Non(anti)commutative Gauge Theories in Harmonic Superspace

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To the Memory of my Mother
Silvana Zaia de Quevedo

Zusammenfassung

In dieser Arbeit werden die Eigenschaften von nicht-Singlett Q-deformierten $N = 2$ supersymmetrischen Eichfeldtheorien untersucht. Nach einer Analyse der durch eine allgemeine Deformation induzierte supersymmetrie Brechung konstruieren wir die nicht-Singlett deformierten Eichtransformationen für sämtliche Komponenten des Vektormultipletts sowie die entsprechende Seiberg-Witten Abbildung. Entsprechend der Wahl unterschiedlicher nicht-Singlett Deformationstensoren werden verschiedene Deformationen der supersymmetrischen Yang-Mills-Wirkung bestimmt. Mittels einer Zerlegung solcher Tensoren lassen sich exakte Wirkungen für den bosonische Sektor der deformierten $N = (1, 0)$ und der vollständigen $N = (1, 1/2)$ erweiterten supersymmetrischen Theorie berechnen. Durch eine sogenannte schwache Wiederherstellung vernachlässigter Freiheitsgrade des Deformationstensors erhalten wir eine neue Wirkung, welcher die balancierte Symmetriebrechung von $N = (1, 1/2)$ nach $N = (1, 0)$ Supersymmetrie beschreibt. Zum Abschluss bestimmen wir die entsprechende Supersymmetrietransformationen für alle betrachtete Fälle.

Als Vorbereitung einer nicht(anti)kommutativen Verallgemeinung von $N = 2$ erweiterten Eichtheorien, werden im ersten Teil dieser Arbeit nichtkommutative Feldtheorien und harmonische Superräume eingeführt. Des Weiteren werden die Eigenschaften nicht(anti)kommutativer $N = 2$ euklidischer Superräume untersucht, insbesondere die Struktur der von Q-Defomationen induzierten Brechungen der Supersymmetrie. Als einfaches Beispiel wird die singlett-deformierte Supersymmetrische Yang-Mills-Theorie vorgestellt.

Im zweiten Teil dieser Arbeit beschäftigen wir uns hauptsächlich mit non-Singlett Q-Deformationen von Eichtheorien. Die Konstruktion exakter Eichtransformationen und Seiberg-Witten Abbildungen mittels einer Zerlegung des Deformationstensor erfolgt mit Hilfe eines neu entwickelten Algorithmus zur Lösung harmonischer Gleichungen. Unter anderen werden dadurch deformierte supersymmetrische Yang-Mills-Wirkungen und die zugehörigen Supersymmetrietransformationen bestimmt.

Schlagworte: Nicht-Singlett Q-Deformationen, Nichtkommutative Eichtheorie, Harmonischer Superraum.

Abstract

In this work we study the properties of non-singlet Q-deformed N = 2 supersymmetric gauge theories, from a field theoretical point of view. Starting from the supersymmetry breaking pattern induced by a general deformation matrix, we embark on the construction of the non-singlet deformed gauge transformation laws for all vector multiplet fields and their corresponding minimal Seiberg-Witten map. Several deformed super-Yang-Mills actions in components corresponding to different choices of the non-singlet deformation tensor are built. For a particular decomposition ansatz of such tensor, we obtain exact actions describing the bosonic sector of the deformed $N = (1, 0)$ and the full action for enhanced $N = (1, 1/2)$ residual supersymmetry. A tuned supersymmetry breaking of this enhanced action down to the $N = (1, 0)$ case is found by weakly restoring some discarded degrees of freedom of the deformation. Finally we find the associated residual supersymmetry transformations for the cases studied.

The first part of this work, gives an overview of noncommutativity in quantum field theory and of harmonic superspace as needed to define noncommutative generalizations of extended gauge field theories. A study of general properties of non(anti)commutative structures in $N = 2$ euclidean superspace and the (super)symmetry breaking pattern induced by Q-deformations will follow. In addition, singlet-deformed super-Yang-Mills is given as an example.

The second part deals with non-singlet Q-deformations of gauge theories. We will introduce a decomposition ansatz for the deformation matrix, allowing the exact study of the deformed gauge transformations, and develop a general algorithm to solve the harmonic equations associated to this decomposition. A close expression for the gauge transformations of component fields is derived, along with the corresponding minimal Seiberg-Witten map to an equivalent commutative gauge theory. Finally we will build deformed super-Yang-Mills actions and their corresponding supersymmetry transformations for relevant cases of the deformation matrix.

KEYWORDS: Non-singlet Q-deformations, Noncommutative Gauge Theories, Harmonic Superspace.

Contents

Preface

This dissertation is not intended to be an introductory text. It assumes knowledge of Quantum Field Theory, Supersymmetry and Group theory, especially Cartan's treatment of coset spaces. The first part of this work pretends to establish the fundamentals and conceptual setup for the second part instead of giving a profound introductory review. Novel (published and unpublished) results and interpretarions are nevertheless not only confined to the second part. An effort has been made to keep the body of the work compact, by restricting technical details and material not essential for the discourse to the appendixes that constitute the third part of the dissertation. Chapters include a short introductory section whose style is thought as to convey a comprehensive view of the work just by reading them alone.

Introduction

Noncommutativity is a long-established idea in physics and mathematics. In the operator sense, its interplay with the classical geometrical objects of (phase)space was already of importance in the early days and during the development of quantum mechanics [1, 2]. As far back as in the late 1940's [3, 4], noncommutative generalisations of the position coordinates in quantum field theory were proposed in order to treat divergences, but introduced conceptual difficulties as nonlocality and Lorentz symmetry breaking. As the project of incorporating space noncommutativity into quantum field theory appeared around the time in which a consistent renormalization scheme was developed, the interest in this idea quickly faded for physicists. For mathematicians instead, it became the cornerstone of a generalized geometry, called *Noncommutative geometry*, in which the algebra of functions over a certain space is taken to be noncommutative. By being free of the prejudices derived by physical considerations, mathematicians were able to formulate, from the standpoint provided by this marriage between geometry and algebra, the first noncommutative field and gauge theories in a clear and consistent way [5]. Strangely enough, they helped physicists to understand the relevance of such theories in their own, and not as loose alterations of quantum field theories.

Supersymmetry has similarly a long tradition of more than three decades. It is very well-grounded in quantum field theory since the work of Haag, Łopuszański and Sohnius [6] established it as *the* symmetry of the S-Matrix. Supersymmetry also has striking regularizing properties that led to the discovery of the first set of ultraviolet-finite local quantum field theories in four dimensions [7, 8], where its characteristic mixing of bosons and fermions produces the "miraculous" cancellation of divergences. It played a crucial rôle in the solution of the hierarchy problem in grand unification theories[9, 10, 11], and provided a new approach to the search for a theory of all interactions including gravity [12, 13, 14]. Additionally, manifestly supersymmetric invariant theories can be formulated on an extension of spacetime known as *superspace* [15, 16] for which, from a strictly mathematical point of view, it is very natural to pursue a noncommutative generalization.

The frame where this two concepts naturally merge is provided by *string theory*, which requires supersymmetry for its sound formulation and simultaneously seems to enclose noncommutativity by actually *producing* it in some particular setups. The appearance of string theory brought forth the only known consistent scheme for the regularization of a quantum theory of gravity, and hence revived the interest in unification theories.

Introducing noncommutative spacetime coordinates implies uncertainty relations on them that set a fundamental length scale. Under this scale, the idea of a point blurs and singular objects can turn into smeared out entities. A field theory defined on such a space should become divergence-free, effectively imposing an ultraviolet cut-off equivalent to the introduction of a lattice. Monopole configurations, for instance, could have finite energy. Analogously, the principle underlying the stringy plan of quantum gravity regularization is precisely nonlocality, since it replaces pointlike interactions by smooth two-dimensional junctions. Additionally, string theory carries a length scale l_s. It is generally believed that the structure of spacetime in a quantum field theory describing gravity must change at Planck scale due to an uncertainty principle: If we try to measure positions with accuracies comparable to this scale, the energy and momentum needed will deform spacetime itself significantly enough to destroy the resolution of the measure [17]. If string theory is to describe physics at such high energy regimes, such phenomena are to be expected.

One of the most celebrated results of string theory is the appearance of general relativity from the limit where a string is immersed in a background of modes smaller in length than the string itself. This supports the belief that the geometry of spacetime is an emerging property of string dynamics. A more recent discovery from the study of M(atrix) theory [18, 19], and D-branes in certain backgrounds [20, 21, 22, 23, 24], is that geometries derived from strings can be noncommutatively deformed when the background fields are large in string length units, that is, when stringy effects become comparatively important. A background consisting of a constant magnetic Neveu-Schwarz field, for example, will produce a low energy dynamics of D3 brane excitations governed by noncommutative $N = 4$ super Yang-Mills theory [25]. Furthermore, the fact that both an ordinary and a noncommutative field theory can be obtained from different regularizations of this setup suggested the existence of a map between them [25], that has been called *Seiberg-Witten map*.

As was discovered soon afterwards, noncommutative spacetime as a consequence of stringy phenomena was not the end of the story. A similar analysis for Ramond–Ramond backgrounds like the *graviphoton* [26, 27] produced a nontrivial modification of the algebra of coordinates, this time involving the *extended* part of superspace, constituted of Graßmann-odd objects. As their algebra is characterized by *anticommutators* in addition of commutators, the resulting geometry has come to be called *non(anti)commutative*. The possibilities when constructing these generalized geometries from string setups, resulted in a reconsideration of spacetime (and superspace) non(anti)commutativity in quantum field theories and conferred such generalisations more physical relevance.

Early works on the subject of non(anti)commutativity started in the context of a proposed fermionic substructure of spacetime [28, 29], in quantum gravity [30, 31], and on the more mathematical study of quantum deformations of supersymmetry [32, 33]. More general studies that followed [34, 35, 36], explored the possible allowed deformations of superspace and established the restrictions that particular deformation structures and space signatures impose on non(anti)commutativity itself. A distinction was then made between D- and *Q-deformations*, constructed out of spinor derivatives and supercharges, respectively. When string-inspired non(anti)commutative deformations started to appear [37, 38, 39], the attention focused on Q-deformations since these were directly implied from the few worked-out examples. The subject soon expanded to include their impact on *extended* superspace [36, 35], and further on *harmonic* superspace [40, 41]. Nevertheless, interesting features of non(anti)commutativity *per se*, that link it to soft/dynamical supersymmetry breaking [42, 43], BPS-solutions [44], quantum deformations of supersymmetry [45, 46, 47] and target space geometry of sigma models [48, 49], have presently made very tempting to continue the study of the physical consequences thereof, postponing the issue of a precise relation to specific string backgrounds.

Non(anti)commutative deformations are controlled by a *deformation matrix* that specifies the deformed algebra of Graßmann coordinates in extended superspace. In the case of Q-deformations, which have a severe impact on the action of symmetry generators on the algebra of superfields, the resulting supersymmetry breaking pattern is tuned by this matrix. In contrast to the $N = 1$ case, where this consists simply in a partial breaking to $N = 1/2$ [34, 37], the extended supersymmetry case is richer in structure. In $N = 2$, for example, different decompositions of the deformation matrix, classified by their tensor properties under the R-symmetry group $SU(2)$, lead to $N = 1$ or $N = 3/2$ residual supersymmetry. For so-called *singlet* deformations, where this matrix becomes an $SU(2)$ singlet, several deformed field models of the hypermultiplet [42] and super-Yang-Mills theory [50, 51, 52], have been studied. The general choice of deformation was also explored perturbatively for super-Yang-Mills up to some orders in the deformation matrix parameters [53, 54, 55]. A remarkable quantum property of these theories is that even when the parameters of the deformation have negative mass dimension and should induce divergences by naive power-counting, all studied examples have turned out to be renormalizable. This subject is at the present under intensive study, and its scope covers non(anti)commutative $N = (1/2, 0)$ Wess-Zumino and Yang-Mills models [56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69] and also the $N = 2$ Yang-Mills and the neutral hypermultiplet, as studied in the pioneering work [70].

The main goal of this work is to study the properties of *non-singlet* Q-deformed $N = 2$ supersymmetric gauge theories, from a field theoretical point of view. The program includes the study of the supersymmetry breaking pattern induced by a general deformation matrix, the construction of gauge transformation laws for all vector multiplet fields and their corresponding minimal Seiberg-Witten map, the derivation of the different deformed super-Yang-Mills actions *in components* that follow from the pattern, and finally the construction of the associated residual supersymmetry transformations.

In the first part of this work, we will start by giving an overview of noncommutativity in quantum field theory from its origin in the Weyl quantization formalism, where we will introduce the Moyal product that deforms the algebra of functions in phase space. This product will allow us to make a connection to the study of algebra deformations and of noncommutativity as a way to generalize field theories. A comment on the stringy origin of noncommutativity will be included. In §2, follows the introduction of harmonic superspace as needed to define extended gauge field theories. We will then devote §3 to the subject of non(anti)commutativity in general. There, the requirements that supersymmetry imposes on it, and the peculiarities of Q-deformation will be explained. The breaking pattern induced by these deformations will then be derived. Singlet-deformed super-Yang-Mills is given as an example.

The second part of this work will deal exclusively with non-singlet Q-deformations of gauge theories. In §4 we will introduce a decomposition ansatz for the deformation matrix, allowing the exact study of the deformed gauge transformations. A general algorithm to solve the harmonic equations associated to this decomposition is developed. After giving the close form of the gauge transformations, we will derive the corresponding minimal Seiberg-Witten map to an equivalent *anticommutative* $U(1)$ gauge theory. Chapter §5 will deal with the construction of deformed actions and corresponding supersymmetry transformations for relevant cases of the deformation matrix.

A third part of the dissertation includes only appendixes that cover the notation and conventions followed, the technical details of the calculations supporting the discussion in the main body of the work, and the common tools used to perform them.

Part I

Fundamentals of Non(anti)commutativity

Chapter 1

Noncommutativity in Field Theory

In quantum mechanics, uncertainty relations between phase space quantities is a consequence of the noncommutativity of the operators associated to them. In a classical work by Weyl [1] he proposed a quantization prescription based on the idea of mapping functions of phase space to their related quantum operators. This naturally led to the idea of introducing noncommutativity in the algebra of phase space functions, further developed by Moyal [2]. As the classical variables should be recovered in the limit $\hbar \to 0$, when quantum effects cease to be important, the noncommutativity was thought to be *parametrized* by this scale and conceived as a *deformation* of an otherwise commuting algebra of functions. It became clear that these functions must obey strong restrictions for a deformation conceived as a series expansion of their products to converge. In the seminal papers of Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [71, 72] it was shown that one could understand quantization as a *formal* deformation, without worrying to much about the convergence and the precise construction of Hilbert spaces.

In the context of quantum field theories, noncommutative generalizations are to appear when the algebra of functions over the standard configuration space of a theory is replaced by a noncommutative one. One would like such effects to be present at a certain quantum scale, representing the coordinate uncertainty that must disappear when we zoom out back to lower energies. This leads to the idea of noncommutativity as a *deformation* of smooth continuous space spacetime, implemented through an anticommutative product as in the theory of deformation quantization.

A simple way to deform the algebra of spacetime functions is by using an associative but noncommutative product '$\star$' called *Moyal product* [2]

$$f(x)g(x) \longrightarrow f(x) \star g(x), \quad \text{such that} \quad [x^\mu, x^\nu]_\star \neq 0. \tag{1.1}$$

Spacetime noncommutativity introduces nonlocality that can smear out singular objects. In quantum field theories it will enter through interaction terms that, generally speaking,

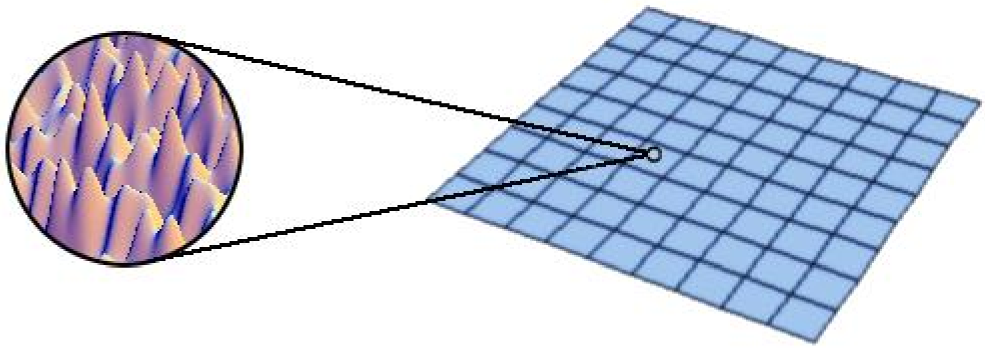

Figure 1.1: Noncommutativity as a small scale deformation of spacetime

produce vertex diagrams with a phase factor depending on the momenta, which were expected to regulate divergences. In addition, even more bizarre behaviour appears, as exemplified by the breaking of Lorentz invariance, and the so called *UV/IR mixing* between the high and low energy regimes. Introducing noncommutativity suggests useful nontrivial generalizations of relevant theories. It can provide a free parameter to do perturbative expansions, and in some cases allows a whole new set of exact solutions not present in the commutative limit[73, 74]. Though a unique noncommutative generalization of a given theory may not exist, a selection scheme could be nevertheless based on the preservation of symmetries for a particular theory.

Recent applications of noncommutative field theory have been related but not limited to the strings scenario. In condensed matter theory, for instance, a model of electrons in a magnetic field due to Landau provides a simple example of noncommuting coordinates which has a direct analogous in string theory. Also the theory of the quantum Hall effect [75] has received interesting inputs from noncommutative geometry [76] and noncommutative field theory [77]. A noncommutative generalization of the standard model [78] has been studied in the framework of non stringy particle phenomenology [33], were even experimental test have been suggested [79]. In cosmology, an inflation mechanism without inflaton but relying on spacetime noncommutativity was proposed in [80]. They also appear in the theory of quantum algebras associated to massive superparticles [81, 82].

In this chapter we will see how do deformations in the algebra of functions of a field theory appear from the Weyl quantization formalism, where the *Moyal star product* is introduced. Typical features as the smearing out of the product of compact functions and the occurrence of non local effects relating different energy regimes are presented. Afterwards, in §1.2, will see how the deformations of the algebra of functions are related to the symmetries of a theory. In particular, we will see how the Moyal product can be defined in terms of a Poisson structure, built out of the generators of spacetime translations.

This will be the basis to define *non(anti)commutative* deformations of superspace in §3, where *supercharges* will be used instead as translation generators. Most of the eccentric consequences of noncommutativity in quantum field theory, as nonlocality and UV/IR mixing (see for example [83] and references therein), will be absent when we particularize to the kind of non(anti)commutative deformations which are the main object of this work. However, in section §1.3 we will comment on the essential quantum theoretical effects of noncommutativity, taking ϕ^4 theory as an example. Finally, as complementary motivation to the study of noncommutative (and later on non(anti)commutative) deformations, we will comment in §1.4 how this deformations appear from string theory when strings attached to D-branes are placed in particular backgrounds.

1.1 Weyl Quantization and the Star Product

For a physicist, the most familiar example of a noncommutative algebra comes from quantum mechanics, where the operators $\hat{q}$, $\hat{p}$ corresponding to conjugated classical variables q, p, satisfy

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar \tag{1.2}$$

Through the so called Weyl transform it is possible to quantize the system by mapping functions of phase space $f(q,p)$ into their corresponding quantum operators $\hat{O}_f(\hat{q},\hat{p})$

$$\hat{O}_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int d\sigma\, d\tau\, dq\, dp\; e^{-\mathrm{i}\tau(\hat{q}-q)-\mathrm{i}\sigma(\hat{p}-p)} f(q,p) \tag{1.3}$$

We can try to find a function $h(p,q)$ in phase space whose Weyl transform corresponds to the product of operators $\hat{O}_h = \hat{O}_f\hat{O}_g$. As we may expect, the composition law of Weyl transforms induces a noncommutative product of functions in phase space $h = f \star g$ which reflects the fact that operators in quantum mechanics do not commute $\hat{O}_f\hat{O}_g \neq \hat{O}_g\hat{O}_f$, and reduces to the ordinary product when $\hbar \to 0$. In what follows we will consider this map in more detail to see how the star product arises from noncommutativity of quantum operators.

We start by considering the above commutation relation (1.2) as the Lie algebra of what has come to be called the *Weyl-Heisenberg group*[1]. Elements of this group are defined by

$$U(\tau,\sigma) = \exp\left[-\mathrm{i}(\tau\hat{q} + \sigma\hat{p})\right], \tag{1.4}$$

and are unitary transformations that represent translations in phase space

$$\begin{aligned} U(\tau,\sigma)\,\hat{q}\,\bar{U}(\tau,\sigma) &= \hat{q} - \hbar\sigma \\ U(\tau,\sigma)\,\hat{p}\,\bar{U}(\tau,\sigma) &= \hat{p} + \hbar\tau \end{aligned} \tag{1.5}$$

The key idea of Weyl is to define quantum operators as elements of this group's algebra

$$\hat{O}_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int d\sigma\, d\tau\; U(\tau,\sigma)\tilde{f}(\tau,\sigma) \tag{1.6}$$

Taking $\tilde{f}(q,p)$ to be the Fourier transform of a function in phase space

$$\tilde{f}(\tau,\sigma) = \int dq\, dp\; e^{\mathrm{i}(\tau q+\sigma p)} f(q,p), \tag{1.7}$$

we arrive to the Weyl transform mentioned above (1.3). Using Glauber's identity we can restate it as

$$\hat{O}_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int d\sigma\, d\tau\, dq\, dp\; e^{-\mathrm{i}\tau(\hat{q}-q)}e^{-\mathrm{i}\sigma(\hat{p}-p)}e^{\frac{1}{2}\mathrm{i}\hbar\tau\sigma} f(q,p) \tag{1.8}$$

The factor $e^{\frac{1}{2}\mathrm{i}\hbar\tau\sigma}$ can be interpreted as the one arising from commuting the $\hat{q}$'s and $\hat{p}$'s as needed to obtain a Weyl ordered operator from the possible products of q and p appearing in the original phase space function. Additionally the transform has the proper classical limit $\hbar \to 0$,

$$\hat{O}_f(\hat{q},\hat{p}) = \int dq\, dp\; \delta(\hat{q}-q)\delta(\hat{p}-p) f(q,p) = f(\hat{q},\hat{p}). \tag{1.9}$$

Having a complete quantization prescription, lets us now take a closer look to the composition law of the Weyl transform. First note that the Weyl-Heisenberg group represents phase space translations up to a phase

$$U(\tau_1,\sigma_1)U(\tau_2,\sigma_2) = e^{-\frac{\mathrm{i}}{2}\hbar(\tau_1\sigma_2-\sigma_1\tau_2)}U(\tau_1+\tau_2,\sigma_1+\sigma_2). \tag{1.10}$$

So that the product of two operators correspond to the following transform

$$\begin{aligned}
\hat{O}_f(\hat{q},\hat{p})\hat{O}_g(\hat{q},\hat{p}) &= \frac{1}{(2\pi)^4}\int d\sigma_1 d\tau_1 d\sigma_2 d\tau_2\, U(\tau_1+\tau_2,\sigma_1+\sigma_2)e^{-\frac{\mathrm{i}}{2}\hbar(\tau_1\sigma_2-\sigma_1\tau_2)}\tilde{f}(\tau_1,\sigma_1)\tilde{g}(\tau_2,\sigma_2)\\
&= \frac{1}{(2\pi)^2}\int d\sigma\, d\tau\; U(\tau,\sigma)\\
&\quad\times\left[\frac{1}{(2\pi)^2}\int d\sigma'\, d\tau'\; e^{\frac{\mathrm{i}}{2}\hbar(\sigma\tau'-\tau\sigma')}\tilde{f}(\tfrac{\tau}{2}+\tau',\tfrac{\sigma}{2}+\sigma')\tilde{g}(\tfrac{\tau}{2}-\tau',\tfrac{\sigma}{2}-\sigma')\right]
\end{aligned}$$

In the last integral it is possible to recognize the Fourier transform of the quantity

$$e^{\frac{\mathrm{i}}{\hbar}(\partial_p\partial_{q'}-\partial_q\partial_{p'})} f(q,p)g(q',p')|_{(q,p)=(q',p')} \equiv f\star g(p,q) \tag{1.11}$$

which defines a bilinear map over the algebra of smooth functions in phase space, that is, some noncommutative product.

This discussion can be directly generalized for general complex valued Schwarz functions on d-dimensional Euclidean spaces, for which it is always possible to define a Fourier transform

$$\tilde{f}(k) = \int d^d x\, e^{-\mathrm{i}k\cdot x} f(x). \tag{1.12}$$

We will introduce again commutation relations for the operators $\hat{x}^m$

$$[\hat{x}^m, \hat{x}^n] = \mathrm{i}\theta^{mn}, \tag{1.13}$$

with θ^{mn} taken to be a constant antisymmetric invertible matrix. The quantization formalism provides us then with the quantum operator or *Weyl symbol* associated to the function f

$$\hat{O}_f = \int \frac{d^d k}{(2\pi)^d}\, e^{\mathrm{i}k\cdot\hat{x}}\, \tilde{f}(k) \tag{1.14}$$

which can be rewritten explicitly in terms of the Weyl map $\hat{\Delta}(x)$ as

$$\hat{O}_f = \int d^d x\, f(x)\hat{\Delta}(x), \qquad \hat{\Delta}(x) = \int \frac{d^d k}{(2\pi)^d}\, e^{\mathrm{i}k\cdot\hat{x}} e^{-\mathrm{i}k\cdot x}. \tag{1.15}$$

If the operators $\hat{x}^m$ commute with each other, then the map $\hat{\Delta}(x)$ reduces to a delta function $\delta^d(\hat{x}-x)$, and the symbol of a function f reduces to the original "classical" function $\hat{O}_f = f(\hat{x})$. Again, using Glauber's identity

$$e^{\mathrm{i}k\cdot\hat{x}} e^{\mathrm{i}k'\cdot\hat{x}} = e^{-\frac{\mathrm{i}}{2}\theta^{mn}k_m k'_n} e^{\mathrm{i}(k+k')\cdot\hat{x}}, \tag{1.16}$$

we can compute the function whose symbol is associated to products of operators. A composition law relating the noncommutative product of operators with a noncommutative product of functions will then be given by

$$\hat{O}_f \hat{O}_g = \hat{O}_{f\star g}, \tag{1.17}$$

where $\star$ represents the *Moyal star product*

$$\begin{aligned} f(x)\star g(x) &\equiv f(x)\exp\left(\frac{\mathrm{i}}{2}\overleftarrow{\partial}_m \theta^{mn}\overrightarrow{\partial}_n\right) g(x) \\ &= \int \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\, \tilde{f}(k)\tilde{g}(k-k') e^{-\frac{\mathrm{i}}{2}\theta^{mn}k_m k'_n} e^{\mathrm{i}k'\cdot x} \end{aligned} \tag{1.18}$$

As we will see in the next section, this noncommutative yet associative product is a particular case of the star product usually found in deformation quantization [71].

The introduction of nonlocal behaviour of noncommutative quantum field theories through this product produces a plethora of unexpected and novel properties. Singular pointlike sources, for instance, are smeared out and made to interact over an extended finite region.

Let us take a closer look into this phenomenon by calculating explicitly the integral representation of the star product of two functions, starting from the product of two Weyl maps,

$$\begin{aligned}\hat{\Delta}(x)\hat{\Delta}(y) &= \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\, e^{\mathrm{i}(k+k')\cdot\hat{x}}e^{-\frac{\mathrm{i}}{2}\theta^{mn}k_mk'_n}e^{-\mathrm{i}k\cdot x-\mathrm{i}k'\cdot y}\\ &= \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\int d^dz\, e^{\mathrm{i}(k+k')\cdot z}\hat{\Delta}(z)e^{-\frac{\mathrm{i}}{2}\theta^{mn}k_mk'_n}e^{-\mathrm{i}k\cdot x-\mathrm{i}k'\cdot y}\\ &= \frac{1}{\pi^d|\det\theta|}\int d^dz\; \hat{\Delta}(z)e^{-2\mathrm{i}(\theta^{-1})_{mn}(x-z)^m(y-z)^n}, \end{aligned} \tag{1.19}$$

where we tacitly assume invertibility of θ. We see then that the Weyl maps are orthonormal

$$\mathrm{Tr}\left(\hat{\Delta}(x)\hat{\Delta}(y)\right) = \delta^d(x-y), \tag{1.20}$$

and we can in this case define the inverse Weyl transform, which is precisely the *Wigner distribution function* [84] taking symbols into functions

$$f(x) = \mathrm{Tr}\left(\hat{O}_f\hat{\Delta}(x)\right). \tag{1.21}$$

The integral representation of the star product of two functions is then given by

$$\mathrm{Tr}\left(\hat{O}_f\hat{O}_g\hat{\Delta}(x)\right) = \frac{1}{\pi^d|\det\theta|}\int d^dy\, d^dz\; f(y)g(z)\exp\left[-2\mathrm{i}(\theta^{-1})_{mn}(x-y)^m(x-z)^n\right], \tag{1.22}$$

which shows how point interactions get distributed over a region in space. We can define a scale of deformation by rotating θ^{mn} into a skew diagonal form

$$\theta^{mn} = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_{d/2} \\ & & & -\theta_{d/2} & 0 \end{pmatrix}, \tag{1.23}$$

and taking the operator norm of θ^{mn}

$$\|\theta\| = \max_{1\le i\le d/2}|\theta_i|. \tag{1.24}$$

Now we can state precisely that in particular, compact functions that vanish outside a region of size $l \ll \sqrt{\|\theta\|}$ have a star product that is non vanishing over a region of typical size $\|\theta\|/l$. For example, the star product of point sources is nonzero in every point of space

$$\delta^d(x)\star\delta^d(y) = \frac{1}{\pi^d|\det\theta|}. \tag{1.25}$$

In more physical terms, the fields with a small typical size —that is very high energy— interact instantaneously over long distances through the star product, having profound consequences for the quantum theory. One would expect that nonlocal effects turn negligible for energies below the $\sqrt{\|\theta\|}$ deformation scale, but in fact high energy virtual particles contribute to low energy processes producing the UV/IR regime mixing. An ultraviolet cutoff Λ will control the standard high momentum divergences just to produce infrared singularities to be controlled by a low momenta cutoff $1/\|\theta\|\Lambda$.

Our final goal in this section is to move on to the analysis of deformed quantum field theories. As these involve derivatives of fields, the Weyl symbol corresponding to field derivatives will be needed. This is easily seen to correspond to the anti-Hermitian linear derivation $\hat{\partial}_m$ defined by

$$[\hat{\partial}_m, \hat{x}^n] = \delta^n_m, \qquad [\hat{\partial}_m, \hat{\partial}_n] = 0. \tag{1.26}$$

It acts on the Weyl map through a commutator

$$[\hat{\partial}_m, \hat{\Delta}] = -\hat{\partial}_m \hat{\Delta}, \tag{1.27}$$

and therefore the symbol of a derivative is precisely

$$\hat{O}_{\partial_m f} = \int d^d x\, \partial_m f(x) \hat{\Delta}(x) = [\hat{\partial}_m, \hat{O}_f] \tag{1.28}$$

The generators of translations are also given in terms of derivations $\hat{\partial}_m$ as unitary operators

$$U(v) = e^{v\cdot\hat{\partial}}, \qquad U(v)\hat{\Delta}(x)\bar{U}(v) = \hat{\Delta}(x+v), \qquad \text{for } v \in \mathbb{R}^d \tag{1.29}$$

From this it is clear that the trace of the Weyl map $\mathrm{Tr}\,\hat{\Delta}(x)$ is independent of $x \in \mathbb{R}^d$ and therefore we can choose a normalization $\mathrm{Tr}\,\hat{\Delta}(x) = 1$ leading to

$$\mathrm{Tr}\,\hat{O}_f = \int d^d x\, f(x), \tag{1.30}$$

and implying the cyclicity of the star product under the integral as a consequence of the cyclicity of the trace

$$\int d^d x\, f_1(x) \star \cdots \star f_M(x) = \mathrm{Tr}\left(\hat{O}_{f_1} \cdots \hat{O}_{f_M}\right). \tag{1.31}$$

In particular the star product of two functions under the integral is commutative

$$\int d^d x\, f(x) \star g(x) = \int d^d x\, f(x) g(x). \tag{1.32}$$

Weyl's quantization formalism can be also used in presence of more general commutators resulting in non constant deformation parameters or even other operators [85]. A particularly relevant situation appears when quantizing open strings in the presence of a nonconstant B-field. There the set of commutators involving both coordinates and momenta will depend on operator functions of such variables, and will constitute an algebra of pseudo-differential operators on noncommutative space [86, 87, 88, 89]. Instead of Moyal product, the star product will be given by the Kontsevich formula [90]. When the B-field is a closed two form $dB = 0$ then non constant Poisson tensors θ appear, to keep the Kontsevich product associative.

1.2 Star Product from Algebra Deformations

Apart from its formulation in the context of the Weyl formalism, the star product also has a natural interpretation in *deformation quantization*. Instead of promoting phase space variables to operators allowing nontrivial commutation relations, one seeks to generalize the algebra of functions on this space by introducing nontrivial products. To assure a proper classical limit, the products depend on some quantum scale $\hbar$ in such a way as to recover the standard algebra of functions in the case $\hbar \to 0$. As the geometry of a manifold can be defined in terms of the properties of the algebra of functions on it, one can understand quantization as a deformation of geometry rather than a promotion of physical quantities to operators in a Hilbert space.

General deformations of the algebra of functions on a manifold M, can be defined in terms of a formal series whose first order corresponds to the original undeformed algebra

$$f \star_\lambda g = fg + \sum_{i=1}^{\infty} \lambda^i C_i(f,g), \qquad f, g : M \longrightarrow \mathbb{C} \tag{1.33}$$

As a formal power series, the Moyal product (1.18) corresponds to

$$f \star g = fg + \sum_{i=1}^{\infty} \frac{1}{i!} \left(\frac{\mathrm{i}}{2}\right)^i \theta^{m_1 n_1} \cdots \theta^{m_i n_i} \partial_{m_1} \cdots \partial_{m_i} f \partial_{n_1} \cdots \partial_{n_i} g \tag{1.34}$$

with θ^{mn} constant. Modulo some redefinitions of f and g, the Moyal product is the only deformation of the algebra of functions on $\mathbb{R}^d$ whose formal series has local differential bilinears of f and g as coefficients, and coincides at first order with the Poisson bracket of functions [71, 72, 91]

$$f \star g = fg + \frac{\mathrm{i}}{2} \theta^{mn} \partial_m f \partial_n g + \mathrm{O}(\theta^2). \tag{1.35}$$

If one starts with a more general *Poisson structure*

$$P_i(f,g) = P^{m_1 n_1} \cdots P^{m_i n_i} \nabla_{m_1} \cdots \nabla_{m_i} f \, \nabla_{n_1} \cdots \nabla_{n_i} g, \tag{1.36}$$

defined in terms of a more general derivation ∇_m and of some antisymmetric matrix P^{mn}, and tries to obtain a smooth deformation of the algebra

$$f \star_\lambda g = fg + \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} P_i(f,g), \tag{1.37}$$

then requiring associativity will impose severe restrictions on the derivative and on P_i itself. Only a *flat torsion-free* derivative and a Poisson structure with constant P^{nm} will render the product associative, taking it back to the Moyal case (1.34), which can therefore be written equivalently as

$$f \star g = f e^P g, \tag{1.38}$$

where the Poisson structure P is simply

$$P = -\frac{\mathrm{i}}{2} \overleftarrow{\partial}_m \theta^{mn} \overrightarrow{\partial}_n \tag{1.39}$$

Noncommutativity of the coordinates on the manifold follows then from the action of this structure

$$[x^m, x^n]_\star \equiv x^m \star x^n - x^n \star x^m = \mathrm{i}\theta^{mn}. \tag{1.40}$$

The conditions on the Poisson structure can also be interpreted as a consequence of translation invariance in flat spacetime since the only nontrivial coordinate deformation consistent with it has constant deformation matrix. More explicitly, if we take θ^{mn} to be a local function of the coordinates $\theta^{mn}(x)$ under translations $x \mapsto x + a$ we obtain

$$[x'^m, x'^n] = [x^m + a^m, x^n + a^n] = [x^m, x^n], \quad \Rightarrow \quad \theta^{mn}(x+a) = \theta^{mn}(x) \tag{1.41}$$

For spacetime symmetries to be preserved, the deformed coordinate algebra must be also covariant. The deformed algebra will in general break invariance under Lorentz transformations $x^m \mapsto x'^m = \Lambda^m_n x^n$ for dimension $d > 2$

$$[x'^m, x'^n] = \left[\Lambda^m_p x^p, \Lambda^n_q x^q\right] = \Lambda^m_p \left[x^p, x^q\right] \Lambda^n_q = \mathrm{i}\Lambda^m_p \theta^{pq} \Lambda^n_q \neq \mathrm{i}\theta^{mn} \tag{1.42}$$

In $d = 2$ every antisymmetric matrix is proportional to the Lorentz invariant Ricci tensor ϵ^{mn}, meaning that the equality could hold.

In general one should analyze how a deformation affects the algebra of the *symmetry generators* of the theory. Within the Poincare algebra, Lorentz and translation generators L_{mn}, P_m satisfy

$$\begin{aligned} [L_{mn}, L_{rs}] &= \eta_{nr} L_{ms} - \eta_{mr} L_{ns} + \eta_{ns} L_{mr} - \eta_{ms} L_{nr} \\ [L_{mn}, P_r] &= \eta_{nr} P_m - \eta_{mr} P_n \\ [P_m, P_n] &= 0. \end{aligned} \tag{1.43}$$

Being the derivative, the standard representation of momentum translations in spacetime $P_m = \partial_m$, one can write the Poisson structure as

$$P = -\overleftarrow{P}_m \theta^{mn} \overrightarrow{P}_n, \tag{1.44}$$

Where the factor of $\frac{\mathrm{i}}{2}$ has been absorbed into θ^{mn} for convenience. Due to the trivial commutation relation of the momentum generators one can easily check that the translation algebra does not get deformed

$$[P_m, P_n]_\star = [P_m, P_n] = 0. \tag{1.45}$$

On the other hand, as translation and Lorentz generators do not commute, it follows that the Poisson structure cannot simply "pass through" the M_{mn} and the Lorentz algebra will pick up a deformation

$$[L_{mn}, L_{rs}]_\star = [L_{mn}, L_{rs}] + \mathrm{O}(\theta). \tag{1.46}$$

One concludes that the noncommutative deformation of a theory will not be Lorentz invariant.

Deformed theories can be nevertheless invariant under another kind of symmetry realized not in terms of classical but *quantum groups*. For quantum groups, the parameters of the transformation will also obey a deformed algebra, and will therefore restore its original structure.

1.3 Generalizing Field Theories

As a first example of a deformed quantum field theory, let us take a look at a simple noncommutative generalization of the ϕ^4 theory in $\mathbb{R}^d$

$$S_{\phi^4_\star} = \int d^d x \left[\frac{1}{2} \partial\phi \cdot \partial\phi + \frac{m^2}{2}\phi^2 + \frac{g^2}{4!} \phi \star \phi \star \phi \star \phi \right] \tag{1.47}$$

Note that the substitution of standard multiplication by star products only affects the interaction terms of the theory due to the cyclicity property of the star product (1.32). As the free part of the theory is undeformed, bare propagators of noncommutative ϕ^4 theory are identical to the standard ones.

In contrast, the interaction vertex in momentum space picks up a phase factor depending on the external momenta

$$\int d^d x \, \phi \star \phi \star \phi \star \phi = \int \prod_{r=1}^{4} \frac{d^d k_r}{(2\pi)^d} \tilde{\phi}(k_r) (2\pi)^d \delta(\textstyle\sum_{r=1}^4 k_r) \underbrace{\prod_{r<s} \exp\left(-\frac{\mathrm{i}}{2} \theta^{mn} k_{r\,m} k_{s\,n} \right)}_{V(k_1,k_2,k_3,k_4)} . \tag{1.48}$$

Though local in each order of a θ expansion, this vertex describes a nonlocal interaction.

As in this setup momentum conservation implies that the vertex $V(k_1, k_2, k_3, k_4)$ is invariant only under cyclic permutations of momenta k_r, it would be very nice to device a way to introduce this information into the diagrammatica to avoid keeping track of the order explicitly. Fortunately some tools are already available from the theory of the large N limit of $U(N)$ Yang-Mills theory or analogously from matrix models [92, 93, 94]. The idea is to substitute lines by oriented ribbons, so that the line corresponding to momentum k carries two "momentum indexes" l_a, l_b

$$\frac{1}{(l_a - l_b)^2 + m^2} \tag{1.49}$$

Being built out of ribbons, the noncommutative Feynman diagrams are graphs drawn over Riemman surfaces of general genus. If is possible to draw a graph over the surface of plane or a sphere, i.e. without crossing lines, it is called *planar*. For such a graph consisting of L loops with $k_1, \ldots, k_n$ external cyclically ordered momenta, we can take $k_a = l_{m_a} - l_{m_{a+1}}$ with m_a running from 1 to $L+1$ and organized in a cyclical way such as $l_{m_{a+1}} = l_{m_1}$. This construction will take care of the order of incoming momenta and will automatically impose momentum conservation on each vertex because adjacent ribbon edges in a vertex have opposite momentum indexes. With this, it is possible to establish [95] that for any planar graph one obtains an overall phase factor depending only on the ordered external momenta $p_1, \ldots, p_M$

$$V_{\mathrm{P}}(p_1, \ldots, p_M) = \prod_{a<b} e^{-\frac{\mathrm{i}}{2}\theta^{mn} p_{a\,m} p_{b\,n}} \tag{1.50}$$

That the only contribution of noncommutativity in planar graphs is this factor, indicates that their UV behaviour is essentially the same as the undeformed one [96, 97, 98]. Even when our expectations of improving renormalizability in this way are gone, there are still some important differences coming from the nonplanar sector of the deformed theory, where we have to take in account momentum propagators crossing each other. The contribution in this case can be written in terms of the planar one

$$V_{\mathrm{NP}}(p_1, \ldots, p_M) = V_{\mathrm{Planar}}(p_1, \ldots, p_M) \prod_{a,b} e^{-\frac{\mathrm{i}}{2}\cap_{ab}\theta^{mn} p_{a\,m} p_{b\,n}}, \tag{1.51}$$

where $\cap_{ab}$ is a matrix counting the crossings. In this case phase oscillations do render all one-loop diagrams UV finite, but acquire a singular IR behaviour. Let us clear this up by looking at the one-particle irreducible two-point function up to one loop

$$\Pi(p) = \Pi^{(0)}(p) + g^2 \Pi^{(1)}_{\mathrm{P}}(p) + g^2 \Pi^{(1)}_{\mathrm{NP}}(p)) + \mathrm{O}(g^4) \tag{1.52}$$

Where

$$\begin{aligned}\Pi^{(0)}(p) &= p^2 + m^2 \\ \Pi^{(1)}_{\mathrm{P}}(p) &= \frac{1}{3}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} \\ \Pi^{(1)}_{\mathrm{NP}}(p) &= \frac{1}{6}\int \frac{d^dk}{(2\pi)^d}\frac{e^{\mathrm{i}\theta^{mn}k_m p_n}}{k^2+m^2}\end{aligned} \tag{1.53}$$

are the bare mass, planar and non planar propagators respectively. As in the limit $p \to 0$ the one loop propagators coincide up to a factor $\Pi^{(1)}_{\mathrm{P}}(p) = 2\Pi^{(1)}_{\mathrm{NP}}(0)$ we can write the full propagator in terms of the nonplanar contribution

$$\Pi(p) = p^2 + m^2 + 2g^2\Pi^{(1)}_{\mathrm{NP}}(0) + g^2\Pi^{(1)}_{\mathrm{NP}}(p) + \mathrm{O}(g^4). \tag{1.54}$$

The main consequence is that the behaviour of the nonplanar propagator fully determines the singularities of the one loop two point function. Momentum regularization with a cutoff of Λ leads to

$$\Pi^{(1)}_{\mathrm{NP}}(p) = \frac{1}{96\pi^2}\left(\Lambda^2_{\mathrm{eff}} - m^2\ln\frac{\Lambda^2_{\mathrm{eff}}}{m^2}\right) + \mathrm{O}(1), \tag{1.55}$$

where the effective cutoff is given in terms of the momentum cutoff as

$$\Lambda^2_{\mathrm{eff}} = \frac{1}{\frac{1}{\Lambda^2} + \delta_{kl}\theta^{mk}\theta^{ln}p_m p_n} \tag{1.56}$$

As the momentum cutoff Λ goes to infinity, one sees that noncommutativity clearly improves the UV behaviour of the propagator by keeping it finite but only to replace the UV divergence by a singular IR behaviour

$$\lim_{\Lambda\to\infty}\Lambda^2_{\mathrm{eff}} = \frac{1}{\delta_{kl}\theta^{mk}\theta^{ln}p_m p_n}, \tag{1.57}$$

this strange phenomenon is called *UV/IR mixing* and has no analog in commutative field theory. Here the singular behaviour at $p = 0$ is the result of *high energy contributions* which turns the standard exponentially decaying correlators of massive scalar particles by polynomial interactions, that is, long-range power-law forces.

1.4 String Theory and Noncommutativity

As an example of the kind of unconventional stringy effects on geometry that motivate noncommutativity in gauge field theories, we are going to study open strings ending on *Dp*-branes in a background of a constant Neveu-Schwarz *B*-field.

We will choose a particularly simple setup, with $B_{ij} \neq 0$ only for $i, j = 1, \dots, r$, and the flat metric $g_{ij} = 0$ for $i = 1, \dots, r, j \neq 1, \dots, r$ where r is precisely the *rank* of the B-field. In this case the worldsheet action of the string is written as

$$S = \frac{1}{4\pi l_s^2} \int_\Sigma (g_{ij}\partial_a x^i \partial^a x^j - 2\mathrm{i}\pi l_s^2 B_{ij} \varepsilon^{ab} \partial_a x^i \partial_b x^j) \tag{1.58}$$

l_s being the *string length* and Σ its euclidean worldsheet. The presence of Dp-branes is now essential, since it is possible to integrate by parts the term including the *constant* B-field to obtain a boundary contribution

$$S = \frac{1}{4\pi l_s^2} \int_\Sigma g_{ij}\partial_a x^i \partial^a x^j - \frac{\mathrm{i}}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_\parallel x^j, \tag{1.59}$$

where $\partial_\parallel$ is a tangential derivative along the worlsheet boundary $\partial\Sigma$. Components of B_{ij} not on the brane, can be simply gauged away, meaning we can assume $r \leq 1 + p$. In simple words, without Dp-branes, the Neveu-Schwarz field can be decoupled from the dynamics. For the coordinates along the Dp-branes, boundary conditions are derived from the equations of motion of this system

$$g_{ij}\partial_\perp x^j + 2\mathrm{i}\pi l_s^2 B_{ij} \partial_\parallel x^j \big|_{\partial\Sigma} = 0, \tag{1.60}$$

which include a derivative $\partial_\perp$, normal to the boundary $\partial\Sigma$. This equation interpolates between Neumann conditions in the case $l_s^2 B \to 0$ and Dirichlet conditions for $l_s^2 B \gg 1$, or equivalently $g \to 0$. When the Dirichlet term dominates, the string boundaries are fixed to points in the Dp-brane for B_{ij} invertible, due to $\partial_\parallel x^j \big|_{\partial\Sigma} = 0$. It is in this regime, when the background fields are large in string length units, and stringy effects become comparatively important, the commutator of coordinates on the boundary turns out to be nontrivial.

To see this one considers the classical approximation of string theory for which $\partial\Sigma$ is a disc which can be conformally mapped to the upper half plane. In the corresponding complex coordinates $z = t + \mathrm{i}y$, $y \geq 0$, with $\partial = \partial/\partial z, \bar\partial = \partial/\partial \bar z$, the equations of motion (1.60) are

$$g_{ij}(\partial - \bar\partial)x^j + 2\mathrm{i}\pi l_s^2 B_{ij}(\partial + \bar\partial)x^j \big|_{z=\bar z} = 0, \tag{1.61}$$

The propagator for this boundary conditions is [99, 100, 101]

$$\langle x^i(z) x^j(z')\rangle = -l_s^2 \Bigg(g^{ij} \log|z - z'| - g^{ij} \log|z - \bar z'| + G^{ij} \log|z - \bar z'|^2 + \\ + \frac{1}{2\pi l_s^2} \theta^{ij} \log \frac{|z - \bar z'|}{|\bar z - z'|} + D^{ij} \Bigg), \tag{1.62}$$

where G^{ij} and θ^{ij} are the symmetric and antisymmetric parts of a matrix

$$G^{ij} = \left(\frac{1}{g + 2\pi l_s^2 B}\right)_S^{ij}, \qquad \theta^{ij} = \left(\frac{1}{g + 2\pi l_s^2 B}\right)_S^{ij}., \tag{1.63}$$

and D^{ij} is a constant depending on B but independent of the coordinates. If the low energy limit $l_s^2 \to 0$ is considered, by keeping B fixed, evaluating the propagator at the boundary which is identified to the real line one obtains

$$\langle x^i(t) x^j(0)\rangle = \frac{\mathrm{i}}{2}\theta^{ij}\,\mathrm{Sign}(t), \qquad \theta^{ij} = \left(\frac{1}{B}\right)^{ij}, \qquad i, j = 1, \ldots, r. \tag{1.64}$$

In conformal field theory, the short distance behaviour of operator products coincide with their commutators when replacing time by operator ordering. In this way it is found [24] that the quantity θ^{ij} is related to such commutator according to

$$[x^i(t), x^j(0)] = T\left(x^i(t-0)x^j(0) - x^i(t+0)x^j(0)\right) = \mathrm{i}\theta^{ij} \tag{1.65}$$

In general, normal ordered operators will satisfy

$$\lim_{t\to 0^+} : f(x(t)) :: g(x(0)) := : f(x(0)) \star g(x(0)) :, \tag{1.66}$$

where $\star$ is precisely the Moyal product of functions as defined in (1.18). Another relation to algebra deformations appears [102] when considering a sigma model with the boundary interaction of (1.59), which is a special case of the theory used by Kontsevich in the study of deformation quantization [90].

Deformations of *superspace* instead appear naturally in a graviphoton background field [26, 27]. In particular, Ooguri and Vafa[27] study the consequences of this phenomena at field theory level. The source of inspiration of Q-deformed supersymmetric analysis is its relation to the extention of the Dijkgraaf-Vafa [103] conjecture including non-planar diagrams in the partition function, which is characterized by the presence of the graviphoton field strength.

Chapter 2

Harmonic Superspace

Since our main goal is to describe deformations of extended supersymmetric theories, we will introduce the natural framework in which they are constructed: harmonic superspace [104]. During the 1970s, when the superspace approach was developed, it was realized that manifestly supersymmetric invariant theories could be easily formulated in terms of an extension of standard Minkowskii $\mathbb{M}^4$ or Euclidean $\mathbb{R}^4$ space which was given the name *superspace*. Even though the ideas coming from $N = 1$ superspace rapidly developed into a powerful technique in quantum field theory, it was very disappointing not to find a straightforward generalization of the methods to find suitable *off-shell* manifestly invariant theories of *extended* ($N \geq 2$) supersymmetry, but instead a no-go theorem discarding them [105, 106]. Manifestly invariant theories are not just wanted for aesthetic but also for practical reasons, as quantization of supersymmetric theories is greatly simplified by covariant quantization techniques, which in the context of non(anti)commutative deformations are also deemed to be necessary. Harmonic superspace entered the scene in the middle 1980's circumventing the no-go theorem by introducing an infinite set of auxiliary fields into the analysis, allowing the proper off-shell formulation of manifestly invariant extended supersymmetry theories [107].

In this chapter we will explain how does harmonic superspace suits the purpose of describing $N = 2$ theories without incurring in the overconstrained formulations of standard superspace. Very much in the same way as standard Minkowskii space is built as a coset space of the Poincaré and its Lorentz subgroup, we will show in §2.2, how harmonic superspace is constructed as a coset of the super-Poincaré times $SU(2)$ and Lorentz times $U(1)$ groups. Harmonic superspace coordinates and their corresponding covariant derivatives are also built following Cartan's procedure. As the resulting space corresponds to standard superspace augmented by a sphere, fields in this space can be expanded in *spin weighted spherical harmonics* [108], whose relation with standard harmonic variables will be given in §2.3. Some properties we later use are established in that section. The main object of this work, the $N = (1,1)$ gauge theory action in harmonic superspace, is formulated in §2.4. Some of the concepts and techniques that are to be generalized to the non(anti)commutative case will be then shown. In particular, the method to gauge away the infinite extra degrees of freedom coming from the harmonic expansion.

2.1 The Convenience of Harmonic Superspace

First we note that superspace can be thought as a natural generalization of the *coset* construction of standard spaces. To put this into more concrete terms[1], we should remember that Minkowskii space is the coset of the Poincaré group $\mathcal{P}$ with its Lorentz subgroup $\mathcal{L} = SO(3,1)$.

$$\mathbb{M}^4 = \frac{\mathcal{P}}{\mathcal{L}} = (x^{\alpha\dot{\alpha}}). \tag{2.1}$$

It is then natural to replace here the Poincaré group with the simplest $N = 1$ super-Poincaré group $Su\mathcal{P}$. By doing so, one arrives to $N = 1$ *superspace*

$$\mathbb{M}^{4|4} \equiv \frac{Su\mathcal{P}}{\mathcal{L}} = (x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}). \tag{2.2}$$

The superindex $4|4$ indicates that we are dealing with 4 standard ($x^{\alpha\dot{\alpha}}$) and 4 extended fermionic Graßmann odd ($\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$) coordinates.

In standard classical and quantum field theories invariant under Poincaré group, the tensor properties of a field completely determine their transformation laws. Regardless of the particular theory we are describing, a scalar field, for example, is the one that transforms as

$$f'(x') = f(x). \tag{2.3}$$

whenever the coordinates of spacetime transform under the Poincaré group. We say that a field theory whose action and equations of motion possess a particular symmetry is *manifestly invariant* under it, if such symmetry is realized on fields *geometrically* by coordinate transformations as in (2.3). To devise a manifestly supersymmetric invariant theory one will then need *superfields*, that is functions of superspace $\Phi(x, \theta, \bar{\theta})$ whose transformation laws under supersymmetry are completely determined by their tensor properties. Apart from the standard transformation rules under the Poincaré group, which we already know well, it will be necessary to include also *supertranslations* defined in terms of some anticommuting infinitesimal parameters ϵ^α and $\bar{\epsilon}^{\dot{\alpha}}$ as

$$\delta x^{\alpha\dot{\alpha}} = \mathrm{i}(\epsilon^\alpha \bar{\theta}^{\dot{\alpha}} - \theta^\alpha \bar{\epsilon}^{\dot{\alpha}}), \qquad \delta\theta^\alpha = \epsilon^\alpha, \qquad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}}. \tag{2.4}$$

[1] From here on, we will mainly use spinor notation for vectors in Minkowskii and in Euclidean space, as defined in appendix §A.

A *scalar* superfield is defined as that transforming according to

$$\Phi'(x', \theta', \bar{\theta}') = \Phi(x, \theta, \bar{\theta}). \tag{2.5}$$

The next step to build a theory with superfields would be to construct its action, while being aware of the standard field content that a superfield should have. In this case we should check which representation of the supersymmetry algebra are we dealing with. This is done by the well known method of induced representations (see for example [109]). In short this consists in setting the algebra generators *on-shell* —for us particularly the *supercharges* Q_α, $\bar{Q}_{\dot{\alpha}}$— by selecting fixed time-like ($P^2 < 0$) or light-like ($P^2 = 0$) momenta. This allow us to boost them into convenient frames with standard momenta invariant under the little group $SO(3)$ or $ISO(2)$ for massive and masless particles respectively. With this simplification, creation and annihilation operators can be written easily in terms of the supercharges. Finally physical states are built in the usual way by applying the creation operator to a suitably defined vacuum. Supersymmetry is very strictive when it comes to its irreducible representations, and will force a supermultiplet to contain exactly the same number of fermionic and bosonic degrees of freedom, and only fields of the same mass.

Though is a standard procedure to find irreducible representations of supersymmetry realized over *on-shell* physical states, to study the corresponding *off-shell* realizations over fields is challenging even with the aid of superfields. Supersymmetry irreducible representations are contained *inside* the superfields as their power series of the scalar superfield in the extended coordinates θ and $\bar{\theta}$ reveals

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) = &A(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + (\theta\theta)\, F(x) + (\bar{\theta}\bar{\theta})\, G(x) \\ &+ \theta^\alpha \bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) + (\theta\theta)\bar{\theta}_{\dot{\alpha}} \bar{\kappa}^{\dot{\alpha}}(x) + (\bar{\theta}\bar{\theta})\theta^\alpha \lambda_\alpha(x) + (\bar{\theta}\bar{\theta})(\theta\theta) B(x).\end{aligned} \tag{2.6}$$

Note that this power expansion is truncated to a polynomial due to the anticommuting nature of the extended coordinates which makes them nilpotent. Also, invariance under the Poincaré subgroup assure that all *superfield components*, that is the coefficients of the extended coordinates θ and $\bar{\theta}$, are usual fields. We obtain thus only a *finite* set of components, all transforming properly under the super-Poincaré group. Superfields are however *highly reducible* representations of supersymmetry an contain much more components than those needed. As an example let us consider the simplest supersymmetric theory in four dimensions: the $N = 1$ matter multiplet. On-shell it contains a spinor $\psi^\alpha(x)$ and a complex scalar field $A(x)$, and can be described in $\mathbb{M}^{4|4}$ precisely by the $N = 1$ scalar superfield above. The problem is that this superfield contains not only both these fields but additional scalars, spinors and vectors which we have to discard by means of a *covariant irreducibility constraint*

$$\bar{D}_{\dot{\alpha}} \Phi = 0, \tag{2.7}$$

defined in terms of a covariant spinor derivative

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - 2\mathrm{i}\theta^{\alpha}\partial_{\alpha\dot{\alpha}} \tag{2.8}$$

The irreducibility constraint is solved by a field with less field content

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) =& A(x) + \theta^{\alpha}\psi_{\alpha}(x) + (\theta\theta)\,F(x) \\ &+ \mathrm{i}\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}A(x) + \frac{\mathrm{i}}{2}(\theta\theta)\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^{\alpha}(x) + \frac{1}{4}(\bar{\theta}\bar{\theta})(\theta\theta)\Box A(x).\end{aligned} \tag{2.9}$$

However, there is still an extra field $F(x)$ not appearing in the on-shell spectrum. It turns out that this field is essential for the off-shell formulation of the theory. $F(x)$ has dimension 2 and therefore can only appear without any derivatives in an action, meaning it is an *auxiliary field* that can be eliminated from the theory using its algebraic equations of motion. There are several deep reasons *not* to do that. First of all they allow a *linear* realization of supersymmetry on the fields

$$\begin{aligned}\delta A &= -\epsilon^{\alpha}\psi_{\alpha}, \\ \delta\psi_{\alpha} &= -2\mathrm{i}\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}A - 2\epsilon_{\alpha}F, \\ \delta F &= -\mathrm{i}\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^{\alpha},\end{aligned} \tag{2.10}$$

which is totally independent of the model, and *closes off-shell*, that is, we do not have to impose equations of motion as a constraint in order to obtain the supersymmetry algebra. Without auxiliary fields it therefore a very discouraging endeavour to write supersymmetric interacting theories. Their kinetic, mass and interaction terms will not be separately invariant, only the action as a whole will, and it is only found after a very involved Noether procedure. All these cause great difficulties with a particular impact on the analysis of the UV behaviour of the theory. On the other hand, introducing auxiliary fields and manifestly invariant quantization procedures greatly facilitated the analysis of the quantum aspects of this theories. Auxiliary fields are also fundamental in the description of supersymmetry breaking. As a summary, a theory is not written down off-shell in terms of superfields with auxiliary fields included only because they are beautiful and compact, but because they provide essential tools to calculate and interpret the quantum behaviour of supersymmetric theories.

A key problem arises when trying to generalize this kind of procedure to the $N = 2$ case. As an example let us show what happens with the Fayet-Sohnius matter multiplet, consisting of four scalar fields organized in a $SU(2)$ doublet f^i and two isosinglet spinor fields ψ^{α}, $\bar{\kappa}^{\dot{\alpha}}$. The corresponding superfield containing these is the isodoublet superfield $q^i(x,\theta,\bar{\theta})$ [110] which is a function of $N = 2$ superspace $\mathbb{M}^{4|8}$. The extra degrees of freedom appearing on this superfield are set to zero in a consistent way through the corresponding irreducibility constraint in extended superspace, which reads

$$D_\alpha^{(i}q^{j)} = \bar{D}_{\dot\alpha}^{(i}q^{j)} = 0, \tag{2.11}$$

leaving only the physical fields and their derivatives

$$q^i(x,\theta,\bar\theta) = f^i(x) + \theta^{i\alpha}\psi_\alpha(x) + \bar\theta^i_{\dot\alpha}\bar\kappa^{\dot\alpha}(x) + \text{derivative terms.} \tag{2.12}$$

The crucial point here is that the covariant irreducibility constraint automatically puts the fields on the free mass shell

$$\Box f^i(x) = \partial_{\alpha\dot\alpha}\psi^\alpha = \partial_{\alpha\dot\alpha}\bar\kappa^{\dot\alpha} = 0. \tag{2.13}$$

It is impossible to relax the constraint (2.11) in the framework of standard superspace as it was proven in a celebrated no-go theorem [105, 106]. Therefore, if an off-shell formulation of this theory is intended, one must look for extensions of superspace itself.

For the extended case we can say in general that *the only existing procedure to formulate off-shell manifestly supersymmetric theories in terms of unconstrained superfields is by means of harmonic superspace*, which consists of augmenting the $N = 2$ superspace by a sphere

$$\mathbb{HM}^{4+2|8} = \mathbb{M}^{4|8} \times S^2 \tag{2.14}$$

Harmonic superspace circumvents the no-go theorem preventing the construction of manifestly supersymmetric off shell actions, by introducing an infinite set of auxiliary degrees of freedom. In the case of the $N = 2$ matter hypermultiplet, this extra degrees of freedom appear as an infinite set of auxiliary fields in the the harmonic expansion of the analytic superfield q^+. As we will see later on, the $N = 2$ Yang-Mills theory has instead a *finite* set of auxiliary fields, but infinitely many pure gauge degrees of freedom which can be gauged away.

2.2 Coset Construction

Harmonic superspace is conceived as a natural generalization of real $N = 2$ superspace $\mathbb{M}^{4|8}$, which is also a coset space in the same sense as the Minkowskii or the $N = 1$ superspace before. Apart from the Lorentz subgroup $SO(3,1)$, the general super-Poincaré also includes $SU(N)$ transformations as part of their automorphisms. Representing the groups in term of the algebra that generates them, we can write

$$\mathbb{M}^{4|4N} = \frac{\{so(3,1), P_{\alpha\dot\alpha}, Q^i_\alpha, \bar{Q}_{i\alpha}\}}{\{so(3,1)\}} = \frac{\{so(3,1), P_{\alpha\dot\alpha}, Q^i_\alpha, \bar{Q}_{i\alpha}, su(N)\}}{\{so(3,1), su(N)\}}, \tag{2.15}$$

with $so(3,1)$ and $su(N)$ being the Lorentz and $SU(N)$ algebras, and $Q^i_\alpha, \bar{Q}_{i\alpha}$ the generators of supertranslations. Harmonic superspace as a coset space is defined by keeping only the $U(1)$ part of the $SU(2)$ subgroup in the coset structure of $N=2$ superspace

$$\mathbb{HM}^{4+2|8} \equiv \frac{\{so(3,1), P_{\alpha\dot{\alpha}}, Q^i_\alpha, \bar{Q}_{i\dot{\alpha}}, su(2)\}}{\{so(3,1), u(1)\}} = \mathbb{M}^{4|8} \times \frac{SU(2)}{U(1)}, \tag{2.16}$$

From the known fact that the coset space $SU(2)/U(1)$ corresponds to the sphere S^2, we can see immediately that the topology of this space is $\mathbb{M}^{4|8} \times S^2$. The coset (2.16) can be parametrized as follows

$$\Omega = \exp \mathrm{i}\left(-\tfrac{1}{2}x^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}} + \theta^\alpha_i Q^i_\alpha + \bar{\theta}^i_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}_i\right) \exp \mathrm{i}\left(\xi T^{++} + \bar{\xi} T^{--}\right). \tag{2.17}$$

here $T^{\pm\pm}$ belong to the $su(2)$ algebra together with the $U(1)$ generator T^0

$$\left[T^{++}, T^{--}\right] = T^0, \qquad \left[T^0, T^{\pm\pm}\right] = \pm 2T^{\pm\pm}. \tag{2.18}$$

The resulting parametrization of harmonic superspace obtained its called *central basis* and has coordinates given by $(x^{\alpha\dot{\alpha}}, \theta_{i\alpha}, \bar{\theta}^i_{\dot{\alpha}}, \xi, \bar{\xi})$. The $SU(2)$ algebra is represented on the supercharges by means of the Pauli matrices τ^i as

$$\left[T^0, Q^i\right] = (\tau^3)^i_j Q^j, \qquad \left[T^{\pm\pm}, Q^i\right] = (\tau^{\pm\pm})^i_j Q^j, \tag{2.19}$$

where

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tau^{++} = \frac{1}{2}(\tau^1 + \mathrm{i}\tau^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tau^{--} = \frac{1}{2}(\tau^1 - \mathrm{i}\tau^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.20}$$

Transformation rules for the coordinates of this space can be easily derived from the action of elements of the group on this coset following Cartan's procedure. Under supertranslations, the superspace coordinates transform in the usual way but leave the new coordinates unchanged

$$\delta x^m = \mathrm{i}(\epsilon^i \sigma^m \bar{\theta}_i - \theta^i \sigma^m \bar{\epsilon}_i), \qquad \delta\theta_{i\alpha} = \epsilon_{i\alpha}, \qquad \delta\bar{\theta}^i_{\dot{\alpha}} = \bar{\epsilon}^i_{\dot{\alpha}}, \qquad \delta\xi = \delta\bar{\xi} = 0. \tag{2.21}$$

Additionally we must consider transformations respect to $SU(2)$, under which the odd coordinates $\theta_i, \bar{\theta}^i$ behave like isospinors, as their indexes suggest. The new coordinates ξ, $\bar{\xi}$ however transform in a nonlinear way under $SU(2)$ and is convenient to introduce a new parametrization in terms of *harmonic variables* defined as follows

$$u^+_i = \left[\exp \mathrm{i}\left(\xi\tau^{++} + \bar{\xi}\tau^{--}\right)\right]^1_i, \qquad u^-_i = \left[\exp \mathrm{i}\left(\xi\tau^{++} + \bar{\xi}\tau^{--}\right)\right]^2_i. \tag{2.22}$$

in which the coordinates of the central base are given by

$$(X, u) \equiv (x^{\alpha\dot{\alpha}}, \theta_{i\alpha}, \bar{\theta}^i_{\dot{\alpha}}, u^\pm_i). \tag{2.23}$$

These harmonic variables, or *harmonics* for short, satisfy the conditions

$$u_i^+ \, u_j^- - u_j^+ \, u_i^- = \varepsilon_{ij}, \qquad \overline{u^{+i}} = u_i^- . \tag{2.24}$$

Defining the matrix

$$\mathbb{u} \equiv \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_1^- \end{pmatrix}, \tag{2.25}$$

it is easy to characterize the harmonics as $SU(2)$ variables, since conditions (2.24) correspond to

$$\det(\mathbb{u}) = 1, \qquad \mathbb{u}^\dagger = \mathbb{u}^{-1}. \tag{2.26}$$

The transformation law of harmonics under $SU(2)$ will be now much simpler

$$u_i'^{\pm} = \Lambda_i^j u_i^{\pm} e^{\pm \mathrm{i}\psi(\Lambda,\xi,\bar{\xi})}, \qquad \Lambda_i^j \in SU(2) \tag{2.27}$$

where ψ is a local phase factor of the induced $U(1)$ transformation.

The names of the harmonic variables are seen to be chosen to represent how they transform under this induced $U(1)$ group which should not be confused with with a gauge symmetry group. Heuristically one can think that the geometrical information contained the sphere $S^2 \sim SU(2)/U(1)$ added to the space is encoded in the $SU(2)$ harmonic variables and in the presence of a $U(1)$ *charge*. The symmetrized products of $u_i^\pm$

$$u^{+\cdots+-\cdots-}_{i_1\cdots i_m j_1\cdots j_n} \equiv u^+_{(i_1} \ldots u^+_{i_n} u^-_{j_1} \ldots u^-_{j_m)}, \tag{2.28}$$

are *spin weighted spherical harmonics* and form a complete basis of functions on the sphere S^2. One can indeed *project* any $SU(2)$ tensor onto the sphere using this basis. The transformation properties of the projected tensor under the induced $U(1)$ group are then determined by its total charge. As an example one can take a tensor t^{ij} and project it using u_{ij}^{++} to $t^{++} = t^{ij} u_{ij}^{++}$. Under the induced $U(1)$ transformation t^{++} will pick up a local phase factor $e^{2\mathrm{i}\psi(\Lambda,\xi,\bar{\xi})}$ of charge 2.

One can compare the properties of harmonics with what happens in general relativity where a vierbein e_μ^a turn the spacetime index μ into the coordinate independent index a. Harmonic variables $u_i^\pm$ resemble zweibeins because they turn the $SU(2)$ index i into a $U(1)$ index $\pm$, but as they do not transform under a $U(1)$ group independent of $SU(2)$ they are not true vielbeins (that of general relativity e_μ^a transforms under diffeomorphisms and an independent Lorentz group). Nevertheless it is possible to include an extra phase degree of freedom to turn harmonic variables into true $SU(2) \to U(1)$ zweibeins.

Harmonic variables can be used to project the standard supercoordinates onto the sphere and obtain a new set of coordinates called the *analytical basis* given by

$$(X_{\mathrm{A}}, u) \equiv (x_{\mathrm{A}}^{\alpha\dot{\alpha}}, \theta^{\alpha\pm}, \bar{\theta}^{\dot{\alpha}\pm}, u_i^\pm) = (\zeta, \theta^-, \bar{\theta}^-, u) \tag{2.29}$$

where

$$\begin{aligned}\theta^{\alpha\pm} =&\theta^{\alpha\, k} u^{\pm}_k, \quad \bar\theta^{\dot\alpha\pm} = \bar\theta^{\dot\alpha\, k} u^{\pm}_k\,, \\ x^{\alpha\dot\alpha}_{\mathrm{A}} =& x^{\alpha\dot\alpha} - 4\mathrm{i}\theta^{\alpha\,(i}\,\bar\theta^{\dot\alpha\, j)} u^-_i\, u^+_j\,.\end{aligned} \tag{2.30}$$

The analytical basis can be also defined in terms of a coset space as follows

$$\mathbb{HM}^{4+2|4}_A \equiv \frac{\{so(3,1), P_{\alpha\dot\alpha}, Q^i_\alpha, \bar Q_{i\alpha}, T^{\pm\pm}, T^0\}}{\{so(3,1), Q^+_\alpha, \bar Q^+_{\dot\alpha}, T^0\}}. \tag{2.31}$$

Its corresponding parametrization can be read from

$$\Omega = \exp\mathrm{i}\left(\xi T^{++} + \bar\xi T^{--}\right)\exp\mathrm{i}\left(-\tfrac{1}{2}x^{\alpha\dot\alpha}_{\mathrm{A}} P_{\alpha\dot\alpha} - \theta^{+\alpha}_{\mathrm{A}} Q^-_\alpha - \bar\theta^{+\dot\alpha}_{\mathrm{A}}\bar Q^-_{\dot\alpha}\right)\exp\mathrm{i}\left(\theta^{-\alpha}_{\mathrm{A}} Q^+_\alpha + \bar\theta^{-\dot\alpha}_{\mathrm{A}}\bar Q^+_{\dot\alpha}\right). \tag{2.32}$$

Covariant derivatives can also be found from the coset construct with help of the *Mauer-Cartan* form by a standard procedure[2]

$$\begin{aligned} D^+_\alpha &= \partial_{-\alpha}, \qquad & D^-_\alpha &= -\partial_{+\alpha} + 2\mathrm{i}\bar\theta^{-\dot\alpha}\partial_{\alpha\dot\alpha}, \\ \bar D^+_{\dot\alpha} &= \bar\partial_{-\dot\alpha}, \qquad & \bar D^-_{\dot\alpha} &= -\bar\partial_{+\dot\alpha} - 2\mathrm{i}\theta^{-\alpha}\partial_{\alpha\dot\alpha}. \end{aligned} \tag{2.33}$$

where

$$\partial_{\pm\alpha} = \frac{\partial}{\theta^{\pm\alpha}}, \quad \text{and} \quad \bar\partial_{\pm\dot\alpha} = \frac{\partial}{\bar\theta^{\pm\dot\alpha}}.$$

By using this derivatives in superfields on analytical harmonic superspace, one can define a new kind of Graßmann analyticity analogous to chirality in the standard case. Recall that *chiral superspace* $\mathbb{M}^{4|2N}_L$ is in some sense a subspace of superspace[3] $\mathbb{M}^{4|2N}$ consisting only on left chiral Graßmann coordinates. It is also possible to interpret (2.31) as a way to factor out the Q^+ and $\bar Q^+$ generators and dropping the dependence on the θ^- and $\bar\theta^-$ coordinates. The analogous of the chirality conditions are then the *harmonic Graßmann analyticity conditions*

$$D^+_\alpha\Phi(X_{\mathrm{A}}, u) = \bar D^+_{\dot\alpha}\Phi(X_{\mathrm{A}}, u) = 0 \tag{2.34}$$

which are directly solved by a superfield depending only on positively charged spinor variables $\Phi(\zeta, u)$, where

$$(\zeta, u) \equiv (x^{\alpha\dot\alpha}_{\mathrm{A}}, \theta^{\alpha+}, \bar\theta^{\dot\alpha+}, u^\pm_i). \tag{2.35}$$

As our primary aim will be to describe Q-deformations, we introduce here a superspace suitable for this purpose, the *combined chiral-analytic basis* or *left-chiral basis* of harmonic superspace. $(X_L, u) = (x^{\alpha\dot\alpha}_L, \theta^{\pm\alpha}; u^\pm_i)$ defined by the coordinate change

$$x^{\alpha\dot\alpha}_L = x^{\alpha\dot\alpha}_A + 4\mathrm{i}\theta^{-\alpha}\bar\theta^{+\dot\alpha}, \tag{2.36}$$

[2] An example of an explicit calculation of this kind can be found in [111]

[3] It is truly a subspace of the *complexification* of superspace $\mathbb{M}^{4|2N}$

The corresponding coset space is the one constructed by factoring out the right supercharges as in standard chiral superspace

$$\mathbb{HM}_L^{4+2|4} \equiv \frac{\{so(3,1), P_{\alpha\dot{\alpha}} Q^i_\alpha, \bar{Q}_{i\alpha}, T^{\pm\pm}, T^0\}}{\{so(3,1), \bar{Q}^{\pm}_{\dot{\alpha}}, T^0\}}. \tag{2.37}$$

Covariant derivatives of harmonics can be read out from the Maurer-Cartan forms on the sphere in terms of harmonics.

$$e^{\mathrm{i}\xi\tau^{++}+\bar{\xi}\tau^{--}}\, de^{\mathrm{i}\xi\tau^{++}+\bar{\xi}\tau^{--}} = \frac{\mathrm{i}}{2}\left(\omega^3\tau^3 + \omega^{--}\tau^{++} + \omega^{++}\tau^{--}\right), \qquad \omega^{\pm\pm} = \omega^1 \pm \mathrm{i}\omega^2 \tag{2.38}$$

where

$$\omega^{\pm\pm} = \mp 2\mathrm{i} u^{\pm j} du^{\pm}_j, \qquad \omega^3 = 2\mathrm{i} u^{+j} du^-_j = 2\mathrm{i} u^- du^+_j \tag{2.39}$$

In the central basis, the covariant derivative of an harmonic function of charge q

$$Df^{(q)}(u) = \left(d - \frac{\mathrm{i}q}{2}\omega^3\right) f^{(q)} \tag{2.40}$$

may be then rewritten in terms of the 1-forms in (2.38) to obtain

$$Df^{(q)}(u) = \frac{\mathrm{i}}{2}[\omega^3(D^0 - q) + \omega^{++}\partial^{--} + \omega^{--}\partial^{++}]f^{(q)}(u) \tag{2.41}$$

with

$$D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}, \qquad \partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}. \tag{2.42}$$

The covariant derivatives on the sphere coincide with $\partial^{\pm\pm}$ due to

$$Df^{(q)}(u) = \left(\omega^{++}D^{--} + \omega^{--}D^{++}\right) f^{(q)} \qquad \Longrightarrow \qquad D^{\pm\pm} = \partial^{\pm\pm}, \tag{2.43}$$

and in general form a $SU(2)$ algebra

$$\left[D^{++}, D^{--}\right] = D^0, \quad \left[D^0, D^{\pm\pm}\right] = \pm 2D^{\pm\pm}.$$

In the analytic basis, the covariant derivative (2.40) will pick up extra factors coming from the 1-forms ω^m, $\omega^{\pm\alpha}$, $\omega^{\pm\dot{\alpha}}$ in the expression for $\Omega^{-1}d\Omega$ with the analytic parametrization (2.32) resulting in

$$\begin{aligned} D^0_{\mathrm{A}} &= D^0 + \theta^{+\alpha}\partial_{+\alpha} - \theta^{+\alpha}\partial_{+\alpha} + \bar{\theta}^{+\dot{\alpha}}\bar{\partial}_{+\dot{\alpha}} - \bar{\theta}^{-\dot{\alpha}}\bar{\partial}_{-\dot{\alpha}}, \\ D^{++}_{\mathrm{A}} &= \partial^{++} - 2\mathrm{i}\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{+\alpha}\partial_{-\alpha} + \bar{\theta}^{+\dot{\alpha}}\bar{\partial}_{-\dot{\alpha}}, \\ D^{--}_{\mathrm{A}} &= \partial^{--} - 2\mathrm{i}\theta^{-\alpha}\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{-\alpha}\partial_{+\alpha} + \bar{\theta}^{-\dot{\alpha}}\bar{\partial}_{+\dot{\alpha}}. \end{aligned} \tag{2.44}$$

For completeness we write the main operators in the combined chiral-analytic coordinates

$$\begin{aligned} D^+_\alpha &= \partial_{-\alpha} + 2\mathrm{i}\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \qquad & D^-_\alpha &= -\partial_{+\alpha} + 2\mathrm{i}\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\ \bar{D}^+_{\dot{\alpha}} &= \bar{\partial}_{-\dot{\alpha}}, & \bar{D}^-_{\dot{\alpha}} &= -\bar{\partial}_{+\dot{\alpha}}, \end{aligned} \tag{2.45a}$$

$$
\begin{aligned}
D^{++} =&\partial^{++} + \theta^{+\alpha}\partial_{-\alpha} + \bar{\theta}^{+\dot{\alpha}}\bar{\partial}_{-\dot{\alpha}}, \\
D^{--} =&\partial^{--} + \theta^{-\alpha}\partial_{+\alpha} + \bar{\theta}^{-\dot{\alpha}}\bar{\partial}_{+\dot{\alpha}},
\end{aligned}
\tag{2.45b}
$$

$$
Q^{+}_{\alpha} =\partial_{-\alpha}, \qquad\qquad Q^{-}_{\alpha} = -\,\partial_{+\alpha}. \tag{2.45c}
$$

2.3 Harmonic Variables and Spherical Functions

Now that it has been shown how do the main harmonic superspaces are constructed, it remains to study the properties of fields on them, that is *harmonic superfields*.

It is known that it is possible to describe functions in a coset space G/H in terms of functions of G that are homogeneous on H. The particular example of interest to us are of course functions of the sphere $S^2 \sim SU(2)/U(1)$. It can be shown that any square integrable function on the sphere can be expanded in a series of harmonics as follows

$$
f^{(q)}(u) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} f^{\,(i_1 i_2 \dots\, i_{n+q} j_1 j_2 \dots\, j_n)} u^{+\,\cdots\,+\,-\,\cdots\,-}_{i_1 \cdots i_{n+q} j_1 \cdots j_n}, & q \geq 0, \\ \displaystyle\sum_{n=0}^{\infty} f^{\,(i_1 i_2 \dots\, i_n j_1 j_2 \dots\, j_{n-q})} u^{+\,\cdots\,+\,-\cdots\,-}_{i_1 \cdots i_n j_1 \cdots j_{n-q}}, & q < 0 \end{cases} \tag{2.46}
$$

Where we have used (2.28). The coefficients $f^{(i_1 \cdots j_1 \cdots)}$ are irreducible representations of $SU(2)$, and the function $f^{(q)}$ is homogeneous of degree q in the local $U(1)$ phase $e^{\mathrm{i}\psi}$, as can be seen by multiplying harmonics by their corresponding phase factors

$$
u^{\pm}_i \longrightarrow u^{\pm}_i e^{\pm \mathrm{i}\psi} \qquad \Longrightarrow \qquad f^{(q)} \longrightarrow f^{(q)} e^{\mathrm{i}q\psi} \tag{2.47}
$$

Though harmonics have been traditionally identified with Jacobi polynomials [104], we have found an unconventional approach to this relation, since a better understanding of its geometrical meaning can be obtained by realizing that (2.46) corresponds precisely to the expansion of a spin $2q$ square-integrable function on the sphere, in spin-weighted spherical harmonics of spin $2q$ [112, 113]. A spin n square-integrable function ${}_nF(\vartheta, \varphi) \in L^2(S^2)$ of standard polar and azimuthal angles on the sphere, is defined by their behaviour under rotations of the tangent plane at the point (ϑ, φ) on the sphere, by an angle ψ [108].

$$
{}_nF'(\vartheta, \varphi) = e^{\mathrm{i}n\psi}\,{}_nF(\vartheta, \varphi). \tag{2.48}
$$

As said before, any function of this kind can be expanded uniquely as

$$
{}_nF(\vartheta, \varphi) = \sum_{\substack{l \in \mathbb{N} \\ |m| \leq l}} {}_n\hat{F}_{lm}(\vartheta, \varphi)\; {}_nY_{lm}(\vartheta, \varphi) \tag{2.49}
$$

in terms of spin weighted spherical harmonics ${}_nY_{lm}(\vartheta,\varphi)$. These functions are defined as

$$ {}_nY_{lm}(\vartheta,\varphi) = (-1)^n\sqrt{\frac{2l+1}{4\pi}}\,d^l_{m,-n}(\vartheta)e^{\mathrm{i}m\varphi}, \tag{2.50}$$

where d^l_{mn} are the Wigner d-functions

$$ d^l_{mn}(\vartheta) = \sum_{t=\max(0,q)}^{\min(r,s)}(-1)^t\frac{[r!(r-q)!(s-q)!s!]^{1/2}}{(r-t)!(s-t)!t!(t-q)!}\left(\frac{\cos\theta}{2}\right)^{r+s-2t}\left(\frac{\sin\theta}{2}\right)^{2t-q}. \tag{2.51}$$

Where, for short,

$$ q = m-n, \qquad r = l+m, \qquad s = l-n \tag{2.52}$$

Orthonormality and completeness relation for this functions read

$$ \int_{S^2} d\Omega\ {}_nY^*_{lm}(\vartheta,\varphi){}_nY_{l'm'}(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'} \tag{2.53}$$

and

$$ \sum_{\substack{l\in\mathbb{N}\\ |m|\le l}} {}_nY^*_{lm}(\vartheta',\varphi'){}_nY_{lm}(\vartheta,\varphi) = \delta(\cos\theta'-\cos\theta)\delta(\varphi'-\varphi). \tag{2.54}$$

The coefficients of the expansion (2.49) are obtained by

$$ {}_n\hat{F}_{lm} = \int_{S^2} d\Omega\ {}_nY^*_{lm}(\vartheta,\varphi){}_nF(\vartheta,\varphi), \qquad l\ge|n|,\ |m|\le l. \tag{2.55}$$

Harmonics avoid using a precise parametrization of S^2 allowing us to deal with global functions on the sphere. It is also very convenient from the field theoretical point of view to have manifest $SU(2)$ covariance in the coefficients of the expansion.

We are ready to construct a harmonic superfield expansion in the analytic frame. Being the natural generalization of a chiral superfield in harmonic superspace which is heavily used in the description of supersymmetric gauge theories, we will take again as an example the harmonic Graßmann analytic field (2.34), but with a general $U(1)$ charge q. Its superfield expansion is now

$$\begin{aligned}\Phi^{(q)}(\zeta,u) =&\phi^{(q)}(x_{\mathrm{A}},u) + \theta^+\psi^{(q-1)}(x_{\mathrm{A}},u) + \bar\theta^+\bar\chi^{(q-1)}(x_{\mathrm{A}},u)\\ &+ (\theta^+)^2\,M^{(q-2)}(x_{\mathrm{A}},u) + (\bar\theta^+)^2\,N^{(q-2)}(x_{\mathrm{A}},u) + \theta^{+\alpha}\bar\theta^{+\dot\alpha}A^{(q-2)}_{\alpha\dot\alpha}(x_{\mathrm{A}},u)\\ &+ (\bar\theta^+)^2\theta^+\lambda^{(q-3)}(x_{\mathrm{A}},u) + (\theta^+)^2\bar\theta^+\kappa^{(q-3)}(x_{\mathrm{A}},u) + (\theta^+)^2(\bar\theta^+)^2\,D^{(q-4)}(x_{\mathrm{A}},u).\end{aligned} \tag{2.56}$$

Observe that each field component of this expansion is an harmonic function of definite $U(1)$ charge which can be further expanded according to (2.46), and will therefore contain *infinitely many degrees of freedom* in the form of a tower of irreducible $SU(2)$ representations. It turns out that these extra fields are precisely what is needed to overcome the no-go theorem on manifestly invariant off-shell $N = 2$ theories. As we will see, imposing constraints analogous to (2.11) to a *harmonic hypermultiplet* will properly reduce its field content *without putting the theory on-shell.* From the infinite tower of fields, the number of *physical fields* will remain finite whereas the additional degrees of freedom will consist of the necessary auxiliary or just pure gauge fields.

By means of the following rules one can define an invariant integral on $SU(2)$

$$\begin{aligned} &\text{a)} \quad \int du\, f^{(q)}(u) = 0 \quad \text{if} \quad q \neq 0, \\ &\text{b)} \quad \int du\, u^{+\cdots+-\cdots-}_{i_1\cdots i_{n+q}j_1\cdots j_n} = 0, \\ &\text{c)} \quad \int du\, 1 = 1, \\ &\text{d)} \quad \int du\, \mathrm{D}^{++}(f^{(q)}(u)) = \int du\, \mathrm{D}^{--}(f^{(q)}(u)) = 0. \end{aligned} \tag{2.57}$$

To prove this properties, one has to select a particular parametrization like the Euler angles or stereographic coordinates to map them into identities from ordinary calculus on the sphere S^2 [104]. However, the independent rules suffice to solve all problems we will face within harmonic analysis. A practical observation is that any object with $SU(2)$ indices contracted with $u_i^{\pm}$ produces an object with $U(1)$ indices, for example we can say that $x^{\alpha\dot{\alpha}}$ has $U(1)$ charge $q = 0$, θ^+ has $U(1)$ charge $q = 1$ and θ^- has $U(1)$ charge $q = -1$.

2.4 $N = 2$ Gauge Theory

Gauge fields are defined by their interaction with matter through the minimal coupling, linking charge conservation with the concepts of internal symmetry and curvature. Let us take the standard free action for a spin-$\frac{1}{2}$ matter field

$$S_{\psi_{\text{free}}} = -\frac{\mathrm{i}}{2}\int d^4x\, \psi^\alpha \partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \tag{2.58}$$

where ψ^α is taken to transform globally under some representation of an internal symmetry group. In order to keep the action invariant under a *local* transformation,

$$\psi'_\alpha(x) = e^{\mathrm{i}\lambda(x)}\psi_\alpha(x), \tag{2.59}$$

one needs to introduce a *compensating gauge field* $A_{\alpha\dot{\alpha}}(x)$ into the derivative

$$\partial_{\alpha\dot{\alpha}}\psi^\beta \to (\partial_{\alpha\dot{\alpha}} + \mathrm{i}A_{\alpha\dot{\alpha}}(x))\psi^\beta \equiv \mathcal{D}_{\alpha\dot{\alpha}}\psi^\beta. \tag{2.60}$$

This field is a *connection* in the geometrical sense, as can be seen from its inhomogeneous transformation law

$$A'_{\alpha\dot{\alpha}}(x) = -\mathrm{i}e^{\mathrm{i}\lambda}\partial_{\alpha\dot{\alpha}}e^{-\mathrm{i}\lambda} + e^{\mathrm{i}\lambda}A_{\alpha\dot{\alpha}}e^{-\mathrm{i}\lambda}, \tag{2.61}$$

which assures gauge invariance of the action defined in terms of the *covariant derivative* $\mathcal{D}_{\alpha\dot{\alpha}}$

$$S_{\psi_{\rm mc}} = -\frac{\rm i}{2}\int d^4x\,\psi^\alpha \mathcal{D}_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, \tag{2.62}$$

and gives the well known infinitesimal transformation law for the vector field

$$\delta A_{\alpha\dot{\alpha}}(x) = -\partial_{\alpha\dot{\alpha}}\lambda(x) \tag{2.63}$$

The *curvature* or *field strength tensor* is a gauge covariant object constructed exclusively out of the connection through the commutator of the covariant derivative

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = F_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + F_{\dot{\alpha}\dot{\beta}}\varepsilon_{\alpha\beta} \tag{2.64}$$

The standard Yang-Mills action is constructed with the simplest scalar that can be made out of such curvature

$$S_{\rm YM} = \frac{1}{16}\,{\rm Tr}\int d^4x\,\left(F^{\alpha\beta}F_{\alpha\beta} + F^{\dot{\alpha}\dot{\beta}}F_{\dot{\alpha}\dot{\beta}}\right). \tag{2.65}$$

A natural generalization from this idea, the $N = 2$ *superconnection*, will arise from minimally coupling the simplest $N = 2$ free matter action which corresponds to the hypermultiplet q^+. This action in analytic superspace (2.35) is written as [104]

$$S_{q^+_{\rm free}} = -\int du\,d\zeta^{(-4)}\,\tilde{q}^+{\rm D}^{++}q^+, \tag{2.66}$$

where the conjugation ˜ is defined in §A. First the gauge transformation parameter is turned into a local field of the analytic coordinates (ζ, u)

$$q^{+\prime} = e^{{\rm i}\lambda}q^+, \qquad \lambda = \lambda(\zeta, u) \tag{2.67}$$

Minimal coupling is constructed by means of a compensating gauge superfield

$${\rm D}^{++} \to \mathcal{D}^{++} \equiv {\rm D}^{++} + {\rm i}V^{++}(\zeta, u) \tag{2.68}$$

By this particular choice of coordinates, we assure the Graßmann analyticity of both the hypermultiplet and the minimally coupled action.

$$S_{q^+_{\rm mc}} = -\int du\,d\zeta^{(-4)}\,\tilde{q}^+\mathcal{D}^{++}q^+. \tag{2.69}$$

The inhomogeneous transformation law of this superfield which renders the action invariant is

$$V^{++\prime} = -{\rm i}e^{{\rm i}\lambda}{\rm D}^{++}e^{-{\rm i}\lambda} + e^{{\rm i}\lambda}V^{++}e^{-{\rm i}\lambda}. \tag{2.70}$$

The essential diference with the standard superspace formalism is that the introduction of harmonic superspace valued fields provide us with infinitely many degrees of freedom.

This is precisely what is needed in order to define a theory for this gauge prepotential that is not overconstrained in the sense of (2.11).

To get a taste of the techniques to handle this extra fields, we give the abelian Wess-Zumino gauge fixing prescription as an example. The idea here is to gauge away the extra degrees of freedom using the transformation

$$\delta V^{++} = \mathrm{D}^{++}\lambda \tag{2.71}$$

on the components of the fields

$$V^{++} = v^{++}(x,u) + \theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}A_{\alpha\dot{\alpha}}(x,u) + \cdots \tag{2.72a}$$

$$\lambda = \lambda(x,u) + \theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\lambda^{--}_{\alpha\dot{\alpha}}(x,u) + \cdots\,, \tag{2.72b}$$

giving

$$\delta v^{++} = \partial^{++}\lambda, \qquad \delta A_{\alpha\dot{\alpha}} = \frac{1}{2}\partial^{++}\lambda^{--}_{\alpha\dot{\alpha}} + \partial_{\alpha\dot{\alpha}}\lambda. \tag{2.73}$$

Now, from the full harmonic expansion of these fields, one can gauge away v^{++} fully using all harmonic components of λ but the first. More precisely, as v^{++} has an expansion in fields with isospin greater than 1, the isospin 0 component of λ remains free. This fixation has of course an impact on the variation of the vector field, but again, these freedoms can be fully gauged away into $\lambda^{--}_{\alpha\dot{\alpha}}$ which only contains components with isospin greater than 1. What remains is the standard expression for the physical component, that is

$$\delta A_{\alpha\dot{\alpha}}(x) = \partial_{\alpha\dot{\alpha}}\lambda(x) \tag{2.74}$$

Proceeding analogously for the rest of the components, one arrives to the Wess-Zumino gauge for the superconnection

$$\begin{aligned} V^{++}_{\mathrm{WZ}}(\zeta,u) =&(\theta^+)^2\bar{\phi}(x) + (\bar{\theta}^+)^2\phi(x) + \theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}A_{\alpha\dot{\alpha}}(x) \\ &+ 4(\bar{\theta}^+)^2\theta^{+\alpha}\psi^i_\alpha(x)u^-_i + 4(\theta^+)^2\bar{\theta}^+{}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}i}(x)u^-_i + 3(\theta^+)^2(\bar{\theta}^+)^2D^{ij}(x)u^-_iu^-_j\,. \end{aligned} \tag{2.75}$$

The field content of this harmonic superfield coincides precisely with the $N = 2$ Yang-Mills off-shell supermultiplet, namely a gauge vector $A_{\alpha\dot{\alpha}}$, a doublet of Weyl spinors ψ^i_α, a complex scalar field ϕ, and a triplet of auxiliary fields $D^{(ij)}$.

The road to a super Yang-Mills action in terms of unconstrained fields is not a straightforward generalization as one may think at first glance. It is possible though [104] to define a field strength tensor $\mathcal{W}$ in terms of a non-analytical superfield V^{--} which *is not* the prepotential defined above,

$$\mathcal{W} \equiv -\frac{1}{4}(\bar{D}^+)^2V^{--} \equiv \mathcal{A} + \bar{\theta}^+_{\dot{\alpha}}\tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2\tau^{--} \tag{2.76}$$

This field is linked with the prepotential through a constraint between covariant derivatives

$$\left[\mathcal{D}^{++}, \mathcal{D}^{--}\right] = \mathcal{D}_0, \quad \Rightarrow \quad D^{++}V^{--} - D^{--}V^{++} + \mathrm{i}\left[V^{++}, V^{--}\right] = 0, \tag{2.77}$$

which is a nonlinear differential equation that can be perturbatively solved using harmonic distributions

$$V^{--}(X, u) = \sum_{n=1}^{\infty} \int du_1 \ldots du_n \, (-\mathrm{i})^{n+1} \frac{V^{++}(X, u_1) \ldots V^{++}(X, u_n)}{(u^+ u_1^+)(u_1^+ u_2^+) \ldots (u_n^+ u^+)}. \tag{2.78}$$

A super Yang-Mills action can be then easily written as

$$S_{\mathrm{YM}}^{N=2} = \frac{1}{4} \operatorname{Tr} \int d^4 x_L d^4 \theta du \, \mathcal{W}^2. \tag{2.79}$$

Gauge invariance can be checked directly from the nontrivial transformation law of the field strength

$$\delta \mathcal{W} = [\mathcal{W}, \lambda] \tag{2.80}$$

or by using the usual expression for the variation of the action [107]

$$\delta S \sim \int d^4 x_L d^4 \theta d^4 \bar{\theta} du \, \delta V^{++} V^{--} \tag{2.81}$$

If we express $\mathcal{W}$ in terms of V^{--} using (2.77) we end up with a harmonic "flatness" equation for the field strength

$$\mathrm{D}^{++}\mathcal{W} + \left[V^{++}, \mathcal{W}\right] = 0, \tag{2.82}$$

It is not necessary to solve completely this equation because only its first component $\mathcal{A}$ contribute to the action

$$S_{\mathrm{YM}}^{N=2} = \frac{1}{4} \operatorname{Tr} \int d^4 x_L d^4 \theta \, \mathcal{A}^2. \tag{2.83}$$

As in the gauge case, the symmetry transformations are obtained by compensating the Wess-Zumino breaking terms coming from the standard supercharge generators. For the undotted generarors, for example,

$$\delta V_{\mathrm{WZ}}^{++} = \left(\epsilon^{+\alpha} \partial_{+\alpha} + \epsilon^{-\alpha} \partial_{-\alpha}\right) V_{\mathrm{WZ}}^{++} - D^{++}\Lambda - \left[V_{\mathrm{WZ}}^{++}\,, \Lambda\right], \tag{2.84}$$

where Λ is the corresponding compensating gauge transformation.

As it will be of interest later on, we particularize again to the Abelian case where in fact only three components of the nonanalytical potential have to be determined by the curvature equation in order to fix $\mathcal{A}$

$$V^{--} = \bar{\theta}^-_{\dot{\alpha}} v^{-\dot{\alpha}} + (\bar{\theta}^-)^2 \mathcal{A} + (\bar{\theta}^+ \bar{\theta}^-) \varphi^{--} + \cdots \tag{2.85}$$

After substituting the solution into (2.83) we obtain the well known Abelian $N = 2$ Super Maxwell action

$$S_{\mathrm{M}}^{N=2} = \int d^4x_L \left[-\frac{1}{2}\phi\Box\bar{\phi} + \frac{1}{4}D^2 - \frac{1}{16}F^2 + \mathrm{i}\Psi^{i\,\alpha}\partial_{\alpha\dot{\alpha}}\bar{\Psi}_i^{\dot{\alpha}} \right]. \tag{2.86}$$

which is clearly a free theory.

Chapter 3

Non(anti)commutativity

We have seen that for standard Euclidean or Minkowskii space the Moyal product preserves the algebra of translations, as one can read from (1.45). Keeping in mind that we plan to describe deformations of $N = 2$ theories, we should determine to what extent deformations of the algebra of functions on superspace will affect the symmetry properties of a theory of fields defined on it. We will be particularly interested in the preservation of invariance under supertranslations, and the natural generalization of the Poisson structure (1.44) in terms of *supercharges*.

The presence of both bosonic and *fermionic* coordinates in superspace $x^M = (x, \theta)$, enriches the algebra of coordinates by introducing *anticommutators*

$$[x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] \qquad [x^{\alpha\dot{\alpha}}, \theta^{\beta}] \qquad [x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] \qquad \boxed{\{\theta^{\alpha}, \theta^{\beta}\} \qquad \{\theta^{\alpha}, \bar{\theta}^{\dot{\beta}}\} \qquad \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\}}$$

allowing a bigger set of possible deformations, including those breaking the noncommutative nature of the Graßmann variables. These kind of deformations are called *non(anti)commutative.*

The possible non(anti)commutative deformations of the algebra of supercoordinates will be restricted by the particular properties we require the geometry to have. Invariance under generic translations is usually the basic requirement, followed by the associativity of the fundamental coordinate algebra. Additionally, the conjugation rules for coordinates and fields severely restrict the deformation and are heavily dependent on the kind of base space-time supermanifold. As an example, the conjugation rules for spinorial variables in $N = 1$ super Minkowskii, relate θ and $\bar{\theta}$, and rule out the possibility of having a non trivial anti commutator of fermionic coordinates if we ask for associativity. It is possible though to have such nontrivial deformations of superspace if we allow for reality conditions on fermionic coordinates, separating effectively the behaviour of θ and $\bar{\theta}$. This reality conditions are distinctive of Euclidean manifolds with extended supersymmetry.

3.1 Constraints on Non(anti)commutativity

Naively we would expect non(anti)commutativity to appear already in $N = 1$ super Minkowskii when a deformation on the graded algebra of coordinates $Z^A = (x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, is introduced in the following form

$$[Z^A, Z^B\} = P^{AB}(Z). \tag{3.1}$$

Explicitly,

$$\begin{aligned}
&\{\theta^\alpha, \theta^\beta\} = A^{\alpha\beta}(x,\theta,\bar{\theta}), && \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = B^{\alpha\beta}(x,\theta,\bar{\theta}), && \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \bar{A}^{\dot{\alpha}\dot{\beta}}(x,\theta,\bar{\theta}),\\
&[x^{\alpha\dot{\alpha}}, \theta^\beta] = \mathrm{i}C^{\alpha\dot{\alpha}\,\beta}(x,\theta,\bar{\theta}), && [x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = D^{\alpha\dot{\alpha}\,\beta\dot{\beta}}(x,\theta,\bar{\theta}), && [x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = \mathrm{i}\bar{C}^{\alpha\dot{\alpha}\,\dot{\beta}}(x,\theta,\bar{\theta}).
\end{aligned} \tag{3.2}$$

The conjugation rules for the coordinates $(\theta^\alpha)^\dagger = \bar{\theta}^{\dot{\alpha}}$, restrict the functions to be conjugate of each other

$$(A^{\alpha\beta})^\dagger = \bar{A}^{\dot{\alpha}\dot{\beta}}, \qquad (B^{\alpha\dot{\alpha}})^\dagger = B^{\alpha\dot{\alpha}}, \qquad (C^{\alpha\dot{\alpha}\,\beta})^\dagger = \bar{C}^{\alpha\dot{\alpha}\,\dot{\beta}}, \qquad (D^{\alpha\dot{\alpha}\,\beta\dot{\beta}})^\dagger = D^{\alpha\dot{\alpha}\,\beta\dot{\beta}}, \tag{3.3}$$

reflecting the relation between the coordinates themselves. When invariance of the algebra under generic (super)translations is required

$$[Z'^A, Z'^B\} = P^{AB}(Z') = P^{AB}(Z), \tag{3.4}$$

then the fermionic coordinates allow only constant deformations [35], and the other non vanishing commutators depend only on extended coordinates and are given in terms of the former constants,

$$\begin{aligned}
&\{\theta^\alpha, \theta^\beta\} = A^{\alpha\beta}, && \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = B^{\alpha\beta}, && \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \bar{A}^{\dot{\alpha}\dot{\beta}},\\
&[x^{\alpha\dot{\alpha}}, \theta^\beta] = \mathrm{i}C^{\alpha\dot{\alpha}\,\beta}(\theta,\bar{\theta}), && [x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = D^{\alpha\dot{\alpha}\,\beta\dot{\beta}}(\theta,\bar{\theta}), && [x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = \mathrm{i}\bar{C}^{\alpha\dot{\alpha}\,\dot{\beta}}(\theta,\bar{\theta}).
\end{aligned} \tag{3.5}$$

Further requiring associativity of the algebra, one is left with [35],

$$\begin{aligned}
&\{\theta^\alpha, \theta^\beta\} = 0, && \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0, && \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0,\\
&[x^{\alpha\dot{\alpha}}, \theta^\beta] = \mathrm{i}C^{\alpha\dot{\alpha}\,\beta}, && [x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = D^{\alpha\dot{\alpha}\,\beta\dot{\beta}}(\theta,\bar{\theta}), && [x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = \mathrm{i}\bar{C}^{\alpha\dot{\alpha}\,\dot{\beta}}.
\end{aligned} \tag{3.6}$$

Meaning that the most general associative deformation of $N = 1$ super Minkowskii consistent with the standard spinor conjugation rule does not allow deformation of the fermionic sector of the algebra. Therefore, we should not expect a natural connection with a background field like the graviphoton in the framework of $N = 1$ super Minkowskii.

Not until we relax the constraints imposed by the conjugation rules are we capable of producing deformations in the fermionic sector. What we are looking for, are reality conditions over fermionic coordinates, a characteristic of Euclidean manifolds with extended supersymmetry[1]. A nontrivial anticommutation relation between spinor coordinates can be obtained in $N = 2$ Euclidean superspace, where we can define symplectic Majorana-Weyl spinors, and impose

$$(\theta_i^\alpha)^* = \theta_\alpha^i, \qquad (\bar\theta^{i\,\dot\alpha})^* = \bar\theta_{i\,\dot\alpha} \tag{3.7}$$

In this case, the most general associative algebra consistent with superspace translations is twofold [35], either

$$\begin{aligned}
\{\theta_i^\alpha, \theta_j^\beta\} &= A_{1\,ij}^{\alpha\beta}, \qquad \{\theta_i^\alpha, \bar\theta^{j\dot\beta}\} = 0, \qquad \{\bar\theta^{i\dot\alpha}, \bar\theta^{j\dot\beta}\} = 0,\\
[x^{\alpha\dot\alpha}, \theta^\beta] &= \mathrm{i}C_{1\,i}^{\alpha\dot\alpha\,\beta} - \frac{1}{2}A_{1\,ij}^{\alpha\beta}\bar\theta^{j\dot\alpha}, \qquad [x^{\alpha\dot\alpha}, \bar\theta^{i\dot\beta}] = 0,\\
[x^{\alpha\dot\alpha}, x^{\beta\dot\beta}] &= \mathrm{i}D^{\alpha\dot\alpha\,\beta\dot\beta} + \frac{\mathrm{i}}{2}(C_{1\,i}^{\beta\dot\beta\,\alpha}\bar\theta^{i\dot\alpha} - C_{1\,i}^{\alpha\dot\alpha\,\beta}\bar\theta^{i\dot\beta}) - \frac{1}{4}\bar\theta^{i\dot\alpha}A_{1\,ij}^{\alpha\beta}\bar\theta^{j\dot\beta},
\end{aligned} \tag{3.8}$$

or

$$\begin{aligned}
\{\bar\theta^{i\dot\alpha}, \bar\theta^{j\dot\beta}\} &= A_2^{ij\,\dot\alpha\dot\beta}, \qquad \{\theta_i^\alpha, \bar\theta^{j\dot\beta}\} = 0, \qquad \{\theta_i^\alpha, \theta_j^\beta\} = 0,\\
[x^{\alpha\dot\alpha}, \bar\theta^{i\dot\beta}] &= \mathrm{i}C_2^{i\,\alpha\dot\alpha\,\dot\beta} - \frac{1}{2}A_2^{ij\,\dot\alpha\dot\beta}\theta_j^\alpha, \qquad [x^{\alpha\dot\alpha}, \theta^\beta] = 0,\\
[x^{\alpha\dot\alpha}, x^{\beta\dot\beta}] &= \mathrm{i}D^{\alpha\dot\alpha\,\beta\dot\beta} + \frac{\mathrm{i}}{2}(C_2^{i\,\beta\dot\beta\,\dot\alpha}\theta_i^\alpha - C_2^{i\,\alpha\dot\alpha\,\dot\beta}\theta_i^\beta) - \frac{1}{4}\theta_i^\alpha A_2^{ij\,\dot\alpha\dot\beta}\theta_j^\beta.
\end{aligned} \tag{3.9}$$

This shows how do deformations of the fermionic coordinates may appear naturally in superspace. The constants C_1 and C_2 will break the R-Symmetry of $N = 2$ but we can simply set them to zero if we like.

A more tricky way of obtaining this kind of deformations [37, 61] is to double the fermionic degrees of freedom in order to formally define a kind of $N = 1$ Euclidean superspace out of $N = 2$ spinor variables

$$\begin{aligned}
\theta^\alpha &\equiv \theta_1^\alpha - \theta_2^\alpha, \qquad & \bar\theta^\alpha &\equiv \theta_1^\alpha + \theta_2^\alpha,\\
\theta^{\dot\alpha} &\equiv \bar\theta^{1\dot\alpha} - \bar\theta^{2\dot\alpha}, \qquad & \bar\theta^{\dot\alpha} &\equiv \theta^{1\dot\alpha} + \bar\theta^{2\dot\alpha}.
\end{aligned} \tag{3.10}$$

We then select the $N = 1$ subspace consisting only of undotted variables, and introduce an alternate pseudoconjugation that does not relate θ^α with $\theta^{\dot\alpha}$, but instead

$$\begin{aligned}
(\theta^\alpha)^* &= \mathrm{i}\bar\theta_\alpha, \qquad & (\bar\theta^\alpha)^* &= -\mathrm{i}\theta_\alpha,\\
(\theta_\alpha)^* &= -\mathrm{i}\bar\theta^\alpha, \qquad & (\bar\theta_\alpha)^* &= \mathrm{i}\theta^\alpha.
\end{aligned} \tag{3.11}$$

For a chiral representation of the supersymmetry, where

$$\begin{aligned}
Q_\alpha &= \mathrm{i}\partial_\alpha + \theta^{\dot\alpha}\partial_{\alpha\dot\alpha}, \qquad & Q_{\dot\alpha} &= \mathrm{i}\partial_{\dot\alpha}\\
D_\alpha &= \partial_\alpha, \qquad & D_{\dot\alpha} &= \partial_{\dot\alpha} + \mathrm{i}\theta^\alpha\partial_{\alpha\dot\alpha},
\end{aligned} \tag{3.12}$$

[1] A nice treatment of spinors in spaces with arbitrary dimensions and signatures is given in chapter 3 of [114]

the supertranslations take the form

$$\delta x^{\alpha\dot{\alpha}} = -\mathrm{i}\epsilon^{\alpha}\theta^{\dot{\alpha}}, \qquad \delta\theta^{\alpha} = \epsilon^{\alpha}, \qquad \delta\theta^{\dot{\alpha}} = \epsilon^{\dot{\alpha}}, \tag{3.13}$$

and the non(anti)commutative algebra reduces to just one nontrivial term

$$\{\theta^{\alpha}, \theta^{\beta}\} = C^{\alpha\beta}. \tag{3.14}$$

Requiring again invariance under supertranslations and associativity, we are left with $C^{\alpha\beta}$ constant. This kind of deformation is called a *D-deformation* because it induces a change in the algebra of covariant derivatives leaving the algebra of the supercharges intact. For the representation chosen, such a D-deformation will look like

$$\{D_{\alpha}, D_{\beta}\} = 0, \qquad \{D_{\alpha}, D_{\dot{\alpha}}\} = \mathrm{i}\partial_{\alpha\dot{\alpha}}, \qquad \{D_{\dot{\alpha}}, D_{\dot{\beta}}\} = -C^{\alpha\beta}\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}}, \tag{3.15}$$

and will clearly pose a problem when attempting to construct chiral fields for which the condition $D_{\dot{\alpha}}\phi = 0$ is affected. On the other hand, supersymmetry is preserved totally by this kind of deformations.

Another possibility is to take the *anti*chiral representation

$$\begin{aligned} Q_{\alpha} &= \mathrm{i}\partial_{\alpha}, & Q_{\dot{\alpha}} &= \mathrm{i}\partial_{\dot{\alpha}} + \theta^{\alpha}\partial_{\alpha\dot{\alpha}}, \\ D_{\alpha} &= \partial_{\alpha} + \mathrm{i}\theta^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & D_{\dot{\alpha}} &= \partial_{\dot{\alpha}}, \end{aligned} \tag{3.16}$$

whose corresponding supertranslations take the form

$$\delta x^{\alpha\dot{\alpha}} = -\mathrm{i}\epsilon^{\dot{\alpha}}\theta^{\alpha}, \qquad \delta\theta^{\alpha} = \epsilon^{\alpha}, \qquad \delta\theta^{\dot{\alpha}} = \epsilon^{\dot{\alpha}}. \tag{3.17}$$

In this case we are left with a $N = 1$ generalization of the deformed algebra (3.8),

$$\begin{aligned} &\{\theta^{\alpha}, \theta^{\beta}\} = C^{\alpha\beta}, && \{\theta^{\alpha}, \theta^{\dot{\beta}}\} = 0, && \{\theta^{\dot{\alpha}}, \theta^{\dot{\beta}}\} = 0, \\ &[x^{\alpha\dot{\alpha}}, \theta^{\beta}] = -\mathrm{i}C^{\alpha\beta}\theta^{\dot{\alpha}}, && [x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = \theta^{\dot{\alpha}}C^{\alpha\beta}\theta^{\dot{\beta}}, && [x^{\alpha\dot{\alpha}}, \theta^{\dot{\beta}}] = 0. \end{aligned} \tag{3.18}$$

Here the situation is then the opposite of that of (3.12), as the algebra of covariant derivatives remains the same while the supersymmetry algebra gets deformed

$$\{Q_{\alpha}, Q_{\beta}\} = 0, \qquad \{Q_{\alpha}, Q_{\dot{\alpha}}\} = \mathrm{i}\partial_{\alpha\dot{\alpha}}, \qquad \{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} = -C^{\alpha\beta}\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}}. \tag{3.19}$$

These so called *Q-deformations* break supersymmetry, but in turn preserve chirality. As in this particular case the deformation of the algebra affects only the dotted supercharges, it is often said that $N = 1$ breaks to $N = \frac{1}{2}$ supersymmetry.

With a suitable change of variables

$$x_L^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - \mathrm{i}\theta^{\alpha}\theta^{\dot{\alpha}}, \tag{3.20}$$

the coordinate algebra (3.18) reduces back to the simple case (3.14) with $C^{\alpha\beta}$ constant. As the commutation relations of bosonic coordinates vanish, the Poisson structure in the Moyal-like product over chiral superfields will be nilpotent

$$\Phi \star \Psi = \Phi \exp(-\overleftarrow{\partial}_\alpha C^{\alpha\beta} \overrightarrow{\partial}_\beta)\Psi = \Phi\Psi - \Phi \overleftarrow{\partial}_\alpha C^{\alpha\beta} \overrightarrow{\partial}_\beta \Psi - \frac{1}{2} P^2 \partial^2 \Phi \partial^2 \Psi. \tag{3.21}$$

This is our first example of what are called *nilpotent deformations*. It is the common lore that noncommutativity introduces nonlocality in quantum field theory, but this kind of Poisson structures have the remarkable property of rendering their corresponding Moyal product polynomial, thus producing local actions.

3.2 Nilpotent deformations of $N = 2$ Superspace

As mentioned before, the conjugation relations of Minkowskii space are too restrictive to allow this kind of nilpotent deformations. We turn instead to Euclidean space, which is invariant under $Spin(4) = SU(2)_L \times SU(2)_R$, and where the independence of left and right spinors release this constraints. Let us start with chiral coordinates in $N = (1, 1)$ superspace

$$z_L \equiv (x_L^{\alpha\dot{\alpha}}, \theta_k^\alpha, \bar{\theta}^{\dot{\alpha}k}), \qquad x_L^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + 2\mathrm{i}\theta_k^\alpha \bar{\theta}^{\dot{\alpha}k}. \tag{3.22}$$

The $N = (1, 1)$ Euclidean superalgebra consists of the supercharges Q, the momenta P, and the generators of the group of automorphisms, which factors into the Euclidean space spinor group $Spin(4)$ and the R-Symmetry group $SU(2) \times O(1, 1)$. A differential representation of this algebra, given in terms of the chiral coordinates is given by

$$\begin{array}{lll}
Q_\alpha^k = \partial_\alpha^k, \quad \bar{Q}_{\dot{\alpha}k} = \bar{\partial}_{\dot{\alpha}k} - 2\mathrm{i}\theta_k^\alpha \partial_{\alpha\dot{\alpha}}, \quad P_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}, & & \Big\} \ \text{Supertranslations} \\[2mm]
L_{\alpha\beta} = -\frac{1}{2} x_{L\,(\alpha\dot{\alpha}} \partial_{\beta)}^{\dot{\alpha}} + \theta_{(\alpha k} \partial_{\beta)}^k & SU(2)_L & \multirow{2}{*}{\Bigg\} \ \text{Euclidean } Spin(4)} \\
R^{\dot{\alpha}\dot{\beta}} = \frac{1}{2} x_L^{\alpha(\dot{\alpha}} \partial_\alpha^{\dot{\beta})} + \bar{\theta}^{(\dot{\alpha}k} \bar{\partial}_k^{\dot{\beta})} & SU(2)_R & \\[2mm]
T_{ij} = -\theta_{(i}^\alpha \partial_{\alpha j)} + \bar{\theta}_{(i}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha} j)} & SU(2) & \multirow{2}{*}{\Bigg\} \ \text{R-Symmetry}} \\
O = \theta_k^\alpha \partial_\alpha^k - \bar{\theta}^{\dot{\alpha}k} \bar{\partial}_{\dot{\alpha}k} & O(1,1) &
\end{array} \tag{3.23}$$

Using this representation, a *chiral nilpotent deformation* will be determined by the following Poisson structure

$$P = -\overleftarrow{Q}_\alpha^i C_{ij}^{\alpha\beta} \overrightarrow{Q}_\beta^j = -\overleftarrow{\partial}_\alpha^i C_{ij}^{\alpha\beta} \overrightarrow{\partial}_\beta^j, \tag{3.24}$$

where the matrix deformation parameter $C_{ij}^{\alpha\beta} = C_{ji}^{\beta\alpha}$ is constant, as required by the associativity of the Moyal product. The corresponding graded commutation relations for superspace coordinates are

$$\begin{aligned} &\{\theta_i^\alpha, \theta_j^\beta\} = C_{ij}^{\alpha\beta}, && \{\theta_i^\alpha, \bar{\theta}_j^{\dot{\beta}}\} = 0, && \{\bar{\theta}_i^{\dot{\alpha}}, \bar{\theta}_j^{\dot{\beta}}\} = 0, \\ &[x^{\alpha\dot{\alpha}}, \theta^\beta] = 0, && [x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = 0, && [x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = 0. \end{aligned} \tag{3.25}$$

The Poisson operator is in this case also constructed only in terms of supercharges which, being Graßmann odd, render it nilpotent $P^5 = 0$. On the other hand, considering that these supercharges do not commute with all $N = (1,1)$ generators (3.23), it is obvious that some symmetries will be broken by the deformation. The resulting supersymmetry breaking pattern will depend on our particular selection of $C_{ij}^{\alpha\beta}$, for example, the Poisson structure for a general matrix will not "commute" with any generator containing θ_i^α due to the action of ∂_i^α within the Moyal product. As a consequence, symmetries generated by $L^{\alpha\beta}$, T_{ij}, and $\bar{Q}_{\dot{\alpha}k}$ will be broken, implying a breaking of half the supersymmetry $N = (1,1) \to N = (1,0)$ and of the automorphisms group $Spin(4) \times O(1,1) \times SU(2) \to SU(2)_R$.

We can explore more precisely how a symmetry is deformed by looking at the transformation laws of covariant objects. The way a general superfield A transforms under a particular symmetry is given by the action of the generator of such symmetry G_a and measured by its infinitesimal parameter ϵ^a as in

$$\delta_\epsilon A = -\epsilon^a G_a A. \tag{3.26}$$

In the undeformed case such transformations fulfil naturally the Leibniz rule

$$\delta_\epsilon (AB) = \delta_\epsilon A B + A \delta_\epsilon B, \tag{3.27}$$

but deformations can destroy this property because in general

$$\delta_\epsilon (A \star B) \neq \delta_\epsilon A \star B + A \star \delta_\epsilon B. \tag{3.28}$$

Whenever the equality holds, the symmetry will be preserved by the deformation. This motivates the definition of a *commutator* between the operator $\epsilon \cdot G$ and the bilinear Poisson structure as

$$A[\epsilon \cdot G, P]B \equiv -C_{ij}^{\alpha\beta} \left([\epsilon \cdot G, \partial_\alpha^i] A \, \partial_\beta^j B + \partial_\alpha^i A \, [\epsilon \cdot G, \partial_\beta^j] B \right). \tag{3.29}$$

Using this on the generator O of the algebra (3.23) we immediately see that

$$A[O, P]B = 2APB, \tag{3.30}$$

meaning that the $O(1,1)$ factor of R-symmetry is broken for any kind of Q-deformation.

We can further refine the scheme of breaking by decomposing the matrix $C^{\alpha\beta}_{ij}$. First we can separate it [36, 40] into its $(1,1)$ and $(3,3)$ parts under $SU(2)_L \times SU(2)$, which are referred to as *singlet* and *non singlet* parts respectively

$$C^{\alpha\beta}_{ij} = \underbrace{\varepsilon^{\alpha\beta}\varepsilon_{ij}I}_{\text{(1,1) singlet}} + \underbrace{C^{(\alpha\beta)}_{(ij)}}_{\text{(3,3) nonsinglet}} \tag{3.31}$$

A Poisson operator containing only the singlet component of the deformation matrix will commute with $L_{\alpha\beta}$ and T_{ij} thus restoring spacetime $Spin(4)$ and $SU(2)$ symmetries. A purely non singlet deformation will have in general the same symmetry breaking pattern as the full matrix $C^{\alpha\beta}_{ij}$ unless we further decompose it into what we call [115] the *product ansatz*

$$C^{(\alpha\beta)}_{(ij)} = b_{ij}c^{\alpha\beta}. \tag{3.32}$$

Since the general matrix has rank 3, it can be shown that this is a particular case of the non-singlet deformation matrix decomposition

$$C^{(\alpha\beta)}_{(ij)} = \sum_{r=1}^{3} b^{(r)}_{ij} c^{\alpha\beta}_{(r)}. \tag{3.33}$$

which contains the complete information of its nine original degrees of freedom.

For the product ansatz (3.32), the commutator of P with the $SU(2)_L$ generator will result in

$$A[\lambda \cdot L, P]B = -\frac{1}{2} b_{ij} c^{\alpha}_{\gamma} \lambda^{\gamma\beta} \partial^{i}_{(\alpha} A\, \partial^{j}_{\beta)} B. \tag{3.34}$$

If we select a transformation parameter parallel to the deformation matrix $\lambda^{\alpha\beta} \propto c^{\alpha\beta}$, we obtain a vanishing commutator, meaning that the subgroup arising from choosing a preferential direction of $SU(2)_L$, that is $U(1)_L$, is preserved. A completely analogous argument for the automorphisms group will take us to the preservation of its $U(1)$ subgroup. On the other hand the supersymmetry generator $\bar{Q}_{\dot{\alpha}k}$ produces the following commutator with P

$$A[\bar{\epsilon} \cdot \bar{Q}, P]B = 2\mathrm{i} b_{ij} c^{\alpha\beta} \bar{\epsilon}^{\dot{\alpha}i} \left(\partial_{\alpha\dot{\alpha}} A\, \partial^{j}_{\beta} B - \partial^{j}_{\beta} A\, \partial_{\alpha\dot{\alpha}} B \right). \tag{3.35}$$

From this we can see that turning off components of the b_{ij} matrix could preserve half the supersymmetry. For instance $(b_{11}, b_{12}, b_{22}) = (1,0,0)$ allows commutation of $\bar{Q}^{2}_{\dot{\alpha}}$ generators with P. Schematically, the breaking pattern of the supersymmetry algebra for particular deformation matrix decompositions is expressed in the following diagram

Deformation Type — Automorphisms Preserved — Residual SUSY

$$
\begin{tikzcd}
C_{ij}^{\alpha\beta} \arrow[rr, dotted] \arrow[d] \arrow[dr] & & SU(2)_R & N=(1,0) \\
C_{(ij)}^{(\alpha\beta)} \arrow[d] & I\varepsilon^{\alpha\beta}\varepsilon_{ij} \arrow[r, dotted] & Spin(4)\times SU(2) & N=(1,0) \\
b_{ij}c^{\alpha\beta} \arrow[rr, dotted] & & SU(2)_R\times U(1)_L\times U(1) & N=(1,0),(1,\frac{1}{2})
\end{tikzcd}
$$

3.3 Q-deformations of $N=(1,1)$ Harmonic Superspace

Since the structure of the deformation matrix has been established, we continue by incorporating the deformations treated in §3.2 into the framework of harmonic superspace, where we will introduce it into several theories. We will follow closely what was done in §1.3 for ordinary spacetime, namely the modification of the algebra of superfields by a Moyal product. This will profoundly affect the gauge group of the theory, to the extent of producing generalizations of $U(1)$ theories that are *non-Abelian* in character.

As already mentioned previously, the deformed formulations of superspace are realized via two well known Poisson operators defined in terms of the supersymmetric charges (Q-deformations) or covariant derivative operators (D-deformations). Q-deformations break the supersymmetry, but preserve chirality and Graßmann harmonic analyticity. D-deformations are supersymmetry-preserving, but in turn break chirality and Graßmann analyticity. In what follows, we will focus our attention in the nilpotent Q-deformations defined by (3.24),

$$P=-\overleftarrow{Q}^i_\alpha C^{\alpha\beta}_{ik}\overrightarrow{Q}^k_\beta. \tag{3.36}$$

We have seen that the deformation matrix can split (3.31) into singlet and non-singlet parts

$$P=-I\overleftarrow{Q}^i_\alpha\varepsilon^{\alpha\beta}\varepsilon_{ik}\overrightarrow{Q}^k_\beta.-\overleftarrow{Q}^i_\alpha\hat{C}^{\alpha\beta}_{ik}\overrightarrow{Q}^k_\beta. \tag{3.37}$$

Here we have defined

$$\hat{C}^{\alpha\beta}_{ij}\equiv C^{(\beta\alpha)}_{(ij)}, \tag{3.38}$$

as a shorthand for the constant $SU(2)_L\times SU(2)$ tensor, symmetric under independent exchange of Latin and Greek indices $\hat{C}^{\beta\alpha}_{ij}=\hat{C}^{\alpha\beta}_{ji}$, representing the nonsinglet component of the deformation matrix (3.31). The singlet term will preserve $SO(4)\times SU(2)$ spacetime and automorphisms whereas the nonsinglet term will in general break them down to $SU(2)_R$.

The product of five supercharges Q^i_α will always contain at least one product with repeated indices that will vanish due to anticommutativity. What directly follows is the nilpotency of the Poisson structure. This important property advantageously renders the Moyal product polynomial

$$A \star B = A\,B + A\,P\,B + \frac{1}{2}\,A\,P^2 B + \frac{1}{6}\,A\,P^3 B + \frac{1}{24} A\,P^4 B, \tag{3.39}$$

and assures the locality of the resulting deformed functions. Working in the left-chiral basis (2.45) will reduce the form of the supercharges and their harmonic projections to

$$Q^i_\alpha = \partial^i_\alpha\,, \qquad Q^\pm_\alpha = Q^i_\alpha u^\pm_i = \pm\partial_{\mp\alpha}. \tag{3.40}$$

Furthermore, the harmonic projections of the deformation matrix are

$$C^{\pm\pm\,\alpha\beta} = \hat{C}^{\pm\pm\,\alpha\beta}\,, \qquad C^{\pm\mp\,\alpha\beta} = \hat{C}^{\pm\mp\,\alpha\beta} \pm I\,\varepsilon^{\alpha\beta}\,, \tag{3.41}$$

from which we obtain the harmonic Poisson operator

$$\begin{aligned} P = &- \overleftarrow{\partial}_{+\alpha}\hat{C}^{++\alpha\beta}\overrightarrow{\partial}_{+\beta} - \overleftarrow{\partial}_{+\alpha}\left(\hat{C}^{+-\alpha\beta} + I\varepsilon^{\alpha\beta}\right)\overrightarrow{\partial}_{-\beta} \\ &- \overleftarrow{\partial}_{-\alpha}\left(\hat{C}^{+-\alpha\beta} - I\varepsilon^{\alpha\beta}\right)\overrightarrow{\partial}_{+\beta} - \overleftarrow{\partial}_{-\alpha}\hat{C}^{--\alpha\beta}\overrightarrow{\partial}_{-\beta}. \end{aligned} \tag{3.42}$$

Using this operator it is possible to fully expand the Moyal product (3.39) above for two general $U(1)$ superfields A and B with Graßmann parity a and b respectively. As this is a straightforward but long calculation, we content ourselves here with its first term

$$\begin{aligned} APB = -\,(-1)^a\Big\{&\hat{C}^{++\alpha\beta}\partial_{+\alpha}A\,\partial_{+\beta}B + \left(\hat{C}^{+-\alpha\beta} + I\varepsilon^{\alpha\beta}\right)\partial_{+\alpha}A\,\partial_{-\beta}B \\ &+ \left(\hat{C}^{+-\alpha\beta} - I\varepsilon^{\alpha\beta}\right)\partial_{-\alpha}A\,\partial_{+\beta}B\, + \hat{C}^{--\alpha\beta}\partial_{-\alpha}A\,\partial_{-\beta}B\Big\}, \end{aligned} \tag{3.43}$$

and leave the full expression for the appendix §B.1. In the same fashion, from this products it is a simple task to obtain deformed (anti)commutators between these fields

$$\begin{aligned} [A,B\}_\star = &\left(APB \pm BPA\right) + \frac{1}{2}\left(AP^2B \pm BP^2A\right) \\ &+ \frac{1}{6}\left(AP^3B \pm BP^3A\right) + \frac{1}{24}\left(AP^3B \pm BP^3A\right) \\ \equiv\; &\mathrm{A}^\pm_1 + \frac{1}{2}\mathrm{A}^\pm_2 + \frac{1}{6}\mathrm{A}^\pm_3 + \frac{1}{24}\mathrm{A}^\pm_4\,, \end{aligned} \tag{3.44}$$

but we limit ourselves to write the first order contribution

$$\begin{aligned} \mathrm{A}^\pm_1 = -(-1)^a\left[1 \mp (-1)^{ab}\right]\Big[&(\hat{C}^{+-\alpha\beta} + I\varepsilon^{\alpha\beta})(\partial_{+\alpha}A\,\partial_{-\beta}B + \partial_{-\beta}A\,\partial_{+\alpha}B) \\ &+\hat{C}^{++\alpha\beta}\partial_{+\alpha}A\,\partial_{+\beta}B + \hat{C}^{--\alpha\beta}\partial_{-\alpha}A\,\partial_{-\beta}B\Big], \end{aligned} \tag{3.45}$$

leaving the other orders for the appendix §B.1.

For A and B both Graßmann-even analytic Abelian superfields, we get a particularly simple commutator

$$\begin{aligned}[A,B]_\star = -2\Big[& I\left(\partial^\alpha_- A\partial_{+\alpha}B - \partial^\alpha_+ A\partial_{-\alpha}B\right) + (\partial_{-\alpha}A\partial_{-\beta}B)\hat{C}^{--\alpha\beta} \\ &+ (\partial_{+\alpha}A\partial_{+\beta}B)\hat{C}^{++\alpha\beta} + \left(\partial_{-\alpha}A\partial_{+\beta}B + \partial_{+\alpha}A\partial_{-\beta}B\right)\hat{C}^{+-\alpha\beta}\Big], \\ &- \frac{1}{2}\,[\partial_{-\alpha}(\partial_+)^2 A\,\partial_{-\beta}(\partial_+)^2 B]M^{++\,\alpha\beta}\,, \end{aligned} \tag{3.46}$$

where

$$\begin{aligned} M^{++\,(\alpha\beta)} = {}& \hat{C}^{+-\,(\alpha\gamma)}\hat{C}^{++}_{(\gamma\mu)}\hat{C}^{+-\,(\mu\beta)} - \hat{C}^{++\,(\alpha\gamma)}\hat{C}^{++}_{(\gamma\mu)}\hat{C}^{--\,(\mu\beta)} \\ &- I\left[\hat{C}^{++\,\alpha}_{\gamma}\hat{C}^{+-\,(\gamma\beta)} + \hat{C}^{++\,\beta}_{\gamma}\hat{C}^{+-\,(\gamma\alpha)}\right] + I^2\,\hat{C}^{++\,(\alpha\beta)}\,. \end{aligned} \tag{3.47}$$

Note that for the special choice (3.32) of $\hat{C}$

$$\hat{C}^{ij}_{\alpha\beta} = b^{ij}c_{ij}. \tag{3.48}$$

the expression (3.47) drastically simplifies to

$$M^{++\,\alpha\beta} = c^{\alpha\beta}b^{++}(I^2 - \frac{1}{4}c^2b^2)\,, \qquad c^2 = c^{\alpha\beta}c_{\alpha\beta}\,,\ \ b^2 = b^{ik}b_{ik}\,, \tag{3.49}$$

and vanishes under the particular relation between the deformation parameters

$$I^2 = \frac{1}{4}c^2b^2\,. \tag{3.50}$$

With these tools at hand, the next task is to deform theories by introducing the star product in the algebra of functions, which will affect their construction in a non-linear way.

Let us consider the deformations of the theories for a non Abelian gauge prepotential V^{++} derived from the minimal coupling of the hypermultiplet. As in the Minkowskii case (2.66) we have

$$\int du\, d\zeta^{(-4)}\ \tilde{q}^+\mathrm{D}^{++}q^+, \tag{3.51}$$

From the cyclicity of the Moyal product, it is clear that deformations enter only on interaction terms where there is more than one multiplication of fields. Therefore standard free hypermultiplet actions on harmonic superspace are left unchanged. Introducing gauge fields will in contrast change the whole picture, due to the presence of the Moyal product in the covariant derivative

$$\mathrm{D}^{++}q^+ \to \mathcal{D}^{++}q^+ \equiv \mathrm{D}^{++}q^+ + V^{++}\star q^+, \tag{3.52}$$

and in the gauge transformation law

$$\delta_a V^{++} = \mathrm{D}^{++}\Lambda_a + [V^{++}, \Lambda_a]_\star. \tag{3.53}$$

It is very important to note that even in the Abelian case where fields would normally commute, the star commutator will not vanish, thereby inducing non Abelian behaviour.

For singlet deformations one can impose the Wess-Zumino gauge on the prepotential without any changes in the residual gauge parameter. This does not mean that there are no deformed contributions to the gauge transformations, but that one can use exactly the same field Λ_a as in the *undeformed* case to gauge away extra degrees of freedom [52]. In general however the Wess-Zumino gauging procedure as shown in §2.4 does not remain unaffected by the deformation. Non singlet deformations for example will break the gauge by introducing a mixing in the components of the deformed contribution. In such cases one is led to modify the gauge parameter $\Lambda_a \to \Lambda_a + \Delta\Lambda_a$ to compensate this effects.

Now let us see how the gauge action gets deformed. For simplicity we will follow a discussion in terms of Abelian fields, knowing that it can be readily generalized *mutatis mutandis* to the non Abelian case. The full invariant action in chiral superspace [52] is given by

$$S = \frac{1}{4}\int d^4x\, d^4\theta\, du\, \mathcal{W}^2, \tag{3.54}$$

with $\mathcal{W}$ being the covariant chiral superfield. As this is a second order interaction theory, formally it does not get deformed due to the cyclicity of the Moyal product. Nevertheless, the superfield strength is a *curvature*

$$\mathrm{D}^{++}\mathcal{W} + [V^{++}_{\mathrm{WZ}}, \mathcal{W}]_\star = 0, \tag{3.55}$$

and therefore will receive contributions from the Moyal commutator. A very nontrivial property that Q-deformed theories share with their undeformed limit is that the only field contributing to the invariant action is the first component $\mathcal{A}$ of

$$\mathcal{W} = \mathcal{A} + \bar\theta^+_{\dot\alpha}\tau^{-\dot\alpha} + (\bar\theta^+)^2\tau^{--}. \tag{3.56}$$

This has been verified for the singlet case in [52], but as we will show next, it holds *regardless* of the particular kind of deformation. We start from the expression for the action in components

$$S = \frac{1}{4}\int d^4x\, d^4\theta\, du\, \left[\mathcal{A}^2 + 2\bar\theta^+_{\dot\alpha}\tau^{-\dot\alpha}\mathcal{A} + (\bar\theta^+)^2\left(2\mathcal{A}\tau^{--} - \frac{1}{2}(\tau^-)^2\right)\right], \tag{3.57}$$

and prove that contributions other than the first term vanish. First, the superfield strength must be written in terms of a non analytic gauge potential

$$\mathcal{W} = -\frac{1}{4}(\bar D^+)^2 V^{--}, \tag{3.58}$$

with V^{--} containing the following components

$$V^{--} = v^{--} + \bar{\theta}^{+}_{\dot{\alpha}} v^{(-3)\,\dot{\alpha}} + \bar{\theta}^{-}_{\dot{\alpha}} v^{-\,\dot{\alpha}} + (\bar{\theta}^{-})^2 \mathcal{A} + (\bar{\theta}^{+}\bar{\theta}^{-})\varphi^{--} + \bar{\theta}^{-\dot{\alpha}}\bar{\theta}^{+\dot{\beta}}\varphi^{--}_{\dot{\alpha}\dot{\beta}} + (\bar{\theta}^{+})^2 v^{(-4)} \\ + (\bar{\theta}^{-})^2 \bar{\theta}^{+}_{\dot{\alpha}} \tau^{-\,\dot{\alpha}} + (\bar{\theta}^{+})^2 \bar{\theta}^{-}_{\dot{\alpha}} \tau^{(-3)\,\dot{\alpha}} + (\bar{\theta}^{+})^2 (\bar{\theta}^{-})^2 \tau^{--}. \tag{3.59}$$

The curvature equation for $\mathcal{W}$ can be cast into the so-called *harmonic flatness equation* for V^{--}

$$\mathrm{D}^{++} V^{--} - \mathrm{D}^{--} V^{++}_{\mathrm{WZ}} + \left[V^{++}_{\mathrm{WZ}}, V^{--}\right]_{\star} = 0. \tag{3.60}$$

This equation relates V^{--} with V^{++}_{WZ}, and allows the determination of the components of $\mathcal{W}$. To compute the superfield $\mathcal{A}$ for instance, we only need the following components of the equation

$$\nabla^{++} v^{-\,\dot{\alpha}} - v^{+\,\dot{\alpha}} = 0, \tag{3.61a}$$

$$\nabla^{++} \mathcal{A} = 0, \tag{3.61b}$$

$$\nabla^{++} \varphi^{--} + 2(\mathcal{A} - v) + \frac{1}{2}\left\{v^{+\,\dot{\alpha}}, v^{-}_{\dot{\alpha}}\right\}_{\star} = 0, \tag{3.61c}$$

where we have introduced the *chiral covariant derivative*,

$$\nabla^{++} = \mathrm{D}^{++} + \left[v^{++}, \right]_{\star}. \tag{3.62}$$

This derivative includes $v^{++} = (\theta^{+})^2\,\bar{\phi}$ which is precisely the first component of the prepotential V^{++}_{WZ}. The flatness equation (3.60), when applied to the gauge covariant chiral superfield $\mathcal{W}$, leads to the following equations

$$\nabla^{++} \mathcal{A} = 0, \tag{3.63}$$

$$\nabla^{++} \tau^{-\dot{\alpha}} + \left[v^{-\dot{\alpha}}, \mathcal{A}\right]_{\star} = 0, \tag{3.64}$$

$$\nabla^{++} \tau^{--} - \frac{1}{2}\left\{v^{+}_{\dot{\alpha}}, \tau^{-\dot{\alpha}}\right\}_{\star} + [v, \mathcal{A}]_{\star} = 0 \tag{3.65}$$

From (3.61b), (3.64), and (3.61a) we can solve for $\tau^{-\dot{\alpha}}$

$$\tau^{-\dot{\alpha}} = \left[\mathcal{A}, v^{-\dot{\alpha}}\right]_{\star}. \tag{3.66}$$

Analogously, we can solve for τ^{--} from equations (3.61b), (3.61c), and (3.66),

$$\tau^{--} = \frac{1}{2}\left[\mathcal{A}, \varphi^{--}\right]_{\star} + \frac{1}{2} v^{-}_{\dot{\alpha}} \star \mathcal{A} \star v^{-\dot{\alpha}} + \frac{1}{4}\left\{v^{-}_{\dot{\alpha}} \star v^{-\dot{\alpha}}, \mathcal{A}\right\}_{\star}. \tag{3.67}$$

Inserting these fields (3.66), (3.67) into the invariant action (3.57), we see directly that the terms containing $\tau^{-\dot{\alpha}}$ and τ^{--} cancel out under the integral so that the invariant action is reduced to

$$S = \frac{1}{4}\int d^4x\, d^4\theta\, du\, \mathcal{A}^2. \tag{3.68}$$

The basic property used is that the covariant derivative obeys Leibniz' rule over the star product

$$\nabla^{++}(A \star B) = (\nabla^{++}A) \star B + A \star (\nabla^{++}B). \tag{3.69}$$

This follows from the fact that for Q-deformations, the Poisson structure —and therefore also the star operator— commutes with the harmonic derivative D^{++}. It is remarkable that this result is independent of the particular kind of Poisson structure, and that it can be simply generalized to the non Abelian case as it has been shown for singlet Q-deformations in [52].

One of the great advantages of working in harmonic superspace is the *manifest* supersymmetry of formalisms. Even in the presence of deformations partially breaking this symmetry, a manifestly covariant action like (3.68) is invariant under the residual part of the supersymmetry *by construction*. Nevertheless it is of physical interest to have a definite expression for the supersymmetry transformation laws of the fields in the Q-deformed case even when we know from the very beginning that the action is invariant under them.

Following the Wess-Zumino prescription as in §2.4, the unbroken supersymmetry transformation laws can be read from the action of the residual supersymmetry generators on $V^{++}_{\rm WZ}$ and the corresponding compensating gauge transformation with parameter Λ_ϵ

$$\delta_\epsilon V^{++}_{\rm WZ} = \left(\epsilon^{+\alpha}\partial_{+\alpha} + \epsilon^{-\alpha}\partial_{-\alpha}\right) V^{++}_{\rm WZ} - D^{++}\Lambda_\epsilon - \left[V^{++}_{\rm WZ}\, ,\, \Lambda_\epsilon\right]_\star . \tag{3.70}$$

Again as for gauge transformations, the undeformed parameter Λ_ϵ is suited to compensate Wess-Zumino breaking terms only in the singlet case. Non singlet Q-deformations will also require some correction terms to be added to the parameter $\Lambda_\epsilon \to \Lambda_\epsilon + \Delta\Lambda_\epsilon$.

3.4 Singlet Deformations

The simplest example of the Q-deformation of a theory with extended supersymmetry is given by the non(anti)commutative version of $N = (1,1)$ super-Maxwell [52]. This is constructed from the $U(1)$ gauge harmonic $N = (1,1)$ superpotential in the Wess-Zumino gauge, which written in the chiral basis is given by

$$\begin{aligned}
&V^{++}_{\rm WZ} = v^{++} + \bar\theta^+_{\dot\alpha} v^{+\dot\alpha} + (\bar\theta^+)^2 v \\
&v^{++} = (\theta^+)^2\bar\phi\,, \quad v^{+\dot\alpha} = 2\theta^{+\alpha}A^{\dot\alpha}_\alpha + 4(\theta^+)^2\bar\Psi^{-\dot\alpha} - 2\mathrm{i}(\theta^+)^2\theta^{-\alpha}\partial^{\dot\alpha}_\alpha\bar\phi\,, \\
&v = \phi + 4\theta^+\Psi^- + 3(\theta^+)^2 D^{ij}u^{--}_{ij} - \mathrm{i}(\theta^+\theta^-)\partial^{\alpha\dot\alpha}A_{\alpha\dot\alpha} + \theta^{-\alpha}\theta^{+\beta}F_{\alpha\beta} \\
&- (\theta^+)^2(\theta^-)^2\Box\bar\phi + 4\mathrm{i}\,(\theta^+)^2\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha}\,,
\end{aligned} \tag{3.71}$$

For singlet deformations the corresponding gauge parameter from the undeformed case (2.72b), also gauges away the proper degrees of freedom. In the chiral basis it is written as

$$\Lambda_0 = \mathrm{i}a + 2\theta^{-\alpha}\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}}a - \mathrm{i}(\theta^-)^2(\bar{\theta}^+)^2\Box\, a\,. \tag{3.72}$$

Since $\partial_{+\alpha}\Lambda_0 = 0$, only the first term in the general formula for the commutator (3.44) contributes to the gauge transformation (3.53)

$$\begin{aligned}\delta_a V^{++} =& 2(1+4I\bar{\phi})\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\,\partial_{\alpha\dot{\alpha}}a - 4I\,(\bar{\theta}^+)^2\,A^{\alpha\dot{\alpha}}\,\partial_{\alpha\dot{\alpha}}\,a - 16I\,(\bar{\theta}^+)^2\,\theta^{+\alpha}\bar{\Psi}^{-\dot{\alpha}}\,\partial_{\alpha\dot{\alpha}}a\\ &- 4\mathrm{i}I(\theta^+\theta^-)(\bar{\theta}^+)^2[(\partial a\cdot\partial\bar{\phi}) + 2\bar{\phi}\,\Box a] + 8\mathrm{i}I(\bar{\theta}^+)^2\theta^{+(\alpha}\theta^{-\beta)}\partial_{\alpha\dot{\alpha}}a\partial^{\dot{\alpha}}_{\beta}\bar{\phi}.\end{aligned} \tag{3.73}$$

In terms of components, the transformation is

$$\begin{aligned}&\delta_a\phi = -4IA_{\alpha\dot{\alpha}}\partial^{\alpha\dot{\alpha}}a, \quad \delta_a\bar{\phi} = 0\,, \quad \delta_a A_{\alpha\dot{\alpha}} = (1{+}4I\bar{\phi})\partial_{\alpha\dot{\alpha}}a\,,\\ &\delta_a\Psi^k_\alpha = -4I\bar{\Psi}^{k\dot{\alpha}}\partial_{\alpha\dot{\alpha}}a\,, \quad \delta_a\bar{\Psi}^k_{\dot{\alpha}} = 0\,, \quad \delta_a D^{kl} = 0\,.\end{aligned} \tag{3.74}$$

Now, we can use the curvature equations (3.55) and (3.60) to derive the form of the potential V^{--}, whose relevant component is [52]

$$\begin{aligned}\mathcal{A}(z_c,u) =& \left[\phi + 2I\,A^2\,\frac{1}{1{+}4I\bar{\phi}} + 8I^3(\partial\bar{\phi})^2\,\frac{1}{1{+}4I\bar{\phi}}\right] - (\theta^+)^2(\theta^-)^2\,\Box\bar{\phi}\\ &+ 2\theta^{+\alpha}\left[\Psi^-{}_\alpha + \frac{4I}{1{+}4I\bar{\phi}}\bar{\Psi}^{-\dot{\alpha}}\,A_{\alpha\dot{\alpha}}\right] - \frac{2}{1{+}4I\bar{\phi}}\theta^{-\alpha}\left[\Psi^+{}_\alpha + \frac{4I}{1{+}4I\bar{\phi}}\bar{\Psi}^{+\dot{\alpha}}\,A_{\alpha\dot{\alpha}}\right]\\ &+ (\theta^+)^2\left[\frac{8I}{1{+}4I\bar{\phi}}(\bar{\Psi}^-)^2 + D^{ij}u^{--}_{ij}\right] + \frac{(\theta^-)^2}{(1{+}4I\bar{\phi})^2}\left[\frac{8I}{1{+}4I\bar{\phi}}(\bar{\Psi}^+)^2 + D^{ij}u^{++}_{ij}\right]\\ &- \frac{2(\theta^+\theta^-)}{1+4I\bar{\phi}}\left[\frac{8I}{1{+}4I\bar{\phi}}(\bar{\Psi}^+\bar{\Psi}^-) + D^{ij}u^{+-}_{ij}\right] + \theta^{+\alpha}\theta^{-\beta}\left(F_{\alpha\beta} - 4I\,\frac{\partial_{(\alpha\dot{\alpha}}\bar{\phi}A^{\dot{\alpha}}_{\beta)}}{1{+}4I\bar{\phi}}\right)\\ &+ 2\mathrm{i}(\theta^-)^2\,\theta^{+\alpha}\partial_{\alpha\dot{\alpha}}\left(\frac{\bar{\Psi}^{+\dot{\alpha}}}{1{+}4I\bar{\phi}}\right) + 2\mathrm{i}(\theta^+)^2\,(1{+}4I\bar{\phi})\,\theta^{-\alpha}\partial_{\alpha\dot{\alpha}}\left(\frac{\bar{\Psi}^{-\dot{\alpha}}}{1{+}4I\bar{\phi}}\right).\end{aligned} \tag{3.75}$$

After a field redefinition,

$$\begin{aligned}\varphi &= (1{+}4I\bar{\phi})^{-2}[\phi + 2I(1{+}4I\bar{\phi})^{-1}[(A^2 + 4I^2(\partial\bar{\phi})^2]]\,,\\ a_{\alpha\dot{\alpha}} &= (1{+}4I\bar{\phi})^{-1}A_{\alpha\dot{\alpha}}, \quad \bar{\psi}^k_{\dot{\alpha}} = (1{+}4I\bar{\phi})^{-1}\bar{\Psi}^k_{\dot{\alpha}}\,,\\ \psi^k_\alpha &= (1{+}4I\bar{\phi})^{-2}[\Psi^k_\alpha + 4I(1{+}4I\bar{\phi})^{-1}A_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}k}]\,,\\ d^{kl} &= (1{+}4I\bar{\phi})^{-2}[D^{kl} + 8I(1{+}4I\bar{\phi})^{-1}\bar{\Psi}^k_{\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}l}]\,,\end{aligned} \tag{3.76}$$

it is possible to plug (3.75) into the integral (3.68) to obtain a readable action

$$S = \frac{1}{4}\int d^4x_L d^4\theta\ \mathcal{A}^2 = \int d^4x\ L = \int d^4x\ (1+4I\bar{\phi})^2 L_0 \tag{3.77}$$

Which is simply a factor times a *free* action

$$L_0 = -\frac{1}{2}\varphi\Box\bar{\phi} - \frac{1}{16}f_{\alpha\beta}f^{\alpha\beta} - \mathrm{i}\psi^\alpha_i\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}i} + \frac{1}{4}d^{ij}d_{ij} \tag{3.78}$$

with $f_{\alpha\beta} = 2\mathrm{i}\partial_{(\alpha\dot{\alpha}} a^{\dot{\alpha}}_{\beta)}$. It is not possible to further disentangle the interaction of $\bar{\phi}$ with this gauge field. This is a remarkable yet typical behaviour of nonanticommutatively deformed gauge theories: one starts with a free theory, and the deformation will then introduce interactions.

The supersymmetry transformations can be obtained simply by calculating the star commutator in (3.70). The result in components is

$$\begin{aligned}
&\delta_\epsilon \phi = 2(\epsilon^k \Psi_k)\,, \quad \delta_\epsilon \bar{\phi} = 0\,, \quad \delta_\epsilon A_{\alpha\dot{\alpha}} = \epsilon^k_\alpha \bar{\Psi}_{k\dot{\alpha}}\,, \\
&\delta_\epsilon \Psi^k_\alpha = -\epsilon_{\alpha l}\mathcal{D}^{kl} + \frac{1}{2}(1+4I\bar{\phi})F_{\alpha\beta}\epsilon^{k\beta} - 2\mathrm{i}I\epsilon^k_\alpha A\cdot\partial\bar{\phi}\,, \\
&\delta_\epsilon \bar{\Psi}^k_{\dot{\alpha}} = -\mathrm{i}(1+4I\bar{\phi})\epsilon^{k\alpha}\partial_{\alpha\dot{\alpha}}\bar{\phi}\,, \quad \delta_\epsilon D^{kl} = \mathrm{i}\partial^{\alpha\dot{\alpha}}[\epsilon^{(k}_\alpha \bar{\Psi}^{l)}_{\dot{\alpha}}(1+4I\bar{\phi})]
\end{aligned} \tag{3.79}$$

One can use the field redefinitions to put this residual unbroken supersymmetry in its *standard* realization over components,

$$\begin{aligned}
&\delta_\epsilon \varphi = 2(\epsilon^k \psi_k)\,, \quad \delta_\epsilon \bar{\phi} = 0\,, \quad \delta_\epsilon a_{\alpha\dot{\alpha}} = \epsilon^k_\alpha \bar{\psi}_{k\dot{\alpha}}\,, \\
&\delta_\epsilon \psi^k_\alpha = -\epsilon_{\alpha l} d^{kl} + \tfrac{1}{2} f_{\alpha\beta}\epsilon^{k\beta}\,, \quad \delta_\epsilon \bar{\psi}^k_{\dot{\alpha}} = -\mathrm{i}\epsilon^{k\alpha}\partial_{\alpha\dot{\alpha}}\bar{\phi}\,, \\
&\delta_\epsilon d^{kl} = \mathrm{i}\partial^{\alpha\dot{\alpha}}[\epsilon^{(k}_\alpha \bar{\psi}^{l)}_{\dot{\alpha}}]\,.
\end{aligned} \tag{3.80}$$

The calculations for the non Abelian case follow closely the Abelian case and present clear similarities. In particular, there is evidence from an expansion up to second order in I, that suggests a factorization like (3.77)

This simple example already shows some of the features we will encounter in the case of non-singlet deformations, where the bosonic part of the action also admits a factorization like (3.77) which has manifest Lorentz and R-symmetry. Non-singlet deformations are richer in structure, and include a case which breaks the supersymmetry down to $N = (1, 1/2)$. Although the main scheme to determine the transformations and actions is shared between the singlet and non-singlet case, the latter involve surprisingly arduous calculations due to the breaking of the Wess-Zumino gauge induced by the undeformed compensating parameters. Example of singlet D-deformations of non abelian super-Yang-Mills can be found in [116, 117].

Part II

Non Singlet Q-deformed $N = (1, 1)$ Gauge Theories

Chapter 4

Gauge Transformations and Seiberg-Witten Map

In the remaining chapters we will deal exclusively with non singlet Q-deformed gauge theories, starting with their deformed gauge transformations and their respective Seiberg-Witten map. As we explained in §3.3, deformations enter the gauge transformation laws through a star anticommutator that introduces non-Abelian behaviour even in $U(1)$ gauge theories. As opposed to the singlet case where the gauge parameter was exactly the same as in the commutative case, the Wess-Zumino gauge prescription is deeply affected by non-singlet deformations and requires a new compensating gauge parameter. Instead of starting from scratch, we will consider the problem of correcting the Wess-Zumino breaking terms coming from naively using the same gauge parameter as in the undeformed case. Though this additional compensating parameter could be obtained order by order in a series expansion [53, 53], we will see that a particular decomposition of the deformation matrix corresponding to the maximal supersymmetry preservation, allows its *exact* determination [115, 118]. The harmonic equations that appear can be solved by exploiting their formal similarity with those of harmonic coupled oscillators. In §4.2 we pursue further this similarity and develop a general algorithm to solve the kind of harmonic coupled equations that, as we will see later on when calculating the gauge action, are a typical feature of this non-singlet deformation matrix. By means of this formalism, we calculate the exact deformed gauge transformation law for the component fields of the $N = (1, 1)$ vector multiplet. Finally, we will construct the corresponding Seiberg-Witten map [25, 119] which puts the gauge transformations into their canonical form.

4.1 Compensating Gauge Parameter

The gauge transformation for the superpotential as shown in (3.71) acquire Wess-Zumino breaking terms from the star commutator. To see this, we consider that part of gauge transformation of $V^{++}_{\rm WZ}$ which corresponds to the usual parameter Λ_0 as in (3.72). Since $\partial_{+\alpha}\Lambda_0 = 0$, only the first term in the general formula (3.44) contributes to the present case,

$$\begin{aligned}
\delta_0 V^{++} &= \mathrm{D}^{++}\Lambda_0 + [V^{++},\Lambda_0]_\star \\
&= 2\,\theta^{+\alpha}\bar\theta^{+\dot\alpha}\,\partial_{\alpha\dot\alpha}a - 2\mathrm{i}\,(\theta^+\theta^-)(\bar\theta^+)^2\,\Box\, a + 8\bar\phi\,\theta^+_\alpha\left(\bar\theta^+_{\dot\alpha}\,\partial^{\dot\alpha}_\beta a + \mathrm{i}\,\theta^-_\beta\,(\bar\theta^+)^2\,\Box\, a\right)\hat C^{+-\alpha\beta} \\
&+ 4(\bar\theta^+)^2\left(A^{\dot\alpha}_\alpha + 4\,\theta^+_\alpha\,\bar\Psi^{-\dot\alpha} + \mathrm{i}\,(\theta^+\theta^-)\,\partial^{\dot\alpha}_\alpha\bar\phi + 2\mathrm{i}\,\theta^+_{(\alpha}\theta^-_{\gamma)}\partial^{\gamma\dot\alpha}\bar\phi\right)\partial_{\beta\dot\alpha}a\,\hat C^{+-\alpha\beta} \\
&- 4\mathrm{i}\,(\theta^+)^2(\bar\theta^+)^2\,\partial_{\alpha\dot\alpha}a\,\partial^{\dot\alpha}_\beta\bar\phi\;\hat C^{--\alpha\beta}.
\end{aligned} \tag{4.1}$$

The breaking of the Wess-Zumino gauge becomes evident from component transformations,

$$\delta A_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha}a + 4\bar\phi\hat C^{+-}{}_\alpha^\beta\partial_{\beta\dot\alpha}a. \tag{4.2}$$

Note that the RHS should be independent of harmonics as the gauge field $A_{\alpha\dot\alpha}$. On that account we are led to properly modify the residual gauge freedom parameter by adding to Λ_0 the following terms

$$\begin{aligned}
\Delta\Lambda &= \theta^+_\alpha\bar\theta^+_{\dot\alpha}\,\partial^{\dot\alpha}_\beta\,a\,B^{--\alpha\beta} + (\bar\theta^+)^2\partial_{\beta\dot\beta}\,a\,A^{\dot\beta}_\alpha G^{--\alpha\beta} + (\theta^+)^2(\bar\theta^+)^2\,\Box a P^{(-4)} \\
&+ (\bar\theta^+)^2\theta^+_\alpha\left[\bar\Psi^{-\dot\beta}\partial_{\beta\dot\beta}\,a\,H^{--\alpha\beta} + \bar\Psi^{+\dot\beta}\partial_{\beta\dot\beta}\,a\,G^{(-4)\alpha\beta}\right] + (\theta^+)^2(\bar\theta^+)^2\,\partial_{\alpha\dot\alpha}a\,\partial^{\dot\alpha}_\beta\,\bar\phi\,B^{(-4)\alpha\beta} \\
&+ \mathrm{i}\,\theta^+_\alpha\theta^-_\beta\,(\bar\theta^+)^2\,\Box\,a\,B^{--\alpha\beta} + \mathrm{i}\,\theta^+_\alpha\theta^-_\gamma\,(\bar\theta^+)^2\,\partial_{\beta\dot\lambda}a\,\partial^{\gamma\dot\lambda}\bar\phi\,\frac{d}{d\bar\phi}B^{--\alpha\beta}\,.
\end{aligned} \tag{4.3}$$

For the time being, the components of the compensating superfield (4.3) are arbitrary functions of harmonics, of the field $\bar\phi$ and deformation parameters, to be calculated from imposing Wess-Zumino gauge. Note that these coefficients involve both the symmetric and antisymmetric pieces. The correction term to $\delta_0 V^{++}$, will then be

$$\hat\delta V^{++} = \mathrm{D}^{++}\Delta\Lambda + [V^{++},\Delta\Lambda]_\star\,. \tag{4.4}$$

Once again, from the structure of $\Delta\Lambda$ we conclude that only the lowest order term in (3.44) contributes into the star-commutator in (4.4). It is also easy to see that the term $\sim \hat C^{--\alpha\beta}$ is vanishing. So we are left with

$$\begin{aligned}
[V^{++},\Delta\Lambda]_\star &= -2I\left(\partial^\alpha_- V^{++}\partial_{+\alpha}\Delta\Lambda - \partial^\alpha_+ V^{++}\partial_{-\alpha}\Delta\Lambda\right) - 2\left(\partial_{+\alpha}V^{++}\partial_{+\beta}\Delta\Lambda\right)\hat C^{++\alpha\beta} \\
&- 2\left(\partial_{-\alpha}V^{++}\partial_{+\beta}\Delta\Lambda + \partial_{+\alpha}V^{++}\partial_{-\beta}\Delta\Lambda\right)\hat C^{+-\alpha\beta} \equiv \mathcal{A}^{++} + \mathcal{B}^{++} + \mathcal{C}^{++}\,.
\end{aligned} \tag{4.5}$$

After some calculations we find

$$\begin{aligned}
\mathcal{A}^{++} &= 2\mathrm{i}I\,(\theta^+)^2(\bar\theta^+)^2\left[\bar\phi\,\Box a\left(\varepsilon_{\alpha\beta}B^{--\alpha\beta}\right)-\partial_{\alpha\dot\alpha}\bar\phi\,\partial^{\dot\alpha}_{\beta}a\left(B^{--\alpha\beta}+\bar\phi\frac{d}{d\bar\phi}B^{--\alpha\beta}\right)\right] && (4.6)\\
\mathcal{B}^{++} &= -2\left[(\bar\theta^+)^2\,A_{\alpha\dot\alpha}\,\partial^{\dot\alpha}_{\gamma}a\,B^{--\ \gamma}_{\beta}-2\theta^+_\alpha\bar\theta^+_{\dot\beta}\,\bar\phi\,\partial^{\dot\beta}_{\gamma}a\,B^{--\ \gamma}_{\beta}\right]\\
&\quad -2(\bar\theta^+)^2\theta^+_\alpha\left(\bar\Psi^{+\dot\rho}\,\partial_{\rho\dot\rho}a\,\bar\phi\,G^{-4\ \rho}_{\beta}+\bar\Psi^{-\dot\rho}\,\partial_{\rho\dot\rho}a\,\bar\phi\,H^{--\ \rho}_{\beta}+2\,\bar\Psi^{-\dot\rho}\,\partial_{\rho\dot\rho}a\,B^{--\ \rho}_{\beta}\right)\Big]\,\hat C^{++\alpha\beta}\\
&+\ (\text{terms with }\ \theta^-), && (4.7)\\
\mathcal{C}^{++} &= -2\mathrm{i}\,(\theta^+)^2(\bar\theta^+)^2\left[\Box\,a\,\bar\phi\,B^{--}_{(\alpha\beta)}-\partial_{\alpha\dot\rho}\bar\phi\,\partial^{\dot\rho}_{\rho}a\left(B^{--\ \rho}_{\beta}+\bar\phi\frac{d}{d\bar\phi}B^{--\ \rho}_{\beta}\right)\right]\hat C^{+-\alpha\beta}\\
&+\ (\text{terms with }\ \theta^-). && (4.8)
\end{aligned}$$

The full gauge transformations of the fields induced when including the ansatz (4.3) take the following from

$$\delta V^{++}=\delta_0 V^{++}+\hat\delta V^{++}\,. \qquad (4.9)$$

Its corresponding component expansion is

$$\begin{aligned}
&\delta A_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}a+4\partial_{\beta\dot\alpha}a\,\bar\phi\,\hat C^{+-\beta}_{\alpha}+2\partial_{\beta\dot\alpha}a\,\bar\phi\,B^{--\ \beta}_{\rho}\hat C^{++\ \rho}_{\alpha}+\frac{1}{2}\,\partial_{\beta\dot\alpha}a\,\partial^{++}B^{--\ \beta}_{\alpha}\,, && (4.10)\\
&\delta\phi=\partial_{\alpha\dot\alpha}a\,A^{\dot\alpha}_{\beta}\partial^{++}G^{--\beta\alpha}+4\partial_{\alpha\dot\alpha}a\,A^{\dot\alpha}_{\beta}\,\hat C^{+-\alpha\beta}+2\,\partial_{\alpha\dot\alpha}a\,A^{\dot\alpha}_{\rho}\,B^{--\ \alpha}_{\beta}\hat C^{++\rho\beta}\,, && (4.11)\\
&\delta\Psi^-_\alpha=-4\,\bar\Psi^{-\dot\alpha}\,\partial_{\beta\dot\alpha}a\,\hat C^{+-\beta}_{\alpha}-\ \bar\Psi^{-\dot\alpha}\,\partial_{\beta\dot\alpha}a\,\bar\phi\,H^{--\ \beta}_{\rho}\hat C^{++\rho}_{\alpha}-2\,\bar\Psi^{-\dot\alpha}\,\partial_{\beta\dot\alpha}a\,B^{--\ \beta}_{\rho}\hat C^{++\rho}_{\alpha}\\
&-\frac{1}{4}\bar\Psi^{-\dot\alpha}\,\partial_{\beta\dot\alpha}a\,\partial^{++}H^{--\ \beta}_{\alpha}-\bar\Psi^{+\dot\alpha}\,\partial_{\beta\dot\alpha}a\,\bar\phi\,G^{-4\ \beta}_{\rho}\,\hat C^{++\rho}_{\alpha}-\frac{1}{4}\bar\Psi^{+\dot\alpha}\,\partial_{\beta\dot\alpha}a\,H^{--\ \beta}_{\alpha}\\
&-\frac{1}{4}\bar\Psi^{+\dot\alpha}\,\partial_{\beta\dot\alpha}a\,\partial^{++}G^{-4\ \beta}_{\alpha}\equiv\partial_{\beta\dot\alpha}a\left(\bar\Psi^{-\dot\alpha}\,D^{\ \beta}_{\alpha}+\bar\Psi^{+\dot\alpha}\,D^{--\ \beta}_{\alpha}\right)\,, && (4.12)\\
&\delta D^{--}=-\frac{4}{3}\mathrm{i}\partial_{\alpha\dot\alpha}a\,\partial^{\dot\alpha}_{\beta}\bar\phi\,\hat C^{--\alpha\beta}-\frac{2}{3}\mathrm{i}\left[\bar\phi\,\Box a\,B^{--}_{(\alpha\rho)}-\partial_{\alpha\dot\alpha}\bar\phi\,\partial^{\dot\alpha}_{\beta}a\left(B^{--\ \beta}_{\rho}+\bar\phi\frac{d}{d\bar\phi}B^{--\ \beta}_{\rho}\right)\right]\hat C^{+-\alpha\rho}\\
&+\frac{1}{3}\left[\partial_{\alpha\dot\alpha}a\,\partial^{\dot\alpha}_{\beta}\bar\phi\left(\partial^{++}B^{(-4)\alpha\beta}\right)+\frac{\mathrm{i}}{2}\Box a\,\left(\varepsilon_{\alpha\beta}B^{--\alpha\beta}\right)+\frac{\mathrm{i}}{2}\,\partial_{\alpha\dot\alpha}a\,\partial^{\dot\alpha}_{\beta}\bar\phi\,\left(\frac{d}{d\bar\phi}B^{--\beta\alpha}\right)\right]\\
&+\frac{1}{3}\,\Box a\left(\partial^{++}P^{(-4)}\right)\,. && (4.13)
\end{aligned}$$

We should require that the above transformations preserve the Wess-Zumino gauge. This amounts to imposing the proper harmonic dependence of the fields through the conditions

$$\begin{aligned}
&(a)\ \partial^{++}\delta A_{\alpha\dot\alpha}=0\,,\quad (b)\ \partial^{++}\delta\phi=0\,,\quad (c)\ (\partial^{++})^2\delta\Psi^-_\alpha=0\ \leftrightarrow\ \partial^{--}\delta\Psi^-_\alpha=0\,,\\
&(d)\ (\partial^{++})^3\delta D^{--}=0\ \leftrightarrow\ \partial^{--}\delta D^{--}=0\,. && (4.14)
\end{aligned}$$

These conditions fix the unknown harmonic functions in terms of $\bar\phi$, deformation parameters and harmonics. After solving them, one can find the explicit form of gauge variations. Unfortunately, it is very difficult to find the closed solution in the case of generic deformation parameters. As an example let us first look at the condition (4.14) for $A_{\alpha\dot\alpha}$, which amounts to a harmonic equation for the ansatz field $B^{--\alpha\beta}$

$$(\partial^{++})^2B^{--\ \beta}_{\alpha}+4\,\bar\phi\,\hat C^{++\rho}_{\alpha}\,\partial^{++}B^{--\ \beta}_{\rho}+8\,\bar\phi\,\hat C^{++\beta}_{\alpha}=0\,. \qquad (4.15)$$

One could decompose this tensors into their symmetric and antisymmetric parts and use the traceless property of $SU(2)$ symmetric tensors,

$$B^{--}_{\alpha\beta} = B^{--\,\alpha\beta}_{\mathrm{s}} + \varepsilon^{\alpha\beta}\, B^{--}_{\mathrm{a}}\,, \qquad B^{--\;\alpha}_{\mathrm{s}\,\alpha} = 0, \tag{4.16}$$

to separate the above equation and obtain

$$(\partial^{++})^2 B^{--}_{\mathrm{a}} - 2\bar{\phi}\hat{C}^{++\alpha\beta}\partial^{++}B^{--}_{\mathrm{s}\,\alpha\beta} = 0, \tag{4.17a}$$
$$(\partial^{++})^2 B^{--\alpha\beta}_{\mathrm{s}} + 4\bar{\phi}\,\hat{C}^{++\alpha\beta}\,\partial^{++}B^{--}_{\mathrm{a}} + 4\bar{\phi}\,\hat{C}^{++(\alpha\rho}\partial^{++}B^{--\beta)}_{\mathrm{s}\,\rho} + 8\bar{\phi}\,\hat{C}^{++\alpha\beta} = 0. \tag{4.17b}$$

The antisymmetric part can be fully decoupled from this system by substituting the second equation into the derivative of the first. After a change of variables

$$\mathcal{G} = \partial^{++}B^{--}_{\mathrm{a}} + 2\,, \tag{4.18}$$

we get

$$(\partial^{++})^2\mathcal{G} + 8\bar{\phi}^2\,(\hat{C}^{++})^2\,\mathcal{G} = 0\,, \quad (\hat{C}^{++})^2 \equiv \hat{C}^{++\alpha\beta}\hat{C}^{++}_{\alpha\beta}\,. \tag{4.19}$$

Even when it is possible to solve this scalar equation completely, general closed solutions for $B^{--}_{\alpha\beta}$ are very hard to find without further simplifications. Similar harmonic equations appear also when trying to solve the curvature equations to obtain the form of the action, and deducing a compensating parameter for the residual ssupersymmetric transformations.

Eqs. (4.17), (4.19) can be solved by iterations to any order in the deformation parameter. Expressions up to 2nd order are given in [53]. Closed solutions for these and remaining constraints in (4.14) can be found for the simplified *product* decomposition (3.48) of the deformation parameter

$$\hat{C}^{\alpha\beta}_{ik} \equiv c^{\alpha\beta}b_{ik}.$$

This in turn allows us to decompose tensor fields in the $\{c^{\alpha\beta}, \varepsilon^{\alpha\beta}\}$ base, i.e.

$$B^{--\,\alpha\beta} = \hat{B}^{--}c^{\alpha\beta} + \check{B}^{--}\varepsilon^{\alpha\beta}. \tag{4.20}$$

Which melts equation (4.17) down to a simple coupled system

$$\begin{aligned} &(\partial^{++})^2\check{B}^{--} - \frac{\sqrt{2c^2}}{2}\,\kappa^{++}\,\partial^{++}\hat{B}^{--} = 0\,, \\ &(\partial^{++})^2\hat{B}^{--} + \frac{2}{\sqrt{2c^2}}\,\kappa^{++}\,(\partial^{++}\check{B}^{--} + 2) = 0\,, \end{aligned} \tag{4.21}$$

with

$$\kappa^{++} = 2\bar{\phi}\sqrt{2c^2}b^{++}. \tag{4.22}$$

After some work, this equations can be solved using only harmonic analysis techniques like solving for all powers in the harmonic expansions and using the harmonic integral to find integration constants, as done in [115] for $A_{\alpha\dot{\alpha}}$ and ϕ. We will follow a more general and powerful method to solve this kind of equations, as they will also appear when calculating the component action from the harmonic curvature equation, and compensating the supersymmetric transformations in §5.

4.2 Developing an Algorithm

The usual technique to calculate the compensated variations above has the drawback of being heavily dependent in the particular form of the variation being analyzed. It can be also particularly cumbersome for $U(1)$-charged fields[1] whose corresponding variations are not simple harmonic independent expressions, but have to undergone some symmetrization process to be established. In other words, calculating variations by standard techniques is more of an art than an algorithm. As we have a great number of similar calculations of increasing complexity ahead, one would like to develop a standard procedure suitable to be fed into a computer to tackle all possible equations.

To get a hint of how can we do that, we note that (4.19) resembles a harmonic oscillator equation with "frequency" κ^{++}

$$(\partial^{++})^2\mathcal{G} + (\kappa^{++})^2\,\mathcal{G} = 0. \tag{4.23}$$

We can then propose that $\mathcal{G}$ is some power expansion in a quantity Z whose harmonic derivative corresponds to κ^{++}, that is

$$Z \equiv 2\sqrt{2c^2}\bar{\phi}b^{+-}, \qquad \kappa^{++} = \partial^{++}Z \tag{4.24}$$

and

$$\mathcal{G}(Z) = \sum_{n=0} a_n Z^n. \tag{4.25}$$

This ansatz has the virtue of turning the harmonic equation (4.23) into a formal ODE in Z

$$\mathcal{G}''(Z) + \mathcal{G}(Z) = 0, \tag{4.26}$$

which is directly solved by

$$\mathcal{G}(Z) = C_1 \cos Z + C_2 \sin Z, \tag{4.27}$$

[1] In this section the $U(1)$ does not refer to the gauge group but the phase factor induced by a $SU(2)$ transformation on harmonics as explained in §2.2.

where the functions involved are to be understood as their series expansions in Z. The integration constant C_1 can be determined by integrating harmonically this and noting that (4.18) contains a total derivative

$$\int du\, \mathcal{G}(Z) = \int du\, \left(\partial^{++} B_{\mathrm{a}}^{--} + 2\right) = 2. \tag{4.28}$$

Using the techniques described in §C.1 we obtain

$$\int du\, \mathcal{G}(Z) = C_1 \int du\, \cos Z + C_2 \int du\, \sin Z = C_1 \frac{\sinh X}{X}, \tag{4.29}$$

where

$$X \equiv 2\bar{\phi}\sqrt{b^2 c^2}. \tag{4.30}$$

Comparing these tho integrals we get

$$C_1 = \frac{2X}{\sinh X}. \tag{4.31}$$

The other integration constant would be than fixed by the equation of the oscillator coupled to $\mathcal{G}$ in (4.17). It is therefore very tempting to use ansätze like (4.25) to solve the harmonic equations that appear.

The typical problem that we will find not only while calculating the gauge transformations but also the supersymmetric ones and the corresponding invariant action, involves solving a harmonic equation of the following kind

$$\begin{aligned}(\partial^{++})^n g^{(2m)} - \kappa^{++} (\partial^{++})^{n-1} f^{(2m)} &= (\kappa^{++})^{n+m} \left(P_1(Z) + P_2(Z) \cos Z + P_3(Z) \sin Z\right), \\ (\partial^{++})^n f^{(2m)} + \kappa^{++} (\partial^{++})^{n-1} g^{(2m)} &= (\kappa^{++})^{n+m} \left(P_4(Z) + P_5(Z) \cos Z + P_6(Z) \sin Z\right),\end{aligned} \tag{4.32}$$

where P_i are polynomial in Z.

It is very easy to follow the same procedure we used for $\mathcal{G}$ to solve this general equations when f and g are 0-$U(1)$ charged functions, since the series solution of the equations will precisely coincide with that of the ordinary coupled oscillators with a source, which is the reason why we made the change of variables (4.18), and considered the solution simply as a function of Z. Positively even charged functions can be made to fit into this picture if we consider them a product of κ^{++} with a function of Z whose series expansion will also have the same solution as its ODE analog. A different situation arises for negatively-$U(1)$ charged fields, because in this case we cannot simply multiply by the inverse of the "frequency" $\frac{1}{\kappa^{++}}$ since this is not a function but an ill defined harmonic distribution.

It is nevertheless possible to solve harmonic equations for negatively even charged functions following the analogy of ODEs of the variable Z. The goal will be of course to build something to help us solve the equations, that is some function of Z whose derivative correspond to $\partial^{++} f^{--}$.

$$\partial^{++} f^{--} = \frac{\partial}{\partial Z} f(Z) = f'(Z). \tag{4.33}$$

Inspired in the usual harmonic symmetrization of $\kappa^{++}\kappa^{--}$ given in (C.1)

$$\kappa^{++}\kappa^{--} = X^2 + Z^2, \tag{4.34}$$

we propose the ansatz

$$f^{--} = \kappa^{--} \frac{f(Z)}{X^2 + Z^2}, \tag{4.35}$$

which can be well defined whenever the function $\frac{f(Z)}{X^2+Z^2}$ is regular as a series expansion in Z. The conditions one must ask from this function can be deduced by considering the following harmonic equation

$$\partial^{++} f^{--} = j(Z). \tag{4.36}$$

As the harmonic expansion of $f^{--} = f_0^{ij} u_{ij}^{--} + \cdots$ starts with a charged object, such equation only makes sense if the source j does not have a constant term in its harmonic expansion. The absence of such a term is assured by what we call *consistency* condition

$$\int du\, \partial^{++} f^{--} = 0 \tag{4.37}$$

which fixes the first coefficient of the source as a power series expansion in Z. To understand what is formally represented by the ansatz (4.35) as a power series in Z, we will use it in the harmonic equation (4.36) to obtain

$$\frac{\partial}{\partial Z} f(X, Z) = j \tag{4.38}$$

meaning that we can solve f in terms of the antiderivative of the source with respect to Z and a constant term

$$f(X, Z) = \int dZ\, j(Z) + \text{const} = J(X, Z) + \alpha(X). \tag{4.39}$$

The consistency condition over f looks like

$$\int du\, \frac{\partial}{\partial Z} f(X, Z) = 0, \tag{4.40}$$

and will assure consistency of (4.37) with harmonic analysis. This is not sufficient for our needs, as one may readily check by trying a simple function like $f = Z^2$ which makes f^{--} clearly irregular. A condition imposing regularity on f^{--} can be found by looking at the following equation

$$\partial^{++}(Z f^{--}) = \frac{\partial}{\partial Z}(Z f) = \alpha + \frac{\partial}{\partial Z}(Z f). \tag{4.41}$$

The LHS is a total derivative, so its integral must vanish and we can solve for

$$\alpha(X) = -\int du \, \frac{\partial}{\partial Z}(Zf(X,Z)). \tag{4.42}$$

As we will see, this makes the function regular in the undeformed limit, and additionally fix the integration constant, solving f. This is totally in accordance to standard harmonic analysis where the equation

$$(\partial^{++})^n f^{(-n)} = j^{(n)}(u) \qquad n \geq 0 \tag{4.43}$$

is totally fixed. This completely solves the problem and assures the regularity of the solution, therefore we refer to the following as the *regularity* condition

$$\int du \, \frac{\partial}{\partial Z}(Zf(X,Z)) = 0. \tag{4.44}$$

To show that the solution is regular, we start by assuming that the antiderivative of j has an analytic series in Z, which is true for the cases studied,

$$J = \sum_{n=0}^{\infty} a_n(X) Z^n = \sum_{n=0}^{\infty} a_{2n}(X)\, Z^{2n} + \sum_{n=0}^{\infty} a_{2n+1}(X)\, Z^{2n+1} \tag{4.45}$$

From (4.42) and the harmonic integrals in §C.1, we obtain

$$\alpha = -\sum_{n=0}^{\infty} (-1)^n a_{2n}(X) X^{2n} \tag{4.46}$$

With this pieces we can build up f (4.39)

$$f(X,Z) = \sum_{n=0}^{\infty} \left[a_{2n}(X)\, Z^{2n} + a_{2n+1}(X) Z^{2n+1} - (-1)^n\, a_{2n}(X)\, X^{2n} \right] \tag{4.47}$$

On the other hand, consistency condition (4.40) implies

$$\int du \, \frac{\partial}{\partial Z} f = \sum_{n=0}^{\infty} (2n+1) a_{2n+1}(X)\, (-1)^n \frac{X^{2n}}{2n+1} = 0 \tag{4.48}$$

that is, we can substract

$$Z \sum_{n=0}^{\infty} (-1)^n\, a_{2n+1}(X)\, X^{2n} \tag{4.49}$$

from f, leaving

$$f(X,Z) = \sum_{n=0}^{\infty} (a_{2n}(X) + Z a_{2n+1})(Z^{2n} - (-1)^n\, X^{2n}) \tag{4.50}$$

The term $Z^{2n} - (-1)^n X^{2n}$ appearing in each term of the series has roots in $X = \pm\, \mathrm{i}\, Z$ and therefore, using the polynomial factor theorem, has $X^2 + Z^2 = (X + \mathrm{i}Z)(X - \mathrm{i}Z)$ as one of its factors. This means we can formally write

$$f(X, Z) = (X^2 + Z^2)U(X, Z), \tag{4.51}$$

with $U(X, Z)$ having a regular undeformed limit, whenever we impose consistency and regularity conditions

$$\int du\, f'(Z) = 0, \qquad \int du\, (Zf(Z))' = 0. \tag{4.52}$$

Inserting f into (4.35), we see that for well behaved sources, the solution has the form

$$f^{--} = \kappa^{--}\, U(X, Z) \tag{4.53}$$

This can be readily generalized for even negatively charged function as needed. In summary, choosing a suitable ansatz like

$$f^{(2m)} = \begin{cases} (\kappa^{++})^m f(Z) & m \geq 0, \\ \left(\dfrac{\kappa^{--}}{X^2 + Z^2}\right)^{|m|} f(Z) & m < 0. \end{cases} \tag{4.54}$$

and consistency and regularity conditions (4.52), one can solve the typical equation (4.32) using simple ODE techniques, and later fix the integration constants with help of the harmonic integrals in §C.1.

4.3 Example: The Variations of $A_{\alpha\dot{\alpha}}$ and Ψ^i_α

To see the methods of the last section in action, we are going to determine two related variations corresponding to the fields $A_{\alpha\dot{\alpha}}$ and Ψ^i_α. This choice is based on the fact that the compensating $A_{\alpha\dot{\alpha}}$ determines the field $B^{--\alpha\beta}$ that contributes to all other variations, and that Ψ^i_α is essentially very hard to obtain without the method developed in the last section.

We start by going back to equation (4.21). Introducing the ansatz

$$\check{B}^{--} = \frac{\kappa^{--}}{X^2 + Z^2}\check{B}(Z), \qquad \hat{B}^{--} = \frac{\kappa^{--}}{X^2 + Z^2}\hat{B}(Z), \tag{4.55}$$

the coupled equations reduce to formal ODEs on Z

$$\check{B}'' - \frac{\sqrt{2c^2}}{2}\,\hat{B}' = 0\,, \qquad \hat{B}'' + \frac{2}{\sqrt{2c^2}}\,(\check{B}' + 2) = 0\,. \tag{4.56}$$

which are directly solved by

$$\check{B} = C_3 - 2Z + C_4 \cos Z + C_5 \sin Z\,, \qquad \hat{B} = \frac{2}{\sqrt{2c^2}}(C_6 + C_5 \cos Z - C_4 \sin Z)\,., \quad (4.57)$$

If we impose the consistency and regularity conditions (4.52) on this functions, we fix the integration constants

$$C_3 = 0, \qquad C_4 = 0, \qquad C_5 = \frac{2X}{\sinh X}, \qquad C_6 = -2X \coth X,. \quad (4.58)$$

Which, inserted back into the functions in (4.57) on the ansatz above, give the solution

$$\hat{B}^{--} = \sqrt{\frac{2}{c^2}} \frac{\kappa^{--}}{X^2+Z^2} \frac{2X}{\sinh X}(\cos Z - \cosh X), \quad (4.59a)$$

$$\check{B}^{--} = \frac{\kappa^{--}}{X^2+Z^2}\left(\frac{X \sin Z}{\sinh X} - Z\right). \quad (4.59b)$$

This can be directly plugged into (4.10) to obtain the variation of $A_{\alpha\dot{\alpha}}$.

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} a\,(X \coth X) = \partial_{\alpha\dot{\alpha}} a \left(1 + \frac{X^2}{3} - \frac{X^4}{45} + \dots\right). \quad (4.60)$$

Which clearly shows how helpful this method is, as all harmonic calculations can be simply translated into ODE language in which they can be automatically carried out by a computer. If we desire, using simple trigonometric identities we can recast the solution (4.59) in a manifestly regular way

$$\hat{B}^{--} = 2\bar{\phi} b^{--} X \operatorname{csch}\frac{X}{2} \operatorname{sech}\frac{X}{2} \frac{\sinh\frac{1}{2}(X+\mathrm{i}Z)}{\frac{1}{2}(X+\mathrm{i}Z)} \frac{\sinh\frac{1}{2}(X-\mathrm{i}Z)}{\frac{1}{2}(X-\mathrm{i}Z)}, \quad (4.61a)$$

$$\check{B}^{--} = \frac{\mathrm{i}\kappa^{++}}{\sinh X}\left(\cosh\frac{1}{2}(X+\mathrm{i}Z)\frac{\sinh\frac{1}{2}(X-\mathrm{i}Z)}{\frac{1}{2}(X-\mathrm{i}Z)} - \cosh\frac{1}{2}(X-\mathrm{i}Z)\frac{\sinh\frac{1}{2}(X+\mathrm{i}Z)}{\frac{1}{2}(X+\mathrm{i}Z)}\right). \quad (4.61b)$$

Now we can proceed in a similar fashion for the variation (4.12) of Ψ^i_α. The condition (4.14c) amounts to the following constraints on the matrices $D_\alpha^{\ \beta}$ and $D_\alpha^{--\ \beta}$ defined in (4.12):

$$(a)\ \ (\partial^{++})^2 D_\alpha^{\ \beta} = 0\,, \quad (b)\ \ (\partial^{++})^2 D_\alpha^{--\ \beta} + 2\partial^{++} D_\alpha^{\ \beta} = 0\,, \quad (4.62)$$

which in turn imply

$$D_\alpha^{\ \beta}(u) = D_{0\alpha}^{\ \beta} + D_{0\ \alpha}^{(ik)\ \beta} u_i^+ u_k^-\,, \quad D_\alpha^{--\ \beta}(u) = -D_{0\ \alpha}^{(ik)\ \beta} u_i^- u_k^-\,. \quad (4.63)$$

A solution of this kind allows immediate harmonic reduction of the variation into a correct Wess-Zumino preserving form

$$\delta\Psi^i_\alpha = (\varepsilon^{ij} D_0^{\alpha\beta} + D_0^{ij\,\alpha\beta})\bar{\Psi}_{j\dot{\alpha}} \partial^{\dot{\alpha}}_{\beta} a \quad (4.64)$$

To deduce the unknown functions we start with Eq. (4.62a) for

$$D_\alpha^{\ \beta} = -4\hat{C}_\alpha^{+-\beta} - \frac{1}{4}\partial^{++}H_\alpha^{--\ \beta} - \bar{\phi}\,\hat{C}_\alpha^{++\rho}\,H_\rho^{--\ \beta} - 2\,\hat{C}_\alpha^{++\rho}\,B_\rho^{--\ \beta}\,. \tag{4.65}$$

Explicitly, this equation reads

$$(\partial^{++})^3 H_\alpha^{--\ \beta} + 4\,\bar{\phi}\,\hat{C}_\alpha^{++\rho}\,(\partial^{++})^2 H_\rho^{--\ \beta} + 8\,\hat{C}_\alpha^{++\rho}\,(\partial^{++})^2 B_\rho^{--\ \beta} = 0\,, \tag{4.66}$$

which, for the Ansatz (3.48) and under the definitions

$$H^{--\ \alpha\beta} = \check{H}^{--}\varepsilon^{\alpha\beta} + \hat{H}^{--}c^{\alpha\beta}\,, \tag{4.67}$$

amounts to the following coupled system of equations

$$\begin{aligned}
&(\partial^{++})^3\check{H}^{--} - \sqrt{\frac{c^2}{2}}\kappa^{++}(\partial^{++})^2\left[\hat{H}^{--} + \frac{2}{\bar{\phi}}\hat{B}^{--}\right] = 0\,,\\
&(\partial^{++})^3\hat{H}^{--} + \sqrt{\frac{2}{c^2}}\kappa^{++}(\partial^{++})^2\left[\check{H}^{--} + \frac{2}{\bar{\phi}}\check{B}^{--}\right] = 0,
\end{aligned} \tag{4.68}$$

which corresponds precisely to an equation of the general form (4.32) we mentioned. We therefore use the ansatz (4.35) to solve for $H^{--\,\alpha\beta}$, and afterwards apply the consistency and regularity condition on the resulting functions, to find

$$\begin{aligned}
\hat{H}^{--} =&\sqrt{\frac{2}{c^2}}\frac{\kappa^{--}}{X^2+Z^2}\frac{1}{X\bar{\phi}}\Bigg\{-4\frac{X^2}{\sinh X}\left(X\cos Z\coth X - \frac{X}{\sinh X} + Z\sin Z\right)\\
&+\bar{\phi}\left[(Z\sinh X - X\sin Z\coth X)C_7 + X^2\left(\frac{\cosh X - \cos Z}{\sinh X}\right)C_8\right]\Bigg\}
\end{aligned} \tag{4.69a}$$

$$\begin{aligned}
\check{H}^{--} =&\frac{\kappa^{--}}{X^2+Z^2}\Bigg\{\frac{4}{\bar{\phi}}\frac{X}{\sinh X}\left(Z\cos Z - \frac{X\cosh X\sin Z}{\sinh X}\right)\\
&+(\cos Z - \cosh X)C_8 + \left(Z - \frac{X\sin Z}{\sinh X}\right)C_7\Bigg\}.
\end{aligned} \tag{4.69b}$$

This solution is defined up to some integration constants C_7, C_8 still to be determined. Inserting back into (4.65), and splitting in the usual way $D^{\alpha\beta} = \check{D}\varepsilon^{\alpha\beta} + \hat{D}c^{\alpha\beta}$, we get

$$\hat{D} = \frac{1}{2\sqrt{2c^2}}\left[\left(\cosh X - \frac{\sinh X}{X}\right)C_8 - ZC_7\right]\,, \tag{4.70a}$$

$$\check{D} = \frac{X}{\bar{\phi}\sinh X}\left(\frac{X}{\sinh X} - \cosh X\right) + \frac{Z\sinh X}{4X}C_8 - \frac{1}{4}\left(1 - \frac{X\cosh X}{\sinh X}\right)C_7. \tag{4.70b}$$

To fix the integration constants we have to look at the definition of $D^{--\ \alpha\beta}$ in (4.12),

$$D^{--\ \alpha\beta} = \bar{\phi}b^{++}c^\alpha_\gamma G^{(-4)\ \gamma\beta} - \frac{1}{4}H^{--\ \alpha\beta} - \frac{1}{4}\partial^{++}G^{(-4)\ \alpha\beta}. \tag{4.71}$$

Solving for $G^{(-4)\,\alpha\beta}$ and imposing consistency and regularity conditions on it, one obtains

$$C_7 = \frac{4X^2}{\bar{\phi}} \frac{1 - X\coth X}{X^2 - 2X\cosh X \sinh X + \sinh^2 X}, \qquad C_8 = 0 \tag{4.72}$$

and

$$\hat{D} = \sqrt{\frac{2}{c^2}} \frac{Z}{\bar{\phi}} \frac{(1 - X\coth X)}{1 - 2\cosh X \frac{\sinh X}{X} + \frac{\sinh^2 X}{X^2}}, \tag{4.73}$$

$$\check{D} = \frac{1}{\bar{\phi}} \left(\frac{X^2}{\sinh^2 X} - X\coth X \right) + \frac{1}{\bar{\phi}} \frac{(1 - X\coth X)^2}{1 - 2\cosh X \frac{\sinh X}{X} + \frac{\sinh^2 X}{X^2}}. \tag{4.74}$$

Finally, the gauge variation of Ψ_α is found to be

$$\begin{aligned} \delta\Psi^i_\alpha = & \Bigg[2\sqrt{b^2c^2}(X \operatorname{csch}^2 X - \coth X)\varepsilon^{ij}\varepsilon^{\alpha\beta} \\ & + \frac{2X^2 \operatorname{csch}^2 X(1 - X\coth X)}{X^2 \operatorname{csch}^2 X + 1 - 2X\coth X} \left(2b^{ij}c^{\alpha\beta} - \frac{\sqrt{b^2c^2}}{X}(1 - X\coth X)\varepsilon^{ij}\varepsilon^{\alpha\beta} \right) \Bigg] \bar{\Psi}_{j\dot{\alpha}}\, \partial^{\dot{\alpha}}_{\beta} a\,. \end{aligned} \tag{4.75}$$

4.4 The Minimal Seiberg-Witten Map

It is known that both commutative *and* noncommutative Yang-Mills fields arise from the same setup of open strings in the presence of Dp-branes and a constant B-field, when using two different regularization prescriptions [25]. Based on this, Seiberg and Witten proposed the existence of a map from ordinary to noncommutative Yang-Mills fields, that takes the gauge equivalence from one case to the other. This map can *not* lead to an isomorphism between the two gauge groups as can be seen by taking the commutative group to be abelian, i.e. having (2.74) as its gauge transformation

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}\lambda$$

The noncommutative counterpart transforms as

$$\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}\lambda + \mathrm{i}[\lambda, A_{\alpha\dot{\alpha}}]_\star, \tag{4.76}$$

corresponding to a non-Abelian gauge group which obviously cannot be isomorphic to the case before.

The Seiberg-Witten map consists of a simultaneous local field redefinition and gauge reparametrization that takes two fields related by a gauge transformation $A = e^{\mathrm{i}\lambda}A'$ and maps them into noncommutative fields that are also gauge equivalent $\hat{A} = e^{\mathrm{i}\hat{\lambda}(\lambda,A)}\hat{A}'$, this time through a parameter that depends on *both* λ and A, therefore not defining an isomorphism.

Four our particular kind of deformation, it is also possible to find the map that takes the gauge transformations to the ordinary abelian case. The full expressions for the non(anti)commutative non-abelian Q-deformed transformation are summarized as follows

$$\delta\,\bar{\phi} = 0, \qquad \delta\bar{\Psi}^k_{\dot{\alpha}} = 0, \tag{4.77a}$$

$$\delta\, A_{\alpha\dot{\alpha}} = X \coth X \partial_{\alpha\dot{\alpha}} a\,, \tag{4.77b}$$

$$\delta\,\phi = 2\sqrt{b^2c^2}\left(\frac{1 - X\coth X}{X}\right) A^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} a\,, \tag{4.77c}$$

$$\begin{aligned}\delta\Psi^i_{\alpha} =& \Bigg\{ 2\sqrt{b^2c^2}(X\,\mathrm{csch}^2 X - \coth X)\varepsilon^{ij}\varepsilon^{\alpha\beta} \\ &+ \frac{2X^2\,\mathrm{csch}^2 X(1 - X\coth X)}{X^2\,\mathrm{csch}^2 X + 1 - 2X\coth X}\left[2b^{ij}c^{\alpha\beta} - \frac{\sqrt{b^2c^2}}{X}(1 - X\coth X)\varepsilon^{ij}\varepsilon^{\alpha\beta}\right]\Bigg\}\bar{\Psi}_{j\dot{\alpha}}\partial^{\dot{\alpha}}_{\beta} a,\end{aligned} \tag{4.77d}$$

$$\delta D_{ij} = 2\mathrm{i} b_{ij} c^{\alpha\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi}\,\partial^{\dot{\alpha}}_{\beta} a. \tag{4.77e}$$

From which we can directly propose the minimal map which take us back to the standard abelian gauge transformations

$$A_{\alpha\dot{\alpha}} = \widetilde{A}_{\alpha\dot{\alpha}} X\coth X, \tag{4.78a}$$

$$\phi = \widetilde{\phi} + \sqrt{b^2c^2}\widetilde{A}^2(1 - X\coth X)\coth X, \tag{4.78b}$$

$$\begin{aligned}\delta\Psi^i_{\alpha} =& \widetilde{\Psi}^{i\alpha} + \Bigg\{ 2\sqrt{b^2c^2}(X\,\mathrm{csch}^2 X - \coth X)\varepsilon^{ij}\varepsilon^{\alpha\beta} \\ &+ \frac{2X^2\,\mathrm{csch}^2 X(1 - X\coth X)}{X^2\,\mathrm{csch}^2 X + 1 - 2X\coth X}\left[2b^{ij}c^{\alpha\beta} - \frac{\sqrt{b^2c^2}}{X}(1 - X\coth X)\varepsilon^{ij}\varepsilon^{\alpha\beta}\right]\Bigg\}\bar{\Psi}_{j\dot{\alpha}}\,\widetilde{A}^{\dot{\alpha}}_{\beta}\,,\end{aligned} \tag{4.78c}$$

$$D_{ij} = \widetilde{D}_{ij} + 2\mathrm{i} b_{ij} c^{\alpha\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi}\,\widetilde{A}^{\dot{\alpha}}_{\beta} \tag{4.78d}$$

The gauge field strength $F_{\alpha\beta} = 2\mathrm{i}\partial_{(\alpha\dot{\alpha}}A^{\dot{\alpha}}_{\beta)}$ is redefined as

$$F_{\alpha\beta} = \widetilde{F}_{\alpha\beta}\,X\coth X + 4\mathrm{i}\sqrt{b^2c^2}\widetilde{A}_{(\beta\dot{\alpha}}\partial^{\dot{\alpha}}_{\alpha)}\bar{\phi}\left(\coth X - X\,\mathrm{csch}^2 X\right) \tag{4.79}$$

where obviously $\widetilde{F}_{\alpha\beta} = 2\mathrm{i}\partial_{(\alpha\dot{\alpha}}\widetilde{A}^{\dot{\alpha}}_{\beta)}$. In the next chapter, we will see that, after performing the Seiberg-Witten map transforms the deformed non-Abelian actions to standard $U(1)$ gauge invariant ones.

The natural occurrence of *hyperbolic* functions in the expressions is very remarkable, and its origin is still unclear. Perhaps by relating non-singlet deformations to a particular string background or by studying the hyper-Kähler geometry of the associated deformed hypermultiplet, one could elucidate this appearance.

Chapter 5

Invariant Actions and Residual Supersymmetry

We have seen that the deformation of the super Yang-Mills field strength stems from the star commutator present in the curvature equation defining it. Constructing the deformed version of the corresponding gauge action is therefore equivalent to find solutions of the *deformed harmonic flatness condition* (3.60). As we shown in §3.3, only the first coefficient of the superfield strength (3.56) contributes to the action, reducing the amount of equations in components to be solved to thirty! Solutions have been found for the vector- and hypermultiplet in the singlet case in [52, 42] respectively. From here on we focus on the non singlet deformed action of Abelian gauge theory where non-Abelian interactions appear due to the presence of non(anti)commutativity. All calculations are worked out *on component fields*.

Despite the considerable effort that has been taken to obtain approximate actions for the full set of deformation parameters [54, 53, 55], obtaining exact expressions is a very difficult task. Even when the deformation is chosen to be decomposable (3.32), and applying the methods from previous sections, the resulting expressions for the relevant component of the superfield strength are much more complicated than in the singlet case. Instead of writing down such objects, we will follow the more physically interesting path of analyzing different sectors of the theory for meaningful cases of the deformation. Additionally, though the manifestly covariant formalism of harmonic superspace assures the invariance of the resulting actions under the residual *unbroken* supersymmetry by construction, it is important to obtain the corresponding transformations of the fields, at least for the cases studied. On that account, the main tasks throughout this chapter will be the determination of the gauge action from the curvature equation and the deduction of the corresponding supersymmetry transformations.

We will start by describing the generalities involved in solving the curvature equations and give a very simple example. Afterwards, we will see how the decomposition of the non singlet deformation allows the construction of an exact bosonic action. This action already presents some of the characteristic features of non(anti)commutative deformations, as it has an *interaction* parametrized by the deformation, in fact, the action can be factorized as the *free* bosonic part of $N = 2$ Maxwell theory times a *hyperbolic* function of scalar fields and the deformation parameters [115]. In §5.3 we will present the exact action coming from a deformation preserving 3/4 of the original supersymmetry. As we will see in §5.4, we can interpret b_{ij} as a set of supersymmetry breaking tuning parameters distinguishing between different theories with $\mathcal{N} = (1, 0)$, $\mathcal{N} = (1, 1/2)$ and $\mathcal{N} = (1/2, 1/2)$ supersymmetry, some of them with very simple Lagrangians [120].

The generalities of the calculation of the residual supersymmetry transformation of component fields is briefly described in §5.5, where the differences with the singlet case are highlighted. Interesting cases are shown, like the subalgebra for which readable expressions can be obtained in the general decomposed matrix ansatz, and the full expressions corresponding to the actions presented .

5.1 Solving the Curvature Equations

To explicitly build an action we first expand $\mathcal{A}$ into its components

$$\begin{aligned}\mathcal{A} = \mathcal{A}_1 + \theta^{-\alpha}\mathcal{A}^{+}_{2\,\alpha} + \theta^{+\alpha}\mathcal{A}^{-}_{3\,\alpha} + (\theta^-)^2\mathcal{A}^{++}_4 + (\theta^-\theta^+)\mathcal{A}_5 + \theta^{-\alpha}\theta^{+\beta}\mathcal{A}_{6\,\alpha\beta} + (\theta^+)^2\mathcal{A}^{--}_7 \\ + (\theta^-)^2\theta^{+\alpha}\mathcal{A}^{+}_{8\,\alpha} + (\theta^+)^2\theta^{-\alpha}\mathcal{A}^{-}_{9\,\alpha} + (\theta^-)^2(\theta^+)^2\mathcal{A}_{10}.\end{aligned} \tag{5.1}$$

Assembling the $(\theta^-)^2(\theta^+)^2$ components into the action and integrating in the Graßmann variables, one obtains

$$S = \frac{1}{4}\int d^4x_L\, du\, \left[2\mathcal{A}_1\mathcal{A}_{10} - (\mathcal{A}^+_2\mathcal{A}^-_9) - (\mathcal{A}^-_3\mathcal{A}^+_8) + 2\mathcal{A}^{++}_4\mathcal{A}^{--}_7 - \frac{1}{2}\mathcal{A}^2_5 - \frac{1}{4}(\mathcal{A}_6\cdot\mathcal{A}_6)\right]. \tag{5.2}$$

As we mentioned before, in order to determine the precise form of each component, one has to solve the minimal set of curvature equation components (3.61). To see how this system of 30 harmonic equations is the most economical choice, we start by looking at (3.61b)

$$\nabla^{++}\mathcal{A} = \mathrm{D}^{++}\mathcal{A} + [v^{++}, \mathcal{A}]_\star = 0. \tag{5.3}$$

Being $\mathcal{A}(x_L, \theta^\pm)$ a chiral field, the star commutator involved reduces from the full expressions of appendix §B.1 to a simpler expression from which we get

$$\nabla^{++}\mathcal{A} = \left[\partial^{++} - \left(\varepsilon^{\alpha\beta} + 4\bar{\phi}b^{+-}c^{\alpha\beta}\right)\theta^+_\alpha\partial_{-\beta} - 4\bar{\phi}b^{++}\,c^{\alpha\beta}\theta^+_\alpha\partial_{+\beta}\right]\mathcal{A}\,. \tag{5.4}$$

The hint on how to proceed comes from looking at the homogeneous components of the curvature equation for $\mathcal{A}$,

$$\partial^{++}\mathcal{A}_1 = 0, \qquad \partial^{++}\mathcal{A}^{+}_{2\,\alpha} = 0, \qquad \partial^{++}\mathcal{A}^{++}_4 = 0, \qquad \partial^{++}\mathcal{A}_{10} = 0, \tag{5.5}$$

This equations directly reveal that $\mathcal{A}_1$ and $\mathcal{A}_{10}$ are independent of harmonics, $\mathcal{A}^{++}_4$ is of the form $\mathcal{A}^{++}_4 = \mathcal{A}^{ij}_4 u^+_i u^+_j$, and $\mathcal{A}^{+}_{2\,\alpha}$ is of the form $\mathcal{A}^{+}_{2\,\alpha} = \mathcal{A}^{i}_{2\,\alpha} u^+_i$ with both $\mathcal{A}^{ij}_4$ and $\mathcal{A}^{i}_{2\,\alpha}$ independent of harmonics. It is obvious then that the curvature equation for $\mathcal{A}$ will only determine its components up to some harmonic integration constants like these which will have to be fixed by other curvature equations. From all the remaining coefficients of the curvature equation, (3.61c) contains the less number of unknown fields to be determined

$$\mathrm{D}^{++}\varphi^{--} + [v^{++}, \varphi^{--}]_\star + 2(\mathcal{A} - v) + \frac{1}{2}\left\{v^{+\,\dot\alpha}, v^-_{\dot\alpha}\right\}_\star = 0, \tag{5.6}$$

since v and v^{++}, and $v^{+\dot\alpha}$ are coefficients of the gauge prepotential in the Wess-Zumino gauge, which we already know from (3.71). The missing piece of this system of coupled equations is thus $v^-_{\dot\alpha}$, which can be fixed from only one extra curvature equation (3.61a),

$$\mathrm{D}^{++}v^{-\,\dot\alpha} + [v^{++}, v^{-\,\dot\alpha}]_\star - v^{+\,\dot\alpha} = 0, \tag{5.7}$$

To obtain the complete set of equations to be solved, it is necessary to expand the fields $v^{-\,\dot\alpha}$ and φ^{--}, in the same fashion as in (5.1)

$$v^{-\,\dot\alpha} = w_1^{-\,\dot\alpha} + \theta^{-\alpha} w_{2\,\alpha}^{\;\;\dot\alpha} + \cdots, \qquad \varphi^{--} = \varphi_1^{--} + \theta^{-\alpha}\varphi^{-}_{2\,\alpha} + \cdots \tag{5.8}$$

and take in account the presence of the star anticommutator $\{v^{+\dot\alpha}, v^-_{\dot\alpha}\}_\star$ in (5.6).

$$\begin{aligned}\{v^{+\dot\alpha}, v^-_{\dot\alpha}\}_\star =& 2\Big[c^{\alpha\beta} b^{+-}(\partial_{+\alpha}v^{+\dot\alpha}\partial_{-\beta}v^-_{\dot\alpha} + \partial_{-\beta}v^{+\dot\alpha}\partial_{+\alpha}v^-_{\dot\alpha}) \\ &+ c^{\alpha\beta}\,(b^{++}\partial_{+\alpha}v^{+\dot\alpha}\partial_{+\beta}v^-_{\dot\alpha} + b^{--}\partial_{-\alpha}v^{+\dot\alpha}\partial_{-\beta}v^-_{\dot\alpha})\Big] \\ &- \frac{1}{8}b^2\,c^2\Big[b^{++}\,c^{\alpha\beta}\partial^2_+\partial_{-\alpha}v^{+\dot\alpha}\partial^2_+\partial_{-\beta}v^-_{\dot\alpha} - c^{\alpha\beta}b^{+-}\partial^2_+\partial_{-\alpha}v^{+\dot\alpha}\partial^2_-\partial_{+\beta}v^-_{\dot\alpha}\Big]\end{aligned} \tag{5.9}$$

Complete expressions for the relevant curvature equation components are given in §B.2. Most of the resulting equations are to be splitted in systems of two or more coupled systems to obtain closed expressions as in the last chapter, so we are left with a total of circa 60 equations to solve. Here is when we really profit from the general algorithm developed in §4.2, as all these equations have the typical form (4.32) and can be solved in most cases with help of a computer.

As a sample of the kind of calculation we are dealing with, we will construct the first term of the action, namely

$$\frac{1}{2}\int d^4x_L\,du\,\mathcal{A}_1\mathcal{A}_{10} \tag{5.10}$$

The involved fields satisfy homogeneous harmonic equations (5.5) that tell us they are independent of harmonics. Knowing this, we extract from (5.6) the following equations

$$\begin{aligned}
&\partial^{++}\varphi_1^{--} + 2(\mathcal{A}_1 - \phi) + 2c^{\alpha\beta}b^{+-}A^{\dot\alpha}_{\alpha}w_{2\beta\dot\alpha} + 2b^{++}\,c^{\alpha\beta}A^{\dot\alpha}_{\alpha}w^{--}_{3\beta\dot\alpha}\\
&\quad + 2\mathrm{i}b^2\,c^2\left[b^{++}\,c^{\alpha\beta}\partial^{\dot\alpha}_{\alpha}\bar\phi w^{--}_{9\beta\dot\alpha} - b^{+-}\,c^{\alpha\beta}\partial^{\dot\alpha}_{\alpha}\bar\phi w_{8\beta\dot\alpha}\right] = 0, \qquad (5.11a)\\
&\partial^{++}\varphi^{--}_{10} + 2\mathcal{A}_{10} + 2\Box\bar\phi = 0. \qquad (5.11b)
\end{aligned}$$

From the last of this equations one finds directly

$$\varphi^{--}_{10} = 0, \qquad \mathcal{A}_{10} = -\Box\bar\phi. \tag{5.12}$$

To solve (5.11a) in turn, we require the knowledge of some coefficients of the solution $v^-_{\dot\alpha}$ of (5.7). We content ourselves with the calculation of two of the coefficients, coming from the following equations

$$\begin{aligned}
&\partial^{++}w_2^{\alpha\dot\alpha} = 0,\\
&\partial^{++}w_3^{--\,\alpha\dot\alpha} + w_2^{\alpha\dot\alpha} + 4\bar\phi c^{\alpha\beta}(b^{+-}w_{2\,\beta}{}^{\dot\alpha} + b^{++}w^{--}_{3\,\beta}{}^{\dot\alpha}) - 2A^{\alpha\dot\alpha} = 0,
\end{aligned} \tag{5.13}$$

From the first equation we know that $w_2^{\alpha\dot\alpha}$ is independent of harmonics. Introducing the following ansatz

$$w_2^{\alpha\dot\alpha} = \left(\check w_2\varepsilon^{\alpha\beta} + \hat w_2 c^{\alpha\beta}\right)A^{\dot\alpha}_{\beta}, \qquad w_3^{--\alpha\dot\alpha} = \left(\check w_3^{--}\varepsilon^{\alpha\beta} + \hat w_3^{--}c^{\alpha\beta}\right)A^{\dot\alpha}_{\beta}, \tag{5.14}$$

we are then able to turn the system into the form (4.32)

$$\begin{aligned}
&\partial^{++}\check w_3^{--} - \frac{\sqrt{2c^2}}{2}\hat w_3^{--} = 2 - \check w_2 + \frac{\sqrt{2c^2}}{2}Z\hat w_2, \qquad (5.15a)\\
&\partial^{++}\hat w_3^{--} + \frac{2}{\sqrt{2c^2}}\check w_3^{--} = -\hat w_2 - \frac{\sqrt{2c^2}}{2}Z\check w_2. \qquad (5.15b)
\end{aligned}$$

Again, this are directly solved using the ansatz (4.54), from which we obtain

$$\begin{aligned}
&\check w_2 = 2X\tanh X, \qquad \hat w_2 = 0 \qquad (5.16a)\\
&\check w_3^{--} = \frac{\kappa^{--}}{X^2+Z^2}\frac{2}{\cosh X}\left(\sin Z - Z\sinh X\right) \qquad (5.16b)\\
&\hat w_3^{--} = \frac{\kappa^{--}}{X^2+Z^2}\frac{2}{\cosh X}\frac{2}{\sqrt{2c^2}}\left(\cos Z - \cosh X\right) \qquad (5.16c)
\end{aligned}$$

This can also be done for the rest of the components of $v^-_{\dot\alpha}$, for which one has to propose a reasonable decomposition to reduce the equations involved to the standard form, and then solve the equivalent system of ODEs. The general solution is given in B.2.1. For our case, we feed back the resulting expressions into (5.11a) and repeat the whole procedure, this time for φ_1^{--}. However, for our present purposes it will suffice to integrate that equation harmonically to obtain

$$\mathcal{A}_1 = \phi + \frac{1}{2}\bar{\phi}^{-1}\left(1 - \frac{\tanh X}{X}\right) A^2 + (b^2\, c^2)^{3/2} \tanh X \partial_{\alpha\dot{\alpha}}\bar{\phi}\partial^{\alpha\dot{\alpha}}\bar{\phi}, \tag{5.17}$$

The first term of the Lagrangian is then

$$\frac{1}{2}\mathcal{A}_1\mathcal{A}_{10} = -\frac{1}{2}\phi\Box\bar{\phi} - \frac{1}{2}\left[\frac{1}{2}\bar{\phi}^{-1}\left(1 - \frac{\tanh X}{X}\right) A^2 + (b^2\, c^2)^{3/2} \tanh X \partial_{\alpha\dot{\alpha}}\bar{\phi}\partial^{\alpha\dot{\alpha}}\bar{\phi}\right]\Box\bar{\phi} \tag{5.18}$$

This suggests we could simply repeat the procedure for the rest of the components of the action, but in practice the situation is much more complicated. Solving the whole set of curvature equations by brute force in the computer leads to very complicated functions of X. Even when these involve only polynomials and hyperbolic functions, the resulting rational expressions are surprisingly cumbersome. As the calculation for singlet deformations suggests, a very complicated field redefinition like (3.76) could simplify these components into readable objects, but it seems to be very nontrivial in this case. In what follows, we will first limit the analysis to the bosonic sector, where such redefinitions are easier to find, and later on, we will explore relevant cases of the deformation that include an exact action with $N = (1, 1/2)$ residual supersymmetry, and a limit which exhibits a tuned $N = (1, 1/2) \longrightarrow N = (1, 0)$ breaking.

5.2 The Exact Bosonic Sector

By focusing on the bosonic sector of the action, discarding fermionic terms, the explicit form of the exact action becomes simple enough to propose a field redefinition leading to a very compact action. We start then by dropping the fermionic terms $\mathcal{A}^{+}_{2\,\alpha}, \mathcal{A}^{-}_{3\,\alpha}, \mathcal{A}^{+}_{8\,\alpha}$, and $\mathcal{A}^{-}_{9\,\alpha}$, reducing (5.2) to

$$S_{\text{bos}} = \frac{1}{4}\int d^4x_L\, du\, \left[2\mathcal{A}_1\mathcal{A}_{10} + 2\mathcal{A}_4^{++}\mathcal{A}_7^{--} - \frac{1}{2}\mathcal{A}_5^2 - \frac{1}{4}\mathcal{A}_6^2\right]. \tag{5.19}$$

Even when the procedure to find the other components of the action is in some way the same as was for $\mathcal{A}_1$ and $\mathcal{A}_{10}$, one has to come up with a peculiar ansatz to find them. Inspired by a series solution to the problem which takes in account the interplay of all curvature equations up to second order, we can propose the following particular form for the component fields

$$\begin{aligned}
\mathcal{A}_{6\alpha\beta} &= g_1\, F_{\alpha\beta} + g_2\, c_{\alpha\beta} + g_3\, \mathcal{F}_{\alpha\beta} + g_4\, G_{\alpha\beta} + g_5\, \mathcal{G}_{\alpha\beta},\\
\varphi^{--}_{6\alpha\beta} &= h_1^{--}\, F_{\alpha\beta} + h_2^{--}\, c_{\alpha\beta} + h_3^{--}\, \mathcal{F}_{\alpha\beta} + h_4^{--}\, G_{\alpha\beta} + h_5^{--}\, \mathcal{G}_{\alpha\beta},\\
\mathcal{A}_4^{ij} &= \alpha_1\, D^{ij} + \alpha_2\, b^{ij} + \alpha_2\, \mathcal{D}^{ij},
\end{aligned} \tag{5.20}$$

where

$$G_{\alpha\beta} = A_{(\alpha\dot{\alpha}}\partial^{\dot{\alpha}}_{\beta)}\bar{\phi}, \qquad \mathcal{G}_{\alpha\beta} = c_{(\alpha}{}^{\gamma}G_{\gamma\beta)}, \qquad \mathcal{F}_{\alpha\beta} = c_{(\alpha}{}^{\gamma}F_{\gamma\beta)}, \qquad \mathcal{D}^{ij} = b^{(ik}D^{j)}_{k}. \tag{5.21}$$

The functions g_i, h_i^{--} can depend only on $\bar{\phi}, b^{ij}$ and $c_{\alpha\beta}$ and the harmonics. Similarly α_i can depend only on $\bar{\phi}, b^{ij}$ and $c_{\alpha\beta}$ but must be harmonic independent due to the condition $\partial^{++}\mathcal{A}_4^{++} = 0$. With this, the curvature equations for the bosonic components split into several parts, the one involving $\mathcal{A}_7^{--}$, for instance is

$$\partial^{++}\mathcal{A}_7^{--} + A_5 + 2\bar{\phi}b^{+-}\left(g_1\,(c\cdot F) + c^2\,g_2 + g_4\,(c\cdot G)\right) = 0 \tag{5.22}$$

Where we use the following notation for the traces with the deformation parameters

$$c\cdot F = c^{\alpha\beta}F_{\alpha\beta}, \tag{5.23}$$

which works analogously on $(c\cdot G)$, $(F\cdot G)$, $(b\cdot D)$ and so on. We will also use

$$A\cdot\partial\bar{\phi} = A_{\alpha\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\bar{\phi}, \qquad \partial\cdot A = \partial^{\alpha\dot{\alpha}}A_{\alpha\dot{\alpha}}. \tag{5.24}$$

We can further reduce the amount of unknown fields involved in the action. Integrating by parts $\mathcal{A}_4^{++} = \partial^{++}\mathcal{A}^{+-}$, and substituting the expression for $\partial^{++}\mathcal{A}_7^{--}$ and the ansatz for $A_{6\alpha\beta}$ in (5.19), we obtain

$$\begin{aligned} S = \int d^4x_L\,du\,\Big[&\frac{1}{2}\mathcal{A}_1\mathcal{A}_{10} + \frac{1}{2}\mathcal{A}_4^{++}\mathcal{A}_5 - \bar{\phi}b^{+-}\mathcal{A}_4^{+-}[g_1(c\cdot F) + g_2c^2 + g_4(c\cdot A\partial\bar{\phi})] \\ &- \frac{1}{8}\mathcal{A}_5^2 - \frac{1}{16}\left(g_1^2 + \frac{c^2}{2}g_3^2\right)F^2 - \frac{1}{16}\left(g_4^2 + \frac{c^2}{2}g_5^2\right)(A\partial\bar{\phi})^2 - \frac{1}{16}g_2^2c^2 \\ &- \frac{1}{16}\left(2g_1\,g_2 - \frac{1}{2}g_3^2(c\cdot F)\right)(c\cdot F) - \frac{1}{16}\left(2g_2\,g_4 - \frac{1}{2}g_5^2(c\cdot A\partial\bar{\phi})\right)(c\cdot A\partial\bar{\phi}) \\ &- \frac{1}{16}\left(2g_1\,g_4 - c^2g_3\,g_5\right)(F\cdot A\partial\bar{\phi}) + \frac{1}{16}g_3\,g_5(c\cdot F)(c\cdot A\partial\bar{\phi}) - \frac{1}{8}\left(g_3\,g_4 - g_1\,g_5\right)(F\cdot cA\partial\bar{\phi})\Big] \end{aligned} \tag{5.25}$$

The problem is then reduced to finding A_4^{ij}, A_5 and the functions g_i. As the ansatz proposed directly turns the curvature equations for $\mathcal{A}$ into the standard form (4.32), this functions are solved by the methods we known from former sections. We leave then the details for the appendix §B.2.2, and present directly the resulting action,

$$
\begin{aligned}
S = \int d^4x \Bigg[& -\frac{1}{2}\phi\Box\bar{\phi} - \frac{1}{2}\left[\frac{1}{2}\bar{\phi}^{-1}\left(1 - \frac{\tanh X}{X}\right) A^2 + (b^2\, c^2)^{3/2}(\partial\bar{\phi})^2 \tanh X\right] \Box\bar{\phi} \\
& + \frac{1}{4}\frac{D^2}{\cosh^2 X} - \frac{1}{16}F^2 \frac{\sinh^2 X}{X^2} + \left\{ \frac{1}{2}\bar{\phi}(D\cdot b)(F\cdot c) + \frac{1}{4} b^2 (F\cdot c)^2 \bar{\phi}^2 \frac{\sinh^2 X}{X^2} \right\} \frac{\tanh^2 X}{X^2} \\
& - \mathrm{i}\left\{ (D\cdot b)(G\cdot c)\frac{\tanh X}{X} + b^2 (G\cdot c)(F\cdot c)\bar{\phi}\frac{\sinh^2 X}{X^2}\frac{\tanh X}{X} \right\} \left(\frac{\tanh X}{X} - \frac{2}{\cosh^2 X} \right) \\
& - (G\cdot c)^2 b^2 \frac{\sinh^2 X}{X^2}\left(\frac{\tanh X}{X} - \frac{2}{\cosh^2 X} \right)^2 + \frac{G^2 b^2 c^2}{\cosh^2 X}\left(\frac{\cosh X \sinh X - X}{X^2} \right)^2 \\
& + \mathrm{i}\sqrt{b^2 c^2}\,\frac{(F\cdot G)}{2}\frac{\tanh X}{X}\left(\frac{\cosh X \sinh X - X}{X^2} \right) \Bigg]
\end{aligned}
\tag{5.26}
$$

Using the minimal Seiberg-Witten map (4.78), this Lagrangian gets reduced to a manifestly $U(1)$ gauge invariant form

$$
\begin{aligned}
S = \int d^4x \Bigg\{ & -\frac{1}{2}\tilde{\phi}\Box\bar{\phi} - \frac{1}{2}(b^2\, c^2)^{3/2}(\partial\bar{\phi})^2 \tanh X \Box\bar{\phi} \\
& + \frac{1}{4}\frac{\widetilde{D}^2}{\cosh^2 X} - \frac{1}{16}\widetilde{F}^2 \cosh^2 X + \frac{1}{4} b^2 (\widetilde{F}\cdot c)^2 \bar{\phi}^2 \frac{\sinh^2 X}{X^2} + \frac{1}{2}\bar{\phi}(\widetilde{D}\cdot b)(\widetilde{F}\cdot c)\frac{\tanh X}{X} \Bigg\}
\end{aligned}
\tag{5.27}
$$

Now it is much more easy to find a field redefinition that will further reduce this expression,

$$
\begin{aligned}
d^{ij} &= \frac{1}{\cosh^2 X}\widetilde{D}^{ij} + \bar{\phi}(\widetilde{F}\cdot c) b^{ij}\frac{\tanh X}{X}, \\
a_{\alpha\dot{\alpha}} &= \widetilde{A}_{\alpha\dot{\alpha}}, \\
\varphi &= \frac{1}{\cosh^2 X}\left[\tilde{\phi} + (b^2 c^2)^{3/2}(\partial\bar{\phi})^2 \tanh X\right].
\end{aligned}
$$

Finally the action is mapped to

$$
S = \int d^4x \, \cosh^2 X \left[-\frac{1}{2}\varphi\Box\bar{\phi} + \frac{1}{4} d^{ij} d_{ij} - \frac{1}{16} f^{\alpha\beta} f_{\alpha\beta} \right].
$$

From the structure of $f_{\alpha\beta} = 2\mathrm{i}\partial_{(\alpha\dot{\alpha}} a^{\dot{\alpha}}_{\beta)}$ it is easy to see that there is no way to absorb the function $\cosh^2 X$ in a field redefinition.

5.3 Exact $\mathcal{N} = (1, 1/2)$ supersymmetry action in components

A similar compact expression for the full supersymmetric action seems to be very hard to find. The complicated functions obtained by solving the harmonic equations in a computer result into a very lengthy action in components which for our further analysis is not necessary to present. We will focus in the more significant case for which $b^2 = 0$, and only 1/4 of the original $N = (1, 1)$ supersymmetry is broken. As a more general selection of b_{ij} leads to the breaking of half the supersymmetry, this particular $b^2 = 0$ choice is said to *enhance* the symmetry of the general case.

By taking the limit of the expressions in appendix §B.2.3, we obtain

$$
\begin{aligned}
S = \int d^4x_L \Big[&- \frac{1}{2}\phi\Box\bar{\phi} - \frac{1}{16}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{4}D^2 + \mathrm{i}\Psi^{k\alpha}\partial_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}}_k + \mathrm{i}b_{ij}D^{ij}\, c^{\alpha\beta}\partial_{(\alpha\dot{\alpha}}\bar{\phi}A^{\dot{\alpha}}_{\beta)} \\
&+ \frac{1}{2}\bar{\phi}b_{ij}D^{ij}c^{\alpha\beta}F_{\alpha\beta} + \frac{4\mathrm{i}}{3}b_{ij}c^{\alpha\beta}A_{\alpha\dot{\alpha}}\bar{\Psi}^{i\dot{\alpha}}\partial_{\beta\dot{\beta}}\bar{\Psi}^{j\dot{\beta}} - 4\mathrm{i}b_{ij}c^{\alpha}_{\beta}\Psi^{i\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi}\bar{\Psi}^{j\dot{\alpha}} \\
&- \frac{4}{3}\mathrm{i}b_{ij}c^{\beta}_{\alpha}\phi\Psi^{i\alpha}\partial_{\beta\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}} + c^{\alpha\beta}F_{\alpha\beta}b_{ij}\bar{\Psi}^{i}_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}} - 4\,c^2(b_{ij}\bar{\Psi}^{i}_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}})^2 \\
&- \frac{32}{9}\bar{\phi}c^2b_{ij}D^{ij}b_{kl}\bar{\Psi}^{k}_{\dot{\alpha}}\bar{\Psi}^{l\dot{\alpha}}\Big].
\end{aligned}
\tag{5.28}
$$

It is remarkable that we can decouple the interaction between the scalar field $\bar{\phi}$ and the gauge field and still have a deformed action, contrary to what happens in the singlet case where such decoupling would destroy the deformation [52]. This suggests that in the general fermionic case it will not be possible to obtain a factorizable form like (5.2). Observe also that even in this case, second order terms in the deformation parameters appear.

The gauge transformations for this kind of deformations are obtained from (4.77) by imposing $b^2 = 0$,

$$
\delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}a\,, \qquad \delta\phi = 0\,, \qquad \delta\Psi^i_\beta = -\frac{4}{3}b^{ij}c^{\alpha}_{\beta}\bar{\Psi}^{\dot{\alpha}}_j\,\partial_{\alpha\dot{\alpha}}a, \qquad \delta D_{ij} = 2\mathrm{i}b_{ij}c^{\alpha\beta}\partial_{(\alpha\dot{\alpha}}\bar{\phi}\partial^{\dot{\alpha}}_{\beta)}a
\tag{5.29}
$$

Correspondingly, the minimal Seiberg-Witten map becomes

$$
\Psi^i_\beta = \widetilde{\Psi}^i_\beta - \frac{4}{3}b^{ij}c^{\alpha}_{\beta}\bar{\Psi}^{\dot{\alpha}}_j\,A_{\alpha\dot{\alpha}}, \qquad D_{ij} = \widetilde{D}_{ij} - 2\mathrm{i}b_{ij}c^{\alpha\beta}A_{(\alpha\dot{\alpha}}\partial^{\dot{\alpha}}_{\beta)}\bar{\phi}\,.
\tag{5.30}
$$

Moreover, we can further redefine $\widetilde{\Psi}^{k\alpha}$ and $\widetilde{D}^{ij}$

$$
\widetilde{\Psi}^{k\beta} = \psi^{k\beta} - \frac{4}{3}\mathrm{i}b^k_i c^{\beta}_{\alpha}\phi\widetilde{\Psi}^{i\alpha}
\tag{5.31}
$$

$$
\widetilde{D}^{ij} = d^{ij} - \bar{\phi}\,b^{ij}c^{\alpha\beta}F_{\alpha\beta} + \frac{64}{9}\bar{\phi}c^2b^{ij}b_{kl}\bar{\Psi}^{k}_{\dot{\alpha}}\bar{\Psi}^{l\dot{\alpha}}
\tag{5.32}
$$

to finally obtain the simple expression

$$
\begin{aligned}
S = \int d^4x_L \Big[&- \frac{1}{2}\phi\Box\bar{\phi} - \frac{1}{16}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{4}d^2 + \mathrm{i}\psi^{k\alpha}\partial_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}}_k - 4\mathrm{i}b_{ij}c^{\alpha}_{\beta}\Psi^{i\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi}\bar{\Psi}^{j\dot{\alpha}} \\
&+ c^{\alpha\beta}F_{\alpha\beta}b_{ij}\bar{\Psi}^{i}_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}} - 4\,c^2(b_{ij}\bar{\Psi}^{i}_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}})^2\Big].
\end{aligned}
\tag{5.33}
$$

Again, the non(anti)commutativity introduces interactions in the theory, as we can see from the last three terms, which are not removable under field redefinitions. An action with the same features, particularly the last two terms, has been proposed from a string theoretical point of view [38].

In the theory of type-II strings in $\mathbb{R}^4$ with their worldsheets ending on D3-branes, the presence of a background graviphoton field induces non(anti)commutativity of the coordinates on the boundary. As the low energy field theory on the branes is super Yang-Mills, it is argued that the presence of the graviphoton will produce a similar kind of non(anti)commutative deformations of super-Yang-Mills as the ones studied here. Additionally, a Yukawa-like interaction potential is also present in (5.33).

As the last term $4\,c^2(b_{ij}\bar{\Psi}^i_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}})^2$ is irremovable, there are second order corrections to the first order result given in [55].

5.4 $\mathcal{N}=(1,1/2)\to\mathcal{N}=(1,0)$ supersymmetry breaking

Starting from the $b^2=0$ case, one can *weakly* turn on components of b_{ij} selectively to describe the breaking of the enhanced 1/4 supersymmetry present in the former section. One can consider for example the following limit,

$$b_{11}=1,\qquad b_{12}=0,\qquad b_{22}\ll 1. \tag{5.34}$$

In this regime we may consider perturbative corrections to the enhanced supersymmetry action by including $b^2\neq 0$ contributions. From the full solution in §B.2.3, and applying the corresponding Seiberg-Witten map, we obtain the following action up to first order in b^2

$$\begin{aligned} S=\int d^4x_L\Big[&-\frac{1}{2}\phi\Box\bar{\phi}-\frac{1}{16}\tilde{F}^2+\frac{1}{4}\tilde{D}^2+\mathrm{i}\tilde{\Psi}^{k\alpha}\partial_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}}_k-4\mathrm{i}b_{ij}c^{\alpha}_{\beta}\tilde{\Psi}^{i\beta}\partial_{\alpha\dot{\beta}}\bar{\phi}\bar{\Psi}^{j\dot{\beta}}\\ &-\frac{b^2c^2}{6}\bar{\phi}^2\tilde{F}^2+b^2c^2\bar{\phi}^2\tilde{D}^2+c^{\alpha\beta}\tilde{F}_{\alpha\beta}b_{ij}\bar{\Psi}^i_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}}+4\,c^2(b_{ij}\bar{\Psi}^i_{\dot{\alpha}}\bar{\Psi}^{j\dot{\alpha}})^2\\ &+\frac{\bar{\phi}}{2}b_{ij}\tilde{D}^{ij}c^{\alpha\beta}\tilde{F}_{\alpha\beta}+\frac{\bar{\phi}^2b^2}{4}(c^{\alpha\beta}\tilde{F}_{\alpha\beta})^2-2\mathrm{i}\bar{\phi}b^2c^2\tilde{\Psi}^{i\alpha}\partial_{\gamma\dot{\beta}}\bar{\phi}\bar{\Psi}^{\dot{\beta}}_i\\ &-\frac{32}{9}\bar{\phi}c^2b_{ij}\tilde{D}^{ij}b_{kl}\bar{\Psi}^k_{\dot{\alpha}}\bar{\Psi}^{l\dot{\alpha}}+\mathrm{O}(b^3)\Big]. \end{aligned} \tag{5.35}$$

Its most important feature is the non trivial interaction term

$$\frac{b^2\,c^2}{6}\bar{\phi}^2\widetilde{F}^{\alpha\beta}\widetilde{F}_{\alpha\beta}, \tag{5.36}$$

which appear also in [52, 115] and can not be disentangled by a redefinition of the fields. Due to (5.34), action (5.35) can be interpreted as the weak coupling limit of an interacting theory between $\bar{\phi}$ and the gauge field, where the coupling parameter is precisely b_{22}.

As pointed out in [115], reduction of $\mathcal{N}=(1,1)$ to $\mathcal{N}=(1/2,1/2)$ superspace allows a complementary interpretation. We can choose one of the extended coodinates $\theta^{1\alpha}=\theta^{\alpha}$ to be the left Graßmann coordinate of some $\mathcal{N}=(1/2,1/2)$ superspace. As already mentioned in §3, this choice is not consistent with the standard conjugation rules in superspace, but with the *pseudo*conjugation

$$(\theta^{\alpha})^{*}=\varepsilon_{\alpha\beta}\theta^{\beta}, \qquad (\bar{\theta}^{\dot{\alpha}})^{*}=\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}} \tag{5.37}$$

as it is shown in [40]. One can then use the residual automorphism $U(1)$ and $O(1,1)$ symmetries of the $\mathcal{N}=(1,1)$ superalgebra to rotate $b_{ik}\equiv(b_{11},b_{22},b_{12})=(1,b_{22},0)$. With this choice, the purely nonsinglet deformation operator (3.24) in the decomposition ansatz is reduced to

$$P=-\overleftarrow{\partial_{\alpha}}c^{\alpha\beta}\overrightarrow{\partial_{\beta}}-b_{22}\overleftarrow{\partial^{2}_{\alpha}}c^{\alpha\beta}\overrightarrow{\partial^{2}_{\beta}} \tag{5.38}$$

In other words, it can be expressed as a sum of the mutually commuting chiral Poisson operators on two different $\mathcal{N}=(1/2,1/2)$ subspaces of $\mathcal{N}=(1,1)$ superspace. The component b_{22} is then interpreted as the *ratio* of such independent deformations. In the limit $b_{22}\rightarrow 0$, we fall back into the case where half of the supersymmetry is broken in one of the subspaces, while for $b_{22}\neq 0$, supersymmetry is broken down to $\mathcal{N}=(0,1/2)$ in both subspaces. The parameter b_{22} measures the breakdown of the second $\mathcal{N}=(0,1/2)$ supersymmetry which is implicit in the $\mathcal{N}=(1/2,1/2)$ superfield formulation based on the superspace $(x^{m},\theta^{\alpha},\bar{\theta}^{\dot{\alpha}})$. Again, we remark that within the standard complex conjugation the reduction to $\mathcal{N}=(1/2,1/2)$ superspace is not possible.

5.5 Residual Supersymmetry

One of the great advantages of using a covariant formalism like that of harmonic superspace is that supersymmetry is enforced by geometry itself. For this reason, the actions we built in the last sections are supersymmetric by construction. For completeness, it is nevertheless important to obtain the deformed expressions for the unbroken sector of the supersymmetry for the actions presented. Very much in the same way as for the action itself, a general deformation parameter b_{ij} will lead to very cumbersome supersymmetric transformation of component fields, that are drastically simplified in the cases analysed. We will see that it is nevertheless possible to construct a simple sub-algebra of the variations corresponding to $\bar{\Psi}^{-}_{\dot{\alpha}}$ and $A_{\alpha\dot{\alpha}}$ even in the general case.

As shown in §3.2, for a general choice of the parameter b_{ij} the "undotted" sector of the supersymmetry transformations is preserved. Let us briefly take a glance at the action of the corresponding supercharges on the prepotential,

$$\begin{aligned}(\epsilon^{+\alpha}\partial_{+\alpha}+\epsilon^{-\alpha}\partial_{-\alpha})V^{++}_{\mathrm{WZ}} &= 2(\epsilon^+\theta^+)\bar\phi + 2\epsilon^{+\alpha}\bar\theta^{+\dot\alpha}A_{\alpha\dot\alpha} + 8(\epsilon^+\theta^+)(\bar\theta^+\bar\Psi^+)\\ &- 2\mathrm{i}(\theta^+)^2\epsilon^{-\alpha}\bar\theta^{+\dot\alpha}\partial_{\alpha\dot\alpha}\bar\phi - 4\mathrm{i}(\epsilon^+\theta^+)\theta^{-\alpha}\bar\theta^{+\dot\alpha}\partial_{\alpha\dot\alpha}\bar\phi + 4(\bar\theta^+)^2(\epsilon^+\Psi^-)\\ &+ 6(\bar\theta^+)^2(\epsilon^+\theta^+)D^{ij}u^{--}_{ij} - \mathrm{i}(\bar\theta^+)^2(\epsilon^-\theta^+)(\partial\cdot A) + (\bar\theta^+)^2\epsilon^{-\alpha}\theta^{+\beta}F_{\alpha\beta}\\ &+ (\bar\theta^+)^2\big[-\mathrm{i}(\epsilon^+\theta^-)(\partial\cdot A) + \theta^{-\alpha}\epsilon^{+\beta}F_{\alpha\beta} + 8\mathrm{i}(\epsilon^+\theta^+)\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha}\\ &+ 4\mathrm{i}(\theta^+)^2\epsilon^{-\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha} - 2(\epsilon^+\theta^+)(\theta^-)^2\Box\bar\phi - 2(\theta^+)^2(\epsilon^-\theta^-)\Box\bar\phi\big]\end{aligned} \tag{5.39}$$

In the undeformed case, the supersymmetry transformations generated by these charges are compensated by the following gauge transformation

$$\delta_0 V^{++}_{\mathrm{WZ}} = (\epsilon^{+\alpha}\partial_{+\alpha}+\epsilon^{-\alpha}\partial_{-\alpha})V^{++}_{\mathrm{WZ}} - \mathrm{D}^{++}\Lambda_\epsilon, \tag{5.40}$$

where the matching parameter written in chiral coordinates is [52]

$$\Lambda_\epsilon = \lambda_\epsilon + (\bar\theta^+\lambda^-_\epsilon) + (\bar\theta^+)^2\lambda^{--}_\epsilon \tag{5.41}$$

with

$$\begin{aligned}\lambda_\epsilon &= 2(\epsilon^-\theta^+)\bar\phi\\ \lambda^{-\dot\alpha}_\epsilon &= 4\mathrm{i}(\epsilon^-\theta^+)\theta^-_\alpha\partial^{\alpha\dot\alpha}\bar\phi - 2\epsilon^-_\alpha A^{\alpha\dot\alpha} + 4(\epsilon^-\theta^+)\bar\Psi^{-\dot\alpha}\\ \lambda^{--}_\epsilon &= 2(\epsilon^-\Psi^-) + 2\mathrm{i}\epsilon^{-\alpha}\theta^{-\beta}\partial^{\dot\alpha}_\beta A_{\alpha\dot\alpha} - 2(\epsilon^-\theta^+)(\theta^-)^2\Box\bar\phi\\ &\quad + 4\mathrm{i}(\epsilon^-\theta^+)\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha} + 2(\epsilon^-\theta^+)D^{--}.\end{aligned} \tag{5.42}$$

from this, one obtains the set of undeformed supersymmetry transformations

$$\begin{aligned}&\delta_0\bar\phi = 0 &&\delta_0\phi = 2\epsilon^{i\alpha}\Psi_{i\alpha} &&\delta_0 A_{\alpha\dot\alpha} = 2\epsilon^i_\alpha\bar\Psi_{i\dot\alpha}\\ &\delta_0\bar\Psi^i_{\dot\alpha} = -\mathrm{i}\epsilon^{i\alpha}\partial_{\alpha\dot\alpha}\bar\phi &&\delta\Psi^j_\alpha = -\epsilon_{i\alpha}D^{ij} - \frac{1}{2}\epsilon^{j\beta}F_{\alpha\beta} &&\delta_0 D^{ij} = 2\mathrm{i}\epsilon^{(i\alpha}\partial_{\alpha\dot\alpha}\bar\Psi^{j)\dot\alpha}\end{aligned} \tag{5.43}$$

The deformation introduces a star commutator as in the non-Abelian case, and will require a different compensating gauge parameter Λ to assure Wess-Zumino gauge preservation,

$$\delta V^{++}_{\mathrm{WZ}} = \left(\epsilon^{+\alpha}\partial_{+\alpha}+\epsilon^{-\alpha}\partial_{-\alpha}\right)V^{++}_{\mathrm{WZ}} - D^{++}\Lambda - \left[V^{++}_{\mathrm{WZ}}\,,\,\Lambda\right]_\star \tag{5.44}$$

Following the procedure used for gauge transformations, one starts with the parameter Λ_ϵ used in the undeformed case, and proceeds to correct the unwanted contributions coming from the star products involved. A remarkable difference from the singlet case is again the appearance of extra Wess-Zumino breaking terms not only in the variation of the field components but also in the Graßmann sector. This non-singlet contributions coming from the star commutator on Λ_ϵ are to be denoted $\check{\delta}V^{++}_{\rm WZ}$, and in term of superfield components have the form

$$\begin{aligned}
&\check{\delta}A_{\alpha\dot\alpha} = -24\bar\phi\, c_\alpha^\beta b^{++}\,\epsilon^-_\beta \bar\Psi^-_{\dot\alpha}\,,\\
&\check{\delta}\phi = 8c^{\alpha\beta}b^{++}\,\epsilon^-_\beta\left[A_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha} - 2\bar\phi\Psi^-_\alpha\right]\,,\\
&\check{\delta}\Psi^{-\alpha} = 8\epsilon^-_\beta\left[(\bar\Psi^-)^2 - \bar\phi D^{--}\right]c^{\alpha\beta}b^{++}\\
&\qquad + 2\mathrm{i}b^{+-}\left[\bar\phi(c^{\alpha\beta}(\partial\cdot A) - \mathrm{i}\mathcal{F}^{\alpha\beta})\epsilon^-_\beta - (c\cdot G)\epsilon^{-\alpha}\right]c^{\gamma\beta}b^{+-}\,,\\
&\check{\delta}\bar\Psi^-_{\dot\alpha} = 4\mathrm{i}\,\bar\phi\partial_{\alpha\dot\alpha}\bar\phi\,\epsilon^-_\beta\, c^{\alpha\beta}b^{+-}\,,\\
&\check{\delta}D^{--} = -8\mathrm{i}\,\left[\partial_{\alpha\dot\alpha}\bar\phi\,\epsilon^-_\beta\bar\Psi^{-\dot\alpha} + \bar\phi\epsilon^-_\alpha\partial_{\beta\dot\alpha}\bar\Psi^{-\dot\alpha}\right]c^{\alpha\beta}b^{+-}\,,
\end{aligned}\tag{5.45}$$

being ϵ^i_α the Grassmann $\mathcal{N}=(1,0)$ transformation parameter in $\epsilon^-_\alpha = \epsilon^i_\alpha u^-_i$. Clearly, this variations violate the Wess-Zumino gauge due to the appearance of harmonic variables in the RHS. In addition, there are linear terms in the Graßmann variables which can not be compensated using the singlet parameter

$$\check{\delta}V^{++}_{\rm WZ} = 8\theta^+_\alpha\epsilon^-_\beta(\bar\phi)^2\hat{C}^{++\alpha\beta} - 8\bar\theta^+_{\dot\rho}A^{\dot\rho}_\alpha\epsilon^-_\beta\bar\phi\hat{C}^{++\alpha\beta} + \cdots \tag{5.46}$$

We will then need to correct not only the harmonic, but also in the Graßmann sector by extending the gauge parameter to a suitable one. Thus we rewrite (5.44) in the following way

$$\begin{aligned}
\delta V^{++}_{\rm WZ} =& \left(\epsilon^{+\alpha}\partial_{+\alpha} + \epsilon^{-\alpha}\partial_{-\alpha}\right)V^{++}_{\rm WZ} - D^{++}(\Lambda_\epsilon + F_g) - \left[V^{++}_{\rm WZ}\,,\,(\Lambda_\epsilon + F_g)\right]_\star\\
=&\delta_0 V^{++}_{\rm WZ} + \check{\delta}V^{++}_{\rm WZ} + \hat{\delta}V^{++}_{\rm WZ},
\end{aligned}\tag{5.47}$$

where

$$\hat{\delta}V^{++}_{\rm WZ} = -D^{++}(F_g) - \left[V^{++}_{\rm WZ}\,,\,F_g\right]_\star. \tag{5.48}$$

The general superfield F_g is to be calculated component by component in order to achieve the correct transformation laws for the multiplet. As the undeformed part of the variation does not depend on the harmonic variables, we only need to consider its deformed contribution and the compensating part in order to to correct the harmonic dependence of the full variation. Therefore, it is useful to define the following notation

$$\tilde{\delta} = \check{\delta} + \hat{\delta}, \tag{5.49}$$

representing the two terms of the variation that must be balanced to restore the gauge. The idea is to choose the minimal set of components of F_g needed to eliminate the improper harmonic and Grassmann dependence appearing in (5.47) and (5.46) respectively. We will name the components of this superfield as follows

$$F_g = F + \bar\theta^+\bar F^- + (\bar\theta^+)^2 F^{--},$$

where

$$\begin{aligned}
F &= f + \theta^+ f^- + (\theta^+)^2 f^{--} \\
\bar{F}^{-\dot\alpha} &= 2\mathrm{i}\theta^-_\alpha\, \partial^{\alpha\dot\alpha} f + 2\mathrm{i}\theta^-_\alpha\, \theta^{+\beta}\partial^{\alpha\dot\alpha}\, f^-_\beta + \bar{g}^{-\dot\alpha} + 2\mathrm{i}(\theta^+)^2\, \theta^-_\alpha\, \partial^{\alpha\dot\alpha} f^{--} \\
&\quad - \theta^+_\alpha\, b^{--\alpha\dot\alpha} + (\theta^+)^2\, \bar{g}^{(-3)\dot\alpha} \\
F^{--} &= -(\theta^-)^2\, \Box f - (\theta^-)^2(\theta^+ \Box f^-) + \mathrm{i}\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar{g}^{-\dot\alpha} - (\theta^+)^2(\theta^-)^2\, \Box f^{--} \\
&\quad + g^{--} + \mathrm{i}\theta^{+\alpha}\theta^{-\beta}\partial^{\dot\alpha}_\beta b^{--}_{\alpha\dot\alpha} + (\theta^+ f^{(-3)}) \\
&\quad + \mathrm{i}(\theta^+)^2\theta^{-\alpha}\partial_{\alpha\dot\alpha}\bar{g}^{(-3)\dot\alpha} + (\theta^+)^2\, X^{(-4)}
\end{aligned} \tag{5.50}$$

Before writing the variation induced by this field in its full length, we can discard one of its components corresponding to

$$\hat\delta\bar\phi = -\partial^{++} f^{--}. \tag{5.51}$$

As there was no Wess-Zumino breaking term in the variation of this particular field, the condition $\partial^{++}\delta\bar\phi = 0$ implies $\partial^{++}\hat\delta\bar\phi = 0$, which immediately fixes $f^{--} = 0$. This reduces the complexity of the resulting nontrivial supersymmetry compensating variations,

$$\hat\delta A_{\alpha\dot\alpha} = -\frac{1}{2}\partial^{++} b^{--}_{\alpha\dot\alpha} + \mathrm{i}\partial_{\alpha\dot\alpha} f - \left[2\bar\phi\, b^{--}_{\gamma\dot\alpha} - 8\bar\Psi^-_{\dot\alpha}\, f^-_\gamma\right]\, \hat{C}^{++\gamma}{}_{\alpha} + 4\mathrm{i}\,\bar\phi\,\partial_{\gamma\dot\alpha} f\, \hat{C}^{+-\gamma}{}_{\alpha}, \tag{5.52a}$$

$$\hat\delta\bar\Psi^-_{\dot\alpha} = -\frac{1}{4}\partial^{++}\bar{g}^{(-3)}_{\dot\alpha} - \frac{\mathrm{i}}{4}\partial_{\alpha\dot\alpha}\left[f^{-\alpha} - 4 f^-_\gamma\, \bar\phi \hat{C}^{+-\alpha\gamma}\right], \tag{5.52b}$$

$$\hat\delta\phi = -\partial^{++} g^{--} + \left[2\, A_{\alpha\dot\alpha}\, b^{--\dot\alpha}_\beta + 8\Psi^-_\alpha\, f^-_\beta\right] \hat{C}^{++\alpha\beta} - 4\mathrm{i} A_{\alpha\dot\alpha}\partial^{\dot\alpha}_\beta f \hat{C}^{+-\alpha\beta}, \tag{5.52c}$$

$$\begin{aligned}
\hat\delta\Psi^{-\alpha} &= -\frac{1}{4}\partial^{++} f^{(-3)\alpha} + \frac{\mathrm{i}}{4}\partial^{\alpha\dot\alpha}\bar{g}^-_{\dot\alpha} + \left[\frac{1}{2} f^-_\beta\, F_\gamma{}^{\alpha} - \mathrm{i} A_{\gamma\dot\gamma}\partial^{\dot\gamma}_\beta f^{-\alpha}\right] \hat{C}^{+-\gamma\beta} \\
&\quad + \left[\frac{\mathrm{i}}{2} f^-_\beta\, \partial^{\gamma\dot\gamma} A_{\gamma\dot\gamma} - \mathrm{i}\partial_{\beta\dot\beta}\bar{g}^{-\dot\beta}\bar\phi + 4\mathrm{i}\bar\Psi^{-\dot\beta}\, \partial_{\beta\dot\beta} f\right] \hat{C}^{+-\alpha\beta} \\
&\quad + \left[-f^{(-3)}_\beta\, \bar\phi - \bar{g}^{(-3)\dot\beta}\, A_{\beta\dot\beta} - 3\, f^-_\beta\, D^{--} + 2\bar\Psi^{-\dot\beta}\, b^{--}_{\beta\dot\beta}\right] \hat{C}^{++\alpha\beta},
\end{aligned} \tag{5.52d}$$

$$\begin{aligned}
\hat\delta D^{--} &= -\frac{1}{3}\partial^{++} X^{(-4)} + \frac{\mathrm{i}}{6}\partial^{\alpha\dot\alpha} b^{--}_{\alpha\dot\alpha} - \frac{4}{3}\,\partial_{\alpha\dot\alpha}\bar\phi\,\partial^{\dot\alpha}_\beta f \hat{C}^{--\alpha\beta} \\
&\quad + \frac{1}{3}\left[-2\mathrm{i}\partial_{\alpha\dot\alpha}(\bar\phi\, b^{--\dot\alpha}_\beta) - 8\mathrm{i}\partial_{\beta\dot\beta}\left(f^-_\alpha\, \bar\Psi^{-\dot\beta}\right)\right] \hat{C}^{+-\alpha\beta}.
\end{aligned} \tag{5.52e}$$

We will then use the extra freedom provided by the contributions above, to enforce the Wess-Zumino gauge preserving condition on the variation.

As mentioned before, in addition to the variations, there also appear Wess-Zumino violating terms in the Graßmann sector, as terms linear in θ and $\bar{\theta}$ in $\check{\delta}V^{++}_{\mathrm{WZ}}$, giving the following equations for the compensating parameter

$$\partial^{++}f^{-\alpha} + 4\hat{C}^{++\alpha\beta}\bar{\phi}f^{-}_{\beta} + 8\epsilon^{-}_{\beta}(\bar{\phi})^2\hat{C}^{++\alpha\beta} = 0, \tag{5.53a}$$

$$\partial^{++}\bar{g}^{-}_{\dot{\alpha}} + 4\,A_{\alpha\dot{\alpha}}f^{-}_{\beta}\hat{C}^{++\alpha\beta} + 8\epsilon^{-}_{\beta}A_{\alpha\dot{\alpha}}\bar{\phi}\hat{C}^{++\alpha\beta} = 0, \tag{5.53b}$$

Which is solved by

$$f^{-\alpha} = \epsilon^{-}_{\beta}f^{\alpha\beta} + \epsilon^{+}_{\beta}f^{--\,\alpha\beta} = \epsilon^{-}_{\beta}(c^{\alpha\beta}\hat{f} + \varepsilon^{\alpha\beta}\check{f}) + \epsilon^{+}_{\beta}(c^{\alpha\beta}\hat{f}^{--} + \varepsilon^{\alpha\beta}\check{f}^{--}), \tag{5.54}$$

and

$$\bar{g}^{-}_{\dot{\alpha}} = A_{\alpha\dot{\alpha}}\bar{\phi}^{-1}f^{-\alpha}, \tag{5.55}$$

where (see appendix §B.3.1)

$$\check{f} = 2\bar{\phi}\left(\frac{\sinh X}{X}\cos Z - 1\right), \tag{5.56a}$$

$$\hat{f} = -\frac{4\bar{\phi}}{\sqrt{2c^2}}\frac{\sinh X}{X}\sin Z, \tag{5.56b}$$

$$\check{f}^{--} = \frac{\kappa^{--}}{X^2+Z^2}2\bar{\phi}\left(\cosh X\sin Z - \frac{\sinh X}{X}Z\cos Z\right), \tag{5.56c}$$

$$\hat{f}^{--} = \frac{4\bar{\phi}}{\sqrt{2c^2}}\frac{\kappa^{--}}{X^2+Z^2}\left(\cosh X\cos Z + \frac{\sinh X}{X}Z\sin Z - 1\right). \tag{5.56d}$$

Except for the triplet of fields $\bar{\phi}$, $A_{\alpha\dot{\alpha}}$ and $\bar{\Psi}^{i}_{\dot{\alpha}}$, the expressions for the transformation laws for a general deformation parameter b_{ij} are very complicated. We will limit ourselves to present the exact expressions for this closed subalgebra as calculated in §B.3.2. The residual supersymmetric variation of $\bar{\phi}$ is trivial, the other two transform according to

$$\delta_{\epsilon}\bar{\Psi}^{i}_{\dot{\alpha}} = \left[4\mathrm{i}\bar{\phi}\cosh X\frac{\sinh X}{X}c^{\alpha\beta}b^{ij} - \mathrm{i}\cosh^2 X\varepsilon^{\alpha\beta}\varepsilon^{ij}\right]\epsilon_{j\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi}, \tag{5.57a}$$

$$\delta A_{\alpha\dot{\alpha}} = \left[8\bar{\phi}\,b_{ij}c_{\alpha\beta} + 2X\coth X\varepsilon_{ij}\varepsilon_{\alpha\beta}\right]\epsilon^{i\beta}\bar{\Psi}^{j}_{\dot{\alpha}}. \tag{5.57b}$$

Which form a closed algebra

$$[\delta_{\eta},\delta_{\epsilon}]\,A_{\alpha\dot{\alpha}} = 2\mathrm{i}(\epsilon\cdot\eta)X\coth X\partial_{\alpha\dot{\alpha}}\bar{\phi}. \tag{5.58}$$

As it can be verified by comparison with the residual gauge transformation

$$\delta_{a}A_{\alpha\dot{\alpha}} = X\coth X\partial_{\alpha\dot{\alpha}}a. \tag{5.59}$$

For the actions presented in this chapter it is not necessary to give the full expressions of the variations but their second order expansions in the variable X. That is

$$\delta\bar{\phi} = 0, \tag{5.60a}$$

$$\delta A_{\alpha\dot{\alpha}} = \left[2\left(1+\frac{4}{3}b^2c^2\bar{\phi}^2\right)\varepsilon_{\alpha\beta}\varepsilon_{ij} + 8\bar{\phi}\,c_{\alpha\beta}b_{ij}\right]\epsilon^{i\beta}\bar{\Psi}^j_{\dot{\alpha}} + \mathrm{O}(b^3), \tag{5.60b}$$

$$\delta\bar{\Psi}^i_{\dot{\alpha}} = -\mathrm{i}\left[(1+4b^2c^2\bar{\phi}^2)\varepsilon^{\alpha\beta}\varepsilon^{ij} - 4\bar{\phi}\,c^{\alpha\beta}b^{ij}\right]\epsilon_{j\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi} + \mathrm{O}(b^3), \tag{5.60c}$$

$$\delta\phi = \Psi^i_\alpha\epsilon^j_\beta\left[2\varepsilon^{\alpha\beta}\varepsilon_{ij} + \frac{16}{3}\bar{\phi}c^{\alpha\beta}b_{ij}\right] + A^{\dot{\alpha}}_\alpha\bar{\Psi}^i_{\dot{\alpha}}\epsilon^j_\beta\left[\frac{40}{3}c^{\alpha\beta}b_{ij}\right] + \mathrm{O}(b^3), \tag{5.60d}$$

$$\delta D^{ij} = 2\mathrm{i}\partial^{\alpha\dot{\alpha}}\left[\left(1+\frac{1}{3}b^2c^2\bar{\phi}^2\right)\epsilon^{(i}_\alpha\bar{\Psi}^{j)}_{\dot{\alpha}} + 4\mathrm{i}\phi\epsilon^{((k\beta}\bar{\Psi}^{i)}_{\dot{\alpha}}c_{\alpha\beta}b^{j)}_k - 4\mathrm{i}\bar{\phi}^2c^2b^{ij}\epsilon^k_\alpha\bar{\Psi}^l_{\dot{\alpha}}b_{kl}\right] + \mathrm{O}(b^3), \tag{5.60e}$$

$$\begin{aligned}
\delta\Psi^{i\alpha} =& \Bigg(-D^{ij}\epsilon^{\alpha\beta} + \Bigg\{\left(\frac{1}{2}+\frac{10}{9}b^2c^2\bar{\phi}^2\right)F^{\alpha\beta} - 2\mathrm{i}b^2c^2\bar{\phi}G^{\alpha\beta} \\
&+ \frac{2}{9}\mathrm{i}b^2c^2\bar{\phi}^2\left[8(A\cdot\partial\bar{\phi}) + \bar{\phi}(\partial\cdot A)\right]\varepsilon^{\alpha\beta} + \frac{2}{3}\Big[4(b\cdot\bar{\Psi}\bar{\Psi}) - 4\bar{\phi}(b\cdot D) \\
&- \frac{\sqrt{2c^2}}{3}\mathrm{i}b^2\bar{\phi}(A\cdot\partial\bar{\phi}) - \frac{5}{3}b^2\bar{\phi}^2(c\cdot F) - \frac{2}{3}\mathrm{i}b^2\bar{\phi}^2(c\cdot G)\Big]c^{\alpha\beta}\Bigg\}\varepsilon^{ij} \\
&+ \Bigg\{\frac{2}{3}\mathcal{F}^{\alpha\beta} + \frac{1}{3}\left[\frac{40}{3}c^2\bar{\phi}(b\cdot\bar{\Psi}\bar{\Psi}) - \frac{28}{3}c^2\bar{\phi}^2(b\cdot D) + \frac{\sqrt{2c^2}}{2}\mathrm{i}(A\cdot\partial\bar{\phi}) - 2\mathrm{i}\bar{\phi}(c\cdot G)\right]\varepsilon^{\alpha\beta} \\
&- \frac{2}{3}\mathrm{i}\left[2(A\cdot\partial\bar{\phi}) + \bar{\phi}(\partial\cdot A)\right]c^{\alpha\beta}\Bigg\}b^{ij}\Bigg)\epsilon_{j\beta} + \mathrm{O}(b^3)
\end{aligned} \tag{5.60f}$$

For the particular $N = (1, 1/2)$ case, the limit $b^2 = 0$ of equations (5.60) lead to the *exact* supersymmetry transformations. In contrast to the perturbative methods used in [53, 54, 55], we have found this solutions *exactly* and afterwards performed a series expansion to obtain the results above. The extra residual transformation for the dotted supercharge in the enhanced case is given in [55].

Chapter 6

Conclusions and Outlook

In this work we studied the impact of Q-deformations on $N = 2$ gauge theories, especially for the non-singlet structure of the deformation matrix. By properly defining the commutator of symmetry generators with the Poisson structure that governs deformations, we were able to measure to what extent they break the Leibniz rule for symmetry transformations. This was used to construct the precise (super)symmetry breaking pattern coming from particular selections of the deformation matrix.

Out of the generic non-singlet deformation $\hat{C}^{ij}_{\alpha\beta}$ that fully breaks spacetime $SU(2)_L$ and R-symmetry, we chose a maximally symmetry preserving ansatz $\hat{C}^{ij}_{\alpha\beta} = b^{ij}c_{\alpha\beta}$ that leaves unbroken the $U(1)_L$ and $U(1)$ subgroups of automorphisms. Such product decomposition for the deformation matrix provides a unique possibility to exactly determine the deformed gauge transformations, in contrast to the generic case [53, 54, 55]. We developed a general algorithm to solve the typical harmonic equations related to this ansatz, through a formal analogy with ordinary differential equations. Using this methods we determined the exact expressions for the gauge transformations. Quite remarkably, these are given in terms of rational hyperbolic functions of the variable $X = 2\sqrt{b^2c^2}\bar{\phi}$ which is invariant under the *full* supersymmetry algebra (including its Lorentz and automorphism generators). Through the construction of the minimal Seiberg-Witten map, we were able to find the U(1) gauge transformations for the undeformed equivalent system field theory.

By solving the deformed curvature equations for meaningful cases of the deformation matrix b^{ij}, we were able to derive the corresponding deformed super-Yang-Mills actions with partially broken supersymmetry. We showed that the gauge action is governed exclusively by the first component $\mathcal{A}$ of the superfield strength $\mathcal{W}$ independent of the kind of deformation considered. For a general b^{ij} we constructed the exact bosonic sector of the action which, through a suitable field redefinition, factors into the free bosonic part of $N = 2$ super-Maxwell times a hyperbolic function of X. A striking feature of the result is its manifest Lorentz and R-symmetry invariance. This action shares similarities with the singlet case [50, 52], where the gauge field also interacts non-trivially with the scalar $\bar{\phi}$ from the vector multiplet, and the action factorizes into a free part times the polynomial $(1 + 4I\bar{\phi})^2$. For the maximally supersymmetry preserving choice $b^2 = 0$, where we have an $\mathcal{N} = (1, 0) \longrightarrow \mathcal{N} = (1, 1/2)$ *enhancement* with respect to the more general case, we obtained the full exact action. Its Lagrangian possesses Yukawa-like terms, interaction terms comparable to those in the $\mathcal{N} = 1 + \frac{1}{2}$ deformed D3-brane low energy action constructed in [38], and additional second order terms in the deformation parameters which can not be removed by a redefinition of the fields. Contrary to what happens in the singlet case, decoupling the interaction between the scalar $\bar{\phi}$ and the gauge field does *not* destroy the deformation. Additionally we studied the behavior of the action upon weakly restoring selected components of b^{ij}, as to describe a tuned breaking of the enhanced supersymmetry present in the former case. The non-trivial interactions between $\bar{\phi}$ and the gauge field that arise allow for the interpretation of the additional degrees of freedom from b^{ij} as coupling constants, in particular a term proportional to $b^2 c^2 \bar{\phi}^2 \widetilde{F}^2$ characteristic of $\mathcal{N} = (1, 1)$ gauge multiplet Q-deformation appears. We completed the analysis of the full actions by working out the supersymmetry transformations corresponding to the cases studied. In addition, a particular closed subalgebra is calculated exactly for the general form of the product ansatz.

It is fair to say that without the manifestly supersymmetric harmonic superfield approach, the structures emerging from the deformation would hardly be achievable at the component level.

It would be very natural to continue the study by addressing the issues of renormalization properties, non-abelian extensions, and instantonic solution of the theories obtained. Also of great interest would be to consider the most general non-singlet deformation, perhaps as a perturbation around the decomposition ansatz rather than around the undeformed limit, or by using a different kind of decomposition like a linear combination of deformations already studied. Of particular relevance is the study of the deformations treated here on models including hypermultiplets, along the lines of [52, 42]. This would link to the subject of hyper-Kähler geometries and allow for the study of this deformation on models with matter, which are relevant for more phenomenological applications like providing a specific mechanism of soft supersymmetry breaking. Finally we could try to establish what particular kind of string background produces non-singlet deformations of superspace. This could shed some light on the mysterious appearance of the hyperbolic functions from the product decomposition.

Part III

Appendixes

Appendix A

Notation and conventions

The units throughout follow the canonical choice

$$\hbar = c = 1 \tag{A.1}$$

Only in a discussion of Weyl quantization in §1.1 will $\hbar$ appear explicitely. Throughout the text, Greek and Latin indices are spinorial and $SU(2)$, respectively and are both rised and lowered with the usual antisymmetric tensor $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$, ε_{ij} for which we use the conventions

$$\varepsilon_{12} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = 1, \qquad \varepsilon^{ik}\varepsilon_{kj} = \delta^i_j, \qquad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha_\beta, \qquad \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon_{\dot{\gamma}\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}}. \tag{A.2}$$

Dotted and undotted spinor indices are raised and lowered in the same way

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \qquad \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}, \qquad \bar{\chi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}}. \tag{A.3}$$

The convetional contraction of a pair of spinors is defined as follows

$$(\psi\lambda) \equiv \psi^\alpha\lambda_\alpha, \qquad (\bar{\chi}\bar{\xi}) \equiv \bar{\chi}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}, \qquad \psi^2 \equiv (\psi\psi), \qquad \bar{\psi}^2 \equiv (\bar{\psi}\bar{\psi}). \tag{A.4}$$

Even when most of the work is done on Euclidean space, the Minkowskii metric

$$n^{mn} = \mathrm{diag}(+,-,-,-) \tag{A.5}$$

appears in the standard discussion of harmonic superspace in §2, where we follow the conventions of [104].

With exception of the first chapter, where latin indices around m are used to denote general spacetime indices in d dimensions, vectors are always represented by the two-component spinor formalism. Euclidean (also Minkowskiian) four-vectors are written as bi-spinors defined by

$$x^{\alpha\dot{\alpha}} = \bar{\sigma}_m^{\alpha\dot{\alpha}} x^m, \tag{A.6}$$

where the sigma matrices include the identity and the standard Pauli matrices

$$\sigma^m = (\mathbb{1}, \boldsymbol{\sigma}), \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

and $\bar{\sigma}^m = (1, \pm\boldsymbol{\sigma})$ depending on the signature of spacetime (positive for Euclidean). Whenever the bi-spinor notation is used, we use a convention for vector and tensor contractions that differs with respect to [104],

$$(x \cdot y) \equiv x^{\alpha\dot{\alpha}} y_{\alpha\dot{\alpha}}, \qquad x^2 = (x \cdot x), \qquad (F \cdot G) \equiv F^{\alpha\beta} G_{\alpha\beta}, \qquad F^2 = (F \cdot F). \tag{A.8}$$

Our conventions concerning harmonic superspace are the natural Euclidean $\mathbb{R}^{4|8}$ generalization of Wess-Bagger,

$$\theta^{\pm}_{\alpha}\theta^{\pm}_{\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}(\theta^{\pm}\theta^{\pm}), \quad \bar{\theta}^{\pm}_{\dot{\alpha}}\bar{\theta}^{\pm}_{\dot{\beta}} = -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\pm}\bar{\theta}^{\pm} \quad \theta^{+}_{\alpha}\theta^{-}_{\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}(\theta^{+}\theta^{-}) + \frac{1}{2}(\theta^{k}{}_{\alpha}\,\theta_{k\,\beta}) \tag{A.9}$$

Appendix B

Technical Details

B.1 Moyal product for $N = (1,1)$ Q-deformations

In this section we expose the full structure of the Q-deformed product and (anti)commutators of two arbitrary $U(1)$ superfields A and B with $\mathbb{Z}_2$ gradings a and b respectively, as defined in section §3.3. The deformed terms of (3.39) are

$$\begin{aligned} APB = &-(-1)^a\Big\{\hat{C}^{++\alpha\beta}\partial_{+\alpha}A\,\partial_{+\beta}B + \left(\hat{C}^{+-\alpha\beta} + I\varepsilon^{\alpha\beta}\right)\partial_{+\alpha}A\,\partial_{-\beta}B \\ &+ \left(\hat{C}^{+-\alpha\beta} - I\varepsilon^{\alpha\beta}\right)\partial_{-\alpha}A\,\partial_{+\beta}B + \hat{C}^{--\alpha\beta}\partial_{-\alpha}A\,\partial_{-\beta}B\Big\}, \end{aligned} \tag{B.1}$$

$$\begin{aligned} AP^2B = &-\frac{1}{4}(\hat{C}^{++})^2\partial_+^2A\,\partial_+^2B + \hat{C}^{++\alpha}{}_{\mu}\hat{C}^{+-\mu\beta}\left(\partial_+^2A\,\partial_{+\alpha}\partial_{-\beta}B + \partial_{+\alpha}\partial_{-\beta}A\,\partial_+^2B\right) \\ &-\frac{1}{2}\left(I^2 + \frac{1}{2}(\hat{C}^{+-})^2\right)\left(\partial_+^2A\,\partial_-^2B + \partial_-^2A\,\partial_+^2B\right) \\ &-2\left[\hat{C}^{++\alpha\gamma}\hat{C}^{--\beta\delta} - (\hat{C}^{+-\alpha\delta} + I\varepsilon^{\alpha\delta})(\hat{C}^{+-\beta\gamma} - I\varepsilon^{\beta\gamma})\right]\partial_{+\alpha}\partial_{-\beta}A\,\partial_{+\gamma}\partial_{-\delta}B \\ &-\frac{1}{4}(\hat{C}^{--})^2\partial_-^2A\,\partial_-^2B + \hat{C}^{+-\alpha}{}_{\mu}\hat{C}^{--\mu\beta}\left(\partial_-^2A\,\partial_{+\alpha}\partial_{-\beta}B + \partial_{+\alpha}\partial_{-\beta}A\,\partial_-^2B\right), \end{aligned} \tag{B.2}$$

$$
\begin{aligned}
AP^3B = -\frac{3}{2}(-1)^a \Bigg\{ & \left[\hat{C}^{+-\alpha\mu}\hat{C}^{++}_{\mu\nu}\hat{C}^{+-\nu\beta} + I^2\hat{C}^{++\alpha\beta} - 2I\hat{C}^{++(\alpha}{}_{\gamma}\hat{C}^{+-\gamma\beta)} - \frac{1}{2}(\hat{C}^{++})^2\hat{C}^{--\alpha\beta} \right] \\
& \qquad \times \partial_+^2\partial_{-\alpha}A\,\partial_+^2\partial_{-\beta}B \\
& + \left[\hat{C}^{--\alpha\mu}\hat{C}^{+-}{}_{\mu\nu}\hat{C}^{++\nu\beta} + I\hat{C}^{--\alpha\mu}\hat{C}^{++\beta}{}_{\mu} - \left(I^2 + \frac{1}{2}(\hat{C}^{+-})^2 \right) (\hat{C}^{+-\alpha\beta} - I\varepsilon^{\alpha\beta}) \right] \\
& \qquad \times \partial_+^2\partial_{-\alpha}A\,\partial_-^2\partial_{+\beta}B \\
& + \left[\hat{C}^{++\alpha\mu}\hat{C}^{+-}{}_{\mu\nu}\hat{C}^{--\nu\beta} - I\hat{C}^{++\alpha\mu}\hat{C}^{--\beta}{}_{\mu} - \left(I^2 + \frac{1}{2}(\hat{C}^{+-})^2 \right) (\hat{C}^{+-\alpha\beta} + I\varepsilon^{\alpha\beta}) \right] \\
& \qquad \times \partial_-^2\partial_{+\alpha}A\,\partial_+^2\partial_{-\beta}B \\
& + \left[\hat{C}^{+-\alpha\mu}\hat{C}^{--}{}_{\mu\nu}\hat{C}^{+-\nu\beta} + I^2\hat{C}^{--\alpha\beta} - 2I\hat{C}^{--(\alpha}{}_{\gamma}\hat{C}^{+-\gamma\beta)} - \frac{1}{2}(\hat{C}^{--})^2\hat{C}^{++\alpha\beta} \right] \\
& \qquad \times \partial_-^2\partial_{+\alpha}A\,\partial_-^2\partial_{+\beta}B \Bigg\},
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
AP^4B = -\frac{3}{2}\Bigg[& \hat{C}^{++\alpha}{}_{\beta}\hat{C}^{+-\beta}{}_{\gamma}\hat{C}^{--\gamma}{}_{\delta}\hat{C}^{+-\delta}{}_{\alpha} - \left(I^2 + \frac{1}{2}(\hat{C}^{+-})^2 \right) - \frac{1}{4}(\hat{C}^{++})^2(\hat{C}^{--})^2 \\
& + I^2\hat{C}^{++}{}_{\alpha\beta}\hat{C}^{--\alpha\beta} + 2I\hat{C}^{++\alpha}{}_{\beta}\hat{C}^{+-\beta}{}_{\gamma}\hat{C}^{--\gamma}{}_{\alpha} \Bigg] \partial_+^2\partial_-^2A\,\partial_+^2\partial_-^2B.
\end{aligned} \tag{B.4}
$$

The list of terms corresponding to the (anti)commutator (3.44)

$$
\begin{aligned}
\mathrm{A}_1^{\pm} = -(-1)^a\left[1 \mp (-1)^{ab}\right] \Big[& (\hat{C}^{+-\alpha\beta} + I\varepsilon^{\alpha\beta})(\partial_{+\alpha}A\,\partial_{-\beta}B + \partial_{-\beta}A\,\partial_{+\alpha}B) \\
& + \hat{C}^{++\alpha\beta}\partial_{+\alpha}A\,\partial_{+\beta}B + \hat{C}^{--\alpha\beta}\partial_{-\alpha}A\,\partial_{-\beta}B \Big],
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
\mathrm{A}_2^{\pm} = &-\left[1 \pm (-1)^{ab}\right]\Bigg\{\frac{1}{4}(\hat{C}^{++})^2 \partial_+^2 A\, \partial_+^2 B + \frac{1}{4}(\hat{C}^{--})^2 \partial_-^2 A\, \partial_-^2 B \\
&- \hat{C}^{++\alpha}{}_{\mu}\hat{C}^{+-\mu\beta}\left(\partial_+^2 A\, \partial_{+\alpha}\partial_{-\beta}B + \partial_{+\alpha}\partial_{-\beta}A\, \partial_+^2 B\right) \\
&- \hat{C}^{+-\alpha}{}_{\mu}\hat{C}^{--\mu\beta}\left(\partial_-^2 A\, \partial_{+\alpha}\partial_{-\beta}B + \partial_{+\alpha}\partial_{-\beta}A\, \partial_-^2 B\right), \\
&+ \frac{1}{2}\left(I^2 + (\hat{C}^{+-})^2\right)\left(\partial_+^2 A\, \partial_-^2 B + \partial_-^2 A\, \partial_+^2 B\right) \\
&+ 2\left[\hat{C}^{++\alpha\gamma}\hat{C}^{--\beta\delta} - (\hat{C}^{+-\alpha\delta} + I\varepsilon^{\alpha\delta})(\hat{C}^{+-\beta\gamma} - I\varepsilon^{\beta\gamma})\right]\partial_{+\alpha}\partial_{-\beta}A\, \partial_{+\gamma}\partial_{-\delta}B\Bigg\}
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
\mathrm{A}_3^{\pm} = &-\frac{3}{2}(-1)^a\left[1 \mp (-1)^{ab}\right] \\
&\Bigg\{\left[\hat{C}^{+-\alpha\mu}\hat{C}^{++}{}_{\mu\nu}\hat{C}^{+-\nu\beta} + I^2\hat{C}^{++\alpha\beta} - 2I\hat{C}^{++(\alpha}{}_{\gamma}\hat{C}^{+-\gamma\beta)} - \frac{1}{2}(\hat{C}^{++})^2\hat{C}^{--\alpha\beta}\right] \\
&\qquad \times \partial_+^2\partial_{-\alpha}A\, \partial_+^2\partial_{-\beta}B \\
&+ \left[\hat{C}^{--\alpha\mu}\hat{C}^{+-}{}_{\mu\nu}\hat{C}^{++\nu\beta} + I\hat{C}^{--\alpha\mu}\hat{C}^{++\beta}{}_{\mu} - \left(I^2 + \frac{1}{2}(\hat{C}^{+-})^2\right)(\hat{C}^{+-\alpha\beta} - I\varepsilon^{\alpha\beta})\right] \\
&\qquad \times \left(\partial_+^2\partial_{-\alpha}A\, \partial_-^2\partial_{+\beta}B + \partial_-^2\partial_{+\beta}A\, \partial_+^2\partial_{-\alpha}B\right) \\
&+ \left[\hat{C}^{+-\alpha\mu}\hat{C}^{--}{}_{\mu\nu}\hat{C}^{+-\nu\beta} + I^2\hat{C}^{--\alpha\beta} - 2I\hat{C}^{--(\alpha}{}_{\gamma}\hat{C}^{+-\gamma\beta)} - \frac{1}{2}(\hat{C}^{--})^2\hat{C}^{++\alpha\beta}\right] \\
&\qquad \times \partial_-^2\partial_{+\alpha}A\, \partial_-^2\partial_{+\beta}B\Bigg\},
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
\mathrm{A}_4^{\pm} = &-\frac{3}{2}\left[1 \pm (-1)^{ab}\right]\Bigg[\hat{C}^{++\alpha}{}_{\beta}\hat{C}^{+-\beta}{}_{\gamma}\hat{C}^{--\gamma}{}_{\delta}\hat{C}^{+-\delta}{}_{\alpha} - \left(I^2 + \frac{1}{2}(\hat{C}^{+-})^2\right) \\
&- \frac{1}{4}(\hat{C}^{++})^2(\hat{C}^{--})^2 + I^2\hat{C}^{++}{}_{\alpha\beta}\hat{C}^{--\alpha\beta} + 2I\hat{C}^{++\alpha}{}_{\beta}\hat{C}^{+-\beta}{}_{\gamma}\hat{C}^{--\gamma}{}_{\alpha}\Bigg]\partial_+^2\partial_-^2 A\, \partial_+^2\partial_-^2 B.
\end{aligned} \tag{B.8}
$$

B.2 Non-singlet Curvature Equations in Components

The full component expansion of the relevant curvature equations is obtained by expanding the superfields involved. Using (5.1)

$$\mathcal{A} = \mathcal{A}_1 + \theta^{-\alpha}\mathcal{A}^{+}_{2\,\alpha} + \theta^{+\alpha}\mathcal{A}^{-}_{3\,\alpha} + (\theta^-)^2\mathcal{A}^{++}_4 + (\theta^-\theta^+)\mathcal{A}_5 + \theta^{-\alpha}\theta^{+\beta}\mathcal{A}_{6\,\alpha\beta} + (\theta^+)^2\mathcal{A}^{--}_7 \\ + (\theta^-)^2\theta^{+\alpha}\mathcal{A}^{+}_{8\,\alpha} + (\theta^+)^2\theta^{-\alpha}\mathcal{A}^{-}_{9\,\alpha} + (\theta^-)^2(\theta^+)^2\mathcal{A}_{10},$$

we break up (5.3)

$$\mathrm{D}^{++}\mathcal{A} + [v^{++}, \mathcal{A}]_\star = 0.$$

into

$$\partial^{++}\mathcal{A}_1 = 0, \qquad \partial^{++}\mathcal{A}_{10} = 0, \tag{B.9a}$$

$$\begin{aligned} &\partial^{++}\mathcal{A}^{+}_{2\,\alpha} = 0, \\ &\partial^{++}\mathcal{A}^{-}_{3\,\alpha} + \mathcal{A}^{+}_{2\,\alpha} - 4\bar{\phi}c_{\alpha\beta}(b^{+-}\mathcal{A}^{+\,\beta}_2 + b^{++}\mathcal{A}^{-\,\beta}_3) = 0, \end{aligned} \tag{B.9b}$$

$$\begin{aligned} &\partial^{++}\mathcal{A}^{++}_4 = 0, \\ &\partial^{++}\mathcal{A}_5 + 2\mathcal{A}^{++}_4 - 2\bar{\phi}b^{++}\, c^{\alpha\beta}\, \mathcal{A}_{6\alpha\beta} = 0, \\ &\partial^{++}\mathcal{A}_{6\alpha\beta} + 4\bar{\phi}(2b^{+-}\mathcal{A}^{++}_4 + b^{++}\mathcal{A}_5)\, c_{\alpha\beta} + 4b^{++}\bar{\phi}\mathcal{A}_{6(\alpha\gamma}c^{\gamma}_{\beta)} = 0, \\ &\partial^{++}\mathcal{A}^{--}_7 + \mathcal{A}_5 + 2\bar{\phi}b^{+-}c^{\alpha\beta}\mathcal{A}_{6\alpha\beta} = 0, \end{aligned} \tag{B.9c}$$

$$\begin{aligned} &\partial^{++}\mathcal{A}^{+}_{8\,\alpha} - 4\bar{\phi}b^{++}c_{\alpha\beta}\mathcal{A}^{+\,\beta}_8 = 0, \\ &\partial^{++}\mathcal{A}^{-}_{9\,\alpha} - (\varepsilon_{\alpha\beta} + 4\bar{\phi}b^{+-}c_{\alpha\beta})\mathcal{A}^{+\,\beta}_8 = 0. \end{aligned} \tag{B.9d}$$

Expanding $v^{-\,\dot{\alpha}}$ in a similar way

$$v^{-\,\dot{\alpha}} = w^{-\,\dot{\alpha}}_1 + \theta^{-\alpha}w_{2\,\alpha}^{\;\;\dot{\alpha}} + \theta^{+\alpha}w^{--\,\dot{\alpha}}_{3\,\alpha} + (\theta^-)^2 w^{+\,\dot{\alpha}}_4 + (\theta^-\theta^+)w^{-\,\dot{\alpha}}_5 + \theta^{-\alpha}\theta^{+\beta}w^{-\,\dot{\alpha}}_{6\,\alpha\beta} \\ + (\theta^+)^2 w^{(-3)\,\dot{\alpha}}_7 + (\theta^-)^2\theta^{+\alpha}w_{8\,\alpha}^{\;\;\dot{\alpha}} + (\theta^+)^2\theta^{-\alpha}w^{--\,\dot{\alpha}}_{9\,\alpha} + (\theta^-)^2(\theta^+)^2 w^{-\,\dot{\alpha}}_{10}, \tag{B.10}$$

we express equation (5.7)

$$\mathrm{D}^{++}v^{-\,\dot{\alpha}} + [v^{++}, v^{-\,\dot{\alpha}}]_\star - v^{+\,\dot{\alpha}} = 0,$$

as

$$\partial^{++}w^{-\dot{\alpha}}_1 = 0, \qquad \partial^{++}w^{-\dot{\alpha}}_{10} = 0. \tag{B.11a}$$

$$\begin{aligned}&\partial^{++}w_2^{\alpha\dot\alpha}=0,\\&\partial^{++}w_3^{--\,\alpha\dot\alpha}+w_2^{\alpha\dot\alpha}+4\bar\phi c^{\alpha\beta}(b^{+-}w_{2\,\beta}{}^{\dot\alpha}+b^{++}w_{3\,\beta}^{--\,\dot\alpha})-2A^{\alpha\dot\alpha}=0,\end{aligned}\tag{B.11b}$$

$$\begin{aligned}&\partial^{++}w_4^{+\,\dot\alpha}=0,\\&\partial^{++}w_5^{-\,\dot\alpha}+2w_4^{+\,\dot\alpha}-2\bar\phi b^{++}\,c^{\alpha\beta}\,w_{6\,\alpha\beta}^{-\,\dot\alpha}=0,\\&\partial^{++}w_{6\,\alpha\beta}^{-\,\dot\alpha}+4\bar\phi c_{\alpha\beta}(2b^{+-}w_4^{+\,\dot\alpha}+b^{++}w_5^{-\,\dot\alpha})+4b^{++}\bar\phi c_{(\alpha}{}^{\gamma}w_{6\,\beta)\gamma}^{-\,\dot\alpha}=0,\\&\partial^{++}w_7^{(-3)\,\dot\alpha}+w_5^{-\,\dot\alpha}+2\bar\phi b^{+-}c^{\alpha\beta}w_{6\,\alpha\beta}^{-\,\dot\alpha}-4\bar\Psi^{-\,\dot\alpha}=0,\end{aligned}\tag{B.11c}$$

$$\begin{aligned}&\partial^{++}w_8^{\alpha\dot\alpha}+4\bar\phi b^{++}c^{\alpha\beta}w_{8\,\beta}{}^{\dot\alpha}=0,\\&\partial^{++}w_9^{--\,\alpha\dot\alpha}-(\varepsilon^{\alpha\beta}-4\bar\phi b^{+-}c^{\alpha\beta})w_{8\,\beta}{}^{\dot\alpha}+2\mathrm{i}\partial^{\alpha\dot\alpha}\bar\phi=0.\end{aligned}\tag{B.11d}$$

Similarly, we decompose

$$\begin{aligned}\varphi^{--}=\varphi_1^{--}&+\theta^{-\alpha}\varphi_{2\,\alpha}^{-}+\theta^{+\alpha}\varphi_{3\,\alpha}^{(-3)}+(\theta^-)^2\varphi_4+(\theta^-\theta^+)\varphi_5^{--}+\theta^{-\alpha}\theta^{+\beta}\varphi_{6\,\alpha\beta}^{--}\\&+(\theta^+)^2\varphi_7^{(-4)}+(\theta^-)^2\theta^{+\alpha}\varphi_{8\,\alpha}^{-}+(\theta^+)^2\theta^{-\alpha}\varphi_{9\,\alpha}^{(-3)}+(\theta^-)^2(\theta^+)^2\varphi_{10}^{--},\end{aligned}\tag{B.12}$$

to obtain the components of (5.6)

$$\mathrm{D}^{++}\varphi^{--}+[v^{++},\varphi^{--}]_\star+2(\mathcal{A}-v)+\frac{1}{2}\left\{v^{+\,\dot\alpha},v^{-}_{\dot\alpha}\right\}_\star=0,$$

namely

$$\begin{aligned}&\partial^{++}\varphi_1^{--}+2(\mathcal{A}_1-\phi)+2c^{\alpha\beta}A_\alpha^{\dot\alpha}(b^{+-}w_{2\,\beta\dot\alpha}+b^{++}w_{3\,\beta\dot\alpha}^{--})\\&\qquad\qquad\qquad\qquad-2\mathrm{i}b^2c^2c^{\alpha\beta}\partial_\alpha^{\dot\alpha}\bar\phi(b^{+-}w_{8\,\beta\dot\alpha}-b^{++}w_{9\,\beta\dot\alpha}^{--})=0,\\&\partial^{++}\varphi_{10}^{--}+2(\mathcal{A}_{10}+\Box\bar\phi)=0.\end{aligned}\tag{B.13a}$$

$$\begin{aligned}&\partial^{++}\varphi_{2\,\alpha}^{-}+2\mathcal{A}_{2\,\alpha}^{+}+2A^{\beta\dot\alpha}\left[c_{\alpha\beta}(2b^{+-}w_{4\,\dot\alpha}^{+}+b^{++}w_{5\,\dot\alpha}^{-})+b^{++}c_\beta^\gamma w_{6\,\gamma\alpha\dot\alpha}^{-}\right]=0,\\&\partial^{++}\varphi_{3\,\alpha}^{(-3)}+\varphi_{2\,\alpha}^{-}-4\bar\phi c_{\alpha\beta}(b^{+-}\varphi_2^{-\,\beta}+b^{++}\varphi_3^{(-3)\,\beta})+2(\mathcal{A}_{3\,\alpha}^{-}-4\Psi^{-}{}_{\alpha})\\&+2A^{\beta\dot\alpha}\left[c_{\alpha\beta}(b^{+-}w_{5\,\dot\alpha}^{-}+2b^{++}w_{7\,\dot\alpha}^{(-3)})-c_\beta^\gamma b^{+-}w_{6\,\gamma\alpha\dot\alpha}^{-}\right]-8\bar\Psi^{-\,\dot\alpha}c_\alpha^\gamma(b^{+-}w_{2\,\gamma\dot\alpha}+b^{++}w_{3\,\gamma\dot\alpha}^{--})=0,\end{aligned}\tag{B.13b}$$

$$\partial^{++}\varphi_4 + 2\mathcal{A}_4^{++} + 2b^{++}c^{\alpha\beta}A_\alpha^{\dot\alpha}w_{8\,\beta\dot\alpha} = 0,$$

$$\partial^{++}\varphi_5^{--} + 2\varphi_4 - 2\bar\phi b^{++}c^{\alpha\beta}\varphi_{6\,\alpha\beta}^{--} + 2[\mathcal{A}_5 + \mathrm{i}(\partial\cdot A)] + 2c_\alpha^\beta A^{\alpha\dot\alpha}(b^{+-}w_{8\,\beta\dot\alpha} + b^{++}w_{9\,\beta\dot\alpha}^{--})$$
$$+ 4b^{++}\bar\Psi^{-}{}_{\dot\alpha}c^{\alpha\beta}w_{6\,\alpha\beta}^{-\,\dot\alpha} + 2\mathrm{i}c^{\alpha\beta}\partial_\alpha^{\dot\alpha}\bar\phi(b^{+-}w_{2\,\beta\dot\alpha} + b^{++}w_{3\,\beta\dot\alpha}^{--}) = 0,$$

$$\partial^{++}\varphi_{6\,\alpha\beta}^{--} + 4\bar\phi c_{\alpha\beta}(2b^{+-}\varphi_4 + b^{++}\varphi_5^{--}) + 4b^{++}\bar\phi c_{(\alpha}{}^{\gamma}\varphi_{6\,\beta)\gamma}^{--} + 2(\mathcal{A}_{6\,\alpha\beta} - F_{\alpha\beta})$$
$$+ 4A^{\gamma\dot\alpha}c_{\gamma(\alpha}(b^{+-}w_{8\,\beta)\dot\alpha} + b^{++}w_{9\,\beta)\dot\alpha}^{--}) - 8b^{++}\bar\Psi^{-}{}_{\dot\alpha}c_{(\alpha}^{\gamma}w_{6\,\beta)\gamma}^{-\,\dot\alpha} + 8c_{\alpha\beta}\bar\Psi^{-}{}_{\dot\alpha}(2b^{+-}w_4^{+\,\dot\alpha} + b^{++}w_5^{-\,\dot\alpha})$$
$$- 4\mathrm{i}c_{(\alpha}^{\gamma}\partial_{\beta)}^{\dot\alpha}\bar\phi(b^{+-}w_{2\,\gamma\dot\alpha} + b^{++}w_{3\,\gamma\dot\alpha}^{--}) = 0,$$

$$\partial^{++}\varphi_7^{(-4)} + \varphi_5^{--} + 2\bar\phi b^{+-}c^{\alpha\beta}\varphi_{6\,\alpha\beta}^{--} + 2(\mathcal{A}_7^{--} - 3D^{--}) + 2b^{+-}c^{\alpha\beta}(A_\alpha^{\dot\alpha}w_{9\,\beta\dot\alpha}^{--} + 2\bar\Psi^{-\dot\alpha}w_{6\,\alpha\beta\dot\alpha}^{-})$$
$$- 2\mathrm{i}c^{\alpha\beta}\partial_\alpha^{\dot\alpha}\bar\phi(b^{--}w_{2\,\beta\dot\alpha} + b^{+-}w_{3\,\beta\dot\alpha}^{--}) = 0, \tag{B.13c}$$

$$\partial^{++}\varphi_{8\,\alpha}^{-} - 4\bar\phi b^{++}c_{\alpha\beta}\varphi_8^{-\,\beta} + 2\mathcal{A}_{8\,\alpha}^{+} - 8c_\alpha^\beta b^{++}\bar\Psi^{-\dot\alpha}w_{8\,\beta\dot\alpha} + 2\mathrm{i}c_\alpha^\beta\partial_\beta^{\dot\alpha}\bar\phi(2b^{+-}w_{4\,\dot\alpha}^{+} + b^{++}w_{5\,\dot\alpha}^{-})$$
$$- 2\mathrm{i}c_\alpha^\beta\partial^{\gamma\dot\alpha}\bar\phi b^{++}w_{6\,\beta\gamma\dot\alpha}^{-} = 0,$$

$$\partial^{++}\varphi_{9\,\alpha}^{(-3)} - (\varepsilon_{\alpha\beta} - 4\bar\phi b^{+-}c_{\alpha\beta})\varphi_8^{-\,\beta} + 2(\mathcal{A}_{9\,\alpha}^{-} - 4\mathrm{i}\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha}) - 8b^{+-}c_{\alpha\beta}\bar\Psi^{-}{}_{\dot\alpha}w_8^{\beta\dot\alpha}$$
$$+ 2\mathrm{i}b^{+-}c^{\beta\gamma}(\partial_\beta^{\dot\alpha}\bar\phi w_{6\,\alpha\gamma\dot\alpha}^{-} - \partial_\alpha^{\dot\alpha}\bar\phi w_{6\,\beta\gamma\dot\alpha}^{-}) + 2\mathrm{i}c_\alpha^\beta\partial_\beta^{\dot\alpha}\bar\phi(2b^{--}w_{4\,\dot\alpha}^{+} + b^{+-}w_{5\,\dot\alpha}^{-}) = 0. \tag{B.13d}$$

B.2.1 Solving for $v_{\dot\alpha}^{-}$

The general technique to solve this kind of equations has been exemplified in §5.1. By an analogous procedure and choosing suitable ansätze, one can obtain the full solution for the rest of the components of this field. The first and last equations of (B.11) directly set $w_1^{-\dot\alpha}$ and $w_{10}^{-\dot\alpha}$ to zero. The complete set of fields are given by

$$w_1^{-\,\dot\alpha} = w_{10}^{-\,\dot\alpha} = 0, \qquad w_2^{\alpha\dot\alpha} = 2\frac{\tanh X}{X}A^{\alpha\dot\alpha}, \qquad w_4^{+\,\dot\alpha} = -2\,\mathrm{sech}^2 X\bar\Psi^{+\dot\alpha} \tag{B.14a}$$

$$w_3^{--\,\alpha\dot\alpha} = \frac{\kappa^{--}}{X^2+Z^2}\left[\frac{2}{\sqrt{2c^2}}(\cos Z - \cosh X)c^{\alpha\beta} + \left(\sin Z - Z\frac{\sinh X}{X}\right)\varepsilon^{\alpha\beta}\right]2\,\mathrm{sech}\,X A_\beta^{\dot\alpha} \tag{B.14b}$$

$$w_5^{-\,\dot\alpha} = 4\cos Z\,\mathrm{sech}\,X\bar\Psi^{-\dot\alpha} + \frac{\kappa^{--}\bar\Psi^{+\dot\alpha}}{X^2+Z^2}4\,\mathrm{sech}\,X\,[Z(\mathrm{sech}\,X - \cos Z) + X\tanh X\sin Z] \tag{B.14c}$$

$$w^{-\ \dot\alpha}_{6\alpha\beta} = \sqrt{\frac{2}{c^2}}\left[-4\sin Z\,\mathrm{sech}\,X\,\bar\Psi^{-\dot\alpha} + \frac{\kappa^{--}\bar\Psi^{+\dot\alpha}}{X^2+Z^2}4\,\mathrm{sech}\,X(X\tanh X\cos Z + Z\sin Z)\right]c_{\alpha\beta} \tag{B.14d}$$

$$w^{(-3)\ \dot\alpha}_{7} = \frac{\kappa^{--}\bar\Psi^{-\dot\alpha}}{X^2+Z^2}4Z(1-\cos Z\,\mathrm{sech}\,X) - \frac{(\kappa^{--})^2\bar\Psi^{+\dot\alpha}}{(X^2+Z^2)^2}2\big[X^2\tanh^2 X \tag{B.14e}$$

$$+\,2ZX\tanh X\,\mathrm{sech}\,X\sin Z + Z^2(1+\mathrm{sech}^2 X - 2\cos Z\,\mathrm{sech}\,X)\big] \tag{B.14f}$$

$$w^{\alpha\dot\alpha}_{8} = \left[-\frac{2}{\sqrt{2c^2}}\sin Z c^{\alpha\beta} + \cos Z\varepsilon^{\alpha\beta}\right]2\mathrm{i}\,\mathrm{sech}\,X\,\partial^{\dot\alpha}_{\beta}\bar\phi \tag{B.14g}$$

$$w^{--\ \alpha\dot\alpha}_{9} = \frac{\kappa^{--}\,\mathrm{sech}\,X}{X^2+Z^2}\left[-\frac{2}{\sqrt{2c^2}}(Z\sin Z + X\sinh X)c^{\alpha\beta} + Z(\cos Z - \cosh X)\varepsilon^{\alpha\beta}\right]2\mathrm{i}\,\partial^{\dot\alpha}_{\beta}\bar\phi \tag{B.14h}$$

The natural splitting of this fields in the bases $c^{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ is useful to further solve the flatness equation of φ^{--} (5.6), and therefore to fix the integration constants in the determination of $\mathcal{A}$.

B.2.2 Non-singlet Q-deformed Bosonic Action in Components

The ansatz (5.20)

$$\begin{aligned}
\mathcal{A}_{6\alpha\beta} &= g_1\,F_{\alpha\beta} + g_2\,c_{\alpha\beta} + g_3\,\mathcal{F}_{\alpha\beta} + g_4\,G_{\alpha\beta} + g_5\,\mathcal{G}_{\alpha\beta},\\
\varphi^{--}_{6\alpha\beta} &= h^{--}_1\,F_{\alpha\beta} + h^{--}_2\,c_{\alpha\beta} + h^{--}_3\,\mathcal{F}_{\alpha\beta} + h^{--}_4\,G_{\alpha\beta} + h^{--}_5\,\mathcal{G}_{\alpha\beta},\\
\mathcal{A}^{ij}_4 &= \alpha_1\,D^{ij} + \alpha_2\,b^{ij} + \alpha_2\,\mathcal{D}^{ij},
\end{aligned} \tag{B.15}$$

splits naturally the curvature equations for $\mathcal{A}$ that are relevant for the bosonic action into

$$\begin{aligned}
&\partial^{++}\mathcal{A}^{++}_4 = 0\\
&\partial^{++}\mathcal{A}_5 + 2\,\mathcal{A}^{++}_4 - 2\bar\phi b^{++}\ \big[g_1\,(c\cdot F) + g_2\,c^2 + g_4\,(c\cdot G)\big] = 0\\
&\partial^{++}g_2 + 8\bar\phi b^{+-}\,\mathcal{A}^{++}_4 + 2\bar\phi b^{++}\ [g_3\,(c\cdot F) + g_5\,(c\cdot G) + 2\,\mathcal{A}_5] = 0\\
&\partial^{++}g_1 - 2\bar\phi b^{++}\,c^2\,g_3 = 0\\
&\partial^{++}g_3 + 4\bar\phi b^{++}\,g_1 = 0\\
&\partial^{++}g_4 - 2\bar\phi b^{++}\,c^2\,g_5 = 0\\
&\partial^{++}g_5 + 4\bar\phi b^{++}\,g_4 = 0\\
&\partial^{++}\mathcal{A}^{--}_7 + A_5 + 2\bar\phi b^{+-}\ \big(g_1\,(c\cdot F) + c^2\,g_2 + g_4\,(c\cdot G)\big) = 0
\end{aligned} \tag{B.16}$$

This equations are solvable up to integration constants that are fixed by (5.6), which is also split by inserting (B.15). In particular, the coefficients $(\theta^-)^2$, $\theta^{-\alpha}\theta^{+\beta}$ and $(\theta^+)^2$ of

(5.6) comprise the following coupled system of equations

$$\partial^{++}\varphi_4 + 2\mathcal{A}_4^{++} + 2b^{++}c^{\alpha}_{\beta}A_{\alpha\dot{\alpha}}w_8^{\beta\dot{\alpha}} = 0 \tag{B.17a}$$

$$\begin{aligned}\partial^{++}\varphi_5^{--} &- 2\bar{\phi}b^{++}\left[h_2^{--}c^2 + h_1^{--}(c\cdot F) + h_4^{--}(c\cdot G)\right] + 2\mathcal{A}_5 + 2\varphi_4 + 2\mathrm{i}(\partial\cdot A)\\ &- b^{++}\left[c^2\,(A\cdot\partial\bar{\phi})(\hat{w}_9^{--} - \mathrm{i}\hat{w}_3^{--}) + 2(c\cdot G)(\check{w}_9^{--} - \mathrm{i}\check{w}_3^{--})\right]\\ &+ b^{+-}\left[2(c\cdot G)(\check{w}_8 + \mathrm{i}w_2) - c^2\,(A\cdot\partial\bar{\phi})\hat{\omega}_8\right] = 0\end{aligned} \tag{B.17b}$$

$$\begin{aligned}\partial^{++}h_2^{--} &+ 2\phi b^{++}\left[2\varphi_5^{--} + h_3^{--}(c\cdot F) + h_5^{--}(c\cdot G)\right] + 2g_2 + 8\bar{\phi}b^{+-}\varphi_4\\ &- 2b^{++}\left[4(c\cdot G)\hat{\omega}_9^{--} + (A\cdot\partial\bar{\phi})(\check{w}_9^{--} + \mathrm{i}\check{w}_3^{--})\right]\\ &- 2b^{+-}\left[2(c\cdot G)\hat{w}_8 - (A\cdot\partial\bar{\phi})(\check{w}_8 - \mathrm{i}w_2)\right] = 0\end{aligned} \tag{B.17c}$$

$$\partial^{++}h_1^{--} + 2(g_1 - 1) - 2b^{++}\bar{\phi}\,c^2h_3^{--} = 0 \tag{B.17d}$$

$$\partial^{++}h_3^{--} + 2\,g_3 + 4b^{++}\bar{\phi}h_1^{--} = 0 \tag{B.17e}$$

$$\partial^{++}h_4^{--} + 2\,g_4 - 2b^{++}\bar{\phi}\,c^2h_5^{--} - 2b^{++}\,c^2(\hat{w}_9^{--} + \mathrm{i}\hat{w}_3^{--}) + 2b^{+-}\,c^2\hat{w}_8 = 0 \tag{B.17f}$$

$$\partial^{++}h_5^{--} + 2\,g_5 + 4b^{++}\bar{\phi}h_1^{--} + 4b^{++}(\check{w}_9^{--} + \mathrm{i}\check{w}_3^{--}) - 4b^{+-}(\check{w}_8 - \mathrm{i}w_2) = 0 \tag{B.17g}$$

$$\begin{aligned}\partial^{++}\varphi_7^{(-4)} &+ \varphi_5^{--} + 2\mathcal{A}_7^{--} - 6D^{--} - 2\mathrm{i}b^{--}\,(c\cdot G)w_2\\ &+ 2b^{+-}\,\left[h_2^{--}c^2 + h_1^{--}(c\cdot F) + h_4^{--}(c\cdot G)\right]\\ &+ b^{+-}\left[c^2\,(A\cdot\partial\bar{\phi})(\hat{w}_9^{--} - \mathrm{i}\hat{w}_3^{--}) + 2(c\cdot G)(\check{w}_9^{--} - \mathrm{i}\check{w}_3^{--})\right] = 0\end{aligned} \tag{B.17h}$$

where we have adopted a conventional notation for any $SU(2)$ tensor field $\Phi^{\alpha\beta}$,

$$\Phi^{\alpha\beta} = \check{\Phi}\varepsilon^{\alpha\beta} + \hat{\Phi}c^{\alpha\beta}. \tag{B.18}$$

More explicitly, the scalar functions commonly denoted by the letter w, are related to the solution of the flatness equation (5.7) found in B.2.1, as follows

$$w_2^{\alpha\dot{\alpha}} = w_2A^{\alpha\dot{\alpha}}, \quad w_3^{--\alpha\dot{\alpha}} = w_3^{--\alpha\beta}A^{\dot{\alpha}}_{\beta}, \quad w_8^{\alpha\dot{\alpha}} = w_8^{\alpha\beta}\partial^{\dot{\alpha}}_{\beta}\bar{\phi}, \quad w_9^{--\alpha\dot{\alpha}} = w_9^{--\alpha\beta}\partial^{\dot{\alpha}}_{\beta}\bar{\phi}, \tag{B.19}$$

where $w_3^{--\alpha\beta}$, $w_8^{\alpha\beta}$ and $w_9^{--\alpha\beta}$ are decomposed according to (B.18).

Equations (B.16) and (B.17) are equations of the standard form (4.32), whose solutions fully determine $\mathcal{A}_4^{ij}$, $\mathcal{A}_5$ and $\mathcal{A}_{6\alpha\beta}$. The first two are given by

$$\mathcal{A}_4^{ij} = \frac{\sigma\,b^{ij}}{\cosh^3 X} + \frac{D^{ij}}{\cosh^2 X} \tag{B.20}$$

where

$$\sigma = \frac{\sinh X}{X^2}\left[-2\mathrm{i}(c\cdot G)(\cosh X\sinh X - 2X) + (c\cdot F)\bar{\phi}\cosh X\sinh X\right], \tag{B.21}$$

and

$$\begin{aligned}\mathcal{A}_5 =& -\frac{2\sigma\, b^{+-}}{\cosh^3 X} + \frac{\sqrt{2b^2}\sin Z}{X^2\cosh^3 X}\Big(-2\mathrm{i}\cosh^2 X(\cosh X\sinh X - 1)(c\cdot G) \\ & - X^2\sinh X\sigma + \bar{\phi}\cosh^3 X\sinh X(c\cdot F)\Big) - 2\frac{D^{+-}}{\cosh X}\cos Z \\ & - \frac{\bar{\phi}\sqrt{2c^2}b^{--}\,D^{++}}{(X^2+Z^2)}[X\sinh X\sin Z + Z(1-\cosh X\cos Z)]. \end{aligned} \tag{B.22}$$

The components of $\mathcal{A}_{6\alpha\beta}$ (see (5.20)) are

$$g_1 = \left(\frac{\sinh X}{X}\,,\,0\right) \tag{B.23a}$$

$$\begin{aligned} g_2 =& -\frac{2\bar{\phi}\,b^2\left[(X^2+Z^2)\sigma + 8\bar{\phi}^2c^2\cosh X b^{--}\,D^{++}\right]}{X^2\cosh^3 X(X^2+Z^2)}\left[Z(0\,,\,2\cosh X) + (2X\sinh X\,,\,0)\right], \\ & + \sqrt{\frac{2}{c^2}}\left(0\,,\,\frac{2D^{+-}}{\cosh X} + \frac{2b^{+-}\sigma}{\cosh^2 X}\right), \end{aligned} \tag{B.23b}$$

$$g_3 = -\sqrt{\frac{2}{c^2}}\left(0\,,\,\frac{\sinh X}{X}\right), \tag{B.23c}$$

$$g_4 = \frac{2\mathrm{i}\sqrt{b^2c^2}}{X^2\cosh X}(X - \cosh X\sinh X\,,\,0) \tag{B.23d}$$

$$g_5 = \frac{2\mathrm{i}\sqrt{b^2c^2}}{X^2\cosh X}(0\,,\,\cosh X\sinh X - X) \tag{B.23e}$$

The series expansion of the solution coincides up to second order in b with previous known results

$$\sigma = 2\mathrm{i}(c\cdot G) + (c\cdot F)\bar{\phi} + X^2\left[\frac{5}{6}(c\cdot F)\bar{\phi} - \mathrm{i}(c\cdot G)\right] + \cdots \tag{B.24}$$

$$\begin{aligned} \mathcal{A}_4^{ij} =& D^{ij} + b^{ij}[2\mathrm{i}(c\cdot A\partial\bar{\phi}) + \bar{\phi}(c\cdot F)] + \cdots \qquad \mathcal{A}_5 = -2D^{+-} - 4\mathrm{i}b^{+-}(c\cdot A\partial\bar{\phi}) + \cdots \\ g_1 =& 1 + \cdots \qquad g_2 = 8\bar{\phi}(b^{+-}D^{+-} - b^{--}D^{++}) + \cdots \qquad g_3 = -4\bar{\phi}b^{+-} + \cdots \\ g_4 =& \mathrm{O}(b^2), \qquad g_5 = \mathrm{O}(b^3). \end{aligned} \tag{B.25}$$

Inserting (B.20), (B.22) and (B.23) in (5.25) and taking the harmonic integral we obtain the action as presented in (5.26).

B.2.3 The Full Deformed Action in Components

The pure bosonic coefficients where calculated in §B.2.2, where we only considered $\mathcal{A}_1$, $\mathcal{A}_4^{++}$, $\mathcal{A}_5$, $\mathcal{A}_{6\,\alpha\beta}$, $\mathcal{A}_7^{--}$ and $\mathcal{A}_{10}$. To solve the full case, including the fermions, we must take into account all components of $\mathcal{A}$

The solution of $\mathcal{A}_{2\alpha}^{+}$ and $\mathcal{A}_{3\alpha}^{-}$.

The series solution of the subsystem of equations including $\mathcal{A}_{2\alpha}^{+}$, $\mathcal{A}_{3\alpha}^{-}$, $\varphi_{2\alpha}^{-}$, $\varphi_{3\alpha}^{(-3)}$ gives

$$w^0_{2\alpha\dot\alpha} = 2A_{\alpha\dot\alpha}, \quad w^{0+}_{4\dot\alpha} = -2\bar\Psi^+_{\dot\alpha}, \quad w^{0-}_{5\dot\alpha} = 4\bar\Psi^-_{\dot\alpha}, \quad w^0_{8\alpha\dot\alpha} = 2\mathrm{i}\partial_{\alpha\dot\alpha}\bar\phi, \tag{B.26}$$

$$w^{1--}_{3\,\alpha\dot\alpha} = -4\bar\phi c_\alpha^\beta A_{\beta\dot\alpha} b^{--}, \qquad w^{1-\dot\alpha}_{6\alpha\beta} = 16\bar\phi c_{\alpha\beta}\bar\Psi^{i\dot\alpha} b_i^-, \tag{B.27}$$

$$w^1_{8\alpha\dot\alpha} = -8\mathrm{i}\bar\phi c_\alpha^\beta \partial_{\beta\dot\alpha}\bar\phi b^{+-}, \qquad w^{1--}_{9\alpha\dot\alpha} = -8\mathrm{i}\bar\phi c_\alpha^\beta \partial_{\beta\dot\alpha}\bar\phi b^{--}. \tag{B.28}$$

$$\mathcal{A}^{0+}_{2\alpha} = -2\Psi^+_\alpha, \qquad \mathcal{A}^{0-}_{3\alpha} = 2\Psi^-_\alpha, \qquad \varphi^{0-}_{2\alpha} = 4\Psi^-_\alpha, \qquad \varphi^{0(-3)}_{3\alpha} = 0. \tag{B.29}$$

$$\begin{aligned}
\mathcal{A}^{1+}_{2\alpha} &= +\frac{8}{3} c_{\alpha\beta} A^{\beta\dot\alpha}\bar\Psi^i_{\dot\alpha} b_i^+ - \frac{8}{3}\bar\phi c_{\alpha\beta}\Psi^{i\beta} b_i^+ \\
\mathcal{A}^{1-}_{3\alpha} &= -\frac{8}{3} c_{\alpha\beta} A^{\beta\dot\alpha}\bar\Psi^i_{\dot\alpha} b_i^- - \frac{16}{3}\bar\phi c_{\alpha\beta}\Psi^{i\beta} b_i^- \\
\varphi^{1-}_{2\alpha} &= +\frac{8}{3} c_{\alpha\beta} A^{\beta\dot\alpha}\bar\Psi^i_{\dot\alpha} b_i^- + \frac{16}{3}\bar\phi c_{\alpha\beta}\Psi^{i\beta} b_i^- \\
\varphi^{1(-3)}_{3\alpha} &= +\frac{8}{3} c_{\alpha\beta} A^{\beta\dot\alpha} b^{--}\bar\Psi^-_{\dot\alpha} + \frac{16}{3}\bar\phi c_{\alpha\beta} b^{--}\Psi^{-\beta} \\
\mathcal{A}^{2+}_{2\alpha} &= +\frac{8}{3}\bar\phi\, b^2 c^2 \left(2A^{\dot\alpha}_\alpha\bar\Psi^+_{\dot\alpha} + \bar\phi\Psi^+_\alpha\right) \\
\mathcal{A}^{2-}_{3\alpha} &= -\frac{8}{3}\bar\phi\, b^2 c^2 A^{\dot\alpha}_\alpha\bar\Psi^-_{\dot\alpha} + 8\bar\phi^2 c^2 \Psi^i_\alpha b_i^{(+} b^{--)} \\
\varphi^{2-}_{2\alpha} &= -\frac{16}{3}\bar\phi\, b^2 c^2 \left(A^{\dot\alpha}_\alpha\bar\Psi^-_{\dot\alpha} - \bar\phi\Psi^-_\alpha\right) + 8\bar\phi\, c^2 A^{\dot\alpha}_\alpha\bar\Psi^i_{\dot\alpha} b_i^{(+} b^{--)} \\
\varphi^{2(-3)}_{3\alpha} &= -\frac{16}{3}\bar\phi\, c^2 A^{\dot\alpha}_\alpha\left[\bar\Psi^{(+}_{\dot\alpha} b^{--} b^{--)} + \frac{13}{10}\bar\Psi^i_{\dot\alpha} b_i^{(-} b^{--)}\right] \\
&\quad - \frac{8}{3}\bar\phi^2 c^2\left[\Psi^{(+}_\alpha b^{--} b^{--)} + \frac{14}{5}\Psi^i_\alpha b_i^{(-} b^{--)}\right]
\end{aligned} \tag{B.30}$$

This inspires the ansatz for a general solution which turns out to be

$$\mathcal{A}^+_{2\alpha} = A_{\alpha\beta}\Psi^{+\beta} + B_{\alpha\beta}A^{\beta\dot\alpha}\bar\Psi^+_{\dot\alpha} + F^{++}_{\alpha\beta}\Psi^{-\beta} + G^{++}_{\alpha\beta}A^{\beta\dot\alpha}\bar\Psi^-_{\dot\alpha} \tag{B.31a}$$

$$\mathcal{A}^-_{3\alpha} = H_{\alpha\beta}\Psi^{-\beta} + J_{\alpha\beta}A^{\beta\dot\alpha}\bar\Psi^-_{\dot\alpha} + H^{--}_{\alpha\beta}\Psi^{+\beta} + J^{--}_{\alpha\beta}A^{\beta\dot\alpha}\bar\Psi^+_{\dot\alpha} \tag{B.31b}$$

from (B.9b) we immediately have

$$\partial^{++}A_{\alpha\beta} + F^{++}_{\alpha\beta} = 0, \qquad \partial^{++}B_{\alpha\beta} + G^{++}_{\alpha\beta} = 0 \tag{B.32}$$

$$\partial^{++}F^{++}_{\alpha\beta} = 0, \qquad \partial^{++}G^{++}_{\alpha\beta} = 0 \tag{B.33}$$

the general solution for this system is

$$F^{++}_{\alpha\beta} = F^{1}_{\alpha\beta}b^{++}, \qquad G^{++}_{\alpha\beta} = G^{1}_{\alpha\beta}b^{++} \tag{B.34}$$

$$A_{\alpha\beta} = F^{0}_{\alpha\beta} - F^{1}_{\alpha\beta}\, b^{+-}, \qquad B_{\alpha\beta} = G^{0}_{\alpha\beta} - G^{1}_{\alpha\beta}\, b^{+-} \tag{B.35}$$

where $F^0_{\alpha\beta}$, $F^1_{\alpha\beta}$, $G^0_{\alpha\beta}$ and $G^1_{\alpha\beta}$ are harmonic constants leaving (B.31a) as

$$\mathcal{A}^{+}_{2\alpha} = (F^{0}_{\alpha\beta} - F^{1}_{\alpha\beta}\, b^{+-})\Psi^{+\beta} + (G^{0}_{\alpha\beta} - G^{1}_{\alpha\beta}\, b^{+-})A^{\beta\dot\alpha}\bar\Psi^{+}_{\dot\alpha} + F^{1}_{\alpha\beta}b^{++}\Psi^{-\beta} + G^{1}_{\alpha\beta}b^{++}A^{\beta\dot\alpha}\bar\Psi^{-}_{\dot\alpha} \tag{B.36}$$

splitting the unknown functions in the basis $\hat f c^{\alpha\beta} + \check f \varepsilon^{\alpha\beta}$ and solving for them, we obtain $\mathcal{A}^{+}_{2\alpha}$

$$\hat F^0 = 0, \quad \check F^0 = 2\,\frac{\sinh X}{X\cosh X}, \quad \hat F^1 = 8\,\bar\phi\,\frac{(X\cosh X - \sinh X)}{X^3\cosh X}, \quad \check F^1 = 0, \tag{B.37}$$

$$\hat G^0 = 0, \quad \check G^0 = 4\sqrt{b^2c^2}\,\frac{(X - \cosh X\sinh X)}{X\cosh^2 X}, \quad \hat G^1 = \frac{8\,(\sinh X - X\cosh X)}{X^3\cosh X}, \quad \check G^1 = 0 \tag{B.38}$$

and for $\mathcal{A}^{-}_{3\alpha}$,

$$\hat H = -\hat F^1 b^{+-} + \left(\sqrt{\frac{2}{c^2}}\check F^0\cosh X - \sqrt{\frac{b^2}{2}}\hat F^1\right)\sin Z \sim \frac{16}{3}\bar\phi b^{+-} - \frac{4}{15}\bar\phi X^2 b^{+-} + \mathrm{O}(b^4) \tag{B.39a}$$

$$\check H = \left(\hat F^1\sqrt{\frac{c^2}{2}}X\sinh X - \check F^0\cosh X\right)\cos Z \sim -2 + 8\bar\phi^2c^2(b^{+-})^2 + \mathrm{O}(b^3) \tag{B.39b}$$

$$\hat J = -\hat G^1 b^{+-} + \left(\sqrt{\frac{2}{c^2}}\check G^0\cosh X - \sqrt{\frac{b^2}{2}}\hat G^1\right)\sin Z \sim \frac{8}{3}b^{+-} - \frac{42}{15}X^2 b^{+-} + \mathrm{O}(b^4) \tag{B.39c}$$

$$\check J = \left(\hat G^1\sqrt{\frac{c^2}{2}}X\sinh X - \check G^0\cosh X\right)\cos Z \sim \frac{4}{3}\bar\phi b^2c^2 + \mathrm{O}(b^3) \tag{B.39d}$$

$$\hat{H}^{--} = \frac{\kappa^{--}}{X^2+Z^2}\Bigg\{X^2\left(\hat{F}^1\cosh X - \sqrt{\frac{2}{c^2}}\check{F}^0\frac{\sinh X}{X}\right)\cos Z + \hat{F}^1 Z^2$$

$$+ Z\left(\hat{F}^1 X\sinh X - \sqrt{\frac{2}{c^2}}\check{F}^0\cosh X\right)\sin Z\Bigg\} \sim -\frac{16}{3}\bar{\phi}b^{--} + \mathrm{O}(b^2) \qquad \text{(B.40a)}$$

$$\check{H}^{--} = \frac{\kappa^{--}}{X^2+Z^2}\Bigg\{X^2\left(\sqrt{\frac{2}{c^2}}\hat{F}^1\cosh X - \check{F}^0\frac{\sinh X}{X}\right)\sin Z - \hat{F}^0 Z$$

$$+ Z\left(\hat{F}^0\cosh X - \sqrt{\frac{2}{c^2}}\check{F}^1 X^2\sinh X\right)\cos Z\Bigg\} \sim -8\bar{\phi}^2c^2b^{+-}b^{--} + \mathrm{O}(b^3) \qquad \text{(B.40b)}$$

$$\hat{J}^{--} = \frac{\kappa^{--}}{X^2+Z^2}\Bigg\{X^2\left(\hat{G}^1\cosh X - \sqrt{\frac{2}{c^2}}\check{G}^0\frac{\sinh X}{X}\right)\cos Z + \hat{G}^1 Z^2$$

$$+ Z\left(\hat{G}^1 X\sinh X - \sqrt{\frac{2}{c^2}}\check{G}^0\cosh X\right)\sin Z\Bigg\} \sim -\frac{8}{3}\bar{\phi}b^{--} + \mathrm{O}(b^2) \qquad \text{(B.40c)}$$

$$\check{J}^{--} = \frac{\kappa^{--}}{X^2+Z^2}\Bigg\{X^2\left(\sqrt{\frac{2}{c^2}}\hat{G}^1\cosh X - \check{G}^0\frac{\sinh X}{X}\right)\sin Z - \hat{G}^0 Z$$

$$+ Z\left(\hat{G}^0\cosh X - \sqrt{\frac{2}{c^2}}\check{G}^1 X^2\sinh X\right)\cos Z\Bigg\} \sim \frac{32}{9}X^2\bar{\phi}\,c^2b^{+-}b^{--} + \mathrm{O}(b^5) \qquad \text{(B.40d)}$$

Solution of the fermionic contribution to $\mathcal{A}_4^{++}$, $\mathcal{A}_5$, $\mathcal{A}_{6\,\alpha\beta}$, $\mathcal{A}_{8\alpha}^{+}$ and $\mathcal{A}_{9\alpha}^{-}$.

The procedure to obtain the rest of the components is similar than the one in last subsection, therefore we report here just the results. We name $\mathcal{A}_{4f}^{++}$, $\mathcal{A}_{5f}$, $\mathcal{A}_{6f\,\alpha\beta}$ the fermionic part of the quantities whose bosonic part we know exactly from (B.20), (B.22) and (B.23). First we define some functions frequently appearing along the calculations

$$\alpha = \frac{(X+\cosh X\sinh X)\tanh X}{X^2\cosh X} \qquad \text{(B.41a)}$$

$$\beta = \frac{X - 2X\cosh^2 X + \cosh X\sinh X}{X^2\cosh^3 X} \qquad \text{(B.41b)}$$

$$\gamma = \frac{3X - 4X\cosh^2 X + 3\cosh X\sinh X - 2\cosh^3 X\sinh X}{X^2\cosh^4} \qquad \text{(B.41c)}$$

With this we can write

$$
\begin{aligned}
\mathcal{A}_{4f}^{++} =& 2c^2 b^{++} b_{ij} \bar\Psi^i \bar\Psi^j \frac{\gamma}{X} - \sqrt{b^2c^2}\bar\Psi^+\bar\Psi^+ \frac{\beta}{\cosh X} && \text{(B.42a)}\\
\mathcal{A}_5 =& 2\sqrt{2c^2} b^{++}\bar\Psi^-\bar\Psi^- \alpha \sin Z - \sqrt{2c^2} b_{ij}\bar\Psi^i\bar\Psi^j \gamma \sinh X \sin Z \\
&+ \bar\Psi^+\bar\Psi^- \Big\{ 4\sqrt{2c^2} b^{+-}\alpha \sin Z + 2\sqrt{b^2\,c^2}\beta \cos Z \Big\} \\
&+ \bar\Psi^+\bar\Psi^+ \frac{2\sqrt{2c^2}b^{--}}{X^2+Z^2} \Big\{ X^2\,(\beta-\alpha)\sin Z - 8\bar\phi\, c^2 (b^{+-})^2 \alpha \sin Z - 2\sqrt{2c^2} b^{+-}\beta \cos Z \\
&+ 2\sqrt{2c^2} b^{+-}\frac{\beta X}{\cosh X} \Big\} - 4\bar\phi\, c^2 b^{+-} b_{ij}\bar\Psi^i\bar\Psi^j \gamma && \text{(B.42b)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{6f\;\alpha\beta} =& \Big\{ \bar\Psi^+\bar\Psi^+ \frac{b^{--}}{X^2+Z^2}\left[4\beta X Z \sin Z + 4X^2(\beta \tanh X - \alpha)\cos Z - 32\bar\phi^2 (b^{+-})^2 \alpha \cos Z\right] \\
&\bar\Psi^-\bar\Psi^+ \left[-2\sqrt{2b^2}\beta \sin Z + 4\sqrt{2c^2}\alpha \cos Z \right] \\
&- 4\bar\Psi^-\bar\Psi^- b^{++}\alpha \cos Z - 2b_{ij}\bar\Psi^i\bar\Psi^j \gamma \sinh X \cos Z \Big\}\, c_{\alpha\beta} && \text{(B.42c)}
\end{aligned}
$$

The last fermionic components are $\mathcal{A}^+_{8\alpha}$ and $\mathcal{A}^-_{9\alpha}$ which also have frequent complicated expressions in common

$$
\delta = -2\cosh X + (2+X^2)\frac{\sinh X}{X}, \qquad \rho = 1 - 7\cosh^2 X + \left(\frac{3}{X} + X\right)\sinh(2X) \tag{B.43a}
$$

$$
\eta = -3 + 7\cosh^2 X - 4(2+X^2)\frac{\cosh X \sinh X}{X} + (4+X^4)\frac{\sinh^2 X}{X^2} \tag{B.43b}
$$

$$
\zeta = -5 - 2X^2 + (17+2X^2)\cosh^2 X - 9X\cosh X \sinh X + \frac{6\sinh X(\sinh X - 3X\cosh X)}{X^2} \tag{B.43c}
$$

$$
\begin{aligned}
\mathcal{A}_8^{+\alpha} =& 8\mathrm{i}\bar\phi b^{++}\frac{\delta}{\eta}\cos Z c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\Psi^{-\dot\alpha} + 2\mathrm{i}\kappa^{++}\frac{\delta}{\eta}\sin Z \partial^\alpha_{\dot\alpha}\bar\Psi^{-\dot\alpha} + 2\mathrm{i}\sqrt{2c^2}b^{++}\frac{\zeta \sin Z}{\eta \cosh X}\partial^\alpha_{\dot\alpha}\bar\phi\bar\Psi^{-\dot\alpha} \\
&+ \left[2\mathrm{i}\sqrt{\frac{2}{c^2}}\frac{\sin Z}{\sinh X}\left(\frac{\delta^2}{\eta} - 1\right) - 8\mathrm{i}\bar\phi\frac{\delta}{\eta}\cos Z \right] c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\Psi^{+\dot\alpha} \\
&+ \left[-2\mathrm{i}Z \sin Z \frac{\delta}{\eta} + \frac{2\mathrm{i}\cos Z}{\sinh X}\left(1 - \frac{\delta^2}{\eta}\right)\right] \partial^\alpha_{\dot\alpha}\bar\Psi^{+\dot\alpha} + 4\mathrm{i}b^{++}\frac{\zeta\cos Z}{\eta\cosh X}c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\phi\bar\Psi^{-\dot\alpha} \\
&- \left[2\mathrm{i}\sqrt{2b^2}\left(\frac{\eta\rho - \cosh X\delta\zeta}{\eta\cosh^2 X \sinh X}\right)\sin Z + 4\mathrm{i}b^{+-}\frac{\zeta\cos Z}{\eta\cosh X}\right] c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\phi\bar\Psi^{+\dot\alpha} \\
&+ \left[2\mathrm{i}\sqrt{b^2c^2}\left(\frac{\eta\rho - \cosh X\delta\zeta}{\eta\cosh^2 X \sinh X}\right)\cos Z - 2\sqrt{2c^2}\mathrm{i}b^{+-}\frac{\zeta\sin Z}{\eta\cosh X}\right]\partial^\alpha_{\dot\alpha}\bar\phi\bar\Psi^{+\dot\alpha}
\end{aligned} \tag{B.44}
$$

$$
\begin{aligned}
\mathcal{A}_9^{-\alpha} =& 4\mathrm{i}b^{+-}\frac{\delta}{\eta}\cos Z c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\Psi^{-\dot\alpha} + 4\mathrm{i}b^{+-}\frac{\zeta\cos Z}{\cosh X\eta}c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\phi\bar\Psi^{-\dot\alpha} \\
&+ \left[2\mathrm{i}X\coth X + 2\mathrm{i}X\frac{\delta(\sinh^2 X - \cosh X\delta)}{\eta\sinh X} + 4\mathrm{i}\sqrt{2c^2}b^{+-}\frac{\delta}{\eta}\sin Z\right]\partial^\alpha_{\dot\alpha}\bar\Psi^{-\dot\alpha} \\
&+ \frac{2\mathrm{i}\sqrt{b^2c^2}}{\cosh X\sinh X}\left[\rho + \frac{(\sinh^2 X - \delta\cosh X)\zeta}{\eta} + Z\sin Z\frac{\zeta\sinh X}{X\eta}\right]\partial^\alpha_{\dot\alpha}\bar\phi\bar\Psi^{-\dot\alpha} \\
&- \frac{8\mathrm{i}\bar\phi b^{--}}{X^2+Z^2}\left[X^2\left(\frac{\delta^2 - \cosh X\delta}{\eta} - 1\right) + \frac{XZ\sin Z}{\sinh X}\left(\frac{\delta^2}{\eta} - 1\right) - Z^2\frac{\delta}{\eta}\cos Z\right]c^\alpha_\gamma\partial^\gamma_{\dot\alpha}\bar\Psi^{+\dot\alpha} \\
&+ \frac{16\mathrm{i}\bar\phi^2\,c^2b^{--}b^{+-}}{\sinh X(X^2+Z^2)}\Bigg\{\cosh X\left[\frac{\delta}{\eta}(-2X\cosh X + 2\sinh X + X^2\sinh X) - X\right] \\
&+ \frac{\delta}{\eta}\sinh X(X - Z\sin Z) + X\cos Z\left(1 - \frac{\delta^2}{\eta}\right)\Bigg\}\partial^\alpha_{\dot\alpha}\bar\Psi^{+\dot\alpha} \\
&- \frac{4\mathrm{i}b^{--}}{\cosh X(X^2+Z^2)}\Bigg\{X^2\left[\frac{\rho}{\cosh X} + (\cosh X - \delta)\frac{\zeta}{\eta}\right] + Z^2\cos Z\frac{\zeta}{\eta} \\
&+ \frac{XZ\sin Z}{\cosh X\sinh X}\left(1 + \cosh X\delta\frac{\zeta}{\eta}\right)\Bigg\}c^{\alpha\gamma}\partial_{\gamma\dot\alpha}\bar\phi\bar\Psi^{+\dot\alpha} \\
&+ \frac{8\mathrm{i}\bar\phi b^{--}b^{+-}}{X^2+Z^2}\Bigg\{\frac{\zeta}{\eta}\left(2 + X^2 - 2X\coth X - X\frac{\zeta}{\eta}\tanh X\right) - \frac{X\rho}{\cosh X\sinh X} \\
&- Z\frac{\zeta\sin Z}{\cosh X\eta} + \left(\frac{\rho}{(\cosh X)^2\sinh X} - \frac{X\delta\zeta}{\cosh X\sinh X\eta}\right)\Bigg\}\partial^\alpha_{\dot\alpha}\bar\phi\bar\Psi^{+\dot\alpha}
\end{aligned}
\tag{B.45}
$$

B.3 Residual Supersymmetry transformations

B.3.1 The Graßmann Sector

We start from (5.53)

$$
\begin{aligned}
\partial^{++}f^{-\alpha} + 4\hat C^{++\alpha\beta}\bar\phi f^-_\beta + 8\epsilon^-_\beta(\bar\phi)^2\hat C^{++\alpha\beta} =& 0, && \text{(B.46a)}\\
\partial^{++}g^-_{\dot\alpha} + 4\,A_{\alpha\dot\alpha}f^-_\beta\hat C^{++\alpha\beta} + 8\epsilon^-_\beta A_{\alpha\dot\alpha}\bar\phi\hat C^{++\alpha\beta} =& 0, && \text{(B.46b)}
\end{aligned}
$$

To solve the first of these equations we propose the following ansatz

$$
f^{-\alpha} = \epsilon^-_\beta f^{\alpha\beta} + \epsilon^+_\beta f^{--\,\alpha\beta} = \epsilon^-_\beta(c^{\alpha\beta}\hat f + \varepsilon^{\alpha\beta}\check f) + \epsilon^+_\beta(c^{\alpha\beta}\hat f^{--} + \varepsilon^{\alpha\beta}\check f^{--}). \tag{B.47}
$$

which maps the problem to the standard form (4.32),

$$\begin{aligned}
\partial^{++}\hat{f} + \frac{2}{\sqrt{2c^2}}\kappa^{++}(\check{f}+2\bar{\phi}) &= 0, && \text{(B.48a)}\\
\partial^{++}\check{f} - \frac{\sqrt{2c^2}}{2}\kappa^{++}\hat{f} &= 0, && \text{(B.48b)}\\
\partial^{++}\hat{f}^{--} + \hat{f} + \frac{2}{\sqrt{2c^2}}\kappa^{++}\check{f}^{--} &= 0, && \text{(B.48c)}\\
\partial^{++}\check{f}^{--} + \check{f} - \frac{\sqrt{2c^2}}{2}\kappa^{++}\hat{f}^{--} &= 0, && \text{(B.48d)}
\end{aligned}$$

Meaning we can use again the same methods as before to solve it. That is, propose an ansatz of the form (4.54), solve the equivalent system of ODEs and impose the consistency and regularity conditions (4.52) to obtain

$$\begin{aligned}
\check{f} &= 2\bar{\phi}\left(\frac{\sinh X}{X}\cos Z - 1\right), && \text{(B.49a)}\\
\hat{f} &= -\frac{4\bar{\phi}}{\sqrt{2c^2}}\frac{\sinh X}{X}\sin Z, && \text{(B.49b)}\\
\check{f}^{--} &= \frac{\kappa^{--}}{X^2+Z^2}2\bar{\phi}\left(\cosh X\sin Z - \frac{\sinh X}{X}Z\cos Z\right), && \text{(B.49c)}\\
\hat{f}^{--} &= \frac{4\bar{\phi}}{\sqrt{2c^2}}\frac{\kappa^{--}}{X^2+Z^2}\left(\cosh X\cos Z + \frac{\sinh X}{X}Z\sin Z - 1\right). && \text{(B.49d)}
\end{aligned}$$

This solution can be written in a manifestly regular way using trigonometric identities,

$$\begin{aligned}
f^{-\alpha} =& 2\bar{\phi}\,\epsilon^{-}_{\beta}\left[\varepsilon^{\alpha\beta}\left(\frac{\sinh X}{X}\cos Z - 1\right) - c^{\alpha\beta}\sqrt{\frac{2}{c^2}}\frac{\sinh X}{X}\sin Z\right]\\
&- 4\bar{\phi}^2 b^{--}\epsilon^{+}_{\beta}\Bigg\{\varepsilon^{\alpha\beta}\sqrt{\frac{c^2}{2}}\frac{\mathrm{i}}{X}\left[\frac{\sinh(X+\mathrm{i}Z)}{X+\mathrm{i}Z} - \frac{\sinh(X-\mathrm{i}Z)}{X-\mathrm{i}Z}\right]\\
&\quad - c^{\alpha\beta}\frac{1}{X}\left(\frac{1}{X-\mathrm{i}Z}\left[\cosh(X-\mathrm{i}Z)-1\right] + \frac{1}{X+\mathrm{i}Z}\left[\cosh(X+\mathrm{i}Z)-1\right]\right)\Bigg\},
\end{aligned} \tag{B.50}$$

Repeating the procedure for the equation (B.46b), involving $\bar{g}^{-}_{\dot{\alpha}}$, we obtain directly

$$\bar{g}^{-}_{\dot{\alpha}} = A_{\alpha\dot{\alpha}}\bar{\phi}^{-1}f^{-\alpha}. \tag{B.51}$$

B.3.2 Closed Sub-algebra

We will now construct a sub-algebra of the residual supersymmetry involving the fields $\bar{\phi}$, $\bar{\Psi}^i_{\dot{\alpha}}$ and $A_{\alpha\dot{\alpha}}$, for which the deformed expressions are particularly simple even for a general deformation parameter b_{ij}. From (5.45) and the new compensating parameter, we get

$$\tilde{\delta}\bar{\Psi}^{-}_{\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}\left[2\mathrm{i}\bar{\phi}^2\epsilon^{-}_{\beta}c^{\alpha\beta}b^{+-} - \frac{\mathrm{i}}{4}f^{-\alpha} + \mathrm{i}f^{-}_{\beta}\bar{\phi}c^{\alpha\beta}b^{+-}\right] - \frac{1}{4}\partial^{++}\bar{g}^{(-3)}_{\dot{\alpha}}, \tag{B.52}$$

on which the Wess-Zumino gauge preserving condition must be imposed after introducing the following Ansatz

$$\bar{g}^{(-3)}_{\dot{\alpha}} = \epsilon^{-}_{\beta}\partial_{\alpha\dot{\alpha}}P^{--\alpha\beta} + \epsilon^{+}_{\beta}\partial_{\alpha\dot{\alpha}}P^{(-4)\alpha\beta}. \tag{B.53}$$

that turns $\tilde{\delta}\bar{\Psi}^{-}_{\dot{\alpha}}$ into

$$\tilde{\delta}\bar{\Psi}^{-}{}_{\dot{\alpha}} = \epsilon^{-}{}_{\beta}\partial_{\alpha\dot{\alpha}}K^{\alpha\beta} + \epsilon^{+}{}_{\beta}\partial_{\alpha\dot{\alpha}}K^{--\alpha\beta}, \tag{B.54}$$

where

$$K^{\alpha\beta} = 2\mathrm{i}\bar{\phi}^2 b^{+-}c^{\alpha\beta} - \frac{\mathrm{i}}{4}f^{\alpha\beta} - \mathrm{i}f^{\gamma\beta}\bar{\phi}b^{+-}c^{\alpha}_{\gamma} - \frac{1}{4}\partial^{++}P^{--\alpha\beta}, \tag{B.55a}$$

$$K^{--\alpha\beta} = -\frac{\mathrm{i}}{4}f^{--\alpha\beta} - \mathrm{i}f^{--\gamma\beta}\bar{\phi}b^{+-}c^{\alpha}_{\gamma} - \frac{1}{4}P^{--\alpha\beta} - \frac{1}{4}\partial^{++}P^{(-4)\alpha\beta}. \tag{B.55b}$$

By imposing the condition

$$\left(\partial^{++}\right)^2\delta\bar{\Psi}^{-}_{\dot{\alpha}} = 0, \tag{B.56}$$

we obtain the following set of equations

$$(\partial^{++})^2 K^{\alpha\beta} = 0, \tag{B.57a}$$

$$2\,\partial^{++}K^{\alpha\beta} + (\partial^{++})^2 K^{--\alpha\beta} = 0. \tag{B.57b}$$

Decomposing the fields in the usual way, i.e.

$$K^{\alpha\beta} = \hat{K}c^{\alpha\beta} + \check{K}\varepsilon^{\alpha\beta}, \qquad K^{--\alpha\beta} = \hat{K}^{--}c^{\alpha\beta} + \check{K}^{--}\varepsilon^{\alpha\beta}. \tag{B.58}$$

One is back to the standard form (4.32), which is solved by

$$\begin{aligned}
\check{K} &= \frac{\mathrm{i}}{2}\bar{\phi}\left(1 - \cosh X\frac{\sinh X}{X}\right), & \check{K}^{--} &= 0,\\
\hat{K} &= 2\mathrm{i}\bar{\phi}^2\left(\frac{\sinh X}{X}\right)^2 b^{+-}, & \hat{K}^{--} &= -2\mathrm{i}\bar{\phi}^2\left(\frac{\sinh X}{X}\right)^2 b^{--}.
\end{aligned} \tag{B.59}$$

Feeding this solution back to (B.54), we obtain the compensated part of the variation

$$\tilde{\delta}\bar{\Psi}^{i}_{\dot{\alpha}} = \left[4\mathrm{i}\bar{\phi}\cosh X\frac{\sinh X}{X}c^{\alpha\beta}b^{ij} - \mathrm{i}\sinh^2 X\varepsilon^{\alpha\beta}\varepsilon^{ij}\right]\epsilon_{j\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi} \tag{B.60}$$

Including the undeformed contribution, we get the full variation

$$\delta_{\epsilon}\bar{\Psi}^{i}_{\dot{\alpha}} = \left[4\mathrm{i}\bar{\phi}\cosh X\frac{\sinh X}{X}c^{\alpha\beta}b^{ij} - \mathrm{i}\cosh^2 X\varepsilon^{\alpha\beta}\varepsilon^{ij}\right]\epsilon_{j\beta}\partial_{\alpha\dot{\alpha}}\bar{\phi}, \tag{B.61}$$

Now the task is to solve

$$\partial^{++}\delta A_{\alpha\dot\alpha} = 0. \tag{B.62}$$

In this case we propose to consider the terms depending on $b^{--}_{\alpha\dot\alpha}$ in equation (5.52a) as the minimal terms needed to correct this variation, i.e.

$$\tilde\delta A_{\alpha\dot\alpha} = 24\bar\Psi^-_{\dot\alpha}\epsilon^-_\beta\,\bar\phi\,\hat C^{++\,\beta}_{\ \ \alpha} - \frac{1}{2}\partial^{++}b^{--}_{\alpha\dot\alpha} + \left[-2\bar\phi\,b^{--}_{\beta\dot\alpha} + 8\bar\Psi^-_{\dot\alpha}f^-_\beta\right]\hat C^{++\,\beta}_{\ \ \alpha}. \tag{B.63}$$

Thereupon, we use the following Ansatz

$$b^{--}_{\alpha\dot\alpha} = \epsilon^{-\beta}\bar\Psi^-_{\dot\alpha}\,A_{\alpha\beta} + \epsilon^{(-\beta}\bar\Psi^{+)}_{\dot\alpha}\,B^{--}_{\alpha\beta} + \epsilon^{+\beta}\bar\Psi^+_{\dot\alpha}\,C^{(-4)}_{\alpha\beta} + \epsilon^{k\beta}\bar\Psi_{k\dot\alpha}\,E^{--}_{\alpha\beta}. \tag{B.64}$$

We remark that this selection is also suitable to correct the variation of the remaining fields, where the compensating component $b^{--}_{\alpha\dot\alpha}$ is present. In this particular case, (B.63) acquires the form

$$\tilde\delta A_{\alpha\dot\alpha} = \epsilon^{-\beta}\bar\Psi^-_{\dot\alpha}\,F^{++}_{\alpha\beta} + \epsilon^{(-\beta}\bar\Psi^{+)}_{\dot\alpha}\,G_{\alpha\beta} + \epsilon^{+\beta}\bar\Psi^+_{\dot\alpha}\,H^{--}_{\alpha\beta} + \epsilon^{k\beta}\bar\Psi_{k\dot\alpha}\,I_{\alpha\beta}, \tag{B.65}$$

where

$$F^{++}_{\alpha\beta} = 24\bar\phi\,b^{++}\,c_{\alpha\beta} - \frac{1}{2}\partial^{++}A_{\alpha\beta} + 2b^{++}c^\gamma_\alpha\left(4f_{\gamma\beta} - \bar\phi\,A_{\gamma\beta}\right), \tag{B.66a}$$

$$G_{\alpha\beta} = -A_{\alpha\beta} - \frac{1}{2}\partial^{++}B^{--}_{\alpha\beta} + 2b^{++}c^\gamma_\alpha\left(4f^{--}_{\gamma\beta} - \bar\phi\,B^{--}_{\gamma\beta}\right), \tag{B.66b}$$

$$H^{--}_{\alpha\beta} = -\frac{1}{2}B^{--}_{\alpha\beta} - \frac{1}{2}\partial^{++}C^{(-4)}_{\alpha\beta} - 2\bar\phi b^{++}c^\gamma_\alpha C^{(-4)}_{\gamma\beta}, \tag{B.66c}$$

$$I_{\alpha\beta} = -\frac{1}{2}\partial^{++}E^{--}_{\alpha\beta} + 2b^{++}c^\gamma_\alpha\left(2f^{--}_{\gamma\beta} - \bar\phi\,E^{--}_{\gamma\beta}\right). \tag{B.66d}$$

Imposing the Wess-Zumino gauge (B.62) amounts to

$$\partial^{++}F^{++}_{\alpha\beta} = 0, \qquad 2F^{++}_{\alpha\beta} + \partial^{++}G_{\alpha\beta} = 0, \qquad G_{\alpha\beta} + \partial^{++}H^{--}_{\alpha\beta} = 0, \qquad \partial^{++}I_{\alpha\beta} = 0. \tag{B.67}$$

which is in general solved by

$$F^{++}_{\alpha\beta} = F^{ij}_{\alpha\beta}u^+_{(i}u^+_{j)}, \qquad G_{\alpha\beta} = -2F^{+-}_{\alpha\beta}, \qquad H^{--}_{\alpha\beta} = F^{--}_{\alpha\beta}. \tag{B.68}$$

and by taking $I^{\alpha\beta}$ to be independent of harmonics. The variation (B.63) will then be just

$$\tilde\delta A_{\alpha\dot\alpha} = \epsilon^{-\beta}\bar\Psi^-_{\dot\alpha}F^{++}_{\alpha\beta} - 2\epsilon^{(-\beta}\bar\Psi^{+)}_{\dot\alpha}\,F^{+-}_{\alpha\beta} + \epsilon^{+\beta}\bar\Psi^+_{\dot\alpha}\,F^{--}_{\alpha\beta} + \epsilon^{k\beta}\bar\Psi_{k\dot\alpha}\,I_{\alpha\beta}, \tag{B.69}$$

which is independent of the harmonic variables, as can be seen after using the reduction identities (2.24),

$$\tilde\delta A_{\alpha\dot\alpha} = \left(F_{ij\,\alpha\beta} + \varepsilon_{ij}I_{\alpha\beta}\right)\epsilon^{i\beta}\bar\Psi^j_{\dot\alpha}. \tag{B.70}$$

To find the explicit form of this variation we just have to solve for $F_{ij\,\alpha\beta}$ and $I_{\alpha\beta}$. Let us tackle the latter, by replacing (B.66d) into its corresponding equation in (B.67)

$$(\partial^{++})^2 E^{--}_{\alpha\beta} - 4b^{++}c_\alpha^\gamma \partial^{++}\left(2f^{--}_{\gamma\beta} - \bar{\phi}\, E^{--}_{\gamma\beta}\right) = 0. \tag{B.71}$$

Splitting in the usual way $E^{--\alpha\beta} = \hat{E}^{--}c^{\alpha\beta} + \check{E}^{--}\varepsilon^{\alpha\beta}$, the equations to solve are of the standard form

$$(\partial^{++})^2\check{E}^{--} - \frac{\sqrt{2c^2}}{2}\kappa^{++}\partial^{++}(\hat{E}^{--} - 2\bar{\phi}^{-1}\hat{f}^{--}) \;=\; 0 \tag{B.72a}$$

$$(\partial^{++})^2\hat{E}^{--} + \frac{2}{\sqrt{2c^2}}\kappa^{++}\partial^{++}(\check{E}^{--} - 2\bar{\phi}^{-1}\check{f}^{--}) \;=\; 0. \tag{B.72b}$$

Reinserting their solution on (B.66d) we obtain

$$I_{\alpha\beta} = -2(1 - X\coth X)\varepsilon_{\alpha\beta}. \tag{B.73}$$

The same procedure can be used for the other equations, we solve

$$\begin{aligned} \hat{F}^{++} &= 24\bar{\phi}b^{++} - \frac{1}{2}\partial^{++}\hat{A} + 2b^{++}(4\check{f} - \bar{\phi}\check{A}), \\ \check{F}^{++} &= -\frac{1}{2}\partial^{++}\check{A} - c^2b^{++}(4\hat{f} - \bar{\phi}\hat{A}), \end{aligned} \tag{B.74}$$

$$\begin{aligned} -2\hat{F}^{+-} &= -\hat{A} - \frac{1}{2}\partial^{++}\hat{B}^{--} + 2b^{++}(4\check{f}^{--} - \bar{\phi}\check{B}^{--}), \\ -2\check{F}^{+-} &= -\check{A} - \frac{1}{2}\partial^{++}\check{B}^{--} - c^2b^{++}(4\hat{f}^{--} - \bar{\phi}\hat{B}^{--}), \end{aligned} \tag{B.75}$$

$$\begin{aligned} \hat{F}^{--} &= -\frac{1}{2}\hat{B}^{--} - \frac{1}{2}\partial^{++}\hat{C}^{(-4)} - 2\bar{\phi}b^{++}\check{C}^{(-4)}, \\ \check{F}^{--} &= -\frac{1}{2}\check{B}^{--} - \frac{1}{2}\partial^{++}\check{C}^{(-4)} + c^2\bar{\phi}b^{++}\hat{C}^{(-4)}, \end{aligned} \tag{B.76}$$

for $A_{\alpha\beta}$, $B^{--}_{\alpha\beta}$ and $C^{(-4)}_{\alpha\beta}$, imposing the usual consistency and regularity conditions to get

$$F^{ij}_{\alpha\beta} = 8\bar{\phi}\, b^{ij}c_{\alpha\beta}. \tag{B.77}$$

That is,

$$\tilde{\delta}A_{\alpha\dot{\alpha}} = \left[8\bar{\phi}\, b_{ij}c_{\alpha\beta} - 2(1 - X\coth X)\varepsilon_{ij}\varepsilon_{\alpha\beta}\right]\epsilon^{i\beta}\bar{\Psi}^j_{\dot{\alpha}}. \tag{B.78}$$

adding the undeformed contribution we obtain the full variation of the field

$$\delta A_{\alpha\dot{\alpha}} = \left[8\bar{\phi}\, b_{ij}c_{\alpha\beta} + 2X\coth X\varepsilon_{ij}\varepsilon_{\alpha\beta}\right]\epsilon^{i\beta}\bar{\Psi}^j_{\dot{\alpha}}. \tag{B.79}$$

Which together with $\delta\bar{\Psi}^i_\alpha$ forms a closed algebra

$$[\delta_\eta, \delta_\epsilon]\, A_{\alpha\dot{\alpha}} = 2\mathrm{i}(\epsilon\cdot\eta)X\coth X\partial_{\alpha\dot{\alpha}}\bar{\phi}. \tag{B.80}$$

Remember that the residual gauge transformation is

$$\delta_r A_{\alpha\dot{\alpha}} = X\coth X\partial_{\alpha\dot{\alpha}}a. \tag{B.81}$$

B.3.3 Supersymmetry Enhancement and Breaking

In what follows we will concentrate on the supersymmetry variations under which the actions (5.28) and (5.35) are invariant. Taking advantage of the approach we have developed, it will be enough for both cases to limit the extent of the solution as to include only up to second order terms in the variable X. As we have seen this kind of calculations time and again, we will restrict the discussion to proposing the ansatz needed to put the field compensating equations into the standard form and then simply state the series solution. For $D^{ij}u^{--}_{ij}$, for instance we start from

$$\begin{aligned}\tilde{\delta}D^{ij}u^{--}_{ij} =& -8\mathrm{i}\partial_{\alpha\dot\alpha}\left[\bar\phi\epsilon^{-}_{\beta}\bar\Psi^{-\dot\alpha}\right]b^{+-}c^{\alpha\beta}-\frac{1}{3}\partial^{++}X^{(-4)}+\frac{\mathrm{i}}{6}\partial^{\alpha\dot\alpha}b^{--}_{\alpha\dot\alpha}\\ &-\frac{1}{3}\mathrm{i}\partial_{\alpha\dot\alpha}\left[2\,\bar\phi\, b^{--\dot\alpha}_{\gamma}+8f^{-}_{\gamma}\bar\Psi^{-\dot\alpha}\right]b^{+-}c^{\gamma\alpha}.\end{aligned} \tag{B.82}$$

The proper Ansatz for X^{-4} is

$$X^{(-4)}=\partial^{\alpha\dot\alpha}\left[e^{--\beta}_{\dot\alpha}\,J^{--}_{\alpha\beta}+e^{+-\beta}_{\dot\alpha}\,J^{(-4)}_{\alpha\beta}+e^{++\beta}_{\dot\alpha}\,J^{(-6)}_{\alpha\beta}+e^{\beta}_{\dot\alpha}\,\tilde J^{(-4)}_{\alpha\beta}\right], \tag{B.83}$$

where

$$e^{ij}_{\beta\dot\alpha}=\epsilon^{(i}_{\beta}\bar\Psi^{j)}_{\dot\alpha},\qquad e_{\beta\dot\alpha}=\epsilon^{i}_{\beta}\bar\Psi_{i\dot\alpha} \tag{B.84}$$

Introducing the known values of f^{-}_{α} and $b^{--}_{\alpha\dot\alpha}$, $\delta\, D^{--}$ takes the form

$$\tilde\delta\, D^{--}=\partial^{\alpha\dot\alpha}\left[e^{--\beta}_{\dot\alpha}\,M_{\alpha\beta}+e^{+-\beta}_{\dot\alpha}\,M^{--}_{\alpha\beta}+e^{++\beta}_{\dot\alpha}\,M^{(-4)}_{\alpha\beta}+e^{\beta}_{\dot\alpha}\,\tilde M^{--}_{\alpha\beta}\right], \tag{B.85}$$

where

$$M_{\alpha\beta} = 8\mathrm{i}\bar\phi b^{+-}c_{\alpha\beta}+\frac{\mathrm{i}}{6}A_{\alpha\beta}-\frac{2\mathrm{i}}{3}b^{+-}c^{\gamma}_{\alpha}(4f_{\gamma\beta}-\bar\phi A_{\gamma\beta})-\frac{1}{3}\partial^{++}J^{--}_{\alpha\beta}, \tag{B.86}$$

$$M^{--}_{\alpha\beta} = \frac{\mathrm{i}}{6}B^{--}_{\alpha\beta}-\frac{2\mathrm{i}}{3}b^{+-}c^{\gamma}_{\alpha}(4f^{--}_{\gamma\beta}-\bar\phi B^{--}_{\gamma\beta})-\frac{1}{3}\partial^{++}J^{(-4)}_{\alpha\beta}, \tag{B.87}$$

$$M^{(-4)}_{\alpha\beta} = \frac{\mathrm{i}}{6}C^{(-4)}_{\alpha\beta}+\frac{2\mathrm{i}}{3}\bar\phi\, b^{+-}c^{\gamma}_{\alpha}C^{(-4)}_{\gamma\beta}-\frac{1}{3}\partial^{++}J^{(-6)}_{\alpha\beta}, \tag{B.88}$$

$$\tilde M^{--}_{\alpha\beta} = \frac{\mathrm{i}}{6}E^{--}_{\alpha\beta}-\frac{2\mathrm{i}}{3}b^{+-}c^{\gamma}_{\alpha}(2f^{--}_{\gamma\beta}-\bar\phi E^{--}_{\gamma\beta})-\frac{1}{3}\partial^{++}\tilde J^{(-4)}_{\alpha\beta}. \tag{B.89}$$

From the condition $(\partial^{++})^3\delta\, D^{--}=0$ we obtain the following set of equations

$$\begin{aligned}&(\partial^{++})^3\,M_{\gamma\alpha}=0,\\ &6(\partial^{++})^2\,M_{\gamma\alpha}+(\partial^{++})^3\,M^{--}_{\gamma\alpha}=0,\\ &6\partial^{++}\,M_{\gamma\alpha}+3(\partial^{++})^2\,M^{--}_{\gamma\alpha}+(\partial^{++})^3\,M^{(-4)}_{\gamma\alpha}=0,\\ &(\partial^{++})^3\tilde M^{--}_{\gamma\alpha}=0.\end{aligned} \tag{B.90}$$

We now split $M_{\alpha\beta}=\hat M c_{\alpha\beta}-\check M\varepsilon_{\alpha\beta}$ in the standard way, to obtain

$$\tilde\delta D^{ij}=-\partial^{\alpha\dot\alpha}\Big\{\left[D_1 e^{ij}_{\beta\dot\alpha}+8\bar\phi^2c^2D_3(b\cdot e_{\beta\dot\alpha})b^{ij}\right]\delta^{\beta}_{\alpha}-4\bar\phi\left[D_2\tilde e^{ij}_{\beta\dot\alpha}\right]c^{\beta}_{\alpha}\Big\} \tag{B.91}$$

where

$$\tilde{e}^{ij}_{\alpha\dot{\alpha}} = \kappa^{(ik} e^{j)}_{k\alpha\dot{\alpha}}, \qquad \text{and} \qquad (\kappa \cdot e_{\alpha\dot{\alpha}}) = \kappa_{ij} \cdot e^{ij}_{\alpha\dot{\alpha}}. \tag{B.92}$$

After solving for D_1, D_2, D_3, the series expansion of the variation up to second order in b^{ij} is

$$\delta D^{ij} = 2\mathrm{i}\partial^{\alpha\dot{\alpha}} \left[\left(1 + \frac{1}{3} b^2 c^2 \bar{\phi}^2 \right) \epsilon^{(i}_{\alpha} \bar{\Psi}^{j)}_{\dot{\alpha}} + 4\mathrm{i}\phi\epsilon^{((k\beta} \bar{\Psi}^{i)}_{\dot{\alpha}} c_{\alpha\beta} b^{j)}_{k} - 4\mathrm{i}\bar{\phi}^2 c^2 b^{ij} \epsilon^{k}_{\alpha} \bar{\Psi}^{l}_{\dot{\alpha}} b_{kl} \right] + \mathrm{O}(b^3) \tag{B.93}$$

For the field ϕ in turn, combining the results in (5.45) with (5.52c), we have

$$\tilde{\delta}\phi = 8\left(2\Psi^{-}_{\alpha}\epsilon^{-}_{\beta}\bar{\phi} - A_{\alpha\dot{\alpha}}\bar{\Psi}^{-\dot{\alpha}}\epsilon^{-}_{\beta}\right)\hat{C}^{++\alpha\beta} - \partial^{++}g^{--} + \left(2A_{\alpha\dot{\alpha}}b^{--\,\dot{\alpha}}_{\beta} + 8\Psi^{-}_{\alpha}f^{-}_{\beta}\right)\hat{C}^{++\alpha\beta}, \tag{B.94}$$

which includes terms already present in the variation of $A_{\alpha\dot{\alpha}}$, whose u-dependence was already treated in the preceding section. It is helpful then, to simply substitute (B.63) in the equation above, and choose g^{--} to be of the form

$$g^{--} = -\frac{\bar{\phi}^{-1}}{2} A^{\alpha\dot{\alpha}} b^{--}_{\alpha\dot{\alpha}} + G^{--}, \tag{B.95}$$

where G^{--} is still an unknown function. In this way we get

$$\tilde{\delta}\phi = -\partial^{++}G^{--} - \bar{\phi}^{-1}A^{\alpha\dot{\alpha}}\tilde{\delta}A_{\alpha\dot{\alpha}} + 8b^{++}c^{\alpha\beta}(A_{\alpha\dot{\alpha}}\bar{\Psi}^{-\dot{\alpha}} + \bar{\phi}\Psi^{-}_{\alpha})(2\epsilon^{-}_{\beta} + \bar{\phi}^{-1}f^{-}_{\beta}). \tag{B.96}$$

We take now the following Ansatz for the function G^{--}

$$G^{--} = \hat{e}^{--}_{\alpha\beta}G^{\alpha\beta} + \hat{e}^{+-}_{\alpha\beta}G^{--\,\alpha\beta} + \hat{e}^{++}_{\alpha\beta}G^{(-4)\,\alpha\beta} + \hat{e}_{\alpha\beta}\tilde{G}^{--\,\alpha\beta}, \tag{B.97}$$

where

$$\begin{aligned} \hat{e}^{ij}_{\alpha\beta} &= (A_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}} + \bar{\phi}\Psi_{\alpha})^{(i}\epsilon^{j)}_{\beta}, \\ \hat{e}_{\alpha\beta} &= \left(A_{\alpha\dot{\alpha}}\bar{\Psi}^{i\dot{\alpha}} + \bar{\phi}\Psi^{i}_{\alpha}\right)\epsilon_{i\beta}. \end{aligned} \tag{B.98}$$

Inserting the Ansatz in (B.96), it takes the form

$$\tilde{\delta}\phi = \hat{e}^{--}_{\alpha\beta}N^{++\alpha\beta} + \hat{e}^{+-}_{\alpha\beta}N^{\alpha\beta} + \hat{e}^{++}_{\alpha\beta}N^{--\alpha\beta} + \hat{e}_{\alpha\beta}\tilde{N}^{\alpha\beta} - \bar{\phi}^{-1}A^{\alpha\dot{\alpha}}\tilde{\delta}A_{\alpha\dot{\alpha}}, \tag{B.99}$$

where

$$N^{++\alpha\beta} = -\partial^{++}G^{\alpha\beta} + 8b^{++}(2c^{\alpha\beta} - \bar{\phi}^{-1}c^{\alpha}_{\gamma}f^{\gamma\beta}), \tag{B.100a}$$

$$N^{\alpha\beta} = -\partial^{++}G^{--\alpha\beta} - 2G^{\alpha\beta} - 8b^{++}\bar{\phi}^{-1}c^{\alpha}_{\gamma}f^{--\gamma\beta}, \tag{B.100b}$$

$$N^{--\alpha\beta} = -\partial^{++}G^{(-4)\alpha\beta} - \partial^{++}G^{--\alpha\beta}, \tag{B.100c}$$

$$\tilde{N}^{\alpha\beta} = -\partial^{++}\tilde{G}^{--\alpha\beta} + 4b^{++}\bar{\phi}^{-1}c^{\alpha}_{\gamma}f^{--\gamma\beta}. \tag{B.100d}$$

In this case the variations $\tilde{\delta} A_{\alpha\dot{\alpha}}$ and $\tilde{\delta}\phi$ do not contain their undeformed components. As usual, we will add them later. Now, solving $\partial^{++}\delta\phi = 0$ and expanding in power series, we obtain

$$\delta\phi = \Psi^i_\alpha \epsilon^j_\beta \left[2\varepsilon^{\alpha\beta}\varepsilon_{ij} + \frac{16}{3}\bar{\phi} c^{\alpha\beta} b_{ij}\right] + A^{\dot{\alpha}}_\alpha \bar{\Psi}^i_{\dot{\alpha}} \epsilon^j_\beta \left[\frac{40}{3} c^{\alpha\beta} b_{ij}\right] + \mathrm{O}(b^3) \tag{B.101}$$

Finally, for $\Psi^{-\alpha}$ one has

$$\begin{aligned}
\tilde{\delta}\Psi^{-\alpha} =& 8\epsilon^-_\beta \left[(\bar{\Psi}^-)^2 - \bar{\phi} D^{--}\right] c^{\alpha\beta} b^{++} \\
&+ 2\mathrm{i} b^{+-} \left[\bar{\phi}(c^{\alpha\beta}(\partial\cdot A) - \mathrm{i}\mathcal{F}^{\alpha\beta})\epsilon^-_\beta - (c\cdot G)\epsilon^{-\alpha}\right] c^{\gamma\beta} b^{+-} \\
&- \frac{1}{4}\partial^{++} f^{(-3)\alpha} + \frac{\mathrm{i}}{4}\partial^{\alpha\dot{\alpha}} \bar{g}^-_{\dot{\alpha}} \\
&+ \left[-f^{(-3)}_\beta \bar{\phi} - \bar{g}^{(-3)\dot{\beta}} A_{\beta\dot{\beta}} - 3 f^-_\beta D^{--} + 2\bar{\Psi}^{-\dot{\beta}} b^{--}_{\beta\dot{\beta}}\right] b^{++} c^{\alpha\beta} \\
&+ \left[\frac{\mathrm{i}}{2} f^-_\beta \partial^{\gamma\dot{\gamma}} A_{\gamma\dot{\gamma}} - \mathrm{i}\partial_{\beta\dot{\beta}} \bar{g}^{-\dot{\beta}} \bar{\phi}\right] b^{+-} c^{\alpha\beta} \\
&+ \left[\frac{1}{2} f^-_\beta F_\gamma^{\ \alpha} - \mathrm{i} A_{\gamma\dot{\gamma}} \partial^{\dot{\gamma}}_\beta f^{-\alpha}\right] b^{+-} c^{\gamma\beta}.
\end{aligned} \tag{B.102}$$

Inserting (B.51), and (B.64) and (B.47) and using

$$f^{(-3)}_\alpha = \epsilon^{-\beta} U^{--}_{\alpha\beta} + \epsilon^{+\beta} U^{(-4)}_{\alpha\beta}, \tag{B.103}$$

the variation (B.102) acquires the form

$$\delta\Psi^{-\alpha} = \epsilon^-_\beta W^{\alpha\beta} + \epsilon^+_\beta W^{--\alpha\beta}, \tag{B.104}$$

where

$$\begin{aligned}
W^{\alpha\beta} =& \frac{1}{4}\partial^{++} U^{--\alpha\beta} - b^{++} c^\alpha_\gamma \bar{\phi} U^{--\gamma\beta} \\
&+ b^{++}(\bar{\Psi}^-\bar{\Psi}^-) 2(4c^{\alpha\beta} + c^\alpha_\gamma \mathcal{A}^{\gamma\beta}) + b^{++}(\bar{\Psi}^+\bar{\Psi}^-) c^\alpha_\gamma (B^{--\gamma\beta} - 2E^{--\gamma\beta}) \\
&+ D^{ij} u^{--}_{ij} b^{++} + (3c^\alpha_\gamma f^{\gamma\beta} - 8\bar{\phi} c^{\alpha\beta}) + \mathrm{i}(\partial\cdot A)\left[\frac{1}{4} g^{\alpha\beta} + b^{+-}(2\bar{\phi} c^{\alpha\beta} - c^\alpha_\gamma f^{\gamma\beta})\right] \\
&+ (c\cdot G)\left[\frac{\mathrm{i}}{2} b^{+-}\bar{\phi}\frac{d}{d\bar{\phi}} g^{\alpha\beta} - \mathrm{i} b^{+-}\frac{d}{d\bar{\phi}} f^{\alpha\beta} - 2\mathrm{i} b^{+-}\varepsilon^{\alpha\beta} - \frac{1}{2} b^{++}\frac{d}{d\bar{\phi}} P^{--\alpha\beta}\right] \\
&+ (A\cdot\partial\bar{\phi})\left[\frac{\mathrm{i}}{4}\frac{d}{d\bar{\phi}} g^{\alpha\beta} - \frac{\mathrm{i}}{2} b^{+-} c^\alpha_\gamma \bar{\phi}\frac{d}{d\bar{\phi}} g^{\gamma\beta} + \frac{1}{2} b^{++} c^\alpha_\gamma \frac{d}{d\bar{\phi}} P^{--\gamma\beta}\right] \\
&+ b^{+-}[-2\bar{\phi}\mathcal{F}^{\alpha\beta} + \mathcal{F}^\alpha_\gamma f^{\gamma\beta}] + \mathrm{i} b^{+-}\mathcal{G}^\alpha_\gamma \bar{\phi}\frac{d}{d\bar{\phi}} g^{\gamma\beta} + b^{++}\mathcal{G}^\alpha_\gamma \frac{d}{d\bar{\phi}} P^{--\gamma\beta},
\end{aligned} \tag{B.105}$$

and

$$
\begin{aligned}
W^{--\alpha\beta} =& \frac{1}{4}U^{--\alpha\beta} + \frac{1}{4}\partial^{++}U^{(-4)\alpha\beta} - b^{++}c^{\alpha}_{\gamma}\bar{\phi}U^{(-4)\gamma\beta} \\
&+ b^{++}c^{\alpha}_{\gamma}\left[(\bar{\Psi}^{-}\bar{\Psi}^{-})(B^{--\gamma\beta} + 2E^{--\gamma\beta}) + (\bar{\Psi}^{+}\bar{\Psi}^{-})2C^{(-4)-\gamma\beta}\right] \\
&+ 3D^{ij}u^{--}_{ij}b^{++}c^{\alpha}_{\gamma}f^{--\gamma\beta} + \frac{\mathrm{i}}{4}(\partial\cdot A)g^{--\alpha\beta} \\
&+ (c\cdot G)\left[\frac{\mathrm{i}}{2}b^{+-}\bar{\phi}\frac{d}{d\bar{\phi}}g^{--\alpha\beta} - \mathrm{i}b^{+-}\frac{d}{d\bar{\phi}}f^{--\alpha\beta} - \frac{1}{2}b^{++}\frac{d}{d\bar{\phi}}P^{(-4)\alpha\beta}\right] \\
&+ (A\cdot\partial\bar{\phi})\left[\frac{\mathrm{i}}{4}\frac{d}{d\bar{\phi}}g^{--\alpha\beta} - \frac{\mathrm{i}}{2}b^{+-}c^{\alpha}_{\gamma}\bar{\phi}\frac{d}{d\bar{\phi}}g^{--\gamma\beta} + \frac{1}{2}b^{++}c^{\alpha}_{\gamma}\frac{d}{d\bar{\phi}}P^{(-4)\gamma\beta}\right] \\
&- b^{+-}\mathcal{F}^{\alpha}_{\gamma}f^{--\gamma\beta} + \mathrm{i}b^{+-}\mathcal{G}^{\alpha}_{\gamma}\bar{\phi}\frac{d}{d\bar{\phi}}g^{--\gamma\beta} + b^{++}\mathcal{G}^{\alpha}_{\gamma}\frac{d}{d\bar{\phi}}P^{(-4)\gamma\beta},
\end{aligned}
\tag{B.106}
$$

Forgetting for a moment the complexity of the functions involved, the condition $(\partial^{++})^2\delta\Psi^{-\alpha} = 0$, amounts to

$$(\partial^{++})^2 W^{\alpha\gamma} = 0, \tag{B.107}$$

$$2\,\partial^{++}W^{\alpha\gamma} + (\partial^{++})^2 W^{--\alpha\gamma} = 0. \tag{B.108}$$

which has a simple solution

$$W^{\alpha\gamma} = p^{\alpha\gamma} + r^{\alpha\gamma}\,Z, \tag{B.109}$$

$$W^{--\alpha\gamma} = -\kappa^{--}\,r^{\alpha\gamma}, \tag{B.110}$$

for $p_{\alpha\gamma}$ and $r_{\alpha\gamma}$ independent of harmonics, thus

$$\delta\Psi^{j\alpha} = \epsilon^{j}_{\gamma}\,p^{\alpha\gamma} - \epsilon^{i}_{\gamma}\,\kappa^{j}_{i}\,r^{\alpha\gamma}. \tag{B.111}$$

Using the same methods as before, and expanding in power series one obtains

$$
\begin{aligned}
\delta\Psi^{i\alpha} =& \Bigg(-D^{ij}\epsilon^{\alpha\beta} + \Bigg\{\left(\frac{1}{2} + \frac{10}{9}b^2c^2\bar{\phi}^2\right)F^{\alpha\beta} - 2\mathrm{i}b^2c^2\bar{\phi}G^{\alpha\beta} \\
&+ \frac{2}{9}\mathrm{i}b^2c^2\bar{\phi}^2\left[8(A\cdot\partial\bar{\phi}) + \bar{\phi}(\partial\cdot A)\right]\varepsilon^{\alpha\beta} + \frac{2}{3}\Bigg[4(b\cdot\bar{\Psi}\bar{\Psi}) - 4\bar{\phi}(b\cdot D) \\
&- \frac{\sqrt{2c^2}}{3}\mathrm{i}b^2\bar{\phi}(A\cdot\partial\bar{\phi}) - \frac{5}{3}b^2\bar{\phi}^2(c\cdot F) - \frac{2}{3}\mathrm{i}b^2\bar{\phi}^2(c\cdot G)\Bigg]c^{\alpha\beta}\Bigg\}\varepsilon^{ij} \\
&+ \Bigg\{\frac{2}{3}\mathcal{F}^{\alpha\beta} + \frac{1}{3}\left[\frac{40}{3}c^2\bar{\phi}(b\cdot\bar{\Psi}\bar{\Psi}) - \frac{28}{3}c^2\bar{\phi}^2(b\cdot D) + \frac{\sqrt{2c^2}}{2}\mathrm{i}(A\cdot\partial\bar{\phi}) - 2\mathrm{i}\bar{\phi}(c\cdot G)\right]\varepsilon^{\alpha\beta} \\
&- \frac{2}{3}\mathrm{i}\left[2(A\cdot\partial\bar{\phi}) + \bar{\phi}(\partial\cdot A)\right]c^{\alpha\beta}\Bigg\}b^{ij}\Bigg)\epsilon_{j\beta} + \mathrm{O}(b^3).
\end{aligned}
\tag{B.112}
$$

Appendix C

Useful Formulæ

C.1 Harmonic integrals

In this Appendix we give explicit formulas for some harmonic integrals used throuought the calculations. As said in §2.3, integration over harmonics is fully specified by the two rules [104]

$$\int du\, 1 = 1, \qquad \int du\, u^{+\cdots+-\cdots-}_{i_1\cdots i_n j_1\cdots j_m} = 0.$$

This means, in particular, that the harmonic integral of any object of the form $\partial^{++} f^{--}$ or $\partial^{--} f^{++}$ equals to zero, i.e. one can integrate by parts. Using this property and equation

$$b^{++}b^{--} - (b^{+-})^2 = \lambda \equiv \frac{1}{2}\, b^2\,, \qquad b^2 = b^{ik} b_{ik}\,, \tag{C.1}$$

that follow from the reduction identities of harmonic superspace (2.24),it is then easy to show that

$$(b^{+-})^{2k+1} = \frac{1}{2(k+1)}\partial^{++}\left\{b^{--}\left[(b^{+-})^{2k} - \lambda(b^{+-})^{2k-2} + \cdots + (-1)^k\lambda^k\right]\right\}, \tag{C.2}$$

$$\begin{aligned}(b^{+-})^{2(k+1)} &= \frac{1}{2k+3}\partial^{++}\left\{b^{--}\left[(b^{+-})^{2k+1} - \lambda(b^{+-})^{2k-1} + \cdots + (-1)^k\lambda^k b^{+-}\right]\right\} \\ &\quad + \frac{1}{2k+3}(-1)^{k+1}\lambda^{k+1}\end{aligned} \tag{C.3}$$

Therefore any odd power of $b^{+-} = b^{(ik)}u^+_i u^-_k$ is a total harmonic derivative, and its integral vanishes. This provides a simple way of obtaining integrals of functions of b^{+-} from its Taylor expansions. Formally it is equivalent to a simple integral

$$\int du\, f(b^{+-}) = \frac{1}{2\mathrm{i}\sqrt{\lambda}}\int_{-\mathrm{i}\sqrt{\lambda}}^{\mathrm{i}\sqrt{\lambda}} dx\, f(x) \tag{C.4}$$

From which we derive

$$\int du\, Z^{2n+1} = 0, \qquad \int du\, Z^{2n} = (-1)^n \frac{X^{2n}}{2n+1} \tag{C.5}$$

$$\int du\, Z^{2n} \cos Z = (-1)^n (2n)! \left[\sum_{k=0}^{n} \frac{X^{2k}}{(2k)!} \frac{\sinh X}{X} - \sum_{k=0}^{n-1} \frac{X^{2k}}{(2k+1)!} \cosh X \right] \tag{C.6}$$

$$\int du\, Z^{2n+1} \sin Z = (-1)^n (2n+1)! \left[\sum_{k=0}^{n} \frac{X^{2k}}{(2k)!} \frac{\sinh X}{X} - \sum_{k=0}^{n} \frac{X^{2k}}{(2k+1)!} \cosh X \right] \tag{C.7}$$

Particular useful cases are

$$\int du\, \cos Z = \frac{\sinh X}{X}\,, \tag{C.8}$$

$$\int du\, Z \sin Z = \frac{\sinh X - X \cosh X}{X}\,, \tag{C.9}$$

$$\int du\, Z^2\, \cos Z = \frac{2X \cosh X - 2\, \sinh X - X^2\, \sinh X}{X}\,, \tag{C.10}$$

$$\int du\, Z^3\, \sin Z = \frac{X^3 \cosh X - 3\, X^2 \sinh X + 6\, X\, \cosh X - 6\, \sinh X}{X}\,, \tag{C.11}$$

$$\int du\, Z^4\, \cos Z = -4(6 + X^2) \cosh X + \frac{(24 + 12X^2 + X^4) \sinh X}{X}\,. \tag{C.12}$$

Another type of harmonic integrals needed to calculate the full action involve general $SU(2)$ tensors A^{ij}, B^{ij}. For a general function $f(Z)$ and its antiderivative

$$F(Z) = \int dZ\, f(Z),$$

we have

$$\int du\, A^{+-}\, f(Z) = \frac{A \cdot \kappa}{2\, X^2} \int Z\, f(Z), \tag{C.13}$$

and

$$\int du\, (A^{+-})^2\, f(Z) = \left[\frac{A^2}{2X^2} - \frac{(A \cdot \kappa)^2}{4X^4} \right] \int du\, Z\, F(Z) + \frac{(A \cdot \kappa)^2}{4X^4} \int du\, Z^2\, f(Z). \tag{C.14}$$

Using this we can show

$$\int du\, A^{+-}\, (Z)^{2n+1} = (-1)^{n+1} \frac{1}{2n+3} \frac{A \cdot \kappa}{2} X^{2n}, \qquad \int du\, A^{+-}\, (Z)^{2n} = 0 \tag{C.15}$$

and

$$\int du\, (A^{+-})^2\, (Z)^{2n} = (-1)^{n+1} \frac{[A^2\, X^2 + n\, (A \cdot \kappa)^2]}{2\, (2n+1)\, (2n+3)} X^{2n-2}, \qquad \int du\, (A^{+-})^2\, (Z)^{2n+1} = 0. \tag{C.16}$$

Some particular integrals are

$$\int du\ (A^{+-})^2\ \cos Z = \left[\frac{A^2}{2X^2} - \frac{(A\cdot\kappa)^2}{4X^4}\right]\int du\ Z\ \sin Z + \frac{(A\cdot\kappa)^2}{4X^4}\int du\ Z^2\ \cos Z \tag{C.17}$$

$$\begin{aligned}\int du\ (A^{+-})^2\ Z\sin Z = &\left[\frac{A^2}{2X^2} - \frac{(A\cdot\kappa)^2}{4X^4}\right]\int du\ \left[-Z\ \sin Z + Z^2\cos(Z)\right] \\ &- \frac{(A\cdot\kappa)^2}{4X^4}\int du\ Z^3\ \sin Z\end{aligned} \tag{C.18}$$

Integrating by parts, we also find

$$\begin{aligned}&\int du\,(A^{--})^2(\kappa^{++})^2 f(Z) = \\ &\frac{1}{2}\int du\,(A^{+-})^2\left[2(X^2+3Z^2)f(Z) + 4Z(X^2+Z^2)f'(Z) + \frac{1}{2}(X^2+Z^2)^2 f''(Z)\right]\end{aligned} \tag{C.19}$$

Due to the remarkable properties of the $SU(2)$ tensors, it is easy to show by direct matrix multiplication that

$$A^{ij}\kappa_{jk}B^{kl}\kappa_{li} = (A\cdot\kappa)(B\cdot\kappa) - X^2(A\cdot B) \tag{C.20}$$

This leads to a direct generalization of some integrals calculated before

$$\int du\ A^{+-}B^{+-}Z^{2n} = (-1)^{n+1}\frac{[(A\cdot B)\ X^2 + n\ (A\cdot\kappa)(B\cdot\kappa)]}{2\ (2n+1)\ (2n+3)}X^{2n-2}. \tag{C.21}$$

$$\begin{aligned}\int du\ A^{+-}B^{+-}\ f(Z) = &\left[\frac{(A\cdot B)}{2X^2} - \frac{(A\cdot\kappa)(B\cdot\kappa)}{4X^4}\right]\int du\ Z\ F(Z) \\ &+ \frac{(A\cdot\kappa)(B\cdot\kappa)}{4X^4}\int du\ Z^2\ f(Z)\end{aligned} \tag{C.22}$$

$$\int du\,A^{\pm\pm}B^{\mp\mp}f(Z) = \int du\,A^{+-}B^{+-}f(Z) + \frac{1}{2}(A\cdot B)\int du\,f(Z) \pm \frac{(A\cdot B\cdot\kappa)}{2X^2}\int du\,Zf(Z) \tag{C.23}$$

Where

$$(A\cdot B\cdot\kappa) \equiv A^{ik}B_k^j\kappa_{ij} \tag{C.24}$$

C.2 Symmetrized products

Some symmetrized products of $SU(2)$ tensors obtained by using the reduction identities (2.24)

$$
\begin{aligned}
b^{++}A^{+-} &= b^{(++}A^{+-)} + \frac{1}{2}b^{(+i}A_i^{+)}, & b^{--}A^{+-} &= b^{(+-}A^{--)} - \frac{1}{2}b^{(-i}A_i^{-)},\\
b^{+-}A^{++} &= b^{(++}A^{+-)} - \frac{1}{2}b^{(+i}A_i^{+)}, & b^{+-}A^{--} &= b^{(+-}A^{--)} + \frac{1}{2}b^{(-i}A_i^{-)},\\
b^{++}A^{--} &= b^{(++}A^{--)} + b^{(+i}A_i^{-)} + \frac{1}{3}(b\cdot A), & b^{(+i}A_i^{+)} &= b^{++}A^{+-} - b^{+-}A^{++},\\
b^{+-}A^{+-} &= b^{(++}A^{--)} - \frac{1}{6}(b\cdot A), & b^{(+i}A_i^{-)} &= \frac{1}{2}\left(b^{++}A^{--} - b^{--}A^{++}\right),\\
b^{--}A^{++} &= b^{(++}A^{--)} - b^{(+i}A_i^{-)} + \frac{1}{3}(b\cdot A), & b^{(-i}A_i^{-)} &= b^{+-}A^{--} - b^{--}A^{+-},
\end{aligned}
\tag{C.25}
$$

C.3 Properties of $SU(2)$ Symmetric Tensors

The conventions used in definig the $SU(2)$ tensors in (5.21), allow us to state some useful properties of this kind of objects in a very compact fashion

$$
\begin{aligned}
\mathcal{F}_{\alpha\gamma}\, c^{\gamma}_{\beta} &= \frac{1}{2}c\cdot F\, c_{\alpha\beta} - \frac{c^2}{2}\, F_{\alpha\beta}, & \mathcal{F}^2 &= \frac{1}{2}\, c^2\, F^2 - \frac{1}{2}\,(c\cdot F)^2\\
\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta} &= \frac{1}{2}c^2(F\cdot G) - \frac{1}{2}\,(c\cdot F)(c\cdot G), & \mathcal{D}^2 &= \frac{1}{2}\, b^2\, D^2 - \frac{1}{2}\,(b\cdot D)^2\\
\mathcal{F}_{\alpha\gamma}c^{\alpha\gamma} &= 0, & \mathcal{F}_{\alpha\gamma}F^{\alpha\gamma} &= 0,\\
\mathcal{G}_{\alpha\gamma}c^{\alpha\gamma} &= 0, & \mathcal{G}_{\alpha\gamma}G^{\alpha\gamma} &= 0,\\
\mathcal{D}_{ij}b^{ij} &= 0, & \mathcal{D}_{ij}D^{ij} &= 0.
\end{aligned}
\tag{C.26}
$$

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