X}$ of X and $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ over $\mathcal{X}$ in the following way: Since X is proper, any point $x_v \in X(K_v)$ lifts uniquely to a point $\widetilde{x}_v \in \mathcal{X}(\mathfrak{o}_v)$. Then $\widetilde{x}_v^* \widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}(\widetilde{x}_v)$ is a free $\mathfrak{o}_v$-module of rank 1 in $x_v^* \mathcal{L} = \mathcal{L}(x_v)$, and we take the unique norm $\|\cdot\|_v$ on $\mathcal{L}(x)$ that assigns to any generator of $\widetilde{\mathcal{L}}(\widetilde{x}_v)$ the norm 1. Since any two models are isomorphic over almost all finite places v, this is independent of the choice of a model.

There are several methods to construct adelic metrics:

- *Pull-backs.* Let $f \colon X \to X'$ be a morphism between smooth, projective K-varieties, and let $\mathcal{L}$ be a line bundle on X', equipped with an adelic metric. Then we get an adelic metric on $f^*\mathcal{L}$ in the following way: Locally, any section of $f^*\mathcal{L}$ has the form $s' = h \cdot f^* s$ for local sections s of $\mathcal{L}$ and h of X. We set $\|s'(x_v)\|_v = |h(x)|_v \, \|s(f(x))\|_v$.

- *Tensor products and inverses.* If $\mathcal{L}$ and $\mathcal{L}'$ are metrized line bundles, there is an induced metric on $\mathcal{L} \otimes \mathcal{L}'$ defined by $\|s \otimes s'\|_v = \|s\|_v \|s'\|_v$ and an induced metric on $\mathcal{L}^\vee$ defined by $\|h(x)\|_v = |(h(s))(x)|_v \|s(x)\|_v^{-1}$, independent of the choice of a local section s of $\mathcal{L}$ that does not vanish in x.

- *Basepoint free bundles.* There is a canonical adelic metric on $\mathcal{O}_{\mathbb{P}^n_K}(1)$: A local section $s \in \Gamma(V, \mathcal{O}_{\mathbb{P}^n_K}(1))$ on an open subvariety $V \subset \mathbb{P}^n_K$ is a homogeneous rational function in $x_0, \dots, x_n$ of degree 1 that does not have a pole on V. Thus, for any point
$$x = (x_0 : \cdots : x_n) \in V(K_v),$$
the norm $|s(x)|_v \cdot \max_i\{|x_i|_v\}^{-1}$ is well-defined. This metric is defined by the integral model $\mathbb{P}^n_{\mathfrak{o}_K}$ at all finite places. Using this, we can associate a metric with any base point free line bundle $\mathcal{L}$ together with a set of global sections $s_0, \dots, s_n$ that do not vanish simultaneously: We have a morphism
$$f\colon X \to \mathbb{P}^n, x \mapsto (s_0(x) : \cdots : s_n(x))$$
with $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then the pull-back construction gives a metric induced by
$$\|f^*s(x)\|_v = \frac{|s(f(x))|}{\max\{|s_0(x)|_v, \dots, |s_n(x)|_v\}} \tag{2.1}$$
for rational functions s as above.

Since every line bundle on a projective variety is a quotient of very ample bundles, this allows the construction of metrics on any bundle.

Example 2.2.1. Returning to $X = \mathbb{P}^1 \times \mathbb{P}^1$, $D = \Delta_{\mathbb{P}^1}$, we can describe the metric on $\mathcal{O}_X(1,1)$ induced by the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. A local section of $\mathcal{O}_X(1,1)$ is a homogeneous rational function f of bidegree $(1,1)$. Its norm at a point $(\underline{x}, \underline{y}) = ((x_0 : x_1), (y_0 : y_1)$ is then
$$\left\|f(\underline{x}, \underline{y})\right\|_v = \frac{\left|f(\underline{x}, \underline{y})\right|_v}{\max\{|x_0|_v, |x_1|_v\} \max\{|y_0|_v, |y_1|_v\}}.$$

2.2.2 Heights

A line bundle $\mathcal{L}$ on a smooth, projective variety X over a number field K together with an adelic metric determines a height function
$$H\colon X(K) \to \mathbb{R}_{\geq 0}, \quad x \mapsto \prod_v \|s(x)\|_v^{-1},$$
where s is a section that does not vanish in x. Since $\|s(x)\|_v = 1$ for almost all v, the height is well-defined, and since $\prod_v |\alpha|_v = 1$ for all $\alpha \in K^\times$, it does not depend on the choice of s. Moreover, different metrics $\|\cdot\|$, $\|\cdot\|'$ on the same line bundle define equivalent height functions H and H', that is, the quotient $H(x)/H'(x)$ is bounded from both above and below.

The number of rational points of bounded height $\#\{x \in X(K) \mid H(x) \leq B\}$ is finite if the line bundle $\mathcal{L}$ is ample. This still holds outside a closed subvariety if $\mathcal{L}$ is big.

Example 2.2.2. In order to construct a height associated with an arbitrary (not necessarily base point free) line bundle, write it as a quotient $\mathcal{L} = \mathcal{A} \otimes \mathcal{B}^{-1}$ of very ample bundles. Take bases $a_0, \dots, a_r$ and $b_0, \dots, b_r$ of the global sections of A and B (or simply sets of sections without a common base point). Then

$$H = \prod_v \frac{\max\{|a_i|_v \mid i = 1, \dots, r\}}{\max\{|b_j|_v \mid j = 1, \dots, s\}}$$

is a height function associated with $\mathcal{L}$, since $\prod_v |s(x)|_v = 1$ for any section s not vanishing in the respective image of x.

If $\mathcal{L}$ is not base point free, it does not suffice to take global sections of $\mathcal{L}$ and consider the maximum of their absolute values. There is, however, an inequality: If $\mathcal{L}$ has global sections, take a basis $s_0, \dots, s_n$ of them. Then we can complete $\{s_i b_j\}_{i,j}$ to a basis of the global sections of $\mathcal{A}$, and get

$$H(x) \geq \prod_v \max_{i=0,\dots,n} \{|s_i|_v\}.$$

If x is not contained in the base locus, the right hand side is $H(f(x))$ for the rational map $f\colon X \dashrightarrow \mathbb{P}^n$ associated with $\mathcal{L}$. From this, we can recover the above fact: Assume that $\mathcal{L}$ is big. By replacing it with a suitable power and taking the n-th root of an associated height function, we can assume that f is birational. Let U a Zariski open subvariety on which f is an isomorphism. Since there are only finitely many points of bounded height on $\mathbb{P}^n$, there are only finitely many points of bounded height on U.

Example 2.2.3. On $\mathbb{P}^1 \times \mathbb{P}^1$, the metric we just defined induces a height function, which coincides with the height function we used for counting, since both the metric and our height were induced by the Segre embedding. To explicitly verify this, take a point $P = ((x_0 : x_1), (y_0 : y_1))$ with coprime coordinates. Then, since x_0 and x_1 are coprime, at least one has p-adic absolute value 1, so we get $\max\{|x_0|_p, |x_1|_p\} = 1$ for all finite primes p, and the same for y. Thus, we get $H(P) = \max\{|x_0|, |x_1|\} \max\{|y_0|, |y_1|\}$ for the height function induced by the metric.

2.2.3 Tamagawa measures

An adelic metric on the canonical bundle ω_X of a smooth, projective variety X over a number field K induces a Borel measure on the K_v-points $X(K_v)$ for all places v, called a *Tamagawa measure*. In local coordinates $x_1, \dots, x_n$, it is given by

$$\mathrm{d}\tau_{X,v} = \|\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n\|_v^{-1} \,\mathrm{d}\mu_v.$$

For finite places $v < \infty$, we consider the modified measure

$$\mathrm{d}\tau_{(X,D),v} = \|1_D \otimes \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n\|_v^{-1} \,\mathrm{d}\mu_v,$$

induced by a metric on the log-canonical bundle $\omega_X(D)$, and its restriction $\tau_{U,v}$ to U. Here, 1_D denotes the canonical section of $\mathcal{O}_X(D)$, corresponding to 1 under the canonical embedding $\mathcal{O}_X(D) \to \mathcal{K}_X$. Since $\mathcal{U}(\mathfrak{o}_v)$ is a compact subset of $U(K_v)$, the norm $\|1_D\|_v^{-1}$ is bounded on $\mathcal{U}(\mathfrak{o}_v)$, and its volume is finite.

Over finite places v, we further multiply these measures with convergence factors associated with $\operatorname{Pic}(U_{\overline{K}})$ and $E(U_{\overline{K}})$ as Galois modules. In our situation, the Galois action on both modules is trivial, and we get the powers

$$\left(1 - \frac{1}{\# k_v}\right)^{\operatorname{rk}\operatorname{Pic}(U) - \operatorname{rk} E(U)}$$

of the local factors at $s = 1$ of the Dedekind zeta function ζ_K of K. These make the product

$$\prod_{v<\infty} \left(1 - \frac{1}{\# k_v}\right)^{\operatorname{rk}\operatorname{Pic}(U) - \operatorname{rk} E(U)} \tau_{U,v}(\mathcal{U}(\mathfrak{o}_{K_v}))$$

absolutely convergent [CLT10a, Theorem 2.5]. Finally, this product is multiplied with the principal value of the corresponding L-function, in this case $\rho_K^{\operatorname{rk}\operatorname{Pic} U - \operatorname{rk} E(U)}$, where

$$\rho_K = \frac{2^r (2\pi)^s \operatorname{Reg}_K h_k}{\# \mu_K \sqrt{|d_K|}}$$

is the principal value of the Dedekind zeta function, with the numbers r and s of real and complex places, the regulator Reg_K, the class number h_K, the group μ_K of roots of unity, and the discriminant d_K of K.

Example 2.2.4. Again returning to $\mathbb{P}^1 \times \mathbb{P}^1$, we first notice that we only have a metric on a line bundle isomorphic to the log-anticanonical bundle, note on the log-anticanonical bundle itself (which in turn would induce a metric on the canonical bundle). To get a metric that induces Tamagawa measures, we need an isomorphism between ω_X and $\mathcal{O}_X(-2,-2)$. Up to constants, there are unique non-vanishing sections of these two line bundles on $X - V(x_0 y_0) \cong \mathbb{A}^2$: the section $\mathrm{d}(x_1/x_0) \wedge \mathrm{d}(y_1/y_0)$ of ω_X and the section $1/x_0^2 y_0^2$ of $\mathcal{O}_X(-2,-2)$. Hence, there is an isomorphism between the two bundles mapping one to the other. Similarly, there is an isomorphism $\mathcal{O}_X(1,1) \cong \mathcal{O}_X(\Delta_{\mathbb{P}^1})$ identifying $x_0 y_1 - x_1 y_0$ and the canonical section $1_{\Delta_{\mathbb{P}^1}}$. With these isomorphisms, we get a metric on $\omega_X(\Delta)$ satisfying

$$\begin{aligned}
\left\| \mathrm{d}x \wedge \mathrm{d}y \otimes 1_{\Delta^1_{\mathbb{P}}} \right\|_v &= \left| \frac{x_0 y_1 - x_1 y_0}{x_0^2 y_0^2} \right|_v \max\{|x_0|_v, |x_1|_v\} \max\{|y_0|_v, |y_1|_v\} \\
&= |x - y|_v \max\{1, |x|_v\} \max\{1, |y|_v\}
\end{aligned}$$

at every place v. With this description, we can compute the Tamagawa numbers at finite places p. Recall that the set of integral points had a description by four disjoint regions. On the first one, we have $|x| < 1$, $|y| \geq 1$, so $\|\mathrm{d}x \wedge \mathrm{d}y \otimes 1_D\|_p = |y^2|$ (using the ultrametric triangle inequality), and so its volume is

$$\int_{\substack{x \in \mathbb{Q}_p \\ |x|<1}} \int_{\substack{y \in \mathbb{Q}_p \\ |y|>1}} \frac{1}{|y|^2} \,\mathrm{d}x \,\mathrm{d}y = \mu_p(\{x \in \mathbb{Q}_p \mid |x| < 1\}) \sum_{\delta \geq 0} \left(1 - \frac{1}{p}\right) p^\delta \frac{1}{p^{2\delta}} = \frac{1}{p},$$

and the second volume is the same. By similar calculations, the volumes of the third and fourth region are

$$\frac{2}{p} - \frac{2}{p^2} \quad \text{and} \quad 1 - \frac{3}{p} + \frac{2}{p^2}.$$

In total, this gives us a volume of

$$\tau_{U,p}(\mathcal{U}(\mathbb{Z}_p)) = 1 + \frac{1}{p},$$

which, after multiplying it with the convergence factor $(1 - \frac{1}{p})$, coincides with the factor at p of the Euler product in our asymptotic formula.

While these measures depend on the choice of isomorphism $\mathcal{O}_X(1,1) \cong \omega_X(D)$, the formula in the end will not: Automorphisms of a line bundle on a projective variety are the morphisms arising by multiplication with a constant λ, so choosing a different isomorphism would multiply all measures by $|\lambda|_v$. Since, in the end, we will multiply all volumes, the result does not depend on λ by the product formula.

2.3 Clemens complexes and associated data

Integral points tend to accumulate near the boundary, with more points lying near intersections of several components of the boundary divisor. For this reason, combinatorial data on the boundary, encoded in Clemens complexes, appears in asymptotic formulas for the number of integral points of bounded height.

2.3.1 Clemens complexes

The *geometric Clemens complex* $\mathcal{C}_{\overline{K}}(D)$ is a partially ordered set defined as follows: Let $\mathcal{A}$ be an index set for the set of irreducible components of D (which are the same as the irreducible components of $D_{\overline{K}}$ by our assumptions); denote by D_α the irreducible component of D corresponding to $\alpha \in \mathcal{A}$, and, for any $A \subset \mathcal{A}$, by D_A the intersection $\bigcap_{\alpha \in A} D_\alpha$. Then the geometric Clemens complex consists of all pairs (A, Z), such that A is a non-empty subset of $\mathcal{A}$, and Z is an irreducible component of $(D_A)_{\overline{K}}$. Its ordering is given by $(A, Z) \prec (A', Z')$ if $A \subset A'$ and $Z \supset Z'$. In other words, we add a vertex for every component of D; if the intersection of a set of components is non-empty, we glue one simplex to the corresponding set of vertices for every geometric component of the intersection. In the following, we will often suppress Z from the notation.

For an archimedean place v, we will also be interested in the *K_v-analytic Clemens complex* $\mathcal{C}_v^{\mathrm{an}}(D)$. It is the subset consisting of all pairs (A, Z) such that Z is defined over K_v and has a K_v-rational point. (Note that this depends on v and not just on the isomorphism class of K_v.) By the assumptions in the beginning of this chapter, we have the following:

Lemma 2.3.1. *If a face (A, Z) of the geometric Clemens complex is part of the K_v-analytic Clemens complex $\mathcal{C}_v^{\mathrm{an}}(D)$, then so are all of its subfaces (A', Z').*

Proof. Such a subface is given by data $A' = \{D_1, \dots, D_r\} \subset A$ and an irreducible component $Z' \subset D_{A'}$ with $Z' \supset Z$. Since $Z(K_v) \neq \emptyset$, it contains a K_v-point P, that is, a point P invariant under the action of the Galois group of K_v; since $Z' \supset Z$, the point P is also on Z'. For contradiction, assume now that Z' is not defined over K_v. Since the D_i are all defined over K, they are invariant under the Galois action, and thus the conjugates ${}^\sigma Z'$ of Z' under the action of the Galois group are also contained in all D_{A_i}, hence also in $D_{A'}$. Since

P is contained in the intersection of all conjugates (and there is more than one by the assumption), $D_{A'}$ is singular in P, and D does not have strict normal crossings, contradicting our assumptions at the beginning of this chapter. $\square$

Note that, since D_A is smooth for every face A, the set $D_A(K_v)$ is a smooth K_v-manifold. We will often be interested in maximal faces of the analytic Clemens complex with respect to the ordering. (Such faces need not be maximal-dimensional, that is, maximal with respect to their number of vertices.) Geometrically, maximal faces are faces A such that $D_A(K_v)$ intersects no other divisor component. We denote the set of maximal faces by $\mathcal{C}_v^{\mathrm{an},\max}(D)$; if the K_v-analytic Clemens complex is empty at a place v, then the empty set is its unique maximal face.

2.3.2 The measure associated with a maximal face

Let v be an archimedean place, and let $A \in \mathcal{C}_v^{\mathrm{an},\max}(D)$ be a maximal face of the K_v-analytic Clemens complex, that is, a maximal subset of the irreducible components whose intersection D_A has a K_v-rational point. Denote by

$$\Delta_A = D - \sum_{\alpha \in A} D_\alpha$$

its "complement". We are interested in a measure τ_{D_A} on $D_A(K_v)$ defined as follows [CLT10a, 2.1.12]: A metric on $\omega_X(\sum_{\alpha\in A} D_\alpha)$ defines a metric on ω_{D_A} and thus a Tamagawa measure τ on $D_A(K_v)$ by repeated use of the adjunction isomorphism (since D is assumed to have strict normal crossings). We consider the modified measure

$$\|1_{\Delta_A}\|^{-1}_{\mathcal{O}(\Delta_A),v}\,\tau.$$

This measure only depends on the metrization of the log-canonical bundle $\omega_X(D)$: This metric induces a metrization of $\omega_{D_A}(\Delta_A)$, via the adjunction isomorphism, and the above measure is equal to

$$\left\|1_{\Delta'_A} \otimes \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_s\right\|^{-1}_{\omega_{D_A}(\Delta_A),v} \mathrm{d}(x_1,\dots,x_s),$$

with local coordinates $x_1,\dots,x_s$. Note that the maximality of A guarantees that $\|1_{\Delta_A}\|_v^{-1}$ does not have a pole on $D_A(K_v)$, and is thus bounded on $D_A(K_v)$ since this set is compact.

These measures are further renormalized by a factor $c_{K_v}^{\#A}$, where $c_\mathbb{R} = 1$, and $c_\mathbb{C} = 2\pi$ is the volume of the unit ball in the archimedean local field with respect to the Haar measure we are using. We thus get a *residue measure*

$$\tau_{D_A} = c_{K_v}^{\#A}\,\|1_{\Delta_A}\|^{-1}_{\mathcal{O}(\Delta_A),v}\,\tau$$

on every D_A. See [CLT10a, 3.1.1, 4.1] for more details.

Example 2.3.2. Once more returning to $\mathbb{P}^1\times\mathbb{P}^1$, we first note that the Clemens complex is just the single vertex $\Delta = \Delta_{\mathbb{P}^1}$, and thus has a single maximal face. Following (loc. cit.), we compute the residue measure at this maximal face. We have a local coordinate $z = x + y$ of Δ, and a local equation $z' = x - y$ of Δ, satisfying $\mathrm{d}z' \wedge \mathrm{d}z = 2\,\mathrm{d}x \wedge \mathrm{d}y$. In these new coordinates, we have

$$\|\mathrm{d}x \wedge \mathrm{d}y \otimes 1_\Delta\|_{\omega_X(\Delta),\infty} = |z| \max\left\{1, \left|\frac{z+z'}{2}\right|\right\} \max\left\{1, \left|\frac{z-z'}{2}\right|\right\}.$$

Thus, we get

$$\|\mathrm{d}z\|_{\omega_\Delta,\infty} = \lim_{z'\to 0} \frac{1}{|z|} \|\mathrm{d}z' \wedge \mathrm{d}z \otimes 1_D\|_{\omega_X(\Delta),\infty} = 2\max\left\{1, \left|\frac{z}{2}\right|\right\}^2.$$

Integrating its inverse yields

$$\int_{|z|\leq 2} \frac{1}{2}\,\mathrm{d}z + \int_{|z|>2} \frac{2}{|z|^2}\,\mathrm{d}z = 4,$$

and after renormalizing with $c_\mathbb{R} = 2$, we get $\tau_{\Delta,\infty}(\Delta(\mathbb{R})) = 8$.

2.3.3 A divisor group

Let $\underline{A} = (A_v)_{v|\infty} \in \prod_v \mathcal{C}_v^{\mathrm{an,max}}(D)$ be a tuple of maximal faces A_v of the K_v-analytic Clemens complexes for all archimedean places of K. We set

$$\Delta_{\underline{A}} = \sum_{\substack{\alpha\notin A_v\\ \text{for all } v|\infty}} D_\alpha \qquad \text{and} \qquad U_{\underline{A}} = X - \Delta_{\underline{A}},$$

so $\Delta_{\underline{A}} \subset D$ is again the “complement” of $\underline{A}$. With Δ_{A_v} as before for the maximal face A_v at a place v and $U_{A_v} = X - \Delta_{A_v}$, we have inclusions

$$\Delta_{\underline{A}} \subset \Delta_{A_v} \subset D \qquad \text{and} \qquad U \subset U_{A_v} \subset U_{\underline{A}} \subset X.$$

We associate some data with $\underline{A}$ analogous to groups defined by Chambert-Loir and Tschinkel for toric varieties [CLT10b, 3.5], using the full set of divisors instead of invariant ones: We let

$$\mathrm{Div}(U;\underline{A}) = \mathrm{Div}(U) \oplus \bigoplus_{v|\infty} \mathbb{Z}^{A_v} \quad \text{and} \quad \mathrm{Pic}(U;\underline{A}) = \mathrm{Div}(U;\underline{A})/\operatorname{im}(\mathrm{div}_{\underline{A}}),$$

where $\mathrm{div}_{\underline{A}} \colon \mathcal{K}_X \to \mathrm{Div}(U;\underline{A})$ maps a rational function f to

$$\left(\mathrm{div}_U(f), \left(\sum_{\alpha\in A_v} \mathrm{ord}_{D_\alpha}(f) D_\alpha\right)_v\right).$$

Since $\mathrm{div}_{\underline{A}}$ is compatible with the standard divisor function, we have canonical homomorphisms $\pi_{\underline{A}}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(U;\underline{A})$ and $\sigma_{\underline{A}}\colon \mathrm{Pic}(U;\underline{A}) \to \mathrm{Pic}(U)$. The first one maps the class of a prime divisor $[E]$ to the class of $(E\cap U, (\mathbb{1}_{E\in A_v} E)_v)$, where $\mathbb{1}_{E\in A_v}$ is 1 if E is a component of D and belongs to the maximal face A_v, and 0 otherwise. The second homomorphism $\sigma_{\underline{A}}$ maps the class of $(E,(E_v)_v) \in \mathrm{Div}(U;\underline{A})$ to $[E]$.

If K has only one archimedean place, these constructions simplify to the Picard group $\mathrm{Pic}(U;\underline{A}) = \mathrm{Pic}(U_{\underline{A}})$ of $U_{\underline{A}}$ and the pullback homomorphisms $\mathrm{Pic}(X) \to \mathrm{Pic}(U_{\underline{A}})$ and $\mathrm{Pic}(U_{\underline{A}}) \to \mathrm{Pic}(U)$.

In the context of asymptotic formulas, we will be interested in the two numbers

$$b_{\underline{A}} = \operatorname{rk}\mathrm{Pic}(U) - \operatorname{rk} E(U) + \sum_{v|\infty} \# A_v \quad \text{and}$$

$$b'_{\underline{A}} = \operatorname{rk}\mathrm{Pic}(U;\underline{A})$$

connected to the exponent of $\log B$ and a factor of the leading constant associated with $\mathrm{Pic}(U;\underline{A})$.

Lemma 2.3.3. *We have*

$$b_{\underline{A}} = \operatorname{rk}\operatorname{Pic} X - \#\mathcal{A} + \sum_{v\mid\infty} \#A_v \text{ and}$$
$$b'_{\underline{A}} = b_{\underline{A}} + \operatorname{rk} E(U_{\underline{A}}).$$

Proof. For the first assertion, note that there is an exact sequence

$$0 \to E(U) \to \mathrm{CH}^0(D) \to \operatorname{Pic}(X) \to \operatorname{Pic}(U) \to 0. \tag{2.2}$$

Indeed, the part on the right is the localization sequence for Chow groups. Exactness on the left follows form the fact that a relation making a divisor supported on $\operatorname{supp}(D)$ linearly trivial has to come from a meromorphic section whose only zeroes and poles are on $\operatorname{supp}(D)$, that is, invertible regular functions on U. The only such functions mapping to 0 in $\mathrm{CH}^0(D)$ are regular and invertible on X, hence invertible constants, and we get $E(U)$ on the left. The assertion then follows with $\operatorname{rk}\mathrm{CH}^0(D) = \#\mathcal{A}$.

The second assertion will follow from the exactness of a sequence

$$0 \to E(U_{\underline{A}}) \to E(U) \to \bigoplus_{v\mid\infty} \mathbb{Z}^{A_v} \to \operatorname{Pic}(U;\underline{A}) \to \operatorname{Pic}(U) \to 0. \tag{2.3}$$

The homomorphism to $\operatorname{Pic}(U)$ is the map $\sigma_{\underline{A}}$ defined above; its kernel is generated by divisors supported outside U, that is, on $\underline{A}$. If such a divisor E is linearly equivalent to 0 in $\operatorname{Pic}(U;\underline{A})$, this equivalence is induced by a section which has corresponding zeroes and poles on E, but no zeroes and poles on U; again, we can exclude constants. Finally, the invertible regular functions on U not inducing such a relation, and thus mapping to 0 in the middle group are those which do not have a zero or pole on any $\underline{A}$, i.e., those that are regular and invertible on $U_{\underline{A}}$. □

Remark 2.3.4. We always have $b_{\underline{A}} = b'_{\underline{A}}$ for e.g. toric varieties [CLT10b, 3.7.1 with the remark before Lemma 3.8.5], partial equivariant compactifications of vector groups, and semisimple groups (since their effective cones are simplicial and generated by invariant divisors). Both numbers play a role in asymptotic formulas, and we will see in Lemma 2.4.4 that they are equal whenever we can expect a tuple $\underline{A}$ of maximal faces to contribute to an asymptotic formula.

Remark 2.3.5. Due to our assumptions on X and D, it does not matter whether we work over K or $\overline{K}$: For a group $\operatorname{Pic}(U_{\overline{K}};\underline{A})$ similarly defined over $\overline{K}$ we would have a canonical isomorphism $\operatorname{Pic}(U_{\overline{K}};\underline{A}) \cong \operatorname{Pic}(U;\underline{A})$. Indeed, we can consider the two exact sequences in the proof of Lemma 2.3.3 over both K and $\overline{K}$ together with the obvious homomorphisms between them. The splitness assumptions imply that the homomorphisms $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\overline{K}})$ and $\mathrm{CH}^0(D) \to \mathrm{CH}^0(D_{\overline{K}})$ are isomorphisms, so using the five lemma three times yields $\operatorname{Pic}(U_{\overline{K}};\underline{A}) \cong \operatorname{Pic}(U;\underline{A})$.

2.3.4 A convex cone

Assume that $\operatorname{Pic}(U;\underline{A})$ is torsion-free – in Lemma 2.4.6 we will see that this holds whenever there is no obstruction to the Zariski density of integral points "near $\underline{A}$". It is thus a lattice in $V_{\underline{A}} = \operatorname{Pic}(U;\underline{A})_{\mathbb{R}}$, and we consider this vector space together with its *effective cone* $\Lambda_{\underline{A}} \subset \operatorname{Pic}(U;\underline{A})_{\mathbb{R}}$ generated by the images of effective divisors $\operatorname{Div}_{\geq 0}(U) \oplus \bigoplus \mathbb{Z}_{\geq 0}^{A_v}$. If K has only one archimedean place, this is simply the effective cone $\overline{\operatorname{Eff}}_{U_{\underline{A}}}$ of $U_{\underline{A}}$.

We can equip the dual vector space $V_A^\vee = \operatorname{Pic}(U;\underline{A})_{\mathbb{R}}^\vee$ with the Haar measure λ normalized by the dual lattice $\operatorname{Pic}(U;\underline{A})^\vee$. The characteristic function of $\Lambda_{\underline{A}}$ is defined via

$$\mathcal{X}_{\Lambda_{\underline{A}}}(\mathcal{L}) = \int_{\Lambda_{\underline{A}}^\vee} e^{-\langle \mathcal{L}, t\rangle} \,\mathrm{d}t.$$

It is finite in the interior of $\Lambda_{\underline{A}}$. Since the log-anticanonical bundle $\omega_X(D)^\vee$ is big, its image is in the interior of $\Lambda_{\underline{A}}$, and we set

$$\alpha_{\underline{A}} = \frac{1}{(b'_{\underline{A}} - 1)!} \mathcal{X}_{\Lambda_{\underline{A}}}(\pi(\omega_X(D)^\vee)),$$

where $\pi\colon \operatorname{Pic}(X) \to \operatorname{Pic}(U;\underline{A})$ is the canonical map. This value is non-zero if and only if $\Lambda_{\underline{A}}$ is strictly convex. In Lemma 2.4.3, we will see that if this is not the case, then there is an obstruction to the Zariski density of integral points "near $\underline{A}$", and the face $\underline{A}$ should not contribute to an asymptotic formula.

The constant $\alpha_{\underline{A}}$ can alternatively be described as a volume: Equip the hyperplanes $H_a = \{t \in V^\vee \mid \langle \mathcal{L}, t\rangle = a\}$ with measures λ_A normalized such that

$$\int_{V_A^\vee} f \,\mathrm{d}\lambda = \int_{\mathbb{R}} \left(\int_{H_a} f \,\mathrm{d}\lambda_{H_a} \right) \mathrm{d}a$$

for all functions f on $V_A^\vee$ with compact support. Then (cf. [Vin63, Chapter 1, § 2])

$$\begin{aligned} \alpha_{\underline{A}} &= \operatorname{vol}\{t \in \Lambda_{\underline{A}}^\vee \mid \langle \omega_X(D)^\vee, t\rangle = 1\} \\ &= b'_A \operatorname{vol}\{t \in \Lambda_{\underline{A}}^\vee \mid \langle \omega_X(D)^\vee, t\rangle \leq 1\}. \end{aligned}$$

If the cone $\Lambda_{\underline{A}}$ is *smooth*, that is, generated by a $\mathbb{Z}$-basis $r_1, \dots, r_{b'_{\underline{A}}}$ of $\operatorname{Pic}(U;\underline{A})$, this further simplifies: If $\pi(\omega(D)^\vee)$ has the representation $(a_1, \dots, a_{b'_{\underline{A}}})$ in this basis, we have

$$\alpha_{\underline{A}} = \frac{1}{(b'_{\underline{A}} - 1)!} \prod_{1 \leq i \leq b'_{\underline{A}}} \frac{1}{a_i}.$$

Example 2.3.6. To finish the geometric interpretation for $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$, let us compute α_Δ. First of all, we note that $b_\Delta = 2 - 1 + 1 = 2$ by Lemma 2.3.3. Since there is only one maximal face, we have $U_\Delta = X$ for this face, so $\operatorname{Pic}(U;\Delta) = \operatorname{Pic}(X) = \mathbb{Z}^2$, and its effective cone $\Lambda_A = \overline{\operatorname{Eff}}_X = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ is smooth. Then we have $b'_A = 2 = b_A$, the log-anticanonical class is $(1,1)$, and we get $\alpha_A = 1$. In total, we now have a geometric interpretation of our asymptotic formula for the number of integral points of bounded height on $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta_{\mathbb{P}^1}$, similar to other results on integral points:

$$\begin{aligned} N(B) = \alpha_\Delta \tau_{D_\Delta,\infty}(D_\Delta(\mathbb{R})) \prod_p \left(\left(1 - \frac{1}{p}\right)^{\operatorname{rk}(\operatorname{Pic} U)} \tau_{U,p}(\mathcal{U}(\mathbb{Z}_p)) \right) B(\log B)^{b_\Delta - 1} \\ + O(B). \end{aligned}$$

(Note that on a field with only one archimedean place we always have $E(U) = 0$ if the set of integral points is Zariski dense by the second argument in the proof of Lemma 2.4.4.) More examples will follow in the next section.

2.4 An obstruction

Let $\underline{A} \in \prod_v \mathcal{C}_v^{\mathrm{an,max}}(D)$ be a tuple of maximal faces of the Clemens complex, and consider the regular sections $\mathcal{O}_X(U_{\underline{A}})$ on $U_{\underline{A}}$. These provide an obstruction to the Zariski density of integral points near $\underline{A}$: If $\mathcal{O}_X(U_{\underline{A}}) \neq K$, that is, if there are non-constant sections on $U_{\underline{A}}$, there are no integral points that are simultaneously near all D_{A_v}, except possibly on a finite set of strict subvarieties. Since we should exclude such subvarieties if they were to contribute to an asymptotic formula, there cannot be a contribution of "points near $\underline{A}$" to an asymptotic formula in this case. In this case, we will say that there is an obstruction to the Zariski density of integral points near $\underline{A}$.

Proposition 2.4.1. *Let $\underline{A} \in \prod_v \mathcal{C}_v^{\mathrm{an,max}}(D)$ be a tuple of maximal faces of the analytic Clemens complexes such that $\mathcal{O}_X(U_{\underline{A}}) \neq K$. Then there is a dense Zariski open subset $V \subset X$ and an analytic neighbourhood U_v of D_{A_v} in $X(K_v)$ for every archimedean place v such that*

$$\{x \in \mathcal{U}(\mathfrak{o}_K) \cap V(K) \mid x \in U_v \text{ for all } v \mid \infty\} = \emptyset.$$

Proof. Let s be a non-constant section in $H^0(U_{\underline{A}}, \mathcal{O}_X)$. After multiplying with a suitable constant, we can assume it is a section of the integral model. Let v be an infinite place. Then, by the maximality assumption, $D_{A_v}(K_v)$ does not intersect $\Delta_{\underline{A}}(K_v)$, so $|s|_v$ is continuous on the compact set $D_{A_v}(K_v)$ and attains its maximum M_v. Let

$$U_v = \{x \in X(K_K) \mid |s(x)|_v < 2M_v\}.$$

Since $s(x) \in \mathfrak{o}_K$ for integral points $x \in \mathcal{U}(\mathfrak{o}_K)$, it can attain only finitely many values α in the box defined by the M_v. Every integral point lying in all of the U_v must thus lie on one of the finitely many subvarieties $\overline{V(s-\alpha)}$. □

If there is such an obstruction for a maximal face $\underline{A}$, points near $D_{\underline{A}'}$ for subfaces $\underline{A}'$ are similarly obstructed – except possibly near a larger face $\underline{B} \supset \underline{A}'$, in which case we would expect that their number is described by invariants attached to $\underline{B}$, or, more precisely, maximal faces containing $\underline{B}$.

Lemma 2.4.2. *Let $\underline{A} \in \prod_v \mathcal{C}_v^{\mathrm{an,max}}(D)$ be a tuple of maximal faces of the Clemens complex such that $\mathcal{O}_X(U_{\underline{A}}) \neq K$. Let $\underline{A}' \subset \underline{A}$ be a subface. For every place v, let $B_{1,v}, \dots, B_{r_v,v}$ be the faces containing A'_v as a strict subface. For every i and v, let $U_{i,v}$ be an arbitrary analytic open neighbourhood of $D_{B_{i,v}}(K_v)$ in $X(K_v)$. Then there exists an analytic neighbourhood U_v of*

$$D_{A'_v} - \bigcup_{i=1}^{n} U_{i,v}$$

in $X(K_v)$ for every $v \mid \infty$ and a dense Zariski open subvariety $V \subset X$ such that

$$\{\mathcal{U}(\mathfrak{o}_K) \cap V(K) \mid x \in U_v, \text{ but } x \notin U_{i,v} \text{ for all } v \mid \infty \text{ and } i = 1, \dots, n\} = \emptyset.$$

Proof. The proof works analogous to the last one (and the above proposition can be regarded as a special case, taking $\underline{A}' = \underline{A}$): Let $s \in \mathcal{O}_X(U_{\underline{A}})$ be non-constant. The poles of $|s|_v$ on $D_{A'_v}(K_v)$ are entirely contained in its intersection with $\Delta_{\underline{A}}(K_v)$, so, again, $|s|_v$ is continuous on the compact set $D_{A'_v} - \bigcap_{i=1}^n U_{i,v}$, and attains its maximum M_v. As before, we get open subsets

$$U_v = \{x_v \in X(K_v) \mid |s(x)|_v \leq 2M_v\}$$

and a finite set of subvarieties of the form $\overline{V(s-\alpha)}$. □

This obstruction can be triggered if some of the objects defined in the previous section behave pathologically: If the cone $\Lambda_{\underline{A}}$ fails to be strictly convex, if $\mathrm{Pic}(U; \underline{A})$ has torsion elements, or if its rank $b'_{\underline{A}}$ is not equal to the exponent $b_A = \mathrm{rk}\,\mathrm{Pic}\,U - \mathrm{rk}\,E(U) + \sum_v \# A_v$, then there is an obstruction to the Zariski density of integral points near $\underline{A}$.

For the first case, if the cone $\Lambda_{\underline{A}}$ whose characteristic function appears in asymptotic formulas is not strictly convex, it yields a factor $\alpha_{\underline{A}} = 0$. An example of this happening, which also has an impact on the exponent of $\log B$ in an asymptotic formula, is analyzed in Chapter 5.

Lemma 2.4.3. *Let $\underline{A} \in \prod_v \mathcal{C}_v^{\mathrm{an},\max}(D)$ be a tuple of maximal faces such that $\Lambda_{\underline{A}}$ is not strictly convex. Then $\mathcal{O}_X(U_{\underline{A}}) \neq K$.*

Proof. That $\Lambda_{\underline{A}}$ is not strictly convex means that it contains a line through 0, that is, we can find two effective divisors $(E, (E_v)_v)$ and $(E', (E'_v)_v) \in \mathrm{Div}(U; \underline{A})$ with $E + E' \sim 0$. Hence there exists a rational function which vanishes on all E, E_v, E', E'_v (and thus is non-constant), and whose only poles are outside $U_{\underline{A}}$. □

We have defined two constants $b_{\underline{A}}$ and $b'_{\underline{A}}$ arising in asymptotic formulas, which coincide for toric varieties and all varieties studied in the next chapters. While this does not hold in general, there is an obstruction whenever they differ.

Lemma 2.4.4. *Let $\underline{A} \in \prod_v \mathcal{C}_v^{\mathrm{an},\max}(D)$ be a tuple of maximal faces such that $b_{\underline{A}} \neq b'_{\underline{A}}$. Then $\mathcal{O}_X(U_{\underline{A}}) \neq K$. If, in addition, K has only one infinite place, then $\mathcal{U}(\mathfrak{o}_K)$ is not Zariski dense for any integral model $\mathcal{U}$ of U.*

Proof. We have seen in Lemma 2.3.3 that this happens if and only if there is a non-trivial invertible regular function $s \in E(U_{\underline{A}})$, so, in particular, there is a non-trivial regular function on $U_{\underline{A}}$.

Next, assume that K has only one infinite place, that is, that the group of units $\mathfrak{o}_K^\times$ is finite, and let $s \in E(U_{\underline{A}})$ be such an invertible regular function. After multiplying s and s^{-1} with appropriate constants, we get regular sections s and s' on $\mathcal{U}$ such that $ss' = a \in \mathfrak{o}_K$. For a rational point $x \in \mathcal{U}(\mathfrak{o}_K)$, the value $s(x)$ then has to be a divisor of a, of which there are only finitely many. The integral point x must thus lie on one of the finitely many subvarieties $\overline{V(s-\alpha)}_{\alpha | a}$ of X. □

Example 2.4.5. Consider $\mathbb{P}^n$ and the three hyperplanes $V(x_0)$, $V(x_1)$, and $V(x_0 + x_1)$. Their sum does not have strict normal crossings, which we can remedy by blowing up $V(x_0, x_1)$. Call the resulting variety X, and consider the pair (X, D) with $D = H_1 + H_2 + H_3 + E$, where the H_i are the strict transforms of the three hyperplanes and E is the exceptional divisor. For $n \geq 3$, the log-anticanonical bundle is big (though never nef), and we have $U = X - D \cong \mathbb{A}^n - V(x_1) - V(x_1+1)$. The geometric and every K_v-analytic Clemens complex is then a "star", with the vertex corresponding to E connected to the other three vertices H_i. If we take $\underline{A} = (A_v)_v$ with the same maximal face $A = \{E, H_i\}$ (for some fixed i) for all infinite places, we have $U_{\underline{A}} \cong \mathbb{A}^{n-1} \times \mathbb{G}_{\mathrm{m}}$. Hence

$$b_{\underline{A}} = \operatorname{rk}\operatorname{Pic}(U) - \operatorname{rk} E(U) + \sum_{v\mid\infty} \#A = 0 - 2 + 2(r+s) = 2(r+s) - 2.$$

On the other hand, using (2.3),

$$0 \to E(U_{\underline{A}}) \to E(U) \to (\mathbb{Z}^A)^{\oplus(r+s)} \to \operatorname{Pic}(U; \underline{A}) \to 0$$

is exact, with the groups to the left having ranks 1, 2, and $2(r+s)$, respectively, so $b'_{\underline{A}} = 2(r+s)-1$, and there is an obstruction. In fact, the set of integral points is not dense: Every integral point lies on one of the subvarieties $\{ax_0 - bx_1 = 0\}$ parametrized by the finitely many solutions $a, b \in \mathfrak{o}_K^\times$ of the unit equation $a + b = 1$.

Lemma 2.4.6. *Let $\underline{A}$ be a tuple of maximal faces such that* $\operatorname{Pic}(U; \underline{A})$ *is not torsion free. Then $\mathcal{O}_X(U_{\underline{A}}) \neq K$.*

Proof. We consider the morphism $\pi_{\underline{A}}\colon \operatorname{Pic}(X) \to \operatorname{Pic}(U; \underline{A})$. It fits into an exact sequence

$$\mathrm{CH}^0(\Delta_{\underline{A}}) \to \operatorname{Pic}(X) \to \operatorname{Pic}(U; \underline{A}) \to \bigoplus_v \mathbb{Z}^{A_v}/\mathbb{Z}^{\mathcal{A}-\operatorname{supp}\Delta_{\underline{A}}} \to 0. \tag{2.4}$$

Indeed, its kernel is generated by divisors supported on $\Delta_{\underline{A}}$, hence the image of the pull-back map $\mathrm{CH}^0(\Delta_{\underline{A}}) \to \operatorname{Pic}(X)$; for exactness on the right, note that $\operatorname{Pic}(X) = (\operatorname{Div}(U) \oplus \mathbb{Z}^{\mathcal{A}})/\operatorname{im}(\operatorname{div}_X)$, so the cokernel of $\pi_{\underline{A}}$ is indeed $\bigoplus_v \mathbb{Z}^{A_v}/\mathbb{Z}^{\mathcal{A}-\operatorname{supp}\Delta_{\underline{A}}}$, after omitting the part $\mathbb{Z}^{\operatorname{supp}\Delta_{\underline{A}}}$ mapped to 0 by $\pi_{\underline{A}}$.

The rightmost group is torsion free: An element $(nD_v)_v$ is in the image of $\mathbb{Z}^{\mathcal{A}}$ if and only if there is a divisor D such that $nD_v = i^{-1}(D)$ for all $i_v\colon U_{A_v} \to U_{\underline{A}}$; in particular, D is divisible by n on $\bigcap A_v$, and thus the class of $(D_v)_v$ is already 0. Hence, every non-zero torsion element $T \in \operatorname{Pic}(U; \underline{A})$ has to be the image of a (non-zero) element $\widetilde{T} \in \operatorname{Pic}(X)$ such that $n\widetilde{T} \in \operatorname{im}(\mathbb{Z}^{\operatorname{supp}\Delta_{\underline{A}}})$, i.e., there are b_α such that $n\widetilde{T} + \sum b_\alpha D_\alpha \sim 0$. Consider

$$\widetilde{T}' = T + \sum \left\lceil \frac{b_\alpha}{n} \right\rceil D_\alpha.$$

The divisor $\widetilde{T}'$ is non-zero and in the effective cone, so, using our assumptions on X, it is represented by an effective $\mathbb{Q}$-divisor E. The image of $\widetilde{T}' = [E]$ is still T, so the image of $[nE]$ is trivial. Working with a suitable multiple of nE that is integral, this means that there is a rational function s vanishing on the support of E, and which can only have poles on $\Delta_{\underline{A}}$. Since the image of $[E]$ in $\operatorname{Pic}(U; \underline{A})$ is non-zero, the support of E cannot be contained in the support of $\Delta_{\underline{A}}$. Hence s is non-constant and regular on $U_{\underline{A}}$, and we have $\mathcal{O}_X(U_{\underline{A}}) \neq K$. $\square$

Example 2.4.7. Let us consider $\mathbb{P}^n$ together with a divisor D having two components: The quadric hypersurface $Q = \{x_0^2 = \sum_{i=1}^n x_i^2\}$, and the hyperplane $H = \{x_0 = 0\}$. If $n \geq 3$, the log-anticanonical bundle is ample. The intersection $Q \cap H$ does not contain any $\mathbb{R}$-points, so, for totally real fields, every K_v-analytic Clemens complex consists of two vertices at every place v. Consider the set $\underline{A}$ consisting of the face H at every place. Since the Picard group of $U = \mathbb{P}^n - D$ is trivial, (2.3) allows us to compute

$$\mathrm{Pic}(U; \underline{A}) \cong \mathbb{Z}^r/(2, \dots, 2) \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Note that if $K = \mathbb{Q}$, there are only finitely many points corresponding to the solutions of $x_1^2 + \cdots + x_n^2 = 2$, while for larger fields, we get the sets of solutions of $x_1^2 + \cdots + x_n^2 = 1 + u$ for units $u \in \mathfrak{o}_K^\times$.

In [JS17], Jahnel and Schindler describe obstructions at archimedean places. For an archimedean place v, the complement U of a very ample divisor D is called *weakly obstructed at* v if there is a connected component U' of $U(K_v)$, a constant $c > 0$, an integer $d > 0$, and a finite set of rational functions of the form $s_i = f_i/1_D^d$ with $f_i \in H^0(X, \mathcal{O}_X(D)^{\otimes d})$ not multiples of 1_D^d (that is, non-constant regular functions s_i on U) such that, for every point $x \in U'$, there is at least one s_i with $|s_i|_v < c$.

Lemma 2.4.8. *Let v be an archimedean place of K, and assume that $U(K_v)$ is connected. Then the following are equivalent:*

- *U is weakly obstructed at v, and*
- *$\mathcal{O}_X(U_A) \neq K$ for all maximal faces A of the K_v-analytic Clemens complex.*

Proof. Let c be a constant and $s_1, \dots, s_n$ be regular functions on U such that, for every $x \in U(K_v)$ we have $|s_i(x)|_v < c$ for some i. Since, by assumption, every point z on the boundary is a limit of points on U, we have $|s_i(x)| \leq c$. Take a point x on $D_A(K_v)$ for a maximal face A. Then there is an s_i with $|s_i(x) \leq c|$. Then $|s_i(x)| < 2c$ is a neighbourhood of x, so, for all $\alpha \in A$, it intersects all $D_\alpha(K_v)$ in an open subset. In particular, s_i cannot have a pole on any of the D_α, and is thus regular on U_A.

For the other direction, we take a non-trivial $s_A \in \mathcal{O}_X(U_A)$ for all maximal faces A. For every point x on the boundary, at least one of the s_A is regular in x. Moreover, all of them are regular on U, and thus $\{|s_i| < c\}_{i,c}$ covers the compact set $X(K_v)$, and there is a finite subcover. We can then take c as the maximal constant used in this subcover. $\square$

Remark 2.4.9. Over fields with only one infinite place, integral points are not Zariski dense if U is weakly obstructed at ∞ by [JS17, Theorem 2.6]. In a more general setting, this does not need to be the case, even if U is obstructed at every archimedean place: If we take $\underline{A} = (A_v)_v$ with different faces A_v for different archimedean places, we have $U_{A_v} \subsetneq U_{\underline{A}}$, and the regular sections $\mathcal{O}_X(U_{\underline{A}})$ might be trivial, even though $\mathcal{O}_X(U_{A_v}) \neq K$ for all archimedean places v. However, the following generalizes said result to fields with more than one infinite place, providing an obstruction to the Zariski density of integral points.

Proposition 2.4.10. *Assume that $\mathcal{O}_X(U_{\underline{A}}) \neq K$ for all tuples of maximal faces. Then $\mathcal{U}(\mathfrak{o}_K)$ is not Zariski dense for any integral model $\mathcal{U}$ of U.*

Proof. For every $\underline{A}$, let $s_{\underline{A}} \in \mathcal{O}_X(U_{\underline{A}}) - K$; after multiplying with a suitable constant, we can assume that $s_{\underline{A}}$ is regular on $\mathcal{U}$. Let $x = (x_v)_v \in \prod_{v|\infty} X(K_v)$. For all archimedean places v such that $x_v \in D(K_v)$, let A'_v be the face of the Clemens complex that is maximal under those with $x \in D_{A'}$. Let A_v be a maximal face containing A'_v. For all v such that $x_v \in U(K_v)$, let A_v be an arbitrary maximal face, and let $\underline{A} = (A_v)_v$. Then $x_v \in U_{A_v}(K_v) \subset U_{\underline{A}}(K_v)$ for all v; hence $s_{\underline{A}}$ is regular in all x_v, and $\left|s_{\underline{A}}(x_v)\right|_v$ is finite for all v. The open sets

$$\{\left|s_{\underline{A}}\right|_v \leq c\}_{\underline{A},c}$$

cover the compact set $\prod_{v|\infty} X(K_v)$, and there is a finite subcover; let c be the maximal constant needed for this finite subcover. Let I be the finite set of $\alpha \in K$ with $|\alpha|_v \leq c$ for all v. Now, for every integral point $x \in \mathcal{U}(\mathbb{Z})$, there has to be a maximal face $\underline{A}$ such that $\left|s_{\underline{A}}(x)\right|_v \leq c$ for all $v \mid \infty$; since $s_{\underline{A}}$ is a regular section of $\mathcal{U}$, we even have $s_{\underline{A}}(x) \in I$. This means that every integral point is on one of the finitely many strict subvarieties

$$\left\{ V(s_{\underline{A}} - \alpha) \;\middle|\; \underline{A} \in \prod_{v|\infty} \mathcal{C}_v^{\mathrm{an},\max}(D),\ \alpha \in I \right\},$$

and the set of integral points is not Zariski dense. □

This obstruction always vanishes after a suitable base change:

Lemma 2.4.11. *There is a finite extension $L \supset K$ such that there is a tuple $\underline{A} = (A_w)_w$ of maximal faces of the analytic Clemens complex $\mathcal{C}^{\mathrm{an}}_{L_w}(D_L)$ at every archimedean place w of L with $\mathcal{O}_X((U_L)_{\underline{A}}) \neq L$.*

Proof. Let $A_1, \dots, A_n$ be the maximal faces of the geometric Clemens complex $\mathcal{C}_{\overline{K}}(D)$, and let $L \supset K$ be an extension with at least n complex places $w_1, \dots, w_n$. Then $\mathcal{C}^{\mathrm{an}}_{L_{w_i}}(D_L) = \mathcal{C}_{\overline{K}}(D)$ for these places, and we can take the tuple $\underline{A} = (A_w)_w$ with $A_{w_i} = A_i$ for these n complex places and A_w an arbitrary maximal face for all other places. Since every D_i belongs to at least one maximal face of the Clemens complex, we have $(U_L)_{\underline{A}} = X_L$, hence $\mathcal{O}_{X_L}((U_L)_{\underline{A}}) = L$. □

Remark 2.4.12. This analysis means that, when studying a variety with Zariski dense integral points, there will always be at least one maximal face for which the objects of the previous section are well-behaved, and, in particular, there always exists a collection of maximal faces $\underline{A}$ with $\alpha_{\underline{A}} \neq 0$. The converse is however far from true: There are varieties with unobstructed tuples of maximal faces whose integral points still are not Zariski dense. For instance, integral points in Example 2.4.5 are never Zariski dense, but the above lemma shows that there is an unobstructed tuple of faces over sufficiently large number fields.

2.5 Asymptotic formulas

These definitions allow the interpretation of asymptotic formulas. Keep all the assumptions on (X, D) from the beginning of this chapter, which included X and D being *split*. Let $\mathcal{U}$ be an integral model of U, and assume that $\mathcal{U}(\mathbb{Z})$ is

Zariski dense. Let H be the height function associated with a metric on the log-anticanonical bundle $\omega_X(D)^\vee$. We are interested in the asymptotic behaviour of the number

$$N(B) = \{x \in \mathcal{U}(\mathfrak{o}_K) \cap V(K) \mid H(x) \leq B\}$$

of integral points of bounded height whose generic point is on a suitable subset V of X. If strong approximation holds (using the set of connected components at archimedean places, cf. e.g. [CTWX18]), we might then expect an asymptotic expansion for $N(B)$ of the form

$$c_\infty c_{\text{fin}} B(\log B)^{b-1}(1 + o(1)), \tag{2.5}$$

where

$$c_\infty = \frac{1}{|d_K|^{\dim U/2}} \sum_{\underline{A} \in \mathcal{C}^{\max,\circ}(D)} \alpha_{\underline{A}} \prod_{v|\infty} \tau_{D_{A_v},v}(D_{A_v}(K_v)) \qquad \text{and}$$

$$c_{\text{fin}} = \rho_K^{\operatorname{rk}\operatorname{Pic} U - \operatorname{rk} E(U)} \prod_{v<\infty} \left(1 - \frac{1}{\# k_v}\right)^{\operatorname{rk}\operatorname{Pic} U - \operatorname{rk} E(U)} \tau_{U,v}(\mathcal{U}(\mathfrak{o}_{K_v})).$$

Here, the number b in the exponent of $\log B$ is the maximal value of $b_{\underline{A}} = b'_{\underline{A}}$ attained on tuples $\underline{A}$ of maximal faces with $\mathcal{O}_X(U_{\underline{A}}) \neq K$, i.e., on tuples without an obstruction. The sum runs over the set

$$\mathcal{C}^{\max,\circ}(D) = \left\{ \underline{A} \in \prod_{v|\infty} \mathcal{C}_v^{\text{an},\max}(D) \;\middle|\; \mathcal{O}_X(U_{\underline{A}}) \neq K,\ b_{\underline{A}} = b \right\}$$

of tuples $\underline{A}$ on which this maximum b is attained, that is, the set of "maximal-dimensional tuples" under those without an obstruction. The results in the previous section guarantee that the sum does not run over the empty set and that the factors are non-zero.

In a more general setting, the volume has to be that of a suitable subset of adelic points instead of a product of volumes, the factor ρ_K is the principal value of a different L-function, and additional factors appear in the constant, related to failures of strong approximation, to non-splitness, and to cohomological invariants (similar to the case of rational points). It is unclear to the author what the shape of such a factor for arbitrary (X, D) should be, and under which conditions it should be different from 1. Note that the Brauer group, whose order appears in Manin's conjecture for rational points, might not be trivial even for split U.

We can compare (2.5) to results in the framework by Chambert-Loir and Tschinkel. We note a difference in the case of toric varieties, and list the additional factors appearing in these asymptotic formulas.

- The formula above agrees with the [CLT12, Theorem 3.5.6] on partial equivariant compactifications of vector groups, since the obstruction never occurs in these cases, and since the cones $\Lambda_{\underline{A}}$ are all smooth, satisfying

$$\alpha_{\underline{A}} = \frac{1}{(b-1)!} \left(\prod_{\alpha \notin \mathcal{A}} \frac{1}{\rho_\alpha} \right) \left(\prod_{v|\infty} \prod_{\alpha \in A_v} \frac{1}{\rho_\alpha - 1} \right)$$

with the description $-K_X = \sum_{\alpha\in\mathcal{D}} \rho_\alpha D_\alpha$ of the anticanonical divisor as a sum of the boundary components $\{D_\alpha\}_{\alpha\in\mathcal{D}}$.

- Similarly, in the case of partial equivariant compactifications of split semi-simple groups [TBT13], the obstruction does not occur, and the cones are smooth with a similar description of $\alpha_{\underline{A}}$, making the formulas compatible. An additional factor is part of the asymptotic formula (18) in op. cit.: the number $|\chi_{S,D,\lambda}(G)|$ of certain automorphic characters of the underlying group G, related to strong approximation on G.

- The formula (2.5) is not compatible with [CLT10b, Theorem 3.11.5] on toric varieties; it modifies the exponent $b-1$ of $\log B$ and the index set of the sum. Our formula above agrees with the asymptotic formula we determine in Chapter 5. The formula in loc. cit. contains additional factors

$$\frac{|A(T,U,K)^*|}{|A(T)^*|}\frac{\left|H^1(\Gamma,\mathrm{Pic}(X_E))\right|}{|H^1(\Gamma,M_E)|}:$$

  two groups of automorphic characters, related to weak and strong approximation on T, and cohomology groups from the action of the Galois group (which is trivial in the split case). Moreover, the volume is taken on the subset of the adelic points cut out by these automorphic characters.

- The formula (2.5) with this general version of $\alpha_{\underline{A}}$ (using all divisors instead of only torus-invariant ones in op. cit.) is defined for the non-toric varieties that we study in in Chapters 3 and 4, and agrees with the asymptotic formulas determined in these two chapters.

Chapter 3

Integral points on a Fano threefold

3.1 Introduction

The aim of this chapter is to provide an asymptotic formula for the number of integral points of bounded height on a certain Fano threefold. Fano threefolds were classified by Iskovskih, Mori and Mukai [Isk77, MM82]. For these, Manin proved a lower bound for the number of rational points [Man93]. Those Fano threefolds that are toric or additive and for which Manin's conjecture is thus known have been classified by Batyrev [Bat81] and Huang–Montero [HM18], respectively. Besides such results for general classes of varieties, Manin's conjecture for Fano threefolds remains open.

We prove an asymptotic formula for a Fano threefold that does not belong to any of the classes for which an asymptotic formula for the number of integral points is known (cf. Lemma 3.2.4 and Remark 3.2.5). More precisely, we are interested in a pair (X, D), where X is in particular Fano, has Picard number 2 and is of type 30 in the classification of Fano threefolds [MM82]. Let $\pi\colon X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3 = \operatorname{Proj} \mathbb{Q}[a,b,c,d]$ in the smooth conic $C = \mathrm{V}(a^2+bc, d)$. We will provide asymptotic formulas for the number of integral points on $X - D_i$, where D_1 is the preimage $\pi^{-1}(\mathrm{V}(b))$ of a plane intersecting C twice in one rational point and D_2 is the preimage $\pi^{-1}(\mathrm{V}(a))$ of a plane intersecting C in two rational points. Up to $\mathbb{Q}$-automorphism, these are precisely the planes intersecting C in rational points. To construct integral models $\mathcal{U}_i$ of $U_i = X - D_i$, we consider the blow-up $\mathcal{X}$ of $\mathbb{P}^3_{\mathbb{Z}}$ in $V(a^2+bc, d)$ and define $\mathcal{U}_1 = \mathcal{X} - \overline{D_1}$, $\mathcal{U}_2 = \mathcal{X} - \overline{D_2}$.

We describe their integral points explicitly by a universal torsor in Section 3.2. In Section 3.3, we construct a log-anticanonical height function

$$H\colon X(\mathbb{Q}) \to \mathbb{R}_{>0},$$

measures $\tau_{(X,D_i),p}$ on $X(\mathbb{Q}_p)$ and $\tau_{D_i,\infty}$ on $D_i(\mathbb{R})$ renormalized with convergence factors defined in [CLT10a], and a renormalization factor $c_{\mathbb{R}}$. We continue with a description of a constant α and the exponent of $\log B$ in the expected asymptotic. In Sections 3.4 and 3.5, we prove an asymptotic formula for the number of integral points of bounded height on $\mathcal{U}_1$ and $\mathcal{U}_2$. A comparison of these formulas with the computations in the preceding section yields the following result:

Theorem 3.1.1. *For $i \in \{1,2\}$, let X, $\mathcal{X}$, D_i, $\mathcal{U}_i$, and H be as above. Let $V_1 = X - \pi^{-1}(V(abd))$ and $V_2 = X - \pi^{-1}(V(abcd))$. Then the number of integral points of bounded height $N_i(B) = \#\{x \in \mathcal{U}_i(\mathbb{Z}) \cap V_i(\mathbb{Q}) \mid H(x) \leq B\}$ satisfies the asymptotic formula*

$$N_i(B) = c_{i,\mathrm{fin}} c_{i,\infty} B \log B(1 + o(1)),$$

where

$$c_{i,\mathrm{fin}} = \prod_p \left(1 - \frac{1}{p}\right)^{\mathrm{rk}\,\mathrm{Pic}\,X} \tau_{U,p}(\mathcal{U}_i(\mathbb{Z}_p)),$$
$$c_{i,\infty} = \alpha_{D_i} \tau_{D_i,\infty}(D_i(\mathbb{R})),$$

and all constants are associated with the unique maximal face D_i of the Clemens complex. Moreover, the exponent $1 = b_{D_i} - 1 = b'_{D_i} - 1$ agrees with the definitions in the previous chapter. More explicitly, we have

$$N_1(B) = \frac{20}{3\zeta(2)} B \log B + O(B) \quad \textit{and}$$
$$N_2(B) = \frac{20}{3} \prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) B \log B + O(B(\log\log B)^2).$$

3.2 A universal torsor

The Cox ring of X over $\overline{\mathbb{Q}}$ is

$$R(X_{\overline{\mathbb{Q}}}) = \bigoplus_{d \in \mathrm{Pic}(X_{\overline{\mathbb{Q}}})} H^0(X, \mathcal{L}_d),$$

where $(\mathcal{L}_d)_d$ is a suitable system of representatives of every class in the geometric Picard group; its ring structure is induced by the sum and tensor product of sections. By [DHH$^+$15, Theorem 4.5, Case 30] (which contains a typo in the degrees of x and y), it is

$$R(X_{\overline{\mathbb{Q}}})] = \overline{\mathbb{Q}}[a, b, c, x, y, z]/(a^2 + bc - yz),$$

and its grading by $\mathrm{Pic}(X_{\overline{\mathbb{Q}}}) \cong \mathrm{Pic}(X) \cong \mathbb{Z}^2$ is

a	b	c	x	y	z
1	1	1	1	2	0
0	0	0	−1	−1	1

The pullbacks of planes along π correspond to degree $[1,0]$, the exceptional divisor E to degree $[0,1]$, and the anticanonical bundle thus to degree $[4,-1]$.

Lemma 3.2.1. *The variety*

$$T_{\overline{\mathbb{Q}}} = \mathrm{Spec}\, R(X_{\overline{\mathbb{Q}}}) - V(I_{\mathrm{irr}}),$$

where $I_{\mathrm{irr}} = (a, b, c, z)(x, y)$, is a universal torsor over $X_{\overline{\mathbb{Q}}}$.

Proof. In addition to the ring itself, we argue using the bunch of cones Φ associated with X [ADHL15, 3.2]. It consists of all cones $\mathrm{Cone}(\{\deg(t) \mid t \in M\})$ generated by the degrees $\deg(t) \in \mathrm{Pic}(X)_{\mathbb{Q}}$ of a subset $M \subset \{a, b, c, x, y, z\}$ of the generators satisfiying the following: We have $\prod_{t \in M} t \notin \sqrt{(t \mid t \notin M)}$, that is, the equation $a^2 + bc - yz$ has a solution with $t = 0$ for $t \notin M$ and $t \neq 0$ for $t \in M$; and we have $\omega^\vee \in \mathrm{Cone}(\{\deg(t) \mid t \in M\})$. The bunch of cones is thus

$$\Phi = \left\{\mathrm{Cone}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\right), \mathrm{Cone}\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\right), \mathrm{Cone}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\-1\end{smallmatrix}\right]\right), \mathrm{Cone}\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\-1\end{smallmatrix}\right]\right)\right\},$$

given by, for example, the generators $\{b, x\}$, $\{z, x\}$, $\{a, y\}$, and $\{a, y, z\}$, respectively (these are all possible cones containing the anticanonical bundle); the condition is seen to hold by considering the solution $(0, 1, 0, 1, 0, 0)$, $(0, 0, 0, 1, 0, 1)$, $(0, 1, 0, 0, 1, 0)$, or $(1, 0, 0, 0, 1, 1)$, respectively. Indeed, by [ADHL15, Theorem 3.2.1.9 (ii)], X is defined by a bunched ring with a maximal bunch, which can only be the bunch Φ just defined.

The irrelevant ideal I_{irr} is generated by all elements of the form $\prod_{t \in M} t$ such that M is a subset of the generators satisfying $\mathrm{Cone}(\deg(t) \mid t \in M) \in \Phi$. This yields

$$I_{\mathrm{irr}} = (ax, bx, cx, zx, ay, by, cy, zy) = (a, b, c, z)(x, y)$$

since the minimal subsets suffice. □

Denote by $p\colon T \to X$ a morphism rendering T a universal torsor. We note that the composition of morphisms $T \to X \to \mathbb{P}^3$ maps $(a, b, c, x, y, z) \to (a : b : c : xz)$, that $V(x) \subset T$ is the preimage of the strict transform of $V(c) \subset \mathbb{P}^3$, the subvariety $V(y) \subset T$ is the preimage of the strict transform of $V(a^2 + bc)$, and that $V(z)$ is the preimage of the exceptional divisor $E \subset X$. Next, we construct an integral model of this torsor. Consider the ring

$$R_{\mathbb{Z}} = \mathbb{Z}[a, b, c, x, y, z]/(a^2 + bc - yz)$$

and the ideal $I_{\mathrm{irr},\mathbb{Z}} = (a, b, c, z)(x, y) \subset R_{\mathbb{Z}}$.

Lemma 3.2.2. *The scheme $\mathcal{T} = \mathrm{Spec}\, R_{\mathbb{Z}} - V(I_{\mathrm{irr},\mathbb{Z}})$ is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^2$-torsor over $\mathcal{X}$.*

Proof. We note that removing yz from the generators of I_{irr} does not change the radical of the ideal and that the degrees of the two factors of any of the remaining generators $f_i \in \{ax, bx, cx, zx, ay, by, cy\}$ form a basis of the Picard group. Thus, [FP16, Theorem 3.3] shows that $\mathcal{T} = \mathrm{Spec}\, R_{\mathbb{Z}} - V(I_{\mathrm{irr},\mathbb{Z}})$ is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^2$-torsor over the $\mathbb{Z}$-scheme $\mathcal{X}'$ obtained by gluing the spectra of the degree-0-parts $R_{\mathbb{Z}}[f_i^{-1}]^{(0)}$ of the localizations in the generators f_i of the irrelevant ideal.

This integral model $\mathcal{X}'$ of $X_{\overline{\mathbb{Q}}}$ coincides with the blow-up $\mathcal{X}$. Indeed, we can embed both the Cox ring $R_{\mathbb{Z}}$ and the Rees algebra

$$A = \bigoplus_{n \geq 0} I^n = \mathbb{Z}[a, b, c, d][(a^2 + bc)\xi, d\xi]$$

for $I = (a^2 + bc, d)$ into the field $\mathbb{Q}(a, b, c, d, \xi) = \mathrm{Frac}(A)$, where the first embedding maps $z \mapsto \xi^{-1}$, $x \mapsto d\xi$, and $y \mapsto (a^2 + bc)\xi$. The blow-up is then given by gluing the spectra of the seven rings $A_{s,t} \subset \mathrm{Frac}(A)$ arising the following way: First take the degree-0-part (with respect to the usual grading of $\mathbb{Z}[a, b, c, d]$, not considering the natural grading of the Rees algebra) of the localizations of A in $s \in \{a, b, c, d\}$, then further localize in one of the generators $t \in \{\frac{a^2+bc}{s^2}\xi,\ \frac{d}{s}\xi\}$ ($t = \xi$ suffices for $s = d$) of the Rees algebra and take the degree-0-part with respect to the grading induced by the natural grading of the Rees algebra. The rings $R_{\mathbb{Z}}[f_i^{-1}]^{(0)}$ for $f_i = ax, bx, cx, zx, ay, by, cy$ coincide with the rings $A_{s,t}$ for

$$(s,t) = (a, \frac{d}{a}\xi), (b, \frac{d}{b}\xi), (c, \frac{d}{c}\xi), (d, \xi), (a, (a^2+bc)\xi), (b, (a^2+bc)\xi), (c, (a^2+bc)\xi),$$

so the two schemes defined by the blow-up and [FP16, Construction 3.1] coincide. $\square$

Lemma 3.2.3. *The morphism p induces a 4-to-1-correspondence between integral points on $\mathcal{X}$ and*

$$\mathcal{T}(\mathbb{Z}) = \left\{ (a,b,c,x,y,z) \in \mathbb{Z}^6 \;\middle|\; {\scriptstyle a^2+bc-yz=0 \atop \gcd(a,b,c,z)=\gcd(x,y)=1} \right\}, \tag{3.1}$$

between integral points on $\mathcal{U}_1$ and

$$\mathcal{T}_1(\mathbb{Z}) = \left\{ (a,b,c,x,y,z) \in \mathbb{Z}^6 \;\middle|\; {\scriptstyle a^2+bc-yz=0 \atop b=\pm1,\gcd(x,y)=1} \right\}, \tag{3.2}$$

and between integral points on $\mathcal{U}_2$ and

$$\mathcal{T}_2(\mathbb{Z}) = \left\{ (a,b,c,x,y,z) \in \mathbb{Z}^6 \;\middle|\; {\scriptstyle a^2+bc-yz=0 \atop a=\pm1,\gcd(x,y)=1} \right\}. \tag{3.3}$$

Proof. The fiber $f^{-1}(P)$ of any point $P \in \mathcal{X}(\mathbb{Z})$ is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^2$-torsor. Since such torsors are parametrized by $H^2_{\mathrm{fppf}}(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_{\mathrm{m}}^2) = \mathrm{Cl}(\mathbb{Z})^2 = 1$, all fibers are isomorphic to $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^2$, and we get a 4-to-1-correspondence between integral points on the torsor $\mathcal{T}$ and those on $\mathcal{X}$.

Since $\mathcal{T}$ is quasi-affine, its integral points have a description as lattice points satisfying the equation of the Cox ring and coprimality conditions given by the irrelevant ideal. Points on the preimages of $\mathcal{U}_1$ and $\mathcal{U}_2$ under the morphism $p\colon \mathcal{T} \to \mathcal{X}$ are defined by the additional condition $(b) = 1$ and $(a) = 1$, respectively. $\square$

We conclude this section with some observations on the geometry of X.

Lemma 3.2.4. *There is no action of $\mathbb{G}_{\mathrm{a}}^3$ on X with an open orbit under which D_1 or D_2 are invariant, and neither is X toric.*

Proof. Since $\mathbb{G}_{\mathrm{a}}^3$ has to act continuously on $\mathrm{Pic}(X)$, the exceptional divisor has to be invariant and we thus get an action on $\mathbb{P}^3 - C$. If one of the planes not containing C is invariant, the action further restricts to the complement $\mathbb{A}^3 - C$ of a conic in $\mathbb{A}^3$. Since the action needs to have an open orbit, we would get an open immersion $\mathbb{A}^3 \to \mathbb{A}^3 - C$, an impossibility by Ax–Grothendieck.

Since its Cox ring is not polynomial, X cannot be toric, cf. [HK00]. $\square$

Remark 3.2.5. The total variety X is a compactification of $\mathbb{G}_{\mathrm{a}}^3$, as classified by [HM18] (induced by the action of $\mathbb{G}_{\mathrm{a}}^3$ on $\mathbb{P}^3$, where the group acts trivially on the plane $V(d)$ and by addition on the complement). Manin's conjecture for rational points [CLT02] and asymptotics for integral points on some open subvaries [CLT12] are known due to Chambert-Loir and Tschinkel: The admissible divisors D are the exceptional divisor, the strict transform of $V(d)$, and their sum. Even though X is an equivariant compactification of $\mathbb{G}_\mathrm{a}^3$, the pairs (X, D_i) are neither partial equivariant compactifications of $\mathbb{G}_\mathrm{a}^3$ nor toric by the previous lemma. Our result is thus not a special case of [CLT10b] or [CLT12].

Lastly, we can describe the geometric Picard group with the information we gathered in the proof of Lemma 3.2.1: The pseudo-effective cone is generated by the degrees of the generators of the Cox ring, so $\overline{\mathrm{Eff}}(X) = \mathrm{Cone}(E, H - E)$. The semiample cone is the intersection of all cones in Φ and thus $\mathrm{SAmple}(X) = \mathrm{Cone}(H, 2H - E)$. In particular, the log-anticanonical bundles

$$\omega(D_1)^\vee \cong \omega(D_2)^\vee \cong \mathcal{O}_X(3H - E)$$

are in its interior, and thus ample.

3.3 Metrics, a height function, and Tamagawa measures

Adelic metrics

We endow certain line bundles on X with adelic metrics. For fixed $d \in \mathrm{Pic}(X)$, the elements of degree d in the Cox rings are the global sections of a line bundle $\mathcal{L}_d$ with isomorphism class d (such that $\mathcal{L}_d \otimes \mathcal{L}_e = \mathcal{L}_{d+e}$ by the construction of the Cox ring). We consider the bundles $\mathcal{L}_{[3,-1]}$ and $\mathcal{L}_{[1,0]}$ that are isomorphic to the log-anticanonical bundles $\omega(D_1)^\vee \cong \omega(D_2)^\vee$ and the pullback of the tautological bundle $\pi^*\mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$. Neither of the sets

$$\{a^2x, b^2x, c^2x, z^2x^3, ay, by, cy\} \quad \text{and} \quad \{a, b, c, xz\}$$

of sections of these bundles can vanish simultaneously, so (2.1) gives us the metrics

$$(s, (a : b : c : x : y : z)) \mapsto \frac{|s(a, b, c, x, y, z)|_v}{\max\{|a^2x|_v, |b^2x|_v, |c^2x|_v, |z^2x^3|_v, |ay|_v, |by|_v, |cy|_v\}} \tag{3.4}$$

on $\mathcal{L}_{[3,-1]}$ and

$$(t, (a : b : c : x : y : z)) \mapsto \frac{|t(a, b, c, x, y, z)|_v}{\max\{|a|_v, |b|_v, |c|_v, |xz|_v\}} \tag{3.5}$$

on $\mathcal{L}_{[1,0]}$, where $(a:b:c:x:y:z)$ is the image of $(a, b, c, x, y, z) \in T(\mathbb{Q}_v)$, that is, a point in *Cox coordinates*) in $X(\mathbb{Q}_v)$, $s \in R(X)$ has degree $[3, -1]$, and $t \in R(X)$ has degree $[1, 0]$.

A height function

The chosen metric on $\mathcal{L}_{[3,-1]}$ defines a log-anticanonical height function on X, which we can easily describe in Cox coordinates: Since X is proper, every rational point in $X(\mathbb{Q})$ extends to a unique integral point in $\mathcal{X}(\mathbb{Z})$, which in turn corresponds to four integral points $(a, b, c, x, y, z) \in \mathcal{T}(\mathbb{Z})$ by Lemma 3.2.3.

By the coprimality condition and the equation, no prime can divide all of the monomials in the denominator of (3.4). Thus we get

$$H(a:b:c:x:y:z) = \max\{\left|a^2x\right|, \left|b^2x\right|, \left|c^2x\right|, \left|z^2x^3\right|, |ay|, |by|, |cy|\} \tag{3.6}$$

for the image $(a:b:c:x:y:z) \in X(\mathbb{Q})$ of $(a,b,c,x,y,z) \in \mathcal{T}(\mathbb{Z})$, with the usual real absolute value.

Tamagawa measures

To explicitly calculate Tamagawa volumes on our variety X, we need metrics on the bundles ω, $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$, not just on bundles isomorphic to them. To this end, we choose isomorphisms between those bundles and the bundles $\mathcal{L}_{[4,-1]}$ and $\mathcal{L}_{[1,0]}$, and identify sections corresponding under those isomorphisms. Up to scalar, the canonical section 1_{D_1} (resp. 1_{D_2}) is the unique section of $\mathcal{O}(D_1)$ (resp. $\mathcal{O}(D_2)$) corresponding to D_1 (resp. D_2). This also holds for the elements b (resp. a) of the degree-$[1,0]$-part of the Cox ring (regarded as the global sections of the bundle $\mathcal{L}_{[1,0]}$), so there are isomorphisms with $1_{D_1} \mapsto b$ and $1_{D_2} \mapsto a$, and we will use these. For the (anti-)canonical bundle, we consider the chart

$$f\colon V \to \mathbb{A}^3,\ (a:b:c:x:y:z) \mapsto \left(\frac{a}{xz}, \frac{b}{xz}, \frac{c}{xz}\right)$$

and its inverse

$$g\colon \mathbb{A}^3 \to V,\ (a,b,c) \mapsto (a:b:c:1:a^2+bc:1),$$

where $V = X - V(xz) = \pi^{-1}(V(d)) \cong \mathbb{A}^3$. The sections $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c$ and $\frac{\mathrm{d}}{\mathrm{d}a} \wedge \frac{\mathrm{d}}{\mathrm{d}b} \wedge \frac{\mathrm{d}}{\mathrm{d}c}$ of the canonical and anticanonical bundle have neither zeroes or poles on $\mathbb{A}^3 \cong V$, and their tensor product is 1. Up to scalar, they are the only sections with this property. Since the analogous property holds for $x^{-4}z^{-3}$ and x^4z^3, we can fix isomorphisms identifying $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c$ with $x^{-4}z^{-3}$ and $\frac{\mathrm{d}}{\mathrm{d}a} \wedge \frac{\mathrm{d}}{\mathrm{d}b} \wedge \frac{\mathrm{d}}{\mathrm{d}c}$ with x^4z^3.

Lemma 3.3.1. *For any prime p, we have*

$$\tau_{(X,D_1),p}(\mathcal{U}_1(\mathbb{Z}_p)) = 1 + \frac{1}{p} = \frac{\#\mathcal{U}_1(\mathbb{F}_p)}{p^3} \quad \text{and}$$
$$\tau_{(X,D_2),p}(\mathcal{U}_2(\mathbb{Z}_p)) = 1 + \frac{1}{p} - \frac{1}{p^2} = \frac{\#\mathcal{U}_2(\mathbb{F}_p)}{p^3}.$$

Proof. Under the above chart, the integral points $\mathcal{U}_1(\mathbb{Z}_p) \cap V(\mathbb{Q}_p)$ correspond to the set

$$\left\{\left(\frac{a}{d}, \frac{1}{d}, \frac{c}{d}\right) \;\middle|\; a,c,d \in \mathbb{Z}_p\right\} = \{(a,b,c) \in \mathbb{Q}_p^3 \mid |b| \geq 1, |a|, |c| \leq |b|\}$$

and, analogously, $\mathcal{U}_2(\mathbb{Z}_p) \cap V(\mathbb{Q}_p)$ corresponds to the set

$$\{(a,b,c) \in \mathbb{Q}_p^3 \mid |a| \geq 1, |b|, |c| \leq |a|\}.$$

On $\mathcal{U}_1(\mathbb{Z}_p) \cap V(\mathbb{Q}_p)$, we have $\|f_{D_1}\|_{\mathcal{O}(D_1)} = \|b\|_{\mathcal{O}(D_1)} = \frac{|b|}{\max\{|a|,|b|,|c|,|xz|\}} = 1$ and

$$\begin{aligned}
&\|\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c\|_{\omega_X} \\
&\quad = \frac{\max\{\left|a^2x\right|, \left|b^2x\right|, \left|c^2x\right|, \left|z^2x^3\right|, |ay|, |by|, |cy|\} \max\{|a|, |b|, |c|, |xz|\}}{\left|x^4z^3\right|} \\
&\quad = \max\{\left|b^2\right|, 1, \left|b(a^2+bc)\right|\} |b| .
\end{aligned}$$

This means that

$$\mathrm{d}f_*\tau_{(X,D_1),p} = (\max\{\left|b^2\right|, 1, \left|b(a^2+bc)\right|\})^{-1} |b| \,\mathrm{d}\mu_p$$

on $g^{-1}(\mathcal{U}_1(\mathbb{Z}_p))$ and, by an analogous argument, that

$$\mathrm{d}f_*\tau_{(X,D_2),p} = (\max\{\left|a^2\right|, 1, \left|a(a^2+bc)\right|\})^{-1} |a| \,\mathrm{d}\mu_p$$

on $g^{-1}(\mathcal{U}_2(\mathbb{Z}_p))$.

With these descriptions we can explicitly compute the volumes. In the first case, we get

$$\begin{aligned}
\tau_{(X,D_1),p}(\mathcal{U}_1(\mathbb{Z}_p)) &= \int_{\substack{|b|\geq 1\\ |a|,|c|\leq|b|}} \frac{1}{|b| \max\{|b^2|, |b(a^2+bc)|\}} \,\mathrm{d}(a,b,c) \\
&= \int_{\substack{\left|a^2+bc\right|\leq|b|\\ |b|\geq 1\\ |a|,|c|\leq|b|}} \frac{1}{|b|^3} \,\mathrm{d}(a,b,c) + \int_{\substack{\left|a^2+bc\right|>|b|\\ |b|\geq 1\\ |a|,|c|\leq|b|}} \frac{1}{|b|^2 \left|a^2+bc\right|} \,\mathrm{d}(a,b,c).
\end{aligned}$$

The first of these terms is

$$\begin{aligned}
&\int_{\substack{\left|\frac{a^2}{b}+c\right|\leq 1\\ |b|\geq 1, |a|\leq|b|}} \frac{1}{|b|^3} \,\mathrm{d}(a,b,c) = \int_{\substack{|b|\geq 1\\ |a|\leq|b|}} \frac{1}{|b|^3} \,\mathrm{d}(a,b) = \int_{|b|\geq 1} \frac{1}{|b|^2} \,\mathrm{d}b \\
&\quad = \sum_{k\geq 0} \frac{1}{p^{2k}} \left(1-\frac{1}{p}\right) p^k = 1,
\end{aligned}$$

while the second is

$$\begin{aligned}
&\int_{\substack{1<\left|\frac{a^2}{b}+c\right|\leq|b|\\ |b|\geq 1, |a|\leq|b|}} \frac{1}{|b|^3 \left(\frac{a^2}{b}+c\right)} \,\mathrm{d}(a,b,c) = \int_{\substack{|b|\geq 1\\ |a|\leq|b|}} \frac{1}{|b|^3} \sum_{k=1}^{|v(b)|} \frac{1}{p^k} \left(1-\frac{1}{p}\right) p^k \,\mathrm{d}(a,b) \\
&\quad = \left(1-\frac{1}{p}\right) \int_{\substack{|b|\geq 1\\ |a|\leq|b|}} \frac{|v(b)|}{|b|^3} \,\mathrm{d}(a,b) = \left(1-\frac{1}{p}\right) \int_{|b|\geq 1} \frac{|v(b)|}{|b|^2} \,\mathrm{d}b \\
&\quad = \left(1-\frac{1}{p}\right) \sum_{k\geq 0} \frac{k}{p^{2k}} \left(1-\frac{1}{p}\right) p^k = \frac{1}{p},
\end{aligned}$$

so $\tau_p(\mathcal{U}_1(\mathbb{Z}_p)) = 1 + \frac{1}{p} = \frac{\#\mathcal{U}_1(\mathbb{F}_p)}{p^3}$. The volume $\tau_{(X,D_2),p}(\mathcal{U}_2(\mathbb{Z}_p))$ is calculated similarly. □

In both cases, the Clemens complex of D_i is simply a vertex, and we are interested in the residue measure τ_{D_i}, as well as the constant α_{D_i} associated with its unique maximal face D_i.

Lemma 3.3.2. *We have $\tau_{D_1,\infty}(D_1(\mathbb{R})) = \tau_{D_2,\infty}(D_2(\mathbb{R})) = 40$.*

Proof. The adjunction isomorphism induces a metrization of ω_{D_1} via

$$\|\mathrm{d}a \wedge \mathrm{d}c\|_{\omega_{D_1}} = \|\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c\|_{\omega_X} \|b\|^{-1}_{\mathcal{O}(-D_1)}. \tag{3.7}$$

Since $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c$ corresponds to $x^4z^3 \in R(X)$, the first factor of (3.7) is

$$\begin{aligned}&\frac{\max\{\left|a^2x\right|, \left|b^2x\right|, \left|c^2x\right|, \left|z^2x^3\right|, |ay|, |by|, |cy|\}\max\{|a|, |b|, |c|, |xz|\}}{|x^4z^3|}\\ &= \max\{\left|a^2\right|, \left|c^2\right|, 1, \left|a^3\right|, \left|a^2c\right|\}\max\{|a|, |c|, 1\},\end{aligned}$$

when evaluated in $(a : 0 : c : 1 : a^2 : 1) \in V \cap D_1$. On the affine variety V, regarding b as an element of $\Gamma(V, \mathcal{O}_V(-D_1)) \subset \mathcal{O}_V(V)$, using the canonical trivialization of $\mathcal{O}(-D_1)$ outside D_1 and the fact that 1_{D_1} corresponds to $b \in R(X)$ under our chosen isomorphism, we get

$$\begin{aligned}&\lim_{b\to 0}\left(\frac{|b|}{\|1_{D_1}\|_{\mathcal{O}(D_1)}}\right)^{-1} = \lim_{b\to 0}\left(\frac{|b|\max\{|a|, |b|, |c|, |xz|\}}{|b|}\right)^{-1}\\ &= \max\{|a|, |c|, 1\}^{-1}\end{aligned}$$

for the second factor. We thus have explicit descriptions

$$\mathrm{d}f_*\tau'_{D_1,\infty} = \|\mathrm{d}a \wedge \mathrm{d}c\|^{-1}_{\omega_{D_1}} \mathrm{d}(a,c) = \frac{1}{\max\{|a^2|, |c^2|, 1, |a^3|, |a^2c|\}} \mathrm{d}(a,c)$$

and, by an analogous argument,

$$\mathrm{d}f_*\tau'_{D_2,\infty}\,\mathrm{d}(b,c) = \frac{1}{\max\{|b^2|, |c^2|, 1, |b^2c|, |bc^2|\}} \mathrm{d}(b,c)$$

of the unnormalized Tamagawa measures $\tau'_{D_1,\infty}$ and $\tau'_{D_2,\infty}$ with respect to the Lebesgue measure. For the volume of the first divisor, we now get

$$\begin{aligned}\tau'_{D_1,\infty}(D_1(\mathbb{R})) &= \int_{|a|,|a^2c|\le 1} \frac{1}{\max\{|c^2|, 1\}} \mathrm{d}(a,c) + \int_{\substack{|a|\ge|c|\\ |a|>1}} \frac{1}{|a^3|} \mathrm{d}(a,c)\\ &+ \int_{\substack{|c|>|a|\\ \left|a^2c\right|>1}} \frac{1}{\max\{|c^2|, |a^2c|\}} \mathrm{d}(a,c).\end{aligned}$$

The first term of this expression is $\frac{20}{3}$ by (3.10) below, the second is $\int_{|a|} \frac{2}{|a^2|} = 4$ and the third is

$$\int_{\substack{|c|>|a|,\left|a^2c\right|>1\\ \left|a^2\right|>|c|}} \frac{1}{|a^2c|} \mathrm{d}(a,c) + \int_{\substack{|c|>a,\left|a^2c\right|>1\\ \left|a^2\right|\le|c|}} \frac{1}{|c^2|} \mathrm{d}(a,c). \tag{3.8}$$

In (3.8), the first term is

$$\int_{\substack{|c|\ge 1\\ |c|^{1/2}<|a|<|c|}} \frac{1}{|a^2c|} \mathrm{d}(a,c) = \int_{|c|\ge 1} \frac{2}{|c|}(|c|^{1/2} - |c|^{-1})\,\mathrm{d}c = 4$$

and the second is

$$\int_{a\in\mathbb{R}} \frac{2}{\max\{|a^2|, |a^{-2}|\}} \,\mathrm{d}a = \int_{|a|\le 1} 2\,|a|^2 \,\mathrm{d}a + \int_{|a|>1} \frac{2}{|a^2|} \,\mathrm{d}a = \frac{16}{3}.$$

Thus, (3.8) is $\frac{28}{3}$ and $\tau'_{D_1,\infty}(D_1(\mathbb{R})) = \frac{20}{3} + 4 + \frac{28}{3} = 20$.

For the other divisor, we get $\tau'_{D_2,\infty}(D_2(\mathbb{R})) = 20$ by similar arguments, and have to normalize both volumes by multiplying with $c_{\mathbb{R}} = 2$. $\square$

In both cases, we have $U_{D_i} = X$ for the unique maximal face D_i of the Clemens complex, so $\mathrm{Pic}(U; D_i) = \mathrm{Pic}(X)$. Hence $b_{D_i} = 2 = b'_{D_i}$. In order to compute α, we consider the pseudo-effective cone $\overline{\mathrm{Eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$ and its characteristic function $\mathcal{X}_{\overline{\mathrm{Eff}}(X)}(\mathcal{L})$. Since the cone is smooth and generated by E and $H - E$, and $\omega(D_i)^\vee \cong \mathcal{O}_X(3(H-E) + 2E)$ for both $i = 1$ and $i = 2$, we get

$$\alpha_{D_i} = \frac{1}{(b-1)!}\mathcal{X}_{\overline{\mathrm{Eff}}(X)}(\omega(D_i)^\vee) = \frac{1}{6}$$

for both divisors D_1 and D_2.

3.4 Integral points on $X - D_1$

We study the number

$$N_1(B) = \#\{x \in \mathcal{U}_1(\mathbb{Z}) \cap V_1(\mathbb{Q}) \mid H(x) \le B\}$$

of integral points of bounded height on $\mathcal{U}_1 = \mathcal{X} - V(b)$ that, as rational points, are in the complement V_1 of the subvariety $V(abxz) = \pi^{-1}(V(abd))$.

Using the 4-to-1-correspondence (3.2) with integral points on the universal torsor $\mathcal{T}_1$ and noticing the symmetry in the two values ± 1 of b in (3.2), this description of integral points on the universal torsor yields the formula

$$N_1(B) = \frac{1}{2}\,\#\left\{(a,c,x,y,z) \in \mathbb{Z}^5 \;\middle|\; \substack{a^2+c-yz=0,\ \gcd(x,y)=1,\\ H(a,1,c,x,y,z)\le B,\ a,x,z\ne 0}\right\},$$

where

$$H(a,b,c,x,y,z) = \max\left\{\left|a^2x\right|, \left|b^2x\right|, \left|c^2x\right|, \left|z^2x^3\right|, |ay|, |by|, |cy|\right\}$$

by (3.6). Solving the equation, we can simplify this to

$$\frac{1}{2}\,\#\left\{(a,x,y,z) \in \mathbb{Z}^4 \;\middle|\; \substack{\gcd(x,y)=1,\ \widetilde{H}_1(a,x,y,z)\le B,\\ a,x,z\ne 0}\right\},$$

where

$$\begin{aligned}\widetilde{H}_1(a,x,y,z) &= H(a,1,yz-a^2,x,y,z)\\ &= \max\{\left|a^2x\right|, |x|, \left|(yz-a^2)^2x\right|, \left|z^2x^3\right|, |ay|, |y|, \left|(yz-a^2)y\right|\}.\end{aligned}$$

Lemma 3.4.1. *We have*

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha}\sum_{x',z\in\mathbb{Z}_{\ne 0}}\int_{\substack{|a^2\alpha x'|,|a(a^2+c)z^{-1}|,\\ |c^2\alpha x'|,|c(a^2+c)z^{-1}|,\\ |\alpha^3x^3z^2|\le B,|a|\ge 1}} \frac{1}{|z|}\,\mathrm{d}(a,c) + O(B).$$

Proof. A Möbius inversion yields

$$N_1(B) = \sum_{\alpha>0} \mu(\alpha) \sum_{a,x',z\in\mathbb{Z}_{\neq 0}} \#\{y' \in \mathbb{Z} \mid \widetilde{H}_1(a, \alpha x', \alpha y', z) \le B\},$$

and we can estimate the number of points by the volume

$$\#\{y \in \mathbb{Z} \mid \widetilde{H}_1(a, \alpha x', \alpha y', z) \le B\} = \int_{\widetilde{H}_1(a,\alpha x',\alpha y',z)\le B} \mathrm{d}y' + R(\alpha, a, x', z; B),$$

where we denote the integral by $V_1(\alpha, a, x', z; B)$. Since all integrands f and the regions defined by height functions are semialgebraic, we get $|R(\alpha, a, x', z; B)| \ll 1$ and similar error terms of the form $C \max_\xi |f|$ when replacing the sum over a variable ξ by an integral in the next steps using [DF14, Lemma 3.6]. Using this and the height conditions $\left|\alpha a^2 x'\right|, \left|\alpha^3 z^2 x'^3\right| < B$, we can bound the sum over the error terms by

$$\sum_{\substack{\alpha>0,\\ a,x',z\in\mathbb{Z}_{\neq 0}}} |\mu(\alpha) R(\alpha, a, x', z; B)| \ll \sum_{\substack{\alpha>0,\\ x'\in\mathbb{Z}_{\neq 0}}} \frac{B}{|\alpha x'|^2} \ll B,$$

and get $N_1(B) = \sum_\alpha \mu(\alpha) \sum_{a,x',z} \int_{\widetilde{H}_1(a,\alpha x',\alpha y',z)\le B} \mathrm{d}y' + O(B)$. Turning to the variable a next we estimate the sum $\sum_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B)$ by the integral $V_2(\alpha, x', z; B) = \int V_1(\alpha, a, x', z; B)\, \mathrm{d}a$, introducing an error term $R_2(\alpha, x', z; B)$. Using the height condition $\left|\alpha y' z - a^2\right| \le B$ to estimate the integral over y' and $\alpha^3 x^3 z^2$ to estimate the sum, we can bound the total error by

$$\begin{aligned} \sum_{\substack{\alpha>0,\\ x',z\in\mathbb{Z}_{\neq 0}}} |R_2(\alpha, x', z; B)| &\ll \sum_{\substack{\alpha>0,\\ x',z\in\mathbb{Z}_{\neq 0}}} \sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B) \\ &\ll \sum_{\substack{\alpha>0,\\ x',z\in\mathbb{Z}_{\neq 0}}} \frac{B^{1/2}}{\alpha\, |x'|^{1/2}\, |z|} \ll \sum_{\substack{\alpha>0,\\ z\in\mathbb{Z}_{\neq 0}}} \frac{B^{2/3}}{\alpha^{3/2}\, |z|^{4/3}} \ll B^{2/3}. \end{aligned}$$

A change of variables $c = \alpha y' z - a^2$ now gives us the description

$$V_2(\alpha, x', z; B) = \int_{|a|\ge 1} V_1(\alpha, a, x', z; B)\, \mathrm{d}a = \int_{\substack{\left|a^2\alpha x'\right|, \left|a(a^2+c)z^{-1}\right|,\\ \left|c^2\alpha x'\right|, \left|c(a^2+c)z^{-1}\right|,\\ \left|\alpha^3 x^3 z^2\right| \le B, |a|\ge 1}} \frac{1}{|\alpha z|}\, \mathrm{d}(a, c)$$

of the main term. □

Lemma 3.4.2. *We have*

$$N_1(B) = \frac{1}{2} \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2} \int_{\substack{\left|a^2 x\right|, \left|a^3 z^{-1}\right|, \left|c^2 x\right|,\\ \left|a^2 c z^{-1}\right|, \left|x^3 z^2\right| \le B,\\ |a|,|z|\ge 1, |x|\ge\alpha}} \frac{1}{|z|}\, \mathrm{d}(a, c, x, z) + O(B). \tag{3.9}$$

Proof. We first want to replace the two instances of a^2+c by a^2 in the inequalities defining the region for the volume function V_2 of the previous lemma, to get a new volume function $V_2'(\alpha, x', z; B)$. The error we introduce when replacing $\left|a(a^2+c)z^{-1}\right|$ by $\left|a^3 z^{-1}\right|$ is bounded by the integral over the region

$$B - \left|\frac{ac}{z}\right| \leq \left|\frac{a^3}{z}\right| \leq B + \left|\frac{ac}{z}\right|, \qquad \text{i.e.,} \qquad \left|a^2 - \frac{B\,|z|}{|a|}\right| \leq |c|\,.$$

With a change of variables $a' = a^2 - B\,|z|\,|a|^{-1}$, where

$$\left|\frac{\mathrm{d}a'}{\mathrm{d}a}\right| = 2\,|a| + \frac{B\,|z|}{|a|^2} \geq \sqrt{|a'|},$$

we get a bound for the total error:

$$\begin{aligned}
|R_1(B)| &\ll \sum_{\alpha>0} \frac{2}{\alpha} \sum_{x',z\in\mathbb{Z}_{\neq 0}} \int_{\substack{|a'|\leq|c|,|\alpha c^2 x'|,\\ |\alpha^3 x'^3 z^2|\leq B}} \frac{1}{\sqrt{|a'|}\,|z|}\,\mathrm{d}(a',c)\\
&\ll \sum_{\alpha>0} \frac{1}{\alpha} \sum_{x',z\in\mathbb{Z}_{\neq 0}} \int_{\substack{|\alpha^3 x'^3 z^2|,\\ |\alpha c^2 x'|\leq B}} \frac{\sqrt{|c|}}{|z|}\,\mathrm{d}c \ll \sum_{\alpha>0} \frac{1}{\alpha} \sum_{\substack{x',z\in\mathbb{Z}_{\neq 0}\\ |\alpha^3 x'^3 z^2|\leq B}} \frac{B^{3/4}}{\alpha^{3/4}\,|x|^{3/4}\,|z|}\\
&\ll \sum_{\substack{\alpha>0\\ z\in\mathbb{Z}_{\neq 0}}} \frac{B^{5/6}}{\alpha^{7/4}\,|z|^{7/6}} \ll B^{5/6}.
\end{aligned}$$

When modifying the other inequality, the error we introduce is bounded by an integral over a similar region, and, after an analogous change of variables, we get the same bound.

Next, we estimate the summation over z. Using the height conditions $|a| \leq B^{1/3}\,|z|^{1/3}$ and $|c| \leq B^{1/2}\,|\alpha x|^{-1/2}$, we can bound the volume

$$V_2'(\alpha, x', z; B) \ll B^{5/6}\,|\alpha x|^{-1/2}\,|z|^{-2/3}\,.$$

Replacing the sum over z by an integral, we introduce an error

$$|R_2(B)| \ll \sum_{\alpha>0} \frac{1}{\alpha^{3/2}} \sum_{1\leq|x'|\leq B^{1/3}} \frac{B^{5/6}}{|x|^{1/2}} \ll B.$$

For $V_3(\alpha, x'; B) = \int_{|z|\geq 1} V_2'(a, x', z; B)\,\mathrm{d}z$, we get an upper bound

$$V_3(\alpha, x'; B) \ll \int_{|\alpha^3 x'^3 z^2|\leq B} \frac{B^{5/6}}{\alpha^{1/2}\,|x'|^{1/2}\,|z|^{2/3}}\,\mathrm{d}z \ll \frac{B}{\alpha\,|x'|}.$$

Finally, replacing the sum over x' by an integral $\int_{|x'|\geq 1} V_3(\alpha, x'; B)$ introduces an error term

$$|R_3(B)| \ll \sum_{\alpha>0} \frac{B}{\alpha^2} \ll B,$$

and a change of variables $x = \alpha x'$ completes the proof. □

Proposition 3.4.3. *The number of integral points of bounded height on $\mathcal{U}_2$ satisfies the asymptotic formula*

$$N_1(B) = \frac{20}{3\zeta(2)} B \log B + O(B).$$

Proof. We first remove the condition $|a| \geq 1$ in (3.9) and get an error term

$$|R_1(B)| \ll \sum_{\alpha} \frac{1}{\alpha^2} \int_{\substack{|c^2 x|,|x^3 z^2| \leq B,\\ |x| \geq \alpha}} \mathrm{d}(c,x,z) \ll \sum_{\alpha} \frac{1}{\alpha^2} \int_{|x|\geq\alpha} \frac{B}{|x|^2}\,\mathrm{d}x$$
$$\ll \sum_{\alpha} \frac{B}{\alpha^3} \ll B.$$

By a change of variables $a \mapsto az^{-1/3}B^{-1/3}$, $c \mapsto cz^{-1/3}B^{-1/3}$, $x \mapsto az^{2/3}B^{-1/3}$, we now have

$$N_1(B) = \frac{1}{2} \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2} \int_{\substack{|a^2 x|,|c^2 x|,|a|,\\ |a^2 c|,|x|\leq 1,\\ 1\leq |z| \leq |x|^{3/2} B^{1/2} \alpha^{-3/2}}} \frac{B}{|z|}\,\mathrm{d}(a,c,x,z) + O(B)$$
$$= \frac{1}{2} \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2} \int_{\substack{|a^2 x|,|c^2 x|,|a|,\\ |a^2 c|,|x|\leq 1}} B\left(\log\left(B\,|x|^3\,\alpha^{-3}\right)\right)\mathrm{d}(a,c,x) + R_2(B) + O(B)$$
$$= \frac{1}{2} \sum_{\alpha>0} \frac{\mu(\alpha) B\log B}{\alpha^2} \int_{\substack{|a^2 x|,|c^2 x|,|a|,\\ |a^2 c|,|x|\leq 1}} \mathrm{d}(a,c,x) + R_3(B) + O(B)$$
$$= \frac{B\log B}{\zeta(2)} \int_{|a|,|a^2 c|\leq 1} \frac{1}{\max\{1,|a^2|,|c^2|\}}\,\mathrm{d}(a,c) + O(B)$$

since the error terms are

$$|R_2(B)| \ll \sum_{\alpha\geq 1} \frac{B}{\alpha^2} \int_{\substack{|x|^{3/2}\alpha^{-3/2}B^{1/2},\\ |a|,|c^2 x|\leq 1}} \left|\log\left(B\,|x|^3\,\alpha^{-3}\right)\right| \mathrm{d}(a,c,x)$$
$$\ll \sum_{\alpha\geq 1} \frac{B}{\alpha^2} \int_{|x|\leq \alpha B^{-1/3}} 3\left|\log\left(B^{1/3}\,|x|\,\alpha^{-1}\right)\right| \frac{1}{|x|^{1/2}}\,\mathrm{d}x$$
$$\ll \sum_{\alpha\geq 1} \frac{B}{\alpha^2} \frac{\alpha^{1/2}}{B^{1/6}} \left(2 - \log\left(B^{1/3}\alpha^{-1}B^{-1/3}\alpha\right)\right) \ll B^{5/6}$$

and

$$|R_3(B)| \ll \sum_{\alpha\geq 1} \frac{B}{\alpha^2} \int_{|a|,|c^2 x|,|x|\leq 1} 3\left|\log\left(\frac{|x|}{\alpha}\right)\right| \mathrm{d}(a,c,x)$$
$$\ll \sum_{\alpha\geq 1} \frac{B}{\alpha^2} \int_{|x|\leq 1} \frac{1}{|x|^{1/2}} \left|\log\left(\frac{|x|}{\alpha}\right)\right| \mathrm{d}x \ll \sum_{\alpha\geq 1} \frac{B}{\alpha^2}(2+\log(\alpha)) \ll B.$$

Finally, we note that the integral evaluates to

$$\int_{|a|,|a^2 c|\leq 1} \frac{1}{\max\{1,|c^2|\}}\,\mathrm{d}(a,c) = \int_{c\in\mathbb{R}} \frac{2\min\{1,|c|^{-1/2}\}}{\max\{|c^2|,1\}}\,\mathrm{d}c$$
$$= \int_{|c|\leq 1} 2\,\mathrm{d}c + \int_{|x|>1} \frac{2}{|c|^{5/2}}\,\mathrm{d}c = \frac{20}{3}, \tag{3.10}$$

and arrive at the asymptotic expression. □

3.5 Integral points on $X - D_2$

We count the number

$$N_2(B) = \#\{x \in \mathcal{U}_1(\mathbb{Z}) \cap V_2(\mathbb{Q}) \mid H(x) \le B\}$$

of integral points of bounded height on $\mathcal{U}_2 = \mathcal{X} - \overline{V(a)}$, that, as rational points, are in the complement V_2 of $V(abcxz) = \pi^{-1}(V(abcd))$. With the 4-to-1-correspondence to integral points on the torsor, and noticing the symmetry in the two possible values $a = \pm 1$ of a in (3.3), we get

$$N_2(B) = \frac{1}{2}\,\#\left\{(b,c,x,y,z) \in \mathbb{Z}^5 \;\middle|\; \substack{1+bc-yz=0,\ \gcd(x,y)=1,\\ H(1,b,c,x,y,z)\le B,\\ b,c,x,z\neq 0}\right\}. \tag{3.11}$$

Lemma 3.5.1. *We have*

$$N_2(B) = \sum_{b,x,z\in\mathbb{Z}_{\neq 0}} \theta_1(b,x,z)V_1(b,x,z;B) + O(B),$$

where

$$V_1(b,x,z;B) = \frac{1}{2}\int_{\substack{\widetilde{H}_2(b,c,x,z)\le B\\ |b|,|c|,|x|,|z|\ge 1}} \frac{1}{|z|}\,\mathrm{d}c$$

with

$$\begin{aligned}\widetilde{H}_2(b,c,x,z) &= H(1,b,c,x,(1+bc)z^{-1},z)\\ &= \max\left\{|x|,\left|b^2x\right|,\left|c^2x\right|,\left|z^2x^3\right|,\left|\frac{(1+bc)}{z}\right|,\left|\frac{b(1+bc)}{z}\right|,\left|\frac{c(1+bc)}{z}\right|\right\},\end{aligned}$$

and $\theta_1(b,x,z) = \prod_p \theta_1^{(p)}(b,x,z)$ *with*

$$\theta_1^{(p)}(b,c,z) = \begin{cases} 0, & p \mid b, p \mid z,\\ 1-\frac{1}{p}, & p\nmid b, p\mid x,\\ 1, & \textit{else.}\end{cases}$$

Proof. Using a Möbius inversion to remove the condition $\gcd(x,y) = 1$ in (3.11), and setting $y' = \frac{y}{\alpha}$, we get

$$N_2(B) = \frac{1}{2}\sum_{b,x,z\in\mathbb{Z}_{\neq 0}}\sum_{\alpha\mid x}\mu(\alpha)\widetilde{N_2}(\alpha,b,x,z;B),$$

where

$$\widetilde{N_2}(\alpha,b,x,z;B) = \#\left\{(c,y')\in\mathbb{Z}^2 \;\middle|\; \substack{c\neq 0,\ 1+bc-y'\alpha z=0,\\ H(1,b,c,x,\alpha y',z)\le B}\right\}.$$

To estimate $\widetilde{N}_2$, we first note that $\widetilde{N_2}(\alpha,b,x,z;B) = 0$ whenever αz and b are not coprime. If they are coprime, we estimate

$$\begin{aligned}\widetilde{N_2}(\alpha,b,x,z;B) &= \#\left\{c\in\mathbb{Z}_{\neq 0} \;\middle|\; \substack{bc\equiv -1 \pmod{\alpha z},\\ \widetilde{H}_2(b,c,x,z)\le B}\right\}\\ &= \int_{\substack{\widetilde{H}_2(b,c,x,z)\le B\\ |c|\ge 1}}\frac{1}{|\alpha z|}\,\mathrm{d}c + R(\alpha,b,x,z;B).\end{aligned}$$

Analogously to the first case, we get an error term $|R(\alpha, b, x, z; B)| \ll 1$. This inequality together with the height conditions $\left|b^2 x\right| \leq B$ and $\left|z^2 x^3\right| \leq B$ allows us to bound the summation over the error terms:

$$\sum_{\substack{b,x,z \in \mathbb{Z}_{\neq 0} \\ \left|b^2 x\right|, \left|z^2 x^3\right| \leq B}} \sum_{\substack{\alpha | x \\ (b, \alpha z)=1}} |\mu(\alpha) R(b, x, z; B)| \ll \sum_{\substack{b,x,z \in \mathbb{Z}_{\neq 0} \\ \left|b^2 x\right|, \left|z^2 x^3\right| \leq B}} 2^{\omega(x)}$$
$$\ll \sum_{x \in \mathbb{Z}_{\neq 0}} \frac{2^{\omega(x)} B}{|x|^2} \ll B.$$

We arrive at

$$N_2(B) = \sum_{b,x,z} \sum_{\substack{\alpha | x \\ (b, \alpha z)=1}} \frac{\mu(\alpha)}{\alpha} V_1(b, x, z; B) + O(B),$$

where

$$V_1(b, x, z; B) = \frac{1}{2} \int_{\substack{\widetilde{H}_2(b,c,x,z) \leq B \\ |b|,|c|,|x|,|z| \geq 1}} \frac{1}{|z|} \, \mathrm{d}c.$$

Using the multiplicativity of μ and gcd, we can factor the sum over α

$$\sum_{\substack{\alpha | x \\ (b, \alpha z)=1}} \frac{\mu(\alpha)}{\alpha} = \prod_p \begin{cases} 0, & p \mid b, p \mid z, \\ 1 - \frac{1}{p}, & p \nmid b, p \mid x, \\ 1, & \text{else} \end{cases}$$

to get a description of the arithmetic term θ_1. □

Lemma 3.5.2. *We have*

$$N_2(B) = \sum_{b,z} \theta_2(x, z) V_2(x, z; B) + O(B \left(\log \log B\right)^2),$$

where

$$V_2(x, z; B) = \frac{1}{2} \int_{\substack{\widetilde{H}_2(b,c,x,z) \leq B \\ |b|,|c|,|x|,|z| \geq 1}} \frac{1}{|z|} \, \mathrm{d}(b, c)$$

and $\theta_2(x, z) = \prod_p \theta_2^{(p)}$ *with*

$$\theta_2^{(p)} = \begin{cases} (1 - \frac{1}{p})^2, & p \mid x, z, \\ 1 - \frac{1}{p} + \frac{1}{p^2}, & p \mid x, p \nmid z, \\ 1 - \frac{1}{p}, & p \nmid x, p \mid z, \\ 1, & p \nmid x, z. \end{cases}$$

Proof. Using the height conditions $\left|c^2 x\right|, \left|b(1 + bc) z^{-1}\right| \leq B$ to estimate the integral, we can bound the volume function by the geometric average

$$V_1(b, x, z; B) \ll \frac{1}{|z|} \left(\frac{B^{1/2}}{|x|^{1/2}}\right)^{2/3} \left(\frac{B\,|z|}{|b|^2}\right)^{1/3} = \frac{B^{2/3}}{|b|^{2/3} |x|^{1/3} |z|^{2/3}}$$
$$= \frac{B}{|bxz|} \left(\frac{B}{|b^2 x|}\right)^{-1/6} \left(\frac{B}{|z^2 x^3|}\right)^{-1/6}.$$

Since the integral is zero whenever $|b^2x| \geq B$ or $|z^2x^3| \geq B$, the assertion follows by [Der09, Proposition 3.9] with $r = 0$, $s = 2$. (In the notation of loc. cit. we consider the ordering $\eta_0 = b, \eta_1 = x, \eta_2 = z$ of the variables, take $a_1 = a_2 = 1/6$, and $k_{i,j}$ to be the exponents in these two height conditions. Note that θ_1 satisfies [Der09, Definition 7.8], and hence the requirements of the proposition.). $\square$

Lemma 3.5.3. *We have*

$$N_2(B) = \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) \int_{\substack{\widetilde{H}_2(b,c,x,z) \leq B \\ |b|,|c|,|x|,|z| \geq 1}} \frac{1}{|z|} \,\mathrm{d}(b,c,x,z) + O(B(\log\log B)^2).$$

Proof. Using the same estimate for the integral over c as in the previous lemma and estimating the integral over b using the height condition $|b^2x| \leq B$, we get the bound

$$\begin{aligned} V_2(x,z;B) &\ll \int_{1 \leq |b| \leq B^{1/2}|x|^{-1/2}} \frac{B^{2/3}}{|b|^{2/3}\,|x|^{1/3}\,|z|^{2/3}} \ll \frac{B^{5/6}}{|x|^{1/2}|z|^{2/3}} \\ &\ll \frac{B}{|xz|}\left(\frac{B}{|x|^3\,|z|^2}\right)^{-1/6} \end{aligned}$$

for the volume function V_2. Since $V_2(b,z;B) = 0$ whenever $|z^2x^3| > B$, we get an asymptotic formula by [Der09, Proposition 4.3] (with $r = s = 1$). We are only left to see that the constant is indeed

$$\begin{aligned} &\prod_p \left(\frac{1}{p^2}\left(1-\frac{1}{p}\right)^2 + \frac{1}{p}\left(1-\frac{1}{p}\right)\left(2 - \frac{2}{p} + \frac{1}{p^2}\right) + \left(1 - \frac{1}{p}\right)^2\right) \\ &\quad = \prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right). \end{aligned}$$

$\square$

Proposition 3.5.4. *We have*

$$N_2(B) = cB\log(B) + O(B(\log\log B)^2),$$

where

$$c = \frac{20}{3}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right).$$

Proof. We have to estimate the integral in the previous lemma. We first want to replace $(1+bc)$ by bc in the height conditions. In the case of the condition $b(1+bc)/z$, this leaves us with an error term $R_1(B)$ that can be bounded by the integral over the region defined by $B - \left|\frac{b}{z}\right| \leq \left|\frac{b^2c}{z}\right| \leq B + \left|\frac{b}{z}\right|$, i.e., $\left|\frac{Bz}{b^2}\right| - \frac{1}{|b|} \leq |c| \leq \left|\frac{Bz}{b^2}\right| + \frac{1}{|b|}$, and the remaining height conditions, so that

$$|R_1(B)| \ll \int_{\substack{|b^2x|,|x^2z^3| \\ |b|,|z| \geq 1}} \frac{1}{|bz|} \,\mathrm{d}(b,x,z) \ll \int \frac{B^{1/2}}{|bz|^{3/2}} \,\mathrm{d}(b,z) \ll B^{1/2}.$$

The condition $c(1+bc)/z$ can be dealt with analogously. Next, we remove the condition $|b| \geq 1$, where we get an error term

$$|R_2(B)| \ll \int_{\substack{|c| \leq \frac{B^{1/2}}{|x^{1/2}|}, |x| \leq \frac{B^{1/3}}{|z|^{2/3}} \\ |z| \geq 1}} \frac{1}{|z|} \,\mathrm{d}(c,x,z) \ll \int_{\substack{|x| \leq \frac{B^{1/3}}{|z|^{2/3}} \\ |z| \geq 1}} \frac{B^{1/2}}{|x|^{1/2}\,|z|} \,\mathrm{d}(x,z)$$
$$\ll \int_{\substack{|x| \leq \frac{B^{1/3}}{|z|^{2/3}} \\ |z| \geq 1}} \frac{B^{2/3}}{|z|^{4/3}} \,\mathrm{d}(z) \ll B^{2/3},$$

and subsequently remove $|c| \geq 1$ analogously. Thus, we can estimate the integral in the previous lemma as $V_3(B) + O(B^{2/3})$, where

$$V_3(B) = \int_{\substack{|b^2x|, |c^2x|, |x^3z^2|, \\ |b^2cz^{-1}|, |bc^2z^{-1}| \leq B, \\ |x|, |z| \geq 1}} \frac{1}{|z|} \,\mathrm{d}(b,c,x,z).$$

By a change of variables $b \mapsto B^{-1/3}bz^{-1/3}$, $c \mapsto B^{-1/3}cz^{-1/3}$, $x \mapsto B^{-1/3}xz^{2/3}$, we get

$$\begin{aligned} V_3(B) &= B \int_{\substack{|b^2x|, |c^2x|, |x|, \\ |b^2c|, |bc^2| \leq 1, \\ 1 \leq |z| \leq B^{1/2}|x|^{3/2}}} \frac{1}{|z|} \,\mathrm{d}(b,c,x,z) \\ &= 2B \int_{\substack{|b^2x|, |c^2x|, |x|, \\ |b^2c|, |bc^2| \leq 1}} \log\left(B^{1/2}\,|x|^{3/2}\right) \mathrm{d}(b,c,x) + R_3(B) \\ &= B \log B \int_{\substack{|b^2x|, |c^2x|, |x|, \\ |b^2c|, |bc^2| \leq 1}} \mathrm{d}(b,c,x) + R_3(B) + R_4(B) \\ &= 2B \log B \int_{|b^2c|, |bc^2| \leq 1} \max\{|b^2|, |c^2|, 1\}^{-1} \,\mathrm{d}(b,c) \\ &\quad + R_3(B) + R_4(B). \end{aligned} \tag{3.12}$$

The error terms are

$$|R_3(B)| \ll B \int_{\substack{|x| \leq B^{-1/3} \\ |b^2c|, |bc^2| \leq 1}} \left|\log(B^{1/2}\,|x|^{3/2})\right| \mathrm{d}(b,c,x)$$
$$\ll B \int_{|b^2c|, |bc^2| \leq 1} \frac{3}{2} B^{-1/3} \,\mathrm{d}(b,c) \ll B^{2/3} \left(\int_{|c| \leq 1} \frac{1}{\sqrt{|c|}} \,\mathrm{d}c + \int_{|c| > 1} \frac{1}{c^2} \,\mathrm{d}c \right)$$
$$\ll B^{2/3}$$

and

$$|R_4(B)| \ll B \int_{\substack{|b^2c|, |bc^2|, \\ |x| \leq 1}} \left|\log\left(|x|^{3/2}\right)\right| \mathrm{d}(b,c,x) \ll \int_{|b^2c|, |bc^2| \leq 1} \mathrm{d}(b,c) \ll B.$$

The integral at the end of (3.12) then further evaluates to

$$\int_{|b^2c|,|bc^2|\le 1} \frac{1}{\max\{|b^2|, |c^2|, 1\}} \,\mathrm{d}(b,c) = 2\int_{\substack{|b|\ge|c| \\ |b^2c|\le 1}} \frac{1}{\max\{|b^2|, 1\}} \,\mathrm{d}(b,c)$$

$$= 2\int \frac{2\min\{|b|, |b^{-2}|\}}{\max\{|b^2|, 1\}} \,\mathrm{d}b = \int_{|b|\le 1} 4\,|b| \,\mathrm{d}b + \int_{|b|>1} \frac{4}{|b^4|} \,\mathrm{d}b = \frac{20}{3},$$

and we get the desired asymptotic. □

Chapter 4

Integral points on a singular quartic del Pezzo surface

This chapter is joint work with Ulrich Derenthal.

4.1 Introduction

Let $S \subset \mathbb{P}^4_{\mathbb{Q}}$ be the quartic del Pezzo surface over $\mathbb{Q}$ defined by

$$x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0.$$

It has an A_1-singularity $Q_1 = (0:1:0:0:0)$ and an A_3-singularity $Q_2 = (0:0:0:0:1)$. See [Der09, Section 8] for a proof of Manin's conjecture for S over $\mathbb{Q}$, [FP16] for a proof over arbitrary number fields. We are interested in the number of integral points on $S \setminus Q_i$ of bounded log-anticanonical height. More precisely, let $\mathcal{S} \subset \mathbb{P}^4_{\mathbb{Z}}$ be the integral model of S defined by the same equations. An integral point on $S \setminus Q_i$ is a rational point $\mathbf{x} \in S(\mathbb{Q})$ such that the corresponding integral point in $\mathcal{S}(\mathbb{Z})$ does not meet the closure $\overline{Q_i}$ of Q_i in $\mathcal{S}(\mathbb{Z})$; in other words, writing $\mathbf{x} = (x_0 : \cdots : x_4)$ with coprime integral coordinates, $(x_0 : \cdots : x_4) \not\equiv Q_i \pmod p$ for all primes p. We set $U_i = S \setminus Q_i$ and $\mathcal{U}_i = \mathcal{S} \setminus \overline{Q_i}$, and have the sets $\mathcal{U}_i(\mathbb{Z})$ of integral points. Let V be the complement of the three lines

$$\{x_0 = x_1 = x_2 = 0\},\ \{x_0 + x_3 = x_1 = x_2 = 0\} \text{ and } \{x_0 = x_2 = x_3 = 0\}$$

on S. We consider the following height functions: For an integral point $x = (x_0, \ldots, x_4) \in \mathcal{U}_1(\mathbb{Z})$, we set

$$H_1(x) = \max\{|x_0|, |x_2|, |x_3|, |x_4|\},$$

and for an integral point $x \in \mathcal{U}_2(\mathbb{Z})$, we set

$$H_2(x) = \max\{|x_0|, |x_1|, |x_2|, |x_3|\}.$$

In Lemma 4.2.1 we will see that these two height functions are log-anticanonical on the minimal desingularization of S.

$$\begin{tikzcd}
E_9 \arrow[rrrr, dash] \arrow[dd, dash] \arrow[dr, dash] & & & & E_1 \arrow[dr, dash] & \\
& E_7 \arrow[r, dash] \arrow[dl, dash] & E_5 \arrow[r, dash] & E_6 \arrow[r, dash] & E_4 \arrow[r, dash] & E_3 \arrow[dl, dash] \\
E_8 \arrow[rrrr, dash] & & & & E_2 &
\end{tikzcd}$$

Figure 4.1: Configuration of the divisors E_i.

Theorem 4.1.1. *Let $N_i(B) := \#\{x \in \mathcal{U}_i(\mathbb{Z}) \cap V(\mathbb{Q}) \mid H_i(x) \leq B\}$. Then*

$$N_1(B) = \frac{13}{540}\left(\prod_p \left(1-\frac{1}{p}\right)^5 \left(1+\frac{5}{p}\right)\right) B(\log B)^5 + O(B(\log B)^4 \log\log B),$$

$$N_2(B) = \frac{1}{32}\left(\prod_p \left(1-\frac{1}{p}\right)^3 \left(1+\frac{3}{p}\right)\right) B(\log B)^4 + O(B(\log B)^3 \log\log B).$$

Interpreting them on the minimal desingularization $\widetilde{S}$ of S, these asymptotic formulas satisfy

$$N_i(B) = c_{i,\infty} c_{i,\mathrm{fin}} B(\log B)^{b_i-1}(1+o(1)),$$

with

$$c_{i,\infty} = \sum_A \alpha_A \tau_{D_A,\infty}(D_A(\mathbb{R})),$$

$$c_{i,\mathrm{fin}} = \prod_p \left(1-\frac{1}{p}\right)^{\mathrm{rk}\,\mathrm{Pic}(U_i)} \tau_{U_i,p}(\mathcal{U}_i(\mathbb{Z}_p)),$$

where the sum runs over all maximal dimensional faces of the Clemens complex $\mathcal{C}^{\mathrm{an}}_{\mathbb{R}}(D_i)$ of the preimage of Q_i, and b_i is the exponent associated with these faces.

4.2 Counting

We use the notation of [Der09, Section 8], in particular the numbering of the divisors E_i corresponding to the generators of the Cox ring.

Let $\widetilde{S}$ be the minimal desingularization of S as in [Der14]. Let the divisor $D_1 = E_7$ be the (-2)-curve on $\widetilde{S}$ corresponding to the singularity Q_1 on S, and the divisor $D_2 = E_3 + E_4 + E_6$ the sum of the (-2)-curves corresponding to Q_2, and let $\widetilde{U}_i = X - D_i$. Let $\widetilde{V} \subset \widetilde{S}$ be the complement of the negative curves $E_1, \dots, E_7$.

The Cox ring of $\widetilde{S}$ is $\mathbb{Q}[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2\eta_7)$; in the basis $l_0, \dots, l_5$ of $\mathrm{Pic}(\widetilde{S})$, its grading is given by

$$\begin{array}{ccccccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 & \eta_9 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{array},$$

and its irrelevant ideal is $I_{\mathrm{irr}} = \prod(\eta_i, \eta_j)$, where the product runs over all pairs i, j such that there is no edge between E_i and E_j in Figure 4.1. The sections

$$\{\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8, \eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6, \eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2\eta_7, \eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7^2, \eta_7\eta_8\eta_9\}$$

have anticanonical degree and define the morphism $\widetilde{S} \to S$.

The desingularization $\widetilde{S}$ is given by a certain sequence of five blow-ups of $\mathbb{P}^2$ in rational points. Let $\widetilde{\mathcal{S}}$ be the integral model defined by the same sequence of blow-ups of $\mathbb{P}^2_{\mathbb{Z}}$, and let $\widetilde{\mathcal{U}}_i = \mathcal{S} - \overline{D_i}$. Consider the open subscheme $\mathcal{Y}$ of the spectrum of $R_{\mathbb{Z}} = \mathbb{Z}[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2\eta_7)$ defined as the complement of $I_{\mathrm{irr}} \cap R_{\mathbb{Z}}$. By [FP16, 4.1], it is a $\mathbb{G}^6_{\mathrm{m},\mathbb{Z}}$-torsor over $\widetilde{\mathcal{S}}$ via a morphism $\pi : \mathcal{Y} \to \widetilde{\mathcal{S}}$. The same sections as above now also induce a morphism $\widetilde{\mathcal{S}} \to \mathcal{S}$. In particular, this morphism induces bijections

$$\widetilde{\mathcal{U}}_i(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q}) \longleftrightarrow \mathcal{U}_i(\mathbb{Z}) \cap V(\mathbb{Q}).$$

Lemma 4.2.1. *The log-anticanonical bundles $\omega_{\widetilde{S}}(D_i)^\vee$ are big and nef. Neither of the sets of monomials in $\operatorname{Cox}(\widetilde{S})$ of degree $\omega_{\widetilde{S}}(D_i)^\vee$*

$$\{\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8, \eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2, \eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7, \eta_8\eta_9\} \text{ and}$$
$$\{\eta_2\eta_5\eta_7\eta_8, \eta_1^2\eta_2^2\eta_3^2\eta_4, \eta_4\eta_5^4\eta_6^2\eta_7^2\}$$

has a common zero on $\widetilde{S}$.

Proof. For the first set, assume that $\eta_8\eta_9 = 0$. Then $\eta_3, \dots, \eta_6$ have to be non-zero, since the corresponding divisors $E_3, \dots, E_6$ share an edge with neither E_8 nor E_9 in Figure 4.1 (i.e., the corresponding divisors do not intersect). If $\eta_8 = 0$, then $\eta_1 \neq 0$ (since E_8 does not share an edge with E_1), and only one of the sections η_2 and η_7 can vanish (since E_2 and E_7 do not share an edge); hence the second or third section is non-zero. Analogously, if $\eta_9 = 0$, then $\eta_2 \neq 0$, and only one of η_1 and η_7 can vanish; again, the second or third section is non-zero.

For the second set, assume that its first section vanishes. If η_2 vanishes, the third section cannot vanish; if η_5 or η_7 vanishes, the second section cannot vanish; if η_8 vanishes, only one of η_2 and η_7 can vanish, while the other factors of the second and third section cannot vanish, so one of these two sections does not vanish.

In particular, this means that both log-anticanonical divisors are base point free, hence nef. Moreover, the height conditions in [Der09, Lemma 8.3] show that

$$-K = E_1 + E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 + E_7 = E_3 + 2E_4 + 4E_5 + 3E_6 + 2E_7,$$

hence

$$\begin{aligned}
-K - D_1 &= \frac{1}{2}(E_1 + E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6) \\
&\qquad + \frac{1}{2}(E_3 + 2E_4 + 4E_5 + 3E_6 + E_7), \\
-K - D_2 &= E_1 + E_2 + E_3 + E_4 + 2E_5 + E_6 + E_7,
\end{aligned}$$

which are positive linear combinations of all negative curves (i.e., of the generators of the effective cone), so both log-anticanonical divisors are big. □

These two sets of sections then define an adelic metric on line bundles isomorphic to $\omega_{\widetilde{S}}(D_i)^\vee$, as well as log-anticanonical height functions H_1 and H_2. These give us an explicit description of our counting problem: Integral points on $\mathcal{Y}$ are lattice points $(\eta_1, \dots, \eta_9) \in \mathbb{Z}^9$ satisfying the equation in the Cox ring and the coprimality condition induced by the irrelevant ideal

$$\gcd(\eta_i, \eta_j) = 1 \qquad \text{if } E_i \text{ and } E_j \text{ do not share an edge in Figure 4.1.} \tag{4.1}$$

The log-anticanonical height of the image of such a point is the maximum of the absolute values of the sections in Lemma 4.2.1. Integral points on $\pi^{-1}(\widetilde{\mathcal{U}}_1) \subset \mathcal{Y}$ are precisely those satisfying $\eta_7 \in \{\pm 1\}$, and integral points on $\pi^{-1}(\widetilde{\mathcal{U}}_2)$ are those satisfying $\eta_3, \eta_4, \eta_6 \in \{\pm 1\}$. In total, we get the following:

Lemma 4.2.2. *Let*

$$\begin{aligned}
H_1(\eta_1, \dots, \eta_6, \eta_8) &:= \max \left\{ \begin{matrix} |\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8|, |\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|, \\ |\eta_3\eta_4^2\eta_5^4\eta_6^3|, |(\eta_2\eta_8^2 + \eta_4\eta_5^3\eta_6^2\eta_8)/\eta_1| \end{matrix} \right\}, \\
H_2(\eta_1, \eta_2, \eta_5, \eta_7, \eta_8) &:= \max\{|\eta_2\eta_5\eta_7\eta_8|, |\eta_1^2\eta_2^2|, |\eta_5^4\eta_7^2|\}.
\end{aligned}$$

Then these log-anticanonical heights coincide with the heights defined in the introduction, and we have

$$\begin{aligned}
N_1(B) &= \frac{1}{2^5}\# \left\{ (\eta_1, \dots, \eta_6, \eta_8, \eta_9) \in \mathbb{Z}^8 \;\middle|\; \begin{array}{l} \eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2 = 0, \\ H_1(\eta_1, \dots, \eta_6, \eta_8) \le B, \\ \eta_1 \cdots \eta_6 \ne 0, \ (4.1) \textit{ holds} \end{array} \right\}, \\
N_2(B) &= \frac{1}{2^3}\# \left\{ (\eta_1, \eta_2, \eta_5, \eta_7, \eta_8, \eta_9) \in \mathbb{Z}^6 \;\middle|\; \begin{array}{l} \eta_1\eta_9 + \eta_2\eta_8 + \eta_5^3\eta_7 = 0, \\ H_2(\eta_1, \eta_2, \eta_5, \eta_7, \eta_8) \le B, \\ \eta_1\eta_2\eta_5\eta_7 \ne 0, \ (4.1) \textit{ holds} \end{array} \right\}.
\end{aligned}$$

Proof. Since $\mathcal{Y}$ is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^6$-torsor over $\widetilde{\mathcal{S}}$, the morphism π induces a 2^6-to-1-correspondence between integral points on $\mathcal{Y}$ and $\widetilde{\mathcal{S}}$. We can solve the equation by η_9 to remove the last variable from the height function, and remove those variables which are $\in \{\pm 1\}$. By symmetry, we can assume $\eta_7 = 1$ in the first case, and $\eta_3 = \eta_4 = \eta_6 = 1$ in the second case. The preimage of $\widetilde{V}$ in the universal torsor is the complement of $\eta_1 \cdots \eta_7 = 0$, so π induces a 2^5-to-1-, resp. 2^3-to-1-correspondence between the above sets and points on $\widetilde{\mathcal{U}}_1(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q})$, resp. $\widetilde{\mathcal{U}}_2(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q})$ of log-anticanonical height $\le B$.

To see that these height functions coincide with the ones defined in the introduction, we note that, for example, for a point in $\pi^{-1}(\widetilde{\mathcal{U}}_1)(\mathbb{Z})$ we have $\eta_7 \in$

$\{\pm 1\}$, and thus $|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8| = |\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8| = |x_0|$. We get analogous identities for the other coordinates. There is no section corresponding to x_2 in the second height function, but, for integral points on $\pi^{-1}(\widetilde{\mathcal{U}}_2)$, we have $|x_2| = \sqrt{|\eta_1^2\eta_2^2|\,|\eta_5^4\eta_7^2|} = \sqrt{|x_1x_3|}$, hence it can never contribute to the maximum. Together with the remark before Lemma 4.2.1, we get the comparison between the subsets of $\mathcal{Y}(\mathbb{Z})$ in the lemma, and integral points in $\mathcal{U}_i(\mathbb{Z}) \cap V(\mathbb{Q})$ of height $\leq B$. $\square$

Lemma 4.2.3. *We have*

$$N_1(B) = \frac{1}{2^5} \sum_{\eta_1,\dots,\eta_6 \in \mathbb{Z}_{\neq 0}} \theta_1(\eta_1,\dots,\eta_6,1) V_{1,1}(\eta_1,\dots,\eta_6;B) + O(B\log B),$$
$$N_2(B) = \frac{1}{2^3} \sum_{\eta_1,\eta_2,\eta_5,\eta_7 \in \mathbb{Z}_{\neq 0}} \theta_1(\eta_1,\eta_2,1,1,\eta_5,1,\eta_7) V_{2,1}(\eta_1,\eta_2,\eta_5,\eta_7;B) + O(B\log B)$$

with

$$V_{1,1}(\eta_1,\dots,\eta_6;B) := \int_{H_1(\eta_1,\dots,\eta_6,\eta_8)\leq B} \frac{\mathrm{d}\eta_8}{|\eta_1|},$$
$$V_{2,1}(\eta_1,\eta_2,\eta_5,\eta_7;B) := \int_{H_2(\eta_1,\eta_2,\eta_5,\eta_7,\eta_8)\leq B} \frac{\mathrm{d}\eta_8}{|\eta_1|},$$

and θ_1 as in [Der09, Lemma 8.4], namely

$$\theta_1(\eta_1,\dots,\eta_7) = \prod_p \theta_{1,p}(I_p(\eta_1,\dots,\eta_7))$$

where $I_p(\eta_1,\dots,\eta_7) = \{i \in \{1,\dots,7\} : p \mid \eta_i\}$ and

$$\theta_{1,p}(I) = \begin{cases} 1, & I = \emptyset, \{1\}, \{2\}, \{7\} \\ 1-\frac{1}{p}, & I = \{4\},\{5\},\{6\},\{1,3\},\{2,3\},\{3,4\},\{4,6\},\{5,6\},\{5,7\}, \\ 1-\frac{2}{p}, & I = \{3\}, \\ 0, & \text{else.} \end{cases}$$

Proof. The proof is as in [Der09, Lemma 8.4], with slightly different height functions and some $\eta_i = 1$, which leads to different error terms. In the first case, using the second height condition, it is

$$\ll \sum_{\eta_1,\dots,\eta_6} 2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6)} \ll \sum_{\eta_2,\dots,\eta_6} \frac{2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6)}B}{|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|} \ll B\log B.$$

In the second case, using the second and the third height condition, it is

$$\ll \sum_{\eta_1,\eta_2,\eta_5,\eta_7} 2^{\omega(\eta_5)} \ll \sum_{\eta_1,\eta_5} \frac{2^{\omega(\eta_5)}B}{|\eta_1\eta_5^2|} \ll B\log B. \qquad \square$$

Lemma 4.2.4. *We have*

$$N_1(B) = \frac{1}{2^5}\left(\prod_p \omega_{1,p}\right) V_{1,0}(B) + O(B(\log B)^4 \log\log B),$$
$$N_2(B) = \frac{1}{2^3}\left(\prod_p \omega_{2,p}\right) V_{2,0}(B) + O(B(\log B)^3 \log\log B),$$

where

$$\omega_{1,p} = \left(1-\frac{1}{p}\right)^5\left(1+\frac{5}{p}\right),$$
$$\omega_{2,p} = \left(1-\frac{1}{p}\right)^3\left(1+\frac{3}{p}\right),$$

and

$$V_{1,0}(B) = \int_{|\eta_1|,\dots,|\eta_6|\geq 1} V_{1,1}(\eta_1,\dots,\eta_6;B)\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_6,$$
$$V_{2,0}(B) = \int_{|\eta_1|,|\eta_2|,|\eta_5|,|\eta_7|\geq 1} V_{2,1}(\eta_1,\eta_2,\eta_5,\eta_7;B)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_5\,\mathrm{d}\eta_7.$$

Proof. In the first case, as in [Der09, Lemma 8.5], we have

$$V_{1,1}(\eta_1,\dots,\eta_6;B) \ll \frac{B^{1/2}}{|\eta_1\eta_2|^{1/2}} = \frac{B}{|\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6|}\left(\frac{B}{|\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|}\right)^{-1/2}.$$

In the second case, we use

$$V_{2,1}(\eta_1,\eta_2,\eta_5,\eta_7;B) \ll \frac{B}{|\eta_1\eta_2\eta_5\eta_7|}.$$

Therefore, [Der09, Proposition 4.3, Corollary 7.10] gives the result. ☐

Lemma 4.2.5. *We have* $|V'_{1,0}(B) - V_{1,0}(B)| \ll B(\log B)^4$*, where*

$$V'_{1,0}(B) = \int_{\substack{|\eta_2|,|\eta_4|,|\eta_5|,|\eta_6|\geq 1,\\ |\eta_2\eta_4^2\eta_5^2\eta_6^2|\leq B}} W(\eta_2,\eta_4,\eta_5,\eta_6,B)\frac{B\,\mathrm{d}\eta_2\,\mathrm{d}\eta_4\,\mathrm{d}\eta_5\,\mathrm{d}\eta_6}{|\eta_2\eta_4\eta_5\eta_6|}, \tag{4.2}$$

with

$$W(\eta_2,\eta_4,\eta_5,\eta_6,B) = \int_{\substack{|x_0|,|x_1|,|x_2|,\frac{|x_0(x_0+x_2)|}{|x_1|}\leq 1\\ |x_2|\geq\frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},\frac{|\eta_2x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}} \frac{\mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}x_2}{|x_1x_2|},$$

Proof. We can introduce $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2| \leq B$ without changing $V_{1,0}(B)$, since this inequality follows from $|\eta_1| \geq 1$ and the second height condition. Afterwards, we can remove the condition $|\eta_1| \geq 1$ from $V_{1,0}(B)$ while changing the integral by $\ll B(\log B)^4$. Indeed, the integral over η_8 is $\ll \frac{B^{1/2}}{|\eta_1\eta_2|^{1/2}}$ (as in the bound for V_1 above). Now we use $|\eta_1| \leq 1$ (for the new piece of the integral) and $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2| \leq B$ (our new condition) to show that the integral over η_1,η_2 is $\ll \frac{B}{|\eta_3\eta_4\eta_5\eta_6|}$. Integrating over $1 \leq |\eta_i| \leq B$ (where the upper bound follows from $|\eta_2|,\dots,|\eta_6|\geq 1$ and the new condition) for $i=3,4,5,6$, we get an error $\ll B(\log B)^4$, as claimed. The volume now has the form

$$\int_{|\eta_2|,\dots,|\eta_6|\geq 1,\ |\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|,H(\eta_1,\dots,\eta_6,\eta_8)\leq B} \frac{1}{|\eta_1|}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_6\,\mathrm{d}\eta_8+O(B(\log B)^4).$$

A change of variables with $\eta_8=\frac{B}{\eta_2\eta_3\eta_4\eta_5\eta_6}x_0$ and $\eta_1=\frac{B}{\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2}x_1$ turns $\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_8}{|\eta_1|}$ into $\frac{B\,\mathrm{d}x_0\,\mathrm{d}x_1}{|x_1\eta_2\eta_3\eta_4\eta_5\eta_6|}$. A further change of variables $\eta_3=\frac{B}{\eta_4^2\eta_5^4\eta_6^3}x_2$ transforms $\frac{\mathrm{d}\eta_3}{|\eta_3|}$ to $\frac{\mathrm{d}x_2}{|x_2|}$. These substitutions turn the first height condition into $|x_0|\leq 1$, the second height condition into $|x_1|\leq 1$, the third height condition into $|x_2|\leq 1$, and the fourth height condition into $|x_0(x_0+x_2)/x_1|\leq 1$. The inequality $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|\leq B$ becomes $\frac{|\eta_2x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1$, and $|\eta_3|\geq 1$ becomes $|x_2|\geq\frac{|\eta_4^2\eta_5^4\eta_6^3|}{B}$. The condition $|\eta_2\eta_4^2\eta_5^2\eta_6^2|\leq B$ is implied by the $|\eta_2\eta_3\eta_4^2\eta_5^2\eta_6^2|\leq B$ and $\eta_3\geq 1$. ☐

Lemma 4.2.6. *We have*

$$W(\eta_2,\eta_4,\eta_5,\eta_6,B)=\int_{\substack{|x_2|\leq 1\\ |x_2|\geq\frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},\\ \frac{|\eta_2x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}} 8\frac{\mathrm{d}x_2}{|x_2|}+O(1).$$

Proof. As a first step, we integrate over x_1 to get

$$W(\eta_2,\eta_4,\eta_5,\eta_6,B)=\int_{\substack{|x_0(x_0+x_2)|\\ |x_0|,|x_2|,\leq 1\\ |x_2|\geq\frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},\\ \frac{|\eta_2x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}}(-2\log|x_0|-2\log|x_0+x_2|)\frac{\mathrm{d}x_0\,\mathrm{d}x_2}{|x_2|},\quad(4.3)$$

and will integrate the two terms individually. To determine the integral over the first one, we remove the condition $|x_0(x_0+x_2)|\leq 1$, introducing an error of at most

$$|R_1(\eta_2,\eta_4,\eta_5,\eta_6,B)|\leq 4\int_{\substack{x_0,|x_2|\leq 1,x_0\geq 0\\ |x_0(x_0+x_2)|\geq 1}}-\log x_0\frac{\mathrm{d}x_0\,\mathrm{d}x_2}{|x_2|},$$

by using the symmetry in the signs of x_0 and x_2. The last inequality implies that x_2 has a distance of at least $1/|x_0|$ (which is ≥ 1) from $-x_0$. Since $x_0>0$ and $x_2>-1$, it cannot be smaller, and thus $-x_0+1/x_0\leq x_2\leq 1$ holds. We thus get

$$\begin{aligned}|R_1(\eta_2,\eta_4,\eta_5,\eta_6,B)|&\ll\int_{0\leq x_0\leq 1}-\log x_0\left(\int_{-x_0+\frac{1}{x_0}}^{1}\frac{\mathrm{d}x_2}{|x_2|}\right)\mathrm{d}x_0\\ &\ll\int_{0\leq x_0\leq 1}\left|\log x_0\log\left(-x_0+\frac{1}{x_0}\right)\right|\mathrm{d}x_0\ll 1.\end{aligned}$$

We can now integrate the first term in (4.3) over x_0 and get

$$\int_{\substack{|x_0|,|x_2|,|x_0(x_0+x_2)|\leq 1\\ |x_2|\geq\frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},\frac{|\eta_2x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}}-2\log|x_0|\frac{\mathrm{d}x_0\,\mathrm{d}x_2}{|x_2|}=\int_{\substack{|x_2|,\frac{|\eta_2x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1\\ |x_2|\geq\frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},}}4\frac{\mathrm{d}x_2}{|x_2|}+O(1).$$

To treat the second term, we begin with a change of variables $x_0' = x_0 + x_2$, and add the condition $|x_0'| \leq 1$, introducing an error of at most

$$|R_2(\eta_2, \eta_4, \eta_5, \eta_6, B)| \leq \int_{\substack{|x_0'-x_2|,|x_2|,|x_0'(x_0'-x_2)|\leq 1\\ x_0'>1}} 4 \log x_0' \frac{\mathrm{d}x_0' \, \mathrm{d}x_2}{|x_2|}, \tag{4.4}$$

again using the symmetry of the integral. The third condition implies $|x_2 - x_0'| \leq 1/|x_0'| < 1$, i.e., $x_0' - 1/x_0' < x_2$, and thus we get

$$\begin{aligned}|R_2(\eta_2, \eta_4, \eta_5, \eta_6, B)| &\ll \int_{x_0'>1} \log x_0' \left(\int_{x_0' - \frac{1}{x_0'}}^{1} \frac{\mathrm{d}x_2}{|x_2|} \right) \mathrm{d}x_0' \\ &\ll \int_{1<x_0'\leq 2} \log x_0' \left| \log \left(x_0' - \frac{1}{x_0'} \right) \right| \mathrm{d}x_0' \ll 1.\end{aligned}$$

(For the second inequality, note that $x_0' - 1/x_0' \leq 1$ implies $x_0' \leq 2$.) Thus, the second term of (4.3) is

$$\int_{\substack{|x_0'(x_0'-x_2)|,|x_0'|,|x_0'-x_2|,|x_2|\leq 1\\ |x_2|\geq \frac{|\eta_4^2\eta_5^4\eta_6^3|}{B}, \frac{|\eta_2 x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}} -2\log|x_0'| \frac{\mathrm{d}x_0' \, \mathrm{d}x_2}{|x_2|} + O(1).$$

The condition $|x_0'(x_0' - x_2)| \leq 1$ is implied by the second and third condition, so we can remove it. Removing $|x_0' - x_2| \leq 1$ introduces an error of at most

$$|R_3(\eta_2, \eta_4, \eta_5, \eta_6, B)| \leq \int_{\substack{x_0',|x_2|\leq 1,\\ |x_0'-x_2|>1\\ x_0'\geq 0}} -2\log x_0' \frac{\mathrm{d}x_0' \, \mathrm{d}x_2}{|x_2|}$$

by the symmetry of the integral. The conditions imply $-1 \leq x_2 \leq x_0' - 1$ and thus

$$\begin{aligned}|R_3(\eta_2, \eta_4, \eta_5, \eta_6, B)| &\ll \int_{0\leq x_0'\leq 1} - \log x_0' \left(\int_{-1}^{x_0'-1} \frac{\mathrm{d}x_2}{|x_2|} \right) \mathrm{d}x_0' \\ &\ll \int_{0\leq x_0'\leq 1} \log x_0' \log |x_0' - 1| \, \mathrm{d}x_0' \ll 1.\end{aligned}$$

Thus, the integral of the second summand of (4.3) is

$$\int_{\substack{|x_0'|,|x_2|\leq 1\\ |x_2|\geq \frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},\\ \frac{|\eta_2 x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}} -2\log|x_0'| \frac{\mathrm{d}x_0' \, \mathrm{d}x_2}{|x_2|} + O(1) = \int_{\substack{|x_2|\leq 1\\ |x_2|\geq \frac{|\eta_4^2\eta_5^4\eta_6^3|}{B},\\ \frac{|\eta_2 x_2^2|B}{|\eta_4^2\eta_5^6\eta_6^4|}\leq 1}} 4 \frac{\mathrm{d}x_2}{|x_2|} + O(1). \tag{4.5}$$

Adding (4.4) and (4.5) now yields the desired result. □

Lemma 4.2.7. *We have*

$$\begin{aligned}V_{1,0}(B) &= 2^5 C_1 B (\log B)^5 + O(B \log B^4), \\ V_{2,0}(B) &= 2^3 C_2 B (\log B)^4,\end{aligned}$$

with

$$C_1 = 8 \operatorname{vol} \left\{ (t_2, \dots, t_6) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{aligned} & t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 \leq 1, \\ & t_3 + 2t_4 + 4t_5 + 3t_6 \leq 1 \end{aligned} \right\},$$

$$C_2 = 4 \operatorname{vol} \left\{ (t_1, t_2, t_5, t_7) \in \mathbb{R}^4_{\geq 0} \;\middle|\; \begin{aligned} & 2t_1 + 2t_2 \leq 1 \\ & 4t_5 + 2t_7 \leq 1 \end{aligned} \right\}.$$

Proof. A change of variables inverse to the last one in the proof of Lemma 4.2.5 gives

$$W(\eta_2, \eta_4, \eta_5, \eta_6, B) = \int_{\substack{|\eta_3| \geq 1 \\ |\eta_2 \eta_3^2 \eta_4^2 \eta_5^2 \eta_6^2|, \\ |\eta_3 \eta_4^2 \eta_5^4 \eta_6^3| \leq B}} 8 \frac{\mathrm{d}\eta_3}{|\eta_3|} + O(1).$$

Plugging this back into (4.2) and removing the inequality $|\eta_2 \eta_4^2 \eta_5^2 \eta_6^2| \leq B$ implied by the others, we get

$$V_{1,0}(B) = \int_{\substack{|\eta_2|, |\eta_3|, |\eta_4|, |\eta_5|, |\eta_6| \geq 1, \\ |\eta_3 \eta_4^2 \eta_5^4 \eta_6^3|, |\eta_2 \eta_3^2 \eta_4^2 \eta_5^2 \eta_6^2| \leq B}} \frac{8B \,\mathrm{d}\eta_2 \,\mathrm{d}\eta_3 \,\mathrm{d}\eta_4 \,\mathrm{d}\eta_5 \,\mathrm{d}\eta_6}{|\eta_2 \eta_3 \eta_4 \eta_5 \eta_6|} + O(B(\log B)^4),$$

since integrating the error term yields

$$\int_{\substack{|\eta_2|, |\eta_4|, |\eta_5|, |\eta_6| \geq 1, \\ |\eta_4^2 \eta_5^4 \eta_6^3|, |\eta_2 \eta_4^2 \eta_5^2 \eta_6^2| \leq B}} 1 \cdot \frac{B \,\mathrm{d}\eta_2 \,\mathrm{d}\eta_4 \,\mathrm{d}\eta_5 \,\mathrm{d}\eta_6}{|\eta_2 \eta_4 \eta_5 \eta_6|} \ll B(\log B)^4$$

using $1 \leq |\eta_i| \leq B$. Restricting to $\eta_i \geq 1$ introduces a factor of 2^5. Substituting $\eta_i = B^{t_i}$ turns $\mathrm{d}\eta_i / \eta_i$ to $\log B \,\mathrm{d}t_i$, and we thus arrive at

$$V_{1,0}(B) = 2^5 \int_{\substack{t_2, t_3, t_4, t_5, t_6 \geq 0 \\ t_3 + 2t_4 + 4t_5 + 3t_6, \\ t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 \leq 1}} 8 \,\mathrm{d}t_2 \,\mathrm{d}t_3 \,\mathrm{d}t_4 \,\mathrm{d}t_5 \,\mathrm{d}t_6 B(\log B)^5 + O(B(\log B)^4).$$

For the second case, using the first height condition in the first step, assuming $\eta_i > 0$ (with a factor 2 by symmetry) and transforming $\eta_i = B^{t_i}$ (with $\mathrm{d}\eta_i = B^{t_i} \log B \,\mathrm{d}t_i$) in the second step, we get

$$\begin{aligned} V_{2,0}(B) &= 2 \int_{|\eta_i| \geq 1, \; |\eta_1^2 \eta_2^2|, |\eta_5^4 \eta_7^2| \leq B} \frac{B \,\mathrm{d}\eta_1 \,\mathrm{d}\eta_2 \,\mathrm{d}\eta_5 \,\mathrm{d}\eta_7}{|\eta_1 \eta_2 \eta_5 \eta_7|} \\ &= 2^3 B(\log B)^4 \int_{t_1, t_2, t_5, t_7 \geq 0, \; 2t_1 + 2t_2 \leq 1, \; 4t_5 + 2t_7 \leq 1} \mathrm{d}t_1 \,\mathrm{d}t_2 \,\mathrm{d}t_5 \,\mathrm{d}t_7 \end{aligned}$$

for the volume. □

Plugging this into Lemma 4.2.4, we get

$$N_1(B) = C_1 \prod_p \omega_{1,p} \, B(\log B)^5 + O(B(\log B)^4 \log\log B),$$

$$N_1(B) = C_2 \prod_p \omega_{2,p} \, B(\log B)^4 + O(B(\log B)^3 \log\log B),$$

with $\omega_{i,p}$ and C_i as before. The main theorem will follow using that we have $C_i = c_{i,\infty}$ by Lemma 4.3.5 and that

$$\omega_{i,p} = \left(1 - \frac{1}{p}\right)^{\operatorname{rk} \operatorname{Pic} U_i} \tau_{U_i,p}(\mathcal{U}_i(\mathbb{Z}_p))$$

by Lemma 4.3.2.

4.3 The leading constant

In order to compute the Tamagawa volumes, we work with the chart $f\colon V' = \widetilde{S} - V(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6) \to \mathbb{A}^2$

$$(\eta_1 : \eta_2 : \eta_3 : \eta_4 : \eta_5 : \eta_6 : \eta_7 : \eta_8 : \eta_9) \mapsto \left(\eta_7 \cdot \frac{\eta_5^2\eta_6}{\eta_1\eta_2\eta_3}, \eta_8 \cdot \frac{1}{\eta_1\eta_3\eta_4\eta_5\eta_6}\right)$$

and its inverse $g\colon \mathbb{A}^2 \to \widetilde{S}$

$$(x, y) \mapsto (1 : 1 : 1 : 1 : 1 : 1 : x : y : -x - y).$$

Note that the two elements

$$\eta_7 \cdot \frac{\eta_5^2\eta_6}{\eta_1\eta_2\eta_3} \quad \text{and} \quad \eta_8 \cdot \frac{1}{\eta_1\eta_3\eta_4\eta_5\eta_6}$$

have degree 0 in the field of fractions of the Cox ring. The rational map they define is thus invariant under the torus action and descends to $\widetilde{S}$.

Lemma 4.3.1. *The images of the sets of p-adic integral points are*

$$f(\widetilde{\mathcal{U}}_1(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) = \{(x, y) \in \mathbb{Z}_p^2 \mid |x| \geq 1 \text{ or } |xy^2| \geq 1\},$$
$$f(\widetilde{\mathcal{U}}_2(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) = \{(x, y) \in \mathbb{Z}_p^2 \mid |y| \leq 1,\ |xy^2| \leq 1, \text{ or } |x + y| \leq 1\}.$$

Proof. Consider the image (x, y) of an integral point

$$(\eta_1 : \cdots : \eta_9) = \pi(\eta_1, \dots, \eta_9)$$

(given in *Cox coordinates*), where $(\eta_1, \dots, \eta_9)$ is an integral point of the torsor over $\mathcal{U}_1$. Assume $|x| < 1$. Then $\eta_5 \notin \mathbb{Z}_p^\times$ or $\eta_6 \notin \mathbb{Z}_p^\times$ (since $\eta_7 \in \mathbb{Z}_p^\times$). In both cases, the coprimality conditions imply $\eta_8 \in \mathbb{Z}_p^\times$, and thus $|xy^2| = |\eta_7\eta_8^2/\eta_1^3\eta_2\eta_3^3\eta_4^2\eta_6| \leq 1$.

On the other hand, let us consider a point (x, y) in the above set and construct an integral point $(\eta_1, \dots, \eta_9)$ on the torsor with $f(\pi(\eta_1, \dots, \eta_9)) = (x, y)$. If $|x| < 1$, we distinguish two cases for $|y|$:

1. If $1/|x| \leq |y| < 1/|x|^2$, let $\eta_5 = xy$, $\eta_6 = 1/xy^2$, $\eta_9 = -1 - x/y$, and the remaining coordinates be 1. Then $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, since $|x/y| \leq |x|^{1/2} < 1$, and thus the coprimality conditions are satisfied.

2. If $1/|x| \leq |y|$, let $\eta_4 = 1/xy$, $\eta_6 = x$, $\eta_9 = -1 - x/y$, and let all the other coordinates be 1. Since $|x/y| \leq |x|^2 < 1$, we again have $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, and thus the coprimality conditions hold.

If $|x| \geq 1$, we distinguish three cases for $|y|$.

1. If $|y| < 1$, let $\eta_2 = 1/x$, $\eta_8 = y$, $\eta_9 = -1 - y/x$, and the remaining coordinates be 1. Then $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, since $|y/x| < 1$.

2. If $1 \leq |y| < |x|$, let $\eta_3 = 1/y$, $\eta_2 = y/x$, $\eta_9 = -1 - y/x$, and the remaining coordinates be 1. Again, we have $|y/x| < 1$, so that $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$.

3. Finally, if $|x| \leq |y|$, let $\eta_3 = 1/x$, $\eta_4 = x/y$, $\eta_1 = -1 - x/y$, and the remaining coordinates be 1. If $|y| > |x|$, we have $\eta_1 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$; if $|x| = |y|$, we have $\eta_4 \in \mathbb{Z}_p^\times$. In both cases, the coprimality conditions on the torsor are satisfied.

We now turn to $\mathcal{U}_2$. Let (x, y) be in the image of the set of integral points. If $|y| > 1$, we have either $|\eta_5| < 1$ or $|\eta_1| < 1$. In the first case, we get $|xy^2| = |\eta_7\eta_8^2/\eta_1^3\eta_2| = \eta_7| \leq 1$ (since all other variables have to be units); for the second case, we note that

$$x + y = \frac{\eta_4\eta_5^3\eta_6^2\eta_7 + \eta_2\eta_8}{\eta_1\eta_2\eta_3\eta_4\eta_6\eta_7} = -\frac{\eta_1\eta_9}{\eta_1\eta_2\eta_3\eta_4\eta_6\eta_7},$$

and thus $|x + y| = |\eta_9| \leq 1$ (since all other variables have to be units).

On the other hand, let (x, y) be in the set on the right hand side in the statement of the lemma. We want to construct an integral point on the torsor lying above (x, y). If $|y| \leq 1$ and $|x| \leq 1$, let $\eta_8 = y$, $\eta_7 = x$, $\eta_9 = -x-y$, and the remaining variables be 1, which satisfies the coprimality-conditions. If $|y| \leq 1$ and $|x| > 1$, let $\eta_8 = y$, $\eta_2 = 1/x$, $\eta_9 = -1 - y/x$, and the remaining variables be 1. Then $\eta_9 \in -1 - p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, so $(\eta_1, \dots, \eta_9)$ is integral. Let now $|y| > 1$. If $|xy^2| \leq 1$, let $\eta_5 = 1/y$, $\eta_7 = xy^2$, $\eta_9 = -1 - xy$, and the remaining variables be 1; again, $\eta_9 \in \mathbb{Z}_p^\times$. Finally, if $|x + y| \leq 1$, let $\eta_1 = 1/x$, $\eta_9 = -x - y$, and $\eta_8 = -\eta_1\eta_9 - 1$ and the remaining variables be 1. Then $\eta_8 \in \mathbb{Z}_p^\times$, so $(\eta_1, \dots, \eta_9)$ is integral, and, since $\eta_8/\eta_1 = (-\eta_1\eta_9 - 1)/\eta_1 = x + y - x = y$, it indeed lies above (x, y). $\square$

Lemma 4.3.2. *We have*

$$\tau_{U_1,p}(\mathcal{U}_1(\mathbb{Z}_p)) = 1 + \frac{5}{p} \qquad \textit{and} \qquad \tau_{U_2,p}(\mathcal{U}_2(\mathbb{Z}_p)) = 1 + \frac{3}{p}.$$

Proof. We again start with the first case: We want to integrate $\|(\mathrm{d}x \wedge \mathrm{d}y) \otimes 1_{E_7}\|^{-1}_{\omega_{\widetilde{S}}(D)}$ over the set of integral points. To make sense of this, we need a metric on the log-anticanonical bundle, not just on a line bundle isomorphic to it. To this end, we consider the isomorphism between the canonical bundle $\omega_{\widetilde{S}}$ and the line bundle $\mathcal{L}$ whose meromorphic sections are elements of degree $\omega_{\widetilde{S}}$ of the field of fractions of Cox ring that maps $\mathrm{d}x \wedge \mathrm{d}y$ to $\eta_3^{-1}\eta_5^2\eta_6$; in addition, we consider the isomorphisms between $\mathcal{O}(E_i)$ and the line bundles whose sections are elements of the Cox ring mapping 1_{E_i} to η_i. In Cox coordinates, this norm is

$$\frac{|\eta_3\eta_5^{-2}\eta_6^{-1}|}{|\eta_7| \max\{|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8|, |\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|, |\eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7|, |\eta_8\eta_9|\}}. \tag{4.6}$$

Evaluating this at the image of (x, y) and integrating over the set of integral points yields

$$\tau_{U_1,p}(\mathcal{U}(\mathbb{Z}_p)) = \int_{\substack{x,y \in \mathbb{Q}_p, \\ |x| \geq 1 \text{ or } |xy^2| \geq 1}} \frac{1}{|x| \max\{1, |x|, |y|, |y(y + x)|\}} \,\mathrm{d}x\,\mathrm{d}y$$

for the Tamagawa volumes at finite places.

Let us compute this volume. Subdividing the domain of integration into the regions with $|x| > |y|$, $|x| = |y|$, and $|x| < |y|$ in order to simplify the denominator, we get

$$\begin{aligned}\int_{\substack{|y|<|x|,\\|x|\geq 1}} \frac{1}{|x|\max\{|x|,|xy|\}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\substack{|y|=|x|,\\|x|\geq 1}} \frac{1}{|x|\max\{|x|,|y(y+x)|\}}\,\mathrm{d}x\,\mathrm{d}y\\ + \int_{\substack{|x|<|y|,\\|xy^2|\geq 1}} \frac{1}{|xy^2|}\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{4.7}$$

after simplifying the description of the domains ($|x| < 1$ would imply $|xy^2| \leq |x|^3 < 1$ in the first two cases; $|y|^2 < 1/|x|$ would imply $|y|^2 < 1/|x| \leq 1 \leq |x|^2 < |y|^2$ in the third case).

The first of the integrals in (4.7) is

$$\begin{aligned}\int_{|x|\geq 1} \frac{1}{|x|^2}\int_{|y|<|x|} \frac{1}{\max\{1,|y|\}}\,\mathrm{d}y\,\mathrm{d}x &= \int_{|x|\geq 1}\frac{1}{|x|^2}\left(\frac{1}{p}+\int_{1\leq|y|<|x|}\frac{1}{|y|}\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_{|x|\geq 1}\frac{1}{|x|^2}\left(\frac{1}{p}+\left(1-\frac{1}{p}\right)|v(x)|\right)\mathrm{d}x = \frac{1}{p}+\sum_{\delta\geq 0}\left(1-\frac{1}{p}\right)^2\frac{\delta}{p^\delta} = \frac{2}{p},\end{aligned} \tag{4.8}$$

while the second integral is

$$\int_{\substack{|y+x|\leq\frac{1}{p}\\|x|\geq 1}} \frac{1}{|x|^2} + \int_{\substack{|y+x|\geq 1\\|x|\geq 1,|y|=|x|}} \frac{1}{|xy(x+y)|}. \tag{4.9}$$

The first integral in (4.9) is $\frac{1}{p}\int_{|x|\geq 1}\frac{1}{|x|^2}\,\mathrm{d}x = \frac{1}{p}$. Turning to the second one, we note that $|x| = |y|$ is implied by the ultrametric triangle inequality if $|x+y| < |x|$. The set of $y \in \mathbb{Q}_p$ with $|x+y| = |x|$ and $|y| = |x|$ has volume $|x| - 2|x|/p$, since the two sets $\{y \mid |y-0| < |x|\}$ and $\{y \mid |y+x| < |x|\}$ have volume $|x|/p$ and are disjoint, since $|y| < |x|$ implies $|y+x| = |x|$. We thus get

$$\begin{aligned}&\int_{|x|\geq 1}\frac{1}{|x|^2}\left(\sum_{0\leq\delta<|v(x)|}\left(1-\frac{1}{p}\right)\frac{p^\delta}{p^\delta}+\left(1-\frac{2}{p}\right)\frac{|x|}{|x|}\right)\mathrm{d}x\\ &= \int_{|x|\geq 1}\frac{1}{|x|^2}\left(\left(1-\frac{1}{p}\right)|v(x)|+\left(1-\frac{2}{p}\right)\right)\mathrm{d}x = \frac{1}{p}+1-\frac{2}{p} = 1-\frac{1}{p},\end{aligned}$$

computing the integral over x similarly as in (4.8). The second integral in (4.7) thus evaluates to 1. Finally, the third integral in (4.7) is

$$\begin{aligned}\int\frac{1}{|y|^2}\int_{1/|y|^2\leq|x|<|y|}\frac{1}{|x|}\,\mathrm{d}x\,\mathrm{d}y &= \int_{|y|\geq 1}\frac{1}{|y|^2}\sum_{-2|v(y)|\leq\delta<|v(y)|}\left(1-\frac{1}{p}\right)\mathrm{d}y\\ &= \int_{|y|\geq 1}\left(1-\frac{1}{p}\right)\frac{3|v(y)|}{|y|^2} = \frac{3}{p},\end{aligned}$$

again computed analogously to the previous ones. Adding the three terms in (4.7), we arrive at

$$\tau_{U_1,p}(\mathcal{U}(\mathbb{Z}_p)) = \frac{2}{p}+1+\frac{3}{p} = 1+\frac{5}{p}.$$

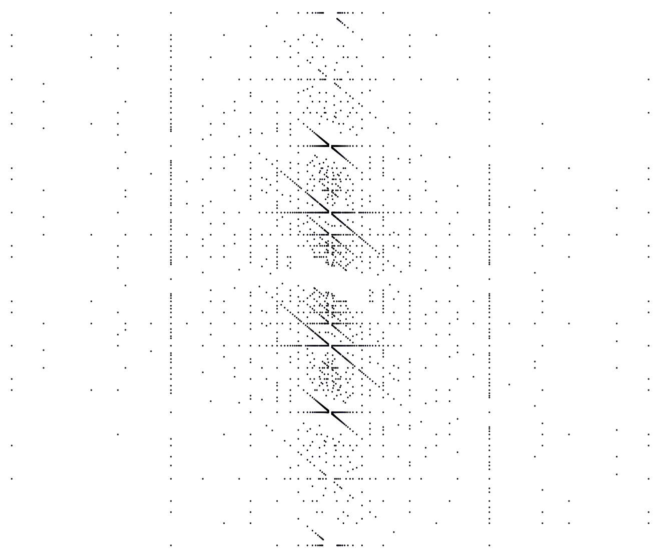

Figure 4.2: Integral points on $\mathcal{U}_1$ of height ≤ 90, viewed along the local chart f of $\widetilde{S}$. The boundary divisor E_7 is the central vertical line; the diagonal and the horizontal line which appear to be "missing" are E_8 and E_9. (If e.g. $\eta_8 = 0$, then the coprimality conditions imply $\eta_2, \dots, \eta_6 = \pm 1$, and then the equation implies $\eta_1, \eta_9 \in \pm 1$, leaving us only with the two points $(\pm 1, 0)$.)
Some horizontal and diagonal lines look accumulating, but in fact are not: They contain $\sim c'B$ points, which is less than the $cB(\log B)^5$ points on U; the constants c' can however be up to 2, while the constant c in our main theorem is numerically $\approx .0003$.

In the second case, we can analogously determine the norm

$$\|(\mathrm{d}x \wedge \mathrm{d}y) \otimes 1_{E_3} \otimes 1_{E_4} \otimes 1_{E_6}\|^{-1}_{\omega_{\widetilde{S}}(D_2)}$$

in Cox coordinates:

$$\frac{1}{|\eta_3\eta_4\eta_6|} \frac{|\eta_3\eta_5^{-2}\eta_6^{-1}|}{\max\{|\eta_2\eta_5\eta_7\eta_8|, |\eta_1^2\eta_2^2\eta_3^2\eta_4|, |\eta_4\eta_5^4\eta_6^2\eta_7^2|\}}. \tag{4.10}$$

Integrating this over $\mathcal{U}_2(\mathbb{Z}_p)$, using the same local chart as before, yields

$$\begin{aligned}&\int_{|y|\leq 1,\ |xy^2|\leq 1,\ \text{or}\ |x+y|\leq 1} \frac{1}{\max\{1, |xy|, |x^2|\}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{|y|\leq 1} \frac{1}{\max\{1, |x^2|\}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\substack{|y|>1,\\ |x+y|\leq 1}} \frac{1}{|y^2|}\,\mathrm{d}x\,\mathrm{d}y + \int_{\substack{|y|>1,\\ |x|\leq 1/|y|^2}} \mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

(since $|x| = |y|$ in the second case). The first integral is then

$$1 + \int_{|x|>1} \frac{1}{|x^2|}\,\mathrm{d}x = 1 + \frac{1}{p},$$

while both the second and third are

$$\int_{|y|>1} \frac{1}{|y^2|}\,\mathrm{d}y = \frac{1}{p}. \qquad \square$$

The remaining parts of the constant are associated with maximal faces of the Clemens complex. The first divisor $D_1 = E_7$ is geometrically irreducible, so the Clemens complex consists of just one vertex, which we will simply name E_7. The second divisor has three vertices corresponding to its components, and two 1-simplices, which we will call A_1 and A_2, added between the intersecting exceptional curves.

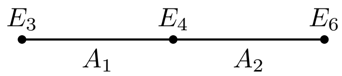

Figure 4.3: The Clemens complex of $E_3 + E_4 + E_6$.

Lemma 4.3.3. *We have*

$$\tau_{E_7,\infty}(E_7(\mathbb{R})) = 8 \quad \textit{and} \quad \tau_{A_1,\infty}(D_{A_1}(\mathbb{R})) = \tau_{A_2,\infty}(D_{A_2}(\mathbb{R})) = 4.$$

Proof. There is a metric on ω_{E_7} induced by the metrization of $\omega_{\widetilde{S}}(E_7)$ via the adjunction isomorphism. We can then compute the unnormalized Tamagawa volume of E_7 by integrating

$$\|\,\mathrm{d}y\|_{\omega_{E_7}}^{-1} = \lim_{x\to 0}\left(|x|\|(\mathrm{d}x \wedge \mathrm{d}y)\otimes 1_{E_7}\|_{\omega_{\widetilde{S}}(E_7)}^{-1}\right).$$

Again evaluating (4.6) in the image of (x, y), we get the volume

$$\begin{aligned}\tau'_{E_7,\infty}(E_7(\mathbb{R})) &= \int_{\mathbb{R}} \lim_{x\to 0} \frac{|x|}{|x|\max\{1,|x|,|y|,|y(y+x)|\}}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{1}{\max\{1,|y^2|\}}\,\mathrm{d}y = 4,\end{aligned}$$

which we renormalize by multiplying with $c_{\mathbb{R}} = 2$.

For the second case, we work in neighbourhoods of the two intersection points $D_{A_1} = E_3 \cap E_4$ and $D_{A_2} = E_4 \cap E_6$. The Tamagawa measures on these points are simply real numbers. In order to compute them, we consider the charts

$$\begin{aligned}&g'\colon \mathbb{A}^2 \to \widetilde{S},\ (a,b) \mapsto (1:1:a:b:1:1:1:1:-1-b) \ \text{ and} \\ &g''\colon \mathbb{A}^2 \to \widetilde{S},\ (c,d) \mapsto (1:1:1:c:1:d:1:1:-1-cd).\end{aligned}$$

Since $\|\,\mathrm{d}x \wedge \mathrm{d}y\| = |\det(J_{f\circ g'})|\|\,\mathrm{d}a \wedge \mathrm{d}b\|$, we get the norms

$$\begin{aligned}&\|(\mathrm{d}a \wedge \mathrm{d}b)\otimes 1_{E_3}\otimes 1_{E_4}\otimes 1_{E_6}\|_{\omega_{\widetilde{S}}(D_2)} = \max\{|a^3b^2|,|ab|,|ab^2|\} \text{ and} \\ &\|(\mathrm{d}c \wedge \mathrm{d}d)\otimes 1_{E_3}\otimes 1_{E_4}\otimes 1_{E_6}\|_{\omega_{\widetilde{S}}(D_2)} = \max\{|c^2d|,|cd|,|c^2d^3|\}.\end{aligned}$$

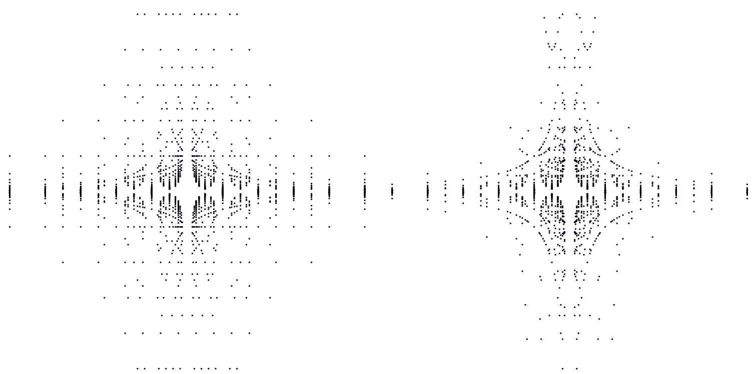

Figure 4.4: Integral points on $\mathcal{U}_2$ of height ≤ 60. On the left: in a neighbourhood of $E_3 \cap E_4$, viewed along the local chart g' of $\widetilde{S}$. The divisor E_3 is the central vertical line, and E_4 is the central horizontal line. On the right: In a neighbourhood of $E_4 \cap E_6$, viewed along the local chart g'' (with flipped axes) of $\widetilde{S}$. The divisor E_4 again is the central horizontal line, and E_6 is the central vertical line. The two charts overlap, and fully cover E_4.

Analogously to the first case we now arrive at

$$\tau'_{D_{A_1},\infty} = \lim_{(a,b)\to(0,0)} \frac{|ab|}{\max\{|a^3b^2|, |ab|, |ab^2|\}} = 1,$$

and, similarly, $\tau'_{D_{A_2},\infty} = 1$ for the unnormalized measures on the points $D_{A_i}(\mathbb{R})$, which we multiply with $c_{\mathbb{R}}^2 = 4$. $\square$

Lemma 4.3.4. *We have*

$$\alpha_1 = \operatorname{vol}\left\{(t_2,\dots,t_6) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 \leq 1, \\ t_3 + 2t_4 + 4t_5 + 3t_6 \leq 1 \end{array}\right\},$$

$$\alpha_{A_1} = \operatorname{vol}\left\{(t_1,t_2,t_5,t_7) \in \mathbb{R}^4_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 \leq 2t_5 + t_7 \\ 4t_5 + 2t_7 \leq 1 \end{array}\right\},$$

$$\alpha_{A_2} = \operatorname{vol}\left\{(t_1,t_2,t_5,t_7) \in \mathbb{R}^4_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 \geq 2t_5 + t_7 \\ 2t_1 + 2t_2 \leq 1 \end{array}\right\}.$$

Proof. In the first case, we have $\operatorname{Pic}(U_1; E_7) = \operatorname{Pic}(X)$ and $\Lambda_{E_7} = \overline{\operatorname{Eff}_X}$, whose dual is the nef cone of X. To determine α_1, we have to compute the volume, normalized as in 2.3.4, of the intersection of the nef cone with the hyperplane of $\mathbb{R}$-divisor classes having intersection number 1 with $-K - D_1$. The data in [Der14] shows that $[E_7] = [E_1+E_2+E_3-2E_5-E_6]$ and $-K = [2E_1+2E_2+3E_4+2E_4+E_6]$ in $\operatorname{Pic}(\widetilde{S})$. Therefore, $[-K-D_1] = [E_1+E_2+2E_3+2E_4+2E_5+2E_6]$. Working with the dual basis of $E_1,\dots,E_6$, we obtain

$$\alpha_1 = \operatorname{vol}\left\{(t_1,\dots,t_6) \in \mathbb{R}^6_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 + t_3 - 2t_5 - t_6 \geq 0, \\ t_1 + t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 = 1 \end{array}\right\}$$

and eliminate t_1.

In the second case, there are two constants α_{A_i}, $i = 1, 2$ associated with the maximal faces $A_{1,2}$ of the Clemens complex. The two divisor groups are $\mathrm{Pic}(U_2; A_1) = \mathrm{Pic}(U_{A_1})$ and $\mathrm{Pic}(U_2; A_2) = \mathrm{Pic}(U_{A_2})$, where $U_{A_1} = X - E_6$ and $U_{A_2} = X - E_3$. The constant α_{A_1} is the volume of the intersection of the dual cone $\Lambda_{A_1}^\vee$ of the effective cone of U_{A_1} with the hyperplane H_{A_1} defined by $\langle \cdot, K_{A_1} \rangle = 1$, where K_{A_1} is the pullback of the log-anticanonical class. The Picard group of U_{A_1} is $\mathrm{Pic}(\widetilde{S})/\langle E_6 \rangle$; a basis is given by the classes of E_1, E_2, E_4, E_5, E_7 modulo E_6, and its effective cone is generated by the classes of $E_1, \dots, E_5, E_7$. Since $[-K - D_2] = [E_4 + 4E_5 + 2E_6 + 2E_7]$ and $[E_3] = [-E_1 - E_2 + 2E_5 + E_6 + E_7]$, and working modulo E_6, we obtain

$$\alpha_{A_1} = \mathrm{vol} \left\{ (t_1, t_2, t_4, t_5, t_7) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} -t_1 - t_2 + 2t_5 + t_7 \geq 0, \\ t_4 + 4t_5 + 2t_7 = 1 \end{array} \right\},$$

and eliminate t_4.

The computation of α_{A_2} is similar. Here, our basis is given by the classes of E_1, E_2, E_4, E_5, E_7 modulo E_3, and we use $[E_6] = [E_1 + E_2 + E_3 - 2E_5 - E_7]$ and $-K - D_2 = [2E_1 + 2E_2 + 2E_3 + E_4]$ to obtain

$$\alpha_{A_2} = \mathrm{vol} \left\{ (t_1, t_2, t_4, t_5, t_7) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 - 2t_5 - t_7 \geq 0, \\ 2t_1 + 2t_2 + t_4 = 1 \end{array} \right\};$$

again, we eliminate t_4. □

Lemma 4.3.5. *In total, we get archimedean contributions*

$$\begin{aligned} c_{1,\infty} &= \alpha_1 c_{\mathbb{R}} \tau_{E_7,\infty}(E_7(\mathbb{R})) \\ &= 8 \,\mathrm{vol} \left\{ (t_2, \dots, t_6) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 \leq 1, \\ t_3 + 2t_4 + 4t_5 + 3t_6 \leq 1 \end{array} \right\} = \frac{13}{4320}, \\ c_{2,\infty} &= \alpha_{A_1} c_{\mathbb{R}}^2 \tau_{A_1,\infty}(D_{A_1}(\mathbb{R})) + \alpha_{A_2} c_{\mathbb{R}}^2 \tau_{A_2,\infty}(D_{A_2}(\mathbb{R})) \\ &= 4 \,\mathrm{vol} \left\{ (t_1, t_2, t_5, t_7) \in \mathbb{R}^4_{\geq 0} \;\middle|\; \begin{array}{l} 2t_1 + 2t_2 \leq 1 \\ 4t_5 + 2t_7 \leq 1 \end{array} \right\} = \frac{1}{32} \end{aligned}$$

to the expected constant.

Proof. The two polytopes whose volumes are α_{A_1} and α_{A_2} fit together to the one stated. Using the formula [DEJ14, (1.1)] and Magma, we explicitly compute the volumes. □

Chapter 5

Integral points on a toric variety

5.1 Introduction

The aim of this chapter is to provide an asymptotic formula for the number of integral points of bounded height on a certain toric variety X. Integral points on toric varieties were treated by Chambert-Loir and Tschinkel in [CLT10b], however, our result contradicts part of this work.[1]

More precisely, let X be the toric variety obtained by first blowing up $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (with coordinates $a_0, a_1, b_0, b_1, c_0, c_1$ and the standard torus action) in the line $l_1 = V(a_1, b_1)$, and then blowing up the resulting variety in the strict transform of the line $l_2 = V(a_1, c_1)$ and denote by $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the composition of these two blow-ups; let $T = \pi^{-1}(V(a_0a_1b_0b_1c_0c_1))$ be the open orbit of the torus action. Let us consider the divisor $D = H + E_1 + E_2$, where E_1 is the strict transform of the exceptional divisor of the first blowup, E_2 is the exceptional divisor of the second blowup, and H is the preimage of the plane $V(a_0)$, and let $U = X - D$. We are interested in the integral points on a suitable model $\mathcal{U}$ of U. More precisely, using the log-anticanonical height function H defined after Lemma 5.2.1, we study the number

$$N(B) = \{x \in \mathcal{U}(\mathbb{Z}) \cap T(\mathbb{Q}) \mid H(x) \leq B\}$$

of integral points bounded height. After parametrizing the set of integral points using a universal torsor in Section 5.2, we determine an asymptotic formula in Section 5.3. The exponent of $\log B$ is smaller by 1 than the one given in [CLT10b], which is explained by an obstruction to the existence of integral points on a certain part of X: Chambert-Loir's and Tschinkel's asymptotic formula is associated with the one-dimensional face $\{E_1, E_2\}$ of the Clemens complex. There is a function obstructing to the existence of integral points near E_1 and E_2, which also makes the leading constant of their asymptotic formula vanish. In Section 5.4, we compare our formula to the one given by Chambert-Loir and Tschinkel in greater detail and get a very similar geometric

[1] This is due to a gap in the proof of Lemma 3.11.4 of which the authors were already aware and because of which they no longer believed in the correctness of the final result of their preprint.

interpretation to theirs, associated with the maximal, but only zero-dimensional face H of the Clemens complex.

Theorem 5.1.1. *We have*

$$N(B) = c_\infty c_{\mathrm{fin}} B(\log B)^{b_H - 1}(1 + o(1)),$$

with

$$c_\infty = \alpha_H \tau_{H,\infty}(H(\mathbb{R})),$$
$$c_{\mathrm{fin}} = \prod_p \left(1 - \frac{1}{p}\right)^{\operatorname{rk}\operatorname{Pic} U} \tau_{U,p}(\mathcal{U}(\mathbb{Z}_p)),$$

where, with the notation of Chapter 2, all constants are associated with the maximal, but not maximal-dimensional, face H of the Clemens complex. More explicitly, we have

$$N(B) = cB(\log B)^2 + O(B \log B (\log\log B)^3),$$

where

$$c = 4\prod_p \left(\left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right)\right).$$

5.2 Passage to a universal torsor

The fan Σ_X of X can be obtained by starting with the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then subdividing it by adding the ray $\rho_x = \mathbb{R}(-1,-1,0)$, then further subdividing it by adding the ray $\rho_y = \mathbb{R}(-1,0,-1)$. The Picard group of X is

$$\operatorname{Pic}(X) = \mathbb{Z}\pi^*[H_1] + \mathbb{Z}\pi^*[H_2] + \mathbb{Z}\pi^*[H_3] + \mathbb{Z}[E_1] + \mathbb{Z}[E_2] \cong \mathbb{Z}^5,$$

where H_1, H_2, and H_3 are planes of degree $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

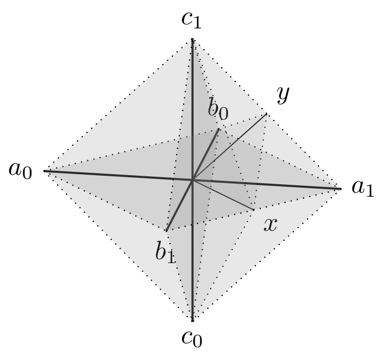

Figure 5.1: The fan Σ_X of X, its rays labeled with the corresponding generators of the Cox ring.

The Cox ring of X is $R_X = \mathbb{Q}[a_0, a_1, b_0, b_1, c_0, c_1, x, y]$, its generators corresponding to the rays of Σ_X. Its grading by $\operatorname{Pic}(X)$ (under the above isomorphism) is

$$\begin{array}{cccccccc} a_0 & a_1 & b_0 & b_1 & c_0 & c_1 & x & y \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}.$$

The irrelevant ideal is generated by the set $\{\prod g \mid \rho_g \not\subset \sigma\}_{\sigma \in \Sigma^{(\max)}}$; it is thus

$$\begin{aligned} I_{\mathrm{irr}} = (&a_1b_1c_1xy,\ a_1b_0c_1xy,\ a_1b_1c_0xy,\ a_1b_0c_0xy,\\ &a_0b_1c_1xy,\ a_0b_0b_1c_1y,\ a_0a_1b_0c_1y,\ a_0b_1c_0c_1x,\\ &a_0a_1b_1c_0x,\ a_0a_1b_0c_0y,\ a_0b_0b_1c_0c_1,\ a_0a_1b_0b_1c_0), \end{aligned}$$

and we get a universal torsor $Y = \operatorname{Spec} R_X - V(I_{\mathrm{irr}})$. The fan Σ_X similarly defines a *toric $\mathbb{Z}$-scheme* $\mathcal{X}$ (cf. e.g. [Dem70]) with $\mathcal{X}_{\mathbb{Q}} \cong X$. The scheme

$$\mathcal{Y} = \operatorname{Spec} R_{X,\mathbb{Z}} - V(I_{\mathrm{irr},\mathbb{Z}}),$$

where $R_{X,\mathbb{Z}} = \mathbb{Z}[a_0,a_1,b_0,b_1,c_0,c_1,x,y]$ and $I_{\mathrm{irr},\mathbb{Z}} = I_{\mathrm{irr}} \cap R_{X,\mathbb{Z}}$, is a $\mathbb{G}^5_{\mathrm{m},\mathbb{Z}}$-torsor over $\mathcal{X}$ (cf. [Sal98]). In particular, there is a 32-to-1-correspondence between rational points on X, respectively integral points on $\mathcal{X}$, and the set

$$\mathcal{Y}(\mathbb{Z}) = \left\{ (a_0,a_1,b_0,b_1,c_0,c_1,x,y) \in \mathbb{Z}^5 \;\middle|\; \begin{array}{c} \gcd(a_1b_1c_1xy, a_1b_0c_1xy, a_1b_1c_0xy, \\ a_1b_0c_0xy, a_0b_1c_1xy, a_0b_0b_1c_1y, \\ a_0a_1b_0c_1y, a_0b_1c_0c_1x, a_0a_1b_1c_0x, \\ a_0a_1b_0c_0y, a_0b_0b_1c_0c_1, a_0a_1b_0b_1c_0)=1 \end{array} \right\}.$$

of integral points on the torsor $\mathcal{Y}$.

Lemma 5.2.1. *The log-anticanonical bundle is big, i.e., in the interior of the effective cone, but it is not nef. It has the description $\omega_X(D)^\vee \cong \mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ as a quotient of base point free bundles, where the class of $\mathcal{L}_1$ is $(2,2,2,-2,-2)$, and the class of $\mathcal{L}_2$ is $(1,0,0,0,0)$ under the above isomorphism $\operatorname{Pic}(X) \cong \mathbb{Z}^5$.*

Proof. Let $D = E_1 + E_2 + H$ be the sum of the exceptional divisors $E_1 = V(x)$, $E_2 = V(y)$ and the invariant plane $H = V(a_0)$ not meeting E_1 or E_2. The log-anticanonical class $\omega_X(D)$ corresponds to

$$(1,2,2,-2,-2) = \sum_{g \text{ generator of } R_X} \deg(g) - \deg(x) - \deg(y) - \deg(a_0)$$

under the above isomorphism $\operatorname{Pic}(X) \cong \mathbb{Z}^5$. It is not base point free, since b_1c_1 divides all of its global sections. Since the same holds for all its multiples, it is not semi-ample (and thus not nef, since the two notions coincide on toric varieties). It is, however, big: The effective cone is generated by the degrees of the generators of the Cox ring, and hence

$$\begin{aligned}&(1,2,2,-2,-2)\\ &\quad = \tfrac{\deg(a_0)+3\deg(a_1)+\deg(b_0)+7\deg(b_1)+\deg(c_0)+7\deg(c_1)+2\deg(x)+2\deg(y)}{4}.\end{aligned}$$

is in its interior. Consider the description $(2,2,2,-2,-2) - (1,0,0,0,0)$ as a difference of base point free classes. We have sets

$$\{a_1^2b_0^2c_0^2, a_1^2b_1^2c_0^2x^2, a_1^2b_0^2c_1^2y^2, a_1^2b_1^2c_1^2x^2y^2, a_0^2b_1^2c_1^2\} \text{ and } \{a_0, a_1xy\}$$

of sections corresponding to these to classes. The sections in neither of these sets can vanish simultaneously, so both classes are indeed base point free, and these choices of sections induce metrics and a height function. ☐

Lemma 5.2.2. *We have a 4-to-1-correspondence between the set of integral points $\mathcal{U}(\mathbb{Z}) \cap T(\mathbb{Q})$ and the set*

$$\{(a_0, b_1, b_2, c_1, c_2) \in \mathbb{Z}^5_{\neq 0} \mid \gcd(a_1 b_0 c_0, a_1 b_0 c_1, a_1 b_1 c_0, b_1 c_1) = 1\} \subset \mathcal{Y}(\mathbb{Z}).$$

The log-anticanonical height of the image of a point $(a_1, b_0, b_1, c_0, c_1)$ in the above set is

$$H(a_1, b_0, b_1, c_0, c_1) = |a_1| \max\{\left|b_0^2\right|, \left|b_1^2\right|\} \max\{\left|c_0^2\right|, \left|c_1^2\right|\}. \tag{5.1}$$

Proof. The integral points in the preimage of $\mathcal{U}$ are precisely the points in $\mathcal{Y}(\mathbb{Z})$ satisfying $a_0, x, y \in \{\pm 1\}$. After plugging these values into the coprimality condition of the universal torsor, the condition simplifies to

$$\gcd(a_1 b_0 c_0, a_1 b_0 c_1, a_1 b_1 c_0, b_1 c_1) = 1. \tag{5.2}$$

The generic point of such an integral point lies on the torus if and only if all coordinates are non-zero.

The choice of global sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ induces metrics of these line bundles and, consequently, on the line bundle $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ isomorphic to the log-anticanonical bundle. This metrization thus induces a log-anticanonical height function. Its value on the image of a point $(a_0, a_1, b_0, b_1, c_0, c_1, x, y) \in \mathcal{Y}(\mathbb{Z})$ is

$$\frac{\max\{\left|a_1^2 b_0^2 c_0^2\right|, \left|a_1^2 b_1^2 c_0^2 x^2\right|, \left|a_1^2 b_0^2 c_1^2 y^2\right|, \left|a_1^2 b_1^2 c_1^2 x^2 y^2\right|, \left|a_0^2 b_1^2 c_1^2\right|\}}{\max\{|a_0|, |a_1 x y|\}}.$$

Using the facts that $|a_0| = |x| = |y| = 1$ and $|a_1| \geq 1$, this simplifies to the stated height function. Using the symmetry in $a_0, x, y = \pm 1$, we can assume that all of them are 1, making the 32-to-1-correspondence a 4-to-1-correspondence. □

5.3 Counting

In other words, we now have a new description

$$N(B) = \frac{1}{4} \#\{(a_1, b_0, b_1, c_0, c_1) \in \mathbb{Z}^5_{\neq 0} \mid H(a_1, b_0, b_1, c_0, c_1) \leq B, \text{ (5.2) holds}\}.$$

of the counting function, with the height function H in (5.1).

Lemma 5.3.1. *We have*

$$N(B) = \prod_p \left(\left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right) \right) V(B) + O(B \log B (\log \log B)^3)$$

with

$$V(B) = \frac{1}{4} \int_{\substack{|a_1|, |b_0|, |b_1|, |c_0|, |c_1| \geq 1, \\ |a_1| \max\{\left|b_0^2\right|, \left|b_1^2\right|\} \max\{\left|c_0^2\right|, \left|c_1^2\right|\} \leq B}} \mathrm{d}(a_1, b_0, b_1, c_0, c_1).$$

Proof. We can further rephrase the counting problem as follows:

$$N(B) = \frac{1}{4} \sum_{\substack{a_1, b_0, b_1, c_0, c_1 \in \mathbb{Z}_{\neq 0} \\ H(a_1, b_0, b_1, c_0, c_1) \leq B}} \theta(a_1, b_0, b_1, c_0, c_1),$$

where $\theta = \mathbb{1}_{\{\gcd(a_1 b_0 c_0, a_1 b_0 c_1, a_1 b_1 c_0, b_1 c_1)=1\}} = \prod_p \theta^{(p)}$ with

$$\theta^{(p)}(a_1, b_0, b_1, c_0, c_1) = \begin{cases} 0, & \text{if } p \mid a_1 b_0 c_0, a_1 b_0 c_1, a_1 b_1 c_0, b_1 c_1, \\ 1, & \text{else.} \end{cases}$$

We first want to replace the sum over b_0 by an integral. The height conditions imply that

$$\left|a_1 b_0^2 c_0^2\right|, \left|a_1 b_1^2 c_0^2\right|, \left|a_1 b_0 b_1 c_1^2\right| \leq B,$$

since the latter one is the geometric average of two terms in the height function. We have

$$1 = \frac{B}{|a_1 b_0 b_1 c_0 c_1|} \left(\frac{B}{|a_1 b_0^2 c_0^2|}\right)^{-1/4} \left(\frac{B}{|a_1 b_1^2 c_0^2|}\right)^{-1/4} \left(\frac{B}{|a_1 b_0 b_1 c_1^2|}\right)^{-1/2},$$

and note that the function θ satisfies Definition 7.9 in [Der09]. We use [Der09, Proposition 3.9] with $r = 1$, $s = 3$ and get

$$N(B) = \sum_{\substack{a_1, b_1, \\ c_0, c_1 \in \mathbb{Z}_{\neq 0}}} \theta_1(a_1, b_1, c_0, c_1) V_1(a_1, b_1, c_0, c_1; B) + O(B \log B (\log \log B)^3),$$

where $V_1(a_1, b_1, c_0, c_1; B) = \frac{1}{4} \int_{\substack{|b_0| \geq 1 \\ H(a_0, b_0, b_1, c_0, c_1) \leq B}} \mathrm{d}b_0$ and $\theta_1 = \prod_p \theta_1^{(p)}$ with

$$\theta_1^{(p)}(a_1, b_1, c_0, c_1) = \begin{cases} 0, & \text{if } p \mid a_1 c_0, a_1 c_1, b_1 c_1, \\ 1 - \frac{1}{p}, & \text{if } p \mid b_1, p \nmid a_1 \text{ and } (p \nmid c_0 \text{ or } p \nmid c_1), \\ 1, & \text{if } p \nmid b_1 \text{ and } (p \nmid c_1 \text{ or } p \nmid a_1 c_0). \end{cases}$$

Using the geometric average of the two height conditions involving b_0, we can bound V_1 by

$$V_1(a_1, b_1, c_0, c_1; B) \ll \sqrt{\frac{B}{|a_1 c_0 c_1|}} = \frac{B}{|a_1 b_1 c_0 c_1|} \left(\frac{B}{|a_1 b_1^2 c_0^2|}\right)^{-1/4} \left(\frac{B}{|a_1 b_1^2 c_1^2|}\right)^{-1/4}.$$

Since $\left|a_1 b_1^2 c_0^2\right|$ and $\left|a_1 b_1^2 c_1^2\right|$ are bounded by B, applying [Der09, Proposition 3.9] once more (with $r = 1$, $s = 2$) yields

$$N(B) = \sum_{a_1, b_1, c_1 \in \mathbb{Z}_{\neq 0}} \theta_2(a_1, b_1, c_1) V_2(a_1, b_1, c_1; B) + O(B \log B (\log \log B)^3),$$

where $V_2(a_1, b_1, c_1; B) = \frac{1}{4} \int_{\substack{|b_0|, |c_0| \geq 1 \\ H(a_0, b_0, b_1, c_0, c_1) \leq B}} \mathrm{d}(b_0, c_0)$ and $\theta_2 = \prod_p \theta_2^{(p)}$ with

$$\theta_2^{(p)}(a_1, b_1, c_0, c_1) = \begin{cases} 0, & \text{if } p \mid a_1, b_1 c_1 \\ \left(1 - \frac{1}{p}\right)^2, & \text{if } p \mid b_1 c_1,\ p \nmid a_1 \\ 1 - \frac{1}{p}, & \text{if } p \mid b_1,\ p \nmid a_1 c_1 \\ 1 - \frac{1}{p}, & \text{if } p \mid c_1,\ p \nmid a_1 b_1 \\ 1, & \text{if } p \nmid b_1 c_1. \end{cases}$$

To complete the summations, we use the fact that the height conditions imply $\left|a_1 b_0^2 c_0 c_1\right| \le B$, and get an upper bound

$$\begin{aligned} V_2(a_1, b_1, c_1; B) &\ll \int_{\substack{|c_0|\ge 1\\ \left|a_1 b_1^2 c_0^2\right| \le B}} \sqrt{\frac{B}{|a_1 c_0 c_1|}}\, \mathrm{d}c_0 \ll \frac{B^{3/4}}{|a_1|^{3/4}\, |b_1|^{1/2}\, |c_1|^{1/2}} \\ &= \frac{B}{|a_1 b_1 c_1|} \left(\frac{B}{\left|a_1 b_1^2 c_1^2\right|}\right)^{-1/4} \end{aligned}$$

for V_1. Since $\left|a_1 b_1^2 c_1^2\right| \le B$, [Der09, Proposition 4.3] yields the desired result, for which are only left to check that the constant is indeed $\prod_p c_p$ with

$$\begin{aligned} c_p &= \frac{1}{p^2}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p}\right)^2 + 2\frac{1}{p}\left(1-\frac{1}{p}\right)^2\left(1-\frac{1}{p}\right) + \left(1-\frac{1}{p}\right)^2 \\ &= \left(1-\frac{1}{p^2}\right)\left(1+\frac{2}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right). \end{aligned}$$

□

Proposition 5.3.2. *We have*

$$N(B) = cB(\log B)^2 + O(B\log B(\log\log B)^3),$$

where

$$c = 4\prod_p \left(\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right)\right).$$

Proof. We just have to provide an asymptotic for $V(B)$. The error we introduce when removing the condition $|a_1| \ge 1$ in the integral, while keeping the condition $\max\{\left|b_0^2\right|, \left|b_1^2\right|\}\max\{\left|c_0^2\right|, \left|c_1^2\right|\} \le B$ implied by the others, is at most

$$\begin{aligned} &2\int_{\substack{|c_0|,|c_1|\ge 1,\\ \max\{\left|b_0^2\right|,\left|b_1^2\right|\}\max\{\left|c_0^2\right|,\left|c_1^2\right|\}\le B}} \mathrm{d}(b_0, b_1, c_0, c_1) \\ &\quad \ll \int_{|c_0|,|c_1|\ge 1} \frac{B}{\max\{\left|c_0^2\right|, \left|c_1^2\right|\}}\, \mathrm{d}(c_0, c_1) \ll B\log B. \end{aligned}$$

Using the symmetry of the integral, we get

$$V(B) = \int_{\substack{|b_0|,|b_1|,|c_0|,|b_0|\ge 1\\ |b_0|\le|b_1|,|c_0|\le|c_1|,\\ \left|b_1^2 c_1^2\right|\le B}} \frac{B}{\left|b_1^2\right|\left|c_1^2\right|}\, \mathrm{d}(b_0, b_1, c_0, c_1) + O(B\log B).$$

Removing $|b_0| \ge 1$ introduces an error of at most

$$2\int_{\substack{|b_1|,|c_1|\ge 1\\ |c_0|\le|c_1|\le\sqrt{B}/|b_1|}} \frac{B}{\left|b_1^2\right|\left|c_1^2\right|}\, \mathrm{d}(b_1, c_0, c_1) \ll \int_{1\le|c_1|\le B} \frac{B}{|c_1|}\, \mathrm{d}c_1 \ll B\log B,$$

as, analogously, does removing $|c_0| \ge 1$. We thus have

$$\begin{aligned} V(B) &= 4\int_{\substack{|b_1|,|c_1|\ge 1,\\ |b_1|\le\sqrt{B}/|c_1|}} \frac{B}{|b_1|\,|c_1|}\, \mathrm{d}(b_1, c_1) + O(B\log B) \\ &= 4\int_{1\le|c_1|\le\sqrt{B}} \frac{B\log B}{|c_1|} + O(B\log B) = 4B(\log B)^2 + O(B\log B). \end{aligned}$$

□

5.4 Interpretation of the result

Remember that the *Clemens complex* associated with a split toric variety (X, D) over $\mathbb{Q}$ encodes the incidence properties of the irreducible components of the boundary divisor D in the following way: It consists of vertices $\{D'\}$ for every irreducible component D' of D, and we glue an s-simplex $\{D_0, \dots, D_s\}$ to the complex whenever the intersection $\bigcap_{i=0}^{s} D_i$ is non-empty. With a maximal face A, we associated the lattice $\Lambda_A = \mathrm{Pic}(X - \bigcup_{D' \notin A} D')$ and its effective cone $\overline{\mathrm{Eff}}_A \subset \Lambda_A$. For our variety, this means that the Clemens complex consists of a 1-simplex $A = \{E_1, E_2\}$ and an isolated vertex $\{H\}$ (which we will also simply denote by H). Integral points tend to accumulate around the boundary divisor; their number is dominated by those points lying near the intersection of a maximal number of boundary components. It is for this reason that the dimension of the Clemens complex is part of the exponent in the main theorem of [CLT10b].

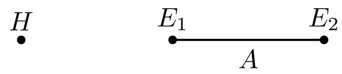

Figure 5.2: The Clemens complex of D

For the toric variety X, this does not hold. There is an obstruction to the existence of points near the intersection $E_1 \cap E_2$ (and even to the existence of integral points near $E_1 \cup E_2$): Let us consider the rational function $f = a_1 xy/a_0$ (in fact, a character of T) on X. It is a non-constant regular function on $U_A = X - H$, so there is an obstruction in the sense of Proposition 2.4.1.

Concretely, this means the following: The function f is a regular in a neighbourhood of $E_1 \cap E_2$, vanishing on $E_1 \cap E_2$. If a point p is near $E_1 \cap E_2$, $|f(p)|$ should thus be small. However, since f is a regular function on $\mathcal{U}$, its value is an integer at any integral point in $\mathcal{U}(\mathbb{Z})$ – and thus $|f(p)| \geq 1$ except for points on the subvariety $\{f = 0\}$. This means that the only integral points that are close to $E_1 \cap E_2$ can be points on this subvariety (which we excluded in our counting problem). For this reason we cannot expect a contribution of the maximal face A of the Clemens complex to our asymptotic formula. Since f is even regular on neigbourhoods of both E_1 and E_2, there can in fact be no integral points near either of those divisors and we cannot expect a contribution of those two non-maximal faces. The existence of this function also has an effect on the Picard group. That f vanishes on E_1, E_2, and $H' = V(a_1)$, and that it has a pole on H means that we have $[E_1] + [E_2] + [H'] = [H]$ in $\mathrm{Pic}(X)$, and thus $[E_1] + [E_2] + [H'] = 0$ in $\mathrm{Pic}(X - H)$. All three classes are non-trivial, hence the effective cone of $X - H$ contains a plane. It is thus not strictly convex, and its characteristic function is identically 0.

Since a value of the characteristic function is a factor of the leading constant in op. cit., this means that, for this variety, the leading constant is zero, contrary to their claim in Lemma 3.11.4. In particular, this variety is an example for the obstruction in Section 2.4, and, more precisely, the situation considered in Lemma 2.4.3. The exponent of $\log B$ in our Proposition 5.3.2 is one less than the one given by Chambert-Loir and Tschinkel. We can however interpret our asymptotic formula analogously to the formula given by Chambert-Loir and

Tschinkel: There is no obstruction at the only remaining maximal face H of the Clemens complex. Substituting this face for the maximal dimensional face A of the Clemens complex, we get the correct asymptotic formula. Summarizing, the situation is as follows:

Proposition 5.4.1.

1. *The cone $\Lambda_A = \overline{\mathrm{Eff}}_{X-H} \subset \mathrm{Pic}(X-H)_{\mathbb{R}}$, associated with the unique maximal-dimensional face A of the Clemens complex, is not strictly convex.*

2. *The cone Λ_H, associated with the unique other maximal face H, is strictly convex. The constant associated with this face is $\alpha_H = 1/8$, and the exponent associated with it is $b_H = b'_H = 3$.*

3. *We have*
$$N(B) = cB(\log B)^{b_H-1}(1+o(1)),$$
where
$$c = \alpha_H \tau_{H,\infty}(H(\mathbb{R})) \prod_p \left(\left(1-\frac{1}{p}\right)^{\mathrm{rk}\,\mathrm{Pic}(U)} \tau_{U,p}(\mathcal{U}(\mathbb{Z}_p)) \right).$$

Proof. The Picard group $\mathrm{Pic}(U;A) = \mathrm{Pic}(X-H)$ is the quotient
$$\mathrm{Pic}(X)/[H] \cong \mathbb{Z}^5/(1,0,0,0,0)/ \cong \mathbb{Z}^4.$$
The effective cone is generated by the classes of the torus-invariant prime divisors
$$(0,0,-1,-1),\ (1,0,0,0),\ (1,0,-1,0),\ (0,1,0,0),$$
$$(0,1,0,-1),\ (0,0,1,0),\ \text{and } (0,0,0,1),$$
and thus contains the plane $\{(0,0,x,y) \mid x,y \in \mathbb{R}\}$; in particular, it is not strictly convex.

The Picard group $\mathrm{Pic}(U;H) = \mathrm{Pic}(U_H)$ for $U_H = X - V(x) - V(y)$ is the quotient
$$\mathrm{Pic}(X)/\left\langle \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\rangle \cong \mathbb{Z}^3.$$
Its rank is $b'_H = 3$, so it coincides with
$$b_H = \mathrm{rk}\,\mathrm{Pic}(U) + \mathrm{Pic}\,E(U) + \#H = 2+0+1.$$
The effective cone $\Lambda_H = \mathrm{Eff}_{U_H}$ is smooth and generated by
$$(1,0,0),\ (0,1,0),\ \text{and } (0,0,1).$$
The image of the log-anticanonical class in this quotient is $(1,2,2)$. The characteristic function of Λ_A thus evaluates to $1/4$, and we get
$$\alpha_H = \frac{1}{(b'_H-1)!}\frac{1}{4} = \frac{1}{8}.$$
Comparing this, together with the descriptions of the Tamagawa volumes in Lemma 5.4.3, to Proposition 5.3.2 gives the stated asymptotic formula. □

To compute the Tamagawa volumes, we consider the chart

$$\begin{aligned} X - V(a_1b_1c_1xy) &\rightarrow \mathbb{A}^3, \\ (a_0 : a_1 : b_0 : b_1 : c_0 : c_1 : x : y) &\mapsto \left(\frac{a_0}{a_1xy}, \frac{b_0}{b_1x}, \frac{c_0}{c_1y}\right) \end{aligned}$$

and its inverse $\mathbb{A}^3 \to X$

$$(a, b, c) \mapsto (a : 1 : b : 1 : c : 1 : 1 : 1).$$

Lemma 5.4.2. *Under this chart, the integral points $\mathcal{U}(\mathbb{Z}_p)$ correspond to*

$$\{(a, b, c) \in \mathbb{Z}_p^3 \mid \textit{either } |a| = 1, \textit{ or } |a| > 1 \textit{ and } |b|, |c| \leq 1\}.$$

Proof. Let (a, b, c) be the image of a point $(a_0, a_1, b_0, b_1, c_0, c_1, x, y) \in \mathcal{Y}'(\mathbb{Z}_p)$. Since $a_0 \in \mathbb{Z}_p^\times$, we have $|a| = |a_0/a_1xy| \geq 1$. If $|a| \geq 1$, then $|a_1| < 1$, since $x, y \in \mathbb{Z}_p^\times$. The coprimality conditions then imply $b_1c_1 \in \mathbb{Z}_p^\times$, and thus $|b| = |b_0|, |c| = |c_0| \leq 1$.

On the other hand, consider a point (a, b, c) in the above set. If $|a| = 1$, let $a_1 = x = y = 1$ and $a_0 = a^{-1}$. If $|b| \leq 1$, let $b_0 = b$ and $b_1 = 1$, else, let $b_0 = 1$ and $b_1 = b^{-1}$, and set c_0, c_1 analogously. Then $(a_0, a_1, b_0, b_1, c_0, c_1, x, y) \in \mathcal{Y}'(\mathbb{Z}_p)$ maps to (a, b, c). If $|a| > 1$, let $a_0 = a$, $b_0 = b$, $c_0 = c$, and the remaining coordinates be 1. Again, $(a_0, a_1, b_0, b_1, c_0, c_1, x, y)$ is integral and maps to (a, b, c). $\square$

Lemma 5.4.3. *We have*

$$\tau_{H,\infty}(H(\mathbb{R})) = 16 \quad \textit{and} \quad \tau_{U,p}(\mathcal{U}(\mathbb{Z}_p)) = 1 + \frac{2}{p} - \frac{1}{p^2} - \frac{1}{p^3}$$

for all primes p.

Proof. Similarly to the previous varieties, we choose the isomorphism from ω_X to the bundle $\mathcal{L}_{\omega_X}$ whose sections are elements of degree ω_X in the Cox ring that maps $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c$ to $a_1^{-2}b_1^{-2}c_1^{-2}x^{-1}y^{-1}$. For the archimedean volume, we want to integrate $\|1_{E_1}1_{E_2}\,\mathrm{d}b \wedge \mathrm{d}c\|_{\omega_H(E_1+E_2)} = \left\|a^{-1}1_{E_1}1_{E_2}\,\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c\right\|_{\omega_X(D)}$ on $H(\mathbb{R})$ (regarding $a^{-1} \in \Gamma(U, \mathcal{O}(-H)) \subset \Gamma(U, \mathcal{K}_{\mathbb{A}^3})$). Outside H, we have $a^{-1} = a^{-1}1_H$, where the first factor is a section in $\Gamma(\mathbb{A}^3 - H, \mathcal{O}_{\mathbb{A}^3})$, and thus

$$\|1_{E_1}1_{E_2}\,\mathrm{d}b \wedge \mathrm{d}c\|_{\omega_H(E_1+E_2)} = \lim_{a \to 0}\left(\left|a^{-1}\right| \|1_H 1_{E_1}1_{E_2}\,\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c\|_{\omega_X(D)}\right)$$

The norm $\|1_H1_{E_1}1_{E_2}\,\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c\|_{\omega_X(D)}$ is

$$\frac{\max\{|a_0|, |a_1xy|\}}{|a_0xy| \max\{|a_1^2b_0^2c_0^2|, |a_1^2b_1^2c_0^2x^2|, |a_1^2b_0^2c_1^2y^2|, |a_1^2b_1^2c_1^2x^2y^2|, |a_0^2b_1^2c_1^2|\}}$$

at a point $(a_0 : a_1 : b_0 : b_1 : c_0 : c_1 : x : y)$ given in homogeneous coordinates. Evaluation this in the image of a point (a, b, c) now yields

$$\|\mathrm{d}b \wedge \mathrm{d}c\| = \lim_{a \to 0} \frac{|a| \max\{1, |a|\}}{|a| \max\{|b^2c^2|, |c^2|, |b^2|, 1, |a^2|\}} = \frac{1}{\max\{1, |b|^2\} \max\{1, |c|^2\}}$$

Integrating now gives the archimedean Tamagawa volume

$$\tau_{(H,E_1+E_2),\infty}(H(\mathbb{R})) = \int_{\mathbb{R}^2} \frac{1}{\max\{1,|b|^2\}\max\{1,|c|^2\}}\,\mathrm{d}b\,\mathrm{d}c = 16,$$

which has to be renormalized with the factor $c_{\mathbb{R}} = 2$.

For the Tamagawa volumes at the non-archimedean places, we integrate $\|1_H 1_{E_1} 1_{E_2}\,\mathrm{d}a\wedge\mathrm{d}b\wedge\mathrm{d}c\|_{\omega_X(D)}$ over $\mathcal{U}(\mathbb{Z}_p)$. Using the same description as above, this yields

$$\left(1-\frac{1}{p}\right)\int_{b,c\in\mathbb{Q}_p} \frac{1}{\max\{1,|b|^2\}\max\{1,|c|^2\}}\,\mathrm{d}b\,\mathrm{d}c + \int_{\substack{|a|>1\\|b|,|c|\leq 1}} \frac{1}{|a|^2}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$$

The first integral is

$$\left(\int_{b\in\mathbb{Q}_p}\frac{1}{\max\{1,|b|^2\}}\,\mathrm{d}b\right)^2 = \left(1+\int_{|b|>1}\frac{1}{|b|^2}\,\mathrm{d}b\right)^2 = \left(1+\frac{1}{p}\right)^2,$$

and the second is

$$\int_{|a|>1}\frac{1}{|a|^2}\,\mathrm{d}a = \frac{1}{p},$$

so, in total, we get

$$\tau_{U,p}(\mathcal{U}(\mathbb{Z}_p)) = \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^2 + \frac{1}{p} = 1+\frac{2}{p}-\frac{1}{p^2}-\frac{1}{p^3}. \qquad \square$$

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