Cubic fourfolds with ADE singularities and K3 surfaces

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Abstract

In this thesis, we will give a partial classification of cubic fourfolds by their isolated ADE singularities. We have a correspondence between cubic fourfolds and complete $(2,3)$-intersections in $\mathbb{P}^4$ having both certain isolated ADE singularities. The minimal model for a complete $(2,3)$-intersection in $\mathbb{P}^4$ with isolated ADE singularities is a quasi-polarized K3 surface of degree 6. We will prove that the existence of certain lattice embeddings into the K3 lattice is a necessary and sufficient condition for the existence of these singular cubic fourfolds and complete $(2,3)$-intersections, respectively. We will determine all direct sums of negative definite irreducible ADE lattices such that their direct sum with the rank one lattice whose generator has self-intersection number 6 admits a primitive embedding into the K3 lattice. This will prove the existence of complete $(2,3)$-intersections in $\mathbb{P}^4$ lying on smooth quadrics and having exactly these ADE singularities and their corresponding cubic fourfolds. Finally, we will show that we have an isomorphism between the moduli space of cubic fourfolds with certain ADE singularities and the moduli space of quasi-polarized K3 surfaces of degree 6 such that the quasi-polarization induces a birational map from the K3 surface into $\mathbb{P}^4$ whose image is a complete $(2,3)$-intersection in $\mathbb{P}^4$ having certain ADE singularities.

Key words: Cubic fourfolds, ADE singularities, K3 surfaces, quadratic forms, moduli spaces of K3 surfaces.

Kurzzusammenfassung

In dieser Doktorarbeit wird eine partielle Klassifikation von kubischen Vierfaltigkeiten anhand ihrer isolierten ADE Singularitäten gegeben. Es gibt eine Korrespondenz zwischen kubischen Vierfaltigkeiten und vollständigen $(2,3)$-Durchschnitten in $\mathbb{P}^4$ mit jeweils bestimmten isolierten ADE Singularitäten. Das minimale Model eines vollständigen $(2,3)$-Durchschnitts in $\mathbb{P}^4$ mit isolierten ADE Singularitäten ist eine quasi-polarisierte K3 Fläche vom Grad 6. Wir werden zeigen, dass die Existenz bestimmter Gittereinbettungen in das K3 Gitter eine notwendige und hinreichende Bedingung für die Existenz dieser kubischen Vierfaltigkeiten bzw. dieser vollständigen $(2,3)$-Durchschnitte in $\mathbb{P}^4$ ist. Wir werden alle direkten Summen von negativ definiten irreduziblen ADE Gittern bestimmen, sodass deren direkte Summe mit einem Gitter vom Rang eins, dessen Erzeuger Selbstschnitt 6 hat, eine primitive Einbettung in das K3 Gitter besitzt. Dies wird die Existenz derjenigen vollständigen $(2,3)$-Durchschnitte in $\mathbb{P}^4$ beweisen, die auf glatten Quadriken liegen und exakt diese ADE Singularitäten haben, sowie den korrespondierenden kubischen Vierfaltigkeiten. Schließlich werden wir beweisen, dass der Modulraum der kubischen Vierfaltigkeiten mit bestimmten ADE Singularitäten isomorph ist zum Modulraum bestimmter quasi-polarisierter K3 Flächen vom Grad 6, sodass die Quasi-Polarisierung eine birationale Abbildung von der K3 Fläche in den $\mathbb{P}^4$ induziert, deren Bild ein vollständiger $(2,3)$-Durchschnitt mit bestimmten ADE Singularitäten in $\mathbb{P}^4$ ist.

Schlagwörter: kubische Vierfaltigkeiten, ADE Singularitäten, K3 Flächen, quadratische Formen, Modulräume von K3 Flächen.

Introduction

Cubic hypersurfaces have been a central theme in Algebraic Geometry throughout the last centuries. Starting from the famous result of A. Cayley and G. Salmon in [Cay49] and [Sal49] that a smooth cubic surface contains exactly 27 lines, to the proof of C. H. Clemens and P. A. Griffith that any smooth cubic threefold is irrational in [CG72], to more recent investigations on the rationality/irrationality of cubic fourfolds (see for instance [Has00]).

Cubic fourfolds are of particular interest for at least two reasons. First, the rationality of smooth cubic fourfolds is still an open problem in Algebraic Geometry and it is conjectured that a very general smooth cubic fourfold is irrational. However, while some classes of rational cubic fourfolds have been described in [Fan43], [Tre84], [Tre93], and [BD85], no smooth cubic fourfold has yet been proven to be irrational. Second, smooth cubic fourfolds are related to hyperkähler manifolds (see [BD85] and [LSV17]), which are themselves of interest to algebraic geometers. Surprisingly, the period map for smooth cubic fourfolds behaves similarly as the period map for K3 surfaces as investigated in [Voi86], [Voi08], and [Laz10]. Furthermore, since the monodromy groups associated to ADE singularities of cubic fourfolds are finite, the period map on smooth cubic fourfolds extends to cubic fourfolds with isolated ADE singularities.

The ADE singularities or simple hypersurface singularities were classified by V. I. Arnol'd in the famous ADE list in [Arn72]. In the case of surfaces, they are precisely rational double points and there are various ways to characterize them (see [Dur79]).

The central topic of this thesis is the study of possible isolated ADE singularities on cubic fourfolds. More precisely, we give a partial classification of cubic fourfolds by their ADE singularities.

In the past, people have already succesfully classified other projective varieties by their ADE singularities: The classification of cubic surfaces by their ADE singularities was done in the 19th century by L. Schläfli in [Sch63]; a more modern and geometric proof was given by J. W. Bruce and C. T. C. Wall in [BW79]. The classification of cubic threefolds was done about fifteen years ago by R. Laza in the (unpublished) notes [Laz05]. A partial classification of quartic surfaces by their ADE singularities was given by T. Urabe in [Ura87] and [Ura88] which was completed by J.-G. Yang in [Yan96] and a partial classification of complete $(2, 2, 2)$-intersections in $\mathbb{P}^5$ by their ADE singularities by L.-Z. Tang in [Tan93].

The strategies in [BW79] and [Laz05] to classify all cubic surfaces and threefolds by their isolated ADE singularities, respectively, are similar. The authors use that we can associate to a cubic hypersurface X in $\mathbb{P}^n$ with only isolated ADE singularities a complete $(2, 3)$-intersection in $\mathbb{P}^{n-1}$ and prove then the existence of certain ADE singularities on the cubic by showing the existence of corresponding ADE singularities on the complete $(2, 3)$-intersection.

More precisely: In homogeneous coordinates $(x_0 : \ldots : x_n)$ on $\mathbb{P}^n$ such that one marked ADE singularity p of X is the point $(1 : 0 : \ldots : 0) \in \mathbb{P}^n$, we have

$$X\colon x_0 f_2(x_1, \ldots, x_n) + f_3(x_1, \ldots, x_n) = 0 \subseteq \mathbb{P}^n,$$

where f_2 and f_3 are homogeneous polynomials of degree 2 and 3, respectively. Then, X induces the complete $(2, 3)$-intersection

$$S_p\colon f_2(x_1, \ldots, x_n) = f_3(x_1, \ldots, x_n) = 0 \subseteq \mathbb{P}^{n-1}.$$

We also have a more geometric and coordinate-free description of S_p. Indeed, the complete $(2,3)$-intersection S_p is the image of the union of all lines in X through the point p under the projection of $\mathbb{P}^n$ through p onto the hyperplane $\{x_0 = 0\} \cong \mathbb{P}^{n-1}$.
We use the above strategy in the four dimensional case, as well, and relate the problem of finding certain combinations of ADE singularities on cubic fourfolds to finding them on complete $(2,3)$-intersections in $\mathbb{P}^4$.

Since the minimal model for a complete $(2,3)$-intersection in $\mathbb{P}^4$ with at most isolated ADE singularities is a K3 surface with a quasi-polarization of degree 6, we obtain consequently a geometric correspondence between cubic fourfolds with isolated ADE singularities and quasi-polarized K3 surfaces of degree 6.

The minimal models of quartic surfaces in $\mathbb{P}^3$ and complete $(2,2,2)$-intersections in $\mathbb{P}^5$ with at most isolated ADE singularities are quartic and octic K3 surface, respectively. In [Ura87] and [Tan93], the authors investigated that by the surjectivity of the period map, the question if a certain combination of ADE singularities can occur on these quartic surfaces and complete $(2,2,2)$-intersections, respectively, is transformed into a question about lattices.

We follow this idea and relate the existence of certain combinations of isolated ADE singularities on complete $(2,3)$-intersections in $\mathbb{P}^4$ to the existence of certain lattice embeddings into the K3 lattice. Using V. V. Nikulin's Theorem on the existence of primitive lattice embeddings in [Nik80] and the theory of quadratic forms as formulated by R. Miranda and D. R. Morrison in [MM09], we determine computer-aided certain possible combinations of ADE singularities on those complete $(2,3)$-intersections in $\mathbb{P}^4$ which lie on smooth quadrics.

The maximal number of A_1 singularities which we can find with our methods on a cubic fourfold with no other singularities is 11. Further, the maximal combinations of A_1 and A_2 singularities with respect to their Milnor number which we can here find on a cubic fourfold with no other singularities are $3A_1 + 6A_2$, $5A_1 + 5A_2$, and $7A_1 + 4A_2$.

In [Has00, 4.2], B. Hassett related the moduli space of cubic fourfolds with a single A_1 singularity to the moduli space of K3 surfaces with a very ample line bundle of degree 6. Here, we relate the moduli space of cubic fourfolds with a certain combination of isolated ADE singularities to the moduli space of certain quasi-polarized K3 surfaces of degree 6.

Indeed, R. Laza showed in [Laz09] that cubic fourfolds with at most isolated ADE singularities are stable in the sense of D. Mumford's Geometric Invariant Theory (GIT). Using this result, we construct the moduli space of cubic fourfolds with a certain combination of isolated ADE singularities as GIT quotients. Further, we construct the moduli space of certain quasi-polarized K3 surfaces of degree 6 as the moduli space of certain lattice polarized K3 surfaces. Finally, we show that both moduli spaces are isomorphic.

Structure of the thesis and results

In **Chapter 1**, we will recall basics of ADE singularities on complex analytic spaces. In particular, we will focus on properties of ADE singularities on complex analytic surfaces.

In **Chapter 2**, we will recall essential definitions related to symmetric bilinear and quadratic forms, and quadratic modules. In particular, we will study lattices and introduce ADE lattices and the K3 lattice as examples.

In **Chapter 3**, we will recall basics of (quasi-polarized) K3 surfaces. In particular, we will study complete linear systems on K3 surface and discuss when a linear system $|L|$ on a K3 surface S induces a birational map φ_L from the K3 surface onto its image in the projective space. We will see that if $|L|$ is fixed part free and φ_L is birational onto its image, the existence of certain irreducible ADE lattices in $\mathrm{Pic}(S)$ will imply the existence of ADE singularities of corresponding type on $\varphi_L(S)$. Further, if $L^2 = 6$, the surface $\varphi_L(S) \subseteq \mathbb{P}^4$ will be a complete $(2,3)$-intersection. Finally, we will define the period domain and the period map for K3 surfaces and recall the theorem on the surjectivity of the period map.

In **Chapter 4**, we will study complete $(2,3)$-intersections in $\mathbb{P}^4$ for each possible rank of the underlying quadric individually. For such a complete $(2,3)$-intersection S, we will, depending on the rank of the underlying quadric, construct a certain hyperplane section which passes through those singularities of S lying on the singular locus of the quadric. Furthermore, we will classify the types of those singularities. In particular, we will understand in this chapter the geometry of complete $(2,3)$-intersections in $\mathbb{P}^4$.

In **Chapter 5**, we will study cubic hypersurfaces in $\mathbb{P}^n$ with isolated ADE singularities and explain how to associate to them complete $(2,3)$-intersections in $\mathbb{P}^{n-1}$. In particular, we explain how ADE singularities on cubic hypersurfaces correspond to ADE singularities on the associated complete $(2,3)$-intersections.

In **Chapter 6**, we will state and prove our first Main Theorem which establishes a correspondence between the existence of firstly cubic fourfolds with certain ADE singularities, secondly complete $(2,3)$-intersections with certain ADE singularities in $\mathbb{P}^4$, and thirdly embeddings of certain lattices into the K3 lattice:

For $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ an ADE singularity type and a positive integer n, denote by $\sigma(\mathbf{T})$ the ADE singularities on the exceptional divisor of the blowing-up of an n-dimensional $\mathbf{T}$ singularity. Let $\mathrm{corank}_{\mathbf{T}}$ be $n+1$ minus the rank of the Hessian matrix of the analytic function defining $\mathbf{T}$ in the origin. We note that $\mathrm{corank}_{\mathbf{T}}$ is invariant with respect to different dimensions of $\mathbf{T}$. Let $\Gamma_{\sigma(\mathbf{T})}$ as in Table 6.1 be the weighted graph which we obtain by extending the Dynkin diagram associated to $\sigma(\mathbf{T})$ in a certain way. Let $\Lambda(\Gamma_{\sigma(\mathbf{T})})$ be the lattice associated to $\Gamma_{\sigma(\mathbf{T})}$ and $h_{\mathbf{T}} \in \Lambda(\Gamma_{\sigma(\mathbf{T})})$ a certain linear combination of the vertices of $\Gamma_{\sigma(\mathbf{T})}$.

Main Theorem 1. *For* $\big((a_1,\dots,a_n),(d_4,\dots,d_m),(e_6,e_7,e_8)\big) \in {\mathbb{Z}_{\geq 0}}^n \times {\mathbb{Z}_{\geq 0}}^{m-3} \times {\mathbb{Z}_{\geq 0}}^3$, *let*

$$\mathbf{G} := \sum_{i=1}^{n} a_i \mathbf{A}_i + \sum_{j=4}^{m} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k$$

be a formal finite sum of ADE singularity types,

$$\Gamma_{\mathbf{G}} := \sum_{i=1}^{n} a_i \mathcal{A}_i + \sum_{j=4}^{m} d_j \mathcal{D}_j + \sum_{k=6}^{8} e_k \mathcal{E}_k$$

a Dynkin diagram with connected components $\mathcal{A}_i$, $\mathcal{D}_j$, and $\mathcal{E}_k$, and $\Lambda(\Gamma_{\mathbf{G}})$ the associated lattice.

The following are equivalent:

1. *There exists a cubic fourfold X in $\mathbb{P}^5$ with a singularity of type $\mathbf{T}$ and such that all other singularities of X correspond to $\mathbf{G}$.*

2. *There exists a complete $(2,3)$-intersection S in $\mathbb{P}^4$ of a quadric Q of* $\operatorname{corank}(Q) = \operatorname{corank}_{\mathbf{T}}$ *in $\mathbb{P}^4$ and a cubic Y such that the singularities of S that lie on the singular locus of Q are of type $\sigma(\mathbf{T})$ as in Table 6.1 and such that all other singularities of S correspond to $\mathbf{G}$.*

3. *There exists an embedding*

$$i\colon \Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})}) \hookrightarrow L_{K3}$$

such that the following conditions a), b), and c) hold:
Let $\operatorname{Sat}_{L_{K3}}(i)$ be the saturation of $\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$ in L_{K3} with respect to i.

   a) If $x \in \operatorname{Sat}_{L_{K3}}(i)$ with $i(h_{\mathbf{T}}).x = 0$ and $x^2 = -2$, then $x \in i\big(\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})\big)$.

   b) There exists no element $x \in \operatorname{Sat}_{L_{K3}}(i)$ with $i(h_{\mathbf{T}}).x = 1$ and $x^2 = 0$.

   c) There exists no element $x \in \operatorname{Sat}_{L_{K3}}(i)$ with $i(h_{\mathbf{T}}).x = 2$ and $x^2 = 0$.

In **Chapter 7**, we introduce finite bilinear and quadratic forms and define discriminant bilinear and quadratic forms. For an odd prime p, we will define the normal form of quadratic forms and finite quadratic forms over $\mathbb{Z}_p$. We will see that knowing the normal form of a finite quadratic form (G, q_p) over $\mathbb{Z}_p$, we can construct a quadratic $\mathbb{Z}_p$-module (L, Q_p) such that the rank of L coincides with the length $l(G)$ of G and such that the discriminant form induced by (L, Q_p) is isomorphic to (G, q_p). Finally, we will state Nikulin's Theorem on the existence of lattice embeddings.

In **Chapter 8**, we describe an algorithm to determine all ADE lattices Λ such that $\langle 6\rangle \oplus \Lambda$ can be embedded primitively into the K3 lattice L_{K3}. We wrote a code based on this algorithm to be implemented in the computer-algebra software Wolfram Mathematica which gives us the full list of these ADE lattices Λ. Independently from our computation, S. Brandhorst found the same list with an algorithm implemented in the computer-algebra software Sage. We will then be able to prove our second main result:

Main Theorem 2. *Let*

$$\mathbf{G} \coloneqq \sum_{i=1}^{19} a_i \mathbf{A}_i + \sum_{j=4}^{19} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k$$

be a formal sum of ADE singularities such that the ADE lattice

$$\Lambda \coloneqq \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k$$

is one of the 2942 elements in the list in Appendix C. The following hold:

1. *There exists a complete $(2,3)$-intersection S of a smooth quadric and a cubic in $\mathbb{P}^4$ such that S has singularities of type $\mathbf{G}$.*

2. *There exists a cubic fourfold with ADE singularities of type $\mathbf{G}$ and an $\mathbf{A}_1$ singularity.*

In **Chapter 9**, we will firstly recall the notion of lattice polarized K3 surfaces. For a combination $\mathbf{G}$ of ADE singularities, $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ an ADE singularity type, and $\Lambda(\Gamma_{\sigma(\mathbf{T})})$ and $\Lambda(\Gamma_{\mathbf{G}})$ as above, let

$$i\colon \Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}}) \hookrightarrow L_{K3}$$

be an embedding into the K3 lattice which is unique up to automorphisms of L_{K3} and $\mathrm{Sat}_{L_{K3}}(i)$ the saturation of $\Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})$ in L_{K3} with respect to i.

We will construct the moduli space $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ of all quasi-polarized K3 surfaces $(\widetilde{S}, L_{\mathbf{T}})$ of degree 6 such that

1. $\varphi_{L_{\mathbf{T}}} : \widetilde{S} \to \mathbb{P}^4$ is birational onto its image
2. $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ is contained in a quadric $Q \subseteq \mathbb{P}^4$ of $\mathrm{corank}(Q) = \mathrm{corank}_{\mathbf{T}}$ such that
   a) the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ lying on $\mathrm{Sing}(Q)$ correspond to $\sigma(\mathbf{T})$
   b) the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ not lying on $\mathrm{Sing}(Q)$ correspond to $\mathbf{G}$

as an open subset of the moduli space of certain $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surfaces. Likewise, we will construct the moduli space of all cubic fourfolds $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$ having singularities corresponding to $\mathbf{G}$ and $\mathbf{T}$. Finally, we will prove our third Main Theorem.

Main Theorem 3. *We have an isomorphism of quasi-projective varieties*

$$\mathcal{M}^{\mathbf{T}+\mathbf{G}} \xrightarrow{\sim} \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}.$$

Contents

1 ADE singularities

In this chapter, we will define ADE singularities of complex analytic spaces and state basic properties of those. In particular, we will recall that on a surface we can identify an ADE singularity with the Dynkin diagram associated to the exceptional divisor of the minimal resolution of this ADE singularity. This chapter provides a foundation to the following chapters where we study ADE lattices and ADE singularities on both cubic fourfolds and complete $(2,3)$-intersections in $\mathbb{P}^4$.

1.1 Basic notation, definitions, and properties

Let X be a complex analytic space of dimension d.

Let p be a singularity of X and assume that the germ $(X,p) \subseteq (\mathbb{C}^{d+1},p)$ is an isolated hypersurface singularity. The *(analytic) type* of p is the equivalence class of the germ (X,p) with respect to local analytic isomorphisms. We say that X has an *ADE singularity of type* $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ in p if the analytic type of p is the equivalence class of the germ defined by the following equation $\mathbf{T}$ on $\mathbb{C}^{d+1}$ at $(0,\dots,0) \in \mathbb{C}^{d+1}$:

$$
\begin{aligned}
&\mathbf{A}_i\colon && x_1^2 + \ldots + x_{d-1}^2 + x_d^2 + x_{d+1}^{i+1} = 0 && (i \geq 1)\\
&\mathbf{D}_j\colon && x_1^2 + \ldots + x_{d-1}^2 + x_d^2 x_{d+1} + x_{d+1}^{j-1} = 0 && (j \geq 4)\\
&\mathbf{E}_6\colon && x_1^2 + \ldots + x_{d-1}^2 + x_d^4 + x_{d+1}^3 = 0 &&\\
&\mathbf{E}_7\colon && x_1^2 + \ldots + x_{d-1}^2 + x_d^3 x_{d+1} + x_{d+1}^3 = 0 &&\\
&\mathbf{E}_8\colon && x_1^2 + \ldots + x_{d-1}^2 + x_d^5 + x_{d+1}^3 = 0, &&
\end{aligned}
$$

where $x_1,\dots,x_{d+1}$ are analytic coordinates on $\mathbb{C}^{d+1}$. We call the germ defined by the equation $\mathbf{T}$ in $\mathbb{C}^{d+1}$ at $(0,\dots,0) \in \mathbb{C}^{d+1}$ a $\mathbf{T}$ *type*. We will call a singularity p simply an *ADE singularity* if it is an ADE singularity of any type $\mathbf{T}$. Let

$$\mathbf{G} \coloneqq \sum_{i\geq 1} a_i \mathbf{A}_i + \sum_{j\geq 4} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k$$

be a (formal) sum of ADE types. If X has a_i isolated singularities of type $\mathbf{A}_i$ $(i \geq 1)$, d_j isolated singularities of type $\mathbf{D}_j$ $(j \geq 4)$, and e_k isolated singularities of type $\mathbf{E}_k$ $(8 \geq k \geq 6)$, we say that the *singularities of X correspond to* $\mathbf{G}$.

A direct computation shows that an ADE singularity is resolved by finitely many blowing-ups in finitely many points. Indeed, in Table 1.1 we can find for a singularity of type $\mathbf{T}$ on X the singularities $\sigma(\mathbf{T})$ occurring on the exceptional divisor of the blowing-up $\pi_p\colon \mathrm{Bl}_p X \to X$ of X in p.

We say that a complex space germ (X, p) defined by $\mathbf{T}$ is *adjacent* to the complex space germ (X', p') defined by $\mathbf{T}'$ (up to analytic isomorphism) if the germ (X, p) can be deformed by an arbitrarily small deformation into the germ (X', p'). For *ADE* singularities, the adjacencies are known, see [AGLV98, Chap. 2.2.7].

$\mathbf{T}$	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_{n\geq 3}$	$\mathbf{D}_4$	$\mathbf{D}_{n\geq 5}$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
$\sigma(\mathbf{T})$	$\emptyset$	$\emptyset$	$\mathbf{A}_{n-2}$	$3\mathbf{A}_1$	$\mathbf{A}_1 + \mathbf{D}_{n-2}$	$\mathbf{A}_5$	$\mathbf{D}_6$	$\mathbf{E}_7$

Table 1.1: Singularities corresponding to $\sigma(\mathbf{T})$ on the exceptional divisor of the blowing-up of a singularity of type $\mathbf{T}$. We understand $\mathbf{D}_3$ as $\mathbf{A}_3$. See [DR01, Lemma 2.1].

1.2 *ADE* singularities on surfaces

Let C be a curve on a smooth surface with components $C_1, \dots, C_s$. The *(weighted) graph* associated to $C_1, \dots, C_s$ is the graph whose vertices are the curves C_i with *weights* $C_i.C_i$ and such that two vertices C_i and C_j are joint by $C_i.C_j$ edges.

If S is a surface, it is well known that we can identify the *ADE* type of a singularity p on S by its weighted graph associated to the exceptional divisor of the minimal resolution of p:

Theorem 1.2.1 ([Dur79, Theorem A]). *Let S be a normal surface with a singularity p. Let $\pi\colon \widetilde{S} \to (S, p)$ be the minimal resolution of the germ (S, p) whose exceptional divisor $E := \pi^{-1}(p)$ is the union of the irreducible curves $E_1, \dots, E_s$. Then, p has type $\mathbf{T} = \mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}$, or $\mathbf{E}_{8\geq k\geq 6}$ if and only if the weighted graph associated to $E_1, \dots, E_s$ is the Dynkin diagram $\mathcal{T} = \mathcal{A}_{i\geq 1}, \mathcal{D}_{j\geq 4}$, or $\mathcal{E}_{8\geq k\geq 6}$, respectively, listed in [Dur79, Table 1].*

We will refer to the graph associated to the irreducible curves in the exceptional divisor of the minimal resolution of an *ADE* singularity p as in Theorem 1.2.1 for short as the *Dynkin diagram of the minimal resolution of* p.

We call a disjoint finite union of connected Dynkin diagrams of type *ADE* again a *Dynkin diagram.*

If Γ is a Dynkin diagram with a_i, d_j, and e_k connected components $\mathcal{A}_i$ $(i \geq 1)$, $\mathcal{D}_j$ $(j \geq 4)$, and $\mathcal{E}_k$ $(8 \geq k \geq 6)$, we will write Γ as the (formal) sum

$$\Gamma = \sum_{i\geq 1} a_i \mathcal{A}_i + \sum_{j\geq 4} d_j \mathcal{D}_j + \sum_{k=6}^{8} e_k \mathcal{E}_k.$$

We note one further characterization of *ADE* singularities on surfaces:

Theorem 1.2.2 ([Dur79, Theorem A]). *Let S be a normal surface with a singularity p and $\pi\colon \widetilde{S} \to (S, p)$ the minimal resolution of the germ (S, p). Then, p has ADE type if and only if p is a rational singularity, i.e. the higher direct image sheaf $R^i\pi_*\mathcal{O}_{\widetilde{S}}$ is trivial for all $i > 0$.*

2 Bilinear forms, quadratic forms, and quadratic modules

In this chapter, we will introduce symmetric bilinear forms, quadratic forms, and quadratic modules and then define a lattice as an integral non-degenerate bilinear form. In particular, we are interested in the lattices which we associate to the Dynkin diagrams of the minimal resolutions of ADE singularities and the K3 lattice. This chapter provides a basis for the chapters where we study ADE singularities on complete $(2,3)$-intersections in $\mathbb{P}^4$ in terms of lattices.

2.1 Basic notation, definitions, and properties

Let R be a commutative ring with 1.

A *symmetric bilinear form over* R is a pair $(L, \langle\, ,\, \rangle_L)$, where L is an R-module and

$$\langle\, ,\, \rangle_L \colon L \times L \to R$$

is a function which is symmetric and R-bilinear.

For simplicity and by abuse of notation, we will often write L instead of $(L, \langle\, ,\, \rangle_L)$ and the associated function $\langle\, ,\, \rangle_L$ is assumed to be given.

We will call $(L, \langle\, ,\, \rangle_L)$ *non-degenerate* if $\langle\, ,\, \rangle_L$ is non-degenerate. For $x, y \in L$, we will write $x.y$ and x^2 instead of $\langle x, y\rangle_L$ and $\langle x, x\rangle_L$, respectively.

A *quadratic form over* R is a pair (L, Q_L), where L is an R-module and Q_L is a function such that

1. $Q_L(rl) = r^2 Q_L(l)$ for all $r \in R$ and $l \in L$
2. $\langle\, ,\, \rangle_{Q_L} \colon L \times L \to R$, $(x, y) \mapsto Q_L(x+y) - Q_L(x) - Q_L(y)$ is a symmetric bilinear form over R.

Remark 2.1.1. Note that we defined here the quadratic form as in [MM09, Chap. I.4.1]; in the literature one can find more often the requirement that $\langle x, y\rangle_{Q_L} = \frac{1}{2}\big(Q_L(x+y) - Q_L(x) - Q_L(y)\big)$.

In the cases we will consider in the following chapters, a symmetric bilinear form will induce a unique quadratic form and vice versa:

Lemma 2.1.2 ([MM09, Chap. I, Corollary 2.4])**.** *Assume that* 2 *is not a zero divisor in* R*. Let* $(L, \langle\, ,\, \rangle_L)$ *be a symmetric bilinear over* R *such that there exists a quadratic form* (L, Q_L) *over* R *with* $\langle\, ,\, \rangle_L = \langle\, ,\, \rangle_{Q_L}$*. Then,* (L, Q_L) *is uniquely determined.*

For two quadratic forms (L_1, Q_{L_1}) and (L_2, Q_{L_2}), the direct sum $(L_1 \oplus L_2, Q_{L_1} + Q_{L_2})$ is the *orthogonal direct sum* (i.e. for $x_1 \in L_1$ and $x_2 \in L_2$, $(Q_{L_1} + Q_{L_2})(x_1 + x_2) = Q_{L_1}(x_1) + Q_{L_2}(x_2)$).

A *homomorphism* $(L_1, Q_{L_1}) \to (L_2, Q_{L_2})$ *between two quadratic forms* is an R-module homomorphism $\phi\colon L_1 \to L_2$ such that $Q_{L_2} \circ \phi = Q_{L_1}$.

A *quadratic R-module* is a non-degenerate quadratic form (L, Q_L) over R such that L is a finitely generated free R-module. Let $\langle\, , \rangle_{Q_L}$ be the bilinear function associated to Q_L and let $s_1, \dots, s_n$ be a basis of L. The *intersection matrix* of (L, Q_L) (or equivalently of $(L, \langle\, , \rangle_{Q_L})$) is the symmetric $n \times n$ matrix

$$M_{(L,Q_L)} \coloneqq (\langle s_i, s_j \rangle_{Q_L})_{i,j=1,\dots,n} \in \mathrm{Mat}_n(R).$$

On the other hand, the intersection matrix determines the bilinear function $\langle\, , \rangle_{Q_L}$. Indeed, let $e_1, \dots, e_n$ be the standard basis on R^n and $\phi\colon L \to R^n$, $s_i \mapsto e_i$ the coordinate isomorphism, then $\langle x, x' \rangle_{Q_L} = \phi(x)^T M_{(L,Q_L)} \phi(x')$.

If (L, Q_L) is a quadratic R-module, the *discriminant*

$$\mathrm{disc}(L) \coloneqq \det(M_{(L,Q_L)}) \in R/(R^\times)^2$$

of (L, Q_L) is the determinant in $R/(R^\times)^2$ of the intersection matrix $M_{(L,Q_L)}$ with respect to an arbitrary basis of L.

Lemma 2.1.3. *For a direct sum $(L_1 \oplus L_2, Q_{L_1 \oplus L_2})$ of quadratic R-modules, we have* $\mathrm{disc}(L_1 \oplus L_2) = \mathrm{disc}(L_1) \cdot \mathrm{disc}(L_2)$.

Proof. The intersection matrix $M_{L_1 \oplus L_2, Q_{L_1 \oplus L_2}}$ is a block diagonal matrix with blocks given by $M_{L_1,Q_{L_1}}$ and $M_{L_2,Q_{L_2}}$. Hence, $\det(M_{L_1 \oplus L_2, Q_{L_1 \oplus L_2}}) = \det(M_{L_1,Q_{L_1}}) \cdot \det(M_{L_2,Q_{L_2}})$. □

2.2 Lattices

We call a non-degenerate symmetric bilinear form $(L, \langle\, , \rangle_L)$ over $\mathbb{Z}$ a *lattice* if L is a finitely generated free $\mathbb{Z}$-module.

The lattice L is called *even* if $x^2 \in 2\mathbb{Z}$ for all $x \in L$ and *odd* otherwise. We say that the lattice L is *unimodular* if $\mathrm{disc}(L) = \pm 1$.

The *rank* $\mathrm{rank}(L)$ of a lattice L is the rank of its underlying free $\mathbb{Z}$-module.

We call $(L', \langle\, , \rangle_{L'})$ a *sublattice* of $(L, \langle\, , \rangle_L)$ if L' is a $\mathbb{Z}$-submodule of L and $\langle\, , \rangle_{L'}$ is the restriction of $\langle\, , \rangle_L$ to L'. The lattice L is called *irreducible* if it cannot be written as the orthogonal direct sum of two proper sublattices.

Let $i\colon L_1 \hookrightarrow L$ be an injective homomorphism. Then, we say that i is a *primitive embedding* and $i(L_1)$ is a *primitive sublattice* of L if the cokernel of i is torsion free. We call

$$\mathrm{Sat}_L(i) \coloneqq \{x \in L;\, mx \in i(L_1) \text{ for some } m \in \mathbb{Z}\}$$

the *saturation* of L_1 in L. The lattice $\mathrm{Sat}_L(i)$ is the smallest primitive sublattice of L containing $i(L_1)$.

The *signature* of L is the pair (n_+, n_-), where n_+ is the number of positive eigenvalues and n_- the number of negative eigenvalues of the extension of $\langle\,,\,\rangle_L$ to the real vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$. The lattice L is *positive definite* if $n_- = 0$, *negative definite* if $n_+ = 0$, and *indefinite* otherwise.

An element $x \in L$ is *primitive* if the intersection of $x\mathbb{Q}$ with L in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by x, i.e. x cannot be written in the form $x = my$ with $m > 1$.

The following three definitions will be only needed at the end of Section 9.4:

An element $x \in L$ is *isotropic* if $x^2 = 0$. The *divisibility* of $x \in L$ is the positive integer $div(x)$ such that $\langle x, L\rangle_L = div(x)\mathbb{Z}$. We then call an isotropic primitive element $x \in L$ *m-admissible* if $div(x) = m$ and there exists an isotropic primitive element $y \in L$ with $\langle x, y\rangle_L = m$ and $div(y) = m$.

We will refer in the sequel to the following lattices:

Example 2.2.1. 1. $\langle m\rangle$ denotes the rank 1 lattice with intersection matrix (m).

2. The *hyperbolic plane* U is the even, unimodular, indefinite rank two lattice with intersection matrix
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
U has signature $(1, 1)$.

3. The lattice $\Lambda(\Gamma)$ associated to a weighted graph Γ: The underlying free $\mathbb{Z}$-module of $\Lambda(\Gamma)$ is generated by the vertices of Γ and the underlying bilinear form is given by the intersection matrix defined by the vertices of Γ. For simplicity, if Γ is one of the Dynkin diagrams $\mathcal{T} = \mathcal{A}_{i\geq 1}, \mathcal{D}_{j\geq 4}$, or $\mathcal{E}_{8\geq k\geq 6}$, we will denote the associated negative definite lattice $\Lambda(\Gamma)$ by $T = A_{i\geq 1}, D_{j\geq 4}$, or $E_{8\geq k\geq 6}$, respectively. By [Ebe13, Theorem 1.2], the lattice T is irreducible.

   For instance, the A_2 lattice is defined by the intersection matrix
$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$
   We will call a lattice Λ which is the orthogonal direct sum of irreducible ADE lattices for short *ADE lattice*.

4. The *K3 lattice*
$$L_{K3} := 3U \oplus 2E_8$$
   is the unique even and unimodular lattice of signature $(3, 19)$.

3 K3 surfaces

In this chapter, we study K3 surfaces. After recalling all necessary definitions, we will investigate under which conditions the complete linear system induced by a quasi-polarization L on a K3 surface S, defines a birational morphism φ_L from S onto its image in the projective space. We will show that if φ_L is birational, the existence of certain ADE lattices in the Picard group will imply the existence of corresponding ADE singularities on $\varphi_L(S)$ in the projective space. In particular, if $L^2 = 6$, we will see that $\varphi_L(S)$ is a complete $(2,3)$-intersection in $\mathbb{P}^4$. Finally, we will prove the existence of a K3 surface having a certain Picard group. This chapter is a foundation to the following chapters where we relate the existence of embeddings of ADE lattices into the K3 lattice to the existence of complete $(2,3)$-intersections in $\mathbb{P}^4$ having corresponding ADE singularities.

3.1 Basic notation, definitions, and properties

A *K3 surface* is a smooth complex projective surface S with trivial canonical bundle ω_S and $H^1(S, \mathcal{O}_S) = 0$.

Let S be a K3 surface.

The exponential sequence induces the exact sequence

$$0 \to \operatorname{Pic}(S) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \xrightarrow{\exp^*} H^2(S,\mathcal{O}_S).$$

Since $H^2(S,\mathbb{Z})/c_1\big(\operatorname{Pic}(S)\big)$ injects into $H^2(S,\mathcal{O}_S)$ and since $H^2(S,\mathcal{O}_S) \cong \mathbb{C}$ is torsion-free, the embedding $c_1 \colon \operatorname{Pic}(S) \hookrightarrow H^2(S,\mathbb{Z})$ is primitive. We will identify $\operatorname{Pic}(S)$ with its image in $H^2(S,\mathbb{Z})$.

Let $L \in \operatorname{Pic}(S)$.

The Riemann-Roch Theorem yields

$$h^0(S,L) + h^0(S,L^\vee) \geq 2 + \frac{1}{2}L^2, \tag{3.1}$$

where $L^\vee \in \operatorname{Pic}(S)$ is the dual line bundle of L. Hence, we can conclude:

Lemma 3.1.1. *Assume that $L^2 \geq -2$. Then, either L or $L^\vee \in \operatorname{Pic}(S)$ is effective.*

We say that L is *nef* (*ample*) if $L.C \geq 0$ ($L.C > 0$) for all curves C on S (for the general definition of ample and nef line bundles on schemes see [Laz04, 1.2, 1.4]). We call L *big and nef* if L is nef and $L^2 > 0$.

We call L a *quasi-polarization of degree d* if L is big and nef such that $L^2 = d$ and L is primitive, i.e. there exists no line bundle $L' \in \operatorname{Pic}(S)$ such that $L = (L')^k$ for $k \geq 2$. We call two quasi-polarized K3 surfaces (S, L) and (S', L') *isomorphic* if their exists an isomorphism $\phi\colon S \to S'$ between the K3 surfaces preserving the quasi-polarization, i.e. $L = \phi^* L'$.

3.2 Linear systems on K3 surfaces

Let S be a surface and L a line bundle on S. Write $|L|$ for the complete linear system on S given by L, i.e. the space of all effective divisors linearly equivalent to L. We can show that we have $|L| = \mathbb{P}(H^0(S,L))$.

We follow [Huy16, Chap. 2.1.1] and call a divisor F on S the *fixed part* of $|L|$ if F is the biggest effective divisor on S contained in all elements of $|L|$, i.e. F is the one-dimensional part of the base locus of $|L|$. We call a point $p \in S$ a *fixed point* of $|L|$ if p is contained in every element of $|L|$. The *mobile part* $M := L(-F)$ of L is fixed part free and has only finitely many fixed points. Further, the mobile part is nef and satisfies $M^2 \geq 0$. We can then decompose L into its mobile and fixed part and write $L = M + F$.

Assume now that S is a K3 surface.

We call a curve C on S a (-2)-*curve* if C is irreducible and $C^2 = -2$. It is known ([Huy16, Chap. 2.1, p. 23]) that a (-2)-curve C is in fact smooth and rational, i.e. $C \cong \mathbb{P}^1$.

Lemma 3.2.1 ([Huy16, Chap. 2, Lemma 1.3])**.** *The fixed part F of a linear system on S is a linear combination of (-2)-curves, i.e. $F = \sum_{i=1}^n a_i C_i$ with $a_i \geq 0$ and C_i a (-2)-curve $(i = 1, \dots, n)$.*

Lemma 3.2.2 ([Huy16, Chap. 2, Corollary 1.5])**.** *Let L be a line bundle on S with $L^2 \geq 0$ and such that $L.C \geq 0$ for all (-2)-curves C on S. Then, L is nef unless there exists no (-2)-curve on S in which case L or $L^\vee$ is nef.*

The restriction of the intersection product on $H^2(S,\mathbb{R})$ to $H^{1,1}(S,\mathbb{R}) := H^2(S,\mathbb{R}) \cap H^1(S,\Omega_S^1)$ has signature $(1,19)$. Hence, the subspace $\{x \in H^{1,1}(S,\mathbb{R});\, x.x > 0\}$ has two connected components. Let $\mathcal{C}_S$ be the connected component that contains one and hence all Kähler classes. We call $\mathcal{C}_S$ the *positive cone* of S.

For $R \in H^2(S,\mathbb{Z})$ with $R^2 = -2$, we have a reflection

$$s_R \colon H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{Z}),\ P \mapsto P + (P.R)R$$

called *Picard-Lefschetz reflection*. We note that s_R preserves the intersection form.

Proposition 3.2.3 ([Huy16, Chap. 8, Corollary 2.9])**.** *For a line bundle L on S with $L^2 > 0$ such that $L \in \mathcal{C}_S$, there exist finitely many (-2)-curves $C_1, \dots, C_n \in \operatorname{Pic}(S)$ such that $(s_{C_1} \circ \dots \circ s_{C_n})(L)$ is nef.*

Theorem 3.2.4 ([May72, Proposition 1, 5], [Nik91, Proposition 0.1])**.** *Let L be a nef line bundle on S. Then, one of the following holds:*

1. *$L^2 > 0$, $|L|$ is fixed point free. A generic member of $|L|$ is an irreducible curve and we have $\dim |L| = 1 + L^2/2 > 0$.*
2. *$L^2 > 0$, $|L| = m|E| + F$ with $m > 1$, where $|E|$ is an elliptic pencil, F is a (-2)-curve, and $E.F = 1$. Then, $m = \dim |L|$ and F is the fixed part of $|L|$.*
3. *$L^2 = 0$, $|L| = \emptyset$.*
4. *$L^2 = 0$, $|L| = m|E|$ with $m \geq 1$ and $|E|$ is an elliptic pencil.*

Remark 3.2.5. Note that in case 4. in Theorem 3.2.4, a general member of $|E|$ is in particular irreducible, see [Huy16, Chap. 2, Proposition 3.10].

If $L^2 \geq 0$, inequality (3.1) implies that (after possibly replacing L by $L^\vee$) L has more than one global section. Hence, the linear system $|L|$ on S induces a rational map

$$\varphi_L \colon S \dashrightarrow \mathbb{P}^{\dim |L|}$$

which is a morphism outside its base locus.

Proposition 3.2.6. *Let L be a nef line bundle on S with $L^2 \geq 4$. Then, φ_L fails to be a birational morphism onto a surface of degree L^2 in $\mathbb{P}^{\dim |L|}$ if and only if one of the following holds:*

1. *There exists $E \in \operatorname{Pic}(S)$ such that $E^2 = 0$ and $L.E = 1$.*
2. *There exists $E \in \operatorname{Pic}(S)$ such that $E^2 = 0$ and $L.E = 2$.*

Proof. Assume that φ_L fails to be birational onto its image in $\mathbb{P}^{\dim |L|}$. By [SD74, (4.1)], the complete linear system $|L|$ then has either a fixed part or φ_L is of degree 2 and its image has degree $L^2/2$. Since $L^2 \geq 4$, by [SD74, Theorem 5.2] the latter case can only occur if item 2. holds. If $|L|$ has a fixed part, we have $|L| = m|E| + F$ for a (-2)-curve F and an elliptic pencil $|E|$ such that $E.F = 1$ by Theorem 3.2.4. Hence, $L.E = 1$, i.e. item 1. holds.

Then, assume that there exists $E \in \operatorname{Pic}(S)$ with $E^2 = 0$ such that $L.E = 1$ or 2. We assume to the contrary that φ_L is birational onto its image in $\mathbb{P}^{\dim |L|}$. By (3.1), we have $h^0(S, E) > 0$ or $h^0(S, E^\vee) > 0$. However, if $h^0(S, E^\vee) > 0$, we obtain for $A \in |E^\vee|$ that $L.A = -L.E = -1$ or -2 in contradiction to L being nef. Hence, we have $h^0(S, E) > 0$, i.e. E is effective. Let $M + \Gamma$ be a general member in $|E|$, where $|M|$ is the mobile part and Γ the fixed part of $|E|$. Since $|M|$ is fixed part free and $M^2 \geq 0$, every irreducible component of M has by Theorem 3.2.4 a non-negative self-intersection number. Since L is nef, we have $L.\Gamma \geq 0$ and $L.M \geq 0$. However, $L.M = 0$ would imply $M^2 < 0$ by the Hodge-Index Theorem (see [SD74, (4.2)]) which is absurd. Hence, $L.M > 0$. Then, $L.E = L.M + L.\Gamma = 1$ or 2 implies that $L.M = 1$ or 2. Since φ_L is by assumption birational onto its image and generically one-to-one on M, we deduce that $\varphi_L(M)$ is a curve in $\mathbb{P}^{\dim |L|}$ with degree ≤ 2. By [Mum95, Corollary 5.13], an irreducible component of $\varphi_L(M)$ is then isomorphic to $\mathbb{P}^1$. Hence, M has an irreducible component which is isomorphic to $\mathbb{P}^1$. This is a contradiction to M having only irreducible components with non-negative self-intersection number according to Theorem 3.2.4. Therefore, the assumption must be wrong and φ_L is not birational onto its image. $\square$

Remark 3.2.7. We will call a line bundle as in item 1. in Proposition 3.2.6 *unigonal* and a line bundle as in item 2. *hyperelliptic*.

Remark 3.2.8. We note that item 1. in Proposition 3.2.6 is redundant. Indeed, if φ_L fails to be birational and $|L|$ has a fixed part, we argue as in the proof above that we have $E \in \operatorname{Pic}(\widetilde{S})$ such that $E^2 = 0$ and $E.F = 1$. For $E' := 2E$, we then have $E'^2 = 0$ and $L.E' = 2$. Hence, E' satisfies item 2. Conversely, the existence of a line bundle $E \in \operatorname{Pic}(\widetilde{S})$ as in item 2. implies that φ_L is not birational as shown in the proof above.

3.3 (-2)-curves on K3 surfaces

Let (S, L) be a quasi-polarized K3 surface with $L^2 > 0$.

Define

$$R_L \coloneqq \{[C] \in \mathrm{Pic}(S);\, C^2 = -2, L.C = 0\}.$$

Then, R_L is a finite root system (see [Bou07, Chap. VI §1] for the definition of root system). Let

$$\Delta_L \coloneqq \{[C] \in \mathrm{Pic}(S);\, C \ (-2)\text{-curve}, L.C = 0\}.$$

By [SS19, Lemma 11.17], every element in R_L can be written as a non-negative sum of elements in Δ_L. Hence, Δ_L is a basis (sometimes called fundamental system) of the root system R_L (see also [Bou07, Chap. VI §1] for the definition of a basis of a root system).

Proposition 3.3.1. *Let $\Delta_L^1, \dots, \Delta_L^n$ be the connected components of Δ_L. The intersection matrix of the (-2)-curves in Δ_L^i $(i = 1, \dots, n)$ is described by the Dynkin diagram $\mathcal{A}_{n\geq 1}$, $\mathcal{D}_{n\geq 4}$, or $\mathcal{E}_{8\geq n\geq 6}$.*

Proof. Let $\Delta_L^i = \cup_{j=1}^m C_j^i$, where all C_j^i are (-2)-curves. By the Hodge-Index Theorem (see [Har77, Chap. V, Theorem 1.9]), the intersection matrix $(C_r^i.C_s^i)_{1\leq i,j\leq m}$ is negative definite. One then computes the possible intersection products $C_r^i.C_s^i$ for all $r, s = 1, \dots, m$ (see [BHPVdV04, Chap. III.2.iii)]). □

Theorem 3.3.2. *Let Δ_L^i $(i = 1, \dots, n)$ be as in Proposition 3.3.1. There exists a projective normal surface S' and a morphism*

$$\theta \colon S \to S'$$

such that θ maps each Δ_L^i to an ADE singularity p_i and $\theta \colon S \setminus \cup_{i=1}^n \Delta_L^i \to S' \setminus \cup_{i=1}^n p_i$ is an isomorphism. The singularity types of the p_i are determined by the Dynkin diagrams associated to Δ_L^i.

Proof. The existence of θ follows from [Art62, Theorem 2.7]. By Proposition 3.3.1, the (-2)-curves in Δ_L^i $(i = 1, \dots, n)$ are the vertices of a Dynkin diagram $\mathcal{A}_{n\geq 1}$, $\mathcal{D}_{n\geq 4}$, or $\mathcal{E}_{8\geq n\geq 6}$ and by Theorem 1.2.1, the singularity p_i has type $\mathbf{A}_{n\geq 1}$, $\mathbf{D}_{n\geq 4}$, or $\mathbf{E}_{8\geq n\geq 6}$, respectively. □

Definition 3.3.3. We call the morphism θ in Theorem 3.3.2 the *contraction morphism* of the connected components $\Delta_L^1, \dots, \Delta_L^n$ of Δ_L.

The next proposition states that we can identify the normal surface S' in Theorem 3.3.2 with the image $\varphi_L(S)$ of S under φ_L in $\mathbb{P}^4$.

Proposition 3.3.4 ([SD74, Theorem 6.1 (iii)])**.** *Assume that L is a fixed part free line bundle on S such that $\varphi_L \colon S \to \mathbb{P}^{\dim |L|}$ is birational onto its image. Then, φ_L admits a factorization $\varphi_L = u_L \circ \theta$ by the contraction morphism θ and an embedding $u_L \colon S' \to \mathbb{P}^{\dim |L|}$. Further, if $L^2 = 6$, the surface $\varphi_L(S)$ is the complete $(2,3)$-intersection of a quadric and a cubic in $\mathbb{P}^4$.*

Corollary 3.3.5. *Assume that L is a fixed part free line bundle on S with $L^2 > 0$ such that $\varphi_L\colon S \to \mathbb{P}^{\dim|L|}$ is birational onto its image. Let K be the lattice in* $\mathrm{Pic}(S)$ *generated by the elements in the root system R_L. Assume that*

$$K := \bigoplus_{i\geq 1} a_i A_i \oplus \bigoplus_{j\geq 4} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k.$$

Then, $\varphi_L(S) \subseteq \mathbb{P}^{\dim|L|}$ has ADE singularities corresponding to

$$\mathbf{G} := \sum_{i\geq 1} a_i \mathbf{A}_i + \sum_{j\geq 4} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k.$$

Proof. Let $\Delta_L^1, \ldots, \Delta_L^n$ be the connected components of Δ_L. By Proposition 3.3.1, the intersection matrix of the (-2)-curves in Δ_L^i is described by a connected Dynkin diagram. Let $\Gamma' := \sum_{i\geq 1} a_i' \mathcal{A}_i + \sum_{j\geq 4} d_j' \mathcal{D}_j + \sum_{8\geq k\geq 6} e_k' \mathcal{E}_k$ be the union of all Dynkin diagrams associated to the union of the Δ_L^i and let $\Lambda(\Gamma') = \bigoplus_{i\geq 1} a_i' A_i \oplus \bigoplus_{j\geq 4} d_j' D_j \oplus \bigoplus_{8\geq k\geq 6} e_k' E_k$ be the associated ADE lattice.

Since Δ_L is the basis of R_L, we have $K = \Lambda(\Gamma')$, i.e.

$$\bigoplus_{i\geq 1} a_i A_i \oplus \bigoplus_{j\geq 4} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k = \bigoplus_{i\geq 1} a_i' A_i \oplus \bigoplus_{j\geq 4} d_j' D_j \oplus \bigoplus_{k=6}^{8} e_k' E_k.$$

We claim that $a_i = a_i'$ $(i \geq 1)$, $d_j = d_j'$ $(j \geq 4)$, $e_k = e_k'$ $(8 \geq k \geq 6)$. Indeed, let M be an irreducible ADE lattice in the left-hand direct sum. Suppose that M is not contained in any irreducible ADE lattice in $\Lambda(\Gamma')$. Since M is contained $\Lambda(\Gamma')$, this would imply that M is the orthogonal direct sum of two sublattices of M. However, this is absurd since M is irreducible. Consequently, M is contained in one irreducible ADE lattice N in $\Lambda(\Gamma')$. Conversely, the same argument gives that the ADE lattice N has to be contained in an irreducible ADE lattice M' in K. Since N contains M, it follows that the irreducible ADE lattice M is contained in the irreducible ADE lattice M'. Since M was a direct summand in K, this forces $M = M'$. Consequently, it follows that any irreducible ADE lattices in K coincides with an irreducible ADE lattices in $\Lambda(\Gamma')$ and vice versa. In conclusion, $a_i = a_i'$ $(i \geq 1)$, $d_j = d_j'$ $(j \geq 4)$, $e_k = e_k'$ $(8 \geq k \geq 6)$. By Theorem 3.3.2, there exists a projective normal surface S' whose singularities correspond to $\mathbf{G}$ and a contraction morphism $\theta\colon S \to S'$. By Proposition 3.3.4, we have a factorization $\varphi_L = u_L \circ \theta$ through an embedding $u_L\colon S' \to \mathbb{P}^4$. Hence, $\varphi_L(S)$ has singularities corresponding to $\mathbf{G}$. □

3.4 Periods of K3 surfaces

For a K3 surface S, the integral cohomology $H^2(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module. The intersection form on $H^2(S, \mathbb{Z})$ turns it into a lattice of signature $(3, 19)$. Since this lattice is even and unimodular, it is isometric to the K3 lattice

$$L_{K3} = 3U \oplus 2E_8$$

independent of the choice of S (see [Mil58, Corollary §1]). We refer to an isometry $\phi\colon H^2(S,\mathbb{Z}) \to L_{K3}$ as a *marking* of S and to a pair (S,ϕ) as a *marked K3 surface*. For $H^2(S,\mathbb{C}) = H^2(S,\mathbb{Z}) \otimes \mathbb{C}$, we have the Hodge decomposition

$$H^2(S,\mathbb{C}) = H^2(S,\mathcal{O}_S) \oplus H^1(S,\Omega^1_S) \oplus H^0(S,\Omega^2_S). \tag{3.2}$$

Since S is a K3 surface, $\mathrm{Pic}(S)$ is isomorphic to $H^2(S,\mathbb{Z}) \cap H^1(S,\Omega^1_S)$ under the embedding $c_1\colon \mathrm{Pic}(S) \hookrightarrow H^2(S,\mathbb{Z})$. Let ω_S be a generator of the one-dimensional $\mathbb{C}$-vector space $H^2(X,\mathcal{O}_S)$. We note in particular that ω_S is uniquely determined up to a scalar multiple in $\mathbb{C}^*$. Hence, a marked K3 surface (S,ϕ) determines uniquely a point $[\phi(\omega_S)] = \phi(\omega_S) \mod \mathbb{C}^* \in \mathbb{P}(L_{K3} \otimes_{\mathbb{Z}} \mathbb{C})$ which we call the *period point* of (S,ϕ). We will call the 20-dimensional connected complex manifold

$$\Omega_{L_{K3}} \coloneqq \{[x] \in \mathbb{P}(L_{K3} \otimes \mathbb{C});\, x^2 = 0, x.\overline{x} > 0\} \tag{3.3}$$

the *period domain* of L_{K3}. We note that the period point $[\phi(\omega_S)]$ is contained in $\Omega_{L_{K3}}$. Further, for each $x \in H^2(S,\mathbb{Z}) \cap H^1(S,\Omega^1_S)$, we have $x.\omega_S = 0$ by the Hodge decomposition (3.2). Hence, we deduce

Lemma 3.4.1. *We have* $\mathrm{Pic}(S) = \{x \in H^2(S,\mathbb{Z});\, x.\omega_S = 0\}$.

Let $\pi\colon \mathcal{S} \to \mathcal{U}$ be a flat family of K3 surfaces with central fiber $S \coloneqq \pi^{-1}(0) \in \mathcal{S}$ over $0 \in \mathcal{U}$. For a sufficiently small contractible open neighborhood $U \subseteq \mathcal{U}$ of $0 \in \mathcal{U}$, a marking $\phi\colon H^2(S,\mathbb{Z}) \to L_{K3}$ can be extended to a marking $\phi_U\colon R^2\pi_*\mathbb{Z} \to (L_{K3})_U$ in a unique way, where $(L_{K3})_U$ is the constant sheaf on U with fiber L_{K3}. We obtain a holomorphic map $\rho\colon U \to \Omega_{L_{K3}}$, $u \mapsto [\phi_U(\omega_{\mathcal{S}_u})]$ called the *period map* associated to the family $\pi\colon \mathcal{S} \to U$. By the following theorem, the period map is surjective:

Theorem 3.4.2 (Horikawa-Shah-Kulikov-Persson-Pinkham-Todorov-Looijenga, for a proof see [BHPVdV04, Chap. VIII, Theorem 14.1])**.** *For every element* $[x]$ *in* $\Omega_{L_{K3}}$*, there exists a marked K3 surface* (S,ϕ) *such that* $[x]$ *is the period point of* (S,ϕ).

4 Complete $(2,3)$-intersections in $\mathbb{P}^4$

In this chapter, we will study complete $(2,3)$-intersections in $\mathbb{P}^4$. Since projective quadrics are determined up to isomorphism by their rank, we will consider these intersections for each possible rank of the underlying quadric individually. We will firstly study certain pencils of planes on quadrics in $\mathbb{P}^4$ and construct with these certain hyperplane sections of complete $(2,3)$-intersections in $\mathbb{P}^4$. Finally, we will determine which ADE singularities of the complete $(2,3)$-intersection in $\mathbb{P}^4$ can lie on the singular locus of the underlying quadric. The minimal model of a complete $(2,3)$-intersection in $\mathbb{P}^4$ with isolated ADE singularities is a K3 surface. The results in this chapter will explain the geometry of complete $(2,3)$-intersections in $\mathbb{P}^4$, which we need to understand for the following chapters.

4.1 Quadrics in $\mathbb{P}^4$

4.1.1 Basic notation, definitions, and properties

Let $(x_0 : \ldots : x_n)$ be homogeneous coordinates on $\mathbb{P}^n$.

A *quadric* Q in $\mathbb{P}^n$ is the zero locus of a non-trivial quadratic homogeneous polynomial, i.e.

$$Q : \sum_{i,j=0}^{n} a_{ij} x_i x_j = 0 \subseteq \mathbb{P}^n.$$

For $M_Q := (a_{ij})_{i,j} \in \mathrm{Mat}_{n+1}(\mathbb{C})$, we denote by

$$\mathrm{rank}(Q) := \mathrm{rank}(M_Q)$$

the *rank* of Q and by

$$\mathrm{corank}(Q) := (n+1) - \mathrm{rank}(Q)$$

the *corank* of Q.

We recall from linear algebra that over the complex numbers two quadrics in $\mathbb{P}^n$ are isomorphic if their ranks (or coranks) coincide. Hence, we can classify the quadrics in $\mathbb{P}^n$ by their coranks.

The linear subspace of $\mathbb{P}^n$ corresponding to the kernel of the matrix M_Q in $\mathbb{C}^{n+1}$ is the singular locus $\mathrm{Sing}(Q)$ of Q. More precisely:

Lemma 4.1.1 ([GH94, Chap. 6.1, p. 734])**.** *A quadric $Q \subseteq \mathbb{P}^n$ of corank k is the cone through a $(k-1)$-dimensional linear subspace $\Lambda \subseteq Q \subseteq \mathbb{P}^n$ over a smooth quadric in $\mathbb{P}^{n-k}$ and Λ is the singular locus of Q. In particular, Q is smooth if and only if Q has corank 0 in $\mathbb{P}^n$.*

For a quadric $Q \subseteq \mathbb{P}^n$ of corank k in $\mathbb{P}^n$ and a smooth point $x \in Q$ (the existence of x implies that $k \leq n-1$), we denote by

$$\mathbb{T}_x Q \subseteq \mathbb{P}^n$$

the projective tangent space to Q at x. Then, the tangent hyperplane section $\mathbb{T}_x Q \cap Q \subseteq \mathbb{P}^{n-1}$ of Q is a quadric of corank $k+1$ in $\mathbb{P}^{n-1}$. Indeed, the singular locus of $\mathbb{T}_x Q \cap Q$ is the span of the singular locus of Q and x, i.e.

$$\dim \operatorname{Sing}(\mathbb{T}_x Q \cap Q) = \dim \operatorname{Sing}(Q) + 1 = (k-1) + 1 = k.$$

Hence, $\operatorname{corank}(\mathbb{T}_x Q \cap Q) = k+1$ in $\mathbb{P}^{n-1}$ by Lemma 4.1.1.

Lemma 4.1.2 ([Har92, Lecture 22, p. 285])**.** *A smooth quadric in $\mathbb{P}^3$ is isomorphic to the image of the Segre embedding*

$$\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3,\ \big((x_0 : x_1), (y_0 : y_1)\big) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1). \tag{4.1}$$

For $\alpha, \beta \in \mathbb{P}^1$, define the lines $l_{1,\alpha} \coloneqq \sigma(\{\alpha\} \times \mathbb{P}^1)$ and $l_{2,\beta} \coloneqq \sigma(\mathbb{P}^1 \times \{\beta\})$. The quadric has hence the two rulings $\{l_{1,\alpha}\}_{\alpha \in \mathbb{P}^1}$ and $\{l_{2,\beta}\}_{\beta \in \mathbb{P}^1}$ and through every point in it passes exactly one line from each of the rulings.

4.1.2 Planes on Quadrics in $\mathbb{P}^4$

We collect now results on planes on quadrics of corank 0, 1, and 2 in $\mathbb{P}^4$ and deduce these in the latter two cases from results on linear spaces on smooth quadrics in $\mathbb{P}^3$ and $\mathbb{P}^2$, respectively.

4.1.2.1 Quadrics of corank 0 in $\mathbb{P}^4$

Lemma 4.1.3. *A quadric in $\mathbb{P}^4$ is smooth if and only if it contains no planes in $\mathbb{P}^4$.*

Proof. Smooth quadrics in $\mathbb{P}^n$ contain no planes (see [GH94, Chap. 6.1, Proposition]) so this holds in particular for $n = 4$.

Let now Q be a quadric of corank k in $\mathbb{P}^4$ containing no planes. By Lemma 4.1.1, the singular locus of Q is a linear subspace Λ of dimension $k-1$ and Q is the cone through Λ over a smooth quadric Q' in $\mathbb{P}^{4-k}$. If $k \geq 3$, the singular locus of the quadric contains a plane. If $k = 2$, the singular locus of Q is a line and the plane spanned by the singular line and a point in $Q' \subseteq Q$ is contained in Q. If $k = 1$, the singular locus of Q is a point and we have an isomorphism $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} Q'$ by Lemma 4.1.2. The plane spanned by the singular point and $\sigma(\mathbb{P}^1 \times \{\mathrm{pt}\})$ in Q then is a plane in Q. Consequently, we must have $k = 0$, i.e. Q is smooth. □

4.1.2.2 Quadrics of corank 1 in $\mathbb{P}^4$

Let Q be a quadric of corank 1 in $\mathbb{P}^4$ with vertex p. By Lemma 4.1.1, Q is the cone over a smooth quadric Q' in $\mathbb{P}^3$. By Lemma 4.1.2, we have two rulings $\{l_{1,\alpha}\}_{\alpha \in \mathbb{P}^1}$ and $\{l_{2,\beta}\}_{\beta \in \mathbb{P}^1}$ on Q'. For $\alpha, \beta \in \mathbb{P}^1$, let

$$\Pi_{1,\alpha} \coloneqq \text{ plane spanned by } p \text{ and the line } l_{1,\alpha} \subseteq \mathbb{P}^4$$

$$\Pi_{2,\beta} := \text{ plane spanned by } p \text{ and the line } l_{2,\beta} \subseteq \mathbb{P}^4.$$

We obtain two pencils of planes $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ on Q, see Figure 4.1.

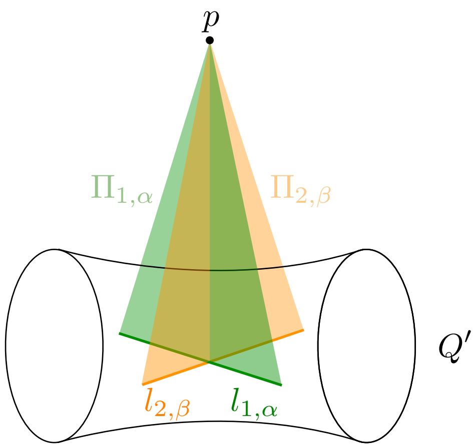


Figure 4.1: Cone through p over the smooth quadric surface Q'.

Lemma 4.1.4. *Every line in Q through p is contained in a unique plane in each of the pencils $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$.*

Proof. By Lemma 4.1.2, through every point in Q' passes a unique line from each of the rulings $\{l_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{l_{2,\beta}\}_{\beta\in\mathbb{P}^1}$. Hence, we can deduce that through each line in Q through p passes a unique plane from each of the pencils $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$. □

4.1.2.3 Quadrics of corank 2 in $\mathbb{P}^4$

Let Q be a quadric of corank 2 in $\mathbb{P}^4$. By Lemma 4.1.1, Q is the cone through a line l over a smooth quadric $Q' \subseteq \mathbb{P}^2$ and l is the singular locus of Q. The quadric Q' is isomorphic to $\mathbb{P}^1$. For $t \in Q' \cong \mathbb{P}^1$, let then

$$\Pi_t := \text{ plane in } Q \text{ spanned by } l \text{ and } t \subseteq \mathbb{P}^4.$$

We obtain the pencil $\{\Pi_t\}_{t\in\mathbb{P}^1}$ of planes on Q, see Figure 4.2.

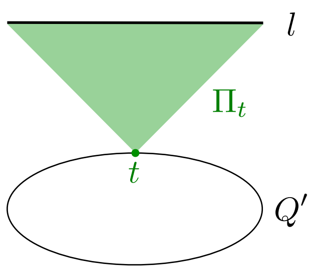


Figure 4.2: Cone through l over the smooth quadric curve Q'.

Lemma 4.1.5. *Through any point in Q passes a plane in the pencil $\{\Pi_t\}_{t\in\mathbb{P}^1}$ which is unique if the point is smooth.*

Proof. Obviously, all singular points of Q are contained in all the planes in $\{\Pi_t\}_{t\in\mathbb{P}^1}$. If t_0 is a smooth point of Q, the plane Π through $\mathrm{Sing}(Q)$ and t_0 intersects Q' in a single point. Indeed, if Π intersected Q' in two different points, the line joining those points would be contained in Q' which is absurd since Q' is by definition an irreducible curve of degree 2 in $\mathbb{P}^2$. Hence, Π is uniquely determined and contained in $\{\Pi_t\}_{t\in\mathbb{P}^1}$. $\square$

4.2 Basic properties of complete $(2,3)$-intersections in $\mathbb{P}^4$

Recall that an m-dimensional variety $V \subseteq \mathbb{P}^n$ is a *complete* $(d_1,\dots,d_{n-m})$*-intersection* if there exist $n-m$ homogeneous polynomials $f_i(x_0,\dots,x_n)$ of degree d_i $(1 \le i \le n-m)$ in $\mathbb{C}[x_0,\dots,x_n]$ generating all homogeneous polynomials in $\mathbb{C}[x_0,\dots,x_n]$ which are vanishing on V.

Lemma 4.2.1 ([GH94, Chap. 4.5, p. 592])**.** *Let S be a complete $(2,3)$-intersection in $\mathbb{P}^4$. Then, the quadric $Q \subseteq \mathbb{P}^4$ containing S is uniquely determined and the cubic in $\mathbb{P}^4$ containing S is uniquely determined modulo those cubics containing the quadric Q.*

Lemma 4.2.2. *Let S be a complete $(2,3)$-intersection in $\mathbb{P}^4$ with at most isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of S. Then, $\widetilde{S}$ is a K3 surface. The line bundle $L := \pi^*\big(\mathcal{O}_S(1)\big)$ on $\widetilde{S}$ is nef and the map $\varphi_L\colon \widetilde{S} \to S$ induced by L coincides with π. Furthermore, we have* $\deg L = L^2 = 6$.

Proof. The surface S has only isolated ADE singularities and these are precisely rational double points by Theorem 1.2.2. By [Rei87, 1.5], we can naturally extend the definition of the canonical bundle on smooth surfaces to those with rational double points (see [Pan15, Theorem 1] for more details). Since S is a complete $(2,3)$-intersection in $\mathbb{P}^4$, we then compute using [Har77, Chap. II, Ex. 8.4 (e)] that $\omega_S = \mathcal{O}_S$. Further, by [Rei87, 1.9, Example (1)], we have $\omega_{\widetilde{S}} = \pi^*\omega_S$. Hence, $\omega_{\widetilde{S}} = \pi^*\mathcal{O}_S = \mathcal{O}_{\widetilde{S}}$. Again, since S has only rational double points, we have $R^i\pi_*\mathcal{O}_{\widetilde{S}} = 0$ for all $i > 0$. Therefore, $\Gamma(S, R^1\pi_*\mathcal{O}_{\widetilde{S}}) = H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = 0$. Consequently, $\widetilde{S}$ is a K3 surface.

The minimal model $\widetilde{S}$ is in particular quasi-compact and separated. Hence, we can apply the projection formula for a (-2)-curve C on $\widetilde{S}$ and obtain that $\pi^*\mathcal{O}_{\mathbb{P}^4}(1).C = \mathcal{O}_{\mathbb{P}^4}(1).\pi_*C$ on $\mathbb{P}^4$. Since the hyperplane bundle $\mathcal{O}_{\mathbb{P}^4}(1)$ is very ample, it is in particular nef. Hence, $\mathcal{O}_{\mathbb{P}^4}(1).\pi_*C \ge 0$. In conclusion, $L = \pi^*\mathcal{O}_S(1) = \pi^*\mathcal{O}_{\mathbb{P}^4}(1)_{|S}$ is nef. Likewise, the projection formula implies that we cannot have a curve E on $\widetilde{S}$ with the property that $\pi^*\mathcal{O}_S(1).E = 1$ or 2. Therefore, the map $\varphi_L\colon \widetilde{S} \to S$ is birational by Proposition 3.2.6. Then, Proposition 3.3.4 implies that φ_L coincides with π.

For a general hyperplane H in $\mathbb{P}^4$, the hyperplane section $H \cap S$ of S is a curve of degree 6. By Bertini's Theorem, $H \cap S$ passes through none of the singularities of S. Hence, $\pi^*(H \cap S) \in \mathrm{Div}(\widetilde{S})$ has degree 6 as well. Therefore, $\deg L = L^2 = 6$. $\square$

4.3 Hyperplane sections of complete $(2,3)$-intersections in $\mathbb{P}^4$

Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$.

We will construct in the following certain hyperplane sections of S depending on the corank of Q in $\mathbb{P}^4$.

We will need the following auxiliary result:

Lemma 4.3.1. *Let $Q \subseteq \mathbb{P}^4$ be a quadric of corank 1 or 2 in $\mathbb{P}^4$, let $Y \subseteq \mathbb{P}^4$ a cubic such that Q and Y have no singularities in common, and let $S := Q \cap Y \subseteq \mathbb{P}^4$. For a pencil of planes $\{\Pi_s\}_{s\in\mathbb{P}^1}$ in Q as in Subsection 4.1.2.2 or 4.1.2.3, let $C_s := \Pi_s \cap Y \subseteq S$. Then, the general curve in $\{C_s\}_{s\in\mathbb{P}^1}$ is smooth in $p \in Y \cap \operatorname{Sing}(Q)$.*

Proof. Firstly, note that $\operatorname{Sing}(Q)$ is contained in all planes in Q and hence all planes Π_s for all $s \in \mathbb{P}^1$. Consequently, those singularities of S lying on the singular locus of Q are contained in C_s for all $s \in \mathbb{P}^1$, i.e. $\operatorname{Sing}(S) \cap \operatorname{Sing}(Q) \subseteq \cap_{s\in\mathbb{P}^1} C_s$.

By assumption, the cubic Y is smooth in p since Q is singular at p. Further, Π_s is smooth in all points as a plane. Hence, the curve $C_s := Y \cap \Pi_s$ is smooth in p if and only if the affine tangent spaces T_pY and $T_p\Pi_s$ of Y and Π_s in p, respectively, intersect transversally, i.e.

$$T_p\mathbb{P}^4 = T_pY + T_p\Pi_s. \tag{4.2}$$

Since Y and Π_s are both smooth in p, we have $\dim T_pY = \dim Y = 3$ and $\dim T_p\Pi_s = \dim \Pi_s = 2$, so equation (4.2) holds if and only if $T_p\Pi_s \not\subseteq T_pY$.

Assume that we had for all $s \in \mathbb{P}^1$

$$T_p\Pi_s \subseteq T_pY.$$

By construction of the pencil of planes $\{\Pi_s\}_{s\in\mathbb{P}^1}$ in Q in Subsection 4.1.2.2 or 4.1.2.3, we have

$$\bigcup_{s\in\mathbb{P}^1} \Pi_s = Q.$$

Consequently, the tangent spaces of the planes Π_s at p span the tangent cone of Q, i.e.

$$\sum_{s\in\mathbb{P}^1} T_p\Pi_s = T_pQ,$$

so by assumption

$$T_pQ \subseteq T_pY.$$

Since Q is singular at p, we have $3 = \dim Q < \dim T_pQ \leq 4$. Hence, $\dim T_pQ = 4$. However, the four-dimensional space T_pQ cannot be contained in the three-dimensional space T_pY.

Consequently, the assumption must have been wrong and there exists a plane Π_s such that $T_p\Pi_s \not\subseteq T_pY$. Zariski closed proper subsets in $\mathbb{P}^1$ are finite. Since the open set

$$\{s \in \mathbb{P}^1;\, T_p\Pi_s \not\subseteq T_pY\} = \mathbb{P}^1 \setminus \{s \in \mathbb{P}^1;\, T_p\Pi_s \subseteq T_pY\}$$

is non-empty, it is Zariski-dense in $\mathbb{P}^1$. Hence, the general plane Π_s is not contained in T_pY. In conclusion, the general cubic curve in $\{C_s\}_{s\in\mathbb{P}^1}$ is smooth in p. □

By the following Lemma 4.3.2, the assumption that a singularity p of S is not a singularity of both Q and Y is satisfied if p is a hypersurface singularity and therefore in particular if p is an ADE singularity.

Lemma 4.3.2. *Let p be a singularity of a complete $(2,3)$-intersection $S \subseteq \mathbb{P}^4$ of a quadric Q and a cubic Y in $\mathbb{P}^4$. Then, p is a hypersurface singularity of S if and only if it is not a singularity of both the quadric Q and the cubic Y.*

Proof. Assume that the hypersurface singularity p is a singularity of both Q and Y. The germ (S,p) is locally analytically isomorphic to the germ $(V,\mathbf{0}) \subseteq (\mathbb{C}^3,\mathbf{0})$, where V is a surface in $\mathbb{C}^3$ and $\mathbf{0} := (0,0,0)$. Since $\mathbf{0}$ is a singularity of V, we have $3 \geq \dim T_{\mathbf{0}}V > \dim V = 2$. Therefore, $\dim T_{\mathbf{0}}V = 3$. On the other hand, we have $4 \geq \dim T_pQ > \dim Q = 3$ and $4 \geq \dim T_pY > \dim Y = 3$ which forces $\dim T_pQ = \dim T_pY = 4$. Furthermore, $\dim(T_pQ + T_pY) \leq \dim T_p\mathbb{P}^4 = 4$. Consequently, $\dim T_pS = \dim T_pQ + \dim T_pY - \dim(T_pQ + T_pY) \geq 4 + 4 - 4 = 4$. Therefore, $\dim T_{\mathbf{0}}V \neq \dim T_pS$ which is a contradiction to (S,p) and $(V,\mathbf{0})$ being locally analytically isomorphic.

On the other hand, assume that p is a smooth point of Q or Y and assume without loss of generality that Q is smooth at p. Then, locally analytically at p the quadric Q is isomorphic to a hyperplane $H \cong \mathbb{C}^3$ in $\mathbb{P}^4$. If g is the cubic polynomial defining Y, the surface S is therefore locally analytically at p on the hyperplane $H \cong \mathbb{C}^3$ defined by g. Hence, (S,p) is a hypersurface singularity. □

4.3.1 Q has corank 1 in $\mathbb{P}^4$

Let $Q \subseteq \mathbb{P}^4$ be a quadric of corank 1 in $\mathbb{P}^4$ with vertex p and let $Y \subseteq \mathbb{P}^4$ be a cubic such that $S := Q \cap Y$ is a complete $(2,3)$-intersection in $\mathbb{P}^4$ having at most isolated ADE singularities. By Lemma 4.3.2, this implies that Q and Y have no common singularities.

Let $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ be the two pencils of planes on Q as in Lemma 4.1.4.

For $\alpha,\beta \in \mathbb{P}^1$, we define the plane cubic curves on S

$$C_{1,\alpha} := \Pi_{1,\alpha} \cap Y \subseteq S \quad \text{and} \quad C_{2,\beta} := \Pi_{2,\beta} \cap Y \subseteq S$$

and obtain two pencils of plane cubic curves $\{C_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{C_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ on S.

Lemma 4.3.3. *Let Π_1 and Π_2 be the planes in the pencils $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$, spanned by p and $l_1 \in \{l_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $l_2 \in \{l_{2,\beta}\}_{\beta\in\mathbb{P}^1}$, respectively, as defined in subsection 4.1.2.2 and x the intersection point of l_1 and l_2. Let $C_1 := \Pi_1 \cap Y$ and $C_2 := \Pi_2 \cap Y$. The divisor $C_1 + C_2$ on $S \subseteq \mathbb{P}^4$ is supported on $\mathbb{T}_xQ \cap S$. In particular, $C_1 + C_2$ is a hyperplane section of S.*

Proof. $\mathbb{T}_xQ \cap Q$ is a quadric of corank 2 in $\mathbb{P}^3$ whose singular locus is the line l_x joining x and the vertex p of Q. By Lemma 4.1.4, there are unique planes in $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ containing l_x which then must be Π_1 and Π_2 as they both contain l_x. Hence, $\mathbb{T}_xQ \cap Q \cap Y = \mathbb{T}_xQ \cap S$ is the union of the curves $C_1 := \Pi_1 \cap Y$ and $C_2 := \Pi_2 \cap Y$.

Let m_1 and m_2 be the positive integers such that $m_1C_1 + m_2C_2 = \mathbb{T}_xQ \cap S$ as divisors on S. We claim that the planes Π_1 and Π_2 are not contained in Y. Indeed, if one of the planes was contained in Y, the complete $(2,3)$-intersection S would contain this plane, as well. Therefore, the smooth minimal model $\widetilde{S}$ of S would be rational which is absurd since $\widetilde{S}$ is a K3 surface by Lemma 4.2.2. Hence, the hyperplane section $\mathbb{T}_xQ \cap S \subseteq \mathbb{P}^4$ of S is a curve of degree 6 by Bezout's Theorem. Using that C_1 and C_2 are cubics, we have

$\deg(m_1C_1+m_2C_2) = 3(m_1+m_2)$. Since $\mathbb{T}_xQ\cap S$ has degree 6, it follows that $m_1 = m_2 = 1$. In conclusion, $\mathbb{T}_xQ \cap Y = C_1 + C_2 \in \mathrm{Div}(S)$. $\square$

Let

$$\pi^{(1)}\colon S^{(1)} := \mathrm{Bl}_pS \to S$$

be the blowing-up of S in p with exceptional divisor $E_S^{(1)}$ and let $C_{1,\alpha}^{(1)}$ and $C_{2,\beta}^{(1)}$ be the strict transforms of $C_{1,\alpha}$ and $C_{2,\beta}$ in $S^{(1)}$.

Lemma 4.3.4. *We can find $\alpha, \beta \in \mathbb{P}^1$ such that the following conditions are all satisfied:*

(1) $C_{1,\alpha}$ and $C_{2,\beta}$ are smooth in p

(2) $C_{1,\alpha}^{(1)}$ and $C_{2,\beta}^{(1)}$ are both contained in the smooth locus of $S^{(1)}$

(3) $C_{1,\alpha}^{(1)} \cap C_{2,\beta}^{(1)} \cap E_S^{(1)} = \emptyset$.

If p is of type $\mathbf{A}_{n\geq 2}$, we have $E_S^{(1)} = E_1^{(1)} \cup E_{n-2}^{(1)}$, where $E_1^{(1)}$ and $E_{n-2}^{(1)}$ are irreducible curves intersecting transversally in a singularity of type $\mathbf{A}_{n-2}$ of $S^{(1)}$.

(4) After exchanging $E_1^{(1)}$ by $E_{n-2}^{(1)}$ if necessary, $C_{1,\alpha}^{(1)}$ intersects $E_1^{(1)}$ but not $E_{n-2}^{(1)}$ and $C_{2,\beta}^{(1)}$ intersects $E_{n-2}^{(1)}$ but not $E_1^{(1)}$ and the intersection point of $E_1^{(1)}$ with $E_{n-2}^{(1)}$ is contained in neither $C_{1,\alpha}^{(1)}$ nor $C_{2,\beta}^{(1)}$, see Figure 4.3.

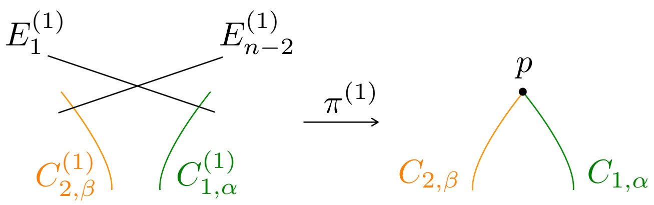


Figure 4.3: Assume that p is of type $\mathbf{A}_{n\geq 2}$. The curves $C_{1,\alpha}$ and $C_{2,\beta}$ satisfy condition (4) in Lemma 4.3.4.

Proof. We claim firstly that the set

$$I_1 := \{(\alpha,\beta) \in \mathbb{P}^1 \times \mathbb{P}^1;\, C_{1,\alpha} \text{ or } C_{2,\beta} \text{ are singular in } p\}$$

is a proper closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, by Lemma 4.3.1, the general curves in $\{C_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{C_{2,\beta}\}_{\beta\in\mathbb{P}^1}$, respectively, are smooth in p. Hence, only finitely many curves in each family are singular in p, i.e. I_1 is a proper closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$.

We claim secondly that the set

$$I_2 := \{(\alpha,\beta) \in \mathbb{P}^1 \times \mathbb{P}^1;\, C_{1,\alpha}^{(1)} \text{ or } C_{2,\beta}^{(1)} \text{ contains a singularity of } S^{(1)} \text{ outside } E_S^{(1)}\}$$

is a proper closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, since S has only isolated singularities, for only finitely many choices of α and $\beta \in \mathbb{P}^1$ the curves $C_{1,\alpha}$ and $C_{2,\beta} \subseteq S$ contain a singularity of S different from p. Hence, for only finitely many choices of α and β the strict transforms $C_{1,\alpha}^{(1)}$ and $C_{2,\beta}^{(1)}$ in $S^{(1)}$ of the curves $C_{1,\alpha}$ and $C_{2,\beta}$ contain a singularity of $S^{(1)}$ outside $E_S^{(1)}$, i.e. I_2 is a proper closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$.

We claim thirdly that the set

$$I_3 \coloneqq \{(\alpha,\beta) \in \mathbb{P}^1 \times \mathbb{P}^1;\, C_{1,\alpha}^{(1)} \cap C_{2,\beta}^{(2)} \cap E_S^{(1)} \neq \emptyset\}$$

is a proper closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$ and prove this in the following by an explicit computation in coordinates on $\mathbb{P}^4$.

The quadric $Q \subseteq \mathbb{P}^4$ is up to isomorphism uniquely determined by its rank. Hence, we can choose homogeneous coordinates $(v : w : x : y : z)$ on $\mathbb{P}^4$ such that Q is the image of the Segre embedding σ in (4.1):

$$Q : \, xy - zw = 0 \subseteq \mathbb{P}^4$$

and thus $p = (1 : 0 : 0 : 0 : 0) \in \mathbb{P}^4$ is the singular point of Q.

Until the rest of the proof, let $\alpha, \beta \in \mathbb{P}^1 \setminus \{(0:1),(1:0)\}$. We then identify $\alpha \coloneqq (\alpha_0 : \alpha_1)$ and $\beta \coloneqq (\beta_0 : \beta_1)$ with

$$a_\alpha \coloneqq \frac{\alpha_1}{\alpha_0} \quad \text{and} \quad b_\beta \coloneqq \frac{\beta_1}{\beta_0} \in \mathbb{C} \setminus \{0\}, \tag{4.3}$$

respectively.

In coordinates, the lines in the rulings $\{l_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{l_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ are given by

$$\begin{aligned} l_{1,\alpha} &= \sigma(\{\alpha\} \times \mathbb{P}^1) : y - a_\alpha w = z - a_\alpha x = 0 \subseteq \mathbb{P}^3 \\ l_{2,\beta} &= \sigma(\mathbb{P}^1 \times \{\beta\}) : z - b_\beta y \ = x - b_\beta w = 0 \subseteq \mathbb{P}^3. \end{aligned}$$

Hence,

$$\begin{aligned} \Pi_{1,\alpha} &= \text{ plane spanned by } l_{1,\alpha} \text{ and } p \text{ in } \mathbb{P}^4 : y - a_\alpha w = z - a_\alpha x = 0 \subseteq \mathbb{P}^4 \\ \Pi_{2,\beta} &= \text{ plane spanned by } l_{2,\beta} \text{ and } p \text{ in } \mathbb{P}^4 : z - b_\beta y \ = x - b_\beta w = 0 \subseteq \mathbb{P}^4. \end{aligned}$$

There are $a_1, \ldots, a_4 \in \mathbb{C}$ and homogeneous complex polynomials $f_2(w,x,y,z)$ and $f_3(w,x,y,z)$ in w,x,y,z of degree 2 and 3, respectively, such that the cubic $Y \subseteq \mathbb{P}^4$ has the form

$$Y : \, v^2(a_1w + a_2x + a_3y + a_4z) + vf_2(w,x,y,z) + f_3(w,x,y,z) = 0 \subseteq \mathbb{P}^4.$$

Indeed, Y contains the vertex $p = (1 : 0 : 0 : 0 : 0)$ of Q. Therefore, the polynomial defining Y has no summand v^3.

The cubic Y is smooth in p since Y and Q have by assumption no common singularities. Hence, at least one of the coefficients $a_1, \ldots, a_4$ is non-zero and we will assume in the following without loss of generality that

$$a_4 \neq 0.$$

Consequently, we have on $\mathbb{P}^4$

$$\begin{aligned} S = Q \cap Y \quad &: \begin{cases} xy - zw = 0 \\ v^2(a_1w + a_2x + a_3y + a_4z) + vf_2(w,x,y,z) + f_3(w,x,y,z) = 0 \end{cases} \\ C_{1,\alpha} = \Pi_{1,\alpha} \cap Y \quad &: \begin{cases} y - a_\alpha w = z - a_\alpha x = 0 \\ v^2(a_1w + a_2x + a_3y + a_4z) + vf_2(w,x,y,z) + f_3(w,x,y,z) = 0 \end{cases} \\ C_{2,\beta} = \Pi_{2,\beta} \cap Y \quad &: \begin{cases} z - b_\beta y = x - b_\beta w = 0 \\ v^2(a_1w + a_2x + a_3y + a_4z) + vf_2(w,x,y,z) + f_3(w,x,y,z) = 0. \end{cases} \end{aligned}$$

Let $\pi^{(1)}\colon \mathbb{P}^4 \times \mathbb{P}^3 \supseteq (\mathbb{P}^4)^{(1)} \to \mathbb{P}^4$ be the blowing-up of $\mathbb{P}^4$ in p and $(w_1 : x_1 : y_1 : z_1)$ homogeneous coordinates on $\mathbb{P}^3$.

On the affine chart $\mathbb{P}^4 \times \mathbb{A}^3 \subseteq \mathbb{P}^4 \times \mathbb{P}^3$ defined by $w_1 \neq 0$, we have

$$(\mathbb{P}^4)^{(1)} = \{\big((v : w : x : y : z), (1, x_1, y_1, z_1)\big) \in \mathbb{P}^4 \times \mathbb{A}^3;\ x = x_1 w,\ y = y_1 w,\ z = z_1 w\}.$$

We compute the strict transforms $S^{(1)}$, $C_{1,\alpha}^{(1)}$, and $C_{2,\beta}^{(1)}$ of S, $C_{1,\alpha}$, and $C_{2,\beta}$, respectively, in $(\mathbb{P}^4)^{(1)}$:

$$S^{(1)} : \begin{cases} z_1 = x_1 y_1 \\ v^2(a_1 + a_2 x_1 + a_3 y_1 + a_4 z_1) + vw f_2(1, x_1, y_1, z_1) + w^2 f_3(1, x_1, y_1, z_1) = 0 \end{cases}$$

with exceptional divisor $E_S^{(1)} \subseteq S^{(1)}$

$$E_S^{(1)} : w = x = y = z = 0,\ z_1 = x_1 y_1,\ v^2\Big(\frac{a_1}{a_4} + \frac{a_2}{a_4} x_1 + \frac{a_3}{a_4} y_1 + x_1 y_1\Big) = 0$$

$$C_{1,\alpha}^{(1)} : \begin{cases} y_1 - a_\alpha = z_1 - a_\alpha x_1 = 0 \\ v^2(a_1 + a_2 x_1 + a_3 y_1 + a_4 z_1) + vw f_2(1, x_1, y_1, z_1) + w^2 f_3(1, x_1, y_1, z_1) = 0 \end{cases}$$

$$C_{2,\beta}^{(1)} : \begin{cases} z_1 - b_\beta y_1 = x_1 - b_\beta = 0 \\ v^2(a_1 + a_2 x_1 + a_3 y_1 + a_4 z_1) + vw f_2(1, x_1, y_1, z_1) + w^2 f_3(1, x_1, y_1, z_1) = 0. \end{cases}$$

A point $\big((v : w : x : y : z), (1, x_1, y_1, z_1)\big) \in \mathbb{P}^4 \times \mathbb{A}^3$ is contained in $C_{1,\alpha}^{(1)} \cap C_{2,\beta}^{(1)} \cap E_S^{(1)}$ if and only if

$$\begin{cases} w = x = y = z = 0 \\ x_1 = b_\beta,\ y_1 = a_\alpha,\ z_1 = a_\alpha b_\beta \\ v^2\big(\frac{a_1}{a_4} + \frac{a_2}{a_4} b_\beta + \frac{a_3}{a_4} a_\alpha + a_\alpha b_\beta\big) = 0. \end{cases}$$

A direct computation of the blowing-ups of S, $C_{1,\alpha}$, and $C_{2,\beta}$ on the other charts of $\mathbb{P}^3$ as above shows that all points of $C_{1,\alpha}^{(1)} \cap C_{2,\beta}^{(1)} \cap E_S^{(1)}$ are contained in the chart $w_1 \neq 0$ as $a_\alpha, b_\beta \neq 0$.

Hence, with the definitions in (4.3)

$$\begin{aligned} I_3 \setminus \big(\{(1:0),(0:1)\}^2\big) = \big\{\big((\alpha_0 : \alpha_1), (\beta_0 : \beta_1)\big) &\in \big(\mathbb{P}^1 \setminus \{(1:0),(0:1)\}\big)^2; \\ &\frac{a_1}{a_4} + \frac{a_2}{a_4} b_\beta + \frac{a_3}{a_4} a_\alpha + a_\alpha b_\beta = 0\big\} \end{aligned}$$

and this is a proper closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$.

In conclusion, $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \big(I_1 \cup I_2 \cup I_3 \cup \{(1:0),(0:1)\}^2\big)$ is a non-empty open subset of $\mathbb{P}^1 \times \mathbb{P}^1$ and for each $(\alpha, \beta) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \big(I_1 \cup I_2 \cup I_3 \cup \{(1:0),(0:1)\}^2\big)$ the curves $C_{1,\alpha}$ and $C_{2,\beta}$ satisfy conditions (1)-(3).

This finalizes the proof if p is of type $\mathbf{A}_1$.

Claim 4.3.5. *The exceptional divisor $E_S^{(1)}$ is reducible if and only if $\frac{a_2}{a_4} \cdot \frac{a_3}{a_4} = \frac{a_1}{a_4}$.*

Proof. Assume that $E_S^{(1)}$ is reducible, i.e. $\frac{a_1}{a_4} + \frac{a_2}{a_4} x_1 + \frac{a_3}{a_4} y_1 + x_1 y_1 = 0 \subseteq \mathbb{A}^2$ is reducible. We homogenize the equation by w_1 and obtain the projective quadric

$$q : \frac{a_1}{a_4} w_1^2 + \frac{a_2}{a_4} x_1 w_1 + \frac{a_3}{a_4} y_1 w_1 + x_1 y_1 = 0 \subseteq \mathbb{P}^2.$$

Then, $q \subseteq \mathbb{P}^2$ is reducible if and only if the discriminant $\mathrm{Disc}(q)$ of q is zero. We have

$$\mathrm{Disc}(q) = \begin{vmatrix} \frac{a_1}{2a_4} & \frac{a_2}{2a_4} & \frac{a_3}{2a_4} \\ \frac{a_2}{2a_4} & 0 & \frac{1}{2} \\ \frac{a_3}{2a_4} & \frac{1}{2} & 0 \end{vmatrix} = \frac{1}{4}(\frac{a_2}{a_4} \cdot \frac{a_3}{a_4} - \frac{a_1}{a_4}).$$

Hence, $q \subseteq \mathbb{P}^2$ is reducible if and only if $\frac{a_2}{a_4} \cdot \frac{a_3}{a_4} - \frac{a_1}{a_4} = 0$. $\square$

Then, assume that p is of type $\mathbf{A}_{n\geq 2}$. We claim that condition (4) holds, as well. Indeed, if p is of type $\mathbf{A}_{n\geq 2}$, the exceptional divisor $E_S^{(1)}$ is reducible. Therefore, we have by Claim 4.3.5: $\frac{a_1}{a_4} = \frac{a_2}{a_4} \cdot \frac{a_3}{a_4}$. Hence,

$$E_S^{(1)} : w = \frac{a_1}{a_4} + \frac{a_2}{a_4}x_1 + \frac{a_3}{a_4}y_1 + x_1 y_1 = (\frac{a_3}{a_4} + x_1)(\frac{a_2}{a_4} + y_1) = 0.$$

Let

$$E_1^{(1)} : w = \frac{a_3}{a_4} + x_1 = 0 \text{ and } E_{n-2}^{(1)} : w = \frac{a_2}{a_4} + y_1 = 0.$$

For $(\alpha, \beta) \notin I_3 \cup \{(1:0),(0:1)\}^2$, we have

$$\frac{a_1}{a_4} + \frac{a_2}{a_4}b_\beta + \frac{a_3}{a_4}a_\alpha + a_\alpha b_\beta = (\frac{a_2}{a_4} + a_\alpha)(\frac{a_3}{a_4} + b_\beta) \neq 0.$$

Hence, $a_\alpha \neq -\frac{a_2}{a_4}$ and $b_\beta \neq -\frac{a_3}{a_4}$. We see that $C_{1,\alpha}^{(1)}$ intersects $E_1^{(1)}$ in

$$\big((v:w:x:y:z),(1:x_1:y_1:z_1)\big) = \big((1:0:0:0:0),(1:-\frac{a_3}{a_4}:a_\alpha:-\frac{a_3}{a_4}a_\alpha)\big)$$

but not $E_{n-2}^{(1)}$ as we have $y_1 = -\frac{a_2}{a_4}$ on $E_{n-2}^{(1)}$ but $y_1 = a_\alpha$ on $C_{1,\alpha}^{(1)}$ and $-\frac{a_2}{a_4} \neq a_\alpha$. On the other hand, $C_{2,\beta}^{(1)}$ intersects $E_{n-2}^{(1)}$ in

$$\big((v:w:x:y:z),(1:x_1:y_1:z_1)\big) = \big((1:0:0:0:0),(1:b_\beta:-\frac{a_2}{a_4}:-\frac{a_2}{a_4}b_\beta)\big)$$

but not $E_1^{(1)}$ since we have $x_1 = -\frac{a_3}{a_4}$ on $E_1^{(1)}$ but $x_1 = b_\beta$ on $C_{2,\beta}^{(1)}$ and $-\frac{a_3}{a_4} \neq b_\beta$. Further, $E_{n-2}^{(1)} \cap E_1^{(1)} : w = 0, x_1 = -\frac{a_3}{a_4}, y_1 = -\frac{a_2}{a_4}$ is contained in neither $C_{1,\alpha}^{(1)}$ nor $C_{2,\beta}^{(1)}$.

This finalizes the proof of Lemma 4.3.4. $\square$

4.3.2 Q has corank 2 in $\mathbb{P}^4$

Let $Q \subseteq \mathbb{P}^4$ be a quadric of corank 2 in $\mathbb{P}^4$. More precisely, let Q be the cone through its singular line $l := \mathrm{Sing}(Q)$ over a smooth quadric Q' in $\mathbb{P}^2$. Let $Y \subseteq \mathbb{P}^4$ be a cubic such that $S := Q \cap Y$ is a complete $(2,3)$-intersection in $\mathbb{P}^4$ having at most isolated ADE singularities. By Lemma 4.3.2, this implies that Q and Y have no common singularities.

Let $\{\Pi_t\}_{t\in\mathbb{P}^1}$ be the pencils of planes on Q defined in Subsection 4.1.2.3.

For $t \in \mathbb{P}^1$ we define the plane cubic curves on S

$$C_t := \Pi_t \cap Y \subseteq S$$

and obtain a pencils of plane cubic curves $\{C_t\}_{t\in\mathbb{P}^1}$ on S.

Lemma 4.3.6. *Let $t \in \mathbb{P}^4$ be a smooth point of Q and $\mathbb{T}_t Q$ the projective tangent space on Q at t. Then, $2C_t$ is the divisor on S supported on $C_t = \mathbb{T}_t Q \cap S$.*

Proof. $\mathbb{T}_t Q \cap Q$ is a quadric of corank 3 in $\mathbb{P}^3$, i.e. a double plane containing t which must be Π_t by Lemma 4.1.5 since Π_t contains t. The plane Π_t is not contained in Y since S contained otherwise a plane and hence the smooth minimal model $\widetilde{S}$ for S was rational which is absurd since $\widetilde{S}$ is a K3 surface by Lemma 4.2.2. Consequently, $C_t := \Pi_t \cap S = \mathbb{T}_t Q \cap S$. Let m be the positive integer such that $mC_t = \mathbb{T}_t Q \cap S$ as divisors on S. The curve $\mathbb{T}_t Q \cap S$ in $\mathbb{P}^4$ has degree 6. Since C_t has degree 3, we must have $m = 2$. □

Lemma 4.3.7. *We can choose $t \in Q'$ such that the following two conditions are satisfied.*

1. *C_t contains no singularity of S that is not lying on the singular line l of Q.*
2. *C_t is smooth in all points $p \in Y \cap l$.*

Proof. Indeed, the set

$$I_1 := \{t \in Q';\, C_t \text{ contains a singularity of } S \text{ outside of } l\}$$

is finite since S has only isolated singularities. Further, the set

$$I_2 := \{t \in Q';\, C_t \text{ is singular in some } p \in Y \cap l\}$$

is finite. Indeed, we have

$$\bigcup_{t \in Q'} \Pi_t = Q.$$

Hence, by Lemma 4.3.1 the general curve C_t is smooth in $p \in Y \cap l$. Hence, only finitely many curves C_t are singular in $p \in Y \cap l$, i.e. I_2 is finite.

In conclusion, there exists $t \in Q' \setminus (I_1 \cup I_2)$. □

4.4 Possible ADE singularities of a complete $(2,3)$-intersection on the singular locus of the underlying quadric

Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$. Assume that S has only isolated ADE singularities. By Lemma 4.3.2, this implies that Q and Y have no common singularities.

We will discuss in this section which combinations of ADE singularities of S can lie on the singular locus $\mathrm{Sing}(Q)$ of Q.

4.4.1 Q has corank 1 in $\mathbb{P}^4$

Let S be the complete $(2,3)$-intersection of a quadric Q of corank 1 in $\mathbb{P}^4$ with vertex p and Y a cubic in $\mathbb{P}^4$.

Lemma 4.4.1. *Assume that the vertex p of Q is contained in S. Then, p is a singularity of type $\mathbf{A}_{n\geq 1}$ on S.*

Proof. Let $(v : w : x : y : z)$ be homogeneous coordinates on $\mathbb{P}^4$. Since two projective quadrics of the same rank are isomorphic, we can assume that $Q : wx + yz = 0$, i.e. $p = (1 : 0 : 0 : 0 : 0) \in \mathbb{P}^4$. Since Q and Y have by assumption no common singularities, Y is smooth in p. Then, the projective tangent space $\mathbb{T}_pY$ of Y at p is a hyperplane in $\mathbb{P}^4$. Since $p \in \mathbb{T}_pY$, we have $\mathbb{T}_pY : \alpha w + \beta x + \gamma y + \epsilon z = 0$ for $\alpha, \beta, \gamma, \epsilon \in \mathbb{C}$. One of $\alpha, \beta, \gamma, \epsilon$ is not equal to zero. Assume without loss of generality that $\alpha \neq 0$. Now consider the chart $\mathbb{C}^3$ of $\mathbb{P}^4$ given by $v \neq 0$. There exists an analytic coordinate transformation ϕ of $\mathbb{C}^4$ such that $T_pY = \phi(Y)$ locally around p. Further, $\phi(Q) : wx + yz + f(w, x, y, z) = 0$, where f is a power series in w, x, y, z with monomials of degree ≥ 3. Then, $Q \cap Y$ is locally around p given by $T_pY \cap \phi(Q) : yz - \frac{\beta}{\alpha}x^2 - \frac{\gamma}{\alpha}xy - \frac{\epsilon}{\alpha}xz + f(\frac{\beta}{\alpha}x - \frac{\gamma}{\alpha}y - \frac{\epsilon}{\alpha}z, x, y, z) = 0 \subseteq \mathbb{C}^3$ which describes by the classification of ADE singularities (see [GLS07, Chap. I, Theorem 2.48]) a singularity of type $\mathbf{A}_{n\geq 1}$ in the origin since the corank of the Hessian matrix of the defining power series is 0 or 1 in $\mathbb{C}^3$. □

4.4.2 Q has corank 2 in $\mathbb{P}^4$

Let S be the complete $(2,3)$-intersection of a quadric Q of corank 2 in $\mathbb{P}^4$ and Y a cubic in $\mathbb{P}^4$. Let l be the singular line of Q. Since Q and Y have by assumption no common singularities, l is not contained in Y. Let $\{\Pi_t\}_{t\in\mathbb{P}^1}$ be the pencil of planes in Q defined in 4.1.2.3 and $\{C_t \coloneqq \Pi_t \cap Y\}_{t\in\mathbb{P}^1}$ the induced pencil of plane cubic curves on S.

Recall the definition of the intersection multiplicity of closed subschemes at a point on a smooth surface in [Ful98, Chap. 8.2].

We show in the next lemma that all plane cubic curves C_t in $\{C_t\}_{t\in\mathbb{P}^1}$ intersect the singular line l of Q in the same points with the same multiplicities.

Lemma 4.4.2. *For each $t \in \mathbb{P}^1$, we have $l \cap C_t = l \cap Y = \mathrm{Sing}(S)$. Moreover, the intersection multiplicities $l.C_t$ on the planes Π_t are independent of $t \in \mathbb{P}^1$.*

Proof. Let $C_t, C_{t'} \in \{C_t\}_{t\in\mathbb{P}^1}$. By definition $C_t = \Pi_t \cap Y$ and $C_{t'} = \Pi_{t'} \cap Y$. Since l is contained in both Π_t and $\Pi_{t'}$, we have $C_t \cap l = \Pi_t \cap Y \cap l = Y \cap l = \Pi_{t'} \cap Y \cap l = C_{t'} \cap l$. Further, for $p \in C_t \cap l = C_{t'} \cap l$, the intersection multiplicities $(C_t.l)_p = (\Pi_t \cap Y.l)_p$ and $(C_{t'}.l)_p = (\Pi_{t'} \cap Y.l)_p$ on Π_t and $\Pi_{t'}$, respectively, are well-defined. By Lemma A.0.1, we have $(\Pi_t \cap Y.l)_p = (\Pi_{t'} \cap Y.l)_p$. Therefore, $(C_t.l)_p = (C_{t'}.l)_p$. Further, since all points on l are singularities of Q, those points on l contained in Y are singularities of S. □

Lemma 4.4.3. *Let C_t in $\{C_t\}_{t\in\mathbb{P}^1}$ be a curve on the plane Π_t and $p \in C_t \cap l$ such that on Π_t we have $(C_t.l)_p = 1$. Then, S has a singularity of type $\mathbf{A}_1$ in p.*

Proof. We claim firstly that $l \nsubseteq \mathbb{T}_pY$. Assume conversely that $l \subseteq \mathbb{T}_pY$. Since $p \in l$ and since l is contained in all planes in $\{\Pi_t\}_{t\in\mathbb{P}^1}$ whose union is the quadric Q, the line l then is contained in the tangent space of one of the curves $C_t = \Pi_t \cap Y$, i.e. $l \subseteq \mathbb{T}_pC_t$. However, by Lemma 4.4.2, we have $(C_t.l)_p = 1$ which contradicts $l \subseteq \mathbb{T}_pC_t$. Hence, $l \nsubseteq \mathbb{T}_pY$.

The intersection $\mathbb{T}_pY \cap Q$ is a quadric in $\mathbb{P}^3$. Since l is not contained in $\mathbb{T}_pY$, the quadric $\mathbb{T}_pY \cap Q$ is only singular at p. Hence, $\mathbb{T}_pY \cap Q$ is a quadric of corank 1 in $\mathbb{P}^3$ with vertex p. The analytic type of p on $\mathbb{T}_pY \cap Q$ is hence type $\mathbf{A}_1$, i.e. the singularity p has type $\mathbf{A}_1$. Since Y is smooth in p, for an appropriate analytic coordinate change ϕ in a small neighborhood around p, we have $\mathbb{T}_pY = \phi(Y)$. Applying this coordinate change to $S = Y \cap Q$, we obtain that $Y \cap Q$ is in a small neighborhood around p via ϕ isomorphic to $\mathbb{T}_pY \cap \phi(Q)$. As in the proof of Lemma 4.4.1, we show that $\mathbb{T}_pY \cap \phi(Q)$ is the zero locus of a power series in $\mathbb{P}^3$ whose quadratic terms are given by Q and all other terms are of higher order. Consequently, $\mathbb{T}_pY \cap \phi(Q)$ has type $\mathbf{A}_1$ in p. $\square$

Let $C \in \{C_t\}_{t\in\mathbb{P}^1}$ be contained in the plane $\Pi \in \{\Pi_t\}_{t\in\mathbb{P}^1}$. Since C and l are contained in the plane Π, we can apply Bezout's Theorem and obtain

$$C.l = \sum_{p\in C\cap l} (C.l)_p = 3. \tag{4.4}$$

We now establish how often we need to blow-up S over the singularities of S on l such that the strict transform of C under these blowing-ups does not contain any of the singularities on the exceptional divisor in the last blowing-up step. We fix some notation to which we will also refer in a subsequent chapter:

Notation 4.4.4. Let $p \in C \cap l$. By (4.4), we have $m := (C.l)_p \leq 3$.

$$(\mathbb{P}^4)^{(0)} := \mathbb{P}^4, \quad S^{(0)} := S, \quad C^{(0)} := C, \quad l^{(0)} := l, \quad p^{(0)} := p$$

and for $i = 1, \dots, m$ let iteratively

$$\pi^{(i)} \colon (\mathbb{P}^4)^{(i)} \to (\mathbb{P}^4)^{(i-1)}$$

be the blowing-up of $(\mathbb{P}^4)^{(i-1)}$ in $p^{(i-1)}$, where for $i \geq 2$, we let

$$p^{(i-1)} \in C^{(i-1)} \cap l^{(i-1)} \cap E_{\mathbb{P}^4}^{(i-1)}$$

and $E_{\mathbb{P}^4}^{(i-1)}$ is the exceptional divisor of $\pi^{(i-1)}$ in $(\mathbb{P}^4)^{(i-1)}$ and $S^{(i-1)}$, $C^{(i-1)}$, and $l^{(i-1)}$ are the strict transforms of $S^{(i-2)}$, $C^{(i-2)}$, and $l^{(i-2)}$ in $(\mathbb{P}^4)^{(i-1)}$, respectively.

Note that $p^{(i-1)}$ is uniquely determined since the blowing-up $\pi^{(i)}$ is by construction an isomorphism on $C^{(i-1)} \setminus (C^{(i-1)} \cap E_{\mathbb{P}^4}^{(i-1)})$ and $l^{(i-1)} \setminus (l^{(i-1)} \cap E_{\mathbb{P}^4}^{(i-1)})$ onto $C \setminus \{p^{(i-2)}\}$ and $l \setminus \{p^{(i-2)}\}$, respectively, so $C^{(i-1)}$ and $l^{(i-1)}$ intersect $E_{\mathbb{P}^4}^{(i-1)}$ in the same point $p^{(i-1)}$.

Lemma 4.4.5. *The point $p^{(i-1)} \in C^{(i-1)} \cap l^{(i-1)} \cap E_{\mathbb{P}^4}^{(i-1)}$ is a singularity of $S^{(i-1)}$ and $C^{(m)} \cap l^{(m)} \cap E_{\mathbb{P}^4}^{(m)} = \emptyset$. Further, $p^{(m-1)}$ is of type $\mathbf{A}_1$ on $S^{(m-1)}$.*

Proof. The strict transform $Q^{(i)}$ of Q in $(\mathbb{P}^4)^{(i)}$ has singular locus $l^{(i)}$, hence $p^{(i)} \in C^{(i)} \cap l^{(i)} \cap E_{\mathbb{P}^4}^{(i)} \subseteq S^{(i)}$ is a singular point of $S^{(i)}$.

Both C and l are contained in the plane $\Pi \subseteq \mathbb{P}^4$. For $i = 1, \dots, m$, let $\Pi^{(i)}$ be the strict transform of Π in $(\mathbb{P}^4)^{(i)}$. By Lemma A.0.4, we have $C^{(1)}.l^{(1)} = C.l - 1$. Then, blowing-up iteratively $\Pi^{(i)}$ in $p^{(i)} \in C^{(i)} \cap l^{(i)} \cap E_{\mathbb{P}^4}^{(i)}$ gives $C^{(m)}.l^{(m)} = C.l - m = 3 - m$. Since the blowing-ups are isomorphisms outside their exceptional divisors and since $\sum_{q\in C\cap l,\, q\neq p} (C.l)_q = C.l - (C.l)_p = 3 - m$, it follows that $C^{(m)} \cap l^{(m)} \cap E_{\mathbb{P}^4}^{(m)} = \emptyset$.

We show that $p^{(m-1)}$ is of type $\mathbf{A}_1$ on $S^{(m-1)}$. Indeed, we have $C^{(m-1)}.l^{(m-1)} = 3 - (m-1)$. Since $\sum_{q\in C\cap l;\, q\neq p} (C.l)_q = 3 - m$, we must have $(C^{(m-1)}.l^{(m-1)})_{p^{(m-1)}} = 1$. By Lemma 4.4.3, it follows that $p^{(m-1)}$ is of type $\mathbf{A}_1$. $\square$

Lemma 4.4.6. *All possible ADE singularities of S lying on the singular line l of Q are: $3\mathbf{A}_1$, $\mathbf{A}_1 + \mathbf{D}_{n-2}$ $(n \geq 5)$, $\mathbf{A}_5$, $\mathbf{D}_6$, and $\mathbf{E}_7$.*

Proof. By Lemma 4.4.2, C and l intersect in the singular point of S lying on l.

Assume that C and l intersect in three different singularities p_1, p_2, and p_3. By (4.4), this implies that for $i = 1, 2, 3$, we have $(C.l)_{p_i} = 1$. By Lemma 4.4.5, this means that the singularities p_i have type $\mathbf{A}_1$ on S , i.e. C and l intersect in three $\mathbf{A}_1$ singularities.

Then, assume that C and l intersect in p_1 with multiplicity one and in p_2 with multiplicity two. By Lemma 4.4.5, this means that p_1 is of type $\mathbf{A}_1$. Further, Lemma 4.4.5 implies that on the exceptional divisor of the blowing-up of p_2 must lie an $\mathbf{A}_1$ singularity. According to Table 1.1, the only *ADE* singularities which have an $\mathbf{A}_1$ singularity on the exceptional divisor after blow-up, are of type $\mathbf{A}_3$, $\mathbf{D}_4$, and $\mathbf{D}_{n\geq 5}$. In conclusion, p_2 must have singularity type $\mathbf{D}_{n\geq 3}$.

Finally, assume that C and l intersect in p_1 with multiplicity three. Blowing-up two times over p_1, we must obtain an $\mathbf{A}_1$ singularity on the exceptional divisor of the second blowing-up by Lemma 4.4.5. Again, according to Table 1.1, the only *ADE* singularities having an $\mathbf{A}_1$ singularity on the exceptional divisor of a second blowing-up over them are of type $\mathbf{A}_5$, $\mathbf{D}_6$, or $\mathbf{E}_7$. Hence, p_1 is of type $\mathbf{A}_5$, $\mathbf{D}_6$, or $\mathbf{E}_7$. □

5 Cubic hypersurfaces with isolated ADE singularities

In this chapter, we will study cubic hypersurfaces. We will explain how to associate to a cubic hypersurface in $\mathbb{P}^n$ with only isolated ADE singularities a complete $(2,3)$-intersection in $\mathbb{P}^{n-1}$ and how the ADE singularities of the cubic hypersurface are related to the ADE singularities of this complete $(2,3)$-intersection. This will enable us to prove in the following chapters that the existence of a cubic fourfold with a certain combination of isolated ADE singularities is equivalent to the existence of a complete $(2,3)$-intersection in $\mathbb{P}^4$ with certain isolated ADE singularities.

5.1 Basic notation, definitions, and properties

Let $(x_0 : \ldots : x_n)$ be homogenous coordinates on $\mathbb{P}^n$ $(n \geq 2)$.

Let X be a cubic hypersurface in $\mathbb{P}^n$ and assume that X is singular in $p \in X$. After a linear change of coordinates, we can assume that $p = (1 : 0 : \ldots : 0) \in \mathbb{P}^n$.

Lemma 5.1.1 ([Wal, §2], [Hav16, 2.1])**.** *In the chosen coordinates, the equation defining X has the form*

$$x_0 f_2(x_1, \ldots, x_n) + f_3(x_1, \ldots, x_n) = 0,$$

where f_2 and f_3 are homogenous polynomials of degree 2 and 3 in $\mathbb{C}[x_1, \ldots, x_n]$, respectively.

We write Q and Y for the quadric and cubic in $\mathbb{P}^{n-1}$ defined by f_2 and f_3, respectively, as in Lemma 5.1.1 and refer to the form of X as the *normal form* of X with respect to the chosen coordinates.

Let $\pi_p \colon \mathbb{P}^n \dashrightarrow \{x_0 = 0\} \cong \mathbb{P}^{n-1}$, $(x_0 : \ldots : x_n) \mapsto (0 : x_1 : \ldots : x_n)$ be the projection through p onto the hyperplane $\mathbb{P}^{n-1}$ given by $\{x_0 = 0\} \subseteq \mathbb{P}^n$. Let $F_p \subseteq X$ be the union of all lines in X passing through p. Define

$$S_p \coloneqq \pi_p\big(F_p\big) \subseteq \mathbb{P}^{n-1}$$

as the image of F_p under π_p in $\mathbb{P}^{n-1}$.

Lemma 5.1.2 ([CG72, Lemma 6.5], [Hav16, 2.1])**.** *Assume that X has only isolated singularities and a double point p. Then, S_p is the complete $(2,3)$-intersection in $\mathbb{P}^{n-1}$ defined as*

$$S_p : \ f_2(x_1, \ldots, x_n) = f_3(x_1, \ldots, x_n) = 0 \subseteq \mathbb{P}^{n-1}.$$

Lemma 5.1.2 shows in particular that F_p is the cone in X through p over the complete $(2,3)$-intersection S_p.

The definition of S_p does not depend on the choice of the hyperplane $H \subseteq \mathbb{P}^n$ with $p \notin H$ onto which we project F_p:

Lemma 5.1.3. *The quadric Q and the complete $(2,3)$-intersection S_p are uniquely determined by p and do not depend on the choice of the hyperplane $H \subseteq \mathbb{P}^n$ with $p \notin H$ onto which we project F_p through p, while the cubic Y is only determined modulo Q.*

Proof. Let $H : x_0 + \sum_{i=1}^n a_i x_i = 0 \subseteq \mathbb{P}^n$ and

$$\pi_p^H : \mathbb{P}^n \dashrightarrow H \cong \mathbb{P}^{n-1},\ (x_0 : \ldots : x_n) \mapsto (-\sum_{i=1}^n a_i x_i : x_1 : \ldots : x_n)$$

be the projection of X onto H through p. Let $q \coloneqq (-\sum_{i=1}^n a_i x_i : x_1 : \ldots : x_n) \in \mathbb{P}^n$ be a point in H and $(\lambda - \mu \sum_{i=1}^n a_i x_i : \mu x_1 : \mu x_1 : \ldots : \mu x_n)$ the line connecting p and q parametrized by $(\lambda : \mu) \in \mathbb{P}^1$. This line is contained in X if and only if

$$\begin{aligned} 0 &= (\lambda - \mu \sum_{i=1}^n a_i x_i) f_2(\mu x_1, \ldots, \mu x_n) + f_3(\mu x_1, \ldots, \mu x_n) \\ &= \lambda \mu^2 f_2(x_1, \ldots, x_n) + \mu^3 \big(f_3(x_1, \ldots, x_n) - (\sum_{i=1}^n a_i x_i) f_2(x_1, \ldots, x_n)\big) \end{aligned}$$

for all choices of $(\lambda : \mu) \in \mathbb{P}^1$, in particular for $(0 : 1)$ which gives

$$f_3(x_1, \ldots, x_n) - (\sum_{i=1}^n a_i x_i) f_2(x_1, \ldots, x_n) = 0$$

and for $(1 : 0)$ which gives

$$f_2(x_1, \ldots, x_n) = 0.$$

Hence, the projection of F_p onto H is isomorphic to the zero locus

$$f_2(x_1, \ldots, x_n) = f_3(x_1, \ldots, x_n) - (\sum_{i=1}^n a_i x_i) f_2(x_1, \ldots, x_n) = 0 \subseteq \mathbb{P}^{n-1}.$$

In conclusion, we see that S_p and the quadric on which S_p is lying are uniquely determined, and the cubic is uniquely determined up to the quadric. $\square$

Hence, Lemma 5.1.3 shows that S_p can be defined without choosing coordinates on $\mathbb{P}^n$.

5.2 ADE singularities on cubic hypersurfaces and complete $(2,3)$-intersections

We follow the notation in Section 5.1.

Assume that the cubic hypersurface $X \subseteq \mathbb{P}^n$ has only ADE singularities, in particular p is an ADE singularity. Let $\pi^{(1)} : \mathrm{Bl}_p X \to X$ be the blowing-up of X in p with exceptional divisor $E \coloneqq (\pi^{(1)})^{-1}(p) \subseteq \mathrm{Bl}_p X$.

Lemma 5.2.1. *E is isomorphic to the quadric $Q \subseteq \mathbb{P}^{n-1}$.*

Proof. E is the projectivized tangent cone to X at p and the latter is defined as the zero locus of f_2 in $\{x_0 = 0\} \cong \mathbb{P}^{n-1}$. □

We now establish that an ADE singularity of type $\mathbf{T}$ on S_p induces a unique singularity with a certain singularity type on $\mathrm{Bl}_p X$:

Proposition 5.2.2 ([Wal, §2])**.** *Let $q \in S_p$. If q is a singularity of both Q and Y, then X is singular along the line $\overline{pq}$ connecting p and q. This means in particular that X has non-isolated singularities. Then, assume that q is not a singularity of both Q and Y and assume that q is of ADE type $\mathbf{T}$ in the locally smooth scheme Q or Y.*

(i) If Q is smooth at q, the cubic hypersurface X has exactly two singularities p and p' on the line $\overline{pq}$ and p' has type $\mathbf{T}$.

(ii) If Q is singular at q, the line $\overline{pq}$ intersects X only in p and the blowing-up $Bl_p X$ has a singularity of type $\mathbf{T}$ at q.

We now enhance the result in Proposition 5.2.2 and show that actually each singularity on $\mathrm{Bl}_p X$ is induced by a singularity on S_p and determine the location of those singularities. This establishes that the singularities of S_p are in one-to-one correspondence with the singularities of $\mathrm{Bl}_p X$ including the singularity type.

Corollary 5.2.3. *The singularities of $X \setminus \{p\}$ correspond, including their singularity type, one-to-one to those singularities of S_p which are not contained in the singular locus of Q. The singularities of $Bl_p X$ on E correspond, including their singularity type, one-to-one to those singularities of S_p which are contained in the singular locus of Q.*

Proof. We give firstly a one-to-one correspondence between the singularities of $X \setminus \{p\}$ and those singularities of S_p which are not lying on the singular locus of Q.

By item (i) in Proposition 5.2.2, given a singularity p' on S_p, the cubic X has a unique singularity $q' \neq p$ on the line $\overline{pp'}$.

Conversely, for an ADE singularity $q := (q_0 : \ldots : q_n) \in X$ with $q \neq p$, the line $\overline{pq}$ must be contained in X. Indeed, p and q are both double points of X so $\overline{pq}$ intersects X with multiplicity 4. Since X has degree 3, this means that $\overline{pq}$ must be contained in X. We claim that the image of $\overline{pq}$ under the projection π_p of $\mathbb{P}^n$ through p onto the hyperplane $\mathbb{P}^{n-1}$ given by $\{x_0 = 0\}$ is a singularity of S_p. In fact, the line $\overline{pq}$ is given by $(\lambda - \mu q_0 : \mu q_1 : \ldots : \mu q_n)$, where $(\lambda : \mu) \in \mathbb{P}^1$. Then, $\pi_p\big((\lambda - \mu q_0 : \mu q_1 : \ldots : \mu q_n)\big) = (q_1 : \ldots : q_n)$. Since q is a singularity of X, we have

$$0 = q_0 f_2(q_1, \ldots, q_n) + f_3(q_1, \ldots, q_n) \tag{5.1}$$

$$0 = q_0 \frac{\partial}{\partial x_i} f_2(q_1, \ldots, q_n) + \frac{\partial}{\partial x_i} f_3(q_1, \ldots, q_n) \text{ for all } i = 1, \ldots, n \tag{5.2}$$

$$0 = f_2(q_1, \ldots, q_n). \tag{5.3}$$

Equations (5.1) and (5.3) give that $(q_1 : \ldots : q_n) \in S_p$. By equation (5.2), we have $\frac{\partial}{\partial x_i} f_3(q_1, \ldots, q_n) = -q_0 \frac{\partial}{\partial x_i} f_2(q_1, \ldots, q_n)$ for all $i = 1, \ldots, n$. Hence, the Jacobian matrix of the polynomials f_2 and f_3 has at $(q_1 : \ldots : q_n)$ not full rank. Therefore, $(q_1 : \ldots : q_n)$ is a singularity of S_p. However, $(q_1 : \ldots : q_n)$ is not a singularity of Q. Otherwise, $(q_1 : \ldots : q_n)$ would also be a singularity of Y by (5.2) and hence X would have non-isolated singularities by Proposition 5.2.2 which is false by assumption.

The construction above establishes a one-to-one correspondence between the singularities of $X \setminus \{p\}$ and those singularities of S_p which are not lying on the singular locus of Q. Moreover, by item (i) in Proposition 5.2.2, corresponding singularities have the same singularity types.

We show by a direct computation that a singularity q of $\mathrm{Bl}_p X$ is contained in E if and only if it naturally corresponds to a singularity of S_p lying on the singular locus of Q.

Indeed, let $\pi^{(1)} \colon \mathbb{P}^n \times \mathbb{P}^{n-1} \supseteq (\mathbb{P}^n)^{(1)} \to \mathbb{P}^n$ be the blowing-up of $\mathbb{P}^n$ in p and $(y_1 : \ldots : y_n)$ homogeneous coordinates on $\mathbb{P}^{n-1}$. Assume without loss of generality that q is contained in the affine chart $\mathbb{P}^n \times \mathbb{A}^{n-1} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by $y_1 \neq 0$. We have

$$(\mathbb{P}^n)^{(1)} := \{\big((x_0 : \ldots : x_n), (1, y_2, \ldots, y_n)\big) \in \mathbb{P}^n \times \mathbb{A}^{n-1};\ x_i = y_i x_1 \text{ for all } i = 2, \ldots, n\}.$$

The strict transform of X in $(\mathbb{P}^n)^{(1)}$ is given by

$$\mathrm{Bl}_p X :\ x_0 f_2(1, y_2, \ldots, y_n) + x_1 f_3(1, y_2, \ldots, y_n) = 0$$

and the exceptional divisor $E \subseteq \mathrm{Bl}_p X$ by

$$E = \{\big((1 : 0 : \ldots : 0), (1, y_2, \ldots, y_n)\big) \in \mathbb{P}^n \times \mathbb{A}^{n-1};\ f_2(1, y_2, \ldots, y_n) = 0\}.$$

Note that with respect to the projection $\mathrm{pr}_2 \colon \mathbb{P}^n \times \mathbb{A}^{n-1} \to \mathbb{A}^{n-1}$, the exceptional divisor E is isomorphic to Q on $\mathbb{A}^{n-1}$ (this proves in particular Lemma 5.2.1 in coordinates).

Assume that we have in coordinates $q = \big((w_0 : \ldots : w_n), (1, r_2, \ldots, r_n)\big) \in \mathrm{Bl}_p X$. Since q is a singularity of $\mathrm{Bl}_p X$, it is a zero of all partial derivatives of the function defining $\mathrm{Bl}_p X$ on this chart, i.e.

$$0 = f_2(1, r_2, \ldots, r_n) \tag{5.4}$$
$$0 = f_3(1, r_2, \ldots, r_n) \tag{5.5}$$
$$0 = w_0 \cdot \frac{\partial f_2}{\partial y_i}(1, r_1, \ldots, r_n) + w_1 \cdot \frac{\partial f_3}{\partial y_i}(1, r_1, \ldots, r_n) \quad \text{for all } i = 2, \ldots, n. \tag{5.6}$$

Equations (5.4) and (5.5) give that the image of q under pr_2 is contained in S_p.

Now assume that q is contained in E, i.e. $q = \big((1 : 0 : \ldots : 0), (1, r_2, \ldots, r_n)\big)$. Equation (5.6) gives

$$0 = 1 \cdot \frac{\partial f_2}{\partial y_i}(1, r_1, \ldots, r_n) + 0 \cdot \frac{\partial f_3}{\partial y_i}(1, r_1, \ldots, r_n) = \frac{\partial f_2}{\partial y_i}(1, r_1, \ldots, r_n)$$

for all $i = 2, \ldots, n$. Hence, the image $(1, r_1, \ldots, r_n)$ of q under the projection pr_2 is a singularity of Q.

Conversely, assume that $(1, r_1, \ldots, r_n)$ is a singularity of Q. Then, for all $i = 2, \ldots, n$

$$\frac{\partial f_2}{\partial y_i}(1, r_1, \ldots, r_n) = 0. \tag{5.7}$$

Furthermore, $(1, r_1, \ldots, r_n)$ cannot be a singularity of Y, as well, since X had otherwise non-isolated singularities by Proposition 5.2.2. Therefore, plugging (5.7) into equation (5.6), we obtain $w_1 = 0$. This gives $w_i = r_i w_1 = 0$ for all $i = 1, \ldots, n$. Therefore, $q = \big((1 : 0 : \ldots : 0), (1, r_2, \ldots, r_n)\big)$, i.e. $q \in E$ (this also proves in particular partly item (ii) in Proposition 5.2.2).

The computations are similar on the other charts of the blowing-up.

In conclusion, we see that all singularities of $\mathrm{Bl}_p X$ on E correspond to singularities of S_p on the singular locus of Q. Furthermore, by item (ii) in Proposition 5.2.2, the corresponding singularities have the same singularity types. □

In Table 1.1, we recorded for an ADE singularity of type $\mathbf{T}$ on X the singularities $\sigma(\mathbf{T})$ that occur on the exceptional divisor E.

6 Cubic fourfolds and K3 surfaces with isolated ADE singularities

In this chapter, we prove the first Main Theorem which states that the existence of a cubic fourfold with certain isolated ADE singularities is equivalent to both the existence of complete $(2,3)$-intersections in $\mathbb{P}^4$ with certain isolated ADE singularities and embeddings of certain lattices into the K3 lattice. To prove the Main Theorem, we will firstly prove an auxiliary technical proposition where we compute the pull-back of a certain hyperplane section of a complete $(2,3)$-intersection in $\mathbb{P}^4$ to the smooth minimal model of this complete $(2,3)$-intersection.

6.1 Main Theorem 1

Main Theorem 1. *Let* $\mathbf{T} \in \{\mathbf{A}_{i\geq 1},\, \mathbf{D}_{j\geq 4},\, \mathbf{E}_{8\geq k\geq 6}\}$ *be an ADE singularity type.*

For $\big((a_1,\ldots,a_n),(d_4,\ldots,d_m),(e_6,e_7,e_8)\big) \in \mathbb{Z}_{\geq 0}{}^n \times \mathbb{Z}_{\geq 0}{}^{m-3} \times \mathbb{Z}_{\geq 0}{}^3$, *let*

$$\mathbf{G} \coloneqq \sum_{i=1}^{n} a_i \mathbf{A}_i + \sum_{j=4}^{m} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k$$

be a formal sum of ADE singularity types and

$$\Gamma_{\mathbf{G}} \coloneqq \sum_{i=1}^{n} a_i \mathcal{A}_i + \sum_{j=4}^{m} d_j \mathcal{D}_j + \sum_{k=6}^{8} e_k \mathcal{E}_k$$

a Dynkin diagram with connected components $\mathcal{A}_i$, $\mathcal{D}_j$, and $\mathcal{E}_k$.

The following are equivalent:

1. *There exists a cubic fourfold X in $\mathbb{P}^5$ with a singularity of type $\mathbf{T}$ and such that all other singularities of X correspond to $\mathbf{G}$.*
2. *There exists a complete $(2,3)$-intersection S in $\mathbb{P}^4$ of a quadric Q of* $\operatorname{corank}(Q) = \operatorname{corank}_{\mathbf{T}}$ *as in Table 6.1 and a cubic Y such that the singularities of S that lie on the singular locus of Q are of type $\sigma(\mathbf{T})$ as in Table 6.1 and such that all other singularities of S correspond to $\mathbf{G}$.*
3. *Let $\Gamma_{\sigma(\mathbf{T})}$ be a weighted graph as in Table 6.1. Let $\Lambda(\Gamma_{\mathbf{G}})$ and $\Lambda(\Gamma_{\sigma(\mathbf{T})})$ be the lattices associated to the weighted graphs $\Gamma_{\mathbf{G}}$ and $\Gamma_{\sigma(\mathbf{T})}$. Let $h_{\mathbf{T}} \in \Lambda(\Gamma_{\sigma(\mathbf{T})})$ be the sum of the vertices of $\Gamma_{\sigma(\mathbf{T})}$ as in Table 6.1. There exists an embedding*

   $$i\colon \Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}}) \hookrightarrow L_{K3}$$

   such that the following conditions a), b), and c) hold:

a) If $x \in \mathrm{Sat}_{L_{K3}}(i)$ with $i(h_{\mathbf{T}}).x = 0$ and $x^2 = -2$, then $x \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$.

b) There exists no element $x \in \mathrm{Sat}_{L_{K3}}(i)$ with $i(h_{\mathbf{T}}).x = 1$ and $x^2 = 0$.

c) There exists no element $x \in \mathrm{Sat}_{L_{K3}}(i)$ with $i(h_{\mathbf{T}}).x = 2$ and $x^2 = 0$.

Remark 6.1.1. By Lemmas 4.4.1 and 4.4.6, we consider in 2. all types of singularities that a complete $(2,3)$-intersection in $\mathbb{P}^4$ can possibly have on the singular locus of the underlying quadric.

6.2 Proof of Main Theorem 1

To prove Main Theorem 1, we show the following auxiliary proposition:

Proposition 6.2.1. *Let $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ be an ADE singularity type and* $\mathrm{corank}_{\mathbf{T}}$ *and $\sigma(\mathbf{T})$ as in Table 6.1.*

Let S be a complete $(2,3)$-intersection of a quadric Q of $\mathrm{corank}(Q) = \mathrm{corank}_{\mathbf{T}}$ *and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities. Assume that all singularities of S lying on $\mathrm{Sing}(Q)$ are of type $\sigma(\mathbf{T})$. Let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities of S.*

Then, there exists a hyperplane section $C_{\mathbf{T}}$ of S such that $h_{\mathbf{T}} := \pi^*(C_{\mathbf{T}}) \in \mathrm{Div}(\widetilde{S})$ is the formal sum of curves on $\widetilde{S}$ as in Table 6.1 and the associated weighted graph to these curves is $\Gamma_{\sigma(\mathbf{T})}$ in Table 6.1.

6.3 An auxiliary step in the proof of Main Theorem 1

As outlined in Chapter 4.1, a projective quadric is up to isomorphism uniquely determined by its rank. Hence, we prove Proposition 6.2.1 for all possible coranks of the quadric Q in $\mathbb{P}^4$ individually.

6.3.1 The quadric Q has corank 0 in $\mathbb{P}^4$

Proposition 6.3.1. *Let S be a complete $(2,3)$-intersection of a quadric Q of* $\mathrm{corank}(Q) = 0$ *and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities. Let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Then, there exists a hyperplane section $C_{\mathbf{A}_1}$ of S such that $h_{\mathbf{A}_1} := \pi^*(C_{\mathbf{A}_1}) \in \mathrm{Div}(\widetilde{S})$ is an irreducible curve on $\widetilde{S}$.

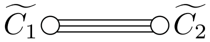
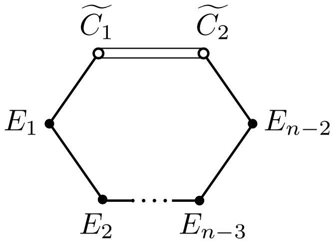
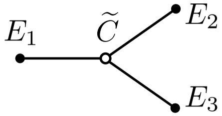
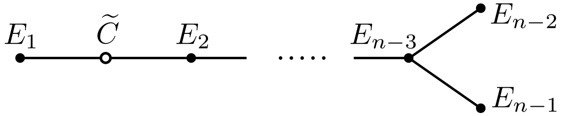
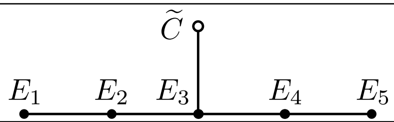
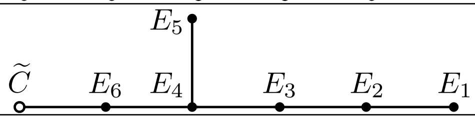
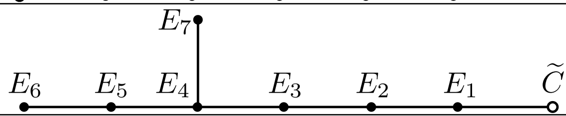
Proof. Since S has only isolated ADE singularities, Bertini's Theorem [Har77, Chap. II, Theorem 8.18, Remark 8.18.1] implies that for a general hyperplane H in $\mathbb{P}^4$ the curve

$$C := H \cap S \subseteq S$$

is irreducible, smooth, and contains none of the singularities of S. Therefore, we have

$$\pi^* C = \widetilde{C} \in \mathrm{Div}(\widetilde{S}),$$

Table 6.1

$\mathbf{T}$	$\mathrm{corank}_{\mathbf{T}}$	$\sigma(\mathbf{T})$	$\Gamma_{\sigma(\mathbf{T})}$	$h_{\mathbf{T}}$
$\mathbf{A}_1$	0	$\emptyset$	$\diamond\, \widetilde{C}$	$\widetilde{C}$
$\mathbf{A}_2$	1	$\emptyset$		$\widetilde{C}_1 + \widetilde{C}_2$
$\mathbf{A}_{n\geq 3}$	1	$\mathbf{A}_{n-2}$		$\widetilde{C}_1 + \widetilde{C}_2 + E_1 + \ldots + E_{n-2}$
$\mathbf{D}_4$	2	$3\mathbf{A}_1$		$2\widetilde{C} + E_1 + E_2 + E_3$
$\mathbf{D}_{n\geq 5}$	2	$\mathbf{A}_1 + \mathbf{D}_{n-2}$		$2\widetilde{C} + E_1 + 2E_2 + \ldots + 2E_{n-4} + 2E_{n-3} + E_{n-2} + E_{n-1}$
$\mathbf{E}_6$	2	$\mathbf{A}_5$		$2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5$
$\mathbf{E}_7$	2	$\mathbf{D}_6$		$2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 4E_4 + 2E_5 + 3E_6$
$\mathbf{E}_8$	2	$\mathbf{E}_7$		$2\widetilde{C} + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7$

The vertices of the graphs in column $\Gamma_{\sigma(\mathbf{T})}$ are curves intersecting each other according to the number of edges joining them. The vertex $\diamond$ has self-intersection number 6, the vertices $\circ$ self-intersection number 0, and the vertices $\bullet$ self-intersection number (-2).

where $\widetilde{C}$ is the strict transform of C in $\widetilde{S}$ under the minimal resolution π of all singularities on S. Further, $\widetilde{C}$ is irreducible since C is irreducible. In conclusion, $h_{\mathbf{A}_1} := \pi^*(C)$ is an irreducible curve on $\widetilde{S}$. □

Remark 6.3.2. In the proof of Proposition 6.3.1, we actually did not use the assumption that the quadric Q in which the complete $(2,3)$-intersection is contained is of $\operatorname{corank}(Q) = 0$ in $\mathbb{P}^4$.

6.3.2 The quadric Q has corank 1 in $\mathbb{P}^4$

6.3.2.1 General setting and notation

We fix some notation which we will need in the following.

Let S be a complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ with the property that S has only isolated ADE singularities. By Lemma 4.3.2, this implies that Q and Y have no common singularities. In particular, the results in Sections 4.3 and 4.4 hold for this choice of S.

Assume that the quadric Q has corank 1 in $\mathbb{P}^4$. We then recall from Subsections 4.1.1, 4.1.2, and 4.3.1: By Lemma 4.1.1, Q is the cone through p over a smooth quadric Q' in $\mathbb{P}^3$ and p is the only singular point of Q. By Lemma 4.1.2, there are two rulings $\{l_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{l_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ on Q' such that through every point in Q' passes exactly one line from each of the rulings. For $\alpha, \beta \in \mathbb{P}^1$, let

$$\begin{aligned}\Pi_{1,\alpha} &:= \text{plane spanned by } p \text{ and } l_{1,\alpha} \subseteq \mathbb{P}^4\\ \Pi_{2,\beta} &:= \text{plane spanned by } p \text{ and } l_{2,\beta} \subseteq \mathbb{P}^4.\end{aligned}$$

Both $\Pi_{1,\alpha}$ and $\Pi_{2,\beta}$ are then contained in the quadric $Q \subseteq \mathbb{P}^4$ such that we obtain two pencils of planes $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ on Q. Let

$$C_{1,\alpha} := \Pi_{1,\alpha} \cap Y \subseteq S \text{ and } C_{2,\beta} := \Pi_{2,\beta} \cap Y \subseteq S$$

be the cubic curves on S lying on the planes $\Pi_{1,\alpha}$ and $\Pi_{2,\beta}$, respectively. We then have the pencils $\{C_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{C_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ on S.

For α_s, $\beta_s \in \mathbb{P}^1$ such that conditions (1)-(4) in Lemma 4.3.4 are satisfied, write

$$\begin{array}{llllll}\Pi_1 := & \Pi_{1,\alpha_s}, & \Pi_2 := & \Pi_{2,\alpha_s}, & l & := l_{\alpha_s,\beta_s}\\ C_1 := & C_{1,\alpha_s}, & C_2 := & C_{2,\beta_s}. & & \end{array} \tag{6.1}$$

Let $\pi\colon \widetilde{S} \to S$ the minimal resolution of all singularities on S and $\widetilde{C_1}$ and $\widetilde{C_2}$ the strict transforms in $\widetilde{S}$ under π of C_1 and C_2, respectively.

Lemma 6.3.3. *We have* $\widetilde{C_1}^2 = \widetilde{C_2}^2 = 0$.

Proof. Let $i = 1, 2$.

We compute the arithmetic genus $p_a(C_i)$ of C_i. By definition, we have

$$p_a(C_i) = 1 - \chi(C_i, \mathcal{O}_{C_i}),$$

where $\chi(C_i, \mathcal{O}_{C_i})$ is the Euler characteristic of C_i. Since $\dim H^0(C_i, \mathcal{O}_{C_i}) = 1$, we obtain

$$p_a(C_i) = \dim H^1(C_i, \mathcal{O}_{C_i}).$$

We claim that we have $\dim H^1(C_i, \mathcal{O}_{C_i}) = 1$. Indeed, the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{C_i} \to 0$$

induces the long exact sequence on cohomology

$$\cdots \to \underbrace{H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})}_{=0} \to H^1(C_i, \mathcal{O}_{C_i}) \to H^2\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)\big) \to \underbrace{H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})}_{=0} \to \cdots .$$

Consequently, $H^1(C_i, \mathcal{O}_{C_i}) \cong H^2\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)\big)$. Since $\dim H^2\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)\big) = 1$, we obtain $p_a(C_i) = \dim H^1(C_i, \mathcal{O}_{C_i}) = 1$.

By Lemma 4.3.4, C_i is smooth in p and contains no singularities of S different from p. Hence, $\widetilde{C_i} \cong C_i$ so $p_a(C_i) = p_a(\widetilde{C_i})$. We get

$$p_a(\widetilde{C_i}) = 1. \tag{6.2}$$

On the other hand, by the adjunction formula, we have

$$p_a(\widetilde{C_i}) = 1 + \frac{1}{2} \deg \Big(\big(\omega_{\widetilde{S}} \otimes_{\mathcal{O}_{\widetilde{S}}} \mathcal{O}_{\widetilde{S}}(\widetilde{C_i})\big)_{|\widetilde{C_i}} \Big).$$

Since $\widetilde{S}$ is a K3 surface, the canonical bundle $\omega_{\widetilde{S}}$ is trivial. Hence,

$$p_a(\widetilde{C_i}) = 1 + \frac{1}{2} \deg \big(\mathcal{O}_{\widetilde{S}}(\widetilde{C_i})_{|\widetilde{C_i}}\big) = 1 + \frac{1}{2}\widetilde{C_i}^2,$$

so

$$\widetilde{C_i}^2 = 2p_a(\widetilde{C_i}) - 2 = 0.$$

We conclude from (6.2) that $\widetilde{C_i}^2 = 0$. □

In the following subsections, we compute the pull-back $\pi^*(C_1 + C_2) \in \mathrm{Div}(\widetilde{S})$ explicitly.

6.3.2.2 Assumption: $\mathsf{T} = \mathsf{A}_2$ (thus $\sigma(\mathsf{T}) = \emptyset$)

We prove Proposition 6.2.1 in case $\mathrm{corank}(Q) = 1$, $\mathbf{T} = \mathbf{A}_2$, and thus $\sigma(\mathbf{A}_2) = \emptyset$:

Proposition 6.3.4. *Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi : \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 1 in $\mathbb{P}^4$ and the singular point p of Q is not contained in S.

Let C_1 and C_2 be the plane cubic curves on S and $\widetilde{C_1}$ and $\widetilde{C_2}$ the strict transforms of C_1 and C_2 under π in $\widetilde{S}$ as in (6.1).

Then, for the hyperplane section $C_1 + C_2$ of S, we have $h_{\mathbf{A}_2} := \pi^*(C_1 + C_2) = \widetilde{C_1} + \widetilde{C_2} \in$ $\mathrm{Div}(\widetilde{S})$. The lattice in $\mathrm{Div}(\widetilde{S})$ with basis $\widetilde{C_1}, \widetilde{C_2}$ has the intersection matrix:

$$\begin{array}{c c} & \begin{array}{cc} \widetilde{C_1} & \widetilde{C_2} \end{array} \\ \begin{array}{c} \widetilde{C_1} \\ \widetilde{C_2} \end{array} & \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \end{array} \tag{6.3}$$

Proof. We proved in Lemma 4.3.3 that the divisor $C_1 + C_2$ on S is a hyperplane section of S. The curves C_1 and C_2 satisfy condition (2) in Lemma 4.3.4 by their choice in (6.1). Since the singular locus of Q is not contained in Y, this means that C_1 and C_2 are contained in the smooth locus S° of S. Hence, the total transforms of C_1 and C_2 in $\widetilde{S}$ under the minimal resolution π coincide with the strict transforms $\widetilde{C_1}$ and $\widetilde{C_2}$ under π. Consequently,

$$\pi^*(C_1 + C_2) = \widetilde{C_1} + \widetilde{C_2}.$$

By Lemma 6.3.3, we have

$$\widetilde{C_1}^2 = \widetilde{C_2}^2 = 0. \tag{6.4}$$

Again, since C_1 and C_2 are both contained in S°, they are isomorphic to $\widetilde{C_1}$ and $\widetilde{C_2}$ via π. Hence,

$$\widetilde{C_1}.\widetilde{C_2} = C_1.C_2. \tag{6.5}$$

Since $C_2 = Y \cap \Pi_2 = S^\circ \cap \Pi_2$, we have

$$C_1.C_2 = C_1.(S^\circ \cap \Pi_2). \tag{6.6}$$

Since C_1 is contained in both S° and Π_1, Lemma A.0.1 implies

$$C_1.(S^\circ \cap \Pi_2) = C_1.(\Pi_1 \cap \Pi_2). \tag{6.7}$$

The line $l := \Pi_1 \cap \Pi_2$ intersects the cubic C_1 on the plane Π_1 in three points by Bezout's Theorem. Hence,

$$C_1.(\Pi_1 \cap \Pi_2) = C_1.l = 3. \tag{6.8}$$

Equations (6.5)-(6.8) together give

$$\widetilde{C_1}.\widetilde{C_2} = 3. \tag{6.9}$$

In conclusion, the lattice with basis $\widetilde{C_1}$ and $\widetilde{C_2}$ has by (6.4) and (6.9) the intersection matrix (6.3) with respect to this basis. □

6.3.2.3 Assumption: $\mathsf{T} = \mathsf{A}_n$ for $n \geq 3$ (thus $\sigma(\mathsf{T}) = \mathsf{A}_{n-2}$)

We prove Proposition 6.2.1 in case $\operatorname{corank}(Q) = 1$, for $n \geq 3$, $\mathbf{T} = \mathbf{A}_n$, and thus $\sigma(\mathbf{A}_n) = \mathbf{A}_{n-2}$:

Proposition 6.3.5. *Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 1 in $\mathbb{P}^4$ and the singular point p of Q is contained in S.

Let C_1 and C_2 be the plane cubic curves on S and $\widetilde{C_1}$ and $\widetilde{C_2}$ the strict transforms of C_1 and C_2, respectively, under π in $\widetilde{S}$.

Then, for the hyperplane section $C_1 + C_2$ of S, we have $h_{\mathbf{A}_n} := \pi^*(C_1 + C_2) = \widetilde{C_1} + \widetilde{C_2} + E_1 + \ldots + E_{n-2} \in \operatorname{Div}(\widetilde{S})$, where $E_1, \ldots, E_{n-2}$ are (-2)-curves on $\widetilde{S}$.

The lattice in $\operatorname{Div}(\widetilde{S})$ with basis $\widetilde{C_1}, \widetilde{C_2}, E_1 \ldots, E_{n-2}$ has the intersection matrix:

$$
\begin{array}{c}
\\
\widetilde{C_1}\\
\widetilde{C_2}\\
E_1\\
E_2\\
\vdots\\
\\
E_{n-3}\\
E_{n-2}
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
\widetilde{C_1} & \widetilde{C_2} & E_1 & E_2 & \cdots\cdots & E_{n-3} & E_{n-2}
\end{array}\\
\left(\begin{array}{cc|ccccc}
0 & 2 & 1 & 0 & \cdots\cdots & 0 & 0\\
2 & 0 & 0 & 0 & \cdots\cdots & 0 & 1\\
\hline
1 & 0 & -2 & 1 & & & \\
0 & 0 & 1 & \ddots & \ddots & & \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \\
 & & & & \ddots & \ddots & \\
0 & 0 & & & & \ddots & 1\\
0 & 1 & & & & 1 & -2
\end{array}\right)\\
\qquad\qquad\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{A_{n-2}}
\end{array}
\tag{6.10}
$$

Proof. We proved in Lemma 4.3.3 that the divisor $C_1 + C_2$ on S is a hyperplane section of S.

By Lemma 6.3.3, we have

$$\widetilde{C_1}^2 = \widetilde{C_2}^2 = 0. \tag{6.11}$$

Let $\pi^{(1)} \colon (\mathbb{P}^4)^{(1)} := \mathrm{Bl}_p\mathbb{P}^4 \to \mathbb{P}^4$ be the blowing-up of $\mathbb{P}^4$ in p. Let

$$S^{(1)}, \quad Y^{(1)}, \quad \Pi_i^{(1)}, \quad l^{(1)}, \quad \text{and} \quad C_i^{(1)} = \Pi_i^{(1)} \cap Y^{(1)} \quad (i = 1, 2)$$

be the strict transforms of S, Y, Π_i, $l := \Pi_1 \cap \Pi_2$, and C_i, respectively under $\pi^{(1)}$ in $(\mathbb{P}^4)^{(1)}$.

We recall that C_1 and C_2 satisfy condition (2) in Lemma 4.3.4 by their choice in (6.1). Hence,

$$C_1^{(1)} \text{ and } C_2^{(1)} \text{ are contained in the smooth locus } (S^{(1)})^\circ \text{ of } S^{(1)}. \tag{6.12}$$

By (6.12), $C_1^{(1)}$ and $C_2^{(1)}$ are isomorphic to the strict transforms $\widetilde{C_1}$ and $\widetilde{C_2}$ of C_1 and C_2, respectively under π in $\widetilde{S}$. Hence,

$$\widetilde{C_1}.\widetilde{C_2} = C_1^{(1)}.C_2^{(1)}. \tag{6.13}$$

Further, we have $C_2^{(1)} = (S^{(1)})^\circ \cap \Pi_2^{(1)}$. Since $C_1^{(1)}$ is contained in both $(S^{(1)})^\circ$ and $\Pi_1^{(1)}$, Lemma A.0.1 gives

$$C_1^{(1)}.C_2^{(1)} = C_1^{(1)}.\big((S^{(1)})^\circ \cap \Pi_2^{(1)}\big) = C_1^{(1)}.(\Pi_2^{(1)} \cap \Pi_1^{(1)}) = C_1^{(1)}.l^{(1)}. \tag{6.14}$$

Consequently, by Lemma A.0.3

$$C_1^{(1)}.l^{(1)} = C_1.l - 1. \tag{6.15}$$

Since C_1 and l_1 lie on the plane Π_1, we can apply Bezout's Theorem and obtain

$$C_1.l - 1 = 3 - 1 = 2. \tag{6.16}$$

Equations (6.13)-(6.16) together give

$$\widetilde{C_1}.\widetilde{C_2} = 2. \tag{6.17}$$

Again, by the choice of the curves C_1 and C_2, the $\mathbf{A}_{n-2}$ singularity p is the only singularity of S which is contained in C_1 and C_2. Hence, the divisor $\pi^*(C_1+C_2)$ on $\widetilde{S}$ is supported on the union of $\widetilde{C_1}$, $\widetilde{C_2}$, and the strict transforms $E_1,\dots,E_{n-2}$ in $\widetilde{S}$ of the exceptional curves of the minimal resolution of p. Hence, the weighted graph with vertices $E_1,\dots,E_{n-2}$ is the Dynkin diagram of type $\mathcal{A}_{n-2}$ and we chose the notation such that it is given by Figure 6.1.

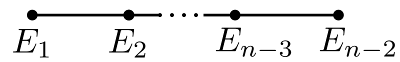


Figure 6.1: Dynkin diagram corresponding to the $\mathbf{A}_{n-2}$ singularity p.

We compute the intersection numbers of $\widetilde{C_1}$ and $\widetilde{C_2}$ with $E_1,\dots,E_{n-2}$. Let

$$E^{(1)}_{\mathbb{P}^4} \subseteq (\mathbb{P}^4)^{(1)}, \quad E^{(1)}_S := E^{(1)}_{\mathbb{P}^4} \cap S^{(1)} \subseteq S^{(1)}, \quad \text{and} \quad E^{(1)}_{\Pi_i} := E^{(1)}_{\mathbb{P}^4} \cap \Pi^{(1)}_i \subseteq \Pi^{(1)}_i \quad (i=1,2)$$

be the exceptional divisors of the blowing-up of $\mathbb{P}^4$, S, and Π_i in p, respectively.

By (6.12) and since $(S^{(1)})^\circ$ is isomorphic to its strict transform $\widetilde{(S^{(1)})^\circ}$ in $\widetilde{S}$, Lemma A.0.2 gives

$$\widetilde{C_i}.\widetilde{E^{(1)}_S} = (\widetilde{C_i}_{|\widetilde{(S^{(1)})^\circ}}).(\widetilde{E^{(1)}_S}_{|\widetilde{(S^{(1)})^\circ}}). \tag{6.18}$$

Again, by (6.12),

$$\widetilde{C_i} \cong C^{(1)}_i.$$

Hence,

$$(\widetilde{C_i}_{|\widetilde{(S^{(1)})^\circ}}).(\widetilde{E^{(1)}_S}_{|\widetilde{(S^{(1)})^\circ}}) = C^{(1)}_i.(E^{(1)}_S{}_{|(S^{(1)})^\circ}). \tag{6.19}$$

We have $E^{(1)}_S{}_{|(S^{(1)})^\circ} = E^{(1)}_{\mathbb{P}^4} \cap (S^{(1)})^\circ$. Moreover, $C^{(1)}_i$ is contained in both $(S^{(1)})^\circ$ and $\Pi^{(1)}_i$. Hence, by Lemma A.0.1

$$C^{(1)}_i.(E^{(1)}_S{}_{|(S^{(1)})^\circ}) = C^{(1)}_i.E^{(1)}_{\Pi_i}. \tag{6.20}$$

By Lemma A.0.3, we have

$$C^{(1)}_i.E^{(1)}_{\Pi_i} = 1. \tag{6.21}$$

Putting together equations (6.18)–(6.21), we obtain

$$\widetilde{C_i}.\widetilde{E^{(1)}_S} = 1. \tag{6.22}$$

If p is an $\mathbf{A}_{3-2} = \mathbf{A}_1$ singularity, $\widetilde{E^{(1)}_S}$ is irreducible. We write $E_1 := \widetilde{E^{(1)}_S}$ and therefore

$$\widetilde{C_1}.E_1 = \widetilde{C_2}.E_1 = 1. \tag{6.23}$$

If p is an $\mathbf{A}_{n-2}$ ($n \geq 4$) singularity, we have $E_S^{(1)} = E_1^{(1)} \cup E_{n-2}^{(1)}$, where the strict transforms $\widetilde{E_1^{(1)}}$ and $\widetilde{E_{n-2}^{(1)}}$ of $E_1^{(1)}$ and $E_{n-2}^{(1)}$, respectively, in the minimal model $\widetilde{S}$ are two irreducible (-2)-curves. By choice, C_1 and C_2 satisfy condition (4) in Lemma 4.3.4 (after exchanging $\widetilde{E_1^{(1)}}$ by $\widetilde{E_{n-2}^{(1)}}$ if necessary). Therefore,

$$\widetilde{C_1}.\widetilde{E_1^{(1)}} = \widetilde{C_2}.\widetilde{E_{n-2}^{(1)}} = 1 \text{ and } \widetilde{C_1}.\widetilde{E_{n-2}^{(1)}} = \widetilde{C_2}.\widetilde{E_1^{(1)}} = 0.$$

Studying the resolution of an $\mathbf{A}_{n-2}$ singularity, we see that after possibly exchanging E_1 by E_2, we have $E_1 = \widetilde{E_1^{(1)}}$ and $E_{n-2} = \widetilde{E_{n-2}^{(1)}}$ in Figure 6.1.

Hence, we obtain

$$\widetilde{C_1}.E_1 = \widetilde{C_2}.E_{n-2} = 1. \tag{6.24}$$

If p is an $\mathbf{A}_{n-2}$ singularity with $n \geq 5$, the exceptional divisors $E_1^{(1)}$ and $E_{n-2}^{(1)}$ intersect in an $\mathbf{A}_{n-4}$ singularity which is contained in neither $C_1^{(1)}$ nor $C_2^{(1)}$ again by the choice of C_1 and C_2 satisfying condition (4) in Lemma 4.3.4. Hence, the strict transforms $\widetilde{C_1}$ and $\widetilde{C_2}$ in the minimal model $\widetilde{S}$ intersect no further exceptional divisors, i.e.

$$\widetilde{C_1}.E_j = \widetilde{C_2}.E_j = 0 \quad (j = 2, \ldots, n-3). \tag{6.25}$$

We have

$$h_{\mathbf{A}_n} \coloneqq \pi^*(C_1 + C_2) = \widetilde{C_1} + \widetilde{C_2} + r_1 E_1 + \ldots + r_{n-2} E_{n-2} \in \operatorname{Div}(\widetilde{S}),$$

where $r_1, \ldots, r_{n-2}$ are positive integers. By Lemma 4.2.2, the divisor $h_{\mathbf{A}_n}$ has degree 6. The divisor $h \coloneqq \widetilde{C_1} + \widetilde{C_2} + E_1 + \ldots + E_{n-2}$ has degree 6, as well. Let $h' \coloneqq (r_1 - 1)E_1 + \ldots + (r_{n-2} - 1)E_{n-2}$. For all $i = 1, \ldots, n-1$, we have $h.E_i = 0$. Therefore,

$$h.h' = 0.$$

This gives

$$6 = h_{\mathbf{A}_n}^2 = h^2 + 2h.h' + h'^2 = 6 + h'^2. \tag{6.26}$$

Since h' is contained in the negative definite lattice A_{n-2}, equation (6.26) can only hold if the divisor h' is trivial.

In conclusion,

$$h = h_{\mathbf{A}_n} = \widetilde{C_1} + \widetilde{C_2} + E_1 + \ldots + E_{n-2}.$$

By equations (6.11), (6.17), and (6.23) if $n = 3$ and equations (6.11), (6.17), (6.24), (6.25), and the intersection numbers in Figure 6.1 if $n \geq 4$, the lattice with basis $\widetilde{C_1}$, $\widetilde{C_2}$, $E_1, \ldots, E_{n-2}$ has with respect to this basis the intersection matrix (6.10). □

6.3.3 The quadric Q has corank 2 in $\mathbb{P}^4$

6.3.3.1 General setting and notation

We fix some notation which we will need in the following.

Let S be a complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities. By Lemma 4.3.2, this implies that Q and Y have no common singularities. In particular, the results in Sections 4.3 and 4.4 hold for this choice of S.

Assume that the quadric Q has corank 2 in $\mathbb{P}^4$. We recall from Subsections 4.1.1, 4.1.2, and 4.3.2: By Lemma 4.1.1, Q is the cone through a line l over a smooth quadric $Q' \cong \mathbb{P}^1$ in $\mathbb{P}^2$ and l is the singular locus of Q. For $t \in Q' \subseteq Q$, let

$$\Pi_t := \text{plane spanned by } t \text{ and } l \subseteq \mathbb{P}^4.$$

The planes Π_t are contained in the quadric Q and by Lemma 4.1.5, $\{\Pi_t\}_{t\in\mathbb{P}^1}$ is a pencil of planes on Q such that through any non-singular point of Q passes a unique plane in this pencil. For $t \in \mathbb{P}^1$, let

$$C_t := \Pi_t \cap Y \subseteq S$$

be the cubic curve lying on the plane Π_t. We then have a pencil $\{C_t\}_{t\in\mathbb{P}^1}$ on S.

For $t_s \in Q'$ such that conditions (1) and (2) in Lemma 4.3.7 are satisfied, write

$$C := C_{t_s}, \quad \Pi := \Pi_{t_s}. \tag{6.27}$$

Let $\pi\colon \widetilde{S} \to S$ the minimal resolution of all singularities on S and $\widetilde{C}$ the strict transform of C in $\widetilde{S}$ under π.

Lemma 6.3.6. *We have* $\widetilde{C}^2 = 0$.

Proof. As in Lemma 6.3.3, simply replace C_1 by C. □

We recall the notation from Subsection 4.4.2:

Since C and l are contained in the plane Π, we can apply Bezout's Theorem and obtain $C.l = \sum_{p\in C\cap l}(C.l)_p = 3$ and hence, for $p \in C \cap l$, we have $m := (C.l)_p \leq 3$.

We define successive blowing-ups of $\mathbb{P}^4$ over p: Let

$$(\mathbb{P}^4)^{(0)} := \mathbb{P}^4, \quad S^{(0)} := S, \quad C^{(0)} := C, \quad l^{(0)} := l, \quad p^{(0)} := p$$

and for $i = 1, \ldots, m$ let iteratively

$$\pi^{(i)}\colon (\mathbb{P}^4)^{(i)} \to (\mathbb{P}^4)^{(i-1)}$$

be the blowing-up of $(\mathbb{P}^4)^{(i-1)}$ in $p^{(i-1)}$, where for $i \geq 2$, we let $p^{(i-1)}$ be the unique point in $C^{(i-1)} \cap l^{(i-1)} \cap E^{(i-1)}_{\mathbb{P}^4}$ (see Section 4.4.2) and $E^{(i-1)}_{\mathbb{P}^4}$ is the exceptional divisor of $\pi^{(i-1)}$ in $(\mathbb{P}^4)^{(i-1)}$ and $S^{(i-1)}$, $C^{(i-1)}$, and $l^{(i-1)}$ are the strict transforms of $S^{(i-2)}$, $C^{(i-2)}$, and $l^{(i-2)}$ in $(\mathbb{P}^4)^{(i-1)}$, respectively.

Let

$$(\mathbb{P}^4)^{(3)} \xrightarrow{\pi^{(3)}} \ldots \xrightarrow{\pi^{(m+1)}} (\mathbb{P}^4)^{(m)}$$

be the successive blowing-up of $(\mathbb{P}^4)^{(m)}$ over all points in $C \cap l$ different from p and for $i = m{+}1, \ldots, 3$, let $S^{(i)}$ and $C^{(i)}$ be the strict transforms of S and C in $(\mathbb{P}^4)^{(i)}$, respectively.

For $i = 1, 2, 3$, let $E^{(i)}_S := E^{(i)}_{\mathbb{P}^4} \cap S^{(i)}$ and let $\widetilde{C}$ and $\widetilde{E^{(i)}_S}$ be the strict transforms of C and $E^{(i)}_S$, respectively, in the minimal model $\widetilde{S}$ under π.

Lemma 6.3.7. *We have*

$$\widetilde{C}.\widetilde{E_S^{(m)}} = 1, \quad \widetilde{C}.\widetilde{E_S^{(i)}} = 0 \quad (i < m).$$

Proof. Let $(S^{(3)})^\circ$ be the smooth locus of $S^{(3)}$. Let

$$\widetilde{(S^{(3)})^\circ} \quad \text{and} \quad \widetilde{E_S^{(i)}}$$

be the strict transforms of $(S^{(3)})^\circ$ and $E_S^{(i)}$ in the minimal model $\widetilde{S}$ for S.

Since C contains by choice in (6.27) no singularities of S that are not lying on l and since $C^{(3)} \cap l^{(3)} = \emptyset$, by applying successively Lemma 4.4.5, we obtain that $\widetilde{C}$ is contained in $\widetilde{(S^{(3)})^\circ}$. Hence,

$$\widetilde{C}.\widetilde{E_S^{(i)}} = \widetilde{C}.\big(\widetilde{E_S^{(i)}} \cap \widetilde{(S^{(3)})^\circ}\big) \tag{6.28}$$

by Lemma A.0.2 and

$$\widetilde{C} \cong C^{(3)}.$$

For $1 \leq i \leq 3$, let $E_\Pi^{(i)} := E_{\mathbb{P}^4}^{(i)} \cap \Pi^{(i)}$. For $3 \geq j > i$, we denote

$$E_{\mathbb{P}^4}^{(i,j)}, \quad E_S^{(i,j)}, \quad \text{and} \quad E_\Pi^{(i,j)}$$

the strict transforms of $E_{\mathbb{P}^4}^{(i)}$, $E_S^{(i)}$, and $E_\Pi^{(i)}$ in $(\mathbb{P}^4)^{(j)}$, $S^{(j)}$, and $\Pi^{(j)}$, respectively. Then,

$$\widetilde{E_S^{(i)}} \cap \widetilde{(S^{(3)})^\circ} \cong E_S^{(i,3)} \cap (S^{(3)})^\circ.$$

Therefore,

$$\widetilde{C}.\big(\widetilde{E_S^{(i)}} \cap \widetilde{(S^{(3)})^\circ}\big) = C^{(3)}.\big(E_S^{(i,3)} \cap (S^{(3)})^\circ\big). \tag{6.29}$$

We have $E_{\mathbb{P}^4}^{(i,3)} \cap (S^{(3)})^\circ = E_S^{(i,3)} \cap (S^{(3)})^\circ$ and $E_{\mathbb{P}^4}^{(i,3)} \cap \Pi^{(3)} = E_\Pi^{(i,3)}$. Besides, $C^{(3)}$ is contained in both $(S^{(3)})^\circ$ and $\Pi^{(3)}$. Hence, by Lemma A.0.1

$$C^{(3)}.\big(E_S^{(i,3)} \cap (S^{(3)})^\circ\big) = C^{(3)}.E_\Pi^{(i,3)}. \tag{6.30}$$

By Lemma A.0.3, we then have

$$C^{(3)}.E_\Pi^{(m,3)} = 1 \quad \text{and} \quad C^{(3)}.E_\Pi^{(i,3)} = 0 \text{ for } i < m. \tag{6.31}$$

In conclusion, equations (6.28)-(6.31) together give

$$\widetilde{C}.\widetilde{E_S^{(m)}} = 1 \quad \text{and} \quad \widetilde{C}.\widetilde{E_S^{(i)}} = 0 \text{ for } i < m. \qquad \square$$

6.3.3.2 Assumption: $\mathsf{T} = \mathsf{D}_4$ (thus $\sigma(\mathsf{T}) = 3\mathsf{A}_1$)

We prove Proposition 6.2.1 in case $\operatorname{corank}(Q) = 2$, $\mathbf{T} = \mathbf{D}_4$, and thus $\sigma(\mathbf{D}_4) = 3\mathbf{A}_1$, i.e.:

Proposition 6.3.8. *Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 2 in $\mathbb{P}^4$ and the singularities of S lying on the singular line l of Q are of type $3\mathbf{A}_1$.

Let C be the plane cubic curve on S and $\widetilde{C}$ the strict transform of C under π in $\widetilde{S}$ as in (6.27).

Then, for the hyperplane section $2C$ of S, we have $h_{\mathbf{D}_4} := \pi^*(2C) = 2\widetilde{C} + E_1 + E_2 + E_3 \in \mathrm{Div}(\widetilde{S})$, where E_1, E_2, E_3 are (-2)-curves on $\widetilde{S}$. The lattice in $\mathrm{Div}(\widetilde{S})$ with basis $\widetilde{C}$, E_1, E_2, E_3 has the intersection matrix:

$$\begin{array}{c c} & \begin{array}{cccc} \widetilde{C} & E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} \widetilde{C} \\ E_1 \\ E_2 \\ E_3 \end{array} & \left(\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & \boxed{-2} & 0 & 0 \\ 1 & 0 & \boxed{-2} & 0 \\ 1 & 0 & 0 & \boxed{-2} \end{array}\right) \\ & \begin{array}{cccc} & \underbrace{}_{A_1} & \underbrace{}_{A_1} & \underbrace{}_{A_1} \end{array} \end{array} \tag{6.32}$$

Proof. Let C be the cubic curve as in Definition 6.27. We proved in Lemma 4.3.6 that the divisor $2C$ is a hyperplane section of S.

By Lemma 6.3.6, we have

$$\widetilde{C}^2 = 0. \tag{6.33}$$

The cubic curve C and the singular line l of Q both lie on the plane Π. By Bezout's Theorem, we have: $C.l = \sum_{p \in C \cap l} (C.l)_p = 3$. Since the singularities of S lying on l are three $\mathbf{A}_1$ singularities p_1, p_2, and p_3, we deduce $(C.l)_{p_i} = 1$ $(i = 1, 2, 3)$.

Since C contains no singularity of S different from p_1, p_2, and p_3, the pull-back $\pi^*(2C)$ on $\widetilde{S}$ is supported on the union of $\widetilde{C}$ with the strict transforms E_1, E_2, and E_3 in $\widetilde{S}$ of the exceptional curves of the minimal resolution of p_1, p_2, and p_3.

For $i = 1, 2, 3$, let $\pi^{(i)}\colon S^{(i)} \to S^{(i-1)}$ be the successive blowing-up of $S^{(i-1)}$ in p_i with $S^{(0)} := S$ and exceptional divisors $E_S^{(i)} \subseteq S^{(i)}$. Then, $E_i = \widetilde{E_S^{(i)}}$ is the strict transform of $E_S^{(i)}$ in $\widetilde{S}$ under the minimal resolution π of all singularities on S. Since the singularities p_1, p_2, and p_3 are of type $\mathbf{A}_1$, the E_i are irreducible curves with

$$E_i^2 = -2. \tag{6.34}$$

By Lemma 6.3.7, we have

$$\widetilde{C}.E_i = 1. \tag{6.35}$$

Further, since the singularities p_1, p_2, and p_3 are isolated from each other,

$$E_i.E_j = 0 \quad \text{for } i \neq j. \tag{6.36}$$

We have

$$h_{\mathbf{D}_4} := \pi^*(2C) = 2\widetilde{C} + r_1E_1 + r_2E_2 + r_2E_3 \in \mathrm{Div}(\widetilde{S}),$$

where r_1, r_2, r_3 are non-negative integers. The divisor $h_{\mathbf{D}_4}$ has degree 6 by Lemma 4.2.2. On the other hand, the divisor $h := 2\widetilde{C} + E_1 + E_2 + E_3$ has degree 6, as well. Let $h' := (r_1 - 1)E_1 + (r_2 - 1)E_2 + (r_3 - 1)E_3$. For all $i = 1, 2, 3$, we have $h.E_i = 0$. Hence, $h.h' = 0$. This gives

$$6 = h_{\mathbf{D}_4}^2 = h^2 + 2h.h' + h'^2 = 6 + h'^2. \tag{6.37}$$

Since h' is contained in the negative definite lattice $A_1 \oplus A_1 \oplus A_1$, equation (6.37) can only hold if h' is trivial. Consequently,

$$h = h_{\mathbf{D}_4} = 2\widetilde{C} + E_1 + E_2 + E_3$$

and by equations (6.33), (6.34), (6.35), and (6.36), the lattice with basis $\widetilde{C}$, E_1, E_2, and E_3 has with respect to this basis the intersection matrix (6.32). $\square$

6.3.3.3 Assumption: $\mathsf{T} = \mathsf{D}_n$ (thus $\sigma(\mathsf{T}) = \mathsf{A}_1 + \mathsf{D}_{n-2}$ $(n \geq 5)$)

We prove Proposition 6.2.1 in case $\mathrm{corank}(Q) = 2$, $\mathbf{T} = \mathbf{D}_n$ $(n \geq 5)$, and thus $\sigma(\mathbf{D}_n) = \mathbf{A}_1 + \mathbf{D}_{n-2}$ (where $\mathbf{D}_3 := \mathbf{A}_3$):

Proposition 6.3.9. *Let S be the complete $(2, 3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 2 in $\mathbb{P}^4$ and the singularities of S lying on the singular line l of Q are of type $\mathbf{A}_1 + \mathbf{D}_{n-2}$.

Let C be the plane cubic curve on S and $\widetilde{C}$ the strict transform of C under π in $\widetilde{S}$ as in (6.27).

Then, for the hyperplane section $2C$ of S, we have

$$h_{\mathbf{D}_n} := \pi^*(2C) = 2\widetilde{C} + E_1 + 2E_2 + \ldots + 2E_{n-3} + E_{n-2} + E_{n-1}$$

on $\widetilde{S}$, where $E_1, \ldots, E_{n-1}$ are (-2)-curves on $\widetilde{S}$. Consequently, the lattice in $\mathrm{Div}(\widetilde{S})$ with basis $\widetilde{C}$, E_1, $\ldots, E_{n-1}$ has the intersection matrix:

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \widetilde{C} & E_1 & E_2 & E_3 & \cdots\cdots & E_{n-3} & E_{n-2} & E_{n-1} \end{array} \\
\begin{array}{c} \widetilde{C} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{n-3} \\ E_{n-2} \\ E_{n-1} \end{array} &
\left(\begin{array}{cccccccc}
0 & 1 & 1 & 0 & \cdots\cdots & 0 & 0 & 0 \\
1 & \boxed{-2} & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & & & & \\
0 & 0 & 1 & \ddots & \ddots & & & \\
\vdots & \vdots & & \ddots & \ddots & \ddots & & \\
0 & 0 & & & \ddots & -2 & 1 & 1 \\
0 & 0 & & & & 1 & -2 & 0 \\
0 & 0 & & & & 1 & 0 & -2
\end{array}\right) \\
 & \begin{array}{cc} \quad A_1 \quad & \qquad\qquad D_{n-2} \qquad\qquad \end{array}
\end{array} \tag{6.38}$$

Proof. We proved in Lemma 4.3.6 that $2C$ is a hyperplane section of S.

By Lemma 6.3.6, we have

$$\widetilde{C}^2 = 0. \tag{6.39}$$

By assumption, the only singularities of S lying on the singular line l of the quadric Q are an $\mathbf{A}_1$ singularity p_1 and a $\mathbf{D}_{n-2}$ singularity p_2. Moreover, by choice of C in (6.27), p_1 and p_2 are the only singularities of S contained in C. Hence, the pull-back $\pi^*(2C)$ to $\widetilde{S}$ is supported on the union of $\widetilde{C}$ with the exceptional divisors $\pi^{-1}(p_1)$ and $\pi^{-1}(p_2)$ of the minimal resolution of p_1 and p_2, respectively. The exceptional divisors $\pi^{-1}(p_1) \in \mathrm{Div}(\widetilde{S})$ of the $\mathbf{A}_1$ singularity p_1 is supported on an irreducible curve E_1 such that

$$E_1^2 = -2. \tag{6.40}$$

The exceptional divisor $\pi^{-1}(p_2) \in \mathrm{Div}(\widetilde{S})$ of the $\mathbf{D}_{n-2}$ singularity p_2 is supported on the union of the irreducible curves $E_2, \ldots, E_{n-1}$ in $\widetilde{S}$ whose corresponding weighted graph is a Dynkin diagram of type $\mathcal{D}_{n-2}$ and we chose the notation such that this is the graph in Figure 6.2.

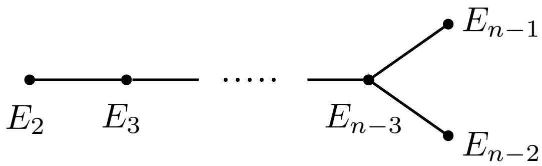


Figure 6.2: Dynkin diagram corresponding to the $\mathbf{D}_{n-2}$ singularity p_2 on C.

Further, since p_1 and p_2 are isolated

$$E_1.E_j = 0 \quad \text{for all } j = 2, \ldots, n-1. \tag{6.41}$$

The cubic curve C and the singular line l of Q both lie on the plane Π. By Bezout's Theorem, we have $C.l = \sum_{p \in C \cap l} (C.l)_p = 3$. Since an $\mathbf{A}_1$ singularity is resolved after one blowing-up, Lemma 4.4.5 implies that $(C.l)_{p_1} = 1$ and hence $(C.l)_{p_2} = 2$.

Let $\pi^{(1)} \colon (\mathbb{P}^4)^{(1)} \to \mathbb{P}^4$ be the blowing-up of $\mathbb{P}^4$ in the $\mathbf{A}_1$ singularity p_1 with exceptional divisor $E_{\mathbb{P}^4}^{(1)}$ and $S^{(1)}$ the strict transform of S in $(\mathbb{P}^4)^{(1)}$ under $\pi^{(1)}$. Let $E_S^{(1)} := E_{\mathbb{P}^4}^{(1)} \cap S^{(1)}$. Let $\widetilde{E_S^{(1)}}$ be the strict transform of $E_S^{(1)}$ in $\widetilde{S}$. Then, $E_1 = \widetilde{E_S^{(1)}}$ and by Lemma 6.3.7

$$\widetilde{C}.E_1 = 1. \tag{6.42}$$

Let $\pi^{(2)} \colon (\mathbb{P}^4)^{(2)} \to (\mathbb{P}^4)^{(1)}$ be the blowing-up of $(\mathbb{P}^4)^{(1)}$ in the $\mathbf{D}_{n-2}$ singularity p_2 with exceptional divisor $E_{\mathbb{P}^4}^{(2)}$ and $C^{(2)}$ and $l^{(2)}$ the strict transforms of $C^{(1)}$ and $l^{(1)}$ in $(\mathbb{P}^4)^{(2)}$ under $\pi^{(2)}$, respectively. Let $p_2^{(2)} \in C^{(2)} \cap l^{(2)} \cap E_{\mathbb{P}^4}^{(2)}$ and let $\pi^{(3)} \colon (\mathbb{P}^4)^{(3)} \to (\mathbb{P}^4)^{(2)}$ be the blowing-up of $(\mathbb{P}^4)^{(2)}$ in $p_2^{(2)}$ with exceptional divisor $E_{\mathbb{P}^4}^{(3)}$. Let $S^{(3)}$, $C^{(3)}$, and $l^{(3)}$ be the strict transforms of $S^{(2)}$, $C^{(2)}$, and $l^{(2)}$ in $(\mathbb{P}^4)^{(3)}$ under $\pi^{(3)}$, respectively, and let $E_S^{(3)} := E_{\mathbb{P}^4}^{(3)} \cap S^{(3)}$. For $i = 2, 3$, let $\widetilde{E_S^{(i)}}$ be the strict transform of $E_S^{(i)}$ in $\widetilde{S}$. We have by Lemma 6.3.7

$$\widetilde{C}.\widetilde{E_S^{(3)}} = 1 \quad \text{and} \quad \widetilde{C}.\widetilde{E_S^{(2)}} = 0 \tag{6.43}$$

and $C^{(3)}$ is contained in the smooth locus of $S^{(3)}$. Consequently, $\widetilde{C}$ intersects only the divisor $\widetilde{E_S^{(3)}} \in \mathrm{Div}(\widetilde{S})$ in $\pi^{-1}(p_2)$. Hence, we need to determine to which of the curves E_i in Figure 6.2 the divisor $\widetilde{E_S^{(3)}}$ corresponds.

If $n = 5$, the singularity p_2 has type $\mathbf{D}_{n-2} = \mathbf{A}_3$. Therefore, the exceptional divisor $E_S^{(2)}$ of the blowing-up of p_2 is the union of two irreducible curves $E_{S,1}^{(2)}$ and $E_{S,2}^{(2)}$ intersecting in a singularity of type $\mathbf{A}_1$. This must be the singularity $p_2^{(2)}$ on $S^{(2)}$ contained in $C^{(2)}$ and $l^{(2)}$. The exceptional divisor $E_S^{(3)}$ of the blowing-up of $p_2^{(2)}$ is irreducible and separates the strict transforms $E_{S,1}^{(2,3)}$ and $E_{S,2}^{(2,3)}$ in $S^{(3)}$ of $E_{S,1}^{(2)}$ and $E_{S,2}^{(2)}$, respectively. The strict transforms $\widetilde{E_S^{(3)}}$, $\widetilde{E_{S,1}^{(2,3)}}$, and $\widetilde{E_{S,2}^{(2,3)}}$ in $\widetilde{S}$ of $E_S^{(3)}$, $E_{S,1}^{(2,3)}$, and $E_{S,2}^{(2,3)}$, respectively, then are the vertices of a Dynkin diagram of type $\mathcal{A}_3$, see Figure 6.3 for an illustration of the blowing-up process.

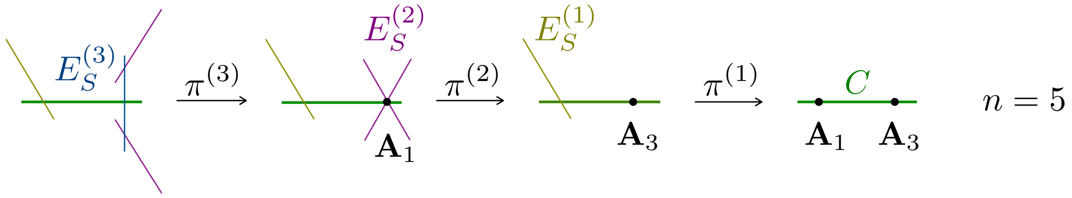


Figure 6.3: Blowing-up over the $\mathbf{A}_1$ and $\mathbf{A}_3$ singularity on C.

In particular, we see that $\widetilde{E_S^{(3)}} = E_2$, $\widetilde{E_{S,1}^{(2,3)}} = E_3$, and $\widetilde{E_{S,2}^{(2,3)}} = E_4$ in Figure 6.2 after exchanging possibly E_3 by E_4. Further, $\widetilde{E_S^{(1)}} = E_1$.

If $n = 6$, the singularity p_2 has type $\mathbf{D}_{n-2} = \mathbf{D}_4$. Therefore, the exceptional divisor $E_S^{(2)}$ of the blowing-up of p_2 is one irreducible curve on which lie three $\mathbf{A}_1$ singularities of $S^{(2)}$. One of these must be the singularity $p_2^{(2)}$ contained in $C^{(2)}$ and $l^{(2)}$. The exceptional divisor $E_S^{(3)}$ of the blowing-up of $p_2^{(2)}$ is irreducible and intersects the strict transform $E_S^{(2,3)}$ of $E_S^{(2)}$ in $S^{(3)}$ on which the two $\mathbf{A}_1$ singularities which have not been blown-up are lying, see Figure 6.4 for an illustration of the blowing-up process.

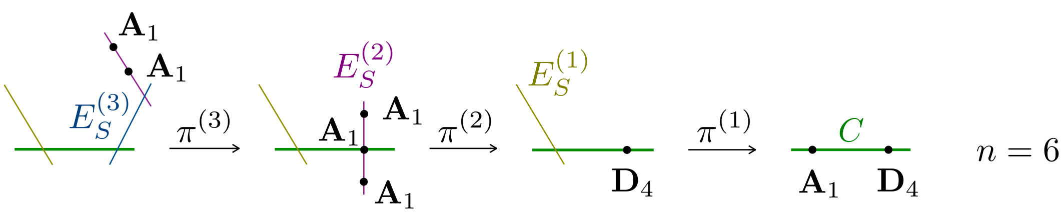


Figure 6.4: Blowing-up over the $\mathbf{A}_1$ and $\mathbf{D}_4$ singularity on C.

In particular, we have $\widetilde{E_S^{(3)}} = E_2$ and $\widetilde{E_S^{(2)}} = E_3$ in Figure 6.2 after exchanging possibly E_2 by E_4 or E_5. Further, $\widetilde{E_S^{(1)}} = E_1$.

Assume finally that $n \geq 7$ and the singularity p_2 has type $\mathbf{D}_{n-2}$. Therefore, the exceptional divisor $E_S^{(2)}$ of the blowing-up of p_2 is one irreducible curve on which lie an $\mathbf{A}_1$ singularity and a $\mathbf{D}_{n-2}$ singularity of $S^{(2)}$. One of these must be the singularity $p_2^{(2)}$ contained in $C^{(2)}$ and $l^{(2)}$. By Lemma 4.4.5, $(C^{(2)}.l^{(2)})_{p_2^{(2)}} = 1$ and hence $p_2^{(2)}$ is of type $\mathbf{A}_1$.

The exceptional divisor $E_S^{(3)}$ of the blowing-up of $p_2^{(2)}$ is irreducible and intersects the strict transform $E_S^{(2,3)}$ of $E_S^{(2)}$ in $S^{(3)}$ on which the $\mathbf{D}_{n-4}$ singularity which has not been blown-up yet is lying, see Figure 6.5 for an illustration of the blowing-up process.

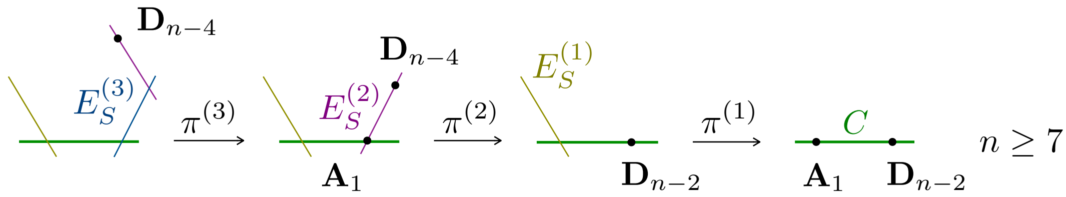


Figure 6.5: Blowing-up over the $\mathbf{A}_1$ and $\mathbf{D}_{n-2}$ ($n \geq 7$) singularity on C.

In particular, $\widetilde{E_S^{(2)}} = E_3$ and $\widetilde{E_S^{(3)}} = E_2$ as in Figure 6.2, and $\widetilde{E_S^{(1)}} = E_1$.

In conclusion, for all $n \geq 3$, we have

$$\widetilde{C}.E_1 = \widetilde{C}.E_2 = 1, \text{ and } \widetilde{C}.E_i = 0 \quad \text{for } i = 3, \ldots, n-1. \tag{6.44}$$

Then,

$$h_{\mathbf{D}_{n\geq 5}} := \pi^*(2C) = 2\widetilde{C} + r_1E_1 + r_2E_2 + \ldots + r_{n-4}E_{n-4} + r_{n-3}E_{n-3} + r_{n-2}E_{n-2} + r_{n-1}E_{n-1},$$

where $r_1, \ldots, r_{n-1}$ are positive integers. By Lemma 4.2.2, $h_{\mathbf{D}_{n\geq 5}}$ has degree 6. For $h = 2\widetilde{C} + E_1 + 2E_2 + \ldots + 2E_{n-4} + 2E_{n-3} + E_{n-2} + E_{n-1} \in \mathrm{Div}(\widetilde{S})$, we have $h^2 = 6$. As in (6.26), we show that $h' = (r_1 - 1)E_1 + (r_2 - 1)E_2 + \ldots + (r_{n-4} - 1)E_{n-4} + (r_{n-3} - 1)E_{n-3} + (r_{n-2} - 1)E_{n-2} + (r_{n-1} - 1)E_{n-1} \in \mathrm{Div}(\widetilde{S})$ must be trivial since it is contained in the negative definite lattice $A_1 \oplus D_{n-2}$. Hence,

$$h_{\mathbf{D}_{n\geq 5}} = h = 2\widetilde{C} + E_1 + 2E_2 + \ldots + 2E_{n-4} + 2E_{n-3} + E_{n-2} + E_{n-1}$$

and by equations (6.39), (6.41), (6.44), and the intersection numbers in Figure 6.2, the lattice with basis $\widetilde{C}$, E_1, E_2, $\ldots$, E_{n-1} has with respect to this basis the intersection matrix (6.38). □

6.3.3.4 Assumption: $\mathsf{T} = \mathsf{E}_6, \mathsf{E}_7$, or E_8 (thus $\sigma(\mathsf{T}) = \mathsf{A}_5, \mathsf{D}_6$, or E_7, respectively)

Then, S contains exactly one singularity p of type $\mathbf{A}_5, \mathbf{D}_6$, or $\mathbf{E}_7$ on the singular locus l of Q. Both C and l lie in the plane Π. By Bezout's Theorem, C and l intersect in p with multiplicity three, i.e. $(C.l) = (C.l)_p = 3$.

6.3.3.5 Assumption: $\mathsf{T} = \mathsf{E}_6$ (thus $\sigma(\mathsf{T}) = \mathsf{A}_5$)

We prove Proposition 6.2.1 in case $\mathrm{corank}(Q) = 2$, $\mathbf{T} = \mathbf{E}_6$, and thus $\sigma(\mathbf{E}_6) = \mathbf{A}_5$:

Proposition 6.3.10. *Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 2 in $\mathbb{P}^4$ and the singularities of S lying on the singular line l of Q are of type $\mathbf{A}_5$.

Let C be the plane cubic curve on S and $\widetilde{C}$ the strict transform of C under π in $\widetilde{S}$ as in (6.27).

Then, for the hyperplane section $2C$ of S, we have $h_{\mathbf{E}_6} := \pi^*(2C) = 2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 \in \mathrm{Div}(\widetilde{S})$ on $\widetilde{S}$, where $E_1, \ldots, E_5$ are (-2)-curves on $\widetilde{S}$. The lattice in $\mathrm{Div}(\widetilde{S})$ *with basis $\widetilde{C}$, $E_1, \ldots,$ E_5 has the intersection matrix:*

$$\begin{array}{c} \\ \widetilde{C} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccc} \widetilde{C} & E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\
\left(\begin{array}{c|ccccc}
0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 \\
1 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 1 & -2
\end{array}\right) \\
\hspace{2em}\underbrace{\hspace{10em}}_{A_5}
\end{array} \tag{6.45}$$

Proof. We proved in Lemma 4.3.6 that $2C$ is a hyperplane section of S.

By Lemma 6.3.6, we have
$$\widetilde{C}^2 = 0. \tag{6.46}$$

By assumption, the only singularity of S lying on the singular line l of Q is an $\mathbf{A}_5$ singularity p. Since C contains by choice no singularity of S different from p , the pull-back $\pi^*(2C)$ is supported on the union of $\widetilde{C}$, and the exceptional divisor $\pi^{-1}(p)$ of the minimal resolution of p, i.e. the union of the smooth irreducible curves $E_1, \ldots, E_5$ intersecting in a Dynkin diagram of type $\mathcal{A}_5$ and we chose the notation such that this is the graph in Figure 6.6.

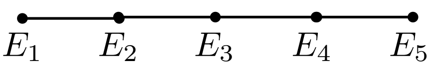


Figure 6.6: Dynkin diagram corresponding to the $\mathbf{A}_5$ singularity p.

We use Notation 4.4.4 for $m = 3$.

By Lemma 6.3.7, we have

$$\widetilde{C}.\widetilde{E_S^{(3)}} = 1 \quad \text{and} \quad \widetilde{C}.\widetilde{E_S^{(1)}} = \widetilde{C}.\widetilde{E_S^{(2)}} = 0.$$

We now determine to which of the curves E_i in Figure 6.6 the divisor $\widetilde{E_S^{(3)}}$ corresponds. By Table 1.1 and our knowledge of the exceptional divisors of ADE singularities in Theorem 1.2.1:

1. The exceptional divisor $E_S^{(1)}$ of the blowing-up of S in the $\mathbf{A}_5$ singularity p contains only the $\mathbf{A}_3$ singularity $p^{(1)}$ and $E_S^{(1)}$ is the union of two irreducible curves $E_S^{(1)} = E_{1,S}^{(1)} \cup E_{5,S}^{(1)}$ intersecting in $p^{(1)}$.
2. The exceptional divisor $E_S^{(2)}$ of the blowing-up of $S^{(1)}$ in the $\mathbf{A}_3$ singularity $p^{(1)}$ contains an $\mathbf{A}_1$ singularity $p^{(2)}$ and the divisor $E_S^{(2)}$ is the union of two irreducible curves $E_S^{(2)} = E_{2,S}^{(2)} \cup E_{4,S}^{(2)}$ intersecting in $p^{(2)}$.
3. The exceptional divisor $E_S^{(3)}$ of the blowing-up of $S^{(2)}$ in the $\mathbf{A}_1$ singularity $p^{(2)}$ is contained in the smooth locus of $S^{(3)}$ and the divisor $E_S^{(3)}$ is irreducible.

See Figure 6.7 for an illustration of the blowing-up process.

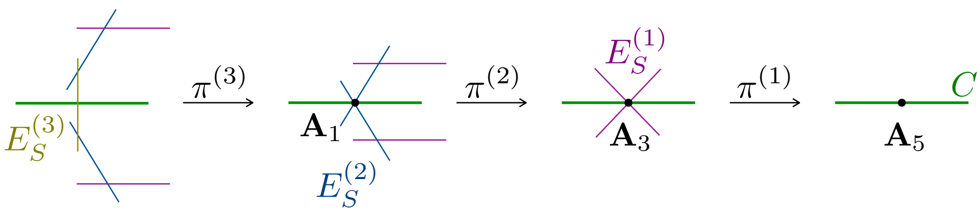


Figure 6.7: Blowing-up over the $\mathbf{A}_5$ singularity p on C.

Hence, we see that

$$\widetilde{E_{1,S}^{(1)}} = E_1, \quad \widetilde{E_{5,S}^{(1)}} = E_5, \quad \widetilde{E_{2,S}^{(2)}} = E_2, \quad \widetilde{E_{4,S}^{(2)}} = E_4, \quad \widetilde{E_S^{(3)}} = E_3$$

in Figure 6.6 up to exchanging E_1 by E_5 and E_2 by E_4 if necessary, i.e.

$$\widetilde{C}.E_3 = 1 \quad \text{and} \quad \widetilde{C}.E_i = 0 \quad (i = 1, 2, 4, 5). \tag{6.47}$$

Then,

$$h_{\mathbf{E}_6} := \pi^*(2C) = 2\widetilde{C} + r_1E_1 + r_2E_2 + r_3E_3 + r_4E_4 + r_5E_5,$$

where $r_1, \ldots, r_5$ are positive integers and $h_{\mathbf{E}_6}^2 = 6$ by Lemma 4.2.2. For $h := 2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5$, we have $h^2 = 6$. As in (6.26), we show that $h' = (r_1 - 1)E_1 + \ldots + (r_5 - 1)E_5 \in \mathrm{Div}(\widetilde{S})$ must be trivial since it is contained in the negative definite lattice A_5. Hence,

$$h_{\mathbf{E}_6} = h = 2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5.$$

By equations (6.46), (6.47), and the intersection numbers in Figure 6.6, the lattice with basis $\widetilde{C}$, E_1, $\ldots$, E_5 has with respect to this basis the intersection matrix (6.45). □

6.3.3.6 Assumption: $\mathsf{T} = \mathsf{E}_7$ (thus $\sigma(\mathsf{T}) = \mathsf{D}_6$)

We prove Proposition 6.2.1 in case $\mathrm{corank}(Q) = 2$, $\mathbf{T} = \mathbf{E}_7$, and thus $\sigma(\mathbf{E}_7) = \mathbf{D}_6$:

Proposition 6.3.11. *Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 2 in $\mathbb{P}^4$ and the singularities of S lying on the singular line l of Q are of type $\mathbf{D}_6$.

Let C be the plane cubic curve on S and $\widetilde{C}$ the strict transform of C under π in $\widetilde{S}$ as (6.27).

Then, for the hyperplane section $2C$ of S, we have

$$h_{\mathbf{E}_7} := \pi^*(2C) = 2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 4E_4 + 2E_5 + 3E_6$$

on $\widetilde{S}$, where $E_1, \dots, E_6$ are (-2)-curves on $\widetilde{S}$. The lattice in $\mathrm{Div}(\widetilde{S})$ with basis $\widetilde{C}$, $E_1, \dots, E_6$ has the intersection matrix:

$$\begin{array}{c} \\ \widetilde{C} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \widetilde{C} & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \left(\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right) \end{array} \tag{6.48}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{D_6}$$

Proof. We proved in Lemma 4.3.6 that $2C$ is a hyperplane section of S.

By Lemma 6.3.6, we have

$$\widetilde{C}^2 = 0. \tag{6.49}$$

By assumption, the only singularity of S lying on the singular line l of Q is a $\mathbf{D}_6$ singularity p. Since C contains by choice no singularity of S different from p , the pull-back $\pi^*(2C)$ is supported on the union of $\widetilde{C}$, and the exceptional divisor $\pi^{-1}(p)$ of the minimal resolution of p, i.e. the union of the smooth irreducible curves $E_1, \dots, E_6$ intersecting in the Dynkin diagram of type $\mathcal{D}_6$ and we chose the notation such that this is the graph in Figure 6.8.

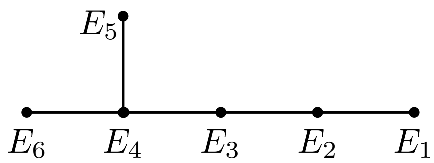


Figure 6.8: Dynkin diagram corresponding to the $\mathbf{D}_6$ singularity p on C.

We use Notation 4.4.4 for $m = 3$.

By Lemma 6.3.7, we have

$$\widetilde{C}.\widetilde{E_S^{(3)}} = 1 \quad \text{and} \quad \widetilde{C}.\widetilde{E_S^{(1)}} = \widetilde{C}.\widetilde{E_S^{(2)}} = 0.$$

We now determine to which curve E_i in Figure 6.8 the divisor $\widetilde{E_S^{(3)}}$ corresponds. By Table 1.1 and our knowledge of the exceptional divisors of ADE singularities in Theorem 1.2.1:

1. The exceptional divisor $E_S^{(1)}$ of the blowing-up $S^{(1)}$ of S in p is irreducible and contains a $\mathbf{D}_4$ singularity and an $\mathbf{A}_1$ singularity. We claim that $p^{(1)}$ must be of type $\mathbf{D}_4$. Indeed, if $p^{(1)} \in C^{(1)} \cap l^{(1)} \cap E_{\mathbb{P}^4}^{(1)}$ was of type $\mathbf{A}_1$, the strict transform $C^{(2)}$ of C in $S^{(2)}$ would be contained in the smooth locus of $S^{(2)}$ but by Lemma 4.4.5 this is not the case since $C^{(2)}.l^{(2)} = 1$.
2. The exceptional divisor $E_S^{(2)}$ of the blowing-up $S^{(2)}$ of $S^{(1)}$ in $p^{(1)}$ is irreducible and contains three $\mathbf{A}_1$ singularities. One of these $\mathbf{A}_1$ singularities, say $p^{(2)}$, is contained in the strict transform $C^{(2)}$ of $C^{(1)}$ in $S^{(2)}$ since $C^{(2)}$ is not contained in the smooth locus of $S^{(2)}$, again by Lemma 4.4.5.
3. The exceptional divisor $E_S^{(3)}$ of the blowing-up $S^{(3)}$ of $S^{(2)}$ in the $\mathbf{A}_1$ singularity $p^{(2)}$ is irreducible and smooth.

See Figure 6.9 for an illustration of the blowing-up process.

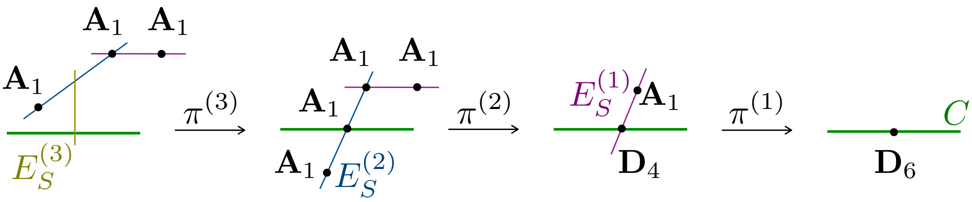


Figure 6.9: Blowing-up over the $\mathbf{D}_6$ singularity p on C.

Hence, $\widetilde{E_S^{(3)}} = E_6$, $\widetilde{E_S^{(2)}} = E_4$, and $\widetilde{E_S^{(1)}} = E_2$ in Figure 6.8 after exchanging possibly E_6 by E_5 so

$$\widetilde{C}.E_6 = 1 \quad \text{and} \quad \widetilde{C}.E_i = 0 \quad (i = 1, \ldots, 5). \tag{6.50}$$

Then,

$$h_{\mathbf{E}_7} := \pi^*(2C) = 2\widetilde{C} + r_1E_1 + r_2E_2 + r_3E_3 + r_4E_4 + r_5E_5 + r_6E_6,$$

where $r_1, \ldots, r_6$ are positive integers and $h_{\mathbf{E}_7}^2 = 6$ by Lemma 4.2.2. For $h := 2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 4E_4 + 2E_5 + 3E_6$, we have $h^2 = 6$. As in (6.26), we show that $h' = (r_1 - 1)E_1 + \ldots + (r_6 - 1)E_6 \in \mathrm{Div}(\widetilde{S})$ must be trivial since it is contained in the negative definite lattice D_6. Hence,

$$h_{\mathbf{E}_7} = h = 2\widetilde{C} + E_1 + 2E_2 + 3E_3 + 4E_4 + 2E_5 + 3E_6.$$

By equations (6.49), (6.50), and the intersection numbers in Figure 6.8, the lattice with basis $\widetilde{C}$, E_1, …, E_6 has with respect to this basis the intersection matrix (6.48). □

6.3.3.7 Assumption: $\mathsf{T} = \mathsf{E}_8$ (thus $\sigma(\mathsf{E}_8) = \mathsf{E}_7$)

We prove Proposition 6.2.1 in case $\mathrm{corank}(Q) = 2$, $\mathbf{T} = \mathbf{E}_8$, and thus $\sigma(\mathbf{E}_8) = \mathbf{E}_7$:

Proposition 6.3.12. *Let S be the complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that S has only isolated ADE singularities and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of all singularities on S.*

Assume that Q has corank 2 in $\mathbb{P}^4$ and the singularities of S lying on the singular line l of Q are of type $\mathbf{E}_7$.

Let C be the plane cubic curve on S and $\widetilde{C}$ the strict transform of C under π in $\widetilde{S}$ as in (6.27).

Then, for the hyperplane section $2C$ of S, we have

$$h_{\mathbf{E}_8} := \pi^*(2C) = 2\widetilde{C} + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7$$

on $\widetilde{S}$, where $E_1, \dots, E_7$ are (-2)-curves on $\widetilde{S}$. The lattice in $\mathrm{Div}(\widetilde{S})$ with basis $\widetilde{C}$, $E_1, \dots, E_7$ has, with respect to this basis, the intersection matrix:

$$\begin{array}{c} \\ \widetilde{C} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} \widetilde{C} & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 \end{array} \\ \left(\begin{array}{c|ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{array}\right) \\ \qquad\qquad E_7 \end{array} \tag{6.51}$$

Proof. We proved in Lemma 4.3.6 that $2C$ is a hyperplane section of S.

By Lemma 6.3.6, we have

$$\widetilde{C}^2 = 0. \tag{6.52}$$

By assumption, the only singularity of S lying on the singular line l of Q is an $\mathbf{E}_7$ singularity p. Since C contains by choice no singularity of S different from p , the pull-back $\pi^*(2C)$ is supported on the union of $\widetilde{C}$, and the exceptional divisor $\pi^{-1}(p)$ of the minimal resolution of p, i.e. the union of the smooth irreducible curves $E_1, \dots, E_7$ intersecting in a Dynkin diagram of type $\mathcal{E}_7$ and we chose the notation such that this is the graph in Figure 6.10.

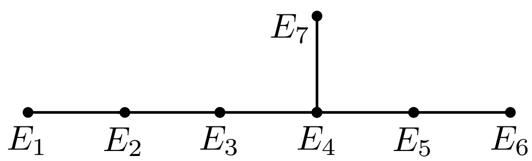


Figure 6.10: Dynkin diagram corresponding to the $\mathbf{E}_7$ singularity p on C.

We use Notation 4.4.4 for $m = 3$.

By Lemma 6.3.7, we have

$$\widetilde{C}.\widetilde{E_S^{(3)}} = 1, \text{ and } \quad \widetilde{C}.\widetilde{E_S^{(1)}} = \widetilde{C}.\widetilde{E_S^{(2)}} = 0.$$

We now determine to which of the curves E_i in Figure 6.10 the divisor $\widetilde{E_S^{(3)}}$ corresponds. By Table 1.1 and our knowledge of the exceptional divisors of ADE singularities in Theorem 1.2.1:

1. The exceptional divisor $E_S^{(1)}$ of the blowing-up $S^{(1)}$ of S in p is irreducible and contains a $\mathbf{D}_6$ singularity $p^{(1)}$.
2. The exceptional divisor $E_S^{(2)}$ of the blowing-up $S^{(2)}$ of $S^{(1)}$ in $p^{(1)}$ is irreducible and contains an $\mathbf{A}_1$ singularity and a $\mathbf{D}_4$ singularity. Since $C^{(2)}.l^{(2)} = 1$, the singularity $p^{(2)}$ has type $\mathbf{A}_1$ by Lemma 4.4.5.
3. The exceptional divisor $E_S^{(3)}$ of the blowing-up $S^{(3)}$ of $S^{(2)}$ in the $\mathbf{A}_1$ singularity $p^{(2)}$ is irreducible and smooth.

See Figure 6.11 for an illustration of the blowing-up process.

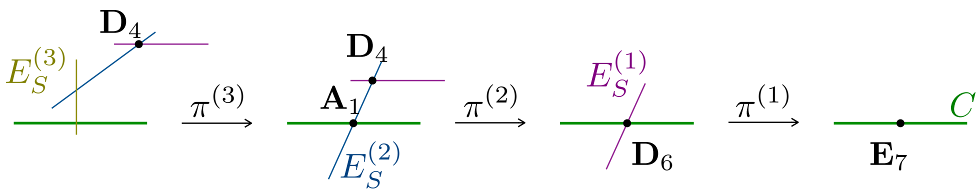


Figure 6.11: Blowing-up over the $\mathbf{E}_7$ singularity p on C.

Hence, $E_1 = \widetilde{E_S^{(3)}}$, $E_2 = \widetilde{E_S^{(2)}}$ and $E_7 = \widetilde{E_S^{(1)}}$ in Figure 6.10 so

$$\widetilde{C}.E_1 = 1 \quad \text{and} \quad \widetilde{C}.E_i = 0 \quad (i = 2, \dots, 7). \tag{6.53}$$

Then,

$$h_{\mathbf{E}_8} := \pi^*(2C) = 2\widetilde{C} + r_1E_1 + r_2E_2 + r_3E_3 + r_4E_4 + r_5E_5 + r_6E_6 + r_7E_7,$$

where $r_1, \dots, r_7$ are positive integers and $h_{\mathbf{E}_8}^2 = 6$ by Lemma 4.2.2. For $h := 2\widetilde{C} + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7$, we have $h^2 = 6$. As in (6.26), we show that $h' = (r_1 - 1)E_1 + \ldots + (r_7 - 1)E_7 \in \mathrm{Div}(\widetilde{S})$ must be trivial since it is contained in the negative definite lattice E_7. Hence,

$$h_{\mathbf{E}_8} = h = 2\widetilde{C} + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7.$$

By equations (6.52), (6.53), and the intersection numbers in Figure 6.10, the lattice with basis $\widetilde{C}$, E_1, $\dots$, E_7 has with respect to this basis the intersection matrix (6.51). □

This finishes the proof of Proposition 6.2.1.

Remark 6.3.13. In the situation of Proposition 6.2.1, let $h_{\mathbf{T}} = \pi^*(C_{\mathbf{T}}) \in \mathrm{Div}(\widetilde{S})$ be the pull-back of the hyperplane section $C_{\mathbf{T}}$ of S under the minimal resolution $\pi \colon \widetilde{S} \to S$ of all singularities on S. Let Z be the fundamental cycle (see [BHPVdV04, Chap. III.3, p. 95]) which is supported on the exceptional divisor of the ADE singularities of type $\sigma(\mathbf{T})$ of S which are contained in $C_{\mathbf{T}}$. Then, we have $h_{\mathbf{T}} \geq Z$.

6.4 Proof of Main Theorem 1

(1) ⇒ (2): Let $X \subseteq \mathbb{P}^5$ be a cubic fourfold with only isolated ADE singularities and such that one singularity p of X has type $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ and the combination of all other singularities of X corresponds to $\mathbf{G}$.

Let $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$ be homogeneous coordinates on $\mathbb{P}^5$.

After a linear change of coordinates, we can assume that $p = (1 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^5$. By Lemma 5.1.1, X then is defined by

$$X : x_0 f_2(x_1, x_2, x_3, x_4, x_5) + f_3(x_1, x_2, x_3, x_4, x_5) = 0 \subseteq \mathbb{P}^5,$$

where f_2 and f_3 are homogenous polynomials of degree 2 and 3, respectively, defining a quadric Q of $\operatorname{corank}(Q) = \operatorname{corank}_{\mathbf{T}}$ and a cubic Y in $\mathbb{P}^4$. By Lemma 5.1.2,

$$S_p : f_2(x_1, x_2, x_3, x_4, x_5) = f_3(x_1, x_2, x_3, x_4, x_5) = 0 \subseteq \mathbb{P}^4$$

is a complete $(2, 3)$-intersection in $\mathbb{P}^4$. Let $\pi_p : \mathrm{Bl}_p X \to X$ be the blowing-up of X in p with exceptional divisor $E := \pi_p^{-1}(p)$ in $\mathrm{Bl}_p X$. Then, $\mathrm{Bl}_p X$ has on E singularities of type $\sigma(\mathbf{T})$, where $\sigma(\mathbf{T})$ is as in Table 1.1 and the types of all singularities outside E are given by $\mathbf{G}$. Hence, by Corollary 5.2.3, S_p has singularities of type $\sigma(\mathbf{T})$ lying on the singular locus of Q and the combination of all other singularities of S_p corresponds to $\mathbf{G}$.

(2) $\Rightarrow$ (1): Let S be a complete $(2, 3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that the singularities of S lying on the singular locus of Q are of type $\sigma(\mathbf{T})$, where for $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ we let $\sigma(\mathbf{T})$ be as in Table 6.1 and such that the combination of all other singularities on S corresponds to $\mathbf{G}$.

Let $(x_1 : x_2 : x_3 : x_4 : x_5)$ be homogeneous coordinates on $\mathbb{P}^4$.

Assume that Q and Y are defined by homogeneous polynomials f_2 and f_3 of degree 2 and 3 in $\mathbb{C}[x_1, \ldots, x_5]$, respectively, i.e.

$$S = Q \cap Y : f_2(x_1, x_2, x_3, x_4, x_5) = f_3(x_1, x_2, x_3, x_4, x_5) = 0 \subseteq \mathbb{P}^4.$$

Let $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$ be homogeneous coordinates on $\mathbb{P}^5$.

We then define the cubic fourfold

$$X : x_0 f_2(x_1, x_2, x_3, x_4, x_5) + f_3(x_1, x_2, x_3, x_4, x_5) = 0 \subseteq \mathbb{P}^5.$$

Let $p := (1 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^5$. Let $\pi_p : \mathrm{Bl}_p X \to X$ be the blowing-up of X in p with exceptional divisor $E := \pi_p^{-1}(p) \subseteq \mathrm{Bl}_p X$. By Corollary 5.2.3, the singularities on $\mathrm{Bl}_p X \setminus E$ correspond to those singularities of S that are not lying on the singular locus of Q including their singularity type. Hence, the combination of all singularities of $X \setminus \{p\}$ corresponds to $\mathbf{G}$. Further, again by Corollary 5.2.3, the singularities of $\mathrm{Bl}_p X$ on E correspond to those singularities of S that lie on the singular locus of Q including their singularity type. Hence, X has singularities of type $\sigma(\mathbf{T})$ on E and therefore p is a singularity of type $\mathbf{T}$ according to Table 1.1.

(2) $\Rightarrow$ (3): Let S be a complete $(2, 3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$ such that for $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ the singularities of S lying on the singular locus of Q are of type $\sigma(\mathbf{T})$ as in Table 6.1 and such that the combination of all other singularities on S corresponds to $\mathbf{G}$. In particular, we see that S has only isolated ADE singularities. Let

$$\pi : \widetilde{S} \to S$$

be the minimal resolution of all singularities on S. By Lemma 4.2.2, $\widetilde{S}$ is a K3 surface. By Lemmas 4.3.4 and 4.3.7, for each choice of $\mathbf{T}$ there exists a hyperplane section $C_{\mathbf{T}}$

of S passing only through the singularities of type $\sigma(\mathbf{T})$ of S on the singular locus of Q. Further, by Proposition 6.2.1, $h_{\mathbf{T}} := \pi^*(C_{\mathbf{T}}) \in \mathrm{Div}(\widetilde{S})$ is the formal sum of curves on $\widetilde{S}$ whose associated weighted graph is the graph $\Gamma_{\sigma(\mathbf{T})}$ in Table 6.1. Let $L_{\mathbf{T}}$ be the line bundle associated to the divisor $h_{\mathbf{T}}$ on $\widetilde{S}$, i.e. $L_{\mathbf{T}} = \pi^*\mathcal{O}_S(1) \in \mathrm{Pic}(\widetilde{S})$. By Lemma 4.2.2, $L_{\mathbf{T}}$ is nef and the induced map $\varphi_{L_{\mathbf{T}}}\colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image. The line bundles associated to the curves on $\widetilde{S}$ in Proposition 6.2.1 with associated weighted graph $\Gamma_{\sigma(\mathbf{T})}$ generate a lattice $\Lambda(\Gamma_{\sigma(\mathbf{T})})$ in $\mathrm{Pic}(\widetilde{S})$. The exceptional (-2)-curves on $\widetilde{S}$ from the minimal resolution of the singularities of S corresponding to $\mathbf{G}$ span a Dynkin diagram $\Gamma_{\mathbf{G}}$ according to Theorem 1.2.1. Let $\Lambda(\Gamma_{\mathbf{G}})$ be the sublattice of $\mathrm{Pic}(\widetilde{S})$ defined by the line bundles associated to the exceptional (-2)-curves generating $\Gamma_{\mathbf{G}}$. Since all singularities on S are isolated, we have an orthogonal direct sum $\Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})$ which is a sublattice of $\mathrm{Pic}(\widetilde{S})$. Let

$$\phi\colon H^2(\widetilde{S},\mathbb{Z}) \to L_{K3}$$

be a marking of $\widetilde{S}$. By restricting ϕ, we obtain an embedding

$$i\colon \Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})}) \hookrightarrow L_{K3}$$

and the inclusion defines a primitive embedding ι of the saturation of $\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$ in the K3 lattice with respect to i into the K3 lattice

$$\iota\colon \mathrm{Sat}_{L_{K3}}(i) \hookrightarrow L_{K3}.$$

We now show that Items (3a),(3b), and (3c) hold:

Let

$$\Delta := \{\mathcal{O}(C) \in \mathrm{Pic}(\widetilde{S});\ C \text{ irreducible curve in the exceptional divisor of } \pi\}$$

and

$$M := \text{free } \mathbb{Z}\text{-module generated by } \Delta \text{ in } \mathrm{Pic}(\widetilde{S}).$$

By definition, M is a lattice isomorphic to $\Lambda\big(\sigma(\mathcal{T})\big) \oplus \Lambda(\Gamma_{\mathbf{G}})$, where $\sigma(\mathcal{T})$ is the Dynkin diagram corresponding to the ADE singularities $\sigma(\mathbf{T})$. Let

$$R := \{E \in M;\ E^2 = -2\}.$$

We have $L_{\mathbf{T}}.E = 0$ for all $E \in \Delta$ and hence also for all $E \in R$ since Δ is a basis of M and $R \subseteq M$. Define further the root system

$$R' := \{E \in \mathrm{Pic}(\widetilde{S});\ L_{\mathbf{T}}.E = 0, E^2 = -2\}.$$

We have $R \subseteq R'$ and we claim that we even have an equality: Indeed, let

$$\theta\colon \widetilde{S} \to S'$$

be the contraction of all (-2)-curves on $\widetilde{S}$ as in Definition 3.3.3. By Proposition 3.3.4, we have $S \cong S'$, i.e. $R = R'$.

Let $x \in \mathrm{Sat}_{L_{K3}}(i)$ such that $i(h_{\mathbf{T}}).x = 0$ and $x^2 = -2$. We have

$$F := \phi^{-1}(x) \in \phi^{-1}\big(\mathrm{Sat}_{L_{K3}}(i)\big) = \mathrm{Sat}_{H^2(\widetilde{S},\mathbb{Z})}(\phi^{-1} \circ i).$$

Hence, there is an integer $n \geq 1$ such that $nF \in \Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$. However, we have a primitive embedding $\operatorname{Pic}(\widetilde{S}) \hookrightarrow H^2(\widetilde{S},\mathbb{Z})$, i.e. $H^2(\widetilde{S},\mathbb{Z})/\operatorname{Pic}(\widetilde{S})$ is torsion free, and $\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})}) \subseteq \operatorname{Pic}(\widetilde{S})$. Hence, we obtain $F \in \operatorname{Pic}(\widetilde{S})$. Further, $L_{\mathbf{T}}.F = 0$ and $F^2 = -2$ since ϕ is an isometry, i.e. $F \in R' = R \subseteq \Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$. In conclusion, $x = \phi(F) \in \phi\big(\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})\big) = i\big(\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})\big)$, i.e. item (3a) holds.

The existence of elements $x, x' \in \operatorname{Sat}_{L_{K3}}(i)$ such that $x'^2 = x^2 = 0$ and $i(h_{\mathbf{T}}).x = 1$ and $i(h_{\mathbf{T}}).x' = 2$ would imply the existence of line bundles $E := \phi^{-1}(x), E' := \phi^{-1}(x') \in \operatorname{Sat}_{H^2(\widetilde{S},\mathbb{Z})}(\phi^{-1} \circ i)$ such that $E^2 = E'^2 = 0$ and $L_{\mathbf{T}}.E = 1$ and $L_{\mathbf{T}}.E' = 2$, respectively. As above, we have E, $E' \in \operatorname{Pic}(\widetilde{S})$. However, Proposition 3.2.6 would then imply that $\varphi_{L_{\mathbf{T}}}$ does not map $\widetilde{S}$ birationally onto S which is a contradiction. Consequently, items (3b) and (3c) hold, as well.

This concludes the proof of (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2)

This step in the proof is inspired by [Ura87, Theorem 1.15].

Let

$$i \colon \Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}}) \hookrightarrow L_{K3}$$

be an embedding and $\operatorname{Sat}_{L_{K3}}(i)$ the saturation of $\Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})$ in L_{K3} with respect to i such that that items (3a)-(3c) hold.

We construct a period point $\omega \in \Omega_{L_{K3}}$ such that

$$\operatorname{Sat}_{L_{K3}}(i) = \{x \in L_{K3};\, \omega.x = 0\}. \tag{6.54}$$

The lattice $\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$ has to have rank $r \leq 22$ as it admits an embedding into L_{K3}. Therefore, we must have $\mathbf{T} \in \{\mathbf{A}_{1\leq i\leq 22}, \mathbf{D}_{4\leq j\leq 22}, \mathbf{E}_{6\leq k\leq 8}\}$. Computer-aided, we determine that the signature of $\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$ is $(1, r-1)$. Let $N := \big(\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})\big)^{\perp}_{L_{K3}}$ be the orthogonal complement of the lattice $\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})$ in L_{K3} with respect to i. The lattice N has signature $(3-1, 19-(r-1)) = (2, 20-r)$. Let $t := 22 - r$ be the rank of N and $e_1, \ldots, e_t$ a basis of N such that $e_t^2 > 0$. We can always find such an e_t since N is indefinite if $r < 20$ and positive definite if $r = 20$. Let $r_1, \ldots, r_{t-1} \in \mathbb{R}$ such that $r_1, \ldots, r_{t-1}, 1$ are linearly independent over $\mathbb{Q}$. We choose a sufficiently large positive rational number r_t such that for

$$v = \sum_{i=1}^{t} r_i e_i \in N \otimes_{\mathbb{Z}} \mathbb{R}$$

we have

$$v^2 = (\sum_{i=1}^{t-1} r_i e_i)^2 + 2 \sum_{i=1}^{t-1} r_i r_t (e_i.e_t) + r_t^2 e_t^2 > 0.$$

Let $x \in L_{K3}$. We note that

$$0 = x.v = \sum_{i=1}^{t} r_i(x.e_i) \iff 0 = x.e_i \text{ for } i = 1, \ldots, t \iff x \in \operatorname{Sat}_{L_{K3}}(i). \tag{6.55}$$

The first equivalence holds since $x.e_i \in \mathbb{Z}$ for all $i = 1, \ldots, t$ and $r_1, \ldots, r_{t-1}, 1$ are $\mathbb{Q}$-linearly independent, while the second equivalence holds since $e_1, \ldots, e_t$ is a basis of N.

Let $N' := \{x \in N \otimes_{\mathbb{Z}} \mathbb{R};\ v.x = 0\}$. The symmetric bilinear form on the $\mathbb{R}$-vector space N' has signature $(2-1, t-2) = (1, t-2)$. Since N' is indefinite if $t > 2$ and positive definite if $t = 2$, we can find $x_0' \in N'$ such that $(x_0')^2 > 0$. For $x_0 := \sqrt{\frac{v^2}{(x_0')^2}}x_0' \in N'$, we then have $x_0^2 = v^2$ and define

$$\omega := v + ix_0 \in L_{K3} \otimes_{\mathbb{Z}} \mathbb{C}.$$

We have $\omega^2 = v^2 + 2i(v.x_0) - x_0^2 = 0$ and $\omega.\overline{\omega} = v^2 + x_0^2 = 2v^2 > 0$. Consequently, the image $[\omega]$ of ω in $\mathbb{P}(L_{K3} \otimes_{\mathbb{Z}} \mathbb{C})$ is contained in the period domain $\Omega_{L_{K3}}$. We claim that with this choice of ω, equation (6.54) holds. Indeed, let $x \in \mathrm{Sat}_{L_{K3}}(i)$, then $v.x = 0$ by (6.55). We have an $n \geq 1$ such that $nx \in i\big(\Lambda(\Gamma_{\mathbf{G}}) \oplus \Lambda(\Gamma_{\sigma(\mathbf{T})})\big)$, therefore $x_0.x = \frac{1}{n}(x_0.nx) = 0$ as $x_0 \in N \otimes_{\mathbb{Z}} \mathbb{R}$. Consequently, we have $\omega.x = (v + ix_0).x = v.x + i(x_0.x) = 0$, i.e. $x \in \{x \in L_{K3};\ \omega.x = 0\}$. On the other hand, assume that $x \in \{x \in L_{K3};\ \omega.x = 0\}$. Then, $0 = \omega.x = (v + ix_0).x = v.x + i(x_0.x)$ which only holds if $v.x = x_0.x = 0$. Hence, we have $x \in \mathrm{Sat}_{L_{K3}}(i)$ by (6.55).

By Theorem 3.4.2, there exists a marked K3 surface $(\widetilde{S}, \phi)$ such that $[\omega]$ is the period point of $(\widetilde{S}, \phi)$. Then, let $\eta \in H^{0,2}(\widetilde{S})$ such that $\phi(\eta) = \omega$. By Lemma 3.4.1, ϕ induces an isomorphism

$$\phi\colon \mathrm{Pic}(\widetilde{S}) \xrightarrow{\sim} \mathrm{Sat}_{L_{K3}}(i). \tag{6.56}$$

Let $L_{\mathbf{T}} = \phi^{-1}\big(i(h_{\mathbf{T}})\big) \in \mathrm{Pic}(\widetilde{S})$. Then, $L_{\mathbf{T}}^2 = h_{\mathbf{T}}^2 = 6$. Since $[\omega] = [-\omega]$ in $\mathbb{P}(L_{K3} \otimes_{\mathbb{Z}} \mathbb{C})$, the marked K3 surfaces $(\widetilde{S}, \phi)$ and $(\widetilde{S}, -\phi)$ define the same period point in $\Omega_{L_{K3}}$. Thus, after replacing $(\widetilde{S}, \phi)$ by $(\widetilde{S}, -\phi)$ if necessary, we can assume that $L_{\mathbf{T}}$ belongs to the positive cone $\mathcal{C}_{\widetilde{S}}$ containing the Kähler class. By Proposition 3.2.3, for a finite number of elements $F_1, \ldots, F_r \in \mathrm{Pic}(\widetilde{S})$ with $F_i^2 = -2$ $(i = 1, \ldots, r)$, the image $(s_{F_1} \circ \cdots \circ s_{F_r})(L_{\mathbf{T}})$ of the line bundle $L_{\mathbf{T}}$ under the Picard-Lefschetz-reflection $s_{F_1} \circ \cdots \circ s_{F_r}$ is nef. Since $\eta.F = 0$ for all $F \in \mathrm{Pic}(\widetilde{S})$ with $F^2 = -2$, we have $\omega = \phi(\eta) = (\phi \circ s_{F_1} \circ \cdots \circ s_{F_r})(\eta)$, i.e. $(\widetilde{S}, \phi)$ and $(\widetilde{S}, \phi \circ s_{F_1} \circ \cdots \circ s_{F_r})$ define the same period. After replacing $(\widetilde{S}, \phi)$ by $(\widetilde{S}, \phi \circ s_{F_1} \circ \cdots \circ s_{F_r})$, we can assume that $L_{\mathbf{T}}$ is nef. By items (3b) and (3c), $L_{\mathbf{T}}$ does not satisfy items (1) and (2) in Proposition 3.2.6, i.e. we have a birational morphism $\varphi_{L_{\mathbf{T}}}\colon \widetilde{S} \to \mathbb{P}^4$ of $\widetilde{S}$ onto its image. By Theorem 3.3.2, we know that the contraction $\theta\colon \widetilde{S} \to S'$ defines a surface S' whose singularities are described by the root system

$$R_{\mathbf{T}} := \{F \in \mathrm{Pic}(\widetilde{S});\ F^2 = -2, L_{\mathbf{T}}.F = 0\}.$$

Further, Proposition 3.3.4 gives that $\varphi_{L_{\mathbf{T}}}$ factors through θ and furthermore that $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ is a complete (2,3)-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$.

We will now show for each $\mathbf{T}$ individually that

$$S := \varphi_{L_{\mathbf{T}}}(\widetilde{S}) \subseteq \mathbb{P}^4$$

lies on a quadric Q such that S has singularities of type $\sigma(\mathbf{T})$ on the singular locus of Q and all other singularities of S correspond to $\mathbf{G}$.

Assumption: $\mathbf{T} = \mathbf{A}_1$

Let $\widetilde{C}$ be the vertex of the graph $\Gamma_{\sigma(\mathbf{A}_1)}$ in Table 6.1 and $h_{\mathbf{A}_1} = \widetilde{C}$. Then, $\widetilde{C}$ is a basis of the lattice $\Lambda(\Gamma_{\sigma(\mathbf{A}_1)})$. By means of the isomorphism

$$\phi\colon \mathrm{Pic}(\widetilde{S}) \xrightarrow{\sim} \mathrm{Sat}_{L_{K3}}(i),$$

we may assume that $\widetilde{C}$ is a divisor on $\widetilde{S}$ and $[\widetilde{C}]$ is its numerical equivalence class in $\operatorname{Pic}(\widetilde{S})$. We have

$$L_{\mathbf{A}_1} = \phi^{-1}\big(i(h_{\mathbf{A}_1})\big) = [\widetilde{C}] \in \operatorname{Pic}(\widetilde{S}).$$

1. We show that the singularities of $S := \varphi_{L_{\mathbf{A}_1}}(\widetilde{S}) \subseteq \mathbb{P}^4$ correspond to $\mathbf{G}$:

Let $M_{\mathbf{A}_1}$ be the lattice in $\operatorname{Pic}(\widetilde{S})$ generated by the root system

$$R_{\mathbf{A}_1} := \{F \in \operatorname{Pic}(\widetilde{S});\ F^2 = -2, L_{\mathbf{A}_1}.F = 0\}.$$

We claim that we have an isomorphism

$$\phi\colon M_{\mathbf{A}_1} \xrightarrow{\sim} \Lambda(\Gamma_{\mathbf{G}}). \tag{6.57}$$

Indeed, let $F \in \operatorname{Pic}(\widetilde{S})$ such that $F^2 = -2$ and $L_{\mathbf{A}_1}.F = 0$. Then, $\phi(F)^2 = -2$ and $i(h_{\mathbf{A}_1}).\phi(F) = 0$. Hence, by assumption (3a) in Main Theorem 1, $\phi(F) \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{A}_1)}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$. Then, write $F = a\widetilde{C} + F'$, where $\phi(F') \in i\big(\Lambda(\Gamma_{\mathbf{G}})\big)$ and $a \in \mathbb{Z}$. Since $0 = L_{\mathbf{A}_1}.F = L_{\mathbf{A}_1}.(a\widetilde{C} + F') = 6a$, we obtain $a = 0$. Hence, $F = F' \in \phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big)$. Obviously, we have $\phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big) \subseteq M_{\mathbf{A}_1}$. This proves the claim.

By Corollary 3.3.5, the singularities of S then are of type $\mathbf{G}$.

2. We show that S is contained in a quadric of corank zero in $\mathbb{P}^4$:

The quadric Q has corank ≤ 2 in $\mathbb{P}^4$. Indeed, if Q had corank ≥ 3 in $\mathbb{P}^4$, the singular locus of Q would have dimension ≥ 2 and therefore the cubic Y would intersect the singular locus of Q in a variety of dimension ≥ 1. Hence, S would be singular along this variety in contradiction to the fact that S has only isolated singularities corresponding to $\mathbf{G}$.

If Q had corank one in $\mathbb{P}^4$, by Proposition 6.2.1, $\operatorname{Pic}(\widetilde{S})$ would contain two classes of curves $\widetilde{C_1}$ and $\widetilde{C_2}$ with $\widetilde{C_1}^2 = \widetilde{C_2}^2 = 0$ and such that $\widetilde{C_1}.\widetilde{C_2} > 0$. Further, the lattice $\Lambda(\Gamma_{\mathbf{G}})$ generated by the exceptional (-2)-curves of the resolution of the singularities corresponding to $\mathbf{G}$ is contained in $\operatorname{Pic}(\widetilde{S})$. Since $\Lambda(\Gamma_{\mathbf{G}})$ is negative definite, neither $\widetilde{C_1}$ nor $\widetilde{C_2}$ can be contained in $\Lambda(\Gamma_{\mathbf{G}})$. Hence, the rank of $\operatorname{Pic}(\widetilde{S})$ would be $\geq \operatorname{rank}\big(\Lambda(\Gamma_{\mathbf{G}})\big) + 2$ in contradiction to $\operatorname{rank}\big(\operatorname{Pic}(\widetilde{S})\big) = \operatorname{rank}\big(\Lambda(\Gamma_{\sigma(\mathbf{A}_1)}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big) = \operatorname{rank}\big(\Lambda(\Gamma_{\mathbf{G}})\big) + 1$.

If Q had corank two in $\mathbb{P}^4$, again by Proposition 6.2.1, $L_{\mathbf{A}_1}$ would be the class of $2\widetilde{C} + F$, where $\widetilde{C}$ is a curve on $\widetilde{S}$ such that $\widetilde{C}^2 = 0$ and $L_{\mathbf{A}_1}.\widetilde{C} = 3$ and F is a linear combination of (-2)-curves on $\widetilde{S}$ such that $L_{\mathbf{A}_1}.F = 0$. By definition, we have $F \in M_{\mathbf{A}_1} \cong \Lambda(\Gamma_{\mathbf{G}})$, therefore $F.\widetilde{C} = 0$. This implies $3 = L_{\mathbf{A}_1}.\widetilde{C} = (2\widetilde{C} + F).\widetilde{C} = 0$ which is a contradiction.

Consequently, Q must have corank 0 in $\mathbb{P}^4$.

In conclusion, S is a complete $(2,3)$-intersection lying on a quadric of corank 0 in $\mathbb{P}^4$ such that all singularities of S correspond to $\mathbf{G}$.

Assumption: $\mathbf{T} = \mathbf{A}_2$

The proof is inspired by [SZ07, Proposition 7.1].

Let $\widetilde{C_1}$ and $\widetilde{C_2}$ be the vertices of the graph $\Gamma_{\sigma(\mathbf{A}_2)}$ in Table 6.1 and $h_{\mathbf{A}_2} = \widetilde{C_1} + \widetilde{C_2}$. Then, $\widetilde{C_1}, \widetilde{C_2}$ is a basis of the lattice $\Lambda(\Gamma_{\sigma(\mathbf{A}_2)})$. By means of the isomorphism

$$\phi \colon \operatorname{Pic}(\widetilde{S}) \xrightarrow{\sim} \operatorname{Sat}_{L_{K3}}(i),$$

we may assume that $\widetilde{C_1}$ and $\widetilde{C_2}$ are divisors on $\widetilde{S}$ and $[\widetilde{C_1}]$ and $[\widetilde{C_2}]$ are their numerical equivalence classes in $\operatorname{Pic}(\widetilde{S})$.

We have

$$L_{\mathbf{A}_2} = \phi^{-1}\big(i(h_{\mathbf{A}_2})\big) = [\widetilde{C_1}] + [\widetilde{C_2}] \in \operatorname{Pic}(\widetilde{S}).$$

1. We show that the singularities of $S := \varphi_{L_{\mathbf{A}_2}}(\widetilde{S}) \subseteq \mathbb{P}^4$ correspond to $\mathbf{G}$:

Let $M_{\mathbf{A}_2}$ be the lattice in $\operatorname{Pic}(\widetilde{S})$ generated by the root system

$$R_{\mathbf{A}_2} := \{F \in \operatorname{Pic}(\widetilde{S});\ F^2 = -2, L_{\mathbf{A}_2}.F = 0\}.$$

We claim that we have an isomorphism

$$\phi \colon M_{\mathbf{A}_2} \xrightarrow{\sim} \Lambda(\Gamma_{\mathbf{G}}). \tag{6.58}$$

Indeed, let $F \in \operatorname{Pic}(\widetilde{S})$ such that $F^2 = -2$ and $L_{\mathbf{A}_2}.F = 0$. Then, $\phi(F)^2 = -2$ and $i(h_{\mathbf{A}_2}).\phi(F) = 0$. Hence, by assumption (3a) in Main Theorem 1, $\phi(F) \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{A}_2)}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$. Then, write $F = a\widetilde{C_1} + b\widetilde{C_2} + F'$, where $\phi(F') \in i\big(\Lambda(\Gamma_{\mathbf{G}})\big)$ and $a, b \in \mathbb{Z}$. Since $0 = L_{\mathbf{A}_2}.F = 3a + 3b$, we obtain $a = -b$. Then,

$$-2 = (a\widetilde{C_1} + b\widetilde{C_2} + F')^2 = -6a^2 + F'^2. \tag{6.59}$$

Since $\Lambda(\Gamma_{\mathbf{G}})$ is negative definite, we have $F'^2 \leq 0$. Thus, equation (6.59) can only hold if $a = 0$. Hence, $F = F' \in \phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big)$. Obviously, we have $\phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big) \subseteq M_{\mathbf{A}_2}$. This proves the claim.

By Corollary 3.3.5, the singularities of S then are of type $\mathbf{G}$.

2. We show that S is contained in a quadric of corank one in $\mathbb{P}^4$:

Let $i = 1, 2$ and assume that $\widetilde{C_i}$ is a general member in $|\widetilde{C_i}|$.

By Lemma 3.1.1, either $\widetilde{C_i}$ or $-\widetilde{C_i}$ is effective. However, if $-\widetilde{C_i}$ was effective, we had $L_{\mathbf{A}_2}.(-\widetilde{C_i}) = -3$ in contradiction to the fact that $L_{\mathbf{A}_2}$ is nef. Hence, $\widetilde{C_i}$ must be effective.

We claim that $|\widetilde{C_i}|$ is fixed point free and in particular nef. Indeed, assume that we have

$$|\widetilde{C_i}| = |M_i| + F_i,$$

where $|M_i|$ is the mobile part of $|\widetilde{C_i}|$ and F_i the fixed part. Let $\widetilde{C_i} = M_i + F_i$. Assume that $\varphi_{L_{\mathbf{A}_2}}(F_i)$ is one-dimensional. The curve $\varphi_{L_{\mathbf{A}_2}}(M_i) \subseteq \mathbb{P}^4$ then has degree one or two, i.e. has an irreducible component which is isomorphic to $\mathbb{P}^1$. It follows that S contains a continuous family of rational curves. Hence, S is uniruled. Since $\widetilde{S}$ and S are birational it follows that also $\widetilde{S}$ is uniruled, a contradiction to the fact that $\widetilde{S}$ is a K3 surface. Consequently, $\varphi_{L_{\mathbf{A}_2}}(F_i)$ must be a set of points in $\mathbb{P}^4$. Let $F_{i,1}, \ldots, F_{i,n}$ be the irreducible components of F_i. For $j = 1, \ldots, n$, we have $F_{i,j}^2 = -2$ by Lemma 3.2.1. Since $L_{\mathbf{A}_2}.F_i = 0$, we have also $L_{\mathbf{A}_2}.F_{i,j} = 0$. Hence, $[F_{i,j}] \in \phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big)$ by (6.58). Therefore, $\widetilde{C_i}.F_{i,j} = 0$ which gives $\widetilde{C_i}.F_i = 0$. Consequently, $M_i^2 = (\widetilde{C_i} - F_i)^2 = F_i^2 < 0$ since F_i is by assumption contained in the negative definite lattice $\phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big)$. However, this is absurd since $|M_i|$ is nef as the mobile part of $|\widetilde{C_i}|$ and therefore $M_i^2 \geq 0$. Hence, $|\widetilde{C_i}|$ is fixed part free. If $|\widetilde{C_i}|$ had fixed points, the curves in $|\widetilde{C_i}|$ would intersect in these points which is absurd since we have for all $\widetilde{C_i} \in |\widetilde{C_i}|$ that $\widetilde{C_i}^2 = 0$. Hence, $|\widetilde{C_i}|$ is fixed point free and therefore in particular nef.

We claim that $|\widetilde{C_i}|$ is an elliptic pencil. Indeed, since $|\widetilde{C_i}|$ is nef, it follows by Theorem 3.2.4 that $|\widetilde{C_i}| = m_i|\widetilde{C_i}'|$ for a positive integer m_i and an elliptic pencil $|\widetilde{C_i}'|$ on $\widetilde{S}$. Note that by Proposition 3.3.4, the map $\varphi_{L_{\mathbf{A}_2}}$ is generically one-to-one on a general member $\widetilde{C_i'}$ in $|\widetilde{C_i'}|$ since $\widetilde{C_i'}$ is irreducible and $\widetilde{C_i'}^2 = 0$. We have $3 = L_{\mathbf{A}_2}.\widetilde{C_i} = m_i.(L_{\mathbf{A}_2}.\widetilde{C_i}')$. This equation only holds if $m_i = 1$ and $L_{\mathbf{A}_2}.\widetilde{C_i}' = 3$ or $m_i = 3$ and $L_{\mathbf{A}_2}.\widetilde{C_i}' = 1$. The latter case would imply that $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_i}')$ is isomorphic to $\mathbb{P}^1$. Since $\varphi_{L_{\mathbf{A}_2}}$ is generically one-to-one on $\widetilde{C_i'}$, this would give that $\widetilde{C_i'}$ is isomorphic to $\mathbb{P}^1$ which is absurd. Consequently, $|\widetilde{C_i}|$ is an elliptic pencil.

We claim that the curves in $|\widetilde{C_i}|$ are mapped by $\varphi_{L_{\mathbf{A}_2}}$ onto plane cubic curves such that we obtain a pencil of planes in Q. Since $\widetilde{C_i}$ is general in $|\widetilde{C_i}|$ and $|\widetilde{C_i}|$ is an elliptic pencil, $\widetilde{C_i}$ is irreducible, see Remark 3.2.5. Since $L_{\mathbf{A}_2}.\widetilde{C_i} = ([\widetilde{C_1}]+[\widetilde{C_2}]).\widetilde{C_i} = 3$, the curve $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_i})$ has degree 3. If $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_i})$ was not planar, it would be the twisted cubic which is isomorphic to $\mathbb{P}^1$. Since $\varphi_{L_{\mathbf{A}_2}}$ is generically one-to-one on $\widetilde{C_i}$, this would imply that $\widetilde{C_i}$ is isomorphic to $\mathbb{P}^1$. This is absurd since $\widetilde{C_i}$ is a general member in $|\widetilde{C_i}|$ and therefore, by Theorem 3.2.4, has no component with self-intersection number (-2). Consequently, $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_i})$ is an irreducible plane cubic curve. Let $\{\widetilde{C_{1,\alpha}}\}_{\alpha\in\mathbb{P}^1}$ and $\{\widetilde{C_{2,\beta}}\}_{\beta\in\mathbb{P}^1}$ be the families of curves induced by the one dimensional linear systems $|\widetilde{C_1}|$ and $|\widetilde{C_2}|$, respectively. The images $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_{1,\alpha}})$ and $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_{2,\beta}})$ are plane cubic curves in S so in particular contained in planes $\Pi_{1,\alpha}$ and $\Pi_{2,\beta}$ in $\mathbb{P}^4$. Hence, we obtain two pencils of planes $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ on $\mathbb{P}^4$. These planes are contained in Q and not in Y since the curves $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_{1,\alpha}}) = \Pi_{1,\alpha}\cap S$ and $\varphi_{L_{\mathbf{A}_2}}(\widetilde{C_{2,\beta}}) = \Pi_{2,\beta}\cap S$ had otherwise not degree 3. Write $C_{1,\alpha} \coloneqq \varphi_{L_{\mathbf{A}_2}}(\widetilde{C_{1,\alpha}}) = \Pi_{1,\alpha}\cap Y$ and $C_{2,\beta} \coloneqq \varphi_{L_{\mathbf{A}_2}}(\widetilde{C_{2,\beta}}) = \Pi_{2,\beta}\cap Y$.

We claim that Q can only have corank one or two in $\mathbb{P}^4$. Indeed, since Q contains planes, it cannot be smooth by Lemma 4.1.3. Further, if Q had corank strictly larger than 2, the complete $(2,3)$-intersection $S \subseteq \mathbb{P}^4$ would have non-isolated singularities on the singular locus of Q. However, we already know that S has only isolated singularities corresponding to $\mathbf{G}$.

We claim that Q has corank 1 in $\mathbb{P}^4$. Indeed, if Q has corank 2 in $\mathbb{P}^4$, the families $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ coincide. Consequently, the pencils $\{C_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{C_{2,\beta}\}_{\alpha\in\mathbb{P}^1}$ coincide, as well. Thus, $|\widetilde{C_1}| = |\widetilde{C_2}|$, in contradiction to $\widetilde{C_1}.\widetilde{C_2} = 3$. Hence, the assumption must be wrong and Q has corank 1 in $\mathbb{P}^4$.

3. We show that the vertex p of Q is not contained in S:

Indeed, if p was contained in S, it would be an ADE singularity on S and for all $\alpha, \beta \in \mathbb{P}^1$

the curves $C_{1,\alpha}$ and $C_{2,\beta}$ would contain p. Then, $\varphi^*_{L_{\mathbf{A}_2}}(C_{1,\alpha})$ and $\varphi^*_{L_{\mathbf{A}_2}}(C_{2,\beta}) \in \mathrm{Div}(\widetilde{S})$ would contain the exceptional divisor E from the minimal resolution of p and $\widetilde{C_{1,\alpha}}$ and $\widetilde{C_{2,\beta}}$ would intersect this exceptional divisor. We claim that this does not happen. Indeed, let E_0 be a (-2)-curve in E on $\widetilde{S}$: Since $|\widetilde{C_1}|$ and $|\widetilde{C_2}|$ are nef, we have $\widetilde{C_1}.E_0 \geq 0$, $\widetilde{C_2}.E_0 \geq 0$. Since $0 = L_{\mathbf{A}_2}.E_0 = \widetilde{C_1}.E_0 + \widetilde{C_2}.E_0$, we obtain $\widetilde{C_1}.E_0 = \widetilde{C_2}.E_0 = 0$. Hence, $\widetilde{C_1}$ and $\widetilde{C_2}$ do not intersect E. Therefore, p is not contained in Q.

In conclusion, S is a complete $(2,3)$-intersection lying on a quadric of corank 1 in $\mathbb{P}^4$ such that the singular locus of Q is not contained in S and all other singularities of S correspond to $\mathbf{G}$.

Assumption: $\mathbf{T} = \mathbf{A}_n$ for $n \geq 3$

Let $\widetilde{C_1}, \widetilde{C_2}, E_1, \ldots, E_{n-2}$ be the vertices of the graph $\Gamma_{\sigma(\mathbf{A}_n)}$ in Table 6.1 and $h_{\mathbf{A}_n} = \widetilde{C_1} + \widetilde{C_2} + E_1 + \ldots + E_{n-2}$. Then, $\widetilde{C_1}, \widetilde{C_2}, E_1, \ldots, E_{n-2}$ is a basis of the lattice $\Lambda(\Gamma_{\sigma(\mathbf{A}_n)})$. By means of the isomorphism

$$\phi\colon \mathrm{Pic}(\widetilde{S}) \xrightarrow{\sim} \mathrm{Sat}_{L_{K3}}(i),$$

we may assume that $\widetilde{C_1}, \widetilde{C_2}, E_1, \ldots, E_{n-2}$ are divisors on $\widetilde{S}$ and $[\widetilde{C_1}], [\widetilde{C_2}], [E_1], \ldots, [E_{n-2}]$ are their numerical equivalence classes in $\mathrm{Pic}(\widetilde{S})$.

We have

$$L_{\mathbf{A}_n} = \phi^{-1}\big(i(h_{\mathbf{A}_n})\big) = [\widetilde{C_1}] + [\widetilde{C_2}] + [E_1] + \ldots + [E_{n-2}] \in \mathrm{Pic}(\widetilde{S}).$$

1. We show that the singularities of $S := \varphi_{L_{\mathbf{A}_n}}(\widetilde{S}) \subseteq \mathbb{P}^4$ correspond to $\sigma(\mathbf{T}) + \mathbf{G}$:

Let $M_{\mathbf{A}_n}$ be the lattice in $\mathrm{Pic}(\widetilde{S})$ generated by the root system

$$R_{\mathbf{A}_n} := \{F \in \mathrm{Pic}(\widetilde{S});\ F^2 = -2, L_{\mathbf{A}_n}.F = 0\}.$$

The subgraph of $\Gamma_{\sigma(\mathbf{A}_n)}$ generated by $E_1, \ldots, E_{n-2}$ is of type $\mathcal{A}_{n-2}$ and the associated lattice is $\Lambda(\mathcal{A}_{n-2}) = A_{n-2}$. We claim that we have an isomorphism

$$\phi\colon M_{\mathbf{A}_n} \to i\big(\Lambda(\mathcal{A}_{n-2}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big). \tag{6.60}$$

Indeed, let $F \in \mathrm{Pic}(\widetilde{S})$ such that $F^2 = -2$ and $L_{\mathbf{A}_n}.F = 0$. It follows that $\phi(F)^2 = -2$ and $i(h_{\mathbf{A}_n}).\phi(F) = 0$. By assumption (3a) in Main Theorem 1, $\phi(F) \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{A}_n)}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$. Then, write $F = a\widetilde{C_1} + b\widetilde{C_2} + e_1E_1 + \ldots + e_{n-2}E_{n-2} + F'$, where $\phi(F') \in i\big(\Lambda(\Gamma_{\mathbf{G}})\big)$ and $a, b, e_1, \ldots, e_{n-2} \in \mathbb{Z}$. Since $0 = L_{\mathbf{A}_n}.F = 3a + 3b$, we obtain $a = -b$. Further, since $F^2 = -2$ and by inequality $2e_ie_{i+1} \leq e_i^2 + e_{i+1}^2$ for $i = 1, \ldots, n-3$, we obtain

$$\begin{aligned}
-2 &= (a\widetilde{C_1} - a\widetilde{C_2} + e_1E_1 + \ldots + e_{n-2}E_{n-2} + F')^2 \\
&= -4a^2 + 2a(e_1 - e_{n-2}) - 2(e_1^2 + \ldots + e_{n-2}^2) + 2(e_1e_2 + \ldots + e_{n-3}e_{n-2}) + F'^2 \\
&\leq -4a^2 + 2a(e_1 - e_{n-2}) - 2(e_1^2 + \ldots + e_{n-2}^2) + e_1^2 + 2(e_2^2 + \ldots + e_{n-3}^2) + e_{n-2}^2 + F'^2 \\
&= -2a^2 - \big(2a^2 - 2a(e_1 - e_{n-2}) + e_1^2 + e_{n-2}^2\big) + F'^2 \\
&= -2a^2 - (a - e_1)^2 - (a + e_{n-2})^2 + F'^2
\end{aligned}$$

which only holds if $1 = a = e_1 = -e_{n-2}$ and $F' = 0$, or if $a = 0$. However, in the first case, we have $F^2 = -4 + (e_1E_1 + \ldots + e_{n-2}E_{n-2})^2 + F'^2 < -4$ which is absurd. Hence, we must have $a = 0$ and therefore $F = e_1E_1 + \ldots + e_{n-2}E_{n-2} + F'^2$. On the other hand, we have obviously $i\big(\Lambda(\mathcal{A}_{n-2}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big) \subseteq \phi(M_{\mathbf{A}_n})$.

By Corollary 3.3.5, the singularities of S are of type $\sigma(\mathbf{T}) + \mathbf{G}$.

2. We show that S is contained in a quadric of corank one in $\mathbb{P}^4$:

Let $i = 1, 2$ and assume that $\widetilde{C_i}$ is a general member in $|\widetilde{C_i}|$.

As in the case $\mathbf{T} = \mathbf{A}_2$, step 2. above, we can show that the divisor $\widetilde{C_i} \in \mathrm{Div}(\widetilde{S})$ is effective.

We can write

$$|\widetilde{C_i}| = |M_i| + F_i,$$

where $|M_i|$ is the mobile part of $|\widetilde{C_i}|$ and F_i the fixed part. Let $\widetilde{C_i} = M_i + F_i$. As in the case $\mathbf{T} = \mathbf{A}_2$, step 2. we can show that $\varphi_{L_{\mathbf{A}_n}}(F_i)$ is a point in S, i.e. $L_{\mathbf{A}_n}.F_i = 0$. Let $F_{i,1}, \ldots, F_{i,n}$ be the irreducible components of F_i. For $j = 1, \ldots, n$, we have $F_{i,j}^2 = -2$ by Lemma 3.2.1. Since $L_{\mathbf{A}_n}.F_i = 0$, we have also $L_{\mathbf{A}_n}.F_{i,j} = 0$. Hence, $[F_{i,j}] \in M_{\mathbf{A}_n} = \phi^{-1}\Big(i\big(\Lambda(\mathcal{A}_{n-2}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)\Big)$ by (6.60). Therefore, also $[F_i] \in \phi^{-1}\Big(i\big(\Lambda(\mathcal{A}_{n-2}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)\Big)$. The mobile part $|M_i|$ is by definition nef. Similarly as in the case $\mathbf{T} = \mathbf{A}_2$, step 2., we show that $|M_i|$ is an elliptic pencil. By Theorem 3.2.4, M_i has no irreducible component which has self-intersection number (-2). Since $\Lambda(\Gamma_{\mathbf{G}})$ is negative definite, this gives $\phi([M_i]) \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{A}_n)})\big)$. Since $\phi([\widetilde{C_i}]) \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{A}_n)})\big)$ as a part of its basis, we have $\phi([F_i]) \in i\big(\Lambda(\mathcal{A}_{n-2})\big)$.

Let $\{M_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{M_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ be the families of curves induced by the one-dimensional linear systems $|M_1|$ and $|M_2|$, respectively. As in the case $\mathbf{T} = \mathbf{A}_2$, step 2. above, we show that $|M_i|$ induces two families $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ of planes on Q and such that $C_{1,\alpha} := \varphi_{L_{\mathbf{A}_n}}(M_{1,\alpha}) = \Pi_{1,\alpha} \cap Y$ and $C_{2,\beta} := \varphi_{L_{\mathbf{A}_n}}(M_{2,\beta}) = \Pi_{2,\beta} \cap Y$ are plane cubic curves on S. Again, as in the case $\mathbf{T} = \mathbf{A}_2$, step 2., we can deduce that S lies on a quadric of corank 1 in $\mathbb{P}^4$.

3. We show that the vertex of Q is an $\mathbf{A}_{n-2}$ singularity on S:

Let $M_i \in |M_i|$. If $M_i.F_i = 0$, we have $0 = \widetilde{C_i}^2 = (M_i + F_i)^2 = M_i^2 + 2M_i.F_i + F_i^2 = F_i^2$ and since $\phi([F_i])$ is contained in the negative definite lattice $i\big(\Lambda(\mathcal{A}_{n-2})\big)$, it follows $F_i = 0$, i.e. $|\widetilde{C_i}|$ is fixed part free and $|\widetilde{C_i}| = |M_i|$. However, the curves in $|\widetilde{C_i}|$ intersect the divisors supported on the union of $E_1, \ldots, E_{n-2} \in \mathrm{Div}(\widetilde{S})$ once.

On the other hand, if $F_i \neq 0$, we obtain consequently that M_i intersects F_i and the support of F_i is contained in the union of $E_1, \ldots, E_{n-2} \in \mathrm{Div}(\widetilde{S})$.

Since the curves $E_1, \ldots, E_{n-2}$ are contracted by $\varphi_{L_{\mathbf{A}_n}}$ to a singularity of type $\mathbf{A}_{n-2}$ of $S \subseteq \mathbb{P}^4$ by Corollary 3.3.5, this singularity then must be contained in all plane cubic curves in $\{C_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{C_{2,\beta}\}_{\beta\in\mathbb{P}^1}$. Since the only common intersection point of all the planes in $\{\Pi_{1,\alpha}\}_{\alpha\in\mathbb{P}^1}$ and $\{\Pi_{2,\beta}\}_{\beta\in\mathbb{P}^1}$ containing the curves $C_{1,\alpha}$ and $C_{2,\beta}$ is the vertex of Q, the $\mathbf{A}_{n-2}$ singularity must be the vertex of Q.

In conclusion, S is a complete $(2,3)$-intersection lying on a quadric of corank 1 in $\mathbb{P}^4$ such that the singular locus of Q is an $\mathbf{A}_{n-2}$ singularity in S and all other singularities of S correspond to $\mathbf{G}$.

Assumption: $\mathbf{T} = \mathbf{D}_n$ for $n \geq 4$, $\mathbf{E}_6, \mathbf{E}_7$, or $\mathbf{E}_8$

Let $k := \operatorname{rank}\big(\Lambda(\Gamma_{\sigma(\mathbf{T})})\big) - 1$.

Let $\widetilde{C}, E_1, \ldots, E_k$ be the vertices of the graph $\Gamma_{\sigma(\mathbf{T})}$ in Table 6.1 and $r_1, \ldots, r_k$ positive integers such that $h_{\mathbf{T}} = \widetilde{C} + r_1 E_1 + \ldots + r_k E_k$ as in in Table 6.1. Then, $\widetilde{C}, E_1, \ldots, E_k$ is a basis of the lattice $\Lambda(\Gamma_{\sigma(\mathbf{T})})$. By means of the isomorphism

$$\phi\colon \operatorname{Pic}(\widetilde{S}) \xrightarrow{\sim} \operatorname{Sat}_{L_{K3}}(i),$$

we may assume that $\widetilde{C}, E_1, \ldots, E_k$ are divisors on $\widetilde{S}$ and $[\widetilde{C}], [E_1], \ldots, [E_k]$ are their numerical equivalence classes in $\operatorname{Pic}(\widetilde{S})$.

We then have

$$L_{\mathbf{T}} = \phi^{-1}\big(i(h_{\mathbf{T}})\big) \in \operatorname{Pic}(\widetilde{S}).$$

1. We show that the singularities of $S := \varphi_{L_{\mathbf{T}}}(\widetilde{S}) \subseteq \mathbb{P}^4$ correspond to $\sigma(\mathbf{T}) + \mathbf{G}$:

Let $M_{\mathbf{T}}$ be the lattice in $\operatorname{Pic}(\widetilde{S})$ generated by the root system

$$R_{\mathbf{T}} := \{F \in \operatorname{Pic}(\widetilde{S});\ F^2 = -2, L_{\mathbf{T}}.F = 0\}.$$

Denote the subgraph of $\Gamma_{\sigma(\mathbf{T})}$ generated by $E_1, \ldots, E_k$ by $\sigma(\mathcal{T})$ and let $\Lambda\big(\sigma(\mathcal{T})\big)$ be the associated sublattice of $\Lambda(\Gamma_{\sigma(\mathbf{T})})$. We claim that we have an isomorphism

$$\phi\colon M_{\mathbf{T}} \to i\big(\Lambda\big(\sigma(\mathcal{T})\big) \oplus \Lambda(\Gamma_{\mathbf{G}})\big). \tag{6.61}$$

Indeed, let $F \in \operatorname{Pic}(\widetilde{S})$ such that $F^2 = -2$ and $L_{\mathbf{T}}.F = 0$. It follows that $\phi(F)^2 = -2$ and $i(h_{\mathbf{T}}).\phi(F) = 0$. By assumption (3a) in Main Theorem 1, $\phi(F) \in i\big(\Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$. Write $F = a\widetilde{C} + e_1 E_1 + \ldots + e_k E_k + F'$ for integers $a, e_1, \ldots, e_k$ and $\phi(F') \in i\big(\Lambda(\Gamma_{\mathbf{G}})\big)$. Then, $0 = L_{\mathbf{T}}.F = L_{\mathbf{T}}.(a\widetilde{C} + e_1 E_1 + \ldots + e_k E_k + F') = a(L_{\mathbf{T}}.\widetilde{C}) = 3a$, i.e. $a = 0$. Hence, $F = e_1 E_1 + \ldots + e_k E_k + F'$. On the other hand, we have obviously $i\big(\Lambda(\sigma(\mathcal{T})) \oplus \Lambda(\Gamma_{\mathbf{G}})\big) \subseteq \phi(M_{\mathbf{T}})$.

By Corollary 3.3.5, the singularities of S then are of type $\sigma(\mathbf{T}) + \mathbf{G}$.

2. We will show that S is contained in a quadric of corank two in $\mathbb{P}^4$:

Assume that $\widetilde{C}$ is a general member in $|\widetilde{C}|$.

As in the case $\mathbf{T} = \mathbf{A}_2$, step 2. above, we can choose $\widetilde{C}$ to be a curve on $\widetilde{S}$.

We determine the fixed part of $|\widetilde{C}|$. Indeed, assume that we have

$$|\widetilde{C}| = |M_{\mathbf{T}}| + F_{\mathbf{T}},$$

where $|M_{\mathbf{T}}|$ is the mobile part of $|\widetilde{C}|$ and $F_{\mathbf{T}}$ the fixed part. Assume that $\widetilde{C} = M_{\mathbf{T}} + F_{\mathbf{T}}$. As in the case $\mathbf{T} = \mathbf{A}_2$, step 2., we show that $\varphi_{L_{\mathbf{T}}}$ contracts $F_{\mathbf{T}}$. Let $F_{\mathbf{T},i}$ be an irreducible component of $F_{\mathbf{T}}$. By Lemma 3.2.1, we have $F_{\mathbf{T},i}^2 = -2$. Since $L_{\mathbf{T}}.F_{\mathbf{T}} = 0$, we have $L_{\mathbf{T}}.F_{\mathbf{T},i} = 0$. Therefore, we obtain by (6.61) that $\phi([F_{\mathbf{T},i}]) \in i\big(\Lambda(\sigma(\mathcal{T})) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$ and hence also $\phi([F_{\mathbf{T}}]) \in i\big(\Lambda(\sigma(\mathcal{T})) \oplus \Lambda(\Gamma_{\mathbf{G}})\big)$. As in the case $\mathbf{T} = \mathbf{A}_2$, step 2., we show that we have $\phi([F_{\mathbf{T}}]) \in i\big(\Lambda(\sigma(\mathcal{T}))\big)$.

As in the case $\mathbf{T} = A_2$, step 2. above, we show that $|M_{\mathbf{T}}|$ is an elliptic pencil on $\widetilde{S}$ inducing a family $\{\Pi_t\}_{t\in\mathbb{P}^1}$ of planes on the quadric Q. For $t \in \mathbb{P}^1$, let $C_t := \Pi_t \cap Y$. We obtain a family $\{C_t\}_{t\in\mathbb{P}^1}$ of plane cubic curves on S.

We claim that Q has corank 2 in $\mathbb{P}^4$. Indeed, if Q had corank one, we would find two different families of planes in Q. Let $\{\Pi'_t\}_{t\in\mathbb{P}^1}$ be a family of planes in Q. None of the planes is contained in Y since S would otherwise contain a plane and hence $\widetilde{S}$ would be rational which contradicts the fact that $\widetilde{S}$ is a K3 surface. Therefore, $\{\Pi'_t\}_{t\in\mathbb{P}^1}$ induces a family of plane cubic curves $\{C'_t := \Pi'_t \cap Y\}_{t\in\mathbb{P}^1}$ on S. Let C'_t be a curve in $\{C'_t\}_{t\in\mathbb{P}^1}$. The pull-back $\varphi^*_{L_{\mathbf{T}}}(C'_t) \in \mathrm{Div}(\widetilde{S})$ to $\widetilde{S}$ has degree 3, i.e. $L_{\mathbf{T}}.\varphi^*_{L_{\mathbf{T}}}(C'_t) = 3$. We can assume that

$$\varphi^*_{L_{\mathbf{T}}}(C'_t) = a\widetilde{C} + e_1E_1 + \ldots + e_kE_k + eF'$$

for $a, e_1, \ldots, e_k, e \in \mathbb{Q}$, and F' a divisor whose class is contained in $\phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big)$. Then, $3 = L_{\mathbf{T}}.\varphi^*_{L_{\mathbf{T}}}(C') = 3a$ gives $a = 1$, i.e. $\varphi^*_{L_{\mathbf{T}}}(C'_t) = \widetilde{C} + e_1E_1 + \ldots + e_kE_k + eF'$. Further, since $\varphi_{L_{\mathbf{T}}}$ contracts $E_1, \ldots, E_k$, and F' to singularities on S, we must have $C'_t = \varphi_{L_{\mathbf{T}}}\big(\varphi^*_{L_{\mathbf{T}}}(C'_t)\big) = \varphi_{L_{\mathbf{T}}}(\widetilde{C}) \in \{C_t\}_{t\in\mathbb{P}^1}$. Therefore, the family $\{C'_t\}_{t\in\mathbb{P}^1}$ coincides with the family $\{C_t\}_{t\in\mathbb{P}^1}$. Hence, we do not find two different families of planes in Q, i.e. Q must have corank 2 instead of 1.

3. We show that the singularities of S lying on the singular locus of Q are of type $\sigma(\mathbf{T})$:

Since the planes in $\{\Pi_t\}_{t\in\mathbb{P}^1}$ intersect only in the singular line l of Q, all cubic curves in $\{C_t\}_{t\in\mathbb{P}^1}$ pass through (counted with multiplicity) the three points in $l \cap Y$ on the singular line of Q which are singularities of S.

We show that the curves in the mobile part $|M_{\mathbf{T}}|$ of $|\widetilde{C}|$ intersect each connected component of the union of the divisors $E_1, \ldots, E_k$ on $\widetilde{S}$:

Let $M_{\mathbf{T}}$ be a general member in $|M_{\mathbf{T}}|$.

Let $\mathbf{T} = \mathbf{D}_4$. Write the fixed part of $|\widetilde{C}|$ as $F_{\mathbf{D}_4} = F_{1,\mathbf{D}_4} + F_{2,\mathbf{D}_4} + F_{3,\mathbf{D}_4}$, where $F_{i,\mathbf{D}_4}$ is supported on E_i or $F_{i,\mathbf{D}_4} = 0$ for $i = 1, 2, 3$. We have

$$0 = \widetilde{C}^2 = (M_{\mathbf{D}_4} + F_{1,\mathbf{D}_4} + F_{2,\mathbf{D}_4} + F_{3,\mathbf{D}_4})^2 = \sum_{i=1}^{3} 2M_{\mathbf{D}_4}.F_{i,\mathbf{D}_4} + F^2_{i,\mathbf{D}_4}$$

and we see that this equation can only hold if $M_{\mathbf{D}_4}.F_{i,\mathbf{D}_4} \geq 1$ for the non-trivial $F_{i,\mathbf{D}_4}$ $(i = 1, 2, 3)$ using that the classes of $F_{1,\mathbf{D}_4}, F_{2,\mathbf{D}_4}$, and $F_{3,\mathbf{D}_4}$ are contained in the even, negative definite lattice $\phi^{-1}\Big(i\big(\Lambda(\sigma(\mathcal{D}_4))\big)\Big)$. On the other hand, if $F_{i,\mathbf{D}_4} = 0$ for some $i = 1, 2, 3$, we have $F_{\mathbf{D}_4}.E_i = 0$ and therefore $M_{\mathbf{D}_4}.E_i = (\widetilde{C} - F_{\mathbf{D}_4}).E_i = 1$ by definition of the intersection matrix $\Lambda\big(\sigma(\mathcal{D}_4)\big)$.

If $\mathbf{T} = \mathbf{D}_n$ $(n \geq 5)$, write $F_{\mathbf{D}_n} = F_{1,\mathbf{D}_n} + F_{2,\mathbf{D}_n}$, where $F_{1,\mathbf{D}_n}$ is supported on E_1 or $F_{1,\mathbf{D}_n} = 0$ and the support of $F_{2,\mathbf{D}_n}$ is contained in the union of $E_2, \ldots, E_{n-1}$ or $F_{2,\mathbf{D}_n} = 0$. Similarly as above, we have

$$0 = \widetilde{C_{\mathbf{D}_n}}^2 = (M_{\mathbf{D}_n} + F_{1,\mathbf{D}_n} + F_{2,\mathbf{D}_n})^2 = 2M_{\mathbf{D}_n}.F_{1,\mathbf{D}_n} + 2M_{\mathbf{D}_n}.F_{2,\mathbf{D}_n} + F^2_{1,\mathbf{D}_n} + F^2_{2,\mathbf{D}_n}$$

and this equation can only hold if $M_{\mathbf{D}_n}.F_{i,\mathbf{D}_n} \geq 1$ for the non-trivial $F_{i,\mathbf{D}_n}$ $(i = 1, 2)$. On the other hand, if $F_{i,\mathbf{D}_n} = 0$ for $i = 1$ or 2, we have $M_{\mathbf{D}_n}.E_1 = (\widetilde{C} - F_{\mathbf{D}_n}).E_1 = 1$ or $M_{\mathbf{D}_n}.E_2 = (\widetilde{C} - F_{\mathbf{D}_n}).E_2 = 1$, respectively, similarly as above.

If $\mathbf{T} = \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$, the support of $F_{\mathbf{T}}$ is contained in the union of $E_1, \ldots, E_k$ with $k = 5, 6, 7$, respectively, or $F_{\mathbf{T}} = 0$. Similarly as above, we show that we have $M_{\mathbf{T}}.F_{\mathbf{T}} \geq 1$ if $F_{\mathbf{T}} \neq 0$. If $F_{\mathbf{T}} = 0$, we have for $\mathbf{T} = \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ that $M_{\mathbf{T}}.E_i = \widetilde{C}.E_i = 1$ with $i = 3, 6, 1$, respectively, and $M_{\mathbf{T}}.E_j = \widetilde{C}.E_j = 0$ for $j = 1, \ldots, k$ with $j \neq i$ by definition of the intersection matrix $\Lambda\big(\sigma(\mathcal{T})\big)$.

Hence, for all choices of $\mathbf{T} = \mathbf{D}_{n\geq 4}, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$, the curves in $|M_{\mathbf{T}}|$ intersect each connected component of the union of the divisors $E_1, \ldots, E_k$ on $\widetilde{S}$.

By Corollary 3.3.5, the connected components of the union of the divisors $E_1, \ldots, E_k$ are contracted by $\varphi_{L_{\mathbf{T}}}$ to singularities of type $\sigma(\mathbf{T})$ on S and since the curves in $|M_{\mathbf{T}}|$ intersect with these connected components, the plane cubic curves in $\{C_t\}_{t\in\mathbb{P}^1}$ intersect in these singularities. Since the only intersection points of the curves in $\{C_t\}_{t\in\mathbb{P}^1}$ are on the singular line of Q, we can conclude that S has singularities of type $\sigma(\mathbf{T})$ on the singular line of Q. Further, the curves in in $\{C_t\}_{t\in\mathbb{P}^1}$ do not intersect with any divisor class in $\phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big) \subseteq \mathrm{Pic}(\widetilde{S})$ since the class of $\widetilde{C}$ is not contained in $\phi^{-1}\Big(i\big(\Lambda(\Gamma_{\mathbf{G}})\big)\Big)$. Hence, the singularities of type $\mathbf{G}$ are not lying on the singular line of Q.

In conclusion, S is a complete $(2,3)$-intersection lying on a quadric of corank 2 in $\mathbb{P}^4$ such that the singularities of S lying on the singular locus of Q are of type $\sigma(\mathbf{T})$ and all other singularities of S correspond to $\mathbf{G}$.

This conclues the proof of (3) $\Rightarrow$ (2).

7 Existence of primitive lattice embeddings

In this chapter, it is our goal to state Nikulin's Theorem on the existence of certain lattice embeddings. To do so, we will define firstly finite bilinear and quadratic forms and discriminant bilinear and quadratic forms. We will study quadratic forms and finite quadratic forms over the p-adic integers $\mathbb{Z}_p$. For odd primes, we will define their normal forms. Then, we will explain how to construct a quadratic $\mathbb{Z}_p$-module L_p, given a finite quadratic form G_p in normal form over $\mathbb{Z}_p$ such that the rank of L_p is the length of G_p and such that the discriminant quadratic form of L_p is isomorphic to G_p. We then will state Nikulin's Theorem which provides necessary and sufficient conditions for the existence of a primitive embedding of an even lattice into an even unimodular lattice. Finally, we will state a sufficient condition when this embedding is unique up to automorphism. The results in this chapter will be needed in the following chapter where we will give an algorithm to determine all ADE lattices Λ such that $\langle 6\rangle \oplus \Lambda$ can be embedded primitively into the K3 lattice. This algorithm will be based on Nikulin's Theorem.

7.1 Finite symmetric bilinear forms and finite quadratic forms

Let G be a finite abelian group and $\langle\, ,\, \rangle\colon G \times G \to \mathbb{Q}/\mathbb{Z}$ a symmetric bilinear function. We call a pair $(G, \langle\, ,\, \rangle)$ a *finite symmetric bilinear form*.

If $q\colon G \to \mathbb{Q}/\mathbb{Z}$ is a map such that

1. $q(rg) = r^2 q(g)$ for all $r \in \mathbb{Z}$ and all $g \in G$
2. the function $\langle\, ,\, \rangle_q\colon G \times G \to \mathbb{Q}/\mathbb{Z}$ defined by $\langle g, g'\rangle_q = q(g+g') - q(g) - q(g') \mod \mathbb{Z}$ is a symmetric bilinear form on G,

we call the pair (G, q) a *finite quadratic form* and $\langle\, ,\, \rangle_q$ the bilinear form *associated to* q.

We denote the minimal number of generators of G by $l(G)$ and call it the *length* of G.

Remark 7.1.1. Note that we defined here the finite quadratic form as in [MM09, Chap. I, Definition 2.1]; in the literature, it is usually required that $\langle g, g'\rangle_q = \frac{1}{2}\big(q(g+g') - q(g) - q(g')\big)$ mod $\mathbb{Z}$.

7.2 The discriminant form of a lattice

Let $(L, \langle\, ,\, \rangle_L)$ be a lattice. The $\mathbb{Z}$-module

$$L^\vee = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q};\ \langle x, y\rangle_L \in \mathbb{Z} \text{ for all } y \in L\}$$

together with the natural extension $\langle\, ,\, \rangle_{L^\vee}\colon L^\vee \times L^\vee \to \mathbb{Q}$ of $\langle\, ,\, \rangle_L$ to $L^\vee$ is the *dual lattice* of L. The cokernel of the natural inclusion $i\colon L \hookrightarrow L^\vee$ is the *discriminant group*

$$A(L) \coloneqq L^\vee / i(L).$$

The *discriminant bilinear form* is the pair $(A(L), b_{A(L)})$, where

$$b_{A(L)}\colon A(L) \times A(L) \to \mathbb{Q}/\mathbb{Z}$$

defined by $b_{A(L)}(x,y) = \langle x,y\rangle_{L^\vee} \mod \mathbb{Z}$. Similarly, let (L, Q_L) be the quadratic form associated to $(L, \langle\, ,\, \rangle_L)$. Then, the finite quadratic form $(A(L), q_L)$, where

$$q_L\colon A(L) \to \mathbb{Q}/\mathbb{Z}$$

defined by $q_L(x) = Q_{L^\vee}(x) \mod \mathbb{Z}$ is the *discriminant quadratic form* of L.

Lemma 7.2.1. *For the orthogonal sum $L_1 \oplus L_2$ of two lattices (L_1, b_{L_1}), (L_2, b_{L_2}), we have $A(L_1 \oplus L_2) = A(L_1) \oplus A(L_2)$ and*

$$b_{A(L_1\oplus L_2)} = b_{A(L_1)} \oplus b_{A(L_2)} \text{ and } q_{A(L_1\oplus L_2)} = q_{A(L_1)} \oplus q_{A(L_2)}.$$

The following discriminant groups will be used in the sequel where $n \geq 1$:

L	$\langle 6\rangle$	A_n	D_{2n+2}	D_{2n+1}	E_6	E_7	E_8
$A(L)$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{0\}$

Table 7.1: Discriminant groups of ADE lattices, see [MM09, Chap. II, Table 7.2].

7.3 Quadratic forms and finite quadratic forms over $\mathbb{Z}_p$

Let p be a prime number. We will in the following always denote by $\mathbb{Q}_p$ and $\mathbb{Z}_p$ the p-adic numbers and p-adic integers, respectively.

For a finite group G, we denote

$$G_p \coloneqq \{x \in G;\, p^k x = 0 \text{ for some } k \geq 0\}$$

the *p-primary part* of G.

Let (G, q) be a finite quadratic form over $\mathbb{Z}$ and $q_p\colon G_p \to (\mathbb{Q}/\mathbb{Z})_p$, $x \mapsto q(x)$ the restriction of q to G_p.

Lemma 7.3.1. *We have a group isomorphism $G_p \cong G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ such that*

$$\begin{tikzcd} G \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[r, "q\otimes \mathbb{Z}_p"] \arrow[d, "\cong"] & \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[d, "\cong"] \\ G_p \arrow[r, "q_p"] & \mathbb{Q}_p/\mathbb{Z}_p \end{tikzcd} \tag{7.1}$$

commutes, where $q \otimes \mathbb{Z}_p\colon G \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $g \otimes \alpha \mapsto q(g) \otimes \alpha^2$. Hence, the finite quadratic forms $(G \otimes_{\mathbb{Z}} \mathbb{Z}_p, q \otimes \mathbb{Z}_p)$ and (G_p, q_p) are isomorphic over $\mathbb{Z}_p$.

Proof. For $x \in (\mathbb{Q}/\mathbb{Z})_p$, there exists a positive integer k such that $p^k x \in \mathbb{Z}$. Write $[x]_p := \sum_{i=-k}^{-1} c_i p^i$ for the p-fraction part of x. Then,

$$(\mathbb{Q}/\mathbb{Z})_p \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p,\ x \mapsto [x]_p \mod \mathbb{Z}_p$$

is an isomorphism. Hence, we have $q_p \colon G_p \to \mathbb{Q}_p/\mathbb{Z}_p$. By [Gra03, Chap. III.1.2.3], we have an isomorphism of groups

$$\lambda \colon G \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} G_p,\ g \otimes \alpha \mapsto \alpha g.$$

Then, we have for $g \otimes \alpha \in G \otimes_{\mathbb{Z}} \mathbb{Z}_p$:

$$\lambda\big((q \otimes \mathbb{Z}_p)(g \otimes \alpha)\big) = \lambda\big(q(g) \otimes \alpha^2\big) = \alpha^2 q(g) = q(\alpha g) = q\big(\lambda(g \otimes \alpha)\big) = q_p\big(\lambda(g \otimes \alpha)\big).$$

Hence, diagram (7.1) commutes. □

We call (G_p, q_p) a *finite quadratic form over* $\mathbb{Z}_p$. Likewise, the definition for discriminant quadratic forms then extends to discriminant quadratic forms over $\mathbb{Z}_p$.

The following example of a quadratic form over $\mathbb{Z}_p$ and their discriminant quadratic forms will be needed in the next chapter:

Example 7.3.2.

1. For an odd prime p and $a \in \mathbb{Z}_p \backslash \{0\}$, we write $a = p^k u$ with $u \in \mathbb{Z}_p^\times$ and $k \geq 0$. Let

$$\chi \colon \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \to \{\pm 1\},\ u \mapsto \begin{cases} 1 & \text{if } u \text{ is a square mod } p \\ -1 & \text{if } u \text{ is not a square mod } p \end{cases}$$

be the Legendre symbol. Then, the finite quadratic form $W_{p,k}^{\epsilon}$ over $\mathbb{Z}_p$ with $\epsilon = \chi(u)$ is the rank one lattice with intersection matrix $(p^k u)$. The discriminant of $W_{p,k}^{\chi(u)}$ is given by

$$\operatorname{disc}(W_{p,k}^{\chi(u)}) = p^k u \mod (\mathbb{Z}_p^\times)^2. \tag{7.2}$$

2. For a prime p and $k \geq 1$, let $G := \mathbb{Z}/p^k\mathbb{Z}$ and let $a \in \mathbb{Z}$ such that $\gcd(a, p^k) = 1$ and $ap^k \in 2\mathbb{Z}$. For the generator g of G and $r \in \mathbb{Z}$, let $q \colon G \to \mathbb{Q}/\mathbb{Z}$ with $q(rg) = \frac{r^2 a}{2p^k}$. This definition is well defined since $q(p^k g) = \frac{p^{2k} a}{2p^k} = \frac{ap^k}{2} \in \mathbb{Z}$.
For an odd prime p, let

$$\chi \colon G^\times/(G^\times)^2 \to \{\pm 1\}, u \mapsto \begin{cases} 1 & \text{if } u \text{ is a square mod } p \\ -1 & \text{if } u \text{ is not a square mod } p \end{cases}$$

be the Legendre symbol.

For $p = 2$, let

$$\begin{aligned} \chi \colon & (\mathbb{Z}/2\mathbb{Z})^\times/\ \big((\mathbb{Z}/2\mathbb{Z})^\times\big)^2 \to (\mathbb{Z}/2\mathbb{Z})^\times = \{1\} \text{ is the identity map} \\ \chi \colon & (\mathbb{Z}/4\mathbb{Z})^\times/\ \big((\mathbb{Z}/4\mathbb{Z})^\times\big)^2 \to (\mathbb{Z}/4\mathbb{Z})^\times = \{1, 3\} \text{ is the identity map} \\ \chi \colon & (\mathbb{Z}/2^k\mathbb{Z})^\times/\big((\mathbb{Z}/2^k\mathbb{Z})^\times\big)^2 \to (\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\} \text{ is the mod 8 map.} \end{aligned}$$

Then, we denote the finite quadratic form (G, q) over $\mathbb{Z}$ by $w_{p,k}^{\chi(a \bmod p^k)}$, inducing the finite quadratic form (G_p, q_p) over $\mathbb{Z}_p$, where $G_p = \mathbb{Z}/p^k\mathbb{Z}$ and $q_p \colon G_p \to \mathbb{Q}_p/\mathbb{Z}_p$ with $q_p(rg) = q(rg)$ for g a generator of G_p and $r \in \mathbb{Z}$. We will refer to (G_p, q_p) as the finite quadratic form $w_{p,k}^{\chi(a \bmod p^k)}$ over $\mathbb{Z}_p$.

7.3.1 Normal form decompositions of quadratic forms and finite quadratic forms over $\mathbb{Z}_p$, p odd

Let p be an odd prime.

Let (G, q) be a finite quadratic form over $\mathbb{Z}_p$.

Definition 7.3.3. We say that a decomposition of (G, q) is given in *normal form* over $\mathbb{Z}_p$ if

$$(G, q) = \bigoplus_{k \geq 1} \left((w^1_{p,k})^{\oplus n(k)} \oplus (w^{-1}_{p,k})^{\oplus m(k)}\right),$$

where $n(k)$ and $m(k)$ are non-negative integers for each k.

Let (L, Q) be a quadratic $\mathbb{Z}_p$-module.

Definition 7.3.4. We say that a decomposition of (L, Q) is given in *normal form* over $\mathbb{Z}_p$ if

$$(L, Q) = \bigoplus_{k \geq 0} \left((W^1_{p,k})^{\oplus n(k)} \oplus (W^{-1}_{p,k})^{\oplus m(k)}\right),$$

where $n(k)$ and $m(k)$ are non-negative integers for each k.

Remark 7.3.5. In the definition of a normal form of a finite quadratic form over $\mathbb{Z}_p$ and quadratic $\mathbb{Z}_p$-module in [MM09, Chap. IV, Definition 2.2, 2.6], it is furthermore requested that $m(k) \leq 1$ for each k. With these stronger definitions, we can show that if q and Q are non-degenerate, (G, q) and (L, Q), respectively, have unique normal form decompositions by [MM09, Chap. IV, Proposition 2.4, 2.7]. Obviously, a normal form decomposition as in [MM09] is in particular a normal form as defined here.

Proposition 7.3.6 ([MM09, Chap. IV, Corollary 2.10])**.** *For a finite quadratic form (G, q) over $\mathbb{Z}_p$, there exists an up to isomorphism unique quadratic $\mathbb{Z}_p$-module (L, Q) such that rank$(L) = l(G)$ and the discriminant form of (L, Q) is isomorphic to (G, q).*

Corollary 7.3.7. *Let (G, q) be a finite quadratic form over $\mathbb{Z}_p$ in its normal form*

$$(G, q) \coloneqq \bigoplus_{k \geq 1} \left((w^1_{p,k})^{\oplus n(k)} \oplus (w^{-1}_{p,k})^{\oplus m(k)}\right).$$

The up to isomorphism uniquely determined $\mathbb{Z}_p$-module (L, Q) such that rank$(L) = l(G)$ and such that the discriminant form of (L, Q) is isomorphic to (G, q) is

$$\bigoplus_{k \geq 1} \left((W^1_{p,k})^{\oplus n(k)} \oplus (W^{-1}_{p,k})^{\oplus m(k)}\right).$$

Proof. For $i = \pm 1$, we have $\operatorname{rank}(W^i_{p,k}) = 1$ and $l(w^i_{p,k}) = l(\mathbb{Z}/p^k\mathbb{Z}) = 1$. Further, the discriminant quadratic form of $W^i_{p,k}$ is

$$\left(A(W^i_{p,k}), q_{W^i_{p,k}}\right) = (\mathbb{Z}/p^k\mathbb{Z}, q_{W^i_{p,k}})$$

and $(\mathbb{Z}/p^k\mathbb{Z}, q_{W^i_{p,k}})$ is simply the finite quadratic form $w^i_{p,k}$. By Lemma 7.2.1, the discriminant form of

$$\bigoplus_{k \geq 1} \left((W^1_{p,k})^{\oplus n(k)} \oplus (W^{-1}_{p,k})^{\oplus m(k)}\right)$$

then is

$$\bigoplus_{k\geq 1}\big((w^1_{p,k})^{\oplus n(k)}\oplus(w^{-1}_{p,k})^{\oplus m(k)}\big).$$

By Proposition 7.3.6, the quadratic form $\bigoplus_{k\geq 1}\big((W^1_{p,k})^{\oplus n(k)}\oplus(W^{-1}_{p,k})^{\oplus m(k)}\big)$ is up to isomorphism unique with these properties. □

Remark 7.3.8. Likewise, there exists the notion of normal form for finite quadratic forms over $\mathbb{Z}_2$ and quadratic $\mathbb{Z}_2$-modules and a version of Proposition 7.3.6 over $\mathbb{Z}_2$, see [MM09, Chap. IV.4, IV.5].

7.4 Primitive embeddings into unimodular lattices

For a finite quadratic form (G,q) over $\mathbb{Z}$, the induced finite quadratic form $(G\otimes_{\mathbb{Z}}\mathbb{Z}_p, q\otimes\mathbb{Z}_p)$ over $\mathbb{Z}_p$ is by Lemma 7.3.1 isomorphic to the finite quadratic form (G_p,q_p) over $\mathbb{Z}_p$ on the p-primary part G_p of G. Let $K(q_p)$ be the unique quadratic $\mathbb{Z}_p$-module of rank $l(G_p)$ and with discriminant form isomorphic to (G_p,q_p). Note that $K(q_p)$ exists for odd primes p by Proposition 7.3.6 and for $p=2$ by [MM09, Chap. IV, Corollary 5.6].

We recall V. V. Nikulin's Theorem about the existence of primitive lattice embeddings into even unimodular lattices:

Theorem 7.4.1 ([Nik80, Theorem 1.12.2 $(a)\Leftrightarrow(d)$])**.** *The following properties are equivalent:*

1. *There exists a primitive embedding of an even lattice (M,Q) with signature (m_+,m_-) and discriminant form $(A(M),q)$ into an even unimodular lattice L with signature (l_+,l_-).*
2. *The following conditions are all satisfied:*
   a) $l_+-l_-\equiv 0 \mod 8$
   b) $l_--m_-\geq 0$, $l_+-m_+\geq 0$
   c) $(l_-+l_+)-(m_-+m_+)\geq l\big(A(M)\big)$
   d) *If p is an odd prime and $(l_-+l_+)-(m_-+m_+)=l\big(A(M)_p\big)$, then we have* $(-1)^{l_+-m_+}|A(M)|\equiv \operatorname{disc}\big(K(q_p)\big) \mod (\mathbb{Z}_p^\times)^2$
   e) *If $(l_-+l_+)-(m_-+m_+)=l\big(A(M)_2\big)$ and $\omega^\epsilon_{2,k}$ does not split off q_2 for some k, then we have* $|A(M)|\equiv\pm\operatorname{disc}\big(K(q_2)\big) \mod (\mathbb{Z}_2^\times)^2$.

Remark 7.4.2. We note that V. V. Nikulin gives in [Nik80, §2] different definitions for quadratic forms and finite quadratic forms than we do in Sections 2.1 and 7.1, respectively, see Remarks 2.1.1 and 7.1.1. However, every quadratic form and every finite quadratic form in Nikulin's definition corresponds naturally to a quadratic form and finite quadratic form, respectively, defined here and vice versa. Furthermore, this correspondence respects naturally the decomposition of the quadratic forms and finite quadratic forms into direct summands. Moreover, for both Nikulin's definition and the definition here, the definitions of the bilinear forms associated to the quadratic forms coincide. Hence, we compute for both quadratic forms the same discriminants. Therefore, we may use the definitions made here for Nikulin's Theorem in [Nik80, Theorem 1.12.2].

Imposing a stronger condition on the lattices L and M as in Theorem 7.4.1, we can guarantee that a primitive embedding $M \hookrightarrow L$ is even unique up to automorphisms of L.

Theorem 7.4.3 ([Dol83, Theorem 1.4.8])**.** *A primitive embedding of an even lattice M of signature (m_+, m_-) into an even lattice L of signature (l_+, l_-) is unique up to an automorphism of L provided: $(l_- + l_+) - (m_- + m_+) \geq l\big(A(M)\big) + 2$.*

8 Finding certain primitive lattice embeddings into the K3 lattice

In this chapter, we want to find all those ADE lattices Λ such that $\langle 6\rangle \oplus \Lambda$ has a primitive embedding into the K3 lattice. We will present an algorithm which enables us to determine these lattices Λ computer-aided. Using Main Theorem 1, the existence of these lattice embeddings will imply the existence of cubic fourfolds as well as complete $(2,3)$-intersections in $\mathbb{P}^4$ both with certain ADE singularities.

8.1 Algorithm to compute certain primitive lattice embeddings into L_{K3}

Theorem 8.1.1. *Let Λ be a direct sum of irreducible ADE lattices. Then, there exists a primitive embedding $\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$ if and only if Λ is one of the 2942 lattices in Appendix C. Further, all lattices Λ in Appendix C marked with an asterisk (*) have the property that the embedding $\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$ is unique up to an automorphism of L_{K3}.*

Proof. The lattice $\langle 6\rangle \oplus \Lambda$ is even and note furthermore that the K3 lattice L_{K3} is both even and unimodular. Hence, Theorem 7.4.1 gives us necessary and sufficient conditions such that $\langle 6\rangle \oplus \Lambda$ can be embedded primitively into L_{K3}. Further, Theorem 7.4.3 gives us a sufficient condition such that this embedding is unique up to an automorphism of L_{K3}. The algorithm below determines all lattices Λ such that for $\langle 6\rangle \oplus \Lambda$ all conditions (2a)-(2e) in Theorem 7.4.1 hold. These can be found in the list in Appendix C. The algorithm furthermore identifies those for which the condition in Theorem 7.4.3 holds, as well. These are the lattices Λ in Appendix C marked with an asterisk (*). $\square$

Remark 8.1.2. Independently from us, S. Brandhorst found the complete list of 2942 ADE lattice Λ in Appendix C such that we have a primitive embedding $\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$ by means of the computer-algebra software Sage.

We now describe the algorithm mentioned in the proof of Theorem 8.1.1 based on Theorem 7.4.1 to determine all possible direct sums of ADE lattices Λ such that we have a primitive embedding

$$\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$$

and on Theorem 7.4.3 to determine some embeddings which are unique up to automorphisms of L_{K3}. The algorithm is implemented in the computer-algebra software Wolfram Mathematica (version 11.1.1.0), find the code in Appendix B. Summarized, the algorithm determines step-by-step the set of all ADE lattices Λ such that the lattices $\langle 6\rangle \oplus \Lambda$ satisfy the necessary and sufficient conditions (2a)-(2e) in Theorem 7.4.1. In the final step we obtain the list of ADE lattices Λ such that there exists a primitive embedding

$\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$. Imposing a stronger condition in (2c), lattices Λ such that the primitive embedding $\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$ is unique up to an automorphism of L_{K3} are determined simultaneously.

We now describe the algorithm structured by the following Subsections 8.1.1-8.1.5 in more detail:

8.1.1 Check condition (2a) in Theorem 7.4.1

Condition (2a) in Theorem 7.4.1 is always satisfied in our case since the K3 lattice L_{K3} has signature $(3, 19)$ so

$$19 - 3 = 16 \equiv 0 \mod 8.$$

8.1.2 Check condition (2b) in Theorem 7.4.1

Let

$$\Lambda := \bigoplus_{i\geq 1} a_i A_i \oplus \bigoplus_{j\geq 4} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k$$

be an ADE lattice. The lattice $\langle 6\rangle \oplus \Lambda$ has signature

$$(1, \sum_{i\geq 1} a_i i + \sum_{j\geq 4} d_j j + \sum_{k=6}^{8} e_k k).$$

Hence, it satisfies condition (2b) in Theorem 7.4.1 if and only if

$$19 \geq \sum_{1\geq i} a_i i + \sum_{j\geq 4} d_j j + \sum_{k=6}^{8} e_k k.$$

In particular, this means that $1 \leq i \leq 19$, $4 \leq j \leq 19$, and $6 \leq k \leq 8$. Consequently, the set of all lattices satisfying condition (2b) in Theorem 7.4.1 is given by

$$\mathsf{listb} := ([0, 19] \cap \mathbb{Z})^{19} \times ([0, 19] \cap \mathbb{Z})^{16} \times ([0, 19] \cap \mathbb{Z})^{3}.$$

Just to find all tuples in listb more time efficiently, we use an iteration in the code in Appendix B which is justified by the following Lemma:

Lemma 8.1.3. *Let n and r be positive integers with $r \leq n$. Let*

$$\mathrm{L}_{r,n} := \{(a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n;\ \sum_{i=1}^{n} a_i i = r\}$$

and for $i = 1, \dots, n-1$

$$\mathrm{step}_i \colon \mathrm{L}_{r,n} \to (\mathbb{Z}_{\geq 0})^n,\ (a_1, \dots, a_n) \mapsto (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1} + 1, a_{i+2}, \dots, a_n).$$

Let

$$\mathrm{L}_{r,n}^{\mathrm{step}} := \bigcup_{(a_1,\dots,a_n)\in \mathrm{L}_{r,n}} \{\mathrm{step}_i\big((a_1, \dots, a_n)\big);\ \textit{for } i = 1, \dots, n-1 \textit{ with } a_i \neq 0\}.$$

Then,

$$\mathrm{L}_{r+1,n} = \{(r + 1, 0, \dots, 0) \in \mathbb{Z}^n\} \cup \mathrm{L}_{r,n}^{\mathrm{step}}. \tag{8.1}$$

Proof. Let $(a_1, \ldots, a_n) \in \mathrm{L}_{r+1,n}$ and assume that $a_s \neq 0$ for some $2 \leq s \leq n$. We have $\sum_{i=1}^n a_i i = r+1$. Then, $(a_1, \ldots, a_{s-2}, a_{s-1}+1, a_s - 1, a_{s+1}, \ldots, a_n) \in \mathrm{L}_{r,n}$ since

$$\begin{aligned} & a_1 + \ldots + a_{s-2}(s-2) + (a_{s-1}+1)(s-1) + (a_s - 1)s + a_{s+1}(s+1) + \ldots + a_n n \\ =& a_1 + \ldots + a_{s-2}(s-2) + a_{s-1}(s-1) + a_s s + a_{s+1}(s+1) + \ldots + a_n n + (s-1) - s \\ =& r + 1 - 1 = r. \end{aligned}$$

Hence, $(a_1, \ldots, a_n) \in \mathrm{L}^{\mathrm{step}}_{r,n}$. If $(a_1, \ldots, a_n) \in \mathrm{L}_{r+1,n}$ such that $a_s = 0$ for all $2 \leq s \leq n$, then $(a_1, \ldots, a_n) = (r+1, 0, \ldots, 0)$.

Assume conversely that $(a_1, \ldots, a_n) \in \{(r+1, 0, \ldots, 0)\} \cup \mathrm{L}^{\mathrm{step}}_{r,n}$. Obviously $(r+1, 0, \ldots, 0) \in \mathrm{L}_{r+1,n}$. If $(a_1, \ldots, a_n) \in \mathrm{L}^{\mathrm{step}}_{r,n}$, we have $a_s \neq 0$ for some $s \geq 2$ such that $(a_1, \ldots, a_{s-2}, a_{s-1}+1, a_s - 1, a_{s+1}, \ldots, a_n) \in \mathrm{L}_{r,n}$. Hence,

$$\begin{aligned} & \sum_{i=1}^n a_i i \\ &= \big(a_1 + \ldots + a_{s-2}(s-2) + (a_{s-1}+1)(s-1) + (a_s - 1)s + a_{s+1}(s+1) + \ldots + a_n n\big) + 1 \\ &= r + 1 \end{aligned}$$

so $(a_1, \ldots, a_n) \in \mathrm{L}_{r+1,n}$. □

Following the notation in Lemma 8.1.3, we define the set

$$\mathsf{listab[r]} := L_{r,19}$$

which contains all tuples $(a_1, \ldots, a_{19})$ such that $\sum_{i=1}^{19} a_i i = r$. Lemma 8.1.3 now enables us to compute listab[r] iteratively by using that listab[r] $= \{(r, 0, \ldots, 0) \in \mathbb{Z}^{19}\} \cup \mathrm{L}^{\mathrm{step}}_{r-1,19}$, where $\mathrm{L}^{\mathrm{step}}_{r-1,19}$ can be computed by means of listab[r-1]. This turns out to be faster than a direct computation of listab[r].

We then define

$$\begin{aligned} \mathsf{listdb[r]} &:= \{(d_1, \ldots, d_{19}) \in \mathsf{listab[r]};\ d_1 = d_2 = d_3 = 0\} \\ \mathsf{listeb[r]} &:= \{(e_1, \ldots, e_{19}) \in \mathsf{listab[r]};\ e_1 = \ldots = e_5 = e_9 = \ldots = e_{19} = 0\}. \end{aligned}$$

Consequently,

$$\mathsf{listb[r]} := \{\big((a_1, \ldots, a_{19}), (d_4, \ldots, d_{19}), (e_6, e_7, e_8)\big) \in \mathsf{listab[i]} \times \mathsf{listdb[j]} \times \mathsf{listeb[k]};\ i+j+k = r\}$$

and

$$\mathsf{listb} := \cup_{r=1}^{19} \mathsf{listb[r]}.$$

8.1.3 Check condition (2c) in Theorem 7.4.1

Let Λ be an ADE lattice in listb, i.e. $\langle 6 \rangle \oplus \Lambda$ satisfies condition (2b) in Theorem 7.4.1. In particular, Λ has the form

$$\Lambda := \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k.$$

The signature of $\langle 6\rangle \oplus \Lambda$ is

$$(1, \sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k).$$

Hence, it satisfies condition (2c) if and only if

$$(3+19) - (1 + \sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k) \geq l\big(A(\langle 6\rangle \oplus \Lambda)\big).$$

Consequently, the set of all lattices Λ such that $\langle 6\rangle \oplus \Lambda$ satisfies condition (2c) is given by

$$\begin{aligned}\mathsf{listbc} := \Big\{ \Lambda := \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k \in \mathsf{listb};\\ 21 - (\sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k) \geq l\big(A(\langle 6\rangle \oplus \Lambda)\big)\Big\}.\end{aligned}$$

The set of all lattices Λ such that $\langle 6\rangle \oplus \Lambda$ satisfies additionally the assumptions in Theorem 7.4.3 is given by

$$\begin{aligned}\mathsf{listbcu} := \Big\{ \Lambda := \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k \in \mathsf{listb};\\ 19 - (\sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k) \geq l\big(A(\langle 6\rangle \oplus \Lambda)\big)\Big\}.\end{aligned}$$

We now present how we compute the length $l\big(A(\langle 6\rangle \oplus \Lambda)\big)$ of the discriminant group $A(\langle 6\rangle \oplus \Lambda)$ in the code in Appendix B. Indeed, by the following Lemma 8.1.4, the length $l\big(A(\langle 6\rangle \oplus \Lambda)\big)$ is just the maximum of the lengths of the p-primary parts $A(\langle 6\rangle \oplus \Lambda)_p$ of $A(\langle 6\rangle \oplus \Lambda)$:

Lemma 8.1.4. *Let G be a finite abelian group. Then,*

$$l(G) = \max_{p \text{ prime}} \big(l(G_p)\big),$$

where G_p is the p-primary part of G. More explicitly, let p_0 be a prime such that $l(G) = l(G_{p_0})$ and

$$G_{p_0} = \mathbb{Z}/p_0^{s_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_0^{s_n}\mathbb{Z}$$

for $s_1, \ldots, s_n \in \mathbb{Z}_{\geq 1}$, then $l(G) = l(G_{p_0}) = n$.

Proof. Since G_p is a subgroup of G for all primes p, we have $l(G_p) \leq l(G)$, in particular $\max_p \big(l(G_p)\big) \leq l(G)$. The group G has the invariant factor decomposition

$$G = \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_n\mathbb{Z}$$

with $d_i | d_{i+1}$ for $i = 1, \ldots, n-1$. Then, G has at most n generators, i.e. $l(G) \leq n$. Let p_0 be a prime dividing d_1. We have positive integers $s_1, \ldots, s_n$ such that

$$G_{p_0} = \mathbb{Z}/p_0^{s_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_0^{s_n}\mathbb{Z},$$

where $p_0^{s_i}|d_i$ and $p_0^{s_i+1} \nmid d_i$ for $i = 1, \ldots, n$. Hence, $l(G_{p_0}) \leq n$. We have a surjective morphism

$$\pi \colon G_{p_0} \to (\mathbb{Z}/p_0\mathbb{Z})^n,\ (x_1, \ldots, x_n) \mapsto (x_1 \bmod p_0, \ldots, x_n \bmod p_0).$$

Assume then that $g_1, \ldots, g_m$ generate G_{p_0} with $m < n$. Since π is a surjective morphism, $\pi(g_1), \ldots, \pi(g_m)$ must generate $(\mathbb{Z}/p_0\mathbb{Z})^n$. Since every element $x \in (\mathbb{Z}/p_0\mathbb{Z})^n$ satisfies $p_0 x = 0$, it can be written as $x = \big(a_1\pi(g_1), \ldots, a_m\pi(g_m)\big)$ with $0 \leq a_1, \ldots, a_m \leq p_0 - 1$. However, then $(\mathbb{Z}/p_0\mathbb{Z})^n$ had cardinality p_0^m which is false. Hence, the assumption must be wrong and we have $l(G_{p_0}) = n$. In conclusion, $n = l(G_{p_0}) \leq l(G) \leq n$ so $n = l(G_{p_0}) = l(G)$. Hence, $l(G) = \max_p \big(l(G_p)\big)$. □

Using Lemma 7.2.1 and Table 7.1, we deduce that the discriminant group of $\langle 6\rangle \oplus \Lambda$ is given by

$$\begin{aligned} A(\langle 6\rangle \oplus \Lambda) = \mathbb{Z}/6\mathbb{Z} \,\oplus\, & \bigoplus_{1=i}^{19} a_i\mathbb{Z}/(i+1)\mathbb{Z} \\ \oplus\, & \bigoplus_{j=2}^{9} d_{2j}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \,\oplus\, d_{2j+1}\mathbb{Z}/4\mathbb{Z} \\ \oplus\, & e_6\mathbb{Z}/3\mathbb{Z} \,\oplus\, e_7\mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

For all primes p, the p-primary parts of $A(\langle 6\rangle \oplus \Lambda)$ are given by

$$\begin{aligned} A(\langle 6\rangle \oplus \Lambda)_2 &= \mathbb{Z}/2\mathbb{Z} \\ &\oplus \bigoplus_{i=0}^{4} a_{4i+1}\mathbb{Z}/2\mathbb{Z} \\ &\oplus a_3\mathbb{Z}/2^2\mathbb{Z} \,\oplus\, a_7\mathbb{Z}/2^3\mathbb{Z} \,\oplus\, a_{11}\mathbb{Z}/2^2\mathbb{Z} \,\oplus\, a_{15}\mathbb{Z}/2^4\mathbb{Z} \,\oplus\, a_{19}\mathbb{Z}/2^2\mathbb{Z} \\ &\oplus \bigoplus_{j=2}^{9} d_{2j}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \,\oplus\, d_{2j+1}\mathbb{Z}/2^2\mathbb{Z} \\ &\oplus e_7\mathbb{Z}/2\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_3 &= \mathbb{Z}/3\mathbb{Z} \\ &\oplus a_2\mathbb{Z}/3\mathbb{Z} \,\oplus\, a_5\mathbb{Z}/3\mathbb{Z} \,\oplus\, a_{11}\mathbb{Z}/3\mathbb{Z} \,\oplus\, a_{14}\mathbb{Z}/3\mathbb{Z} \\ &\oplus a_8\mathbb{Z}/3^2\mathbb{Z} \,\oplus\, a_{17}\mathbb{Z}/3^2\mathbb{Z} \\ &\oplus e_6\mathbb{Z}/3\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_5 &= a_4\mathbb{Z}/5\mathbb{Z} \,\oplus\, a_9\mathbb{Z}/5\mathbb{Z} \,\oplus\, a_{14}\mathbb{Z}/5\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_7 &= a_6\mathbb{Z}/7\mathbb{Z} \,\oplus\, a_{13}\mathbb{Z}/7\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_{11} &= a_{10}\mathbb{Z}/11\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_{13} &= a_{12}\mathbb{Z}/13\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_{17} &= a_{16}\mathbb{Z}/17\mathbb{Z} \\ A(\langle 6\rangle \oplus \Lambda)_{19} &= a_{18}\mathbb{Z}/19\mathbb{Z} \end{aligned}$$

and for all primes $p > 19$, $A(\langle 6\rangle \oplus \Lambda)_p = \{0\}$. Hence, by Lemma 8.1.4,

$$\begin{aligned} l\big(A(\langle 6\rangle \oplus \Lambda)\big) &= \max_{p \text{ prime}} \Big(l\big(A(\langle 6\rangle \oplus \Lambda)_p\big)\Big) \\ &= \max\big(\quad 1 + \sum_{i=0}^{9} a_{2i+1} + 2(\sum_{j=2}^{9} d_{2j}) + \sum_{j=2}^{9} d_{2j+1} + e_7, \\ &\qquad\quad 1 + a_2 + a_5 + a_8 + a_{11} + a_{14} + a_{17} + e_6, \\ &\qquad\quad a_4 + a_9 + a_{14}, \\ &\qquad\quad a_6 + a_{13}, \\ &\qquad\quad a_{10}, \\ &\qquad\quad a_{12}, \\ &\qquad\quad a_{16}, \\ &\qquad\quad a_{18}\big). \end{aligned}$$

8.1.4 Check condition (2d) in Theorem 7.4.1

Let Λ be an ADE lattice in listbc, i.e. $\langle 6\rangle \oplus \Lambda$ satisfies conditions (2b) and (2c) in Theorem 7.4.1. In particular, Λ has the form

$$\Lambda := \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k.$$

Let p be an odd prime.

To check condition (2d), we assume that we chose Λ in listbc such that

$$(19+3) - (\sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k + 1) = l\big(A(\langle 6\rangle \oplus \Lambda)_p\big). \tag{8.2}$$

Let $\big(K(q_p), Q_p\big)$ be the unique quadratic $\mathbb{Z}_p$-module of rank $l\big(A(\langle 6\rangle\oplus\Lambda)_p\big)$ and such that the discriminant form of $\big(K(q_p), Q_p\big)$ is isomorphic to the finite quadratic form $\big(A(\langle 6\rangle\oplus\Lambda)_p, q_p\big)$ over $\mathbb{Z}_p$. Recall that $\big(K(q_p), Q_p\big)$ exists by Proposition 7.3.6.

We have to check condition (2d) for the primes $p = 3, 5, 7$ only since we find computer-aided (lines 117-131 in the code in Appendix B) that just for those primes there exists a lattice $\Lambda \in$ listc such that equation (8.2) holds for $\langle 6\rangle \oplus \Lambda$.

The lattice $\langle 6\rangle \oplus \Lambda$ satisfies then condition (2d) if and only if for $p = 3, 5, 7$ we have

$$(-1)^{3-1}|A(\Lambda)| = |A(\Lambda)| \equiv \operatorname{disc}\big(K(q_p)\big) \mod (\mathbb{Z}_p^\times)^2. \tag{8.3}$$

We now compute the discriminant $\operatorname{disc}\big(K(q_p)\big)$ of $\big(K(q_p), Q_p\big)$.

By Lemma 7.2.1, we have for a prime number p a decomposition of the finite quadratic form:

$$\begin{aligned} A(\langle 6\rangle \oplus \Lambda)_p &= A(\langle 6\rangle)_p \oplus \bigoplus_{i=1}^{19} a_i A(A_i)_p \oplus \bigoplus_{j=4}^{19} d_j A(D_j)_p \oplus \bigoplus_{k=6}^{8} e_k A(E_k)_p \\ q_{A(\langle 6\rangle\oplus\Lambda)_p} &= q_{A(\langle 6\rangle)_p} \oplus \bigoplus_{i=1}^{19} a_i q_{A(A_i)_p} \oplus \bigoplus_{j=4}^{19} d_j q_{A(D_j)_p} \oplus \bigoplus_{k=6}^{8} e_k q_{A(E_k)_p}. \end{aligned} \tag{8.4}$$

Hence, we compute for each prime $p = 3, 5, 7$ separately in the following Subsections 8.1.4.1-8.1.4.3 the normal form of the finite quadratic form $\big(A(M)_p, q_{A(M)_p}\big)$ over $\mathbb{Z}_p$ for

$$M \in \{\langle 6\rangle,\ A_i\,(1 \le i \le 19),\ D_j(4 \le j \le 19),\ E_k(6 \le k \le 8)\}.$$

We associate then to $\big(A(\langle 6\rangle \oplus \Lambda)_p, q_{A(\langle 6\rangle\oplus\Lambda)_p}\big)$ the quadratic form $\big(K(q_p), Q_p\big)$ over $\mathbb{Z}_p$ using Corollary 7.3.7.

The discriminant of $\big(K(q_p), Q_p\big)$ is then (see (2.1.3)) the product of the discriminants of the direct summands $W_{p,k}^{\pm 1}$ in the normal form of $\big(K(q_p), Q_p\big)$, see Example 7.3.2.1.

8.1.4.1 Computing the discriminant of $\big(K(q_3), Q_3\big)$

According to Table 7.1, only the discriminant groups of the lattices

$$M \in \{\langle 6\rangle, A_2, A_5, A_8, A_{11}, A_{14}, A_{17}, E_6\}$$

have a non-trivial 3-primary part. The quadratic functions Q_M on the lattices M induce on the discriminant groups $A(M)$ the quadratic functions $q_{A(M)}$ given by (see [MM09, Chap. II, Table 7.2]):

$$\begin{aligned}
q_{A(\langle 6\rangle)}&\colon & \mathbb{Z}/6\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},\ rg &\mapsto \frac{r^2}{2\cdot 6} & &\\
q_{A(A_n)}&\colon & \mathbb{Z}/(n+1)\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},\ rg &\mapsto -\frac{nr^2}{2(n+1)} & &\text{for } n = 2,5,8,11,14,17\\
q_{A(E_6)}&\colon & \mathbb{Z}/3\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},\ rg &\mapsto \frac{r^2}{2\cdot 3}. & &
\end{aligned}$$

We compute $\big(A(M)_3, q_{A(M)_3}\big)$ over $\mathbb{Z}_3$:

$$\begin{aligned}
A(\langle 6\rangle)_3 &= \mathbb{Z}/3\mathbb{Z}, & q_{A(\langle 6\rangle)_3} &: A(\langle 6\rangle)_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto \frac{r^2}{2\cdot 6} \equiv \frac{2r^2}{2\cdot 3} \mod \mathbb{Z}_3\\
A(A_2)_3 &= \mathbb{Z}/3\mathbb{Z}, & q_{A(A_2)_3} &: A(A_2)_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto -\frac{2r^2}{2\cdot 3} \equiv \frac{4r^2}{2\cdot 3} \mod \mathbb{Z}_3\\
A(A_5)_3 &= \mathbb{Z}/3\mathbb{Z}, & q_{A(A_5)_3} &: A(A_5)_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto -\frac{5r^2}{2\cdot 6} \equiv \frac{2r^2}{2\cdot 3} \mod \mathbb{Z}_3\\
A(A_8)_3 &= \mathbb{Z}/3^2\mathbb{Z}, & q_{A(A_8)_3} &: A(A_8)_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto -\frac{8r^2}{2\cdot 3^2} \equiv \frac{10r^2}{2\cdot 3^2} \mod \mathbb{Z}_3\\
A(A_{11})_3 &= \mathbb{Z}/3\mathbb{Z}, & q_{A(A_{11})_3} &: A(A_{11})_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto -\frac{11r^2}{2\cdot 12} \equiv \frac{4r^2}{2\cdot 3} \mod \mathbb{Z}_3\\
A(A_{14})_3 &= \mathbb{Z}/3\mathbb{Z}, & q_{A(A_{14})_3} &: A(A_{14})_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto -\frac{14r^2}{2\cdot 15} \equiv \frac{2r^2}{2\cdot 3} \mod \mathbb{Z}_3\\
A(A_{17})_3 &= \mathbb{Z}/3^2\mathbb{Z}, & q_{A(A_{17})_3} &: A(A_{17})_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto -\frac{17r^2}{2\cdot 18} \equiv \frac{14r^2}{2\cdot 3^2} \mod \mathbb{Z}_3\\
A(E_6)_3 &= \mathbb{Z}/3\mathbb{Z}, & q_{A(E_6)_3} &: A(E_6)_3 \to \mathbb{Q}_3/\mathbb{Z}_3, rg \mapsto \frac{2r^2}{2\cdot 3} \mod \mathbb{Z}_3.
\end{aligned}$$

According to Definition 7.3.3, the normal forms of all these discriminant groups over $\mathbb{Z}_3$ have the form $w_{3,k}^{\epsilon}$ with

$$q_3\colon w_{3,k}^{\epsilon} \to \mathbb{Q}_3/\mathbb{Z}_3,\ rg \mapsto \frac{r^2u}{2\cdot 3^k},$$

where $(u, p^k) = 1$, $up^k \in 2\mathbb{Z}$ and $\chi(u) = \epsilon$.

We obtain:

M	$A(M)$	$A(M)_3$	$\big(A(M)_3, q_{A(M)_3}\big)$	$K(q_{A(M)_3})$	$\lvert A(M)\rvert$	$\mathrm{disc}\big(K(q_{A(M)_3})\big)$
$\langle 6\rangle$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$w_{3,1}^{-1}$	$W_{3,1}^{-1}$	6	$3\cdot 2 \mod (\mathbb{Z}_3^\times)^2$
A_2	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$w_{3,1}^{1}$	$W_{3,1}^{1}$	3	$3 \mod (\mathbb{Z}_3^\times)^2$
A_5	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$w_{3,1}^{-1}$	$W_{3,1}^{-1}$	6	$3\cdot 2 \mod (\mathbb{Z}_3^\times)^2$
A_8	$\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/3^2\mathbb{Z}$	$w_{3,2}^{1}$	$W_{3,2}^{1}$	9	$3^2 \mod (\mathbb{Z}_3^\times)^2$
A_{11}	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$w_{3,1}^{1}$	$W_{3,1}^{1}$	12	$3 \mod (\mathbb{Z}_3^\times)^2$
A_{14}	$\mathbb{Z}/15\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$w_{3,1}^{-1}$	$W_{3,1}^{-1}$	15	$3\cdot 2 \mod (\mathbb{Z}_3^\times)^2$
A_{17}	$\mathbb{Z}/18\mathbb{Z}$	$\mathbb{Z}/3^2\mathbb{Z}$	$w_{3,2}^{-1}$	$W_{3,2}^{-1}$	18	$3^2\cdot 14 \mod (\mathbb{Z}_3^\times)^2$
E_6	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$w_{3,1}^{-1}$	$W_{3,1}^{-1}$	3	$3\cdot 2 \mod (\mathbb{Z}_3^\times)^2$

Table 8.1: Quadratic forms over $\mathbb{Z}_3$ on discriminant groups

Hence,

$$A(\Lambda)_3 = w_{3,1}^{-1} \oplus a_2 w_{3,1}^{1} \oplus a_5 w_{3,1}^{-1} \oplus a_8 w_{3,2}^{1} \oplus a_{11} w_{3,1}^{1} \oplus a_{14} w_{3,1}^{-1} \oplus a_{17} w_{3,2}^{-1} \oplus e_6 w_{3,1}^{-1}$$

in normal form. The associated quadratic $\mathbb{Z}_3$-module $\big(K(q_3), Q_3\big)$ is then given by Corollary 7.3.7 by

$$K(q_3) = W_{3,1}^{-1} \oplus a_2 W_{3,1}^{1} \oplus a_5 W_{3,1}^{-1} \oplus a_8 W_{3,2}^{1} \oplus a_{11} W_{3,1}^{1} \oplus a_{14} W_{3,1}^{-1} \oplus a_{17} W_{3,2}^{-1} \oplus e_6 W_{3,1}^{-1}.$$

The discriminant of $\big(K(q_3), Q_3\big)$ is then

$$\mathrm{disc}\big(K(q_3)\big) = (3{\cdot}2){\cdot}3^{a_2}{\cdot}(3{\cdot}2)^{a_5}{\cdot}(3^2)^{a_8}{\cdot}3^{a_{11}}{\cdot}(3{\cdot}2)^{a_{14}}{\cdot}(3^2{\cdot}14)^{a_{17}}{\cdot}(3{\cdot}2)^{e_6} \mod (\mathbb{Z}_3^\times)^2. \quad (8.5)$$

8.1.4.2 Computing the discriminant of $\big(K(q_5), Q_5\big)$

According to Table 7.1, only the discriminant groups of the lattices

$$M \in \{A_4,\ A_9,\ A_{14},\ \text{and } A_{19}\}$$

have a non-trivial 5-primary part. The quadratic functions Q_M on the lattices M induce on the discriminant groups $A(M)$ the quadratic functions $q_{A(M)}$ given by (see [MM09, Chap. II, Table 7.2]):

$$q_{A(A_n)} \colon \mathbb{Z}/(n+1)\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},\ rg \mapsto -\frac{nr^2}{2(n+1)} \qquad \text{for } n = 4, 9, 14, 19.$$

We compute $\big(A(M)_5, q_{A(M)_5}\big)$ over $\mathbb{Z}_5$:

$$\begin{aligned}
A(A_4)_5 &= \mathbb{Z}/5\mathbb{Z}, & q_{A(A_4)_5} &: A(A_4)_5 \to \mathbb{Q}_5/\mathbb{Z}_5,\ rg \mapsto -\frac{4r^2}{2\cdot 5} \equiv \frac{6r^2}{2\cdot 5} && \text{mod } \mathbb{Z}_5\\
A(A_9)_5 &= \mathbb{Z}/5\mathbb{Z}, & q_{A(A_9)_5} &: A(A_9)_5 \to \mathbb{Q}_5/\mathbb{Z}_5,\ rg \mapsto -\frac{9r^2}{2\cdot 10} \equiv \frac{8r^2}{2\cdot 5} && \text{mod } \mathbb{Z}_5\\
A(A_{14})_5 &= \mathbb{Z}/5\mathbb{Z}, & q_{A(A_{14})_5} &: A(A_{14})_5 \to \mathbb{Q}_5/\mathbb{Z}_5,\ rg \mapsto -\frac{14r^2}{2\cdot 15} \equiv \frac{2r^2}{2\cdot 5} && \text{mod } \mathbb{Z}_5\\
A(A_{19})_5 &= \mathbb{Z}/5\mathbb{Z}, & q_{A(A_{19})_5} &: A(A_{19})_5 \to \mathbb{Q}_5/\mathbb{Z}_5,\ rg \mapsto -\frac{19r^2}{2\cdot 20} \equiv \frac{4r^2}{2\cdot 5} && \text{mod } \mathbb{Z}_5.
\end{aligned}$$

According to Definition 7.3.3, the normal forms of all these discriminant groups over $\mathbb{Z}_5$ have the form $w^{\epsilon}_{5,k}$ with

$$q_{w^{\epsilon}_{5,k}} : w^{\epsilon}_{5,k} \to \mathbb{Q}_5/\mathbb{Z}_5,\ rg \mapsto \frac{r^2 u}{2\cdot 5^k},$$

where $(u, 5^k) = 1$, $u5^k \in 2\mathbb{Z}$ and $\chi(u) = \epsilon$.

We obtain:

M	$A(M)$	$A(M)_5$	$\big(A(M)_5, q_{A(M)_5}\big)$	$K(q_{A(M)_5})$	$\lvert A(M)\rvert$	$\mathrm{disc}\big(K(q_{A(M)_5})\big)$
A_4	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$w^1_{5,1}$	$W^1_{5,1}$	5	$5 \mod (\mathbb{Z}_5^{\times})^2$
A_9	$\mathbb{Z}/10\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$w^{-1}_{5,1}$	$W^{-1}_{5,1}$	10	$5\cdot 8 \mod (\mathbb{Z}_5^{\times})^2$
A_{14}	$\mathbb{Z}/15\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$w^{-1}_{5,1}$	$W^{-1}_{5,1}$	15	$5\cdot 2 \mod (\mathbb{Z}_5^{\times})^2$
A_{19}	$\mathbb{Z}/20\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$w^1_{5,1}$	$W^1_{5,1}$	20	$5 \mod (\mathbb{Z}_5^{\times})^2$

Table 8.2: Quadratic forms over $\mathbb{Z}_5$ on discriminant groups

Hence,

$$A(\Lambda)_5 = a_4 w^1_{5,1} \oplus a_9 w^{-1}_{5,1} \oplus a_{14} w^{-1}_{5,1} \oplus a_{19} w^1_{5,1}$$

in normal form. The associated quadratic $\mathbb{Z}_5$-module $\big(K(q_5), Q_5\big)$ is then given by Corollary 7.3.7 by

$$K(q_5) = a_4 W^1_{5,1} \oplus a_9 W^{-1}_{5,1} \oplus a_{14} W^{-1}_{5,1} \oplus a_{19} W^1_{5,1}.$$

The discriminant of $\big(K(q_5), Q_5\big)$ is then

$$\mathrm{disc}\big(K(q_5)\big) = 5^{a_4} \cdot (5\cdot 8)^{a_9} \cdot (5\cdot 2)^{a_{14}} \cdot 5^{a_{19}} \mod (\mathbb{Z}_5^{\times})^2. \tag{8.6}$$

8.1.4.3 Computing the discriminant of $\big(K(q_7), Q_7\big)$

According to Table 7.1, only the discriminant groups of the lattices

$$M \in \{A_6,\ A_{13},\}$$

have a non-trivial 7-primary part. The quadratic functions Q_M on the lattices M induce on the discriminant groups $A(M)$ the quadratic functions $q_{A(M)}$ given by (see [MM09, Chap. II, Table 7.2]):

$$q_{A(A_n)}\colon \mathbb{Z}/(n+1)\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},\ rg \mapsto -\frac{nr^2}{2(n+1)} \quad \text{for } n = 6, 13.$$

We compute $\big(A(M)_7, q_{A(M)_7}\big)$ over $\mathbb{Z}_7$:

$$\begin{aligned} A(A_6)_7 &= \mathbb{Z}/7\mathbb{Z}, & q_{A(A_6)_7} &: A(A_6)_7 \to \mathbb{Q}_7/\mathbb{Z}_7,\ rg \mapsto -\frac{6r^2}{2\cdot 7} \equiv \frac{8r^2}{2\cdot 7} \mod \mathbb{Z}_7 \\ A(A_{13})_7 &= \mathbb{Z}/7\mathbb{Z}, & q_{A(A_{13})_7} &: A(A_{13})_7 \to \mathbb{Q}_7/\mathbb{Z}_7,\ rg \mapsto -\frac{13r^2}{2\cdot 14} \equiv \frac{18r^2}{2\cdot 7} \mod \mathbb{Z}_7. \end{aligned}$$

According to Definition 7.3.3, the normal forms of all these discriminant groups over $\mathbb{Z}_7$ have the form $w^{\epsilon}_{7,k}$ with

$$q_{w^{\epsilon}_{7,k}}\colon w^{\epsilon}_{7,k} \to \mathbb{Q}_7/\mathbb{Z}_7,\ rg \mapsto \frac{r^2u}{2\cdot 7^k},$$

where $(u, 7^k) = 1$, $u7^k \in 2\mathbb{Z}$ and $\chi(u) = \epsilon$.

We obtain:

M	$A(M)$	$A(M)_7$	$\big(A(M)_7, q_{A(M)_7}\big)$	$K(q_{A(M)_7})$	$\lvert A(M)\rvert$	$\operatorname{disc}\big(K(q_{A(M)_7})\big)$
A_6	$\mathbb{Z}/7\mathbb{Z}$	$\mathbb{Z}/7\mathbb{Z}$	$\omega^1_{7,1}$	$W^1_{7,1}$	7	$7 \mod (\mathbb{Z}_7^\times)^2$
A_{13}	$\mathbb{Z}/14\mathbb{Z}$	$\mathbb{Z}/7\mathbb{Z}$	$\omega^1_{7,1}$	$W^1_{7,1}$	14	$7 \mod (\mathbb{Z}_7^\times)^2$

Table 8.3: Quadratic forms over $\mathbb{Z}_7$ on discriminant groups

Hence,

$$A(\Lambda)_7 = a_6 w^1_{7,1} \oplus a_{13} w^1_{7,1}$$

in normal form. The associated quadratic $\mathbb{Z}_7$-module $K(q_7)$ is then given by Corollary 7.3.7 by

$$K(q_7) = a_6 W^1_{7,1} \oplus a_{13} W^1_{7,1}.$$

The discriminant of $\big(K(q_7), Q_7\big)$ is then

$$\operatorname{disc}\big(K(q_7)\big) = 7^{a_6}\cdot 7^{a_{13}} \mod (\mathbb{Z}_7^\times)^2. \tag{8.7}$$

8.1.4.4 Check condition (8.3)

The cardinality of the discriminant group $A(\Lambda)$ is

$$|A(\Lambda)| = \Big(\prod_{i=1}^{19}(i+1)^{a_i}\Big)\cdot\Big(\prod_{j=4}^{19}4^{d_j}\Big)\cdot 2^{e_6}\cdot 3^{e_7}. \tag{8.8}$$

For odd primes p, as a consequence of Hensel's Lemma, an element in $x \in \mathbb{Z}_p^\times$ is a square root in $\mathbb{Z}_p^\times$ if and only if x is a square root mod p in $(\mathbb{Z}/p\mathbb{Z})^\times$, see [Eis95, Chap. 7.2, p. 184]. Hence, equation (8.3) holds if and only if for all squares u mod p in $(\mathbb{Z}/p\mathbb{Z})^\times$ we have

$$\frac{|A(\Lambda)| - u \cdot \operatorname{disc}\big(K(q_p)\big)}{v_p(|A(\Lambda)|)} \equiv 0 \mod p$$

for all possible choices of $u \in \big((\mathbb{Z}/p\mathbb{Z})^\times\big)^2$, where v_p is the p-adic valuation on $\mathbb{Z}$.

We compute

$$\big((\mathbb{Z}/3\mathbb{Z})^\times\big)^2 = \{1\}, \quad \big((\mathbb{Z}/5\mathbb{Z})^\times\big)^2 = \{1,\, 4\}, \quad \big((\mathbb{Z}/7\mathbb{Z})^\times\big)^2 = \{1,\, 2,\, 4\} \tag{8.9}$$

and

$$\begin{aligned} v_3\big(|A(\Lambda)|\big) &= a_2 + a_5 + 2a_8 + a_{11} + a_{14} + 2a_{17} + e_7, \\ v_5\big(|A(\Lambda)|\big) &= a_4 + a_9 + a_{14}, \\ v_7\big(|A(\Lambda)|\big) &= a_6 + a_{13}. \end{aligned} \tag{8.10}$$

Consequently, the set of all lattices Λ such that $\langle 6\rangle \oplus \Lambda$ satisfies conditions (2a)-(2d) is given by

$$\mathsf{result} := \left\{ \begin{aligned} &\Lambda := \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k \in \mathsf{listbc};\ \text{for } p = 3,5,7: \\ &\quad \text{if} \quad 21 - (\sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k) = l\big(A(\langle 6\rangle \oplus \Lambda)_p\big), \\ &\quad \text{then} \quad \frac{|A(\Lambda)| - u \cdot \operatorname{disc}\big(K(q_p)\big)}{v_p\big(|A(\Lambda)|\big)} \equiv 0 \bmod p \quad \text{for } u \in \big((\mathbb{Z}/p\mathbb{Z})^\times\big)^2 \end{aligned} \right\},$$

where $|A(\Lambda)|$ has been computed in (8.8), $\big((\mathbb{Z}/p\mathbb{Z})^\times\big)^2$ in (8.9), $\operatorname{disc}\big(K(q_p)\big)$ in (8.5), (8.6), and (8.7), and $v_p\big(|A(\Lambda)|\big)$ in (8.10). The set of all lattices in result such that the assumptions in Theorem 7.4.3 holds, as well, is

$$\mathsf{resultu} := \left\{ \begin{aligned} &\Lambda := \bigoplus_{i=1}^{19} a_i A_i \oplus \bigoplus_{j=4}^{19} d_j D_j \oplus \bigoplus_{k=6}^{8} e_k E_k \in \mathsf{listbcu};\ \text{for } p = 3,5,7: \\ &\quad \text{if} \quad 21 - (\sum_{1=i}^{19} a_i i + \sum_{4=j}^{19} d_j j + \sum_{k=6}^{8} e_k k) = l\big(A(\langle 6\rangle \oplus \Lambda)_p\big), \\ &\quad \text{then} \quad \frac{|A(\Lambda)| - u \cdot \operatorname{disc}\big(K(q_p)\big)}{v_p\big(|A(\Lambda)|\big)} \equiv 0 \bmod p \quad \text{for } u \in \big((\mathbb{Z}/p\mathbb{Z})^\times\big)^2 \end{aligned} \right\}.$$

8.1.5 Check condition (2e) in Theorem 7.4.1

We claim that for all ADE lattices Λ, the lattice $\langle 6\rangle \oplus \Lambda$ satisfies condition (2e) in Theorem 7.4.1. Indeed, the discriminant group of $\langle 6\rangle \oplus \Lambda$ is given by $A(\langle 6\rangle \oplus \Lambda) =$

$A(\langle 6\rangle) \oplus A(\Lambda)$. By Lemma 7.3.1, the finite quadratic form $\big(A(\langle 6\rangle), q_{A(\langle 6\rangle)}\big)$ over $\mathbb{Z}_2$ is given by $\big(A(\langle 6\rangle)_2, q_{A(\langle 6\rangle)_2}\big)$, where

$$A(\langle 6\rangle)_2 = (\mathbb{Z}/6\mathbb{Z})_2 = \mathbb{Z}/2\mathbb{Z}, \quad q_2\colon A(\langle 6\rangle)_2 \to \mathbb{Q}_2/\mathbb{Z}_2,\ rg \mapsto \frac{r^2}{2\cdot 6} \equiv \frac{3r^2}{2\cdot 2} \mod \mathbb{Z}_2.$$

Hence, $\big(A(\langle 6\rangle)_2, q_{A(\langle 6\rangle)_2}\big)$ is the finite quadratic form $w^3_{1,2}$ over $\mathbb{Z}_2$. Consequently, $w^3_{1,2}$ splits off the quadratic function $q_{A(\langle 6\rangle\oplus\Lambda)}$ on $A(\langle 6\rangle \oplus \Lambda)$ over $\mathbb{Z}_2$. Hence, for all choices of Λ, we do not need to check condition (2e).

In conclusion, the set `result` contains all ADE lattices Λ such that there exists a primitive embedding $\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$ and the set `resultu` a subset of lattices in `resultu` such that $\langle 6\rangle \oplus \Lambda \hookrightarrow L_{K3}$ is uniquely determined up to an automorphism of L_{K3}.This concludes the algorithm.

8.2 Main Theorem 2

Main Theorem 2. *Let*

$$\mathbf{G} := \sum_{i=1}^{19} a_i\mathbf{A}_i + \sum_{j=4}^{19} d_j\mathbf{D}_j + \sum_{k=6}^{8} e_k\mathbf{E}_k$$

be a formal sum of ADE singularities such that the ADE lattice

$$\Lambda := \bigoplus_{i=1}^{19} a_iA_i \oplus \bigoplus_{j=4}^{19} d_jD_j \oplus \bigoplus_{k=6}^{8} e_kE_k$$

is one of the 2942 elements in the list in Appendix C. The following hold:

1. *There exists a complete* $(2,3)$*-intersection* S *of a smooth quadric and a cubic in* $\mathbb{P}^4$ *such that* S *has singularities of type* $\mathbf{G}$.
2. *There exists a cubic fourfold with ADE singularities of type* $\mathbf{G}$ *and an* $\mathbf{A}_1$ *singularity.*

Proof. By choice of Λ, we have a primitive embedding $i\colon \Lambda \oplus \langle 6\rangle \hookrightarrow L_{K3}$ into the K3 lattice and let h be the generator of the rank one lattice $\langle 6\rangle$. In particular $h^2 = 6$.

Since i is primitive, the saturation $\mathrm{Sat}_{L_{K3}}(i)$ of $\langle 6\rangle \oplus \Lambda$ in L_{K3} is isomorphic to $\langle 6\rangle \oplus \Lambda$ with respect to i.

We claim, item (3) in Main Theorem 1 is satisfied: Let $x \in \langle 6\rangle \oplus \Lambda$ with $h.x = 0$ and write $x = nh + g$, where $n \in \mathbb{Z}$ and $g \in \Lambda$. Then, $0 = h.x = h.(nh + g) = 6n$ gives $n = 0$. Hence, $x \in \Lambda$. Consequently, all $x \in \langle 6\rangle \oplus \Lambda$ with $h.x = 0$ and $x^2 = -2$ are contained in Λ. Further, assume that we have $h.x = 1$ (or $h.x = 2$). Then, $1 = h.x = h.(nh + g) = nh^2 + h.g = 6n$ (or $2 = 6n$). However, this equation holds for no $n \in \mathbb{Z}$. Hence, such an x does not exist. In particular, there exists no $x \in \langle 6\rangle \oplus \Lambda$ with $h.x = 1$ (or $h.x = 2$) and $x^2 = 0$.

Consequently, by implications (3)⇒(1) and (3)⇒(2) in Main Theorem 1, there exists a cubic fourfold having singularities of type $\mathbf{G}$ and an $\mathbf{A}_1$ singularity and a complete $(2,3)$-intersection S of a smooth quadric and a cubic in $\mathbb{P}^4$ such that S has singularities of type $\mathbf{G}$, respectively. □

The lattice $10A_1$ is the lattice with largest rank in the list in Appendix C which has only A_1 lattices as direct summands. Hence, we obtain the following:

Corollary 8.2.1. *The following exist:*

1. *A complete* $(2,3)$*-intersection of a smooth quadric and a cubic in* $\mathbb{P}^4$ *with precisely 10* A_1 *singularities.*
2. *A cubic fourfold with precisely 11* A_1 *singularities.*

Proof. The list in Appendix C contains the lattice $10A_1$. Hence, by Main Theorem 2, there exists a complete $(2,3)$-intersection of a smooth quadric and a cubic in $\mathbb{P}^4$ with 10 A_1 singularities and a cubic fourfold with 11 A_1 singularities. □

Remark 8.2.2. We note that Corollary 8.2.1 does not necessarily give the maximal number of A_1 singularities which can occur on a complete $(2,3)$-intersection in $\mathbb{P}^4$ and a cubic fourfold, respectively. Indeed, Varchenko's bound for the maximal number of singularities which can occur on a cubic fourfold is 15 (see [Var84, Theorem on the Upper Bound, p. 2781]) and hence a cubic fourfold with more than 11 but strictly less than 16 A_1 singularities could exist.

The lattices $2A_1 \oplus 6A_2$, $4A_1 \oplus 5A_2$, and $6A_1 \oplus 4A_2$ are the lattices with largest rank in the list in Appendix C which have only A_1 and A_2 lattices as direct summands. Therefore, we obtain:

Corollary 8.2.3. *The following exist:*

1. *A complete* $(2,3)$*-intersection of a smooth quadric and a cubic in* $\mathbb{P}^4$ *with precisely:*
   a) *2* A_1 *and 6* A_2 *singularities.*
   b) *4* A_1 *and 5* A_2 *singularities.*
   c) *6* A_1 *and 4* A_2 *singularities.*
2. *A complete* $(2,3)$*-intersection of a quadric of corank 1 and a cubic in* $\mathbb{P}^4$ *with precisely:*
   a) *3* A_1 *and 5* A_2 *singularities.*
   b) *5* A_1 *and 4* A_2 *singularities.*
   c) *7* A_1 *and 3* A_2 *singularities.*
3. *A cubic fourfold with precisely:*
   a) *3* A_1 *and 6* A_2 *singularities.*
   b) *5* A_1 *and 5* A_2 *singularities.*
   c) *7* A_1 *and 4* A_2 *singularities.*

Proof. The list in Appendix C contains the lattices $2A_1 \oplus 6A_2$, $4A_1 \oplus 5A_2$, and $6A_1 \oplus 4A_2$. Hence, by Main Theorem 2, there exist complete $(2,3)$-intersections of smooth quadrics and cubics in $\mathbb{P}^4$ whose singularities are precisely of type $2A_1 + 6A_2$, $4A_1 + 5A_2$, and $6A_1 + 4A_2$. Moreover, there exist three cubic fourfolds with singularities of type $3A_1 + 6A_2$, $5A_1 + 5A_2$, and $7A_1 + 4A_2$. By implication (1) $\Rightarrow$ (2) in Main Theorem 1, we have furthermore the existence of complete $(2,3)$-intersections of quadrics of corank 1 and cubics in $\mathbb{P}^4$ with singularities precisely of type $3A_1 + 5A_2$, $5A_1 + 4A_2$, and $7A_1 + 3A_2$. □

Remark 8.2.4. We note that Corollary 8.2.3 does not necessarily give the maximal number of A_1 and A_2 singularities which can occur on a complete $(2,3)$-intersection in $\mathbb{P}^4$ and a cubic fourfold, respectively.

9 Correspondence between the moduli space of cubic fourfolds and quasi-polarized K3 surfaces of degree 6

In this chapter, we will firstly define lattice polarized K3 surfaces and then recall the construction of the moduli space of big and nef lattice polarized K3 surfaces. We will then construct the moduli space of those quasi-polarized K3 surfaces $(\widetilde{S}, L)$ such that the map $\varphi_L\colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image and such that $\varphi_L(\widetilde{S})$ has a certain configuration of ADE singularities, as the moduli space of certain lattice polarized K3 surfaces. Secondly, we will construct the moduli space of cubic fourfolds with certain ADE singularities. Finally, we will prove Main Theorem 3, which says that both moduli spaces are isomorphic.

9.1 Lattice polarized K3 surfaces

9.1.1 Basic notation and definitions

Let M be an even lattice of signature $(1, t)$ with $t \geq 0$.

An *M-polarized K3 surface* is a pair $(\widetilde{S}, j)$, where $\widetilde{S}$ is a K3 surface and $j\colon M \hookrightarrow \mathrm{Pic}(\widetilde{S})$ is a primitive embedding. We say that an M-polarized K3 surface $(\widetilde{S}, j)$ is *big and nef* if there exists an isomorphism class of a line bundle in $j\big(M\big)$ which is big and nef. Two M-polarized K3 surfaces $(\widetilde{S}, j)$ and $(\widetilde{S}', j')$ are *isomorphic* if there exists an isomorphism $f\colon \widetilde{S} \to \widetilde{S}'$ such that $j = f^* \circ j'$.

We note that for $t = 0$, an M-polarized K3 surface is simply a quasi-polarized K3 surface defined in Chapter 3 and all results here specialize to the results for quasi-polarized K3 surfaces.

9.1.2 Periods of lattice polarized K3 surfaces

Let M be an even lattice of signature $(1, t)$ with $t \geq 0$ which is embeddable into the K3 lattice L_{K3}. We fix a primitive embedding $i_M\colon M \hookrightarrow L_{K3}$ and identify M with its image $i_M(M)$ in L_{K3}.

We call a pair $(\widetilde{S}, \phi)$ a *marked M-polarized K3 surface* if $\widetilde{S}$ is a K3 surface and $\phi\colon H^2(\widetilde{S}, \mathbb{Z}) \to L_{K3}$ is a marking such that $\phi^{-1}(M) \subseteq \mathrm{Pic}(\widetilde{S})$. It follows that for $j_\phi \coloneqq {\phi^{-1}}_{|M}\colon M \hookrightarrow \mathrm{Pic}(\widetilde{S})$ the pair $(\widetilde{S}, j_\phi)$ is an M-polarized K3 surface and we call a marked M-polarized K3 surface *big and nef* if $(\widetilde{S}, j_\phi)$ is big and nef. Two marked M-polarized K3 surfaces

(S,ϕ) and (S',ϕ') are called *isomorphic* if there exists an isomorphism $f\colon S \to S'$ such that $\phi' = \phi \circ f^*$.

Denote by $F_{M,\mathrm{m}}$ the fine moduli space of marked M-polarized K3 surfaces (see [Dol96, §3]) and by $F^{\mathrm{bn}}_{M,\mathrm{m}}$ the subset of all isomorphism classes of big and nef marked M-polarized K3 surfaces.

Let $M^{\perp}_{L_{K3}}$ be the orthogonal complement of M in L_{K3} with respect to i_M. Let $\Omega_{L_{K3}}$ be the period domain defined in Section 3.4. Then,

$$\Omega(M) \coloneqq \{[x] \in \mathbb{P}(M^{\perp}_{L_{K3}} \otimes_{\mathbb{Z}} \mathbb{C});\ x^2 = 0, x.\overline{x} > 0\} \subseteq \Omega_{L_{K3}}$$

is the *period domain of big and nef M-polarized K3 surfaces*, a complex $\big(20 - \mathrm{rank}(M)\big)$-dimensional manifold with two connected components each of which is a bounded symmetric domain of type IV.

Let $(\widetilde{S},\phi)$ be a marked M-polarized K3 surface. We have a Hodge decomposition

$$H^2(\widetilde{S},\mathbb{C}) = H^2(\widetilde{S},\mathcal{O}_{\widetilde{S}}) \oplus H^1(\widetilde{S},\Omega^1_{\widetilde{S}}) \oplus H^0(\widetilde{S},\Omega^2_{\widetilde{S}}).$$

For a generator ω of the 1-dimensional $\mathbb{C}$-vector space $H^2(\widetilde{S},\mathcal{O}_{\widetilde{S}})$, we let $[\phi(\omega)] \coloneqq \phi(\omega) \bmod \mathbb{C}^* \in \mathbb{P}(M^{\perp}_{L_{K3}} \otimes_{\mathbb{Z}} \mathbb{C})$. We can show that $[\phi(\omega)] \in \Omega(M)$ and call $[\phi(\omega)]$ the *period point* of the marked M-polarized K3 surface $(\widetilde{S},\phi)$.

Let $O(L_{K3})$ be the automorphism group of L_{K3} and

$$O(L_{K3}, M) \coloneqq \{g \in O(L_{K3});\ g_{|M} = \mathrm{id}_{|M}\}$$

the subgroup of $O(L_{K3})$ fixing M point-wise. The group $O(L_{K3}, M)$ acts on $F_{M,\mathrm{m}}$ by sending a marked M-polarized K3 surface $(\widetilde{S},\phi)$ and an automorphism $\sigma \in O(L_{K3}, M)$ to $(\widetilde{S},\sigma\circ\phi)$ without changing the isomorphism class of the M-polarized K3 surface $(\widetilde{S}, j_\phi)$.

Let $O(M^{\perp}_{L_{K3}})$ be the automorphism group of $M^{\perp}_{L_{K3}}$ and O_M be the image of the injection $O(L_{K3}, M) \to O(M^{\perp}_{L_{K3}})$ obtained by restricting an element in $O(L_{K3}, M)$ to $M^{\perp}_{L_{K3}}$.

Proposition 9.1.1 ([Dol96, Proposition 3.3])**.** *O_M is an arithmetic subgroup of the indefinite orthogonal group $O\big(2, 19 - \mathrm{rank}(M)\big)$.*

The group O_M acts properly-discontinuously on $\Omega(M)$. Hence, $\Omega(M)/O_M$ is a complex algebraic variety of dimension $20 - \mathrm{rank}(M)$.

Theorem 9.1.2 ([Dol96, Remark 3.4], [HT15, 3.1])**.** *Assume that the embedding $i\colon M \hookrightarrow L_{K3}$ is unique up to an automorphism of L_{K3}.*

The elements of the quotient set

$$F^{\mathrm{bn}}_{M} \coloneqq F^{\mathrm{bn}}_{M,\mathrm{m}}/O(L_{K3}, M)$$

are the isomorphism classes of big and nef M-polarized K3 surfaces. Furthermore, we have a bijection

$$\rho\colon F^{\mathrm{bn}}_{M} \xrightarrow{\mathrm{bij}} \mathcal{F}^{\mathrm{bn}}_{M} \coloneqq \Omega(M)/O_M$$

defined by the period map.

We refer to $\mathcal{F}^{\mathrm{bn}}_{M}$ as in Theorem 9.1.2 as a coarse moduli space of big and nef M-polarized K3 surfaces.

9.2 Moduli spaces of K3 surfaces with a certain Picard group

We define in the next two Subsections 9.2.1 and 9.2.2 isomorphism classes of certain quasi-polarized K3 surfaces and certain lattice polarized K3 surfaces. In Subsection 9.2.3, we show that we have a correspondence between the two sets of isomorphism classes. In Subsection 9.2.4, we construct then the moduli space of these polarized K3 surfaces as a moduli space of the corresponding lattice polarized K3 surfaces.

For $\mathbf{T} \in \{\mathbf{A}_{i\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$, let the following be defined as in Table 6.1: The formal sum of ADE singularity types $\sigma(\mathbf{T})$, the positive integer $\mathrm{corank}_{\mathbf{T}}$, the weighted graph $\Gamma_{\sigma(\mathbf{T})}$ with associated lattice $\Lambda(\Gamma_{\sigma(\mathbf{T})})$, and the linear combination $h_{\mathbf{T}} \in \Lambda(\Gamma_{\sigma(\mathbf{T})})$ of the vertices of $\Gamma_{\sigma(\mathbf{T})}$.

Let

$$\big((a_1, \dots, a_n), (d_4, \dots, d_m), (e_6, e_7, e_8)\big) \in (\mathbb{Z}_{\geq 0})^n \times (\mathbb{Z}_{\geq 0})^{m-3} \times (\mathbb{Z}_{\geq 0})^3.$$

9.2.1 Isomorphism classes of certain quasi-polarized K3 surfaces of degree 6

Let $(\widetilde{S}, L_{\mathbf{T}})$ be a polarized K3 surface of degree 6 such that $\varphi_{L_{\mathbf{T}}} \colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image. By Proposition 3.3.4, $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ is a complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$.

Let

$$\mathbf{G} \coloneqq \sum_{i=1}^{n} a_i \mathbf{A}_i + \sum_{j=4}^{m} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k$$

be a formal sum of ADE singularity types.

Definition 9.2.1. Let $K^{\circ}_{\sigma(\mathbf{T}),\mathbf{G}}$ be the set of all isomorphism classes of quasi-polarized K3 surfaces $(\widetilde{S}, L_{\mathbf{T}})$ of degree 6 such that

1. $\varphi_{L_{\mathbf{T}}} \colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image
2. $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ is contained in a quadric $Q \subseteq \mathbb{P}^4$ of $\mathrm{corank}(Q) = \mathrm{corank}_{\mathbf{T}}$ such that
    a) the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ lying on $\mathrm{Sing}(Q)$ correspond to $\sigma(\mathbf{T})$
    b) the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ not lying on $\mathrm{Sing}(Q)$ correspond to $\mathbf{G}$.

9.2.2 Isomorphism classes of certain lattice polarized K3 surfaces

For $\big((a_1, \dots, a_n), (d_4, \dots, d_m), (e_6, e_7, e_8)\big) \in {\mathbb{Z}_{\geq 0}}^n \times {\mathbb{Z}_{\geq 0}}^{m-3} \times {\mathbb{Z}_{\geq 0}}^3$, let

$$\mathbf{G} \coloneqq \sum_{i=1}^{n} a_i \mathbf{A}_i + \sum_{j=4}^{m} d_j \mathbf{D}_j + \sum_{k=6}^{8} e_k \mathbf{E}_k$$

be a formal sum of ADE singularity types and

$$\Gamma_{\mathbf{G}} \coloneqq \sum_{i=1}^{n} a_i \mathcal{A}_i + \sum_{j=4}^{m} d_j \mathcal{D}_j + \sum_{k=6}^{8} e_k \mathcal{E}_k$$

a Dynkin diagram with connected components $\mathcal{A}_i$, $\mathcal{D}_j$, and $\mathcal{E}_k$. Let $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \coloneqq \Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})$ be the associated lattice such that we have an embedding (not necessarily primitive or unique)

$$i\colon \Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}.$$

Let

$$\mathrm{Sat}_{L_{K3}}(i) \subseteq L_{K3}$$

be the saturation of $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}$ in L_{K3} with respect to i. Then, $L_{K3}/\mathrm{Sat}_{L_{K3}}(i)$ is torsion-free by definition of the saturation. Hence, the inclusion defines a primitive embedding

$$\iota\colon \mathrm{Sat}_{L_{K3}}(i) \hookrightarrow L_{K3}.$$

Definition 9.2.2. Let $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ be the set of all isomorphism classes of $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surfaces $(\widetilde{S}, j)$ such that for $L_{\mathbf{T}} \coloneqq j\big(i(h_{\mathbf{T}})\big)$ we have

1. for all $E \in \mathrm{Pic}(\widetilde{S})$ with $L_{\mathbf{T}}.E = 0$ and $E^2 = -2$, we have $E \in j\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big)$
2. there exists no $E \in \mathrm{Pic}(\widetilde{S})$ such that $L_{\mathbf{T}}.E = 1$ and $E^2 = 0$
3. there exists no $E \in \mathrm{Pic}(\widetilde{S})$ such that $L_{\mathbf{T}}.E = 2$ and $E^2 = 0$.

9.2.3 Correspondence between isomorphism classes of certain quasi-polarized and lattice polarized K3 surfaces

We keep the notation and definitions made previously in Subsection 9.2 and will make in the following furthermore the assumption:

The embedding $i\colon \Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}$ defined in Subsection 9.2.2 is unique up to an automorphism of L_{K3}. (9.1)

Such lattices $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}$ exist. Indeed, in Theorem 8.1.1 we determined 1607 ADE lattices $\Lambda(\Gamma_{\mathbf{G}})$ such that we have a primitive embedding $\Lambda_{\sigma(\mathbf{A}_1),\Gamma} \coloneqq \langle 6 \rangle \oplus \Lambda \hookrightarrow L_{K3}$ which is unique up to an automorphism of L_{K3}.

By assumption (9.1), we have then a correspondence between the sets of isomorphism classes in Definition 9.2.1 and 9.2.2 in the last two subsections:

Lemma 9.2.3. *We have a natural bijection* $K^{\circ}_{\sigma(\mathbf{T}),\mathbf{G}} \xrightarrow{\text{bij}} F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$.

Proof. We claim that a bijection $K^{\circ}_{\sigma(\mathbf{T}),\mathbf{G}} \to F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is defined by $\Sigma\colon [(\widetilde{S}, L_{\mathbf{T}})] \mapsto [(\widetilde{S}, j_{\phi})]$ for a marking $\phi\colon H^2(\widetilde{S}, \mathbb{Z}) \to L_{K3}$ with $\phi(L_{\mathbf{T}}) = i(h_{\mathbf{T}})$ with $h_{\mathbf{T}}$ as in Table 6.1, where $[(\widetilde{S}, L_{\mathbf{T}})]$ is the isomorphism class of the quasi-polarized K3 surface $(\widetilde{S}, L_{\mathbf{T}})$ and $[(\widetilde{S}, j_{\phi})]$ the isomorphism class of the $\mathrm{Sat}_{L_{K3}}(i)$-lattice polarized K3 surface $(\widetilde{S}, j_{\phi})$ with $j_{\phi} \coloneqq \phi^{-1}{}_{|\mathrm{Sat}_{L_{K3}}(i)}\colon \mathrm{Sat}_{L_{K3}}(i) \hookrightarrow \mathrm{Pic}(\widetilde{S})$.

We show that the map Σ is well-defined:

We prove that the lattice $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}$ is contained in $\mathrm{Pic}(\widetilde{S})$. Indeed, in the proof of (2) $\Rightarrow$ (3) in Main Theorem 1 we showed that for a specific hyperplane section $C_{\mathbf{T}}$ of $S \coloneqq \varphi_{L_{\mathbf{T}}}(\widetilde{S}) \subseteq \mathbb{P}^4$ the pull-back $\varphi^{*}_{L_{\mathbf{T}}}(C_{\mathbf{T}}) \in \mathrm{Div}(\widetilde{S})$ is the linear combination of curves in $\mathrm{Div}(\widetilde{S})$ such that the weighted graph associated to these curves is $\Gamma_{\sigma(\mathbf{T})}$ as in Table 6.1. Furthermore, $L_{\mathbf{T}}$ is the line bundle on $\widetilde{S}$ associated to $\varphi^*_{L_{\mathbf{T}}}(C_{\mathbf{T}})$. Let $\Lambda(\Gamma_{\sigma(\mathbf{T})})$ be the lattice in $\mathrm{Pic}(\widetilde{S})$ associated to $\Gamma_{\sigma(\mathbf{T})}$. Further, we showed that the weighted graph associated to the exceptional divisor in $\widetilde{S}$ of the minimal resolution of all singularities corresponding to $\mathbf{G}$ is the graph $\Gamma_{\mathbf{G}}$ and spans the lattice $\Lambda(\Gamma_{\mathbf{G}})$ in $\mathrm{Pic}(\widetilde{S})$. Hence, the corresponding lattice $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \coloneqq \Lambda(\Gamma_{\sigma(\mathbf{T})}) \oplus \Lambda(\Gamma_{\mathbf{G}})$ is contained in $\mathrm{Pic}(\widetilde{S})$.

The marking $\phi\colon H^2(\widetilde{S},\mathbb{Z}) \to L_{K3}$ with $\phi(L_{\mathbf{T}}) = i(h_{\mathbf{T}})$ restricts to an embedding

$$\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}}\colon \Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}$$

and the inclusion defines naturally a primitive embedding

$$\mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}}) \hookrightarrow L_{K3}$$

of the saturation of $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}$ into L_{K3} with respect to $\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}}$. We prove that

$$t_\phi \coloneqq \phi^{-1}_{\mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})}\colon \mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}}) \hookrightarrow \mathrm{Pic}(\widetilde{S}) \tag{9.2}$$

defines a primitive embedding. Indeed, let $x \in \mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})$, i.e. $x \in L_{K3}$ and there is $n_x \geq 1$ such that $n_x x \in \phi(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$. Since $\phi(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}) \subseteq \phi\big(\mathrm{Pic}(\widetilde{S})\big)$, we obtain $n_x\phi^{-1}(x) \in \mathrm{Pic}(\widetilde{S})$. However, $H^2(\widetilde{S},\mathbb{Z})/\mathrm{Pic}(\widetilde{S})$ is torsion-free and hence $t_\phi(x) = \phi^{-1}(x) \in \mathrm{Pic}(\widetilde{S})$. Therefore, the map is well-defined. Further, the embedding is primitive. Indeed, let $x \in \mathrm{Pic}(\widetilde{S})$ such that for $n_x \geq 1$, we have $n_x x \in t_\phi\big(\mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})\big)$, i.e. $n_x\phi(x) \in \mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})$. However, $\phi(x) \in L_{K3}$ and $L_{K3}/\mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})$ is torsion-free so $\phi(x) \in \mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})$, i.e. $x \in t_\phi\big(\mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}})\big)$.

By assumption (9.1), the embedding $i\colon \Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}$ is unique up to an automorphism of L_{K3}. Hence, $\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}} = \lambda \circ i$ for an automorphism λ of L_{K3} inducing an isomorphism

$$\lambda_{|\mathrm{Sat}_{L_{K3}}(i)}\colon \mathrm{Sat}_{L_{K3}}(i) \to \mathrm{Sat}_{L_{K3}}(\phi_{|\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}}).$$

Therefore, we have a primitive embedding

$$j_\phi = t_\phi \circ \lambda_{|\mathrm{Sat}_{L_{K3}}(i)}\colon \mathrm{Sat}_{L_{K3}}(i) \hookrightarrow \mathrm{Pic}(\widetilde{S}).$$

Consequently, $(\widetilde{S}, j_\phi)$ is a $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surface and the isomorphism class of $(\widetilde{S}, j_\phi)$ is independent of the choice of the marking ϕ.

We showed in (2) $\Rightarrow$ (3) in Main Theorem 1 that 1.-3. in Definition 9.2.2 hold.

In conclusion, the isomorphism class $[(\widetilde{S}, j_\phi)]$ of $\big(\widetilde{S}, j_\phi\big)$ is contained in $F^\circ_{\mathrm{Sat}_{L_{K3}}(i)}$, i.e. the map Σ is well-defined.

We claim that $\Theta\colon F^\circ_{\mathrm{Sat}_{L_{K3}}(i)} \to K^\circ_{\sigma(\mathbf{T}),\mathbf{G}}$, $[(\widetilde{S}, j)] \mapsto [\Big(\widetilde{S}, j\big(i(h_{\mathbf{T}})\big)\Big)]$ with $h_{\mathbf{T}} \in \Lambda(\Gamma_{\sigma(\mathbf{T})})$ as in Table 6.1, is inverse to Σ.

We show that Θ is well-defined:

Let $(\widetilde{S}, j)$ be an element in the isomorphism class $[(\widetilde{S}, j)] \in F^\circ_{\mathrm{Sat}_{L_{K3}}(i)}$. We have a primitive embedding $j\colon \mathrm{Sat}_{L_{K3}}(i) \hookrightarrow \mathrm{Pic}(\widetilde{S})$ and items 1.-3. in Definition 9.2.2 hold.

Let $L_{\mathbf{T}} \coloneqq j\big(i(h_{\mathbf{T}})\big)$. Note that for an effective Hodge isometry $\alpha\colon H^2(\widetilde{S},\mathbb{Z}) \to H^2(\widetilde{S},\mathbb{Z})$ the $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surfaces $(\widetilde{S}, j)$ and $(\widetilde{S}, \alpha \circ j)$ are isomorphic. We claim that we can choose α such that $\alpha(L_{\mathbf{T}}) \in \mathrm{Pic}(\widetilde{S})$ is nef. Indeed, by replacing j by $-j$ if necessary, we can assume that $L_{\mathbf{T}}$ is contained in the positive cone $\mathcal{C}_{\widetilde{S}}$. Then, by Proposition 3.2.3, we have (-2)-curves $C_1, \dots, C_n \in \mathrm{Pic}(\widetilde{S})$ such that the image $(s_{C_1} \circ \dots \circ s_{C_n})(L_{\mathbf{T}})$ of $L_{\mathbf{T}}$ under the Picard-Lefschetz reflection $s_{C_1} \circ \dots \circ s_{C_n}$ is nef. In conclusion, taking $\alpha \coloneqq s_{C_1} \circ \dots \circ s_{C_n} \circ (\pm \mathrm{id})$, we can assume that $L_{\mathbf{T}}$ is nef.

Since items 2. and 3. in Definition 9.2.2 hold, there exists no element $E \in \mathrm{Pic}(\widetilde{S})$ with $E^2 = 0$ and $L_{\mathbf{T}}.E \in \{1, 2\}$. Therefore, Proposition 3.2.6 implies that the map $\varphi_{L_{\mathbf{T}}} : \widetilde{S} \to \mathbb{P}^4$ is birational onto its image. By Proposition 3.3.4, the image $S \coloneqq \varphi_{L_{\mathbf{T}}}(\widetilde{S}) \subseteq \mathbb{P}^4$ of $\widetilde{S}$ under $\varphi_{L_{\mathbf{T}}}$ is a complete $(2,3)$-intersection in $\mathbb{P}^4$.

Let $M_{\mathbf{T}}$ be the $\mathbb{Z}$-module generated by the root system $R_{L_{\mathbf{T}}} \coloneqq \{C \in \mathrm{Pic}(\widetilde{S});\, C^2 = -2, L_{\mathbf{T}}.C = 0\}$. We claim that $M_{\mathbf{T}} = j\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big)$. By definition of $L_{\mathbf{T}}$, we have $j\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big) \subseteq M_{\mathbf{T}}$. Further, since $[(\widetilde{S}, j)]$ satisfies item 1. in Definition 9.2.2, there exists no C in $\mathrm{Pic}(\widetilde{S})$ such that $C^2 = -2$, $C.L_{\mathbf{T}} = 0$, and $C \notin j\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big)$. Hence, $j\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big) = M_{\mathbf{T}}$. By Corollary 3.3.5, S has singularities of type $\sigma(\mathbf{T}) + \mathbf{G}$ corresponding to the Dynkin diagram $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}} = \Lambda(\Gamma_{\sigma(\mathbf{T})}) + \Lambda(\Gamma_{\mathbf{G}})$.

Following the proof of (2) $\Rightarrow$ (3) in Main Theorem 1, we see that S is contained in a quadric Q of $\mathrm{corank}(Q) = \mathrm{corank}_{\mathbf{T}}$ in $\mathbb{P}^4$ such that all singularities of S on the singular locus of Q are of type $\sigma(\mathbf{T})$ and all other singularities of S are of type $\mathbf{G}$.

We show that Σ and Θ are inverse to each other:

Let $[(\widetilde{S}, L_{\mathbf{T}})] \in K^{\circ}_{\sigma(\mathbf{T}),\mathbf{G}}$. We have $\Sigma([(\widetilde{S}, L_{\mathbf{T}})]) = [(\widetilde{S}, j_\phi)]$ for a marking ϕ of $\widetilde{S}$ such that $\phi(L_{\mathbf{T}}) = i(h_{\mathbf{T}})$. Then, $\Theta([(\widetilde{S}, j_\phi)]) = [\Big(\widetilde{S}, j_\phi\big(i(h_{\mathbf{T}})\big)\Big)] = [(\widetilde{S}, L_{\mathbf{T}})]$. Therefore, $\Theta \circ \Sigma = \mathrm{id}_{K^{\circ}_{\sigma(\mathbf{T}),\mathbf{G}}}$.

Let $[(\widetilde{S}, j)] \in F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$. We have $\Theta([(\widetilde{S}, j)]) = [\Big(\widetilde{S}, j\big(i(h_{\mathbf{T}})\big)\Big)]$. Then, for a marking ϕ of $\widetilde{S}$ such that $\phi\Big(j\big(i(h_{\mathbf{T}})\big)\Big) = i(h_{\mathbf{T}})$, we have $\Sigma([\Big(\widetilde{S}, j\big(i(h_{\mathbf{T}})\big)\Big)]) = [(\widetilde{S}, j_\phi)]$. Since the embedding $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}$ is uniquely determined up to an automorphism of L_{K3}, we have $[(\widetilde{S}, j)] = [(\widetilde{S}, j_\phi)]$. Therefore, also $\Sigma \circ \Theta = \mathrm{id}_{F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}}$. $\square$

9.2.4 Moduli space of certain polarized K3 surfaces as the moduli space of certain lattice polarized K3 surfaces

We keep the notation and assumptions made at the beginning of Subsection 9.2 and in Subsection 9.2.2. Let

$$\begin{aligned}
\Delta_n &\coloneqq \{x \in L_{K3} \setminus i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}});\quad i(h_{\mathbf{T}}).x = 0, x^2 = -2\}\\
\Delta_u &\coloneqq \{x \in L_{K3};\quad i(h_{\mathbf{T}}).x = 1, x^2 = 0\}\\
\Delta_h &\coloneqq \{x \in L_{K3};\quad i(h_{\mathbf{T}}).x = 2, x^2 = 0\}.
\end{aligned}$$

Remark 9.2.4. The indices n, u, and h should remind us of *n*odal, *u*nigonal, and *h*yperelliptic classes in the Picard group of a K3 surface, respectively.

For $\varepsilon \in \Delta_n, \Delta_u$, or Δ_h, let

$$\varepsilon^\perp \coloneqq \{x \in (\mathrm{Sat}_{L_{K3}}(i))^\perp_{L_{K3}} \otimes_\mathbb{Z} \mathbb{C};\, \varepsilon.x = 0\}$$

be the orthogonal complement of ε in L_{K3} and

$$H_{\Delta_n} \coloneqq \bigcup_{\varepsilon\in\Delta_n} \varepsilon^\perp, \qquad H_{\Delta_u} \coloneqq \bigcup_{\varepsilon\in\Delta_u} \varepsilon^\perp, \qquad H_{\Delta_h} \coloneqq \bigcup_{\varepsilon\in\Delta_h} \varepsilon^\perp.$$

We define then the following subset of the period domain $\Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$:

$$\Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)^\circ \coloneqq \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big) \setminus \Big(\big(H_{\Delta_n} \cup H_{\Delta_u} \cup H_{\Delta_h}\big) \cap \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)\Big).$$

We note that $\big(H_{\Delta_n} \cup H_{\Delta_u} \cup H_{\Delta_h}\big) \cap \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$ is a countable union of hyperplanes in $\Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$. However, we claim the number of the $O_{\mathrm{Sat}_{L_{K3}}(i)}$-orbits of the hyperplanes in $\big(H_{\Delta_n} \cup H_{\Delta_u} \cup H_{\Delta_h}\big) \cap \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$ is finite. Indeed, by Eichler's criterion (see [GHS13, Lemma 7.5]), there are only finitely many $O\big(L_{K3}, \mathrm{Sat}_{L_{K3}}(i)\big)$-orbits of elements with a fixed length in L_{K3}. Since $O\big(L_{K3}, \mathrm{Sat}_{L_{K3}}(i)\big)$ and $O_{\mathrm{Sat}_{L_{K3}}(i)}$ are isomorphic, we have consequently only finitely many $O_{\mathrm{Sat}_{L_{K3}}(i)}$-orbits of hyperplanes $\varepsilon^\perp$ in $\big(\mathrm{Sat}_{L_{K3}}(i)\big)^\perp_{L_{K3}} \otimes_\mathbb{Z} \mathbb{C}$ with $\varepsilon \in \Delta_n \cup \Delta_u \cup \Delta_h$ having a fixed length.

Consequently, $O_{\mathrm{Sat}_{L_{K3}}(i)} \setminus \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)^\circ$ is the complement of the finitely many orbits of hyperplanes $\varepsilon^\perp$ in the moduli space $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)} \coloneqq O_{\mathrm{Sat}_{L_{K3}}(i)} \setminus \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$ of big and nef $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surfaces constructed in Theorem 9.1.2 and hence is in particular a quasi-projective variety, i.e.

$$\mathcal{F}^\circ_{\mathrm{Sat}_{L_{K3}}(i)} \coloneqq O_{\mathrm{Sat}_{L_{K3}}(i)} \setminus \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)^\circ$$

is an open subvariety of $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$.

Proposition 9.2.5. *$\mathcal{F}^\circ_{\mathrm{Sat}_{L_{K3}}(i)}$ is a coarse moduli space of all quasi-polarized K3 surface $(\widetilde{S}, L_{\mathbf{T}})$ of degree 6 such that:*

1. *$\varphi_{L_{\mathbf{T}}} \colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image*
2. *$\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ is contained in a quadric $Q \subseteq \mathbb{P}^4$ of $\mathrm{corank}(Q) = \mathrm{corank}_{\mathbf{T}}$ such that*
   a) the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ lying on $\mathrm{Sing}(Q)$ are of type $\sigma(\mathbf{T})$
   b) the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ not lying on $\mathrm{Sing}(Q)$ correspond to $\mathbf{G}$,

i.e. with Definition 9.2.1, we have a bijection

$$K^\circ_{\sigma(\mathbf{T}),\mathbf{G}} \xrightarrow{\text{bij}} \mathcal{F}^\circ_{\mathrm{Sat}_{L_{K3}}(i)}. \tag{9.3}$$

Proof. By Lemma 9.2.3, we have a bijection

$$K^\circ_{\sigma(\mathbf{T}),\mathbf{G}} \xrightarrow{\text{bij}} F^\circ_{\mathrm{Sat}_{L_{K3}}(i)}, \tag{9.4}$$

where $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is the set of isomorphism classes of $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surfaces in Definition 9.2.2.

Let $(\widetilde{S}, j)$ be a $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surface whose class is contained in $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$. We saw in the proof of (3) $\Rightarrow$ (2) in Main Theorem 1 that for a marking $\phi\colon H^2(\widetilde{S}, \mathbb{Z}) \to L_{K3}$, the line bundle $L_{\mathbf{T}} \coloneqq \phi^{-1}\big(i(h_{\mathbf{T}})\big) \in \mathrm{Pic}(\widetilde{S})$ with $h_{\mathbf{T}} \in \Lambda_{\sigma(\mathbf{T}),\mathbf{G}}$ as in Table 6.1, is big and nef. Hence, $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is a subset of the set $F^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$ of all big and nef $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surfaces introduced in Subsection 9.1.2.

We now show that the bijection $\rho\colon F^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)} \xrightarrow{\text{bij}} \mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$ in Theorem 9.1.2, given by the period map ρ, descends to a bijection:

$$\rho^{\circ} \coloneqq \rho_{|F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}} \colon F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)} \xrightarrow{\text{bij}} \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}. \tag{9.5}$$

We prove firstly that $\rho^{\circ}(F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}) \subseteq \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$.

Indeed, let $(\widetilde{S}, j)$ be a $\mathrm{Sat}_{L_{K3}}(i)$-polarized K3 surface in $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$. Let ϕ be a marking for $\widetilde{S}$ such that $j_{\phi} \coloneqq \phi^{-1}_{|\mathrm{Sat}_{L_{K3}}(i)} = j$ and hence $j_{\phi}\big(\mathrm{Sat}_{L_{K3}}(i)\big) = j\big(\mathrm{Sat}_{L_{K3}}(i)\big) \subseteq \mathrm{Pic}(\widetilde{S})$ (note that such a ϕ actually exists since the embedding $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}$ is unique up to an automorphism of L_{K3} by assumption (9.1), see [Dol96, p. 2606]). Let $\omega_{\widetilde{S}}$ be the generator of the 1-dimensional $\mathbb{C}$-vector space $H^0(\widetilde{S}, \Omega^2_{\widetilde{S}})$. Let $[\phi(\omega_{\widetilde{S}})] \in \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$ be the period point of the marked $\mathrm{Sat}_{L_{K3}}(i)$-polarized $K3$ surface $(\widetilde{S}, \phi)$.

We have to show that $[\phi(\omega_{\widetilde{S}})] \notin H_{\Delta_n} \cup H_{\Delta_u} \cup H_{\Delta_h}$:

Indeed, if $[\phi(\omega_{\widetilde{S}})] \in H_{\Delta_n}$, we have an $\varepsilon \in \Delta_n$ such that $\varepsilon.[\phi(\omega_{\widetilde{S}})] = 0$, i.e. $E \coloneqq \phi^{-1}(\varepsilon) \in \mathrm{Pic}(\widetilde{S})$. By definition, $\varepsilon \in L_{K3} \setminus i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$, $\varepsilon^2 = -2$ and $\varepsilon.i(h_{\mathbf{T}}) = 0$. Therefore, $E \in \mathrm{Pic}(\widetilde{S}) \setminus j_{\phi}\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big)$ with $E^2 = -2$ and $E.L_{\mathbf{T}} = 0$. Since the isomorphism class of $\widetilde{S}$ is contained in $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ and hence satisfies condition 1. above, such an E and therefore such an ε cannot exist.

Likewise, if $[\phi(\omega_{\widetilde{S}})] \in H_{\Delta_u}$ (or $[\phi(\omega_{\widetilde{S}})] \in H_{\Delta_h}$) we have an $\varepsilon \in \Delta_u$ (or $\varepsilon \in \Delta_h$) such that $\varepsilon.[\phi(\omega_{\widetilde{S}})] = 0$, i.e. $E \coloneqq \phi^{-1}(\varepsilon) \in \mathrm{Pic}(\widetilde{S})$. By definition, $\varepsilon \in L_{K3}$, $\varepsilon^2 = 0$, and $\varepsilon.i(h_{\mathbf{T}}) = 1$ (or $\varepsilon.i(h_{\mathbf{T}}) = 2$). Therefore, $E \in \mathrm{Pic}(\widetilde{S})$ with $E^2 = 0$ and $E.L_{\mathbf{T}} = 1$ (or $E.L_{\mathbf{T}} = 2$). Again, since the isomorphism class of $\widetilde{S}$ is contained in $F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ and therefore satisfies conditions 2. (and 3.) above, such an E and therefore such an ε cannot exist.

Consequently, $[\phi(\omega_{\widetilde{S}})] \in \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)^{\circ}$.

Moreover, two markings as above differ by an element in the group of automorphisms $O_{\mathrm{Sat}_{L_{K3}}(i)}$.

In conclusion, we obtain $\rho^{\circ}(F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}) \subseteq O_{\mathrm{Sat}_{L_{K3}}(i)} \setminus \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)^{\circ} = \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$.

We prove secondly that ρ° is surjective:

Indeed, for $x \in \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$, we have by the surjectivity of the period map a K3 surface $\widetilde{S}$ and a marking ψ for $\widetilde{S}$ such that x is the period point of the marked K3 surface $(\widetilde{S}, \psi)$.

Since $x \in \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$, we have $x.\mathrm{Sat}_{L_{K3}}(i) = 0$. Therefore, $\psi^{-1}\big(\mathrm{Sat}_{L_{K3}}(i)\big) \subseteq \mathrm{Pic}(\widetilde{S})$. Consequently, $j_\psi := \psi^{-1}{}_{|\mathrm{Sat}_{L_{K3}}(i)} : \mathrm{Sat}_{L_{K3}}(i) \hookrightarrow \mathrm{Pic}(\widetilde{S})$ defines a primitive embedding.

We claim that $(\widetilde{S}, j_\psi) \in F^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$:

Indeed, assume that we have $E \in \mathrm{Pic}(\widetilde{S}) \setminus j_\psi\big(i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})\big)$ with $j_\psi\big(i(h_{\mathbf{T}})\big).E = 0$ and $E^2 = -2$. Then, $\varepsilon := \psi(E) \in L_{K3} \setminus i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$, $i(h_{\mathbf{T}}).\varepsilon = 0$, and $\varepsilon^2 = -2$ in contradiction to the fact that $x \notin H_n$, i.e. there exists no such ε.

Likewise, assume that we have $E \in \mathrm{Pic}(\widetilde{S})$ with $E^2 = 0$ and $j_\psi\big(i(h_{\mathbf{T}})\big).E = 1$ (or $j_\psi\big(i(h_{\mathbf{T}})\big).E = 2$). Then, $\varepsilon := \psi(E) \in L_{K3} \setminus i(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$, $\varepsilon^2 = 0$, and $i(h_{\mathbf{T}}).\varepsilon = 1$ (or $i(h_{\mathbf{T}}).\varepsilon = 2$), in contradiction to the fact that $x \notin H_h$ (or $x \notin H_u$), i.e. there exists no such ε.

In conclusion, ρ° is bijective.

By (9.4) and (9.5), we have a bijection $K^{\circ}_{\sigma(\mathbf{T}),\mathbf{G}} \xrightarrow{\text{bij}} \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$. This concludes the proof. $\square$

Lemma 9.2.6. *The quasi-projective variety* $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ *has dimension* $20 - \mathrm{rank}(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$.

Proof. The period domain $\Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$ has dimension $20 - \mathrm{rank}\big(\mathrm{Sat}_{L_{K3}}(i)\big) = 20 - \mathrm{rank}(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$. As $O_{\mathrm{Sat}_{L_{K3}}(i)}$ acts properly-discontinuously on $\Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$, this implies that the quotient $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)} = \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)/O_{\mathrm{Sat}_{L_{K3}}(i)}$ has dimension $20 - \mathrm{rank}(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$ and since $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is an open subvariety of $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$, it has dimension $20 - \mathrm{rank}(\Lambda_{\sigma(\mathbf{T}),\mathbf{G}})$, as well. $\square$

Remark 9.2.7. Proposition 9.2.5 proves in particular implication (3) $\Rightarrow$ (2) in Main Theorem 1 in case we have a unique embedding $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}} \hookrightarrow L_{K3}$. Indeed, we showed that the points in the moduli space $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ parametrize in this case quasi-polarized K3 surfaces as in item 2. in Main Theorem 1.

9.3 Moduli spaces of cubic fourfolds with isolated ADE singularities

Let $\mathbf{G}$ be a finite formal sum of ADE singularity types.

We denote $M^{\mathbf{G}}$ the set of all isomorphism classes of cubic fourfolds having only singularities corresponding to $\mathbf{G}$.

The projective space $\mathbb{P}\big(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))\big) \cong \mathbb{P}^{55}$ parametrizes all cubic fourfolds. We denote by $[X]$ the point in $\mathbb{P}^{55}$ associated to a cubic fourfold $X \subseteq \mathbb{P}^5$.

For each $[X] \in \mathbb{P}^{55}$, fix a small open neighborhood $U([X]) \subseteq \mathbb{P}^{55}$ of $[X]$ such that all points in $U([X])$ correspond to cubic fourfolds whose singularities are adjacent to those of X (see Section 1.1 for the definition of adjacent).

Let

$$I_{\mathbf{G}} := \{[X] \in \mathbb{P}^{55};\ \mathrm{Sing}(X) = \mathbf{G}\}$$

be the set of all points in $\mathbb{P}^{55}$ associated to cubic fourfolds with singularities corresponding to $\mathbf{G}$. Denote

$$\Sigma_{\mathbf{G}} := \{\mathbf{G}' \text{ formal sum of } ADE \text{ singularity tpes; } \mathbf{G}' \text{ is adjacent to } \mathbf{G} \text{ and } \mathbf{G}' \neq \mathbf{G}\}$$

the set of all possible combinations of ADE singularity types which are adjacent but not equal to $\mathbf{G}$. Let

$$I'_{\mathbf{G}} := \bigcup_{\mathbf{G}' \in \Sigma_{\mathbf{G}}} \{[X] \in \mathbb{P}^{55};\ \mathrm{Sing}(X) = \mathbf{G}'\}$$

be the set of all points in $\mathbb{P}^{55}$ associated to cubic fourfolds with singularities adjacent but not equal to $\mathbf{G}$. Then, $\bigcup_{[X]\in I'_{\mathbf{G}}} U([X])$ is an open subset of $\mathbb{P}^{55}$ containing only points in $\mathbb{P}^{55}$ associated to cubic fourfolds whose singularities are adjacent but not equal to $\mathbf{G}$.

Hence, $\mathbb{P}^{55} \setminus \bigcup_{[X]\in I'_{\mathbf{G}}} U([X])$ is closed in $\mathbb{P}^{55}$. Likewise, $\bigcup_{[X]\in I_{\mathbf{G}}} U([X])$ is an open subset of $\mathbb{P}^{55}$ containing only points in $\mathbb{P}^{55}$ such that the singularities of the associated cubic fourfolds are adjacent to $\mathbf{G}$. Consequently,

$$\mathcal{U}^{\mathbf{G}} := \bigcup_{[X]\in I_{\mathbf{G}}} U([X]) \bigcap \Big(\mathbb{P}^{55} \setminus \bigcup_{[X]\in I'_{\mathbf{G}}} U([X])\Big) \subseteq \mathbb{P}^{55}$$

is locally closed in $\mathbb{P}^{55}$, i.e. a quasi-projective variety in $\mathbb{P}^{55}$ and contains only those points in $\mathbb{P}^{55}$ associated to cubic fourfolds with singularities corresponding exactly to $\mathbf{G}$.

Let $(x_0 : \ldots : x_5)$ be coordinates on $\mathbb{P}^5$. For an element g in the special linear group $\mathrm{SL}(6)$ and $[X] \in \mathbb{P}^{55}$ the class of a cubic fourfold $X :\ f(x_0, \ldots, x_5) = 0 \subseteq \mathbb{P}^5$, defined by a homogeneous cubic polynomial f, we let

$$g([X]) :\ f\big(g(x_0, \ldots, x_5)\big) = 0 \subseteq \mathbb{P}^5$$

and obtain hence an action of $\mathrm{SL}(6)$ on $\mathbb{P}^{55}$.

For the action of a reductive group G on a projective variety M together with a linearization of a line bundle over M for this group action, we consider the open subset $M^s \subseteq M$ of G-stable points of M in M (see [MFK94, Chap. 1, Definitions 1.4, 1.7] for the definitions). By Mumford's Geometric Invariant Theory (GIT), we have a quotient $M^s//G$ of M^s by the group G (see [MFK94, Chap. 1, Theorem 1.10]). The group $\mathrm{SL}(n)$ is reductive for a positive integer n. In the following, we will consider the above action of $\mathrm{SL}(6)$ on the projective space $\mathbb{P}^{55}$ together with the natural $\mathrm{SL}(6)$-linearization with respect to the hyperplane bundle $\mathcal{O}_{\mathbb{P}^{55}}(1)$. We have then:

Theorem 9.3.1 ([Laz09, Theorem 1.1])**.** *Let X be a cubic fourfold with only isolated singularities. Then, X is* $\mathrm{SL}(6)$*-stable if and only if X has at most ADE singularities.*

Corollary 9.3.2.

$$\mathcal{M}^{\mathbf{G}} := \mathcal{U}^{\mathbf{G}} /\!\!/ \mathrm{SL}(6)$$

is in the sense of GIT a coarse moduli space of cubic fourfolds with ADE singularities corresponding to $\mathbf{G}$*, i.e. we have a bijection*

$$M^{\mathbf{G}} \xrightarrow{\text{bij}} \mathcal{M}^{\mathbf{G}}.$$

Proof. By definition, all points in $\mathcal{U}^{\mathbf{G}} \subseteq \mathbb{P}^{55}$ parametrize cubic fourfolds with singularities corresponding to $\mathbf{G}$. By Theorem 9.3.1, all these points are stable with respect to the action of SL(6) on $\mathbb{P}^{55}$. Hence, we have a well-defined GIT quotient $\mathcal{U}^{\mathbf{G}} /\!\!/ \mathrm{SL}(6)$ which is a quasi-projective variety, see [Muk03, Corollary 5.15, Example 4.42]. □

Lemma 9.3.3. *Let $\tau := \sum_{p\in\mathbf{G}} \tau(p)$ be the sum of the Tjurina numbers of all singularities in $\mathbf{G}$. Assume that we have $\tau < 16$. Then, $\mathcal{M}^{\mathbf{G}}$ has dimension $20 - \tau$.*

Proof. Let $X_0 \subseteq \mathbb{P}^5$ be a cubic fourfold having only the singularities $p_{X_0,1}, \ldots, p_{X_0,n}$ with ADE types $\mathbf{T}_1, \ldots, \mathbf{T}_n$, respectively, such that $\mathbf{G} = \mathbf{T}_1 + \ldots + \mathbf{T}_n$. Let $U([X_0]) \subseteq \mathbb{P}^{55}$ be an arbitrarily small open neighborhood of $[X_0]$. Let

$$\mathcal{Y} := \{([X], x) \subseteq \mathbb{P}^{55} \times \mathbb{P}^5;\ X \text{ cubic fourfold, } x \in X\}$$

be the universal cubic fourfold. For $i = 1, \ldots, n$, we now construct a deformation of the germ $(X_0, p_{X_0,i})$. Indeed, for an arbitrarily small neighborhood $V(p_{X_0,i}) \subseteq \mathbb{P}^5$ of the singularity $p_{X_0,i}$ of X_0, let

$$\mathcal{Y}_{U([X_0]),i} := \mathcal{Y}_{|U([X_0])\times V(p_{X_0,i})}$$

be the restriction of $\mathcal{Y}$ to $U([X_0]) \times V(p_{X_0,i})$. Then,

$$d_i \colon \mathcal{Y}_{U([X_0]),i} \to U([X_0]),\ ([X], x) \mapsto [X] \tag{9.6}$$

is a deformation of the germ $(X_0, p_{X_0,i})$ over the base point $[X_0] \in U([X_0])$. On the other hand, by [GLS07, Chap. II, Corollary 1.17], we have a semi-universal deformation

$$u_{\mathbf{T}_i} \colon \mathcal{X}_{\mathbf{T}_i} \to \mathbb{C}^{\tau(\mathbf{T}_i)}$$

of the germ $(X_0, p_{X_0,i})$ over the base point $(0, \ldots, 0) \in \mathbb{C}^{\mu(\mathbf{T}_i)}$. Consequently, there exists a morphism

$$\kappa_i \colon U([X_0]) \to \mathbb{C}^{\tau(\mathbf{T}_i)}$$

such that we have a pull-back diagram

$$\begin{tikzcd}
\mathcal{Y}_{U([X_0]),i} \arrow[r, "s_i"] \arrow[d, "d_i"'] & \mathcal{X}_{\mathbf{T}_i} \arrow[d, "u_{\mathbf{T}_i}"] \\
U([X_0]) \arrow[r, "\kappa_i"] & \mathbb{C}^{\tau(\mathbf{T}_i)}
\end{tikzcd}$$

for some morphism s_i. We obtain a commutative diagram

$$\begin{tikzcd}
\prod_{i=1}^n \mathcal{Y}_{U([X_0]),i} \arrow[r, "\prod_{i=1}^n s_i"] \arrow[d, "\prod_{i=1}^n d_i"] & \prod_{i=1}^n \mathcal{X}_{\mathbf{T}_i} \arrow[d, "\prod_{i=1}^n u_{\mathbf{T}_i}"] \\
\prod_{i=1}^n U([X_0]) \arrow[r, "\prod_{i=1}^n \kappa_i"] & \prod_{i=1}^n \mathbb{C}^{\tau(\mathbf{T}_i)}.
\end{tikzcd}$$

For

$$j\colon \quad U([X_0]) \to \prod_{i=1}^{n} U([X_0]), \quad ([X],x) \mapsto \prod_{i=1}^{n}([X],x),$$

let

$$\kappa := \big(\prod_{i=1}^{n}\kappa_i\big)\circ j\colon \quad U([X_0]) \to \prod_{i=1}^{n}\mathbb{C}^{\tau(\mathbf{T}_i)}.$$

We recall

Theorem 9.3.4 ([dPW00, Theorem 1.1])**.** *Let X be a hypersurface of degree d in $\mathbb{P}^n$ with only isolated singularities. Let $\tau(X)$ be the global Tjurina number of the singularities of X. For $d = 3,4$ or $d \geq 5$ set $\delta = 16, 18$ or $\delta = 4(d-1)$, respectively. If $\tau(X) < \delta$, the family of degree d hypersurfaces induces a simultaneous versal deformation of all singularities on X.*

By Theorem 9.3.4, it follows that the morphism κ is a submersion, cf. [CGHL15, 3.4]. Hence, we have

$$\dim \kappa^{-1}(0) = \dim U([X_0]) - \dim(\prod_{i=1}^{n}\mathbb{C}^{\tau(\mathbf{T}_i)}) = 55-\tau.$$

Since fibres of the map $\prod_i^n u_{\mathbf{T}_i}$ over all points different from the central fibre $(0,\dots,0) \in \prod_{i=1}^n \mathbb{C}^{\tau(\mathbf{T}_i)}$ are singularities milder than $\mathbf{T}_1,\dots,\mathbf{T}_n$ and since the diagram commutes, the locus of all points in $U([X_0])$ having only singularities of type $\mathbf{G}$ is $\kappa^{-1}(0)$. Since $U([X_0])$ is an open subset in $\mathbb{P}^{55}$, this gives that the locus $\mathcal{U}^{\mathbf{G}}$ of all cubic fourfolds with ADE singularities of type $\mathbf{G}$ has dimension $55-\tau$. Therefore, the quotient $\mathcal{M}^{\mathbf{G}} = \mathcal{U}^{\mathbf{G}} /\!\!/ \mathrm{SL}(6)$ has dimension $\dim \mathcal{M}^{\mathbf{G}} = 55-\tau-(36-1) = 20-\tau$. $\square$

9.4 Main Theorem 3

In this section, we want to show that the moduli space of cubic fourfolds with a certain combination of ADE singularities constructed in Subsection 9.3 is isomorphic to the moduli space of certain quasi-polarized K3 surfaces constructed in Subsection 9.2.4. We keep the notation made in those subsections.

Let $\mathbf{T} \in \{\mathbf{A}_{n\geq 1}, \mathbf{D}_{j\geq 4}, \mathbf{E}_{8\geq k\geq 6}\}$ be an ADE singularity type. For a tuple of non-negative integers

$$\big((a_1,\dots,a_n),(d_4,\dots,d_m),(e_6,e_7,e_8)\big) \in (\mathbb{Z}_{\geq 0})^n \times (\mathbb{Z}_{\geq 0})^{m-4} \times \mathbb{Z}^3,$$

let

$$\mathbf{G} := \sum_{i=1}^{n} a_i\mathbf{A}_i + \sum_{j=4}^{m} d_j\mathbf{D}_j + \sum_{k=6}^{8} e_k\mathbf{E}_k$$

be a formal finite sum of ADE singularity types, and let

$$\Gamma_{\mathbf{G}} := \sum_{i=1}^{n} a_i\mathcal{A}_i + \sum_{j=4}^{m} d_j\mathcal{D}_j + \sum_{k=6}^{8} e_k\mathcal{E}_k$$

be a finite Dynkin graph such that condition (9.1) holds for the lattice $\Lambda_{\sigma(\mathbf{T}),\mathbf{G}}$.

Let $\mathcal{U}^{\mathbf{T}+\mathbf{G}}$ be the locally closed subspace of all cubic fourfolds in $\mathbb{P}\big(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))\big)$ with isolated *ADE* singularities of type $\mathbf{G}$ and a singularity of type $\mathbf{T}$ and

$$\mathcal{M}^{\mathbf{T}+\mathbf{G}} := \mathcal{U}^{\mathbf{T}+\mathbf{G}} /\!\!/ \mathrm{SL}(6)$$

the coarse moduli space in the sense of GIT of all cubic fourfolds with singularities corresponding to $\mathbf{T}+\mathbf{G}$ constructed in Subsection 9.3.

Let $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ be the moduli space constructed in Subsection 9.2.4 of all quasi-polarized K3 surfaces $(\widetilde{S}, L_{\mathbf{T}})$ with the property that $\varphi_{L_{\mathbf{T}}} \colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image and $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ is contained in a quadric $Q \subseteq \mathbb{P}^4$ of $\mathrm{corank}(Q) = \mathrm{corank}_{\mathbf{T}}$ such that firstly the singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ lying on the singular locus of Q are of type $\sigma(\mathbf{T})$ and secondly those singularities of $\varphi_{L_{\mathbf{T}}}(\widetilde{S})$ not lying on the singular locus of Q correspond to $\mathbf{G}$.

It is our goal in this subsection to prove the following Main Theorem 3.

Main Theorem 3. *We have an isomorphism of quasi-projective varieties*

$$\phi \colon \mathcal{M}^{\mathbf{T}+\mathbf{G}} \to \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}, \quad [X] \mapsto [(\widetilde{S_{p_X}}, \pi^* \mathcal{O}_{S_{p_X}}(1))],$$

where p_X is a singularity of ADE type $\mathbf{T}$ on a cubic fourfold X, S_{p_X} is the image of the union of all lines in X through p_X under the projection of $\mathbb{P}^5$ through p_X onto $\mathbb{P}^4$ as defined in Section 5.1, and $\pi \colon \widetilde{S_{p_X}} \to S_{p_X}$ is the minimal resolution of all singularities on S_{p_X}.

We want to show that in the situation of Main Theorem 3, the minimal model $\widetilde{S_p}$ for the surface S_p is up to isomorphism independent of the choice of a singularity p of type $\mathbf{T}$ on the cubic fourfold X. Before we can prove this, we need one technical preparatory result:

Lemma 9.4.1. *Let $X \subseteq \mathbb{P}^5$ be a cubic fourfold with only isolated ADE singularities and $l_0 \subseteq X$ a line through an ADE singularity p of X. Let $\overline{l\, l_0}$ be the plane in $\mathbb{P}^5$ spanned by l_0 and a general line l in X through p. Then, $\overline{l\, l_0}$ is not contained in X.*

Proof. Assume conversely that for a general line l in X through p the plane $\overline{l\, l_0}$ is contained in X. As in Section 5.1, let $\pi_p \colon \mathbb{P}^5 \dashrightarrow H \cong \mathbb{P}^4$ be the projection of $\mathbb{P}^5$ through p onto a hyperplane $H \subseteq \mathbb{P}^5$ with $p \notin H$, let F_p be the union of all lines in X through p, and let $S_p := \pi_p(F_p) \subseteq \mathbb{P}^4$. By Corollary 5.2.3, S_p has only isolated *ADE* singularities and the minimal model $\widetilde{S_p}$ of S_p is by Lemmas 5.1.2 and 4.2.2 a K3 surface. Since the plane $\overline{l\, l_0}$ is by assumption contained in X, it follows that F_p contains all lines in the plane $\overline{l\, l_0}$ through p and S_p contains the line $H \cap \overline{l\, l_0}$. Since l is general, we have a continuous family of distinct planes in X through p and hence also a continuous family of distinct lines in S_p. This implies that S_p is uniruled. Since $\widetilde{S_p}$ is birational to S_p, this gives that $\widetilde{S_p}$ is uniruled, as well, in contradiction to $\widetilde{S_p}$ being a K3 surface. Hence, the assumption must have been wrong and for a general line l in X through p the plane $\overline{l\, l_0}$ is not contained in X. $\square$

Proposition 9.4.2. *Let X be a cubic fourfold with only isolated ADE singularities and two singularities p_1 and p_2 both of the same ADE type. For $i = 1, 2$, let S_{p_i} be the image of the union F_{p_i} of all lines in X through p_i under the projection π_{p_i} from $\mathbb{P}^5$ through p_i onto $\mathbb{P}^4$ as defined in Section 5.1. Then, S_{p_1} and S_{p_2} are birational.*

Proof. Let l_1 be a general line in X passing through p_1. Let l_0 be the line containing both p_1 and p_2. Since p_1 and p_2 are double points, l_0 intersects p_1 and p_2 with multiplicity 2, hence l_0 intersects X with multiplicity 4. However, since X has degree 3, this means that l_0 must be contained in X. Let $\overline{l_0 l_1}$ be the plane spanned by l_0 and l_1. By Lemma 9.4.1, the plane $\overline{l_0 l_1}$ is not contained in X. Hence, $C = X \cap \overline{l_0 l_1}$ is a plane cubic curve. Since C contains the line l_1, the cubic curve is reducible. Since C contains even a second line, namely l_0, it must be the union of three lines l_0, l_1, and l_2. Since C is singular at p_2, the line l_2 must pass through p_2. Consequently, C is the union of the lines l_0, l_1, and l_2 such that l_0 and l_1 intersect in the singularity p_1 and l_0 and l_2 intersect in the singularity p_2. Hence, l_2 is contained in F_{p_2}. For $i = 1, 2$, now denote by $\mathbf{F}_{p_i}$ the Fano scheme of all lines in X through p_i and by $[l]$ the point in $\mathbf{F}_{p_i}$ corresponding to a line l in F_{p_i}. We now define a rational map

$$\psi\colon \mathbf{F}_{p_1} \dashrightarrow \mathbf{F}_{p_2}$$

with $\psi([l_1]) = [l_2]$. Exchanging p_1 by p_2 in the arguments above, we can define the rational map $\varphi\colon \mathbf{F}_{p_2} \dashrightarrow \mathbf{F}_{p_1}$ with $\varphi([l_2]) = [l_1]$ which is inverse to ψ. Hence, ψ is birational. Since $\mathbf{F}_{p_1}$ and $\mathbf{F}_{p_2}$ are birational to S_{p_1} and S_{p_2} via the projections π_{p_1} and π_{p_2}, respectively, S_{p_1} and S_{p_2} are consequently birational, as well. □

Now we are in the position to prove Main Theorem 3:

Proof of Main Theorem 3. We show firstly that ϕ is well-defined:

Let $[X] \in \mathcal{M}^{\mathbf{T}+\mathbf{G}}$ be the class of a cubic fourfold $X \subseteq \mathbb{P}^5$ with an ADE singularity p_X of type $\mathbf{T}$ and such that all other singularities of X correspond to $\mathbf{G}$.

Let $(x_0 : \ldots : x_5)$ be homogeneous coordinates on $\mathbb{P}^5$.

After a linear change of coordinates, we can assume that $p_X = (1:0:0:0:0:0) \in \mathbb{P}^5$ and then by Lemma 5.1.1

$$X \,:\, x_0 f_2(x_1, \ldots, x_5) + f_3(x_1, \ldots, x_5) = 0,$$

where f_2 and f_3 are homogeneous polynomials of degree 2 and 3 in $\mathbb{C}[x_0, \ldots, x_5]$, respectively. By Lemma 5.1.2, the projection S_{p_X} of the union of all lines in X through p_X onto $\mathbb{P}^4$ is a complete $(2,3)$-intersection in $\mathbb{P}^4$ given by

$$S_{p_X} \,:\, f_2(x_1, \ldots, x_5) = f_3(x_1, \ldots, x_5) = 0 \subseteq \mathbb{P}^4$$

and S_{p_X} is uniquely determined by p_X by Lemma 5.1.3.

Let $\pi_{p_X} : \mathrm{Bl}_{p_X} X \to X$ the blowing-up of X in p_X with exceptional divisor $E \subseteq \mathrm{Bl}_{p_X} X$. By Corollary 5.2.3, the singularities of $\mathrm{Bl}_{p_X} X$ and S_{p_X} are in one-to-one correspondence including the singularity types. More intrinsically, the singularities of $\mathrm{Bl}_{p_X} X$ on E correspond to the singularities of the quadric $Q :\, f_2(x_1, \ldots, x_5) = 0 \subseteq \mathbb{P}^4$ and are of type $\sigma(\mathbf{T})$ with $\sigma(\mathbf{T})$ as in Table 6.1 and the singularities on $\mathrm{Bl}_{p_X} X \setminus E$ correspond to the singularities of S_{p_X} not lying on the singular locus of Q and are of type $\mathbf{G}$.

Let

$$\pi\colon \widetilde{S_{p_X}} \to S_{p_X}$$

be the minimal resolution of all singularities on S_{p_X}. By Lemma 4.2.2, $\widetilde{S_p}$ is a K3 surface and the pull-back $L := \pi^* \mathcal{O}_{S_{p_X}}(1)$ by π of the hyperplane bundle on S_{p_X} to $\widetilde{S_{p_X}}$ has degree

6. Further, the morphism φ_L induced by the linear system $|L|$ is given by π, so φ_L is in particular birational. We have $\varphi_L(\widetilde{S_{p_X}}) = \pi(\widetilde{S_{p_X}}) = S_{p_X}$. Consequently, the isomorphism class $[(\widetilde{S_{p_X}}, L)]$ of the quasi-polarized K3 surface $(\widetilde{S_{p_X}}, L)$ is parametrized by a point in $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$.

Assume then that X has two singularities p_X and p'_X both of type $\mathbf{T}$. Then, S_{p_X} and $S_{p_X'}$ are birational by Proposition 9.4.2. Hence, $\widetilde{S_{p_X}}$ and $\widetilde{S_{p'_X}}$ are isomorphic. Consequently, $\big(\widetilde{S_{p_X}}, \pi^*\mathcal{O}_{S_{p_X}}(1)\big)$ and $\big(\widetilde{S_{p'_X}}, \pi^*\mathcal{O}_{S_{p'_X}}(1)\big)$ are isomorphic.

In conclusion, ϕ is well defined.

We define an inverse map to ϕ:

Let $(\widetilde{S}, L)$ be a quasi-polarized K3 surface of degree 6 such that $\varphi_L\colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image. By Proposition 3.3.4, $S := \varphi_L(\widetilde{S})$ is a complete $(2,3)$-intersection of a quadric Q and a cubic Y in $\mathbb{P}^4$. By Lemma 4.2.1, the quadric Q is uniquely determined up to isomorphism and the cubic Y is uniquely determined up to isomorphism and modulo those cubics containing the quadric. Assume that we have homogeneous coordinates $x_1, \dots, x_5$ on $\mathbb{P}^4$ such that up to isomorphism

$$Q:\ f_2(x_1,\dots,x_5) = 0 \text{ and } Y:\ f_3(x_1,\dots,x_5) + \lambda\, l(x_1,\dots,x_5) f_2(x_1,\dots,x_5) = 0 \subseteq \mathbb{P}^4,$$

where $\lambda \in \mathbb{C}$ and $l(x_1,\dots,x_5)$ is a linear polynomial. Then,

$$X:\ x_0 f_2(x_1,\dots,x_5) + \big(f_3(x_1,\dots,x_5) + \lambda\, l(x_1,\dots,x_5) f_2(x_1,\dots,x_5)\big) = 0 \subseteq \mathbb{P}^5$$

defines a cubic fourfold on X. Therefore, X is isomorphic to

$$x_0 f_2(x_1,\dots,x_5) + f_3(x_1,\dots,x_5) = 0 \subseteq \mathbb{P}^5$$

with respect to the linear coordinate transformation $x_0 \mapsto x_0 - \lambda\, l(x_1,\dots,x_5)$. Hence, we see that the isomorphism class $[X]$ of X does not depend on the choice of the cubic Y in which S is contained. Write $S = S(f_2, f_3)$ and $[X(f_2, f_3)]$ for the isomorphism class of X.

By assumption, the singularities of S lying on Q are of type $\sigma(\mathbf{T})$ and all other singularities of S correspond to $\mathbf{G}$. By Proposition 5.2.2, the singularity $(1:0:0:0:0:0) \in \mathbb{P}^5$ of X is of type $\mathbf{T}$ and all other singularities of X correspond to $\mathbf{G}$. Define then

$$\psi\colon \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)} \to \mathcal{M}^{\mathbf{T}+\mathbf{G}},\ [(\widetilde{S(f_2, f_3)}, L)] \mapsto [X(f_2, f_3)].$$

We check that ϕ and ψ are inverse to each other:

Indeed, let $\phi([X]) = \big(\widetilde{S_{p_X}}, \pi^*\mathcal{O}_{S_{p_X}}(1)\big) \in \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ and write $L := \pi^*\mathcal{O}_{S_{p_X}}(1)$, where $[X] \in \mathcal{M}^{\mathbf{T}+\mathbf{G}}$ is the isomorphism class of the cubic fourfold $X:\ x_0 f_2(x_1,\dots,x_5) + f_3(x_1,\dots,x_5) = 0$. The surface $S_{p_X} := \varphi_L(\widetilde{S_{p_X}})$ is then a complete $(2,3)$-intersection in $\mathbb{P}^4$. By Lemma 4.2.1, S_{p_X} lies on a unique quadric Q and a cubic Y uniquely determined modulo those cubics containing the quadric Q. Hence, $Q:\ f_2(x_1,\dots,x_5) = 0 \subseteq \mathbb{P}^4$ and $Y:\ f_3(x_1,\dots,x_5) + \lambda\, l(x_1,\dots,x_5) f_2(x_1,\dots,x_5) = 0 \subseteq \mathbb{P}^4$, where $\lambda \in \mathbb{C}$ and $l(x_1,\dots,x_5)$ is a linear polynomial. Then, $\psi\big((\widetilde{S_{p_X}}, L)\big)$ is the class of the cubic fourfold $\big(1 + \lambda l(x_1,\dots,x_5)\big) f_2(x_1,\dots,x_5) + f_3(x_1,\dots,x_5) = 0 \subseteq \mathbb{P}^5$ which is simply the isomorphism class $[X]$ of the cubic fourfold X and hence $\psi \circ \phi = \mathrm{id}_{\mathcal{M}^{\mathbf{T}+\mathbf{G}}}$.

On the other hand, let $(\widetilde{S}, L)$ be a quasi-polarized K3 surface of degree 6 such that $\varphi_L\colon \widetilde{S} \to \mathbb{P}^4$ is birational onto its image such that $S \coloneqq \varphi_L(\widetilde{S})$ is the complete $(2,3)$-intersection of the quadric $Q:\ f_2(x_1,\dots,x_5)$ and the cubic $Y:\ f_3(x_1,\dots,x_5)+\lambda\, l(x_1,\dots,x_5)f_2(x_1,\dots,x_5) = 0$ in $\mathbb{P}^4$, where $\lambda \in \mathbb{C}$ and $l(x_1,\dots,x_5)$ is a linear polynomial. We have $\psi\big((\widetilde{S}, L)\big) = [X]$, where $X:\ x_0 f_2(x_1,\dots,x_5) + f_3(x_1,\dots,x_5) = 0 \subseteq \mathbb{P}^5$. Then, $\phi([X])$ is the complete (2,3)-intersection $S:\ f_2(x_1,\dots,x_5) = f_3(x_1,\dots,x_5) = 0$. The minimal resolution of all singularities on S is then simply $\widetilde{S}$. Further, $L = \pi^*\big(\mathcal{O}_S(1)\big)$ so $\phi([X]) = (\widetilde{S}, L)$. Hence, $\phi \circ \psi = \mathrm{id}_{\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}}$.

Finally, the map ϕ is holomorphic since the period map is holomorphic. By Borel's Theorem [Bor72, Theorem 3.10], the defined map is then a morphism of quasi-projective varieties.

We show that ϕ is in fact an isomorphism:

Since the morphism ϕ is surjective, it induces an inclusion of the functions fields

$$\phi^*\colon K(\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}) \hookrightarrow K(\mathcal{M}^{\mathbf{T}+\mathbf{G}}).$$

Further, since ϕ is bijective, all fibers $\phi^{-1}(y)$ with $y \in \mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ of ϕ have cardinality one. By [Har92, Proposition 7.16], the degree $[K(\mathcal{M}^{\mathbf{T}+\mathbf{G}}) : K(\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)})]$ of the field extension equals then one. Hence, ϕ is birational. We note that the quasi-projective variety $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is normal by [Huy16, Chap. 6, Theorem 1.13]. By Zariski's Main Theorem in its original form [Mum99, Chap. III.9, p. 209], the morphism ϕ is then an open immersion. Since ϕ is surjective, it is hence even an isomorphism. $\square$

Corollary 9.4.3. *The isomorphism ϕ in Main Theorem 3 maps the connected components of the moduli spaces $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ and $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$ onto each other. In particular, the moduli space $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$ has at most two connected components.*

Proof. The isomorphism ϕ is in particular a homeomorphism. Hence, ϕ defines a bijection between the connected components of $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ and $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$. The period domain $\Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)$ has two connected components $D_{\mathrm{Sat}_{L_{K3}}(i)}$ and $D'_{\mathrm{Sat}_{L_{K3}}(i)}$. Hence, $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)} \coloneqq \Omega\big(\mathrm{Sat}_{L_{K3}}(i)\big)/O_{\mathrm{Sat}_{L_{K3}}(i)}$ has one connected component if and only if $O_{\mathrm{Sat}_{L_{K3}}(i)}$ interchanges $D_{\mathrm{Sat}_{L_{K3}}(i)}$ and $D'_{\mathrm{Sat}_{L_{K3}}(i)}$ and two connected components otherwise. The first happens if and only if the group $O_{\mathrm{Sat}_{L_{K3}}(i)}$ contains an element with real spinor norm -1 (see [GHS09, Sec. 1]). As $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is a subvariety of $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$, it has then also at most two connected components. Therefore, also $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$ has at most two connected components. $\square$

Remark 9.4.4. If the lattice $\big(\mathrm{Sat}_{L_{K3}}(i)\big)^{\perp}_{L_{K3}}$ contains an m-admissible element with $m \leq 2$, the quasi-projective variety $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$ is irreducible by [Dol96, Proposition 5.6] and $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ is irreducible as an open subvariety of $\mathcal{F}^{\mathrm{bn}}_{\mathrm{Sat}_{L_{K3}}(i)}$. Since $\mathcal{F}^{\circ}_{\mathrm{Sat}_{L_{K3}}(i)}$ and $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$ are isomorphic by Main Theorem 3, in this situation it follows that $\mathcal{M}^{\mathbf{T}+\mathbf{G}}$ is irreducible.

A Intersection theory on surfaces

In this appendix, we will recall basic properties of the intersection pairing on surfaces and compute certain intersection numbers on those.

Lemma A.0.1. *Let $i'\colon S \hookrightarrow \mathbb{P}^4$ and $j'\colon H \hookrightarrow \mathbb{P}^4$ be embeddings of two-dimensional smooth connected subvarieties S and H in $\mathbb{P}^4$. Let $k\colon E \hookrightarrow \mathbb{P}^4$ be an embedding of a three-dimensional variety into $\mathbb{P}^4$. Let $i\colon C \hookrightarrow S$ and $j\colon C \hookrightarrow H$ be embeddings of the curve C in S and H, respectively. Assume that the following diagram commutes:*

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & S \arrow[d, "i'"] & \\ H \arrow[r, "j'"'] & \mathbb{P}^4 & \\ & & E \arrow[ul, "k"] \end{tikzcd}$$

Then, we have

$$j_*C.{j'}^*k_*E = i_*C.{i'}^*k_*E.$$

Proof. We have $j_*C,\ {j'}^*k_*E \in A^1(H)$ so $j_*C.{j'}^*k_*E \in A^2(H) \cong \mathbb{Z}$. On the other hand, $i_*C,\ {i'}^*k_*E \in A^1(S)$ so $i_*C.{i'}^*k_*E \in A^2(S) \cong \mathbb{Z}$. The projection formula gives

$$j_*C.{j'}^*k_*E = j_*(C.j^*{j'}^*k_*E) \text{ and } i_*C.{i'}^*k_*E = i_*(C.i^*{i'}^*k_*E).$$

Since C, $j^*{j'}^*k_*E$, and $j^*{j'}^*k_*E$ are curves, $C.(j^*{j'}^*k_*E)$ and $C.i^*{i'}^*k_*E$ are integers. Hence, $j_*(C.j^*{j'}^*k_*E) = C.j^*{j'}^*k_*E$ and $i_*(C.i^*{i'}^*k_*E) = C.i^*{i'}^*k_*E$. Further, by the commutativity of the diagram, we have $j^*{j'}^* = i^*{i'}^*$. Hence, $j_*C.{j'}^*k_*E = i_*C.{i'}^*k_*E$. □

Lemma A.0.2 ([Ful98, Chap. 8.2])**.** *Let X be quasi-projective variety and D_1 and D_2 closed subvarieties in X. Assume that X° is a smooth open subvariety of X such that $D_1 \cap D_2 \subseteq X^\circ$. Then,*

$$D_1.D_2 = D_{1|X^\circ}.D_{2|X^\circ}.$$

Lemma A.0.3. *Let H be a smooth projective surface and C, $l \in \mathrm{Div}(H)$. Let $p \in C \cap l$ be a smooth point of both C and l. Let*

1. *$H^{(1)} = Bl_pH \xrightarrow{\pi_p} H$ be the blowing-up of H in p with exceptional divisor $E^{(1)} = \pi_p^{-1}(p)$ and let $C^{(1)}$ and $l^{(1)}$ be the strict transforms of C and l in $H^{(1)}$, respectively. Let p_1 be the intersection point of $E^{(1)}$ with $C^{(1)}$.*

2. *$H^{(2)} = Bl_{p_1}H^{(1)} \xrightarrow{\pi_{p_1}} H^{(1)}$ be the blowing-up of $H^{(1)}$ in p_1 with exceptional divisor $E^{(2)} = \pi_{p_1}^{-1}(p_1)$ and let $C^{(2)}$ and $E^{(1,2)}$ be the strict transforms of C and $E^{(1)}$ in $H^{(2)}$, respectively. Let p_2 be the intersection point of $E^{(2)}$ with $C^{(2)}$.*

3. *$H^{(3)} = Bl_{p_2}H^{(2)} \xrightarrow{\pi_{p_2}} H^{(2)}$ be the blowing-up of $H^{(2)}$ in p_2 with exceptional divisor $E^{(3)} = \pi_{p_2}^{-1}(p_2)$ and let $C^{(3)}$, $E^{(1,3)}$, and $E^{(2,3)}$ be the strict transforms of C, $E^{(1)}$ and $E^{(2)}$ in $H^{(3)}$, respectively. Let p_3 be the intersection point of $E^{(3)}$ with $C^{(3)}$.*

We have the following intersection numbers, see Figure A.1:

$$\begin{array}{llll}
\textit{On } H^{(1)}: & C^{(1)}.E^{(1)} = 1, & C^{(1)}.l^{(1)} = C.l - 1, & \\
\textit{On } H^{(2)}: & C^{(2)}.E^{(2)} = 1, & C^{(2)}.E^{(1,2)} = 0, & \\
 & E^{(1,2)}.E^{(3)} = 1, & & \\
\textit{On } H^{(3)}: & C^{(3)}.E^{(3)} = 1, & C^{(3)}.E^{(2,3)} = 0, & C^{(3)}.E^{(1,3)} = 0, \\
 & E^{(1,3)}.E^{(3)} = 0, & E^{(1,3)}.E^{(2,3)} = 1, & E^{(2,3)}.E^{(3)} = 1.
\end{array}$$

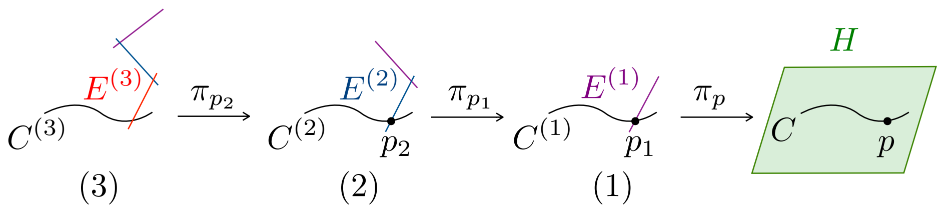


Figure A.1: Iterated blowing-ups of the surface H.

Proof. 1. On $H^{(1)}$, we have by [Har77, Chap. V, Proposition 3.1, 3.2, 3.6]:

(a) $(E^{(1)})^2 = -1$

(b) $(\pi_p^*C).E^{(1)} = (\pi_p^*l).E^{(1)} = 0$

(c) $(\pi_p^*C).(\pi_p^*l) = C.l$

(d) $C^{(1)} = \pi_p^*C - E^{(1)}$, $l^{(1)} = \pi_p^*l - E^{(1)}$.

Hence,

$$C^{(1)}.E^{(1)} = (\pi_p^*C - E^{(1)}).E^{(1)} = 1 \text{ and } C^{(1)}.l^{(1)} = C.l - 1.$$

2. On $H^{(2)}$, we have by [Har77, Chap. V, Proposition 3.1, 3.2, 3.6]:

(a) $(E^{(2)})^2 = -1$

(b) $(\pi_{p_1}^*C^{(1)}).E^{(2)} = (\pi_{p_1}^*E^{(1)}).E^{(2)} = 0$

(c) $(\pi_{p_1}^*E^{(1)}).(\pi_{p_1}^*C^{(1)}) = E^{(1)}.C^{(1)} = 1$

(d) $C^{(2)} = \pi_{p_1}^*C^{(1)} - E^{(2)}$, $E^{(1,2)} = \pi_{p_1}^*E^{(1)} - E^{(2)}$.

Using all these equalities, we compute

$$\begin{aligned}
C^{(2)}.E^{(1,2)} &= (\pi_{p_1}^*C^{(1)} - E^{(2)}).(\pi_{p_1}^*E^{(1)} - E^{(2)}) = 0,\\
C^{(2)}.E^{(2)} &= (\pi_{p_1}^*C^{(1)} - E^{(2)}).E^{(2)} = 1,\\
E^{(1,2)}.E^{(2)} &= (\pi_{p_1}^*E^{(1)} - E^{(2)}).E^{(2)} = 1.
\end{aligned}$$

3. On $H^{(3)}$, we have by [Har77, Chap. V, Proposition 3.1, 3.2, 3.6]:

(a) $(E^{(3)})^2 = -1$

(b) $(\pi_{p_2}^* C^{(2)}).E^{(3)} = (\pi_{p_2}^* E^{(2)}).E^{(3)} = (\pi_{p_2}^* E^{(1,2)}).E^{(3)} = 0$

(c) $(\pi_{p_2}^* C^{(2)}).(\pi_{p_2}^* E^{(2)}) \quad = C^{(2)}.E^{(2)} \quad = 1,$
$(\pi_{p_2}^* C^{(2)}).(\pi_{p_2}^* E^{(1,2)}) = C^{(2)}.E^{(1,2)} = 0$

(d) $C^{(3)} = \pi_{p_2}^* C^{(2)} - E^{(3)}$, $E^{(2,3)} = \pi_{p_2}^* E^{(2)} - E^{(3)}$.

Using all these equalities, we compute

$$\begin{aligned} C^{(3)}.E^{(2,3)} &= (\pi_{p_2}^* C^{(2)} - E^{(3)}).(\pi_{p_2}^* E^{(2)} - E^{(3)}) = 0, \\ C^{(3)}.E^{(3)} &= (\pi_{p_2}^* C^{(2)} - E^{(3)}).E^{(3)} = 1, \\ E^{(2,3)}.E^{(3)} &= (\pi_{p_2}^* E^{(2)} - E^{(3)}).E^{(3)} = 1. \end{aligned}$$

Since $p_2 \in C^{(2)}$ and $C^{(2)}.E^{(1,2)} = 0$, we have $p_2 \notin E^{(1,2)}$.

Hence, $\pi_{p_2}^* E^{(1,2)} = E^{(1,3)}$ and

$$\begin{aligned} C^{(3)}.E^{(1,3)} &= (\pi_{p_2}^* C^{(2)} - E^{(3)}).(\pi_{p_2}^* E^{(1,2)}) = 0, \\ E^{(1,3)}.\ E^{(3)} &= \quad (\pi_{p_2}^* (E^{(1,2)})). \quad E^{(3)} = 0, \\ E^{(2,3)}.E^{(1,3)} &= (\pi_{p_2}^* E^{(2)} - E^{(3)}).(\pi_{p_2}^* E^{(1,2)}) = 1. \end{aligned}$$

□

Lemma A.0.4. *Let H be a smooth surface and D_1, $D_2 \in \mathrm{Div}(H)$. Assume that $D_1.D_2 = m$. Let $p \in D_1 \cap D_2$ be a smooth point of D_1 and D_2. Let $H^{(1)} \to H$ the blowing-up of H in p and let $D_1^{(1)}$ and $D^{(2)}$ be the strict transforms of D_1 and D_2 in $H^{(1)}$. Then, $D_1^{(1)}.D_2^{(1)} = m - 1$.*

Proof. Let $E^{(1)}$ be the exceptional divisor of the blowing-up. By [Har77, Chap. V, Proposition 3.1, 3.2, 3.6], we have

$$(E^{(1)})^2 = -1,\ (\pi_p^* D_1).E^{(1)} = 0,\ (\pi_p^* D_2).E^{(1)} = 0,\ (\pi_p^* D_1).(\pi_p^* D_2) = D_1.D_2$$

and

$$D_1^{(1)} = \pi_p^* D_1 - E^{(1)},\ D_2^{(1)} = \pi_p^* D_2 - E^{(1)}.$$

Hence, $D_1^{(1)}.D_2^{(1)} = (\pi_p^* D_1 - E^{(1)}).(\pi_p^* D_2 - E^{(1)}) = m - 1$. □

B Code to determine all ADE lattices Λ such that $\langle 6\rangle \oplus \Lambda$ has a primitive embedding into the K3 lattice

In this appendix, we give the code to be implemented in the computer algebra software Wolfram Mathematica (Version: 11.1.1.0) to determine the list of all ADE lattices Λ such that the lattice $\langle 6\rangle \oplus \Lambda$ can be embedded primitively into the K3 lattice. The code is based on the algorithm presented in Section 8.1. Find the final list of all ADE lattice Λ as above in Appendix C.

```
(*We realize condition (2b) in Theorem 7.4.1.*)

(*Define function which returns for x := {x_1, ..., x_rmax} the list
    {{x_1 ..., x_{i_k-1}, x_{i_k} - 1, x_{i_k+1} + 1, x_{i_k+2}, ..., x_rmax}; k = 1, ..., rmax} where x_{i_1}, ..., x_{i_r} (
    r ∈ {1, ..., rmax - 1}) are the nonzero entries of x.*)
rmax=19;

operation [x_]:=Block[{tuplerules, nonzeros},
 tuplerules =ArrayRules[x];
nonzeros=Length[tuplerules]−1;
Table[x−UnitVector[rmax,tuplerules [[ i ,1,1]]]+ UnitVector[rmax, tuplerules [[ i ,1,1]]+1],{ i ,1, nonzeros}]
]

(*Define function which 1. finds in tuplelist := {{x^j_1, ..., x^j_rmax}, j = 1, ..., m} the largest entry c := x^j_i,
    2. saves {{c + 1, 0, ..., 0} ∈ Z^rmax} ∪ {operation[x]; x ∈ tuplelist}.*)
 iteration [ tuplelist_ ]:=Block[{ list },
 list ={(Max[tuplelist]+1)UnitVector[rmax,1]};
 list =Flatten[Append[operation[#]&/@tuplelist, list ],1];
 DeleteDuplicates [ list ]
]

(*Define the list step: Define the list step_0 := {{1, 0, ..., 0}},define successively
    step_i := {step_{i-1}, iteration(step_{i-1})},and step := step_{rmax-1}. step is the list whose i−th entry is the lists
    of all (a_1, ..., a_19) ∈ Z_{≥0}^rmax such that 1a_1 + 2a_2 + ... + 19a_19 = i.*)
step={{UnitVector[rmax,1]}};

Do[step=Append[step,iteration[step [[−1]]]];,{ rmax−1}];

 listab =step;

 listdb =listab;

(*formd is list of all {0, 0, 0, d_4, ..., d_rmax}.*)
formd=Join[{0,0,0},Table[_,{i ,4, rmax}]]

(* Lists in listdb contained in formd.*)
Table[ listdb [[ j]]=Cases[ listdb [[ j ]], formd],{ j ,1, rmax}];
```

```
(*Delete the first tree entries of all lists contained in the last defined list .*)
Table[ listdb [[ j]]= listdb [[ j ]][[ All ,4;;−1]],{ j ,1, rmax}];
 listeb =listab ;

(*forme is the list of all {0, 0, 0, 0, 0, e_6, e_7, e_8, 0, . . . , 0}.*)
forme=Join [{0,0,0,0,0}, Table[_,{i ,6,8}], Table[0,{ i ,9, rmax}]]

(* Lists in listeb contained in forme*)
Table[ listeb [[ j]]=Cases[ listeb [[ j ]], forme],{ j ,1, rmax}];

(*Delete the first five and last rmax − 8 entries of all lists in the list defined in the last step.*)
Table[ listeb [[ j]]= listeb [[ j ]][[ All ,6;;8]],{ j ,1, rmax}];

(* List of all triples {a, b, c} with a ∈ {0, . . . , rmax},b ∈ {0, 4, . . . , rmax},c ∈ {0, 6, . . . , rmax} such that
      a + b + c = i.*)
 listcombine=Table[Select[Tuples[{Range[0,rmax],Join[{0},Range[4,rmax]],Join[{0},Range[6,rmax]]}],Total
      [#]==i&],{i,1,rmax}]

(*Define function: For {i, j, k} the list of all {{a_1, . . . , a_19}, {d_4, ..., d_19}, {e_6, e_7, e_8}} with {a_1, . . . , a_19}
      from the i−th element in listab,{0, 0, 0, d_4, ..., d_19} from the j−th element of listdb and
      {0, 0, 0, 0, 0, e_6, e_7, e_8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0} from the k−th element of listeb. We have
      1a_1 + . . . + 19a_19 = i, 4d_4 + . . . + 19d_19 = j, 6e_6 + 7e_7 + 8e_8 = k such that i + j + k ≤ rmax*)
pick [{i_,j_,k_}]:=Block[{atake,dtake,etake},
atake=If[ i==0,{Table[0,{rmax}]},atake=listab[[i ]]];
dtake=If[j==0,{Table[0,{irun,4,rmax}]},dtake=listdb [[ j ]]];
etake=If[k==0,{Table[0,{irun ,6,8}]}, etake= listeb [[ k ]]];
Tuples[{atake, dtake, etake}]
]

(* lists {{a_1, . . . , a_19}, {d_4, ..., d_19}, {e_6, e_7, e_8}} correspond to all ADE lattices
      ⊕_{i=1}^{19} a_i A_i ⊕ ⊕_{j=4}^{19} d_j D_j ⊕ ⊕_{k=6}^{8} e_k E_k of rank r.*)
(*Number of all ADE lattices of rank 1 ≤ r ≤ 19.*)
Table[Length[Sort[Flatten [ pick[#]&/@(listcombine[[r ]]) ,1]]],{ r ,1,19}]
 {1,2,3,6,9,16,24,39,57,88,128,193,276,403,570,815,1137,1599,2207}

(*ADE lattices of rank 1 ≤ r ≤ 19*)
 listb =Table[Sort[Flatten [ pick[#]&/@(listcombine[[r ]]) ,1]],{ r ,1,19}];


(*We realize condition (2c) in Theorem 7.4.1.*)

(*We compute the length of the discriminat group ⟨6⟩ ⊕ Λ for an ADE lattice Λ.*)
l [x_]:=Block[{l2,l3 , l5 , l7 , l11 , l13 , l17 , l19 }, l2=1+Sum[x[[1,2i+1]],{i,0,9}]+Sum[x[[2,2i+1−3]],{i,2,9}]+2
      Sum[x[[2,2i−3]],{ i ,2,9}]+x [[3,2]];
l3=1+x[[1,2]]+x[[1,5]]+x [[1,8]]+ x [[1,11]]+ x [[1,14]]+ x [[1,17]]+ x [[3,1]];
l5=x[[1,4]]+x [[1,9]]+ x [[1,14]]+ x [[1,19]];
l7=x[[1,6]]+x [[1,13]];
l11=x [[1,10]];
l13=x [[1,12]];
l17=x [[1,16]];
l19=x [[1,18]];
Max[l2,l3 , l5 , l7 , l11 , l13 , l17 , l19 ]
]

(*Define function which checks if an ADE lattice Λ satisfies condition (2c) in Theorem 7.4.1.*)
test [x_]:=Block[{r},r=Sum[i x[[1, i ]],{ i,1,19}]+Sum[j x [[2, j −3]],{ j ,4,19}]+Sum[k x[[3,k−5]],{k ,6,8}];
If [21−l[x]>=r,x,False]
]
```

```
(*Define function which checks if an ADE lattice Λ satisfies condition (2c) in Theorem 7.4.1 and such
      that the embedding ⟨6⟩ ⊕ Λ into L_K3, if it exists, is unique up to automorphism of L_K3 according to
      Theorem 7.4.3.*)
testu [x_]:=Block[{r},r=Sum[i x[[1, i ]],{ i,1,19}]+Sum[j x[[2, j −3]],{ j,4,19}]+Sum[k x[[3,k−5]],{k ,6,8}];
 If [19−l[x]>=r,x,False]
]

(* Total number of ADE lattices Λ which satisfy condition (2b) and (2c) in Theorem 7.4.1.*)
Table[DeleteCases[test [#]&/@(listb [[ r ]]) , False]//Length,{r,1,19}]// Total
3032

(* Total number of ADE lattices Λ which satisfy condition (2b) and (2c) in Theorem 7.4.1 and such that the
      conditions in Theorem 7.4.3 holds.*)
Table[DeleteCases[testu[#]&/@(listb [[ r ]]) , False]//Length,{r,1,19}]// Total
1607

(*ADE lattices Λ which satisfy condition (2b) and (2c) in Theorem 7.4.1.*)
 listbc =Table[DeleteCases[test[#]&/@(listb[[ r ]]) , False ],{ r ,1,19}];
 listbcu =Table[DeleteCases[testu[#]&/@(listb[[ r ]]) , False ],{ r ,1,19}];


(*We realize condition (2d) in Theorem 7.4.1.*)

(*For an ADE lattice Λ, we compute the length of the p−part of the discriminant group of ⟨6⟩ ⊕ Λ.*)
lp [p_,x_]:=Block[{error},error :: boole="The␣value␣‘1‘␣is␣not␣allowed␣for␣p";

Switch[p,3,1+x[[1,2]]+ x [[1,5]]+ x [[1,8]]+ x [[1,11]]+ x [[1,14]]+ x [[1,17]]+ x [[3,1]],5, x [[1,4]]+ x [[1,9]]+ x
      [[1,14]]+ x [[1,19]],7, x [[1,6]]+ x [[1,13]],11, x [[1,10]],13, x [[1,12]],17, x [[1,16]],19, x [[1,18]], _,
      Message[error:: boole, p ];]
]

(*Check for a specific prime p,if condition (2d) Theorem 7.4.1 has to be checked.*)
testdTrue [p_,x_]:=Block[{r},r=Sum[i x[[1,i ]],{ i,1,19}]+Sum[j x[[2, j −3]],{ j,4,19}]+Sum[k x[[3,k−5]],{k
      ,6,8}];
 If [21−r==lp[p,x],x, False ]
]

(*For each prime p = 3, 5, 7, 11, 13, 17, 19 compute the number of ADE lattices Λ of rank 1 ≤ r ≤ 19 such
      that we need to check for ⟨6⟩ ⊕ Λ condition (2d) in Theorem 7.4.1.*)
Table[Length[DeleteCases[testdTrue[3,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
Table[Length[DeleteCases[testdTrue[5,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
Table[Length[DeleteCases[testdTrue[7,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
Table[Length[DeleteCases[testdTrue[11,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
Table[Length[DeleteCases[testdTrue[13,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
Table[Length[DeleteCases[testdTrue[17,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
Table[Length[DeleteCases[testdTrue[19,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}]
   {0,0,0,0,0,0,0,0,0,0,0,0,0,1,7,28,66,98,55}
   {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,10,14}
   {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,8}
   {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
   {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
   {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
   {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

(*In particular , this shows that condition (2d) in Theorem 7.4.1 has to be checked only for p = 3, 5, 7.*)(*
      List of ADE lattices for which we need to check (2d) for p = 3.*)
Table[ Print [DeleteCases[testdTrue[3,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}];

(* List of ADE lattices for which we need to check (2d) for p = 5.*)
Table[ Print [DeleteCases[testdTrue[5,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}];
```

```
(* List of ADE lattices for which we need to check (2d) for p = 7.*)
Table[Print [DeleteCases[testdTrue[7,#]&/@(listbc [[ r ]]) , False ]],{ r ,1,19}];

(* List of all ADE lattices for which we need to check condition (2d).*)
textd [p_,r_]:=(DeleteCases[testdTrue[p,#]&/@(listbc[[r ]]) , False ]) ;

(*Define function which gives the cardinality of the discriminant group of ⟨6⟩ ⊕ Λ for an ADE lattice Λ.*)
g[x_]:=Block[{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18,a19,d4,d5,d6,d7,d8,d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19,e6,e7,e8},{{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18,a19},{d4,d5,d6,d7,d8,d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19},{e6,e7,e8}}=x;
{x,6(Product[(i+1)^x[[1,i ]],{ i ,1,19}]) (Product[(4)^x[[2, j−3]],{j ,4,19}]) 2^e7 3^e6}
]

(*Define p−adic valuation.*)
v[p_,x_]:=Block[{ primefactorlist }, If [IntegerQ[x ],, Print ["x␣is␣no␣integer" ]];
 primefactorlist =FactorInteger[x];
 If [MemberQ[primefactorlist[[ All ,1]], p ], Select [ primefactorlist ,#[[1]]==p &][[1,2]],0]
]

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ conditions (2b) and (2c) in Theorem 7.4.1 are satisfied and condition (2d) needs to be checked for p = 3.*)
testd3=Flatten[Table[DeleteCases[testdTrue[3,#]&/@(listbc [[ r ]]) , False ],{ r ,1,19}],1];

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) in Theorem 7.4.1 and the conditon in Theorem 7.4.3 are satisfied and condition (2d) needs to be checked for p = 3.*)
testd3u=Flatten[Table[DeleteCases[testdTrue[3,#]&/@(listbcu [[ r ]]) , False ],{ r ,1,19}],1];

(*Compute the discriminant for the unique 3−adic lattice .*)
d3[tuple_]:=Block[{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18,a19,d4,d5,d6,d7,d8,d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19,e6,e7,e8},{{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18,a19},{d4,d5,d6,d7,d8,d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19},{e6,e7,e8}}=tuple;
{tuple,6*3^a2*6^a5*9^a8*3^a11*6^a14*126^a17*6^e6}
]

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) are satisfied and condition (2d) holds/does not hold for p = 3, as well.*)
Lr3={};
Ln3={};
 If [Mod[((g[#][[2]])−(d3 [#][[2]]) )/3^v[3,g [#][[2]]],3]==0, Lr3=Append[Lr3,#],Ln3=Append[Ln3,#]]&/@testd3;
Length[testd3]
Length[Lr3]
Length[Ln3]
255
186
69

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) and the conditon in Theorem 7.4.3 are satisfied and condition (2d) holds/does not hold for p = 3, as well.*)
Lr3u={};
Ln3u={};
 If [Mod[((g[#][[2]])−(d3 [#][[2]]) )/3^v[3,g [#][[2]]],3]==0, Lr3=Append[Lr3,#],Ln3=Append[Ln3,#]]&/@testd3u;
Length[testd3u]
Length[Lr3u]
Length[Ln3u]
0
```

```
0
0

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) are satisfied and condition (2d) needs to be
    checked for p = 5.*)
testd5=Flatten[Table[DeleteCases[testdTrue[5,#]&/@(listbc [[ r ]]) , False ],{ r ,1,19}],1];

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) in Theorem 7.4.1 and the conditon in
    Theorem 7.4.3 are satisfied and condition (2d) needs to be checked for p = 5.*)
testd5u=Flatten[Table[DeleteCases[testdTrue[5,#]&/@(listbcu [[ r ]]) , False ],{ r ,1,19}],1];

(*Compute the discriminant for a 5−adic lattice .*)
d5[tuple_]:=Block[{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18,a19,d4,d5,d6,d7,d8,
    d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19,e6,e7,e8},{{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13
    ,a14,a15,a16,a17,a18,a19},{d4,d5,d6,d7,d8,d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19},{e6,e7,e8
    }}=tuple;
{tuple,5^a4*40^a9*10^a14*5^a19}
]

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) are satisfied and condition (2d) holds/does
    not holds p = 5.*)
Lr5={};
Ln5={};

 If [Mod[((g[#][[2]])−1(d5[#][[2]]))/5^v[5,g [#][[2]]],5] Mod[((g[#][[2]])−4(d5[#][[2]]))/5^v[5,g
    [#][[2]]],5]==0, Lr5=Append[Lr5,#],Ln5=Append[Ln5,#]]&/@testd5;
Length[testd5]
Length[Lr5]
Length[Ln5]
25
9
16

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) and the condition in Theorem 7.4.3 are
    satisfied and condition (2d) holds/does not hold for p = 5, as well.*)
Lr5u={};
Ln5u={};
 If [Mod[((g[#][[2]])−1(d5[#][[2]]))/5^v[5,g [#][[2]]],5] Mod[((g[#][[2]])−4(d5[#][[2]]))/5^v[5,g
    [#][[2]]],5]==0, Lr5=Append[Lr5,#],Ln5=Append[Ln5,#]]&/@testd5u;
Length[testd5u]
Length[Lr5u]
Length[Ln5u]
0
0
0

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) are satisfied and condition (2d) needs to be
    checked for p = 7.*)
testd7=Flatten[Table[DeleteCases[testdTrue[7,#]&/@(listbc [[ r ]]) , False ],{ r ,1,19}],1];

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) in Theorem 7.4.1 and the condition in
    Theorem 7.4.3 are satisfied and condition (2d) needs to be checked for p = 7.*)
testd7u=Flatten[Table[DeleteCases[testdTrue[7,#]&/@(listbcu [[ r ]]) , False ],{ r ,1,19}],1];

(*Compute the discriminant for a 7−adic lattice .*)
d7[tuple_]:=Block[{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18,a19,d4,d5,d6,d7,d8,
    d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19,e6,e7,e8},{{a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13
    ,a14,a15,a16,a17,a18,a19},{d4,d5,d6,d7,d8,d9,d10,d11,d12,d13,d14,d15,d16,d17,d18,d19},{e6,e7,e8
    }}=tuple;
{tuple,7^a6*7^a13}]
```

```
(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) are satisfied and condition (2d) holds/does
    not holds p = 7.*)
Lr7={};
Ln7={};
If [Mod[((g[#][[2]])−1(d7[#][[2]]))/7^v[7,g [#][[2]]],7] Mod[((g[#][[2]])−2(d7[#][[2]]))/7^v[7,g
    [#][[2]]],7] Mod[((g[#][[2]])−4(d7[#][[2]]))/7^v[7,g [#][[2]]],7]==0, Lr7=Append[Lr7,#],Ln7=
    Append[Ln7,#]]&/@testd7;
Length[testd7]
Length[Lr7]
Length[Ln7]
9
3
6

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b) and (2c) and the conditon in Theorem 7.4.3 are
    satisfied and condition (2d) holds/does not hold for p = 7, as well.*)
Lr7u={};
Ln7u={};
If [Mod[((g[#][[2]])−1(d7[#][[2]]))/7^v[7,g [#][[2]]],7] Mod[((g[#][[2]])−2(d7[#][[2]]))/7^v[7,g
    [#][[2]]],7] Mod[((g[#][[2]])−4(d7[#][[2]]))/7^v[7,g [#][[2]]],7]==0, Lr7=Append[Lr7,#],Ln7=
    Append[Ln7,#]]&/@testd7u;
Length[testd7u]
Length[Lr7u]
Length[Ln7u]
0
0
0

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b), (2c) hold, and (2d) does not hold.*)
Ln=Join[Ln3,Ln5,Ln7];

(*All lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b), (2c) and the conditon in Theorem 7.4.3 hold, and
    (2d) does not hold.*)
Lnu=Join[Ln3u,Ln5u,Ln7u];

(* Cardinality of Ln and Lnu.*)
{Length[Ln]}
{Length[DeleteDuplicates[Ln]]}
{Length[Lnu]}
{Length[DeleteDuplicates[Lnu]]}

(*Delete all duplicates in Ln and Lnu*)
Ln=DeleteDuplicates[Ln];
Lnu=DeleteDuplicates[Lnu];

(*Number of lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b), (2c), and (2d) hold.*)
Complement[Flatten[Table[DeleteCases[test[#]&/@(listbc[[r ]]) , False ],{ r ,1,19}],1], Ln]//Length
2942

(*Number of lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b), (2c), and (2d) hold and the conditon in
    Theorem 7.4.3.*)
Complement[Flatten[Table[DeleteCases[test[#]&/@(listbcu[[r ]]) , False ],{ r ,1,19}],1], Lnu]//Length
1607

(* Lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b), (2c), and (2d) hold.*)
result =Complement[Flatten[Table[DeleteCases[test[#]&/@(listbc[[r]]) , False ],{ r ,1,19}],1], Ln];

(* Lattices Λ such that for ⟨6⟩ ⊕ Λ condition (2b), (2c), and (2d) hold and the conditon in Theorem 7.4.3.*)
resultu =Complement[Flatten[Table[DeleteCases[test[#]&/@(listbcu[[r]]) , False ],{ r ,1,19}],1], Lnu];
```

```
(* resultr and resultu sorted by rank*)
rank[tuple_]:=(Sum[(tuple[[1, i ]] i ),{ i ,1,19}]) +(Sum[(tuple[[2,j−3]](j)),{ j ,4,19}]) +tuple[[3,1]]*6+ tuple
    [[3,2]]*7+ tuple [[3,3]]*8;
resultr =GatherBy[SortBy[result,rank[#]&],rank[#]&]
resultru =GatherBy[SortBy[resultu,rank[#]&],rank[#]&]

(*An ADE lattice Λ = ⊕_{i=1}^{19} a_i A_1 ⊕ ⊕_{j=4}^{19} d_j D_j ⊕ ⊕_{k=6}^{8} e_k E_k in resultr has the form
    {{a1, ..., a19}, {d4, ..., d19}, {e6, e7, e8}}.*)
```

C List of all ADE lattices Λ such that $\Lambda \oplus \langle 6 \rangle$ can be embedded primitively into the K3 lattice

In this appendix, we give the list of all ADE lattices Λ such that $\langle 6 \rangle \oplus \Lambda$ can be embedded primitively into the K3 lattice. The list is obtained computer-aided with the code in Appendix B. The asterisk * infront of a lattice Λ indicates that the lattice $\langle 6 \rangle \oplus \Lambda$ admits a unique embedding into L_{K3} up to automorphisms of L_{K3}.

$\mathrm{rank}(\boldsymbol{\Lambda}) = 1$

1. $^*A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 2$

2. $^*A_2$
3. $^*2A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 3$

4. $^*A_3$
5. $^*A_1 \oplus A_2$
6. $^*3A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 4$

7. $^*D_4$
8. $^*A_4$
9. $^*2A_2$
10. $^*A_1 \oplus A_3$
11. $^*2A_1 \oplus A_2$
12. $^*4A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 5$

13. $^*D_5$
14. $^*A_5$
15. $^*A_2 \oplus A_3$
16. $^*A_1 \oplus D_4$
17. $^*A_1 \oplus A_4$
18. $^*A_1 \oplus 2A_2$
19. $^*2A_1 \oplus A_3$
20. $^*3A_1 \oplus A_2$
21. $^*5A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 6$

22. $^*E_6$
23. $^*D_6$
24. $^*A_6$
25. $^*2A_3$
26. $^*A_2 \oplus D_4$
27. $^*A_2 \oplus A_4$
28. $^*3A_2$
29. $^*A_1 \oplus D_5$
30. $^*A_1 \oplus A_5$
31. $^*A_1 \oplus A_2 \oplus A_3$
32. $^*2A_1 \oplus D_4$
33. $^*2A_1 \oplus A_4$
34. $^*2A_1 \oplus 2A_2$
35. $^*3A_1 \oplus A_3$
36. $^*4A_1 \oplus A_2$
37. $^*6A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 7$

38. $^*E_7$
39. $^*D_7$
40. $^*A_7$
41. $^*A_3 \oplus D_4$
42. $^*A_3 \oplus A_4$
43. $^*A_2 \oplus D_5$
44. $^*A_2 \oplus A_5$
45. $^*2A_2 \oplus A_3$
46. $^*A_1 \oplus E_6$
47. $^*A_1 \oplus D_6$
48. $^*A_1 \oplus A_6$
49. $^*A_1 \oplus 2A_3$
50. $^*A_1 \oplus A_2 \oplus D_4$
51. $^*A_1 \oplus A_2 \oplus A_4$
52. $^*A_1 \oplus 3A_2$
53. $^*2A_1 \oplus D_5$
54. $^*2A_1 \oplus A_5$
55. $^*2A_1 \oplus A_2 \oplus A_3$
56. $^*3A_1 \oplus D_4$
57. $^*3A_1 \oplus A_4$
58. $^*3A_1 \oplus 2A_2$
59. $^*4A_1 \oplus A_3$
60. $^*5A_1 \oplus A_2$
61. $^*7A_1$

$\mathrm{rank}(\boldsymbol{\Lambda}) = 8$

62. $^*E_8$
63. $^*D_8$
64. $^*2D_4$
65. $^*A_8$
66. $^*A_4 \oplus D_4$
67. $^*2A_4$
68. $^*A_3 \oplus D_5$
69. $^*A_3 \oplus A_5$
70. $^*A_2 \oplus E_6$
71. $^*A_2 \oplus D_6$
72. $^*A_2 \oplus A_6$
73. $^*A_2 \oplus 2A_3$
74. $^*2A_2 \oplus D_4$
75. $^*2A_2 \oplus A_4$
76. $^*4A_2$
77. $^*A_1 \oplus E_7$
78. $^*A_1 \oplus D_7$
79. $^*A_1 \oplus A_7$
80. $^*A_1 \oplus A_3 \oplus D_4$
81. $^*A_1 \oplus A_3 \oplus A_4$
82. $^*A_1 \oplus A_2 \oplus D_5$
83. $^*A_1 \oplus A_2 \oplus A_5$
84. $^*A_1 \oplus 2A_2 \oplus A_3$
85. $^*2A_1 \oplus E_6$
86. $^*2A_1 \oplus D_6$
87. $^*2A_1 \oplus A_6$
88. $^*2A_1 \oplus 2A_3$

89. $^*2A_1 \oplus A_2 \oplus D_4$
90. $^*2A_1 \oplus A_2 \oplus A_4$
91. $^*2A_1 \oplus 3A_2$
92. $^*3A_1 \oplus D_5$
93. $^*3A_1 \oplus A_5$
94. $^*3A_1 \oplus A_2 \oplus A_3$
95. $^*4A_1 \oplus D_4$
96. $^*4A_1 \oplus A_4$
97. $^*4A_1 \oplus 2A_2$
98. $^*5A_1 \oplus A_3$
99. $^*6A_1 \oplus A_2$
100. $^*8A_1$

rank(Λ) = 9

101. $^*D_9$
102. $^*D_4 \oplus D_5$
103. $^*A_9$
104. $^*A_5 \oplus D_4$
105. $^*A_4 \oplus D_5$
106. $^*A_4 \oplus A_5$
107. $^*A_3 \oplus E_6$
108. $^*A_3 \oplus D_6$
109. $^*A_3 \oplus A_6$
110. $^*3A_3$
111. $^*A_2 \oplus E_7$
112. $^*A_2 \oplus D_7$
113. $^*A_2 \oplus A_7$
114. $^*A_2 \oplus A_3 \oplus D_4$
115. $^*A_2 \oplus A_3 \oplus A_4$
116. $^*2A_2 \oplus D_5$
117. $^*2A_2 \oplus A_5$
118. $^*3A_2 \oplus A_3$
119. $^*A_1 \oplus E_8$
120. $^*A_1 \oplus D_8$
121. $^*A_1 \oplus 2D_4$
122. $^*A_1 \oplus A_8$
123. $^*A_1 \oplus A_4 \oplus D_4$
124. $^*A_1 \oplus 2A_4$
125. $^*A_1 \oplus A_3 \oplus D_5$
126. $^*A_1 \oplus A_3 \oplus A_5$
127. $^*A_1 \oplus A_2 \oplus E_6$
128. $^*A_1 \oplus A_2 \oplus D_6$
129. $^*A_1 \oplus A_2 \oplus A_6$
130. $^*A_1 \oplus A_2 \oplus 2A_3$
131. $^*A_1 \oplus 2A_2 \oplus D_4$
132. $^*A_1 \oplus 2A_2 \oplus A_4$
133. $^*A_1 \oplus 4A_2$
134. $^*2A_1 \oplus E_7$
135. $^*2A_1 \oplus D_7$
136. $^*2A_1 \oplus A_7$
137. $^*2A_1 \oplus A_3 \oplus D_4$
138. $^*2A_1 \oplus A_3 \oplus A_4$
139. $^*2A_1 \oplus A_2 \oplus D_5$
140. $^*2A_1 \oplus A_2 \oplus A_5$
141. $^*2A_1 \oplus 2A_2 \oplus A_3$
142. $^*3A_1 \oplus E_6$
143. $^*3A_1 \oplus D_6$
144. $^*3A_1 \oplus A_6$
145. $^*3A_1 \oplus 2A_3$
146. $^*3A_1 \oplus A_2 \oplus D_4$
147. $^*3A_1 \oplus A_2 \oplus A_4$
148. $^*3A_1 \oplus 3A_2$
149. $^*4A_1 \oplus D_5$
150. $^*4A_1 \oplus A_5$
151. $^*4A_1 \oplus A_2 \oplus A_3$
152. $^*5A_1 \oplus D_4$
153. $^*5A_1 \oplus A_4$
154. $^*5A_1 \oplus 2A_2$
155. $^*6A_1 \oplus A_3$
156. $^*7A_1 \oplus A_2$
157. $^*9A_1$

rank(Λ) = 10

158. $^*D_{10}$
159. $^*2D_5$
160. $^*D_4 \oplus E_6$
161. $^*D_4 \oplus D_6$
162. $^*A_{10}$
163. $^*A_6 \oplus D_4$
164. $^*A_5 \oplus D_5$
165. $^*2A_5$
166. $^*A_4 \oplus E_6$
167. $^*A_4 \oplus D_6$
168. $^*A_4 \oplus A_6$
169. $^*A_3 \oplus E_7$
170. $^*A_3 \oplus D_7$
171. $^*A_3 \oplus A_7$
172. $^*2A_3 \oplus D_4$
173. $^*2A_3 \oplus A_4$
174. $^*A_2 \oplus E_8$
175. $^*A_2 \oplus D_8$
176. $^*A_2 \oplus 2D_4$
177. $^*A_2 \oplus A_8$
178. $^*A_2 \oplus A_4 \oplus D_4$
179. $^*A_2 \oplus 2A_4$
180. $^*A_2 \oplus A_3 \oplus D_5$
181. $^*A_2 \oplus A_3 \oplus A_5$
182. $^*2A_2 \oplus E_6$
183. $^*2A_2 \oplus D_6$
184. $^*2A_2 \oplus A_6$
185. $^*2A_2 \oplus 2A_3$
186. $^*3A_2 \oplus D_4$
187. $^*3A_2 \oplus A_4$
188. $^*5A_2$
189. $^*A_1 \oplus D_9$
190. $^*A_1 \oplus D_4 \oplus D_5$
191. $^*A_1 \oplus A_9$
192. $^*A_1 \oplus A_5 \oplus D_4$
193. $^*A_1 \oplus A_4 \oplus D_5$
194. $^*A_1 \oplus A_4 \oplus A_5$
195. $^*A_1 \oplus A_3 \oplus E_6$
196. $^*A_1 \oplus A_3 \oplus D_6$
197. $^*A_1 \oplus A_3 \oplus A_6$
198. $^*A_1 \oplus 3A_3$
199. $^*A_1 \oplus A_2 \oplus E_7$
200. $^*A_1 \oplus A_2 \oplus D_7$
201. $^*A_1 \oplus A_2 \oplus A_7$
202. $^*A_1 \oplus A_2 \oplus A_3 \oplus D_4$
203. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_4$
204. $^*A_1 \oplus 2A_2 \oplus D_5$
205. $^*A_1 \oplus 2A_2 \oplus A_5$
206. $^*A_1 \oplus 3A_2 \oplus A_3$
207. $^*2A_1 \oplus E_8$
208. $^*2A_1 \oplus D_8$
209. $^*2A_1 \oplus 2D_4$
210. $^*2A_1 \oplus A_8$
211. $^*2A_1 \oplus A_4 \oplus D_4$
212. $^*2A_1 \oplus 2A_4$
213. $^*2A_1 \oplus A_3 \oplus D_5$
214. $^*2A_1 \oplus A_3 \oplus A_5$
215. $^*2A_1 \oplus A_2 \oplus E_6$
216. $^*2A_1 \oplus A_2 \oplus D_6$
217. $^*2A_1 \oplus A_2 \oplus A_6$
218. $^*2A_1 \oplus A_2 \oplus 2A_3$
219. $^*2A_1 \oplus 2A_2 \oplus D_4$
220. $^*2A_1 \oplus 2A_2 \oplus A_4$
221. $^*2A_1 \oplus 4A_2$
222. $^*3A_1 \oplus E_7$
223. $^*3A_1 \oplus D_7$
224. $^*3A_1 \oplus A_7$
225. $^*3A_1 \oplus A_3 \oplus D_4$
226. $^*3A_1 \oplus A_3 \oplus A_4$
227. $^*3A_1 \oplus A_2 \oplus D_5$
228. $^*3A_1 \oplus A_2 \oplus A_5$
229. $^*3A_1 \oplus 2A_2 \oplus A_3$
230. $^*4A_1 \oplus E_6$
231. $^*4A_1 \oplus D_6$
232. $^*4A_1 \oplus A_6$
233. $^*4A_1 \oplus 2A_3$
234. $^*4A_1 \oplus A_2 \oplus D_4$

235. $^*4A_1 \oplus A_2 \oplus A_4$
236. $^*4A_1 \oplus 3A_2$
237. $^*5A_1 \oplus D_5$
238. $^*5A_1 \oplus A_5$
239. $^*5A_1 \oplus A_2 \oplus A_3$
240. $^*6A_1 \oplus D_4$
241. $^*6A_1 \oplus A_4$
242. $^*6A_1 \oplus 2A_2$
243. $^*7A_1 \oplus A_3$
244. $^*8A_1 \oplus A_2$
245. $10A_1$

rank(Λ) = 11

246. $^*D_{11}$
247. $^*D_5 \oplus E_6$
248. $^*D_5 \oplus D_6$
249. $^*D_4 \oplus E_7$
250. $^*D_4 \oplus D_7$
251. $^*A_{11}$
252. $^*A_7 \oplus D_4$
253. $^*A_6 \oplus D_5$
254. $^*A_5 \oplus E_6$
255. $^*A_5 \oplus D_6$
256. $^*A_5 \oplus A_6$
257. $^*A_4 \oplus E_7$
258. $^*A_4 \oplus D_7$
259. $^*A_4 \oplus A_7$
260. $^*A_3 \oplus E_8$
261. $^*A_3 \oplus D_8$
262. $^*A_3 \oplus 2D_4$
263. $^*A_3 \oplus A_8$
264. $^*A_3 \oplus A_4 \oplus D_4$
265. $^*A_3 \oplus 2A_4$
266. $^*2A_3 \oplus D_5$
267. $^*2A_3 \oplus A_5$
268. $^*A_2 \oplus D_9$
269. $^*A_2 \oplus D_4 \oplus D_5$
270. $^*A_2 \oplus A_9$
271. $^*A_2 \oplus A_5 \oplus D_4$
272. $^*A_2 \oplus A_4 \oplus D_5$
273. $^*A_2 \oplus A_4 \oplus A_5$
274. $^*A_2 \oplus A_3 \oplus E_6$
275. $^*A_2 \oplus A_3 \oplus D_6$
276. $^*A_2 \oplus A_3 \oplus A_6$
277. $^*A_2 \oplus 3A_3$
278. $^*2A_2 \oplus E_7$
279. $^*2A_2 \oplus D_7$
280. $^*2A_2 \oplus A_7$
281. $^*2A_2 \oplus A_3 \oplus D_4$
282. $^*2A_2 \oplus A_3 \oplus A_4$
283. $^*3A_2 \oplus D_5$
284. $^*3A_2 \oplus A_5$
285. $^*4A_2 \oplus A_3$
286. $^*A_1 \oplus D_{10}$
287. $^*A_1 \oplus 2D_5$
288. $^*A_1 \oplus D_4 \oplus E_6$
289. $^*A_1 \oplus D_4 \oplus D_6$
290. $^*A_1 \oplus A_{10}$
291. $^*A_1 \oplus A_6 \oplus D_4$
292. $^*A_1 \oplus A_5 \oplus D_5$
293. $^*A_1 \oplus 2A_5$
294. $^*A_1 \oplus A_4 \oplus E_6$
295. $^*A_1 \oplus A_4 \oplus D_6$
296. $^*A_1 \oplus A_4 \oplus A_6$
297. $^*A_1 \oplus A_3 \oplus E_7$
298. $^*A_1 \oplus A_3 \oplus D_7$
299. $^*A_1 \oplus A_3 \oplus A_7$
300. $^*A_1 \oplus 2A_3 \oplus D_4$
301. $^*A_1 \oplus 2A_3 \oplus A_4$
302. $^*A_1 \oplus A_2 \oplus E_8$
303. $^*A_1 \oplus A_2 \oplus D_8$
304. $^*A_1 \oplus A_2 \oplus 2D_4$
305. $^*A_1 \oplus A_2 \oplus A_8$
306. $^*A_1 \oplus A_2 \oplus A_4 \oplus D_4$
307. $^*A_1 \oplus A_2 \oplus 2A_4$
308. $^*A_1 \oplus A_2 \oplus A_3 \oplus D_5$
309. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_5$
310. $^*A_1 \oplus 2A_2 \oplus E_6$
311. $^*A_1 \oplus 2A_2 \oplus D_6$
312. $^*A_1 \oplus 2A_2 \oplus A_6$
313. $^*A_1 \oplus 2A_2 \oplus 2A_3$
314. $^*A_1 \oplus 3A_2 \oplus D_4$
315. $^*A_1 \oplus 3A_2 \oplus A_4$
316. $^*A_1 \oplus 5A_2$
317. $^*2A_1 \oplus D_9$
318. $^*2A_1 \oplus D_4 \oplus D_5$
319. $^*2A_1 \oplus A_9$
320. $^*2A_1 \oplus A_5 \oplus D_4$
321. $^*2A_1 \oplus A_4 \oplus D_5$
322. $^*2A_1 \oplus A_4 \oplus A_5$
323. $^*2A_1 \oplus A_3 \oplus E_6$
324. $^*2A_1 \oplus A_3 \oplus D_6$
325. $^*2A_1 \oplus A_3 \oplus A_6$
326. $^*2A_1 \oplus 3A_3$
327. $^*2A_1 \oplus A_2 \oplus E_7$
328. $^*2A_1 \oplus A_2 \oplus D_7$
329. $^*2A_1 \oplus A_2 \oplus A_7$
330. $^*2A_1 \oplus A_2 \oplus A_3 \oplus D_4$
331. $^*2A_1 \oplus A_2 \oplus A_3 \oplus A_4$
332. $^*2A_1 \oplus 2A_2 \oplus D_5$
333. $^*2A_1 \oplus 2A_2 \oplus A_5$
334. $^*2A_1 \oplus 3A_2 \oplus A_3$
335. $^*3A_1 \oplus E_8$
336. $^*3A_1 \oplus D_8$
337. $^*3A_1 \oplus 2D_4$
338. $^*3A_1 \oplus A_8$
339. $^*3A_1 \oplus A_4 \oplus D_4$
340. $^*3A_1 \oplus 2A_4$
341. $^*3A_1 \oplus A_3 \oplus D_5$
342. $^*3A_1 \oplus A_3 \oplus A_5$
343. $^*3A_1 \oplus A_2 \oplus E_6$
344. $^*3A_1 \oplus A_2 \oplus D_6$
345. $^*3A_1 \oplus A_2 \oplus A_6$
346. $^*3A_1 \oplus A_2 \oplus 2A_3$
347. $^*3A_1 \oplus 2A_2 \oplus D_4$
348. $^*3A_1 \oplus 2A_2 \oplus A_4$
349. $^*3A_1 \oplus 4A_2$
350. $^*4A_1 \oplus E_7$
351. $^*4A_1 \oplus D_7$
352. $^*4A_1 \oplus A_7$
353. $^*4A_1 \oplus A_3 \oplus D_4$
354. $^*4A_1 \oplus A_3 \oplus A_4$
355. $^*4A_1 \oplus A_2 \oplus D_5$
356. $^*4A_1 \oplus A_2 \oplus A_5$
357. $^*4A_1 \oplus 2A_2 \oplus A_3$
358. $^*5A_1 \oplus E_6$
359. $^*5A_1 \oplus D_6$
360. $^*5A_1 \oplus A_6$
361. $^*5A_1 \oplus 2A_3$
362. $^*5A_1 \oplus A_2 \oplus D_4$
363. $^*5A_1 \oplus A_2 \oplus A_4$
364. $^*5A_1 \oplus 3A_2$
365. $^*6A_1 \oplus D_5$
366. $^*6A_1 \oplus A_5$
367. $^*6A_1 \oplus A_2 \oplus A_3$
368. $7A_1 \oplus D_4$
369. $^*7A_1 \oplus A_4$
370. $^*7A_1 \oplus 2A_2$
371. $8A_1 \oplus A_3$
372. $9A_1 \oplus A_2$

rank(Λ) = 12

373. $^*2E_6$
374. $^*D_{12}$
375. $^*D_6 \oplus E_6$
376. $^*2D_6$
377. $^*D_5 \oplus E_7$
378. $^*D_5 \oplus D_7$
379. $^*D_4 \oplus E_8$

380. $^*D_4 \oplus D_8$
381. $^*3D_4$
382. $^*A_{12}$
383. $^*A_8 \oplus D_4$
384. $^*A_7 \oplus D_5$
385. $^*A_6 \oplus E_6$
386. $^*A_6 \oplus D_6$
387. $^*2A_6$
388. $^*A_5 \oplus E_7$
389. $^*A_5 \oplus D_7$
390. $^*A_5 \oplus A_7$
391. $^*A_4 \oplus E_8$
392. $^*A_4 \oplus D_8$
393. $^*A_4 \oplus 2D_4$
394. $^*A_4 \oplus A_8$
395. $^*2A_4 \oplus D_4$
396. $^*3A_4$
397. $^*A_3 \oplus D_9$
398. $^*A_3 \oplus D_4 \oplus D_5$
399. $^*A_3 \oplus A_9$
400. $^*A_3 \oplus A_5 \oplus D_4$
401. $^*A_3 \oplus A_4 \oplus D_5$
402. $^*A_3 \oplus A_4 \oplus A_5$
403. $^*2A_3 \oplus E_6$
404. $^*2A_3 \oplus D_6$
405. $^*2A_3 \oplus A_6$
406. $^*4A_3$
407. $^*A_2 \oplus D_{10}$
408. $^*A_2 \oplus 2D_5$
409. $^*A_2 \oplus D_4 \oplus E_6$
410. $^*A_2 \oplus D_4 \oplus D_6$
411. $^*A_2 \oplus A_{10}$
412. $^*A_2 \oplus A_6 \oplus D_4$
413. $^*A_2 \oplus A_5 \oplus D_5$
414. $^*A_2 \oplus 2A_5$
415. $^*A_2 \oplus A_4 \oplus E_6$
416. $^*A_2 \oplus A_4 \oplus D_6$
417. $^*A_2 \oplus A_4 \oplus A_6$
418. $^*A_2 \oplus A_3 \oplus E_7$
419. $^*A_2 \oplus A_3 \oplus D_7$
420. $^*A_2 \oplus A_3 \oplus A_7$
421. $^*A_2 \oplus 2A_3 \oplus D_4$
422. $^*A_2 \oplus 2A_3 \oplus A_4$
423. $^*2A_2 \oplus E_8$
424. $^*2A_2 \oplus D_8$
425. $^*2A_2 \oplus 2D_4$
426. $^*2A_2 \oplus A_8$
427. $^*2A_2 \oplus A_4 \oplus D_4$
428. $^*2A_2 \oplus 2A_4$
429. $^*2A_2 \oplus A_3 \oplus D_5$
430. $^*2A_2 \oplus A_3 \oplus A_5$
431. $^*3A_2 \oplus E_6$
432. $^*3A_2 \oplus D_6$
433. $^*3A_2 \oplus A_6$
434. $^*3A_2 \oplus 2A_3$
435. $^*4A_2 \oplus D_4$
436. $^*4A_2 \oplus A_4$
437. $^*6A_2$
438. $^*A_1 \oplus D_{11}$
439. $^*A_1 \oplus D_5 \oplus E_6$
440. $^*A_1 \oplus D_5 \oplus D_6$
441. $^*A_1 \oplus D_4 \oplus E_7$
442. $^*A_1 \oplus D_4 \oplus D_7$
443. $^*A_1 \oplus A_{11}$
444. $^*A_1 \oplus A_7 \oplus D_4$
445. $^*A_1 \oplus A_6 \oplus D_5$
446. $^*A_1 \oplus A_5 \oplus E_6$
447. $^*A_1 \oplus A_5 \oplus D_6$
448. $^*A_1 \oplus A_5 \oplus A_6$
449. $^*A_1 \oplus A_4 \oplus E_7$
450. $^*A_1 \oplus A_4 \oplus D_7$
451. $^*A_1 \oplus A_4 \oplus A_7$
452. $^*A_1 \oplus A_3 \oplus E_8$
453. $^*A_1 \oplus A_3 \oplus D_8$
454. $^*A_1 \oplus A_3 \oplus 2D_4$
455. $^*A_1 \oplus A_3 \oplus A_8$
456. $^*A_1 \oplus A_3 \oplus A_4 \oplus D_4$
457. $^*A_1 \oplus A_3 \oplus 2A_4$
458. $^*A_1 \oplus 2A_3 \oplus D_5$
459. $^*A_1 \oplus 2A_3 \oplus A_5$
460. $^*A_1 \oplus A_2 \oplus D_9$
461. $^*A_1 \oplus A_2 \oplus D_4 \oplus D_5$
462. $^*A_1 \oplus A_2 \oplus A_9$
463. $^*A_1 \oplus A_2 \oplus A_5 \oplus D_4$
464. $^*A_1 \oplus A_2 \oplus A_4 \oplus D_5$
465. $^*A_1 \oplus A_2 \oplus A_4 \oplus A_5$
466. $^*A_1 \oplus A_2 \oplus A_3 \oplus E_6$
467. $^*A_1 \oplus A_2 \oplus A_3 \oplus D_6$
468. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_6$
469. $^*A_1 \oplus A_2 \oplus 3A_3$
470. $^*A_1 \oplus 2A_2 \oplus E_7$
471. $^*A_1 \oplus 2A_2 \oplus D_7$
472. $^*A_1 \oplus 2A_2 \oplus A_7$
473. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus D_4$
474. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus A_4$
475. $^*A_1 \oplus 3A_2 \oplus D_5$
476. $^*A_1 \oplus 3A_2 \oplus A_5$
477. $^*A_1 \oplus 4A_2 \oplus A_3$
478. $^*2A_1 \oplus D_{10}$
479. $^*2A_1 \oplus 2D_5$
480. $^*2A_1 \oplus D_4 \oplus E_6$
481. $^*2A_1 \oplus D_4 \oplus D_6$
482. $^*2A_1 \oplus A_{10}$
483. $^*2A_1 \oplus A_6 \oplus D_4$
484. $^*2A_1 \oplus A_5 \oplus D_5$
485. $^*2A_1 \oplus 2A_5$
486. $^*2A_1 \oplus A_4 \oplus E_6$
487. $^*2A_1 \oplus A_4 \oplus D_6$
488. $^*2A_1 \oplus A_4 \oplus A_6$
489. $^*2A_1 \oplus A_3 \oplus E_7$
490. $^*2A_1 \oplus A_3 \oplus D_7$
491. $^*2A_1 \oplus A_3 \oplus A_7$
492. $^*2A_1 \oplus 2A_3 \oplus D_4$
493. $^*2A_1 \oplus 2A_3 \oplus A_4$
494. $^*2A_1 \oplus A_2 \oplus E_8$
495. $^*2A_1 \oplus A_2 \oplus D_8$
496. $^*2A_1 \oplus A_2 \oplus 2D_4$
497. $^*2A_1 \oplus A_2 \oplus A_8$
498. $^*2A_1 \oplus A_2 \oplus A_4 \oplus D_4$
499. $^*2A_1 \oplus A_2 \oplus 2A_4$
500. $^*2A_1 \oplus A_2 \oplus A_3 \oplus D_5$
501. $^*2A_1 \oplus A_2 \oplus A_3 \oplus A_5$
502. $^*2A_1 \oplus 2A_2 \oplus E_6$
503. $^*2A_1 \oplus 2A_2 \oplus D_6$
504. $^*2A_1 \oplus 2A_2 \oplus A_6$
505. $^*2A_1 \oplus 2A_2 \oplus 2A_3$
506. $^*2A_1 \oplus 3A_2 \oplus D_4$
507. $^*2A_1 \oplus 3A_2 \oplus A_4$
508. $^*2A_1 \oplus 5A_2$
509. $^*3A_1 \oplus D_9$
510. $^*3A_1 \oplus D_4 \oplus D_5$
511. $^*3A_1 \oplus A_9$
512. $^*3A_1 \oplus A_5 \oplus D_4$
513. $^*3A_1 \oplus A_4 \oplus D_5$
514. $^*3A_1 \oplus A_4 \oplus A_5$
515. $^*3A_1 \oplus A_3 \oplus E_6$
516. $^*3A_1 \oplus A_3 \oplus D_6$
517. $^*3A_1 \oplus A_3 \oplus A_6$
518. $^*3A_1 \oplus 3A_3$
519. $^*3A_1 \oplus A_2 \oplus E_7$
520. $^*3A_1 \oplus A_2 \oplus D_7$
521. $^*3A_1 \oplus A_2 \oplus A_7$
522. $^*3A_1 \oplus A_2 \oplus A_3 \oplus D_4$
523. $^*3A_1 \oplus A_2 \oplus A_3 \oplus A_4$
524. $^*3A_1 \oplus 2A_2 \oplus D_5$

525. $^*3A_1 \oplus 2A_2 \oplus A_5$
526. $^*3A_1 \oplus 3A_2 \oplus A_3$
527. $^*4A_1 \oplus E_8$
528. $^*4A_1 \oplus D_8$
529. $4A_1 \oplus 2D_4$
530. $^*4A_1 \oplus A_8$
531. $^*4A_1 \oplus A_4 \oplus D_4$
532. $^*4A_1 \oplus 2A_4$
533. $^*4A_1 \oplus A_3 \oplus D_5$
534. $^*4A_1 \oplus A_3 \oplus A_5$
535. $^*4A_1 \oplus A_2 \oplus E_6$
536. $^*4A_1 \oplus A_2 \oplus D_6$
537. $^*4A_1 \oplus A_2 \oplus A_6$
538. $^*4A_1 \oplus A_2 \oplus 2A_3$
539. $^*4A_1 \oplus 2A_2 \oplus D_4$
540. $^*4A_1 \oplus 2A_2 \oplus A_4$
541. $^*4A_1 \oplus 4A_2$
542. $^*5A_1 \oplus E_7$
543. $^*5A_1 \oplus D_7$
544. $^*5A_1 \oplus A_7$
545. $5A_1 \oplus A_3 \oplus D_4$
546. $^*5A_1 \oplus A_3 \oplus A_4$
547. $^*5A_1 \oplus A_2 \oplus D_5$
548. $^*5A_1 \oplus A_2 \oplus A_5$
549. $^*5A_1 \oplus 2A_2 \oplus A_3$
550. $^*6A_1 \oplus E_6$
551. $6A_1 \oplus D_6$
552. $^*6A_1 \oplus A_6$
553. $6A_1 \oplus 2A_3$
554. $6A_1 \oplus A_2 \oplus D_4$
555. $^*6A_1 \oplus A_2 \oplus A_4$
556. $^*6A_1 \oplus 3A_2$
557. $7A_1 \oplus D_5$
558. $7A_1 \oplus A_5$
559. $7A_1 \oplus A_2 \oplus A_3$
560. $8A_1 \oplus A_4$
561. $8A_1 \oplus 2A_2$

$\mathrm{rank}(\Lambda) = 13$

562. $^*E_6 \oplus E_7$
563. $^*D_{13}$
564. $^*D_7 \oplus E_6$
565. $^*D_6 \oplus E_7$
566. $^*D_6 \oplus D_7$
567. $^*D_5 \oplus E_8$
568. $^*D_5 \oplus D_8$
569. $^*D_4 \oplus D_9$
570. $^*2D_4 \oplus D_5$
571. $^*A_{13}$
572. $^*A_9 \oplus D_4$
573. $^*A_8 \oplus D_5$
574. $^*A_7 \oplus E_6$
575. $^*A_7 \oplus D_6$
576. $^*A_6 \oplus E_7$
577. $^*A_6 \oplus D_7$
578. $^*A_6 \oplus A_7$
579. $^*A_5 \oplus E_8$
580. $^*A_5 \oplus D_8$
581. $^*A_5 \oplus 2D_4$
582. $^*A_5 \oplus A_8$
583. $^*A_4 \oplus D_9$
584. $^*A_4 \oplus D_4 \oplus D_5$
585. $^*A_4 \oplus A_9$
586. $^*A_4 \oplus A_5 \oplus D_4$
587. $^*2A_4 \oplus D_5$
588. $^*2A_4 \oplus A_5$
589. $^*A_3 \oplus D_{10}$
590. $^*A_3 \oplus 2D_5$
591. $^*A_3 \oplus D_4 \oplus E_6$
592. $^*A_3 \oplus D_4 \oplus D_6$
593. $^*A_3 \oplus A_{10}$
594. $^*A_3 \oplus A_6 \oplus D_4$
595. $^*A_3 \oplus A_5 \oplus D_5$
596. $^*A_3 \oplus 2A_5$
597. $^*A_3 \oplus A_4 \oplus E_6$
598. $^*A_3 \oplus A_4 \oplus D_6$
599. $^*A_3 \oplus A_4 \oplus A_6$
600. $^*2A_3 \oplus E_7$
601. $^*2A_3 \oplus D_7$
602. $^*2A_3 \oplus A_7$
603. $^*3A_3 \oplus D_4$
604. $^*3A_3 \oplus A_4$
605. $^*A_2 \oplus D_{11}$
606. $^*A_2 \oplus D_5 \oplus E_6$
607. $^*A_2 \oplus D_5 \oplus D_6$
608. $^*A_2 \oplus D_4 \oplus E_7$
609. $^*A_2 \oplus D_4 \oplus D_7$
610. $^*A_2 \oplus A_{11}$
611. $^*A_2 \oplus A_7 \oplus D_4$
612. $^*A_2 \oplus A_6 \oplus D_5$
613. $^*A_2 \oplus A_5 \oplus E_6$
614. $^*A_2 \oplus A_5 \oplus D_6$
615. $^*A_2 \oplus A_5 \oplus A_6$
616. $^*A_2 \oplus A_4 \oplus E_7$
617. $^*A_2 \oplus A_4 \oplus D_7$
618. $^*A_2 \oplus A_4 \oplus A_7$
619. $^*A_2 \oplus A_3 \oplus E_8$
620. $^*A_2 \oplus A_3 \oplus D_8$
621. $^*A_2 \oplus A_3 \oplus 2D_4$
622. $^*A_2 \oplus A_3 \oplus A_8$
623. $^*A_2 \oplus A_3 \oplus A_4 \oplus D_4$
624. $^*A_2 \oplus A_3 \oplus 2A_4$
625. $^*A_2 \oplus 2A_3 \oplus D_5$
626. $^*A_2 \oplus 2A_3 \oplus A_5$
627. $^*2A_2 \oplus D_9$
628. $^*2A_2 \oplus D_4 \oplus D_5$
629. $^*2A_2 \oplus A_9$
630. $^*2A_2 \oplus A_5 \oplus D_4$
631. $^*2A_2 \oplus A_4 \oplus D_5$
632. $^*2A_2 \oplus A_4 \oplus A_5$
633. $^*2A_2 \oplus A_3 \oplus E_6$
634. $^*2A_2 \oplus A_3 \oplus D_6$
635. $^*2A_2 \oplus A_3 \oplus A_6$
636. $^*2A_2 \oplus 3A_3$
637. $^*3A_2 \oplus E_7$
638. $^*3A_2 \oplus D_7$
639. $^*3A_2 \oplus A_7$
640. $^*3A_2 \oplus A_3 \oplus D_4$
641. $^*3A_2 \oplus A_3 \oplus A_4$
642. $^*4A_2 \oplus D_5$
643. $^*4A_2 \oplus A_5$
644. $^*5A_2 \oplus A_3$
645. $^*A_1 \oplus 2E_6$
646. $^*A_1 \oplus D_{12}$
647. $^*A_1 \oplus D_6 \oplus E_6$
648. $^*A_1 \oplus 2D_6$
649. $^*A_1 \oplus D_5 \oplus E_7$
650. $^*A_1 \oplus D_5 \oplus D_7$
651. $^*A_1 \oplus D_4 \oplus E_8$
652. $^*A_1 \oplus D_4 \oplus D_8$
653. $A_1 \oplus 3D_4$
654. $^*A_1 \oplus A_{12}$
655. $^*A_1 \oplus A_8 \oplus D_4$
656. $^*A_1 \oplus A_7 \oplus D_5$
657. $^*A_1 \oplus A_6 \oplus E_6$
658. $^*A_1 \oplus A_6 \oplus D_6$
659. $^*A_1 \oplus 2A_6$
660. $^*A_1 \oplus A_5 \oplus E_7$
661. $^*A_1 \oplus A_5 \oplus D_7$
662. $^*A_1 \oplus A_5 \oplus A_7$
663. $^*A_1 \oplus A_4 \oplus E_8$
664. $^*A_1 \oplus A_4 \oplus D_8$
665. $^*A_1 \oplus A_4 \oplus 2D_4$
666. $^*A_1 \oplus A_4 \oplus A_8$
667. $^*A_1 \oplus 2A_4 \oplus D_4$
668. $^*A_1 \oplus 3A_4$
669. $^*A_1 \oplus A_3 \oplus D_9$
670. $^*A_1 \oplus A_3 \oplus D_4 \oplus D_5$
671. $^*A_1 \oplus A_3 \oplus A_9$

672. $^*A_1 \oplus A_3 \oplus A_5 \oplus D_4$
673. $^*A_1 \oplus A_3 \oplus A_4 \oplus D_5$
674. $^*A_1 \oplus A_3 \oplus A_4 \oplus A_5$
675. $^*A_1 \oplus 2A_3 \oplus E_6$
676. $^*A_1 \oplus 2A_3 \oplus D_6$
677. $^*A_1 \oplus 2A_3 \oplus A_6$
678. $^*A_1 \oplus 4A_3$
679. $^*A_1 \oplus A_2 \oplus D_{10}$
680. $^*A_1 \oplus A_2 \oplus 2D_5$
681. $^*A_1 \oplus A_2 \oplus D_4 \oplus E_6$
682. $^*A_1 \oplus A_2 \oplus D_4 \oplus D_6$
683. $^*A_1 \oplus A_2 \oplus A_{10}$
684. $^*A_1 \oplus A_2 \oplus A_6 \oplus D_4$
685. $^*A_1 \oplus A_2 \oplus A_5 \oplus D_5$
686. $^*A_1 \oplus A_2 \oplus 2A_5$
687. $^*A_1 \oplus A_2 \oplus A_4 \oplus E_6$
688. $^*A_1 \oplus A_2 \oplus A_4 \oplus D_6$
689. $^*A_1 \oplus A_2 \oplus A_4 \oplus A_6$
690. $^*A_1 \oplus A_2 \oplus A_3 \oplus E_7$
691. $^*A_1 \oplus A_2 \oplus A_3 \oplus D_7$
692. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_7$
693. $^*A_1 \oplus A_2 \oplus 2A_3 \oplus D_4$
694. $^*A_1 \oplus A_2 \oplus 2A_3 \oplus A_4$
695. $^*A_1 \oplus 2A_2 \oplus E_8$
696. $^*A_1 \oplus 2A_2 \oplus D_8$
697. $^*A_1 \oplus 2A_2 \oplus 2D_4$
698. $^*A_1 \oplus 2A_2 \oplus A_8$
699. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus D_4$
700. $^*A_1 \oplus 2A_2 \oplus 2A_4$
701. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus D_5$
702. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus A_5$
703. $^*A_1 \oplus 3A_2 \oplus E_6$
704. $^*A_1 \oplus 3A_2 \oplus D_6$
705. $^*A_1 \oplus 3A_2 \oplus A_6$
706. $^*A_1 \oplus 3A_2 \oplus 2A_3$
707. $^*A_1 \oplus 4A_2 \oplus D_4$
708. $^*A_1 \oplus 4A_2 \oplus A_4$
709. $A_1 \oplus 6A_2$
710. $^*2A_1 \oplus D_{11}$
711. $^*2A_1 \oplus D_5 \oplus E_6$
712. $^*2A_1 \oplus D_5 \oplus D_6$
713. $^*2A_1 \oplus D_4 \oplus E_7$
714. $^*2A_1 \oplus D_4 \oplus D_7$
715. $^*2A_1 \oplus A_{11}$
716. $^*2A_1 \oplus A_7 \oplus D_4$
717. $^*2A_1 \oplus A_6 \oplus D_5$
718. $^*2A_1 \oplus A_5 \oplus E_6$
719. $^*2A_1 \oplus A_5 \oplus D_6$
720. $^*2A_1 \oplus A_5 \oplus A_6$
721. $^*2A_1 \oplus A_4 \oplus E_7$
722. $^*2A_1 \oplus A_4 \oplus D_7$
723. $^*2A_1 \oplus A_4 \oplus A_7$
724. $^*2A_1 \oplus A_3 \oplus E_8$
725. $^*2A_1 \oplus A_3 \oplus D_8$
726. $2A_1 \oplus A_3 \oplus 2D_4$
727. $^*2A_1 \oplus A_3 \oplus A_8$
728. $^*2A_1 \oplus A_3 \oplus A_4 \oplus D_4$
729. $^*2A_1 \oplus A_3 \oplus 2A_4$
730. $^*2A_1 \oplus 2A_3 \oplus D_5$
731. $^*2A_1 \oplus 2A_3 \oplus A_5$
732. $^*2A_1 \oplus A_2 \oplus D_9$
733. $^*2A_1 \oplus A_2 \oplus D_4 \oplus D_5$
734. $^*2A_1 \oplus A_2 \oplus A_9$
735. $^*2A_1 \oplus A_2 \oplus A_5 \oplus D_4$
736. $^*2A_1 \oplus A_2 \oplus A_4 \oplus D_5$
737. $^*2A_1 \oplus A_2 \oplus A_4 \oplus A_5$
738. $^*2A_1 \oplus A_2 \oplus A_3 \oplus E_6$
739. $^*2A_1 \oplus A_2 \oplus A_3 \oplus D_6$
740. $^*2A_1 \oplus A_2 \oplus A_3 \oplus A_6$
741. $^*2A_1 \oplus A_2 \oplus 3A_3$
742. $^*2A_1 \oplus 2A_2 \oplus E_7$
743. $^*2A_1 \oplus 2A_2 \oplus D_7$
744. $^*2A_1 \oplus 2A_2 \oplus A_7$
745. $^*2A_1 \oplus 2A_2 \oplus A_3 \oplus D_4$
746. $^*2A_1 \oplus 2A_2 \oplus A_3 \oplus A_4$
747. $^*2A_1 \oplus 3A_2 \oplus D_5$
748. $^*2A_1 \oplus 3A_2 \oplus A_5$
749. $^*2A_1 \oplus 4A_2 \oplus A_3$
750. $^*3A_1 \oplus D_{10}$
751. $^*3A_1 \oplus 2D_5$
752. $^*3A_1 \oplus D_4 \oplus E_6$
753. $3A_1 \oplus D_4 \oplus D_6$
754. $^*3A_1 \oplus A_{10}$
755. $^*3A_1 \oplus A_6 \oplus D_4$
756. $^*3A_1 \oplus A_5 \oplus D_5$
757. $^*3A_1 \oplus 2A_5$
758. $^*3A_1 \oplus A_4 \oplus E_6$
759. $^*3A_1 \oplus A_4 \oplus D_6$
760. $^*3A_1 \oplus A_4 \oplus A_6$
761. $^*3A_1 \oplus A_3 \oplus E_7$
762. $^*3A_1 \oplus A_3 \oplus D_7$
763. $^*3A_1 \oplus A_3 \oplus A_7$
764. $3A_1 \oplus 2A_3 \oplus D_4$
765. $^*3A_1 \oplus 2A_3 \oplus A_4$
766. $^*3A_1 \oplus A_2 \oplus E_8$
767. $^*3A_1 \oplus A_2 \oplus D_8$
768. $3A_1 \oplus A_2 \oplus 2D_4$
769. $^*3A_1 \oplus A_2 \oplus A_8$
770. $^*3A_1 \oplus A_2 \oplus A_4 \oplus D_4$
771. $^*3A_1 \oplus A_2 \oplus 2A_4$
772. $^*3A_1 \oplus A_2 \oplus A_3 \oplus D_5$
773. $^*3A_1 \oplus A_2 \oplus A_3 \oplus A_5$
774. $^*3A_1 \oplus 2A_2 \oplus E_6$
775. $^*3A_1 \oplus 2A_2 \oplus D_6$
776. $^*3A_1 \oplus 2A_2 \oplus A_6$
777. $^*3A_1 \oplus 2A_2 \oplus 2A_3$
778. $^*3A_1 \oplus 3A_2 \oplus D_4$
779. $^*3A_1 \oplus 3A_2 \oplus A_4$
780. $^*3A_1 \oplus 5A_2$
781. $^*4A_1 \oplus D_9$
782. $4A_1 \oplus D_4 \oplus D_5$
783. $^*4A_1 \oplus A_9$
784. $4A_1 \oplus A_5 \oplus D_4$
785. $^*4A_1 \oplus A_4 \oplus D_5$
786. $^*4A_1 \oplus A_4 \oplus A_5$
787. $^*4A_1 \oplus A_3 \oplus E_6$
788. $4A_1 \oplus A_3 \oplus D_6$
789. $^*4A_1 \oplus A_3 \oplus A_6$
790. $4A_1 \oplus 3A_3$
791. $^*4A_1 \oplus A_2 \oplus E_7$
792. $^*4A_1 \oplus A_2 \oplus D_7$
793. $^*4A_1 \oplus A_2 \oplus A_7$
794. $4A_1 \oplus A_2 \oplus A_3 \oplus D_4$
795. $^*4A_1 \oplus A_2 \oplus A_3 \oplus A_4$
796. $^*4A_1 \oplus 2A_2 \oplus D_5$
797. $^*4A_1 \oplus 2A_2 \oplus A_5$
798. $^*4A_1 \oplus 3A_2 \oplus A_3$
799. $^*5A_1 \oplus E_8$
800. $5A_1 \oplus D_8$
801. $^*5A_1 \oplus A_8$
802. $5A_1 \oplus A_4 \oplus D_4$
803. $^*5A_1 \oplus 2A_4$
804. $5A_1 \oplus A_3 \oplus D_5$
805. $5A_1 \oplus A_3 \oplus A_5$
806. $^*5A_1 \oplus A_2 \oplus E_6$

807. $5A_1 \oplus A_2 \oplus D_6$
808. $^*5A_1 \oplus A_2 \oplus A_6$
809. $5A_1 \oplus A_2 \oplus 2A_3$
810. $5A_1 \oplus 2A_2 \oplus D_4$
811. $^*5A_1 \oplus 2A_2 \oplus A_4$
812. $^*5A_1 \oplus 4A_2$
813. $6A_1 \oplus E_7$
814. $6A_1 \oplus D_7$
815. $6A_1 \oplus A_7$
816. $6A_1 \oplus A_3 \oplus A_4$
817. $6A_1 \oplus A_2 \oplus D_5$
818. $6A_1 \oplus A_2 \oplus A_5$
819. $6A_1 \oplus 2A_2 \oplus A_3$
820. $7A_1 \oplus E_6$
821. $7A_1 \oplus A_6$
822. $7A_1 \oplus A_2 \oplus A_4$
823. $7A_1 \oplus 3A_2$

rank(Λ) = 14

824. $^*2E_7$
825. $^*E_6 \oplus E_8$
826. $^*D_{14}$
827. $^*D_8 \oplus E_6$
828. $^*D_7 \oplus E_7$
829. $^*2D_7$
830. $^*D_6 \oplus E_8$
831. $^*D_6 \oplus D_8$
832. $^*D_5 \oplus D_9$
833. $^*D_4 \oplus D_{10}$
834. $^*D_4 \oplus 2D_5$
835. $^*2D_4 \oplus E_6$
836. $2D_4 \oplus D_6$
837. $^*A_{14}$
838. $^*A_{10} \oplus D_4$
839. $^*A_9 \oplus D_5$
840. $^*A_8 \oplus E_6$
841. $^*A_8 \oplus D_6$
842. $^*A_7 \oplus E_7$
843. $^*A_7 \oplus D_7$
844. $^*2A_7$
845. $^*A_6 \oplus E_8$
846. $^*A_6 \oplus D_8$
847. $^*A_6 \oplus 2D_4$
848. $^*A_6 \oplus A_8$
849. $^*A_5 \oplus D_9$
850. $^*A_5 \oplus D_4 \oplus D_5$
851. $^*A_5 \oplus A_9$
852. $^*2A_5 \oplus D_4$
853. $^*A_4 \oplus D_{10}$
854. $^*A_4 \oplus 2D_5$
855. $^*A_4 \oplus D_4 \oplus E_6$
856. $^*A_4 \oplus D_4 \oplus D_6$
857. $^*A_4 \oplus A_{10}$
858. $^*A_4 \oplus A_6 \oplus D_4$
859. $^*A_4 \oplus A_5 \oplus D_5$
860. $^*A_4 \oplus 2A_5$
861. $^*2A_4 \oplus E_6$
862. $^*2A_4 \oplus D_6$
863. $^*2A_4 \oplus A_6$
864. $^*A_3 \oplus D_{11}$
865. $^*A_3 \oplus D_5 \oplus E_6$
866. $^*A_3 \oplus D_5 \oplus D_6$
867. $^*A_3 \oplus D_4 \oplus E_7$
868. $^*A_3 \oplus D_4 \oplus D_7$
869. $^*A_3 \oplus A_{11}$
870. $^*A_3 \oplus A_7 \oplus D_4$
871. $^*A_3 \oplus A_6 \oplus D_5$
872. $^*A_3 \oplus A_5 \oplus E_6$
873. $^*A_3 \oplus A_5 \oplus D_6$
874. $^*A_3 \oplus A_5 \oplus A_6$
875. $^*A_3 \oplus A_4 \oplus E_7$
876. $^*A_3 \oplus A_4 \oplus D_7$
877. $^*A_3 \oplus A_4 \oplus A_7$
878. $^*2A_3 \oplus E_8$
879. $^*2A_3 \oplus D_8$
880. $2A_3 \oplus 2D_4$
881. $^*2A_3 \oplus A_8$
882. $^*2A_3 \oplus A_4 \oplus D_4$
883. $^*2A_3 \oplus 2A_4$
884. $^*3A_3 \oplus D_5$
885. $^*3A_3 \oplus A_5$
886. $^*A_2 \oplus 2E_6$
887. $^*A_2 \oplus D_{12}$
888. $^*A_2 \oplus D_6 \oplus E_6$
889. $^*A_2 \oplus 2D_6$
890. $^*A_2 \oplus D_5 \oplus E_7$
891. $^*A_2 \oplus D_5 \oplus D_7$
892. $^*A_2 \oplus D_4 \oplus E_8$
893. $^*A_2 \oplus D_4 \oplus D_8$
894. $A_2 \oplus 3D_4$
895. $^*A_2 \oplus A_{12}$
896. $^*A_2 \oplus A_8 \oplus D_4$
897. $^*A_2 \oplus A_7 \oplus D_5$
898. $^*A_2 \oplus A_6 \oplus E_6$
899. $^*A_2 \oplus A_6 \oplus D_6$
900. $^*A_2 \oplus 2A_6$
901. $^*A_2 \oplus A_5 \oplus E_7$
902. $^*A_2 \oplus A_5 \oplus D_7$
903. $^*A_2 \oplus A_5 \oplus A_7$
904. $^*A_2 \oplus A_4 \oplus E_8$
905. $^*A_2 \oplus A_4 \oplus D_8$
906. $^*A_2 \oplus A_4 \oplus 2D_4$
907. $^*A_2 \oplus A_4 \oplus A_8$
908. $^*A_2 \oplus 2A_4 \oplus D_4$
909. $^*A_2 \oplus 3A_4$
910. $^*A_2 \oplus A_3 \oplus D_9$
911. $^*A_2 \oplus A_3 \oplus D_4 \oplus D_5$
912. $^*A_2 \oplus A_3 \oplus A_9$
913. $^*A_2 \oplus A_3 \oplus A_5 \oplus D_4$
914. $^*A_2 \oplus A_3 \oplus A_4 \oplus D_5$
915. $^*A_2 \oplus A_3 \oplus A_4 \oplus A_5$
916. $^*A_2 \oplus 2A_3 \oplus E_6$
917. $^*A_2 \oplus 2A_3 \oplus D_6$
918. $^*A_2 \oplus 2A_3 \oplus A_6$
919. $^*A_2 \oplus 4A_3$
920. $^*2A_2 \oplus D_{10}$
921. $^*2A_2 \oplus 2D_5$
922. $^*2A_2 \oplus D_4 \oplus E_6$
923. $^*2A_2 \oplus D_4 \oplus D_6$
924. $^*2A_2 \oplus A_{10}$
925. $^*2A_2 \oplus A_6 \oplus D_4$
926. $^*2A_2 \oplus A_5 \oplus D_5$
927. $^*2A_2 \oplus 2A_5$
928. $^*2A_2 \oplus A_4 \oplus E_6$
929. $^*2A_2 \oplus A_4 \oplus D_6$
930. $^*2A_2 \oplus A_4 \oplus A_6$
931. $^*2A_2 \oplus A_3 \oplus E_7$
932. $^*2A_2 \oplus A_3 \oplus D_7$
933. $^*2A_2 \oplus A_3 \oplus A_7$
934. $^*2A_2 \oplus 2A_3 \oplus D_4$
935. $^*2A_2 \oplus 2A_3 \oplus A_4$
936. $^*3A_2 \oplus E_8$
937. $^*3A_2 \oplus D_8$
938. $^*3A_2 \oplus 2D_4$
939. $^*3A_2 \oplus A_8$
940. $^*3A_2 \oplus A_4 \oplus D_4$
941. $^*3A_2 \oplus 2A_4$
942. $^*3A_2 \oplus A_3 \oplus D_5$
943. $^*3A_2 \oplus A_3 \oplus A_5$
944. $4A_2 \oplus E_6$
945. $^*4A_2 \oplus D_6$
946. $^*4A_2 \oplus A_6$
947. $^*4A_2 \oplus 2A_3$
948. $5A_2 \oplus D_4$
949. $5A_2 \oplus A_4$
950. $^*A_1 \oplus E_6 \oplus E_7$
951. $^*A_1 \oplus D_{13}$
952. $^*A_1 \oplus D_7 \oplus E_6$
953. $^*A_1 \oplus D_6 \oplus E_7$

954. $^*A_1 \oplus D_6 \oplus D_7$
955. $^*A_1 \oplus D_5 \oplus E_8$
956. $^*A_1 \oplus D_5 \oplus D_8$
957. $^*A_1 \oplus D_4 \oplus D_9$
958. $A_1 \oplus 2D_4 \oplus D_5$
959. $^*A_1 \oplus A_{13}$
960. $^*A_1 \oplus A_9 \oplus D_4$
961. $^*A_1 \oplus A_8 \oplus D_5$
962. $^*A_1 \oplus A_7 \oplus E_6$
963. $^*A_1 \oplus A_7 \oplus D_6$
964. $^*A_1 \oplus A_6 \oplus E_7$
965. $^*A_1 \oplus A_6 \oplus D_7$
966. $^*A_1 \oplus A_6 \oplus A_7$
967. $^*A_1 \oplus A_5 \oplus E_8$
968. $^*A_1 \oplus A_5 \oplus D_8$
969. $A_1 \oplus A_5 \oplus 2D_4$
970. $^*A_1 \oplus A_5 \oplus A_8$
971. $^*A_1 \oplus A_4 \oplus D_9$
972. $^*A_1 \oplus A_4 \oplus D_4 \oplus D_5$
973. $^*A_1 \oplus A_4 \oplus A_9$
974. $^*A_1 \oplus A_4 \oplus A_5 \oplus D_4$
975. $^*A_1 \oplus 2A_4 \oplus D_5$
976. $^*A_1 \oplus 2A_4 \oplus A_5$
977. $^*A_1 \oplus A_3 \oplus D_{10}$
978. $^*A_1 \oplus A_3 \oplus 2D_5$
979. $^*A_1 \oplus A_3 \oplus D_4 \oplus E_6$
980. $A_1 \oplus A_3 \oplus D_4 \oplus D_6$
981. $^*A_1 \oplus A_3 \oplus A_{10}$
982. $^*A_1 \oplus A_3 \oplus A_6 \oplus D_4$
983. $^*A_1 \oplus A_3 \oplus A_5 \oplus D_5$
984. $^*A_1 \oplus A_3 \oplus 2A_5$
985. $^*A_1 \oplus A_3 \oplus A_4 \oplus E_6$
986. $^*A_1 \oplus A_3 \oplus A_4 \oplus D_6$
987. $^*A_1 \oplus A_3 \oplus A_4 \oplus A_6$
988. $^*A_1 \oplus 2A_3 \oplus E_7$
989. $^*A_1 \oplus 2A_3 \oplus D_7$
990. $^*A_1 \oplus 2A_3 \oplus A_7$
991. $A_1 \oplus 3A_3 \oplus D_4$
992. $^*A_1 \oplus 3A_3 \oplus A_4$
993. $^*A_1 \oplus A_2 \oplus D_{11}$
994. $^*A_1 \oplus A_2 \oplus D_5 \oplus E_6$
995. $^*A_1 \oplus A_2 \oplus D_5 \oplus D_6$
996. $^*A_1 \oplus A_2 \oplus D_4 \oplus E_7$
997. $^*A_1 \oplus A_2 \oplus D_4 \oplus D_7$
998. $^*A_1 \oplus A_2 \oplus A_{11}$
999. $^*A_1 \oplus A_2 \oplus A_7 \oplus D_4$
1000. $^*A_1 \oplus A_2 \oplus A_6 \oplus D_5$
1001. $^*A_1 \oplus A_2 \oplus A_5 \oplus E_6$
1002. $^*A_1 \oplus A_2 \oplus A_5 \oplus D_6$
1003. $^*A_1 \oplus A_2 \oplus A_5 \oplus A_6$
1004. $^*A_1 \oplus A_2 \oplus A_4 \oplus E_7$
1005. $^*A_1 \oplus A_2 \oplus A_4 \oplus D_7$
1006. $^*A_1 \oplus A_2 \oplus A_4 \oplus A_7$
1007. $^*A_1 \oplus A_2 \oplus A_3 \oplus E_8$
1008. $^*A_1 \oplus A_2 \oplus A_3 \oplus D_8$
1009. $A_1 \oplus A_2 \oplus A_3 \oplus 2D_4$
1010. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_8$
1011. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus D_4$
1012. $^*A_1 \oplus A_2 \oplus A_3 \oplus 2A_4$
1013. $^*A_1 \oplus A_2 \oplus 2A_3 \oplus D_5$
1014. $^*A_1 \oplus A_2 \oplus 2A_3 \oplus A_5$
1015. $^*A_1 \oplus 2A_2 \oplus D_9$
1016. $^*A_1 \oplus 2A_2 \oplus D_4 \oplus D_5$
1017. $^*A_1 \oplus 2A_2 \oplus A_9$
1018. $^*A_1 \oplus 2A_2 \oplus A_5 \oplus D_4$
1019. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus D_5$
1020. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus A_5$
1021. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus E_6$
1022. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus D_6$
1023. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus A_6$
1024. $^*A_1 \oplus 2A_2 \oplus 3A_3$
1025. $^*A_1 \oplus 3A_2 \oplus E_7$
1026. $^*A_1 \oplus 3A_2 \oplus D_7$
1027. $^*A_1 \oplus 3A_2 \oplus A_7$
1028. $^*A_1 \oplus 3A_2 \oplus A_3 \oplus D_4$
1029. $^*A_1 \oplus 3A_2 \oplus A_3 \oplus A_4$
1030. $^*A_1 \oplus 4A_2 \oplus D_5$
1031. $A_1 \oplus 4A_2 \oplus A_5$
1032. $A_1 \oplus 5A_2 \oplus A_3$
1033. $^*2A_1 \oplus 2E_6$
1034. $^*2A_1 \oplus D_{12}$
1035. $^*2A_1 \oplus D_6 \oplus E_6$
1036. $2A_1 \oplus 2D_6$
1037. $^*2A_1 \oplus D_5 \oplus E_7$
1038. $^*2A_1 \oplus D_5 \oplus D_7$
1039. $^*2A_1 \oplus D_4 \oplus E_8$
1040. $2A_1 \oplus D_4 \oplus D_8$
1041. $^*2A_1 \oplus A_{12}$
1042. $^*2A_1 \oplus A_8 \oplus D_4$
1043. $^*2A_1 \oplus A_7 \oplus D_5$
1044. $^*2A_1 \oplus A_6 \oplus E_6$
1045. $^*2A_1 \oplus A_6 \oplus D_6$
1046. $^*2A_1 \oplus 2A_6$
1047. $^*2A_1 \oplus A_5 \oplus E_7$
1048. $^*2A_1 \oplus A_5 \oplus D_7$
1049. $^*2A_1 \oplus A_5 \oplus A_7$
1050. $^*2A_1 \oplus A_4 \oplus E_8$
1051. $^*2A_1 \oplus A_4 \oplus D_8$
1052. $2A_1 \oplus A_4 \oplus 2D_4$
1053. $^*2A_1 \oplus A_4 \oplus A_8$
1054. $^*2A_1 \oplus 2A_4 \oplus D_4$
1055. $^*2A_1 \oplus 3A_4$
1056. $^*2A_1 \oplus A_3 \oplus D_9$
1057. $2A_1 \oplus A_3 \oplus D_4 \oplus D_5$
1058. $^*2A_1 \oplus A_3 \oplus A_9$
1059. $2A_1 \oplus A_3 \oplus A_5 \oplus D_4$
1060. $^*2A_1 \oplus A_3 \oplus A_4 \oplus D_5$
1061. $^*2A_1 \oplus A_3 \oplus A_4 \oplus A_5$
1062. $^*2A_1 \oplus 2A_3 \oplus E_6$
1063. $2A_1 \oplus 2A_3 \oplus D_6$
1064. $^*2A_1 \oplus 2A_3 \oplus A_6$
1065. $2A_1 \oplus 4A_3$
1066. $^*2A_1 \oplus A_2 \oplus D_{10}$
1067. $^*2A_1 \oplus A_2 \oplus 2D_5$
1068. $^*2A_1 \oplus A_2 \oplus D_4 \oplus E_6$
1069. $2A_1 \oplus A_2 \oplus D_4 \oplus D_6$
1070. $^*2A_1 \oplus A_2 \oplus A_{10}$
1071. $^*2A_1 \oplus A_2 \oplus A_6 \oplus D_4$
1072. $^*2A_1 \oplus A_2 \oplus A_5 \oplus D_5$
1073. $^*2A_1 \oplus A_2 \oplus 2A_5$
1074. $^*2A_1 \oplus A_2 \oplus A_4 \oplus E_6$
1075. $^*2A_1 \oplus A_2 \oplus A_4 \oplus D_6$
1076. $^*2A_1 \oplus A_2 \oplus A_4 \oplus A_6$
1077. $^*2A_1 \oplus A_2 \oplus A_3 \oplus E_7$
1078. $^*2A_1 \oplus A_2 \oplus A_3 \oplus D_7$
1079. $^*2A_1 \oplus A_2 \oplus A_3 \oplus A_7$
1080. $2A_1 \oplus A_2 \oplus 2A_3 \oplus D_4$
1081. $^*2A_1 \oplus A_2 \oplus 2A_3 \oplus A_4$
1082. $^*2A_1 \oplus 2A_2 \oplus E_8$
1083. $^*2A_1 \oplus 2A_2 \oplus D_8$

1084. $2A_1 \oplus 2A_2 \oplus 2D_4$
1085. $^*2A_1 \oplus 2A_2 \oplus A_8$
1086. $^*2A_1 \oplus 2A_2 \oplus A_4 \oplus D_4$
1087. $^*2A_1 \oplus 2A_2 \oplus 2A_4$
1088. $^*2A_1 \oplus 2A_2 \oplus A_3 \oplus D_5$
1089. $^*2A_1 \oplus 2A_2 \oplus A_3 \oplus A_5$
1090. $^*2A_1 \oplus 3A_2 \oplus E_6$
1091. $^*2A_1 \oplus 3A_2 \oplus D_6$
1092. $^*2A_1 \oplus 3A_2 \oplus A_6$
1093. $^*2A_1 \oplus 3A_2 \oplus 2A_3$
1094. $^*2A_1 \oplus 4A_2 \oplus D_4$
1095. $^*2A_1 \oplus 4A_2 \oplus A_4$
1096. $2A_1 \oplus 6A_2$
1097. $^*3A_1 \oplus D_{11}$
1098. $^*3A_1 \oplus D_5 \oplus E_6$
1099. $3A_1 \oplus D_5 \oplus D_6$
1100. $3A_1 \oplus D_4 \oplus E_7$
1101. $3A_1 \oplus D_4 \oplus D_7$
1102. $^*3A_1 \oplus A_{11}$
1103. $3A_1 \oplus A_7 \oplus D_4$
1104. $^*3A_1 \oplus A_6 \oplus D_5$
1105. $^*3A_1 \oplus A_5 \oplus E_6$
1106. $3A_1 \oplus A_5 \oplus D_6$
1107. $^*3A_1 \oplus A_5 \oplus A_6$
1108. $^*3A_1 \oplus A_4 \oplus E_7$
1109. $^*3A_1 \oplus A_4 \oplus D_7$
1110. $^*3A_1 \oplus A_4 \oplus A_7$
1111. $^*3A_1 \oplus A_3 \oplus E_8$
1112. $3A_1 \oplus A_3 \oplus D_8$
1113. $^*3A_1 \oplus A_3 \oplus A_8$
1114. $3A_1 \oplus A_3 \oplus A_4 \oplus D_4$
1115. $^*3A_1 \oplus A_3 \oplus 2A_4$
1116. $3A_1 \oplus 2A_3 \oplus D_5$
1117. $3A_1 \oplus 2A_3 \oplus A_5$
1118. $^*3A_1 \oplus A_2 \oplus D_9$
1119. $3A_1 \oplus A_2 \oplus D_4 \oplus D_5$
1120. $^*3A_1 \oplus A_2 \oplus A_9$
1121. $3A_1 \oplus A_2 \oplus A_5 \oplus D_4$
1122. $^*3A_1 \oplus A_2 \oplus A_4 \oplus D_5$
1123. $^*3A_1 \oplus A_2 \oplus A_4 \oplus A_5$
1124. $^*3A_1 \oplus A_2 \oplus A_3 \oplus E_6$
1125. $3A_1 \oplus A_2 \oplus A_3 \oplus D_6$
1126. $^*3A_1 \oplus A_2 \oplus A_3 \oplus A_6$
1127. $3A_1 \oplus A_2 \oplus 3A_3$
1128. $^*3A_1 \oplus 2A_2 \oplus E_7$
1129. $^*3A_1 \oplus 2A_2 \oplus D_7$
1130. $^*3A_1 \oplus 2A_2 \oplus A_7$
1131. $3A_1 \oplus 2A_2 \oplus A_3 \oplus D_4$
1132. $^*3A_1 \oplus 2A_2 \oplus A_3 \oplus A_4$
1133. $^*3A_1 \oplus 3A_2 \oplus D_5$
1134. $^*3A_1 \oplus 3A_2 \oplus A_5$
1135. $^*3A_1 \oplus 4A_2 \oplus A_3$
1136. $4A_1 \oplus D_{10}$
1137. $4A_1 \oplus 2D_5$
1138. $4A_1 \oplus D_4 \oplus E_6$
1139. $^*4A_1 \oplus A_{10}$
1140. $4A_1 \oplus A_6 \oplus D_4$
1141. $4A_1 \oplus A_5 \oplus D_5$
1142. $4A_1 \oplus 2A_5$
1143. $^*4A_1 \oplus A_4 \oplus E_6$
1144. $4A_1 \oplus A_4 \oplus D_6$
1145. $^*4A_1 \oplus A_4 \oplus A_6$
1146. $4A_1 \oplus A_3 \oplus E_7$
1147. $4A_1 \oplus A_3 \oplus D_7$
1148. $4A_1 \oplus A_3 \oplus A_7$
1149. $4A_1 \oplus 2A_3 \oplus A_4$
1150. $^*4A_1 \oplus A_2 \oplus E_8$
1151. $4A_1 \oplus A_2 \oplus D_8$
1152. $^*4A_1 \oplus A_2 \oplus A_8$
1153. $4A_1 \oplus A_2 \oplus A_4 \oplus D_4$
1154. $^*4A_1 \oplus A_2 \oplus 2A_4$
1155. $4A_1 \oplus A_2 \oplus A_3 \oplus D_5$
1156. $4A_1 \oplus A_2 \oplus A_3 \oplus A_5$
1157. $^*4A_1 \oplus 2A_2 \oplus E_6$
1158. $4A_1 \oplus 2A_2 \oplus D_6$
1159. $^*4A_1 \oplus 2A_2 \oplus A_6$
1160. $4A_1 \oplus 2A_2 \oplus 2A_3$
1161. $4A_1 \oplus 3A_2 \oplus D_4$
1162. $^*4A_1 \oplus 3A_2 \oplus A_4$
1163. $4A_1 \oplus 5A_2$
1164. $5A_1 \oplus D_9$
1165. $5A_1 \oplus A_9$
1166. $5A_1 \oplus A_4 \oplus D_5$
1167. $5A_1 \oplus A_4 \oplus A_5$
1168. $5A_1 \oplus A_3 \oplus E_6$
1169. $5A_1 \oplus A_3 \oplus A_6$
1170. $5A_1 \oplus A_2 \oplus E_7$
1171. $5A_1 \oplus A_2 \oplus D_7$
1172. $5A_1 \oplus A_2 \oplus A_7$
1173. $5A_1 \oplus A_2 \oplus A_3 \oplus A_4$
1174. $5A_1 \oplus 2A_2 \oplus D_5$
1175. $5A_1 \oplus 2A_2 \oplus A_5$
1176. $5A_1 \oplus 3A_2 \oplus A_3$
1177. $6A_1 \oplus E_8$
1178. $6A_1 \oplus A_8$
1179. $6A_1 \oplus 2A_4$
1180. $6A_1 \oplus A_2 \oplus E_6$
1181. $6A_1 \oplus A_2 \oplus A_6$
1182. $6A_1 \oplus 2A_2 \oplus A_4$
1183. $6A_1 \oplus 4A_2$

rank($\boldsymbol{\Lambda}$) = 15

1184. $^*E_7 \oplus E_8$
1185. $^*D_{15}$
1186. $^*D_9 \oplus E_6$
1187. $^*D_8 \oplus E_7$
1188. $^*D_7 \oplus E_8$
1189. $^*D_7 \oplus D_8$
1190. $^*D_6 \oplus D_9$
1191. $^*D_5 \oplus D_{10}$
1192. $^*3D_5$
1193. $^*D_4 \oplus D_{11}$
1194. $^*D_4 \oplus D_5 \oplus E_6$
1195. $D_4 \oplus D_5 \oplus D_6$
1196. $2D_4 \oplus E_7$
1197. $2D_4 \oplus D_7$
1198. $^*A_{15}$
1199. $^*A_{11} \oplus D_4$
1200. $^*A_{10} \oplus D_5$
1201. $^*A_9 \oplus E_6$
1202. $^*A_9 \oplus D_6$
1203. $^*A_8 \oplus E_7$
1204. $^*A_8 \oplus D_7$
1205. $^*A_7 \oplus E_8$
1206. $^*A_7 \oplus D_8$
1207. $A_7 \oplus 2D_4$
1208. $^*A_7 \oplus A_8$
1209. $^*A_6 \oplus D_9$
1210. $^*A_6 \oplus D_4 \oplus D_5$
1211. $^*A_6 \oplus A_9$
1212. $^*A_5 \oplus D_{10}$
1213. $^*A_5 \oplus 2D_5$
1214. $^*A_5 \oplus D_4 \oplus E_6$
1215. $A_5 \oplus D_4 \oplus D_6$
1216. $^*A_5 \oplus A_{10}$
1217. $^*A_5 \oplus A_6 \oplus D_4$
1218. $^*2A_5 \oplus D_5$
1219. $^*3A_5$
1220. $^*A_4 \oplus D_{11}$
1221. $^*A_4 \oplus D_5 \oplus E_6$
1222. $^*A_4 \oplus D_5 \oplus D_6$
1223. $^*A_4 \oplus D_4 \oplus E_7$

1224. $^*A_4 \oplus D_4 \oplus D_7$
1225. $^*A_4 \oplus A_{11}$
1226. $^*A_4 \oplus A_7 \oplus D_4$
1227. $^*A_4 \oplus A_6 \oplus D_5$
1228. $^*A_4 \oplus A_5 \oplus E_6$
1229. $^*A_4 \oplus A_5 \oplus D_6$
1230. $^*A_4 \oplus A_5 \oplus A_6$
1231. $^*2A_4 \oplus E_7$
1232. $^*2A_4 \oplus D_7$
1233. $^*2A_4 \oplus A_7$
1234. $^*A_3 \oplus 2E_6$
1235. $^*A_3 \oplus D_{12}$
1236. $^*A_3 \oplus D_6 \oplus E_6$
1237. $A_3 \oplus 2D_6$
1238. $^*A_3 \oplus D_5 \oplus E_7$
1239. $^*A_3 \oplus D_5 \oplus D_7$
1240. $^*A_3 \oplus D_4 \oplus E_8$
1241. $A_3 \oplus D_4 \oplus D_8$
1242. $^*A_3 \oplus A_{12}$
1243. $^*A_3 \oplus A_8 \oplus D_4$
1244. $^*A_3 \oplus A_7 \oplus D_5$
1245. $^*A_3 \oplus A_6 \oplus E_6$
1246. $^*A_3 \oplus A_6 \oplus D_6$
1247. $^*A_3 \oplus 2A_6$
1248. $^*A_3 \oplus A_5 \oplus E_7$
1249. $^*A_3 \oplus A_5 \oplus D_7$
1250. $^*A_3 \oplus A_5 \oplus A_7$
1251. $^*A_3 \oplus A_4 \oplus E_8$
1252. $^*A_3 \oplus A_4 \oplus D_8$
1253. $A_3 \oplus A_4 \oplus 2D_4$
1254. $^*A_3 \oplus A_4 \oplus A_8$
1255. $^*A_3 \oplus 2A_4 \oplus D_4$
1256. $^*A_3 \oplus 3A_4$
1257. $^*2A_3 \oplus D_9$
1258. $2A_3 \oplus D_4 \oplus D_5$
1259. $^*2A_3 \oplus A_9$
1260. $2A_3 \oplus A_5 \oplus D_4$
1261. $^*2A_3 \oplus A_4 \oplus D_5$
1262. $^*2A_3 \oplus A_4 \oplus A_5$
1263. $^*3A_3 \oplus E_6$
1264. $3A_3 \oplus D_6$
1265. $^*3A_3 \oplus A_6$
1266. $5A_3$
1267. $^*A_2 \oplus E_6 \oplus E_7$
1268. $^*A_2 \oplus D_{13}$
1269. $^*A_2 \oplus D_7 \oplus E_6$
1270. $^*A_2 \oplus D_6 \oplus E_7$
1271. $^*A_2 \oplus D_6 \oplus D_7$
1272. $^*A_2 \oplus D_5 \oplus E_8$
1273. $^*A_2 \oplus D_5 \oplus D_8$
1274. $^*A_2 \oplus D_4 \oplus D_9$
1275. $A_2 \oplus 2D_4 \oplus D_5$
1276. $^*A_2 \oplus A_{13}$
1277. $^*A_2 \oplus A_9 \oplus D_4$
1278. $^*A_2 \oplus A_8 \oplus D_5$
1279. $^*A_2 \oplus A_7 \oplus E_6$
1280. $^*A_2 \oplus A_7 \oplus D_6$
1281. $^*A_2 \oplus A_6 \oplus E_7$
1282. $^*A_2 \oplus A_6 \oplus D_7$
1283. $^*A_2 \oplus A_6 \oplus A_7$
1284. $^*A_2 \oplus A_5 \oplus E_8$
1285. $^*A_2 \oplus A_5 \oplus D_8$
1286. $A_2 \oplus A_5 \oplus 2D_4$
1287. $^*A_2 \oplus A_5 \oplus A_8$
1288. $^*A_2 \oplus A_4 \oplus D_9$
1289. $^*A_2 \oplus A_4 \oplus D_4 \oplus D_5$
1290. $^*A_2 \oplus A_4 \oplus A_9$
1291. $^*A_2 \oplus A_4 \oplus A_5 \oplus D_4$
1292. $^*A_2 \oplus 2A_4 \oplus D_5$
1293. $^*A_2 \oplus 2A_4 \oplus A_5$
1294. $^*A_2 \oplus A_3 \oplus D_{10}$
1295. $^*A_2 \oplus A_3 \oplus 2D_5$
1296. $^*A_2 \oplus A_3 \oplus D_4 \oplus E_6$
1297. $A_2 \oplus A_3 \oplus D_4 \oplus D_6$
1298. $^*A_2 \oplus A_3 \oplus A_{10}$
1299. $^*A_2 \oplus A_3 \oplus A_6 \oplus D_4$
1300. $^*A_2 \oplus A_3 \oplus A_5 \oplus D_5$
1301. $^*A_2 \oplus A_3 \oplus 2A_5$
1302. $^*A_2 \oplus A_3 \oplus A_4 \oplus E_6$
1303. $^*A_2 \oplus A_3 \oplus A_4 \oplus D_6$
1304. $^*A_2 \oplus A_3 \oplus A_4 \oplus A_6$
1305. $^*A_2 \oplus 2A_3 \oplus E_7$
1306. $^*A_2 \oplus 2A_3 \oplus D_7$
1307. $^*A_2 \oplus 2A_3 \oplus A_7$
1308. $A_2 \oplus 3A_3 \oplus D_4$
1309. $^*A_2 \oplus 3A_3 \oplus A_4$
1310. $^*2A_2 \oplus D_{11}$
1311. $^*2A_2 \oplus D_5 \oplus E_6$
1312. $^*2A_2 \oplus D_5 \oplus D_6$
1313. $^*2A_2 \oplus D_4 \oplus E_7$
1314. $^*2A_2 \oplus D_4 \oplus D_7$
1315. $^*2A_2 \oplus A_{11}$
1316. $^*2A_2 \oplus A_7 \oplus D_4$
1317. $^*2A_2 \oplus A_6 \oplus D_5$
1318. $2A_2 \oplus A_5 \oplus E_6$
1319. $^*2A_2 \oplus A_5 \oplus D_6$
1320. $^*2A_2 \oplus A_5 \oplus A_6$
1321. $^*2A_2 \oplus A_4 \oplus E_7$
1322. $^*2A_2 \oplus A_4 \oplus D_7$
1323. $^*2A_2 \oplus A_4 \oplus A_7$
1324. $^*2A_2 \oplus A_3 \oplus E_8$
1325. $^*2A_2 \oplus A_3 \oplus D_8$
1326. $2A_2 \oplus A_3 \oplus 2D_4$
1327. $^*2A_2 \oplus A_3 \oplus A_8$
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1329. $^*2A_2 \oplus A_3 \oplus 2A_4$
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1332. $^*3A_2 \oplus D_9$
1333. $^*3A_2 \oplus D_4 \oplus D_5$
1334. $^*3A_2 \oplus A_9$
1335. $3A_2 \oplus A_5 \oplus D_4$
1336. $^*3A_2 \oplus A_4 \oplus D_5$
1337. $3A_2 \oplus A_4 \oplus A_5$
1338. $3A_2 \oplus A_3 \oplus E_6$
1339. $^*3A_2 \oplus A_3 \oplus D_6$
1340. $^*3A_2 \oplus A_3 \oplus A_6$
1341. $^*3A_2 \oplus 3A_3$
1342. $4A_2 \oplus E_7$
1343. $4A_2 \oplus D_7$
1344. $4A_2 \oplus A_7$
1345. $4A_2 \oplus A_3 \oplus D_4$
1346. $4A_2 \oplus A_3 \oplus A_4$
1347. $5A_2 \oplus D_5$
1348. $^*A_1 \oplus 2E_7$
1349. $^*A_1 \oplus E_6 \oplus E_8$
1350. $^*A_1 \oplus D_{14}$
1351. $^*A_1 \oplus D_8 \oplus E_6$
1352. $^*A_1 \oplus D_7 \oplus E_7$
1353. $^*A_1 \oplus 2D_7$
1354. $^*A_1 \oplus D_6 \oplus E_8$
1355. $A_1 \oplus D_6 \oplus D_8$
1356. $^*A_1 \oplus D_5 \oplus D_9$
1357. $A_1 \oplus D_4 \oplus D_{10}$
1358. $A_1 \oplus D_4 \oplus 2D_5$
1359. $A_1 \oplus 2D_4 \oplus E_6$
1360. $^*A_1 \oplus A_{14}$
1361. $^*A_1 \oplus A_{10} \oplus D_4$
1362. $^*A_1 \oplus A_9 \oplus D_5$
1363. $^*A_1 \oplus A_8 \oplus E_6$
1364. $^*A_1 \oplus A_8 \oplus D_6$
1365. $^*A_1 \oplus A_7 \oplus E_7$
1366. $^*A_1 \oplus A_7 \oplus D_7$
1367. $^*A_1 \oplus 2A_7$
1368. $^*A_1 \oplus A_6 \oplus E_8$
1369. $^*A_1 \oplus A_6 \oplus D_8$
1370. $A_1 \oplus A_6 \oplus 2D_4$

1371. $^*A_1 \oplus A_6 \oplus A_8$

1372. $^*A_1 \oplus A_5 \oplus D_9$

1373. $A_1 \oplus A_5 \oplus D_4 \oplus D_5$

1374. $^*A_1 \oplus A_5 \oplus A_9$

1375. $A_1 \oplus 2A_5 \oplus D_4$

1376. $^*A_1 \oplus A_4 \oplus D_{10}$

1377. $^*A_1 \oplus A_4 \oplus 2D_5$

1378. $^*A_1 \oplus A_4 \oplus D_4 \oplus E_6$

1379. $A_1 \oplus A_4 \oplus D_4 \oplus D_6$

1380. $^*A_1 \oplus A_4 \oplus A_{10}$

1381. $^*A_1 \oplus A_4 \oplus A_6 \oplus D_4$

1382. $^*A_1 \oplus A_4 \oplus A_5 \oplus D_5$

1383. $^*A_1 \oplus A_4 \oplus 2A_5$

1384. $^*A_1 \oplus 2A_4 \oplus E_6$

1385. $^*A_1 \oplus 2A_4 \oplus D_6$

1386. $^*A_1 \oplus 2A_4 \oplus A_6$

1387. $^*A_1 \oplus A_3 \oplus D_{11}$

1388. $^*A_1 \oplus A_3 \oplus D_5 \oplus E_6$

1389. $A_1 \oplus A_3 \oplus D_5 \oplus D_6$

1390. $A_1 \oplus A_3 \oplus D_4 \oplus E_7$

1391. $A_1 \oplus A_3 \oplus D_4 \oplus D_7$

1392. $^*A_1 \oplus A_3 \oplus A_{11}$

1393. $A_1 \oplus A_3 \oplus A_7 \oplus D_4$

1394. $^*A_1 \oplus A_3 \oplus A_6 \oplus D_5$

1395. $^*A_1 \oplus A_3 \oplus A_5 \oplus E_6$

1396. $A_1 \oplus A_3 \oplus A_5 \oplus D_6$

1397. $^*A_1 \oplus A_3 \oplus A_5 \oplus A_6$

1398. $^*A_1 \oplus A_3 \oplus A_4 \oplus E_7$

1399. $^*A_1 \oplus A_3 \oplus A_4 \oplus D_7$

1400. $^*A_1 \oplus A_3 \oplus A_4 \oplus A_7$

1401. $^*A_1 \oplus 2A_3 \oplus E_8$

1402. $A_1 \oplus 2A_3 \oplus D_8$

1403. $^*A_1 \oplus 2A_3 \oplus A_8$

1404. $A_1 \oplus 2A_3 \oplus A_4 \oplus D_4$

1405. $^*A_1 \oplus 2A_3 \oplus 2A_4$

1406. $A_1 \oplus 3A_3 \oplus D_5$

1407. $A_1 \oplus 3A_3 \oplus A_5$

1408. $^*A_1 \oplus A_2 \oplus 2E_6$

1409. $^*A_1 \oplus A_2 \oplus D_{12}$

1410. $^*A_1 \oplus A_2 \oplus D_6 \oplus E_6$

1411. $A_1 \oplus A_2 \oplus 2D_6$

1412. $^*A_1 \oplus A_2 \oplus D_5 \oplus E_7$

1413. $^*A_1 \oplus A_2 \oplus D_5 \oplus D_7$

1414. $^*A_1 \oplus A_2 \oplus D_4 \oplus E_8$

1415. $A_1 \oplus A_2 \oplus D_4 \oplus D_8$

1416. $^*A_1 \oplus A_2 \oplus A_{12}$

1417. $^*A_1 \oplus A_2 \oplus A_8 \oplus D_4$

1418. $^*A_1 \oplus A_2 \oplus A_7 \oplus D_5$

1419. $^*A_1 \oplus A_2 \oplus A_6 \oplus E_6$

1420. $^*A_1 \oplus A_2 \oplus A_6 \oplus D_6$

1421. $^*A_1 \oplus A_2 \oplus 2A_6$

1422. $^*A_1 \oplus A_2 \oplus A_5 \oplus E_7$

1423. $^*A_1 \oplus A_2 \oplus A_5 \oplus D_7$

1424. $^*A_1 \oplus A_2 \oplus A_5 \oplus A_7$

1425. $^*A_1 \oplus A_2 \oplus A_4 \oplus E_8$

1426. $^*A_1 \oplus A_2 \oplus A_4 \oplus D_8$

1427. $A_1 \oplus A_2 \oplus A_4 \oplus 2D_4$

1428. $^*A_1 \oplus A_2 \oplus A_4 \oplus A_8$

1429. $^*A_1 \oplus A_2 \oplus 2A_4 \oplus D_4$

1430. $^*A_1 \oplus A_2 \oplus 3A_4$

1431. $^*A_1 \oplus A_2 \oplus A_3 \oplus D_9$

1432. $A_1 \oplus A_2 \oplus A_3 \oplus D_4 \oplus D_5$

1433. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_9$

1434. $A_1 \oplus A_2 \oplus A_3 \oplus A_5 \oplus D_4$

1435. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus D_5$

1436. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$

1437. $^*A_1 \oplus A_2 \oplus 2A_3 \oplus E_6$

1438. $A_1 \oplus A_2 \oplus 2A_3 \oplus D_6$

1439. $^*A_1 \oplus A_2 \oplus 2A_3 \oplus A_6$

1440. $A_1 \oplus A_2 \oplus 4A_3$

1441. $^*A_1 \oplus 2A_2 \oplus D_{10}$

1442. $^*A_1 \oplus 2A_2 \oplus 2D_5$

1443. $^*A_1 \oplus 2A_2 \oplus D_4 \oplus E_6$

1444. $A_1 \oplus 2A_2 \oplus D_4 \oplus D_6$

1445. $^*A_1 \oplus 2A_2 \oplus A_{10}$

1446. $^*A_1 \oplus 2A_2 \oplus A_6 \oplus D_4$

1447. $^*A_1 \oplus 2A_2 \oplus A_5 \oplus D_5$

1448. $A_1 \oplus 2A_2 \oplus 2A_5$

1449. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus E_6$

1450. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus D_6$

1451. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus A_6$

1452. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus E_7$

1453. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus D_7$

1454. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus A_7$

1455. $A_1 \oplus 2A_2 \oplus 2A_3 \oplus D_4$

1456. $^*A_1 \oplus 2A_2 \oplus 2A_3 \oplus A_4$

1457. $^*A_1 \oplus 3A_2 \oplus E_8$

1458. $^*A_1 \oplus 3A_2 \oplus D_8$

1459. $A_1 \oplus 3A_2 \oplus 2D_4$

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1462. $^*A_1 \oplus 3A_2 \oplus 2A_4$

1463. $^*A_1 \oplus 3A_2 \oplus A_3 \oplus D_5$

1464. $A_1 \oplus 3A_2 \oplus A_3 \oplus A_5$

1465. $A_1 \oplus 4A_2 \oplus E_6$

1466. $A_1 \oplus 4A_2 \oplus D_6$

1467. $A_1 \oplus 4A_2 \oplus A_6$

1468. $A_1 \oplus 4A_2 \oplus 2A_3$

1469. $A_1 \oplus 5A_2 \oplus A_4$

1470. $^*2A_1 \oplus E_6 \oplus E_7$

1471. $^*2A_1 \oplus D_{13}$

1472. $^*2A_1 \oplus D_7 \oplus E_6$

1473. $2A_1 \oplus D_6 \oplus E_7$

1474. $2A_1 \oplus D_6 \oplus D_7$

1475. $^*2A_1 \oplus D_5 \oplus E_8$

1476. $2A_1 \oplus D_5 \oplus D_8$

1477. $2A_1 \oplus D_4 \oplus D_9$

1478. $^*2A_1 \oplus A_{13}$

1479. $2A_1 \oplus A_9 \oplus D_4$

1480. $^*2A_1 \oplus A_8 \oplus D_5$

1481. $^*2A_1 \oplus A_7 \oplus E_6$

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1483. $^*2A_1 \oplus A_6 \oplus E_7$

1484. $^*2A_1 \oplus A_6 \oplus D_7$

1485. $^*2A_1 \oplus A_6 \oplus A_7$

1486. $^*2A_1 \oplus A_5 \oplus E_8$

1487. $2A_1 \oplus A_5 \oplus D_8$

1488. $^*2A_1 \oplus A_5 \oplus A_8$

1489. $^*2A_1 \oplus A_4 \oplus D_9$

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1491. $^*2A_1 \oplus A_4 \oplus A_9$

1492. $2A_1 \oplus A_4 \oplus A_5 \oplus D_4$

1493. $^*2A_1 \oplus 2A_4 \oplus D_5$

1494. $^*2A_1 \oplus 2A_4 \oplus A_5$

1495. $2A_1 \oplus A_3 \oplus D_{10}$

1496. $2A_1 \oplus A_3 \oplus 2D_5$

1497. $2A_1 \oplus A_3 \oplus D_4 \oplus E_6$

1498. $^*2A_1 \oplus A_3 \oplus A_{10}$

1499. $2A_1 \oplus A_3 \oplus A_6 \oplus D_4$

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1501. $2A_1 \oplus A_3 \oplus 2A_5$

1502. $^*2A_1 \oplus A_3 \oplus A_4 \oplus E_6$

1503. $2A_1 \oplus A_3 \oplus A_4 \oplus D_6$

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1541. $^*2A_1 \oplus 3A_2 \oplus D_7$
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1559. $^*3A_1 \oplus 2A_6$
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1574. $3A_1 \oplus A_2 \oplus D_{10}$
1575. $3A_1 \oplus A_2 \oplus 2D_5$
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1577. $^*3A_1 \oplus A_2 \oplus A_{10}$
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1582. $3A_1 \oplus A_2 \oplus A_4 \oplus D_6$
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1588. $^*3A_1 \oplus 2A_2 \oplus E_8$
1589. $3A_1 \oplus 2A_2 \oplus D_8$
1590. $^*3A_1 \oplus 2A_2 \oplus A_8$
1591. $3A_1 \oplus 2A_2 \oplus A_4 \oplus D_4$
1592. $^*3A_1 \oplus 2A_2 \oplus 2A_4$
1593. $3A_1 \oplus 2A_2 \oplus A_3 \oplus D_5$
1594. $3A_1 \oplus 2A_2 \oplus A_3 \oplus A_5$
1595. $3A_1 \oplus 3A_2 \oplus E_6$
1596. $3A_1 \oplus 3A_2 \oplus D_6$
1597. $^*3A_1 \oplus 3A_2 \oplus A_6$
1598. $3A_1 \oplus 3A_2 \oplus 2A_3$
1599. $3A_1 \oplus 4A_2 \oplus D_4$
1600. $3A_1 \oplus 4A_2 \oplus A_4$
1601. $4A_1 \oplus D_{11}$
1602. $4A_1 \oplus D_5 \oplus E_6$
1603. $4A_1 \oplus A_{11}$
1604. $4A_1 \oplus A_6 \oplus D_5$
1605. $4A_1 \oplus A_5 \oplus E_6$
1606. $4A_1 \oplus A_5 \oplus A_6$
1607. $4A_1 \oplus A_4 \oplus E_7$
1608. $4A_1 \oplus A_4 \oplus D_7$
1609. $4A_1 \oplus A_4 \oplus A_7$
1610. $4A_1 \oplus A_3 \oplus E_8$
1611. $4A_1 \oplus A_3 \oplus A_8$
1612. $4A_1 \oplus A_3 \oplus 2A_4$
1613. $4A_1 \oplus A_2 \oplus D_9$
1614. $4A_1 \oplus A_2 \oplus A_9$
1615. $4A_1 \oplus A_2 \oplus A_4 \oplus D_5$
1616. $4A_1 \oplus A_2 \oplus A_4 \oplus A_5$
1617. $4A_1 \oplus A_2 \oplus A_3 \oplus E_6$
1618. $4A_1 \oplus A_2 \oplus A_3 \oplus A_6$
1619. $4A_1 \oplus 2A_2 \oplus E_7$
1620. $4A_1 \oplus 2A_2 \oplus D_7$
1621. $4A_1 \oplus 2A_2 \oplus A_7$
1622. $4A_1 \oplus 2A_2 \oplus A_3 \oplus A_4$
1623. $4A_1 \oplus 3A_2 \oplus D_5$
1624. $4A_1 \oplus 3A_2 \oplus A_5$
1625. $4A_1 \oplus 4A_2 \oplus A_3$
1626. $5A_1 \oplus A_{10}$
1627. $5A_1 \oplus A_4 \oplus E_6$
1628. $5A_1 \oplus A_4 \oplus A_6$
1629. $5A_1 \oplus A_2 \oplus E_8$
1630. $5A_1 \oplus A_2 \oplus A_8$

1631. $5A_1 \oplus A_2 \oplus 2A_4$

1632. $5A_1 \oplus 2A_2 \oplus E_6$

1633. $5A_1 \oplus 2A_2 \oplus A_6$

1634. $5A_1 \oplus 3A_2 \oplus A_4$

rank(Λ) = 16

1635. $^*2E_8$

1636. $^*D_{16}$

1637. $^*D_{10} \oplus E_6$

1638. $^*D_9 \oplus E_7$

1639. $^*D_8 \oplus E_8$

1640. $2D_8$

1641. $^*D_7 \oplus D_9$

1642. $D_6 \oplus D_{10}$

1643. $^*D_5 \oplus D_{11}$

1644. $^*2D_5 \oplus E_6$

1645. $2D_5 \oplus D_6$

1646. $^*D_4 \oplus 2E_6$

1647. $D_4 \oplus D_{12}$

1648. $D_4 \oplus D_6 \oplus E_6$

1649. $D_4 \oplus D_5 \oplus E_7$

1650. $D_4 \oplus D_5 \oplus D_7$

1651. $2D_4 \oplus E_8$

1652. $^*A_{16}$

1653. $^*A_{12} \oplus D_4$

1654. $^*A_{11} \oplus D_5$

1655. $^*A_{10} \oplus E_6$

1656. $^*A_{10} \oplus D_6$

1657. $^*A_9 \oplus E_7$

1658. $^*A_9 \oplus D_7$

1659. $^*A_8 \oplus E_8$

1660. $^*A_8 \oplus D_8$

1661. $A_8 \oplus 2D_4$

1662. $^*2A_8$

1663. $^*A_7 \oplus D_9$

1664. $A_7 \oplus D_4 \oplus D_5$

1665. $^*A_7 \oplus A_9$

1666. $^*A_6 \oplus D_{10}$

1667. $^*A_6 \oplus 2D_5$

1668. $^*A_6 \oplus D_4 \oplus E_6$

1669. $A_6 \oplus D_4 \oplus D_6$

1670. $^*A_6 \oplus A_{10}$

1671. $^*2A_6 \oplus D_4$

1672. $^*A_5 \oplus D_{11}$

1673. $^*A_5 \oplus D_5 \oplus E_6$

1674. $A_5 \oplus D_5 \oplus D_6$

1675. $A_5 \oplus D_4 \oplus E_7$

1676. $A_5 \oplus D_4 \oplus D_7$

1677. $^*A_5 \oplus A_{11}$

1678. $A_5 \oplus A_7 \oplus D_4$

1679. $^*A_5 \oplus A_6 \oplus D_5$

1680. $2A_5 \oplus E_6$

1681. $2A_5 \oplus D_6$

1682. $^*2A_5 \oplus A_6$

1683. $^*A_4 \oplus 2E_6$

1684. $^*A_4 \oplus D_{12}$

1685. $^*A_4 \oplus D_6 \oplus E_6$

1686. $A_4 \oplus 2D_6$

1687. $^*A_4 \oplus D_5 \oplus E_7$

1688. $^*A_4 \oplus D_5 \oplus D_7$

1689. $^*A_4 \oplus D_4 \oplus E_8$

1690. $A_4 \oplus D_4 \oplus D_8$

1691. $^*A_4 \oplus A_{12}$

1692. $^*A_4 \oplus A_8 \oplus D_4$

1693. $^*A_4 \oplus A_7 \oplus D_5$

1694. $^*A_4 \oplus A_6 \oplus E_6$

1695. $^*A_4 \oplus A_6 \oplus D_6$

1696. $^*A_4 \oplus 2A_6$

1697. $^*A_4 \oplus A_5 \oplus E_7$

1698. $^*A_4 \oplus A_5 \oplus D_7$

1699. $^*A_4 \oplus A_5 \oplus A_7$

1700. $^*2A_4 \oplus E_8$

1701. $^*2A_4 \oplus D_8$

1702. $2A_4 \oplus 2D_4$

1703. $^*2A_4 \oplus A_8$

1704. $^*3A_4 \oplus D_4$

1705. $4A_4$

1706. $^*A_3 \oplus E_6 \oplus E_7$

1707. $^*A_3 \oplus D_{13}$

1708. $^*A_3 \oplus D_7 \oplus E_6$

1709. $A_3 \oplus D_6 \oplus E_7$

1710. $A_3 \oplus D_6 \oplus D_7$

1711. $^*A_3 \oplus D_5 \oplus E_8$

1712. $A_3 \oplus D_5 \oplus D_8$

1713. $A_3 \oplus D_4 \oplus D_9$

1714. $^*A_3 \oplus A_{13}$

1715. $A_3 \oplus A_9 \oplus D_4$

1716. $^*A_3 \oplus A_8 \oplus D_5$

1717. $^*A_3 \oplus A_7 \oplus E_6$

1718. $A_3 \oplus A_7 \oplus D_6$

1719. $^*A_3 \oplus A_6 \oplus E_7$

1720. $^*A_3 \oplus A_6 \oplus D_7$

1721. $^*A_3 \oplus A_6 \oplus A_7$

1722. $^*A_3 \oplus A_5 \oplus E_8$

1723. $A_3 \oplus A_5 \oplus D_8$

1724. $^*A_3 \oplus A_5 \oplus A_8$

1725. $^*A_3 \oplus A_4 \oplus D_9$

1726. $A_3 \oplus A_4 \oplus D_4 \oplus D_5$

1727. $^*A_3 \oplus A_4 \oplus A_9$

1728. $A_3 \oplus A_4 \oplus A_5 \oplus D_4$

1729. $^*A_3 \oplus 2A_4 \oplus D_5$

1730. $^*A_3 \oplus 2A_4 \oplus A_5$

1731. $2A_3 \oplus D_{10}$

1732. $2A_3 \oplus 2D_5$

1733. $2A_3 \oplus D_4 \oplus E_6$

1734. $^*2A_3 \oplus A_{10}$

1735. $2A_3 \oplus A_6 \oplus D_4$

1736. $2A_3 \oplus A_5 \oplus D_5$

1737. $2A_3 \oplus 2A_5$

1738. $^*2A_3 \oplus A_4 \oplus E_6$

1739. $2A_3 \oplus A_4 \oplus D_6$

1740. $^*2A_3 \oplus A_4 \oplus A_6$

1741. $3A_3 \oplus E_7$

1742. $3A_3 \oplus D_7$

1743. $3A_3 \oplus A_7$

1744. $4A_3 \oplus A_4$

1745. $^*A_2 \oplus 2E_7$

1746. $^*A_2 \oplus E_6 \oplus E_8$

1747. $^*A_2 \oplus D_{14}$

1748. $^*A_2 \oplus D_8 \oplus E_6$

1749. $^*A_2 \oplus D_7 \oplus E_7$

1750. $^*A_2 \oplus 2D_7$

1751. $^*A_2 \oplus D_6 \oplus E_8$

1752. $A_2 \oplus D_6 \oplus D_8$

1753. $^*A_2 \oplus D_5 \oplus D_9$

1754. $A_2 \oplus D_4 \oplus D_{10}$

1755. $A_2 \oplus D_4 \oplus 2D_5$

1756. $A_2 \oplus 2D_4 \oplus E_6$

1757. $^*A_2 \oplus A_{14}$

1758. $^*A_2 \oplus A_{10} \oplus D_4$

1759. $^*A_2 \oplus A_9 \oplus D_5$

1760. $A_2 \oplus A_8 \oplus E_6$

1761. $^*A_2 \oplus A_8 \oplus D_6$

1762. $^*A_2 \oplus A_7 \oplus E_7$

1763. $^*A_2 \oplus A_7 \oplus D_7$

1764. $^*A_2 \oplus 2A_7$

1765. $^*A_2 \oplus A_6 \oplus E_8$

1766. $^*A_2 \oplus A_6 \oplus D_8$

1767. $A_2 \oplus A_6 \oplus 2D_4$

1768. $^*A_2 \oplus A_6 \oplus A_8$

1769. $^*A_2 \oplus A_5 \oplus D_9$

1770. $A_2 \oplus A_5 \oplus D_4 \oplus D_5$

1771. $^*A_2 \oplus A_5 \oplus A_9$

1772. $A_2 \oplus 2A_5 \oplus D_4$

1773. $^*A_2 \oplus A_4 \oplus D_{10}$

1774. $^*A_2 \oplus A_4 \oplus 2D_5$

1775. $^*A_2 \oplus A_4 \oplus D_4 \oplus E_6$

1776. $A_2 \oplus A_4 \oplus D_4 \oplus D_6$

1777. $^*A_2 \oplus A_4 \oplus A_{10}$

1778. $^*A_2 \oplus A_4 \oplus A_6 \oplus D_4$
1779. $^*A_2 \oplus A_4 \oplus A_5 \oplus D_5$
1780. $A_2 \oplus A_4 \oplus 2A_5$
1781. $^*A_2 \oplus 2A_4 \oplus E_6$
1782. $^*A_2 \oplus 2A_4 \oplus D_6$
1783. $^*A_2 \oplus 2A_4 \oplus A_6$
1784. $^*A_2 \oplus A_3 \oplus D_{11}$
1785. $^*A_2 \oplus A_3 \oplus D_5 \oplus E_6$
1786. $A_2 \oplus A_3 \oplus D_5 \oplus D_6$
1787. $A_2 \oplus A_3 \oplus D_4 \oplus E_7$
1788. $A_2 \oplus A_3 \oplus D_4 \oplus D_7$
1789. $^*A_2 \oplus A_3 \oplus A_{11}$
1790. $A_2 \oplus A_3 \oplus A_7 \oplus D_4$
1791. $^*A_2 \oplus A_3 \oplus A_6 \oplus D_5$
1792. $A_2 \oplus A_3 \oplus A_5 \oplus E_6$
1793. $A_2 \oplus A_3 \oplus A_5 \oplus D_6$
1794. $^*A_2 \oplus A_3 \oplus A_5 \oplus A_6$
1795. $^*A_2 \oplus A_3 \oplus A_4 \oplus E_7$
1796. $^*A_2 \oplus A_3 \oplus A_4 \oplus D_7$
1797. $^*A_2 \oplus A_3 \oplus A_4 \oplus A_7$
1798. $^*A_2 \oplus 2A_3 \oplus E_8$
1799. $A_2 \oplus 2A_3 \oplus D_8$
1800. $^*A_2 \oplus 2A_3 \oplus A_8$
1801. $A_2 \oplus 2A_3 \oplus A_4 \oplus D_4$
1802. $^*A_2 \oplus 2A_3 \oplus 2A_4$
1803. $A_2 \oplus 3A_3 \oplus D_5$
1804. $A_2 \oplus 3A_3 \oplus A_5$
1805. $2A_2 \oplus 2E_6$
1806. $^*2A_2 \oplus D_{12}$
1807. $2A_2 \oplus D_6 \oplus E_6$
1808. $2A_2 \oplus 2D_6$
1809. $^*2A_2 \oplus D_5 \oplus E_7$
1810. $^*2A_2 \oplus D_5 \oplus D_7$
1811. $^*2A_2 \oplus D_4 \oplus E_8$
1812. $2A_2 \oplus D_4 \oplus D_8$
1813. $^*2A_2 \oplus A_{12}$
1814. $2A_2 \oplus A_8 \oplus D_4$
1815. $^*2A_2 \oplus A_7 \oplus D_5$
1816. $2A_2 \oplus A_6 \oplus E_6$
1817. $^*2A_2 \oplus A_6 \oplus D_6$
1818. $^*2A_2 \oplus 2A_6$
1819. $2A_2 \oplus A_5 \oplus E_7$
1820. $2A_2 \oplus A_5 \oplus D_7$
1821. $2A_2 \oplus A_5 \oplus A_7$
1822. $^*2A_2 \oplus A_4 \oplus E_8$
1823. $^*2A_2 \oplus A_4 \oplus D_8$
1824. $2A_2 \oplus A_4 \oplus 2D_4$
1825. $2A_2 \oplus A_4 \oplus A_8$
1826. $^*2A_2 \oplus 2A_4 \oplus D_4$
1827. $^*2A_2 \oplus 3A_4$
1828. $^*2A_2 \oplus A_3 \oplus D_9$
1829. $2A_2 \oplus A_3 \oplus D_4 \oplus D_5$
1830. $^*2A_2 \oplus A_3 \oplus A_9$
1831. $2A_2 \oplus A_3 \oplus A_5 \oplus D_4$
1832. $^*2A_2 \oplus A_3 \oplus A_4 \oplus D_5$
1833. $2A_2 \oplus A_3 \oplus A_4 \oplus A_5$
1834. $2A_2 \oplus 2A_3 \oplus E_6$
1835. $2A_2 \oplus 2A_3 \oplus D_6$
1836. $^*2A_2 \oplus 2A_3 \oplus A_6$
1837. $2A_2 \oplus 4A_3$
1838. $3A_2 \oplus D_{10}$
1839. $3A_2 \oplus 2D_5$
1840. $3A_2 \oplus D_4 \oplus D_6$
1841. $3A_2 \oplus A_{10}$
1842. $3A_2 \oplus A_6 \oplus D_4$
1843. $3A_2 \oplus A_5 \oplus D_5$
1844. $3A_2 \oplus A_4 \oplus E_6$
1845. $3A_2 \oplus A_4 \oplus D_6$
1846. $3A_2 \oplus A_4 \oplus A_6$
1847. $3A_2 \oplus A_3 \oplus E_7$
1848. $3A_2 \oplus A_3 \oplus D_7$
1849. $3A_2 \oplus A_3 \oplus A_7$
1850. $3A_2 \oplus 2A_3 \oplus D_4$
1851. $3A_2 \oplus 2A_3 \oplus A_4$
1852. $4A_2 \oplus E_8$
1853. $4A_2 \oplus D_8$
1854. $4A_2 \oplus 2D_4$
1855. $4A_2 \oplus 2A_4$
1856. $4A_2 \oplus A_3 \oplus D_5$
1857. $^*A_1 \oplus E_7 \oplus E_8$
1858. $^*A_1 \oplus D_{15}$
1859. $^*A_1 \oplus D_9 \oplus E_6$
1860. $A_1 \oplus D_8 \oplus E_7$
1861. $^*A_1 \oplus D_7 \oplus E_8$
1862. $A_1 \oplus D_7 \oplus D_8$
1863. $A_1 \oplus D_6 \oplus D_9$
1864. $A_1 \oplus D_5 \oplus D_{10}$
1865. $A_1 \oplus 3D_5$
1866. $A_1 \oplus D_4 \oplus D_{11}$
1867. $A_1 \oplus D_4 \oplus D_5 \oplus E_6$
1868. $^*A_1 \oplus A_{15}$
1869. $A_1 \oplus A_{11} \oplus D_4$
1870. $^*A_1 \oplus A_{10} \oplus D_5$
1871. $^*A_1 \oplus A_9 \oplus E_6$
1872. $A_1 \oplus A_9 \oplus D_6$
1873. $^*A_1 \oplus A_8 \oplus E_7$
1874. $^*A_1 \oplus A_8 \oplus D_7$
1875. $^*A_1 \oplus A_7 \oplus E_8$
1876. $A_1 \oplus A_7 \oplus D_8$
1877. $^*A_1 \oplus A_7 \oplus A_8$
1878. $^*A_1 \oplus A_6 \oplus D_9$
1879. $A_1 \oplus A_6 \oplus D_4 \oplus D_5$
1880. $^*A_1 \oplus A_6 \oplus A_9$
1881. $A_1 \oplus A_5 \oplus D_{10}$
1882. $A_1 \oplus A_5 \oplus 2D_5$
1883. $A_1 \oplus A_5 \oplus D_4 \oplus E_6$
1884. $^*A_1 \oplus A_5 \oplus A_{10}$
1885. $A_1 \oplus A_5 \oplus A_6 \oplus D_4$
1886. $A_1 \oplus 2A_5 \oplus D_5$
1887. $A_1 \oplus 3A_5$
1888. $^*A_1 \oplus A_4 \oplus D_{11}$
1889. $^*A_1 \oplus A_4 \oplus D_5 \oplus E_6$
1890. $A_1 \oplus A_4 \oplus D_5 \oplus D_6$
1891. $A_1 \oplus A_4 \oplus D_4 \oplus E_7$
1892. $A_1 \oplus A_4 \oplus D_4 \oplus D_7$
1893. $^*A_1 \oplus A_4 \oplus A_{11}$
1894. $A_1 \oplus A_4 \oplus A_7 \oplus D_4$
1895. $^*A_1 \oplus A_4 \oplus A_6 \oplus D_5$
1896. $^*A_1 \oplus A_4 \oplus A_5 \oplus E_6$
1897. $A_1 \oplus A_4 \oplus A_5 \oplus D_6$
1898. $^*A_1 \oplus A_4 \oplus A_5 \oplus A_6$
1899. $^*A_1 \oplus 2A_4 \oplus E_7$
1900. $^*A_1 \oplus 2A_4 \oplus D_7$
1901. $^*A_1 \oplus 2A_4 \oplus A_7$
1902. $^*A_1 \oplus A_3 \oplus 2E_6$
1903. $A_1 \oplus A_3 \oplus D_{12}$
1904. $A_1 \oplus A_3 \oplus D_6 \oplus E_6$
1905. $A_1 \oplus A_3 \oplus D_5 \oplus E_7$
1906. $A_1 \oplus A_3 \oplus D_5 \oplus D_7$
1907. $A_1 \oplus A_3 \oplus D_4 \oplus E_8$
1908. $^*A_1 \oplus A_3 \oplus A_{12}$
1909. $A_1 \oplus A_3 \oplus A_8 \oplus D_4$
1910. $A_1 \oplus A_3 \oplus A_7 \oplus D_5$
1911. $^*A_1 \oplus A_3 \oplus A_6 \oplus E_6$
1912. $A_1 \oplus A_3 \oplus A_6 \oplus D_6$
1913. $^*A_1 \oplus A_3 \oplus 2A_6$
1914. $A_1 \oplus A_3 \oplus A_5 \oplus E_7$
1915. $A_1 \oplus A_3 \oplus A_5 \oplus D_7$
1916. $A_1 \oplus A_3 \oplus A_5 \oplus A_7$
1917. $^*A_1 \oplus A_3 \oplus A_4 \oplus E_8$
1918. $A_1 \oplus A_3 \oplus A_4 \oplus D_8$
1919. $^*A_1 \oplus A_3 \oplus A_4 \oplus A_8$
1920. $A_1 \oplus A_3 \oplus 2A_4 \oplus D_4$
1921. $^*A_1 \oplus A_3 \oplus 3A_4$
1922. $A_1 \oplus 2A_3 \oplus D_9$
1923. $A_1 \oplus 2A_3 \oplus A_9$
1924. $A_1 \oplus 2A_3 \oplus A_4 \oplus D_5$

1925. $A_1 \oplus 2A_3 \oplus A_4 \oplus A_5$
1926. $A_1 \oplus 3A_3 \oplus E_6$
1927. $A_1 \oplus 3A_3 \oplus A_6$
1928. $^*A_1 \oplus A_2 \oplus E_6 \oplus E_7$
1929. $^*A_1 \oplus A_2 \oplus D_{13}$
1930. $^*A_1 \oplus A_2 \oplus D_7 \oplus E_6$
1931. $A_1 \oplus A_2 \oplus D_6 \oplus E_7$
1932. $A_1 \oplus A_2 \oplus D_6 \oplus D_7$
1933. $^*A_1 \oplus A_2 \oplus D_5 \oplus E_8$
1934. $A_1 \oplus A_2 \oplus D_5 \oplus D_8$
1935. $A_1 \oplus A_2 \oplus D_4 \oplus D_9$
1936. $^*A_1 \oplus A_2 \oplus A_{13}$
1937. $A_1 \oplus A_2 \oplus A_9 \oplus D_4$
1938. $^*A_1 \oplus A_2 \oplus A_8 \oplus D_5$
1939. $^*A_1 \oplus A_2 \oplus A_7 \oplus E_6$
1940. $A_1 \oplus A_2 \oplus A_7 \oplus D_6$
1941. $^*A_1 \oplus A_2 \oplus A_6 \oplus E_7$
1942. $^*A_1 \oplus A_2 \oplus A_6 \oplus D_7$
1943. $^*A_1 \oplus A_2 \oplus A_6 \oplus A_7$
1944. $^*A_1 \oplus A_2 \oplus A_5 \oplus E_8$
1945. $A_1 \oplus A_2 \oplus A_5 \oplus D_8$
1946. $A_1 \oplus A_2 \oplus A_5 \oplus A_8$
1947. $^*A_1 \oplus A_2 \oplus A_4 \oplus D_9$
1948. $A_1 \oplus A_2 \oplus A_4 \oplus D_4 \oplus D_5$
1949. $^*A_1 \oplus A_2 \oplus A_4 \oplus A_9$
1950. $A_1 \oplus A_2 \oplus A_4 \oplus A_5 \oplus D_4$
1951. $^*A_1 \oplus A_2 \oplus 2A_4 \oplus D_5$
1952. $^*A_1 \oplus A_2 \oplus 2A_4 \oplus A_5$
1953. $A_1 \oplus A_2 \oplus A_3 \oplus D_{10}$
1954. $A_1 \oplus A_2 \oplus A_3 \oplus 2D_5$
1955. $A_1 \oplus A_2 \oplus A_3 \oplus D_4 \oplus E_6$
1956. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_{10}$
1957. $A_1 \oplus A_2 \oplus A_3 \oplus A_6 \oplus D_4$
1958. $A_1 \oplus A_2 \oplus A_3 \oplus A_5 \oplus D_5$
1959. $A_1 \oplus A_2 \oplus A_3 \oplus 2A_5$
1960. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus E_6$
1961. $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus D_6$
1962. $^*A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_6$
1963. $A_1 \oplus A_2 \oplus 2A_3 \oplus E_7$
1964. $A_1 \oplus A_2 \oplus 2A_3 \oplus D_7$
1965. $A_1 \oplus A_2 \oplus 2A_3 \oplus A_7$
1966. $A_1 \oplus A_2 \oplus 3A_3 \oplus A_4$
1967. $^*A_1 \oplus 2A_2 \oplus D_{11}$
1968. $A_1 \oplus 2A_2 \oplus D_5 \oplus E_6$
1969. $A_1 \oplus 2A_2 \oplus D_5 \oplus D_6$
1970. $A_1 \oplus 2A_2 \oplus D_4 \oplus E_7$
1971. $A_1 \oplus 2A_2 \oplus D_4 \oplus D_7$
1972. $A_1 \oplus 2A_2 \oplus A_{11}$
1973. $A_1 \oplus 2A_2 \oplus A_7 \oplus D_4$
1974. $^*A_1 \oplus 2A_2 \oplus A_6 \oplus D_5$
1975. $A_1 \oplus 2A_2 \oplus A_5 \oplus E_6$
1976. $A_1 \oplus 2A_2 \oplus A_5 \oplus D_6$
1977. $A_1 \oplus 2A_2 \oplus A_5 \oplus A_6$
1978. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus E_7$
1979. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus D_7$
1980. $^*A_1 \oplus 2A_2 \oplus A_4 \oplus A_7$
1981. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus E_8$
1982. $A_1 \oplus 2A_2 \oplus A_3 \oplus D_8$
1983. $A_1 \oplus 2A_2 \oplus A_3 \oplus A_8$
1984. $A_1 \oplus 2A_2 \oplus A_3 \oplus A_4 \oplus D_4$
1985. $^*A_1 \oplus 2A_2 \oplus A_3 \oplus 2A_4$
1986. $A_1 \oplus 2A_2 \oplus 2A_3 \oplus D_5$
1987. $A_1 \oplus 2A_2 \oplus 2A_3 \oplus A_5$
1988. $A_1 \oplus 3A_2 \oplus D_9$
1989. $A_1 \oplus 3A_2 \oplus D_4 \oplus D_5$
1990. $A_1 \oplus 3A_2 \oplus A_9$
1991. $A_1 \oplus 3A_2 \oplus A_4 \oplus D_5$
1992. $A_1 \oplus 3A_2 \oplus A_4 \oplus A_5$
1993. $A_1 \oplus 3A_2 \oplus A_3 \oplus E_6$
1994. $A_1 \oplus 3A_2 \oplus A_3 \oplus D_6$
1995. $A_1 \oplus 3A_2 \oplus A_3 \oplus A_6$
1996. $A_1 \oplus 3A_2 \oplus 3A_3$
1997. $A_1 \oplus 4A_2 \oplus E_7$
1998. $A_1 \oplus 4A_2 \oplus A_7$
1999. $A_1 \oplus 4A_2 \oplus A_3 \oplus A_4$
2000. $2A_1 \oplus 2E_7$
2001. $^*2A_1 \oplus E_6 \oplus E_8$
2002. $2A_1 \oplus D_{14}$
2003. $2A_1 \oplus D_8 \oplus E_6$
2004. $2A_1 \oplus D_7 \oplus E_7$
2005. $2A_1 \oplus 2D_7$
2006. $2A_1 \oplus D_6 \oplus E_8$
2007. $2A_1 \oplus D_5 \oplus D_9$
2008. $^*2A_1 \oplus A_{14}$
2009. $2A_1 \oplus A_{10} \oplus D_4$
2010. $2A_1 \oplus A_9 \oplus D_5$
2011. $^*2A_1 \oplus A_8 \oplus E_6$
2012. $2A_1 \oplus A_8 \oplus D_6$
2013. $2A_1 \oplus A_7 \oplus E_7$
2014. $2A_1 \oplus A_7 \oplus D_7$
2015. $2A_1 \oplus 2A_7$
2016. $^*2A_1 \oplus A_6 \oplus E_8$
2017. $2A_1 \oplus A_6 \oplus D_8$
2018. $^*2A_1 \oplus A_6 \oplus A_8$
2019. $2A_1 \oplus A_5 \oplus D_9$
2020. $2A_1 \oplus A_5 \oplus A_9$
2021. $2A_1 \oplus A_4 \oplus D_{10}$
2022. $2A_1 \oplus A_4 \oplus 2D_5$
2023. $2A_1 \oplus A_4 \oplus D_4 \oplus E_6$
2024. $^*2A_1 \oplus A_4 \oplus A_{10}$
2025. $2A_1 \oplus A_4 \oplus A_6 \oplus D_4$
2026. $2A_1 \oplus A_4 \oplus A_5 \oplus D_5$
2027. $2A_1 \oplus A_4 \oplus 2A_5$
2028. $^*2A_1 \oplus 2A_4 \oplus E_6$
2029. $2A_1 \oplus 2A_4 \oplus D_6$
2030. $^*2A_1 \oplus 2A_4 \oplus A_6$
2031. $2A_1 \oplus A_3 \oplus D_{11}$
2032. $2A_1 \oplus A_3 \oplus D_5 \oplus E_6$
2033. $2A_1 \oplus A_3 \oplus A_{11}$
2034. $2A_1 \oplus A_3 \oplus A_6 \oplus D_5$
2035. $2A_1 \oplus A_3 \oplus A_5 \oplus E_6$
2036. $2A_1 \oplus A_3 \oplus A_5 \oplus A_6$
2037. $2A_1 \oplus A_3 \oplus A_4 \oplus E_7$
2038. $2A_1 \oplus A_3 \oplus A_4 \oplus D_7$
2039. $2A_1 \oplus A_3 \oplus A_4 \oplus A_7$
2040. $2A_1 \oplus 2A_3 \oplus E_8$
2041. $2A_1 \oplus 2A_3 \oplus A_8$
2042. $2A_1 \oplus 2A_3 \oplus 2A_4$
2043. $2A_1 \oplus A_2 \oplus 2E_6$
2044. $2A_1 \oplus A_2 \oplus D_{12}$
2045. $2A_1 \oplus A_2 \oplus D_6 \oplus E_6$
2046. $2A_1 \oplus A_2 \oplus D_5 \oplus E_7$
2047. $2A_1 \oplus A_2 \oplus D_5 \oplus D_7$
2048. $2A_1 \oplus A_2 \oplus D_4 \oplus E_8$
2049. $^*2A_1 \oplus A_2 \oplus A_{12}$
2050. $2A_1 \oplus A_2 \oplus A_8 \oplus D_4$
2051. $2A_1 \oplus A_2 \oplus A_7 \oplus D_5$
2052. $^*2A_1 \oplus A_2 \oplus A_6 \oplus E_6$
2053. $2A_1 \oplus A_2 \oplus A_6 \oplus D_6$
2054. $^*2A_1 \oplus A_2 \oplus 2A_6$
2055. $2A_1 \oplus A_2 \oplus A_5 \oplus E_7$
2056. $2A_1 \oplus A_2 \oplus A_5 \oplus D_7$

2057. $2A_1 \oplus A_2 \oplus A_5 \oplus A_7$
2058. $^*2A_1 \oplus A_2 \oplus A_4 \oplus E_8$
2059. $2A_1 \oplus A_2 \oplus A_4 \oplus D_8$
2060. $^*2A_1 \oplus A_2 \oplus A_4 \oplus A_8$
2061. $2A_1 \oplus A_2 \oplus 2A_4 \oplus D_4$
2062. $^*2A_1 \oplus A_2 \oplus 3A_4$
2063. $2A_1 \oplus A_2 \oplus A_3 \oplus D_9$
2064. $2A_1 \oplus A_2 \oplus A_3 \oplus A_9$
2065. $2A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus D_5$
2066. $2A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$
2067. $2A_1 \oplus A_2 \oplus 2A_3 \oplus E_6$
2068. $2A_1 \oplus A_2 \oplus 2A_3 \oplus A_6$
2069. $2A_1 \oplus 2A_2 \oplus D_{10}$
2070. $2A_1 \oplus 2A_2 \oplus 2D_5$
2071. $2A_1 \oplus 2A_2 \oplus D_4 \oplus E_6$
2072. $^*2A_1 \oplus 2A_2 \oplus A_{10}$
2073. $2A_1 \oplus 2A_2 \oplus A_6 \oplus D_4$
2074. $2A_1 \oplus 2A_2 \oplus A_5 \oplus D_5$
2075. $2A_1 \oplus 2A_2 \oplus 2A_5$
2076. $2A_1 \oplus 2A_2 \oplus A_4 \oplus E_6$
2077. $2A_1 \oplus 2A_2 \oplus A_4 \oplus D_6$
2078. $^*2A_1 \oplus 2A_2 \oplus A_4 \oplus A_6$
2079. $2A_1 \oplus 2A_2 \oplus A_3 \oplus E_7$
2080. $2A_1 \oplus 2A_2 \oplus A_3 \oplus D_7$
2081. $2A_1 \oplus 2A_2 \oplus A_3 \oplus A_7$
2082. $2A_1 \oplus 2A_2 \oplus 2A_3 \oplus A_4$
2083. $2A_1 \oplus 3A_2 \oplus E_8$
2084. $2A_1 \oplus 3A_2 \oplus D_8$
2085. $2A_1 \oplus 3A_2 \oplus A_8$
2086. $2A_1 \oplus 3A_2 \oplus A_4 \oplus D_4$
2087. $2A_1 \oplus 3A_2 \oplus 2A_4$
2088. $2A_1 \oplus 3A_2 \oplus A_3 \oplus D_5$
2089. $2A_1 \oplus 3A_2 \oplus A_3 \oplus A_5$
2090. $2A_1 \oplus 4A_2 \oplus D_6$
2091. $2A_1 \oplus 4A_2 \oplus A_6$
2092. $2A_1 \oplus 4A_2 \oplus 2A_3$
2093. $3A_1 \oplus E_6 \oplus E_7$
2094. $3A_1 \oplus D_{13}$
2095. $3A_1 \oplus D_7 \oplus E_6$
2096. $3A_1 \oplus D_5 \oplus E_8$
2097. $3A_1 \oplus A_{13}$
2098. $3A_1 \oplus A_8 \oplus D_5$
2099. $3A_1 \oplus A_7 \oplus E_6$
2100. $3A_1 \oplus A_6 \oplus E_7$
2101. $3A_1 \oplus A_6 \oplus D_7$
2102. $3A_1 \oplus A_6 \oplus A_7$
2103. $3A_1 \oplus A_5 \oplus E_8$
2104. $3A_1 \oplus A_5 \oplus A_8$
2105. $3A_1 \oplus A_4 \oplus D_9$
2106. $3A_1 \oplus A_4 \oplus A_9$
2107. $3A_1 \oplus 2A_4 \oplus D_5$
2108. $3A_1 \oplus 2A_4 \oplus A_5$
2109. $3A_1 \oplus A_3 \oplus A_{10}$
2110. $3A_1 \oplus A_3 \oplus A_4 \oplus E_6$
2111. $3A_1 \oplus A_3 \oplus A_4 \oplus A_6$
2112. $3A_1 \oplus A_2 \oplus D_{11}$
2113. $3A_1 \oplus A_2 \oplus D_5 \oplus E_6$
2114. $3A_1 \oplus A_2 \oplus A_{11}$
2115. $3A_1 \oplus A_2 \oplus A_6 \oplus D_5$
2116. $3A_1 \oplus A_2 \oplus A_5 \oplus E_6$
2117. $3A_1 \oplus A_2 \oplus A_5 \oplus A_6$
2118. $3A_1 \oplus A_2 \oplus A_4 \oplus E_7$
2119. $3A_1 \oplus A_2 \oplus A_4 \oplus D_7$
2120. $3A_1 \oplus A_2 \oplus A_4 \oplus A_7$
2121. $3A_1 \oplus A_2 \oplus A_3 \oplus E_8$
2122. $3A_1 \oplus A_2 \oplus A_3 \oplus A_8$
2123. $3A_1 \oplus A_2 \oplus A_3 \oplus 2A_4$
2124. $3A_1 \oplus 2A_2 \oplus D_9$
2125. $3A_1 \oplus 2A_2 \oplus A_9$
2126. $3A_1 \oplus 2A_2 \oplus A_4 \oplus D_5$
2127. $3A_1 \oplus 2A_2 \oplus A_4 \oplus A_5$
2128. $3A_1 \oplus 2A_2 \oplus A_3 \oplus E_6$
2129. $3A_1 \oplus 2A_2 \oplus A_3 \oplus A_6$
2130. $3A_1 \oplus 3A_2 \oplus E_7$
2131. $3A_1 \oplus 3A_2 \oplus D_7$
2132. $3A_1 \oplus 3A_2 \oplus A_7$
2133. $3A_1 \oplus 3A_2 \oplus A_3 \oplus A_4$
2134. $4A_1 \oplus 2E_6$
2135. $4A_1 \oplus A_{12}$
2136. $4A_1 \oplus A_6 \oplus E_6$
2137. $4A_1 \oplus 2A_6$
2138. $4A_1 \oplus A_4 \oplus E_8$
2139. $4A_1 \oplus A_4 \oplus A_8$
2140. $4A_1 \oplus 3A_4$
2141. $4A_1 \oplus A_2 \oplus A_{10}$
2142. $4A_1 \oplus A_2 \oplus A_4 \oplus E_6$
2143. $4A_1 \oplus A_2 \oplus A_4 \oplus A_6$
2144. $4A_1 \oplus 2A_2 \oplus E_8$
2145. $4A_1 \oplus 2A_2 \oplus A_8$
2146. $4A_1 \oplus 2A_2 \oplus 2A_4$
2147. $4A_1 \oplus 3A_2 \oplus A_6$

<u>$\mathbf{rank(\Lambda) = 17}$</u>

2148. $^*D_{17}$
2149. $^*D_{11} \oplus E_6$
2150. $D_{10} \oplus E_7$
2151. $^*D_9 \oplus E_8$
2152. $D_8 \oplus D_9$
2153. $D_7 \oplus D_{10}$
2154. $D_6 \oplus D_{11}$
2155. $D_5 \oplus 2E_6$
2156. $D_5 \oplus D_{12}$
2157. $D_5 \oplus D_6 \oplus E_6$
2158. $2D_5 \oplus E_7$
2159. $2D_5 \oplus D_7$
2160. $D_4 \oplus E_6 \oplus E_7$
2161. $D_4 \oplus D_{13}$
2162. $D_4 \oplus D_7 \oplus E_6$
2163. $D_4 \oplus D_5 \oplus E_8$
2164. $^*A_{17}$
2165. $A_{13} \oplus D_4$
2166. $^*A_{12} \oplus D_5$
2167. $A_{11} \oplus E_6$
2168. $A_{11} \oplus D_6$
2169. $^*A_{10} \oplus E_7$
2170. $^*A_{10} \oplus D_7$
2171. $^*A_9 \oplus E_8$
2172. $A_9 \oplus D_8$
2173. $^*A_8 \oplus D_9$
2174. $A_8 \oplus D_4 \oplus D_5$
2175. $^*A_8 \oplus A_9$
2176. $A_7 \oplus D_{10}$
2177. $A_7 \oplus 2D_5$
2178. $A_7 \oplus D_4 \oplus E_6$
2179. $^*A_7 \oplus A_{10}$
2180. $^*A_6 \oplus D_{11}$
2181. $^*A_6 \oplus D_5 \oplus E_6$
2182. $A_6 \oplus D_5 \oplus D_6$
2183. $A_6 \oplus D_4 \oplus E_7$
2184. $A_6 \oplus D_4 \oplus D_7$
2185. $^*A_6 \oplus A_{11}$
2186. $A_6 \oplus A_7 \oplus D_4$

2187. $^*2A_6 \oplus D_5$
2188. $A_5 \oplus 2E_6$
2189. $A_5 \oplus D_{12}$
2190. $A_5 \oplus D_6 \oplus E_6$
2191. $A_5 \oplus D_5 \oplus E_7$
2192. $A_5 \oplus D_5 \oplus D_7$
2193. $A_5 \oplus D_4 \oplus E_8$
2194. $^*A_5 \oplus A_{12}$
2195. $A_5 \oplus A_8 \oplus D_4$
2196. $A_5 \oplus A_7 \oplus D_5$
2197. $A_5 \oplus A_6 \oplus E_6$
2198. $A_5 \oplus A_6 \oplus D_6$
2199. $^*A_5 \oplus 2A_6$
2200. $2A_5 \oplus E_7$
2201. $2A_5 \oplus D_7$
2202. $2A_5 \oplus A_7$
2203. $^*A_4 \oplus E_6 \oplus E_7$
2204. $^*A_4 \oplus D_{13}$
2205. $^*A_4 \oplus D_7 \oplus E_6$
2206. $A_4 \oplus D_6 \oplus E_7$
2207. $A_4 \oplus D_6 \oplus D_7$
2208. $^*A_4 \oplus D_5 \oplus E_8$
2209. $A_4 \oplus D_5 \oplus D_8$
2210. $A_4 \oplus D_4 \oplus D_9$
2211. $^*A_4 \oplus A_{13}$
2212. $A_4 \oplus A_9 \oplus D_4$
2213. $^*A_4 \oplus A_8 \oplus D_5$
2214. $^*A_4 \oplus A_7 \oplus E_6$
2215. $A_4 \oplus A_7 \oplus D_6$
2216. $^*A_4 \oplus A_6 \oplus E_7$
2217. $^*A_4 \oplus A_6 \oplus D_7$
2218. $^*A_4 \oplus A_6 \oplus A_7$
2219. $^*A_4 \oplus A_5 \oplus E_8$
2220. $A_4 \oplus A_5 \oplus D_8$
2221. $A_4 \oplus A_5 \oplus A_8$
2222. $^*2A_4 \oplus D_9$
2223. $2A_4 \oplus D_4 \oplus D_5$
2224. $2A_4 \oplus A_9$
2225. $2A_4 \oplus A_5 \oplus D_4$
2226. $3A_4 \oplus D_5$
2227. $3A_4 \oplus A_5$
2228. $A_3 \oplus 2E_7$
2229. $^*A_3 \oplus E_6 \oplus E_8$
2230. $A_3 \oplus D_{14}$
2231. $A_3 \oplus D_8 \oplus E_6$
2232. $A_3 \oplus D_7 \oplus E_7$
2233. $A_3 \oplus 2D_7$
2234. $A_3 \oplus D_6 \oplus E_8$
2235. $A_3 \oplus D_5 \oplus D_9$
2236. $^*A_3 \oplus A_{14}$
2237. $A_3 \oplus A_{10} \oplus D_4$
2238. $A_3 \oplus A_9 \oplus D_5$
2239. $A_3 \oplus A_8 \oplus E_6$
2240. $A_3 \oplus A_8 \oplus D_6$
2241. $A_3 \oplus A_7 \oplus E_7$
2242. $A_3 \oplus A_7 \oplus D_7$
2243. $A_3 \oplus 2A_7$
2244. $^*A_3 \oplus A_6 \oplus E_8$
2245. $A_3 \oplus A_6 \oplus D_8$
2246. $^*A_3 \oplus A_6 \oplus A_8$
2247. $A_3 \oplus A_5 \oplus D_9$
2248. $A_3 \oplus A_5 \oplus A_9$
2249. $A_3 \oplus A_4 \oplus D_{10}$
2250. $A_3 \oplus A_4 \oplus 2D_5$
2251. $A_3 \oplus A_4 \oplus D_4 \oplus E_6$
2252. $^*A_3 \oplus A_4 \oplus A_{10}$
2253. $A_3 \oplus A_4 \oplus A_6 \oplus D_4$
2254. $A_3 \oplus A_4 \oplus A_5 \oplus D_5$
2255. $A_3 \oplus A_4 \oplus 2A_5$
2256. $^*A_3 \oplus 2A_4 \oplus E_6$
2257. $A_3 \oplus 2A_4 \oplus D_6$
2258. $^*A_3 \oplus 2A_4 \oplus A_6$
2259. $2A_3 \oplus D_{11}$
2260. $2A_3 \oplus D_5 \oplus E_6$
2261. $2A_3 \oplus A_{11}$
2262. $2A_3 \oplus A_6 \oplus D_5$
2263. $2A_3 \oplus A_5 \oplus E_6$
2264. $2A_3 \oplus A_5 \oplus A_6$
2265. $2A_3 \oplus A_4 \oplus E_7$
2266. $2A_3 \oplus A_4 \oplus D_7$
2267. $2A_3 \oplus A_4 \oplus A_7$
2268. $3A_3 \oplus E_8$
2269. $3A_3 \oplus A_8$
2270. $3A_3 \oplus 2A_4$
2271. $^*A_2 \oplus E_7 \oplus E_8$
2272. $^*A_2 \oplus D_{15}$
2273. $A_2 \oplus D_9 \oplus E_6$
2274. $A_2 \oplus D_8 \oplus E_7$
2275. $^*A_2 \oplus D_7 \oplus E_8$
2276. $A_2 \oplus D_7 \oplus D_8$
2277. $A_2 \oplus D_6 \oplus D_9$
2278. $A_2 \oplus D_5 \oplus D_{10}$
2279. $A_2 \oplus 3D_5$
2280. $A_2 \oplus D_4 \oplus D_{11}$
2281. $A_2 \oplus D_4 \oplus D_5 \oplus E_6$
2282. $^*A_2 \oplus A_{15}$
2283. $A_2 \oplus A_{11} \oplus D_4$
2284. $^*A_2 \oplus A_{10} \oplus D_5$
2285. $A_2 \oplus A_9 \oplus E_6$
2286. $A_2 \oplus A_9 \oplus D_6$
2287. $A_2 \oplus A_8 \oplus E_7$
2288. $A_2 \oplus A_8 \oplus D_7$
2289. $^*A_2 \oplus A_7 \oplus E_8$
2290. $A_2 \oplus A_7 \oplus D_8$
2291. $A_2 \oplus A_7 \oplus A_8$
2292. $^*A_2 \oplus A_6 \oplus D_9$
2293. $A_2 \oplus A_6 \oplus D_4 \oplus D_5$
2294. $^*A_2 \oplus A_6 \oplus A_9$
2295. $A_2 \oplus A_5 \oplus D_{10}$
2296. $A_2 \oplus A_5 \oplus 2D_5$
2297. $A_2 \oplus A_5 \oplus A_{10}$
2298. $A_2 \oplus A_5 \oplus A_6 \oplus D_4$
2299. $A_2 \oplus 2A_5 \oplus D_5$
2300. $^*A_2 \oplus A_4 \oplus D_{11}$
2301. $A_2 \oplus A_4 \oplus D_5 \oplus E_6$
2302. $A_2 \oplus A_4 \oplus D_5 \oplus D_6$
2303. $A_2 \oplus A_4 \oplus D_4 \oplus E_7$
2304. $A_2 \oplus A_4 \oplus D_4 \oplus D_7$
2305. $A_2 \oplus A_4 \oplus A_{11}$
2306. $A_2 \oplus A_4 \oplus A_7 \oplus D_4$
2307. $^*A_2 \oplus A_4 \oplus A_6 \oplus D_5$
2308. $A_2 \oplus A_4 \oplus A_5 \oplus E_6$
2309. $A_2 \oplus A_4 \oplus A_5 \oplus D_6$
2310. $A_2 \oplus A_4 \oplus A_5 \oplus A_6$
2311. $^*A_2 \oplus 2A_4 \oplus E_7$
2312. $^*A_2 \oplus 2A_4 \oplus D_7$
2313. $^*A_2 \oplus 2A_4 \oplus A_7$
2314. $A_2 \oplus A_3 \oplus 2E_6$
2315. $A_2 \oplus A_3 \oplus D_{12}$
2316. $A_2 \oplus A_3 \oplus D_6 \oplus E_6$
2317. $A_2 \oplus A_3 \oplus D_5 \oplus E_7$
2318. $A_2 \oplus A_3 \oplus D_5 \oplus D_7$
2319. $A_2 \oplus A_3 \oplus D_4 \oplus E_8$
2320. $^*A_2 \oplus A_3 \oplus A_{12}$
2321. $A_2 \oplus A_3 \oplus A_8 \oplus D_4$
2322. $A_2 \oplus A_3 \oplus A_7 \oplus D_5$
2323. $A_2 \oplus A_3 \oplus A_6 \oplus E_6$
2324. $A_2 \oplus A_3 \oplus A_6 \oplus D_6$
2325. $^*A_2 \oplus A_3 \oplus 2A_6$
2326. $A_2 \oplus A_3 \oplus A_5 \oplus E_7$
2327. $A_2 \oplus A_3 \oplus A_5 \oplus D_7$
2328. $A_2 \oplus A_3 \oplus A_5 \oplus A_7$
2329. $^*A_2 \oplus A_3 \oplus A_4 \oplus E_8$
2330. $A_2 \oplus A_3 \oplus A_4 \oplus D_8$
2331. $A_2 \oplus A_3 \oplus A_4 \oplus A_8$
2332. $A_2 \oplus A_3 \oplus 2A_4 \oplus D_4$
2333. $A_2 \oplus A_3 \oplus 3A_4$
2334. $A_2 \oplus 2A_3 \oplus D_9$

2335. $A_2 \oplus 2A_3 \oplus A_9$
2336. $A_2 \oplus 2A_3 \oplus A_4 \oplus D_5$
2337. $A_2 \oplus 2A_3 \oplus A_4 \oplus A_5$
2338. $A_2 \oplus 3A_3 \oplus E_6$
2339. $A_2 \oplus 3A_3 \oplus A_6$
2340. $2A_2 \oplus E_6 \oplus E_7$
2341. $2A_2 \oplus D_{13}$
2342. $2A_2 \oplus D_6 \oplus E_7$
2343. $2A_2 \oplus D_6 \oplus D_7$
2344. $2A_2 \oplus D_5 \oplus E_8$
2345. $2A_2 \oplus D_5 \oplus D_8$
2346. $2A_2 \oplus D_4 \oplus D_9$
2347. $2A_2 \oplus A_{13}$
2348. $2A_2 \oplus A_9 \oplus D_4$
2349. $2A_2 \oplus A_8 \oplus D_5$
2350. $2A_2 \oplus A_7 \oplus E_6$
2351. $2A_2 \oplus A_7 \oplus D_6$
2352. $2A_2 \oplus A_6 \oplus E_7$
2353. $2A_2 \oplus A_6 \oplus D_7$
2354. $2A_2 \oplus A_6 \oplus A_7$
2355. $2A_2 \oplus A_5 \oplus E_8$
2356. $2A_2 \oplus A_5 \oplus D_8$
2357. $2A_2 \oplus A_4 \oplus D_9$
2358. $2A_2 \oplus A_4 \oplus D_4 \oplus D_5$
2359. $2A_2 \oplus A_4 \oplus A_9$
2360. $2A_2 \oplus 2A_4 \oplus D_5$
2361. $2A_2 \oplus 2A_4 \oplus A_5$
2362. $2A_2 \oplus A_3 \oplus D_{10}$
2363. $2A_2 \oplus A_3 \oplus 2D_5$
2364. $2A_2 \oplus A_3 \oplus A_{10}$
2365. $2A_2 \oplus A_3 \oplus A_6 \oplus D_4$
2366. $2A_2 \oplus A_3 \oplus A_5 \oplus D_5$
2367. $2A_2 \oplus A_3 \oplus A_4 \oplus E_6$
2368. $2A_2 \oplus A_3 \oplus A_4 \oplus D_6$
2369. $2A_2 \oplus A_3 \oplus A_4 \oplus A_6$
2370. $2A_2 \oplus 2A_3 \oplus E_7$
2371. $2A_2 \oplus 2A_3 \oplus D_7$
2372. $2A_2 \oplus 2A_3 \oplus A_7$
2373. $2A_2 \oplus 3A_3 \oplus A_4$
2374. $3A_2 \oplus D_{11}$
2375. $3A_2 \oplus D_5 \oplus D_6$
2376. $3A_2 \oplus D_4 \oplus D_7$
2377. $3A_2 \oplus A_6 \oplus D_5$
2378. $3A_2 \oplus A_4 \oplus E_7$
2379. $3A_2 \oplus A_4 \oplus A_7$
2380. $3A_2 \oplus A_3 \oplus E_8$
2381. $3A_2 \oplus A_3 \oplus D_8$
2382. $3A_2 \oplus A_3 \oplus 2A_4$
2383. $3A_2 \oplus 2A_3 \oplus D_5$
2384. $^*A_1 \oplus 2E_8$
2385. $A_1 \oplus D_{16}$
2386. $A_1 \oplus D_{10} \oplus E_6$
2387. $A_1 \oplus D_9 \oplus E_7$
2388. $A_1 \oplus D_8 \oplus E_8$
2389. $A_1 \oplus D_7 \oplus D_9$
2390. $A_1 \oplus D_5 \oplus D_{11}$
2391. $A_1 \oplus 2D_5 \oplus E_6$
2392. $A_1 \oplus D_4 \oplus 2E_6$
2393. $^*A_1 \oplus A_{16}$
2394. $A_1 \oplus A_{12} \oplus D_4$
2395. $A_1 \oplus A_{11} \oplus D_5$
2396. $^*A_1 \oplus A_{10} \oplus E_6$
2397. $A_1 \oplus A_{10} \oplus D_6$
2398. $A_1 \oplus A_9 \oplus E_7$
2399. $A_1 \oplus A_9 \oplus D_7$
2400. $^*A_1 \oplus A_8 \oplus E_8$
2401. $A_1 \oplus A_8 \oplus D_8$
2402. $A_1 \oplus 2A_8$
2403. $A_1 \oplus A_7 \oplus D_9$
2404. $A_1 \oplus A_7 \oplus A_9$
2405. $A_1 \oplus A_6 \oplus D_{10}$
2406. $A_1 \oplus A_6 \oplus 2D_5$
2407. $A_1 \oplus A_6 \oplus D_4 \oplus E_6$
2408. $^*A_1 \oplus A_6 \oplus A_{10}$
2409. $A_1 \oplus 2A_6 \oplus D_4$
2410. $A_1 \oplus A_5 \oplus D_{11}$
2411. $A_1 \oplus A_5 \oplus D_5 \oplus E_6$
2412. $A_1 \oplus A_5 \oplus A_{11}$
2413. $A_1 \oplus A_5 \oplus A_6 \oplus D_5$
2414. $A_1 \oplus 2A_5 \oplus E_6$
2415. $A_1 \oplus 2A_5 \oplus A_6$
2416. $A_1 \oplus A_4 \oplus 2E_6$
2417. $A_1 \oplus A_4 \oplus D_{12}$
2418. $A_1 \oplus A_4 \oplus D_6 \oplus E_6$
2419. $A_1 \oplus A_4 \oplus D_5 \oplus E_7$
2420. $A_1 \oplus A_4 \oplus D_5 \oplus D_7$
2421. $A_1 \oplus A_4 \oplus D_4 \oplus E_8$
2422. $^*A_1 \oplus A_4 \oplus A_{12}$
2423. $A_1 \oplus A_4 \oplus A_8 \oplus D_4$
2424. $A_1 \oplus A_4 \oplus A_7 \oplus D_5$
2425. $^*A_1 \oplus A_4 \oplus A_6 \oplus E_6$
2426. $A_1 \oplus A_4 \oplus A_6 \oplus D_6$
2427. $^*A_1 \oplus A_4 \oplus 2A_6$
2428. $A_1 \oplus A_4 \oplus A_5 \oplus E_7$
2429. $A_1 \oplus A_4 \oplus A_5 \oplus D_7$
2430. $A_1 \oplus A_4 \oplus A_5 \oplus A_7$
2431. $^*A_1 \oplus 2A_4 \oplus E_8$
2432. $A_1 \oplus 2A_4 \oplus D_8$
2433. $^*A_1 \oplus 2A_4 \oplus A_8$
2434. $A_1 \oplus 3A_4 \oplus D_4$
2435. $A_1 \oplus A_3 \oplus E_6 \oplus E_7$
2436. $A_1 \oplus A_3 \oplus D_{13}$
2437. $A_1 \oplus A_3 \oplus D_7 \oplus E_6$
2438. $A_1 \oplus A_3 \oplus D_5 \oplus E_8$
2439. $A_1 \oplus A_3 \oplus A_{13}$
2440. $A_1 \oplus A_3 \oplus A_8 \oplus D_5$
2441. $A_1 \oplus A_3 \oplus A_7 \oplus E_6$
2442. $A_1 \oplus A_3 \oplus A_6 \oplus E_7$
2443. $A_1 \oplus A_3 \oplus A_6 \oplus D_7$
2444. $A_1 \oplus A_3 \oplus A_6 \oplus A_7$
2445. $A_1 \oplus A_3 \oplus A_5 \oplus E_8$
2446. $A_1 \oplus A_3 \oplus A_5 \oplus A_8$
2447. $A_1 \oplus A_3 \oplus A_4 \oplus D_9$
2448. $A_1 \oplus A_3 \oplus A_4 \oplus A_9$
2449. $A_1 \oplus A_3 \oplus 2A_4 \oplus D_5$
2450. $A_1 \oplus A_3 \oplus 2A_4 \oplus A_5$
2451. $A_1 \oplus 2A_3 \oplus A_{10}$
2452. $A_1 \oplus 2A_3 \oplus A_4 \oplus E_6$
2453. $A_1 \oplus 2A_3 \oplus A_4 \oplus A_6$
2454. $A_1 \oplus A_2 \oplus 2E_7$
2455. $A_1 \oplus A_2 \oplus E_6 \oplus E_8$
2456. $A_1 \oplus A_2 \oplus D_{14}$
2457. $A_1 \oplus A_2 \oplus D_8 \oplus E_6$
2458. $A_1 \oplus A_2 \oplus D_7 \oplus E_7$
2459. $A_1 \oplus A_2 \oplus 2D_7$
2460. $A_1 \oplus A_2 \oplus D_6 \oplus E_8$
2461. $A_1 \oplus A_2 \oplus D_5 \oplus D_9$
2462. $A_1 \oplus A_2 \oplus A_{14}$
2463. $A_1 \oplus A_2 \oplus A_{10} \oplus D_4$
2464. $A_1 \oplus A_2 \oplus A_9 \oplus D_5$
2465. $A_1 \oplus A_2 \oplus A_8 \oplus E_6$
2466. $A_1 \oplus A_2 \oplus A_8 \oplus D_6$
2467. $A_1 \oplus A_2 \oplus A_7 \oplus E_7$
2468. $A_1 \oplus A_2 \oplus A_7 \oplus D_7$
2469. $A_1 \oplus A_2 \oplus 2A_7$
2470. $^*A_1 \oplus A_2 \oplus A_6 \oplus E_8$
2471. $A_1 \oplus A_2 \oplus A_6 \oplus D_8$
2472. $A_1 \oplus A_2 \oplus A_6 \oplus A_8$
2473. $A_1 \oplus A_2 \oplus A_5 \oplus D_9$
2474. $A_1 \oplus A_2 \oplus A_5 \oplus A_9$
2475. $A_1 \oplus A_2 \oplus A_4 \oplus D_{10}$
2476. $A_1 \oplus A_2 \oplus A_4 \oplus 2D_5$
2477. $A_1 \oplus A_2 \oplus A_4 \oplus D_4 \oplus E_6$
2478. $^*A_1 \oplus A_2 \oplus A_4 \oplus A_{10}$
2479. $A_1 \oplus A_2 \oplus A_4 \oplus A_6 \oplus D_4$

2480. $A_1 \oplus A_2 \oplus A_4 \oplus A_5 \oplus D_5$

2481. $A_1 \oplus A_2 \oplus A_4 \oplus 2A_5$

2482. $A_1 \oplus A_2 \oplus 2A_4 \oplus E_6$

2483. $A_1 \oplus A_2 \oplus 2A_4 \oplus D_6$

2484. $^*A_1 \oplus A_2 \oplus 2A_4 \oplus A_6$

2485. $A_1 \oplus A_2 \oplus A_3 \oplus D_{11}$

2486. $A_1 \oplus A_2 \oplus A_3 \oplus D_5 \oplus E_6$

2487. $A_1 \oplus A_2 \oplus A_3 \oplus A_{11}$

2488. $A_1 \oplus A_2 \oplus A_3 \oplus A_6 \oplus D_5$

2489. $A_1 \oplus A_2 \oplus A_3 \oplus A_5 \oplus E_6$

2490. $A_1 \oplus A_2 \oplus A_3 \oplus A_5 \oplus A_6$

2491. $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus E_7$

2492. $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus D_7$

2493. $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_7$

2494. $A_1 \oplus A_2 \oplus 2A_3 \oplus E_8$

2495. $A_1 \oplus A_2 \oplus 2A_3 \oplus A_8$

2496. $A_1 \oplus A_2 \oplus 2A_3 \oplus 2A_4$

2497. $A_1 \oplus 2A_2 \oplus D_{12}$

2498. $A_1 \oplus 2A_2 \oplus D_6 \oplus E_6$

2499. $A_1 \oplus 2A_2 \oplus D_5 \oplus E_7$

2500. $A_1 \oplus 2A_2 \oplus D_5 \oplus D_7$

2501. $A_1 \oplus 2A_2 \oplus D_4 \oplus E_8$

2502. $A_1 \oplus 2A_2 \oplus A_{12}$

2503. $A_1 \oplus 2A_2 \oplus A_7 \oplus D_5$

2504. $A_1 \oplus 2A_2 \oplus A_6 \oplus E_6$

2505. $A_1 \oplus 2A_2 \oplus A_6 \oplus D_6$

2506. $A_1 \oplus 2A_2 \oplus 2A_6$

2507. $A_1 \oplus 2A_2 \oplus A_5 \oplus E_7$

2508. $A_1 \oplus 2A_2 \oplus A_5 \oplus A_7$

2509. $A_1 \oplus 2A_2 \oplus A_4 \oplus E_8$

2510. $A_1 \oplus 2A_2 \oplus A_4 \oplus D_8$

2511. $A_1 \oplus 2A_2 \oplus A_4 \oplus A_8$

2512. $A_1 \oplus 2A_2 \oplus 2A_4 \oplus D_4$

2513. $A_1 \oplus 2A_2 \oplus 3A_4$

2514. $A_1 \oplus 2A_2 \oplus A_3 \oplus D_9$

2515. $A_1 \oplus 2A_2 \oplus A_3 \oplus A_9$

2516. $A_1 \oplus 2A_2 \oplus A_3 \oplus A_4 \oplus D_5$

2517. $A_1 \oplus 2A_2 \oplus A_3 \oplus A_4 \oplus A_5$

2518. $A_1 \oplus 2A_2 \oplus 2A_3 \oplus E_6$

2519. $A_1 \oplus 2A_2 \oplus 2A_3 \oplus A_6$

2520. $A_1 \oplus 3A_2 \oplus A_{10}$

2521. $A_1 \oplus 3A_2 \oplus A_4 \oplus D_6$

2522. $A_1 \oplus 3A_2 \oplus A_4 \oplus A_6$

2523. $A_1 \oplus 3A_2 \oplus A_3 \oplus E_7$

2524. $A_1 \oplus 3A_2 \oplus A_3 \oplus A_7$

2525. $A_1 \oplus 3A_2 \oplus 2A_3 \oplus A_4$

2526. $2A_1 \oplus E_7 \oplus E_8$

2527. $2A_1 \oplus D_{15}$

2528. $2A_1 \oplus D_9 \oplus E_6$

2529. $2A_1 \oplus D_7 \oplus E_8$

2530. $2A_1 \oplus A_{15}$

2531. $2A_1 \oplus A_{10} \oplus D_5$

2532. $2A_1 \oplus A_9 \oplus E_6$

2533. $2A_1 \oplus A_8 \oplus E_7$

2534. $2A_1 \oplus A_8 \oplus D_7$

2535. $2A_1 \oplus A_7 \oplus E_8$

2536. $2A_1 \oplus A_7 \oplus A_8$

2537. $2A_1 \oplus A_6 \oplus D_9$

2538. $2A_1 \oplus A_6 \oplus A_9$

2539. $2A_1 \oplus A_5 \oplus A_{10}$

2540. $2A_1 \oplus A_4 \oplus D_{11}$

2541. $2A_1 \oplus A_4 \oplus D_5 \oplus E_6$

2542. $2A_1 \oplus A_4 \oplus A_{11}$

2543. $2A_1 \oplus A_4 \oplus A_6 \oplus D_5$

2544. $2A_1 \oplus A_4 \oplus A_5 \oplus E_6$

2545. $2A_1 \oplus A_4 \oplus A_5 \oplus A_6$

2546. $2A_1 \oplus 2A_4 \oplus E_7$

2547. $2A_1 \oplus 2A_4 \oplus D_7$

2548. $2A_1 \oplus 2A_4 \oplus A_7$

2549. $2A_1 \oplus A_3 \oplus 2E_6$

2550. $2A_1 \oplus A_3 \oplus A_{12}$

2551. $2A_1 \oplus A_3 \oplus A_6 \oplus E_6$

2552. $2A_1 \oplus A_3 \oplus 2A_6$

2553. $2A_1 \oplus A_3 \oplus A_4 \oplus E_8$

2554. $2A_1 \oplus A_3 \oplus A_4 \oplus A_8$

2555. $2A_1 \oplus A_3 \oplus 3A_4$

2556. $2A_1 \oplus A_2 \oplus E_6 \oplus E_7$

2557. $2A_1 \oplus A_2 \oplus D_{13}$

2558. $2A_1 \oplus A_2 \oplus D_7 \oplus E_6$

2559. $2A_1 \oplus A_2 \oplus D_5 \oplus E_8$

2560. $2A_1 \oplus A_2 \oplus A_{13}$

2561. $2A_1 \oplus A_2 \oplus A_8 \oplus D_5$

2562. $2A_1 \oplus A_2 \oplus A_7 \oplus E_6$

2563. $2A_1 \oplus A_2 \oplus A_6 \oplus E_7$

2564. $2A_1 \oplus A_2 \oplus A_6 \oplus D_7$

2565. $2A_1 \oplus A_2 \oplus A_6 \oplus A_7$

2566. $2A_1 \oplus A_2 \oplus A_5 \oplus E_8$

2567. $2A_1 \oplus A_2 \oplus A_5 \oplus A_8$

2568. $2A_1 \oplus A_2 \oplus A_4 \oplus D_9$

2569. $2A_1 \oplus A_2 \oplus A_4 \oplus A_9$

2570. $2A_1 \oplus A_2 \oplus 2A_4 \oplus D_5$

2571. $2A_1 \oplus A_2 \oplus 2A_4 \oplus A_5$

2572. $2A_1 \oplus A_2 \oplus A_3 \oplus A_{10}$

2573. $2A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus E_6$

2574. $2A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_6$

2575. $2A_1 \oplus 2A_2 \oplus D_{11}$

2576. $2A_1 \oplus 2A_2 \oplus A_{11}$

2577. $2A_1 \oplus 2A_2 \oplus A_6 \oplus D_5$

2578. $2A_1 \oplus 2A_2 \oplus A_5 \oplus A_6$

2579. $2A_1 \oplus 2A_2 \oplus A_4 \oplus E_7$

2580. $2A_1 \oplus 2A_2 \oplus A_4 \oplus D_7$

2581. $2A_1 \oplus 2A_2 \oplus A_4 \oplus A_7$

2582. $2A_1 \oplus 2A_2 \oplus A_3 \oplus E_8$

2583. $2A_1 \oplus 2A_2 \oplus A_3 \oplus A_8$

2584. $2A_1 \oplus 2A_2 \oplus A_3 \oplus 2A_4$

2585. $2A_1 \oplus 3A_2 \oplus D_9$

2586. $2A_1 \oplus 3A_2 \oplus A_9$

2587. $2A_1 \oplus 3A_2 \oplus A_3 \oplus A_6$

2588. $3A_1 \oplus E_6 \oplus E_8$

2589. $3A_1 \oplus A_{14}$

2590. $3A_1 \oplus A_8 \oplus E_6$

2591. $3A_1 \oplus A_6 \oplus E_8$

2592. $3A_1 \oplus A_6 \oplus A_8$

2593. $3A_1 \oplus A_4 \oplus A_{10}$

2594. $3A_1 \oplus 2A_4 \oplus E_6$

2595. $3A_1 \oplus 2A_4 \oplus A_6$

2596. $3A_1 \oplus A_2 \oplus A_{12}$

2597. $3A_1 \oplus A_2 \oplus A_6 \oplus E_6$

2598. $3A_1 \oplus A_2 \oplus 2A_6$

2599. $3A_1 \oplus A_2 \oplus A_4 \oplus E_8$

2600. $3A_1 \oplus A_2 \oplus A_4 \oplus A_8$

2601. $3A_1 \oplus A_2 \oplus 3A_4$

2602. $3A_1 \oplus 2A_2 \oplus A_{10}$

2603. $3A_1 \oplus 2A_2 \oplus A_4 \oplus A_6$

rank(Λ) = 18

2604. D_{18}

2605. $D_{12} \oplus E_6$

2606. $D_{11} \oplus E_7$

2607. $D_{10} \oplus E_8$
2608. $2D_9$
2609. $D_7 \oplus D_{11}$
2610. $D_6 \oplus 2E_6$
2611. $D_5 \oplus E_6 \oplus E_7$
2612. $D_5 \oplus D_{13}$
2613. $D_5 \oplus D_7 \oplus E_6$
2614. $2D_5 \oplus E_8$
2615. $D_4 \oplus E_6 \oplus E_8$
2616. $^*A_{18}$
2617. $A_{14} \oplus D_4$
2618. $A_{13} \oplus D_5$
2619. $A_{12} \oplus E_6$
2620. $A_{12} \oplus D_6$
2621. $A_{11} \oplus E_7$
2622. $A_{11} \oplus D_7$
2623. $^*A_{10} \oplus E_8$
2624. $A_{10} \oplus D_8$
2625. $A_9 \oplus D_9$
2626. $2A_9$
2627. $A_8 \oplus D_{10}$
2628. $A_8 \oplus 2D_5$
2629. $A_8 \oplus A_{10}$
2630. $A_7 \oplus D_{11}$
2631. $A_7 \oplus D_5 \oplus E_6$
2632. $A_7 \oplus A_{11}$
2633. $A_6 \oplus 2E_6$
2634. $A_6 \oplus D_{12}$
2635. $A_6 \oplus D_6 \oplus E_6$
2636. $A_6 \oplus D_5 \oplus E_7$
2637. $A_6 \oplus D_5 \oplus D_7$
2638. $A_6 \oplus D_4 \oplus E_8$
2639. $^*A_6 \oplus A_{12}$
2640. $A_6 \oplus A_8 \oplus D_4$
2641. $A_6 \oplus A_7 \oplus D_5$
2642. $2A_6 \oplus E_6$
2643. $2A_6 \oplus D_6$
2644. $A_5 \oplus E_6 \oplus E_7$
2645. $A_5 \oplus D_{13}$
2646. $A_5 \oplus D_5 \oplus E_8$
2647. $A_5 \oplus A_{13}$
2648. $A_5 \oplus A_8 \oplus D_5$
2649. $A_5 \oplus A_7 \oplus E_6$
2650. $A_5 \oplus A_6 \oplus E_7$
2651. $A_5 \oplus A_6 \oplus D_7$
2652. $A_5 \oplus A_6 \oplus A_7$
2653. $2A_5 \oplus E_8$
2654. $A_4 \oplus 2E_7$
2655. $A_4 \oplus E_6 \oplus E_8$
2656. $A_4 \oplus D_{14}$
2657. $A_4 \oplus D_8 \oplus E_6$
2658. $A_4 \oplus D_7 \oplus E_7$
2659. $A_4 \oplus 2D_7$
2660. $A_4 \oplus D_6 \oplus E_8$
2661. $A_4 \oplus D_5 \oplus D_9$
2662. $A_4 \oplus A_{14}$
2663. $A_4 \oplus A_{10} \oplus D_4$
2664. $A_4 \oplus A_9 \oplus D_5$
2665. $A_4 \oplus A_8 \oplus E_6$
2666. $A_4 \oplus A_8 \oplus D_6$
2667. $A_4 \oplus A_7 \oplus E_7$
2668. $A_4 \oplus A_7 \oplus D_7$
2669. $A_4 \oplus 2A_7$
2670. $^*A_4 \oplus A_6 \oplus E_8$
2671. $A_4 \oplus A_6 \oplus D_8$
2672. $A_4 \oplus A_6 \oplus A_8$
2673. $A_4 \oplus A_5 \oplus D_9$
2674. $A_4 \oplus A_5 \oplus A_9$
2675. $2A_4 \oplus D_{10}$
2676. $2A_4 \oplus 2D_5$
2677. $2A_4 \oplus D_4 \oplus E_6$
2678. $2A_4 \oplus A_{10}$
2679. $2A_4 \oplus A_6 \oplus D_4$
2680. $2A_4 \oplus A_5 \oplus D_5$
2681. $2A_4 \oplus 2A_5$
2682. $3A_4 \oplus D_6$
2683. $A_3 \oplus E_7 \oplus E_8$
2684. $A_3 \oplus D_{15}$
2685. $A_3 \oplus D_9 \oplus E_6$
2686. $A_3 \oplus D_7 \oplus E_8$
2687. $A_3 \oplus A_{15}$
2688. $A_3 \oplus A_{10} \oplus D_5$
2689. $A_3 \oplus A_9 \oplus E_6$
2690. $A_3 \oplus A_8 \oplus E_7$
2691. $A_3 \oplus A_8 \oplus D_7$
2692. $A_3 \oplus A_7 \oplus E_8$
2693. $A_3 \oplus A_7 \oplus A_8$
2694. $A_3 \oplus A_6 \oplus D_9$
2695. $A_3 \oplus A_6 \oplus A_9$
2696. $A_3 \oplus A_5 \oplus A_{10}$
2697. $A_3 \oplus A_4 \oplus D_{11}$
2698. $A_3 \oplus A_4 \oplus D_5 \oplus E_6$
2699. $A_3 \oplus A_4 \oplus A_{11}$
2700. $A_3 \oplus A_4 \oplus A_6 \oplus D_5$
2701. $A_3 \oplus A_4 \oplus A_5 \oplus E_6$
2702. $A_3 \oplus A_4 \oplus A_5 \oplus A_6$
2703. $A_3 \oplus 2A_4 \oplus E_7$
2704. $A_3 \oplus 2A_4 \oplus D_7$
2705. $A_3 \oplus 2A_4 \oplus A_7$
2706. $2A_3 \oplus 2E_6$
2707. $2A_3 \oplus A_{12}$
2708. $2A_3 \oplus A_6 \oplus E_6$
2709. $2A_3 \oplus 2A_6$
2710. $2A_3 \oplus A_4 \oplus E_8$
2711. $2A_3 \oplus A_4 \oplus A_8$
2712. $2A_3 \oplus 3A_4$
2713. $A_2 \oplus 2E_8$
2714. $A_2 \oplus D_{16}$
2715. $A_2 \oplus D_9 \oplus E_7$
2716. $A_2 \oplus D_8 \oplus E_8$
2717. $A_2 \oplus D_7 \oplus D_9$
2718. $A_2 \oplus D_5 \oplus D_{11}$
2719. $A_2 \oplus A_{16}$
2720. $A_2 \oplus A_{12} \oplus D_4$
2721. $A_2 \oplus A_{11} \oplus D_5$
2722. $A_2 \oplus A_{10} \oplus E_6$
2723. $A_2 \oplus A_{10} \oplus D_6$
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Curriculum Vitae

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