$\mathbb{Q}_\ell$-COHOMOLOGY PROJECTIVE PLANES FROM ENRIQUES SURFACES IN ODD CHARACTERISTIC

MATTHIAS SCHÜTT

ABSTRACT. We give a complete classification of $\mathbb{Q}_\ell$-cohomology projective planes with isolated ADE-singularities and numerically trivial canonical bundle in odd characteristic. This leads to a beautiful relation with certain Enriques surfaces which parallels the situation in characteristic zero, yet displays intriguing subtleties.

1. INTRODUCTION

Over the complex numbers, there is a complete understanding of fake projective planes, i.e. smooth projective surfaces whose singular homology resembles that of $\mathbb{P}^2$, with Betti numbers $b_2 = 1, b_1 = b_3 = 0$, while not being isomorphic to $\mathbb{P}^2$ itself. Necessarily, a fake projective plane has general type; this offers one explanation why the classification (due to Prasad–Yeung [20] and Cartwright–Steger [5]) remains group-theoretic in nature. In fact, our understanding of the geometry of fake projective planes is rather rudimentary, and in particular, there are no explicit models known (in terms of equations as opposed to abstract ball quotients).

Another downside of the above picture is that it does not carry over to the positive characteristic side, although also there the notion of a fake projective plane S makes perfect sense, with ℓ-adic cohomology replacing singular homology. It may come surprising at first that this can be helped by allowing for S to be singular. Here we shall restrict to isolated ordinary double points as singularities, partly because these do not affect the canonical divisor K_S (or the dualizing sheaf). We then distinguish whether K_S is positive, numerically trivial or negative. This paper concentrates on the case which is rich, yet accessible in nature, where K_S is numerically trivial. In contrast to the situation over $\mathbb{C}$ which was treated in [22], this leads to two possibilities for the minimal desingularization

$$(1) \qquad\qquad Y \to S.$$

Either Y is a K3 surface or an Enriques surface. Since the former case only occurs in small characteristic by [25], we shall investigate the latter case, i.e. Y is an Enriques surface supporting a root type R of rank 9 corresponding to

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the configuration of exceptional curves in (1). Since Enriques surfaces behave rather differently in characteristic 2 (although many arguments can be adapted for singular Enriques surfaces), we will deal with them separately in future work and consider only fields of odd characteristic here.

Theorem 1.1. *There are 31 root types of rank* 9 *realized through smooth rational curves on Enriques surfaces in odd characteristic (28 in characteristic* 3, *30 in characteristic* 5*). For each the following hold:*

  (i) *the root types are supported on 1-dimensional families of Enriques surfaces;*

 (ii) *the moduli spaces can have up to 3 different components;*

(iii) *each family has rational base and is defined over the prime field;*

(iv) *each family can be parametrized explicitly, see Tables 3 and 4.*

It is intriguing how Theorem 1.1 parallels the complete classification of these Enriques surfaces over $\mathbb{C}$. In fact, we will derive explicit parametrizations of all Enriques surfaces in Theorem 1.1 which do not depend on the characteristic at all (except for the three root types which cease to exist in characteristic 3, and one in characteristic 5). To arrive at the explicit parametrizations, however, we have to be extremely careful because many of the arguments over $\mathbb{C}$ do not carry over directly, for instance those from [8] invoking the orbifold Bogomolov–Miyaoka–Yau inequality and those based on [11] using Nikulin's work on automorphisms on complex K3 surfaces. In consequence, we will sometimes take shortcuts and refer the reader for the detailed arguments in [22], but this will be compensated by the detours which we have to take where the characteristic zero arguments do not work or are not sufficient.

The paper is organized as follows. After setting up the scenery in the Section 2, Section 3 reviews root types on Enriques surfaces and the underlying lattice theory relevant for this paper. In Section 4, we relate this to elliptic fibrations and rational elliptic surfaces. This will already enable us to show that fields of positive characteristic admit no other root types than characteristic zero (Proposition 5.4). The main geometric technique of Enriques involutions of base change type is introduced in Section 6 (with a view towards positive characteristic). We work out an explicit formulation, geared towards our aims, in Section 7 which then calls for a review of the key ingredients, namely extremal rational elliptic surfaces in Section 8. This will put us in the position to develop explicit parametrizations on all Enriques surfaces in question in Section 9 from the information obtained thus far (Tables 3 & 4). This will prove most part of Theorem 1.1. The final section concerns the number of components of the moduli spaces, thus completing the proof of Theorem 1.1. Throughout the paper, we will work out instructive examples in detail to give a clear idea of the methods and techniques involved.

2. Set-up

We work over an algebraically closed field K of characteristic $p > 2$. Let S be a normal projective surface over K with only ADE singularities, i.e. isolated rational double points. For a prime $\ell \neq p$, let $b_i(S)$ be the Betti numbers of S for the ℓ-adic étale cohomology:

$$b_i(S) = \dim \mathrm{H}^i_{\text{ét}}(S, \mathbb{Q}_\ell)$$

Definition 2.1. We call S a $\mathbb{Q}_\ell$-*cohomology projective plane* if

$$b_1(S) = b_3(S) = 0, \quad b_2(S) = 1.$$

We emphasize that, except for the notation, the above notion does not depend on the chosen prime $\ell \neq p$ by the usual comparison theorems.

Let Y denote the minimal desingularization of S as in (1). Due to the mild singularities, the canonical divisors of Y and S are the same, and we distinguish whether K_Y (or K_S) is positive, numerically trivial, or negative. In the first case, Y is of general type, and it seems hard to say anything about Y even over $\mathbb{C}$. Meanwhile in the last case, Y is a del Pezzo surface, and the complex classification from [29] should carry over with a few modifications and exceptions, yielding some isolated surfaces and a one-dimensional family.

In the sequel, we assume that K_Y is numerically trivial. Since the vanishing $b_1(Y) = 0$ is inherited from S, the Enriques–Kodaira classification (extended to positive characteristic by Bombieri and Mumford [3], [4], [17]) dictates that Y is either a K3 surface or an Enriques surface. In either case, Y supports a root type R of maximal rank $b_2(Y) - 1 = 21$ resp. 9 supported on smooth rational curves by assumption. In particular, all of $\mathrm{H}^2_{\text{ét}}(Y, \mathbb{Q}_\ell)$ is algebraic, i.e. Y is supersingular. The root type R will be a direct sum of root lattices A_n, D_k, E_l which are all taken to be negative-definite throughout this paper. Orthogonal sums are indicated by a '+', and likewise by $2R$ etc.

For K3 surfaces, this set-up turns out to be rather restrictive (so restrictive, in fact, that they cannot occur over $\mathbb{C}$ by Lefschetz' $(1,1)$-theorem):

Theorem 2.2 (Shimada). *If Y is a K3 surface supporting a root type of rank 21, then the characteristic satisfies $0 < p \leq 19$.*

In fact, Shimada gives a much more precise result in [25], determining all possible root types starting from $A_{18} + A_3$ and $A_{18} + A_2 + A_1$ in characteristic 19 down to $21A_1$ in characteristic 2. It is interesting to see how the moduli dimensions go up in unison, from zero in characteristics 19 through 11 to full 10-dimensional moduli in characteristic 2.

The case of Enriques surfaces, in contrast, presents a totally different challenge as it does not essentially depend on the characteristic and always leads to one-dimensional families as recorded in Theorem 1.1. Partly, this may be explained because an Enriques surface Y is always supersingular. More precisely, we have

$$H^2_{\text{ét}}(Y, \mathbb{Q}_\ell) \cong \mathrm{Pic}(Y) \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^{10}$$

and, as a lattice,

$$(2) \qquad \mathrm{Num}(Y) = \mathrm{Pic}(Y)/(\text{torsion}) \cong U + E_8$$

by [10] where U denotes the hyperbolic plane. This lattice structure will play a crucial role throughout this paper, both when classifying the possible root types and when working out the explicit parametrizations of the Enriques surfaces proving Theorem 1.1.

3. Maximal root types on Enriques surfaces

It is an elementary exercise to compile a list of all root types R of rank 9, starting from A_9 and D_9 all the way to $9A_1$ (most of which may be found in 3.1 and 3.2). Over $\mathbb{C}$, many of them can be ruled out to occur on Enriques surfaces using the orbifold Bogomolov–Miyaoka–Yau inequality; in fact, in [8] it is shown along these lines that either R has at most 4 orthogonal summands (and satisfies some extra conditions) or $R = 2A_3 + 3A_1$.

In positive characteristic, however, this approach is not available. Here we shall therefore replace it with some general theory of quadratic forms (which is completely characteristic free) and special properties of elliptic fibrations on Y which can be derived purely from R.

We start by reviewing the portion of the theory of quadratic forms which is relevant to our paper. Here we could use Hilbert symbols and local epsilon-invariants following [9], but it will be more beneficial for our purposes to pursue an explicit approach based on the analysis of discriminant forms á la Nikulin [18]. Let us explain how this applies to our precise problem.

Let Y be an Enriques surface containing nine smooth rational curves which support a given root type R of rank 9. Then this gives an embedding

$$(3) \qquad R \hookrightarrow \mathrm{Num}(Y)$$

which naturally extends to the primitive closure

$$R' = (R \otimes \mathbb{Q}) \cap \mathrm{Num}(Y) \hookrightarrow \mathrm{Num}(Y).$$

For later use, we emphasize the following result which will be very useful for our purposes (cf. [22, Lem. 3.1]).

Lemma 3.1. *If $R' \neq R$, then R' is always obtained from R by adding some (-2)-vectors, but these can never be effective nor anti-effective.*

In particular, the lemma implies that the extra classes in R' cannot be represented by smooth rational curves on Y. This is contrary to the K3 case where Riemann–Roch applies to show that any (-2)-divisor is either effective or anti-effective; on an Enriques surface, however, the torsion in $\mathrm{Pic}(Y)$ prevents this implication from holding.

It follows from (2) and (3) that there is a positive class $H \in \mathrm{Num}(Y)$ such that

$$\mathbb{Z}H + R' \hookrightarrow \mathrm{Num}(Y)$$

has finite index

$$d = H^2 = |\operatorname{disc} R'|.$$

More precisely, the discriminant forms of the orthogonal summands have to be compatible, i.e. the discriminant group $(R')^\vee/R'$ together with the induced quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$ and likewise for H (or $\mathbb{Z}H$); since $\operatorname{Num}(Y)$ is unimodular, this translates as

$$(4) \qquad\qquad q_H = -q_{R'}.$$

In particular, we find that

$$(5) \qquad\qquad (R')^\vee/R' \cong H^\vee/H \cong \mathbb{Z}/d\mathbb{Z},$$

i.e. R' has length one (the minimum number of generators of the discriminant group) which turns out to be a severe restriction on the root type R. We illustrate this with a simple example.

Example 3.2. Let $R = D_4 + A_4 + A_1$. Then one readily checks that R has no proper even integral overlattices, i.e. necessarily $R = R'$ if R were to be supported on some Enriques surface Y, but R has length three, giving a contradiction by (5).

The same line of argument applies to $\hat{R} = A_4 + A_1^5$ of length five, since here the only proper even integral overlattice is R.

Example 3.3 ($R = A_5 + 2A_2$). On the positive side, one checks that for $R = A_5 + 2A_2$ of length 3, the only even integral overlattice $R' = E_7 + A_2$ has discriminant form matching $\mathbb{Z}H$ for $H^2 = 6$ up to sign, so R and R' embed into $U + E_8$.

It is easy to imagine (and check!) that this kind of argument suffices to rule out a good number of root types (as was done in [22] which also provides many more explicit examples). In particular, this applies to all root types with at most 4 summands (outside characteristics $3, 5$ where due to degenerations we have to throw in some extra work as shall see in $9.1 - 9.4$). However, without the orbifold Bogomolov–Miyaoka–Yau inequality at our disposal, there is no obvious way to rule out even $9A_1$ in positive characteristic! We shall work around this using an argument which plays out elliptic fibrations against Lemma 3.1. The next section will review the construction. Before we get there, we list the root types which are not excluded by the quadratic forms argument sketched above.

3.1. **Root types from characteristic zero.** By [9] and [22], exactly the following root types of rank 9 are supported on Enriques surfaces in characteristic zero:

$$A_9, A_8 + A_1, A_7 + A_2, A_7 + 2A_1, A_6 + A_2 + A_1, A_5 + A_4, A_5 + A_3 + A_1,$$
$$A_5 + 2A_2, A_5 + A_2 + 2A_1, 2A_4 + A_1, A_4 + A_3 + 2A_1, 3A_3, 2A_3 + A_2 + A_1,$$
$$2A_3 + 3A_1, A_3 + 3A_2, D_9, D_8 + A_1, D_7 + 2A_1, D_6 + A_3, D_6 + A_2 + A_1,$$
$$D_6 + 3A_1, D_5 + A_4, D_5 + A_3 + A_1, D_5 + D_4, D_4 + A_3 + 2A_1, 2D_4 + A_1,$$
$$E_8 + A_1, E_7 + A_2, E_7 + 2A_1, E_6 + A_3, E_6 + A_2 + A_1.$$

We highlight that Theorem 1.1 states that all these 31 root types indeed occur over fields of odd characteristic – except for some non-existence bit in characteristics 3 and 5 (see 9.3, 9.4).

3.2. **Additional root types.** Here we list those 7 root types (with at least 5 orthogonal summands) which admit an embedding into $U + E_8$, but do not occur in characteristic zero due to the orbifold Bogomolow–Miyaoka–Yau inequality.

$$D_5 + 4A_1, D_4 + 5A_1, D_4 + A_2 + 3A_1, A_3 + 6A_1, 4A_2 + A_1, A_2 + 7A_1, 9A_1.$$

4. Elliptic fibrations

We continue to elaborate on ideas from [22] to sort out the possible maximal root types, adjusted for our situation of positive characteristic. Every Enriques surfaces Y admits an elliptic fibrations

$$(6) \qquad\qquad\qquad f : Y \to \mathbb{P}^1.$$

This has two ramified fibers, of multiplicity two, so there cannot be a section (but there always is a bisection given by a smooth rational curve or a curve of arithmetic genus one (the generic case)). Elliptic fibrations are encoded in the lattice $\mathrm{Num}(Y)$ in terms of isotropic vectors. Indeed, pick a primitive isotropic vector $E \in \mathrm{Num}(Y)$ which we may assume to be effective after changing sign, if necessary, by Riemann–Roch. Then the linear system $|2E|$ may still contain some base locus (which consists of (-2)-curves), but after subtracting the base locus, the resulting isotropic vector will induce an elliptic fibration.
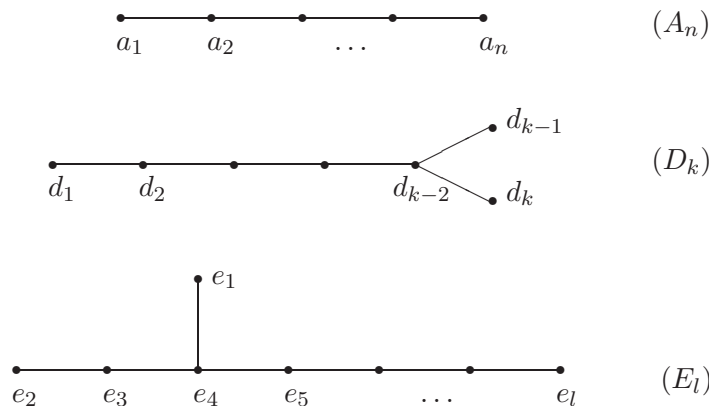
Proposition 4.1. *Let Y be an Enriques surface supporting a root type R of rank 9. Then there is a primitive isotropic vector $E \in \mathrm{Num}(Y)$ such that*

(1) there is a smooth rational curve C on Y with $C.E = 1$,

(2) $R_0 = R \cap E^\perp \subset \mathrm{Num}(Y)$ is a root lattice of rank 8.

The proof of Proposition 4.1 amounts to a case-by-case analysis for all the 38 root types R from 3.1 and 3.2. It purely bases on the gluing data analyzed in Section 3. More precisely, the isotropic vector E will be given using the positive class h and some vector $e^\vee \in (R')^\vee$ which we choose as dual vector of a single vertex $e \in R$. Once this is realized, the second claim of Proposition 4.1 follows readily (with $R_0 = R \setminus \{e\}$). For the first, one may take precisely the smooth rational curve on Y corresponding to the vertex $e \in R$; in particular, $e.E = 1$, so E is primitive as stated. For the details, we slightly modify a table in the

suggestive notation from [22] (to be recalled below) and supplement it by the corresponding data for the 7 additional root types from 3.2.

4.1. **Notation for dual vectors.** The gluing argument from the proof of Proposition 4.1 uses dual vectors of single vertices of ADE root lattices. Below we sketch the notation to be used in Table 1. For details we refer to [22, §2].

Remark 4.2. We note that the choices made in the proof of Proposition 4.1 are far from being unique as Y may contain many isotropic vectors, and many elliptic fibrations.

Convention 4.3. Given a primitive isotropic vector E as above, we will always refer to the associated elliptic fibration as associated to the linear system $|2E|$, regardless of the subtleties of sign and base locus.

4.2. **Jacobian elliptic surface.** To make the whole situation explicit, let us consider the Jacobian of (6). This is a rational elliptic surface $\text{Jac}(f)$ from which Y can be recovered as a torsor (or over $\mathbb{C}$, by a logarithmic transformation). Here this surface turns out to be rather special; namely it will be extremal:

Definition 4.4. A rational elliptic surface $S \to \mathbb{P}^1$ is called *extremal* if its Mordell–Weil group $\text{MW}(S)$ is finite.

Explicit examples will be given in Section 8. For now we content ourselves with the following important consequence of our findings so far:

Corollary 4.5. *Let Y be an Enriques surface supporting a root type R of rank 9. Then Y admits an elliptic fibration with a smooth rational bisection such that $\text{Jac}(Y)$ is an extremal rational elliptic surface.*

Proof. All we need is already contained in Proposition 4.1 whose notation we will employ. For instance, if $|2E|$ already gives an elliptic fibration, then C is a bisection, and the root lattice R_0 is supported on fiber components. Since the same singular fibers feature on $\text{Jac}(Y)$, the claim follows from the Shioda–Tate formula [26].

R	R'	$\lvert d \rvert$	E	R_0
A_9	A_9	10	$a_2^\vee + 2h/5$	$A_7 + A_1$
$A_8 + A_1$	$A_8 + A_1$	18	$(a_3^\vee, 0) + h/3$	$A_5 + A_2 + A_1$
	$E_8 + A_1$	2	$(a_3^\vee, 0) + h$	$A_5 + A_2 + A_1$
$A_7 + A_2$	$E_7 + A_2$	6	$(a_2^\vee, 0) + h/2$	$A_5 + A_2 + A_1$
$A_7 + 2A_1$	$E_8 + A_1$	2	$(a_4^\vee, 0, 0) + h$	$2A_3 + 2A_1$
$A_6 + A_2 + A_1$	$A_6 + A_2 + A_1$	42	$(a_1^\vee, 0, 0) + h/7$	$A_5 + A_2 + A_1$
$A_5 + A_4$	$A_5 + A_4$	30	$(0, a_2^\vee) + h/5$	$A_5 + A_2 + A_1$
$A_5 + A_3 + A_1$	$E_6 + A_3$	24	$(a_2^\vee, 0, 0) + h/3$	$2A_3 + 2A_1$
$A_5 + 2A_2$	$E_7 + A_2$	6	$(0, 0, a_1^\vee) + h/3$	$A_5 + A_2 + A_1$
$A_5 + A_2 + 2A_1$	$E_8 + A_1$	2	$(0, 0, 0, a_1^\vee) + h/2$	$A_5 + A_2 + A_1$
$2A_4 + A_1$	$E_8 + A_1$	2	$(0, 0, a_1^\vee) + h/2$	$2A_4$
$A_4 + A_3 + 2A_1$	$A_4 + D_5$	20	$(a_1^\vee, 0, 0, 0) + h/5$	$2A_3 + 2A_1$
$3A_3$	D_9	4	$(a_2^\vee, 0, 0) + h/2$	$2A_3 + 2A_1$
$2A_3 + A_2 + A_1$	$E_7 + A_2$	6	$(0, 0, a_1^\vee, 0) + h/3$	$2A_3 + 2A_1$
$2A_3 + 3A_1$	$E_8 + A_1$	2	$(0, 0, 0, 0, a_1^\vee) + h/2$	$2A_3 + 2A_1$
$A_3 + 3A_2$	$A_3 + E_6$	12	$(a_1^\vee, 0, 0, 0) + h/4$	$4A_2$
D_9	D_9	4	$d_9^\vee + 3h/4$	A_8
$D_8 + A_1$	$E_8 + A_1$	2	$d_8^\vee + h$	$A_7 + A_1$
$D_7 + 2A_1$	D_9	4	$(d_1^\vee, 0, 0) + h/2$	$D_6 + 2A_1$
$D_6 + A_3$	D_9	4	$(0, a_2^\vee) + h/2$	$D_6 + 2A_1$
$D_6 + A_2 + A_1$	$E_7 + A_2$	6	$(0, a_1^\vee, 0) + h/3$	$D_6 + 2A_1$
$D_6 + 3A_1$	$E_8 + A_1$	2	$(d_2^\vee, 0, 0, 0) + h$	$D_4 + 4A_1$
$D_5 + A_4$	$D_5 + A_4$	20	$(d_5^\vee, 0) + h/4$	$2A_4$
$D_5 + A_3 + A_1$	$E_8 + A_1$	2	$(d_2^\vee, 0, 0) + h$	$2A_3 + 2A_1$
$D_5 + D_4$	D_9	4	$(0, d_1^\vee) + h/2$	$D_5 + A_3$
$D_4 + A_3 + 2A_1$	D_9	4	$(d_1^\vee, 0, 0, 0) + h/2$	$2A_3 + 2A_1$
$2D_4 + A_1$	$E_8 + A_1$	2	$(0, 0, a_1^\vee) + h/2$	$2D_4$
$E_8 + A_1$	$E_8 + A_1$	2	$(e_8^\vee, 0) + h$	$E_7 + A_1$
$E_7 + A_2$	$E_7 + A_2$	6	$(e_3^\vee, 0) + h$	$A_5 + A_2 + A_1$
$E_7 + 2A_1$	$E_8 + A_1$	2	$(e_2^\vee, 0, 0) + h$	$D_6 + 2A_1$
$E_6 + A_3$	$E_6 + A_3$	12	$(e_6^\vee, 0) + h/3$	$D_5 + A_3$
$E_6 + A_2 + A_1$	$E_8 + A_1$	2	$(e_1^\vee, 0, 0) + h$	$A_5 + A_2 + A_1$
$D_5 + 4A_1$	D_9	4	$d_1^\vee + h/2$	$D_4 + 4A_1$
$D_4 + 5A_1$	$E_8 + A_1$	2	$a_1^\vee + h/2$	$D_4 + 4A_1$
$D_4 + A_2 + 3A_1$	$E_7 + A_2$	6	$a_2^\vee + h/3$	$D_4 + 4A_1$
$A_3 + 6A_1$	D_9	4	$a_2^\vee + h/2$	$8A_1$
$4A_2 + A_1$	$E_8 + A_1$	2	$(0, 0, 0, 0, a_1^\vee) + h/2$	$4A_2$
$A_2 + 7A_1$	$A_2 + E_7$	6	$a_2^\vee, + h/3$	$8A_1$
$9A_1$	$E_8 + A_1$	2	$a_1^\vee + h/2$	$8A_1$

TABLE 1. Isotropic vectors and root lattices for the proof of Proposition 4.1

If $|\pm 2E|$ involves some base locus, then subtracting this amounts to Picard–Lefschetz reflections in the corresponding smooth rational curves. In particular, this may affect the (-2)-curves $C_1, \ldots, C_9$ representing R, but in essence this may only add smooth rational curves. Indeed, any C_i is mapped to some effective or anti-effective (-2)-divisor which itself is supported on smooth rational curves. Thus we do obtain a smooth rational bisection (potentially different from C) and a root lattice R_0 supported on fiber components as before. $\square$

5. Ruling out the additional root types

Many readers may be used to the situation where a root lattice R_0 supported on the singular fibers of an elliptic fibration determines the fibers themselves (up to some ambiguity within Kodaira types I_2/III and I_3/IV) as the extended Dynkin diagrams $\tilde{R}_v$ corresponding to the orthogonal summands of R_0. Presently, this will indeed often be the case, but not always (leading to different moduli components, in fact). We shall make this more precise in the following very useful criterion. Then we shall illustrate the method with an example in 5.2 before turning to the additional root types from 3.2 in 5.3.

5.1. **Criterion for fiber configurations.** By inspection, for all data in Table 1, the isotropic vector E involves a fraction of the positive vector h and the dual vector $e^\vee$ of a single curve $e \in R$. In particular, $R_0 = R \setminus \{e\}$. This set-up often already suffices to rule out many fiber configurations for $|2E|$ as we shall record in the following criterion.

Criterion 5.1. If less than two of the curves forming R_0 meet e, then the only possible fibre configuration for $|2E|$ is comprised of $\tilde{R}_v$ fibers for the orthogonal summand R_v of R_0.

Remark 5.2. Note that there still may be multiplicities at the multiplicative fibers involved.

Proof. Assume that the orthogonal summands $R_1, \ldots, R_m$ embed into a singular fiber of type $\tilde{\mathcal{R}}$ by way of the reflection σ. By Corollary 4.5, we may assume that the R_i have total rank one less than $\mathcal{R}$. Now there are two cases to distinguish.

If all R_i are orthogonal to e, then for rank reasons the bisection $B = \sigma(e)$ meets a single simple fiber component Θ of $\tilde{R}$, and the R_v embed as finite index sublattice into $\tilde{\mathcal{R}} \setminus \{\Theta\} = \mathcal{R}$ via σ. But then

$$\bigoplus_{i=1}^m \sigma(R_i) \hookrightarrow \mathcal{R}$$

is primitive by Lemma 3.1; on the other hand, $\mathcal{R}$ is irreducible, of course, so $m = 1$ and $\mathcal{R} = R_1$ as claimed.

On the other hand, if a single curve e' in the R_i meets e, then $e'.e = 1$ implies that either the fiber is multiple and thus multiplicative, so the rank condition

only allows for $\tilde{\mathcal{R}} = \tilde{R}_1$ (i.e. $m = 1$); or B meets two different fiber components Θ, Θ' of which exactly Θ', say, is contained in the support of $\sigma(e')$ while both components are off the images of the other curves in the R_i under σ. That is, the R_i embed already into $\tilde{\mathcal{R}} \setminus \{\Theta\} = \mathcal{R}$, and we conclude as before. $\qquad\square$

Corollary 5.3. *In the above framework, there are never two orthogonal root lattices embedding into $\tilde{E}_8$.*

5.2. **Prototype example ($R = A_5 + 2A_2$ cont'd).** With the primitive isotropic vector E from Table 1, there are several alternatives for singular fibers supporting the root lattice $R_0 = A_5 + A_2 + A_1$:

$$\tilde{A}_5 + \tilde{A}_2 + \tilde{A}_1, \ \tilde{E}_6 + \tilde{A}_2, \ \tilde{E}_7 + \tilde{A}_1, \tilde{E}_8.$$

However, by Criterion 5.1, only the first alternative may be compatible with the extra information provided by the smooth rational bisection B induced by the curve a_2. To check the compatibility, we can simply draw an appropriate diagram of (-2)-curves comprising fiber components and bisection (where the dashed arrows mean that the fiber may be multiple (though not simultaneously when involving the I_3 fiber)).
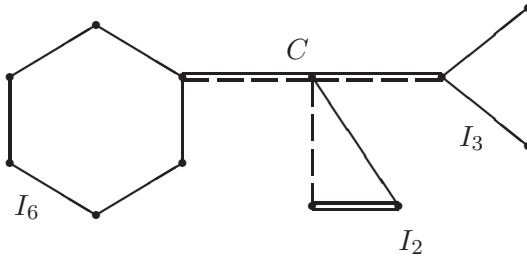

FIGURE 1. Configurations supporting the root type $R = A_5 + 2A_2$

5.3. **Additional root types.** We now turn to the additional root types from 3.2 which we could not rule out in positive characteristic because we cannot apply the orbifold Bogomolow–Miyaoka–Yau inequality. Yet we claim that they do not occur:

Proposition 5.4. *No root type from 3.2 is supported on an Enriques surface in odd characteristic.*

Proof. We first assume that $R \neq 4A_2 + A_1$ (so in particular R is 2-elementary). By Corollary 4.5, the data from Table 1 endows Y with an elliptic fibration whose jacobian is an extremal rational elliptic surface. More precisely, Criterion 5.1 predicts that the singular fibers are given as $\tilde{R}_v$ for the orthogonal summands of R_0. However, this would cause the Euler-Poincaré characteristic to exceed 12, contradiction.

It remains to discuss the root type $R = 4A_2 + A_1$ with bisection B off R_0 induced by the data in Table 1. By Criterion 5.1, the singular fibers are determined as $4\tilde{A}_2$, i.e. Y (and thus $\text{Jac}(Y)$) has four fibers of Kodaira type I_3. This may seem

plausible, yet we claim that it does not occur. In characteristic 3, this follows readily from the classification of semi-stable extremal rational elliptic surfaces in [13] (or from the degeneration of the group scheme of 3-torsion points in characteristic 3, as this causes the Hesse pencil X_{3333} to degenerate, cf. Table 2). Otherwise, we have $p > 3$ and consider the K3 cover X which is endowed with an elliptic fibration with 8 fibers of type I_3 (or potentially some merging to type I_6), and sections O, P which B splits into. In the language of Mordell–Weil lattices [27], P has height 4 (or 5/2 or 1 if there are mergings). Hence the given curves generate a sublattice

$$(7) \qquad\qquad L \hookrightarrow \mathrm{NS}(X)$$

of 3-length 8 (resp. $7, 6$) and rank 19 (resp. $20, 21$). Note that the embedding (7) need not be primitive because there may be three-torsion sections involved, but it any case the primitive closure L' will have 3-length at least 4 (resp. $3, 2$). To see this, just verify that neither the section P nor any of its translates by torsion sections may be 3-divisible, since otherwise the resulting section would have height 4/9 (resp. 5/18 or 1/9), but the height pairing has image in $\frac{1}{6}\mathbb{Z}$ by inspection of the present singular fibers and their correction terms.

On the other hand, since the Tate conjecture is known for elliptic K3 surfaces by [1], either X is supersingular and $\mathrm{NS}(X)$ is p-elementary of rank 22, or by [6], X lifts to characteristic zero with all of $\mathrm{NS}(X)$ which thus embeds into the K3 lattice $\Lambda = 3U + 2E_8$. Either case imposes the rank inequality

$$(\text{3-length})(L') = (\text{3-length})(L^\perp) \leq \mathrm{rank}(L^\perp) = 22 - \mathrm{rank}(L')$$

where the orthogonal complement is taken in $\mathrm{NS}(X)$ in the supersingular case resp. in Λ. Since this inequality never holds presently, we conclude that the root type $R = 4A_2 + A_1$ cannot be supported on an Enriques surface. This completes the proof of Proposition 5.4. $\qquad\square$

Corollary 5.5. *There are no other root types supported on Enriques surfaces in odd characteristic compared to characteristic zero.*

5.4. **Orbifold Bogomolow–Miyaoka–Yau inequality.** As a byproduct of Proposition 5.4 we record the following consequence:

Corollary 5.6. *The orbifold Bogomolow–Miyaoka–Yau inequality holds for any $\mathbb{Q}_\ell$-cohomology projective plane obtained from an Enriques surface.*

Remark 5.7. Of course, the above argument works in characteristic zero, too. Hence, in the classification of maximal root types supported on Enriques surfaces in [9] and [22], there is no formal need to appeal to the orbifold Bogomolov–Miyaoka–Yau inequality.

6. Base change construction

Usually Enriques surfaces are not so easy to get a hand on explicitly, especially with regards to the smooth rational curves which we presently care about. Here one may equally well think about Enriques' original model, Horikawa's model or

the torsor interpretation alluded to in 4.2. Presently, however, the root lattice R of rank 9 endows Y with so much structure that everything can be made totally explicit. This is due to the special elliptic fibration from Corollary 4.5. Notably, π^*E induces an elliptic fibration on the K3 cover

$$\pi^*f : \; X \longrightarrow \mathbb{P}^1$$

which generically would have no section, but presently clearly does as the smooth rational bisection B of $|2E|$ splits into two disjoint smooth rational curves on X which are exchanged by the Enriques involution τ (as exploited in 5.3). We denote them by O and P where the former indicates the chosen zero section of π^*f.

It is exactly this situation where a base change construction of Kondō [12] kicks in (later extended in [7]). Since Kondō's original argument works only over $\mathbb{C}$ as it builds on Nikulin's classification of non-symplectic automorphisms of complex K3 surfaces, we shall give a proof which also works in odd characteristic. (A similar argument is given independently in forthcoming work of Martin [15].)

Proposition 6.1. *There is an involution $\imath$ on X such that*

$$\tau = (\text{translation by } P) \circ \imath.$$

Proof. Let t_P denote translation by P, defined fiberwise and extended to a symplectic automorphism of X. Set

$$\imath = t_{-P} \circ \tau = t_P^{-1} \circ \tau.$$

Then $\imath$ fixes O as a set, with two fixed points in the ramified fibers. We claim that $\imath^2 = 1$. To see this, note that $\imath^2$ fixes fibers as sets and O pointwise. Hence $\imath^2$ acts as an automorphism on the generic fiber X_η of π^*f. In particular,

$$\text{ord}(\imath^2) \in \{1, 2, 3, 4, 6\}.$$

Outside characteristic 3, the order corresponds to the action on the regular 1-form of X_η, i.e. $\imath^2$ has order n if and only if it acts by a primitive n-th root of unity on $H^0(X_\eta, \omega_{X_\eta})$. Since given a minimal Weierstrass form, we locally have $\omega_X = dt \wedge \omega_{X_\eta}$, the same holds for its action on $H^0(X, \omega_X)$. But by definition, $\imath^2$ is symplectic, so $\imath^2 = 1$.

In characteristic 3, this argument may only break down for a wild automorphism, more precisely if $\text{ord}(\imath^2) = 3$ (order 6 is ruled out by the above reasoning because $\imath^2$ is symplectic.) This in turn implies that X_η has zero j-invariant. In particular, π^*f is an isotrivial fibration, and all singular fibers are additive. But then the ramified fibers necessarily are smooth (since otherwise they would have to be multiplicative), and $\imath^2$ fixes just one point on them. In detail, if the generic fiber is given by

$$y^2 = x^3 - \alpha^2 x, \quad \alpha \in K(t),$$

then $\imath^2$ would have to act by

$$(x, y) \mapsto (x + \alpha, y) \quad \text{or} \quad (x - \alpha, y),$$

and this action directly extends to the all smooth fibers including O being the single fixed point. On the other hand, $\imath$ itself acts as an automorphism of a ramified fiber X_0 as it fixes $O|_{X_0}$. In particular, from the definition of $\imath$ we can infer

$$\imath(P|_{X_0}) = -P|_{X_0} \implies \imath^2(P|_{X_0}) = P|_{X_0},$$

i.e. $\imath^2$ has at least two fixed points on the ramified fiber X_0, contradiction. $\quad\square$

With the involution $\imath$ at our disposal, the reasoning from [12] applies to prove that X is in fact a quadratic base change from the rational elliptic surface $\mathrm{Jac}(f)$ (hence the title of this section).

Corollary 6.2. $X/\imath = \mathrm{Jac}(f)$.

Remark 6.3. The base change type construction may also be applied to the problem of Enriques surfaces with finite automorphism group in positive characteristic (much like what Kondō did over $\mathbb{C}$, but now without reference to Torelli type theorems). For details, see upcoming work of Martin [15].

7. Translation into equations

It is about time to collect what we have found so far for Enriques surfaces Y supporting a root type R from (6), and start to take advantage of it following [7]. To this end, we note two crucial observations:

Observation 7.1. (1) the K3 cover X is uniquely determined by the rational elliptic surface $\mathrm{Jac}(f)$ and the base change which in turn only depends on the location of the ramified fibers; both information we can easily read off from Y;

(2) the section P which is used to define the Enriques involution $\tau = t_P \circ \imath$ is anti-invariant for $\imath$.

In consequence, considering (2), the section P is invariant for the composition $\jmath$ of $\imath$ with the hyperelliptic involution on X_η, i.e. it is induced from a section P' on the resolved quotient

$$X' = \widetilde{X/\jmath}.$$

This is yet another K3 surface which happens to be the quadratic twist of $\mathrm{Jac}(f)$ at the ramified fibers. Assume that $\mathrm{Jac}(f)$ is given by a minimal Weierstrass form

$$\mathrm{Jac}(f): \quad y^2 = x^3 + a_2 x^2 + a_4 x + a_6, \quad a_i \in K[t], \quad \deg a_i \le i,$$

not all a_i being scalar multiples of i-th powers of a single linear form in t. Then the quadratic twist X' may be defined by

$$(8) \qquad\qquad X': \quad gy^2 = x^3 + a_2 x^2 + a_4 x + a_6$$

where the quadratic polynomial $g \in K[t]$ encodes the location of the ramified fibers (i.e. it has two distinct zeroes, possibly including ∞ so that it would be linear in that case). Let us check under what condition a section P' on X' will lead to τ being fixed point free. Clearly, P' has to be disjoint from the induced

zero section O', but there is a little more to it. Namely, in order for P not to meet O at the ramified fibers, P' has to meet the ramified fibers of X' (of Kodaira type I_n^* for some $n \in \mathbb{N}_0$) at a component different from O', and in fact in one of the two far components if $n > 0$.

Writing P' componentwise as $P' = (\mathrm{x}, \mathrm{y})$ for a priori rational functions in t, all the above translates as

$$\deg \mathrm{x} = \deg \mathrm{y} = 2$$

in terms of the twisted Weierstrass equation (8), with extra conditions entering if far components of the ramified fibers are met or if other singular fibers are intersected non-trivially (i.e. at different components than O' as will happen frequently in our setting).

In what follows, we shall start with the rational elliptic surface $\mathrm{Jac}(f)$ and aim for endowing its quadratic twist X' with a suitable section P'. Note that this is a codimension one condition in moduli unless P' is the classical example of a two-torsion section (but in that case there will be extra ramification conditions cutting down the number of parameters in our setting).

8. Extremal rational elliptic surfaces

In order to attack determining the Enriques surfaces supporting the root types from 3.1, and thus to prove Theorem 9.2, we turn our attention to Observation 7.1 (1). Extremal rational elliptic surfaces have been studied starting from Beauville (for the semi-stable case over $\mathbb{C}$, cf. [2]). In positive characteristic, the picture changes only slightly, due to Lang [13], [14]. More precisely, the singular fiber types may change (or the surfaces may cease to exist anymore, such as X_{3333} in characteristic 3 as we have used in 5.3; compare Table 2), but the torsion structure of the Mordell–Weil group does not change.

The following table reproduces the classification, following the notation over $\mathbb{C}$ from [16]. The Weierstrass forms are designed for our purposes with a view towards quadratic twists; hence, in order to include equations valid in characteristic 2, major modifications would be in order. Note that the 2-torsion sections are always located at zeros of $f(x)$; here we made sure to always normalise one of them, if any over $K(t)$, to $(0,0)$.

There is one surface, X_{33}, missing from the above table because it is a degeneration of X_{321}. Let us explain this briefly (a similar, but more abstract argument for X_{431} and X_{44} has appeared in [21]). Compared to X_{321} as given in Table 2, X_{33} has the singular fibers of type I_2, I_1 replaced by a III fiber. Both surfaces can be combined in an isotrivial family of rational elliptic surfaces:

$$\mathcal{X}: \quad y^2 = x(x^2 + cx + t), \quad c \in K.$$

Here all surfaces at $c \neq 0$ are isomorphic to X_{321} (since K is algebraically closed) while X_{33} sits at $c = 0$. Recall that for our Enriques surfaces, we are going to apply quadratic base changes to extremal rational elliptic surfaces

notation	Weierstrass eqn. $y^2 = f(x)$	sing. fibers	char(K)	MW
X_{9111}	$x^3 + (tx+1)^2/4$	$I_9/\infty, I_1/t^3 - 27$	$p \neq 3$	$\mathbb{Z}/3\mathbb{Z}$
		$I_9/\infty, II/0$	$p = 3$	
X_{8211}	$x(x^2 + (t^2+2)x + 1)$	$I_8/\infty, I_2/0, I_1/t^2 + 4$	$p \neq 2$	$\mathbb{Z}/4\mathbb{Z}$
X_{6321}	$x(x^2 + (t^2/4 + t - 2)x + 1 - t)$	$I_6/\infty, I_3/0, I_2/1, I_1/-8$	$p \neq 2, 3$	$\mathbb{Z}/6\mathbb{Z}$
		$I_6/\infty, I_3/0, III/1$	$p = 3$	
X_{5511}	$x^3 - tx^2 + ((1-t)x - t)^2/4$	$I_5/0, \infty, I_1/t^2 - 11t - 1$	$p \neq 5$	$\mathbb{Z}/5\mathbb{Z}$
		$I_5/0, \infty, II/3$	$p = 5$	
X_{4422}	$x(x-1)(x-t^2)$	$I_4/0, \infty, I_2/\pm 1$	$p \neq 2$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
X_{3333}	$x^3 + (3tx + t^3 - 1)^2/4$	$I_3/\infty, t^3 - 1$	$p \neq 3$	$(\mathbb{Z}/3\mathbb{Z})^2$
X_{321}	$x(x^2 + x + t)$	$III^*/\infty, I_2/0, I_1/\frac{1}{4}$	$p \neq 2$	$\mathbb{Z}/2\mathbb{Z}$
X_{222}	$x(x-1)(x-t)$	$I_2^*/\infty, I_2/0, 1$	$p \neq 2$	$(\mathbb{Z}/2\mathbb{Z})^2$
X_{141}	$x(x^2 + t(t+2)x + t^2)$	$I_4/\infty, I_1^*/0, I_1/-4$	$p \neq 2$	$\mathbb{Z}/4\mathbb{Z}$
X_λ	$x(x^2 + tx + \lambda t^2)$ $(\lambda \in K^\times)$	$I_0^*/0, \infty$	$p \neq 2$	$(\mathbb{Z}/2\mathbb{Z})^2$

TABLE 2. Extremal rational elliptic surfaces

(Observation 7.1 (1)). Hence the following claim will allow us to tacitly omit X_{33} in the sequel.

Claim 8.1. The family of quadratic base changes of X_{321} contains those of X_{33} in its closure.

Proof. Let us write the base change by

$$(9) \qquad t = \gamma^{-1}\frac{q}{r}$$

for some quadratic polynomials $q, r \in K[s]$ and $\gamma \in K^\times$, say (which, of course, we could still normalize, for instance using Möbius transformations). Applying (9) to X_{321}, we can rescale coordinates by

$$(x, y) \mapsto (\gamma^{-1/2}x, \gamma^{-3/4}y)$$

to arrive at the Weierstrass form

$$y^2 = x\left(x^2 + \gamma^{1/4}x + \frac{q}{r}\right).$$

Extending to $\gamma = 0$, we land exactly in the family of quadratic base changes of X_{33}. $\qquad\square$

9. ONE-DIMENSIONAL FAMILIES

We now come to the root types from 3.1, aiming to prove their existence and eventually deduce Theorem 1.1. The arguments proceed exactly as in characteristic zero (except for a few degenerations in characteristics $3, 5$ which we shall point out below). In particular, we have the same critical result for the root lattice R_0:

Proposition 9.1. *If the elliptic fibration arises from the data in Table 1, then all curves representing R_0 are taken to fiber components by a suitable composition of reflections.*

The proof of the proposition consists of a tedious case-by-case analysis for each root type covering all possible fiber configurations including multiplicities (and often using the obstructions posed by Lemma 3.1). Since the arguments are combinatorial in nature, essentially only using elementary properties of root lattices and Weyl groups, they completely parallel the characteristic zero setting (which was sketched to some extent in [22]). We omit the details for brevity (except for a quick pointer in 9.1).

In particular, Proposition 9.1 allows us to read off how the smooth rational bisection B intersects the singular fibers (depending on the multiplicities). This translates into the configuration of P' and thus yields explicit conditions for the Weierstrass form in (8). Spelling out these conditions, we can solve directly for the resulting families (see e.g. 9.1, 9.2) and obtain the following result (paralleling the characteristic zero case).

Theorem 9.2. *Let Y be an Enriques surface supporting a root lattice R of rank 9. Then Y appears in Table 3 or 4.*

The notation in Tables 3 and 4 should be self-explanatory based on the previous sections.

#	root type	Jac($|2E|$)	section: x(P')	quadr. twist: g
1	A_9	X_{8211}	$\mu t - 1$	$(\mu t - 1)(\mu t + \mu^2 - 1)$
2	$A_8 + A_1$	X_{6321}	$\mu t + 1$	$(\mu t + 1)(\mu t + (2\mu + 1)^2)$
	also #27			
3	$A_7 + A_2$	X_{6321}	$\mu(t - 1)$	$\mu t^2 + 4\mu(\mu + 1)t - 4(\mu + 1)^2$
	also #28			
4	$A_7 + 2A_1$	X_{4422}	μt	$(\mu t - 1)(t - \mu)$
	also #27, #29			
5	$A_6 + A_2 + A_1$	X_{6321}	$-(ut - 4u^2 + 3u - 1)/(u^2(u + 1))$	
				$((u^2 + u)t - (u - 1)^2)(ut - 4u^2 + 3u - 1)$
6	$A_5 + A_4$	X_{6321}	$(t - 1)(\mu t - 1)$	$(\mu t - 1)(\mu(4\mu + 1)t - (2\mu + 1)^2)$
7	$A_5 + A_3 + A_1$	X_{4422}	$\mu t + 1 - \mu$	$(\mu t + 1 - \mu)(t + 1 - \mu)$
	also #30			
8*	$A_5 + 2A_2$	X_{6321}	$27(t + u)(t - 1)/((u - 8)(u + 1)^2)$	$(t + u)(3(u + 4)t - (u - 2)^2)$
	also #28			
9*	$A_5 + A_2 + 2A_1$	X_{6321}	$27(t - u)^2/((u - 1)(u + 8)^2)$	
				$3(u + 11)t^2 - 4(u^2 + 13u + 4)t + 4(u - 4)^2$

TABLE 3. Explicit data for Enriques surfaces supporting the 31 maximal root types (those not existing in char. 3 being starred)

Since the proof of Theorem 9.2 follows the same lines as in characteristic zero, we shall only comment on the degenerate cases (compare [22, §9]).

#	root type	Jac($|2E|$)	section: x(P')	quadr. twist: g
$10^\dagger$	$2A_4 + A_1$	X_{5511}	$-(s^5t^2 - s^3(11s^2 - 15s + 5)t - (s-1)^5)/(5s^2 - 5s + 1)^2$	
			$s^2(4s-1)t^2 - 2(2s-1)(11s^2 - 11s + 2)t - (4s-3)(s-1)^2$	
11	$A_4 + A_3 + 2A_1$	X_{4422}	$4\mu(t + \mu)$	$(t + \mu)(4\mu t - 4\mu^2 - 1)$
12	$3A_3$	X_{4422}	$\mu t^2 - \mu + 1$	$\mu t^2 - \mu + 1$
	also #16, #19			
13	$2A_3 + A_2 + A_1$	X_{4422}	$(\mu t - \mu + 1)^2$	$(\mu t - \mu + 2)(\mu t + t - \mu + 1)$
14	$2A_3 + 3A_1$	X_{4422}	$-(t^2 + 4\mu t + 1)/(4\mu^2 - 1)$	$t^2 + 4\mu t + 1$
15^*	$A_3 + 3A_2$	X_{3333}	$(t-1)((s^3 + 6s^2 + 9s + 3)t + s^3 + 3s^2 - 3)/s^2(2s + 3)$	
			$((6s+3)(s+1)^2t^2 + (8s^3 + 12s^2 - 6s - 6)t + 4s^3 - 6s + 3)$	
16	D_9	X_{9111}	μ	$\mu^2t^2 + 2\mu t + 4\mu^3 + 1$
17	$D_8 + A_1$	X_{8211}	μ	$\mu t^2 + (\mu + 1)^2$
	also #27	X_{8211}	0	$t - \mu$
18	$D_7 + 2A_1$	X_{222}	$t + \mu$	$(t + \mu)(t + \mu - 1)$
19	$D_6 + A_3$	X_{222}	$(\mu t + 1 - \mu)t$	$(\mu t + 1 - \mu)(\mu t + 1)$
	also #16			
20	$D_6 + A_2 + A_1$	X_{222}	$-4\lambda t(\lambda t - 1)$	$(\lambda t - 1)(4\lambda^2 t - 4\lambda + 1)$
21	$D_6 + 3A_1$	X_{222}	0	$t(t - \lambda)$
22	$D_5 + A_4$	X_{5511}	μ	$(\mu + 1)^2t^2 - 2\mu(3\mu + 1)t + \mu^2(4\mu + 1)$
23	$D_5 + A_3 + A_1$	X_{4422}	1	$(t - 1)(t - \lambda)$
24	$D_5 + D_4$	X_{141}	μ	$(\mu + 1)t^2 + 2\mu t + \mu^2$
25	$D_4 + A_3 + 2A_1$	X_{4422}	μ	$t^2 - \mu$
	also #18	X_{4422}	0	$t - \lambda$
26	$2D_4 + A_1$	$X_{-4\mu(4\mu+1)}$	$(t + \mu)^2$	$t^2 + (6\mu + 1)t + \mu^2$
27	$E_8 + A_1$	X_{321}	0	$t(t - \lambda)$
28	$E_7 + A_2$	X_{6321}	0	$t - \lambda$
29	$E_7 + 2A_1$	X_{222}	0	$t(t - \lambda)$
30	$E_6 + A_3$	X_{141}	$(\mu t - 1)t$	$(\mu t - 1)(\mu(\mu + 1)t - 1)$
31^*	$E_6 + A_2 + A_1$	X_{6321}	0	$(t - 1)(t - \lambda)$

TABLE 4. Explicit data for Enriques surfaces supporting the 31 maximal root types (those not existing in char. 3 being starred resp. daggered for char. 5)

9.1. $\boldsymbol{R = A_5 + 2A_2}$. Given an Enriques surface Y supporting the root type $R = A_5 + 2A_2$, we work with the primitive isotropic vector E from Table 2. Then $R_0 = A_5 + A_2 + A_1$, and $\mathrm{Jac}(|2E|) = X_{6321}$ by 5.2.

We now study the smooth rational bisection $B = \sigma(a_1)$ induced by the vertex $a_1 \in A_2$ whose dual vector is involved in E. By definition, B meets the $\tilde{A}_1$ fiber in two different components (unless it is ramified), and all other reducible fibers in a single component – the one which is not contained in the support of the image of the orthogonal summands A_5 and A_2 of R_0 under the reflection σ. Hence $\sigma(A_5)$ is exactly supported on five components of the I_6 fiber, and further reflections take each (-2)-divisor precisely to a fiber component (as stated in

Proposition 9.1). The same applies to $\sigma(A_2)$, and we conclude that the possible intersection patterns are exactly depicted in Figure 1.

In order to pour this set-up into equations as indicated in in Section 7, we first assume that no reducible fiber is multiple. Hence we aim for a quadratic twist X' of X_{6321} at some quadratic polynomial g without roots at $t = 0, 1, \infty$ such that there is a section P' compatible with the intersection pattern of C. That is, we require P' to meet the non-identity component of the I_2 fiber at $t = 1$; on the Weierstrass model (8), P' thus goes through the node at $(x, y, t) = (0, 0, 1)$. In terms of the coordinates of $P' = (\mathrm{x}, \mathrm{y})$, this means

$$(t - 1) \mid \mathrm{x}, \mathrm{y}.$$

Setting $\mathrm{x} = (t - 1)(\mu t + \lambda)$, the right-hand side of (8) evaluates as

$$(10) \qquad\qquad (t - 1)^2 H_{\lambda,\mu}(t)$$

for some degree 4 polynomial $H \in K[\lambda, \mu][t]$ when substituting $x = \mathrm{x}$. For the section P' to lie on X' then simply requires that (10) factors as gy^2, i.e. H admits a square factor (agreeing with y), and the residual quadratic polynomial has to be exactly g (which is not a square!). Put differently, the non-zero polynomial H determines y and g as soon as its discriminant vanishes. But the discriminant of H with respect to t is readily computed:

$$\mathrm{discrim}_t(H) = \mu^6(\lambda + 1)^3(\lambda^3 - 6\lambda^2\mu - 15\lambda\mu^2 - 8\mu^3 - 27\mu^2).$$

If $\mu = 0$, then P' meets the I_6 fiber at $t = \infty$ at a non-identity component, but C meets twice the same component, contradiction. Similarly, if $\lambda = -1$, then P' meets the I_3 fiber at $t = 0$ at a non-identity component, giving the same kind of contradiction. It thus remains to investigate the cubic factor which we transform to the following shape in auxiliary parameters u, v with $\mu = 1/v, \lambda = u\mu$:

$$(11) \qquad\qquad u^3 - 6u^2 - 15u - 27v - 8.$$

If $\mathrm{char}(K) \neq 3$, then solving (11) for v directly leads to

$$\mathrm{x} = 27(t + u)(t - 1)/((u - 8)(u + 1)^2), \quad g = (t + u)(3(u + 4)t - (u - 2)^2).$$

In conclusion, the base change construction from section 6 produces a one-dimensional family of Enriques surfaces, parametrized by u, supporting the root type R.

In contrast, if $\mathrm{char}(K) = 3$, then (11) factors as $(u + 1)^3$, i.e. $\lambda = -\mu = \nu^3$, giving

$$\mathrm{x} = \mu(t - 1)^2, \quad g = (t - 1)((\nu^2 + \nu)t + 2(\nu^2 + \nu + 1)).$$

Note, however, that the fiber at $t = 1$ degenerates to Kodaira type III in characteristic 3 (see Table 2). With g as above, this would be multiple on Y, contradiction. In order words, the involution τ on X (which is still well-defined as before) ceases to be fixed-point free, and the base change construction involving P', g etc. does not produce Enriques surfaces in characteristic 3 anymore.

9.2. **Second moduli component.** What may come as a surprise at first is that we are not yet done with the given root type, in fact. Namely, we still have to take multiple fibers into account. For instance, ramification at the I_3 fiber visibly occurs at $u = 0, 2$ within the above family, and τ stays fixed-point free exactly at the CM point $u = 2$. (For the I_2 or I_6 fiber, this might hold at $u = -1$ or $u = 8$ resp. at $u = \infty$, but this is not so obvious from the given parametrization.) However, the situation changes drastically if we allow for (or, in fact, impose abstractly) ramification at the I_6 fiber: here the section P is forced to meet the opposite component Θ_6 of the resulting I_{12} fiber on X. Hence the theory of Mordell–Weil lattices [27] shows that P has height zero, i.e. it is a torsion section, and in fact of order 2. But then $P = (0,0)$ is already defined on X_{6321}, so substitution into (8) returns zero identically. Hence, imposing the quadratic base change to ramifiy at ∞ yields a second family of Enriques surfaces Y' originating from

$$(12) \qquad\qquad P' = (0,0), \quad g = t - \lambda \quad (\lambda \in K^\times).$$

The main difference to the previous case is that Y' even supports the root type $E_7 + A_2$, i.e. the primitive closure R' of R (as one easily verifies in Figure 1). Obviously this construction works in any characteristic (it can even be adjusted to characteristic 2 as we shall explore in future work).

Remark 9.3. We shall verify in Section 10 that the two families of Enriques surfaces constructed (outside characteristic 3) are in fact distinct.

9.3. **Further degenerations in characteristic** 3. We have already seen in 9.1 that the data under #8 degenerate in characteristic 3 (but by 9.2, the given root type is still supported on the family under #28). There are three more root types which degenerate, and in fact cease to exist on Enriques surfaces in characteristic 3.

For #15, this should be clear, since by Criterion 5.1, the root lattice $R_0 = 4A_2$ from Table 1 implies that $|2E|$ would have four fibers of Kodaira type I_3. However, such a surface does not exist in characteristic 3 (since there cannot be 3-torsion in the Mordell–Weil group of length two).

Meanwhile for #31, this follows directly from the set-up from Table 1 as we shall sketch briefly. By Criterion 5.1, $\mathrm{Jac}(|2E|) = X_{6321}$, and either the I_6 fiber is ramified or the bisection B meets it in opposite components. Consider the sections O, P on the K3 cover X which B splits into.

Claim 9.4. $P = (0,0)$, the two-torsion section.

Proof. Otherwise we may compute the height pairing between P and $(0,0)$ following [27]. Since both meet the I_6 fibers (or the I_{12} fibers in case of ramification) in the same components (opposite the identity component met by O), we obtain

$$\langle P, (0,0) \rangle = 2 - (P.(0,0)) - 3 - \ldots \leq -1.$$

But since $(0,0)$ is torsion, the pairing is zero identically, contradiction. $\qquad\square$

It follows that B (just like P) meets the $\tilde{A}_1$ fiber in both components unless it ramifies. Presently, however, the Kodaira type is III by Table 2 (since we are in characteristic 3), so there cannot be ramification. But then the orthogonal summand A_1 embedding into $\tilde{A}_1$ cannot be perpendicular to B, contradiction. (In characteristic $p > 3$, we infer ramification at the I_2 fiber which exactly gives the family in Table 4.)

It remains to discuss #9. Here the isotropic vector E from Table 1 leads to $\mathrm{Jac}(|2E|) = X_{6321}$, again by Criterion 5.1, with smooth rational bisection meeting each singular fiber in a single component. Searching for the corresponding section P' on the quadratic twist X' simplifies substantially if we take into account that X' (just like X_{6321}) admits a 2-torsion section. With the given data, this means that

- either $\mathrm{x}(P')$ is a square in $k[t]$,

- or $\mathrm{x}(P') \sim g$.

In the first case, write $\mathrm{x}(P') = \mu(t - \lambda)^2$. Substitution into (8) yields a degree 4 polynomial in $k[t]$ which we require to split off a square factor. Therefore its discriminant vanishes which one computes as

$$(13) \qquad (\lambda^3\mu + 15\lambda^2\mu + 48\lambda\mu - 64\mu - 27)(\lambda^2\mu - 1)^3\mu^2$$

If $\mu = 0$, then $P' = (0,0)$ is two-torsion while for $\lambda^2\mu = 1$, it meets the I_3 fiber non-trivially. Hence we infer that the first factor of (13) has to vanish. If $\mathrm{char}(k) \neq 3$, then we can solve for μ and obtain the displayed solution. However, if $\mathrm{char}(k) = 3$, then the first factor degenerates to $\mu(\lambda - 1)^3$. Here $\lambda = 1$ implies that the fiber at $t = 1$ is multiple; but the fiber is additive of type III by Table 2, contradiction.

In the second case where $\mathrm{x}(P')$ encodes the ramified fibers, we can directly solve for the residual degree 4 factor of the RHS of (8) after substituting to form a perfect square – this turns out to be impossible unless P' meets the I_3 fiber non-trivially which we presently ruled out.

In summary, in each of the above cases, we find exactly the families in Table 4, but only outside characteristic 3.

9.4. **Degeneration in characteristic** 5. The analysis of the root type $R = 2A_4 + A_1$ is similar to the last case, but considerably more involved since there are no torsion sections on X' to assist us. For starters, Criterion 5.1 applies to the data in Table 1 to show that $\mathrm{Jac}(|2E|) = X_{5511}$. Write

$$\mathrm{x}(P') = \nu t^2 + \mu t + \lambda$$

and substitute into (8). Then the RHS polynomial r has to split of the square of a quadratic polynomial (which will give $\mathrm{y}(P')$). Hence not only does the discriminant of r with respect to t vanish, but also its discriminant (with respect to either coefficient of $\mathrm{x}(P')$). In detail, we have

$$\Delta(r) = \nu^5\lambda^5 r_1.$$

Since the first two factors correspond to the section P' meeting the fiber at $t = \infty$ resp. $t = 0$ non-trivially, we may continue arguing with the third factor r_1 which gives

$$\Delta(r_1, \lambda) = \nu^5 (\mu - 1)^5 r_2^2 r_3^3.$$

We have already discussed the first factor while $\mu = 1$ leads to r_1 splitting off another factor λ^2. We claim that we can also ignore $r_3 = 0$ for our purposes. This can be read off directly from the definition of the discriminant: here there would be a triple root instead of two double roots. Thus we have to determine when the remaining factor

$$\begin{aligned}
r_2 \;=\;& \mu^5 + 55\mu^4\nu + 960\mu^3\nu^2 + 5060\mu^2\nu^3 - 1920\mu\nu^4 + 176\nu^5 \\
& -625\mu^2\nu^2 - 7500\mu\nu^3 + 2500\nu^4 + 3125\nu^3
\end{aligned}$$

vanishes. If $\mathrm{char}(k) = 5$, then $r_2 = (\mu + \nu)^5$, and $\nu = -\mu$ subsequently implies $\lambda = \mu - 1$ and ramification at $t = 3$. But there the fiber has type II, so τ will not be fixed point free.

On the other hand, if $\mathrm{char}(k) \neq 5$, then r_2 has degree 2 in any homogenizing variable. That is, r_2 defines a hyperelliptic curve C, with branch points affinely given by $3125(4\mu - 1)(\mu + 6)^4$. It follows that the curve C is birational to $\mathbb{P}^1$ over the prime field, and parametrizing C gives the data in Table 4.

9.5.   We emphasize that Theorem 9.2 implies Theorem 1.1 except for part *(ii)* on the number of moduli components. This will be treated in the next section.

10. Component analysis

The families of Enriques surfaces from Tables 3 and 4, and the way they were derived, give us a clear picture of the moduli components: they are determined by the possible fiber configurations, including multiplicities, which are compatible with the data in Table 1. Once this classification is achieved (as recorded in Theorem 9.2), it only remains to rule out potential symmetries.

Proposition 10.1. *If a root type R admits more than one moduli components, then R can be found in Table 5. Moreover, all but one components support some proper overlattice $\hat{R}$ of R.*

The following table lists the root types admitting more than one component together with the configurations of reducible fibers (with multiplicities) and the overlattices $\hat{R}$ supported on the extra families. Note that in characteristic 3 the starred root type $R = A_5 + 2A_2$ has only one component since the first fiber configuration degenerates (as studied in 9.1).

10.1. **Proof of Proposition 10.1.** Deriving the families underlying Table 5 is an easy exercise along the lines of the previous sections. We point out one non-trivial step here: the root type $3A_3$ in fact leads to four fiber configurations (including multiplicities), supported on one-dimensional families, but then two turn out to be equivalent (as checked in [22, §10.2]).

	R	R_0	fibers	$\hat{R}$	fibers
a)	$A_8 + A_1$	$A_7 + A_1$	I_8, I_2	$E_8 + A_1$	$2I_8, I_2$
b)	$A_7 + 2A_1$	$2A_3 + 2A_1$	I_4, I_4, I_2, I_2	$E_7 + 2A_1$	$I_2^*, 2I_2, I_2$
				$E_8 + A_1$	$III^*, 2I_2$
c)	$A_7 + A_2$	$A_5 + A_2 + A_1$	I_6, I_3, I_2	$E_7 + A_2$	$2I_6, I_3, I_2$
d)	$A_5 + A_3 + A_1$	$2A_3 + 2A_1$	I_4, I_4, I_2, I_2	$E_6 + A_3$	I_1^*, I_4
e)	$A_5 + 2A_2 \, (*)$	$A_5 + A_2 + A_1$	I_6, I_3, I_2	$E_7 + A_2$	$2I_6, I_3, I_2$
f)	$3A_3$	$2A_3 + 2A_1$	I_4, I_4, I_2, I_2	$D_6 + A_3$	$2I_4, I_4, I_2, I_2$
				D_9	$I_1^*, 2I_4$
g)	$D_8 + A_1$	$A_7 + A_1$	I_8, I_2	$E_8 + A_1$	$2I_8, I_2$
h)	$D_6 + A_3$	$D_6 + 2A_1$	I_2^*, I_2, I_2	D_9	I_4^*
i)	$D_4 + A_3 + 2A_1$	$2A_3 + 2A_1$	I_4, I_4, I_2, I_2	$D_7 + 2A_1$	$2I_4, I_4, I_2, I_2$

TABLE 5. Root types with more than one moduli components

It remains to verify that the families are indeed distinct (as opposed to the symmetry alluded to above). Over fields of characteristic zero, this can be achieved as follows. The smooth rational curves on the Enriques surface Y equip the K3 cover X with a rank 19-sublattice

$$L \subset \mathrm{NS}(X)$$

generated by fiber components, the two sections O, P mapping to the smooth rational bisection B and torsion sections. By the moduli theory of complex K3 surfaces, it follows that generically $\rho(\mathcal{X}) = 19$. Upon checking that L embeds primitively into $\mathrm{NS}(\mathcal{X})$ (which amounts to verifying that the section mapping to B does not yield a divisible point in $\mathrm{MW}(\mathcal{X})$), we can thus compute the discriminant (of $\mathrm{NS}(\mathcal{X})$) of the generic member of $\mathcal{X}$ based on the formula from [24, (11.22)]. As soon as these discriminants differ, the families have to be distinct.

10.2. In positive characteristic, it is still easy to check that L embeds primitively into $\mathrm{NS}(X)$, but there is a little complication since we could have $\rho(\mathcal{X}) = 22$, i.e. the whole family of K3 surfaces could be supersingular (as happens for the K3 surfaces with maximal root types from Theorem 2.2). We resolve this problem by the following result.

Proposition 10.2. *For each of the above families, the general member (over $\bar{\mathbb{F}}_p$) is an ordinary K3 surface.*

Proof. Assume that the general member X over $\bar{\mathbb{F}}_p$ is not ordinary, i.e. X has height $h \geq 2$ including possible ∞ (the supersingular case). If $h < \infty$, then

$$\rho(X) \leq 22 - 2h \leq 18$$

which is impossible since L has already rank 19. Hence $h = \infty$, and X is supersingular. Denote the Artin-invariant of X by σ such that

$$\mathrm{NS}(X)^\vee / \mathrm{NS}(X) \cong (\mathbb{Z}/p\mathbb{Z})^{2\sigma}.$$

More precisely, $\mathrm{NS}(X)$ is a p-elementary lattice of length at least 4 (since the supersingular K3 surface with Artin invariant $\sigma = 1$ is just a point in moduli by [19]). We continue to argue similar to the proof of Proposition 5.4. We have

$$4 \leq 2\sigma = p\text{-length}(\mathrm{NS}(X)) \leq p\text{-length}(L) + \underbrace{p\text{-length}(L^\perp)}_{\leq \,\mathrm{rank}(L^\perp)=3}.$$

We now have to distinguish whether $p = 3$ or $p > 3$. In the latter case, L always has p-length zero, so we obtain a contradiction. The same holds for $b)$ and $f) - i)$ in characteristic 3 including the second family for $a)$. For the remaining families, we proceed by exhibiting an explicit ordinary member of the family. Note that Proposition 10.2 genuinely concerns K3 surfaces, so we are free to consider those surfaces where the Enriques involution degenerates (as will be quite convenient). In fact, we shall specialize to singular K3 surfaces over $\mathbb{Q}$ (i.e. with $\rho = 20$) and then switch to the smooth reduction mod 3:

$a)$ At $\mu = -\frac{1}{2}$, the I_3 fiber in the first family becomes ramified. Thus

$$h(P) = 4 - 2 \cdot \frac{5}{6} - \frac{3}{2} = \frac{5}{6},$$

and we obtain $\rho(X) = 20$, $d = \mathrm{discr}(\mathrm{NS}(X)) = -20$ by [24, (11.22)].

$c)$ The first family has an additional section Q at $\mu = -2$, namely Q is also induced from a section Q' on X' with $x(Q') = (-1 + 1/\sqrt{2})t + 1$ and

$$h(Q) = 4 - 2 \cdot \frac{5}{6} - 2 \cdot \frac{2}{3} = 1.$$

Since the height of P is unchanged,

$$h(P) = 4 - 2 \cdot \frac{5}{6} - 2 \cdot \frac{1}{2} = \frac{4}{3},$$

it remains to compute the pairing $\langle P, Q \rangle = 2\langle P', Q' \rangle$. Since P', Q' are disjoint and meet one I_0^* fiber at the same non-identity component and the other at different ones, while the I_6 fiber is meet at the same fiber component (for suitable sign choices in the y-coordinate), the latter gives

$$\langle P', Q' \rangle = 2 - 1 - \frac{1}{2} - \frac{5}{6} = -\frac{1}{3}.$$

For the Mordell-Weil lattice on X, we thus obtain

$$\mathrm{MWL}(X) = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 1 \end{pmatrix}$$

of determinant $-8/3$. Hence $\rho(X) = 20$ and $d = -32$, again by [24, (11.22)]. The second family attains a ramified I_3 fiber at $\lambda = 0$ such that $\rho = 20, d = -8$.

$d)$ The first family attains two multiple fibers at $\mu = 0$ while P becomes two-torsion. This yields $\rho = 20, d = -32$.
The second family attains a ramified I_1 fiber at $\mu = -\frac{1}{2}$ which P meets non-trivially. Hence

$$h(P) = 4 - 2 \cdot \frac{5}{4} - \frac{1}{2} = 1,$$

so that again $\rho = 20, d = -32$.

e) The second family agrees with that of *c*).

In either case, it is easily verified that the reduction modulo 3 is good. The proof of Proposition 10.2 will be completed by verifying the following statement:

Claim 10.3. For each of the above K3 surfaces over $\mathbb{Q}$, one has $\rho(X \otimes \bar{\mathbb{F}}_3) = 20$.

Proof. To see the claim, one can use a kind of complementary statement to [23, Prop. 4.1]. Here we can argue with the Shioda–Inose structure following [28] which relates X to a product of two isogenous elliptic curves E, E' with CM in $\mathcal{O}_K$ or an order of conductor 2 for $K = \mathbb{Q}(\sqrt{d})$. Since d is a square modulo 3, the elliptic curves are ordinary, and $\rho((E \times E') \otimes \bar{\mathbb{F}}_3) = 4$ which yields the claim. Alternatively, one can argue purely on the level of lattices that $\mathrm{NS}(X)$ does not admit an embedding into the Néron–Severi lattice of a supersingular K3 surface for any p such that d is a square modulo p. $\square$

10.3.   It follows from Proposition 10.2 that the generic member of the family $\mathcal{X}$ has Picard number 19 and the same discriminant as in characteristic zero (after checking that there is no extra divisibility in $\mathrm{MW}(\mathcal{X})$ in characteristic p). In consequence, the given families are all distinct, and Proposition 10.1 is proven. $\square$

10.4. **Conclusion.** This proves part *(ii)* of Theorem 1.1. In view of 9.5, the proof of Theorem 1.1 is thus complete. $\square$

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INSTITUT FÜR ALGEBRAISCHE GEOMETRIE, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY

E-mail address: schuett@math.uni-hannover.de