On a three-dimensional model for MEMS with hinged boundary conditions

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Abstract

We study a free boundary problem arising from the modeling of an idealized electrostatically actuated MEMS device. In contrast to existing literature, we consider a three-dimensional device involving a hinged elastic plate. The model couples the electrostatic potential to the displacement of the elastic plate, which is caused by a voltage difference that is applied to the device. The electrostatic potential is harmonic in the free domain between the elastic plate and a rigid ground plate. The elastic plate displacement solves a fourth-order parabolic equation with hinged boundary conditions and a right-hand side proportional to the square of the trace of the gradient of the electrostatic potential on the elastic plate. We establish local and global well-posedness of the model in dependence of the applied voltage difference and show that only touchdown of the elastic plate on the ground plate can generate a finite time singularity. Next, we consider stationary solutions and prove that such solutions exist for small voltage values and do not exist for large voltage values. To prove the nonexistence result, we show that the fourth-order elliptic operator with hinged boundary conditions satisfies a positivity preserving property and has a positive eigenpair.

Keywords: MEMS, free boundary problem, hinged plate, well-posedness, positivity preserving property, nonexistence.

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Chapter 1

Introduction

A free boundary problem consists of one or more partial differential equations (PDEs) which have to be solved in a domain with a boundary that is partly unknown a priori; this part is accordingly named a free boundary. Thus, in addition to the standard boundary conditions that are needed to solve the PDEs, an extra condition must be prescribed on the free boundary. One then wishes to determine both the free boundary and the solution of the PDEs. As the behavior of the free boundary depends on the solution, the problem as a whole is going to be a nonlinear problem which greatly complicates the analysis. The theory of free boundary problems has seen great progress in the last forty years. Moreover, many problems in physics, chemistry, biology, mechanics, and other areas can be described as free boundary problems.

The present thesis deals with a free boundary problem that arises in the modeling of microelectromechanical systems (MEMS), that is, integrated electromechanical devices with a size on the order of micrometers. Richard Feynman, in his famous 1959 lecture "There's Plenty of Room at the Bottom" [21], predicted that MEMS will become an important and indispensable field of scientific research in the 21st century:

> "It is a staggeringly small world that is below. In the year 2000, when they look back at this age, they will wonder why it was not until the year 1960 that anybody began seriously to move in this direction."

Nowadays, there are a wide variety of applications for MEMS to report on: MEMS are commonly used as microsensors or microactuators, they appear in accelerometers and gyroscopes, they have commercial applications, e.g., in radio frequency (RF) switches, micropumps for inkjet printer heads and micromirrors for projection displays, and are even used in the medical field, e.g., to measure blood pressure within the body. Many more examples can be found in [65, 74, 85]. A number of different sensing and actuation properties like piezoelectric, piezoresistive, electrostatic, thermal, electromagnetic, and optical have been used in MEMS [65]. Of these, electrostatics is often the preferred technique.

Most of the electrostatically actuated MEMS devices consist of an elastic plate suspended above a rigid ground plate and operate in a similar fashion: Holding both plates at different voltages induces an electric field and hence a Coulomb force, resulting in a mechanical deformation of the elastic plate. When a sufficiently large voltage difference is applied, the elastic plate may touch down on the rigid plate, creating a so-called "pull-in instability". This instability may limit the effectiveness of some devices such as micromirrors and microresonators but be essential for the operation of others such as switches and microvalves. Such pull-in instability was first experimentally observed by Taylor [82] and Nathanson et al. [67]. Understanding and quantifying this instability is a topic of technological and mathematical interest.

An introduction to the mathematical theory of MEMS devices can be found in [74]. Recently, the following idealized model for an electrostatically actuated MEMS device has been proposed, see [19, 53, 74]: There is a thin elastic plate that is clamped on its boundary and lies above a rigid grounded plate. Let $D \subset \mathbb{R}^n$ with $n = 1, 2$ be a bounded smooth domain representing the identical shapes of the two plates and consider a function $u(t, x)$ of time $t > 0$ and $x \in D$ with $u > -1$. The MEMS device will be modeled by the $(n + 1)$-dimensional domain

$$\Omega(u(t)) = \{(x, z) \in \mathbb{R}^{n+1}\,;\, x \in D, -1 < z < u(t, x)\},$$

and its two horizontal boundary components will be denoted by $\mathfrak{G}_{u(t)} = \{z = u(t, x)\}$ and $\mathfrak{G}_{-1} = \{z = -1\}$. The function u models the n-dimensional displacement of the elastic plate from $\mathfrak{G}_0$ when a positive voltage difference is applied to the device $\Omega(0)$. The electrostatic potential in the region between both plates is denoted by ψ and it satisfies Laplace's equation, equals to one on the upper and vanishes on the lower plate, and is assumed to be an affine function of z on the lateral boundary components. The evolution of the elastic plate starts from its initial position $u(0, x) = u^0(x)$ at its initial velocity $\partial_t u(0, x) = u^1(x)$ and is described by a fourth-order damped wave equation with a right-hand side that depends on the square of the trace of the gradient of the electrostatic potential on the elastic plate. By rescaling, there is a parameter $\varepsilon > 0$ called the aspect ratio of the device, that is, the ratio between the vertical and horizontal length scales.

Let ∇' and Δ' denote the gradient and the Laplace operator with respect to $x \in D$ for functions of x and z. Then, the problem reads

$$\varepsilon^2 \Delta' \psi + \partial_z^2 \psi = 0 \qquad \text{in } \Omega(u),\ t > 0 \tag{1.1}$$

$$\psi = \frac{1+z}{1+u} \qquad \text{on } \partial\Omega(u),\ t > 0, \tag{1.2}$$

$$\alpha^2 \partial_t^2 u + \partial_t u + \beta \Delta^2 u - \tau \Delta u = -\lambda \big\{ \varepsilon^2 |\nabla' \psi|^2 + (\partial_z \psi)^2 \big\} \qquad \text{on } \mathfrak{G}_u, t > 0, \tag{1.3}$$

$$u = \partial_\nu u = 0 \qquad \text{on } \partial D,\ t > 0, \tag{1.4}$$

$$u(0, \cdot) = u^0, \quad \partial_t u(0, \cdot) = u^1 \qquad \text{in } D, \tag{1.5}$$

with ν denoting the outward unit normal on ∂D. The parameter α^2 is a ratio of inertial terms and a damping constant, while $\beta > 0$ and $\tau \geq 0$ are related to bending and stretching of the elastic plate, respectively. The Dirichlet boundary conditions (1.4) mean that the elastic plate is clamped. Note that (1.1)-(1.5) is a free boundary problem since the domain

$\Omega(u)$ and its boundary component $\mathfrak{G}_u$ are a priori unknown and depend on the solution (u, ψ). For this reason, equations (1.1) and (1.3) are strongly coupled.

In (1.3), $\lambda > 0$ is a tuning parameter proportional to the square of the applied voltage difference. Accordingly, the pull-in instability is expected to take place if λ is large enough. This, though, is rather well-understood for the "vanishing aspect ratio model", obtained by formally setting $\varepsilon = 0$ in (1.1)-(1.5). Such an assumption is often made in MEMS and allows one to explicitly solve for the potential, that is, $\psi = (1+z)/(1+u)$ in $\overline{\Omega(u)}$, $t > 0$, thus reducing the free boundary problem to the singular evolution equation

$$\alpha^2 \partial_t^2 u + \partial_t u + \beta \Delta^2 u - \tau \Delta u = -\frac{\lambda}{(1+u)^2} \quad \text{in } D,\ t > 0, \tag{1.6}$$

subject to (1.4) and (1.5) solely involving u. A serious difficulty for its study is caused by the lack of a maximum principle in general for the fourth-order operator $\beta\Delta^2 - \tau\Delta$ with Dirichlet boundary conditions. But if one restricts one's attention to the unit ball, then one does have a positivity preserving property (PPP) for the bilaplace operator $\beta\Delta^2$ due to Boggio [9]. That is, $\beta\Delta^2 u \geq 0$ implies $u \geq 0$. Boggio's result was recently extended in [49] to the operator $\beta\Delta^2 - \tau\Delta$ in radial symmetry.

On the contrary, in the case of Navier boundary conditions, i.e.,

$$u = \Delta u = 0 \quad \text{on } \partial D, \tag{1.7}$$

$\beta\Delta^2 - \tau\Delta$ enjoys a PPP in arbitrary smooth domains. This fact has been applied in [19, 34, 60] to prove the existence of a threshold value λ^* of λ so that there is at least one stationary solution to (1.6)-(1.7) for $0 < \lambda < \lambda^*$ and no stationary solution for $\lambda > \lambda^*$. Concerning the evolution equation (1.6) subject to Navier boundary conditions (1.7), it is shown in [32] that there exists a $\overline{\lambda}$ such that, if $0 < \lambda < \overline{\lambda} \leq \lambda^*$, the solution exists globally in time, while, if $\lambda > \lambda^*$, then the solution must touch down to $u = -1$ at some finite time T^*. In addition, some other interesting results have also been obtained by these authors.

When D equals the strip $(-1, 1) \subset \mathbb{R}$ or the unit disk $\mathbb{B}_1 \subset \mathbb{R}^2$, then some results are also available for (1.6) under Dirichlet boundary conditions (1.4). See [19, 47, 61]. The results, however, are less complete and there are still open questions.

When the upper component of the device is modeled by a membrane rather than a plate, that is, when $\beta = 0$, (1.6) reduces to a second-order evolution equation that has been studied extensively in recent years, and there are now many established results concerning the behavior of solutions. See [10, 19, 22, 57] and the references therein for a thorough account. Several variants of the second-order model have also been studied in [73, 75, 58].

Far less is known about the free boundary problem (1.1)-(1.5) with $\varepsilon > 0$, which, due to the present coupling of u and ψ, turns out to be even more involved. For this case, the literature is particularly sparse. A few recent papers, however, take a first step towards demonstrating the presence of the pull-in instability; i.e., there is a number $\lambda^* > 0$ such that when $\lambda > \lambda^*$, (1.1)-(1.5) possesses no stationary solutions and will touch down to

$u = -1$ in finite time. See [52, 48] for the case when D is the strip $(-1, 1)$ and [45, 51] for the parabolic ($\alpha = 0$) version with a two-dimensional convex domain D. It should be mentioned here that the nonexistence result of [51] is limited to the unit disk $\mathbb{B}_1$ due to the absence of PPP in general geometries.

Much more literature can be found on the second-order parabolic ($\alpha = \beta = 0$) equivalent of (1.1)-(1.5) and variants thereof. See [11, 15, 18, 55, 40] and the references therein.

The present thesis offers the following new contributions to the topics introduced so far: There is a rigid ground plate of shape $D \subset \mathbb{R}^2$ and a thin elastic plate of the same shape at rest which is suspended above the rigid one. We assume here that the elastic plate is hinged at its boundary, that is,

$$u = 0 \quad \text{on } \partial D, \tag{1.8}$$

but unlike the clamped case, we do not prescribe $\partial_\nu u = 0$ on ∂D. In that case, the function u additionally satisfies a natural boundary condition, namely

$$\Delta u - (1 - \sigma)\kappa \partial_\nu u = 0 \quad \text{on } \partial D, \tag{1.9}$$

with Poisson ratio $\sigma \in (-1, 1)$ and boundary curvature κ. These boundary conditions are sometimes called "Steklov" due to their first appearance in [79] and may be considered as an intermediate situation between Navier and Dirichlet boundary conditions. A free boundary problem for an idealized electrostatically actuated MEMS device involving a hinged upper plate is derived in detail in Chapter 2. The model consists of a fourth-order parabolic equation with Steklov boundary conditions for the elastic plate displacement u which is coupled to an elliptic equation for the electrostatic potential ψ in the free domain between the elastic and the ground plate. In particular, this model corresponds to the problem (1.1)-(1.5) with $\alpha = 0$ and Steklov boundary conditions (1.8)-(1.9) instead of Dirichlet boundary conditions (1.4). To the best of our knowledge, this model has not been discussed in the literature up to now. In analogy to the model with Dirichlet boundary conditions, we assume that $D \subset \mathbb{R}^2$ is a bounded and convex domain with a sufficiently smooth boundary. In Chapter 3, we show that our model is locally well-posed in time and that solutions exist globally for small values of λ. A criterion for global existence implying that a finite time singularity can only result from touchdown of the elastic plate on the ground plate is obtained in Chapters 4 and 5. In Chapters 6 and 7, we show that the operator $\beta\Delta^2 - \tau\Delta$, subject to the boundary conditions (1.8)-(1.9), satisfies a PPP in D and further establish the existence of a positive eigenpair. Founded on these results, we derive the existence of an asymptotically stable stationary solution for small values of λ and the nonexistence of stationary solutions for large values of λ. This is done in Chapter 8. Appendices A and B collect proofs of auxiliary results and Appendix C contains an alternative proof of the nonexistence result for stationary solutions. At the end of Chapter 2 we give a short overview of the main results and address some open problems.

Chapter 2

A mathematical model for electrostatic MEMS with a hinged top plate

In this chapter, we propose a new mathematical model for the study of the stationary and dynamical behavior of an idealized electrostatically actuated MEMS device. The geometry of the MEMS device is sketched in Figure 2.1.

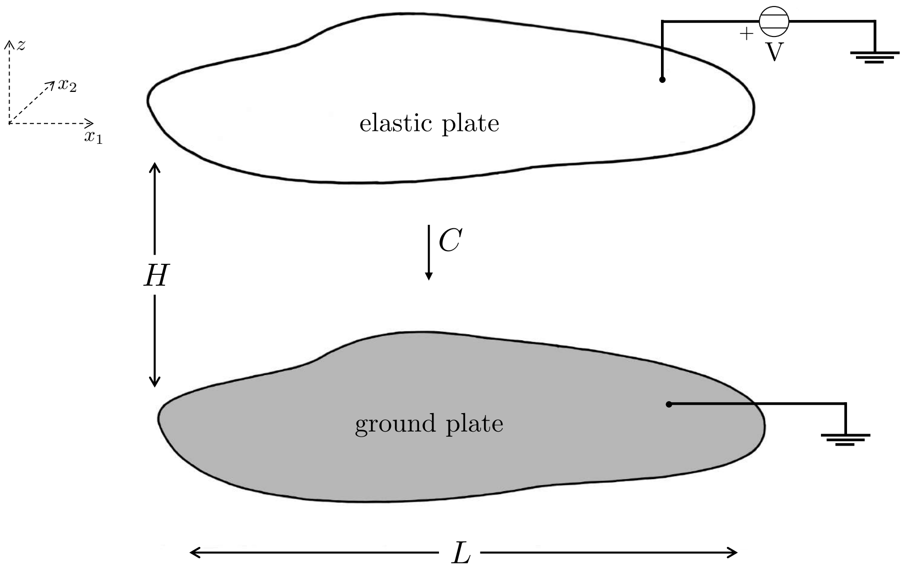

Figure 2.1: Geometry of the idealized MEMS device

The type of MEMS device under consideration consists of a rigid ground plate and a thin elastic plate which is suspended above the rigid one and hinged on its boundary. Both plates are perfect conductors and a dielectric medium, generally air or vacuum, fills the space in between. When a positive voltage difference is applied to the device, an electric field is created in the space between the elastic plate and the rigid plate and a Coulomb force causes a mechanical deformation of the elastic plate, thereby changing the geometry of the device. The induced Coulomb force is varied in strength by varying the applied voltage difference. In practical applications, however, the applied voltage difference has an upper limit, beyond which the electrostatic attractive force is not balanced by the mechanical restoring force in the elastic plate that eventually snaps and touches down on the rigid plate, and the MEMS device collapses. This phenomenon, known as pull-in instability, is a key limiting factor in the effectiveness of our device, and the corresponding voltage is called pull-in voltage. Thus, for designing and manufacturing purposes of the device it is important to know the precise value of the pull-in voltage. The applied voltage difference will appear in the model as a parameter.

The modeling of the above-described MEMS device involves the electrostatic potential between the two plates and the deformation of the elastic plate, which is assumed to be small and only in the vertical direction. To be more specific, let us consider a rigid ground plate of shape $D \subset \mathbb{R}^2$ and a thin elastic plate with the same shape D at rest and being made of a homogeneous isotropic material; see Figure 2.2 for a cross section of the geometry of our MEMS device. The plates are assumed to be perfectly electrically conducting and are separated by air or vacuum as a dielectric (i.e., the relative permittivity is one). The ground plate is located at height $z = -H$ and held at zero voltage, while the elastic plate at rest is located at $z = 0$. When a voltage $V > 0$ is applied to the top plate, an electric field is generated causing a deformation of the top plate from rest whose displacement in the z-direction is modeled by the function $u = u(x)$ for $x = (x_1, x_2) \in D$.

The elastic plate is assumed to be **hinged**, meaning that the vertical position at the boundary is fixed. This gives

$$u(x) = 0, \qquad x \in \partial D.$$

Moreover, we shall see in Section 2.2 below that the hinged plate additionally satisfies a natural boundary condition. Next, let $\psi_u = \psi_u(x, z)$ denote the electrostatic potential defined in the region

$$\Omega(u) := \{(x, z) \in D \times \mathbb{R}\,;\, -H < z < u(x)\}$$

between the two plates. In order to avoid the top plate touching the ground plate and thus making the region $\Omega(u)$ disconnected, we presuppose that $u(x) > -H$ for $x \in D$.

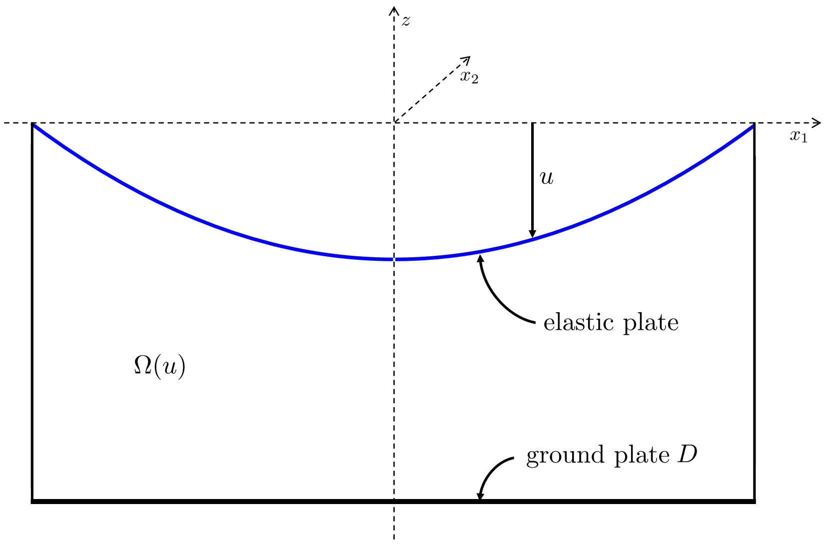

Figure 2.2: Cross section of the idealized MEMS device

The remainder of this chapter is organized as follows. In the section below, we formulate the equations governing the electrostatic potential ψ_u.

In Section 2.2, we derive the governing equations for the vertical displacement u of the elastic plate by applying a variational principle to the total energy $\mathcal{E}$ of the MEMS device. Then, assuming that u is a function of time t, i.e., $u = u(t, x)$, we present the equation describing the motion of the device.

In Section 2.3, we combine the equations for the potential and the plate displacement to get a free boundary problem. It couples a fourth-order semilinear parabolic equation for the displacement u of the elastic plate with the Laplace equation for the electrostatic potential ψ_u in the device. Finally, we rewrite the free boundary problem in a non-dimensional form.

In Section 2.4, we provide a short outline of the subsequent chapters.

2.1 Governing equations for the electrostatic potential ψ_u

By virtue of Gauss's law, see [37, Section 1.7], the electrostatic potential ψ_u for a given displacement u solves the Laplace equation

$$\varepsilon_0 \, \Delta\psi_u = 0 \qquad \text{in } \Omega(u), \tag{2.1}$$

along with the boundary conditions

$$\psi_u(x, -H) = 0, \quad \psi_u(x, u(x)) = V, \qquad x \in D,$$

where the constant ε_0 is the vacuum permittivity. The boundary conditions of ψ_u given a priori only on the ground plate and on the elastic plate are continuously extended to the vertical sides of $\Omega(u)$ by

$$\psi_u(x,z) = \frac{V(H+z)}{H}, \qquad x \in \partial D, \quad z \in (-H,0),$$

and hence

$$\psi_u(x,z) = \frac{V(H+z)}{H+u(x)}, \qquad (x,z) \in \partial\Omega(u). \tag{2.2}$$

2.2 Governing equations for the plate displacement u

2.2.1 Involved energies

In this subsection we give all the energy contributions involved in the MEMS device aiming to derive the Euler-Lagrange equation. The total potential energy $\mathcal{E}$ of the MEMS device is the sum of the mechanical energy $\mathcal{E}_m$ and the electrostatic energy $\mathcal{E}_e$, i.e., for a given displacement u, it holds

$$\mathcal{E}(u) := \mathcal{E}_m(u) + \mathcal{E}_e(u).$$

Electrostatic energy of the MEMS device

According to [56, Section 2.1], the electrostatic energy is given by

$$\mathcal{E}_e(u) := -\frac{\varepsilon_0}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 \, d(x,z), \tag{2.3}$$

where ψ_u is, for a sufficiently smooth $u : \overline{D} \to (-H, \infty)$, the maximizer of the Dirichlet integral

$$-\frac{\varepsilon_0}{2} \int_{\Omega(u)} |\nabla \vartheta|^2 \, d(x,z) \tag{2.4}$$

in the set of functions $\vartheta \in W_2^1(\Omega(u))$ satisfying the boundary condition (2.2).

Clearly, we have $\mathcal{E}_e(u) \leq 0$. Let us note that the functional $\mathcal{E}_e$ depends on the displacement u not only via its domain of integration $\Omega(u)$, but also via the potential ψ_u, which itself depends implicitly on u. We further remark that, given a displacement u, equations (2.1)-(2.2) for the electrostatic potential can be obtained by using the fact that it is a critical point of the functional (2.4) with respect to ϑ.

Mechanical energy of the MEMS device

The mechanical energy $\mathcal{E}_m$ of the device is composed of two terms. The first term corresponds to the energy due to bending and torsion of the plate and the second term is determined by the change of the surface area of the plate.

The Kirchhoff-Love model [39, 62] for the energy corresponding to bending and torsion is given by

$$B \int_D \left(\frac{1}{2} \left(K_1^2 + K_2^2 \right) + \sigma\, K_1 K_2 \right) dx, \tag{2.5}$$

where K_1 and K_2 are the principal curvatures of the graph of u. See also [12, p.250], [24, Section 1.1.2], and [64, Chapter 6]. The parameter σ denotes the Poisson ratio, that is, the ratio between bending and torsion. It is a physical constant that depends on the material of the plate and is usually positive, although auxetic materials have a negative Poisson ratio, see [20]. The Poisson ratio for metals is close to 0.3, see [62, p.105], while for concrete it ranges from 0.1 to 0.2. In any case, see also [20, 62], it always holds true that $-1 < \sigma \leq \frac{1}{2}$. From a mathematical point of view, it suffices to assume that

$$-1 < \sigma < 1.$$

The parameter B defines the flexural rigidity of the plate, a measure of its resistance to deformation. For physical reasons it holds that $B > 0$; see, e.g., [84, Section 2.3].

For a small displacement u, the following approximations hold:

$$\frac{(K_1 + K_2)^2}{2} \approx \frac{(\Delta u)^2}{2} \quad \text{and} \quad K_1 K_2 \approx \det(\nabla^2 u) = \partial_{x_1}^2 u\, \partial_{x_2}^2 u - (\partial_{x_2} \partial_{x_1} u)^2,$$

where $\nabla^2 u$ is the Hessian matrix of u. Therefore,

$$\begin{aligned} \frac{1}{2} \left(K_1^2 + K_2^2 \right) + \sigma K_1 K_2 &= \frac{(K_1 + K_2)^2}{2} - (1 - \sigma) K_1 K_2 \\ &\approx \frac{1}{2} (\Delta u)^2 + (1 - \sigma) \big((\partial_{x_2} \partial_{x_1} u)^2 - \partial_{x_1}^2 u\, \partial_{x_2}^2 u \big) \end{aligned} \tag{2.6}$$

and thus (2.5) becomes

$$B \int_D \left\{ \frac{1}{2} (\Delta u)^2 + (1 - \sigma) \big((\partial_{x_2} \partial_{x_1} u)^2 - \partial_{x_1}^2 u\, \partial_{x_2}^2 u \big) \right\} dx.$$

Next, prestressing the plate and then fixing the horizontal displacement at the boundary results in the term

$$P \int_D \left(\sqrt{1 + |\nabla u|^2} - 1 \right) dx,$$

which we may call the stretching energy. Here, $P \in [0, \infty)$ is the (pre)stress parameter. This term is a result of the larger surface area for nonzero u compared with $u = 0$. For a small displacement u, the approximation $\sqrt{1 + |\nabla u|^2} - 1 \approx |\nabla u|^2 / 2$ leads to the Dirichlet integral

$$\frac{P}{2} \int_D |\nabla u|^2\, dx.$$

Consequently, for a small vertical displacement u, the mechanical energy of the plate D

approximately equals

$$\mathcal{E}_m(u) := B \int_D \left\{ \frac{1}{2} (\Delta u)^2 + (1-\sigma)\big((\partial_{x_2}\partial_{x_1} u)^2 - \partial^2_{x_1} u \, \partial^2_{x_2} u\big) \right\} dx + \frac{P}{2} \int_D |\nabla u|^2 \, dx. \quad (2.7)$$

Let us make a few comments on the mechanical energy (2.7).

Remark 2.2.1 *The mechanical energy is nonnegative. In fact, since $P \geq 0$, we have $\frac{P}{2} \int_D |\nabla u|^2 \, dx \geq 0$. By applying Young's inequality and by using $\sigma \in (-1,1)$ and $B > 0$, we easily obtain that*

$$\begin{aligned}
&B \int_D \left[\frac{1}{2} (\Delta u)^2 + (1-\sigma)\big((\partial_{x_2}\partial_{x_1} u)^2 - \partial^2_{x_1} u \, \partial^2_{x_2} u\big) \right] dx \\
&\quad = B \int_D \left[\frac{1}{2} (\partial^2_{x_1} u)^2 + \frac{1}{2} (\partial^2_{x_2} u)^2 + (1-\sigma)(\partial_{x_2}\partial_{x_1} u)^2 + \sigma \, \partial^2_{x_1} u \, \partial^2_{x_2} u \right] dx \\
&\quad \geq B \int_D \left[\frac{1}{2} (\partial^2_{x_1} u)^2 + \frac{1}{2} (\partial^2_{x_2} u)^2 + (1-\sigma)(\partial_{x_2}\partial_{x_1} u)^2 - \frac{|\sigma|}{2} \Big((\partial^2_{x_1} u)^2 + (\partial^2_{x_2} u)^2 \Big) \right] dx \\
&\quad \geq B \int_D \left[\frac{(1-|\sigma|)}{2} \Big((\partial^2_{x_1} u)^2 + (\partial^2_{x_2} u)^2 \Big) + (1-|\sigma|) \, (\partial_{x_2}\partial_{x_1} u)^2 \right] dx \\
&\quad = \frac{B(1-|\sigma|)}{2} \int_D \left[(\partial^2_{x_1} u)^2 + (\partial^2_{x_2} u)^2 + 2 (\partial_{x_2}\partial_{x_1} u)^2 \right] dx \geq 0,
\end{aligned}$$

and so, $\mathcal{E}_m(u) \geq 0$.

For large deformations of the elastic plate, we cannot simply start with the combination (2.5) of curvatures, but have to consider an energy formulation that takes into account the displacements in all three directions. Let us note:

Remark 2.2.2 *For large deformations, we do not have linear strain-displacement relations resulting in (2.6). Consider a thin plate – a three dimensional body of uniform thickness $h > 0$ that is small compared to the other two dimensions – having a middle surface dividing the thickness. Prior to its deformation, the plate's middle surface is assumed to occupy the region D in the $x_1 x_2$-plane. Assuming moderately large deformations and considering the Kirchhoff-Love hypothesis, we get the (von Karman) nonlinear strain-displacement relations (see [43, Section 1.5] or [76, Section 3.3])*

$$\begin{cases}
\varepsilon_{11} &= \partial_{x_1} w + \frac{1}{2} (\partial_{x_1} u)^2 - z \partial^2_{x_1} u, \\
\varepsilon_{22} &= \partial_{x_2} v + \frac{1}{2} (\partial_{x_2} u)^2 - z \partial^2_{x_2} u, \\
\varepsilon_{12} &= \frac{1}{2} \left(\partial_{x_2} w + \partial_{x_1} v + \partial_{x_1} u \, \partial_{x_2} u - 2z \partial_{x_2}\partial_{x_1} u \right), \\
\varepsilon_{13} &= \varepsilon_{23} = 0,
\end{cases} \quad (2.8)$$

where $w = w(x)$, $v = v(x)$, and $u = u(x)$ denote the components (in the x_1, x_2, and z directions respectively) of the displacement vector at a point $x = (x_1, x_2)$ of the middle *surface D. According to [43, (1.7)] the mechanical energy reads*

$$\frac{E}{2(1-\sigma^2)}\int_{-h/2}^{h/2}\int_D \left(\varepsilon_{11}^2+2\sigma\varepsilon_{11}\varepsilon_{22}+\varepsilon_{22}^2+2(1-\sigma)(\varepsilon_{12}^2+\varepsilon_{13}^2+\varepsilon_{23}^2)\right)dx\,dz,$$

where E is Young's modulus. Using the strains (2.8) and carrying out the integration with respect to z, we find the following expression for the mechanical energy, see [43, Section 1.5.]:

$$\begin{aligned} &B\int_D \left[\frac{1}{2}(\Delta u)^2+(1-\sigma)\big((\partial_{x_2}\partial_{x_1}u)^2-\partial_{x_1}^2 u\,\partial_{x_2}^2 u\big)\right]dx \\ &\quad+\frac{E\,h}{2(1-\sigma^2)}\int_D\left[\Big(\partial_{x_1}w+\frac{1}{2}(\partial_{x_1}u)^2\Big)^2+\Big(\partial_{x_2}v+\frac{1}{2}(\partial_{x_2}u)^2\Big)^2\right. \\ &\quad\left.+2\sigma\Big(\partial_{x_1}w+\frac{1}{2}(\partial_{x_1}u)^2\Big)\Big(\partial_{x_2}v+\frac{1}{2}(\partial_{x_2}u)^2\Big)+\frac{1-\sigma}{2}\Big(\partial_{x_2}w+\partial_{x_1}v+\partial_{x_1}u\,\partial_{x_2}u\Big)^2\right]dx. \end{aligned} \tag{2.9}$$

We note that there is a coupling between the stretching components v, w and the vertical component u. The first term in (2.9), containing the second order derivatives of u, corresponds to pure bending and torsion of the plate and has already been computed in (2.7). The second term is the interaction of u with the stretching components v and w.
In this thesis, however, we are not interested in describing the behavior of the MEMS device under large deformations and thus restrict ourselves to (2.7).

Total energy of the MEMS device

Summarizing, by recalling both (2.3) and (2.7), the total potential energy for a given displacement u is

$$\begin{aligned}\mathcal{E}(u) &= -\frac{\varepsilon_0}{2}\int_{\Omega(u)}|\nabla\psi_u|^2\,d(x,z) \\ &\quad+B\int_D\left\{\frac{1}{2}(\Delta u)^2+(1-\sigma)\big((\partial_{x_2}\partial_{x_1}u)^2-\partial_{x_1}^2 u\,\partial_{x_2}^2 u\big)\right\}dx+\frac{P}{2}\int_D|\nabla u|^2\,dx.\end{aligned}$$

2.2.2 The Euler-Lagrange equation

We derive the Euler-Lagrange equation and the accompanying natural boundary condition by applying variational principles to the energy functional $\mathcal{E}$, that is, by finding its critical points. We need to compute the first variation of $\mathcal{E}$ and to find u such that

$$\delta\mathcal{E}(u;v) := \frac{d}{dr}\mathcal{E}(u+rv)|_{r=0}=0 \qquad \text{for all } v.$$

Since the elastic plate is supposed to be hinged, an appropriate space in order to look for critical points is $\{v\in C^\infty(\overline{D})\,;\, v=0 \text{ on } \partial D\}$.

In the following, we assume that D is a bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary, such that the exterior unit normal ν and the curvature κ of ∂D are well-defined and continuous. For example, $\partial D \in C^2$.

Formal derivation of the first variation of the electrostatic energy

Here, we follow the same approach as the one discussed in [56]. Let us fix a smooth deformation $u : \overline{D} \to \mathbb{R}$ such that $u = 0$ on ∂D and $u > -H$ in D. Let ψ_u be the corresponding solution to (2.1)-(2.2). Note that according to (2.1) the potential ψ_u depends nonlocally on u. Next, let $v \in C^\infty(\overline{D})$ with $v = 0$ on ∂D be arbitrary and set $u_r := u + rv$ for $-r_0 < r < r_0$, where $r_0 > 0$ is chosen sufficiently small so that

$$u_r > -H \quad \text{in } D \;\; \text{for all } r \in (-r_0, r_0).$$

Since u is fixed, we shall write ψ and Ω rather than ψ_u and $\Omega(u)$ in the sequel. In order to compute $\delta\mathcal{E}_e(u; v)$, we introduce, for $r \in (-r_0, r_0)$, the transformation $\Phi(r)$ by

$$\Phi(r)(x, z) := \left(x, z + rv(x)\frac{H + z}{H + u(x)}\right), \qquad (x, z) \in \Omega, \tag{2.10}$$

and notice that

$$\Omega(u_r) = \Phi(r)(\Omega) \quad \text{and} \quad \det(\nabla\Phi(r)) = 1 + \frac{rv}{H + u} > 0.$$

Next, let $\psi(r)$ be the solution to (2.1)-(2.2) in $\Omega(u_r)$, that is,

$$\begin{cases} \Delta\psi(r) = 0 & \text{in } \Omega(u_r), \\ \psi(r)(x, z) = V\frac{H+z}{H+u_r(x)}, & (x, z) \in \partial\Omega(u_r). \end{cases} \tag{2.11}$$

Then, we have $\psi(0) = \psi$. Let us now compute the derivative of

$$\mathcal{E}_e(u_r) = -\frac{\varepsilon_0}{2}\int_{\Omega(u_r)} |\nabla\psi(r)|^2 \, d(x, z)$$

with respect to r. By the Reynolds transport theorem, see, e.g., [36, Theorem 5.2.2] or [7, XII.Theorem 2.11], we deduce that

$$\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = -\varepsilon_0 \int_\Omega \left\{\nabla\psi \cdot \nabla\partial_r\psi(0) + \operatorname{div}\left(\frac{|\nabla\psi|^2}{2}\,\partial_r\Phi(0)\right)\right\} d(x, z).$$

An integration by parts gives

$$\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = \varepsilon_0 \int_\Omega \Delta\psi\,\partial_r\psi(0)\,d(x, z) - \varepsilon_0 \int_{\partial\Omega} \left(\partial_r\psi(0)\nabla\psi + \frac{|\nabla\psi|^2}{2}\,\partial_r\Phi(0)\right) \cdot n_{\partial\Omega}\,dS,$$

where $n_{\partial\Omega}$ is the outward unit normal to $\partial\Omega$. Using that $\Delta\psi = 0$ in Ω, we observe that

$$\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = -\varepsilon_0 \int_{\partial\Omega} \left(\partial_r\psi(0)\nabla\psi + \frac{|\nabla\psi|^2}{2}\,\partial_r\Phi(0)\right)\cdot n_{\partial\Omega}\,dS. \tag{2.12}$$

From (2.10), we see that

$$\partial_r\Phi(0)(x,z) = \left(0, v(x)\,\frac{H+z}{H+u(x)}\right), \qquad (x,z)\in\Omega, \tag{2.13}$$

and thus $\partial_r\Phi(0) = (0,0)$ on $D\times\{-H\}$. It also holds that $\partial_r\psi(0) = 0$ on $D\times\{-H\}$. Since $v = 0$ on ∂D, $\partial_r\Phi(0)$ and $\partial_r\psi(0)$ vanish on $\partial D\times(-H,0)$. Then, (2.12) reduces to

$$\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = -\varepsilon_0 \int_{\mathfrak{G}} \left(\partial_r\psi(0)\nabla\psi + \frac{|\nabla\psi|^2}{2}\,\partial_r\Phi(0)\right)\cdot n_{\mathfrak{G}}\,dS,$$

where

$$n_{\mathfrak{G}} := \frac{1}{\sqrt{1+|\nabla u(x)|^2}}\,(-\nabla u(x),1), \quad x\in D,$$

denotes the outward unit normal on the upper boundary $\mathfrak{G} := \{(x,u(x))\ ;\ x\in D\}$. Hence,

$$\begin{aligned}\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = -\varepsilon_0 \int_D &\Big(\partial_r\psi(0)(x,u(x))\nabla\psi(x,u(x)) \\ &+ \frac{|\nabla\psi(x,u(x))|^2}{2}\,\partial_r\Phi(0)(x,u(x))\Big)\cdot(-\nabla u(x),1)\,dx. \end{aligned}\tag{2.14}$$

If we put $\nabla' := (\partial_{x_1},\partial_{x_2})$ and differentiate the boundary condition $\psi(x,u(x)) = V$ with respect to x, we obtain

$$\nabla'\psi(x,u(x)) = -\nabla u(x)\partial_z\psi(x,u(x)), \qquad x\in D. \tag{2.15}$$

Now recalling from (2.11) that

$$\psi(r)(x,u_r(x)) = V, \qquad x\in D,\ r\in(-r_0,r_0),$$

it follows that

$$\partial_r\psi(0)(x,u(x)) = -\partial_z\psi(x,u(x))\,v(x), \qquad x\in D.$$

This, together with $\partial_r\Phi(0)(x,u(x)) = (0,v(x))$, $x\in D$, and (2.15), then yields that

$$\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = \frac{\varepsilon_0}{2}\int_D \left(1+|\nabla u(x)|^2\right)\left(\partial_z\psi(x,u(x))\right)^2 v(x)\,dx.$$

By (2.15) again,

$$\left(1+|\nabla u(x)|^2\right)\left(\partial_z\psi(x,u(x))\right)^2 = |\nabla\psi(x,u(x))|^2, \qquad x\in D,$$

and hence

$$\frac{d}{dr}\mathcal{E}_e(u_r)|_{r=0} = \frac{\varepsilon_0}{2}\int_D |\nabla\psi(x,u(x))|^2\, v(x)\,dx. \tag{2.16}$$

Remark 2.2.3 *We observe that the above calculations are formal since we did not specify the regularity of $\psi(r)$, neither with respect to r nor with respect to $(x,z) \in \Omega(u_r)$.*

Derivation of the first variation of the mechanical energy

Fix $u \in C^\infty(\overline{D})$ with $u = 0$ on ∂D and $u > -H$ in D. Let v and u_r be as above. We calculate $\delta\mathcal{E}_m(u;v)$ for an arbitrary v as follows: First, we observe that

$$\begin{aligned}
&\frac{d}{dr}\mathcal{E}_m(u_r)|_{r=0}\\
&\quad = B\int_D \Big[\Delta u\,\Delta v + (1-\sigma)\big(2\partial_{x_2}\partial_{x_1}u\,\partial_{x_2}\partial_{x_1}v - \partial^2_{x_1}u\,\partial^2_{x_2}v - \partial^2_{x_2}u\partial^2_{x_1}v\big)\Big]\,dx\\
&\qquad + P\int_D \nabla u\cdot\nabla v\,dx.
\end{aligned} \tag{2.17}$$

Then, we rewrite the right-hand side using integration by parts. Let us start with the second term in the first integral. Two integrations by parts and the boundary condition on u yield that

$$\begin{aligned}
2\int_D \partial_{x_2}\partial_{x_1}u\,\partial_{x_2}\partial_{x_1}v\,dx &= \int_D \partial_{x_2}\partial_{x_1}u\,\partial_{x_2}\partial_{x_1}v\,dx + \int_D \partial_{x_2}\partial_{x_1}u\,\partial_{x_2}\partial_{x_1}v\,dx\\
&= \int_{\partial D} \nu_2\,\partial_{x_2}\partial_{x_1}v\,\partial_{x_1}u\,d\omega + \int_D (\partial^2_{x_2}\partial^2_{x_1}v)\,u\,dx\\
&\quad + \int_{\partial D} \nu_1\partial_{x_2}\partial_{x_1}v\,\partial_{x_2}u\,d\omega + \int_D (\partial^2_{x_2}\partial^2_{x_1}v)\,u\,dx\\
&= \int_{\partial D} \partial_{x_2}\partial_{x_1}v\,(\nu_2\,\partial_{x_1}u + \nu_1\,\partial_{x_2}u)\,d\omega + 2\int_D (\partial^2_{x_2}\partial^2_{x_1}v)\,u\,dx
\end{aligned}$$

with $\nu = (\nu_1,\nu_2)$ denoting the exterior unit normal of ∂D. Again by integration by parts and $u|_{\partial D} = 0$, we get

$$-\int_D \partial^2_{x_1}u\,\partial^2_{x_2}v\,dx = -\int_{\partial D} \nu_1\,\partial^2_{x_2}v\,\partial_{x_1}u\,d\omega - \int_D (\partial^2_{x_2}\partial^2_{x_1}v)\,u\,dx$$

and

$$-\int_D \partial^2_{x_2}u\,\partial^2_{x_1}v\,dx = -\int_{\partial D} \nu_2\,\partial^2_{x_1}v\,\partial_{x_2}u\,d\omega - \int_D (\partial^2_{x_2}\partial^2_{x_1}v)\,u\,dx.$$

Hence, it follows that

$$\int_D \big(2\partial_{x_2}\partial_{x_1}u\,\partial_{x_2}\partial_{x_1}v - \partial^2_{x_1}u\,\partial^2_{x_2}v - \partial^2_{x_2}u\,\partial^2_{x_1}v\big)\,dx$$

$$= \int_{\partial D} \left(\nu_1\, \partial_{x_2}\partial_{x_1} v\, \partial_{x_2} u + \nu_2\, \partial_{x_2}\partial_{x_1} v\, \partial_{x_1} u - \nu_2\, \partial^2_{x_1} v\, \partial_{x_2} u - \nu_1\, \partial^2_{x_2} v\, \partial_{x_1} u\right)\, d\omega.$$

With the counterclockwise oriented tangent vector $s = (s_1, s_2)$ on ∂D and the fact that $\partial_{x_1} u|_{\partial D} = \nu_1 \partial_\nu u + s_1 \partial_s u$, $\partial_{x_2} u|_{\partial D} = \nu_2 \partial_\nu u + s_2 \partial_s u$, and $\partial_s u|_{\partial D} = 0$ since $u|_{\partial D} = 0$, we obtain

$$\begin{aligned}
&\int_{\partial D} \left(\nu_1\, \partial_{x_2}\partial_{x_1} v\, \partial_{x_2} u + \nu_2\, \partial_{x_2}\partial_{x_1} v\, \partial_{x_1} u - \nu_2\, \partial^2_{x_1} v\, \partial_{x_2} u - \nu_1 \partial^2_{x_2} v\, \partial_{x_1} u\right) d\omega \\
&\quad = \int_{\partial D} \left(2\nu_1\nu_2\, \partial_{x_2}\partial_{x_1} v - \nu_2^2\, \partial^2_{x_1} v - \nu_1^2\, \partial^2_{x_2} v\right) \partial_\nu u\, d\omega \\
&\quad = \int_{\partial D} \left(2\nu_1\nu_2\, \partial_{x_2}\partial_{x_1} v + \nu_1^2\, \partial^2_{x_1} v + \nu_2^2\, \partial^2_{x_2} v - \Delta v\right) \partial_\nu u\, d\omega \\
&\quad = \int_{\partial D} \left(\partial^2_\nu v - \Delta v\right) \partial_\nu u\, d\omega.
\end{aligned}$$

In the last step we used $\partial^2_\nu v = \nu_1^2\, \partial^2_{x_1} v + \nu_2^2\, \partial^2_{x_2} v + 2\nu_1\, \nu_2\, \partial_{x_2}\partial_{x_1} v$. Using the relation $\Delta v = \partial^2_\nu v + \partial^2_s v + \kappa\, \partial_\nu v$, see, e.g., [78, Section 4.1], and $\partial_s v = \partial^2_s v = 0$ on ∂D since $v|_{\partial D} = 0$, we get

$$\int_{\partial D} \left(\partial^2_\nu v - \Delta v\right) \partial_\nu u\, d\omega = -\int_{\partial D} \kappa\, \partial_\nu v\, \partial_\nu u\, d\omega.$$

Here, the function κ denotes the signed curvature of the boundary ∂D. Consequently, we deduce that

$$\int_D \left(2\partial_{x_2}\partial_{x_1} u\, \partial_{x_2}\partial_{x_1} v - \partial^2_{x_1} u\, \partial^2_{x_2} v - \partial^2_{x_2} u\, \partial^2_{x_1} v\right) dx = -\int_{\partial D} \kappa\, \partial_\nu v\, \partial_\nu u\, d\omega. \tag{2.18}$$

Integrating the first term on the right-hand side of (2.17) by parts twice and using the boundary condition on v gives

$$\begin{aligned}
\int_D \Delta u\, \Delta v\, dx &= \int_{\partial D} \Delta u\, \partial_\nu v\, d\omega - \int_D \nabla \Delta u \cdot \nabla v\, dx \\
&= \int_{\partial D} \Delta u\, \partial_\nu v\, d\omega + \int_D v\, \Delta^2 u\, dx.
\end{aligned} \tag{2.19}$$

An integration by parts again shows that

$$P \int_D \nabla u \cdot \nabla v\, dx = -P \int_D v\, \Delta u\, dx,$$

and together with (2.18) and (2.19), we finally obtain

$$\frac{d}{dr} \mathcal{E}_m(u_r)|_{r=0} = \int_D \left(B\Delta^2 u - P\Delta u\right) v\, dx + B \int_{\partial D} \left(\Delta u - (1-\sigma)\kappa\, \partial_\nu u\right) \partial_\nu v\, d\omega. \tag{2.20}$$

Remark 2.2.4 *The identity* (2.18) *can be proved under weaker regularity assumptions on*

u and v. See appendix A for details.

The Euler-Lagrange equation and boundary conditions

Gathering (2.16) and (2.20), we get

$$\begin{aligned}\delta\mathcal{E}(u;v) &= \delta\left(\mathcal{E}_e(u;v)+\mathcal{E}_m(u;v)\right)\\ &= \int_D \left(B\Delta^2 u - P\Delta u + \frac{\varepsilon_0}{2}\,|\nabla\psi_u(x,u(x))|^2\right) v\,dx\\ &\quad + B\int_{\partial D}\left(\Delta u-(1-\sigma)\kappa\,\partial_\nu u\right)\partial_\nu v\,d\omega, \end{aligned} \tag{2.21}$$

where v was an arbitrary function in $C^\infty(\overline{D})$ such that $v|_{\partial D}=0$. Setting the first variation equal to zero results in

$$0=\int_D \left(B\Delta^2 u - P\Delta u + \frac{\varepsilon_0}{2}\,|\nabla\psi_u(x,u(x))|^2\right) v\,dx + B\int_{\partial D}\left(\Delta u-(1-\sigma)\kappa\,\partial_\nu u\right)\partial_\nu v\,d\omega$$

for all $v\in C^\infty(\overline{D})$ with $v=0$ on ∂D, and by the fundamental lemma of calculus of variations, first in D and then on ∂D, it follows that

$$0=-B\Delta^2 u + P\Delta u - \frac{\varepsilon_0}{2}\,|\nabla\psi_u(x,u(x))|^2 \quad \text{in } D, \tag{2.22}$$

$$\begin{cases} u &= 0 \quad \text{on } \partial D,\\ B\left(\Delta u-(1-\sigma)\kappa\,\partial_\nu u\right) &= 0 \quad \text{on } \partial D. \end{cases} \tag{2.23}$$

Let us point out that the first boundary condition in (2.23) is an a priori boundary condition, whereas the second boundary condition arises as a natural boundary condition for the energy functional $\mathcal{E}$.

Remark 2.2.5 *In the case where D is a polygonal domain, which is commonly used in engineering, the hinged (or Steklov) boundary conditions (2.23) lead to the Navier boundary conditions $u=B\Delta u=0$ on ∂D with some singularity at corner points (due to "$\kappa=\infty$"). For more details, we refer the reader to [68], where the authors study the linear hinged plate boundary value problem*

$$\Delta^2 u = f \quad \text{in } D, \qquad u=\Delta u-(1-\sigma)\kappa\,\partial_\nu u = 0 \quad \text{on } \partial D,$$

for the special case of a rectangular plate and for the general case of a plate with corners of arbitrary opening angle.

Remark 2.2.6 *Let us briefly consider the case in which the top plate is clamped instead of hinged on the boundary. This means that both the vertical position and the angle at the boundary are fixed, that is,*

$$u=\partial_\nu u = 0 \qquad \text{on } \partial D. \tag{2.24}$$

In this case, we infer from (2.18) that

$$\int_D \left((\partial_{x_2}\partial_{x_1}u)^2 - \partial_{x_1}^2 u\, \partial_{x_2}^2 u \right) dx = 0,$$

and thus the mechanical energy (2.7) simply becomes

$$\mathcal{E}_m(u) = \frac{B}{2}\int_D (\Delta u)^2\, dx + \frac{P}{2}\int_D |\nabla u|^2\, dx.$$

The corresponding Euler-Lagrange equation together with the clamped boundary conditions (2.24) read

$$\begin{cases} 0 = -B\Delta^2 u + P\Delta u - \frac{\varepsilon_0}{2}\,|\nabla\psi_u(x,u(x))|^2 & \textit{in } D, \\ u = \partial_\nu u = 0 & \textit{on } \partial D. \end{cases}$$

Note that the Poisson ratio σ plays no role for clamped boundary conditions.

2.3 Governing equations for (u, ψ_u)

Stationary case

Combining the equations for the electrostatic potential ψ_u and the displacement u, we arrive at the following system of equations:

$$\begin{cases} \Delta\psi_u = 0 & \text{in } \Omega(u), & (2.25) \\ \psi_u(x,z) = V\,\dfrac{H+z}{H+u(x)}, & (x,z)\in\partial\Omega(u), & (2.26) \\ 0 = -B\Delta^2 u + P\Delta u - \dfrac{\varepsilon_0}{2}\,|\nabla\psi_u(x,u(x))|^2 & \text{in } D, & (2.27) \\ u = B\,(\Delta u - (1-\sigma)\kappa\partial_\nu u) = 0 & \text{on } \partial D. & (2.28) \end{cases}$$

Observe that (2.25)-(2.28) is a free boundary problem as the domain $\Omega(u)$ and its boundary component $\mathfrak{G}_u := \{(x,u(x))\,;\, x\in D\}$ have to be determined together with the solution (u,ψ_u). At an equilibrium state of the MEMS device, the mechanical force is equal to the electrostatic force. Regarding the right-hand side of (2.27) as forces on $\mathfrak{G}_u$, i.e.,

$$\text{mechanical force } = -B\Delta^2 u + P\Delta u \quad \text{and} \quad \text{electrostatic force } = -\frac{\varepsilon_0}{2}\,|\nabla\psi_u(x,u(x))|^2,$$

the equilibrium configurations of the device are given by the solutions to the system (2.25)-(2.28), which in turn are the critical points of the total energy $\mathcal{E}$.

Dynamic case

Finally, assume that u also depends on time t, i.e., $u = u(t, x)$. Applying Newton's second law, we obtain that the sum of all forces equals $\frac{m}{|D|}\,\partial_t^2 u$ (inertial force), where m denotes the mass of the plate. With a damping force of the form $-d\partial_t u$, where $d > 0$ is the damping constant, the evolution equation for the displacement u reads

$$\frac{m}{|D|}\,\partial_t^2 u + d\partial_t u + B\Delta^2 u - P\Delta u = -\frac{\varepsilon_0}{2}\,|\nabla\psi_{u(t)}(x, u(t,x))|^2 \qquad \text{in } D, \quad t > 0, \tag{2.29}$$

supplemented with hinged boundary conditions

$$u = B\,(\Delta u - (1-\sigma)\kappa\,\partial_\nu u) = 0 \qquad \text{on } \partial D, \quad t > 0, \tag{2.30}$$

and some initial conditions. The electrostatic potential $\psi_{u(t)} = \psi_{u(t)}(x, z)$ satisfies

$$\Delta\psi_{u(t)} = 0 \qquad \text{in } \Omega(u(t)), \quad t > 0, \tag{2.31}$$

along with the nonhomogeneous Dirichlet boundary condition

$$\psi_{u(t)}(x, z) = V\,\frac{H + z}{H + u(t,x)}\,, \qquad (x, z) \in \partial\Omega(u(t)), \quad t > 0. \tag{2.32}$$

Rescaled equations for the free boundary problem (2.29)-(2.32)

Now, we introduce dimensionless variables in equations (2.31)-(2.32) for ψ_u and (2.29)-(2.30) for u. We scale time based on the strength of damping, the variable x with a characteristic length L of the device, both z and u with the size of the gap H between the ground plate and the undeformed elastic plate, and the electrostatic potential ψ_u with the applied voltage V; so, we define

$$\tilde{t} := \frac{t}{\delta L^4}, \quad \tilde{x} := \frac{x}{L}, \quad \tilde{z} := \frac{z}{H}, \quad \tilde{u} := \frac{u}{H}, \quad \tilde{\psi}_{\tilde{u}} := \frac{\psi_u}{V},$$

and the aspect ratio $\varepsilon := H/L > 0$ of the device. We introduce the sets

$$\tilde{D} := \{\tilde{x} \in \mathbb{R}^2\,;\ L\tilde{x} \in D\}, \qquad \tilde{\Omega}(\tilde{u}(\tilde{t})) := \{(\tilde{x}, \tilde{z}) \in \tilde{D} \times \mathbb{R}\,;\ -1 < \tilde{z} < \tilde{u}(\tilde{t}, \tilde{x})\},$$

and define the parameters

$$\alpha^2 := \frac{m}{|D|\delta L^4} \geq 0, \quad \beta := B > 0, \quad \tau := PL^2 \geq 0, \quad \lambda = \lambda(\varepsilon) := \frac{\varepsilon_0 V^2 L}{2\varepsilon^3} > 0.$$

We then substitute these relations into (2.29)-(2.32) to derive dimensionless equations.

Dropping the tilde symbol, we get for the dimensionless displacement of the elastic plate the evolution equation

$$\begin{aligned}&\alpha^2\partial_t^2 u + \partial_t u + \beta\Delta^2 u - \tau\Delta u\\&\quad = -\lambda\Big\{\varepsilon^2|\nabla'\psi_{u(t)}(x,u(t,x))|^2 + (\partial_z\psi_{u(t)}(x,u(t,x)))^2\Big\}, \qquad x\in D, \quad t>0, \qquad (2.33)\end{aligned}$$

where $\nabla' := (\partial_{x_1}, \partial_{x_2})$. In this thesis, we shall assume that $\alpha << 1$, meaning that the damping force dominates over the inertial force. Rewriting equations (2.30)-(2.32) in terms of dimensionless variables and dropping again the tilde in all expressions, we end up with the following system of equations

$$\begin{cases}\varepsilon^2\Delta'\psi_{u(t)} + \partial_z^2\psi_{u(t)} = 0, & (x,z)\in\Omega(u(t)),\ t>0, \quad (2.34)\\ \psi_{u(t)}(x,z) = \dfrac{1+z}{1+u(t,x)}, & (x,z)\in\partial\Omega(u(t)),\ t>0, \quad (2.35)\\ \partial_t u + \beta\Delta^2 u - \tau\Delta u & \\ \quad = -\lambda\Big\{\varepsilon^2|\nabla'\psi_{u(t)}(x,u(t,x))|^2 + (\partial_z\psi_{u(t)}(x,u(t,x)))^2\Big\}, & x\in D,\ t>0, \quad (2.36)\\ u = \Delta u - (1-\sigma)\kappa\partial_\nu u = 0, & x\in\partial D,\ t>0, \quad (2.37)\\ u(0,x) = u^0(x), & x\in D, \quad (2.38)\end{cases}$$

where $\Delta' := \partial_{x_1}^2 + \partial_{x_2}^2$. Here, $\sigma\in(-1,1)$ is the Poisson ratio as described in Section 2.2 and $u^0(x)$ is the initial position of the elastic plate. This is the **system describing the dynamics of an idealized MEMS device with hinged boundary conditions** that we will consider throughout this thesis. The rescaled total potential energy of the device for a given displacement u is

$$\mathcal{E}(u) := \mathcal{E}_m(u) - \lambda\mathcal{E}_e(u)$$

with rescaled mechanical energy

$$\mathcal{E}_m(u) = \beta\int_D \Big\{\frac{1}{2}(\Delta u)^2 + (1-\sigma)\big((\partial_{x_2}\partial_{x_1}u)^2 - \partial_{x_1}^2 u\,\partial_{x_2}^2 u\big)\Big\}\,dx + \frac{\tau}{2}\int_D |\nabla u|^2\,dx$$

and rescaled electrostatic energy $-\lambda\mathcal{E}_e(u)$, where

$$\mathcal{E}_e(u) = \int_{\Omega(u)} \Big\{\varepsilon^2\,|\nabla'\psi_u|^2 + (\partial_z\psi_u)^2\Big\}\,d(x,z).$$

Let us emphasize that the system (2.34)-(2.38) features a strong coupling between the unknowns u and ψ_u. In fact, the source term in (2.36), governing the evolution of u, is proportional to the square of the gradient trace of ψ_u on the elastic plate. The electrostatic potential in turn solves an elliptic boundary value problem on a domain moving according to the evolution of u. Thus, the source term in (2.36) depends in a nonlocal and nonlinear way on u.

Furthermore, let us point out that (2.34)-(2.38) is only well-defined as long as the top plate does not touch down on the ground plate, that is, the displacement u stays above -1. In fact, if u reaches the value -1 somewhere in D at some time $t > 0$, the region $\Omega(u)$ gets disconnected. In addition, the vertical derivative $\partial_z \psi_u$ appearing in (2.36) becomes singular at such touchdown points as $\psi_u = 1$ along $z = u$ while $\psi_u = 0$ along $z = -1$ (due to (2.35)). This singularity is tuned by the parameter λ which is proportional to the square of the applied voltage V and thus plays an important role in the pull-in instability, which in turn is related to the existence of stationary solutions and to global existence of solutions to (2.34)-(2.38). That will be the topic of the next section. But first let us remark on the vanishing aspect ratio model.

Remark 2.3.1 *We now derive a simplified model from (2.34)-(2.38) by letting the aspect ratio $\varepsilon = H/L$ go to zero, meaning that the vertical dimension of the device is much smaller than its horizontal dimension. Setting formally $\varepsilon = 0$, the system (2.34)-(2.35) can be solved explicitly, yielding the potential*

$$\psi_{u(t)}(x,z) = \frac{1+z}{1+u(t,x)}, \qquad (x,z) \in \Omega(u(t)) \cup \partial\Omega(u(t)),\ t > 0.$$

In view of the evolution equation (2.36) with the hinged boundary conditions (2.37) and the initial condition (2.38), we arrive at the vanishing aspect ratio model

$$\begin{cases} \partial_t u + \beta\Delta^2 u - \tau\Delta u = -\dfrac{\lambda}{(1+u(t,x))^2}, & x \in D,\ t > 0, \quad (2.39) \\ u = \Delta u - (1-\sigma)\kappa\partial_\nu u = 0, & x \in \partial D,\ t > 0, \quad (2.40) \\ u(0,x) = u^0(x), & x \in D. \quad (2.41) \end{cases}$$

In the limit $\varepsilon \to 0$, the coupled problem (2.34)-(2.38) is thus reduced to a single semilinear parabolic equation for the displacement u with a nonlinear source term which is still singular when the top plate touches down on the ground plate, but no longer nonlocal. Let us point out that for Navier boundary conditions

$$u(t,x) = \beta\Delta u(t,x) = 0, \qquad x \in \partial D,\ t > 0,$$

there are several studies of the vanishing aspect ratio model, including a characterization of the pull-in voltage. We refer to [19, 32, 33, 61] and the references therein. The inertial force was also taken into account. It is worthwile to note that the above computation is formal. To give a rigorous justification of the vanishing aspect ratio model, by showing that the solution of the MEMS model (2.34)-(2.38) with $\varepsilon > 0$ converges towards the solution of the vanishing aspect ratio model (2.39)-(2.41) as $\varepsilon \to 0$ is a task for future research. For $D = (-1,1) \subset \mathbb{R}$ and clamped boundary conditions

$$u(t,\pm 1) = \beta\partial_x u(t,\pm 1) = 0, \qquad t > 0,$$

this has been established in [54].

2.4 Brief overview

The free boundary problem for MEMS with clamped boundary conditions, which is system (2.33)-(2.38) with $u = \partial_\nu u = 0$ on ∂D, $t > 0$, instead of (2.37), has been studied in [45, 51] and in [48, 50, 52] for the one-dimensional setting $D = (-1, 1)$. A detailed overview can be found in [53]. In contrast to this situation, our MEMS model with a hinged top plate has not been discussed from the mathematical point of view so far and the challenging task is to refine and extend some of the arguments in the abovementioned literature for the clamped plate as well as to introduce some new methods.

The main contents of this thesis are divided into two major parts.

Part I: Dynamic case

The first part of this thesis is devoted to the dynamics of the plate displacement and the electrostatic potential. In Chapters 3 to 5 we shall establish the following results for the dynamic free boundary problem (2.34)-(2.38):

- **Local-in-time well-posedness.** For any λ, there exists a unique solution (u, ψ_u) on a maximal interval of existence $[0, T_{max})$ with regularity

$$u \in C([0, T_{max}), W_2^4(D)) \cap C^1([0, T_{max}), L_2(D)),$$
$$\psi_{u(t)} \in W_2^2(\Omega(u(t))),$$

  satisfying $u > -1$, provided the initial value $u^0 \in W_2^4(D)$ satisfies $u^0 > -1$ in D and the hinged boundary conditions. We refer to Theorem 3.1.1 in the next chapter for a more precise statement under weaker regularity assumptions on u^0.

- **Global solutions.** The solution (u, ψ_u) exists globally in time, i.e., $T_{max} = \infty$, provided λ and u^0 are sufficiently small. In particular, neither touchdown of the top plate on the ground plate nor a norm blow up of the displacement occurs.

- **Finite time singularity.** If $T_{max} < \infty$, there is touchdown of the top plate in the sense that

$$\liminf_{t \nearrow T_{max}} \min_{\overline{D}} u(t) = -1.$$

It remains an open problem whether a finite time singularity occurs when λ is large enough, as is expected on physical grounds.

Part II: Stationary case

The second part of this thesis focusses on the stationary case of problem (2.34)-(2.38). In Chapters 6 to 8 we shall show the following results:

- **Existence.** For sufficiently small λ, there is a stationary solution that is asymptotically stable.

It remains an open problem whether there is more than one equilibrium configuration of the device, as is physically expected.

- **Positivity preserving property.** The boundary value problem for a hinged plate with stress
$$\begin{cases} \Delta^2 v - \tau\Delta v = f & \text{in } D, \\ v = \Delta v - (1-\sigma)\kappa\,\partial_\nu v = 0 & \text{on } \partial D, \end{cases}$$
is strongly positivity preserving, meaning $0 \not\equiv f \geq 0$ implies $v > 0$.

This general result will be used to prove the following:

- **Nonexistence.** There is an upper threshold for λ above which no stationary solution exists.

The last statement makes sense physically: If the electrostatic attractive force is increased by increasing the applied voltage λ and if that force is greater than the mechanical restoring force, then the equilibrium configuration is lost and the elastic plate might collapse on the lower rigid plate.

Part I

Dynamic case

Chapter 3

Local and global well-posedness of the MEMS model

Our aim in this chapter is to study the parabolic MEMS model introduced in Chapter 2, i.e., the system

$$\begin{cases} \varepsilon^2\Delta'\psi_{u(t)} + \partial_z^2\psi_{u(t)} = 0, & (x,z)\in\Omega(u(t)),\ t>0, \quad (3.1)\\ \psi_{u(t)}(x,z) = \dfrac{1+z}{1+u(t,x)}\,, & (x,z)\in\partial\Omega(u(t)),\ t>0, \quad (3.2)\\ \partial_t u + \beta\Delta^2 u - \tau\Delta u & \\ \quad = -\lambda\Big\{\varepsilon^2|\nabla'\psi_{u(t)}(x,u(t,x))|^2 + (\partial_z\psi_{u(t)}(x,u(t,x)))^2\Big\}, & x\in D,\ t>0, \quad (3.3)\\ u = \Delta u - (1-\sigma)\kappa\partial_\nu u = 0, & x\in\partial D,\ t>0, \quad (3.4)\\ u(0,x) = u^0(x), & x\in D, \quad (3.5) \end{cases}$$

with

$$\Omega(u(t)) := \{(x,z)\in D\times\mathbb{R}\,;\ -1<z<u(t,x)\},$$

$\nabla' := (\partial_{x_1},\partial_{x_2})$, and $\Delta' := \partial_{x_1}^2 + \partial_{x_2}^2$. Throughout the chapter, we assume that $D\subset\mathbb{R}^2$ is a bounded and convex domain with a C^4-boundary ∂D. The function κ is the signed curvature of the boundary ∂D, taken positive on strict convex boundary parts, and ν is the exterior unit normal of ∂D. The parameters

$$\beta>0,\ \tau\ge 0,\ \lambda>0,\ \varepsilon>0,\ \sigma\in(-1,1)$$

and their physical meaning were discussed in Chapter 2. In order to deal with the hinged boundary conditions (3.4), we introduce, for $q\in[1,\infty]$, the function spaces

$$W_{q,B}^{\alpha}(D) := \begin{cases} W_q^{\alpha}(D), & 0\le\alpha\le\frac{1}{q},\\ \big\{v\in W_q^{\alpha}(D)\,;\ v=0 \text{ on } \partial D\big\}\,, & \frac{1}{q}<\alpha\le 2+\frac{1}{q},\\ \big\{v\in W_q^{\alpha}(D)\,;\ v=\Delta v-(1-\sigma)\kappa\partial_\nu v = 0 \text{ on } \partial D\big\}\,, & 2+\frac{1}{q}<\alpha\le 4. \end{cases}$$

Here, we show that (3.1)-(3.5) is locally well-posed in time for any voltage value and that solutions exist globally for small voltage values.

3.1 Main result

The main result in this chapter is:

Theorem 3.1.1 (Well-posedness) *Let $4\xi \in \left(\frac{7}{3}, 4\right) \setminus \{\frac{5}{2}\}$. Consider an initial value $u^0 \in W^{4\xi}_{2,B}(D)$ such that $u^0 > -1$ in D. Then, the following holds:*

(i) **(Local existence)** *For any $\lambda > 0$, there is a unique solution (u, ψ_u) to (3.1)-(3.5) on the maximal interval of existence $[0, T_{max})$ in the sense that*

$$u \in C([0, T_{max}), W^{4\xi}_{2,B}(D)) \cap C((0, T_{max}), W^4_{2,B}(D)) \cap C^1((0, T_{max}), L_2(D)) \tag{3.6}$$

satisfies (3.3)-(3.5) together with

$$u(t, x) > -1, \quad (t, x) \in [0, T_{max}) \times D,$$

and

$$\psi_{u(t)} \in W^2_2(\Omega(u(t))) \tag{3.7}$$

solves (3.1)-(3.2) in $\Omega(u(t))$ for each $t \in [0, T_{max})$. In addition, if $u^0 \in W^4_{2,B}(D)$, then

$$u \in C([0, T_{max}), W^4_{2,B}(D)) \cap C^1([0, T_{max}), L_2(D)).$$

(ii) **(Norm blow up or touchdown)** *If $T_{max} < \infty$, then*

$$\limsup_{t \nearrow T_{max}} \|u(t)\|_{W^{4\xi}_2(D)} = \infty \qquad \text{or} \qquad \liminf_{t \nearrow T_{max}} \left(\min_{x \in \overline{D}} u(t, x)\right) = -1. \tag{3.8}$$

(iii) **(Global existence)** *Given $\rho \in (0, 1/2)$, there are $\lambda_* := \lambda_*(\rho, \varepsilon) > 0$ and $m := m(\rho, \varepsilon) > 0$ such that the solution (u, ψ_u) exists globally in time, i.e., $T_{max} = \infty$, provided that $\lambda \in (0, \lambda_*)$, $\|u^0\|_{W^{4\xi}_2(D)} \le m$, and $u^0 \ge -1 + 2\rho$ in D. In this case,*

$$u \in L_\infty(0, \infty; W^{4\xi}_{2,B}(D))$$

with

$$\inf_{(t,x) \in [0,\infty) \times D} u(t, x) > -1.$$

The first part of this theorem ensures the existence of a unique solution (u, ψ_u), defined on a maximal time interval $[0, T_{max})$, with $u > -1$ and the regularity specified in (3.6)-(3.7) for any $\lambda > 0$. Part (iii) shows that this solution exists globally in time provided that both λ and the initial value u^0 are sufficiently small. The second part of the theorem states that the finiteness of T_{max} implies touchdown of the top plate on the ground plate or blow up of the displacement in $W_2^{4\xi}(D)$. Physically, only the former seems possible. In Chapter 5, however, we will mathematically exclude the norm blow up in finite time.

The proof of Theorem 3.1.1 is based on the following idea: In a first step, the free boundary problem (3.1)-(3.5) is transformed into a problem on a fixed reference domain. For a given displacement $u(t)$, one can uniquely solve the transformed elliptic subproblem for the potential $\psi_{u(t)}$. Then, rewriting the transformed evolution problem for u as an abstract semilinear Cauchy problem, one obtains a solution from the variation of constants formula and the fixed point theorem for contraction mappings.

3.2 The elliptic problem

In this section, we focus our attention on the elliptic problem (3.1)-(3.2) for $\psi_{u(t)}$ and investigate its solvability for a given function $u(t)$. We first transform the problem (3.1)-(3.5) on the a priori unknown domain $\Omega(u(t))$ into a corresponding problem on the fixed domain $\Omega := D \times (0,1)$. In order to do so, we follow the lines of [51, Section 2]:

Let $q \in (2,\infty]$ be given and let $v \in W^2_{q,B}(D)$ be an arbitrary function that takes values in $(-1,\infty)$. We introduce a transformation of coordinates $T_v : \overline{\Omega(v)} \to \overline{\Omega}$ by setting

$$T_v(x,z) := \left(x, \frac{1+z}{1+v(x)}\right), \quad (x,z) \in \overline{\Omega(v)}. \tag{3.9}$$

It is easily seen that T_v is a diffeomorphism $\overline{\Omega(v)} \to \overline{\Omega}$ with the inverse

$$T_v^{-1}(x,\eta) = (x, (1+v(x))\eta - 1), \quad (x,\eta) \in \overline{\Omega}.$$

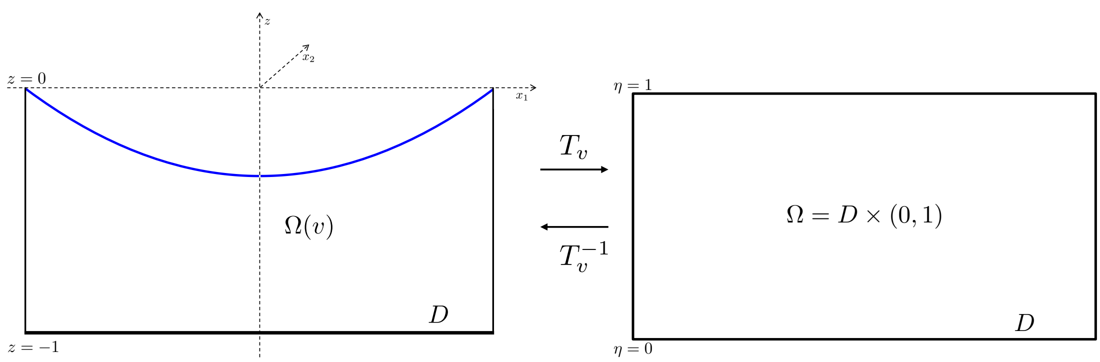

Figure 3.1: Transformation onto a fixed domain

Note that $\Omega(v)$ is a Lipschitz domain and that Ω is convex. Under this transformation of coordinates, the rescaled Laplace operator from (3.1) becomes

$$\mathcal{L}_v w := \varepsilon^2 \Delta' w - 2\varepsilon^2 \eta \frac{\nabla v(x)}{1+v(x)} \cdot \nabla' \partial_\eta w + \frac{1+\varepsilon^2\eta^2|\nabla v(x)|^2}{(1+v(x))^2} \partial_\eta^2 w$$
$$+ \varepsilon^2 \eta \left(2\frac{|\nabla v(x)|^2}{(1+v(x))^2} - \frac{\Delta v(x)}{1+v(x)} \right) \partial_\eta w. \tag{3.10}$$

The subproblem (3.1)-(3.2) is then equivalent to

$$\begin{cases} \left(\mathcal{L}_{u(t)}\phi_{u(t)}\right)(x,\eta) = 0, & (x,\eta) \in \Omega,\ t > 0, \quad (3.11)\\ \phi_{u(t)}(x,\eta) = \eta, & (x,\eta) \in \partial\Omega,\ t > 0, \quad (3.12)\end{cases}$$

for $\phi_{u(t)} = \psi_{u(t)} \circ T_{u(t)}^{-1}$. Furthermore, (3.3)-(3.5) become

$$\begin{cases} \partial_t u + \beta\Delta^2 u - \tau\Delta u = -\lambda \left\{ \frac{1+\varepsilon^2|\nabla u(t,x)|^2}{(1+u(t,x))^2} \left(\partial_\eta \phi_{u(t)}(x,1)\right)^2 \right\}, & x \in D,\ t>0, \quad (3.13)\\ u = \Delta u - (1-\sigma)\kappa\, \partial_\nu u = 0, & x \in \partial D,\ t > 0, \quad (3.14)\\ u(0,x) = u^0(x), & x \in D, \quad (3.15)\end{cases}$$

where we have used

$$\nabla' \phi_{u(t)}(x,1) = (0,0), \quad x \in D,\ t > 0,$$

due to $\phi_{u(t)}(x,1) = 1$ for $x \in D$, $t > 0$, by (3.12). Next, for $\rho \in (0,1)$, we define the set

$$S_q(\rho) := \left\{ v \in W_{q,B}^2(D)\,;\, v > -1+\rho \ \text{in } D \ \text{ and } \|v\|_{W_q^2(D)} < \frac{1}{\rho} \right\}, \tag{3.16}$$

which is open in $W_{q,B}^2(D)$. The closure of $S_q(\rho)$ denoted by $\overline{S}_q(\rho)$ is given by

$$\overline{S}_q(\rho) = \left\{ v \in W_{q,B}^2(D)\,;\, v \geq -1+\rho \ \text{in } D \ \text{ and } \|v\|_{W_q^2(D)} \leq \frac{1}{\rho} \right\}.$$

The next step is to solve the subproblem (3.11)-(3.12) for $u(t) \in \overline{S}_q(\rho)$ given and to discuss some useful properties of the right-hand side of (3.13).

Theorem 3.2.1 *Let $q \in [3,\infty]$ and $\rho \in (0,1)$. Then, for each $v \in \overline{S}_q(\rho)$ there is a unique solution $\phi_v \in W_2^2(\Omega)$ to*

$$\begin{cases} \left(\mathcal{L}_v \phi_v\right)(x,\eta) = 0, & (x,\eta) \in \Omega,\\ \phi_v(x,\eta) = \eta, & (x,\eta) \in \partial\Omega,\end{cases}$$

and there is a constant $C > 0$ only depending on ρ, ε, and D such that

$$\|\phi_{v_1} - \phi_{v_2}\|_{W_2^2(\Omega)} \leq C\|v_1 - v_2\|_{W_q^2(D)}, \quad v_1, v_2 \in \overline{S}_q(\rho).$$

Moreover, the mapping

$$g_\varepsilon : S_q(\rho) \to L_2(D), \quad v \mapsto \frac{1 + \varepsilon^2|\nabla v|^2}{(1+v)^2}\,(\partial_\eta \phi_v(\cdot, 1))^2$$

is analytic, bounded, and globally Lipschitz continuous.

Remark 3.2.2 *We observe that the functions $v \in \overline{S}_q(\rho)$ in Theorem 3.2.1 satisfy the boundary condition $v = 0$ on ∂D. In [51], this theorem is proved for $v \in \overline{S}_q(\rho)$ with $v = \partial_\nu v = 0$ on ∂D, but the arguments are exactly the same in our case. For the sake of completeness we present the proof of Theorem 3.2.1, with some minor modifications, in Appendix B.*

Theorem 3.2.1 implies in particular that, if $u(t) \in \overline{S}_3(\rho)$, then $\psi_{u(t)}$ belongs to $W_2^2(\Omega(u(t)))$ and solves (3.1)-(3.2).

3.3 The semilinear abstract evolution equation

In $L_2(D)$, (3.13)-(3.15) is formulated as the following Cauchy problem:

$$\begin{cases} \partial_t u + Au = -\lambda g_\varepsilon(u), & t > 0, \\ u(0) = u^0. \end{cases} \tag{3.17}$$

Here, the operator $A \in \mathcal{L}(W_{2,B}^4(D), L_2(D))$ is given by

$$Av := (\beta\Delta^2 - \tau\Delta)v, \quad v \in \text{dom}(A) = W_{2,B}^4(D).$$

Observe that the hinged boundary conditions (3.14) are incorporated in the domain of A. Since $C_c^\infty(D)$, the space of $C^\infty(D)$-functions having compact support in D, is dense in $L_2(D)$ and since $C_c^\infty(D) \subset W_{2,B}^4(D)$, we have that $W_{2,B}^4(D)$ is dense in $L_2(D)$. Hence,

$$W_{2,B}^4(D) \overset{d}{\hookrightarrow} L_2(D).$$

Note that once (3.17) is solved, we obtain a solution $\psi_{u(t)}$ to (3.1)-(3.2) by Theorem 3.2.1.

Below we formulate some important properties of A.

3.3.1 Some properties of the fourth-order operator

Let us first look at the following property of A.

Lemma 3.3.1 *It holds that*

$$A \in \mathcal{H}(W^4_{2,B}(D), L_2(D)),$$

meaning that $-A$ *is the generator of a strongly continuous analytic semigroup* $\{e^{-tA}\,;\, t \geq 0\}$ *on* $L_2(D)$.

Proof. We want to apply [5, Remark 4.2 (b)]. Let $D_1 := -i\partial_{x_1}$ and $D_2 := -i\partial_{x_2}$. We can rewrite the operator A as

$$Av = \beta \sum_{k,l=1}^{2} D_k^2 D_l^2 v + \tau \sum_{k=1}^{2} D_k^2 v.$$

The principal symbol of A is given by

$$a_\pi(\xi) := \beta|\xi|^4, \quad \xi \in \mathbb{R}^2. \tag{3.18}$$

Let $\mathbb{S}_1$ be the unit sphere in $\mathbb{R}^2$. Since $\beta > 0$, the spectrum $\sigma(a_\pi(\xi))$ satisfies

$$\sigma(a_\pi(\xi)) \subset \{z \in \mathbb{C}\,;\, \mathrm{Re} z > 0\} \quad \text{for all } \xi \in \mathbb{S}_1.$$

This means that A is normally elliptic (see [5, p.18] for a definition). Furthermore, it follows from (3.14) that the system B of boundary operators is $B := (B_1, B_2)$ with

$$B_1 v = \mathrm{tr}\, v, \quad B_2 v = -\sum_{k=1}^{2} \mathrm{tr}\, D_k^2 v - (1-\sigma) i\, \kappa \sum_{k=1}^{2} \nu_k\, \mathrm{tr}\, D_k v, \tag{3.19}$$

for $v \in W^4_{2,B}(D)$, where tr denotes the trace operator on ∂D. Since $\partial D \in C^4$, we have $\nu \in C^3(\partial D)$ and $\kappa \in C^2(\partial D)$, and hence

$$(1-\sigma) i\, \kappa\, \nu_k \in C^2(\partial D, \mathbb{C}), \quad k = 1, 2.$$

Next, the principal boundary symbol of B is given by

$$b_\pi(\xi) := \left(1, -|\xi|^2\right), \quad \xi \in \mathbb{R}^2. \tag{3.20}$$

Recall from [5, p.18] that (A, B) is said to be normally elliptic if A is normally elliptic and B satisfies the normal complementing condition with respect to A. The latter condition is also called the Lopatinskii-Shapiro condition and requires that, for any $x \in \partial D$, $\xi \in \mathbb{R}^2$ with $\xi \cdot \nu(x) = 0$ and any $\mu \in \mathbb{C}$ with $\mathrm{Re}\, \mu \geq 0$ and $|\mu| + |\xi| \neq 0$, zero is the only exponentially decaying solution of the initial value problem on $[0,\infty)$:

$$\begin{cases} [\mu + a_\pi(\xi + \nu(x) i\, \partial_t)]\, v = 0, \\ b_\pi(\xi + \nu(x) i\, \partial_t)\, v(0) = 0. \end{cases} \tag{3.21}$$

From [5, Remark 4.2(b)] we obtain $A \in \mathcal{H}(W^4_{2,B}(D), L_2(D))$ provided that (A, B) is normally elliptic. So, it remains to check the normal complementing condition. Notice that (3.21) can be written as

$$\begin{cases} [\mu + \beta \left(|\xi|^2 - \partial_t^2\right)^2]\, v(t) = 0, \quad t > 0, & (3.22) \\ v(0) = \partial_t^2 v(0) = 0. & (3.23) \end{cases}$$

If $\mu = 0$, then the general solution to (3.22) is

$$v(t) = (C_1 + C_2 t) e^{|\xi| t} + (C_3 + C_4 t) e^{-|\xi| t}, \qquad t \geq 0,$$

with $C_k \in \mathbb{R}$, $1 \leq k \leq 4$. In this case $\xi \neq 0$. Since the solution v must decay exponentially, we must have $C_1 = C_2 = 0$. Imposing that v also satisfies the initial conditions (3.23), we get $C_3 = C_4 = 0$ and hence $v \equiv 0$.

When $\mu \neq 0$, the characteristic equation of (3.22) is given by

$$r^4 - 2|\xi|^2 r^2 + |\xi|^4 + \frac{\mu}{\beta} = 0.$$

Its roots are

$$r_k = \pm \sqrt{|\xi|^2 \pm i \frac{\sqrt{\mu}}{\sqrt{\beta}}}, \qquad 1 \leq k \leq 4.$$

We recall that the square root of a complex number has a nonnegative real part. Now we write $\sqrt{\mu} = a_\mu + i b_\mu$ with $a_\mu, b_\mu \in \mathbb{R}$, $a_\mu \geq 0$ and note that $\mathrm{Re}\, \mu \geq 0$ implies $a_\mu > 0$. Therefore,

$$r_{1,3} = \pm \sqrt{\left(|\xi|^2 - \frac{b_\mu}{\sqrt{\beta}}\right) + i \frac{a_\mu}{\sqrt{\beta}}}, \qquad r_{2,4} = \pm \sqrt{\left(|\xi|^2 + \frac{b_\mu}{\sqrt{\beta}}\right) - i \frac{a_\mu}{\sqrt{\beta}}},$$

with r_1 and r_2 having positive real part. Moreover, since each root has multiplicity one, the general solution of (3.22) is given by

$$v(t) = \sum_{k=1}^{4} C_k e^{r_k t}, \quad t \geq 0,$$

with $C_k \in \mathbb{R}$, $1 \leq k \leq 4$. Since v must decay exponentially, $C_1 = C_2 = 0$. Invoking (3.23), we find

$$0 = v(0) = C_3 + C_4, \quad 0 = \partial_t^2 v(0) = C_3 r_3^2 + C_4 r_4^2.$$

The initial-value problem (3.22)-(3.23) has for $(\xi, \mu) \in \mathbb{R}^2 \times \{z \in \mathbb{C}\,;\, \mathrm{Re}\, z \geq 0\}$ with $|\mu| + |\xi| \neq 0$ only the trivial solution $v \equiv 0$ if

$$r_4^2 - r_3^2 = (r_4 + r_3)(r_4 - r_3) \neq 0. \tag{3.24}$$

Since $r_4 \neq \pm r_3$, condition (3.24) is fulfilled. Therefore, (A, B) is normally elliptic. ∎

In the next lemma we show that the spectrum of $-A$ is contained in the left half-plane $\{z \in \mathbb{C}\,;\, \mathrm{Re}\, z < 0\}$.

Lemma 3.3.2 *We have*

$$\sigma(-A) \subset \{z \in \mathbb{C}\,;\, \mathrm{Re}\, z < 0\}.$$

Proof. Since $W^4_{2,B}(D)$ embeds compactly in $L_2(D)$ and $A \in \mathcal{H}(W^4_{2,B}(D), L_2(D))$, the operator $-A$ has compact resolvent. For the compact embedding we refer to [1, Theorem 6.3]. In view of [38, Theorem 6.29], the spectrum of $-A$ consists only of isolated eigenvalues of finite multiplicity. If $\mu \in \mathbb{C}$ is such an eigenvalue of $-A$ with a corresponding eigenfunction $\varphi \in W^4_{2,B}(D, \mathbb{C})$, then multiplying the equation

$$-\beta \Delta^2 \varphi + \tau \Delta \varphi = \mu \varphi$$

by $\overline{\varphi}$ and integrating the product in D gives

$$\mu \int_D |\varphi|^2 \, dx = \int_D \left(-\beta \Delta^2 \varphi + \tau \Delta \varphi\right) \overline{\varphi} \, dx. \tag{3.25}$$

Here, $\overline{\varphi}(x) = \overline{\varphi(x)}$ is the complex conjugate of $\varphi(x)$. We write $\varphi : \overline{D} \to \mathbb{C}$ in terms of its real and imaginary parts, say

$$\varphi(x) = a(x) + ib(x), \quad a(x), b(x) \in \mathbb{R}.$$

Note that

$$|\nabla \varphi|^2 = |\nabla a|^2 + |\nabla b|^2, \quad |\partial_\nu \varphi|^2 = (\partial_\nu a)^2 + (\partial_\nu b)^2, \quad |\Delta \varphi|^2 = (\Delta a)^2 + (\Delta b)^2. \tag{3.26}$$

Using integration by parts twice and the boundary conditions for φ, we get

$$\begin{aligned} -\beta \int_D (\Delta^2 \varphi) \, \overline{\varphi} \, dx &= \beta \int_D \nabla \Delta \varphi \cdot \nabla \overline{\varphi} \, dx \\ &= \beta \int_{\partial D} \Delta \varphi \, \partial_\nu \overline{\varphi} \, d\omega - \beta \int_D \Delta \varphi \, \Delta \overline{\varphi} \, dx \\ &= \beta (1 - \sigma) \int_{\partial D} \kappa \, |\partial_\nu \varphi|^2 \, d\omega - \beta \int_D |\Delta \varphi|^2 \, dx. \end{aligned}$$

Integrating by parts and using $\varphi|_{\partial D} = 0$, yields

$$\tau \int_D (\Delta\varphi)\,\overline{\varphi}\,dx = -\tau \int_D |\nabla\varphi|^2\,dx$$

and thus

$$\mu \int_D |\varphi|^2\,dx = \beta(1-\sigma) \int_{\partial D} \kappa\,|\partial_\nu \varphi|^2\,d\omega - \beta \int_D |\Delta\varphi|^2\,dx - \tau \int_D |\nabla\varphi|^2\,dx.$$

Moreover, it follows from Lemma A.0.1 that

$$\begin{aligned}
&\beta(1-\sigma) \int_{\partial D} \kappa\,(\partial_\nu a)^2\,d\omega \\
&\quad = -2\beta(1-\sigma) \int_D \big((\partial_{x_2}\partial_{x_1} a)^2 - \partial_{x_1}^2 a\,\partial_{x_2}^2 a\big)\,dx \\
&\quad = -\beta(1-\sigma) \int_D \big((\partial_{x_1}^2 a)^2 + (\partial_{x_2}^2 a)^2 + 2(\partial_{x_2}\partial_{x_1} a)^2 - (\Delta a)^2\big)\,dx \qquad (3.27)
\end{aligned}$$

and analogously

$$\beta(1-\sigma) \int_{\partial D} \kappa\,(\partial_\nu b)^2\,d\omega = -\beta(1-\sigma) \int_D \big((\partial_{x_1}^2 b)^2 + (\partial_{x_2}^2 b)^2 + 2(\partial_{x_2}\partial_{x_1} b)^2 - (\Delta b)^2\big)\,dx. \qquad (3.28)$$

Thus, by (3.26),

$$\begin{aligned}
\mu \int_D |\varphi|^2\,dx = &-\beta(1-\sigma) \int_D \big(|\partial_{x_1}^2 \varphi|^2 + |\partial_{x_2}^2 \varphi|^2 + 2|\partial_{x_2}\partial_{x_1} \varphi|^2\big)\,dx \\
&- \beta\sigma \int_D |\Delta\varphi|^2\,dx - \tau \int_D |\nabla\varphi|^2\,dx.
\end{aligned}$$

By Young's inequality, we have

$$\frac{1}{2}(\Delta a)^2 \le (\partial_{x_1}^2 a)^2 + (\partial_{x_2}^2 a)^2 + 2(\partial_{x_2}\partial_{x_1} a)^2 \quad \text{in } D \qquad (3.29)$$

and

$$\frac{1}{2}(\Delta b)^2 \le (\partial_{x_1}^2 b)^2 + (\partial_{x_2}^2 b)^2 + 2(\partial_{x_2}\partial_{x_1} b)^2 \quad \text{in } D, \qquad (3.30)$$

so that

$$\begin{aligned}
\mu \int_D |\varphi|^2\,dx &\le -\frac{1}{2}\beta(1-\sigma) \int_D |\Delta\varphi|^2\,dx - \beta\sigma \int_D |\Delta\varphi|^2\,dx - \tau \int_D |\nabla\varphi|^2\,dx \\
&= -\frac{1}{2}\beta(1+\sigma) \int_D |\Delta\varphi|^2\,dx - \tau \int_D |\nabla\varphi|^2\,dx. \qquad (3.31)
\end{aligned}$$

Since $\beta(1+\sigma) > 0$ and $\tau \geq 0$, we deduce

$$\mu \int_D |\varphi|^2 \, dx \leq 0,$$

and hence $\mu \leq 0$. It remains to show that $\mu < 0$. If $\mu = 0$, then, due to (3.31),

$$-\frac{1}{2}\beta(1+\sigma) \int_D |\Delta\varphi|^2 \, dx - \tau \int_D |\nabla\varphi|^2 \, dx = 0,$$

and since $\tau \geq 0$, it follows that

$$-\frac{1}{2}\beta(1+\sigma) \int_D |\Delta\varphi|^2 \, dx = 0. \tag{3.32}$$

So, $\Delta\varphi = 0$ in D. Since $\varphi = 0$ on ∂D, we conclude that $\varphi \equiv 0$, which is a contradiction. Thus, $\mu < 0$. ■

Thanks to this lemma, it follows from [72, Theorem 4.4.3] that the semigroup $\{e^{-tA}\,;\, t \geq 0\}$ has exponential decay, that is, there are $M \geq 1$ and $\varpi > 0$ such that

$$\|e^{-tA}\|_{\mathcal{L}(L_2(D))} \leq M e^{-\varpi t}, \quad t \geq 0.$$

Moreover, it satisfies the following regularizing properties.

Lemma 3.3.3 *There exists $\varpi > 0$ such that the following holds true: If $0 \leq \gamma \leq \alpha \leq 1$ with 4α, $4\gamma \notin \{\frac{1}{2}, \frac{5}{2}\}$, then*

$$\|e^{-tA}\|_{\mathcal{L}(W^{4\gamma}_{2,B}(D), W^{4\alpha}_{2,B}(D))} \leq M e^{-\varpi t} t^{\gamma-\alpha}, \quad t > 0,$$

for some number $M \geq 1$ depending on α and γ.

In the following, we denote by $[\cdot,\cdot]_\theta$ the complex and by $(\cdot,\cdot)_{\theta,q}$, $1 \leq q \leq \infty$, the real interpolation functor for $0 < \theta < 1$. We refer to [6, Section I.2] for a summary of interpolation theory and to [83] for more details and proofs.

Proof. It is easily seen that the system of boundary operators $B := (B_1, B_2)$ given in (3.19) forms a normal system in the sense of [83, Definition 4.3.3 (1)]. Let $\theta \in (0,1)$. Then, by [83, Theorem 4.3.3 (a)], we have

$$\big(L_2(D), W^4_{2,B}(D)\big)_\theta \doteq W^{4\theta}_{2,B}(D) \quad \text{if } 4\theta \notin \left\{\frac{1}{2}, \frac{5}{2}\right\}, \tag{3.33}$$

where

$$(\cdot,\cdot)_\theta := \begin{cases} (\cdot,\cdot)_{\theta,2} & \text{if } 4\theta \notin \{1,2,3\}, \\ [\cdot,\cdot]_\theta & \text{if } 4\theta \in \{1,2,3\}. \end{cases}$$

Let $E_0 := L_2(D)$, $E_1 := W^4_{2,B}(D)$, and put

$$E_\theta := \big(L_2(D), W^4_{2,B}(D)\big)_\theta \doteq W^{4\theta}_{2,B}(D), \quad 4\theta \notin \left\{\frac{1}{2}, \frac{5}{2}\right\}.$$

In view of Lemmas 3.3.1 and 3.3.2, we can apply [6, Theorem V.2.1.3] to conclude that there are $\varpi > 0$ and $M \geq 1$ such that

$$\|e^{-tA}\|_{\mathcal{L}(E_\gamma,E_\alpha)} \leq Me^{-\varpi t}t^{\gamma-\alpha}, \quad t > 0,$$

where $0 \leq \gamma \leq \alpha \leq 1$ with 4α, $4\gamma \notin \{\frac{1}{2}, \frac{5}{2}\}$. The constant M depends on α and γ. ∎

Remark 3.3.4 *Let $p \in (1,\infty)$ and $\theta \in (0,1)$. Repeating the same arguments as in the proof of Lemma 3.3.3, we obtain that*

$$\begin{cases} \Big(L_p(D), W^4_{p,B}(D)\Big)_{\theta,p} \doteq W^{4\theta}_{p,B}(D) & \text{if } 4\theta \in (0,4)\backslash\{\frac{1}{p}, 1, 2, 2+\frac{1}{p}, 3\}, \\ \Big[L_p(D), W^4_{p,B}(D)\Big]_\theta \doteq W^{4\theta}_{p,B}(D) & \text{if } 4\theta \in \{1,2,3\}. \end{cases}$$

We close this subsection with a result concerning maximal L_2-regularity.

Theorem 3.3.5 *Let $4\xi \in (2,4)\backslash\{\frac{5}{2}\}$ and $0 < T < \infty$. Then, for every*

$$(f, u^0) \in L_2(0,T; L_2(D)) \times W^{4\xi}_{2,B}(D),$$

the Cauchy problem

$$\begin{cases} \partial_t u + Au = f(t), & 0 < t \leq T, \\ u(0) = u^0 \end{cases}$$

has a unique solution $u \in L_2(0,T; W^4_{2,B}(D)) \cap W^1_2(0,T; L_2(D))$ and we have the estimate

$$\|u\|_{L_2(0,T;L_2(D))} + \|\partial_t u\|_{L_2(0,T;L_2(D))} + \|Au\|_{L_2(0,T;L_2(D))} \leq C\Big(\|f\|_{L_2(0,T;L_2(D))} + \|u^0\|_{W^{4\xi}_{2,B}(D)}\Big)$$

with $C > 0$ independent of f and u^0. This means that A has maximal L_2-regularity on $[0,T]$.

Proof. We want to apply [6, Theorem III.4.10.8]. Let us show that its assumptions hold. Since $A \in \mathcal{H}(W^4_{2,B}(D), L_2(D))$ with

$$s(-A) := \sup\{\operatorname{Re}\mu\,;\, \mu \in \sigma(-A)\} < 0,$$

we infer from [6, Theorem I.1.4.3] that A satisfies condition [6, III.(4.10.1)]. We next verify that A is self-adjoint and positive definite, that is, $A = A^* \geq m$ for some $m > 0$. We know that A is densely defined on $\mathrm{dom}(A) = W^4_{2,B}(D)$. Moreover, by integration by parts, we have, for $v_1, v_2 \in W^4_{2,B}(D, \mathbb{C})$,

$$\begin{aligned}\langle Av_1, v_2\rangle_{L_2(D)} &= \int_D \left(\beta\Delta^2 v_1 - \tau\Delta v_1\right) \overline{v_2}\, dx \\ &= -\beta \int_D \nabla\Delta v_1 \cdot \nabla\overline{v_2}\, dx + \tau \int_D \nabla v_1 \cdot \nabla\overline{v_2}\, dx \\ &= \beta \int_D \Delta v_1\, \Delta\overline{v_2}\, dx - \beta \int_{\partial D} \Delta v_1\, \partial_\nu \overline{v_2}\, d\omega - \tau \int_D v_1\, \Delta\overline{v_2}\, dx,\end{aligned}$$

where we used $v_1|_{\partial D} = 0$ and $\overline{v_2}|_{\partial D} = 0$. Again, using integration by parts and $v_1|_{\partial D} = 0$, the first integral in the right-hand side can be rewritten in the form

$$\beta \int_D \Delta v_1\, \Delta\overline{v_2}\, dx = \beta \int_{\partial D} \Delta\overline{v_2}\, \partial_\nu v_1\, d\omega + \beta \int_D v_1\, \Delta^2\overline{v_2}\, dx,$$

and hence

$$\langle Av_1, v_2\rangle_{L_2(D)} = \beta \int_{\partial D} \left(\Delta\overline{v_2}\, \partial_\nu v_1\ - \Delta v_1\, \partial_\nu\overline{v_2}\right) d\omega + \int_D v_1 \left(\beta\Delta^2\overline{v_2} - \tau\Delta\overline{v_2}\right)\, dx.$$

Taking into account the second boundary conditions for v_1 and $\overline{v_2}$, we obtain that

$$\langle Av_1, v_2\rangle_{L_2(D)} = \int_D v_1 \left(\beta\Delta^2\overline{v_2} - \tau\Delta\overline{v_2}\right)\, dx = \langle v_1, Av_2\rangle_{L_2(D)}\, .$$

So, A is a symmetric operator in $L_2(D)$. Lemma 3.3.2 shows that $\mathrm{im}(A) = L_2(D)$. Thus A is self-adjoint in $L_2(D)$ and hence $\sigma(A) \subset \mathbb{R}$. By Lemma 3.3.2, we even have $\sigma(A) \subset (0, \infty)$.

By the same reason as above, we have, for $v_1 \in W^4_{2,B}(D, \mathbb{C})$,

$$\langle Av_1, v_1\rangle_{L_2(D)} = \beta \int_D |\Delta v_1|^2\, dx - \beta \int_{\partial D} \Delta v_1\, \partial_\nu\overline{v_1}\, d\omega + \tau \int_D |\nabla v_1|^2\, dx.$$

Using the second boundary condition for v_1, we get

$$\langle Av_1, v_1\rangle_{L_2(D)} = \beta \int_D |\Delta v_1|^2\, dx - \beta(1-\sigma) \int_{\partial D} \kappa\, |\partial_\nu v_1|^2\, d\omega + \tau \int_D |\nabla v_1|^2\, dx.$$

Then, we can argue in a similar way as in the proof of Lemma 3.3.2. Writing v_1 in terms of its real and imaginary parts, say

$$v_1(x) = a(x) + ib(x), \quad a(x), b(x) \in \mathbb{R},$$

it follows from (3.26)-(3.28) that

$$\langle Av_1, v_1\rangle_{L_2(D)}$$

$$= \beta \int_D |\Delta v_1|^2 \, dx + \beta(1-\sigma) \int_{\partial D} \big(|\partial_{x_1}^2 v_1|^2 + |\partial_{x_2}^2 v_1|^2 + 2|\partial_{x_2}\partial_{x_1} v_1|^2 - |\Delta v_1|^2\big) \, dx$$
$$+ \tau \int_D |\nabla v_1|^2 \, dx$$
$$= \beta(1-\sigma) \int_{\partial D} \big(|\partial_{x_1}^2 v_1|^2 + |\partial_{x_2}^2 v_1|^2 + 2|\partial_{x_2}\partial_{x_1} v_1|^2\big) \, dx + \beta\sigma \int_D |\Delta v_1|^2 \, dx$$
$$+ \tau \int_D |\nabla v_1|^2 \, dx.$$

Then, due to (3.29)-(3.30),

$$\begin{aligned}\langle A v_1, v_1 \rangle_{L_2(D)} &\geq \frac{\beta(1-\sigma)}{2} \int_D |\Delta v_1|^2 \, dx + \beta\sigma \int_D |\Delta v_1|^2 \, dx + \tau \int_D |\nabla v_1|^2 \, dx \\ &\geq \frac{\beta(1+\sigma)}{2} \int_D |\Delta v_1|^2 \, dx.\end{aligned}$$

This, together with [28, Theorem 3.1.2.1] (since D is convex), yields that

$$\langle A v_1, v_1 \rangle_{L_2(D)} \geq c \, \|v_1\|^2_{W_2^2(D)} \geq c \, \|v_1\|^2_{L_2(D)}, \quad v_1 \in W^4_{2,B}(D, \mathbb{C}),$$

where $c > 0$ depends only on β, σ, and D. Thus, A is a positive definite self-adjoint operator. Therefore, by [6, Examples III.4.7.3 (a)] it follows that $A \in \mathcal{BIP}(1, 0)$. We refer to [6, Section III.4.7] for a definition of the bounded imaginary powers ($\mathcal{BIP}$) of an operator. Furthermore, according to (3.33) and [6, Section 2.5],

$$W^{4\xi}_{2,B}(D) \hookrightarrow \big(L_2(D), W^4_{2,B}(D)\big)_{\frac{1}{2},2}.$$

Now the theorem follows from [6, Theorem III.4.10.8]. ■

We can now proceed to the proof of Theorem 3.1.1.

3.4 Proof of Theorem 3.1.1

Our proof more or less follows the lines of [51]. Fix $4\xi \in \big(\frac{7}{3}, 4\big) \setminus \{\frac{5}{2}\}$ and consider an initial value $u^0 \in W^{4\xi}_{2,B}(D)$ such that $u^0 > -1$ in D. By the continuous embedding of $W_2^{4\xi}(D)$ in $W_3^2(D)$ and in $C(\overline{D})$, there are $\rho \in (0, \frac{1}{2})$ and a constant $c_1 > 1$ such that

$$\|v\|_{W_3^2(D)} \leq c_1 \|v\|_{W_2^{4\xi}(D)}, \quad v \in W_2^{4\xi}(D),$$

and

$$u^0 \in S_3(2\rho) \ \text{ with } \ \|u^0\|_{W_2^{4\xi}(D)} \leq \frac{1}{2\rho}. \tag{3.34}$$

Furthermore, by Lemma 3.3.3, we have

$$\|e^{-tA}\|_{\mathcal{L}(W^{4\xi}_{2,B}(D))} + t^{\xi}\|e^{-tA}\|_{\mathcal{L}(L_2(D),W^{4\xi}_{2,B}(D))} \leq Me^{-\varpi t}, \quad t \geq 0. \tag{3.35}$$

Let $\rho_0 := \frac{\rho}{Mc_1} \in (0,\rho)$. Recall from Theorem 3.2.1 that, for $v_1, v_2 \in \overline{S}_3(\rho_0)$,

$$\|g_\varepsilon(v_1) - g_\varepsilon(v_2)\|_{L_2(D)} \leq C_L\|v_1 - v_2\|_{W^2_3(D)}, \tag{3.36}$$

and that

$$\|g_\varepsilon(v)\|_{L_2(D)} \leq C_B \quad \text{for all } v \in \overline{S}_3(\rho_0) \tag{3.37}$$

with constants $C_L > 0$ and $C_B > 0$, depending only on ρ, ε, and D.

Part (i): We want to apply the fixed point theorem for contraction mappings. For $0 < T < \infty$, we introduce

$$\mathcal{V}_T := C([0,T], \overline{S}_3(\rho_0))$$

and observe that this set is a nonempty complete metric space. For $v \in \mathcal{V}_T$, we define a function on [0,T]

$$\Lambda(v)(t) := e^{-tA}u^0 - \lambda \int_0^t e^{-(t-s)A}\, g_\varepsilon(v(s))\, ds. \tag{3.38}$$

We want to show that, for any $\lambda > 0$, Λ is a contraction mapping from $\mathcal{V}_T$ into itself, provided that T is sufficiently small, and that the fixed point of Λ is the desired solution of (3.17).

Let us verify that $\Lambda(v) \in \mathcal{V}_T$ for every $v \in \mathcal{V}_T$. Let $v \in \mathcal{V}_T$ and $4\theta \in \left(\frac{7}{3}, 4\xi\right) \setminus \{\frac{5}{2}\}$. Then, according to [6, Theorem II.5.3.1], there is a constant $c_2 > 0$, independent of T, such that, for $0 \leq s \leq t \leq T$,

$$\|\Lambda(v)(t) - \Lambda(v)(s)\|_{W^{4\theta}_{2,B}(D)} \leq c_2(t-s)^{\xi-\theta}\Big(\|u^0\|_{W^{4\xi}_{2,B}(D)} + \|\lambda g_\varepsilon(v)\|_{L_\infty((0,t),L_2(D))}\Big). \tag{3.39}$$

Fix $\eta \in (0, \xi - \theta)$. Observing the continuous embedding $W^{4\theta}_2(D) \hookrightarrow W^2_3(D)$ with embedding constant, say, $c_3 > 0$, and using (3.39), (3.34), and (3.37), we obtain

$$\|\Lambda(v)(t) - \Lambda(v)(s)\|_{W^2_3(D)} \leq c_2\, c_3\, T^{\xi-\theta-\eta}\left(\frac{1}{2\rho} + \lambda\, C_B\right)(t-s)^{\eta}.$$

Hence, if $T > 0$ is sufficiently small, then

$$\|\Lambda(v)(t) - \Lambda(v)(s)\|_{W^2_3(D)} \leq (t-s)^{\eta}, \quad 0 \leq s \leq t \leq T. \tag{3.40}$$

Similar arguments yield that

$$\|\Lambda(v)(t)\|_{W^2_3(D)} \leq c_3\|\Lambda(v)(t)\|_{W^{4\theta}_{2,B}(D)} \leq c_3\left(\|\Lambda(v)(t) - \Lambda(v)(0)\|_{W^{4\theta}_{2,B}(D)} + \|u^0\|_{W^{4\theta}_{2,B}(D)}\right)$$

$$
\begin{aligned}
&\leq c_2\, c_3\, T^{\xi-\theta} \left(\|u^0\|_{W^{4\xi}_{2,B}(D)} + \|\lambda g_\varepsilon(v)\|_{L_\infty((0,t),L_2(D))} \right) + c_3 \|u^0\|_{W^{4\theta}_{2,B}(D)} \\
&\leq c_2\, c_3\, T^{\xi-\theta} \left(\frac{1}{2\rho} + \lambda\, C_B \right) + \frac{c_1}{2\rho}, \quad 0 \leq t \leq T.
\end{aligned}
$$

Moreover, since $u^0 > -1+2\rho$ in D and since $W_2^{4\theta}(D)$ is continuously embedded in $L_\infty(D)$ with embedding constant denoted by $c_4 > 0$, it follows from (3.39) that, for $0 \leq t \leq T$,

$$
\begin{aligned}
\Lambda(v)(t) = u^0 - (\Lambda(v)(0) - \Lambda(v)(t)) &\geq u^0 - \|\Lambda(v)(0) - \Lambda(v)(t)\|_{L_\infty(D)} \\
&\geq u^0 - c_4 \|\Lambda(v)(0) - \Lambda(v)(t)\|_{W^{4\theta}_{2,B}(D)} \\
&\geq -1 + 2\rho - c_2\, c_4\, T^{\xi-\theta} \left(\frac{1}{2\rho} + \lambda\, C_B \right)
\end{aligned}
$$

in D. Hence, Λ maps $\mathcal{V}_T$ into itself, provided that $T > 0$ is sufficiently small.

Let us show that Λ is a contraction mapping of $C([0,T], W_3^2(D))$. Let $v_i \in \mathcal{V}_T$, $i = 1,2$. Applying [6, Theorem II.5.2.1] (or using (3.35)) together with $W_2^{4\xi}(D) \hookrightarrow W_3^2(D)$, we see that there exists a constant $c_5 > 0$, independent of T, such that, for $0 \leq t \leq T$,

$$
\begin{aligned}
\|\Lambda(v_1)(t) - \Lambda(v_2)(t)\|_{W_3^2(D)} &\leq c_1 \|\Lambda(v_1)(t) - \Lambda(v_2)(t)\|_{W^{4\xi}_{2,B}(D)} \\
&\leq c_1\, c_5\, t^{1-\xi}\, \lambda\, \|g_\varepsilon(v_1) - g_\varepsilon(v_2)\|_{L_\infty((0,t),L_2(D))} .
\end{aligned}
$$

This together with (3.36) then yields that

$$
\|\Lambda(v_1) - \Lambda(v_2)\|_{C([0,T],W_3^2(D))} \leq c_1\, c_5\, T^{1-\xi} \lambda\, C_L \|v_1 - v_2\|_{C([0,T],W_3^2(D))},
$$

which shows that Λ is a contraction in $C([0,T], W_3^2(D))$, provided that $T > 0$ is sufficiently small.

Let $T := T(\rho, \lambda, \varepsilon) > 0$ be small enough to make the mapping $\Lambda : \mathcal{V}_T \to \mathcal{V}_T$ a contraction and (3.40) to be satisfied. Then, there exists a unique fixed point $u \in \mathcal{V}_T$ of Λ. This means that u is a mild solution of (3.17) on $[0,T]$. Moreover, by (3.36) and (3.40), we know that $g_\varepsilon(u)$ belongs to $C^\eta([0,T], L_2(D))$. Hence, the linear theory in [5, Theorem 10.1] implies that

$$
u \in C^1((0,T], L_2(D)) \cap C((0,T], W^4_{2,B}(D)) \cap C([0,T], W^{4\xi}_{2,B}(D))
$$

is a solution to (3.17), which then clearly can be extended to some maximal interval of existence $[0, T_{max})$ with $T_{max} \in (0, \infty]$.

We notice that, if we assume that $u^0 \in W^4_{2,B}(D)$, then it follows from [5, Theorem 10.1] that u belongs to $C([0, T_{max}), W^4_{2,B}(D))$ and thus by (3.17) and the property

$g_\varepsilon(u) \in C([0,T_{max}),L_2(D))$ that $u \in C^1([0,T_{max}),L_2(D))$.

Finally, we observe that $\psi_{u(t)} = \phi_{u(t)} \circ T_{u(t)}$ belongs to $W_2^2(\Omega(u(t)))$ and solves (3.1)-(3.2) for each $t \in [0,T_{max})$.

This proves part (i) of Theorem 3.1.1.

Part (ii): Assume (3.8) is false. Then, there is a $\rho \in (0,1)$ and a sequence t_j converging to $T_{max} < \infty$ from below such that

$$\|u(t_j)\|_{W_2^{4\xi}(D)} \le \frac{1}{\rho} \quad \text{and} \quad u(t_j) \ge -1+\rho \ \text{ in } D$$

for all j. One easily verifies that there exists a time $T>0$, independent of j, such that the solution on $[0,t_j]$ can be extended to $[0,t_j+T]$. By choosing t_j such that $t_j > T_{max}-T$, it follows that the solution can be extended beyond T_{max}, which contradicts the definition of T_{max}.

Part (iii): We use a similar argument as in the proof of part (i). We choose $\lambda_* := \lambda_*(\rho,\varepsilon) > 0$ such that

$$\lambda_*\, c_1 M\, \varpi^{\xi-1}\Gamma(1-\xi)\max\{C_B,C_L\} < 1 < \frac{1}{2\rho_0}$$

and

$$\lambda_*\, c_6 M\, \varpi^{\xi-1}\Gamma(1-\xi) C_B \le \frac{\rho_0}{2},$$

where $\Gamma(\cdot)$ is the gamma function and $c_6 > 0$ denotes the embedding constant for the continuous embedding of $W_2^{4\xi}(D)$ in $L_\infty(D)$. Take $m := m(\rho,\varepsilon) > 0$ such that

$$m\, c_6(M+1) \le \frac{\rho_0}{2} \quad \text{and} \quad m\, c_1 M \le \frac{1}{2\rho_0}.$$

We show that, if $\|u^0\|_{W_{2,B}^{4\xi}(D)} \le m$ and if $\lambda \le \lambda_*$, then Λ maps $\mathcal{V}_T$ into itself and is a contraction with respect to the $C([0,T],W_3^2(D))$-norm for any $T>0$. Let T be any positive time.

Let $v \in \mathcal{V}_T$. First, let us observe that thanks to [6, Theorem II.5.3.1], we have $\Lambda(v) \in C([0,T],W_{2,B}^{4\xi}(D))$. Furthermore, using the continuous embedding $W_2^{4\xi}(D) \hookrightarrow W_3^2(D)$, (3.35), and (3.37), we obtain

$$\|\Lambda(v)(t)\|_{W_3^2(D)} \le c_1 \|\Lambda(v)(t)\|_{W_{2,B}^{4\xi}(D)}$$

$$\le c_1 \|e^{-tA}\|_{\mathcal{L}(W_{2,B}^{4\xi}(D))} \|u^0\|_{W_{2,B}^{4\xi}(D)} + c_1 \int_0^t \|e^{-(t-s)A}\|_{\mathcal{L}(L_2(D),W_{2,B}^{4\xi}(D))} ds \|\lambda g_\varepsilon(v)\|_{L_\infty((0,t),L_2(D))}$$

$$\leq c_1 M \|u^0\|_{W^{4\xi}_{2,B}(D)} + c_1 M \int_0^t e^{-\varpi(t-s)}(t-s)^{-\xi}\, ds\, \|\lambda g_\varepsilon(v)\|_{L_\infty((0,t),L_2(D))}$$
$$\leq c_1 M \|u^0\|_{W^{4\xi}_{2,B}(D)} + c_1 M \varpi^{\xi-1}\Gamma(1-\xi)\lambda C_B \quad 0 \leq t \leq T.$$

By similar arguments, we observe that

$$\|\Lambda(v)(t) - u^0\|_{W^{4\xi}_{2,B}(D)} \leq (M+1)\|u^0\|_{W^{4\xi}_{2,B}(D)} + M\varpi^{\xi-1}\Gamma(1-\xi)\lambda C_B,$$

for $0 \leq t \leq T$. It then follows from the continuous embedding of $W_2^{4\xi}(D)$ in $L_\infty(D)$ and $u^0 \geq -1 + 2\rho$ in D that, for $0 \leq t \leq T$,

$$\Lambda(v)(t) = u^0 - \big(u^0 - \Lambda(v)(t)\big) \geq u^0 - c_6\|\Lambda(v)(t) - u^0\|_{W^{4\xi}_{2,B}(D)}$$
$$\geq -1 + 2\rho - c_6(M+1)\|u^0\|_{W^{4\xi}_{2,B}(D)} - c_6 M\varpi^{\xi-1}\Gamma(1-\xi)\lambda C_B$$

in D. Thus, if $\|u^0\|_{W^{4\xi}_{2,B}(D)} \leq m$ and $\lambda \leq \lambda_*$, then Λ maps the set $\mathcal{V}_T$ into itself. Next, let $v_i \in \mathcal{V}_T$, $i = 1, 2$. It follows from (3.35) and (3.36) that

$$\|\Lambda(v_1)(t) - \Lambda(v_2)(t)\|_{W_3^2(D)} \leq c_1\|\Lambda(v_1)(t) - \Lambda(v_2)(t)\|_{W^{4\xi}_{2,B}(D)}$$
$$\leq c_1 \int_0^t \left\|e^{-(t-s)A}\right\|_{\mathcal{L}(L_2(D),W^{4\xi}_{2,B}(D))} ds\, \|\lambda\left(g_\varepsilon(v_1) - g_\varepsilon(v_2)\right)\|_{L_\infty((0,t),L_2(D))}$$
$$\leq c_1 M \int_0^t e^{-\varpi(t-s)}(t-s)^{-\xi}\, ds\, \|\lambda\left(g_\varepsilon(v_1) - g_\varepsilon(v_2)\right)\|_{L_\infty((0,t),L_2(D))}$$
$$\leq c_1 M\varpi^{\xi-1}\Gamma(1-\xi)\lambda C_L\|v_1 - v_2\|_{C([0,T],W_3^2(D))}, \quad 0 \leq t \leq T.$$

This shows that Λ is a contraction in $C([0,T], W_3^2(D))$, provided $\lambda \leq \lambda_*$. We have thus shown that the map $\Lambda : \mathcal{V}_T \to \mathcal{V}_T$ is a contraction for any $T > 0$, provided that $\lambda \leq \lambda_*$ and $\|u^0\|_{W^{4\xi}_{2,B}(D)} \leq m$, and consequently Λ possesses a unique fixed point $u \in \mathcal{V}_T$ for any $T > 0$. According to the definition of $\mathcal{V}_T$, this implies part (iii). ∎

Chapter 4

The energy equality

In this chapter, we will prove a technical theorem about the total potential energy of our MEMS device, which will be used to prove the main theorem of Chapter 5 stating that touchdown of the top plate on the ground plate is the only finite time singularity.

We first recall from Chapter 2 that the total potential energy of the MEMS device is given by

$$\mathcal{E}(u) = \mathcal{E}_m(u) - \lambda \mathcal{E}_e(u).$$

It includes the mechanical energy

$$\mathcal{E}_m(u) = \beta \int_D \left\{ \frac{1}{2}(\Delta u)^2 - (1-\sigma)\det(\nabla^2 u) \right\} dx + \frac{\tau}{2} \int_D |\nabla u|^2 \, dx$$

and the electrostatic energy $-\lambda \mathcal{E}_e(u)$ with

$$\mathcal{E}_e(u) = \int_{\Omega(u)} \left\{ \varepsilon^2 \left| \nabla' \psi_u \right|^2 + (\partial_z \psi_u)^2 \right\} d(x,z).$$

Here, $\nabla^2 u$ denotes the Hessian matrix of u and $\nabla' := (\partial_{x_1}, \partial_{x_2})$. The parameters $\beta, \sigma, \tau, \lambda, \varepsilon$, are the same as in the previous chapters. As in Chapter 3, we assume that $D \subset \mathbb{R}^2$ is a bounded and convex domain with $\partial D \in C^4$.

4.1 Main result

The following theorem states the energy equality for solutions (u, ψ_u) of Theorem 3.1.1.

Theorem 4.1.1 (Energy equality) *Under the assumptions of Theorem 3.1.1,*

$$\mathcal{E}(u(t)) + \int_0^t \|\partial_t u(s)\|^2_{L_2(D)} \, ds = \mathcal{E}(u^0), \qquad t \in [0, T_{max}).$$

The energy equality shows that the evolution problem (3.1)-(3.5) is the L_2-gradient flow for $\mathcal{E}$, a fact which is also indirectly and formally contained in the model derivation.

The idea of proving Theorem 4.1.1 is taken from [48], but in our case the proof is more delicate, since here we are dealing with an arbitrary two-dimensional domain D and hinged boundary conditions on ∂D.

One of the difficulties in the proof of Theorem 4.1.1 is the computation of the derivative $d\mathcal{E}_e(u(t))/dt$. This is due to the fact that the underlying domain $\Omega(u(t))$ varies according to $u(t)$. In Proposition 4.2.1 presented below we use the transformation $T_{u(t)}$ introduced in Chapter 3 to convert $\mathcal{E}_e(u(t))$ to an integral over the fixed cylinder $\Omega := D \times (0,1)$.

Another difficulty arises from the fact that the time regularity of u as stated in Theorem 3.1.1 is not sufficient for a direct computation of the derivative $d\mathcal{E}(u(t))/dt$. This difficulty can be overcome by using an approximation argument based on Proposition 4.2.1.

Now, let us prepare the proof of Theorem 4.1.1.

4.2 A preliminary result on the electrostatic energy

As in [48], in order to prove Theorem 4.1.1, we first establish a preliminary result:

Proposition 4.2.1 *Let $T > 0$, $4\xi \in \left(\frac{7}{3}, 4\right) \setminus \{\frac{5}{2}\}$ and let $v \in C^1([0,T], W^{4\xi}_{2,B}(D))$ be such that*

$$v(t,x) > -1, \quad (t,x) \in [0,T] \times \overline{D}. \tag{4.1}$$

Then,

$$\begin{aligned} &\mathcal{E}_e(v(t_2)) - \mathcal{E}_e(v(t_1)) \\ &\quad = -\int_{t_1}^{t_2} \int_D \left\{ \varepsilon^2 \left| \nabla' \psi_{v(s)}(\cdot, v(s)) \right|^2 + \left(\partial_z \psi_{v(s)}(\cdot, v(s)) \right)^2 \right\} \partial_t v(s) \, dx \, ds \end{aligned} \tag{4.2}$$

for $0 \le t_1 \le t_2 \le T$.

Here, we recall from Section 3.2 that

$$\begin{aligned} &\varepsilon^2 \left| \nabla' \psi_{v(s)}(\cdot, v(s)) \right|^2 + \left(\partial_z \psi_{v(s)}(\cdot, v(s)) \right)^2 \\ &\qquad = \frac{1 + \varepsilon^2 |\nabla v(s)|^2}{(1 + v(s))^2} \left(\partial_\eta \phi_{v(s)}(\cdot, 1) \right)^2 = g_\varepsilon(v(s)). \end{aligned} \tag{4.3}$$

In the case of a one-dimensional interval $D = (-1, 1)$ and clamped boundary conditions on ∂D, Proposition 4.2.1 has been proved in [48, Proposition 2.2].

We now give the proof for the two-dimensional case D, by following similar steps as those in [48]: First, we rewrite the electrostatic energy $\mathcal{E}_e(v(t))$ as an integral over the fixed domain Ω. The resulting electrostatic energy is thus expressed in terms of $\phi_{v(t)}$. We next verify the differentiability of ϕ_v in t. With this result, we show that the transformed electrostatic energy is differentiable in t and compute its derivative. Finally, we transform the obtained derivative back to the original coordinates to get (4.2).

We note that the proof of Proposition 4.2.1 uses only the first boundary condition of $W^{4\xi}_{2,B}(D)$.

Proof. Let $v \in C^1([0,T], W^{4\xi}_{2,B}(D))$ satisfy (4.1). Since $W^{4\xi}_2(D)$ embeds continuously in $W^2_3(D)$ and in $C(\overline{D})$, there is $\rho \in (0,1)$ such that $v(t) \in \overline{S}_3(\rho)$ for all $t \in [0,T]$, and hence Theorem 3.2.1 can be applied. In order to simplify notation, let, for each $t \in [0,T]$, $\phi(t) = \phi_{v(t)} \in W^2_2(\Omega)$ be the solution to (3.11)-(3.12) associated to $v(t)$ and $\psi(t) = \psi_{v(t)} \in W^2_2(\Omega(v(t)))$ be the corresponding solution to (3.1)-(3.2) also associated to $v(t)$. For $(t,x,\eta) \in [0,T] \times \Omega$, we put

$$\Phi(t,x,\eta) := \phi(t,x,\eta) - \eta, \quad V(t,x) := \frac{\nabla v(t,x)}{1+v(t,x)}, \tag{4.4}$$

and denote the components of V by V_1 and V_2. We recall that $\psi(t) = \phi(t) \circ T_{v(t)}$, with the transformation $T_{v(t)}$ as in (3.9). Then, by the change of variables $(x,z) \to (x,\eta)$, $\mathcal{E}_e(v(t))$ is rewritten in the form

$$\mathcal{E}_e(v(t)) = \varepsilon^2 \int_\Omega \left|\nabla'\phi(t) - \eta\,\partial_\eta\phi(t)\,V(t)\right|^2 (1+v(t))\,d(x,\eta) + \int_\Omega \frac{(\partial_\eta\phi(t))^2}{1+v(t)}\,d(x,\eta). \tag{4.5}$$

Next, we set $W^2_{2,B}(\Omega) := W^2_2(\Omega) \cap W^1_{2,B}(\Omega)$ with

$$W^1_{2,B}(\Omega) := \left\{w \in W^1_2(\Omega)\,;\, w = 0 \ \text{ on } \ \partial\Omega\right\}.$$

It follows from (4.4) that, for $0 \le t \le T$, $\Phi(t) \in W^2_{2,B}(\Omega)$ solves

$$\begin{cases} -\mathcal{L}_{v(t)}\Phi(t) = f(t) & \text{in } \ \Omega, \\ \Phi(t) = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{4.6}$$

where

$$f(t,x,\eta) := \varepsilon^2\eta\left[|V(t,x)|^2 - \mathrm{div}V(t,x)\right], \quad (t,x,\eta) \in [0,T] \times \Omega.$$

The operator $\mathcal{L}_{v(t)}$ is given by

$$\mathcal{L}_{v(t)}w = \alpha_1(t)\,\Delta'w + (\alpha_2(t),\alpha_3(t)) \cdot \nabla'\partial_\eta w + \alpha_4(t)\,\partial^2_\eta w + (\alpha_5(t)+\alpha_6(t))\,\partial_\eta w.$$

Here, $\Delta' := \partial^2_{x_1} + \partial^2_{x_2}$ and

$$\alpha_1(t,x,\eta) := \varepsilon^2, \quad \alpha_2(t,x,\eta) := -2\varepsilon^2\eta\,V_1(t,x), \quad \alpha_3(t,x,\eta) := -2\varepsilon^2\eta\,V_2(t,x),$$

$$\alpha_4(t,x,\eta) := \frac{1}{(1+v(t,x))^2} + \varepsilon^2\eta^2\,|V(t,x)|^2, \quad \alpha_5(t,x,\eta) := 2\varepsilon^2\eta\,|V(t,x)|^2,$$

$$\alpha_6(t,x,\eta) := -\varepsilon^2\eta\,\frac{\Delta v(t,x)}{1+v(t,x)}\,, \qquad (t,x,\eta)\in[0,T]\times\Omega.$$

For later use, we write $\mathcal{L}_{v(t)}$ in divergence form:

$$\mathcal{L}_{v(t)}w = \operatorname{div}\left(\boldsymbol{\alpha}(t)\nabla w\right) + \boldsymbol{b}(t)\cdot\nabla w,$$

where

$$\boldsymbol{\alpha}(t) := \begin{pmatrix} \alpha_1(t) & 0 & \alpha_2(t)/2 \\ 0 & \alpha_1(t) & \alpha_3(t)/2 \\ \alpha_2(t)/2 & \alpha_3(t)/2 & \alpha_4(t) \end{pmatrix}$$

and $\boldsymbol{b}(t) := (b_1(t), b_2(t), b_3(t))$ with

$$b_1(t,x,\eta) := \varepsilon^2\,V_1(t,x), \qquad b_2(t,x,\eta) := \varepsilon^2\,V_2(t,x),$$

$$b_3(t,x,\eta) := -\varepsilon^2\eta\,|V(t,x)|^2, \qquad (t,x,\eta)\in[0,T]\times\Omega.$$

Let us next verify the differentiability of Φ in t. We start by briefly recalling some properties of $\mathcal{L}_{v(t)}$; for further details and proofs see Section B.2. We introduce a bounded linear operator $\mathcal{A}(t)\in\mathcal{L}(W^2_{2,B}(\Omega), L_2(\Omega))$ by setting

$$\mathcal{A}(t)w := -\mathcal{L}_{v(t)}w, \quad w\in W^2_{2,B}(\Omega),\ t\in[0,T].$$

For each $t\in[0,T]$, it is seen that $\mathcal{A}(t)$ is invertible and that $\Phi(t) = \mathcal{A}(t)^{-1}f(t)$.

Furthermore, from the time regularity of v, the fact that $v(t)\in\overline{S}_3(\rho)$, and the embeddings

$$W_2^{4\xi}(D)\hookrightarrow W_3^2(D)\hookrightarrow C^1(\overline{D}),$$

it follows by direct computation that

$$\alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5\in C^1([0,T], L_\infty(\Omega)) \quad\text{and}\quad \alpha_6\in C^1([0,T], L_3(\Omega)).$$

So, we easily see that

$$\mathcal{A}\in C^1([0,T], \mathcal{L}(W^2_{2,B}(\Omega), L_2(\Omega)))$$

and that

$$f\in C^1([0,T], L_2(\Omega)). \tag{4.7}$$

Since the map taking an invertible operator to its inverse is continuously differentiable on the space of bounded operators, the mapping

$$[t\mapsto\mathcal{A}(t)^{-1}] : [0,T]\to\mathcal{L}(L_2(\Omega), W^2_{2,B}(\Omega))$$

is continuously differentiable and hence $\|\mathcal{A}(t)^{-1}\|_{\mathcal{L}(L_2(\Omega),W^2_{2,B}(\Omega))}\le C$ for some constant

$C > 0$. Together with (4.7), this implies

$$\Phi \in C^1([0,T], W^2_{2,B}(\Omega))$$

with derivative

$$\partial_t \Phi(t) = \mathcal{A}(t)^{-1}\Big(\partial_t f(t) - \partial_t \mathcal{A}(t)\,\Phi(t)\Big) \in W^2_{2,B}(\Omega), \quad t \in [0,T].$$

Therefore, in view of (4.4), we have

$$\phi \in C^1([0,T], W^2_2(\Omega)) \quad \text{with} \quad \partial_t \phi(t) = \partial_t \Phi(t), \quad t \in [0,T]. \tag{4.8}$$

Let us now again consider (4.5). By direct calculations, we can verify from the fact that $v \in C^1([0,T], W^{4\xi}_{2,B}(D))$ and (4.8) that $\mathcal{E}_e(v) \in C^1([0,T])$ with derivative

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_e(v(t)) \\
&\quad = 2\varepsilon^2 \int_\Omega \Big[\big(\nabla'\phi(t) - \eta\,\partial_\eta\phi(t)\,V(t)\big)\cdot\big(\nabla'\partial_t\phi(t) - \eta\,\partial_\eta\phi(t)\,\partial_t V(t) \\
&\qquad\qquad\qquad\qquad - \eta\,\partial_\eta\partial_t\phi(t)\,V(t)\big)\Big](1+v(t))\,d(x,\eta) \\
&\qquad + \varepsilon^2 \int_\Omega \big|\nabla'\phi(t) - \eta\,\partial_\eta\phi(t)\,V(t)\big|^2\,\partial_t v(t)\,d(x,\eta) \\
&\qquad + 2\int_\Omega \frac{\partial_\eta\phi(t)\,\partial_\eta\partial_t\phi(t)}{1+v(t)}\,d(x,\eta) - \int_\Omega (\partial_\eta\phi(t))^2\,\frac{\partial_t v(t)}{(1+v(t))^2}\,d(x,\eta), \quad t \in [0,T].
\end{aligned} \tag{4.9}$$

We want to write equation (4.9) in a simpler form. For this purpose, we multiply $\mathcal{L}_{v(t)}\phi(t) = 0$ in Ω by $(1+v(t))\partial_t\phi(t)$ and integrate the product over Ω. Then, for $0 \le t \le T$,

$$0 = \int_\Omega (1+v(t))\,\partial_t\phi(t)\,\mathcal{L}_{v(t)}\phi(t)\,d(x,\eta). \tag{4.10}$$

Using the divergence form of $\mathcal{L}_{v(t)}$, integration by parts, and the fact that $\partial_t\phi(t) = 0$ on $\partial\Omega$, we deduce from (4.10) that

$$\begin{aligned}
0 = &-\int_\Omega \nabla\Big((1+v(t))\,\partial_t\phi(t)\Big)\cdot(\boldsymbol{\alpha}(t)\nabla\phi(t))\,d(x,\eta) \\
&+ \int_\Omega (1+v(t))\,\partial_t\phi(t)\,\boldsymbol{b}(t)\cdot\nabla\phi(t)\,d(x,\eta), \quad t \in [0,T].
\end{aligned} \tag{4.11}$$

By the definitions of $b_i(t)$, $i = 1,2,3$, and $V(t)$, we observe that

$$\int_\Omega (1+v(t))\,\partial_t\phi(t)\,\boldsymbol{b}(t)\cdot\nabla\phi(t)\,d(x,\eta)$$

$$= \varepsilon^2 \int_\Omega \partial_t \phi(t)\, \nabla v(t) \cdot \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big)\, d(x,\eta). \tag{4.12}$$

By the definitions of $\alpha_i(t)$, $1 \leq i \leq 4$, and $V(t)$, we have that

$$\begin{aligned}
&- \int_\Omega \nabla\Big((1+v(t))\, \partial_t \phi(t)\Big) \cdot (\boldsymbol{\alpha}(t) \nabla \phi(t))\, d(x,\eta) \\
&= -\varepsilon^2 \int_\Omega \big(\partial_t \phi(t)\, \nabla v(t) + (1+v(t))\, \nabla' \partial_t \phi(t)\big) \cdot \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big)\, d(x,\eta) \\
&\quad + \varepsilon^2 \int_\Omega \eta\, (1+v(t))\, \partial_\eta \partial_t \phi(t)\, V(t) \cdot \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big)\, d(x,\eta) \\
&\quad - \int_\Omega \frac{\partial_\eta \partial_t \phi(t)\, \partial_\eta \phi(t)}{1+v(t)}\, d(x,\eta),
\end{aligned}$$

and together with (4.12), (4.11) becomes

$$\begin{aligned}
0 &= -\varepsilon^2 \int_\Omega \big(\partial_t \phi(t)\, \nabla v(t) + (1+v(t))\, \nabla' \partial_t \phi(t)\big) \cdot \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big)\, d(x,\eta) \\
&\quad + \varepsilon^2 \int_\Omega \eta\, (1+v(t))\, \partial_\eta \partial_t \phi(t)\, V(t) \cdot \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big)\, d(x,\eta) \\
&\quad - \int_\Omega \frac{\partial_\eta \partial_t \phi(t)\, \partial_\eta \phi(t)}{1+v(t)}\, d(x,\eta) \\
&\quad + \varepsilon^2 \int_\Omega \partial_t \phi(t)\, \nabla v(t) \cdot \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big)\, d(x,\eta).
\end{aligned}$$

Rearranging yields

$$\begin{aligned}
0 &= -\varepsilon^2 \int_\Omega (1+v(t)) \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big) \cdot \big(\nabla' \partial_t \phi(t) - \eta\, \partial_\eta \partial_t \phi(t)\, V(t)\big)\, d(x,\eta) \\
&\quad - \int_\Omega \frac{\partial_\eta \partial_t \phi(t)\, \partial_\eta \phi(t)}{1+v(t)}\, d(x,\eta).
\end{aligned}$$

Combining this with (4.9), we get

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}_e(v(t)) &= -2\varepsilon^2 \int_\Omega \big(\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big) \cdot \partial_t V(t)\, \partial_\eta \phi(t)\, \eta\, (1+v(t))\, d(x,\eta) \\
&\quad + \varepsilon^2 \int_\Omega \big|\nabla' \phi(t) - \eta\, \partial_\eta \phi(t)\, V(t)\big|^2\, \partial_t v(t)\, d(x,\eta) \\
&\quad - \int_\Omega (\partial_\eta \phi(t))^2\, \frac{\partial_t v(t)}{(1+v(t))^2}\, d(x,\eta), \quad t \in [0,T].
\end{aligned}$$

Using the transformation $T_{v(t)}$ to write $d\mathcal{E}_e(v(t))/dt$ in terms of $\psi(t)$, we easily obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_e(v(t)) = & -2\varepsilon^2 \int_{\Omega(v(t))} (1+z)\,\partial_z\psi(t)\,\nabla'\psi(t)\cdot\partial_t V(t)\,d(x,z) \\ & + \int_{\Omega(v(t))} \Big(\varepsilon^2|\nabla'\psi(t)|^2 - (\partial_z\psi(t))^2\Big)\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z), \quad t\in[0,T]. \end{aligned} \tag{4.13}$$

We next observe that, for $0 \le t \le T$,

$$\partial_t V(t) = \partial_t\big(\nabla \ln(1+v(t))\big) = \nabla\left(\frac{\partial_t v(t)}{1+v(t)}\right).$$

Then, using integration by parts and $\partial_t v(t) = 0$ on ∂D, the first integral on the right-hand side of (4.13) is rewritten in the form

$$\begin{aligned} & -2\varepsilon^2 \int_{\Omega(v(t))} (1+z)\,\partial_z\psi(t)\,\nabla'\psi(t)\cdot\partial_t V(t)\,d(x,z) \\ & = 2\varepsilon^2 \int_{\Omega(v(t))} (1+z)\Big[\partial_z\psi(t)\,\Delta'\psi(t) + \nabla'\psi(t)\cdot\nabla'\partial_z\psi(t)\Big]\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z) \\ & \quad + 2\varepsilon^2 \int_D \partial_t v(t)\,\partial_z\psi(t,\cdot,v(t))\,\nabla'\psi(t,\cdot,v(t))\cdot\nabla v(t)\,dx. \end{aligned}$$

Since $2\nabla'\psi(t)\cdot\nabla'\partial_z\psi(t) = \partial_z\big(|\nabla'\psi(t)|^2\big)$, we have

$$\begin{aligned} & -2\varepsilon^2 \int_{\Omega(v(t))} (1+z)\,\partial_z\psi(t)\,\nabla'\psi(t)\cdot\partial_t V(t)\,d(x,z) \\ & = \varepsilon^2 \int_{\Omega(v(t))} (1+z)\Big[2\,\partial_z\psi(t)\,\Delta'\psi(t) + \partial_z\big(|\nabla'\psi(t)|^2\big)\Big]\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z) \\ & \quad + 2\varepsilon^2 \int_D \partial_t v(t)\,\partial_z\psi(t,\cdot,v(t))\,\nabla'\psi(t,\cdot,v(t))\cdot\nabla v(t)\,dx. \end{aligned}$$

Therefore, we obtain from (4.13) that, for $0 \le t \le T$,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_e(v(t)) = & \int_{\Omega(v(t))} (1+z)\Big[2\varepsilon^2\,\Delta'\psi(t)\,\partial_z\psi(t) + \varepsilon^2\,\partial_z\big(|\nabla'\psi(t)|^2\big)\Big]\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z) \\ & + \int_{\Omega(v(t))} \Big(\varepsilon^2|\nabla'\psi(t)|^2 - (\partial_z\psi(t))^2\Big)\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z) \\ & + 2\varepsilon^2 \int_D \partial_t v(t)\,\partial_z\psi(t,\cdot,v(t))\,\nabla'\psi(t,\cdot,v(t))\cdot\nabla v(t)\,dx. \end{aligned}$$

Since $\varepsilon^2\Delta'\psi(t) + \partial_z^2\psi(t) = 0$ in $\Omega(v(t))$, it follows that, for $0 \le t \le T$,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_e(v(t)) = &\int_{\Omega(v(t))} (1+z)\Big[-\partial_z\big((\partial_z\psi(t))^2\big) + \varepsilon^2\,\partial_z\big(|\nabla'\psi(t)|^2\big)\Big]\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z)\\ &+ \int_{\Omega(v(t))}\Big(\varepsilon^2|\nabla'\psi(t)|^2 - (\partial_z\psi(t))^2\Big)\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z)\\ &+ 2\varepsilon^2\int_D \partial_t v(t)\,\partial_z\psi(t,\cdot,v(t))\,\nabla'\psi(t,\cdot,v(t))\cdot\nabla v(t)\,dx. \qquad (4.14)\end{aligned}$$

By integration by parts,

$$\begin{aligned}&\int_{\Omega(v(t))} (1+z)\Big[-\partial_z\big((\partial_z\psi(t))^2\big) + \varepsilon^2\,\partial_z\big(|\nabla'\psi(t)|^2\big)\Big]\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z)\\ &\quad= -\int_{\Omega(v(t))}\Big(\varepsilon^2|\nabla'\psi(t)|^2 - (\partial_z\psi(t))^2\Big)\,\frac{\partial_t v(t)}{1+v(t)}\,d(x,z)\\ &\qquad+ \int_D\Big(\varepsilon^2|\nabla'\psi(t,\cdot,v(t))|^2 - (\partial_z\psi(t,\cdot,v(t)))^2\Big)\,\partial_t v(t)\,dx.\end{aligned}$$

Hence, for $0 \le t \le T$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_e(v(t)) = &\int_D\Big(\varepsilon^2|\nabla'\psi(t,\cdot,v(t))|^2 - (\partial_z\psi(t,\cdot,v(t)))^2\Big)\,\partial_t v(t)\,dx\\ &+ 2\varepsilon^2\int_D \partial_t v(t)\,\partial_z\psi(t,\cdot,v(t))\,\nabla'\psi(t,\cdot,v(t))\cdot\nabla v(t)\,dx.\end{aligned}$$

Finally, we use the identity

$$\nabla'\psi(t,x,v(t)) = -\partial_z\psi(t,x,v(t))\,\nabla v(t), \quad (t,x)\in[0,T]\times D,$$

which follows from differentiating the boundary condition $\psi(t,x,v(t)) = 1$, $x \in D$, to deduce that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_e(v(t)) &= -\int_D\big(1+\varepsilon^2|\nabla v(t)|^2\big)\,(\partial_z\psi(t,\cdot,v(t)))^2\,\partial_t v(t)\,dx\\ &= -\int_D\big\{\varepsilon^2|\nabla'\psi(t,\cdot,v(t))|^2 + (\partial_z\psi(t,\cdot,v(t)))^2\big\}\,\partial_t v(t)\,dx, \quad t\in[0,T].\end{aligned}$$

Integrating this equality in $[t_1,t_2]$, we obtain that

$$\mathcal{E}_e(v(t_2)) - \mathcal{E}_e(v(t_1)) = -\int_{t_1}^{t_2}\int_D\big\{\varepsilon^2|\nabla'\psi(s,\cdot,v(s))|^2 + (\partial_z\psi(s,\cdot,v(s)))^2\big\}\,\partial_t v(s)\,dx\,ds,$$

for $0 \le t_1 \le t_2 \le T$, and the proposition is proved. ∎

Remark 4.2.2 *An alternative approach to compute the derivative $d\mathcal{E}_e(v(t))/dt$ and to investigate differentiability properties of $\mathcal{E}_e$ is presented in [44, Section 4]. This approach is based on a transformation that maps $\Omega(w(t))$ onto $\Omega(v(t))$ instead of the transformation $T_{v(t)}$ to a fixed cylinder. Doing this transformation allows one to rewrite $\mathcal{E}_e(w(t))$, for each $w(t)$ in a neighborhood of $v(t)$, as an integral over $\Omega(v(t))$ and then to study the behavior of $\mathcal{E}_e(w(t)) - \mathcal{E}_e(v(t))$ as $w(t) \to v(t)$.*

We are finally in a position to prove the main theorem of this chapter.

4.3 Proof of Theorem 4.1.1

Under the assumptions of Theorem 3.1.1, let (u, ψ_u) be the solution to (3.1)-(3.5). We first notice that we cannot apply Proposition 4.2.1, since we only have $u \in C^1((0, T_{max}), L_2(D))$. To get around this problem, we shall use an approximation argument:

We define the Steklov average u_δ of u by

$$u_\delta(t, x) := \frac{1}{\delta}\int_t^{t+\delta} u(s, x)\, ds, \qquad t \in [0, T_{max}), \quad x \in D, \quad \delta \in (0, T_{max} - t).$$

Now, we fix $T \in (0, T_{max})$, and let $\delta \in (0, T_{max} - T)$. Since $u \in C([0, T + \delta], W^{4\xi}_{2,B}(D))$, we get by the fundamental theorem of calculus that

$$u_\delta \in C^1([0, T], W^{4\xi}_{2,B}(D)) \quad \text{with} \quad \partial_t u_\delta(t) = \frac{u(t + \delta) - u(t)}{\delta}, \quad t \in [0, T]. \tag{4.15}$$

Moreover, for $0 \le t \le T$, it holds that

$$\begin{aligned} \|u_\delta(t) - u(t)\|_{W^{4\xi}_{2,B}(D)} &= \left\| \frac{1}{\delta}\int_t^{t+\delta} (u(s) - u(t))\, ds \right\|_{W^{4\xi}_{2,B}(D)} \\ &\le \max_{s\in[t,t+\delta]} \|u(s) - u(t)\|_{W^{4\xi}_{2,B}(D)} \longrightarrow 0 \quad \text{as} \quad \delta \searrow 0 \end{aligned}$$

and thus

$$u_\delta \to u \quad \text{in} \quad C([0, T], W^{4\xi}_{2,B}(D)) \quad \text{as} \quad \delta \searrow 0. \tag{4.16}$$

We next note that, for any $t_0 \in (0, T)$, we have $u \in C([t_0, T + \delta], W^4_{2,B}(D))$. Then, the estimate

$$\|u_\delta(t) - u(t)\|_{W^4_{2,B}(D)} \le \max_{s\in[t,t+\delta]} \|u(s) - u(t)\|_{W^4_{2,B}(D)}$$

proves that, as $\delta \searrow 0$, $u_\delta(t)$ converges to $u(t)$ in $W^4_{2,B}(D)$ uniformly on $[t_0, T]$ for any $t_0 \in (0, T)$. Since t_0 is arbitrary,

$$u_\delta \to u \quad \text{in} \quad C((0,T], W^4_{2,B}(D)) \qquad \text{as} \quad \delta \searrow 0. \tag{4.17}$$

Since $u \in C^1([t_0, T+\delta], L_2(D))$ for every $t_0 \in (0,T)$, it follows from (4.15) that

$$\partial_t u_\delta(t) = \frac{1}{\delta}\int_t^{t+\delta} \partial_t u(s)\, ds, \qquad t \in [t_0, T].$$

This implies that

$$\|\partial_t u_\delta(t) - \partial_t u(t)\|_{L_2(D)} \to 0 \quad \text{as} \quad \delta \searrow 0,$$

uniformly in $t \in [t_0, T]$, for each $t_0 \in (0,T)$. Since t_0 is arbitrary,

$$\partial_t u_\delta \to \partial_t u \quad \text{in} \quad C((0,T], L_2(D)) \qquad \text{as} \quad \delta \searrow 0. \tag{4.18}$$

Recall next that the mechanical energy with respect to $u_\delta(t)$ is given by

$$\mathcal{E}_m(u_\delta(t)) = \beta \int_D \left\{ \frac{1}{2}(\Delta u_\delta(t))^2 - (1-\sigma)\det(\nabla^2 u_\delta(t)) \right\} dx + \frac{\tau}{2}\int_D |\nabla u_\delta(t)|^2\, dx.$$

By direct calculations, it is easily verified from $u_\delta \in C^1([0,T], W^{4\xi}_{2,B}(D))$ that

$$\mathcal{E}_m(u_\delta) \in C^1([0,T])$$

with derivative

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_m(u_\delta(t)) \\
&= \beta \int_D \Delta u_\delta(t)\, \Delta \partial_t u_\delta(t)\, dx \\
&\quad + \beta(1-\sigma)\int_D \Big(2\,\partial_{x_2}\partial_{x_1} u_\delta(t)\, \partial_{x_2}\partial_{x_1}\partial_t u_\delta(t) - \partial^2_{x_2} u_\delta(t)\, \partial^2_{x_1}\partial_t u_\delta(t) - \partial^2_{x_1} u_\delta(t)\, \partial^2_{x_2}\partial_t u_\delta(t) \Big) dx \\
&\quad + \tau \int_D \nabla u_\delta(t) \cdot \nabla \partial_t u_\delta(t)\, dx, \quad t \in [0,T].
\end{aligned}$$

On account of Lemma A.0.1, we obtain that

$$\begin{aligned}
&\beta(1-\sigma)\int_D \Big(2\,\partial_{x_2}\partial_{x_1} u_\delta(t)\, \partial_{x_2}\partial_{x_1}\partial_t u_\delta(t) - \partial^2_{x_2} u_\delta(t)\, \partial^2_{x_1}\partial_t u_\delta(t) - \partial^2_{x_1} u_\delta(t)\, \partial^2_{x_2}\partial_t u_\delta(t) \Big) dx \\
&\quad = -\beta(1-\sigma)\int_{\partial D} \kappa\, \partial_\nu u_\delta(t)\, \partial_\nu \partial_t u_\delta(t)\, d\omega.
\end{aligned}$$

So,

$$\frac{d}{dt}\mathcal{E}_m(u_\delta(t)) = \beta \int_D \Delta u_\delta(t)\, \Delta\partial_t u_\delta(t)\, dx + \tau \int_D \nabla u_\delta(t) \cdot \nabla \partial_t u_\delta(t)\, dx$$
$$- \beta(1-\sigma) \int_{\partial D} \kappa\, \partial_\nu u_\delta(t)\, \partial_\nu \partial_t u_\delta(t)\, d\omega, \qquad t \in [0,T].$$

Applying integration by parts to the first two terms and using $\partial_t u_\delta(t) = 0$ on ∂D, we have

$$\frac{d}{dt}\mathcal{E}_m(u_\delta(t)) = \beta \int_D \Delta^2 u_\delta(t)\, \partial_t u_\delta(t)\, dx - \tau \int_D \Delta u_\delta(t)\, \partial_t u_\delta(t)\, dx$$
$$+ \beta \int_{\partial D} \Big(\Delta u_\delta(t) - (1-\sigma)\kappa\, \partial_\nu u_\delta(t)\Big) \partial_\nu \partial_t u_\delta(t)\, d\omega, \quad t \in (0,T].$$

Therefore, in view of the second boundary condition for $u_\delta(t)$ (due to $u_\delta(t) \in W^4_{2,B}(D)$ for $t \in (0,T]$), we deduce that

$$\frac{d}{dt}\mathcal{E}_m(u_\delta(t)) = \beta \int_D \Delta^2 u_\delta(t)\, \partial_t u_\delta(t)\, dx - \tau \int_D \Delta u_\delta(t)\, \partial_t u_\delta(t)\, dx, \quad t \in (0,T].$$

Integrating this equality on $[t_1, t_2]$, we obtain that

$$\mathcal{E}_m(u_\delta(t_2)) - \mathcal{E}_m(u_\delta(t_1))$$
$$= \beta \int_{t_1}^{t_2} \int_D \Delta^2 u_\delta(s)\, \partial_t u_\delta(s)\, dx\, ds - \tau \int_{t_1}^{t_2} \int_D \Delta u_\delta(s)\, \partial_t u_\delta(s)\, dx\, ds \tag{4.19}$$

for $0 < t_1 \le t_2 \le T$.

We are now concerned with the limit of (4.19) as $\delta \searrow 0$. By (4.17) and (4.18), we see that, for any $0 < t_1 \le t_2 \le T$, the right-hand side of (4.19) converges to

$$\beta \int_{t_1}^{t_2} \int_D \Delta^2 u(s)\, \partial_t u(s)\, dx\, ds - \tau \int_{t_1}^{t_2} \int_D \Delta u(s)\, \partial_t u(s)\, dx\, ds \tag{4.20}$$

as $\delta \searrow 0$. On the other hand, due to (4.16) and the continuous embedding $W_2^{4\xi}(D) \hookrightarrow W_2^2(D)$, we observe that

$$|\mathcal{E}_m(u_\delta(t_k)) - \mathcal{E}_m(u(t_k))| \longrightarrow 0 \qquad \text{as} \quad \delta \searrow 0,$$

$t_k \in [0,T]$, $k = 1,2$. Together with (4.19) and (4.20), we obtain

$$\mathcal{E}_m(u(t_2)) - \mathcal{E}_m(u(t_1)) = \int_{t_1}^{t_2} \int_D \big(\beta \Delta^2 u(s) - \tau \Delta u(s)\big)\, \partial_t u(s)\, dx\, ds \tag{4.21}$$

for $0 < t_1 \le t_2 \le T$. Furthermore, since $u \in C([0,T], W^{4\xi}_{2,B}(D))$ and since

$W_2^{4\xi}(D) \hookrightarrow W_2^2(D)$, we conclude that

$$\mathcal{E}_m(u(t_1)) \longrightarrow \mathcal{E}_m(u^0) < \infty \qquad \text{as} \quad t_1 \searrow 0.$$

This then shows that (4.21) is valid for $t_1 = 0$.

Next, consider the electrostatic energy. Thanks to Proposition 4.2.1 and (4.3), we have

$$\mathcal{E}_e(u_\delta(t_2)) - \mathcal{E}_e(u_\delta(t_1)) = -\int_{t_1}^{t_2}\int_D g_\varepsilon(u_\delta(s))\partial_t u_\delta(s)\, dx\, ds, \quad 0 \le t_1 \le t_2 \le T. \tag{4.22}$$

We are then interested in the limit of (4.22) as $\delta \searrow 0$. Since $u(t) > -1$ in D, it follows from (4.16) that $u(t)$ and $u_\delta(t)$ belong to $\overline{S}_3(\rho)$ for some $\rho \in (0,1)$ and for $t \in [0,T]$ and $\delta \in (0, \delta_0)$ with $\delta_0 > 0$ sufficiently small. Hence, Theorem 3.2.1 yields

$$\begin{aligned} \|g_\varepsilon(u_\delta)(t) - g_\varepsilon(u)(t)\|_{L_2(D)} &\le C_L \, \|u_\delta(t) - u(t)\|_{W_3^2(D)} \\ &\le C_L\, c_1 \, \|u_\delta(t) - u(t)\|_{W_{2,B}^{4\xi}(D)}, \quad 0 \le t \le T, \end{aligned}$$

where C_L is the constant occurring in (3.36) and $c_1 > 0$ denotes the embedding constant for the embedding $W_2^{4\xi}(D) \hookrightarrow W_3^2(D)$. Then, by using (4.16), we have

$$g_\varepsilon(u_\delta) \to g_\varepsilon(u) \quad \text{in} \quad C([0,T], L_2(D)) \qquad \text{as} \quad \delta \searrow 0. \tag{4.23}$$

From (4.23) it follows that

$$g_\varepsilon(u_\delta) \to g_\varepsilon(u) \quad \text{in} \quad L_2(0,T; L_2(D)) \quad \text{as} \quad \delta \searrow 0.$$

Using this together with (4.18) and Hölder's inequality, we deduce that, for any $t_0 \in (0,T)$,

$$g_\varepsilon(u_\delta)\, \partial_t u_\delta \to g_\varepsilon(u)\, \partial_t u \quad \text{in} \quad L_1(t_0, T; L_1(D)) \qquad \text{as} \quad \delta \searrow 0.$$

Thus, for any $0 < t_1 \le t_2 \le T$,

$$\left| \int_{t_1}^{t_2} \int_D \big[g_\varepsilon(u_\delta(s))\, \partial_t u_\delta(s) - g_\varepsilon(u(s))\, \partial_t u(s) \big]\, dx\, ds \right| \longrightarrow 0 \quad \text{as} \quad \delta \searrow 0. \tag{4.24}$$

In terms of the coordinates $(x, \eta) \in \Omega$, the electrostatic energy reads

$$\begin{aligned} \mathcal{E}_e(u_\delta(t)) = \varepsilon^2 \int_\Omega &\left| \nabla' \phi_\delta(t) - \eta\, \partial_\eta \phi_\delta(t)\, U_\delta(t) \right|^2 (1 + u_\delta(t))\, d(x,\eta) \\ &+ \int_\Omega \frac{(\partial_\eta \phi_\delta(t))^2}{1 + u_\delta(t)}\, d(x,\eta), \end{aligned} \tag{4.25}$$

where

$$\phi_\delta(t) := \phi_{u_\delta(t)} \quad \text{and} \quad U_\delta(t) := \frac{\nabla u_\delta(t)}{1+u_\delta(t)}, \quad t \in [0,T].$$

Finally, we show that

$$|\mathcal{E}_e(u_\delta(t_k)) - \mathcal{E}_e(u(t_k))| \longrightarrow 0 \quad \text{as} \quad \delta \searrow 0, \tag{4.26}$$

$t_k \in [0,T]$, $k = 1,2$. From Theorem 3.2.1 we know that

$$\|\phi_\delta(t) - \phi(t)\|_{W_2^2(\Omega)} \leq C\|u_\delta(t) - u(t)\|_{W_3^2(D)}, \quad 0 \leq t \leq T,$$

with a constant $C > 0$ only depending on ρ, ε, and D. Using the continuous embedding $W_2^{4\xi}(D) \hookrightarrow W_3^2(D)$ and (4.16), we conclude that

$$\phi_\delta \to \phi \quad \text{in} \quad C([0,T], W_2^2(\Omega)) \qquad \text{as} \quad \delta \searrow 0. \tag{4.27}$$

Again, by (4.16) and the embeddings $W_2^{4\xi}(D) \hookrightarrow W_3^2(D) \hookrightarrow C^1(\overline{D})$, we have that

$$U_\delta \to U \quad \text{in} \quad C([0,T], L_\infty(D)) \qquad \text{as} \quad \delta \searrow 0.$$

This, together with (4.27), implies

$$\nabla'\phi_\delta - \eta\,\partial_\eta\phi_\delta\,U_\delta \to \nabla'\phi - \eta\,\partial_\eta\phi\,U \quad \text{in} \quad C([0,T], L_2(\Omega)) \qquad \text{as} \quad \delta \searrow 0;$$

hence, by Hölder's inequality and (4.16),

$$\left|\nabla'\phi_\delta - \eta\,\partial_\eta\phi_\delta\,U_\delta\right|^2 (1+u_\delta) \to \left|\nabla'\phi - \eta\,\partial\phi_\delta\,U\right|^2 (1+u) \quad \text{in} \quad C([0,T], L_1(\Omega)) \qquad \text{as} \quad \delta \searrow 0.$$

Furthermore, from (4.16) and (4.27) it follows that

$$\frac{(\partial_\eta\phi_\delta)^2}{1+u_\delta} \to \frac{(\partial_\eta\phi)^2}{1+u} \quad \text{in} \quad C([0,T], L_1(\Omega)) \qquad \text{as} \quad \delta \searrow 0.$$

Thus, (4.26) is obtained. Therefore, (4.22), together with (4.24) and (4.26), yields that

$$\mathcal{E}_e(u(t_2)) - \mathcal{E}_e(u(t_1)) = -\int_{t_1}^{t_2}\int_D g_\varepsilon(u(s))\,\partial_t u(s)\,dx\,ds, \quad 0 < t_1 \leq t_2 \leq T. \tag{4.28}$$

Now, since $u \in C([0,T], W_{2,B}^{4\xi}(D))$, we can repeat arguments quite similar to those above to prove that

$$\mathcal{E}_e(u(t_1)) \longrightarrow \mathcal{E}_e(u^0) < \infty \qquad \text{as} \quad t_1 \searrow 0.$$

Thus, equation (4.28) also holds true for $t_1 = 0$.

In this way, we have verified that

$$\begin{aligned}\mathcal{E}(u(t_2)) - \mathcal{E}(u(t_1)) &= \mathcal{E}_m(u(t_2)) - \mathcal{E}_m(u(t_1)) - \lambda\Big[\mathcal{E}_e(u(t_2)) - \mathcal{E}_e(u(t_1))\Big] \\ &= \int_{t_1}^{t_2}\int_D \big\{\beta\,\Delta^2 u(s) - \tau\,\Delta u(s) + \lambda\, g_\varepsilon(u(s))\big\}\,\partial_t u(s)\,dx\,ds \\ &= -\int_{t_1}^{t_2} \|\partial_t u(s)\|^2_{L_2(D)}\,ds, \qquad 0 \le t_1 \le t_2 \le T,\end{aligned}$$

where in the last step we used the equation (3.3). We have thus accomplished the proof of Theorem 4.1.1. ■

Theorem 4.1.1 provides a crucial step in the proof of the improved criterion for global existence, which we will discuss in the next chapter.

Chapter 5

Touchdown is the only finite time singularity

In this chapter it is our aim to improve part (ii) of Theorem 3.1.1 by showing that u cannot blow up in $W_2^{4\xi}(D)$ in finite time and hence touchdown of u on the ground plate is the only possible finite time singularity.

5.1 Main result

We assume that $D \subset \mathbb{R}^2$ is a bounded convex domain with $\partial D \in C^{4,\gamma}$ for some $\gamma \in (0, 1)$. The main result in this chapter is:

Theorem 5.1.1 (Touchdown) *Under the assumptions of Theorem 3.1.1, let (u, ψ_u) be the unique solution to (3.1)-(3.5) defined on the maximal interval $[0, T_{max})$. Assume that there are $T_0 > 0$ and $\rho_0 \in (0, 1)$ such that*

$$u(t) \geq -1 + \rho_0 \quad in\, D, \quad t \in [0, T_{max}) \cap [0, T_0]. \tag{5.1}$$

Then, $T_{max} \geq T_0$.

Moreover, if, for each $T > 0$, there is $\rho(T) \in (0, 1)$ such that

$$u(t) \geq -1 + \rho(T) \quad in\, D, \quad t \in [0, T_{max}) \cap [0, T],$$

then $T_{max} = \infty$.

In the case of clamped boundary conditions on ∂D, this theorem has been proved by Laurençot and Walker in [48] for a one-dimensional interval D and in [45] for a two-dimensional convex domain D.

We follow the proof given in [45]. The idea is to use the lower bound (5.1) on u to obtain a lower bound on the total potential energy of the device. The energy equality from Chapter 4 then gives an upper bound on the mechanical energy which in turn implies a $W_2^2(D)$-bound on $u(t)$ for $t \in [0, T_{max}) \cap [0, T_0]$. This leads to an $L_1(D)$-bound on the right-hand side of equation (3.3). We then use semigroup theory and embedding properties of Besov spaces to derive a $W_2^{4\xi}(D)$-bound on $u(t)$ for $t \in [0, T_{max}) \cap [0, T_0]$, which is independent of T_{max}. Part (ii) of Theorem 3.1.1 finishes the proof.

5.2 Proof of Theorem 5.1.1

The second statement follows by applying the first to an arbitrary $T_0 > 0$. So, we restrict our attention to the first.

Let the assumptions of Theorem 3.1.1 be satisfied and let (u, ψ_u) be the solution to (3.1)-(3.5) defined on $[0, T_{max})$. Let $\rho_0 \in (0,1)$ and $T_0 > 0$ be such that (5.1) is satisfied. We want to show that

$$\|u(t)\|_{W_2^{4\xi}(D)} \leq c(\rho_0, T_0), \quad t \in [0, T_{max}) \cap [0, T_0], \tag{5.2}$$

with some constant $c(\rho_0, T_0) > 0$ independent of T_{max}. Then, Theorem 3.1.1 (ii) will imply $T_{max} \geq T_0$.

To verify estimate (5.2), we first recall that $\psi_{u(t)} \in W_2^2(\Omega(u(t)))$ for all $t \in [0, T_{max})$ and that $\Omega(u(t))$ is a Lipschitz domain. Hence, due to [69, Theorem II.5.5], we have $\big[x \mapsto \nabla\psi_{u(t)}(x, u(t,x))\big] \in W_2^{1/2}(D) \hookrightarrow L_4(D)$ and therefore quantity

$$G(u(t))(x) := \varepsilon^2 |\nabla' \psi_{u(t)}(x, u(t,x))|^2 + (\partial_z \psi_{u(t)}(x, u(t,x)))^2, \quad (t,x) \in [0, T_{max}) \times D, \tag{5.3}$$

which appears in the right-hand side of (3.3), belongs to $L_2(D)$ (or see Theorem 3.2.1). Moreover, since

$$\nabla' \psi_{u(t)}(x, u(t,x)) = -\partial_z \psi_{u(t)}(x, u(t,x))\, \nabla u(t,x), \quad (t,x) \in [0, T_{max}) \times D,$$

due to (3.2), (5.3) becomes

$$G(u(t))(x) = \big(1 + \varepsilon^2 |\nabla u(t,x)|^2\big)(\partial_z \psi_{u(t)}(x, u(t,x)))^2.$$

We also need the following two results from [45]. The first lemma provides us with the key estimate on the $L_1(D)$-norm of $G(u(t))$.

Lemma 5.2.1 ([45, Corollary 3.5]) *Let $\rho \in (0,1)$, and let $v \in W_{3,B}^2(D)$ be such that $v \geq -1 + \rho$ in D. Then,*

$$\|G(v)\|_{L_1(D)} \leq \left(4 + \frac{2}{\rho^2}\right) |D| + 4\varepsilon^2 \|\nabla v\|_{L_2(D)}^2.$$

For the next lemma, we recall the total potential energy $\mathcal{E}(v) = \mathcal{E}_m(v) - \lambda \mathcal{E}_e(v)$ with mechanical energy

$$\mathcal{E}_m(v) = \beta \int_D \left\{ \frac{1}{2} (\Delta v)^2 - (1 - \sigma) \det(\nabla^2 v) \right\} dx + \frac{\tau}{2} \int_D |\nabla v|^2 \, dx,$$

where $\nabla^2 v$ is the Hessian matrix of v, and electrostatic energy $-\lambda\mathcal{E}_e(v)$, where

$$\mathcal{E}_e(v) = \int_{\Omega(v)} \left\{\varepsilon^2 |\nabla'\psi_v|^2 + (\partial_z\psi_v)^2\right\} d(x,z). \tag{5.4}$$

Lemma 5.2.2 *Let $\rho \in (0,1)$, and let $v \in W^2_{3,B}(D)$ be such that $v \geq -1+\rho$ in D. Then,*

$$\mathcal{E}(v) \geq \mathcal{E}_m(v) - 3\lambda\varepsilon^2\|\nabla v\|^2_{L_2(D)} - \lambda\left(4 + \frac{1}{2\rho^2}\right)|D|.$$

Proof. We observe that the electrostatic energy in (5.4) is precisely the same as in [45]. Thus, by applying [45, Lemma 3.6], we find that

$$\mathcal{E}(v) = \mathcal{E}_m(v) - \lambda\mathcal{E}_e(v) = \mathcal{E}_m(v) - \lambda|D| + \lambda\int_D v\left(1+\varepsilon^2|\nabla v|^2\right)\partial_z\psi_v(\cdot,v)\,dx.$$

Now we can estimate the last term in the right hand-side by using exactly the same arguments as in the proof of [45, Corollary 3.7]. ∎

Since $W_2^{4\xi}(D)$ embeds continuously in $W_3^2(D)$ and since (5.1) holds, we can apply the above two lemmas with $v = u(t)$.

For the remainder of this section, c denotes a positive constant which depends only on ρ_0, T_0, u^0, β, σ, λ, ε, and D, and may vary from occurrence to occurrence. We emphasize that the constant c is independent of T_{max}.

5.2.1 Auxiliary estimates on the plate displacement

We begin with an $L_2(D)$-bound on $u(t)$.

Lemma 5.2.3 *There is $c > 0$ (depending on T_0) such that*

$$\|u(t)\|_{L_2(D)} \leq c, \quad t \in [0,T_{max}) \cap [0,T_0].$$

Proof. We multiply equation (3.3) by $u(t)$ and integrate the product in D. Then,

$$\begin{aligned} &-\lambda\int_D G(u(t))\,u(t)\,dx \\ &= \frac{1}{2}\frac{d}{dt}\int_D u(t)^2\,dx + \beta\int_D (\Delta^2 u(t))u(t)\,dx - \tau\int_D (\Delta u(t))u(t)\,dx, \quad t \in (0,T_{max}). \end{aligned} \tag{5.5}$$

Two integration by parts and the boundary conditions for $u(t)$ yield

$$-\lambda\int_D G(u(t))\,u(t)\,dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_D u(t)^2\,dx + \beta\int_D (\Delta u(t))^2\,dx - \beta(1-\sigma)\int_{\partial D}\kappa\,(\partial_\nu u(t))^2\,d\omega$$
$$+\,\tau\int_D |\nabla u(t)|^2\,dx.$$

Applying Lemma A.0.1, we obtain that

$$-\,\lambda\int_D G(u(t))\,u(t)\,dx = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L_2(D)} + 2\mathcal{E}_m(u(t)), \quad t\in(0,T_{max}). \tag{5.6}$$

Since $u(t) > -1$ in D and since $G(u(t)) \geq 0$ in D, it follows that

$$-\,\lambda\int_D G(u(t))\,u(t)\,dx \leq \lambda\,\|G(u(t))\|_{L_1(D)}, \quad t\in[0,T_{max}). \tag{5.7}$$

In addition, by Lemma 5.2.1,

$$\|G(u(t))\|_{L_1(D)} \leq c\left(1+\|\nabla u(t)\|^2_{L_2(D)}\right), \quad t\in[0,T_{max})\cap[0,T_0]. \tag{5.8}$$

Furthermore, by integration by parts and Hölder's inequality, we obtain

$$\|\nabla u(t)\|^2_{L_2(D)} = -\int_D (\Delta u(t))u(t)\,dx \leq \|\Delta u(t)\|_{L_2(D)}\,\|u(t)\|_{L_2(D)}. \tag{5.9}$$

Moreover, the inequality

$$\frac{1}{2}(\Delta u(t))^2 \leq (\partial^2_{x_1}u(t))^2 + (\partial^2_{x_2}u(t))^2 + 2(\partial_{x_2}\partial_{x_1}u(t))^2 \ \text{ in } \ D$$

implies that

$$\begin{aligned}
&2\mathcal{E}_m(u(t))\\
&= \beta\int_D \Big\{(\Delta u(t))^2 + (1-\sigma)\big[2(\partial_{x_2}\partial_{x_1}u(t))^2 + (\partial^2_{x_1}u(t))^2 + (\partial^2_{x_2}u(t))^2 - (\Delta u(t))^2\big]\Big\}dx\\
&\quad + \tau\int_D |\nabla u(t)|^2\,dx\\
&\geq \frac{\beta(1+\sigma)}{2}\,\|\Delta u(t)\|^2_{L_2(D)}, \quad t\in[0,T_{max}).
\end{aligned} \tag{5.10}$$

The estimate just established, together with (5.8), (5.9), and Young's inequality yields

$$\|G(u(t))\|_{L_1(D)} \leq \frac{1}{\lambda}\mathcal{E}_m(u(t)) + c\left(1+\|u(t)\|^2_{L_2(D)}\right), \quad t\in[0,T_{max})\cap[0,T_0]. \tag{5.11}$$

Therefore, by (5.6), (5.7) and (5.11),

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L_2(D)} + \mathcal{E}_m(u(t)) \leq c\left(1+\|u(t)\|^2_{L_2(D)}\right), \quad t\in(0,T_{max})\cap(0,T_0],$$

and thus $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L_2(D)} \leq c\left(1+\|u(t)\|^2_{L_2(D)}\right)$. Solving this differential inequality, we conclude that

$$\|u(t)\|^2_{L_2(D)} \leq c, \quad t \in [0, T_{max}) \cap [0, T_0].$$

■

With the aid of Lemma 5.2.2 we obtain the following result:

Lemma 5.2.4 *There is $c > 0$ such that*

$$\mathcal{E}(u(t)) \geq \frac{1}{2}\mathcal{E}_m(u(t)) - c, \quad t \in [0, T_{max}) \cap [0, T_0].$$

Proof. By Lemma 5.2.2 and (5.9),

$$\mathcal{E}(u(t)) \geq \mathcal{E}_m(u(t)) - 3\lambda\varepsilon^2 \|\Delta u(t)\|_{L_2(D)} \|u(t)\|_{L_2(D)} - \lambda|D|\left(4+\frac{1}{2\rho_0^2}\right) \tag{5.12}$$

for $t \in [0, T_{max}) \cap [0, T_0]$. In view of (5.10) and Young's inequality, we get from (5.12) that

$$\mathcal{E}(u(t)) \geq \frac{1}{2}\mathcal{E}_m(u(t)) - c\left(1+\|u(t)\|^2_{L_2(D)}\right), \quad t \in [0, T_{max}) \cap [0, T_0].$$

The assertion now follows from Lemma 5.2.3. ■

Using the energy equality discussed in Chapter 4, we can prove the following estimate.

Corollary 5.2.5 *There is $c > 0$ such that*

$$\frac{\beta(1+\sigma)}{8}\|\Delta u(t)\|^2_{L_2(D)} + \int_0^t \|\partial_t u(s)\|^2_{L_2(D)}\, ds \leq c, \quad t \in [0, T_{max}) \cap [0, T_0].$$

Proof. According to Theorem 4.1.1, we have

$$\mathcal{E}(u(t)) + \int_0^t \|\partial_t u(s)\|^2_{L_2(D)}\, ds = \mathcal{E}(u^0), \quad t \in [0, T_{max}).$$

By virtue of Lemma 5.2.4, it then follows that

$$\mathcal{E}(u^0) \geq \frac{1}{2}\mathcal{E}_m(u(t)) - c + \int_0^t \|\partial_t u(s)\|^2_{L_2(D)}\, ds, \quad t \in [0, T_{max}) \cap [0, T_0],$$

and the assertion follows from (5.10) and the fact that $\mathcal{E}(u^0) < \infty$. ■

We can now prove the following important corollary.

Corollary 5.2.6 *There is $c > 0$ such that*

$$\|G(u(t))\|_{L_1(D)} \leq c, \quad t \in [0, T_{max}) \cap [0, T_0].$$

Proof. By [28, Theorem 3.1.2.1] (since D is convex) and Corollary 5.2.5, we deduce that

$$\|u(t)\|^2_{W_2^2(D)} \leq c, \quad t \in [0, T_{max}) \cap [0, T_0].$$

The assertion then follows from Lemma 5.2.1. ∎

Next, we are going to show that the $L_1(D)$-bound of $G(u(t))$ obtained in Corollary 5.2.6 implies

$$\|u(t)\|_{W_2^{4\xi}(D)} \leq c, \quad t \in [0, T_{max}) \cap [0, T_0]. \tag{5.13}$$

However, first we require an auxiliary result. In what follows, we assume that $4\xi \in \left(\frac{5}{2}, \frac{7}{2}\right)$ and fix $\alpha \in (4\xi - \frac{7}{2}, 0)$. The cases $4\xi \in \left(\frac{7}{3}, \frac{5}{2}\right)$ and $4\xi \in \left[\frac{7}{2}, 4\right)$ can be treated in the same way. From now on, we allow the constant c to depend also on ξ and α, but still not on T_{max}. We explicitly indicate the dependence on any additional parameter.

5.2.2 An auxiliary result on the fourth-order operator in Besov spaces

We first recall that $\partial D \in C^{4,\gamma}$, $\gamma \in (0,1)$. For $s \in (-3-\gamma, 4+\gamma)\backslash\{1,3\}$, we introduce $B^s_{1,1,B}(D)$, i.e., Besov spaces $B^s_{1,1}(D)$ which incorporate the boundary conditions (3.4):

$$B^s_{1,1,B}(D) := \begin{cases} B^s_{1,1}(D), & s \in (-3-\gamma, 1), \\ \left\{v \in B^s_{1,1}(D)\,;\, v = 0 \text{ on } \partial D\right\}, & s \in (1,3), \\ \left\{v \in B^s_{1,1}(D)\,;\, v = \Delta v - (1-\sigma)\kappa\,\partial_\nu v = 0 \text{ on } \partial D\right\}, & s \in (3, 4+\gamma). \end{cases}$$

We want to show that the operator $-A$, given by

$$-Av := -(\beta\Delta^2 - \tau\Delta)v, \quad v \in B^{4+\alpha}_{1,1,B}(D),$$

generates a strongly continuous analytic semigroup $\{e^{-tA}\,;\, t \geq 0\}$ on $B^\alpha_{1,1}(D)$ which satisfies the regularizing property stated in Lemma 5.2.7 below.

In Lemma 3.3.1 and 3.3.3, we have shown that $-A$ restricted to $W^4_{2,B}(D)$ generates a strongly continuous analytic semigroup $\{e^{-tA}\,;\, t \geq 0\}$ on $L_2(D)$ satisfying

$$\|e^{-tA}\|_{\mathcal{L}(W^{4\xi}_{2,B}(D))} \leq M, \quad t \geq 0.$$

We can argue in a similar way as in Lemma 3.3.1 to obtain the following result.

Lemma 5.2.7 *It holds that*

$$A \in \mathcal{H}(B^{4+\alpha}_{1,1,B}(D), B^\alpha_{1,1}(D)). \tag{5.14}$$

Moreover, given $\theta \in (0,1)$ with $\theta \notin \left\{\frac{(1-\alpha)}{4}, \frac{(3-\alpha)}{4}\right\}$, there is a constant $c(\theta) > 0$ (depending *on T_0 but not on T_{max}) such that, for $t \in (0, T_0]$,*

$$\|e^{-tA}\|_{\mathcal{L}(B^{\alpha}_{1,1}(D), B^{4\theta+\alpha}_{1,1,B}(D))} \leq c(\theta)\, t^{-\theta}. \tag{5.15}$$

Proof. To prove (5.14), we want to apply [29, Theorem 2.18]. We first note that $\alpha \in (4\xi - \frac{7}{2}, 0) \subset (-1, 1)$. Let us check that assumptions (m), (n), and (o) of [29, Theorem 2.18] are satisfied. Assumptions (m) and (n) are verified in the same way as in the proof of Lemma 3.3.1. Assumption (o) requires that, for any $x \in \partial D$, $\zeta \in \mathbb{R}^2$, $r \geq 0$ with $\zeta \cdot \nu(x) = 0$ and $(\zeta, r) \neq (0, 0)$, and any $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, zero is the only bounded solution in $[0, \infty)$ to

$$\begin{cases} \left[-\beta\left(|\zeta|^2 - \partial_t^2\right)^2 - re^{i\vartheta}\right] v = 0, \\ v(0) = \partial_t^2 v(0) = 0. \end{cases}$$

But this has already been proved in Lemma 3.3.1. Hence, we can apply [29, Theorem 2.18] and conclude that $-A$ generates a strongly continuous analytic semigroup $\{e^{-tA}\, ;\, t \geq 0\}$ on $B^{\alpha}_{1,1}(D)$. It remains to prove (5.15). By [30, Proposition 4.13], we see that

$$\left(B^{\alpha}_{1,1}(D), B^{4+\alpha}_{1,1,B}(D)\right)_{\theta,1} \doteq B^{4\theta+\alpha}_{1,1,B}(D), \quad 4\theta \in (0,4)\backslash\{1-\alpha, 3-\alpha\},$$

where $(\cdot,\cdot)_{\theta,1}$ denotes the real interpolation functor. Then the desired result follows from [6, Lemma II.5.1.3]. ■

5.2.3 Proof of Theorem 5.1.1

According to [31, Section 4] or [5, Section 5], we have the continuous embeddings

$$B^{4+\alpha}_{1,1,B}(D) \hookrightarrow B^{s}_{1,1,B}(D) \hookrightarrow B^{0}_{1,1}(D) \hookrightarrow L_1(D) \hookrightarrow B^{0}_{1,\infty}(D) \hookrightarrow B^{\alpha}_{1,1}(D)$$

for $s \in (0, 4+\alpha)\backslash\{1, 3\}$. This, together with Corollary 5.2.6, implies

$$\|G(u(t))\|_{B^{\alpha}_{1,1}(D)} \leq c, \quad t \in [0, T_{max}) \cap [0, T_0]. \tag{5.16}$$

We next fix $\theta \in (0,1)$ and $4\xi_1 \in (4\xi, 4)\backslash\{3\}$ so that

$$4\theta + \alpha > 4\xi_1 + 1 > 4\xi + 1.$$

In view of [5, Section 5], we observe that

$$B^{4\theta+\alpha}_{1,1,B}(D) \hookrightarrow B^{4\xi_1}_{2,2,B}(D) \doteq W^{4\xi_1}_{2,B}(D) \hookrightarrow W^{4\xi}_{2,B}(D). \tag{5.17}$$

Using the variation of constants formula

$$u(t) = e^{-tA}u^0 - \lambda \int_0^t e^{-(t-s)A}\, G(u(s))\, ds, \quad t \in [0, T_{max}),$$

we derive from (5.15), (5.16), (5.17), and Lemma 3.3.3 (letting $c_0 > 0$ denote the corresponding embedding constant) that

$$
\begin{aligned}
&\|u(t)\|_{W^{4\xi}_{2,B}(D)} \\
&\le \|e^{-tA}u^0\|_{W^{4\xi}_{2,B}(D)} + \lambda \int_0^t \|e^{-(t-s)A}G(u(s))\|_{W^{4\xi}_{2,B}(D)}\, ds \\
&\le \|e^{-tA}\|_{\mathcal{L}(W^{4\xi}_{2,B}(D))}\, \|u^0\|_{W^{4\xi}_{2,B}(D)} + \lambda c_0 \int_0^t \|e^{-(t-s)A}G(u(s))\|_{B^{4\theta+\alpha}_{1,1,B}(D)}\, ds \\
&\le M\|u^0\|_{W^{4\xi}_{2,B}(D)} + \lambda c_0 \int_0^t \|e^{-(t-s)A}\|_{\mathcal{L}(B^{\alpha}_{1,1}(D), B^{4\theta+\alpha}_{1,1,B}(D))}\, \|G(u(t))\|_{B^{\alpha}_{1,1}(D)}\, ds \\
&\le c(\theta), \qquad t \in [0, T_{max}) \cap [0, T_0].
\end{aligned}
$$

So, we have verified (5.13). Theorem 3.1.1 (ii) completes the proof of Theorem 5.1.1. ■

We have just proved that the top plate certainly touches down on the ground plate when $T_{max} < \infty$.

Part II

Stationary case

Chapter 6

Positivity preserving property for a hinged convex plate with stress

6.1 Introduction

This chapter is an adaptation of

- G. Sweers and K. Vassi, Positivity for a hinged convex plate with stress, SIAM J. Math. Anal., 50 (2018), pp. 1163-1174.

In this chapter, we restrict ourselves to the analysis of the following boundary value problem for a hinged plate with stress:

$$\Delta^2 u - \tau \Delta u = f \;\text{ in }\; D, \qquad u = \Delta u - (1-\sigma)\kappa \partial_\nu u = 0 \;\text{ on }\; \partial D. \tag{6.1}$$

We will prove that this problem is positivity preserving on convex domains, meaning $f \geq 0$ implies $u \geq 0$. This task is motivated by the absence of a general maximum principle for fourth order elliptic equations. The proof relies on optimal estimates for a weighted first Steklov eigenvalue and on an application of the Kreĭn-Rutman theorem for an auxiliary problem.

For convenience, we briefly recall the derivation of (6.1) from Section 2.2.

The model

The energy of a hinged thin plate under the action of a vertical force density $f : D \to \mathbb{R}$ approximately equals to, by a suitable normalization,

$$\mathbb{E}(u) := \int_D \left\{ \frac{1}{2}(\Delta u)^2 + (1-\sigma)\left((\partial_{x_2}\partial_{x_1} u)^2 - \partial_{x_1}^2 u \, \partial_{x_2}^2 u \right) + \frac{1}{2}\tau \left|\nabla u\right|^2 - fu \right\} dx, \tag{6.2}$$

where $D \subset \mathbb{R}^2$ describes the shape of the plate and $u : D \to \mathbb{R}$ its vertical displacement from the rest position. The first two terms in $\mathbb{E}$ form the energy that one may describe as bending respectively torsion. Fixing the vertical position at the boundary gives $u|_{\partial D} = 0$. Fixing the horizontal direction at the boundary introduces the term $\frac{1}{2}\tau \left|\nabla u\right|^2$, a stress term due to an increasing surface for nonzero u. The parameter τ that appears is taken in $[0,\infty)$; the parameter σ is the Poisson ratio of the plate and satisfies $-1<\sigma<1$. For more details, see Section 2.2. The last term is the potential energy from a downward force. Friedrichs [23] was among the first to study the variational formulation for thin plates.

Weak and strong solution

The solution u_f, that we are looking at, for example, for $f\in L_2(D)$, is a minimizer of $\mathbb{E}$ on the space

$$\mathcal{W}:=W_2^2(D)\cap W_{2,B}^1(D),$$

where, as introduced in Chapter 3,

$$W_{p,B}^1(D)=\left\{v\in W_p^1(D)\,;\,v=0\text{ on }\partial D\right\},\quad p\in(1,\infty).$$

We assume that Ω is a bounded domain in $\mathbb{R}^2$ with $\partial\Omega\in C^{2,1}$. One may show, see [23, 68], that $\mathbb{E}$ hence has a unique minimizer on the Hilbert space $\mathcal{W}$, which satisfies

$$\delta\mathbb{E}(u;\varphi)=0\quad\text{for all }\varphi\in\mathcal{W} \tag{6.3}$$

with

$$\delta\mathbb{E}(u;\varphi)=\int_D\Big\{\Delta u\Delta\varphi+(1-\sigma)\left(2\partial_{x_2}\partial_{x_1}u\,\partial_{x_2}\partial_{x_1}\varphi-\partial_{x_1}^2u\,\partial_{x_2}^2\varphi-\partial_{x_2}^2u\,\partial_{x_1}^2\varphi\right)$$
$$+\tau\nabla u\cdot\nabla\varphi-f\varphi\Big\}dx.$$

Integration by parts with smooth functions $u,\varphi\in\mathcal{W}$ shows that (see Lemma A.0.1),

$$\delta\mathbb{E}(u;\varphi)=\langle u,\varphi\rangle_{\mathcal{W},\tau}-\int_D f\varphi\,dx, \tag{6.4}$$

where

$$\langle u,\varphi\rangle_{\mathcal{W},\tau}:=\int_D(\Delta u\,\Delta\varphi+\tau\nabla u\cdot\nabla\varphi)\;dx-(1-\sigma)\int_{\partial D}\kappa\,\partial_\nu u\,\partial_\nu\varphi\,d\omega.$$

Here κ is the signed curvature of the boundary, which is taken positive on strict convex boundary parts, and ν is the exterior unit normal of ∂D. If $u\in\mathcal{W}$ and $\Delta u\in W_2^2(D)$ holds and u is such that $\delta\mathbb{E}(u;\varphi)=0$ for all $\varphi\in\mathcal{W}$, then two integration by parts, starting from (6.4), lead to

$$0=\delta\mathbb{E}(u;\varphi)=\int_D\left(\Delta^2u-\tau\Delta u-f\right)\varphi\,dx+\int_{\partial D}(\Delta u-(1-\sigma)\,\kappa\,\partial_\nu u)\,\partial_\nu\varphi\,d\omega,$$

and by the fundamental lemma of calculus of variations, first in D and then on ∂D, we obtain

$$\begin{cases} \Delta^2 u - \tau\Delta u = f & \text{in } D, \\ \Delta u = (1-\sigma)\kappa\,\partial_\nu u & \text{on } \partial D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{6.5}$$

Note that the boundary condition $u|_{\partial D} = 0$ follows from $u \in W^1_{2,B}(D)$.

Definition 6.1.1 *A function $u \in \mathcal{W}$ satisfying (6.3), we call a weak solution of (6.5).*

When ∂D is smooth enough, a weak solution of (6.3) is a strong solution, i.e., lies in $W^4_2(D)$. If also f is smooth, then it will be a classical solution of (6.5), i.e., lies in $C^4(\overline{D})$. See [24, Theorems 2.20 and 2.19].

Setting $v = -\Delta u$, we may formally rewrite (6.5) as

$$\begin{cases} -\Delta v + \tau v = f & \text{in } D, \\ v = -(1-\sigma)\,\kappa\,\partial_\nu u & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{6.6}$$

For $\partial D \in C^{2,1}$ and $f \in L_2(D)$, we will prove in Proposition 6.3.4 that $u, \Delta u \in C(\overline{D})$. With $v \in C(\overline{D})$, one finds $u \in W^2_p(D)$ for all $p \in (1,\infty)$ and hence, through embedding, that $u \in C^{1,\gamma}(\overline{D})$ for $p > \frac{2}{1-\gamma}$. Thus, $\partial_\nu u$ lies in $C^{0,\gamma}(\partial D)$ for all $\gamma \in (0,1)$.

The problem

First, we fix the following positivity conventions.

Notation *Let $A \subset \mathbb{R}^n$. For functions $\varphi \in C(A)$, we set*

- *$\varphi \geq 0$ when $\varphi(x) \geq 0$ for all $x \in A$;*
- *$\varphi \gneqq 0$ when $\varphi(x) \geq 0$ for all $x \in A$ and $\varphi \not\equiv 0$ in A;*
- *$\varphi > 0$ when $\varphi(x) > 0$ for all $x \in A$.*

For $L_2(A)$-functions the (in)equalities hold almost everywhere.

The operator $T : C(A) \to C(B)$ with $A, B \subset \mathbb{R}^n$, we call

- *positive if $\varphi \geq 0$ implies $T\varphi \geq 0$;*
- *strictly positive if T is positive and if $\varphi \gneqq 0$ implies $T\varphi \gneqq 0$.*

We will use a notion of strong positivity. The definition will need a precise setting and is given later on.

The question that we are interested in is the following. Supposing that u is a solution of (6.5), we ask

Does $f \geq 0$ imply $u \geq 0$?

This is the so-called "positivity preserving property", namely, the property which ensures that if the force density f is of one sign, then also the vertical displacement u has this same sign. For $\tau = 0$ and D convex with $\partial D \in C^{2,1}$, Parini and Stylianou in [71], using [25], showed that (6.5) is strongly positivity preserving, namely, $f \gneq 0$ implies $u > 0$. In [77], Romani considered a semilinear version of (6.2). We will consider $\tau > 0$ and the following theorem is the main result of the present chapter.

Theorem 6.1.3 *Let $D \subset \mathbb{R}^2$ be a bounded and convex domain with a $C^{2,1}$-boundary ∂D. Suppose that $f \in L_2(D)$ satisfies $f \gneq 0$. Then the unique minimizer u_f of $\mathbb{E}$ in $\mathcal{W}$ satisfies $u_f > 0$ in D. Moreover, $u_f \in C^{1,\gamma}(\overline{D})$ for all $\gamma \in (0,1)$ and $-\partial_\nu u_f > 0$ on ∂D.*

6.2 Positivity in a second order system

With the system setting in (6.6) one finds that v is coupled with $\partial_\nu u$ through the boundary and u with v as a source. For a bounded domain $D \in \mathbb{R}^2$ with $\partial D \in C^{2,\gamma}$ for some $\gamma \in (0,1)$ and $\tau \geq 0$, both boundary value problems in (6.6) have well-defined solutions in Hölder as well as in Sobolev space settings for given right-hand sides in the appropriate spaces. When the functions involved are pointwise defined and assuming that D is a convex domain, hence $\kappa \geq 0$, one finds from the maximum principle that

$$\left.\begin{array}{r} -\partial_\nu u \geq 0 \text{ on } \partial D, \\ f \geq 0 \text{ in } D \end{array}\right\} \implies v \geq 0 \text{ in } D \implies u \geq 0 \text{ in } D \implies -\partial_\nu u \geq 0 \text{ on } \partial D. \tag{6.7}$$

Moreover, an inequality that is strict, i.e., $\gneq$, implies strong inequalities, i.e., $>$. However, for $\kappa \geq 0$ the chain of inequalities in (6.7) shows that the coupling in (6.6) is cooperative in nature (see [66]), which means that by Kreĭn-Rutman we have the positivity preserving property, whenever we stay below the first "eigenvalue". Such an ordered setting was employed in [25].

Since our approach strongly depends on properties of the solution operator, it will be convenient to fix the following operators and recall some of their properties:

- We write $w = \mathcal{G}_\tau f$ for the solution of

$$\begin{cases} -\Delta w + \tau w = f & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases} \tag{6.8}$$

  and set $C_0(\overline{D}) := \{v \in C(\overline{D})\,;\, v = 0 \text{ on } \partial D\}$.

Lemma 6.2.1 *Let $D \subset \mathbb{R}^2$ be bounded and $\partial D \in C^{2,\gamma}$. Then for all $f \in C(\overline{D})$, there exists a unique solution $u \in C_0(\overline{D}) \cap W_p^2(D)$ of (6.8) for all $p \in (1,\infty)$. Moreover, setting $u = \mathcal{G}_\tau f$ one finds that*

$$\mathcal{G}_\tau : C(\overline{D}) \to C^{1,\gamma}(\overline{D}) \tag{6.9}$$

is a well-defined compact linear operator.

Proof. Indeed, by [26, Theorems 9.13 and 9.15, Lemma 9.17] there exists for $\tau \geq 0$ a solution operator $\mathcal{S}_\tau$ for (6.8) from $L_p(D)$ to $W_p^2(D) \cap W_{p,B}^1(D)$, and moreover, there is a constant $c > 0$ depending only on τ, p, and D, such that

$$\|\mathcal{S}_\tau f\|_{W_p^2(D)} \leq c \|f\|_{L_p(D)} \text{ for all } f \in L_p(D).$$

Denoting by $I_1 : C(\overline{D}) \to L_p(D)$ the trivial embedding and by $I_2 : W_p^2(D) \cap W_{p,B}^1(D) \to C^{1,\gamma}(\overline{D}) \cap C_0(\overline{D})$ the compact embedding that is guaranteed by a Sobolev embedding whenever $p > \frac{2}{1-\gamma}$, one finds that $\mathcal{G}_\tau = I_2 \mathcal{S}_\tau I_1$ has the desired properties. For the compact embedding we refer to [1, Theorem 6.3]. ■

- We write $w = \mathcal{K}_\tau \psi$ for the solution of

$$\begin{cases} -\Delta w + \tau w = 0 & \text{in } D, \\ w = \psi & \text{on } \partial D. \end{cases} \tag{6.10}$$

Lemma 6.2.2 *Let $D \subset \mathbb{R}^2$ be bounded and $\partial D \in C^{2,\gamma}$. Then for all $\psi \in C(\partial D)$, there exists a (unique) solution $w \in C(\overline{D}) \cap C^{2,\gamma}(D)$ of (6.10). Moreover, setting $w = \mathcal{K}_\tau \psi$ one finds that*

$$\mathcal{K}_\tau : C(\partial D) \to C(\overline{D}) \tag{6.11}$$

is a well-defined bounded linear operator, which is even strictly positive.

Proof. Since $\partial D \in C^{2,\gamma}$ the Perron method [26, Theorem 2.14] and the maximum principle yield a continuous solution operator for $\tau = 0$. The same holds true for $\tau \geq 0$. See [26, Theorem 6.13]. The maximum principle implies $\|\mathcal{K}_\tau \psi\|_{C(\overline{D})} = \max\{|\psi(x)|\,; x \in \partial D\} = \|\psi\|_{C(\partial D)}$ and also that if $\psi \gneq 0$, then $\mathcal{K}_\tau \psi \gneq 0$ holds. ■

- Finally, set $\mathcal{N}w = -(\nu \cdot \nabla w)|_{\partial D}$. For $\partial D \in C^{2,\gamma}$, one has $\nu \in C^{1,\gamma}(\partial D)$ and $\kappa \in C^{0,\gamma}(\partial D)$. So, for $w \in C^{1,\gamma}(\overline{D})$ one finds $\kappa \mathcal{N} w \in C^{0,\gamma}(\partial D)$ and

$$\kappa \mathcal{N} : C^{1,\gamma}(\overline{D}) \to C(\partial D) \tag{6.12}$$

  is well-defined.

With these operators one finds that the system in (6.6) turns into

$$u = \mathcal{G}_0 v = \mathcal{G}_0 \mathcal{G}_\tau f + (1-\sigma)\, \mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N} u. \tag{6.13}$$

Moreover, if the spectral radius $r_{sp} := r\left((1-\sigma)\, \mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}\right)$ is less than 1, then

$$u = \left(\mathcal{I} - (1-\sigma)\, \mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}\right)^{-1} \mathcal{G}_0 \mathcal{G}_\tau f = \sum_{k=0}^{\infty} \left((1-\sigma)\, \mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}\right)^k \mathcal{G}_0 \mathcal{G}_\tau f. \tag{6.14}$$

Since $\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N} : C^{1,\gamma}(\overline{D}) \to C^{1,\gamma}(\overline{D})$ is compact, its spectrum consists of eigenvalues with 0 as the only possible accumulation point. So $r(\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}) = \sup |\mu|$ holds with μ an eigenvalue of $\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}$, i.e.,

$$\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\varphi = \mu\varphi \tag{6.15}$$

for some eigenfunction $\varphi \in C^{1,\gamma}(\overline{D})$. Moreover, such an eigenfunction yields an eigenfunction $\psi := \mathcal{K}_\tau\kappa\mathcal{N}\varphi \in C(\overline{D})$ of

$$\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\psi = \mu\psi \tag{6.16}$$

and vice versa. Hence, in order to have $r_{sp} < 1$, it will be sufficient to find that the in absolute sense largest eigenvalue for (6.16) lies in $\left(0, \frac{1}{2}\right]$. In fact, it will be more convenient to consider

$$\mathcal{T}_\tau := \mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0 : C(\overline{D}) \to C(\overline{D}). \tag{6.17}$$

The first positivity result we collect is as follows.

Lemma 6.2.3 *Let $D \subset \mathbb{R}^2$ be bounded and $\partial D \in C^{2,\gamma}$. Then $\mathcal{N}\mathcal{G}_0 \in \mathcal{L}(C(\overline{D}), C(\partial D))$ is such that*

$$w \gneq 0 \ \ \textit{implies} \ \ \mathcal{N}\mathcal{G}_0 w > 0.$$

Proof. This is a direct consequence of Hopf's boundary point lemma and the regularity of ∂D. ∎

For $\partial D \in C^{2,\gamma}$ and D convex, we have $\kappa \gneq 0$ and with Lemmas 6.2.2 and 6.2.3, the operator $\mathcal{T}_\tau$ in (6.17) is strictly positive. Next, we will show that $\mathcal{T}_\tau$ satisfies a property called "strongly positive".

Definition 6.2.4 *Let $T \in \mathcal{L}(C(\overline{D}))$ be a positive operator.*

- *We call $0 \lneq u_0 \in C(\overline{D})$ a unit for T if for each $0 \lneq u \in C(\overline{D})$, there exists a constant $c_u > 0$ such that $Tu \leq c_u u_0$.*
- *We call T strongly positive with respect to the unit u_0 if for each $0 \lneq u \in C(\overline{D})$, there exists a constant $c_u^* > 0$ such that $Tu \geq c_u^* u_0$.*

Proposition 6.2.5 *Suppose that $D \subset \mathbb{R}^2$ is a bounded convex domain with $\partial D \in C^{2,\gamma}$ for some $\gamma \in (0,1)$. Then, the operator $\mathcal{T}_\tau$ in (6.17) is compact and strongly positive with respect to the unit $u_0 := \mathcal{K}_\tau\kappa 1$.*

Proof. By Lemmas 6.2.1 and 6.2.2 and the definition of $\kappa\mathcal{N}$ in (6.12) it follows that $\mathcal{T}_\tau = \mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0$ is compact. Since $\mathcal{K}_\tau$ and $\kappa\mathcal{N}\mathcal{G}_0$ are strictly positive, so is $\mathcal{T}_\tau$.

We first show that u_0 is a unit. For $u \in C(\overline{D})$ with $u \gneq 0$ the boundedness of $\mathcal{N}\mathcal{G}_0$ implies $\mathcal{N}\mathcal{G}_0 u \leq c_u 1$ for some $c_u \in (0, \infty)$ and hence, using $\kappa \geq 0$ and the positivity of $\mathcal{K}_\tau$, one gets

$$\mathcal{T}_\tau u = \mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0 u \leq \mathcal{K}_\tau\kappa\,(c_u 1) = c_u u_0, \tag{6.18}$$

which implies u_0 is a unit.

For the strong positivity, let $u \in C(\overline{D})$ with $u \gneq 0$. One finds by Lemma 6.2.3 that $\mathcal{N}\mathcal{G}_0 u > 0$ and, since ∂D is compact, that $\mathcal{N}\mathcal{G}_0 u \geq c_u^* 1$ for some $c_u^* > 0$. From Lemma

6.2.2, we know that $\mathcal{K}_\tau$ is positive and since $\kappa \geq 0$, we find

$$\mathcal{T}_\tau u = \mathcal{K}_\tau \kappa \mathcal{N} \mathcal{G}_0 u \geq c_u^* \mathcal{K}_\tau \kappa 1 = c_u^* u_0, \tag{6.19}$$

which shows that $\mathcal{T}_\tau$ is strongly positive with respect to u_0. ∎

Corollary 6.2.6 *Suppose the conditions of Proposition 6.2.5 hold. Then the spectral radius is the in absolute sense largest eigenvalue of (6.17) and is the only eigenvalue with a positive eigenfunction.*

Proof. Set $u_0 := \mathcal{K}_\tau \kappa 1$. Since $\mathcal{T}_\tau u_0 \geq c u_0$ for some $c > 0$, it follows that $\|\mathcal{T}_\tau^n\|_{\mathcal{L}(C(\overline{D}))}^{1/n} \geq c$ and hence that $r(\mathcal{T}_\tau) \geq c > 0$. So, the Kreĭn-Rutman Theorem implies that the spectral radius is the in absolute sense largest eigenvalue, i.e., $\mu_1 = r(\mathcal{T}_\tau)$, all other eigenvalues μ_i of $\mathcal{T}_\tau$ satisfy $|\mu_i| < \mu_1$, and μ_1 corresponds to a positive eigenfunction $\psi_1 \in C(\overline{D})$:

$$\mathcal{T}_\tau \psi_1 = \mu_1 \psi_1$$

(see [42], [80, Appendix 1], [41, Chapter 11]). To find uniqueness of the positive eigenfunction and that the eigenspace is one-dimensional, one uses that for all $u \gneq 0$ one gets

$$c_2 u_0 \geq \mathcal{T}_\tau u \geq c_1 u_0 > 0 \text{ in } D.$$

This implies that $\mathcal{T}_\tau$ is irreducible on $C(\overline{D})$ as described in [80, Appendix 1]. Alternatively one may use the space

$$C_{u_0}(\overline{D}) := \left\{ u \in C(\overline{D})\, ;\, \|u\|_{u_0} := \sup_{x \in D} \left| \frac{u(x)}{u_0(x)} \right| < \infty \right\}$$

as in [3, 2] or [41] and note that $\mathcal{T}_\tau(C(\overline{D})) \subset C_{u_0}(\overline{D})$. ∎

6.2.1 An auxiliary first eigenvalue is small enough

In order to prove that the series in (6.14) converges, we have to show that the spectral radius of $(1-\sigma)\, \mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}$ is less than 1 as $r(\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}) = r(\mathcal{T}_\tau) = \mu_1$ by (6.15) and (6.16). In view of Corollary 6.2.6, it suffices to verify that $\mu_1 < \frac{1}{1-\sigma}$, where μ_1 is the principal eigenvalue of $\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}$. Since $\frac{1}{1-\sigma} > \frac{1}{2}$, it is sufficient to show that $\mu_1 \leq \frac{1}{2}$. Note that $\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 = \mu_1 \varphi_1$ from (6.15) corresponds with

$$\begin{cases} \left(\Delta^2 - \tau \Delta\right) \varphi_1 = 0 & \text{in } D, \\ \varphi_1 = 0 & \text{on } \partial D, \\ -\Delta \varphi_1 = \mu_1^{-1} \kappa \left(-\partial_\nu \varphi_1\right) & \text{on } \partial D. \end{cases} \tag{6.20}$$

For the case $\tau = 0$, $\sigma \in (-1, 1)$, and D convex with $\partial D \in C^{2,1}$, this has been done by Parini and Stylianou in [71] using sharp estimates for the corresponding "weighted first

Steklov eigenvalue" $\delta_1 := \delta_{1,0}$. More precisely, it was proved that $\delta_1 \geq 2$ holds. In [8] it was shown that this bound is sharp, since for $D = \mathbb{B}_1 := \{x \in \mathbb{R}^2 ; |x| < 1\}$ one finds $\delta_1 = 2$. The same proof applies for $\tau > 0$. Indeed, following [71], we define

$$\delta_{1,\tau} := \inf_{0 \not\equiv u \in \mathcal{W}} \boldsymbol{R}(u) \quad \text{for} \quad \boldsymbol{R}(u) := \frac{\displaystyle\int_D \left((\Delta u)^2 + \tau \, |\nabla u|^2 \right) dx}{\displaystyle\int_{\partial D} \kappa \, (\partial_\nu u)^2 \, d\omega} \tag{6.21}$$

and the convention that $\boldsymbol{R}(u) = +\infty$ whenever $\int_{\partial D} \kappa \, (\partial_\nu u)^2 \, d\omega = 0$.

Proposition 6.2.7 *Suppose that $D \subset \mathbb{R}^2$ is bounded and convex with a $C^{2,1}$-boundary. Then for all $\tau \geq 0$ and $\sigma \in (-1, 1)$, one finds that $\delta_{1,\tau} \geq 2$.*

Remark 6.2.8 *If we allow arbitrary $\tau \geq 0$ and $\sigma \in (-1, 1)$, then 2 is optimal. For each $|\sigma| \leq s < 1$ and $\tau > 0$ one obtains $\delta_{1,\tau} \geq c(s, \tau) > 2$.*

Proof. Using the fact that $\tau \, |\nabla u|^2 \geq 0$ and [71, Proposition 2.7], we conclude that $\delta_{1,\tau} \geq 2$. ■

Corollary 6.2.9 *With the assumptions of Proposition 6.2.7, $\tau \geq 0$, and $\sigma \in (-1, 1)$, we have*

$$r \left((1 - \sigma) \, \mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N} \right) < 1.$$

Proof. Let $\mu_1 > 0$ be the principal eigenvalue of $\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N}$ and let $\varphi_1 \in C^{1,\gamma}(\overline{D})$ be its associated eigenfunction, i.e.,

$$\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 = \mu_1 \varphi_1. \tag{6.22}$$

Then $\varphi_1 \in \mathcal{W}$. Using (6.22) and integration by parts, we obtain

$$\begin{aligned}
&\mu_1 \int_D \left((\Delta \varphi_1)^2 + \tau \, |\nabla \varphi_1|^2 \right) dx \\
&\quad = \int_D \left((-\Delta \varphi_1) \left(\mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 \right) + \tau \nabla \varphi_1 \cdot \nabla \left(\mathcal{G}_0 \mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 \right) \right) dx \\
&\quad = - \int_{\partial D} \left(\mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 \right) \partial_\nu \varphi_1 \, d\omega + \int_D \nabla \varphi_1 \cdot \nabla \left(\mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 \right) dx + \tau \int_D \varphi_1 \left(\mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 \right) dx.
\end{aligned}$$

Here we used $\varphi_1|_{\partial\Omega} = 0$ and the definition of $\mathcal{G}_0$. Integrating by parts once more and using $\varphi_1|_{\partial D} = 0$ and the definitions of $\mathcal{K}_\tau$ and $\mathcal{N}$, yields

$$\begin{aligned}
\mu_1 \int_D \left((\Delta \varphi_1)^2 + \tau \, |\nabla \varphi_1|^2 \right) dx &= - \int_{\partial D} \left(\mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 \right) \partial_\nu \varphi_1 \, d\omega + \int_D \varphi_1 \left(-\Delta + \tau \right) \mathcal{K}_\tau \kappa \mathcal{N} \varphi_1 dx \\
&= \int_{\partial D} \kappa \left(\partial_\nu \varphi_1 \right)^2 \, d\omega.
\end{aligned} \tag{6.23}$$

We find from (6.21) that φ_1 satisfies

$$\int_D \left((\Delta\varphi_1)^2 + \tau \left|\nabla\varphi_1\right|^2 \right) dx = \delta_{1,\tau} \int_{\partial D} \kappa \left(\partial_\nu \varphi_1\right)^2 d\omega. \tag{6.24}$$

On account of (6.23) and (6.24), we obtain $\mu_1 = \delta_{1,\tau}^{-1}$. From Proposition 6.2.7 it follows that $\mu_1 \leq \frac{1}{2}$. Since $\sigma \in (-1,1)$, we have $r\left((1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\right) < 1$. ■

Remark 6.2.10 *Instead of estimating the weighted first Steklov eigenvalue as done in [71], one may also try to find a special positive supersolution $\tilde{u}$, that is, $\tilde{u} \in C^{1,\gamma}(\overline{D})$ with $\tilde{u} \gneq 0$ and*

$$\tilde{u} \gneq (1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\tilde{u}.$$

This means that the solution u^* of

$$\begin{cases} -\Delta v^* + \tau v^* = 0 & \text{in } D, \\ v^* = -(1-\sigma)\,\kappa\,\partial_\nu \tilde{u} & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u^* = v^* & \text{in } D, \\ u^* = 0 & \text{on } \partial D, \end{cases} \tag{6.25}$$

satisfies $\tilde{u} \gneq u^*$. In general it is hard to find such a function. For a disk, however, this can be done. See Section 6.4.

6.3 Proof of Theorem 6.1.3

Proposition 6.3.1 *Suppose that $D \subset \mathbb{R}^2$ is a bounded convex domain with a $C^{2,1}$-boundary. Take $u_0 := \mathcal{G}_0 1$. Then*

$$\sum_{k=0}^{\infty} \left((1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\right)^k \mathcal{G}_0 : C(\overline{D}) \to C(\overline{D}) \tag{6.26}$$

is a strongly positive operator with respect to the unit u_0.

Remark 6.3.2 *Note that $\mathcal{G}_\tau : C(\overline{D}) \to C(\overline{D})$ is also strongly positive with respect to the unit $u_0 := \mathcal{G}_\tau 1$. See [2, Lemma 5.3]. Moreover, the operator $\mathcal{G}_\tau : L_2(D) \to \mathcal{W}$ is well-defined and $\mathcal{W}$ is embedded in $C(\overline{D})$. By the maximum principle it even follows that $0 \lneq f \in L_2(D)$ implies $\mathcal{G}_\tau f > 0$ in D. Thus,*

$$\sum_{k=0}^{\infty} \left((1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\right)^k \mathcal{G}_0\mathcal{G}_\tau : L_2(D) \to C(\overline{D})$$

is a positive operator and satisfies

$$f \gneq 0 \text{ in } D \implies \sum_{k=0}^{\infty} \left((1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\right)^k \mathcal{G}_0\mathcal{G}_\tau f > 0 \text{ in } D.$$

This, together with (6.14), immediately implies $u > 0$ in D when $f \gneq 0$.

Proof. Observe that

$$\sum_{k=0}^{\infty}\left((1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\right)^k\mathcal{G}_0=\mathcal{G}_0\sum_{k=0}^{\infty}\left((1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)^k.$$

Since $r\left((1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)=r_{sp}<1$ holds due to Corollary 6.2.9, the series converges. Since $\mathcal{G}_0$ and $(1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0$ are strongly positive operators in the appropriate senses, so is the combination in (6.26). ■

Before stating the next proposition, let us specify the notion of a $\mathcal{C}$-solution to (6.5).

Definition 6.3.3 *For $f\in L_2(D)$ we say that u is a $\mathcal{C}$-solution of (6.5) if $u\in C(\overline{D})$ satisfies*

$$u=\mathcal{G}_0\left(\mathcal{I}-(1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)^{-1}\mathcal{G}_\tau f. \tag{6.27}$$

Proposition 6.3.4 *Suppose that D is a bounded convex domain in $\mathbb{R}^2$ with $\partial D\in C^{2,1}$. If $f\in L_2(D)$, then a $\mathcal{C}$-solution u of (6.5) exists. Moreover, $u\in W_p^2(D)\cap C^{1,\gamma}(\overline{D})$ for all $p\in(1,\infty)$ and $\gamma\in(0,1)$ and u is also a weak solution.*

Proof. For $f\in L_2(D)$, one finds $\mathcal{G}_\tau f\in\mathcal{W}$, which is embedded in $C(\overline{D})$. Since

$$r\left((1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)<1,$$

the following series converges and it holds that

$$z:=\left(\mathcal{I}-(1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)^{-1}\mathcal{G}_\tau f=\sum_{k=0}^{\infty}\left((1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)^k\mathcal{G}_\tau f\in C(\overline{D})\subset L^2(D).$$

Furthermore, we note that

$$\mathcal{G}_0\sum_{k=0}^{\infty}\left((1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)^k\mathcal{G}_\tau=\sum_{k=0}^{\infty}\left((1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\right)^k\mathcal{G}_0\mathcal{G}_\tau.$$

Then, it follows from (6.14) that $u=\mathcal{G}_0 z$. Regularity results for second order elliptic problems imply that $u=\mathcal{G}_0 z\in\mathcal{W}$ and even that $u\in W_p^2(D)\cap C^{1,\gamma}(\overline{D})$ for all $p\in(1,\infty)$ and $\gamma\in(0,1)$. Owing to $r\left((1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0\right)<1$, we can rewrite $u=\mathcal{G}_0 z$ in the form

$$u=\mathcal{G}_0\mathcal{G}_\tau f+(1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}u.$$

For such u and for any $\varphi\in\mathcal{W}$ we have

$$\begin{aligned}&\int_D(\Delta u\,\Delta\varphi+\tau\nabla u\cdot\nabla\varphi)\,dx\\&=\int_D\Big(\left(-\mathcal{G}_\tau f-(1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}u\right)\Delta\varphi+\tau\nabla\left(\mathcal{G}_0\mathcal{G}_\tau f+(1-\sigma)\,\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}u\right)\cdot\nabla\varphi\Big)\,dx.\end{aligned}$$

With two integration by parts, $\varphi|_{\partial D} = 0$, and the definitions of $\mathcal{G}_\tau$, $\mathcal{K}_\tau$, and $\mathcal{N}$, we obtain

$$\begin{aligned} -\int_D (\mathcal{G}_\tau f)\,\Delta\varphi\,dx &= \int_D \nabla(\mathcal{G}_\tau f)\cdot\nabla\varphi\,dx \\ &= -\int_D (\Delta\mathcal{G}_\tau f)\,\varphi\,dx \end{aligned}$$

and

$$\begin{aligned} -(1-\sigma)\int_D (\mathcal{K}_\tau\kappa\mathcal{N}u)\,\Delta\varphi\,dx &= (1-\sigma)\int_{\partial D}\kappa\,\partial_\nu u\,\partial_\nu\varphi\,d\omega + (1-\sigma)\int_D \nabla(\mathcal{K}_\tau\kappa\mathcal{N}u)\cdot\nabla\varphi\,dx \\ &= (1-\sigma)\int_{\partial D}\kappa\,\partial_\nu u\,\partial_\nu\varphi\,d\omega - (1-\sigma)\int_D (\Delta\mathcal{K}_\tau\kappa\mathcal{N}u)\,\varphi\,dx. \end{aligned}$$

Moreover, an integration by parts, $\varphi|_{\partial D} = 0$, and the definition of $\mathcal{G}_0$ yield

$$\tau\int_D \nabla(\mathcal{G}_0\mathcal{G}_\tau f)\cdot\nabla\varphi\,dx = \tau\int_D (\mathcal{G}_\tau f)\,\varphi\,dx$$

and

$$\tau\,(1-\sigma)\int_D \nabla(\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}u)\cdot\nabla\varphi\,dx = \tau\,(1-\sigma)\int_D (\mathcal{K}_\tau\kappa\mathcal{N}u)\,\varphi\,dx.$$

Hence,

$$\begin{aligned} &\int_D (\Delta u\,\Delta\varphi + \tau\nabla u\cdot\nabla\varphi)\;dx \\ &= (1-\sigma)\int_{\partial D}\kappa\,\partial_\nu u\,\partial_\nu\varphi\,d\omega + \int_D ((-\Delta+\tau)\,\mathcal{G}_\tau f)\,\varphi\,dx \\ &\quad + (1-\sigma)\int_D ((-\Delta+\tau)\,\mathcal{K}_\tau\kappa\mathcal{N}u)\,\varphi\,dx \\ &= (1-\sigma)\int_{\partial D}\kappa\,\partial_\nu u\,\partial_\nu\varphi\,d\omega + \int_D f\varphi\,dx, \end{aligned}$$

which shows, together with Lemma A.0.1, that u is a weak solution. ■

Corollary 6.3.5 *Suppose that D is a bounded convex domain in $\mathbb{R}^2$ with $\partial D \in C^{2,1}$. If $f \in L_2(D)$ satisfies $f \gneqq 0$ in D, then the $\mathcal{C}$-solution u satisfies $u > 0$ in D and $-\partial_\nu u > 0$ on ∂D.*

Proof. Since the operator in (6.27) satisfies $\mathcal{G}_0\,(\mathcal{I} - (1-\sigma)\,\mathcal{K}_\tau\kappa\mathcal{N}\mathcal{G}_0)^{-1}\,\mathcal{G}_\tau f > 0$ in D when $f \gneqq 0$, one finds $v > 0$ and $u > 0$ in D. With $v > 0$ on the right hand side of (6.6), it follows from Hopf's boundary point Lemma that $-\partial_\nu u > 0$ on ∂D. ■

6.4 The case of a disk

In the case that $\tau = 0$, one finds an explicit formula for the first eigenfunction φ_1 of $\mathcal{G}_0\mathcal{K}_0\kappa\mathcal{N}$ on $\mathbb{B}_1$, namely,

$$\varphi_1(x) = \tfrac{1}{2}\left(1 - |x|^2\right).$$

Since $\kappa = 1$, it is immediately obvious that $\kappa\mathcal{N}\varphi_1 = 1$ and $\mathcal{K}_0\kappa\mathcal{N}\varphi_1 = 1$. So

$$\varphi^*(x) := \mathcal{G}_0\mathcal{K}_0\kappa\mathcal{N}\varphi_1(x) = \tfrac{1}{4}\left(1 - |x|^2\right).$$

One indeed finds $\varphi^* = \frac{1}{2}\varphi_1$ and hence, with δ_1 from Subsection 6.2.1,

$$\delta_1 = \frac{\varphi_1(x)}{\varphi^*(x)} = 2 \quad \text{and} \quad \mu_1 = \frac{1}{2} < \frac{1}{1-\sigma}.$$

Even in the case $\tau > 0$, the auxiliary eigenfunction and eigenvalue can be computed in the case of the disk. Recall that the first eigenfunction φ_1 and the Steklov eigenvalue $\delta_{1,\tau}$ correspond to (6.20)-(6.21).

Lemma 6.4.1 *Let $D = \mathbb{B}_1$ and $\tau > 0$. Then φ_1 can be written by using I_0, the modified Bessel function of the first kind:*

$$\varphi_1(x) = \frac{I_0(\sqrt{\tau}) - I_0(\sqrt{\tau}\,|x|)}{\tau I_0(\sqrt{\tau})}.$$

Moreover, it holds that

$$\delta_{1,\tau} = \alpha(\sqrt{\tau}) \quad \textit{with} \quad \alpha(t) := \frac{t\; I_0(t)}{I_1(t)}, \tag{6.28}$$

and $\alpha(0) = 2$. Here I_n is the nth modified Bessel function of the first kind. The function $\alpha \in C^\infty(\mathbb{R})$ is strictly increasing on $(0, \infty)$.

Proof. We take a function $\tilde{u}$ with $\kappa\mathcal{N}\tilde{u} = 1$. A direct computation gives

$$\mathcal{K}_\tau\kappa\mathcal{N}\tilde{u} = \mathcal{K}_\tau 1 = \frac{I_0(\sqrt{\tau}\,|\cdot|)}{I_0(\sqrt{\tau})}$$

with

$$I_0(r) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2}\left(\frac{r}{2}\right)^{2m}.$$

Then,

$$\varphi^*(x) := \left(\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}\tilde{u}\right)(x) = \frac{1}{I_0(\sqrt{\tau})}\sum_{m=0}^{\infty} \frac{(2/\sqrt{\tau})^2}{4\left((m+1)!\right)^2}\left(\left(\frac{\sqrt{\tau}}{2}\right)^{2m+2} - \left(\frac{\sqrt{\tau}\,|x|}{2}\right)^{2m+2}\right)$$

$$= \frac{\sum_{m=1}^{\infty} \frac{1}{(m!)^2} \left(\left(\frac{\sqrt{\tau}}{2} \right)^{2m} - \left(\frac{\sqrt{\tau}|x|}{2} \right)^{2m} \right)}{\tau I_0(\sqrt{\tau})}$$
$$= \frac{I_0(\sqrt{\tau}) - I_0(\sqrt{\tau}\,|x|)}{\tau I_0(\sqrt{\tau})}.$$

Note that we did not fix $\tilde{u}$ except for the normal derivative at the boundary. So we may take

$$\tilde{u}(x) = \frac{I_0(\sqrt{\tau}) - I_0(\sqrt{\tau}\,|x|)}{\sqrt{\tau} I_1(\sqrt{\tau})},$$

which, since $I_0' = I_1$ holds, satisfies $\kappa \mathcal{N} \tilde{u} = 1$. Since now $\tilde{u}$ is a multiple of φ^*, we have found the first eigenfunction and

$$\delta_{1,\tau} = \frac{\tilde{u}(x)}{\varphi^*(x)} = \frac{\tau I_0(\sqrt{\tau})}{\sqrt{\tau} I_1(\sqrt{\tau})}.$$

The last claims concerning α follow from

$$\alpha(t) = \frac{t\ I_0(t)}{I_1(t)} = \frac{2 \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{t}{2} \right)^{2m+1}}{\sum_{m=0}^{\infty} \frac{1}{(m!)^2 (m+1)} \left(\frac{t}{2} \right)^{2m+1}},$$

where $\frac{1}{(m!)^2} > \frac{1}{(m!)^2(m+1)}$ for all $m \geq 1$. ■

Remark 6.4.2 *Indeed, this confirms the estimate for the auxiliary eigenvalue in the case of the disk for all $\sigma \in (-1, 1)$, since $\tau \mapsto \frac{I_1(\sqrt{\tau})}{\sqrt{\tau} I_0(\sqrt{\tau})}$ is decreasing and*

$$\lim_{\tau \searrow 0} \frac{I_1(\sqrt{\tau})}{\sqrt{\tau} I_0(\sqrt{\tau})} = \frac{1}{2} < \frac{1}{1-\sigma}.$$

See Figure 6.1.

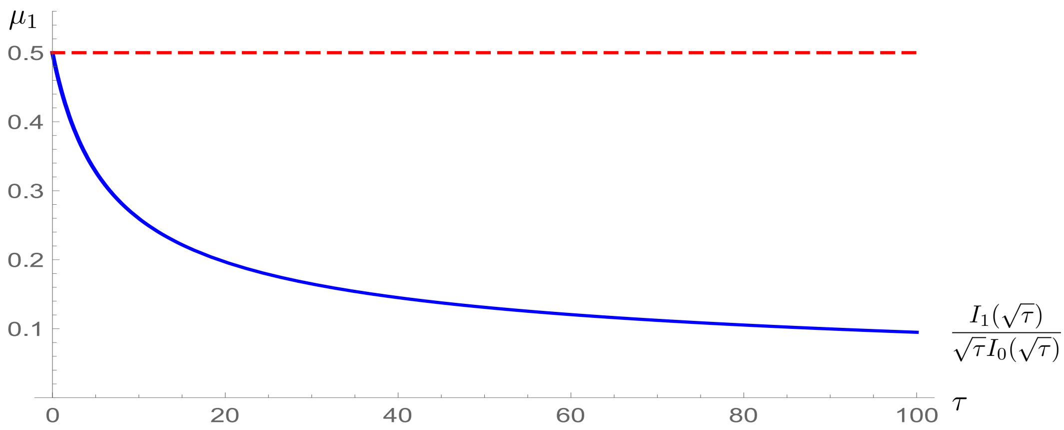

Figure 6.1: The eigenvalue $\mu_1 = \delta_{1,\tau}^{-1}$ of $\mathcal{G}_0\mathcal{K}_\tau\kappa\mathcal{N}$ for the unit disk as a function of τ

Chapter 7

The eigenvalue problem

In this chapter, we study the eigenvalue problem

$$\begin{cases} \beta\Delta^2\varphi - \tau\Delta\varphi = \mu\varphi & \text{in } D \subset \mathbb{R}^2, \\ \varphi = 0 & \text{on } \partial D, \\ \Delta\varphi = (1-\sigma)\kappa\,\partial_\nu\varphi & \text{on } \partial D \end{cases} \tag{7.1}$$

with parameters $\beta > 0$, $\tau \geq 0$, and $\sigma \in (-1, 1)$. Here, again, κ denotes the signed curvature of the boundary ∂D, positive on strict convex boundary parts, and ν the exterior unit normal on ∂D.

We are interested in the existence and uniqueness of $\mu \in \mathbb{R}$ for which (7.1) admits a positive eigenfunction φ. This fundamental result will be derived by combining the positivity preserving property established in Chapter 6 with the Kreĭn-Rutman theorem.

Before stating the main result, let us recall the notation:

$$W^4_{2,B}(D) = \{v \in W^4_2(D)\,;\, v = \Delta v - (1-\sigma)\kappa\,\partial_\nu v = 0 \text{ on } \partial D\}.$$

Theorem 7.0.1 *Let $D \subset \mathbb{R}^2$ be a bounded convex domain with $\partial D \in C^4$. The eigenvalue problem (7.1) admits a unique eigenvalue $\mu_1 \in \mathbb{R}$ which has a positive eigenfunction φ_1. The eigenvalue μ_1 is positive and simple. Moreover, $\varphi_1 \in W^4_{2,B}(D)$ and $\partial_\nu\varphi_1 < 0$ on ∂D.*

Proof. Consider the problem (7.1) with $f = f(x)$ as a right-hand side, i.e.,

$$\begin{cases} \beta\Delta^2\varphi - \tau\Delta\varphi = f & \text{in } D, \\ \varphi = 0 & \text{on } \partial D, \\ \Delta\varphi = (1-\sigma)\kappa\,\partial_\nu\varphi & \text{on } \partial D. \end{cases} \tag{7.2}$$

Then, by [24, Theorem 2.20], there exists a solution operator $\mathcal{S}$ for (7.2) from $L_2(D)$ to $W^4_{2,B}(D)$ and moreover, there is a constant $c > 0$, only depending on τ, β, and D, such that

$$\|\mathcal{S}f\|_{W^4_2(D)} \leq c\|f\|_{L_2(D)} \quad \text{for all } f \in L_2(D).$$

Since the embedding $I_3 : W^4_{2,B}(D) \to L_2(D)$ is compact due to [1, Theorem 6.3], one finds that $L := \mathcal{S}I_3$ is a compact endomorphism of $W^4_{2,B}(D)$. The Banach space $W^4_{2,B}(D)$ is an ordered Banach space with positive cone:

$$(W^4_{2,B}(D))_+ := \{v \in W^4_{2,B}(D)\,;\, v \geq 0 \text{ in } D\}.$$

For the terminology, see [3, 13]. Next, set

$$C^1_0(\overline{D}) := \{v \in C^1(\overline{D})\,;\, v = 0 \text{ on } \partial D\}$$

and observe that $W^4_{2,B}(D)$ is continuously embedded in $C^1_0(\overline{D})$. It is well-known that the space $C^1_0(\overline{D})$ has a positive cone with nonempty interior. The interior points are given by those functions $v \in C^1_0(\overline{D})$ satisfying $v(x) > 0$ for all $x \in D$ and $\partial_\nu v(x) < 0$ for all $x \in \partial D$ (see, e.g., [3] or [13, Chapter 12]). This implies, in particular, that $(W^4_{2,B}(D))_+$ has a nonempty interior. It is easy to see that the interior of $(W^4_{2,B}(D))_+$ is given by

$$\operatorname{int}\big((W^4_{2,B}(D))_+\big) = \{v \in W^4_{2,B}(D)\,;\, v > 0 \text{ in } D \text{ and } \partial_\nu v < 0 \text{ on } \partial D\}.$$

Let us now show that $L\big((W^4_{2,B}(D))_+ \setminus \{0\}\big) \subset \operatorname{int}\big((W^4_{2,B}(D))_+\big)$, i.e.,

$$\text{for any } 0 \lneq f \in W^4_{2,B}(D) \text{ it holds that } Lf > 0 \text{ in } D \text{ and } \partial_\nu(Lf) < 0 \text{ on } \partial D.$$

But this follows directly from Theorem 6.1.3. The sharper version of the Kreĭn-Rutman theorem (see [3, Theorem 3.2] or [14, Theorem 19.3]) then applies to L. Hence the assertion follows. ■

Remark 7.0.2 *Assuming that $\partial D \in C^{4,\gamma}$ for some $\gamma \in (0,1)$, one can show that φ_1 additionally belongs to $C^{4,\gamma}(\overline{D})$; see [24, Theorem 2.19].*

Chapter 8

Stationary solutions of the MEMS model

In the following two sections we will discuss the stationary version of (3.1)-(3.5). It is given by the following system of equations

$$\begin{cases} \varepsilon^2\Delta'\psi_u + \partial_z^2\psi_u = 0, & (x,z) \in \Omega(u), \quad (8.1) \\ \psi_u(x,z) = \dfrac{1+z}{1+u(x)}\,, & (x,z) \in \partial\Omega(u), \quad (8.2) \\ \beta\Delta^2 u - \tau\Delta u = -\lambda\Big\{\varepsilon^2|\nabla'\psi_u(x,u(x))|^2 + (\partial_z\psi_u(x,u(x)))^2\Big\}, & x \in D, \quad (8.3) \\ u = \Delta u - (1-\sigma)\kappa\partial_\nu u = 0, & x \in \partial D. \quad (8.4) \end{cases}$$

We first prove the existence of a stationary solution for small values of the parameter λ, i.e., for small voltage values. Next, we complement this result by a nonexistence theorem for large voltage values.

8.1 Existence of stable stationary solutions

In this section, it is our aim to prove that, for λ sufficiently small, the problem (3.1)-(3.5) possesses a unique stationary solution with $u \in S_3(\rho)$ for some $\rho \in (0,1)$. We recall that the set $S_3(\rho)$ is defined in (3.16). We also show that this stationary solution is exponentially stable. The following theorem is the analog of [51, Theorem 1.2] and [48, Theorem 1.7].

Theorem 8.1.1 (Existence) *Suppose that $D \subset \mathbb{R}^2$ is a bounded convex domain with $\partial D \in C^4$. Let $\rho \in (0,1)$ be fixed.*

(i) There are $\delta = \delta(\rho,\varepsilon) > 0$ and an analytic function $[0,\delta) \to W^4_{2,B}(D)$, $\lambda \mapsto U_\lambda$, such that $(U_\lambda, \Psi_{U_\lambda})$ is for each $\lambda \in (0,\delta)$ the unique stationary solution of (3.1)-(3.5) with $U_\lambda \in S_3(\rho)$ and $\Psi_{U_\lambda} \in W^2_2(\Omega(U_\lambda))$. Moreover, $-1 < U_\lambda \le 0$ in D.

(ii) Let $\lambda \in (0,\delta)$. There are $\varpi_0, r_0, R > 0$ such that for each initial value $u^0 \in W^4_{2,B}(D)$ satisfying $u^0 > -1$ in D and $\|u^0 - U_\lambda\|_{W^4_2(D)} < r_0$, the associated solution (u,ψ_u) to

(3.1)-(3.5) exists globally in time with

$$u \in C([0,\infty), W^4_{2,B}(D)) \cap C^1([0,\infty), L_2(D)),$$
$$\psi_{u(t)} \in W^2_2(\Omega(u(t))), \quad t \geq 0,$$

and $u(t) > -1$ in D for each $t \geq 0$. Moreover,

$$\|u(t) - U_\lambda\|_{W^4_2(D)} + \|\partial_t u(t)\|_{L_2(D)} \leq Re^{-\varpi_0 t}\|u^0 - U_\lambda\|_{W^4_2(D)}, \quad t \geq 0. \tag{8.5}$$

The proof of Theorem 8.1.1 relies on the implicit function theorem for part (i) and the principle of linearized stability for part (ii).

Proof. The proof goes in the same spirit as that of [16, Theorem 3]. To prove (i), we note that $W^4_2(D)$ is continuously embedded in $W^2_3(D)$ and recall that g_ε defined in Theorem 3.2.1 is an analytic map $S_3(\rho) \to L_2(D)$. According to Lemma 3.3.2 (or alternatively [24, Theorem 2.20]), the operator $A = \beta\Delta^2 - \tau\Delta \in \mathcal{L}(W^4_{2,B}(D), L_2(D))$ is invertible. Hence, we find that the map

$$F : \mathbb{R} \times (W^4_{2,B}(D) \cap S_3(\rho)) \to W^4_{2,B}(D), \quad (\lambda, v) \mapsto v + \lambda A^{-1} g_\varepsilon(v)$$

is well-defined and analytic. Moreover, we have

$$F(0,0) = 0 \text{ and } D_v F(0,0) = Id_{W^4_{2,B}(D)}.$$

In view of the implicit function theorem, there is $\delta = \delta(\rho,\varepsilon) > 0$ and an analytic map

$$[\lambda \mapsto U_\lambda] : [0,\delta) \to W^4_{2,B}(D)$$

such that $U_0 = 0$ and $F(\lambda, U_\lambda) = 0$ for $\lambda \in [0,\delta)$. For $\lambda \neq 0$, let Ψ_{U_λ} be the potential associated with U_λ. Then $(U_\lambda, \Psi_{U_\lambda})$ is the unique stationary solution to (3.1)-(3.5) satisfying $U_\lambda \in W^4_{2,B}(D) \cap S_3(\rho)$ and $\Psi_{U_\lambda} \in W^2_2(\Omega(U_\lambda))$ when $\lambda \in (0,\delta)$. The nonpositivity of U_λ follows from the fact that $g_\varepsilon(U_\lambda) \geq 0$ in D and the positivity preserving property for the hinged plate (see Theorem 6.1.3).

We now prove part (ii). Let $\lambda \in (0,\delta)$ and write $v = u - U_\lambda$. From the analyticity of the map g_ε and the continuous embedding $W^4_2(D) \hookrightarrow W^2_3(D)$, we infer that $B_\lambda := \lambda D g_\varepsilon(U_\lambda)$ is a well-defined bounded linear operator from $W^4_{2,B}(D)$ to $L_2(D)$. If we linearize the Cauchy problem (3.17) around the stationary solution U_λ, we obtain

$$\begin{cases} \partial_t v + (A + B_\lambda)v = G_\lambda(v) := -\lambda\big(g_\varepsilon(U_\lambda + v) - g_\varepsilon(U_\lambda) - Dg_\varepsilon(U_\lambda)v\big), & t > 0, \\ v(0) = v^0, \end{cases}$$

where, according to Theorem 3.2.1, the map $G_\lambda \in C^\infty(\mathcal{O}_\lambda, L_2(D))$ is defined on a neighborhood $\mathcal{O}_\lambda$ of zero in $W^4_{2,B}(D)$ such that $U_\lambda + \mathcal{O}_\lambda \subset S_3(\rho)$. Moreover, we find

$$G_\lambda(0) = 0 \text{ and } DG_\lambda(0) = 0.$$

From Lemmas 3.3.1 and 3.3.2 we know that $-A$ is the generator of a strongly continuous analytic semigroup on $L_2(D)$ with a negative spectral bound. Hence, since

$$\|B_\lambda\|_{\mathcal{L}(W^4_{2,B}(D),L_2(D))} \to 0, \text{ as } \lambda \to 0,$$

it follows from [6, Proposition I.1.4.2] that $-(A + B_\lambda)$ is again the generator of a strongly continuous analytic semigroup on $L_2(D)$ with a negative spectral bound, provided λ is sufficiently small. Applying [63, Theorem 9.1.2] and making $\delta > 0$ smaller if necessary, part (ii) follows and the proof of Theorem 8.1.1 is complete. ■

The following is an immediate consequence of (8.5) and the Lipschitz continuity of ϕ_v obtained in Theorem 3.2.1:

Corollary 8.1.2 *Assume that the conditions of Theorem 8.1.1 hold. Then $\phi_{u(t)}$ converges exponentially to ϕ_{U_λ} as $t \to \infty$, i.e.,*

$$\|\phi_{u(t)} - \phi_{U_\lambda}\|_{W^2_2(\Omega)} \le R_1 e^{-\varpi_0 t}\|u^0 - U_\lambda\|_{W^4_2(D)}, \quad t \ge 0,$$

with a positive constant R_1.

8.2 Nonexistence of stationary solutions

We show that there is a threshold for the parameter λ above which no solution to (8.1)-(8.4) exists. We recall that, by Theorem 7.0.1, the operator $\beta\Delta^2 - \tau\Delta$ with hinged boundary conditions has a positive eigenvalue $\mu_1 > 0$ with a corresponding positive eigenfunction $\varphi_1 \in W^4_{2,B}(D)$. The proof of the following theorem relies on the positive eigenpair (μ_1, φ_1).

Theorem 8.2.1 (Nonexistence) *Let $D \subset \mathbb{R}^2$ be a bounded convex domain with $\partial D \in C^4$. Suppose that $\lambda \ge \mu_1$. Then there is no stationary solution (u, ψ_u) to (3.1)-(3.5) such that $u \in W^4_{2,B}(D)$, $\psi_u \in W^2_2(\Omega(u))$, and $u(x) > -1$ for $x \in D$.*

Proof. Consider a stationary solution (u, ψ_u) to (3.1)-(3.5) with regularity $u \in W^4_{2,B}(D)$, $\psi_u \in W^2_2(\Omega(u))$, and satisfying $u > -1$ in D. In order to simplify the notation, we set

$$\gamma_e(x) := \partial_z\psi_u(x, u(x)) \text{ and } G(x) := \big(1 + \varepsilon^2|\nabla u(x)|^2\big)\gamma_e(x)^2, \quad x \in D.$$

Using the identity

$$\nabla'\psi_u(x, u(x)) = -\nabla u(x)\gamma_e(x), \quad x \in D, \tag{8.6}$$

which follows from differentiating the boundary condition $\psi_u(x, u(x)) = 1$, $x \in D$, the function u solves

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = -\lambda\, G & \text{in } D, \\ u = 0 & \text{on } \partial D, \\ \Delta u - (1-\sigma)\,\kappa\partial_\nu u = 0 & \text{on } \partial D. \end{cases} \tag{8.7}$$

Since $\psi_u \in W_2^2(\Omega(u))$ implies that $[x \mapsto \nabla\psi_u(x, u(x))] \in W_2^{1/2}(D) \hookrightarrow L_4(D)$, we get that $G \in L_2(D)$, and since $G \geq 0$, we obtain by Theorem 6.1.3 that

$$u \leq 0 \text{ in } D.$$

We next provide upper and lower bounds for the potential ψ_u.

Lemma 8.2.2 *For* $(x, z) \in \Omega(u)$,

$$0 \leq \psi_u(x, z) \leq 1.$$

Proof. The function $(x, z) \mapsto m$ clearly solves (8.1) for $m = \{0, 1\}$, and furthermore, it holds that $0 \leq \psi_u \leq 1$ on $\partial\Omega(u)$ since $u = 0$ on ∂D. The maximum principle then implies that $0 \leq \psi_u \leq 1$ in $\Omega(u)$. ■

The following lemma is the main ingredient in the proof of Theorem 8.2.1.

Lemma 8.2.3 *For* $(x, z) \in \Omega(u)$, *define* $M(x, z) := 1 + z - u(x)$. *Then*

$$\psi_u(x, z) \leq M(x, z), \quad (x, z) \in \Omega(u), \tag{8.8}$$

and

$$\partial_z\psi_u(x, u(x)) \geq 1, \quad x \in D. \tag{8.9}$$

For $D = (-1, 1) \subset \mathbb{R}$, $\beta = 0$ in (8.3), and for clamped boundary conditions, that is, when u solves

$$\partial_x^2 u(x) = \lambda\Big\{\varepsilon^2(\partial_x\psi_u(x, u(x)))^2 + (\partial_z\psi_u(x, u(x)))^2\Big\},\ x \in (-1, 1), \qquad u(\pm 1) = 0,$$

such a result has been proved by Laurençot and Walker in [46]. In this case, u is clearly convex and the proof then follows from the maximum principle.

A somewhat different approach is needed to prove Lemma 8.2.3. Fortunately, the combination of the boundary conditions for u allows us to rewrite the fourth-order problem (8.7) as a second order system.

Proof. Since $u|_{\partial D} = 0$, the function M obviously satisfies

$$M(x, z) = 1 + z = \psi_u(x, z), \quad x \in \partial D,\ z \in (-1, 0)$$

and

$$M(x, u(x)) = 1 = \psi_u(x, u(x)), \quad x \in D. \tag{8.10}$$

Due to the nonpositivity of u, it also satisfies

$$M(x,-1) = -u(x) \geq 0 = \psi_u(x,-1), \quad x \in D,$$

so that $M \geq \psi_u$ on $\partial\Omega(u)$. In addition, for $(x,z) \in \Omega(u)$, we have

$$-\varepsilon^2 \Delta' M(x,z) - \partial_z^2 M(x,z) = \varepsilon^2 \Delta u(x).$$

To verify that $\Delta u \geq 0$ in D, we rewrite (8.7) as the coupled system

$$\begin{cases} -\beta\Delta v + \tau v = -\lambda G & \text{in } D, \\ v = -(1-\sigma)\,\kappa\,\partial_\nu u & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } D, \\ u = 0 & \text{on } D. \end{cases} \tag{8.11}$$

Since D is convex, hence $\kappa \geq 0$, we find that

$$u \leq 0 \text{ in } D \implies -(1-\sigma)\,\kappa\,\partial_\nu u \leq 0 \text{ on } \partial D,$$

and it follows from $-\lambda G \leq 0$ in D and the maximum principle (for the problem on the left in (8.11)) that $v \leq 0$ in D. Thus, $\Delta u \geq 0$ in D. Hence, as

$$-\varepsilon^2 \Delta' M(x,z) - \partial_z^2 M(x,z) \geq 0 = -\varepsilon^2 \Delta' \psi_u(x,z) - \partial_z^2 \psi_u(x,z), \quad (x,z) \in \Omega(u),$$

we can apply the maximum principle to conclude that $M \geq \psi_u$ in $\Omega(u)$. This, together with (8.10), yields that, for $x \in D$ and $z \in (-1, u(x))$,

$$\frac{\psi_u(x,z) - \psi_u(x,u(x))}{z - u(x)} \geq \frac{M(x,z) - M(x,u(x))}{z - u(x)} = 1.$$

Sending z to $u(x)$, we see that $\partial_z \psi_u(x,u(x)) \geq 1$ for all $x \in D$. ■

Lemma 8.2.3 implies

$$G(x) \geq 1, \quad x \in D,$$

so that by Theorem 6.1.3 we even have $u < 0$ in D. Moreover, we infer by (8.7) that

$$-\beta\Delta^2 u + \tau\Delta u \geq \lambda \quad \text{in } D. \tag{8.12}$$

Multiplying (8.12) by the eigenfunction φ_1 and integrating over D gives

$$\lambda \int_D \varphi_1 \, dx \leq \int_D \left(-\beta\Delta^2 u + \tau\Delta u\right) \varphi_1 \, dx.$$

Integrating by parts and using $\varphi_1|_{\partial D} = 0$, yields

$$\int_D \left(-\beta\Delta^2 u + \tau\Delta u\right) \varphi_1 \, dx = \beta \int_D \nabla\Delta u \cdot \nabla\varphi_1 \, dx - \tau \int_D \nabla u \cdot \nabla\varphi_1 \, dx.$$

With two integration by parts, and taking into account that $u = 0$ on ∂D, we further obtain

$$\int_D \left(-\beta\Delta^2 u + \tau\Delta u\right)\varphi_1\, dx$$
$$= \beta \int_{\partial D} \left(\Delta u\, \partial_\nu \varphi_1 - \Delta\varphi_1\, \partial_\nu u\right) d\omega + \beta \int_D \nabla u \cdot \nabla\Delta\varphi_1\, dx + \tau \int_D u\, \Delta\varphi_1\, dx.$$

From this, using again integration by parts and $u|_{\partial D} = 0$, we infer that

$$\int_D \left(-\beta\Delta^2 u + \tau\Delta u\right)\varphi_1\, dx$$
$$= \beta \int_{\partial D} \left(\Delta u\, \partial_\nu \varphi_1 - \Delta\varphi_1\, \partial_\nu u\right) d\omega + \int_D \left(-\beta\Delta^2\varphi_1 + \tau\Delta\varphi_1\right) u\, dx$$
$$= \int_D \left(-\beta\Delta^2\varphi_1 + \tau\Delta\varphi_1\right) u\, dx.$$

The last step follows from the second boundary condition for u and φ_1. Then

$$\lambda \int_D \varphi_1\, dx \leq \int_D \left(-\beta\Delta^2\varphi_1 + \tau\Delta\varphi_1\right) u\, dx = -\mu_1 \int_D \varphi_1\, u\, dx < \mu_1 \int_D \varphi_1\, dx,$$

since $u > -1$ in D. So $\lambda < \mu_1$, and this completes the proof. ■

Remark 8.2.4 *An alternative proof of Theorem 8.2.1 is contained in Appendix C. It is based on the construction of an appropriate auxiliary problem rather than on the lower bound for $\partial_z\psi_u(x, u(x))$ provided by Lemma 8.2.3. More precisely, we will show that for*

$$\varepsilon < \varepsilon_* := \sqrt{\frac{6}{7\|\Delta\varphi_1\|_{L_1(D)}}} \quad \textit{and} \quad \lambda \geq \frac{\varepsilon^2\left(\mu_1 + \left(K_1 + \dfrac{\beta}{\varepsilon^2}\right)\alpha_\varepsilon\right)^2}{2\alpha_\varepsilon\beta\left(1 - \dfrac{\varepsilon^2}{\varepsilon_*^2}\right)^2},$$

there exists no stationary solution. Here, $\alpha_\varepsilon := \min\{1, \varepsilon^2\}$ and K_1 is a positive constant such that $K_1 > 2\beta$. This result is weaker than Theorem 8.2.1 since one easily verifies that

$$\frac{\varepsilon^2\left(\mu_1 + \left(K_1 + \dfrac{\beta}{\varepsilon^2}\right)\alpha_\varepsilon\right)^2}{2\alpha_\varepsilon\beta\left(1 - \dfrac{\varepsilon^2}{\varepsilon_*^2}\right)^2} > \mu_1.$$

However, it may be that a modification of the estimates in the proof of Theorem C.0.1 could lead to a better outcome.

Part III

Appendix

Appendix A

An important identity

In this appendix we prove identity (2.18) under weaker assumptions.

Lemma A.0.1 ([81, Lemma A.1]) *Let $D \subset \mathbb{R}^2$ be a bounded domain with $\partial D \in C^{2,1}$ and let κ be its signed curvature. Then for*

$$u, \varphi \in W_2^2(D) \cap W_{2,B}^1(D) = \{v \in W_2^2(D)\,;\, v = 0 \text{ on } \partial D\}$$

it holds that

$$\int_D \left(2\partial_{x_2}\partial_{x_1}\varphi\,\partial_{x_2}\partial_{x_1}u - \partial_{x_1}^2\varphi\,\partial_{x_2}^2u - \partial_{x_2}^2\varphi\,\partial_{x_1}^2u\right)dx = -\int_{\partial D} \kappa\,\partial_\nu\varphi\,\partial_\nu u\,d\omega,$$

where ν is the outward unit normal on ∂D.

Proof. For $\partial D \in C^{2,1}$, the functions $\left\{v \in C^\infty(\overline{D})\,;\, v = 0 \text{ on } \partial D\right\}$ lie dense in $W_2^2(D) \cap W_{2,B}^1(D)$. Indeed, one may locally rectify a boundary section by a $C^{2,\gamma}$-diffeomorphism and use a reflection argument. Hence we may assume that $u \in W_2^2(D) \cap W_{2,B}^1(D)$ and $\varphi \in C^\infty(\overline{D})$ with $\varphi = 0$ on ∂D, and so we may use the same arguments as for (2.18). In fact, integrating by parts and using $u|_{\partial D} = 0$, one finds that

$$\begin{aligned}
&\int_D \left(2\partial_{x_2}\partial_{x_1}\varphi\,\partial_{x_2}\partial_{x_1}u - \partial_{x_1}^2\varphi\,\partial_{x_2}^2u - \partial_{x_2}^2\varphi\,\partial_{x_1}^2u\right)dx \\
&= \int_{\partial D} \left(\nu_1\,\partial_{x_2}\partial_{x_1}\varphi\,\partial_{x_2}u + \nu_2\,\partial_{x_2}\partial_{x_1}\varphi\,\partial_{x_1}u - \nu_2\,\partial_{x_1}^2\varphi\,\partial_{x_2}u - \nu_1\,\partial_{x_2}^2\varphi\,\partial_{x_1}u\right)d\omega.
\end{aligned}$$

With the counterclockwise oriented tangent vector $s = (s_1, s_2)$ on ∂D and using the fact that $\partial_{x_1}u|_{\partial D} = \nu_1\partial_\nu u + s_1\partial_s u$, $\partial_{x_2}u|_{\partial D} = \nu_2\partial_\nu u + s_2\partial_s u$ and that $\partial_s u|_{\partial D} = 0$ due to $u|_{\partial D} = 0$, one gets

$$\begin{aligned}
&\int_D \left(2\partial_{x_2}\partial_{x_1}\varphi\,\partial_{x_2}\partial_{x_1}u - \partial_{x_1}^2\varphi\,\partial_{x_2}^2u - \partial_{x_2}^2\varphi\,\partial_{x_1}^2u\right)dx \\
&= \int_{\partial D} \left(2\nu_1\nu_2\,\partial_{x_2}\partial_{x_1}\varphi - \nu_2^2\,\partial_{x_1}^2\varphi - \nu_1^2\,\partial_{x_2}^2\varphi\right)\partial_\nu u\,d\omega \\
&= \int_{\partial D} \left(2\nu_1\nu_2\,\partial_{x_2}\partial_{x_1}\varphi + \nu_1^2\,\partial_{x_1}^2\varphi + \nu_2^2\,\partial_{x_2}^2\varphi - \Delta\varphi\right)\partial_\nu u\,d\omega.
\end{aligned}$$

Taking into account that $\partial_\nu^2\varphi = \nu_1^2\partial_{x_1}^2\varphi + \nu_2^2\partial_{x_2}^2\varphi + 2\nu_1\nu_2\partial_{x_2}\partial_{x_1}\varphi$ and $\Delta\varphi = \partial_\nu^2\varphi + \partial_s^2\varphi + \kappa\partial_\nu\varphi$ (see [78, Section 4.1]), one has

$$\begin{aligned}
&\int_D \left(2\partial_{x_2}\partial_{x_1}\varphi\,\partial_{x_2}\partial_{x_1}u - \partial_{x_1}^2\varphi\,\partial_{x_2}^2 u - \partial_{x_2}^2\varphi\,\partial_{x_1}^2 u\right) dx \\
&= \int_{\partial D} \left(\partial_\nu^2\varphi - \Delta\varphi\right) \partial_\nu u\, d\omega \\
&= \int_{\partial D} \left(-\partial_s^2\varphi - \kappa\partial_\nu\varphi\right) \partial_\nu u\, d\omega.
\end{aligned}$$

Since $\partial_s\varphi = \partial_s^2\varphi = 0$ on ∂D when $\varphi|_{\partial D} = 0$, the proof is complete. ∎

Appendix B

Proof of Theorem 3.2.1

This proof follows that of [51, Proposition 2.1] with minor changes.

By $C_B, C_L, C_1, C_2, \ldots$, we will denote positive constants that depend on ρ, ε, q, D only. These constants are allowed to vary from line to line.

B.1 Some preliminary lemmas

In order to prove Theorem 3.2.1 we need some preliminary lemmas. In this section, we consider the Dirichlet problem

$$\begin{cases} -\mathcal{L}_v \Phi = F & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.1}$$

where $\Omega := D \times (0,1)$ and $\mathcal{L}_v$ is defined in (3.10). We have the following existence and uniqueness result.

Lemma B.1.1 *Let $\rho \in (0,1)$ and $q \in (2,\infty]$. For each $v \in \overline{S}_q(\rho)$ and $F \in L_2(\Omega)$, there is a unique solution*

$$\Phi \in W^1_{2,B}(\Omega) := \left\{ w \in W^1_2(\Omega)\,;\, w = 0 \ \text{on}\ \partial\Omega \right\}$$

to the boundary value problem (B.1). Furthermore, if $v \in \overline{S}_\infty(\rho)$, then $\Phi \in W^2_{2,B}(\Omega) := W^2_2(\Omega) \cap W^1_{2,B}(\Omega)$.

Proof. It follows from the definition of $\overline{S}_q(\rho)$ and Sobolev's embedding theorem that there is $C_1 > 0$ such that

$$1 + v(x) \geq \rho, \quad x \in D, \qquad \|v\|_{C^1(\overline{D})} \leq \frac{C_1}{\rho}, \tag{B.2}$$

for all $v \in \overline{S}_q(\rho)$. Writing $-\mathcal{L}_v$ in divergence form,

$$-\mathcal{L}_v w = -\mathrm{div}\left(\boldsymbol{\alpha}(v)\nabla w\right) - \boldsymbol{b}(v) \cdot \nabla w,$$

where, for $(x,\eta) \in \Omega$,

$$\boldsymbol{\alpha}(v) := \begin{pmatrix} \varepsilon^2 & 0 & -\varepsilon^2\eta\dfrac{\partial_{x_1}v(x)}{1+v(x)} \\ 0 & \varepsilon^2 & -\varepsilon^2\eta\dfrac{\partial_{x_2}v(x)}{1+v(x)} \\ -\varepsilon^2\eta\dfrac{\partial_{x_1}v(x)}{1+v(x)} & -\varepsilon^2\eta\dfrac{\partial_{x_2}v(x)}{1+v(x)} & \dfrac{1+\varepsilon^2\eta^2|\nabla v(x)|^2}{(1+v(x))^2} \end{pmatrix}$$

and

$$\boldsymbol{b}(v) := \left(\varepsilon^2\frac{\partial_{x_1}v(x)}{1+v(x)}, \varepsilon^2\frac{\partial_{x_2}v(x)}{1+v(x)}, -\varepsilon^2\eta\frac{|\nabla v(x)|^2}{(1+v(x))^2}\right),$$

one easily verifies that $-\mathcal{L}_v$ is strictly elliptic with an ellipticity constant depending on ρ and ε but not on v. Moreover, by the definition of $\overline{S}_q(\rho)$ and (B.2), one gets

$$\sum_{i,j=1}^{3} \|\alpha_{ij}(v)\|_{L_\infty(\Omega)} + \sum_{i=1}^{3} \|b_i(v)\|_{L_\infty(\Omega)} \leq C_2, \quad \text{for all } v \in \overline{S}_q(\rho),$$

where $\alpha_{ij}(v)$ denotes the (i,j)-entry of $\boldsymbol{\alpha}(v)$ and $b_i(v)$ the components of $\boldsymbol{b}(v)$. Then, [26, Theorem 8.3] ensures that (B.1) has a unique weak solution $\Phi \in W^1_{2,B}(\Omega)$.

Next, let $v \in \overline{S}_\infty(\rho)$ and set $G := F + \boldsymbol{b}(v)\cdot\nabla\Phi$. Since $\Phi \in W^1_{2,B}(\Omega)$, one has $G \in L_2(\Omega)$. Furthermore, since $\alpha_{ij}(v) \in W^1_\infty(\Omega)$, $1 \leq i,j \leq 3$, and since Ω is convex, [28, Theorem 3.2.1.2] implies that the problem

$$\begin{cases} -\mathrm{div}\,(\boldsymbol{\alpha}(v)\nabla\Psi) = G & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.3}$$

possesses a unique solution $\Psi \in W^2_{2,B}(\Omega)$. From [26, Theorem 8.3], one deduces that (B.3) also admits a unique weak solution in $W^1_{2,B}(\Omega)$, and, hence, according to the definition of G, Φ and Ψ are both weak solutions to (B.3). Thus, $\Phi = \Psi \in W^2_{2,B}(\Omega)$. ∎

What remains to show is that, for $v \in \overline{S}_q(\rho)$, the weak solution of (B.1) has higher regularity and that, in fact, it belongs to $W^2_{2,B}(\Omega)$. For this purpose, we proceed in two steps. First, we derive a uniform estimate for Φ in the anisotropic space

$$X(\Omega) := \left\{w \in W^1_{2,B}(\Omega)\,;\, \partial_\eta w \in W^1_2(\Omega)\right\},$$

and then, using this information, we show that $\Phi \in W^2_2(\Omega)$.

Lemma B.1.2 (Improved regularity: step 1)
Let $\rho \in (0,1)$ and $q \in (2,\infty]$. For each $v \in \overline{S}_q(\rho)$ and $F \in L_2(\Omega)$, the weak solution

$\Phi \in W^1_{2,B}(\Omega)$ *to (B.1) belongs to* $X(\Omega)$*, and there exists* $C_3 > 0$ *such that*

$$\|\Phi\|_{W^1_2(\Omega)} + \|\partial_\eta \Phi\|_{W^1_2(\Omega)} \leq C_3 \|F\|_{L_2(\Omega)}. \tag{B.4}$$

Proof. Since $W^2_\infty(D)$ embeds continuously in $W^2_q(D)$, we prove the lemma only for $v \in \overline{S}_q(\rho)$ with $q \in (2,\infty)$.

Let $\Phi \in W^1_{2,B}(\Omega)$ be the weak solution to (B.1). We first estimate the $W^1_2(\Omega)$-norm of Φ. Using the divergence form of $-\mathcal{L}_v$ and taking $(1+v)\,\Phi$ as a test function in the weak formulation of (B.1), we obtain

$$\int_\Omega (1+v)\,\Phi\, F\, d(x,\eta) = \int_\Omega \nabla\big((1+v)\Phi\big)\cdot\big(\boldsymbol{\alpha}(v)\nabla\Phi\big)\, d(x,\eta) - \int_\Omega (1+v)\,\Phi\, \boldsymbol{b}(v)\cdot\nabla\Phi\, d(x,\eta).$$

The definition of $\boldsymbol{\alpha}(v)$ implies that

$$\begin{aligned}
&\int_\Omega \nabla\big((1+v)\Phi\big)\cdot\big(\boldsymbol{\alpha}(v)\nabla\Phi\big)\, d(x,\eta)\\
&= \varepsilon^2 \int_\Omega \big(\Phi\,\nabla v + (1+v)\,\nabla'\Phi\big)\cdot\left(\nabla'\Phi - \eta\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\right) d(x,\eta)\\
&\quad - \varepsilon^2 \int_\Omega \eta\,(1+v)\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\cdot\left(\nabla'\Phi - \eta\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\right) d(x,\eta) + \int_\Omega \frac{(\partial_\eta\Phi)^2}{1+v}\, d(x,\eta),
\end{aligned}$$

and by the definition of $\boldsymbol{b}(v)$, we have

$$-\int_\Omega (1+v)\,\Phi\,\boldsymbol{b}(v)\cdot\nabla\Phi\, d(x,\eta) = -\varepsilon^2\int_\Omega \Phi\,\nabla v\cdot\left(\nabla'\Phi - \eta\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\right) d(x,\eta).$$

Hence,

$$\int_\Omega (1+v)\,\Phi\, F\, d(x,\eta) = \varepsilon^2 \int_\Omega (1+v)\left|\nabla'\Phi - \eta\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\right|^2 d(x,\eta) + \int_\Omega \frac{(\partial_\eta\Phi)^2}{1+v}\, d(x,\eta).$$

Using Hölder's inequality, we get

$$\int_\Omega (1+v)\,\Phi\, F\, d(x,\eta) \leq \|\,(1+v)\,\Phi\, F\|_{L_1(\Omega)} \leq \|1+v\|_{L_\infty(D)}\|\Phi\|_{L_2(\Omega)}\|F\|_{L_2(\Omega)},$$

and it follows from (B.2) that

$$\begin{aligned}
\varepsilon^2 \int_\Omega (1+v)&\left|\nabla'\Phi - \eta\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\right|^2 d(x,\eta) + \int_\Omega \frac{(\partial_\eta\Phi)^2}{1+v}\, d(x,\eta)\\
&\geq \rho \int_\Omega \left(\varepsilon^2\left|\nabla'\Phi - \eta\,\partial_\eta\Phi\,\frac{\nabla v}{1+v}\right|^2 + \frac{(\partial_\eta\Phi)^2}{(1+v)^2}\right) d(x,\eta).
\end{aligned}$$

Combining the above relations, we infer that

$$\int_\Omega \left(\varepsilon^2 \left| \nabla'\Phi - \eta\, \partial_\eta \Phi \, \frac{\nabla v}{1+v} \right|^2 + \frac{(\partial_\eta \Phi)^2}{(1+v)^2} \right) d(x,\eta) \le \frac{1}{\rho} \, \|1+v\|_{L_\infty(D)} \|\Phi\|_{L_2(\Omega)} \|F\|_{L_2(\Omega)},$$

and, by (B.2) and Young's inequality, that

$$\int_\Omega \left(\varepsilon^2 \left| \nabla'\Phi - \eta\, \partial_\eta \Phi \, \frac{\nabla v}{1+v} \right|^2 + \frac{(\partial_\eta \Phi)^2}{(1+v)^2} \right) d(x,\eta) \le C_4 \left(\|\Phi\|^2_{L_2(\Omega)} + \|F\|^2_{L_2(\Omega)} \right) . \tag{B.5}$$

Using (B.2), it is easy to see by an elementary computation that there exists a constant $m(\rho,\varepsilon) \in (0, \frac{1}{2})$ such that for any $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$,

$$m(\rho,\varepsilon)\, |\zeta|^2 \le \varepsilon^2 \left| (\zeta_1,\zeta_2) - \eta\, \zeta_3 \, \frac{\nabla v}{1+v} \right|^2 + \left(\frac{\zeta_3}{1+v} \right)^2 \quad \text{in } \Omega \tag{B.6}$$

(see also [17]). Setting $\zeta = \nabla \Phi$ in (B.6) and integrating the inequality over Ω, we deduce

$$\|\nabla \Phi\|^2_{L_2(\Omega)} \le \frac{1}{m(\rho,\varepsilon)} \int_\Omega \left(\varepsilon^2 \left| \nabla'\Phi - \eta\, \partial_\eta \Phi \, \frac{\nabla v}{1+v} \right|^2 + \frac{(\partial_\eta \Phi)^2}{(1+v)^2} \right) d(x,\eta),$$

which combined with (B.5), gives

$$\|\Phi\|_{W_2^1(\Omega)} \le C_5 \left(\|\Phi\|_{L_2(\Omega)} + \|F\|_{L_2(\Omega)} \right) .$$

Actually the above estimate can be improved to

$$\|\Phi\|_{W_2^1(\Omega)} \le C_5 \, \|F\|_{L_2(\Omega)} \tag{B.7}$$

by arguing in the same way as in the proof of [16, Eq.(19)].

Let us next estimate the $W_2^1(\Omega)$-norm of $\partial_\eta \Phi$. We first assume that $v \in \overline{S}_q(\rho) \cap W^2_\infty(D)$. Then $\Phi \in W^2_{2,B}(\Omega)$ by Lemma B.1.1. Setting $(\zeta_{x_1}, \zeta_{x_2}) := \nabla' \partial_\eta \Phi$ and $\zeta_\eta := \partial^2_\eta \Phi$, multiplying (B.1) by ζ_η (with $\mathcal{L}_v$ as in (3.10)), and integrating over Ω, we get

$$\begin{aligned}
-\int_\Omega & \zeta_\eta \, F \, d(x,\eta) \\
&= \int_\Omega \zeta_\eta \, \mathcal{L}_v \Phi \, d(x,\eta) \\
&= \varepsilon^2 \int_\Omega \zeta_\eta \, \Delta' \Phi \, d(x,\eta) - 2\varepsilon^2 \int_\Omega \eta \, \zeta_\eta \, \frac{\nabla v}{1+v} \cdot (\zeta_{x_1}, \zeta_{x_2}) \, d(x,\eta) \\
&\quad + \int_\Omega \left(\frac{1 + \varepsilon^2 \eta^2 |\nabla v|^2}{(1+v)^2} \right) \zeta_\eta^2 \, d(x,\eta) + \varepsilon^2 \int_\Omega \eta \left(\frac{2|\nabla v|^2 - \Delta v \, (1+v)}{(1+v)^2} \right) \partial_\eta \Phi \, \zeta_\eta \, d(x,\eta).
\end{aligned}$$

In [51] Laurençot and Walker have shown that for $\Phi \in W^2_{2,B}(\Omega)$,

$$\int_\Omega \partial^2_{x_i}\Phi\,\partial^2_\eta\Phi\,d(x,\eta) = \int_\Omega (\partial_{x_i}\partial_\eta\Phi)^2\,d(x,\eta), \quad i=1,2.$$

From this fact, we deduce that

$$\begin{aligned}
&-\int_\Omega \zeta_\eta\,F\,d(x,\eta)\\
&= \varepsilon^2\int_\Omega (\zeta^2_{x_1}+\zeta^2_{x_2})\,d(x,\eta) - 2\varepsilon^2\int_\Omega \eta\,\zeta_\eta\,\frac{\nabla v}{1+v}\cdot(\zeta_{x_1},\zeta_{x_2})\,d(x,\eta)\\
&\quad+\int_\Omega\left(\frac{1+\varepsilon^2\eta^2|\nabla v|^2}{(1+v)^2}\right)\zeta^2_\eta\,d(x,\eta) + \varepsilon^2\int_\Omega \eta\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)\partial_\eta\Phi\,\zeta_\eta\,d(x,\eta),
\end{aligned}$$

and a rearranging of terms leads to

$$\begin{aligned}
-\int_\Omega \zeta_\eta\,F\,d(x,\eta) &= \int_\Omega\left(\varepsilon^2\left|(\zeta_{x_1},\zeta_{x_2}) - \eta\,\zeta_\eta\,\frac{\nabla v}{1+v}\right|^2 + \left(\frac{\zeta_\eta}{1+v}\right)^2\right)d(x,\eta)\\
&\quad+\varepsilon^2\int_\Omega \eta\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)\partial_\eta\Phi\,\zeta_\eta\,d(x,\eta).
\end{aligned}$$

Using the fact that $\partial_\eta\Phi\,\zeta_\eta = \partial_\eta\big((\partial_\eta\Phi)^2\big)/2$ and integration by parts one easily verifies that

$$\begin{aligned}
&\varepsilon^2\int_\Omega \eta\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)\partial_\eta\Phi\,\zeta_\eta\,d(x,\eta)\\
&\quad= \frac{\varepsilon^2}{2}\int_{\partial\Omega}\eta\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)\big(0,(\partial_\eta\Phi)^2\big)\cdot n_{\partial\Omega}\,dS\\
&\qquad-\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)(\partial_\eta\Phi)^2\,d(x,\eta)\\
&\quad= \frac{\varepsilon^2}{2}\int_D\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)(\partial_\eta\Phi(x,1))^2\,dx\\
&\qquad-\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)(\partial_\eta\Phi)^2\,d(x,\eta)
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_\Omega\left(\varepsilon^2\left|(\zeta_{x_1},\zeta_{x_2}) - \eta\,\zeta_\eta\,\frac{\nabla v}{1+v}\right|^2 + \left(\frac{\zeta_\eta}{1+v}\right)^2\right)d(x,\eta)\\
&\quad= -\int_\Omega \zeta_\eta\,F\,d(x,\eta) + \frac{\varepsilon^2}{2}\int_\Omega\left(\frac{2|\nabla v|^2-\Delta v\,(1+v)}{(1+v)^2}\right)(\partial_\eta\Phi)^2\,d(x,\eta)
\end{aligned}$$

$$- \frac{\varepsilon^2}{2} \int_D \left(\frac{2|\nabla v|^2 - \Delta v\,(1+v)}{(1+v)^2} \right) (\partial_\eta \Phi(x,1))^2 \, dx. \tag{B.8}$$

Next, let $q' := \frac{q}{q-1} \in (1,2)$. By (B.2), Hölder's inequality, and the trace estimate for $\partial_\eta \Phi(\cdot,1)$ (see [51, Lemma 2.4]), we obtain that

$$\begin{aligned}
-\frac{\varepsilon^2}{2} &\int_D \left(\frac{2|\nabla v|^2 - \Delta v\,(1+v)}{(1+v)^2} \right) (\partial_\eta \Phi(x,1))^2 \, dx \\
&\leq \frac{\varepsilon^2}{2\rho^2} \left\| (2|\nabla v|^2 - \Delta v(1+v))(\partial_\eta \Phi(\cdot,1))^2 \right\|_{L_1(D)} \\
&\leq \frac{\varepsilon^2}{2\rho^2} \left\| 2|\nabla v|^2 - \Delta v(1+v) \right\|_{L_q(D)} \|\partial_\eta \Phi(\cdot,1)\|^2_{L_{2q'}(D)} \\
&\leq \frac{\varepsilon^2}{2\rho^2} \Big(2\|\nabla v\|_{L_\infty(D)} \|\nabla v\|_{L_q(D)} + \|1+v\|_{L_\infty(D)} \|\Delta v\|_{L_q(D)} \Big) \|\partial_\eta \Phi(\cdot,1)\|^2_{L_{2q'}(D)} \\
&\leq C_6 \, \|\partial_\eta \Phi\|_{W_2^1(\Omega)}^{\frac{3q'-2}{q'}} \|\partial_\eta \Phi\|_{L_2(\Omega)}^{\frac{2-q'}{q'}} = C_6 \, \|\partial_\eta \Phi\|_{W_2^1(\Omega)}^{\frac{q+2}{q}} \|\partial_\eta \Phi\|_{L_2(\Omega)}^{\frac{q-2}{q}} \\
&= C_6 \left(\|\partial_\eta \Phi\|^2_{L_2(\Omega)} + \|\nabla \partial_\eta \Phi\|^2_{L_2(\Omega)} \right)^{\frac{q+2}{2q}} \|\partial_\eta \Phi\|_{L_2(\Omega)}^{\frac{q-2}{q}}.
\end{aligned} \tag{B.9}$$

From the inequality (B.6), it follows that

$$\|\nabla \partial_\eta \Phi\|^2_{L_2(\Omega)} \leq \frac{1}{m(\rho)} \int_\Omega \left(\left| (\zeta_{x_1}, \zeta_{x_2}) - \eta\, \zeta_\eta \, \frac{\nabla v}{1+v} \right|^2 + \left(\frac{\zeta_\eta}{1+v} \right)^2 \right) d(x,\eta). \tag{B.10}$$

Inserting this into (B.9) and using Young's inequality yields

$$\begin{aligned}
-\frac{\varepsilon^2}{2} &\int_D \left(\frac{2|\nabla v|^2 - \Delta v\,(1+v)}{(1+v)^2} \right) (\partial_\eta \Phi(x,1))^2 \, dx \\
&\leq C_7 \, \|\partial_\eta \Phi\|^2_{L_2(\Omega)} + \frac{\varepsilon^2}{4} \int_\Omega \left| (\zeta_{x_1}, \zeta_{x_2}) - \eta\, \zeta_\eta \, \frac{\nabla v}{1+v} \right|^2 d(x,\eta) \\
&\quad + \frac{1}{4} \int_\Omega \left(\frac{\zeta_\eta}{1+v} \right)^2 d(x,\eta).
\end{aligned} \tag{B.11}$$

To estimate the second integral on the right-hand side of (B.8), we again use (B.2), Hölder's inequality, and the Gagliardo-Nirenberg inequality [70, Theorem 1], since Ω is a bounded Lipschitz domain. We get

$$\begin{aligned}
\frac{\varepsilon^2}{2} &\int_\Omega \left(\frac{2|\nabla v|^2 - \Delta v\,(1+v)}{(1+v)^2} \right) (\partial_\eta \Phi)^2 \, d(x,\eta) \\
&\leq \frac{\varepsilon^2}{2\rho^2} \left\| (2|\nabla v|^2 - \Delta v(1+v))(\partial_\eta \Phi)^2 \right\|_{L_1(\Omega)}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\varepsilon^2}{2\rho^2}\left(2\|\nabla v\|_{L_\infty(D)}\|\nabla v\|_{L_q(D)} + \|1+v\|_{L_\infty(D)}\|\Delta v\|_{L_q(D)}\right)\|\partial_\eta \Phi\|^2_{L_{2q'}(\Omega)}\\
&\leq C_8\,\|\partial_\eta\Phi\|^{\frac{3}{q}}_{W_2^1(\Omega)}\,\|\partial_\eta\Phi\|^{\frac{2q-3}{q}}_{L_2(\Omega)}\\
&= C_8\left(\|\partial_\eta\Phi\|^2_{L_2(\Omega)} + \|\nabla\partial_\eta\Phi\|^2_{L_2(\Omega)}\right)^{\frac{3}{2q}}\|\partial_\eta\Phi\|^{\frac{2q-3}{q}}_{L_2(\Omega)}.
\end{aligned}$$

Then, arguing as above, we conclude that

$$\begin{aligned}
\frac{\varepsilon^2}{2}\int_\Omega &\left(\frac{2|\nabla v|^2 - \Delta v\,(1+v)}{(1+v)^2}\right)(\partial_\eta\Phi)^2\,d(x,\eta)\\
&\leq C_9\,\|\partial_\eta\Phi\|^2_{L_2(\Omega)} + \frac{\varepsilon^2}{4}\int_\Omega\left|(\zeta_{x_1},\zeta_{x_2}) - \eta\,\zeta_\eta\,\frac{\nabla v}{1+v}\right|^2 d(x,\eta) \qquad \text{(B.12)}\\
&\quad + \frac{1}{4}\int_\Omega\left(\frac{\zeta_\eta}{1+v}\right)^2 d(x,\eta).
\end{aligned}$$

From Hölder's and Young's inequalities and (B.2), it follows that

$$-\int_\Omega \zeta_\eta\,F\,d(x,\eta) \leq \|(1+v)\,F\|_{L_2(\Omega)}\left\|\frac{\zeta_\eta}{1+v}\right\|_{L_2(\Omega)} \leq C_{10}\|F\|^2_{L_2(\Omega)} + \frac{1}{4}\int_\Omega\left(\frac{\zeta_\eta}{1+v}\right)^2 d(x,\eta).$$

This, combined with (B.8), (B.11), and (B.12) implies

$$\int_\Omega\left(\varepsilon^2\left|(\zeta_{x_1},\zeta_{x_2}) - \eta\,\zeta_\eta\,\frac{\nabla v}{1+v}\right|^2 + \left(\frac{\zeta_\eta}{1+v}\right)^2\right)d(x,\eta) \leq C_{11}\left(\|F\|^2_{L_2(\Omega)} + \|\partial_\eta\Phi\|^2_{L_2(\Omega)}\right),$$

and since

$$\|\nabla\partial_\eta\Phi\|^2_{L_2(\Omega)} \leq \frac{1}{m(\rho,\varepsilon)}\int_\Omega\left(\varepsilon^2\left|(\zeta_{x_1},\zeta_{x_2}) - \eta\,\zeta_\eta\,\frac{\nabla v}{1+v}\right|^2 + \left(\frac{\zeta_\eta}{1+v}\right)^2\right)d(x,\eta)$$

according to (B.6), we have

$$\|\nabla\partial_\eta\Phi\|^2_{L_2(\Omega)} \leq C_{12}\left(\|F\|^2_{L_2(\Omega)} + \|\partial_\eta\Phi\|^2_{L_2(\Omega)}\right).$$

Thus,

$$\|\partial_\eta\Phi\|^2_{W_2^1(\Omega)} \leq (1+C_{12})\left(\|F\|^2_{L_2(\Omega)} + \|\partial_\eta\Phi\|^2_{L_2(\Omega)}\right),$$

and together with (B.7), this gives (B.4).

We have just shown that, if $v \in \overline{S}_q(\rho)\cap W^2_\infty(D)$, then $\Phi \in X(\Omega)$ and (B.4) holds. Let now $v \in \overline{S}_q(\rho)$. Since $W^2_\infty(D)$ is dense in $W^2_q(D)$, there exists a sequence $(v_n)_n$ in $W^2_\infty(D)$ that converges to v in $W^2_q(D)$ as $n\to\infty$, and since $W^2_q(D)\hookrightarrow C^1(\overline{D})$, we may assume that $v_n \in \overline{S}_q(\frac{1+\rho}{2})$. Then

$$\big(a_{ij}(v_n), b_i(v_n)\big) \longrightarrow \big(a_{ij}(v), b_i(v)\big) \quad \text{in } \ C(\overline{\Omega})$$

for all $1 \leq i, j \leq 3$. Denoting the solution to (B.1) with v_n instead of v by Φ_n, it follows from the above discussion that $\Phi_n \in X(\Omega)$ satisfies

$$\|\Phi_n\|_{W_2^1(\Omega)} + \|\partial_\eta \Phi_n\|_{W_2^1(\Omega)} \leq C_3 \, \|F\|_{L_2(\Omega)}.$$

Since $X(\Omega)$ is a Hilbert space, there exists a subsequence, again denoted by $(\Phi_n)_n$, and a $\Psi \in X(\Omega)$ such that

$$\Phi_n \rightharpoonup \Psi \quad \text{in } \ X(\Omega).$$

Hence, $\|\Psi\|_{X(\Omega)} \leq \liminf_{n\to\infty} \|\Phi_n\|_{X(\Omega)} \leq C_3 \, \|F\|_{L_2(\Omega)}$. Combining the previous convergences and letting $n \to \infty$ in the weak formulation for $-\mathcal{L}_{v_n}\Phi_n = F$ shows that Ψ is a weak solution to (B.1). Consequently, thanks to Lemma B.1.1, Ψ coincides with the unique weak solution Φ to (B.1). ■

To show the $W_2^2(\Omega)$-regularity of our weak solution Φ, we need to assume higher regularity of v.

Lemma B.1.3 (Improved regularity: step 2)
Let $\rho \in (0,1)$ and $q \in [3,\infty]$. For each $v \in \overline{S}_q(\rho)$ and $F \in L_2(\Omega)$, the weak solution $\Phi \in W_{2,B}^1(\Omega)$ to (B.1) belongs to $W_2^2(\Omega)$, and there is a constant $C_{13} > 0$ such that

$$\|\Phi\|_{W_2^2(\Omega)} \leq C_{13} \|F\|_{L_2(\Omega)}. \tag{B.13}$$

Proof. Let $\Phi \in W_{2,B}^1(\Omega)$ be the weak solution to (B.1). For $(x,\eta) \in \Omega$, we set

$$\begin{aligned} J(x,\eta) := 2\varepsilon^2 \eta \, \frac{\nabla v(x)}{1+v(x)} \cdot \nabla' \partial_\eta \Phi(x,\eta) + \left(1 - \frac{1+\varepsilon^2\eta^2|\nabla v(x)|^2}{(1+v(x))^2}\right) \partial_\eta^2 \Phi(x,\eta) \\ - \varepsilon^2 \eta \left(2\frac{|\nabla v(x)|^2}{(1+v(x))^2} - \frac{\Delta v(x)}{1+v(x)}\right) \partial_\eta \Phi(x,\eta). \end{aligned}$$

Then, Φ is also a weak solution of

$$\begin{cases} \varepsilon^2 \Delta' \Phi + \partial_\eta^2 \Phi = J - F & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to Lemma B.1.2, (B.2), and, since $W_q^2(D) \hookrightarrow C^1(\overline{D})$ and $W_2^1(\Omega) \hookrightarrow L_6(\Omega)$, it follows that $J \in L_2(\Omega)$ with

$$\|J\|_{L_2(\Omega)} \leq C_{14} \, \|\partial_\eta \Phi\|_{W_2^1(\Omega)} \leq C_{15} \|F\|_{L_2(\Omega)}. \tag{B.14}$$

Applying [28, Theorem 3.2.1.2], we conclude that $\Phi \in W_{2,B}^2(\Omega)$. Moreover, [28, Theorem

3.1.3.1 and Lemma 3.2.1.1], together with an inspection of the proof of [28, Theorem 3.2.1.2], yield the result that there is a positive constant C_{16} such that

$$\|\Phi\|_{W_2^2(\Omega)} \leq C_{16}\|J - F\|_{L_2(\Omega)},$$

and, hence, by (B.14), we get (B.13). ∎

B.2 Proof of Theorem 3.2.1

The next result states that, for a given displacement $u(t)$, the transformed problem (3.11)-(3.12) on the fixed domain Ω has a unique solution $\phi_{u(t)} \in W_2^2(\Omega)$.

Proposition B.2.1 *Let $\rho \in (0,1)$ and $q \in [3,\infty]$. For each $v \in \overline{S}_q(\rho)$, there exists a unique solution $\phi_v \in W_2^2(\Omega)$ to the boundary value problem*

$$\begin{cases} (\mathcal{L}_v\phi_v)(x,\eta) = 0, & (x,\eta) \in \Omega, \\ \phi_v(x,\eta) = \eta, & (x,\eta) \in \partial\Omega. \end{cases} \tag{B.15}$$

Proof. Setting

$$f_v(x,\eta) := \mathcal{L}_v\eta = \varepsilon^2\eta\left(2\frac{|\nabla v(x)|^2}{(1+v(x))^2} - \frac{\Delta v(x)}{1+v(x)}\right), \quad (x,\eta) \in \Omega, \tag{B.16}$$

it follows from the continuous embedding $W_q^2(D) \hookrightarrow C^1(\overline{D})$ and (B.2) that the function f_v belongs to $L_2(\Omega)$ with

$$\|f_v\|_{L_2(\Omega)} \leq C_{17}. \tag{B.17}$$

So, by Lemmas B.1.1 and B.1.3, we deduce that the problem

$$\begin{cases} \mathcal{L}_v\Phi_v = f_v & \text{in } \Omega, \\ \Phi_v = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $\Phi_v \in W_{2,B}^2(\Omega)$ such that

$$\|\Phi_v\|_{W_2^2(\Omega)} \leq C_{13}\|f_v\|_{L_2(\Omega)}, \tag{B.18}$$

and, hence, the function

$$\phi_v(x,\eta) := \Phi_v(x,\eta) + \eta, \quad (x,\eta) \in \overline{\Omega},$$

solves (B.15). Moreover, it follows from (B.17) and (B.18) that

$$\|\phi_v\|_{W_2^2(\Omega)} \leq \|\Phi_v\|_{W_2^2(\Omega)} + \|\eta\|_{W_2^2(\Omega)} \leq C_{18}. \tag{B.19}$$

∎

We next show that ϕ_v depends Lipschitz continuously on $v \in \overline{S}_q(\rho)$.

Proposition B.2.2 *Let $\rho \in (0,1)$ and $q \in [3,\infty]$. Then there exists a constant $C_{19} > 0$ such that, for all $v_1, v_2 \in \overline{S}_q(\rho)$,*

$$\|\phi_{v_1} - \phi_{v_2}\|_{W_2^2(\Omega)} \leq C_{19}\|v_1 - v_2\|_{W_q^2(D)}. \tag{B.20}$$

Proof. For $v \in \overline{S}_q(\rho)$, we define a second order linear operator $\mathcal{A}(v) \in \mathcal{L}(W_{2,B}^2(\Omega), L_2(\Omega))$ by setting

$$\mathcal{A}(v)w := -\mathcal{L}_v w, \quad w \in W_{2,B}^2(\Omega).$$

Thanks to Lemma B.1.3, $\mathcal{A}(v)$ is invertible and its inverse $\mathcal{A}(v)^{-1} \in \mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))$ satisfies

$$\|\mathcal{A}(v)^{-1}\|_{\mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))} \leq C_{13}. \tag{B.21}$$

For non-zero $w \in W_{2,B}^2(\Omega)$ and $v_1, v_2 \in \overline{S}_q(\rho)$, we get, by Hölder's inequality and the continuity of the map $L_q(\Omega) \cdot W_2^1(\Omega) \hookrightarrow L_2(\Omega)$ with embedding constant, say, $C > 0$,

$$\begin{aligned}
&\frac{\|\mathcal{A}(v_1)w - \mathcal{A}(v_2)w\|_{L_2(\Omega)}}{\|w\|_{W_2^2(\Omega)}} \\
&\quad \leq 2\varepsilon^2 \left\| \frac{\nabla v_1}{1+v_1} - \frac{\nabla v_2}{1+v_2} \right\|_{L_\infty(D)} + \left\| \frac{1+\varepsilon^2\eta^2|\nabla v_1|^2}{(1+v_1)^2} - \frac{1+\varepsilon^2\eta^2|\nabla v_2|^2}{(1+v_2)^2} \right\|_{L_\infty(\Omega)} \\
&\qquad + 2\varepsilon^2 \left\| \frac{|\nabla v_1|^2}{(1+v_1)^2} - \frac{|\nabla v_2|^2}{(1+v_2)^2} \right\|_{L_\infty(D)} + C\varepsilon^2 \left\| \frac{\Delta v_1}{1+v_1} - \frac{\Delta v_2}{1+v_2} \right\|_{L_q(D)}.
\end{aligned}$$

In view of property (B.2) and the continuous embedding of $W_q^1(D)$ in $L_\infty(D)$, we observe that the terms on the right-hand side can be bounded by a positive constant, depending only on ρ, ε, q, and D, times $\|v_1 - v_2\|_{W_q^2(D)}$. Thus,

$$\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\|_{\mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))} \leq C_{20}\|v_1 - v_2\|_{W_q^2(D)},$$

and, hence, due to (B.21) and the second resolvent identity, i.e.,

$$\mathcal{A}(v_1)^{-1} - \mathcal{A}(v_2)^{-1} = \mathcal{A}(v_1)^{-1}\left(\mathcal{A}(v_2) - \mathcal{A}(v_1)\right)\mathcal{A}(v_2)^{-1}, \quad v_1, v_2 \in \overline{S}_q(\rho),$$

we have

$$\begin{aligned}
&\|\mathcal{A}(v_1)^{-1} - \mathcal{A}(v_2)^{-1}\|_{\mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))} \\
&\leq \|\mathcal{A}(v_1)^{-1}\|_{\mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))}\|\mathcal{A}(v_2) - \mathcal{A}(v_1)\|_{\mathcal{L}(W_{2,B}^2(\Omega), L_2(\Omega))}\|\mathcal{A}(v_2)^{-1}\|_{\mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))} \\
&\leq C_{13}^2\,\|\mathcal{A}(v_2) - \mathcal{A}(v_1)\|_{\mathcal{L}(W_{2,B}^2(\Omega), L_2(\Omega))} \\
&\leq C_{13}^2\,C_{20}\,\|v_1 - v_2\|_{W_q^2(D)}.
\end{aligned} \tag{B.22}$$

By similar arguments, one finds that

$$\|f_{v_1} - f_{v_2}\|_{L_2(\Omega)} \leq C_{21}\,\|v_1 - v_2\|_{W_q^2(D)}, \quad v_1, v_2 \in \overline{S}_q(\rho),$$

where f_v is defined in (B.16). This estimate, together with (B.17), (B.21), and (B.22), implies

$$\begin{aligned}\|\phi_{v_1} - \phi_{v_2}\|_{W_2^2(\Omega)} &= \|\Phi_{v_1} - \Phi_{v_2}\|_{W_2^2(\Omega)} = \|\mathcal{A}(v_1)^{-1} f_{v_1} - \mathcal{A}(v_2)^{-1} f_{v_2}\|_{W_2^2(\Omega)} \\ &\leq \|(\mathcal{A}(v_1)^{-1} - \mathcal{A}(v_2)^{-1}) f_{v_1}\|_{W_2^2(\Omega)} + \|\mathcal{A}(v_2)^{-1}(f_{v_1} - f_{v_2})\|_{W_2^2(\Omega)} \\ &\leq C_{22}\,\|v_1 - v_2\|_{W_q^2(D)},\end{aligned}$$

for $v_1, v_2 \in \overline{S}_q(\rho)$. ■

Next, we prove that the transformed right-hand side g_ε depends analytically and Lipschitz continuously on v.

Proposition B.2.3 *Let $\rho \in (0,1)$, $q \in [3,\infty]$ and $v \in \overline{S}_q(\rho)$. Let $\phi_v \in W_2^2(\Omega)$ be the associated unique solution to (B.15). Then the mapping $g_\varepsilon : S_q(\rho) \to L_2(D)$ defined by*

$$g_\varepsilon(v) = \frac{1 + \varepsilon^2|\nabla v|^2}{(1+v)^2}\,(\partial_\eta \phi_v(\cdot,1))^2$$

is analytic, bounded, and globally Lipschitz continuous.

Proof. Since Ω is a bounded Lipschitz domain, it follows from [69, Theorem II.5.5] that, for $v \in \overline{S}_q(\rho)$,

$$\|\partial_\eta \phi_v(\cdot,1)\|_{W_2^{1/2}(D)} \leq C\|\phi_v\|_{W_2^2(\Omega)}, \tag{B.23}$$

which combined with (B.20), gives

$$\|\partial_\eta \phi_{v_1}(\cdot,1) - \partial_\eta \phi_{v_2}(\cdot,1)\|_{W_2^{1/2}(D)} \leq C_{19}\|v_1 - v_2\|_{W_q^2(D)},$$

for $v_1, v_2 \in \overline{S}_q(\rho)$, and, hence, the mapping

$$\overline{S}_q(\rho) \to W_2^{1/2}(D), \quad v \mapsto \partial_\eta \phi_v(\cdot,1)$$

is globally Lipschitz continuous. We infer from the continuity of pointwise multiplication

$$W_2^{1/2}(D) \cdot W_2^{1/2}(D) \hookrightarrow L_2(D), \tag{B.24}$$

guaranteed by $W_2^{1/2}(D) \hookrightarrow L_4(D)$, that

$$\overline{S}_q(\rho) \to L_2(D), \quad v \mapsto (\partial_\eta \phi_v(\cdot,1))^2$$

is globally Lipschitz continuous. Using the continuity of the pointwise multiplications

$$W_q^1(D) \cdot W_q^1(D) \hookrightarrow W_q^1(D), \quad W_q^1(D) \cdot W_\infty^1(D) \hookrightarrow W_q^1(D)$$

(see e.g. [4, Theorem 2.1]), one finds that the mapping

$$\overline{S}_q(\rho) \to W_q^1(D), \quad v \mapsto \frac{1+\varepsilon^2|\nabla v|^2}{(1+v)^2}$$

is globally Lipschitz continuous. Since

$$W_q^1(D) \cdot L_2(D) \hookrightarrow L_2(D)$$

(due to $W_q^1(D) \hookrightarrow L_\infty(D)$), one obtains the global Lipschitz continuity of g_ε. Together with the fact that $0 \in S_q(\rho)$, this gives

$$\begin{aligned}\|g_\varepsilon(v)\|_{L_2(D)} &\le \|g_\varepsilon(v) - g_\varepsilon(0)\|_{L_2(D)} + \|g_\varepsilon(0)\|_{L_2(D)} \\ &\le C_L \, \|v\|_{W_q^2(D)} + \|(\partial_\eta \phi_0(\cdot,1))^2\|_{L_2(D)}\end{aligned}$$

for $v \in \overline{S}_q(\rho)$. It follows from (B.19), (B.23), and (B.24) that

$$\|(\partial_\eta \phi_0(\cdot,1))^2\|_{L_2(D)} \le C_{23}$$

and hence,

$$\|g_\varepsilon(v)\|_{L_2(D)} \le \frac{C_L}{\rho} + C_{23} =: C_B \quad \text{for all } v \in \overline{S}_q(\rho).$$

To prove that g_ε is analytic, we observe that $S_q(\rho)$ is open in $W_{q,B}^2(D)$ and that the mappings

$$\mathcal{A} : S_q(\rho) \to \mathcal{L}(W_{2,B}^2(\Omega), L_2(\Omega)) \quad \text{and} \quad [v \mapsto f_v] : S_q(\rho) \to L_2(\Omega)$$

are analytic. Noting that the map taking an invertible operator to its inverse is analytic on the space of bounded operators, we deduce that $\left[v \mapsto \mathcal{A}(v)^{-1}\right] : S_q(\rho) \to \mathcal{L}(L_2(\Omega), W_{2,B}^2(\Omega))$ is analytic, and thus $[v \mapsto \phi_v] : S_q(\rho) \to W_2^2(\Omega)$ is analytic. This and the above results on pointwise multiplication imply that $g_\varepsilon : S_q(\rho) \to L_2(D)$ is analytic.

■

Appendix C

An alternative proof of Theorem 8.2.1

This appendix presents an alternative proof for the nonexistence of stationary solutions; it is based on an auxiliary oblique derivative problem.

Theorem C.0.1 *Suppose that $D \subset \mathbb{R}^2$ is a bounded convex domain with $\partial D \in C^{4,\gamma}$ for some $\gamma \in (0,1)$. There are $\varepsilon_* > 0$ and a function $\Lambda : (0, \varepsilon_*) \to (0, \infty)$ such that there is no stationary solution (u, ψ_u) to (3.1)-(3.5) for $\varepsilon \in (0, \varepsilon_*)$ and $\lambda \geq \Lambda(\varepsilon)$.*

The idea of the proof comes from Laurençot and Walker [48].

Proof. Let (u, ψ_u) be a stationary solution to (3.1)-(3.5) with $u \in W^4_{2,B}(D)$, $\psi_u \in W^2_2(\Omega(u))$, and $u(x) > -1$ for $x \in D$. Set

$$\gamma_e(x) := \partial_z \psi_u(x, u(x)) \quad \text{and} \quad G(x) := \left(1 + \varepsilon^2 |\nabla u(x)|^2\right) \gamma_e(x)^2$$

for $x \in D$. Then u solves

$$\beta \Delta^2 u - \tau \Delta u = -\lambda G \quad \text{in } D \tag{C.1}$$

with hinged boundary conditions (8.4), and by the nonnegativity of G and the positivity preserving property one finds that

$$-1 < u \leq 0 \quad \text{in } D. \tag{C.2}$$

Let φ_1 be the positive eigenfunction of $\beta\Delta^2 - \tau\Delta$ in $C^{4,\gamma}_B(\overline{D}) := C^{4,\gamma}(\overline{D}) \cap W^4_{2,B}(D)$ satisfying $\partial_\nu \varphi_1 < 0$ on ∂D and normalized by $\|\varphi_1\|_{L_1(D)} = 1$, and let $\mu_1 > 0$ be the corresponding eigenvalue; see Chapter 7.

Let us now modify the calculations in [48]. We begin with the following inequality:

Lemma C.0.2 *One has*

$$\int_D \frac{\varphi_1}{1+u}\, dx \leq \int_{\Omega(u)} \varphi_1 (\partial_z \psi_u)^2 \, d(x,z).$$

Proof. Let $x \in D$, it follows from the boundary conditions for ψ_u and Hölder's inequality that

$$1 = \psi_u(x, u(x)) - \psi_u(x, -1)$$

$$= \int_{-1}^{u(x)} \partial_z \psi_u(x,z)\, dz \leq \left(\int_{-1}^{u(x)} |\partial_z \psi_u(x,z)|^2\, dz \right)^{1/2} \sqrt{1+u(x)}\,,$$

and since $u > -1$ in D,

$$\frac{1}{1+u(x)} \leq \int_{-1}^{u(x)} |\partial_z \psi_u(x,z)|^2\, dz, \quad x \in D. \tag{C.3}$$

Multiplying both sides of (C.3) with φ_1 and integrating over D, we obtain the assertion. ■

The next result is a consequence of (8.1)-(8.2).

Lemma C.0.3 *One has*

$$\begin{aligned}
&\int_D \varphi_1 \left(1+\varepsilon^2 |\nabla u|^2\right) \gamma_e\, dx \\
&\quad = \int_{\Omega(u)} \varphi_1 \left(\varepsilon^2 |\nabla' \psi_u|^2 + (\partial_z \psi_u)^2\right) d(x,z) - \frac{\varepsilon^2}{2} \int_{\Omega(u)} \psi_u^2\, \Delta \varphi_1\, d(x,z) \\
&\quad + \frac{\varepsilon^2}{2} \int_D u\, \Delta \varphi_1\, dx + \frac{\varepsilon^2}{6} \int_D \Delta \varphi_1\, dx.
\end{aligned}$$

Proof. We multiply (8.1) by the function $\varphi_1 \psi_u$ and integrate over $\Omega(u)$ to get

$$0 = \varepsilon^2 \int_{\Omega(u)} \left(\Delta' \psi_u\right) \varphi_1 \psi_u\, d(x,z) + \int_{\Omega(u)} (\partial_z^2 \psi_u)\, \varphi_1 \psi_u\, d(x,z).$$

Integrating the last integral by parts and using the boundary conditions both for φ_1 and ψ_u, yields

$$0 = \varepsilon^2 \int_{\Omega(u)} \left(\Delta' \psi_u\right) \varphi_1 \psi_u\, d(x,z) - \int_{\Omega(u)} \varphi_1 (\partial_z \psi_u)^2\, d(x,z) + \int_D \varphi_1\, \partial_z \psi_u(\cdot,u)\, dx,$$

which can be rewritten as

$$\begin{aligned}
0 &= \varepsilon^2 \int_{\Omega(u)} \varphi_1\, \mathrm{div}'(\psi_u \nabla' \psi_u)\, d(x,z) - \int_{\Omega(u)} \varphi_1 \left(\varepsilon^2 |\nabla' \psi_u|^2 + (\partial_z \psi_u)^2\right) d(x,z) \\
&+ \int_D \varphi_1\, \partial_z \psi_u(\cdot,u)\, dx.
\end{aligned}$$

Again, using integration by parts on the first integral, $\varphi_1|_{\partial D} = 0$, and $\psi_u = 1$ on $\mathfrak{G}_u = \{(x,u(x))\,;\, x \in D\}$, we find

$$\begin{aligned}
0 &= -\varepsilon^2 \int_{\Omega(u)} \psi_u\, \nabla' \psi_u \cdot \nabla \varphi_1\, d(x,z) - \int_{\Omega(u)} \varphi_1 \left(\varepsilon^2 |\nabla' \psi_u|^2 + (\partial_z \psi_u)^2\right) d(x,z) \\
&- \varepsilon^2 \int_D \varphi_1\, \nabla' \psi_u(\cdot,u) \cdot \nabla u\, dx + \int_D \varphi_1\, \partial_z \psi_u(\cdot,u)\, dx,
\end{aligned}$$

and, due to (see (8.6))

$$\nabla'\psi_u(x,u(x)) = -\nabla u(x)\,\gamma_e(x), \quad x\in D,$$

we have

$$0 = -\varepsilon^2\int_{\Omega(u)}\psi_u\,\nabla'\psi_u\cdot\nabla\varphi_1\,d(x,z) - \int_{\Omega(u)}\varphi_1\left(\varepsilon^2|\nabla'\psi_u|^2+(\partial_z\psi_u)^2\right)d(x,z) + \int_D\varphi_1\left(1+\varepsilon^2|\nabla u|^2\right)\gamma_e\,dx. \tag{C.4}$$

Since $\psi_u\,\nabla'\psi_u = \frac{1}{2}\nabla'(\psi_u^2)$, an integration by parts in the first integral gives

$$-\varepsilon^2\int_{\Omega(u)}\psi_u\,\nabla'\psi_u\cdot\nabla\varphi_1\,d(x,z) = \frac{\varepsilon^2}{2}\int_{\Omega(u)}\psi_u^2\,\Delta\varphi_1\,d(x,z)+\frac{\varepsilon^2}{2}\int_D\nabla\varphi_1\cdot\nabla u\,dx - \frac{\varepsilon^2}{6}\int_{\partial D}\partial_\nu\varphi_1\,d\omega,$$

where we used (8.2). Furthermore,

$$-\frac{\varepsilon^2}{6}\int_{\partial D}\partial_\nu\varphi_1\,d\omega = -\frac{\varepsilon^2}{6}\int_D\Delta\varphi_1\,dx,$$

and

$$\frac{\varepsilon^2}{2}\int_D\nabla\varphi_1\cdot\nabla u\,dx = -\frac{\varepsilon^2}{2}\int_D u\,\Delta\varphi_1\,dx.$$

Hence,

$$-\varepsilon^2\int_{\Omega(u)}\psi_u\,\nabla'\psi_u\cdot\nabla\varphi_1\,d(x,z) = \frac{\varepsilon^2}{2}\int_{\Omega(u)}\psi_u^2\,\Delta\varphi_1\,d(x,z)-\frac{\varepsilon^2}{2}\int_D u\,\Delta\varphi_1\,dx - \frac{\varepsilon^2}{6}\int_D\Delta\varphi_1\,dx.$$

Inserting this in (C.4) yields the desired result. ∎

We next consider the auxiliary boundary value problem

$$\begin{cases} -\Delta U + c\,U = u & \text{in } D,\\ \nabla\varphi_1\cdot\nabla U = 0 & \text{on } \partial D,\end{cases} \tag{C.5}$$

where $c\in(2,\infty)$ is a constant coefficient. Since $\partial_\nu\varphi_1<0$ on ∂D, $\nabla\varphi_1$ is nowhere tangential to ∂D, that is, (C.5) is an oblique derivative problem (see [59]). Then, according to [28, Theorem 2.4.2.7], (C.5) admits a unique solution $U\in W_2^2(D)$. Since $\partial D\in C^{4,\gamma}$ and $\varphi_1\in C^{4,\gamma}(\overline{D})$, [28, Remark 2.5.1.2] implies that U lies in $W_2^4(D)$.

Since $\Delta U - cU = -u \geq 0$ in D, it follows from the maximum principle (see [59, Lemma 1.6] or [35, Corollary 11.2.2]) that $U \leq 0$ in D. Similarly, since $u > -1$ in D, we get that $-\frac{1}{2} \leq U$ in D. Hence,

$$-1 < -\frac{1}{2} \leq U \leq 0 \quad \text{in } D. \tag{C.6}$$

Next, taking into account that $\partial_{x_1}\varphi_1|_{\partial D} = \nu_1\partial_\nu\varphi_1 + s_1\partial_s\varphi_1$, $\partial_{x_2}\varphi_1|_{\partial D} = \nu_2\partial_\nu\varphi_1 + s_2\partial_s\varphi_1$, where s represents the counterclockwise oriented tangent vector on ∂D, and that $\partial_s\varphi_1|_{\partial D} = 0$ due to $\varphi_1 = 0$ on ∂D, we deduce that

$$0 = \nabla\varphi_1 \cdot \nabla U = \partial_\nu\varphi_1\, \partial_\nu U \quad \text{on } \partial D. \tag{C.7}$$

As $\partial_\nu\varphi_1 < 0$ on ∂D, (C.7) implies that $\partial_\nu U = 0$ on ∂D, and since D is convex, we can apply [28, Theorem 3.1.1.1] (or [27, Theorem 2.1]) to obtain

$$\int_D |\Delta U|^2\, dx \geq \int_D |D^2U|^2\, dx, \tag{C.8}$$

where $|D^2U|^2 := (\partial^2_{x_1}U)^2 + (\partial^2_{x_2}U)^2 + 2(\partial_{x_2}\partial_{x_1}U)^2$. An integration by parts, together with (C.5) and $\partial_\nu U = 0$ on ∂D, shows that

$$\int_D u\, U\, dx = \int_D (-\Delta U + cU)\, U\, dx = \int_D |\nabla U|^2\, dx + c\int_D |U|^2\, dx,$$

and applying Hölder's inequality we infer that

$$\int_D |\nabla U|^2\, dx + c\int_D |U|^2\, dx \leq \|u\|_{L_2(D)}\, \|U\|_{L_2(D)}.$$

Thus,

$$\|U\|_{L_2(D)} \leq \frac{1}{c}\, \|u\|_{L_2(D)}, \tag{C.9}$$

and

$$\|\nabla U\|^2_{L_2(D)} \leq \frac{1}{c}\, \|u\|^2_{L_2(D)}. \tag{C.10}$$

Combining (C.8)-(C.10), we get

$$\|U\|^2_{W^2_2(D)} \leq \left(\frac{1}{c^2} + \frac{1}{c}\right) \|u\|^2_{L_2(D)} + \|\Delta U\|^2_{L_2(D)}. \tag{C.11}$$

Thanks to Young's inequality, (C.5), and (C.9),

$$\|\Delta U\|^2_{L_2(D)} = \|\Delta U - cU + cU\|^2_{L_2(D)} \leq 2\left(\|\Delta U - cU\|^2_{L_2(D)} + \|cU\|^2_{L_2(D)}\right) \leq 4\, \|u\|^2_{L_2(D)},$$

and hence (C.11) entails

$$\|U\|_{W_2^2(D)} \le \left(\frac{1}{c^2} + \frac{1}{c} + 4\right)^{1/2} \|u\|_{L_2(D)}. \tag{C.12}$$

Let $\alpha \in (0,1]$ be a constant which will be determined later. We now multiply (C.1) by the function $\varphi_1 (1+\alpha U)$ and integrate over D. Three integrations by parts, and using $\varphi_1|_{\partial D} = 0$ and $u|_{\partial D} = 0$, yield

$$\begin{aligned}
\lambda \int_D (1+\alpha U)\varphi_1 \, G \, dx
&= \int_D (1+\alpha U)\, \varphi_1 \left(-\beta\Delta^2 u + \tau\Delta u\right) dx \\
&= \beta \int_D \nabla\big((1+\alpha U)\,\varphi_1\big) \cdot \nabla\Delta u \, dx - \tau \int_D \nabla\big((1+\alpha U)\,\varphi_1\big) \cdot \nabla u \, dx \\
&= \beta \int_{\partial D} (1+\alpha U)\, \partial_\nu\varphi_1 \, \Delta u \, d\omega - \beta \int_D \Delta\big((1+\alpha U)\,\varphi_1\big)\, \Delta u \, dx \\
&\quad + \tau \int_D \Delta\big((1+\alpha U)\,\varphi_1\big)\, u \, dx \\
&= \beta \int_{\partial D} (1+\alpha U)\, \partial_\nu\varphi_1 \, \Delta u \, d\omega - \beta \int_{\partial D} \Delta\big((1+\alpha U)\,\varphi_1\big)\, \partial_\nu u \, d\omega \\
&\quad + \beta \int_D \nabla\Delta\big((1+\alpha U)\,\varphi_1\big) \cdot \nabla u \, dx + \tau \int_D \Delta\big((1+\alpha U)\,\varphi_1\big)\, u \, dx.
\end{aligned}$$

We note that

$$\Delta\big((1+\alpha U)\,\varphi_1\big) = \alpha\varphi_1\Delta U + 2\alpha\nabla U \cdot \nabla\varphi_1 + (1+\alpha U)\Delta\varphi_1. \tag{C.13}$$

This, together with (C.5), integration by parts, and $u|_{\partial D} = 0$, implies

$$\begin{aligned}
\beta \int_D \nabla\Delta\big((1+\alpha U)\,\varphi_1\big) \cdot \nabla u \, dx
&= \alpha\beta c \int_D \nabla(U\varphi_1) \cdot \nabla u \, dx - \alpha\beta \int_D \varphi_1 |\nabla u|^2 \, dx \\
&\quad - \alpha\beta \int_D u \, \nabla u \cdot \nabla\varphi_1 \, dx - \beta \int_D \Big(2\alpha\Delta(\nabla U \cdot \nabla\varphi_1) + \Delta\big((1+\alpha U)\,\Delta\varphi_1\big)\Big)\, u \, dx.
\end{aligned}$$

Again, by (C.5),

$$\begin{aligned}
\tau \int_D \Delta\big((1+\alpha U)\,\varphi_1\big)\, u \, dx
&= \tau \int_D \Big(\alpha c U\varphi_1 - \alpha\varphi_1 u + 2\alpha\nabla U \cdot \nabla\varphi_1 + (1+\alpha U)\Delta\varphi_1\Big)\, u \, dx.
\end{aligned}$$

So,

$$\begin{aligned}
&\lambda \int_D (1+\alpha U)\,\varphi_1\, G\, dx \\
&= \beta \int_{\partial D} (1+\alpha U)\,\partial_\nu \varphi_1\, \Delta u\, d\omega - \beta \int_{\partial D} \Delta\big((1+\alpha U)\,\varphi_1\big)\,\partial_\nu u\, d\omega \\
&\quad + \alpha\beta c \int_D \nabla(U\varphi_1)\cdot\nabla u\, dx - \alpha\beta \int_D \varphi_1 |\nabla u|^2\, dx - \alpha\beta \int_D u\,\nabla u\cdot\nabla\varphi_1\, dx \\
&\quad - \beta \int_D \Big(2\alpha\Delta(\nabla U\cdot\nabla\varphi_1) + \Delta\big((1+\alpha U)\,\Delta\varphi_1\big)\Big)\, u\, dx \qquad \text{(C.14)}\\
&\quad + \tau \int_D \Big(\alpha c U\varphi_1 - \alpha\varphi_1 u + 2\alpha\nabla U\cdot\nabla\varphi_1 + (1+\alpha U)\,\Delta\varphi_1\Big)\, u\, dx.
\end{aligned}$$

We can simplify the right-hand side of this equation by integration by parts. Taking into account that $u|_{\partial D} = 0$ and (C.5), we deduce

$$\begin{aligned}
\alpha\beta c \int_D \nabla(U\varphi_1)\cdot\nabla u\, dx &= -\alpha\beta c \int_D \Delta(U\varphi_1)\, u\, dx \\
&= -\alpha\beta c \int_D \Big(\varphi_1\Delta U + 2\nabla U\cdot\nabla\varphi_1 + U\Delta\varphi_1\Big)\, u\, dx \\
&= -\alpha\beta c^2 \int_D \varphi_1\, U\, u\, dx + \alpha\beta c \int_D \varphi_1\, u^2\, dx \\
&\quad - 2\alpha\beta c \int_D u\,\nabla U\cdot\nabla\varphi_1\, dx - \alpha\beta c \int_D U\, u\,\Delta\varphi_1\, dx, \qquad \text{(C.15)}
\end{aligned}$$

and

$$- \alpha\beta \int_D u\,\nabla u\cdot\nabla\varphi_1\, dx = -\frac{\alpha\beta}{2}\int_D \nabla(u^2)\cdot\nabla\varphi_1\, dx = \frac{\alpha\beta}{2}\int_D u^2\,\Delta\varphi_1\, dx. \qquad \text{(C.16)}$$

Furthermore, by (C.5) and (C.16), we find that

$$\begin{aligned}
&-2\alpha\beta \int_D \Delta(\nabla U\cdot\nabla\varphi_1)\, u\, dx \\
&= -2\alpha\beta \int_D \Big(\nabla\Delta U\cdot\nabla\varphi_1 + \nabla U\cdot\nabla\Delta\varphi_1 \\
&\qquad + 2\big(\partial^2_{x_1} U\,\partial^2_{x_1}\varphi_1 + \partial^2_{x_2} U\,\partial^2_{x_2}\varphi_1 + 2\partial_{x_2}\partial_{x_1} U\,\partial_{x_2}\partial_{x_1}\varphi_1\big)\Big)\, u\, dx \\
&= -2\alpha\beta c \int_D u\,\nabla U\cdot\nabla\varphi_1\, dx - \alpha\beta \int_D u^2\,\Delta\varphi_1\, dx - 2\alpha\beta \int_D u\,\nabla U\cdot\nabla\Delta\varphi_1\, dx \\
&\quad - 4\alpha\beta \int_D \big(\partial^2_{x_1} U\,\partial^2_{x_1}\varphi_1 + \partial^2_{x_2} U\,\partial^2_{x_2}\varphi_1 + 2\partial_{x_2}\partial_{x_1} U\,\partial_{x_2}\partial_{x_1}\varphi_1\big)\, u\, dx. \qquad \text{(C.17)}
\end{aligned}$$

Again by (C.5) we find that

$$
\begin{aligned}
&-\beta \int_D \Delta\big((1+\alpha U)\Delta\varphi_1\big)\, u\, dx \\
&\quad = -\beta \int_D \Big((1+\alpha U)\,\Delta^2\varphi_1 + \alpha\Delta U\,\Delta\varphi_1 + 2\alpha\nabla U\cdot\nabla\Delta\varphi_1\Big)\, u\, dx \\
&\quad = -\beta \int_D u\,(1+\alpha U)\,\Delta^2\varphi_1\, dx - \alpha\beta c \int_D U\, u\,\Delta\varphi_1\, dx + \alpha\beta \int_D u^2\,\Delta\varphi_1\, dx \\
&\qquad - 2\alpha\beta \int_D u\,\nabla U\cdot\nabla\Delta\varphi_1\, dx. \qquad\qquad (C.18)
\end{aligned}
$$

Moreover, using (C.5), (C.13), and the boundary conditions for both u and φ_1, yields

$$
\begin{aligned}
&\beta \int_{\partial D} (1+\alpha U)\,\partial_\nu\varphi_1\,\Delta u\, d\omega - \beta \int_{\partial D} \Delta\big((1+\alpha U)\,\varphi_1\big)\,\partial_\nu u\, d\omega \\
&\qquad = \beta \int_{\partial D} (1+\alpha U)\big(\partial_\nu\varphi_1\,\Delta u - \partial_\nu u\,\Delta\varphi_1\big)\, d\omega = 0. \qquad\qquad (C.19)
\end{aligned}
$$

Therefore, by combining (C.14)-(C.19), we get

$$
\begin{aligned}
&\lambda \int_D (1+\alpha U)\,\varphi_1\, G\, dx \\
&\quad = -\alpha c\,(c\beta-\tau) \int_D \varphi_1\, U\, u\, dx + \alpha\,(c\beta-\tau) \int_D \varphi_1\, u^2\, dx \\
&\qquad - 2\alpha\,(2\beta c-\tau) \int_D u\,\nabla U\cdot\nabla\varphi_1\, dx - 2\alpha\beta c \int_D U\, u\,\Delta\varphi_1\, dx \\
&\qquad - \alpha\beta \int_D \varphi_1|\nabla u|^2\, dx + \frac{\alpha\beta}{2} \int_D u^2\,\Delta\varphi_1\, dx - 4\alpha\beta \int_D u\,\nabla U\cdot\nabla\Delta\varphi_1\, dx \\
&\qquad - 4\alpha\beta \int_D \Big(\partial^2_{x_1}U\,\partial^2_{x_1}\varphi_1 + \partial^2_{x_2}U\,\partial^2_{x_2}\varphi_1 + 2\partial_{x_2}\partial_{x_1}U\,\partial_{x_2}\partial_{x_1}\varphi_1\Big)\, u\, dx \\
&\qquad - \mu_1 \int_D (1+\alpha U)\,\varphi_1\, u\, dx, \qquad\qquad (C.20)
\end{aligned}
$$

where we have used the fact that $(\beta\Delta^2 - \tau\Delta)\varphi_1 = \mu_1\varphi_1$ in D. Next we need to estimate the integrals on the right-hand side of (C.20). From (C.2), (C.6), and the positivity of φ_1, it follows that

$$
-\alpha c\,(c\beta-\tau) \int_D \varphi_1\, U\, u\, dx \le \alpha c\tau \int_D \varphi_1\, U\, u\, dx \le \frac{\alpha c\tau}{2}\,\|\varphi_1\|_{L_1(D)} = \frac{\alpha c\tau}{2}, \qquad (C.21)
$$

and

$$
\alpha\,(c\beta-\tau) \int_D \varphi_1\, u^2\, dx \le \alpha c\beta \int_D \varphi_1\, u^2\, dx < \alpha c\beta\,\|\varphi_1\|_{L_1(D)} = \alpha c\beta. \qquad (C.22)
$$

Similarly, we get

$$-2\alpha\beta c \int_D U\, u\, \Delta\varphi_1\, dx \leq \alpha\beta c\, \|\Delta\varphi_1\|_{L_1(D)} \leq 2\alpha\beta c\, |D|^{1/2}\, \|\varphi_1\|_{W_2^4(D)}, \tag{C.23}$$

and

$$-\mu_1 \int_D (1+\alpha U)\, \varphi_1\, u\, dx \leq -\mu_1 \int_D \varphi_1\, u\, dx < \mu_1\, \|\varphi_1\|_{L_1(D)} = \mu_1. \tag{C.24}$$

By (C.2), we see that

$$\frac{\alpha\beta}{2} \int_D u^2\, \Delta\varphi_1\, dx \leq \frac{\alpha\beta}{2}\, \|\Delta\varphi_1\|_{L_1(D)} \leq \alpha\beta\, |D|^{1/2}\, \|\varphi_1\|_{W_2^4(D)}. \tag{C.25}$$

By virtue of (C.2), Young's inequality and (C.12), we have

$$\begin{aligned}
&-2\alpha\,(2\beta c - \tau) \int_D u\, \nabla U \cdot \nabla\varphi_1\, dx \\
&\qquad \leq 2\alpha\,(2\beta c + \tau) \int_D |\nabla U \cdot \nabla\varphi_1|\, dx \\
&\qquad \leq \alpha\,(2\beta c + \tau)\Big(\|\nabla U\|_{L_2(D)}^2 + \|\nabla\varphi_1\|_{L_2(D)}^2\Big) \\
&\qquad \leq \alpha\,(2\beta c + \tau)\left(\left(\frac{1}{c^2} + \frac{1}{c} + 4\right) \|u\|_{L_2(D)}^2 + \|\varphi_1\|_{W_2^4(D)}^2\right) \\
&\qquad < \alpha\,(2\beta c + \tau)\left(\left(\frac{1}{c^2} + \frac{1}{c} + 4\right) |D| + \|\varphi_1\|_{W_2^4(D)}^2\right).
\end{aligned} \tag{C.26}$$

By the same arguments, we find that

$$\begin{aligned}
&-4\alpha\beta \int_D \big(\partial_{x_1}^2 U\, \partial_{x_1}^2\varphi_1 + \partial_{x_2}^2 U\, \partial_{x_2}^2\varphi_1 + 2\partial_{x_2}\partial_{x_1} U\, \partial_{x_2}\partial_{x_1}\varphi_1\big)\, u\; dx \\
&\qquad \leq 4\alpha\beta \int_D |\partial_{x_1}^2 U\, \partial_{x_1}^2\varphi_1 + \partial_{x_2}^2 U\, \partial_{x_2}^2\varphi_1 + 2\partial_{x_2}\partial_{x_1} U\, \partial_{x_2}\partial_{x_1}\varphi_1|\, dx \\
&\qquad \leq 2\alpha\beta\left(\|D^2 U\|_{L_2(D)}^2 + \|D^2\varphi_1\|_{L_2(D)}^2\right) \\
&\qquad \leq 2\alpha\beta\left(\left(\frac{1}{c^2} + \frac{1}{c} + 4\right) \|u\|_{L_2(D)}^2 + \|\varphi_1\|_{W_2^4(D)}^2\right) \\
&\qquad < 2\alpha\beta\left(\left(\frac{1}{c^2} + \frac{1}{c} + 4\right) |D| + \|\varphi_1\|_{W_2^4(D)}^2\right),
\end{aligned} \tag{C.27}$$

and that

$$\begin{aligned}
&-4\alpha\beta \int_D u\, \nabla U \cdot \nabla\Delta\varphi_1\, dx \\
&\qquad \leq 4\alpha\beta \int_D |\nabla U \cdot \nabla\Delta\varphi_1|\, dx
\end{aligned}$$

$$\begin{aligned}
&\leq 2\alpha\beta\left(\|\nabla U\|_{L_2(D)}^2 + \|\nabla\Delta\varphi_1\|_{L_2(D)}^2\right)\\
&\leq 2\alpha\beta\left(\left(\frac{1}{c^2}+\frac{1}{c}+4\right)\|u\|_{L_2(D)}^2 + 2\|\varphi_1\|_{W_2^4(D)}^2\right)\\
&< 2\alpha\beta\left(\left(\frac{1}{c^2}+\frac{1}{c}+4\right)|D| + 2\|\varphi_1\|_{W_2^4(D)}^2\right). \qquad \text{(C.28)}
\end{aligned}$$

Inserting (C.21)-(C.28) into (C.20) gives

$$\lambda\int_D (1+\alpha U)\,\varphi_1\, G\, dx < \mu_1 + \alpha K_1 - \alpha\beta\int_D \varphi_1 |\nabla u|^2\, dx, \qquad \text{(C.29)}$$

where

$$\begin{aligned}
K_1 := {}& c\,(\tau/2+\beta) + (2\beta\,(c+2)+\tau)\left(\frac{1}{c^2}+\frac{1}{c}+4\right)|D| + |D|^{1/2}\beta\,(2c+1)\|\varphi_1\|_{W_2^4(D)}\\
&+ (2\beta\,(c+3)+\tau)\,\|\varphi_1\|_{W_2^4(D)}^2
\end{aligned}$$

is a positive constant independent of u. This, combined with the fact that

$$\frac{1}{2} \leq 1 - \frac{\alpha}{2} \leq 1+\alpha U \quad \text{in } D,$$

due to (C.6), shows that

$$\frac{\lambda}{2}\int_D \varphi_1\, G\, dx \leq \lambda\int_D (1+\alpha U)\,\varphi_1\, G\, dx < \mu_1 + \alpha K_1 - \alpha\beta\int_D \varphi_1|\nabla u|^2\, dx. \qquad \text{(C.30)}$$

We next give a lower estimate for $\frac{\lambda}{2}\int_D \varphi_1\, G\, dx$. Let $\delta > 0$ be a small number to be determined later. By applying Young's inequality we get

$$\int_D \varphi_1\left(1+\varepsilon^2|\nabla u|^2\right)\gamma_e\, dx \leq \frac{\delta}{2}\int_D \varphi_1\, G\, dx + \frac{1}{2\delta}\int_D \varphi_1\left(1+\varepsilon^2|\nabla u|^2\right)\, dx$$

and hence that

$$\frac{\lambda}{2}\int_D \varphi_1\, G\, dx \geq \frac{\lambda}{\delta}\int_D \varphi_1\left(1+\varepsilon^2|\nabla u|^2\right)\gamma_e\, dx - \frac{\lambda}{2\delta^2}\left(1+\varepsilon^2\int_D \varphi_1|\nabla u|^2\, dx\right).$$

Using Lemmas C.0.2 and C.0.3, we further obtain

$$\begin{aligned}
\frac{\lambda}{2}\int_D \varphi_1\, G\, dx \geq \frac{\lambda}{\delta}\Bigg(&\varepsilon^2\int_{\Omega(u)} \varphi_1\,|\nabla'\psi_u|\, d(x,z) + \int_{\Omega(u)} \varphi_1(\partial_z\psi_u)^2\, d(x,z)\\
&- \frac{\varepsilon^2}{2}\int_{\Omega(u)} \psi_u^2\,\Delta\varphi_1\, d(x,z) + \frac{\varepsilon^2}{2}\int_D u\,\Delta\varphi_1\, dx + \frac{\varepsilon^2}{6}\int_D \Delta\varphi_1\, dx\Bigg)\\
&- \frac{\lambda}{2\delta^2}\left(1+\varepsilon^2\int_D \varphi_1|\nabla u|^2\, dx\right)
\end{aligned}$$

$$\geq \frac{\lambda}{\delta}\int_D \frac{\varphi_1}{1+u}\,dx - \frac{\lambda\varepsilon^2}{2\delta}\int_{\Omega(u)} \psi_u^2\,\Delta\varphi_1\,d(x,z) + \frac{\lambda\varepsilon^2}{2\delta}\int_D u\,\Delta\varphi_1\,dx$$
$$+ \frac{\lambda\varepsilon^2}{6\delta}\int_D \Delta\varphi_1\,dx - \frac{\lambda}{2\delta^2}\left(1+\varepsilon^2\int_D \varphi_1|\nabla u|^2\,dx\right).$$

So, thanks to (C.2), Lemma 8.2.2, and using the positivity of φ_1, we deduce that

$$\begin{aligned}\frac{\lambda}{2}\int_D \varphi_1\,G\,dx &\geq \frac{\lambda}{\delta} - \frac{\lambda\varepsilon^2}{2\delta}\|\Delta\varphi_1\|_{L_1(D)} - \frac{\lambda\varepsilon^2}{2\delta}\|\Delta\varphi_1\|_{L_1(D)} - \frac{\lambda\varepsilon^2}{6\delta}\|\Delta\varphi_1\|_{L_1(D)}\\ &\quad - \frac{\lambda}{2\delta^2} - \frac{\lambda\varepsilon^2}{2\delta^2}\int_D \varphi_1|\nabla u|^2\,dx\\ &= \frac{\lambda}{\delta}\left(1-\frac{7}{6}\varepsilon^2\|\Delta\varphi_1\|_{L_1(D)} - \frac{1}{2\delta}\right) - \frac{\lambda\varepsilon^2}{2\delta^2}\int_D \varphi_1|\nabla u|^2\,dx.\end{aligned}$$

This, combined with (C.30), yields

$$\mu_1 + \alpha K_1 > \frac{\lambda}{\delta}\left(1-\frac{7}{6}\varepsilon^2\|\Delta\varphi_1\|_{L_1(D)} - \frac{1}{2\delta}\right) + \left(\alpha\beta - \frac{\lambda\varepsilon^2}{2\delta^2}\right)\int_D \varphi_1|\nabla u|^2\,dx,$$

and choosing $\delta = \varepsilon\sqrt{\frac{\lambda}{2\alpha\beta}}$ shows that

$$\mu_1 + \alpha K_1 > \frac{\sqrt{2\alpha\beta}}{\varepsilon}\left(1-\frac{7}{6}\varepsilon^2\|\Delta\varphi_1\|_{L_1(D)}\right)\sqrt{\lambda} - \frac{\alpha\beta}{\varepsilon^2}.$$

Finally, taking $\alpha = \alpha_\varepsilon := \min\{1,\varepsilon^2\} \in (0,1]$, we end up with

$$\frac{\varepsilon}{\sqrt{2\alpha_\varepsilon\beta}}\left(\mu_1 + \left(K_1 + \frac{\beta}{\varepsilon^2}\right)\alpha_\varepsilon\right) > \left(1-\frac{7}{6}\varepsilon^2\|\Delta\varphi_1\|_{L_1(D)}\right)\sqrt{\lambda}\,. \tag{C.31}$$

From this we readily infer that λ cannot exceed a threshold value, depending on ε, provided

$$\varepsilon < \varepsilon_* := \sqrt{\frac{6}{7\,\|\Delta\varphi_1\|_{L_1(D)}}}\,.$$

Especially, for $0<\varepsilon<\min\{1,\varepsilon_*/2\}$, we conclude from (C.31) that

$$\lambda < \frac{8\,(\mu_1+K_1+\beta)^2}{9\beta},$$

which is independent of ε. ■

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, vol. 140 of Pure Appl. Math. (Amst.), Elsevier/Academic Press, Amsterdam, 2nd ed., 2003.

[2] H. Amann, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Funct. Anal., 11 (1972), pp. 346–384.

[3] ——, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976), pp. 620–709.

[4] ——, *Multiplication in Sobolev and Besov spaces*, in Nonlinear Analysis, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 1991, pp. 27–50.

[5] ——, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in Function Spaces, Differential Operators and Nonlinear Analysis (Friedrichroda, 1992), vol. 133 of Teubner-Texte Math., Teubner, Stuttgart, 1993, pp. 9–126.

[6] ——, *Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory*, Birkhäuser, Basel, Boston, Berlin, 1995.

[7] H. Amann and J. Escher, *Analysis. III*, Birkhäuser Verlag, Basel, 2009.

[8] E. Berchio, F. Gazzola, and E. Mitidieri, *Positivity preserving property for a class of biharmonic elliptic problems*, J. Diff. Equat., 229 (2006), pp. 1–23.

[9] T. Boggio, *Sulle funzioni di Green d'ordine*, m. Rend. Circ. Mat. Palermo., 20 (1905), pp. 97–135.

[10] N. D. Brubaker and J. A. Pelesko, *Non-linear effects on canonical MEMS models*, European J. Appl. Math., 22 (2011), pp. 455–470.

[11] G. Cimatti, *A free boundary problem in the theory of electrically actuated microdevices*, Appl. Math. Lett., 20 (2007), pp. 1232–1236.

[12] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Interscience Publisher, New York, 1953.

[13] D. Daners and P. Koch Medina, *Abstract evolution equations, periodic problems and applications*, vol. 279 of Pitman Research Notes in Mathematics Series, Longman, Harlow, 1992.

[14] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[15] J. Escher, P. Laurençot, and C. Walker, *Finite time singularity in a free boundary problem modeling MEMS*, C. R. Math. Acad. Sci. Paris, 351 (2013), pp. 807–812.

[16] ——, *A parabolic free boundary problem modeling electrostatic MEMS*, Arch. Ration. Mech. Anal., 211 (2014), pp. 389–417.

[17] ——, *Dynamics of a free boundary problem with curvature modeling electrostatic MEMS*, Trans. Amer. Math. Soc., 367 (2015), pp. 5693–5719.

[18] J. Escher and C. Lienstromberg, *A survey on second-order free boundary value problems modelling MEMS with general permittivity profile*, Discrete Contin. Dyn. Syst. Ser. S, 10 (2017), pp. 745–771.

[19] P. Esposito, N. Ghoussoub, and Y. Guo, *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, vol. 20 of Courant Lect. Notes Math., Courant Institute of Mathematical Sciences, New York, 2010.

[20] K. E. Evans, *Auxetic polymers: a new range of materials*, Endeavour, 15 (1991), pp. 170–174.

[21] R. P. Feynman, *There's plenty of room at the bottom*, IEEE J. Microelectromech. Syst., 1 (1992), pp. 60–66.

[22] G. Flores, G. Mercado, J. A. Pelesko, and N. Smyth, *Analysis of the dynamics and touchdown in a model of electrostatic MEMS*, SIAM J. Appl. Math., 67 (2007), pp. 434–446.

[23] K. Friedrichs, *Die Randwert-und Eigenwertprobleme aus der Theorie der elastischen Platten. (Anwendung der direkten Methoden der Variationsrechnung)*, Math. Ann., 98 (1928), pp. 205–247.

[24] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic Boundary Value Problems*, vol. 1991 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[25] F. Gazzola and G. Sweers, *On positivity for the biharmonic operator under Steklov boundary conditions*, Arch. Ration. Mech. Anal., 188 (2008), pp. 399–427.

[26] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[27] S. Giuffrè, *Strong solvability of boundary value contact problems*, Appl. Math. Optim., 51 (2005), pp. 361–372.

[28] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, 1985.

[29] D. Guidetti, *On elliptic problems in Besov spaces*, Math. Nachr., 152 (1991), pp. 247–275.

[30] ——, *On interpolation with boundary conditions*, Math. Z., 207 (1991), pp. 439–460.

[31] ——, *On elliptic systems in L^1*, Osaka J. Math., 30 (1993), pp. 397–429.

[32] Y. Guo, *Dynamical solutions of singular wave equations modeling electrostatic MEMS*, SIAM J. Appl. Dyn. Syst., 9 (2010), pp. 1135–1163.

[33] Z. Guo, B. Lai, and D. Ye, *Revisiting the biharmonic equation modelling electrostatic actuation in lower dimensions*, Proc. Amer. Math. Soc., 142 (2014), pp. 2027–2034.

[34] Z. Guo and J. Wei, *On a fourth order nonlinear elliptic equation with negative exponent*, SIAM J. Math. Anal., 40 (2009), pp. 2034–2054.

[35] L. L. Helms, *Potential Theory*, Springer, London, 2009.

[36] A. Henrot and M. Pierre, *Shape variation and optimization*, vol. 28, EMS Tracts in Mathematics, 2018.

[37] D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1975.

[38] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.

[39] G. Kirchhoff, *Über das Gleichgewicht und die Bewegung einer elastischen Scheibe*, J. Reine Angew. Math., 40 (1850), pp. 51–88.

[40] M. Kohlmann, *The abstract quasilinear Cauchy problem for a MEMS model with two free boundaries*, Acta Appl. Math., 138 (2015), pp. 171–197.

[41] M. A. Krasnosel'skij, J. A. Lifshits, and A. V. Sobolev, *Positive Linear Systems: The Method of Positive Operators*, vol. 5 of Sigma Ser. Appl. Math., Heldermann Verlag, Berlin, 1989.

[42] M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl., 26 (1950).

[43] J. Lagnese and J.-L. Lions, *Modelling Analysis and Control of Thin Plates*, vol. 6 of Res. Appl. Math., Masson, Paris, 1988.

[44] P. Laurençot and C. Walker, *Shape derivative of the Dirichlet energy for a transmission problem.* (2019, submitted). See `https://arxiv.org/abs/1901.07257`.

[45] ——, *Touchdown is the only finite time singularity in a three-dimensional MEMS model.* (2018, submitted). See `https://arxiv.org/abs/1807.09017`.

[46] ——, *A stationary free boundary problem modeling electrostatic MEMS*, Arch. Ration. Mech. Anal., 207 (2013), pp. 139–158.

[47] ——, *A fourth-order model for MEMS with clamped boundary conditions*, Proc. Lond. Math. Soc., 109 (2014), pp. 1435–1464.

[48] ——, *A free boundary problem modeling electrostatic MEMS: I. Linear bending effects*, Math. Ann., 360 (2014), pp. 307–349.

[49] ——, *Sign-preserving property for some fourth-order elliptic operators in one dimension and radial symmetry*, J. Anal. Math., 127 (2015), pp. 69–89.

[50] ——, *The time singular limit for a fourth-order damped wave equation for MEMS*, in Elliptic and Parabolic Equations, vol. 119 of Springer Proc. Math. Stat., Springer, 2015, pp. 233–246.

[51] ——, *On a three-dimensional free boundary problem modeling electrostatic MEMS*, Interfaces Free Bound., 18 (2016), pp. 393–411.

[52] ——, *A variational approach to a stationary free boundary problem modeling MEMS*, ESAIM Control Optim. Calc. Var., 22 (2016), pp. 417–438.

[53] ——, *Some singular equations modeling MEMS*, Bull. Amer. Math. Soc., 54 (2017), pp. 437–479.

[54] ——, *Vanishing aspect ratio limit for a fourth-order MEMS model*, Ann. Mat. Pura Appl., 196 (2017), pp. 1537–1556.

[55] ——, *Finite time singularity in a MEMS model revisited*, Z. Anal. Anwend., 37 (2018), pp. 209–219.

[56] ——, *Heterogeneous dielectric properties in MEMS models*, SIAM J. Math. Anal., 78 (2018), pp. 504–530.

[57] C. Liang, J. Li, and K. Zhang, *On a hyperbolic equation arising in electrostatic MEMS*, J. Diff. Eqs., 256 (2014), pp. 503–530.

[58] C. Liang and K. Zhang, *Asymptotic stability and quenching behavior of a hyperbolic nonlocal MEMS equation*, Commun. Math. Sci., 13 (2015), pp. 355–368.

[59] G. M. Lieberman, *Oblique Derivative Problems for Elliptic Equations*, World Scientific Publishing Co. Pte. Ltd., 2013.

[60] F. Lin and Y. Yang, *Nonlinear non-local elliptic equation modelling electrostatic actuation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 463 (2007), pp. 1323–1337.

[61] A. E. Lindsay and J. Lega, *Multiple quenching solutions of a fourth order parabolic PDE with a singular nonlinearity modeling a MEMS capacitor*, SIAM J. Appl. Math., 72 (2012), pp. 935–958.

[62] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications, New York, 4th ed., 1944.

[63] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, vol. 16 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Verlag, Basel, 1995.

[64] E. Mansfield, *The Bending and Stretching of Plates*, Cambridge University Press, Cambridge, 1989.

[65] A. F. Marques, R. C. Castelló, and A. M. Shkel, *Modelling the electrostatic actuation of MEMS: state of the art 2005*, Technical Report, Universitat Politècnica de Catalunya, (2005).

[66] E. Mitidieri and G. Sweers, *Weakly coupled elliptic systems and positivity*, Math. Nachr., 173 (1995), pp. 259–286.

[67] H. C. Nathanson, W. E. Newell, R. A. Wickstrom, and J. R. Davis, *The resonant gate transistor*, IEEE Trans. Electron Devices, 14 (1967), pp. 117–133.

[68] S. A. Nazarov, A. Stylianou, and G. Sweers, *Hinged and supported plates with corners*, Z. Angew. Math. Phys., 63 (2012), pp. 929–960.

[69] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.

[70] L. Nirenberg, *An extended interpolation inequality*, Ann. Scuola Norm. Sup. Pisa (3), 20 (1966), pp. 733–737.

[71] E. Parini and A. Stylianou, *On the positivity preserving property of hinged plates*, SIAM J. Math. Anal., 41 (2009), pp. 2031–2037.

[72] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[73] J. A. Pelesko, *Mathematical modeling of electrostatic MEMS with tailored dielectric properties*, SIAM J. Appl. Math., 62 (2002), pp. 888–908.

[74] J. A. Pelesko and D. H. Bernstein, *Modeling MEMS and NEMS*, Chapman & Hall/CRC Press, Boca Raton, 2002.

[75] J. A. Pelesko and A. A. Triolo, *Nonlocal problems in MEMS device control*, J. Eng. Math., 41 (2001), pp. 345–366.

[76] J. N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, Taylor & Francis, Philadelphia, 2nd ed., 2007.

[77] G. Romani, *Positivity for fourth-order semilinear problems related to the Kirchhoff–Love functional*, Anal. PDE, 10 (2017), pp. 943–982.

[78] R. P. Sperb, *Maximum Principles and Their Applications*, vol. 157 of Math. Sci. Engrg., Academic Press, New York, 1981.

[79] W. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. École Norm. Sup., 19 (1902), pp. 191–259.

[80] G. Sweers, *Strong positivity in $C(\overline{\Omega})$ for elliptic systems*, Math. Z., 209 (1992), pp. 251–271.

[81] G. Sweers and K. Vassi, *Positivity for a hinged convex plate with stress*, SIAM J. Math. Anal., 50 (2018), pp. 1163–1174.

[82] G. Taylor, *The coalescence of closely spaced drops when they are at different electric potentials*, Proc. R. Soc. A., 306 (1968), pp. 423–434.

[83] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Johann Ambrosius Barth, Heidelberg, second ed., 1995.

[84] E. Ventsel and T. Krauthammer, *Thin Plates and Shells: Theory, Analysis and Applications*, Maxwell Publishers Inc., New York, 2001.

[85] M. I. Younis, *MEMS Linear and Nonlinear Statics and Dynamics*, Springer, New York, 2011.

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