String Field Theory: Methods And Solutions

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Abstract

We show that superstring field theories are integrable in the sense that their equations of motion can be written as compatibility conditions ("zero-curvature conditions") for certain linear equations. This makes it possible to transfer powerful solution generating techniques for integrable field theories to (open and vacuum) superstring field theories. These techniques should facilitate the task to confirm Sen's conjectures, e. g., by finding classical solutions which correspond to the closed string vacuum in open string field theory.

In a first preparatory step, one particular solution generating technique, the dressing approach, is introduced in a field theory setting. Its use is demonstrated by the construction of several new soliton-like solutions in noncommutative self-dual Yang-Mills theory in 2+2 dimensions. These are interpreted as D-branes in N=2 string theory.

In a second step, the dressing approach is then transferred to Berkovits' WZW-like string field theory. Additionally, a second method for the construction of exact classical solutions is introduced, the splitting technique. Both procedures reduce the nonpolynomial equation of motion to linear equations; solutions of the latter give rise to nonperturbative solutions to the original equations of motion. This discussion applies to N=1 and N=2 strings. In both cases, several classes of solutions are presented explicitly.

In a third step, it is shown that these ideas also carry over to cubic superstring field theory. The transition from the description of string field theory around the open string vacuum to a description around the closed string vacuum is embedded in a natural way into the framework of the dressing approach.

As a (in some respects) simplified model for a string field theory with kinetic operators which mix different world-sheet sectors, N=2 string field theory seems to be a viable candidate. We determine the integration and reflector states as well as the 3-string vertex for the world-sheet fermions in this theory. Since the fermions are of weights 0 and 1, these vertices do not coincide with those for N=1 world-sheet fermions, or ghosts. Our results have applications in Berkovits' hybrid formalism of a covariant superstring field theory in $D = 4$, in the $\eta\xi$ system from the bosonization of N=1 world-sheet ghosts and the twisted bc system used in vacuum string field theory. They pave the way for the concrete computation of solitonic solutions in nonpolynomial string field theory for N=2 strings.

Finally, several appendices on the mathematical background of the zero-curvature condition, twisted N=4 superconformal algebras, and interrelations between the field and the string field theory discussions conclude this work.

Keywords: String field theory, Tachyon condensation, D-branes

Zusammenfassung

Es wird gezeigt, dass Superstringfeldtheorien integrabel sind in dem Sinne, dass ihre Bewegungsgleichungen als Kompatibiltätsbedingungen („zero-curvature conditions“) von linearen Gleichungen aufgefaßt werden können. Somit lassen sich Lösungstechniken von integrablen Feldtheorien auf (offene und Vakuum-) Superstringfeldtheorien übertragen. Diese Techniken lassen sich einsetzen, um die Sen-Vermutungen zu bestätigen – beispielweise indem man klassische Lösungen konstruiert, die das Vakuum für geschlossene Strings aus Sicht der offenen Stringfeldtheorie beschreiben.

Vorbereitend wird dazu eine solche Lösungstechnik, die „dressing-Methode“, in der Feldtheorie eingeführt. Mit ihrer Hilfe werden verschiedene neue solitonartige Lösungen in nichtkommutativer selbstdualer Yang-Mills-Theorie in 2+2 Dimensionen berechnet. Diese werden als D-Branes in einer N=2-Stringtheorie interpretiert.

Im nächsten Schritt wird die dressing-Methode dann auf Berkovits WZW-artige Stringfeldtheorie übertragen. Zusätzlich wird eine zweite Methode für die Konstruktion von exakten klassischen Lösungen vorgestellt, die „splitting-Technik“. Beide Prozeduren reduzieren die nichtpolynomialen Bewegungsgleichungen auf lineare Gleichungen, deren Lösungen wiederum Lösungen der ursprünglichen Bewegungsgleichungen entsprechen. Dieses Verfahren wird für N=1- und N=2-Strings besprochen. Verschiedene Lösungsklassen werden für beide Fälle angegeben.

Im weiteren wird gezeigt, dass sich diese Überlegungen auch auf die kubische Superstringfeldtheorie übertragen lassen. Der Übergang von einer offenen Superstringfeldtheorie auf eine Vakuum-Superstringfeldtheorie ergibt sich hierbei in natürlicher Weise im Rahmen der dressing-Methode.

Als einfaches Modell für eine Stringfeldtheorie mit kinetischen Operatoren, die verschiedene Weltflächen-Sektoren miteinander mischen, bietet sich die N=2-Stringfeldtheorie an. Wir geben den Integrations-, den Reflektor- sowie den 3-String-Vertex für die Weltflächen-Fermionen in dieser Theorie an. Da die Fermionen konforme Gewichte 0 und 1 haben, stimmen diese nicht mit den bekannten Vertizes für N=1-Weltflächen-Fermionen oder -Geistern überein. Unsere Resultate lassen sich auch auf Berkovits Hybridformalismus für eine kovariante Superstringfeldtheorie in vier Dimensionen, auf das $\eta\xi$-System (aus der Bosonisierung der N=1-Weltflächen-Geister) und auf das getwistete bc-System in der Vakuum-Stringfeldtheorie anwenden. Mit ihrer Hilfe lassen sich solitonische Lösungen in der N=2-Stringfeldtheorie konkret berechnen.

Verschiedene Anhänge über den mathematischen Hintergrund der zero-curvature condition, getwistete N=4-superkonforme Algebren und die Beziehungen zwischen feld- und stringfeldtheoretischen Diskussionen beschließen diese Arbeit.

Schlagwörter: Stringfeldtheorie, Tachyonkondensation, D-Branes

Contents

Chapter I

Introduction

Bosonic and non-BPS D-branes carry tachyonic open string excitations on their world-volumes. Sen interpreted these modes as an indication of the instability of such brane configurations and conjectured that the potential of the tachyon field should exhibit a non-trivial minimum allowing the field to condense in a way similar to the Higgs-Kibble mechanism.[1] As the field rolls down to its minimum, the D-brane decays into closed string radiation; Sen conjectured that the negative energy density from the tachyon potential should equal the tension of the unstable brane (up to a sign) [170]. At the endpoint of this condensation process, all open strings should have vanished from the spectrum, leaving us with the closed string vacuum. Apart from that, it was postulated that lower dimensional D-branes in the world-volumes of higher dimensional ones can be interpreted as solitonic lump solutions where the tachyon field asymptotically approaches its minimal value in the directions perpendicular to the lower dimensional brane [170]. In the case of N=1 superstring theory, these conjectures have been used to extend the picture of various string dualities to non-BPS states [165, 166, 167, 168, 169].

Evidence for the above conjectures was collected first in conventional string theory. As a first-quantized on-shell theory, this is actually an inept setting, since the spacetime independent tachyon field has momentum zero and is therefore off-shell. As a candidate for an off-shell formulation of string theory, open string field theory offers the opportunity to test various aspects of Sen's conjectures. Indeed, expanding all string fields around the trivial solution to its equation of motion, string field theory describes the dynamics of open strings on (stable and unstable) D-branes in a second-quantized approach. According to Sen's conjectures, its equation of motion should possess classical solutions corresponding to the closed string vacuum and to lower dimensional D-branes. This applies to the bosonic (Witten's cubic bosonic string field theory) and to the superstring cases (Witten's cubic superstring field theory and Berkovits' nonpolynomial superstring field theory).

It is hard to guess the form of an explicit solution to the equation of motion to either of these theories, mainly for two reasons:

[1] Indeed, (part of) the $U(1)$ gauge symmetry on the worldvolumes of the D-branes is spontaneously broken by tachyon condensation.

Firstly, they are all based on a rather complicated product in the string field algebra. Witten's star product is graphically defined as a an operation which glues the left half of the first string to the right half of the second string; but unfortunately, it turns out to be quite intricate to compute the result of the product of two string fields in an oscillator basis [66, 67, 68]. A possible cure has been offered recently by the proof that the string field algebra is in fact isomorphic to a continuous product of Heisenberg algebras [45, 17, 4, 49, 18, 14, 15]. In a certain basis of this algebra, star products can be computed just like ordinary Moyal products. This makes available all the powerful computational methods developed for noncommutative field theories.

Secondly, the form of the kinetic operator in the above string field theories involves matter as well as ghost fields which renders it impossible to study the equation of motion in each sector separately. As a loophole, Rastelli, Sen, and Zwiebach proposed to consider string field theory around the tachyon vacuum and to describe D-branes from this point of view [151, 152, 154]. It could be shown that, after a singular reparametrization of the world-sheet, the kinetic operator consists solely out of ghosts [151, 80, 59, 144]. Thus, the equation of motion factorizes into a matter and a ghost part, the matter part being simply the condition that the matter string field is a projector of the string field algebra. This discovery triggered many subsequent developments in this area (which will be mentioned partially later in the text). In particular, projectors in the matter sector were determined explicitly [102] and later identified with D-brane solutions of this so-called vacuum string field theory [152]. However, solutions to this theory without the singular reparametrization or to open string field theory which describe the tachyonic ground state (the so-called closed string vacuum), are still missing. A solution generating technique could help out of this predicament.

Indeed, it was shown that nonpolynomial string field theory for N=2 strings is integrable [107] in the sense that its equation of motion derives from a system of linear equations; the same can be proven for N=1 superstring field theories. Taking advantage of this fact, one can try to bring to application the powerful solution-generating techniques available for integrable equations. The goal of this thesis is to present the proof for N=1 superstring field theories, to carry over some of these techniques to string field theory and to attempt to determine exact solutions to its equation of motion.

In a first step, one particular solution generating technique (the so-called dressing approach) is presented in a noncommutative field theory setting in chapter II. These results are mainly based on [I]. Although primarily field-theoretical, the discussion is intended to display connections with string theory and string field theory (for N=2 strings). Compared to string field theory, in the finite-dimensional field theory case it is possible to explain somewhat more stringently the mathematical background of the Lax pair method. We consider the Seiberg-Witten limit of fermionic N=2 string theory with nonvanishing B-field, which is governed by noncommutative self-dual Yang-Mills theory (ncSDYM) in 2+2 dimensions. In this chapter, we construct nonlinear soliton-like and multi-plane wave solutions of the ncSDYM equations corresponding to certain D-brane configurations by employing a solution generating technique, an extension of the so-called dressing approach. The underlying Lax pair is discussed in two different gauges, the unitary and the hermitean gauge. Several examples and applications for both situations are considered, including abelian solutions constructed from GMS-like projectors, noncommutative $U(2)$ soliton-like configurations and interacting plane waves. This concludes the field theory part of this thesis.

In chapter III we give a short introduction to string field theory. After an explanation of the algebraic basics of string field theory we introduce cubic bosonic string field theory, two versions of cubic superstring field theory, and nonpolynomial superstring field theory. This lays the foundation for a further treatment in the following chapters. In the rest of the chapter, we shortly expound upon more modern developments in this area: We present the vacuum versions of the string field theories and work out the importance of projectors in the string field algebra for vacuum string field theories as well as the dressing approach.

After this interlude we show in a second step in chapter IV that the equation of motion for Berkovits' WZW-like open (super)string field theory is integrable in the sense that it can be written as the compatibility condition ("zero-curvature condition") of some linear equations. This enables us to transfer solution-generating techniques known from field theory to superstring field theory. Employing a generalization of solution-generating techniques (the splitting and the dressing methods), we demonstrate how to construct nonperturbative classical configurations of both N=1 superstring and N=2 fermionic string field theories. With and without $u(n)$ Chan-Paton factors, various solutions of the string field equation are presented explicitly. This chapter relies to a large extent on [II].

In a third step these considerations are transferred in chapter V to cubic superstring field theory as well as cubic and nonpolynomial vacuum superstring field theories. It is shown how dressing transformations relate the linear equations in open superstring field theories to those in vacuum superstring field theories. The main references for this chapter are [III, IV].

Chapter VI is devoted to the explicit computation of the identity, the reflector, and the three-string vertex in the world-sheet fermionic part of N=2 string field theory. This defines the interaction in the fermionic sector of this theory in an oscillator language; it paves the way for a more detailed analysis of the dressing equations in nonpolynomial string field theory for N=2 strings. Since this theory shares many characteristic properties with Berkovits' superstring field theory, it is expected that scrutinizing this (in some respects simpler) theory might give a clue of how solutions in the N=1 case look like. As the world-sheet fermionic part of N=2 string field theory is isomorphic to the $\eta\xi$ system from the bosonization of the N=1 superghosts and the twisted bc ghost system in bosonic string theory, the results in this chapter find various applications also in bosonic and N=1 string field theories. This chapter relies to some extent on [V].

Outlook and conclusions are presented in chapter VII, followed by several appendices on the mathematical background of chapter II (appendix A), the conventions used for bosonic, N=1 and N=2 string theories (appendix B) and some background material on the small twisted N=4 superconformal algebra which is important for Berkovits' superstring field theories (appendix C). A theorem on cohomology of vector spaces is proven in appendix D which has several applications throughout the text. The close connection between the field and the string field theory parts is substantiated in appendix E. The bosonic Neumann coefficients and part of the overlap equations for the world-sheet fermion system is finally subject to appendix F.

Chapter II

Field theory

II.1 Introduction

In this chapter, we will introduce the methods used later on in string field theory first in a purely field theoretical setting, namely in noncommutative self-dual Yang Mills theory in 2+2 dimensions. The study of noncommutative field theory has become an important subject in modern theoretical physics. Even though the idea of noncommuting coordinates is a rather old one [174], research in this direction has been boosted only after the discovery that noncommutativity naturally emerges in string theory with a B-field background in a certain zero slope ($\alpha' \to 0$) limit [44, 163, 164]. Of special interest is the study of nonperturbative objects in its low energy field theory limits, like solitons or instantons, which may be interpreted as D-branes in the context of string theory (for a review see [134, 76, 46, 99]). The goal is to gain some insight into the nonperturbative sector of these theories.

The discovery of Ooguri and Vafa that open N=2 string theory at tree level can be identified with self-dual Yang-Mills theory [145] sparked new interest in the study of this area. That noncommutative self-dual Yang-Mills (ncSDYM) appears as the effective field theory describing the open critical N=2 string in 2+2 dimensions with nonvanishing B-field was shown later in [111, 112]. Furthermore, (commutative) self-dual Yang-Mills has been conjectured to be a universal integrable model ([181], see also [86] and references therein), meaning that all (or at least most) of the integrable equations in $d < 4$ can be obtained from the self-dual Yang-Mills equations. Therefore it is worthwhile to study the noncommutative generalization of this theory and, more specifically, plane wave and soliton-like solutions to the ncSDYM equations.

A lot of work concerning noncommutative solitons has been carried out during the past years.[1] In particular, noncommutative solitons and plane waves in an integrable $U(N)$ sigma model in 2+1 dimensions were discussed in [108, 109, 30, 189]. In this chapter we will show that all the cases studied in [108, 109, 30, 189] can be obtained from ncSDYM theory by dimensional reduction, i.e., by demanding that the solutions do not depend on one of the time directions (see

[1] See, e. g., [63, 69, 70, 148, 116, 11, 1, 77, 71, 74, 78, 62, 72, 79, 73, 47, 54, 55, 56, 118, 75, 131].

section II.4.2). The self-duality equations will be regarded as the compatibility conditions of two linear equations (Lax pair). Solutions ψ to residue equations of the latter can then be used to find solutions to the self-duality equations. By employing a solution generating technique, namely a noncommutative extension of the so-called *dressing approach* [192, 50, 182], we are able to compute the aforementioned auxiliary field ψ. Starting from a simple first order pole ansatz for ψ, one can easily construct higher order pole solutions corresponding to multi-soliton configurations.

This chapter is organized as follows: Section II.2 contains a review of some results from N=2 string theory with nonvanishing B-field, motivating the program carried out in this chapter from a string theory point of view and placing it in this context. In section II.3 we introduce our notation and conventions as well as the Moyal-Weyl map as a useful tool for later computations. After that, the dressing approach will be discussed in section II.4. We present various calculations and examples of solutions in this framework in sections II.5 and II.6. The discussion of some mathematical preliminaries like twistor spaces and the moduli space of complex structures on $\mathbb{R}^{2,2}$ is relegated into appendix A.1. An example of an abelian pseudo-instanton which is somewhat detached from the rest of this field theory part will be discussed in appendix A.2. The main reference for this chapter is [I].

II.2 Noncommutativity from string theory

N=0 and N=1 string theories. It is well known for N=0 and N=1 string theories that turning on a B-field in the presence of D-branes modifies the dynamics of open strings [164]. It alters the ordinary Neumann boundary conditions along the brane, which results in a deformation of the space-time metric $G_{\mu\nu}$ seen by open strings. Another consequence is the emergence of space-time noncommutativity in the world-volume of the brane [163],

$$[X^\mu(\tau), X^\nu(\tau)] = \mathrm{i}\theta^{\mu\nu}\,. \tag{II.1}$$

This noncommutativity pertains to the low energy field theory capturing the dynamics of open strings on the brane. In this discussion, $G^{\mu\nu}$ and $\theta^{\mu\nu}$ can be extracted from the closed string metric $g_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$ as the symmetric and antisymmetric part of

$$[(g + 2\pi\alpha' B)^{-1}]^{\mu\nu} = G^{\mu\nu} + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\,. \tag{II.2}$$

In the Seiberg-Witten limit [164]

$$\alpha' \to 0, \quad \text{keeping } G^{\mu\nu} \text{ and } \theta^{\mu\nu} \text{ fixed,} \tag{II.3}$$

open string theory reduces to noncommutative Yang-Mills.[2] The effective open string coupling G_s, which is related to the closed string coupling g_s via $G_s = g_s[(\det G/\det(g + 2\pi\alpha' B)]^{-1/2}$, in this limit reduces to

$$G_s \xrightarrow{\alpha'\to 0} \frac{g^2_{\mathrm{YM}}}{2\pi}\,. \tag{II.4}$$

[2]Alternatively, one can keep α' and $g_{\mu\nu}$ fixed and take $B \to \infty$. This formulation will be useful for the string field theory discussion in section E.2.

Note that bulk effects (due to closed string modes) only decouple from the open string modes if we take the Seiberg-Witten limit.

It is a standard result (cf. [176] and references therein) that soliton solutions in Yang-Mills (-Higgs) theory can be interpreted as lower-dimensional D-brane configurations. These induce an "electric" field $F_{\mu\nu}$ on the brane, thus the B-field in the above formulas has to be replaced by the gauge invariant quantity $\mathcal{F}_{\mu\nu} := B_{\mu\nu} + F_{\mu\nu}$. The noncommutativity parameter $\theta^{\mu\nu}$ will in general be a function of $\mathcal{F}_{\mu\nu}$. Note that, in this work, the back reaction of a nonvanishing gauge field configuration on the open string parameters will be neglected.

N=2 string theory. In the case of (critical) N=2 fermionic string theory in 2+2 dimensions, an analysis of B-field effects was carried out in [111]. In the following, we shall briefly delineate the results of this paper. In critical N=2 string theory with nonvanishing Kähler two-form field $B = (B_{\mu\nu})$, the dynamics of fields on N coincident space-time filling D-branes[3] in the Seiberg-Witten limit is governed by $U(N)$ ncSDYM in the Leznov gauge.[4] As a nontrivial check, the authors of [111] showed the vanishing of the noncommutative field-theory four-point amplitude at tree level. This is in accordance with the expectation from N=2 string theory, which features trivial n-point tree-level scattering amplitudes for $n > 3$, due to a certain kinematical identity in 2+2 dimensions. In this context it is worthwhile to emphasize two points: The failure of the Moyal-Weyl commutator to close in $su(N)$ necessitates the enlargement of the gauge group from $SU(N)$ to $U(N)$ [128]. Furthermore, to obtain ncSDYM in the Yang gauge [190], which will mostly be used in this chapter, one has to consider N=2 string theory restricted to the zero world-sheet instanton sector.

After this brief string theoretic overture, let us now turn to ncSDYM, whose nonperturbative solutions shall concern us for the rest of this chapter.

II.3 Noncommutative self-dual Yang-Mills on $\mathbb{R}^{2,2}$

II.3.1 Notation and conventions

In this chapter we will consider solutions to the self-duality equations for the noncommutative version of $U(N)$ Yang-Mills theory on the space $\mathbb{R}^{2,2}$. We choose coordinates $(x^\mu) = (x, y, \tilde{t}, -t)$ such that the metric will take the form $(\eta_{\mu\nu}) = \mathrm{diag}(+1, +1, -1, -1)$.[5]

Coordinates. The signature $(+ + - -)$ allows for two different choices of isotropic (light-like) coordinates (see appendix A.1). The set of real isotropic coordinates (suitable for the discussion

[3] Due to the absence of R-R forms in the closed string spectrum of N=2 string theory, D-branes are simply defined in parallel to bosonic string theory as submanifolds on which open strings can end.

[4] N=2 string theory with N coincident D2-branes yields an integrable modified $U(N)$ sigma model on noncommutative $\mathbb{R}^{2,1}$ [112] (generalizing the commutative case considered by Ward [182]).

[5] All conventions are chosen to match those of [108, II]. The choice $x^4 = -t$ is motivated by the fact that the hyperplane $\tilde{t} = 0$ then has the same orientation as in earlier work on self-dual Yang-Mills theory dimensionally reduced to this hyperplane.

of the unitary gauge, see section II.4.1) is

$$u := \frac{1}{2}(t+y), \qquad v := \frac{1}{2}(t-y), \tag{II.5a}$$

$$\tilde{u} := \frac{1}{2}(\tilde{t}+x), \qquad \tilde{v} := \frac{1}{2}(\tilde{t}-x)\,, \tag{II.5b}$$

giving rise to

$$\partial_u = \partial_t + \partial_y, \qquad \partial_v = \partial_t - \partial_y, \tag{II.6a}$$

$$\partial_{\tilde{u}} = \partial_{\tilde{t}} + \partial_x, \qquad \partial_{\tilde{v}} = \partial_{\tilde{t}} - \partial_x\,. \tag{II.6b}$$

For the discussion of the hermitean gauge (section II.4.3), the other choice of isotropic coordinates, namely complex ones,

$$z^1 := x + \mathrm{i}y\,, \qquad \bar{z}^1 = x - \mathrm{i}y\,, \tag{II.7a}$$

$$z^2 := \tilde{t} - \mathrm{i}t\,, \qquad \bar{z}^2 = \tilde{t} + \mathrm{i}t\,, \tag{II.7b}$$

turns out to be useful. These yield the following partial derivatives

$$\partial_{z^1} = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)\,, \qquad \partial_{\bar{z}^1} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)\,, \tag{II.8a}$$

$$\partial_{z^2} = \frac{1}{2}(\partial_{\tilde{t}} + \mathrm{i}\partial_t)\,, \qquad \partial_{\bar{z}^2} = \frac{1}{2}(\partial_{\tilde{t}} - \mathrm{i}\partial_t)\,. \tag{II.8b}$$

Star product. The multiplication law used to multiply functions is the standard Moyal-Weyl star product given by

$$(f \star g)(x) := e^{\left[\frac{\mathrm{i}}{2}(\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu})\right]} f(x)g(y)\Big|_{y=x}\,. \tag{II.9}$$

The noncommutativity of the coordinates is encoded in the usual structure of the commutator[6]

$$[x^\mu \overset{\star}{,} x^\nu] = \mathrm{i}\theta^{\mu\nu}\,. \tag{II.10}$$

As a constant antisymmetric matrix, $\theta^{\mu\nu}$ is taken to be

$$(\theta^{\mu\nu}) := \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ \theta^{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & \theta^{43} & 0 \end{pmatrix}, \tag{II.11}$$

where $\theta^{12} = -\theta^{21} =: \theta$ and $\theta^{34} = -\theta^{43} =: \widetilde{\theta}$. Without loss of generality we assume $\theta > 0$, and choose $\widetilde{\theta} \geq 0$, which, in the case $\theta = \widetilde{\theta}$, corresponds to self-dual $\theta^{\mu\nu}$.[7] Note that we are dealing with two time directions which mutually *do not* commute, but that the commutator of one temporal and one spatial coordinate still vanishes.

[6] $[x^\mu \overset{\star}{,} x^\nu] := x^\mu \star x^\nu - x^\nu \star x^\mu$.

[7] In the self-dual case, $\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}$, where $\epsilon_{1234} := 1$.

Yang-Mills theory. The action of noncommutative Yang-Mills theory on $\mathbb{R}^{2,2}$ reads

$$S_{\text{ncYM}} = -\frac{1}{2g_{\text{YM}}^2} \int \mathrm{d}^4 x \, \mathrm{tr}_{u(N)} F_{\mu\nu} \star F^{\mu\nu} \,. \tag{II.12}$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \overset{\star}{,} A_\nu]$. The self-duality equations in x^μ-coordinates are given by

$$F_{12} = F_{34}\,, \quad F_{13} = F_{24} \quad \text{and} \quad F_{14} = -F_{23}\,. \tag{II.13}$$

Due to the Bianchi identities for $F_{\mu\nu}$, each solution to (II.13) will also be a solution to the equations of motion of noncommutative Yang-Mills theory.

II.3.2 Moyal-Weyl map and operator formalism

The Moyal-Weyl map provides us with the possibility to switch between two equivalent noncommutative formalisms. The noncommutativity of the configuration space may be captured by deforming the multiplication law for functions (the Moyal-Weyl- or $\star$-product formalism), which in turn are defined over a commutative space. Equivalently, we may pass to the operator formalism, which often simplifies calculations considerably.

Fock space. In the operator formalism, the coordinates x^μ become operator-valued, thus satisfying $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}$. More specifically, the commutation relations among the coordinates $(x, y, \tilde{t}, -t)$ are:

$$\begin{aligned} \big[\hat{x}, \hat{t}\big] &= \big[\hat{y}, \hat{t}\big] = \big[\hat{x}, \hat{\tilde{t}}\,\big] = \big[\hat{y}, \hat{\tilde{t}}\,\big] = 0\,, && \text{(II.14a)} \\ \big[\hat{x}, \hat{y}\big] &= \mathrm{i}\theta \;\Rightarrow\; \big[\hat{z}^1, \hat{\bar{z}}^1\big] = 2\theta\,, && \text{(II.14b)} \\ \big[\hat{t}, \hat{\tilde{t}}\big] &= \mathrm{i}\widetilde{\theta} \;\Rightarrow\; \big[\hat{z}^2, \hat{\bar{z}}^2\big] = 2\widetilde{\theta}\,. && \text{(II.14c)} \end{aligned}$$

The last two lines lead us to construct creation and annihilation operators (for $\theta, \widetilde{\theta} > 0$):

$$\begin{aligned} a_1 &:= \frac{1}{\sqrt{2\theta}}\,\hat{z}^1, \quad a_2 := \frac{1}{\sqrt{2\widetilde{\theta}}}\,\hat{z}^2\,, && \text{(II.15a)} \\ a_1^\dagger &:= \frac{1}{\sqrt{2\theta}}\,\hat{\bar{z}}^1, \quad a_2^\dagger := \frac{1}{\sqrt{2\widetilde{\theta}}}\,\hat{\bar{z}}^2\,. && \text{(II.15b)} \end{aligned}$$

These operators act, as usual, in a Fock space $\mathcal{H}$ constructed from the action of the two creation operators $a_1^\dagger, a_2^\dagger$ on the vacuum $|0,0\rangle$. We introduce an orthonormal basis for $\mathcal{H}$, i.e., $\{|n_1, n_2\rangle; n_1, n_2 \in \mathbb{N}_0\}$ subject to

$$\begin{aligned} & N_i|n_1, n_2\rangle := a_i^\dagger a_i |n_1, n_2\rangle = n_i |n_1, n_2\rangle\,, \quad i \in \{1, 2\}\,, \\ a_1|n_1, n_2\rangle &= \sqrt{n_1}|n_1 - 1, n_2\rangle\,, \quad a_1^\dagger|n_1, n_2\rangle = \sqrt{n_1 + 1}|n_1 + 1, n_2\rangle\,, \\ a_2|n_1, n_2\rangle &= \sqrt{n_2}|n_1, n_2 - 1\rangle\,, \quad a_2^\dagger|n_1, n_2\rangle = \sqrt{n_2 + 1}|n_1, n_2 + 1\rangle\,. \end{aligned}$$

In the case $\widetilde{\theta} = 0$ we can only introduce a_1 and $a_1^\dagger$; $\mathcal{H}$ will then be a one-oscillator Fock space.

Moyal-Weyl map. It can be shown that there exists a bijective map, which maps functions $f(z^i,\bar z^i)$ (also called Weyl symbols) to operators $\hat f := O_f(a_i,a_i^\dagger)$ (cf. e.g. [76, 71]):

$$\begin{aligned} f(z^i,\bar z^i) \mapsto O_f(a_i,a_i^\dagger) &= -\int \frac{\mathrm{d}^2k_1\mathrm{d}^2k_2}{(2\pi)^4}\mathrm{d}^2z^1\mathrm{d}^2z^2 \\ &\times\ f(z^i,\bar z^i)\mathrm{e}^{-\mathrm{i}\{\bar k_1(\sqrt{2\theta}a_1-z^1)+k_1(\sqrt{2\theta}a_1^\dagger-\bar z^1)+\bar k_2(\sqrt{2\tilde\theta}a_2-z^2)+k_2(\sqrt{2\tilde\theta}a_2^\dagger-\bar z^2)\}}\,, \end{aligned} \tag{II.16}$$

where $\int \frac{\mathrm{d}^2k_1\mathrm{d}^2k_2}{(2\pi)^4}\mathrm{d}^2z^1\mathrm{d}^2z^2 := \int\frac{\mathrm{d}k_1\mathrm{d}\bar k_1}{(2\pi)^2}\mathrm{d}z^1\mathrm{d}\bar z^1\int\frac{\mathrm{d}k_2\mathrm{d}\bar k_2}{(2\pi)^2}\mathrm{d}z^2\mathrm{d}\bar z^2$. Note that this formula implies an ordering prescription, the so-called Weyl ordering. The inverse transformation is given by:

$$\begin{aligned} O_f(a_i,a_i^\dagger) &\mapsto f(z^i,\bar z^i) = 4\pi^2\theta\tilde\theta\int\frac{\mathrm{d}^2k_1\mathrm{d}^2k_2}{(2\pi)^4} \\ &\times\ \mathrm{Tr}_{\mathcal{H}}\left[O_f(a_i,a_i^\dagger)\mathrm{e}^{\mathrm{i}\{\bar k_1(\sqrt{2\theta}a_1-z^1)+k_1(\sqrt{2\theta}a_1^\dagger-\bar z^1)+\bar k_2(\sqrt{2\tilde\theta}a_2-z^2)+k_2(\sqrt{2\tilde\theta}a_2^\dagger-\bar z^2)\}}\right]. \end{aligned} \tag{II.17}$$

It is understood that, under the Moyal-Weyl map,

$$f\star g\mapsto \hat f\hat g. \tag{II.18}$$

Also, an integral $\int\mathrm{d}^4x$ over the configuration space becomes a trace $\mathrm{Tr}_{\mathcal{H}}$ over the Fock space $\mathcal{H}$ (modulo pre-factors) and derivatives are mapped to commutators, e.g.,

$$\partial_x f\mapsto\frac{\mathrm{i}}{\theta}\left[\hat y,\hat f\right],\quad \partial_{z^1}f\mapsto-\frac{1}{\sqrt{2\theta}}\left[a_1^\dagger,\hat f\right], \tag{II.19}$$

and analogously for the other possible combinations. From now on, we will work in the operator formalism; exceptions will be mentioned explicitly. In order to slenderize the notation, hats will be omitted everywhere.

II.4 Dressing approach

As explained in appendix A.1, exact solutions to the self-duality equations (II.13) can be constructed by means of an associated linear system. Solutions to this linear system will be obtained via the so-called *dressing method*. It was originally invented for commutative integrable models as a solution generating technique to construct solutions to the equations of motion (see, e.g. [192, 50, 182]). New solutions can be constructed from a simple vacuum seed solution by recursively applying a dressing transformation. It was shown in [108] that the dressing approach can easily be extended to noncommutative models. In the following we will apply such an extension of the dressing method to construct solutions for the Lax pairs related to the self-duality equations of ncYM on $\mathbb{R}^{2,2}$.

II.4.1 Unitary gauge

Lax pair. Let us start the discussion by considering the Lax pair given in terms of real isotropic coordinates [86]:

$$(\zeta\partial_{\tilde v}+\partial_u)\psi = -(\zeta A_{\tilde v}+A_u)\psi\,, \tag{II.20a}$$

$$(\zeta\partial_v-\partial_{\tilde u})\psi = -(\zeta A_v-A_{\tilde u})\psi\,, \tag{II.20b}$$

where $A = (A_\mu)$ is the antihermitean gauge potential for the self-dual field strength $F = (F_{\mu\nu})$, $\psi \in GL(N, \mathbb{C})$ and $\zeta \in \mathbb{C}P^1$ is the spectral parameter.[8] As shown in appendix A.1, ψ may be chosen to satisfy the following reality condition:

$$\psi(u, v, \tilde{u}, \tilde{v}, \zeta)[\psi(u, v, \tilde{u}, \tilde{v}, \bar{\zeta})]^\dagger = 1\,. \tag{II.21}$$

The compatibility conditions for the linear equations (II.20) are given by the self-duality equations expressed in real isotropic coordinates:

$$\begin{aligned} F_{u\tilde{u}} &= 0\,, &\text{(II.22a)}\\ F_{uv} + F_{\tilde{u}\tilde{v}} &= 0\,, &\text{(II.22b)}\\ F_{v\tilde{v}} &= 0\,. &\text{(II.22c)} \end{aligned}$$

If we require

$$\psi(u, v, \tilde{u}, \tilde{v}, \zeta \to 0) = g_1^{-1}(u, v, \tilde{u}, \tilde{v}) + O(\zeta) \tag{II.23}$$

for some $U(N)$ matrix g_1 and

$$\begin{aligned} A_u &= g_1^{-1}\partial_u g_1\,, &\text{(II.24a)}\\ A_{\tilde{u}} &= g_1^{-1}\partial_{\tilde{u}} g_1\,, &\text{(II.24b)} \end{aligned}$$

then eqs. (II.20) in the limit $\zeta \to 0$ are identically satisfied [190]. Thus, solving (II.20) (without knowing the gauge fields explicitly, simply by exploiting the asymptotics of ψ) amounts to solving (II.22a). In the limit $\zeta \to \infty$, we can read off from (II.20) that

$$\begin{aligned} A_{\tilde{v}} &= g_2^{-1}\partial_{\tilde{v}} g_2\,, &\text{(II.25a)}\\ A_v &= g_2^{-1}\partial_v g_2\,, &\text{(II.25b)} \end{aligned}$$

where $g_2^{-1} := \psi(u, v, \tilde{u}, \tilde{v}, \zeta = \infty) \in U(N)$; clearly, (II.25) solves eq. (II.22c).

Gauge fixing. Note that we can choose a gauge in which A_v and $A_{\tilde{v}}$ vanish: Consider the gauge transformation

$$\psi \mapsto \psi' := g_2\psi\,. \tag{II.26}$$

Its action on the gauge field yields

$$\begin{aligned} A_{\tilde{v}} \mapsto A'_{\tilde{v}} &= g_2 A_{\tilde{v}} g_2^{-1} + g_2\partial_{\tilde{v}} g_2^{-1} = 0\,, &\text{(II.27a)}\\ A_v \mapsto A'_v &= g_2 A_v g_2^{-1} + g_2\partial_v g_2^{-1} = 0\,; &\text{(II.27b)} \end{aligned}$$

this is equivalent to $\psi'(u, v, \tilde{u}, \tilde{v}, \zeta = \infty) = 1$. For the remaining components we find

$$\begin{aligned} A'_{\tilde{u}} &= \Omega^{-1}\partial_{\tilde{u}}\Omega\,, &\text{(II.28a)}\\ A'_u &= \Omega^{-1}\partial_u\Omega\,, &\text{(II.28b)} \end{aligned}$$

[8] For a detailed discussion concerning the appearance of the complexified gauge group, we refer to [119, 149].

with $\Omega^{-1} := g_2 g_1^{-1} = \psi'(u,v,\tilde{u},\tilde{v},\zeta=0)$ (Yang prepotential, cf. [85]). This gauge is called (unitary) Yang gauge.

The gauge-fixed linear equations read

$$(\zeta\partial_{\tilde{v}} + \partial_u)\psi' = -A'_u\psi' \,, \tag{II.29a}$$

$$(\zeta\partial_v - \partial_{\tilde{u}})\psi' = A'_{\tilde{u}}\psi' \,. \tag{II.29b}$$

Moreover, since $g_2 \in U(N)$, the reality condition (II.21) is "preserved" under (II.26):

$$\psi'(u,v,\tilde{u},\tilde{v},\zeta)[\psi'(u,v,\tilde{u},\tilde{v},\bar{\zeta})]^\dagger = g_2 g_2^\dagger = 1 \,. \tag{II.30}$$

In the following we shall omit the primes on the gauge transformed quantities. Using the above expressions (II.28) for $A'_{\tilde{u}}$ and A'_u, the remaining self-duality equation (II.22b) in this gauge takes the form

$$\partial_v(\Omega^{-1}\partial_u\Omega) + \partial_{\tilde{v}}(\Omega^{-1}\partial_{\tilde{u}}\Omega) = 0 \,. \tag{II.31}$$

Action functional. Let us introduce an antisymmetric rank two tensor $\omega^{\mu\nu}$ with components $\omega^{yt} = -\omega^{ty} = -1$, $\omega^{x\tilde{t}} = -\omega^{\tilde{t}x} = -1$. Then $\omega_{\mu\nu}$ coincides with $\bar{f}^2_{\mu\nu}$, the analogue to the 't Hooft tensor in 2+2 dimensions introduced in [86]; it is anti-self-dual. One can interpret $\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu$ as the Kähler form w.r.t. the complex structure $\tilde{J} = -\left(\begin{smallmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^{2,2}$ (σ^3 denotes the third Pauli matrix). In x^μ-coordinates, we can rewrite eq. (II.31) as

$$(\eta^{\mu\nu} - \omega^{\mu\nu})\partial_\mu(\Omega^{-1}\partial_\nu\Omega) = 0 \,. \tag{II.32}$$

In contrast to the metric $\eta_{\mu\nu}$, the Kähler form is not invariant under $SO(2,2)$ rotations; it therefore breaks the rotational invariance of the equation of motion even in the commutative case. A straightforward computation shows that this is the equation of motion for the Nair-Schiff type action [132, 120]

$$S = -\frac{1}{2}\int\limits_{\mathbb{R}^{2,2}} d^4x\, \eta^{\mu\nu}\mathrm{tr}(\partial_\mu\Omega^{-1}\partial_\nu\Omega) - \frac{1}{3}\int\limits_{\mathbb{R}^{2,2}\times[0,1]} \omega \wedge \mathrm{tr}(\tilde{A}\wedge\tilde{A}\wedge\tilde{A}) \,. \tag{II.33}$$

Here the gauge potential $A = \Omega^{-1}d\Omega$ and the Kähler form ω have nonvanishing components only along $\mathbb{R}^{2,2}$; in the Wess-Zumino term, $\tilde{A} = \tilde{\Omega}^{-1}d\tilde{\Omega}$ is defined via a homotopy $\tilde{\Omega}$ from a fixed element Ω_1 from the homotopy class of Ω to Ω, i.e., $\tilde{\Omega}(0) = \Omega_1$, $\tilde{\Omega}(1) = \Omega$. Star products are implicit. Note that the variation w.r.t. $\tilde{\Omega}$ of the Wess-Zumino term is a total divergence. An "energy-momentum" tensor can be easily obtained from this action; however, we do not want to embark on a discussion whether it can serve to give a sensible definition of energy or momentum in 2+2 dimensions. As a simplification, we will sometimes nevertheless speak of soliton solutions if we can verify that the solutions have finite energy in 2+1 dimensional subspaces at asymptotic times.

Dressing approach and ansatz. Note that, due to (II.30), eq. (II.29) can be rewritten as

$$\psi(\zeta\partial_{\tilde{v}} + \partial_u)\psi^\dagger = A_u \,, \tag{II.34a}$$

$$\psi(\zeta\partial_v - \partial_{\tilde{u}})\psi^\dagger = -A_{\tilde{u}} \,. \tag{II.34b}$$

It is possible to solve the gauge-fixed linear equations (II.34) without knowing $A_{\tilde{u}}$ and A_u explicitly, simply by fixing the pole structure[9] of ψ in such a way that the left hand sides of (II.34) are independent of ζ. Inserting an ansatz for ψ, we obtain conditions on its residues which can be solved. From the solution ψ, we may determine $A_{\tilde{u}}$ and A_u via eqs. (II.23) and (II.24). Suppose we have constructed a seed solution ψ_0 by solving some appropriate (gauge-fixed) linear equations, in the present case eqs. (II.29). Then we can look for a new solution of the form

$$\psi_1 = \chi_1 \psi_0 \quad \text{with} \quad \chi_1 = 1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1} P_1 \,, \tag{II.35}$$

where $\mu_1 \in \mathbb{H}_-$ (lower half plane) is a complex constant and where $P_1(u, v, \tilde{u}, \tilde{v})$ is an $N \times N$ matrix independent of ζ. It can be shown that μ_1 may be interpreted as a modulus parametrizing the velocity of the lump solution (see e.g. [182, 108] for the 2+1 dimensional case).

Let us start from the vacuum seed solution $\psi_0 = 1$ (the corresponding gauge potential vanishes). The reality condition (II.30) for ψ_1 is satisfied if we choose P_1 to be a hermitean projector, i.e., $(P_1)^2 = P_1$ and $(P_1)^\dagger = P_1$.[10] The transformation $\psi_0 \mapsto \psi_1$ is called *dressing*. An m-fold repetition of this procedure yields

$$\psi_m = \prod_{p=1}^{m} \left(1 + \frac{\mu_p - \bar{\mu}_p}{\zeta - \mu_p} P_p \right) , \tag{II.36}$$

corresponding to an m-soliton type configuration if all $\mu_p \in \mathbb{H}_-$. For (II.36), the reality condition (II.30) is automatically satisfied if we choose the P_p to be hermitean (not necessarily orthogonal) projectors. We will see below that taking all μ_p to be mutually different will lead us to interacting plane wave and non-interacting solitons, whereas second-order poles in (II.36) (i.e., $\mu_i = \mu_j$ for some $i \neq j$) entail scattering in soliton-like configurations.

First-order pole ansatz. For now, let us restrict to an ansatz (II.36) containing only first-order poles in ζ, i.e., choose all μ_p to be mutually different. Then, performing a decomposition into partial fractions, we can rewrite the multiplicative ansatz (II.36) in the additive form

$$\psi_m = 1 + \sum_{p=1}^{m} \frac{R_p}{\zeta - \mu_p} \,. \tag{II.37}$$

The $N \times N$ matrices $R_p(u, v, \tilde{u}, \tilde{v})$ are constructed from multiplicative combinations of the P_p; as in [108], we take the R_k to be of the form

$$R_p = \sum_{l=1}^{m} T_l \Gamma^{lp} T_p^\dagger \,, \tag{II.38}$$

where the $T_l(u, v, \tilde{u}, \tilde{v})$ are $N \times r$ matrices and the $\Gamma^{lp}(u, v, \tilde{u}, \tilde{v})$ are $r \times r$ matrices for some $r \geq 1$. The ansatz (II.37) has to satisfy the reality condition (II.30). Since the right hand side of the latter is independent of ζ, the poles on the left hand side must be removable. Therefore we

[9] A nontrivial $\psi(\zeta)$ cannot be holomorphic in ζ, since $\zeta \in \mathbb{C}P^1$, which is compact.

[10] This is the simplest solution to the algebraic conditions on P_1 emerging from the reality condition (II.30).

should equate the corresponding residues at $\zeta = \bar{\mu}_k$ and $\zeta = \mu_k$ of the left hand side to zero.[11] This yields

$$\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k}\right) T_k = 0\,. \tag{II.39}$$

These algebraic conditions on T_k imply that the Γ^{lp} invert the matrices

$$\widetilde{\Gamma}_{pk} := \frac{T_p^\dagger T_k}{\mu_p - \bar{\mu}_k}\,, \quad \text{i.e.,} \ \sum_{p=1}^{m} \Gamma^{lp}\widetilde{\Gamma}_{pk} = \delta^l_k\,. \tag{II.40}$$

Furthermore, our ansatz should satisfy the gauge-fixed linear equations (II.34). Putting to zero the residues of the left hand sides of (II.34) at $\zeta = \mu_k$ and $\zeta = \bar{\mu}_k$, we learn that

$$\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k}\right) (\bar{\mu}_k \partial_{\tilde{v}} + \partial_u) R_k^\dagger = 0\,, \tag{II.41a}$$

$$\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k}\right) (\bar{\mu}_k \partial_v - \partial_{\tilde{u}}) R_k^\dagger = 0\,, \tag{II.41b}$$

Thus, we may define new isotropic coordinates (note that μ_k is complex) w_k^1 and w_k^2 in the kernel of the differential operators in (II.41):

$$w_k^1 := \bar{\mu}_k^{-1}\tilde{v} - u \qquad \text{and} \qquad w_k^2 := \bar{\mu}_k^{-1} v + \tilde{u} \tag{II.42a}$$

$$\Rightarrow \quad \overline{w}_k^1 = \mu_k^{-1}\tilde{v} - u \qquad \text{and} \qquad \overline{w}_k^2 = \mu_k^{-1} v + \tilde{u}\,. \tag{II.42b}$$

The Lax operators can be written as antiholomorphic vector fields in terms of these new isotropic coordinates[12]

$$\bar{L}_k^1 := \bar{\mu}_k \partial_{\tilde{v}} + \partial_u = \mu_k^{-1}(\bar{\mu}_k - \mu_k)\frac{\partial}{\partial \overline{w}_k^1}\,, \tag{II.43a}$$

$$\bar{L}_k^2 := \bar{\mu}_k \partial_v - \partial_{\tilde{u}} = \mu_k^{-1}(\bar{\mu}_k - \mu_k)\frac{\partial}{\partial \overline{w}_k^2}\,. \tag{II.43b}$$

As long as T_k is in the kernel of $\bar{L}_k^1$ and $\bar{L}_k^2$, all functions R_k from (II.38) automatically solve eqs. (II.41). Thus, special solutions to (II.41) are given by (II.38) with arbitrary differentiable functions $T_k(w_k^1, w_k^2)$, i.e., $\partial_{\overline{w}_k^1} T_k = 0 = \partial_{\overline{w}_k^2} T_k$ (for each $k = 1, \ldots, m$). By inserting such T_k into

$$\Omega^{-1} = \psi(u, v, \tilde{u}, \tilde{v}, \zeta = 0) = 1 - \sum_{l,p=1}^{m} \frac{T_l \Gamma^{lp} T_p^\dagger}{\mu_p}\,, \tag{II.44}$$

explicit expressions for $A_u, A_{\tilde{u}}$ can be derived from (II.28) and (II.40).

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[11] In fact, the equation for $\zeta = \bar{\mu}_k$ is the hermitean adjoint to the equation for $\zeta = \mu_k$. In general, this will hold for any two residue equations if the points are related by complex conjugation (or, for λ from section II.4.3, by the mapping (A.9)).

[12] In general, $\bar{L}^1(\zeta) := \zeta\partial_{\tilde{v}} + \partial_u$ and $\bar{L}^2(\zeta) := \zeta\partial_v - \partial_{\tilde{u}}$ correspond to the antiholomorphic vector fields introduced in appendix A.1 (in the coordinates $u, v, \tilde{u}, \tilde{v}$). Defining coordinate functions $w^1(\zeta) := \zeta^{-1}\tilde{v} - u$, $w^2(\zeta) := \zeta^{-1}v + \tilde{u}$ in their kernel, we may write $\bar{L}_k^{1,2} = \bar{L}^{1,2}(\zeta = \bar{\mu}_k)$ and $w_k^{1,2} = w^{1,2}(\zeta = \bar{\mu}_k)$. Furthermore we have $\bar{L}^{1,2}(\zeta) = \bar{\zeta}^{-1}(\zeta - \bar{\zeta})\partial_{\overline{w}^{1,2}(\bar{\zeta})}$.

II.4.2 Dimensional reduction to 2+1 dimensions

Dimensional reduction. In order to establish the connection between the solutions obtained above and previous work carried out in 2+1 dimensions[13] (see [108, 109, 30, 189]) we have to perform a dimensional reduction. This can be done by imposing the condition that all fields are independent of one of the time coordinates in $\mathbb{R}^{2,2}$. As a consequence, we may put $\tilde{\theta} = 0$. To be precise, let us impose

$$\partial_{\tilde{t}} T_k = 0\,. \tag{II.45}$$

We switch to the complex isotropic coordinates introduced in (II.42). Using (II.6), we can reexpress $\frac{\partial}{\partial \tilde{t}}$ as

$$\frac{\partial}{\partial \tilde{t}} = \frac{1}{2}\left\{\bar{\mu}_k^{-1}\frac{\partial}{\partial w_k^1} + \mu_k^{-1}\frac{\partial}{\partial \overline{w}_k^1} + \frac{\partial}{\partial w_k^2} + \frac{\partial}{\partial \overline{w}_k^2}\right\}. \tag{II.46}$$

As derived in section II.4.1, eqs. (II.41) are solved by matrices T_k independent of $\overline{w}_k^1$ and $\overline{w}_k^2$; therefore (II.45) reads

$$\left[\bar{\mu}_k^{-1}\frac{\partial}{\partial w_k^1} + \frac{\partial}{\partial w_k^2}\right] T_k(w_k^1, w_k^2) = 0\,, \tag{II.47}$$

i.e., T_k can only be a function of

$$w_k := \nu_k(w_k^2 - \bar{\mu}_k w_k^1) = \nu_k\left(x + \frac{1}{2}(\bar{\mu}_k - \bar{\mu}_k^{-1})y + \frac{1}{2}(\bar{\mu}_k + \bar{\mu}_k^{-1})t\right), \tag{II.48}$$

if it is independent of the second time direction. The normalization constant

$$\nu_k := \left[\frac{4\mathrm{i}}{\mu_k - \bar{\mu}_k - \mu_k^{-1} + \bar{\mu}_k^{-1}} \cdot \frac{\mu_k - \mu_k^{-1} - 2\mathrm{i}}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2\mathrm{i}}\right]^{1/2} \tag{II.49}$$

has been introduced for later convenience. Note that the "co-moving" coordinates w_k become static (i.e., independent of t) when choosing $\mu_k = -\mathrm{i}$; they "degenerate" to the complex coordinates z^1 from (II.7). Conversely, they can be obtained from the "static" coordinates $z^1, \bar{z}^1$ by an inhomogeneous $SU(1,1)$ transformation [108]:

$$\begin{pmatrix} w_k \\ \overline{w}_k \end{pmatrix} = \begin{pmatrix} \cosh\,\tau_k & -e^{\mathrm{i}\vartheta_k}\sinh\,\tau_k \\ -e^{-\mathrm{i}\vartheta_k}\sinh\,\tau_k & \cosh\,\tau_k \end{pmatrix}\begin{pmatrix} z^1 \\ \bar{z}^1 \end{pmatrix} - \sqrt{2\theta}\begin{pmatrix} \beta_k \\ \bar{\beta}_k \end{pmatrix} t\,, \tag{II.50}$$

where

$$\beta_k = -\frac{1}{2}(2\theta)^{-1/2}\nu_k(\bar{\mu}_k + \bar{\mu}_k^{-1})\,, \tag{II.51a}$$

and

$$\cosh\,\tau_k - e^{\mathrm{i}\vartheta_k}\sinh\,\tau_k = \nu_k\,, \quad e^{\mathrm{i}\vartheta_k}\tanh\,\tau_k = \frac{\bar{\mu}_k - \bar{\mu}_k^{-1} - 2\mathrm{i}}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2\mathrm{i}}\,. \tag{II.51b}$$

[13] To recover the linear system of [108], we need to choose the unitary gauge for the linear equations, i.e., Lax pair (II.20) as discussed in section II.4.1.

Recall that a general solution T_k in 2+2 dimensions is an arbitrary function of w^1_k, w^2_k. Hence, dimensional reduction to 2+1 dimensions can be accomplished for T_k depending only on w_k:

$$\partial_{\overline{w}^1_k} T_k = 0 = \partial_{\overline{w}^2_k} T_k \quad \text{and} \quad \partial_{\tilde{t}} T_k = 0 \qquad \Leftrightarrow \qquad T_k = T_k(w_k)\,. \tag{II.52}$$

The Lax operators acting in this 2+1 dimensional subspace are given by:

$$\bar{L}^1_k \;=\; \bar{\mu}_k \partial_{\tilde{t}} - \bar{\mu}_k \partial_x + \partial_u \xrightarrow{2+1} -\bar{\nu}_k(\bar{\mu}_k - \mu_k)\partial_{\overline{w}_k}\,, \tag{II.53a}$$
$$\bar{L}^2_k \;=\; \bar{\mu}_k \partial_v - \partial_{\tilde{t}} - \partial_x \xrightarrow{2+1} \bar{\nu}_k \mu_k^{-1}(\bar{\mu}_k - \mu_k)\partial_{\overline{w}_k}\,. \tag{II.53b}$$

This exactly matches the results of [108].

Note that an alternative reduction can be done if T_k is independent of t (but depends on all other coordinates):

$$\frac{\partial}{\partial t} = -\frac{1}{2}\left\{\frac{\partial}{\partial w^1_k} + \frac{\partial}{\partial \overline{w}^1_k} - \bar{\mu}_k^{-1}\frac{\partial}{\partial w^2_k} - \mu_k^{-1}\frac{\partial}{\partial \overline{w}^2_k}\right\}. \tag{II.54}$$

Then, the condition $\partial_t T_k = 0$ and eqs. (II.41) are satisfied for $T_k = T_k(\widetilde{w}_k)$ with

$$\widetilde{w}_k := \nu_k(w^1_k + \bar{\mu}_k w^2_k) = \nu_k\left(-y + \frac{1}{2}(\bar{\mu}_k - \bar{\mu}_k^{-1})x + \frac{1}{2}(\bar{\mu}_k + \bar{\mu}_k^{-1})\tilde{t}\right). \tag{II.55}$$

Note that a "boost" transformation analogous to (II.50) can be found for the coordinates $\widetilde{w}_k$:

$$\begin{pmatrix} \widetilde{w}_k \\ \overline{\widetilde{w}}_k \end{pmatrix} = \mathrm{i}\begin{pmatrix} \cosh\tau_k & e^{\mathrm{i}\vartheta_k}\sinh\tau_k \\ -e^{-\mathrm{i}\vartheta_k}\sinh\tau_k & -\cosh\tau_k \end{pmatrix}\begin{pmatrix} z^1 \\ \bar{z}^1 \end{pmatrix} - \sqrt{2\theta}\begin{pmatrix} \beta_k \\ \bar{\beta}_k \end{pmatrix}\tilde{t}\,. \tag{II.56}$$

Such $T_k(w_k)$ or $T_k(\widetilde{w}_k)$ lead to Ω which are given by (II.44) and do not depend on $\tilde{t}$ or t, respectively.

Map to operator formalism. If we translate the co-moving coordinates w_k and $\widetilde{w}_k$ into the operator formalism, this yields co-moving creation and annihilation operators:

$$\widehat{w}^\dagger_k = \widehat{\overline{w}}_k \quad \Rightarrow \quad [\widehat{w}_k, \widehat{\overline{w}}_k] = 2\theta\,, \tag{II.57a}$$
$$\widehat{\widetilde{w}}^\dagger_k = \widehat{\overline{\widetilde{w}}}_k \quad \Rightarrow \quad [\widehat{\widetilde{w}}_k, \widehat{\overline{\widetilde{w}}}_k] = 2\theta\,. \tag{II.57b}$$

Note that, in general, the commutators between $\widehat{w}_k$ and $\widehat{\widetilde{w}}_k$ will not vanish. Therefore, derivatives with respect to w_k and $\overline{w}_k$ translate into commutators (cf. (II.19)) of the simple form

$$2\theta\partial_{w_k} = -[\widehat{\overline{w}}_k\,,\,.\,]\,, \quad 2\theta\partial_{\overline{w}_k} = [\widehat{w}_k\,,\,.\,]\,, \tag{II.58}$$

only when acting on functions of $\widehat{w}_k$ and $\widehat{\overline{w}}_k$. Analogous statements hold for derivatives with respect to $\widetilde{w}_k, \overline{\widetilde{w}}_k$. In this framework, the transformations (II.50) and (II.56) may be interpreted as Bogoliubov transformations relating $\hat{z}^1$ and $\hat{\bar{z}}^1$ to the operators in (II.57) [108].

Energy. In 2+1 dimensions it is possible to define the notion of energy in a straightforward manner and to show that it is conserved. Dimensional reduction of the Nair-Schiff type action (II.33) leads to the action for a modified noncommutative sigma model in 2+1 dimensions as presented in [108]. From this, an energy-momentum tensor can easily be derived:

$$T_{cd} = (\delta^a_c\delta^b_d - \frac{1}{2}\eta_{cd}\eta^{ab})\mathrm{tr}(\partial_a\Omega^{-1}\partial_b\Omega)\,, \tag{II.59}$$

a, b, c and d running over x, y, and t. For the proof that T_{cd} is divergence-free we need to apply the equation of motion

$$(\eta^{ab} - \omega^{ab})\partial_a(\Omega^{-1}\partial_b\Omega) = 0 \tag{II.60}$$

obtained by dimensional reduction (by imposing $\partial_{\tilde{t}}(\Omega^{-1}\partial_b\Omega) = 0$) from eq. (II.32). Using the explicit form of $\omega^{\mu\nu}$, it is obvious that one can rewrite eq. (II.60) as

$$(\eta^{ab} + V_c\varepsilon^{cab})\partial_a(\Omega^{-1}\partial_b\Omega) = 0\,, \tag{II.61}$$

where $(V_c) = (V_x, V_y, V_t) = (1, 0, 0)$ manifestly breaks Lorentz-invariance even in the commutative case. With this, one finds that $\int d^2x\, \partial^a T_{at}$ vanishes due to the chosen form of ω_{ab}. For the energy density, one obtains

$$\mathcal{E} = T_{tt} = \frac{1}{2}\mathrm{tr}[(\partial_t\Omega^\dagger)\partial_t\Omega + (\partial_x\Omega^\dagger)\partial_x\Omega + (\partial_y\Omega^\dagger)\partial_y\Omega]\,; \tag{II.62}$$

obviously $\partial_t \int d^2x\, \mathcal{E} = 0$.

Nonabelian soliton in 2+1 dimensions. As an illustrative example, consider a nonabelian one-soliton ($m = 1$) in 2+1 dimensions as described in [108]. Since $m = 1$, we may start from (II.36) with $P_1 \equiv P = T(T^\dagger T)^{-1}T^\dagger$, cf. (II.38) and (II.40). For definiteness, we take the soliton to be embedded into the xyt-plane, i.e., $T_1 \equiv T$ is a function of $w_1 \equiv w$ (cf. (II.48)). Such a function T trivially solves (II.41).

Exemplarily, we briefly review a solution corresponding to a moving $U(2)$ soliton [108]. Using the inverse Moyal-Weyl map, we can deduce from the simplest ansatz $T = \left(\begin{smallmatrix} 1 \\ w \end{smallmatrix}\right)$ that

$$\Omega_\star = 1 - \frac{\bar{\mu} - \mu}{\bar{\mu}} \begin{pmatrix} \frac{2\theta}{w\overline{w}+\theta} & \frac{\sqrt{2\theta}w\overline{w}^2}{(w\overline{w}+\theta)^2} \\ \frac{\sqrt{2\theta}\;w^2\overline{w}}{(w\overline{w}+\theta)^2} & \frac{w\overline{w}+\theta}{w\overline{w}+3\theta} \end{pmatrix}, \tag{II.63}$$

with the ordinary product between w and $\overline{w}$, solves the self-duality equation. With the help of (II.62), the energy of this configuration can be shown to be $E = 8\pi \cosh\eta \sin\varphi$ where $e^{\eta - \mathrm{i}\varphi} = \mu$.

A remark on the interpretation of solitons in terms of D-branes is in order: We start out from ncSDYM on a space-time filling D-brane. If a solution ψ is independent of one coordinate, we are allowed to compactify and subsequently T-dualize this direction. This alters the Neumann boundary conditions for open strings living on the space-time filling branes to Dirichlet boundary conditions. In this case we therefore consider gauge theory on a D2-brane. Although there exists no Hodge self-duality condition in such a three-dimensional gauge theory, we will (in a slight abuse of language) still speak of solitonic solutions (implicitly referring to the four-dimensional gauge theory before T-dualization).

Since $\Omega_\star$ from (II.63) and the corresponding energy density are independent of $\tilde{t}$, a T-dualization in the $\tilde{t}$-direction leads to a gauge configuration on a pair of D2-branes. Taking into account that Ω depends only on two variables $w, \overline{w}$ in three dimensions, we conclude that it corresponds to a D0-brane moving in the world-volume of two D2-branes.

II.4.3 Hermitean gauge

Lax pair. Instead of using ζ, the Riemann sphere $\mathbb{C}P^1$ may alternatively be parametrized by the variable

$$\lambda = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}\,. \tag{II.64}$$

The map $\zeta \mapsto \lambda$ carries the lower half plane in ζ to the exterior of the unit disk $\{|\lambda| > 1\}$ in the λ-plane. In terms of λ and the coordinates $z^1, \bar{z}^1, z^2, \bar{z}^2$ on $\mathbb{R}^{2,2} \cong \mathbb{C}^{1,1}$, the Lax pair (II.20) becomes[14]

$$(\partial_{\bar{z}^1} - \lambda\partial_{z^2})\psi = -(A_{\bar{z}^1} - \lambda A_{z^2})\psi\,, \tag{II.65a}$$
$$(\partial_{\bar{z}^2} - \lambda\partial_{z^1})\psi = -(A_{\bar{z}^2} - \lambda A_{z^1})\psi\,, \tag{II.65b}$$

and its compatibility conditions are the self-duality equations

$$F_{z^1z^2} = 0\,, \tag{II.66a}$$
$$F_{z^1\bar{z}^1} - F_{z^2\bar{z}^2} = 0\,, \tag{II.66b}$$
$$F_{\bar{z}^1\bar{z}^2} = 0\,. \tag{II.66c}$$

Here, ψ may be chosen to satisfy the reality condition

$$\psi(z^1, \bar{z}^1, z^2, \bar{z}^2, \lambda)[\psi(z^1, \bar{z}^1, z^2, \bar{z}^2, \bar{\lambda}^{-1})]^\dagger = 1\,. \tag{II.67}$$

Equations (II.66a) and (II.66c) imply that there exist $g, \tilde{g} \in GL(N, \mathbb{C})$ such that:

$$A_{z^1} = g^{-1}\partial_{z^1} g\,, \qquad A_{z^2} = g^{-1}\partial_{z^2} g\,, \tag{II.68a}$$
$$A_{\bar{z}^1} = \tilde{g}^{-1}\partial_{\bar{z}^1} \tilde{g}\,, \qquad A_{\bar{z}^2} = \tilde{g}^{-1}\partial_{\bar{z}^2} \tilde{g}\,. \tag{II.68b}$$

We read off that a possible choice for g and $\tilde{g}$ is given by

$$g := [\psi(z^i, \bar{z}^i, \lambda \to \infty)]^{-1}\,, \tag{II.69a}$$
$$\tilde{g} := [\psi(z^i, \bar{z}^i, \lambda \to 0)]^{-1}\,. \tag{II.69b}$$

Since we are using antihermitean generators for the gauge group $U(N)$, the $GL(N, \mathbb{C})$-valued fields $g, \tilde{g}$ have to be related:

$$A^\dagger_{z^i} = -A_{\bar{z}^i}\,,\ i = 1, 2 \quad \Rightarrow \quad \tilde{g} = (g^\dagger)^{-1}\,. \tag{II.70}$$

Gauge fixing. As in section II.4.1, we can perform a gauge transformation to set two components of the gauge potential to zero. Contrary to the (unitary) gauge choice there, in the

[14] The conventions for z^1, z^2 are such that for $\mu'_k = \infty$ we obtain holomorphic functions T as solutions of (II.80).

following we set to zero those components which are *not* multiplied by the respective spectral parameter in eqs. (II.65).[15] Explicitly,

$$\psi' = \tilde{g}\psi\,, \tag{II.71a}$$

$$A'_{z^1} = h^{-1}\partial_{z^1}h\,, \qquad A'_{z^2} = h^{-1}\partial_{z^2}h\,, \tag{II.71b}$$

$$A'_{\bar{z}^1} = 0\,, \qquad A'_{\bar{z}^2} = 0\,, \tag{II.71c}$$

where $h := g\tilde{g}^{-1} = gg^\dagger \in GL(N,\mathbb{C})$ is hermitean. This gauge is "asymmetric", i.e., the gauge potential does not obey (II.70), but instead it satisfies $(A'_{z^i})^\dagger = -hA'_{\bar{z}^i}h^{-1} - h\partial_{\bar{z}^i}h^{-1}$. After solving (II.66) we are free to gauge back to a "symmetric" gauge, where (II.70) is restored. This is ensured by the hermiticity of h, which is the remnant of (II.70) in the asymmetric gauge. We will from now on work in the asymmetric gauge and omit all primes on the gauge-transformed quantities.

Now, the gauge-fixed linear equations read

$$(\partial_{\bar{z}^1} - \lambda\partial_{z^2})\psi = \lambda A_{z^2}\psi\,, \tag{II.72a}$$

$$(\partial_{\bar{z}^2} - \lambda\partial_{z^1})\psi = \lambda A_{z^1}\psi\,. \tag{II.72b}$$

Due to (II.70), the reality condition (II.67) transforms into[16]

$$\psi(\lambda)[\psi(\bar{\lambda}^{-1})]^\dagger = \tilde{g}g^{-1} = h^{-1}\,. \tag{II.73}$$

In the asymmetric gauge the remaining self-duality equation (II.66b) reduces to

$$\partial_{\bar{z}^1}(h^{-1}\partial_{z^1}h) - \partial_{\bar{z}^2}(h^{-1}\partial_{z^2}h) = 0\,. \tag{II.74}$$

First-order pole ansatz. Since the reality condition (II.73) is different from the one in the unitary gauge, we are forced to employ a modified ansatz for $\psi(\lambda)$. The first-order pole ansatz for ψ takes the form[17]

$$\psi_m(\lambda) = 1 + \sum_{p=1}^{m}\frac{\lambda\widetilde{R}_p}{\lambda - \mu'_p}\,, \tag{II.75}$$

where

$$\widetilde{R}_p := -\sum_{q=1}^{m}\mu'_p T_p \Gamma^{pq} T_q^\dagger\,. \tag{II.76}$$

The "inverse" matrix $\widetilde{\Gamma} = (\widetilde{\Gamma}_{pk})$, cf. (II.40), here reads:

$$\widetilde{\Gamma}_{pk} = \mu'_p\frac{T_p^\dagger T_k}{1 - \mu'_p\bar{\mu}'_k}\,. \tag{II.77}$$

[15] In this way, we facilitate a comparison with chapter IV.

[16] This is the reason why we call this gauge hermitean. It coincides with the hermitean gauge introduced in [190].

[17] The parameters μ'_p are the images of μ_p under (II.64). However, the ansatz (II.75) is not simply the transform of (II.37).

The matrix-valued function ψ_m should satisfy the linear equations (II.72) and is subject to a reality condition, eq. (II.73). Again, the requirement that the poles at $\lambda = \bar{\mu}_k'^{-1}$ and $\lambda = \mu_k'$ of (II.73) have to be removable yields[18]

$$\Big(1 - \sum_{p=1}^{m} \frac{\widetilde{R}_p}{\mu_p' \bar{\mu}_k' - 1}\Big) T_k = 0\,, \tag{II.78}$$

which is solved by (II.76) with (II.77). Now we exploit the pole structure of the Lax pair which, using (II.73), may be rewritten as

$$\Big[\Big(\frac{1}{\lambda}\partial_{\bar{z}^1} - \partial_{z^2}\Big)\psi\Big]\psi^\dagger \;=\; A_{z^2} h^{-1}\,, \tag{II.79a}$$

$$\Big[\Big(\frac{1}{\lambda}\partial_{\bar{z}^2} - \partial_{z^1}\Big)\psi\Big]\psi^\dagger \;=\; A_{z^1} h^{-1}\,. \tag{II.79b}$$

As before, the right hand sides do not feature poles in λ, therefore taking the residue at $\lambda = \bar{\mu}_k'^{-1}$ and $\lambda = \mu_k'$ leads to the conditions

$$\Big(1 - \sum_{p=1}^{m} \frac{\widetilde{R}_p}{\mu_p' \bar{\mu}_k' - 1}\Big)(\partial_{\bar{z}^1} - \bar{\mu}_k'^{-1}\partial_{z^2})\widetilde{R}_k = 0\,, \tag{II.80a}$$

$$\Big(1 - \sum_{p=1}^{m} \frac{\widetilde{R}_p}{\mu_p' \bar{\mu}_k' - 1}\Big)(\partial_{\bar{z}^2} - \bar{\mu}_k'^{-1}\partial_{z^1})\widetilde{R}_k = 0\,. \tag{II.80b}$$

If we define

$$\eta^1(\lambda) := z^1 + \lambda \bar{z}^2 \qquad \Rightarrow \qquad \bar{\eta}^1(\bar{\lambda}) = \bar{z}^1 + \bar{\lambda} z^2\,, \tag{II.81a}$$

$$\eta^2(\lambda) := z^2 + \lambda \bar{z}^1 \qquad \Rightarrow \qquad \bar{\eta}^2(\bar{\lambda}) = \bar{z}^2 + \bar{\lambda} z^1\,, \tag{II.81b}$$

and denote $\eta_k^i := \eta^i(\lambda = \bar{\mu}_k'^{-1})$, $\bar{\eta}_k^i := \bar{\eta}^i(\bar{\lambda} = \mu_k'^{-1})$, the Lax operators can be written as antiholomorphic vector fields in these coordinates:

$$\bar{L}_k^1 \;=\; \partial_{\bar{z}^1} - \bar{\mu}_k'^{-1}\partial_{z^2} = (1 - |\mu_k'|^{-2})\frac{\partial}{\partial \bar{\eta}_k^1}\,, \tag{II.82}$$

$$\bar{L}_k^2 \;=\; \partial_{\bar{z}^2} - \bar{\mu}_k'^{-1}\partial_{z^1} = (1 - |\mu_k'|^{-2})\frac{\partial}{\partial \bar{\eta}_k^2}\,. \tag{II.83}$$

Functions $T_k = T_k(\eta_k^1, \eta_k^2)$ are in the kernel of $\bar{L}_k^1$ and $\bar{L}_k^2$; therefore $\widetilde{R}_k$ constructed via (II.76) from such T_k automatically satisfy eqs. (II.80). Due to (II.69) and (II.73), we can determine A_{z^1} and A_{z^2} in eq. (II.71) from

$$h^{-1} = 1 + \sum_{p=1}^{m} \widetilde{R}_p\,. \tag{II.84}$$

[18] Note that h^{-1} is independent of λ.

II.5 Configurations without scattering

The aim of this section is to demonstrate the usability of the solution generating technique described in section II.4 in two simple cases. In 2+2 dimensions, we will construct an abelian GMS-like solution of codimension four and a solution representing two nonabelian moving lumps without scattering. The description of configurations with scattering will be relegated to section II.6. Although we do not check their physical properties like tension and fluctuation spectrum, we will refer to them as D-branes.

II.5.1 Abelian GMS-like solution

It is fairly easy to construct $U(1)$ solutions depending on all space-time coordinates (i.e., with codimension four) via the dressing approach in 2+2 dimensions (cf. [82] on euclidean instantons via dressing). To this aim, let us start from the discussion of the dressing approach in the hermitean gauge (section II.4.3). For $m = 1$, we can omit all labels k; a comparison of (II.75)–(II.77) with the multiplicative ansatz shows that $\widetilde{R} = (|\mu'|^2 - 1)P$, where P is a hermitean projector independent of λ. If we choose $\theta = \widetilde{\theta}$ and define harmonic oscillators[19]

$$c_i := \frac{1}{\sqrt{2\theta(1-|\mu'|^{-2})}}\eta^i \quad \text{and} \quad c_i^\dagger := \frac{1}{\sqrt{2\theta(1-|\mu'|^{-2})}}\bar{\eta}^i\,, \tag{II.85}$$

then $[c_i, c_j^\dagger] = \delta_{ij}$; thus, we can easily invert their commutation relations and obtain

$$\sqrt{2\theta(1-|\mu'|^{-2})}\partial_{\bar{\eta}^i} = \big[c_i, \cdot\big]\,. \tag{II.86}$$

With this, we may rewrite (II.80) as

$$(1-P)c_1P \;=\; 0\,, \tag{II.87a}$$

$$(1-P)c_2P \;=\; 0\,. \tag{II.87b}$$

Obviously, these equations can be solved by the projector $P = |0,0\rangle'\,'\langle 0,0|$ onto the new vacuum $|0,0\rangle'$ annihilated by c_1 and c_2.[20] We may use the inverse Moyal-Weyl map (II.18) to transform it to the star formulation:

$$\begin{aligned} P_\star &= \exp\left(-\frac{\eta^1\bar{\eta}^1 + \eta^2\bar{\eta}^2}{\theta(1-|\mu'|^{-2})}\right) \\ &= \exp\left(-\frac{(z^1+\bar{\mu}'^{-1}\bar{z}^2)(\bar{z}^1+\mu'^{-1}z^2)+(z^2+\bar{\mu}'^{-1}\bar{z}^1)(\bar{z}^2+\mu'^{-1}z^1)}{\theta(1-|\mu'|^{-2})}\right)\,. \end{aligned} \tag{II.88}$$

This is the analogue of the GMS-solution [63] in 2+2 dimensions; the projector $P_\star$ is an example for a projector Φ_1 in (E.43). The gauge potential can be derived from (II.71) with $h^{-1} = 1-(1-|\mu'|^2)P_\star$. The computation of the value of the action for this solution turns out to be rather unwieldy.

[19]Recall that $|\mu'| > 1$ since $\mu \in \mathbb{H}_-$ in section II.4.1.

[20]Since the a-oscillators (cf. (II.81) and (II.15)) and the c-oscillators are related by a unitary transformation $c_i = Ua_iU^\dagger$, the (properly normalized) vacuum $|0,0\rangle'$ can naturally be obtained as $|0,0\rangle' = U|0,0\rangle$.

II.5.2 U(2) solitons without scattering

Let us now demonstrate how the additive ansatz (II.37) in the unitary gauge can be employed to construct a solution describing two moving lumps. A detailed description of the asymptotic space-time picture will be given at the end of this section.

Additive ansatz. We work in the star formulation and relax the condition that $\theta = \widetilde{\theta}$. The result of the first dressing step, corresponding to a soliton in 2+1 dimensions, has already been given in section II.4.2. This lump moves w.r.t. t in the xy-plane; its energy (which is well-defined in 2+1 dimensions) was computed in the same section. In the second dressing step, we want to add a soliton confined for large t to the $xy\tilde{t}$-plane. From the preceding discussion in section II.4.1 it is clear that for $m = 2$, we can construct a solution to the self-duality equations using (cf. (II.37))

$$\psi_2 = 1 + \sum_{l,k=1}^{2} \frac{T_l \Gamma^{lk} T_k^\dagger}{\zeta - \mu_k} \tag{II.89}$$

with $T_1 = \binom{1}{w_1}$ and $T_2 = \binom{1}{\bar{w}_2}$. However, it is not obvious that this solution really represents two soliton-like objects, i.e., whether for large t the solution can be integrated over a (spatial) plane in the $xy\tilde{t}$-subspace to give finite energy (and vice versa on a plane in the xyt-subspace at large $\tilde{t}$). To prove this, we compare the additive (first-order pole) and multiplicative ansätze for asymptotic times. Note that the two are only equivalent if the multiplicative ansatz features merely first-order poles in μ_i, that is, if $\mu_1 \neq \mu_2$.

Multiplicative ansatz. In the multiplicative ansatz, $\psi_2 = \chi_2 \chi_1 \psi_0$ may be constructed by two successive dressing steps from a seed solution $\psi_0 = 1$. As in eq. (II.36), we may write

$$\psi_2 = \left(1 + \frac{\mu_2 - \bar{\mu}_2}{\zeta - \mu_2} P_2\right)\left(1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1} P_1\right), \tag{II.90}$$

and this has to coincide with (II.89) for all times. Remember that for hermitean projectors $P_k = \widetilde{T}_k (\widetilde{T}_k^\dagger \widetilde{T}_k)^{-1} \widetilde{T}_k^\dagger$ this ansatz guarantees the reality condition (II.30). The solution ψ_2 is subject to eqs. (II.34):

$$\psi_2 (\zeta \partial_{\bar{v}} + \partial_u) \psi_2^\dagger = A_{2,u}\,, \tag{II.91a}$$

$$\psi_2 (\zeta \partial_v - \partial_{\tilde{u}}) \psi_2^\dagger = -A_{2,\tilde{u}}\,. \tag{II.91b}$$

The removability of the poles of the left hand sides at $\zeta = \bar{\mu}_1$ and $\zeta = \mu_1$ is assured if (for $\mu_1 \neq \mu_2$)

$$(1 - P_1)\bar{L}_1^1 P_1 = 0 \quad \text{and} \quad (1 - P_1)\bar{L}_1^2 P_1 = 0\,, \tag{II.92}$$

and this allows for a solution $\widetilde{T}_1 = T_1 = \binom{1}{w_1}$. Using the inverse Moyal-Weyl map, we obtain for P_1 and its large-time limits

$$P_{1\star} = \begin{pmatrix} \frac{2\theta}{w\bar{w}+\theta} & \frac{\sqrt{2\theta} w \bar{w}^2}{(w\bar{w}+\theta)^2} \\ \frac{\sqrt{2\theta}\; w^2 \bar{w}}{(w\bar{w}+\theta)^2} & \frac{w\bar{w}+\theta}{w\bar{w}+3\theta} \end{pmatrix} \overset{t\to\pm\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: \Pi_{\pm\infty} \tag{II.93}$$

with the ordinary product between w_1 and $\overline{w}_1$, as in (II.63).[21] In contrast, P_2 will in general be a function $P_2(t, \widetilde{w}_2, \overline{\widetilde{w}}_2)$, i.e., $\widetilde{T}_2 \neq T_2$ may also depend on t. Namely, the residue equation of (II.91) at $\zeta = \bar{\mu}_2$ and $\zeta = \mu_2$ yields:

$$(1 - P_2)\left(1 + \frac{\mu_1 - \bar{\mu}_1}{\bar{\mu}_2 - \mu_1} P_1\right) \bar{L}_2^1 \left\{\left(1 + \frac{\bar{\mu}_1 - \mu_1}{\bar{\mu}_2 - \bar{\mu}_1} P_1\right) P_2\right\} = 0\,, \tag{II.94a}$$

$$(1 - P_2)\left(1 + \frac{\mu_1 - \bar{\mu}_1}{\bar{\mu}_2 - \mu_1} P_1\right) \bar{L}_2^2 \left\{\left(1 + \frac{\bar{\mu}_1 - \mu_1}{\bar{\mu}_2 - \bar{\mu}_1} P_1\right) P_2\right\} = 0\,. \tag{II.94b}$$

Due to the asymptotic constancy of P_1 for large $|t|$, we can move the Lax operators next to P_2 in this limit, and a short calculation shows that this leads to

$$(1 - P_2)\partial_{\overline{\widetilde{w}}_2} P_2 = 0 \quad \text{for } |t| \to \infty\,. \tag{II.95}$$

Obviously, we have $\widetilde{T}_2 = T_2$ only asymptotically.

Thus, the energy of the second lump can be computed in the limit $|t| \to \infty$ to give $E_2 = 8\pi \cosh \eta_2 \sin \varphi_2$ as in section II.4.2. Analogously, the energy of the first lump in the limit $|\tilde{t}| \to \infty$ equals $E_1 = 8\pi \cosh \eta_1 \sin \varphi_1$.

For large and fixed $|t|$, the space-time interpretation of the above solution is as follows: Since P_1 is independent of $\tilde{t}$, the first soliton (at a fixed time t) has some definite position in the xy-plane and extends along the $\tilde{t}$-direction (see figure II.1). Moreover, the world-volume of the second soliton in this snapshot corresponds to a tilted line (cf. eq. (II.95)). When t varies in the asymptotic region, the first (vertical) line gets shifted in a direction determined by μ_1, while the second line remains fixed. Generically, the two world-volumes intersect the xy-plane at different points. Since Ω depends on both t and $\tilde{t}$, it is not possible to perform a T-dualization in one of the time directions. Thus, the solution has to be interpreted in terms of tilted D1-branes inside space-time filling D-branes.

II.6 Configurations with scattering

In this section we discuss two different setups entailing configurations with scattering, namely two $U(2)$ soliton-like objects and two noncommutative $U(2)$ plane waves, with world-volumes in 2+1 dimensional subspaces of $\mathbb{R}^{2,2}$. As will become clear in this section, the crucial difference between the two configurations lies in the fact that for these plane waves scattering occurs even if $\mu_1 \neq \mu_2$, whereas soliton-like objects only scatter nontrivially if $\mu_1 = \mu_2$. (In fact, we have already seen in the previous section that the solitonic lumps do not scatter for $\mu_1 \neq \mu_2$.)

II.6.1 U(2) solitons with scattering

The setup is as in section II.5.2; one of the solitonic lumps is evolving with t on the xy-plane. Since a first-order pole ansatz for the auxiliary field ψ_2 in eqs. (II.89) did not lead to scattering, we now scrutinize the multiplicative ansatz with $\mu_1 = \mu_2 = \mu$.

[21]From the asymptotics, we can read off that the two lumps pass through each other without scattering.

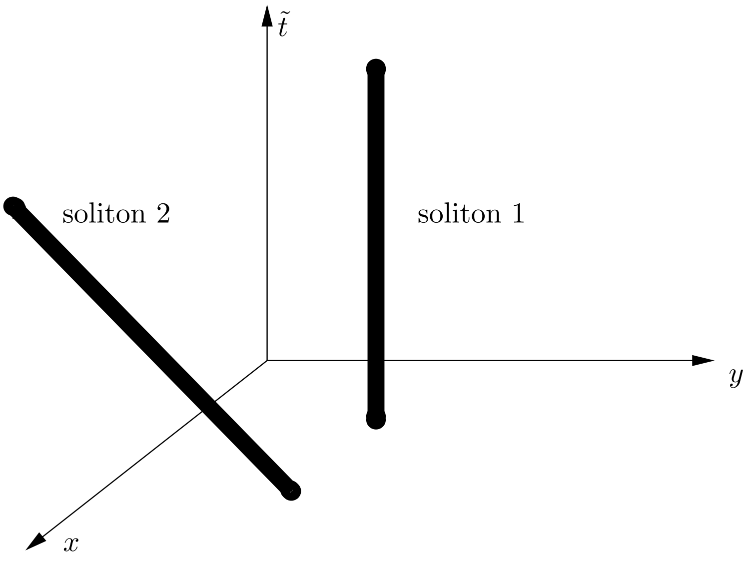

Figure II.1. *Snapshot of the configuration discussed in section II.5.2 for fixed large $|t|$. The support of the solution is concentrated around the solid lines.*

First dressing step. Starting from a seed solution $\psi_0 = 1$, we make the following ansatz for the first dressing step:

$$\psi_1 = 1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_1 \,. \tag{II.96}$$

This automatically fulfills the reality condition (II.30) as long as P_1 is a hermitean projector. The residue condition on the linear equations (II.34) leads to

$$(1 - P_1)\partial_{\overline{w}} P_1 = 0 \,, \tag{II.97}$$

i.e., P_1 varies in a $2+1$ dimensional subspace parametrized by w as defined in (II.48).

Now we set out to find explicit expressions for the components of the gauge potential. First note that, since P_1 is chosen to be independent of $\tilde{t}$, $A_{1,\tilde{u}}$ effectively reduces to $A_{1,x}$. From eqs. (II.28) and using $\psi_2(\zeta = 0) = \Omega^{-1}$ or (II.44), we find[22]

$$A_{1,u} = \bar{\rho}\,(1 - \rho P_1)\,\partial_u P_1 \,, \tag{II.98a}$$

$$A_{1,x} = \bar{\rho}\,(1 - \rho P_1)\,\partial_x P_1 \,, \tag{II.98b}$$

where $\rho = 1 - \bar{\mu}/\mu$ was introduced for convenience.

[22] Alternatively, we could parametrize $A_{1,u}$ and $A_{1,\tilde{u}}$ in terms of the algebra-valued Leznov prepotential ϕ_1 (cf. [85]):

$$A_{1,\tilde{u}} = \partial_v \phi_1, \quad A_{1,u} = -\partial_{\tilde{v}} \phi_1 \,,$$

where $\phi_1 = (\mu - \bar{\mu})P_1$ is defined by the asymptotic condition

$$\psi_1(u, v, \tilde{u}, \tilde{v}, \zeta \to \infty) = 1 + \zeta^{-1}\phi_1(u, v, \tilde{u}, \tilde{v}) + \mathcal{O}(\zeta^{-2}) \,.$$

This also leads to eqs. (II.103).

Second dressing step. The reality condition (II.30) for the new ansatz $\psi_2 = \chi_2\psi_1$ will be satisfied if we choose χ_2 to be of the same functional form as ψ_1, i.e.,

$$\psi_2 = \left(1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_2\right)\left(1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_1\right) \tag{II.99}$$

with a hermitean projector $P_2 = T_2(T_2^\dagger T_2)^{-1}T_2^\dagger$ in general depending on all four coordinates. The corresponding gauge-fixed linear equations (II.34) are:

$$\psi_2(\zeta\partial_{\tilde{v}} + \partial_u)\psi_2^\dagger = A_{2,u}\,, \tag{II.100a}$$

$$\psi_2(\zeta\partial_v - \partial_{\tilde{u}})\psi_2^\dagger = -A_{2,\tilde{u}}\,, \tag{II.100b}$$

which is equivalent to

$$A_{2,u} = \chi_2 A_{1,u}\chi_2^\dagger + \chi_2(\zeta\partial_{\tilde{v}} + \partial_u)\chi_2^\dagger\,, \tag{II.101a}$$

$$-A_{2,\tilde{u}} = \chi_2 A_{1,\tilde{u}}\chi_2^\dagger - \chi_2(\zeta\partial_v - \partial_{\tilde{u}})\chi_2^\dagger\,. \tag{II.101b}$$

Inserting $\chi_2 = 1 + \frac{\mu-\bar{\mu}}{\zeta-\mu}P_2$ and demanding that the right hand sides of eqs. (II.101a) and (II.101b) are free of poles for $\zeta \to \bar{\mu}$ and $\zeta \to \mu$ leads to

$$(1 - P_2)\left\{\rho\partial_{\overline{w}^1} - A_{1,u}\right\}P_2 = 0\,, \tag{II.102a}$$

$$(1 - P_2)\left\{\rho\partial_{\overline{w}^2} + A_{1,x}\right\}P_2 = 0\,. \tag{II.102b}$$

Recall that we defined $\rho = 1 - \bar{\mu}/\mu$. In the following, we shall assume $P_2 = P_2(w, \overline{w}, \tilde{t})$. By appropriately combining eqs. (II.102) and taking into account eqs. (II.98), we obtain the following equations for the projector P_2:

$$(1 - P_2)\left\{\partial_{\overline{w}}P_2 - (\partial_{\overline{w}}P_1)P_2\right\} = 0\,, \tag{II.103a}$$

$$(1 - P_2)\left\{\partial_{\tilde{t}}P_2 + \nu\bar{\rho}(\partial_w P_1)P_2\right\} = 0\,. \tag{II.103b}$$

In the derivation of the second equation we have also made use of the hermitean conjugate of eq. (II.97), that is

$$P_1\partial_w P_1 = 0\,. \tag{II.104}$$

In the operator formalism, all derivatives can be understood as commutators in the sense of (II.57). The projector identities

$$(1 - P_2)P_2 \equiv 0 \quad \text{and} \quad (1 - P_2)T_2 \equiv 0 \tag{II.105}$$

transform eqs. (II.103) into

$$(1 - P_2)\big\{wT_2 - [w, P_1]T_2\big\}(T_2^\dagger T_2)^{-1}T_2^\dagger = 0\,, \tag{II.106a}$$

$$(1 - P_2)\big\{\partial_{\tilde{t}}T_2 - \eta'[\overline{w}, P_1]T_2\big\}(T_2^\dagger T_2)^{-1}T_2^\dagger = 0\,, \tag{II.106b}$$

where $\eta' := \frac{\nu}{2\theta}\bar{\mu}^{-1}(\bar{\mu}-\mu)$ and $\nu = \nu_1 = \nu_2$ from (II.49). Due to (II.105), a sufficient condition for a solution is given by

$$wT_2 - [w, P_1]T_2 = T_2\mathcal{S}_1\,, \tag{II.107a}$$

$$\partial_{\tilde{t}}T_2 - \eta'[\overline{w}, P_1]T_2 = T_2\mathcal{S}_2\,, \tag{II.107b}$$

for some functions $\mathcal{S}_1(w,\overline{w},\tilde{t})$ and $\mathcal{S}_2(w,\overline{w},\tilde{t})$.

Explicit solutions. For the example of $U(2)$ soliton-like configurations, we choose $T_1 = \left(\begin{smallmatrix}1\\ w\end{smallmatrix}\right)$, which is the simplest nontrivial $U(2)$ ansatz compatible with eq. (II.97). In the operator formalism,

$$P_1 = T_1(T_1^\dagger T_1)^{-1}T_1^\dagger = \begin{pmatrix}(1+\overline{w}w)^{-1} & (1+\overline{w}w)^{-1}\overline{w}\\ w(1+\overline{w}w)^{-1} & w(1+\overline{w}w)^{-1}\overline{w}\end{pmatrix}. \tag{II.108}$$

Our task is now to determine a possible solution for T_2. We employ the ansatz

$$T_2 = \begin{pmatrix}u_1(\tilde{t},w,\overline{w})\\ u_2(\tilde{t},w,\overline{w})\end{pmatrix}. \tag{II.109}$$

Setting $\mathcal{S}_1 = w$ and inserting (II.109) into eq. (II.107a) yields

$$\big[w,u_1\big] \;=\; \big[w,(1+\overline{w}w)^{-1}\big](u_1+\overline{w}u_2) + 2\theta(1+\overline{w}w)^{-1}u_2\,, \tag{II.110a}$$

$$\big[w,u_2\big] \;=\; w\big[w,(1+\overline{w}w)^{-1}\big](u_1+\overline{w}u_2) + 2\theta w(1+\overline{w}w)^{-1}u_2\,. \tag{II.110b}$$

The last two equations immediately imply

$$[w, wu_1 - u_2] = 0\,. \tag{II.111}$$

The case $\widetilde{\boldsymbol{\theta}} = \mathbf{0}$. Evidently, if we restrict ourselves to $[t,\tilde{t}] = \mathrm{i}\widetilde{\theta} = 0$,

$$u_2 = wu_1 - f(\tilde{t}, w) \tag{II.112}$$

solves eq. (II.111) with an arbitrary function f (depending only on $\tilde{t}$ and w) yet to be determined. Exploiting eqs. (II.110a) and (II.111), we find a solution

$$u_1 = 1 + (1+\overline{w}w)^{-1}\overline{w}f(\tilde{t},w)\,,\quad u_2 = w - (1+\overline{w}w+2\theta)^{-1}f(\tilde{t},w)\,. \tag{II.113}$$

From (II.107b) we obtain in a similar fashion

$$\partial_{\tilde{t}}u_1 = \eta'\big[\overline{w},(1+\overline{w}w)^{-1}\big](u_1+\overline{w}u_2) \tag{II.114}$$

by setting $\mathcal{S}_2 = 0$. Taking into account eq. (II.114) the explicit $\tilde{t}$-dependence of $f(\tilde{t},w)$ can be easily deduced:

$$f = 2\theta\eta'\left(\tilde{t} + h(w)\right), \tag{II.115}$$

for some function h meromorphic in w. Finally, substituting the results into (II.109) leads to

$$T_2 = \begin{pmatrix}1\\ w\end{pmatrix} + \begin{pmatrix}\overline{w}\\ -1\end{pmatrix}(1+\overline{w}w+2\theta)^{-1}f(\tilde{t},w)\,. \tag{II.116}$$

Translating this to the star formalism, we easily read off that $T_1 = T_2$ at the zero locus of $f_\star(\tilde{t}, w)$; moreover, if we restrict μ to be purely imaginary, $\mu = -\mathrm{i}p$, $p \in (1, \infty)$, Ω degenerates at these points to the identity. If we choose $h_\star = w \star w = w^2$, which corresponds to two moving soliton-like objects, this leads to right angle scattering [109]:

$$f_\star = 0 \quad \Rightarrow \quad w = \pm\sqrt{-\tilde{t}}\,. \tag{II.117}$$

For the points in this locus, w is purely real for $\tilde{t} < 0$, and w is purely imaginary for $\tilde{t} > 0$. Since for the above choice of μ,

$$w = \left(\frac{2}{p + p^{-1}}\right)^{1/2} \left(x + \frac{\mathrm{i}}{2}(p + p^{-1})y + \frac{\mathrm{i}}{2}(p - p^{-1})t\right), \tag{II.118}$$

we see that e.g. for $t = 0$, the point where $\Omega = 1$ moves along the positive x-axis accelerating towards the origin for negative $\tilde{t}$. For positive $\tilde{t}$ it decelerates during its motion along the positive (or negative, depending on the sign in (II.117)) y-axis.

The case $\boldsymbol{\widetilde{\theta} \neq 0}$. If the two time directions do not mutually commute, i.e., $\widetilde{\theta} \neq 0$, eq. (II.107b) can be written as

$$\frac{1}{\mathrm{i}\widetilde{\theta}}[t, T_2] - \eta'[w, P_1]T_2 = T_2\mathcal{S}_2\,. \tag{II.119}$$

Now, we can still solve eq. (II.111) by

$$u_2 = wu_1 - g(\tilde{t}, \overline{w}, w)\,. \tag{II.120}$$

The difference to the case $\widetilde{\theta} = 0$ is that now the vanishing of the commutator (II.111) can only be achieved by a nontrivial choice for $g(\tilde{t}, \overline{w}, w)$, e.g.,

$$g(\tilde{t}, \overline{w}, w) = \tilde{t} + \alpha\overline{w} + h(w)\,, \tag{II.121}$$

where $\alpha := -\frac{\mathrm{i}}{4}(\bar{\mu} + \bar{\mu}^{-1})\frac{\widetilde{\theta}}{\theta}$ and $h(w)$ is again an arbitrary function meromorphic in w. Let us restrict μ again to be purely imaginary, $\mu = -\mathrm{i}p$, $p \in (1, \infty)$, then $\alpha \in \mathbb{R}_+$.

Apparently we also need $\theta \neq 0$; then, the contributions of $[w, \overline{w}]$ and $[w, \tilde{t}]$ add up to zero. If we use the inverse Moyal-Weyl map to translate to the star product and choose $h_\star(w) = w \star w = w^2$, we obtain

$$g_\star(\tilde{t}, \overline{w}, w) = \tilde{t} + \alpha\overline{w} + w^2\,. \tag{II.122}$$

The subsequent calculation is analogous to the case $\widetilde{\theta} = 0$. It turns out that P_1 and P_2 coincide and $\Omega = 1$ at the locus of $g_\star(\tilde{t}, \overline{w}, w)$, i.e., $\tilde{t} + \alpha\overline{w} + w^2 = 0$. If we split w into real and imaginary parts,

$$w = a + \mathrm{i}b\,, \tag{II.123}$$

we can easily read off a and b from eq. (II.118), and the locus where $\Omega = 1$ is given by

$$-\tilde{t} = \alpha a + a^2 - b^2 + \mathrm{i}(2a - \alpha)b\,. \tag{II.124}$$

Since $\tilde{t}$ is real, obviously either $b = 0$ or $a = \alpha/2$. We obtain

$$b = 0 \qquad \Longrightarrow \qquad a = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \tilde{t}} \qquad \text{for} \qquad \tilde{t} \leq \frac{\alpha^2}{4}\,, \tag{II.125a}$$

$$a = \frac{\alpha}{2} \qquad \Longrightarrow \qquad b = \pm\sqrt{\frac{3}{4}\alpha^2 + \tilde{t}} \qquad \text{for} \qquad \tilde{t} \geq -\frac{3}{4}\alpha^2\,. \tag{II.125b}$$

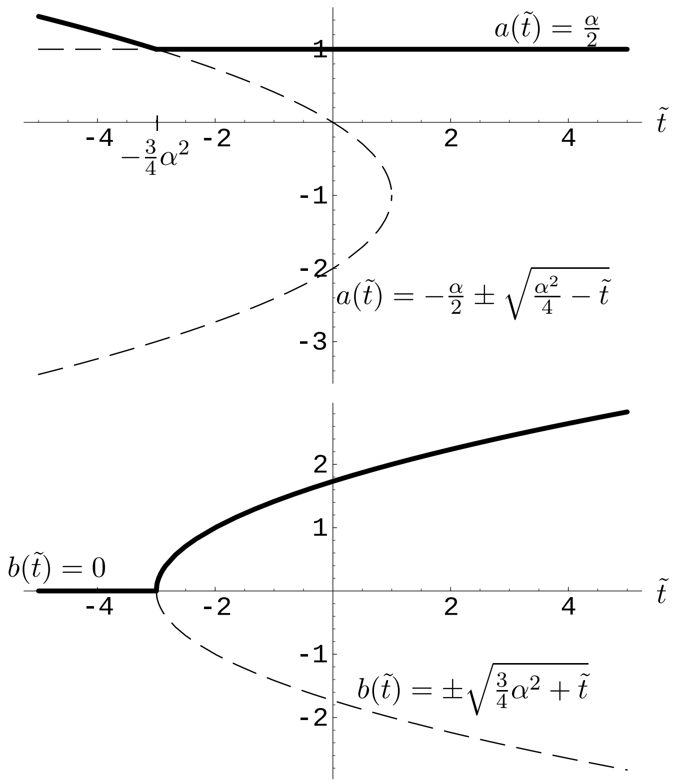

Figure II.2. *Motion of the "point of degeneracy" where $\Omega = 1$ in $\tilde{t}$ (bold lines). Its coordinates a and b are plotted for $\alpha = 2$. For $b(\tilde{t})$, exemplarily the upper branch was chosen.*

This can be interpreted as follows: The "point of degeneracy" where $\Omega = 1$ moves along $a = -\alpha/2 + \sqrt{\alpha^2/4 - \tilde{t}}$ and $b = 0$ as $\tilde{t}$ grows until $\tilde{t} = -3\alpha^2/4$. Then, as $\tilde{t}$ grows larger, it moves along $a = \alpha/2$ and $b = \pm\sqrt{3\alpha^2/4 + \tilde{t}}$ (see figure II.2). With the help of (II.118), it is easy to interpret this motion in the xy plane (for fixed t). Therefore we have shown that it is possible to construct nontrivial configurations with scattering also for the case of noncommuting time directions. More complicated solutions in both cases may be constructed by making different choices for $h(w)$ or by choosing a more sophisticated ansatz for T_1 and T_2.

II.6.2 Colliding plane waves

Beside the soliton-like solutions (discussed above), there is another class of exact solutions to the self-duality equations (II.22), namely extended plane waves. For asymptotic times, each of them has codimension two. In the commutative case, these were constructed and discussed in [43, 100, 157]. In the context of the $U(N)$ sigma model in 2+1 dimensions, this type of solution was first discussed by Leese [117]; the noncommutative generalization was given in [30]. In [71], plane waves were described in (noncommutative) D1-D3 systems. Here we want to show that one can construct noncommutative two-wave solutions in ncSDYM which entail nontrivial scattering even for $\mu_1 \neq \mu_2$.

Additive ansatz. We assume $\mu_1 \neq \mu_2$ henceforth and therefore make a single-pole ansatz for the auxiliary field ψ. In this section, exceptionally all products are understood to be star products (including the inverse and the exponential of coordinates). The calculation is largely parallel to the derivation in section II.5.2 which gives us the opportunity to shorten the description here and to concentrate on the novel features.

We start from the additive ansatz (II.89), but now choose, inspired by [117, 30], the exponential ansätze

$$T_1 = \begin{pmatrix} 1 \\ \mathrm{e}^{b_1 w_1} \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 \\ \mathrm{e}^{b_2 \widetilde{w}_2} \end{pmatrix} \tag{II.126}$$

with $b_1 \in \mathbb{R}_{>0}, b_2 \in \mathbb{R}$. The discussion in section II.4.1 guarantees that this will yield a solution to the self-duality equations. However, it is not obvious that this solution factorizes into two plane waves for asymptotic times; to prove this, we have to compare with the multiplicative ansatz again.

Multiplicative ansatz. The multiplicative ansatz takes the same form as eq. (II.90). It can be easily shown as in section II.5.2 that $P_1 = \widetilde{T}_1(\widetilde{T}_1^\dagger \widetilde{T}_1)^{-1}\widetilde{T}_1^\dagger$ can consistently be constructed from $\widetilde{T}_1 = T_1$, given in (II.126). Let us now scrutinize the $|t| \to \infty$ limits of P_1. For simplicity, we set $\mu_1 = \mathrm{i}p$ strictly imaginary with $p > 1$. Therefore, β_1 in (II.50) is real and

$$\beta_1 = -\frac{1}{2}\theta^{-1/2}(p - p^{-1})(p + p^{-1})^{-1/2} < 0\,. \tag{II.127}$$

If we consider the large t limit, it turns out that w_1 is dominated by the term linear in t, namely:

$$b_1 w_1 \simeq \pm b_1 \sqrt{2\theta}|\beta_1|t, \quad \text{for } t \to \pm\infty\,. \tag{II.128}$$

Thus, P_1 in the large t limit behaves as

$$P_1 = \begin{pmatrix} (1 + e^{b_1\overline{w}_1} e^{b_1 w_1})^{-1} & (1 + e^{b_1\overline{w}_1} e^{b_1 w_1})^{-1} e^{b_1\overline{w}_1} \\ e^{b_1 w_1}(1 + e^{b_1\overline{w}_1} e^{b_1 w_1})^{-1} & e^{b_1 w_1}(1 + e^{b_1\overline{w}_1} e^{b_1 w_1})^{-1} e^{b_1\overline{w}_1} \end{pmatrix},$$

i. e.,

$$P_1 \to \begin{cases} \overset{t\to+\infty}{\longrightarrow} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: \Pi_{+\infty}\,, \\ \overset{t\to-\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: \Pi_{-\infty}\,. \end{cases} \tag{II.129}$$

In these limits, P_1 obviously becomes a constant projector. Again, the Lax operators in (II.94) can be moved next to P_2 in these limits to give (II.95). This concludes the proof that we may write $\widetilde{T}_2 = T_2$ asymptotically.

In addition, we can conclude that this setup entails nontrivial scattering, again by analyzing $\Omega^\dagger = \psi_2(\zeta = 0)$ in the limits $t \to \pm\infty$:

$$\Omega^\dagger\Big|_{t\to+\infty} = \lim_{t\to+\infty} \psi_2(\zeta=0) = (1-\rho_2 P_2)(1-\rho_1 \Pi_{+\infty})\,, \tag{II.130a}$$

$$\Omega^\dagger\Big|_{t\to-\infty} = \lim_{t\to-\infty} \psi_2(\zeta=0) = (1-\rho_2 P_2)(1-\rho_1 \Pi_{-\infty})\,. \tag{II.130b}$$

For convenience, we have set $\rho_k = 1 - \bar{\mu}_k/\mu_k$ for $k = 1, 2$. Clearly, $\Omega^\dagger\Big|_{t\to+\infty}$ and $\Omega^\dagger\Big|_{t\to-\infty}$ are different, which indicates nontrivial scattering behavior.

If we now additionally take $\tilde{t} \to \pm\infty$, we find that P_2 also becomes a constant projector,

$$\lim_{\tilde{t}\to\pm\infty} P_2 = \Pi_{\pm\infty}\,. \tag{II.131}$$

Therefore,

$$\Omega^\dagger\Big|_{t,\tilde{t}\to-\infty} = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \tag{II.132a}$$

and

$$\Omega^\dagger\Big|_{t,\tilde{t}\to+\infty} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \tag{II.132b}$$

where $\gamma := \bar{\mu}_1\bar{\mu}_2\mu_1^{-1}\mu_2^{-1}$. Again, this result shows the existence of scattering in this two-wave configuration.

The above-described solutions represent 1+2 dimensional plane waves in the asymptotic domain, i.e., long before and after the interaction. This can be seen by analyzing the energy density in 2+1 dimensional subspaces, e.g., the energy density for a gauge field constructed from P_1 (at a fixed time $\tilde{t}$) turns out to depend only on one spatial direction [30]. The asymptotic space-time interpretation for this setup can be visualized by the following snapshot for fixed large t (see figure II.3). Since P_1 is independent of $\tilde{t}$, the corresponding wave extends along this direction. The above energy density argument explains its spatial extension. Observe that for this type of solutions, the moduli μ_1, μ_2 not only parametrize the velocities of the plane waves but also their respective parallel directions in the xy-plane (cf. eqs. (II.48) and (II.55) together with (II.126)). When t varies in the asymptotic region, the world-volume of the first plane wave undergoes a parallel shift. Consider a space-like section (i.e., t *and* $\tilde{t}$ fixed). Then, the intersection of the two plane waves with this xy-plane will consist of two lines which generically include some angle determined by the moduli μ_1 and μ_2. For later times t or $\tilde{t}$, the lines corresponding to P_1 and P_2 have changed position in the xy-subspace but kept their directions.

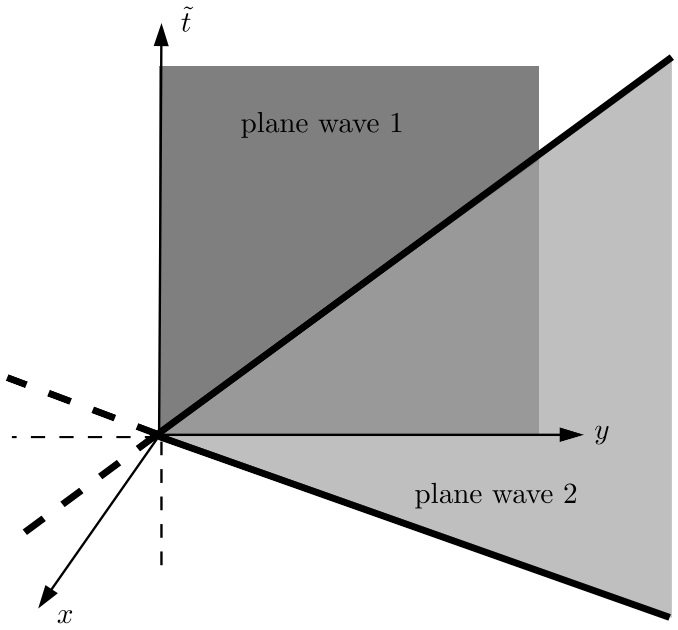

Figure II.3. *Snapshot for fixed large $|t|$ of one octant of the configuration discussed in section II.6.2. For simplicity, the first plane wave was chosen to be static ($\mu_1 = -i$). This choice implies that the energy density for the first wave at fixed $\tilde{t}$ does not depend on y. The support of the solution is concentrated around the grey planes.*

II.7 Conclusions

In this chapter we have discussed exact solutions to the self-duality equations of noncommutative Yang-Mills theory on $\mathbb{R}^{2,2}$. To this aim, a Lax pair has been gauged in two inequivalent ways; appropriate ansätze for the auxiliary field ψ have been discussed. From concrete solutions ψ to the residue equations of the Lax pair explicit expressions for the gauge potentials have been constructed. In appendix E.2 it will be shown that the Lax pair is included in the string field theoretic one; therefore, it seems plausible that this also applies to its solutions. Conversely, our field theoretic solutions could serve as a guideline to construct nonperturbative solutions of N=2 string field theory. It seems reasonable to expect that a similar program could be carried out for N=1 strings.

A GMS-like solution and solutions describing $U(2)$ solitons have been constructed. Moreover, it has been shown that dimensional reduction to 2+1 dimensions leads to results coinciding with those of [108, 109, 30, 189]. Explicitly, the field theory description of D-brane scattering (for plane wave and soliton-like configurations) has been generalized to the 2+2 dimensional case. It would be interesting to trace this description to the string theory level, i.e., compute scattering in the given B-field background by closed string exchange. To corroborate the interpretation of our field theory solutions as lower-dimensional D-branes one could try to compute their Chern characters and examine the fluctuation spectrum around these solutions.

Chapter III

Short introduction to string field theory

III.1 Introduction

String field theory combines graphic ideas and abstract algebraic manipulations into a definition of an off-shell theory of interacting strings. It describes exactly how strings interact; the graphical picture of an interaction vertex ultimately leads to the definition of a rather abstract product on the algebra of string fields. Historically, two different gluing prescriptions for the description of this interaction have been of major importance, namely the endpoint and the midpoint gluing prescriptions. According to the former, two open strings join at their endpoints; the field theory based on this string field product leads to scattering amplitudes in accordance with conformal field theory results. Since the parametrization of the product of three strings obviously depends on the order in which three strings are joined, a reparametrization invariant formulation is required for the product to be associative. A drawback of such theories is, however, the loss of manifest Lorentz invariance (for an introduction and review, see, e. g., [123, 65]). Light-cone string field theory has undergone a renascence in the last years, in particular due to the recent developments in plane wave geometries.

Witten's proposal [186] was to use BRST invariance as a substitute for reparametrization invariance; this admits a covariant formulation of open string field theory which maintains the desirable aspects of reparametrization invariance as long as the BRST current is conserved. The prescription he gave for the gluing of two strings can easily be generalized to higher order vertices; furthermore, an "integration" operation can be naturally defined. These operations are used to formulate an action functional for string field theory, which possesses enough gauge invariance to decouple BRST-trivial states. Witten showed that in the bosonic as well as in the superstring cases, a cubic action functional is sufficient; formally, it takes in both cases the well-known Chern-Simons form. In the superstring case [186], the picture degeneracy of physical states necessitates the introduction of picture changing operators at the interaction points; this was later on shown to give rise to so-called contact term divergences [185, 113].

As an alternative, Berkovits came up with a proposal for a different superstring field theory based on Witten's star product [19]. Its action is nonpolynomial and takes the form of a

Wess-Zumino-Witten action. No picture insertions are necessary; therefore, the theory avoids problems with contact term divergences (however at the cost of a more complicated action functional). Both, Witten's cubic bosonic string field theory as well as Berkovits' nonpolynomial superstring field theory have been shown to reproduce the conformal field theory amplitudes at tree level [187, 138, 23].

In all vertices in string field theories based on Witten's star product, the midpoint of the strings is singled out (this is the reason that, indeed, the proposed vertices are only invariant under midpoint-preserving reparametrizations); the left half of the first string should be glued to the right half of the second string. This leads to N-vertices with manifest cyclicity and associativity.

This chapter is organized as follows: In section III.2, we enlarge on the algebraic underpinnings of cubic string field theory. Their relevance for Berkovits' nonpolynomial string field theory will become clear later. In section III.3, we introduce cubic bosonic string field theory, cubic superstring field theory in two different modifications, and nonpolynomial superstring field theory. This discussion lays the foundation for the forthcoming chapters. In section III.4, we shortly explicate the concepts of vacuum string field theory, which will be scrutinized in chapter V. The importance of projectors of the star algebra becomes clear in this framework; since they also play a crucial role in the dressing approach in chapters IV and V, some known facts on these string fields will be summarized in section III.5.

III.2 Algebraic structure

Before embarking on a more detailed discussion of the different string field theories, let us first briefly explain the algebraic basics of Witten's cubic bosonic and superstring field theories [58].

String fields. An open string is described by a state in the Fock space of its boundary conformal field theory. In the case of a bosonic string, this is a complex vector space[1] $\mathcal{H}$ with a natural $\mathbb{Z}$-grading $\#_{bc}$ given by ghost number (i. e., the zero-mode of the ghost number current $J_{bc} = -bc$, cf. appendix B.1). In the superstring case the grading of this complex vector space is extended to $\mathbb{Z}^3$, where the second $\mathbb{Z}$-grading $\#_{\eta\xi}$ is induced by the zero-mode of the ghost number current $J_{\eta\xi} = \xi\eta$ of the $\eta\xi$-system and the third $\mathbb{Z}$-grading $\#_\phi$ is induced by the zero-mode of the ϕ-charge current $J_\phi = -\partial\phi$. The diagonal of the last two gradings is the *picture grading*, a $\mathbb{Z}$-grading w.r.t. the zero-mode of the current $J_{\text{pic}} = J_\phi + J_{\eta\xi}$. The first and the second $\mathbb{Z}$-gradings can be combined into a further $\mathbb{Z}$-grading $\#_{\text{gh}}$ which is given by the zero-mode of the total ghost number current $J_{\text{gh}} = J_{bc} - J_{\eta\xi}$. It is, however, often useful to work with the original $\mathbb{Z}^3$-grading. The world-sheet statistics of elements of the vector space $\mathcal{H}$ depends on its $\#_{\text{gh}}$-charge. The above-mentioned gradings induce gradings on any tensorial power of $\mathcal{H}$ and $\mathcal{H}^*$ so that one can assign, e. g., ghost numbers to elements of $\mathcal{H}^{\otimes n}$ for any n.

[1] We neglect Chan-Paton labels for the time being. – In string field theory, it will sometimes be necessary to take the closure of $\mathcal{H}$ for the definition of string fields since the star product of two finite linear combinations of basis states in $\mathcal{H}$ generically will include infinite linear combinations. We shall like to avoid a discussion of the subtleties related with this phenomenon and denote both with the same letter $\mathcal{H}$.

Given a basis $\{|\Phi_i\rangle\}$ in $\mathcal{H}$, we can introduce a dual basis $\{\langle\Phi_i|\}$ in the dual vector space $\mathcal{H}^*$ by demanding that

$$\langle\Phi_i|\Phi_j\rangle = \delta_{ij}\,. \tag{III.1}$$

The dual space can be endowed with a grading (which we denote by the same symbols) by

$$\#_{bc}(\langle\Phi_i|) := -\#_{bc}(|\Phi_i\rangle)\,,\quad \#_{\eta\xi}(\langle\Phi_i|) := -\#_{\eta\xi}(|\Phi_i\rangle)\,,\quad \text{and}\quad \#_{\phi}(\langle\Phi_i|) := -\#_{\phi}(|\Phi_i\rangle)\,. \tag{III.2}$$

This choice will be substantiated in appendix B.1, where also the conventions used here can be found.

At the same time, we can associate a space-time field φ^i with each basis vector $|\Phi_i\rangle$ of $\mathcal{H}$ of definite grade. The space-time fields φ^i depend only on the position zero-mode of the string; therefore, their multiplication is point-wise. They form a (graded) algebra $\mathcal{G}$ with a multiplicative action on $\mathcal{H}$; $\mathcal{H}$ is a module over $\mathcal{G}$. A general element of the $\mathcal{G}$-module $\mathcal{H}$ has the form

$$\sum_i |\Phi_i\rangle g^i\,, \tag{III.3}$$

where the g^i are arbitrary elements of $\mathcal{G}$. The subspace of elements $\sum_i |\Phi_i\rangle\varphi^i$ is the space of *string fields* which are the basic ingredients of any string field theory. They are conventionally given in momentum space. For instance, a string field in the bosonic case could take the form

$$\Psi = \int d^{26}p\,[t(p)|0,p\rangle \otimes |{\downarrow}\rangle_{bc} + A_\mu(p)\alpha_{-1}^\mu|0,p\rangle \otimes |{\downarrow}\rangle_{bc} + \ldots]\,. \tag{III.4}$$

Here, t is the tachyon potential and A_μ is the gauge potential for the $U(1)$ gauge field on the brane, which carries the open strings described by our string field theory.

The ghost number grading of $\mathcal{H}$ induces a $\mathbb{Z}$-grading gr on $\mathcal{G}$ which can be adjusted in such a way that the fields corresponding to physical bosons and fermions have even and odd degree, respectively. Now, the module $\mathcal{H}$ in bosonic string field theory has a natural $\mathbb{Z}_2$-degree (the *Grassmann parity*) which is defined as the sum of the degrees $\#_{\text{gh}}$ (on the corresponding vertex operators) and gr for the basis elements:

$$\deg\Big(\sum_i |\Phi_i\rangle\varphi^i\Big) := \sum_i \big(\#_{\text{gh}}(|\Phi_i\rangle) + \tfrac{3}{2} + gr(\varphi^i)\big) \pmod 2\,. \tag{III.5}$$

The shift by $\frac{3}{2}$ is necessitated by the normalization of the ghost number for states (B.10). In the Neveu-Schwarz sector of superstring field theory, $\#_{\text{gh}} + \frac{3}{2}$ in eq. (III.5) has to be replaced by $\#_{bc} + \#_{\eta\xi} + 2 + \#_\phi$. In the Ramond sector, a single state in the Hilbert space of conformal field theory has no definite world-sheet statistics.

Since the vector space $\mathcal{H}$ can be identified with the space of vertex operators via the state-operator correspondence (thereby mapping $|\Phi_i\rangle$ to some vertex operator V_i), we will often conveniently denote string fields as $\Psi = \sum_i \varphi^i V_i$. In the above example (III.4) this means that $\varphi^i \in \{t, A_\mu, \ldots\}$, $\Phi_i \in \{|0,p\rangle \otimes |{\downarrow}\rangle_{bc}, \alpha_{-1}^\mu|0,p\rangle \otimes |{\downarrow}\rangle_{bc}, \ldots\}$, and $V_i \in \{ce^{\mathrm{i}p\cdot X}, c\partial X e^{\mathrm{i}p\cdot X}, \ldots\}$. The state-operator correspondence is one-to-one, thus enabling us to switch to the description most appropriate for the respective structure. Whenever there is a difference between both descriptions of string fields (e. g., concerning ghost numbers[2]), we will make the statement precise.

It is customary to use a third representation of string fields, namely as functionals of the world-sheet fields (which we will denote collectively by X for this purpose). One can easily translate states to this description by contracting them with a bra-eigenvector of all world-sheet oscillators, i. e. symbolically

$$\Psi[X(\sigma)] = \langle X(\sigma)|\Psi\rangle\,. \tag{III.6}$$

This will be substantiated in appendix E.1. For different representations of string fields, we refer to [3]. Sometimes, we stick to the convenient notation $(-1)^{\deg(|\Psi\rangle)} =: (-1)^{|\Psi\rangle} = (-1)^{\Psi}$ for the Grassmannality of $|\Psi\rangle \in \mathcal{H}$. In this section, general string fields will be called Ψ and Υ whereas the Witten string field will be denoted by A.

The dual module $\mathcal{H}^*$ is defined as the space of states

$$\sum_i f^i \langle \Phi_i| \tag{III.7}$$

with $f^i \in \mathcal{G}$ arbitrary. The pairing $\langle \Phi_i|f^i g^j|\Phi_j\rangle = f^i g^j \delta_{ij}$ defines an element of $\mathcal{G}$. The Grassmannality of vectors in $\mathcal{H}^*$ is defined as the Grassmann parity of their dual vectors. In particular, we have $(-1)^{|\downarrow\rangle} = (-1)^{\langle\uparrow|} = -(-1)^{\langle\downarrow|}$. Thus, dual pairings such as $\langle\uparrow|\downarrow\rangle$ are Grassmann-even.

BRST operator. The BRST operator is a Grassmann-odd endomorphism $Q\colon \mathcal{H} \to \mathcal{H}$; it serves as a kinetic operator. In the critical dimension, Q should be nilquadratic:

$$Q^2\Psi = 0\,. \tag{III.8}$$

The action of Q on string fields is defined by the contour integral of the BRST current around the corresponding vertex operator:

$$(Q\Psi)(z) = \oint_z \frac{dw}{2\pi \mathrm{i}}\, J_{BRST}(w)\Psi(z)\,. \tag{III.9}$$

The grading of $\mathcal{H}$ carries over to a grading of End $\mathcal{H}$ which we denote by the same symbols. Then we have

$$\#_{\mathrm{gh}}(Q) = 1\,, \quad \#_{bc}(Q) = 1\,, \quad \text{and} \quad \#_{\mathrm{pic}}(Q) = 0\,. \tag{III.10}$$

In the superstring case the BRST operator can be decomposed into three parts Q_0, Q_1, and Q_2 of ϕ-charge 0,1, and 2, respectively. The $\eta\xi$-charge is always its negative, e. g., $\#_{\eta\xi}(Q_2) = -2$.

BRST invariance will impose severe constraints on the additional structures to be introduced below.

[2]For an explanation of the measuring of ghost numbers etc. for states and for operators, see also appendix B.1.

Graded symmetric form. In order to formulate an action for the string fields, we have to introduce a (graded) symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$. As a bilinear map from $\mathcal{H}\otimes\mathcal{H}$ to $\mathbb{C}$, it is given by a *reflector* element $\langle V_2| \in \mathcal{H}^*\otimes\mathcal{H}^*$ through

$$\langle\Psi,\Upsilon\rangle = {}_{12}\langle V_2|\Psi\rangle_1|\Upsilon\rangle_2\,. \tag{III.11}$$

The subscripts 1 and 2 label the two spaces in the tensor product corresponding to the different strings. If we require that ${}_{12}\langle V_2| = {}_{21}\langle V_2|$, this implies that

$$\langle\Psi,\Upsilon\rangle = (-1)^{\Psi\Upsilon}\langle\Upsilon,\Psi\rangle\,. \tag{III.12}$$

This is just the statement of graded symmetry. The state $\langle V_2|$ will be later on identified with the 2-string vertex. The graded symmetric form can be realized by the prescription that two strings with opposite orientation should be glued together to give a number. The bilinear map itself does not carry any ghost number[3] (this will become clear for the 2-vertex later in chapter VI); with the above conventions, the ghost anomaly on the disk therefore puts a restriction on the ghost numbers of two operators with nonvanishing product:

$$\langle\Psi,\Upsilon\rangle \neq 0 \qquad \Longrightarrow \qquad \#_{bc}(\Psi)+\#_{bc}(\Upsilon)=3\,. \tag{III.13}$$

Obviously, $\langle\cdot,\cdot\rangle$ couples only states of opposite $\mathbb{Z}_2$-grade. For states, we glean that the bc-ghost numbers have to add up to zero. In the superstring case we take $\langle\cdot,\cdot\rangle$ to be neutral w.r.t. $\#_{\eta\xi}$ and $\#_\phi$, therefore we obtain two additional requirements (on operators):

$$\langle\Psi,\Upsilon\rangle \neq 0 \qquad \Longrightarrow \qquad \#_{\eta\xi}(\Psi)+\#_{\eta\xi}(\Upsilon)=1 \quad \text{and} \quad \#_\phi(\Psi)+\#_\phi(\Upsilon)=-2\,. \tag{III.14}$$

Just as the Hilbert space $\mathcal{H}$ is a tensor product of the matter and the ghost parts, one can decompose $\langle V_2| \in \mathcal{H}^*\otimes\mathcal{H}^*$ into its matter and ghost parts. This statement generalizes in the obvious way to the case of superstrings. Then, the reflector state in a fermionic first order system is Grassmann-odd, otherwise even.

The bilinear form is invertible; there exists a $|V_2\rangle \in \mathcal{H}\otimes\mathcal{H}$ such that

$$ {}_{12}\langle V_2|V_2\rangle_{23} = {}_3\quad{}_1\,. \tag{III.15}$$

This implies that for fermionic first order systems with odd background charge (such as the bc and the $\eta\xi$ systems) $|V_2\rangle$ is antisymmetric under interchange of the string labels, since

$$\begin{aligned} {}_{12}\langle V_2| &= {}_{14}\langle V_2|\,{}_{23}\langle V_2|V_2\rangle_{34} \\ = {}_{21}\langle V_2| &= {}_{23}\langle V_2|\,{}_{14}\langle V_2|V_2\rangle_{43}\,. \end{aligned} \tag{III.16}$$

The right hand sides in both lines are equal upon interchange of the Grassmann-odd $\langle V_2|$'s; thus, $|V_2\rangle_{34} = -|V_2\rangle_{43}$.

[3]This is a matter of convention. In [18], e. g., the authors made a different choice.

Explicit forms for $\langle V_2|$ and $|V_2\rangle$ in terms of oscillators for bosonic strings are given, amongst others, in [66, 68, 94]. For superstrings, a two-vertex for the bosonized superghosts has been constructed in [18].

The requirement that the reflector be BRST invariant,[4]

$$ {}_{12}\langle V_2|(Q^{(1)} + Q^{(2)}) = 0\,, \tag{III.17} $$

can be translated into the statement that the BRST charge is self-adjoint (up to a sign) w.r.t. to the graded symmetric form $\langle\cdot,\cdot\rangle$:

$$ \begin{aligned} {}_{12}\langle V_2|Q^{(1)}|\Psi\rangle_1|\Upsilon\rangle_2 = -{}_{12}\langle V_2|Q^{(2)}|\Psi\rangle_1|\Upsilon\rangle_2 = -(-1)^{\Psi}\langle V_2|\Psi\rangle_1 Q^{(2)}|\Upsilon\rangle_2 \\ \Longrightarrow \quad \langle Q\Psi,\Upsilon\rangle = -(-1)^{\Psi}\langle\Psi, Q\Upsilon\rangle\,. \end{aligned} \tag{III.18} $$

The kinetic term in Witten's Chern-Simons like string field theory is proportional to $\langle A, QA\rangle$. From eqs. (III.12) and (III.18) we conclude that the string field A has to be Grassmann-odd in order to have a nonvanishing kinetic term:

$$ \langle A, QA\rangle = (-1)^{A(1+A)}\langle QA, A\rangle = \langle QA, A\rangle = -(-1)^{A}\langle A, QA\rangle\,. \tag{III.19} $$

BPZ conjugation. The above 2-string vertex $\langle V_2|$ defines a linear map from $\mathcal{H}$ to $\mathcal{H}^*$ via

$$ {}_2\langle \mathrm{bpz}(\Psi)| := {}_{12}\langle V_2|\Psi\rangle_1\,. \tag{III.20} $$

The left hand side is the so-called *BPZ conjugate* state to $|\Psi\rangle$; it obviously satisfies $\langle\mathrm{bpz}(\Psi)|\Upsilon\rangle = \langle\Psi,\Upsilon\rangle$. Note that BPZ conjugation is linear, complex numbers are not conjugated. In terms of the state-operator correspondence, the BPZ conjugate can be understood as follows: Let $|0\rangle$ denote a vacuum state (in some oscillator basis) in $\mathcal{H}$ and $\langle 0|$ its dual. Then, a conformal field Φ can be mapped to a state $|\Phi\rangle \in \mathcal{H}$ by $|\Phi\rangle := \lim_{z\to 0}\Phi(z)|0\rangle$; a primary field inserted at the origin (i.e., at $\tau = -\infty$) of the complex plane generates the state $|\Phi\rangle$ from $|0\rangle$. The conformal transformation $I(z) := -1/z$ applied to Φ maps $\tau = -\infty$ to $\tau = \infty$; its application to the bra-vacuum $\langle 0|$ yields the BPZ conjugate state $\langle\mathrm{bpz}(\Phi)|$:

$$ \langle\mathrm{bpz}(\Phi)| := \lim_{z\to\infty}\frac{1}{z^{2h}}\langle 0|\Phi\Big(-\frac{1}{z}\Big)\,, \tag{III.21} $$

where h denotes the weight of Φ. Inserting the mode expansion $\Phi(z) = \sum_n \Phi_n z^{-n-h}$, it is easy to see that BPZ conjugation acts on modes as

$$ \mathrm{bpz}(\Phi_n) = \Phi_{-n}(-1)^{n+h}\,. \tag{III.22} $$

Eq. (III.20) entails that BPZ conjugation acts as a graded antihomomorphism, i. e., the order of the modes is reversed with additional signs for each interchange of Grassmann-odd quantities:

$$ \mathrm{bpz}(\Phi_n\Lambda_m) = \mathrm{bpz}(\Lambda_m)\,\mathrm{bpz}(\Phi_n)\,(-1)^{\Phi\Lambda}\,. \tag{III.23} $$

[4] Since operators sometimes also carry other indices, string labels in this case are denoted as superscripts in brackets.

The 2-string vertex is defined in such a way that the definitions (III.20) and (III.21) coincide: If we parametrize the world-sheet instead with coordinates σ and τ, the inversion I sends σ to $\pi - \sigma$ and τ to $-\tau$ (cf. the conventions in appendix B.1). This latter fact will become important for the overlap conditions which the Neumann matrices have to satisfy, cf. chapter VI.

The state $|V_2\rangle$ induces the inverse map $\mathrm{bpz}^{-1}\colon \mathcal{H}^* \to \mathcal{H}$, $\langle\Psi| \mapsto \big|\mathrm{bpz}^{-1}(\langle\Psi|)\big\rangle$ with

$$\big|\mathrm{bpz}^{-1}(\langle\Psi|)\big\rangle := {}_1\langle\Psi|V_2\rangle_{12}\,(-1)^{\Psi} \,. \tag{III.24}$$

The sign factor ensures that $\mathrm{bpz} \circ \mathrm{bpz}^{-1} = \mathrm{bpz}^{-1} \circ \mathrm{bpz} = \ $. Because of eq. (III.15), this can be rewritten as

$$\big\langle \mathrm{bpz}^{-1}(\langle\Psi|), \Upsilon\big\rangle = {}_{23}\big\langle V_2\big|\mathrm{bpz}^{-1}(\langle\Psi|)\big\rangle_2\big|\Upsilon\big\rangle_3 = {}_{23}\langle V_2|V_2\rangle_{12}\,{}_1\langle\Psi|\Upsilon\rangle_3 = \langle\Psi|\Upsilon\rangle \,. \tag{III.25}$$

This establishes a connection with the dual pairing defined after eq. (III.7).

Hermitean conjugation. Hermitean conjugation[5] hc is a map from $\mathcal{H}$ to $\mathcal{H}^*$ with

$$\overline{\langle \mathrm{hc}(\Psi)|\Upsilon\rangle} = \langle \mathrm{hc}(\Upsilon)|\Psi\rangle \,, \tag{III.26}$$

where the bar denotes complex conjugation in $\mathcal{G}$. Complex conjugation in the Grassmann algebra $\mathcal{G}$ is defined as an antiautomorphism of the algebra, i. e., $\overline{fg} = \overline{g}\,\overline{f}$. On complex numbers, hc acts as complex conjugation. As well as BPZ conjugation, it preserves the Grassmann parity. Formally, we denote the inverse as hc^{-1}. The BRST operator is required to be hermitean w.r.t. hc, i. e.,

$$\mathrm{hc}(Q|\Psi\rangle) = \langle \mathrm{hc}(\Psi)|Q \,. \tag{III.27}$$

This will be important for the reality of the kinetic term.

Star conjugation. The two maps bpz and hc from $\mathcal{H}$ to $\mathcal{H}^*$ and their inverses can be combined in two different ways to give antilinear maps

$$\mathrm{bpz}^{-1} \circ \mathrm{hc}\,,\ \mathrm{hc}^{-1} \circ \mathrm{bpz}\colon \mathcal{H} \to \mathcal{H}\,; \tag{III.28}$$

these two maps invert each other. From (III.25) and (III.26) we learn that

$$\begin{aligned} \overline{\langle\Psi,\Upsilon\rangle} = \overline{\langle \mathrm{bpz}(\Psi)|\Upsilon\rangle} &= \langle \mathrm{hc}(\Upsilon)|\mathrm{hc}^{-1} \circ \mathrm{bpz}(\Psi)\rangle \\ &= \langle \mathrm{bpz}^{-1} \circ \mathrm{hc}(\Upsilon), \mathrm{hc}^{-1} \circ \mathrm{bpz}(\Psi)\rangle \,. \end{aligned} \tag{III.29}$$

Demanding that the two maps in (III.28) be equal, we can define the *star conjugation*

$$* = \mathrm{bpz}^{-1} \circ \mathrm{hc} = \mathrm{hc}^{-1} \circ \mathrm{bpz}\,, \tag{III.30}$$

which acts as an involution and controls the reality of the graded symmetric form:

$$\overline{\langle\Psi,\Upsilon\rangle} = \langle\Upsilon^*, \Psi^*\rangle \,. \tag{III.31}$$

[5] We prefer the symbol hc over † for notational reasons.

From the hermiticity condition (III.27) and the BRST invariance of $|V_2\rangle$, we can conclude that

$$\begin{aligned}(Q|\Psi\rangle)^* &= \mathrm{bpz}^{-1}\circ \mathrm{hc}(Q|\Psi\rangle) = \mathrm{bpz}^{-1}(\langle \mathrm{hc}(\Psi)|Q) \\ &= (-1)^{\Psi}{}_1\langle \mathrm{hc}(\Psi)|Q^{(1)}|V_2\rangle_{12} = -Q^{(2)}{}_1\langle \mathrm{hc}(\Psi)|V_2\rangle_{12} \\ &= -(-1)^{\Psi} Q|\Psi^*\rangle\,,\end{aligned} \tag{III.32}$$

and therefore from (III.31) that

$$\overline{\langle \Psi, Q\Upsilon\rangle} = \langle \Upsilon^*, Q\Psi^*\rangle\,. \tag{III.33}$$

Applying this to the kinetic term $\langle A, QA\rangle$, we see that it is real provided that $A^* = \pm A$. Usually, the string field is taken to be real under (III.30), i. e.,

$$A^* = A\,. \tag{III.34}$$

Star product. As already mentioned in the introductory paragraph, we need a product for the formulation of interaction terms in string field theory. The product will be denoted by $\star$, it maps two elements of $\mathcal{H}$ into a third, turning the $\mathcal{G}$-module $\mathcal{H}$ into an algebra. The product is manifestly associative,

$$\Psi\star(\Upsilon\star\Phi) = (\Psi\star\Upsilon)\star\Phi\,, \tag{III.35}$$

and should satisfy the following property,

$$\langle \Psi, \Upsilon\star\Phi\rangle = \langle \Psi\star\Upsilon, \Phi\rangle\,. \tag{III.36}$$

The latter can in view of eq. (III.12) be rewritten as

$$\langle \Psi, \Upsilon\star\Phi\rangle = (-1)^{\Psi(\Upsilon+\Phi)}\langle \Upsilon, \Phi\star\Psi\rangle\,. \tag{III.37}$$

This guarantees cyclicity up to a sign. It will turn out that Witten's cubic string field action includes an interaction term of the form $\langle A, A\star A\rangle$. It may serve as a check of the signs in eq. (III.37) that this term is manifestly cyclic and therefore in general nonvanishing.

As such, the star product can be realized in terms of an element of $\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}$. With the help of the reflector state in eq. (III.11) we can write the star product in terms of the 3-vertex $\langle V_3| \in \mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}^*$:

$${}_{43}\langle V_2|(|\Psi\rangle\star|\Upsilon\rangle)_4 = {}_{123}\langle V_3|\Psi\rangle_1|\Upsilon\rangle_2\,. \tag{III.38}$$

The 3-vertex can be realized by the midpoint gluing prescription: Take the two incoming strings, fold them by their midpoints and glue the left half of the first string to the right half of the second string. This graphical recipe can be evaluated with conformal field theory methods to give explicit expressions for $\langle V_3|$; we will do this in chapter VI. It is straightforward to construct a 4-vertex from two 3-vertices and one dual reflector state by ${}_{1234}\langle V_4| = {}_{125}\langle V_3|\,{}_{634}\langle V_3|V_2\rangle_{56}$ [57]. This procedure can be applied recursively for the construction of all higher N-vertices $\langle V_N|$.

The graphical recipe mentioned above is reflected in the definition of the star product for the description of string fields as functionals of world-sheet fields: The product between two functionals can be symbolically defined as [186, 3]

$$(\Psi \star \Upsilon)[X(\sigma)] = \int \prod_{\frac{\pi}{2} \leq \sigma \leq \pi} dX'(\sigma)\, dX''(\pi - \sigma)\; \delta[X'(\sigma) - X''(\pi - \sigma)]\Psi[X'(\sigma)]\Upsilon[X''(\sigma)]\,, \tag{III.39}$$

$$\text{with} \qquad X(\sigma) = \begin{cases} X'(\sigma) & \text{for } 0 \leq \sigma \leq \frac{\pi}{2}\,, \\ X''(\sigma) & \text{for } \frac{\pi}{2} \leq \sigma \leq \pi\,. \end{cases}$$

The path integral expressions in eq. (III.39) are understood for all world-sheet fields X^μ, b, c (and ψ^μ, β, γ). It motivates the customary name "delta function overlap". A similar definition exists for all other N-vertices. For concrete computations, they are all rather impractical.

The disk anomalies of the appropriate currents entail that the 3-vertex is charged under the above gradings. We will see from general arguments in chapter VI that the anomaly of the bc-current requires

$$\#_{bc}(\star) = \frac{3}{2}\,. \tag{III.40}$$

We do not specify the further gradings since for superstring field theory, additional mental acrobatics will be necessary; cf. section III.3.2.

The BRST operator acts as a graded derivation of the star algebra:

$$Q(\Psi \star \Upsilon) = (Q\Psi) \star \Upsilon + (-1)^{\Psi}\Psi \star (Q\Upsilon)\,. \tag{III.41}$$

The last equation is the statement that the three-vertex is BRST invariant [114],

$$\langle V_3|(Q^{(1)} + Q^{(2)} + Q^{(3)}) = 0\,. \tag{III.42}$$

Integration operation. The star algebra $\mathcal{H}$ is endowed with a linear evaluation map $\int : \mathcal{H} \to \mathbb{C}$. Due to linearity, this map determines a state $\langle \mathcal{I}| \equiv \langle V_1| \in \mathcal{H}^*$. This *identity* or *integration* state can be used to contract a 3-vertex into a 2-vertex, so that the kinetic term could be formulated without the introduction of a graded symmetric form.[6] The string field $|\mathcal{I}\rangle$ is Grassmann-even. It should be noted that there is some unsolved puzzle concerning the ghost part of the identity string field: It does not act as an identity on all fields in $\mathcal{H}$, since it can be shown that the zero-mode of the reparametrization ghost acts as a derivation on the star algebra, $c_0(\Psi \star \Upsilon) = (c_0\Psi) \star \Upsilon + (-1)^{\Psi}\Psi \star (c_0\Upsilon)$, but $c_0\mathcal{I} \neq 0$, which is in obvious contradiction with the former statement for $\Psi = \Upsilon = \mathcal{I}$ [156, 48, 95, 161].

The BRST invariance of this vertex implies that Q-closed string fields "integrate" to zero:

$$\langle \mathcal{I}|Q = 0 \qquad \Longrightarrow \qquad \int Q\Psi = 0\,. \tag{III.43}$$

[6] Indeed, this is the way it was originally formulated.

The integration operation can be implemented as folding a string by its midpoint and gluing both halves together. The corresponding world-sheet features a curvature singularity at the string midpoint which results in an anomaly just inverse to the star product anomaly:

$$\#_{bc}(\textstyle\int) = -\frac{3}{2}\,; \tag{III.44}$$

the other gradings in the superstring case will be specified below. We will construct operator expressions for the identity string field in the bosonic (and N=2 world-sheet fermionic) sector in chapter VI. This concludes our discussion of the algebraic underpinnings of string field theories based on the Witten star product. We will now describe how string field theory actions may be constructed from these ingredients.

III.3 Three different string field theories

In this section, we want to introduce the action functionals for Witten's bosonic string field theory, cubic superstring field theory, and Berkovits' nonpolynomial superstring field theory. We will discuss their equations of motion (which we finally set out to solve), gauge invariances and problems.

III.3.1 Witten's bosonic string field theory

The formulation of an interacting string field theory requires a kinetic term for string fields that reproduces (at least in a certain gauge) the kinetic terms for the space-time fields in $\mathcal{G}$. It is easy to see that the kinetic term (III.19) satisfies this condition in the Feynman-Siegel gauge [138]. Furthermore, an interaction term of the form

$$S_{\mathrm{int}} = \frac{2g}{3}\langle V_N|(|A\rangle)^{\otimes n} = \frac{2g}{3}\int A^{\star n} \tag{III.45}$$

(with some coupling constant g) is needed. The factor of $\frac{2}{3}$ was chosen for later convenience. Ghost number considerations will single out $n = 3$, restricting the action to a cubic form:

The string field theory action should possess a gauge symmetry that is large enough to decouple BRST-closed states from physical states. Because of the grading (III.40) it is clear that gauge parameters $|\Lambda\rangle$ should have bc-charge $-\frac{3}{2}$ since they should form a closed subalgebra of $\mathcal{H}$. If we assume a gauge invariance under infinitesimal transformations of the form[7]

$$\delta|A\rangle = Q|\Lambda\rangle + g(|A\rangle \star |\Lambda\rangle - |\Lambda\rangle \star |A\rangle)\,, \tag{III.46}$$

string fields $|A\rangle$ will satisfy $\#_{bc}(|A\rangle) = -\frac{1}{2}$ due to (III.10). The kinetic term (III.19) then obviously has ghost number 0. A vanishing ghost number of the interaction term (III.45) can only be achieved for $n\#_{\mathrm{gh}}(|A\rangle) + (n-1)\#_{\mathrm{gh}}(\star) + \#_{\mathrm{gh}}(\int) = 0$, i.e., $n = 3$. It can be shown

[7] Note that this gauge transformation leads to nonabelian gauge transformations of the gauge potential on the brane [138]. All string fields here possibly carry $u(N)$ Chan-Paton labels.

that a linear combination of eqs. (III.19) and (III.45) is not invariant under (III.46) unless the relative coefficient is one. After a rescaling $A \mapsto \frac{\alpha'}{g}A$, the action for bosonic string field theory reads

$$S = \frac{1}{g^2}\left(\frac{1}{\alpha'}\langle V_2|A\rangle Q|A\rangle + \frac{2}{3}\langle V_3|A\rangle|A\rangle|A\rangle\right) \equiv \frac{1}{g^2}\left(\frac{1}{\alpha'}\int A\star QA + \frac{2}{3}\int A\star A\star A\right) . \quad \text{(III.47)}$$

It is easy to see that this action is invariant under the gauge transformations [92]

$$A \mapsto A' = U^\dagger \star QU + U^\dagger \star A \star U \quad \text{(III.48)}$$

with some string field $U = e^\Lambda$. The exponential of string fields is, here and in the following, defined via the Witten star product,

$$U = e^\Lambda = \mathcal{I} + \sum_{k=1}^{\infty} \frac{\Lambda^{\star k}}{k!} \,. \quad \text{(III.49)}$$

Eq. (III.48) generalizes (III.46) to finite gauge transformations.

The equation of motion takes the form

$$\frac{1}{\alpha'}QA + A\star A = 0\,. \quad \text{(III.50)}$$

For convenience, we will set $\alpha' = 1$ from now on.

The computation of the tachyon potential in Witten's bosonic string field theory in the so-called level truncation scheme has been driven to a very high precision [103, 173, 129]. In spite of the recent observation [177, 61] that the resulting energy density for D-branes does not approximate monotonously the value predicted by the Sen conjectures with increasing level (but first exceeds this value and then decreases again), the results are in excellent agreement with Sen's prediction.

In contrast to this numerical success, no analytical solution to eq. (III.50) has been found as of today. In principle, it should be possible to determine solutions describing the closed string vacuum and a situation with multiple D-branes [178].

III.3.2 Witten's cubic superstring field theory

Witten devised an extension of the above bosonic action (III.47) to a superstring field theory (SSFT) [187]. It owes most of its complexity to the inclusion of the Ramond sector and the picture phenomenon [52]. It is formulated entirely within the small Hilbert space.

SSFT in the Neveu-Schwarz sector. A naive ansatz for superstring field theory in the Neveu-Schwarz (NS) sector could be to take the NS string field A in the natural -1 picture; the ghost number (of the operator) $\#_{bc}(A)$ should still be $+1$. This ansatz fails since the cubic term in (III.47) carries ϕ-charge -3 and therefore vanishes if the star product is $\#_\phi$-neutral, cf. (III.14). The obvious cure to this predicament is to modify the star product and the integration map into

$$\Psi * \Upsilon := \mathcal{X}(\Psi \star \Upsilon)\,, \tag{III.51a}$$

$$\oint \Psi := \int Y\Psi\,, \tag{III.51b}$$

where the picture raising and lowering operators are inserted at the string midpoint (i. e., $\sigma = \frac{\pi}{2}$ or $z = \mathrm{i}$). The action for the Neveu-Schwarz sector now reads

$$S = \frac{1}{g^2}\left(\oint A * QA + \frac{2}{3}\oint A * A * A\right)\,. \tag{III.52}$$

The equation of motion is formally the same as in the bosonic case after replacing $\star$ with the modified star product (III.51a),

$$QA + \mathcal{X}(A \star A) = 0\,. \tag{III.53}$$

Since the string field A lives in the small Hilbert space (which does not contain the ξ zero-mode), we additionally have

$$\eta_0 A = 0\,. \tag{III.54}$$

Just as for the BRST operator (cf. eq. (III.9)), the action of η_0 on string fields is defined via contour integration of $\eta(w)$ around the corresponding vertex operator,

$$(\eta_0\Psi)(z) = \oint_z \frac{dw}{2\pi\mathrm{i}}\,\eta(w)\Psi(z)\,. \tag{III.55}$$

SSFT in the Ramond sector. For the extension to the Ramond (R) sector we have to take into account that the product of two Ramond states should be in the Neveu-Schwarz sector. Assuming that the Ramond operator is in the natural $-\frac{1}{2}$ picture and has also ghost number $+1$, we group a NS state A and a R state ψ into a combined system $M = (A, \psi)$ and define the product

$$(A_1, \psi_1)\hat{\star}(A_2, \psi_2) := \big(A_1 * A_2 + \psi_1 \star \psi_2, A_1 * \psi_2 + \psi_1 * A_2\big)\,. \tag{III.56}$$

Two Ramond fields are multiplied by the picture-neutral $\star$. The star products are adjusted in such a way that the product of two picture $(-1, -\frac{1}{2})$ operators has again picture $(-1, -\frac{1}{2})$. Since the integral of a Ramond sector string field must vanish due to Lorentz invariance, a new integration operation is defined by

$$\iint (A, \psi) := \oint A \tag{III.57}$$

with $\oint$ as in eq. (III.51b). It is easy to check that now the action

$$S_{\mathrm{SSFT}} = \frac{1}{g^2}\iint \left(M\hat{\star}QM + \frac{2}{3}M\hat{\star}M\hat{\star}M\right) \tag{III.58}$$

for the combined NSR sector string field $M = (A, \psi)$ is gauge invariant under the gauge transformations

$$\delta M = Q\Lambda + M \hat{\star} \Lambda - \Lambda \hat{\star} M \,. \tag{III.59}$$

However, the proof is formal, and gauge invariance even breaks down already at tree level due to contact term divergences. These divergences arise from the singular operator product expansions of the picture changing operators in (III.51a); they spoil gauge invariance and necessitate the introduction of infinite counterterms in the action [185, 113, 138].

Rewritten in terms of A, ψ and the original $\#_{\rm pic}$-neutral operations $\star$ and $\int$, the action (III.58) reads

$$S_{\rm SSFT} = \frac{1}{g^2} \int \Big(A \star QA + Y\psi \star Q\psi + \tfrac{2}{3} \mathcal{X} A \star A \star A + 2A \star \psi \star \psi \Big) \,. \tag{III.60}$$

This action has been scrutinized in the course of computations of the tachyon potential with the level truncation scheme [42]. Unfortunately, it turns out that the tachyon potential (after the inclusion of the GSO($-$) sector) does not have a minimum in this theory, and that upon inclusion of higher levels the situation does not seem to improve.

Modified cubic superstring field theory. To overcome the above-mentioned contact term problems, a modification of Witten's original action (III.60) was proposed in [8, 9, 150]. Neveu-Schwarz string fields in the small Hilbert space are now taken in the 0-picture; the action for the NS sector reads:

$$S_{\rm SSFT,mod} = \frac{1}{g^2} \int \Big(Y_{-2} A \star QA + \tfrac{2}{3} Y_{-2} A \star A \star A \Big) \,. \tag{III.61}$$

Here, the defining property of the double-step inverse picture changing operator Y_{-2} is [3]

$$\lim_{z \to w} Y_{-2}(z) \mathcal{X}(w) = Y(w) \,. \tag{III.62}$$

It should be BRST invariant, Lorentz invariant (in particular, independent of momentum), and of conformal weight 0. There are two candidates for Y_{-2}, the chiral variant [180]

$$Y_{-2}(z) = -4 e^{-2\phi(z)} - \frac{16}{5} e^{-3\phi} c \partial \xi \psi_\mu \partial X^\mu(z) \,, \tag{III.63}$$

and the nonchiral one [113],

$$Y_{-2}(z, \bar{z}) = Y(z) Y(\bar{z}) \,. \tag{III.64}$$

It is not clear what the physical reason for this ambiguity should be and whether the theories defined with the two different Y_{-2}'s are equivalent off-shell. Furthermore, since the picture changing operators (B.28) and (B.29) have non-trivial kernels one obtains unphysical solutions to the equation of motion, which now reads

$$Y_{-2}(QA + A \star A) = 0 \,. \tag{III.65}$$

Apart from that, the modified theory indeed solves some of the problems of Witten's cubic superstring field theory. For instance, the tachyon potential displays a minimum; level truncation at level (2,4) already yields about 88% of the expected result [7].

III.3.3 Berkovits' nonpolynomial superstring field theory

As an alternative to Witten's superstring field theory, which suffers from contact term divergences, Berkovits proposed a (nonpolynomial) WZW-like action [19] for the NS sector. It is formulated in the large Hilbert space of [52] and based on a twisted small N=4 superconformal algebra. Before motivating Berkovits' result, let us first briefly review some necessary background on twisted small N=4 superconformal algebras:

Small N=4 superconformal algebra. It is well known that the full (i. e., $c=0$) N=1 superconformal algebra of critical N=1 string theory can be embedded into an N=2 superconformal algebra with central charge $c_{N=2}=6$, i. e., the matter central charge of critical N=2 string theory [27]. This embedding is given in appendix C.1. An analogous construction of a nonpolynomial string field theory will also hold for N=2 strings, which naturally comes about with an N=2 superconformal algebra; so we may start our discussion from a given N=2 superconformal algebra generated by an energy-momentum tensor T, two spin 3/2 superpartners $G^{\pm}$ and a $U(1)$ current[8] J. It can be embedded into a small N=4 superconformal algebra with two additional superpartners $\widetilde{G}^{\pm}$ and two spin 1 operators J^{++} and J^{--} supplementing J to an $SU(2)$ (or $SU(1,1)$) current algebra[9]. To this end [138], J can be "bosonized" as $J=\partial H$, and we define

$$J^{++} := e^{H}, \quad J^{--} := e^{-H}, \tag{III.66}$$

where $H(z)$ has the OPE

$$H(z)H(0) \sim \frac{c}{3}\log z \tag{III.67}$$

with a central charge c. Then $\widetilde{G}^{\pm}$ can be defined by

$$\widetilde{G}^{-}(z) := \oint \frac{dw}{2\pi i} J^{--}(w)G^{+}(z) = [J_0^{--}, G^{+}(z)], \tag{III.68}$$

$$\widetilde{G}^{+}(z) := \oint \frac{dw}{2\pi i} J^{++}(w)G^{-}(z) = [J_0^{++}, G^{-}(z)], \tag{III.69}$$

so that $(G^{+},\widetilde{G}^{-})$ and $(\widetilde{G}^{+},G^{-})$ transform as doublets under $SU(2)$ (or $SU(1,1)$). We will see in chapter IV, however, that there is a certain freedom to embed the N=2 superconformal algebra into a small N=4 superconformal algebra, parametrized by $SU(2)$ (or $SU(1,1)$). So, the embedding given above corresponds to a special choice (see section IV.2 for more details).

This small N=4 algebra in general has a nonvanishing central charge. In topological string theories [28], it is removed by "twisting" T by the $U(1)$ current J, i. e. $T \to T + \frac{1}{2}\partial J$, so that the resulting algebra has vanishing central charge, cf. eq. (C.12a).

The realization in terms of matter and ghost multiplets and the operator product expansions of a twisted small N=4 superconformal algebra are given in appendix C.

[8] The labels denote the $U(1)$-charge w. r. t. J.

[9] The case of an $SU(2)$ current algebra applies to N=1 strings, and the case of an $SU(1,1)$ current algebra corresponds to the N=2 string.

N=2 vertex operators. Let us first restrict to N=1 string theory. Neveu-Schwarz vertex operators A in Witten's cubic (unmodified) superstring theory naturally had ghost number $+1$ and were in the -1 picture (and in the small Hilbert space). As such, they were Grassmann-odd. For the construction of a nonpolynomial action, they should be ghost- and picture-neutral; to this aim, one can embed them into the large Hilbert space by mapping them to "N=2 vertex operators"

$$\Phi(z) := \xi A(z)\,. \tag{III.70}$$

As usual, the product is assumed to be normal ordered. Since $\#_{\mathrm{gh}}(\xi) = -1$ and $\#_{\mathrm{pic}}(\xi) = +1$, Φ satisfies the above requirements. Recall that nonvanishing correlation functions (on the disk) in superstring field theory should have operator insertions of ghost charge $\#_{\mathrm{gh}} = 2$, cf. eqs. (III.13) and (III.14). Thus, correlation functions constructed solely from operators (III.70) will vanish; G^+- and $\widetilde{G}^+$-insertions are necessary in order to obtain a nonvanishing result. Obviously, these are the new BRST operators; since unphysical states should decouple, an insertion of G^+ or $\widetilde{G}^+$ into the correlation functions guarantees that correlation functions of BRST-trivial states vanish. Note that after twisting, G^+ and $\widetilde{G}^+$ are currents of weight 1 (and charge $+1$) such that their zero-modes are indeed reasonable candidates for BRST charges. The action of G^+ and $\widetilde{G}^+$ on vertex operators is defined in direct generalization of eqs. (III.9) and (III.55) as an integral

$$(G^+\Phi)(z) = \oint \frac{dw}{2\pi i}\, G^+(w)\Phi(z)\,, \quad (\widetilde{G}^+\Phi)(z) = \oint \frac{dw}{2\pi i}\, \widetilde{G}^+(w)\Phi(z)\,, \tag{III.71}$$

with the integration contour running around z. With this definition,

$$\{G^+, \widetilde{G}^+\} = 0\,, \qquad (G^+)^2 = (\widetilde{G}^+)^2 = 0\,. \tag{III.72}$$

Thus, if we demand

$$0 \stackrel{!}{=} G^+\widetilde{G}^+\Phi(z) = \oint \frac{dw'}{2\pi}\, J_{BRST}(w') \oint \frac{dw}{2\pi}\, \eta(w)\xi(z)A(z) = \{Q, A(z)\}\,, \tag{III.73}$$

this is equivalent to the on-shell condition for A. As argued above, G^+- and $\widetilde{G}^+$-trivial states decouple in correlation functions; therefore one should identify

$$\Phi \sim \Phi + G^+\Lambda + \widetilde{G}^+\tilde{\Lambda}\,. \tag{III.74}$$

The on-shell condition (III.73) together with the equivalence relation (III.74) on on-shell states defines a cohomology, the vector space of physical states. Pure gauge states $\Phi = G^+\Lambda + \widetilde{G}^+\tilde{\Lambda}$ correspond to N=1 vertex operators

$$A = \eta_0\Phi = \widetilde{G}^+\Phi = G^+(-\widetilde{G}^+\Lambda) = Q(-\eta_0\Lambda)\,, \tag{III.75}$$

i. e., BRST-exact states in the N=1 sense.

Nonpolynomial action. The on-shell condition (III.73) can be implemented by the action

$$S_{\mathrm{kin}} = \int \Phi \star G^+\widetilde{G}^+\Phi\,, \tag{III.76}$$

where $\int$ and $\star$ are the picture-neutral integration and product operations. However, it can be shown that it is impossible to construct a purely cubic interaction which is invariant under a non-linear generalization of the gauge transformation (III.74) [19]. Instead, Berkovits proposed a WZW-like action which contains in its Taylor expansion a kinetic term and a cubic interaction similar to Witten's superstring field theory action:

$$S = \frac{1}{2g^2}\,\mathrm{tr}\int\left\{(e^{-\Phi}G^+e^{\Phi})(e^{-\Phi}\widetilde{G}^+e^{\Phi}) - \int_0^1 dt(e^{-\widehat{\Phi}}\partial_t e^{\widehat{\Phi}})\{e^{-\widehat{\Phi}}G^+e^{\widehat{\Phi}}, e^{-\widehat{\Phi}}\widetilde{G}^+e^{\widehat{\Phi}}\}\right\}\,. \qquad \text{(III.77)}$$

Here $e^{\Phi} = \mathcal{I} + \Phi + \frac{1}{2}\Phi\star\Phi + \ldots$ is defined via Witten's midpoint gluing prescription ($\mathcal{I}$ denotes the identity string field), the NS string field Φ now possibly carries $u(n)$ Chan-Paton labels[10] with an extension $\widehat{\Phi}(t)$ interpolating between $\widehat{\Phi}(t=0)=0$ and $\widehat{\Phi}(t=1)=\Phi$.

The action (III.77) in this form also applies to the case of N=2 strings [19, 26]. A discussion of this string field theory is relegated into the subsequent chapters. It will be shown in appendix E.2 that the equation of motion (III.79) in the N=2 case contains the self-duality equation for Yang-Mills theory; this is physically sensible, since the low-energy limit of open N=2 string theory is supposed to describe self-dual Yang-Mills theory. The star product for the world-sheet fermions in this case will be more concretely defined in section VI.5. For the application in the next chapters it might suffice to state that it shares the properties of the bosonic star product, in particular associativity. Furthermore, the realization of the currents G^+ and $\widetilde{G}^+$ in terms of world-sheet fields naturally depends on the degree of world-sheet supersymmetry.

The above action is invariant under the finite gauge transformation [140]

$$e^{\Phi} \mapsto (e^{\Phi})' = \Lambda\star e^{\Phi}\star\tilde{\Lambda} \qquad \text{with} \qquad G^+\Lambda = 0\,, \quad \widetilde{G}^+\tilde{\Lambda} = 0 \qquad \text{(III.78)}$$

generalizing (III.74); and arguments based on this gauge invariance suggest that beyond reproducing the correct four-point tree amplitude all N-point tree amplitudes are correctly reproduced by (III.77). The corresponding equation of motion reads

$$\widetilde{G}^+(e^{-\Phi}G^+e^{\Phi}) = 0\,, \qquad \text{(III.79)}$$

where contour integrations are implied again.

Nonpolynomial string field theory has been scrutinized in the course of computations of the tachyon potential with the level truncation scheme [20, 25, 84, 39]. The results are quite promising already at comparatively low levels: At level $(2,4)$, e. g., the tachyon potential reaches already about 90% of its expected depth. Finally, it should be mentioned that Berkovits extended the Neveu-Schwarz sector action (III.77) to the Ramond sector in [22]. Since the Ramond sector describes spacetime fermions, this is important for checking Sen's conjectures if one wants to

[10] All states in the string field Φ are taken to be GSO(+). For including GSO(−) states (e. g. for the study of tachyon condensation) one has to add internal Chan-Paton labels to Φ and also to G^+ and $\widetilde{G}^+$ (for a review on this subject see [138, 40]).

examine whether supersymmetry is restored in, say, the construction of a BPS brane as a kink via tachyon condensation.[11]

III.4 Vacuum string field theories

An immediate consequence of Sen's conjectures is that there should exist solutions to the equations of motion of string field theories which describe unstable brane configurations that represent the closed string vacuum. Finding solutions even to the bosonic equation of motion (III.50) turned out to be a rather intractable problem, mainly due to the complicated structure of Witten's star product in the oscillator representation (to be reviewed in section VI.2) and the ghost-matter mixing of the BRST operator in (III.50).

III.4.1 Bosonic vacuum string field theory

Rastelli, Sen and Zwiebach proposed a way to circumvent the latter problem by trying to describe D-brane solutions from the point of view of the tachyon vacuum:

Kinetic operator around the tachyon vacuum. Expanding all string fields around a given solution A_1 to the bosonic equation of motion (III.50) according to $A = A_1 + A'$, we see that the equation of motion becomes

$$QA' + A_1 \star A' + A' \star A_1 + A' \star A' = 0\,, \tag{III.80}$$

or, defining a new kinetic operator by $Q'\Psi := Q\Psi + A_1 \star \Psi - (-1)^{A_1}\Psi \star A_1$ for abitrary string fields Ψ,

$$Q'A' + A' \star A' = 0\,. \tag{III.81}$$

Obviously, the equation of motion is form-invariant if we admit a field-dependent redefinition of the kinetic operator.[12] The same is true for the action (III.47) which is simply shifted by a constant. Suppose now that A_1 represents the (unknown) solution describing the tachyon vacuum. In this case the new BRST operator Q' should have vanishing cohomology in order to satisfy the condition that no open strings are left in the closed string vacuum. Furthermore, it should be universal, that is, it should be possible to express it without any reference to the boundary conformal field theory describing the original D-brane.

In general, a nontrivial field redefinition is necessary in order to bring the shifted string field theory action into the canonical form representing the new background [172]. This field redefinition could be used to transform the unknown operator Q' to a simpler form $\mathcal{Q}$, thereby leaving the cubic term in (III.47) invariant. Appropriate redefinitions are of the form

$$\mathcal{A} := e^{-K} A'\,, \tag{III.82}$$

[11] For an earlier work on spacetime supersymmetry in this context, see [191].

[12] This argument will be repeated in chapter V in the framework of the dressing approach.

where K is a linear combination of the generators $K_n := L_n - (-1)^n L_{-n}$ of the midpoint-preserving reparametrizations. As usual, L_n are the modes of the energy-momentum tensor of the coupled matter-ghost system. This guarantees that

$$K(\Psi \star \Upsilon) = (K\,\Psi) \star \Upsilon + \Psi \star (K\Upsilon)\,, \tag{III.83a}$$
$$\langle K\,\Psi, \Upsilon\rangle = -\langle \Psi, K\,\Upsilon\rangle\,, \tag{III.83b}$$

as can be shown using contour deformation arguments. Eq. (III.83) is sufficient for proving that the interaction term is indeed unchanged; the action of string field theory around the tachyon vacuum now takes form

$$S_{\text{VSFT}} = \frac{1}{g^2}\left(\langle \mathcal{A}, \mathcal{Q}\mathcal{A}\rangle + \frac{2}{3}\langle \mathcal{A}, \mathcal{A}\star\mathcal{A}\rangle\right) \tag{III.84}$$

after the subtraction of a constant "zero-point energy". Here,

$$\mathcal{Q} = e^{-K} Q' e^{K} \tag{III.85}$$

should satisfy

$$\mathcal{Q}^2 = 0\,, \tag{III.86a}$$
$$\mathcal{Q}(\Psi \star \Upsilon) = (\mathcal{Q}\Psi) \star \Upsilon + (-1)^{\Psi}\Psi \star (\mathcal{Q}\Upsilon)\,, \tag{III.86b}$$
$$\langle \mathcal{Q}\Psi, \Upsilon\rangle = -(-1)^{\Psi}\langle \Psi, \mathcal{Q}\Upsilon\rangle\,, \tag{III.86c}$$

in order to ensure gauge invariance of the action. These identities hold by virtue of (III.83) and (III.85). The above physical requirements on Q' carry over unchanged to $\mathcal{Q}$.

These requirements can be complied if we take a kinetic operator $\mathcal{Q}$ for the expansion around the tachyon vacuum which is constructed purely from ghost operators [151]. In particular any combination of the ghost number one operators

$$\mathcal{C}_0 := c_0\,, \qquad \mathcal{C}_n := c_n + (-1)^n c_{-n} \quad \text{for } n \neq 0 \tag{III.87}$$

will do the job. First of all, eq. (III.86a) is manifest; the derivation properties (III.86b) and (III.86c) can be easily shown using contour integral arguments [156]; the universality property is also manifest, and the triviality of the cohomology is guaranteed by the existence of contracting homotopy operators $\mathcal{B}_n := \frac{1}{2}(b_n + (-1)^n b_{-n})$ with $\{\mathcal{C}_n, \mathcal{B}_n\} = 1$ (cf. appendix D).[13] The coefficients in this linear combination were determined independently by several authors [80, 59, 144], using a numerical analysis of the equation of motion in the Siegel gauge $b_0\mathcal{A} = 0$.[14] It turned out that the ghost kinetic operator may be written as

$$\mathcal{Q} = \sum_{n=0}^{\infty}(-1)^n \mathcal{C}_{2n} = \frac{1}{2\mathrm{i}}(c(\mathrm{i}) - c(-\mathrm{i}))\,, \tag{III.88}$$

[13] The loophole that a state $|\chi\rangle$ in the kernel of $\mathcal{C}_n$ is trivial (because of $|\chi\rangle = \mathcal{C}_n(\mathcal{B}_n|\chi\rangle)$) unless $\mathcal{B}_n|\chi\rangle$ is infinite was used in [59] to find closed strings at the tachyon vacuum.

[14] Interestingly, the result is such that the product of two string fields Ψ_i in the Siegel gauge $b_0\Psi_i = 0$ is BRST-closed, $\mathcal{Q}(\Psi_1 \star \Psi_2) = 0$ [143].

where $z = \mathrm{i}$ and $z = -\mathrm{i}$ denote the string midpoint (recall that we are using the doubling trick) [59]. A posteriori, Rastelli, Sen and Zwiebach postulated that the ghost kinetic operator could have arisen from a singular reparametrization of the world-sheet which concentrates the string into its midpoint. Under this reparametrization, all fields contained in Q' are transformed according to their conformal weights; it was argued that for such a reparametrization only the vertex operator with lowest weight survives. This is the ghost field c of weight -1 which in this way is concentrated to the string midpoint. The whole argument will be briefly reviewed below in the case of vacuum superstring field theory.

It should be noted, however, that with this choice of a kinetic operator the action (III.84) is multiplied with an infinite normalization factor and has be regularized in some way [59].

Factorization of the equation of motion. A pure ghost kinetic operator allows one to look for factorized solutions of the form $\mathcal{A} = \mathcal{A}_g \otimes \mathcal{A}_m$, where $\mathcal{A}_g$ and $\mathcal{A}_m$ are pure ghost and pure matter string fields, respectively. Since the star product can be taken separately in the ghost and matter sectors, the equation of motion

$$\mathcal{Q}\mathcal{A} = -\mathcal{A} \star \mathcal{A} \tag{III.89}$$

factorizes as

$$\mathcal{Q}\mathcal{A}_g = -\mathcal{A}_g \star \mathcal{A}_g \tag{III.90a}$$

and

$$\mathcal{A}_m = \mathcal{A}_m \star \mathcal{A}_m\,. \tag{III.90b}$$

The latter is the projector property in the matter part; this equation triggered a myriad of papers devoted to the search for projectors in the star algebra. We will shortly review some of the results, most prominently amongst them the sliver state, to provide the necessary background material for the next chapters. Solutions to the former were constructed using so-called bc-twisted conformal field theories in [59], see section III.5.3. Since they are assumed to be universal for all Dp-brane solutions, the ratio of the energies associated with two different D-brane solutions, with matter parts $\mathcal{A}_m$ and $\mathcal{A}'_m$, respectively, is given by

$$\frac{\langle \mathcal{A}'_m, \mathcal{A}'_m\rangle}{\langle \mathcal{A}_m, \mathcal{A}_m\rangle}\,. \tag{III.91}$$

Thus it is possible to compute ratios of D-brane tensions in this theory without knowledge of the ghost solution $\mathcal{A}_g$.

III.4.2 Cubic vacuum superstring field theory

In superstring field theory one has to take the GSO($-$) sector into account for a description of tachyon condensation. The open string tachyon lives in this sector [171]. This is usually accomplished by the introduction of internal Chan-Paton factors, where the conventions are chosen in such a way that the product of two GSO(−) string fields is in the GSO(+) sector, and setting the GSO(−) string to zero reduces the action to the customary GSO(+) action.

The same prescription applies to the vacuum versions of superstring field theory. Let us start with a discussion of cubic vacuum superstring field theory; since Witten's original version was disqualified due to the arguments presented in section III.3.2, we restrict to the 0-picture modification with the non-chiral double step picture lowering operator (III.64).

Internal Chan-Paton factors. We discuss the conventions for internal Chan-Paton factors for open and for vacuum superstring field theory simultaneously. Let labels $\pm$ denote the GSO-sector in which the corresponding string field lives. Hats over a string field indicate that it is tensored with internal Chan-Paton factors. Then, consistent conventions for these Chan-Paton factors are [2, 141]: String fields O of *odd* ghost number are expanded into GSO($\pm$) parts according to

$$\widehat{O} = O_+ \otimes \sigma_3 + O_- \otimes \mathrm{i}\sigma_3 \,, \tag{III.92}$$

where σ_i are the customary Pauli matrices. The same holds true for $\widehat{Q}$, $\widehat{Q}'$ and its vacuum version after the world-sheet reparametrization, $\widehat{\mathcal{Q}}$,

$$\widehat{Q} = Q \otimes \sigma_3 \,, \tag{III.93a}$$

$$\widehat{Q}' = Q_{\text{odd}} \otimes \sigma_3 + Q_{\text{even}} \otimes \mathrm{i}\sigma_2 \,, \tag{III.93b}$$

$$\widehat{\mathcal{Q}} = \mathcal{Q}_{\text{odd}} \otimes \sigma_3 + \mathcal{Q}_{\text{even}} \otimes \mathrm{i}\sigma_2 \,. \tag{III.93c}$$

It will be argued below that the even part of the vacuum BRST operator has the unusual property of being Grassmann-even. It couples GSO(+) to GSO(−) string fields. Naturally, the customary BRST operator Q for the open string vacuum consists of an odd part only. Finally, ghost number *even* string fields have the form

$$\widehat{E} = E_+ \otimes \quad + E_- \otimes \sigma_1 \,. \tag{III.94}$$

It is easy to convince oneself that these Chan-Paton labels have the desired properties.

Cubic superstring field theory for both GSO-sectors. For ease of notation, we slightly modify the notation for the graded symmetric form (III.11):

$$\langle \widehat{Y}_{-2} | \widehat{\Psi}, \widehat{\Upsilon} \rangle := {}_{12}\langle V_2 | \widehat{Y}_{-2} \widehat{\Psi} \rangle_1 | \widehat{\Upsilon} \rangle_2 \,. \tag{III.95}$$

Here, $\widehat{Y}_{-2} = Y(\mathrm{i})Y(-\mathrm{i}) \otimes \sigma_3$ can be inserted in either of the two strings; their midpoints are identified anyway. One can show that the modified graded symmetric form maintains all the required properties for the construction of a cubic action [141]. This leads to the following generalization of eq. (III.61):

$$S_{\text{GSO}\pm} = \frac{1}{g^2} \mathrm{Tr} \left(\langle \widehat{Y}_{-2} | \widehat{A}, \widehat{Q} \widehat{A} \rangle + \tfrac{2}{3} \langle \widehat{Y}_{-2} | \widehat{A}, \widehat{A} \star \widehat{A} \rangle \right) \,. \tag{III.96}$$

The trace Tr is taken over the internal Chan-Paton matrices; and the ghost number 1 string field $\widehat{A} = A_+ \otimes \sigma_3 + A_- \otimes \mathrm{i}\sigma_2$ decomposes into a Grassmann-odd part A_+ consisting of states of integer weights and a Grassmann-even part A_- consisting of states of half-integer weights. As mentioned above, this action reduces consistently to eq. (III.61) if we set $A_- = 0$.

Cubic vacuum superstring field theory. Following the same steps as in the bosonic case, it can be shown that the action for cubic vacuum superstring field theory takes the form

$$S_{\mathrm{GSO}\pm,\,\mathrm{vac}} = \frac{1}{g^2}\mathrm{Tr}\left(\langle \widehat{Y}_{-2}|\widehat{\mathcal{A}}, \widehat{\mathcal{Q}}\widehat{\mathcal{A}}\rangle + \tfrac{2}{3}\langle \widehat{Y}_{-2}|\widehat{\mathcal{A}}, \widehat{\mathcal{A}}\star\widehat{\mathcal{A}}\rangle\right) . \tag{III.97}$$

The fluctuation field $\widehat{\mathcal{A}}$ splits according to eq. (III.92) into GSO(+) and GSO(−) parts. The form of the vacuum kinetic operator will be specified below. The action is invariant under the transformation

$$\delta\widehat{\mathcal{A}} = \widehat{\mathcal{Q}}\widehat{\Lambda} + \widehat{\mathcal{A}}\star\widehat{\Lambda} - \widehat{\Lambda}\star\widehat{\mathcal{A}}\,, \tag{III.98}$$

where $\widehat{\Lambda}$ is an infinitesimal gauge parameter of ghost number 0 and picture number 0, whose internal Chan-Paton structure is given by (III.94). The equation of motion,

$$\widehat{\mathcal{Q}}\widehat{\mathcal{A}} + \widehat{\mathcal{A}}\star\widehat{\mathcal{A}} = 0\,, \tag{III.99}$$

can be written in terms of components as

$$\mathcal{Q}_{\mathrm{odd}}\mathcal{A}_+ + \mathcal{A}_+\star\mathcal{A}_+ + \mathcal{Q}_{\mathrm{even}}\mathcal{A}_- - \mathcal{A}_-\star\mathcal{A}_- = 0\,, \tag{III.100a}$$

$$\mathcal{Q}_{\mathrm{odd}}\mathcal{A}_- + \mathcal{A}_+\star\mathcal{A}_- + \mathcal{Q}_{\mathrm{even}}\mathcal{A}_+ - \mathcal{A}_-\star\mathcal{A}_+ = 0\,. \tag{III.100b}$$

Vacuum kinetic operator. The argument of Rastelli, Sen, and Zwiebach was generalized to the superstring case in [141]. Since there are some additional subtleties and we will need the result in chapter V, let us shortly review the main line of arguments here:

We make the following ansatz for the vacuum BRST operator $\widehat{Q}'$ before the reparametrization:

$$\widehat{Q}' = \sum_r \int_{-\pi}^{\pi} d\sigma a_r(\sigma) O_{+,r}(\sigma)\otimes\sigma_3 + \sum_s \int_{-\pi}^{\pi} d\sigma b_s(\sigma) O_{-,r}(\sigma)\otimes \mathrm{i}\sigma_2\,, \tag{III.101}$$

where a_r and b_r are smooth functions of σ, and $O_{r,+}$, $O_{s,-}$ are local operators of ghost number 1 in the GSO(+) and GSO(−) sectors, respectively. This expression is written on the double cover of the strip, which explains its integration range. We now apply a midpoint preserving world-sheet reparametrization $\sigma \mapsto f(\sigma)$, i. e., $f(\pi-\sigma) = \pi - f(\sigma)$ for $0 \le \sigma \le \pi$ and $f(-\pi-\sigma) = -\pi - f(\sigma)$ for $-\pi \le \sigma \le 0$. Such reparametrizations do not change the structure of the cubic term but change the kinetic term. If $O_{r,+}$ and $O_{s,-}$ denote primary fields of dimension $h_{r,+}$ and $h_{s,-}$, respectively, then $\widehat{Q}'$ under this reparametrization transforms to

$$\sum_r \int_{-\pi}^{\pi} d\sigma a_r(\sigma)(f'(\sigma))^{h_{r,+}} O_{+,r}\big(f(\sigma)\big)\otimes\sigma_3 + \sum_s \int_{-\pi}^{\pi} d\sigma b_s(\sigma)(f'(\sigma))^{h_{s,-}} O_{-,r}\big(f(\sigma)\big)\otimes\mathrm{i}\sigma_2\,. \tag{III.102}$$

Consider now a reparametrization for which $f'(\pm\pi/2)$ is small; then, $\int d\sigma (f'(\sigma))^h$ will receive large contributions from the region around $\sigma = \pm\pi/2$ whenever h is negative. In the supersymmetric Fock space, there are two operators of negative conformal dimension, c and γ (with weights -1 and $-1/2$, respectively). We assume a function f which around $|\sigma| = \pi/2$ behaves as

$$[f'(\sigma)]^{-1} \sim -\frac{1}{\varepsilon_r^2}\delta\left(|\sigma| - \tfrac{\pi}{2}\right) \qquad \text{and} \qquad [f'(\sigma)]^{-\frac{1}{2}} \sim -\frac{1}{\varepsilon_r}\delta\left(|\sigma| - \tfrac{\pi}{2}\right) \tag{III.103}$$

in the singular limit $\varepsilon_r \to 0$. Then $\widehat{\mathcal{Q}}$ is dominated by

$$\begin{aligned} \widehat{\mathcal{Q}} &= \frac{1}{\varepsilon_r^2}\left(a\big(\tfrac{\pi}{2}\big)c\big(\tfrac{\pi}{2}\big) + a\big(-\tfrac{\pi}{2}\big)c\big(-\tfrac{\pi}{2}\big)\right) \otimes \sigma_3 \\ &\quad + \frac{1}{\varepsilon_r}\left(b\big(\tfrac{\pi}{2}\big)\gamma\big(\tfrac{\pi}{2}\big) + b\big(-\tfrac{\pi}{2}\big)\gamma\big(-\tfrac{\pi}{2}\big)\right) \otimes \mathrm{i}\sigma_2 \,. \end{aligned} \tag{III.104}$$

The invariance of the action (III.96) under world-sheet parity $\sigma \mapsto \pi - \sigma$ can be used to determine the leading terms in $\widehat{\mathcal{Q}}$ without knowing the values of $a(\pm\frac{\pi}{2})$ and $b(\pm\frac{\pi}{2})$. It turns out that in terms of z-coordinates,

$$\mathcal{Q}_{\text{odd}} = \frac{1}{2\mathrm{i}\varepsilon_r^2}\big(c(\mathrm{i}) - c(-\mathrm{i})\big) + \frac{1}{2}\oint \frac{dz}{2\pi\mathrm{i}}\, b\gamma^2(z)\,, \tag{III.105a}$$

$$\mathcal{Q}_{\text{even}}^{\text{GSO}(+)} = \frac{1}{2\mathrm{i}\varepsilon_r}\big(\gamma(\mathrm{i}) - \gamma(-\mathrm{i})\big)\,, \tag{III.105b}$$

$$\mathcal{Q}_{\text{even}}^{\text{GSO}(-)} = \frac{1}{2\mathrm{i}\varepsilon_r}\big(\gamma(\mathrm{i}) + \gamma(-\mathrm{i})\big)\,. \tag{III.105c}$$

The second term on the right hand side of eq. (III.105a) was inserted so as to make $\widehat{\mathcal{Q}}$ nilquadratic. It can be shown [141] that $\langle\mathcal{I}|\widehat{\mathcal{Q}} = 0$ after some appropriate regularization, that $\mathcal{Q}$ is still a derivation of the star product, and that it is hermitean in the presence of $\widehat{Y}_{-2}$, cf. eq. (III.95). In [141], first steps for solving the equations of motion (III.100) power by power in ε_r were taken. Very recent results obtained by level truncation [142] seem to indicate that the pure ghost kinetic operator (III.105) fails to describe the theory around the tachyon vacuum.

III.4.3 Nonpolynomial vacuum superstring field theory

As was derived in [96, 126], the nonpolynomial action for fluctuations is of the same form as the action (III.77). After a singular world-sheet reparametrization and the extension to the GSO($-$) sector it reads [141]

$$S = \frac{1}{2g^2}\,\mathrm{Tr}\int\left\{(e^{-\widehat{\Phi}}\widehat{\mathcal{Q}}e^{\widehat{\Phi}})(e^{-\widehat{\Phi}}\widehat{\eta}_0 e^{\widehat{\Phi}}) - \int_0^1 dt(e^{-t\widehat{\Phi}}\partial_t e^{t\widehat{\Phi}})\{e^{-t\widehat{\Phi}}\widehat{\mathcal{Q}}e^{t\widehat{\Phi}}, e^{-t\widehat{\Phi}}\widehat{\eta}_0 e^{t\widehat{\Phi}}\}\right\}\,. \tag{III.106}$$

Here, $\widehat{\Phi}$ denotes the GSO-unprojected Berkovits string field of ghost number 0 and picture number 0. Its expansion in the GSO($\pm$) sectors goes along the lines of (III.94). The only property of the action which was needed in the above derivation of $\widehat{\mathcal{Q}}$ was its twist invariance.

Therefore, it is clear that the form of $\widehat{\mathcal{Q}}$ in nonpolynomial vacuum superstring field theory depends on the twist invariance of eq. (III.77). In [25] it was shown that it possesses the same twist invariance as the cubic action. Thus, $\widehat{\mathcal{Q}}$ is given by (III.105), and $\widehat{\eta}_0$ is assumed to be of the form $\widehat{\eta}_0 = \eta_0 \otimes \sigma_3$. It is trivial to check that, with this definition, $\widehat{\mathcal{Q}}$ and $\widehat{\eta}_0$ anticommute similar to eq. (III.72). Finally, we note that the action (III.106) is invariant under

$$\delta(e^{\widehat{\Phi}}) = (\widehat{\mathcal{Q}}\widehat{\Lambda}) \star e^{\widehat{\Phi}} + e^{\widehat{\Phi}} \star (\widehat{\eta}_0\widehat{\Lambda}) \,, \tag{III.107}$$

and the equation of motion reads

$$\widehat{\eta}_0(e^{-\widehat{\Phi}}\widehat{\mathcal{Q}}e^{\widehat{\Phi}}) = 0 \,. \tag{III.108}$$

III.5 Projectors of the star algebra

The importance of matter projectors in the star algebra is clear from eq. (III.90); they are solutions to the matter part of the vacuum string field equations. Apart from improving our understanding of the string field algebra they are also crucial for the dressing approach (which was presented in the noncommutative field theory case in chapter II and will be generalized to string field theory in chapters IV and V). Therefore we here briefly expound upon the most relevant facts.

Indeed, one projector of the star algebra is already known from section III.2. The identity string field is a projector by definition. We will see below that it belongs to the class of states called wedge states; two members of this class are projectors. The other projector is the *sliver state*, on which we concentrate now.

III.5.1 The sliver

Algebraic construction. The sliver state was first constructed algebraically in [102]. Starting from an ansatz

$$|V_3\rangle_{123} = \exp\Big(-\tfrac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} a_m^{(r)\dagger} \cdot V_{mn}^{rs} \cdot a_n^{(s)\dagger} \Big) |p_1 = 0\rangle_1 \otimes |p_2 = 0\rangle_2 \otimes |p_3 = 0\rangle_3 \tag{III.109}$$

for the bosonic matter 3-vertex at zero momentum in oscillator language (the oscillators are introduced in appendix B.1), Kostelecký and Potting could construct a projector of the star algebra in terms of the Neumann coefficients V_{mn}^{rs} using coherent state techniques.[15] Namely, let

$$M^{rs} := CV^{rs} \quad \text{with} \quad C_{mn} = (-1)^m \delta_{mn} \,, \quad m, n \geq 1 \tag{III.110}$$

and $X := M^{11}$, then the sliver state $|\Xi\rangle$ in the matter sector is constructed as a squeezed state

$$|\Xi\rangle = \mathcal{N}^{26} \exp\Big(-\tfrac{1}{2} a^\dagger \cdot S \cdot a^\dagger \Big) |p = 0\rangle \,, \quad \mathcal{N} = [\det(1 - X)\det(1 + T)]^{1/2} \,, \quad S = CT \,. \tag{III.111}$$

[15] As before, r and s label the Hilbert spaces of the different strings. If the oscillator indices m, n are omitted, we implicitly assume matrix multiplication. The dots denote contraction over the Lorentz indices.

Finally, the matrix T is given by

$$T = (2X)^{-1}\left(1 + X - \sqrt{(1+3X)(1-X)}\right). \tag{III.112}$$

This complicated expression finds a natural explanation [35] in the Moyal formulation of the bosonic sector, cf. section E.1.

Wedge states. Surface states are bra-vectors $\langle\Sigma|$ in the dual Hilbert space $\mathcal{H}^*$ of a boundary conformal field theory with Hilbert space $\mathcal{H}$, which are associated to a particular Riemann surface Σ (with boundary). The surface in question is a disk with one puncture P at the boundary, where local operators ϕ will be inserted. The disk should be endowed with a global parametrization $s\colon H \to \Sigma$ (H denotes the canonical upper half-disk $\{|z| \le 1,\ \mathrm{Im}\ z \ge 0\} \subset \mathbb{C}$) such that $P = s(0)$. Operator insertions on H are made at $z = 0$; looking from the "interaction point" $z = \mathrm{i}$, the arc $\{|z| = 1, \mathrm{Re}\ z \ge 0, \mathrm{Im}\ z \ge 0\} \subset H$ is usually called the left half of the string, the arc $\{|z| = 1, \mathrm{Re}\ z \le 0, \mathrm{Im}\ z \ge 0\} \subset H$ the right half of the string. Then the defining relation for the surface state $\langle\Sigma| \in \mathcal{H}^*$ is

$$\langle\Sigma|\phi\rangle = \langle s \circ \phi(0)\rangle_\Sigma\,, \tag{III.113}$$

where $|\phi\rangle \in \mathcal{H}$ is the state corresponding to the local operator $\phi(z)$ via the state-operator correspondence, $|\phi\rangle = \lim_{z\to 0}\phi(z)|0\rangle$. The correlation function $\langle\,\rangle_\Sigma$ is evaluated on Σ, $s\circ\phi(0) = \big(s'(0)\big)^h\phi(s(0))$ is the conformal transform of ϕ by the map s. It is convenient to introduce the BPZ conjugate ket-state to $\langle\Sigma|$ and denote it with a small abuse of notation by $|\Sigma\rangle$.

We are now ready to introduce the wedge state description of the sliver. Wedge states constitute a subclass of the surface states introduced above. The surface Σ will be embedded as a unit disk $\{w \le 1\}$ in a w-plane, with the puncture P located at $w = 1$. For any $\nu \in$, let the surface Σ_n be parametrized by

$$w_n = s_n(z) \equiv (h(z))^{2/n} = \left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right)^{2/n}, \tag{III.114}$$

where $h(z)$ takes the canonical upper half-disk into a unit half-disk in the $h(z)$-plane, lying in the region $\mathrm{Re}\ h(z) \ge 0$, $|h(z)| \le 1$. The puncture $z = 0$ is mapped to $h(0) = 1$ at the curved side of the half-disk. This half-disk is squeezed by the map $w_n = (h(z))^{2/n}$ into a wedge of angle $2\pi/n$ at $w_n = 0$. The resulting surface state is by convention called $\langle n|$ [153]. Scrutinizing the behavior of wedge states under star multiplication, one finds that they form a closed subalgebra [156]

$$|m\rangle \star |n\rangle = |m + n - 1\rangle\,. \tag{III.115}$$

It can be shown that all wedge states can be expressed in terms of exponentials of the Virasoro generators of the boundary conformal field theory [156]:

$$|n\rangle = \exp\Big(-\frac{n^2-4}{3n^2}L_{-2} + \frac{n^4-16}{30n^4}L_{-4} - \frac{(n^2-4)(176+128n^2+11n^4)}{1890n^6}L_{-6} + \dots\Big)|0\rangle\,. \tag{III.116}$$

Such representations are called *universal*, since they do not depend on the details of the boundary conformal field theory, but are background independent.

We observe that for $n=1$, the surface Σ_1 covers the whole unit disk in the w_1-plane with a cut along the negative real axis. The left half and the right half coincide (are "glued together") along this cut. This identifies the surface state for $n=1$ with the identity,

$$\big\langle n=1\big| = \langle \mathcal{I}| \,. \tag{III.117}$$

In accordance with eq. (III.115), this is a projector. For $|\mathcal{I}\rangle$, a closed expression in terms of L_n was determined in [48].

The state $\langle n=2|$ is the vacuum state, cf. eq. (III.116); the surface Σ_2 is the right half-disk in the w_2 plane. The star product of two $SL(2,\mathbb{R})$ invariant vacua is the state $\langle n=3|$, which is already a nonpolynomial expression in the Fock space oscillators.

The $n\to\infty$ limit of eq. (III.116) is smooth, we find:

$$|n=\infty\rangle = \exp\Big(-\frac{1}{3}L_{-2}+\frac{1}{30}L_{-4}-\frac{11}{1890}L_{-6}+\dots\Big)\,. \tag{III.118}$$

The surface Σ_∞ in this limit is an infinitely slim wedge, its apex lies at the origin. The apparent local coordinate singularity in the $n\to\infty$ limit can be resolved by $SL(2,\mathbb{R})$ invariance. Using conformal field theory techniques similar to the ones applied for the computation of Neumann coefficients (cf. chapter VI) the state $|n=\infty\rangle$ was identified with the sliver $|\Xi\rangle$ given in eq. (III.111) [152]. By (III.115), it can be constructed as an infinite star product of the $SL(2,\mathbb{R})$ invariant vacuum with itself. The sliver state is believed to represent a D25-brane in bosonic vacuum string field theory; a computation of its tension yields the correct value. It was generalized to solutions corresponding to arbitrary Dp-branes; the sliver in the perpendicular directions to the brane is constructed from an oscillator vacuum with $a_0|0\rangle=0$ instead of the momentum zero vacuum $|p=0\rangle$ (see, e. g., [130]). Again, the ratios of tensions agree with the expected values for Dp-branes. The sliver has singular properties: For instance, it could be shown that the left- and right-halves of the open string on the brane described by $|\Xi\rangle$ split and can move independently – the midpoint is fixed to the brane [130, 60].

III.5.2 Other projectors

Butterfly states. Other projectors of the star algebra are the so-called butterfly states $|B_\alpha\rangle$; they belong to a whole family of surface states parametrized by a real parameter $\alpha\in[0,2]$ [59, 60]. They are associated to the whole upper half-plane Σ and can be defined by the map $s_\alpha\colon H\to\Sigma,\ z\mapsto\frac{1}{\alpha}\sin(\alpha\tan^{-1}z)$. As $\alpha\to 0$, we recover the sliver. For $\alpha=1$, this leads to the definition of the butterfly state $|B\rangle$ with the map $s_1(z)=\frac{z}{\sqrt{1+z^2}}$, which in operator form is simply written as

$$|B\rangle = \exp\big(-\frac{1}{2}L_{-2}\big)|0\rangle\,. \tag{III.119}$$

The $|B_\alpha\rangle$ are projectors for all α, $|B_\alpha\rangle\star|B_\alpha\rangle=|B_\alpha\rangle$. The details of their geometrical interpretation are quite intricate and can be found in [60, 162, 127]. In [127], they are also identified with D-brane solutions.

Nothing state. The nothing state is a surface state $|\mathcal{N}\rangle$ defined by the map $s\colon z \mapsto \frac{z}{z^2+1}$. This is the butterfly state with $\alpha = 2$. Again, the surface Σ associated to it is the whole upper half-plane; for a geometrical interpretation, see [60]. An operator expression for the nothing state can be obtained from substituting $L_{-2n} \to (-1)^n L_{-2n}$ in the operator expression for the identity state. In particular, the Fock space representation is given by

$$|\mathcal{N}\rangle = \exp\Big(-\frac{1}{2}\sum_{n=1}^{\infty} a_n^\dagger \cdot a_n^\dagger\Big)|0\rangle \qquad \text{(III.120)}$$

in the bosonic matter part.

It should be noted that all surface state projectors described above have the property that the string midpoints located at $z = \mathrm{i}$ are mapped to $s(\mathrm{i}) \in \{\pm \mathrm{i}\infty\}$ by the defining map $s\colon H \to \Sigma$ in case Σ is the upper half-plane.[16] This will become important in the next subsection.

III.5.3 Further remarks

The *bc* twisted sliver. Rastelli, Sen, and Zwiebach pointed out [59] that the sliver in the so-called *bc*-twisted conformal field theory gives rise to a solution of the ghost part of the vacuum string field equations (III.90). We generalize their argument to any of the above-described surface state projectors: Twisting the *bc* conformal field theory with energy-momentum tensor T and $U(1)$ ghost current J_{bc} by $T \mapsto T' = T - \partial J_{\mathrm{bc}}$ leads to a first order system of fields b', c' with spins $1, 0$. The twisted first order system can be bosonized in terms of a bosonic field φ as $c' = e^{\mathrm{i}\varphi}$, $b' = e^{-\mathrm{i}\varphi}$; in [59], it is shown that the star products in the *bc*- and the $b'c'$-system are related by

$$|B \star C\rangle \propto \sigma^{+'}(\mathrm{i}-\epsilon)\sigma^{-'}(\mathrm{i}-\epsilon)|B \star' C\rangle\,, \qquad \text{(III.121)}$$

where $\sigma^{+'}(z) = e^{\mathrm{i}\varphi/2}(z)$ and $\sigma^{-'}(z) = e^{\mathrm{i}\varphi/2}(\bar z)$ are defined on the double cover of the upper half-plane; the star product in the *bc*-system is denoted by $\star$, the star product in the twisted system by $\star'$. The whole expressoin is understood in the limit $\epsilon \to 0$. Therefore, the ghost equations of motion rewritten in the $\star'$-product read

$$\mathcal{Q}|\mathcal{A}_g\rangle \propto -\sigma^{+'}(\mathrm{i}-\epsilon)\sigma^{-'}(\mathrm{i}-\epsilon)|\mathcal{A}_g \star' \mathcal{A}_g\rangle\,. \qquad \text{(III.122)}$$

We will now take as an ansatz for $\langle\mathcal{A}_g|$ an arbitrary surface state projector $\langle P'|$ in the $b'c'$ conformal field theory defined by a map s to the upper half-plane as described above. This state satisfies (cf. (III.113))

$$\langle P'|\phi\rangle = \langle s\circ\phi(0)\rangle'\,, \qquad \text{(III.123)}$$

where the primed correlation function has to be evaluated in the $b'c'$ conformal field theory on the upper half-plane. This can be used to rewrite eq. (III.122) in terms of correlation functions

[16] The two wedge state projectors $\langle n = 2|$ and $\langle n = \infty|$ were defined via maps to the unit disk. If the unit disk is mapped to the upper half-plane, the same statement holds true.

for arbitrary operators ϕ. By an explicit translation between the bc-oscillators and the $b'c'$-oscillators it is easy to see that $\mathcal{Q} \propto (c'(\mathrm{i}) + c'(-\mathrm{i}))$, cf. eq. (III.88). Therefore, the left-hand side of (III.122) is proportional to

$$\left\langle s \circ \big(\phi(0)(c'(\mathrm{i}) + c'(-\mathrm{i}))\big)\right\rangle' = \left\langle s \circ \phi(0)\big(c'(\mathrm{i}\infty) + c'(-\mathrm{i}\infty)\big)\right\rangle' . \tag{III.124}$$

Here we have made use of the fact that all surface state projectors have the string midpoints mapped to $\pm\mathrm{i}\infty$. Expanding $s(\mathrm{i} + \epsilon) \simeq \mathrm{i}\eta$, the right-hand side of eq. (III.122) is proportional to

$$\left\langle s \circ \big(\phi(0)\sigma^{+'}(\mathrm{i} + \epsilon)\sigma^{-'}(\mathrm{i} + \epsilon)\big)\right\rangle' \propto \left\langle \big(s \circ \phi(0)\sigma^{+'}(\mathrm{i}\eta)\sigma^{+'}(-\mathrm{i}\eta)\big)\right\rangle' , \tag{III.125}$$

where we have used the boundary conditions to relate $\sigma^{-'}(\mathrm{i}\eta)$ to $\sigma^{+'}(-\mathrm{i}\eta)$. Recall that the correlation functions has to be evaluated on the upper half-plane (and not on its double cover), therefore $c'(\mathrm{i}\infty) + c'(-\mathrm{i}\infty)$ can be replaced by $2c'(\mathrm{i}\infty)$ on the right-hand side of eq. (III.124). On the right-side side of eq. (III.125), we can replace $\sigma^{+'}(\mathrm{i}\eta)\sigma^{+'}(-\mathrm{i}\eta)$ by the leading term in the corresponding operator product expansion, i. e. $c'(\mathrm{i}\infty)$. Therefore both sides of eq. (III.122) are proportional to $\langle s \circ \phi(0)c'(\mathrm{i}\infty)\rangle$, which proves that the surface projector state $\langle P'|$ subject to (III.123) solves the vacuum equations for the ghost part. The normalization has to be fixed separately.

In fact, we will determine the Neumann matrices for the $\psi^{\pm}$ system in chapter VI; this system is equivalent to the $b'c'$ system. This paves the way for finding more solutions to the vacuum equations for the ghost part since one is now in the position to determine all possible projectors in the $b'c'$ system.

Projectors in SSFTs. The above proof that the sliver in the twisted ghost system solves the bosonic vacuum string field equations was generalized to the sliver of a doubly twisted [5, 101] and a triply twisted [140] ghost system in the superstring case.

In the case of nonpolynomial vacuum superstring field theory one can show that the full (ghost plus matter) supersliver fails to describe non-BPS D9-branes in type IIA theory. Namely, let Φ be a string field annihilated by η_0 (such as the supersliver and any other surface state, see [140]). Then, e^{Φ} also lies in the kernel of η_0. It satisfies the equation of motion,

$$\eta_0\big(e^{-\Phi} \star \mathcal{Q}e^{\Phi}\big) = (\eta_0 e^{-\Phi}) \star (\mathcal{Q}e^{\Phi}) - e^{-\Phi} \star \mathcal{Q}(\eta_0 e^{\Phi}) = 0 \, , \tag{III.126}$$

but it is gauge-trivial (cf. the generalization of (III.78) to the vacuum case)! A non-trivial solution has to be constructed in the large Hilbert space.

Chapter IV

Solving the string field theory equations

IV.1 Introduction

In the last chapter, Witten's bosonic string field theory and cubic and nonpolynomial superstring field theories as well as their vacuum versions were introduced. In this chapter, we will attack the problem to find solutions to the equation of motion of Berkovits' superstring field theory. Apart from providing a better understanding of the structures of string field theories, finding solutions to the string field equations is important for the study of Sen's conjectures. Namely, solutions to the string field equations describe classical string fields, which convey information about nonperturbative string configurations. However, there are some major obstacles for finding classical solutions to the equations of motion of open string field theories, which already have been mentioned in the course of the last chapter. E. g., the BRST operator usually mixes matter and ghost string fields; this makes it impossible to study matter and ghost sectors separately.

Sen's conjectures have been tested in the framework of Berkovits' superstring field theory in various ways: Using the level-truncation scheme, numerical checks were performed e. g. in [20, 25, 41, 84], and the predicted kink solutions describing lower-dimensional D-branes were found [139]. On the analytic side, a background-independent version of Berkovits' string field theory was proposed in [97, 158], and the ideas of vacuum string field theory and computations of the sliver state of bosonic string field theory were transferred to the superstring case [126, 2, 140]. More recently, there have been some attempts [96, 98, 140] to solve the string field equation (III.79) but, to our knowledge, no general method for finding explicit solutions has been presented so far.

The same applies to N=2 string field theory. However, since N=2 strings have no tachyon in their spectrum, Sen's conjectures are not the prime motivation for a study of their field equations. Nevertheless, it would be interesting to show that lower-dimensional D-branes can be found as solitonic solutions to these equations. Since nonpolynomial string field theories for N=1 and for N=2 strings share most characteristic properties, there is some hope that solutions to the N=2 string field equations may serve as guidelines for the N=1 case.

In general, it would be desirable to have solution generating techniques at hand which allow for finding some (or all) solutions by a well-defined prescription. It is the purpose of this chapter to introduce such a solution generating technique (which is known from the field theory treatment in chapter II) for the case of nonpolynomial (super)string field theory. We will see in the next chapter that this technique can be transferred also to cubic superstring field theory and the vacuum versions of both superstring field theories. This chapter is based mainly on reference [II].

We show that the WZW-like string field equation (III.79) is integrable in the sense that it can be written as the compatibility condition of some linear equations with an extra "spectral parameter" λ. This puts us in the position to parametrize solutions of (III.79) by solutions of linear equations on extended string fields (depending on the parameter λ) and to construct classes of explicit solutions via various solution-generating techniques. This discussion will be independent of any implementation of G^+ and $\widetilde{G}^+$ in terms of matter multiplets and is therefore valid for N=1 superstrings as well as for N=2 strings.

We discuss two related approaches to generating solutions of the WZW-like string field equation (III.79). First, considering the splitting method and using the simplest Atiyah-Ward ansatz for the matrix-valued string field of the associated Riemann-Hilbert problem, we reduce eq. (III.79) to the linear equations $\widetilde{G}^+G^+\rho_k = 0$ with $k = 0, \pm 1$. Here ρ_0 and $\rho_{\pm 1}$ are some string fields parametrizing the field e^Φ. However, our discussion of the splitting approach is restricted to the $n = 2$ case, i. e. $u(2)$ Chan-Paton factors, or a certain embedding of $u(2)$ into $u(2 + l)$. Second, we consider the (related) dressing approach which overcomes this drawback. With this method, new solutions are constructed from an old one by successive application of simple (dressing) transformations. Using them, we write down an ansatz which reduces the nonlinear equation of motion (III.79) to a system of linear equations. Solutions of the latter describe nonperturbative field configurations obeying (III.79). Finally, we present some explicit solutions of the WZW-like string field equation.

This chapter is organized as follows: In section IV.2 we discuss the reality properties of some operators in the superconformal algebra. This issue is crucial for the treatment of the linear equations. We prove the integrability of the WZW-like string field equation in section IV.3. The construction of solutions via solving a Riemann-Hilbert problem and via dressing a seed solution is subject to sections IV.4 and IV.5, respectively. Finally, section IV.6 presents some explicit solutions.

IV.2 Reality properties

Real structures on $sl(2,\mathbb{C})$. We would like to introduce a real structure on the Lie algebra $sl(2,\mathbb{C})$. A real structure σ on a vector space V is by definition an antilinear involution $\sigma\colon V \to V$, i. e. $\sigma(\zeta X + Y) = \bar{\zeta}\sigma(X) + \sigma(Y)$ for $\zeta \in \mathbb{C}$, $X, Y \in V$ and $\sigma^2 = \mathrm{id}_V$ (here we choose $V = sl(2,\mathbb{C})$). There are three choices for a real structure on $sl(2,\mathbb{C})$. Acting onto the defining representation of $sl(2,\mathbb{C})$, we can write them as

$$\sigma_\varepsilon \begin{pmatrix} a & b \\ c & -a \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \tag{IV.1}$$

$$\sigma_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} := \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix} \tag{IV.2}$$

for $\varepsilon = \pm 1$ and $a, b, c \in \mathbb{C}$. We denote the space of fixed points by $V_\mathbb{R}$, i. e. $V_\mathbb{R} := \{X \in V : \sigma(X) = X\}$. For the real structures (IV.1), it is straightforward to check that $V_\mathbb{R} \cong su(2)$ for $\varepsilon = -1$ and $V_\mathbb{R} \cong su(1,1)$ for $\varepsilon = 1$, whereas, for the real structure σ_0, we obtain $V_\mathbb{R} \cong sl(2,\mathbb{R})$. Real linear combinations of vectors within $V_\mathbb{R}$ are again contained in $V_\mathbb{R}$, so they make up real linear subspaces in $sl(2,\mathbb{C})$.[1] Because of the form of eqs. (IV.1) and (IV.2) it is clear that σ preserves the Lie algebra structure, i. e. the $V_\mathbb{R}$ are real Lie subalgebras. In the case of σ_ε, complex conjugation can be "undone" by a conjugation with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix}\right)$ on an element of this linear subspace. Let us already here note that this conjugation matrix is contained within the group $SU(2)$ for $\varepsilon = -1$, but is outside $SU(1,1)$ for $\varepsilon = 1$. This means [83, 81] that conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix}\right)$ is an inner automorphism on $su(2)$ and an outer automorphism on $su(1,1)$.

Bar operation. As stated above, an N=4 superconformal algebra contains as a subalgebra a current algebra generated by J, J^{++}, and J^{--}. Its horizontal Lie algebra, when complexified, is $sl(2,\mathbb{C})$. On $sl(2,\mathbb{C})$, let us introduce a real structure σ_ε from (IV.1), where we choose $\varepsilon = -1$ for N=1 and $\varepsilon = 1$ for N=2 strings. The action of complex conjugation on the N=4 superconformal generators, as determined in [28, 29], looks as follows:

$$J^* = -J\,, \quad (J^{++})^* = J^{--}\,, \quad (G^+)^* = G^-\,, \quad (\widetilde{G}^+)^* = \widetilde{G}^-\,. \tag{IV.3}$$

Now, the signflip of J in (IV.3) necessitates an additional rotation to reestablish the original twist $T \to T + \frac{1}{2}\partial J$ of the N=4 superconformal algebra. As mentioned in the previous paragraph, this can be accomplished by an inner automorphism in the case of N=1 strings and by an outer automorphism of the current Lie algebra in the case of N=2 strings. These automorphisms consist in conjugating an element of the defining representation with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix}\right)$. In the following, we want to scrutinize how the rotation reestablishing the twist acts onto the other elements of the N=4 superconformal algebra.

As explained in [29], the four spin $\frac{3}{2}$ superpartners of the energy-momentum tensor transform under the current group (therefore the current indices $\pm$) as well as under a group[2] of additional automorphisms $SU(2)$ (or $SU(1,1)$). Explicitly,

$$(G^{\tilde{\alpha}\alpha}) = \begin{pmatrix} G^+ & \widetilde{G}^+ \\ \varepsilon\widetilde{G}^- & G^- \end{pmatrix} \tag{IV.4}$$

transforms under current group transformations by left-multiplication (note that the columns of this matrix form doublets under the current algebra, cf. (III.68) and (III.69)). Under additional

[1] In the following, we will be mainly interested in σ_ε. The case of eq. (IV.2) has been used in [26] and [107].

[2] For a discussion of its existence, see [29].

automorphisms it transforms by right-multiplication. Obviously, the rotation reestablishing the original twist acts onto the complex conjugate matrix as

$$\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} G^- & \widetilde{G}^- \\ \varepsilon \widetilde{G}^+ & G^+ \end{pmatrix} = \begin{pmatrix} \varepsilon \widetilde{G}^+ & G^+ \\ \varepsilon G^- & \varepsilon \widetilde{G}^- \end{pmatrix} . \tag{IV.5}$$

Having established the action of this rotation on the N=4 superconformal algebra, we denote the combined operation of star conjugation and "twist-restoring" transformation by a bar, i. e.

$$\overline{G^+} = \varepsilon \widetilde{G}^+ , \quad \overline{\widetilde{G}^+} = G^+ , \quad \overline{\widetilde{G}^-} = G^- , \quad \text{and} \quad \overline{G^-} = \varepsilon \widetilde{G}^- . \tag{IV.6}$$

On elements of the complexified current algebra, the bar operation acts by (IV.1), in particular it acts trivially onto the real $su(2)$ or $su(1,1)$ subalgebra.

Additional automorphisms. The aforementioned additional automorphisms of the N=4 superconformal algebra act on the "G-matrix" from the right [29]. We take them to be elements of $SL(2,\mathbb{C})$ and therefore define

$$\begin{pmatrix} G^+(v) & \widetilde{G}^+(v) \\ \varepsilon\, \widetilde{G}^-(v) & G^-(v) \end{pmatrix} := \begin{pmatrix} G^+ & \widetilde{G}^+ \\ \varepsilon\, \widetilde{G}^- & G^- \end{pmatrix} \begin{pmatrix} v_3 & v_1 \\ v_4 & v_2 \end{pmatrix} \tag{IV.7}$$

for $v_i \in \mathbb{C}$ with $v_2 v_3 - v_1 v_4 = 1$. The action of the additional automorphisms should be compatible with the above-defined bar operation (IV.6), i. e.

$$\overline{G^+(v)} = \varepsilon \widetilde{G}^+(v) , \quad \overline{\widetilde{G}^+(v)} = G^+(v) , \quad \overline{\widetilde{G}^-(v)} = G^-(v) , \quad \text{and} \quad \overline{G^-(v)} = \varepsilon \widetilde{G}^-(v) . \tag{IV.8}$$

These four equations all lead to the same requirements $v_3 = \bar{v}_2 =: u_1$ and $v_4 = \varepsilon \bar{v}_1 =: u_2$, which in effect restricts the group of additional automorphisms from $SL(2,\mathbb{C})$ to $SU(2)$ for $\varepsilon = -1$ and to $SU(1,1)$ for $\varepsilon = 1$:

$$\begin{pmatrix} G^+(u) & \widetilde{G}^+(u) \\ \varepsilon\, \widetilde{G}^-(u) & G^-(u) \end{pmatrix} := \begin{pmatrix} G^+ & \widetilde{G}^+ \\ \varepsilon\, \widetilde{G}^- & G^- \end{pmatrix} \begin{pmatrix} u_1 & \varepsilon \bar{u}_2 \\ u_2 & \bar{u}_1 \end{pmatrix} \tag{IV.9}$$

with $|u_1|^2 - \varepsilon |u_2|^2 = 1$. Note that

$$\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 & \varepsilon \bar{u}_2 \\ u_2 & \bar{u}_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} = \begin{pmatrix} \bar{u}_1 & \varepsilon u_2 \\ \bar{u}_2 & u_1 \end{pmatrix} . \tag{IV.10}$$

Obviously, eq. (IV.8) entails the introduction of the same real structure on the group of additional automorphisms as the one chosen for the current group. Since we are only interested in the ratio of the prefactors of G^+ and $\widetilde{G}^+$, and of G^- and $\widetilde{G}^-$, respectively, we define for later use the combinations

$$\begin{aligned} \widetilde{G}^+(\lambda) &:= \widetilde{G}^+ + \lambda G^+ = \frac{1}{\bar{u}_1} \widetilde{G}^+(u) , \\ \widetilde{G}^-(\lambda) &:= \widetilde{G}^- + \bar{\lambda} G^- = \frac{1}{u_1} \widetilde{G}^-(u) , \\ G^+(\lambda) &:= G^+ + \varepsilon \bar{\lambda} \widetilde{G}^+ = \frac{1}{u_1} G^+(u) , \\ G^-(\lambda) &:= G^- + \varepsilon \lambda \widetilde{G}^- = \frac{1}{\bar{u}_1} G^-(u) . \end{aligned} \tag{IV.11}$$

Here, $(\bar{u}_1 : \varepsilon\bar{u}_2)$ can be regarded as homogeneous coordinates on the sphere $S^2 \cong \mathbb{C}P^1$, and $\lambda \equiv \varepsilon\bar{u}_2/\bar{u}_1$ is a local coordinate for $\bar{u}_1 \neq 0$.

Let us reconsider the action of the bar operation on the matrix (IV.9). Remember that the bar operation consists of a complex conjugation and a subsequent twist-restoring rotation as in (IV.5). Acting on (IV.9), we have

$$\begin{aligned}\begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix}\begin{pmatrix}G^- & \widetilde{G}^-\\ \varepsilon\widetilde{G}^+ & G^+\end{pmatrix}\begin{pmatrix}\bar{u}_1 & \varepsilon u_2\\ \bar{u}_2 & u_1\end{pmatrix} &= \begin{pmatrix}\varepsilon\widetilde{G}^+ & G^+\\ \varepsilon G^- & \varepsilon\widetilde{G}^-\end{pmatrix}\begin{pmatrix}0 & \varepsilon\\ 1 & 0\end{pmatrix}\begin{pmatrix}u_1 & \varepsilon\bar{u}_2\\ u_2 & \bar{u}_1\end{pmatrix}\begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix}\\ &= \begin{pmatrix}G^+ & \widetilde{G}^+\\ \varepsilon\widetilde{G}^- & G^-\end{pmatrix}\begin{pmatrix}\bar{u}_2 & u_1\\ \varepsilon\bar{u}_1 & u_2\end{pmatrix}.\end{aligned} \tag{IV.12}$$

For the first equality, we have used (IV.5) and (IV.10). An additional right-multiplication by $\left(\begin{smallmatrix}0 & \varepsilon\\ 1 & 0\end{smallmatrix}\right)$ transforms the "u-matrix" on the right hand side back to its original form (cf. (IV.10)), thereby taking $\varepsilon\bar{\lambda}^{-1}$ to λ. This mediates a map between operators defined for $|\lambda| < \infty$ and operators defined for $|\lambda| > 0$.

The action of the bar operation on $\widetilde{G}^+(\varepsilon\bar{\lambda}^{-1})$ etc. can be determined from the fact that $\varepsilon\bar{\lambda}^{-1} = u_1/u_2$ and therefore

$$\begin{pmatrix}G^+(\varepsilon\bar{\lambda}^{-1}) & \widetilde{G}^+(\varepsilon\bar{\lambda}^{-1})\\ \varepsilon\widetilde{G}^-(\varepsilon\bar{\lambda}^{-1}) & G^-(\varepsilon\bar{\lambda}^{-1})\end{pmatrix} = \begin{pmatrix}G^+ & \widetilde{G}^+\\ \varepsilon\widetilde{G}^- & G^-\end{pmatrix}\begin{pmatrix}\bar{u}_2 & u_1\\ \varepsilon\bar{u}_1 & u_2\end{pmatrix}\begin{pmatrix}\bar{u}_2^{-1} & 0\\ 0 & u_2^{-1}\end{pmatrix}. \tag{IV.13}$$

Multiplying the complex conjugate matrices from the left by a twist-restoring rotation, we obtain

$$\begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix}\begin{pmatrix}G^- & \widetilde{G}^-\\ \varepsilon\widetilde{G}^+ & G^+\end{pmatrix}\begin{pmatrix}u_2 & \bar{u}_1\\ \varepsilon u_1 & \bar{u}_2\end{pmatrix}\begin{pmatrix}u_2^{-1} & 0\\ 0 & \bar{u}_2^{-1}\end{pmatrix} = \begin{pmatrix}\bar{\lambda}^{-1}G^+(\lambda) & \lambda^{-1}\widetilde{G}^+(\lambda)\\ \varepsilon\bar{\lambda}^{-1}\widetilde{G}^-(\lambda) & \lambda^{-1}G^-(\lambda)\end{pmatrix}. \tag{IV.14}$$

So, "barring" accompanied by the transformation $\lambda \mapsto \varepsilon\bar{\lambda}^{-1}$ maps $\widetilde{G}^+(\lambda)$ (defined for $|\lambda| < \infty$) to $\frac{1}{\lambda}\widetilde{G}^+(\lambda)$ (defined for $|\lambda| > 0$),

$$\overline{\widetilde{G}^+(\varepsilon\bar{\lambda}^{-1})} = \frac{1}{\lambda}\widetilde{G}^+(\lambda). \tag{IV.15}$$

This result will be needed in section IV.3.

For the selection of an N=2 superconformal subalgebra within the N=4 algebra, there is obviously the freedom to choose a linear combination of the J's as the $U(1)$ current. All choices are equivalent through current $SU(2)$- (or $SU(1,1)$-) rotations (acting on the matrices in (IV.4) from the left). In addition, there is the freedom to choose which linear combination of the positively charged G's will be called G^+ [38]. This freedom is parametrized by another $SU(2)$ (or $SU(1,1)$), cf. (IV.9). Since in our case only the ratio of the prefactors of G^+ and $\widetilde{G}^+$ is important, we arrive at generators (IV.11) parametrized by $\lambda \in \mathbb{C}P^1$. So, we obtain a one-parameter family of N=2 superconformal algebras embedded into a small N=4 algebra [28, 29].

IV.3 Integrability of Berkovits' string field theory

In this section we show that the equation of motion (III.79) of Berkovits' string field theory can be obtained as the compatibility condition of some linear equations. In other words, solutions of these linear equations exist iff eq. (III.79) is satisfied. For N=2 strings the integrability of Berkovits' string field theory was shown in [107], and here we extend this analysis to the N=1 case. The payoff for considering such (integrable) models is the availability of powerful techniques for the construction of solutions to the equation(s) of motion.

Sphere $\mathbb{C}P^1$. Let us consider the Riemann sphere $S^2 \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$ and cover it by two coordinate patches

$$\mathbb{C}P^1 = U_+ \cup U_- \,, \quad U_+ := \{\lambda \in \mathbb{C} : |\lambda| < 1+\epsilon\} \,, \quad U_- := \{\lambda \in \mathbb{C}\cup\{\infty\} : |\lambda| > (1+\epsilon)^{-1}\} \tag{IV.16}$$

for some $\epsilon > 0$ with the overlap

$$U_+ \cap U_- \supset S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \,. \tag{IV.17}$$

We will consider $\lambda \in U_+$ and $\tilde{\lambda} \in U_-$ as local complex coordinates on $\mathbb{C}P^1$ with $\tilde{\lambda} = \frac{1}{\lambda}$ in $U_+ \cap U_-$. Recall that $\lambda \in \mathbb{C}P^1$ was introduced in the previous section as a parameter for a family of N=2 subalgebras in the twisted small N=4 superconformal algebra with fermionic currents (IV.11).

Linear system. Taking the string field[3] Φ from (III.77) and operators $\widetilde{G}^+(\lambda)$ from (IV.11), we introduce the following equation:

$$(\widetilde{G}^+ + \lambda G^+ + \lambda A)\Psi = 0 \,, \tag{IV.18}$$

where $A := e^{-\Phi} G^+ e^{\Phi}$ and Ψ is a matrix-valued string field depending not only on X and ψ but also (meromorphically) on the auxiliary parameter $\lambda \in \mathbb{C}P^1$. As in eq. (III.71) the action of G^+ and $\widetilde{G}^+$ on Ψ implies a contour integral of the (fermionic) current around Ψ. Note that G^+ and $\widetilde{G}^+$ are Grassmann-odd and therefore $(G^+)^2 = (\widetilde{G}^+)^2 = G^+\widetilde{G}^+ + \widetilde{G}^+ G^+ = 0$.

If e^{Φ} is given, then (IV.18) is an equation for the field Ψ. Solutions Ψ of this linear equation exist if the term in brackets squares to zero, i. e.

$$\Big(\widetilde{G}^+ + \lambda G^+ + \lambda A\Big)^2 = 0 \quad \Leftrightarrow \quad \lambda^2 (G^+ + A)^2 + \lambda \widetilde{G}^+ A = 0 \tag{IV.19}$$

[3] In this chapter, we will consider string fields as functionals of $X^\mu(\sigma,\tau)$, $\psi^\mu(\sigma,\tau)$, $b(\sigma,\tau)$, $c(\sigma,\tau)$, $\beta(\sigma,\tau)$, and $\gamma(\sigma,\tau)$ for N=1 and of $Z^a(\sigma,\tau), \bar{Z}^{\bar{a}}(\sigma,\tau)$, $\psi^{+a}(\sigma,\tau)$, and $\psi^{-\bar{a}}(\sigma,\tau)$ for N=2. We will often suppress the τ-dependence of the world-sheet fields and only indicate their σ-dependence when necessary. As an abbreviation for this list of world-sheet fields, we use X and ψ and thus denote a string field Φ by $\Phi[X,\psi]$ (in the N=2 case, one may think of X as the real and imaginary parts of Z, similarly with ψ). Throughout this chapter, all string fields are understood to be multiplied via Witten's star product. – In our subsequent discussion, some string fields (such as $\overline{T}_k T_k$) have to be invertible. Without further specification we assume that all string fields are units of the star algebra (which is, e. g., manifest for string fields of the form e^{Φ}). It should, however, be noted that the star algebra is not an integral domain (which can be seen from the fact that it contains non-trivial projectors).

for any λ. Here we have used the Grassmann nature of G^+ and $\widetilde{G}^+$. So, we obtain two equations:

$$G^+A + A^2 \;=\; 0\,, \tag{IV.20}$$
$$\widetilde{G}^+A \;=\; 0\,. \tag{IV.21}$$

The first of these two equations is trivial since G^+ acts as a derivation on the algebra of string fields. The second equation coincides with the equation of motion (III.79).

Chiral string fields. As a special case of eq. (IV.18) one can consider the equations

$$\left(\widetilde{G}^+ + \lambda G^+ + \lambda A\right)\Psi_+ \;=\; 0\,, \tag{IV.22}$$
$$\left(\frac{1}{\lambda}\widetilde{G}^+ + G^+ + A\right)\Psi_- \;=\; 0\,, \tag{IV.23}$$

where Ψ_+ and Ψ_- are invertible matrix-valued string fields depending holomorphically on λ and $\frac{1}{\lambda}$, respectively.

Considering $\lambda \to 0$ in (IV.22), we see that $\widetilde{G}^+\Psi_+(\lambda = 0) = 0$, and one may choose $\Psi_+(\lambda = 0) = \mathcal{I}$. Analogously, taking $\lambda \to \infty$ in (IV.23), we obtain

$$A = e^{-\Phi}G^+e^{\Phi} = \Psi_-(\lambda = \infty)G^+\Psi_-^{-1}(\lambda = \infty)\,, \tag{IV.24}$$

and one may choose $\Psi_-(\lambda = \infty) = e^{-\Phi}$ as a solution thereof. From this we derive the asymptotic behavior of our fields:

$$\Psi_+ = \mathcal{I} + O(\lambda) \quad \text{for } \lambda \to 0\,, \tag{IV.25}$$
$$\Psi_- = e^{-\Phi} + O(\lambda^{-1}) \quad \text{for } \lambda \to \infty\,. \tag{IV.26}$$

We see that Ψ_- may be considered as a λ-augmented solution of eq. (III.79), and all information about e^{Φ} is contained in $\Psi_\pm$.

Suppose that we find solutions Ψ_+ and Ψ_- of eqs. (IV.22) and (IV.23) for a given e^{Φ}. Then one can introduce the matrix-valued string field

$$\Upsilon_{+-} := \Psi_+^{-1}\Psi_- \tag{IV.27}$$

defined for $\lambda \in U_+ \cap U_-$. From eqs. (IV.22) and (IV.23) it follows that

$$\widetilde{G}^+(\lambda)\Upsilon_{+-} \equiv (\widetilde{G}^+ + \lambda G^+)\Upsilon_{+-} = 0\,. \tag{IV.28}$$

String fields annihilated by the operator $\widetilde{G}^+(\lambda)$ will be called *chiral*.

Reality properties. W. r. t. the bar operation from section IV.2, the string field Φ is real[4], i. e. [19] (cf. eq. (III.34))

$$\overline{\Phi[X(\pi-\sigma), \psi(\pi-\sigma)]} = \Phi[X(\sigma), \psi(\sigma)]\,. \tag{IV.29}$$

[4]The world-sheet parity transformation $\sigma \mapsto \pi - \sigma$ is accompanied by a transposition of Chan-Paton matrices. Note that hermitian generators are used for the $u(n)$ Chan-Paton algebra. Therefore, e^{Φ} does not necessarily take values in $U(n)$.

To see the behavior of the extended string field under the bar operation we scrutinize eqs. (IV.22) and (IV.23). We already saw that $\overline{\widetilde{G}^+(\varepsilon\bar{\lambda}^{-1})} = \frac{1}{\lambda}\widetilde{G}^+ + G^+$, and by definition, $\overline{A} = (\overline{G^+ e^{\Phi}})e^{-\overline{\Phi}} = \varepsilon(\widetilde{G}^+ e^{\Phi})e^{-\Phi}$ under $\sigma \mapsto \pi - \sigma$. Then mapping $\lambda \mapsto \varepsilon\bar{\lambda}^{-1}$ and $\sigma \mapsto \pi - \sigma$ in (IV.22) and conjugating, we obtain

$$\Big(\frac{1}{\lambda}\,\widetilde{G}^+ + G^+\Big)\overline{\Psi}_+^{-1} - \frac{1}{\lambda}(\widetilde{G}^+ e^{\Phi})e^{-\Phi}\overline{\Psi}_+^{-1} = 0\,. \tag{IV.30}$$

This coincides with eq. (IV.23) if we set

$$\big(\overline{\Psi}_+\big)^{-1}\Big[X(\pi-\sigma),\psi(\pi-\sigma),\frac{\varepsilon}{\bar{\lambda}}\Big] = e^{\Phi}\Psi_-\Big[X(\sigma),\psi(\sigma),\lambda\Big]\,. \tag{IV.31}$$

Then from (IV.31) it follows that

$$\Upsilon_{+-} = \overline{\Psi}_- e^{\Phi}\Psi_- \tag{IV.32}$$

is real, i.e. $\overline{\Upsilon}_{+-}\big[X(\pi-\sigma),\psi(\pi-\sigma),\varepsilon\bar{\lambda}^{-1}\big] = \Upsilon_{+-}\big[X(\sigma),\psi(\sigma),\lambda\big]$.

Gauge freedom. Recall that gauge transformations of the fields e^{Φ} and A have the form

$$\begin{aligned} e^{\Phi} \mapsto e^{\Phi'} = Be^{\Phi}C \quad \text{with} \quad G^+B = 0\,, \quad \widetilde{G}^+C = 0\,,\\ A \mapsto A' = C^{-1}AC + C^{-1}G^+C\,. \end{aligned} \tag{IV.33}$$

Under C-transformations the fields $\Psi_\pm$ transform as

$$\Psi_\pm \mapsto \Psi'_\pm = C^{-1}\Psi_\pm\,. \tag{IV.34}$$

It is easy to see that the chiral string field Υ_{+-} is invariant under the transformations (IV.34). On the other hand, the field A will remain unchanged after the transformations

$$\Psi_+ \mapsto \Psi_+ h_+\,, \quad \Psi_- \mapsto \Psi_- h_-\,, \tag{IV.35}$$

where h_+ and h_- are chiral string fields depending holomorphically on λ and $\frac{1}{\lambda}$, respectively. In the special case when h_+ and h_- are independent of λ the transformations (IV.35) with $h_+ = \mathcal{I}$, $h_- =: B^{-1}$ induce the B-transformations $e^{\Phi} \mapsto Be^{\Phi}$ in (IV.33). In general, eqs. (IV.35) induce the transformations

$$\Upsilon_{+-} \mapsto h_+^{-1}\Upsilon_{+-}h_- \tag{IV.36}$$

on the space of solutions to eq. (IV.28), and any two solutions differing by a transformation (IV.36) are considered to be equivalent.

Splitting. Up to now, we discussed how to find Υ_{+-} for a given $A = e^{-\Phi}G^+e^{\Phi}$. Consider now the converse situation. Suppose we found a string field Υ_{+-} which depends analytically on $\lambda \in S^1$ and satisfies the linear equation (IV.28). Being real-analytic Υ_{+-} can be extended to a string field depending holomorphically on $\lambda \in U_+ \cap U_-$. Then we can formulate an operator version of the Riemann-Hilbert problem: Split $\Upsilon_{+-} = \Psi_+^{-1}\Psi_-$ into matrix-valued string fields $\Psi_\pm$ depending on $\lambda \in U_+ \cap U_-$ such that Ψ_+ can be extended to a regular (i.e. holomorphic in λ and invertible) matrix-valued function on U_+ and that Ψ_- can be extended to a regular matrix-valued function on U_-. From eq. (IV.28) it then follows that

$$\Psi_+(\widetilde{G}^+ + \lambda G^+)\Psi_+^{-1} = \Psi_-(\widetilde{G}^+ + \lambda G^+)\Psi_-^{-1} = \tilde{A} + \lambda A\,, \tag{IV.37}$$

where $\tilde{A}$ and A are some λ-independent string fields. The last equality follows from expanding Ψ_+ and Ψ_- into power series in λ and λ^{-1}, respectively. If we now choose a function $\Psi_+(\lambda)$ such that[5] $\Psi_+(\lambda = 0) = \mathcal{I}$ then $\tilde{A} = 0$ and

$$A = \Psi_-(\lambda = \infty)G^+\Psi_-^{-1}(\lambda = \infty) \quad \Rightarrow \quad \Psi_-(\lambda = \infty) =: e^{-\Phi}\,. \tag{IV.38}$$

In the general case, we have

$$e^{-\Phi} = \Psi_+^{-1}(\lambda = 0)\Psi_-(\lambda = \infty)\,. \tag{IV.39}$$

So, starting from Υ_{+-} we have constructed a solution e^{Φ} of eq. (III.79).

Suppose that we know a splitting for a given Υ_{+-} and have determined a correspondence $\Upsilon_{+-} \leftrightarrow e^{\Phi}$. Then for any matrix-valued chiral string field $\widetilde{\Upsilon}_{+-}$ from a small enough neighborhood of Υ_{+-} (i. e. $\widetilde{\Upsilon}_{+-}$ is close to Υ_{+-} in some norm) there exists a splitting $\widetilde{\Upsilon}_{+-} = \widetilde{\Psi}_+^{-1}\widetilde{\Psi}_-$ due to general deformation theory arguments. Namely, there are no obstructions to a deformation of a trivial holomorphic vector bundle $\mathcal{E}$ over $\mathbb{C}P^1$ since its infinitesimal deformations are parametrized by the group $H^1(\mathbb{C}P^1, \mathcal{E})$. This cohomology group is trivial because of $H^1(\mathbb{C}P^1, \mathcal{O}) = 0$ where $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}P^1$. But from the correspondence $\widetilde{\Upsilon}_{+-} \leftrightarrow e^{\widetilde{\Phi}}$ it follows that any solution $e^{\widetilde{\Phi}}$ from an open neighborhood (in the solution space) of a given solution e^{Φ} can be obtained from a "free" chiral string field $\widetilde{\Upsilon}_{+-}$. In this sense, Berkovits' string field theory is an integrable theory; in other words, it is completely solvable.

To sum up, we have described a one-to-one correspondence between the gauge equivalence classes of solutions to the nonlinear equation of motion (III.79) and equivalence classes of solutions (chiral string fields) to the auxiliary linear equation (IV.28). The next step is to show how this correspondence helps to solve (III.79).

IV.4 Exact solutions by the splitting approach

Atiyah-Ward ansatz. As described in the previous section, solutions of the string field equations (III.79) can be obtained by splitting a given matrix-valued chiral string functional Υ_{+-}. In general, splitting is a difficult problem but for a large class of special cases it can be achieved. The well known cases (for $n = 2$) are described e. g. by the infinite hierarchy of Atiyah-Ward ansätze [10] generating instantons in four-dimensional $SU(2)$ Yang-Mills theory. These ansätze are easily generalized [110] to the case of noncommutative instantons, the first examples of which were given in [136]. Here, we consider the first Atiyah-Ward ansatz from the above-mentioned hierarchy [10] and discuss its generalization to the string field case.

[5] This can always be achieved by redefining $\Psi_+(\lambda) \mapsto \Psi_+^{-1}(\lambda = 0)\Psi_+(\lambda)$, $\Psi_-(\lambda^{-1}) \mapsto \Psi_+^{-1}(\lambda = 0)\Psi_-(\lambda^{-1})$.

We start from the 2×2 matrix

$$\Upsilon_{+-} = \begin{pmatrix} \rho & \lambda^{-1}\mathcal{I} \\ \varepsilon\lambda\mathcal{I} & 0 \end{pmatrix}\,, \tag{IV.40}$$

where ρ is a real and chiral string field, i. e.

$$\overline{\rho}\left[X, \psi, \frac{\varepsilon}{\overline{\lambda}}\right] = \rho[X, \psi, \lambda] \tag{IV.41}$$

and

$$(\widetilde{G}^+ + \lambda G^+)\rho = 0\,. \tag{IV.42}$$

We assume that ρ depends on $\lambda \in S^1$ analytically and therefore can be extended holomorphically in λ to an open neighborhood $U_+ \cap U_-$ of S^1 in $\mathbb{C}P^1$. From (IV.41) and (IV.42) it follows that the matrix Υ_{+-} in (IV.40) is chiral and real.

Splitting. We now expand ρ into a Laurent series in λ,

$$\rho = \sum_{k=-\infty}^{\infty} \lambda^k \rho_k = \rho_- + \rho_0 + \rho_+\,, \quad \rho_- = \sum_{k<0} \lambda^k \rho_k\,, \quad \rho_+ = \sum_{k>0} \lambda^k \rho_k\,, \tag{IV.43}$$

and obtain from (IV.42) for

$$\rho_k = \oint \frac{d\lambda}{2\pi i} \lambda^{-k-1} \rho \tag{IV.44}$$

the recursion relations

$$\widetilde{G}^+ \rho_{k+1} = -G^+ \rho_k\,. \tag{IV.45}$$

Using (IV.43), one easily checks that

$$\Upsilon_{+-} = \widehat{\Psi}_+^{-1} \widehat{\Psi}_- \tag{IV.46}$$

where

$$\widehat{\Psi}_+^{-1} = \begin{pmatrix} \rho_0 + \rho_+ & -\lambda^{-1}\rho_+ \\ \varepsilon\lambda\mathcal{I} & -\varepsilon\mathcal{I} \end{pmatrix} \rho_0^{-1/2}\,, \qquad \widehat{\Psi}_- = \rho_0^{-1/2} \begin{pmatrix} \rho_0 + \rho_- & \lambda^{-1}\mathcal{I} \\ \lambda\rho_- & \mathcal{I} \end{pmatrix}\,. \tag{IV.47}$$

However, the asymptotic value of $\widehat{\Psi}_+$,

$$\widehat{\Psi}_+(\lambda = 0) = \rho_0^{-1/2} \begin{pmatrix} \mathcal{I} & -\varepsilon\lambda^{-1}\rho_+ \\ \lambda\mathcal{I} & -\varepsilon(\rho_0 + \rho_+) \end{pmatrix}\Bigg|_{\lambda=0} = \rho_0^{-1/2} \begin{pmatrix} \mathcal{I} & -\varepsilon\rho_1 \\ 0 & -\varepsilon\rho_0 \end{pmatrix} \neq \mathbf{1}_2\mathcal{I}\,, \tag{IV.48}$$

shows that this splitting corresponds to a more general gauge than the one used in eq. (III.79) (see [26, 107] for a discussion of this gauge in the case of N=2 strings).

To obtain the asymptotic behavior (IV.25) one may exploit the "gauge freedom" contained in (IV.46) and introduce the fields

$$\Psi_+ := \widehat{\Psi}_+^{-1}(\lambda = 0)\widehat{\Psi}_+ \quad \text{and} \quad \Psi_- := \widehat{\Psi}_+^{-1}(\lambda = 0)\widehat{\Psi}_- \tag{IV.49}$$

which by definition have the right asymptotic behavior. These functionals yield the same chiral string field since

$$\Upsilon_{+-} = \widehat{\Psi}_+^{-1}\widehat{\Psi}_- = \widehat{\Psi}_+^{-1}\widehat{\Psi}_+(\lambda=0)\widehat{\Psi}_+^{-1}(\lambda=0)\widehat{\Psi}_- = \Psi_+^{-1}\Psi_- \,. \tag{IV.50}$$

Explicit solutions. Now, from

$$\Psi_- = \begin{pmatrix} \rho_0 + \rho_- - \lambda\rho_1\rho_0^{-1}\rho_- & \lambda^{-1}\mathcal{I} - \rho_1\rho_0^{-1} \\ -\varepsilon\lambda\rho_0^{-1}\rho_- & -\varepsilon\rho_0^{-1} \end{pmatrix} \tag{IV.51}$$

we can determine a solution of (III.79) with the help of (IV.26),

$$e^{-\Phi} = \Psi_-(\lambda=\infty) = \begin{pmatrix} \rho_0 - \rho_1\rho_0^{-1}\rho_{-1} & -\rho_1\rho_0^{-1} \\ -\varepsilon\rho_0^{-1}\rho_{-1} & -\varepsilon\rho_0^{-1} \end{pmatrix} . \tag{IV.52}$$

A direct calculation shows that this satisfies eq. (III.79) iff ρ_0, ρ_1 and ρ_{-1} satisfy the linear recursion relations (IV.45) for $k=-1,0$. Moreover, substituting (IV.52) into (III.79) yields

$$\widetilde{G}^+ G^+ \rho_0 = 0 \tag{IV.53}$$

which is the analogue of the Laplace equation in the case of instantons in four-dimensional Euclidean space [110, 136]. Note that (IV.53) is just one of an infinite set of equations,

$$\widetilde{G}^+ G^+ \rho_k = 0 \qquad \forall k \in \mathbb{Z}\,, \tag{IV.54}$$

which can easily be obtained from the recursion relations (IV.45). So, the ansatz (IV.40) for Υ_{+-} and its splitting reduce the nonlinear string field theory equation (III.79) to the linear equations (IV.45) which are equivalent to the chirality equation (IV.42).[6]

Finally, notice that in the case of N=1 strings, one may take $|\rho_0\rangle = \xi_0|V\rangle$ where $|V\rangle$ is a state in the "small" Hilbert space [52]. Then eq. (IV.53) reduces to $Q\eta_0(\xi_0|V\rangle) = Q(-\xi_0\eta_0|V\rangle + |V\rangle) = 0$, i. e.

$$Q|V\rangle = 0 \qquad \text{and} \qquad \eta_0|V\rangle = 0\,. \tag{IV.55}$$

This fits in nicely with the discussion in [21].

IV.5 Exact solutions via dressing of a seed solution

Extended solutions. In the previous section we discussed solutions Ψ_+ and Ψ_- of the linear system which are holomorphic in λ and $\frac{1}{\lambda}$, respectively. Now we are interested in those solutions Ψ of eq. (IV.18) which are holomorphic in open neighborhoods of both $\lambda=0$ and $\lambda=\infty$ and

[6] Considering (IV.52) as an ansatz, we can obviously relax the chirality condition (IV.42) and substitute it with the demand that (IV.45) is satisfied for $k=-1,0$. For trivial G^+- and $\widetilde{G}^+$-cohomology, however, this is again equivalent to eq. (IV.45) for all $k\in\mathbb{Z}$.

therefore have poles[7] at finite points $\lambda = \mu_k$, $k = 1, \ldots, m$. Again we see from (IV.18) that $\Psi(\lambda = \infty)$ coincides with $e^{-\Phi}$ up to a gauge transformation and we fix the gauge by putting

$$\Psi^{-1}(\lambda = \infty) = e^{\Phi}\,. \tag{IV.56}$$

The string field $\Psi[X, \psi, \lambda]$ will be called the *extended solution* corresponding to e^{Φ}. Recall that e^{Φ} is a solution of (III.79) where Φ carries $u(n)$ Chan-Paton labels.

The reality properties of extended solutions are derived in very much the same way as the reality properties of Ψ_+ and Ψ_-. Namely, one can easily show that if $\Psi[X, \psi, \lambda]$ satisfies eq. (IV.18) then $e^{-\Phi}\overline{\Psi}^{-1}\left[X, \psi, \varepsilon\bar{\lambda}^{-1}\right]$ satisfies the same equation and therefore

$$\overline{\Psi\Big[X(\pi-\sigma,\tau), \psi(\pi-\sigma,\tau), \frac{\varepsilon}{\bar{\lambda}}\Big]} = \Big[\Psi\left[X(\sigma,\tau), \psi(\sigma,\tau), \lambda\right]\Big]^{-1} e^{-\Phi}\,, \tag{IV.57}$$

or equivalently,

$$\Psi\big[X(\sigma,\tau), \psi(\sigma,\tau), \lambda\big]\,\overline{\Psi\big[X(\pi-\sigma,\tau), \psi(\pi-\sigma,\tau), \frac{\varepsilon}{\bar{\lambda}}\big]} = e^{-\Phi}\,. \tag{IV.58}$$

Using (IV.58), one can rewrite eq. (IV.18) in the form

$$\left[\left(\frac{1}{\lambda}\widetilde{G}^{+} + G^{+}\right)\Psi\left[X, \psi, \lambda\right]\right]\overline{\Psi\left[X, \psi, \frac{\varepsilon}{\bar{\lambda}}\right]} = -A\, e^{-\Phi}\,. \tag{IV.59}$$

Notice that Ψ satisfies the same equation as Ψ_-, and therefore[8] $\Xi := \Psi^{-1}\Psi_-$ is annihilated by the operator $\widetilde{G}^{+}(\lambda)$. Thus,

$$\Psi_- = \Psi\Xi \quad\Rightarrow\quad \Psi_+^{-1} = \overline{\Psi}_- e^{\Phi} = \overline{\Xi}\,\overline{\Psi}e^{\Phi} \tag{IV.60}$$

for some matrix-valued chiral string field Ξ. Moreover, from (IV.57) and (IV.60) we see that

$$\Upsilon_{+-} = \Psi_+^{-1}\Psi_- = \overline{\Xi}\,\overline{\Psi}e^{\Phi}\Psi\Xi = \overline{\Xi}\,\Xi\,. \tag{IV.61}$$

This establishes a connection with the discussion in section IV.4.

Dressing. The dressing method is a recursive procedure generating a new extended solution from an old one. A solution e^{Φ} of the equation of motion (III.79) is obtained from the extended solution via (IV.56). Namely, let us suppose that we have constructed an extended seed solution Ψ_0 by solving the linear equation (IV.18) for a given (seed) solution e^{Φ_0} of eq. (III.79). Then one can look for a new extended solution in the form

$$\Psi_1 = \chi_1\Psi_0 \qquad \text{with} \qquad \chi_1 = \mathcal{I} + \frac{\lambda\alpha_1}{\lambda - \mu_1}\,P_1\,, \tag{IV.62}$$

where α_1 and μ_1 are complex constants and the matrix-valued string field $P_1[X, \psi]$ is independent of λ. The transformation $\Psi_0 \mapsto \Psi_1$ is called *dressing*. Below, we will show explicitly how one

[7] By Liouville's theorem there are no globally defined holomorphic functions on $\mathbb{C}P^1$ besides constants.

[8] This is not to be confused with the sliver introduced in section III.5.1.

can determine Ψ_1 by exploiting the pole structure (in λ) of eq. (IV.59) together with (IV.62). An m-fold repetition of this procedure yields as the new extended solution

$$\Psi_m = \prod_{j=1}^{m} \Big(\mathcal{I} + \frac{\lambda \alpha_j}{\lambda - \mu_j} P_j \Big) \Psi_0 \,. \tag{IV.63}$$

We will choose below the vacuum seed solution $\Phi_0 = 0, \Psi_0 = \mathcal{I}$.

First-order pole ansatz for $\boldsymbol{\Psi}$. Choose the complex constants μ_j in (IV.63) such that they are mutually different. Then using a decomposition into partial fractions, one can rewrite the multiplicative ansatz (IV.63) in the additive form

$$\Psi_m = \Big(\mathcal{I} + \lambda \sum_{q=1}^{m} \frac{R_q}{\lambda - \mu_q} \Big) \Psi_0 \,, \tag{IV.64}$$

where the matrix-valued string fields $R_q[X, \psi]$ are some combinations of (products of) P_j. As already mentioned we now choose the vacuum $\Phi_0 = 0, \Psi_0 = \mathcal{I}$ and consider R_q of the form [193, 50, 183, 108]

$$R_q = - \sum_{p=1}^{m} \mu_q T_p \Gamma^{pq} \overline{T}_q \,, \tag{IV.65}$$

where $T_p[X, \psi]$ are taken to be the $n \times r$ matrices for some $r \geq 1$ and $\Gamma^{pq}[X, \psi]$ are $r \times r$ matrices for which an explicit expression is going to be determined below.

From (IV.64) and (IV.65) it follows that

$$\Psi \;=\; \mathcal{I} - \lambda \sum_{p,q=1}^{m} \mu_q \frac{T_p \, \Gamma^{pq} \, \overline{T}_q}{\lambda - \mu_q} \,, \tag{IV.66}$$

$$\overline{\Psi} \;=\; \mathcal{I} + \sum_{k,\ell=1}^{m} \frac{\varepsilon \, T_\ell \overline{\Gamma}^{k\ell} \overline{T}_k}{\lambda - \varepsilon / \bar{\mu}_\ell} \,. \tag{IV.67}$$

Here we omitted the index m in Ψ_m and $\overline{\Psi}_m$. In accordance with (IV.58) we have to choose Γ^{pq} in such a form that $\Psi\overline{\Psi}$ will be independent of λ. A splitting into partial fractions yields

$$\begin{aligned}
\Psi\overline{\Psi} &= \mathcal{I} + \sum_{k,\ell} \frac{\varepsilon T_\ell \overline{\Gamma}^{k\ell} \overline{T}_k}{\lambda - \varepsilon/\bar{\mu}_\ell} - \lambda \sum_{p,q} \mu_q \frac{T_p \Gamma^{pq} \overline{T}_q}{\lambda - \mu_q} - \sum_{p,q,k,\ell} (\lambda - \mu_q + \mu_q) \varepsilon \mu_q \frac{T_p \Gamma^{pq} \overline{T}_q \, T_\ell \overline{\Gamma}^{k\ell} \overline{T}_k}{(\lambda - \mu_q)(\lambda - \varepsilon/\bar{\mu}_\ell)} \\
&= \mathcal{I} + \sum_{k,\ell} \frac{\varepsilon T_\ell \overline{\Gamma}^{k\ell} \overline{T}_k}{\lambda - \varepsilon/\bar{\mu}_\ell} - \lambda \sum_{p,q} \mu_q \frac{T_p \Gamma^{pq} \overline{T}_q}{\lambda - \mu_q} - \sum_{p,q,k,\ell} \frac{\varepsilon \mu_q T_p \Gamma^{pq} \overline{T}_q \, T_\ell \overline{\Gamma}^{k\ell} \overline{T}_k}{\lambda - \varepsilon/\bar{\mu}_\ell} \\
&\quad - \sum_{p,q,k,\ell} \frac{\varepsilon \mu_q^2 \bar{\mu}_\ell}{\mu_q \bar{\mu}_\ell - \varepsilon} \left(\frac{1}{\lambda - \mu_q} - \frac{1}{\lambda - \varepsilon/\bar{\mu}_\ell} \right) T_p \Gamma^{pq} \overline{T}_q \, T_\ell \overline{\Gamma}^{k\ell} \overline{T}_k \,.
\end{aligned} \tag{IV.68}$$

This motivates us to define

$$\widetilde{\Gamma}_{q\ell} := -\varepsilon \mu_q \frac{\overline{T}_q T_\ell}{\mu_q \bar{\mu}_\ell - \varepsilon} \,, \tag{IV.69}$$

and, as the matrix $\Gamma = (\Gamma^{pq})$ has not yet been specified, to take it to be inverse to $\widetilde{\Gamma} = (\widetilde{\Gamma}_{q\ell})$,

$$\sum_{q=1}^{m} \Gamma^{pq}\widetilde{\Gamma}_{q\ell} = \delta^p{}_\ell \mathcal{I}\,. \tag{IV.70}$$

Upon insertion of eqs. (IV.69) and (IV.70) into (IV.68) nearly all terms cancel each other and we are left with

$$\Psi\overline{\Psi} = \mathcal{I} - \sum_{p,q} \mu_q T_p \Gamma^{pq}\overline{T}_q = e^{-\Phi}\,. \tag{IV.71}$$

This expression is independent of λ and, therefore, we can identify it with $e^{-\Phi}$ as in (IV.58). We see that for the above choice of the Γ-matrices the reality condition is satisfied, and the solution e^{Φ} of eq. (III.79) is parametrized by the matrix-valued string fields T_k, $k = 1, \ldots, m$. Note that (IV.71) coincides with $\Psi\overline{\Psi}|_{\lambda=\infty} = \Psi|_{\lambda=\infty}$.

Pole structure. We are now going to exploit eq. (IV.59) in combination with the ansatz (IV.66). First, it is easy to show that

$$\Psi|_{\lambda=\frac{\varepsilon}{\bar\mu_k}} T_k = \left(\mathcal{I} + \sum_{p,q} \varepsilon\mu_q \frac{T_p\Gamma^{pq}\overline{T}_q}{\mu_q\bar\mu_k - \varepsilon}\right) T_k = T_k - \sum_{p,q} T_p\Gamma^{pq}\widetilde{\Gamma}_{qk} = 0 \tag{IV.72}$$

and

$$\overline{T_k}\,\overline{\Psi}|_{\lambda=\mu_k} = \overline{T}_k \left(\mathcal{I} + \sum_{p,q} \varepsilon \frac{\bar\mu_q T_q \overline{\Gamma^{pq}}\,\overline{T}_p}{\mu_k\bar\mu_q - \varepsilon}\right) = \overline{T}_k - \sum_{p,q} \overline{\widetilde{\Gamma}}_{qk}\overline{\Gamma}^{pq}\,\overline{T}_p = 0\,. \tag{IV.73}$$

Second, note that the right hand side of eq. (IV.59) is independent of λ and therefore the poles on the left hand side have to be removable. Putting to zero the corresponding residue at $\lambda = \frac{\varepsilon}{\bar\mu_k}$ we obtain, due to (IV.72),

$$\Psi|_{\lambda=\frac{\varepsilon}{\bar\mu_k}} \left\{(\varepsilon\bar\mu_k\widetilde{G}^+ + G^+)T_k\right\} \sum_\ell \overline{\Gamma}^{\ell k}\overline{T}_\ell = 0\,. \tag{IV.74}$$

Obviously, a sufficient condition for a solution is

$$\left(\widetilde{G}^+ + \frac{\varepsilon}{\bar\mu_k}\,G^+\right) T_k = T_k \mathcal{Z}_k\,, \tag{IV.75}$$

with an operator $\mathcal{Z}_k$ having the same Grassmann content as the operator $\widetilde{G}^+\big(\frac{\varepsilon}{\bar\mu_k}\big)$. Nilpotency of $\widetilde{G}^+\big(\frac{\varepsilon}{\bar\mu_k}\big)$ implies that[9]

$$\widetilde{G}^+\big(\tfrac{\varepsilon}{\bar\mu_k}\big)\mathcal{Z}_k = \mathcal{Z}_k\mathcal{Z}_k\,. \tag{IV.76}$$

In the same way, the residue at $\lambda = \mu_k$ should vanish,

$$\begin{aligned}\left(\sum_p \mu_k T_p\Gamma^{pk}\right)\left\{\left(\frac{1}{\mu_k}\widetilde{G}^+ + G^+\right)\overline{T}_k\right\}\overline{\Psi}|_{\lambda=\mu_k} = 0\\ \Rightarrow\ (\widetilde{G}^+ + \mu_k G^+)\overline{T}_k = \mathcal{Z}'_k\overline{T}_k\,,\end{aligned} \tag{IV.77}$$

[9] This is suggestive of another "dressing-like" procedure: Start from some trivial $\mathcal{Z}_k$ (e. g., constant or nilpotent) and determine T_k from eq. (IV.75). Then define $\mathcal{Z}'_k := T_k$ and plug this into eq. (IV.75) to get a new T'_k and so on. (I am grateful to Prof. Lechtenfeld for this suggestion.)

with another Grassmann-odd operator $\mathcal{Z}'_k$. Comparing eqs. (IV.75) and (IV.77), we learn that

$$\overline{\mathcal{Z}}_k \overline{T}_k = \varepsilon \mathcal{Z}'_k \overline{T}_k \quad \Rightarrow \quad \mathcal{Z}'_k = \varepsilon \overline{\mathcal{Z}}_k \,. \tag{IV.78}$$

In other words, eqs. (IV.77) are not independent but follow from eq. (IV.75) by conjugation. For every collection $\{T_k, k = 1, \ldots, m\}$ of solutions to eqs. (IV.75) we can determine a solution to eq. (III.79) from eqs. (IV.69)–(IV.71).

Projectors. Now let us consider the simplest case $m = 1$. Then, eq. (IV.66) simplifies to

$$\Psi = \mathcal{I} + \frac{\lambda\varepsilon(|\mu|^2 - \varepsilon)}{\lambda - \mu}\, P \,, \tag{IV.79}$$

where $P := T(\overline{T}T)^{-1}\overline{T}$ is a hermitian projector, $P^2 = P = \overline{P}$, parametrized by an $n \times r$ matrix T. In the abelian ($n = 1$) case r is the rank of the projector P in the Hilbert space $\mathcal{H}$ of string field theory. In the nonabelian ($n > 1$) case $r \leq n$ can be identified with the rank of the projector in the $u(n)$ factor of the $u(n) \otimes \mathcal{H}$ Hilbert space.

From the extended solution Ψ we obtain the solution

$$e^{-\Phi} = \Psi|_{\lambda=\infty} = \mathcal{I} - (1 - \varepsilon|\mu|^2)P \tag{IV.80}$$

of the equation of motion (III.79). Thus, the simplest solutions are parametrized by projectors in the string field theory Hilbert space.

To conclude this section, we summarize the main idea of the dressing approach as follows: One has to extend the string field theory Hilbert space $u(n) \otimes \mathcal{H}$ to $u(n) \otimes \mathcal{H} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, there solve the equations on the extended string field $\Phi[X, \psi, \lambda]$ such that $\Psi^{-1}[X, \psi, \lambda] = e^{\Phi[X,\psi,\lambda]}$, and then project back onto $u(n) \otimes \mathcal{H}$. In this way, one obtains a solution $e^{\Phi} = \Psi^{-1}(\lambda = \infty)$ of the initial equation of motion (III.79), where the extended solution Ψ is parametrized by $T_k[X, \psi, \lambda = \varepsilon\bar{\mu}_k^{-1}]$ with $k = 1, \ldots, m$.

IV.6 Solutions of the linear equations

$\boldsymbol{\widetilde{G}^+(\lambda)}$**-exact solutions.** In the previous section we have shown that in the dressing approach solving the nonlinear string field equation (III.79) reduces to solving the linear equations (IV.75). Solutions T_k, $k = 1, \ldots, m$, of these equations parametrize solutions e^{Φ} of eq. (III.79) (cf. (IV.71)). Obviously, for obtaining some examples of solutions it is sufficient to find solutions for $\mathcal{Z}_k = 0$,

$$\widetilde{G}^+(\varepsilon\bar{\mu}_k^{-1})T_k \equiv \left(\widetilde{G}^+ + \frac{\varepsilon}{\bar{\mu}_k}\, G^+\right) T_k = 0 \,. \tag{IV.81}$$

Here, we present two classes of solutions to these equations.

Recall that $(\widetilde{G}^+(\lambda))^2 = 0$ and, therefore,

$$T_k = \widetilde{G}^+(\varepsilon\bar{\mu}_k^{-1})W_k \tag{IV.82}$$

is a solution of eq. (IV.81) for any string field $W_k \in \mathrm{Mat}(n \times r, \mathbb{C}) \otimes \mathcal{H}$. These solutions are in general nontrivial because they are not annihilated by G^+ and $\widetilde{G}^+$ separately.

This discussion is valid for both N=1 strings ($\varepsilon = -1$) and N=2 strings ($\varepsilon = 1$). Substituting (IV.82) into (IV.71), we get explicit solutions e^{Φ}. In the N=1 case, other obvious solutions are all BRST-closed vertex operators T_k in the small Hilbert space of [52] as they satisfy $[Q, T_k] = 0$ and $[\eta_0, T_k] = 0$ separately.

For the case of N=2 strings we will discuss two classes of explicit solutions of (IV.75) (for both $\mathcal{Z}_k = 0$ and $\mathcal{Z}_k \neq 0$) which in general do not have the form (IV.82).

N=2 string solutions for $\boldsymbol{\mathcal{Z}_k = 0}$. The realization of G^+ and $\widetilde{G}^+$ in terms of the constituents of an N=2 matter multiplet is presented in eqs. (B.35) and (C.3). Using this realization, one can factorize the "G-matrix" in (IV.4) according to

$$\begin{pmatrix} G^+ & \widetilde{G}^+ \\ \widetilde{G}^- & G^- \end{pmatrix} = \begin{pmatrix} \psi^{+1} & -\psi^{+0} \\ -\psi^{-\bar{0}} & \psi^{-\bar{1}} \end{pmatrix} \begin{pmatrix} \partial \bar{Z}^{\bar{1}} & \partial Z^0 \\ \partial \bar{Z}^{\bar{0}} & \partial Z^1 \end{pmatrix} . \tag{IV.83}$$

As in section IV.2, this matrix transforms under current $SU(1,1)$-rotations acting from the left (note that the world-sheet fermions are charged under the current group) and under the additional $SU(1,1)$-rotations as in (IV.9) acting from the right. The latter transform (IV.83) to

$$\begin{pmatrix} G^+(u) & \widetilde{G}^+(u) \\ \widetilde{G}^-(u) & G^-(u) \end{pmatrix} = \begin{pmatrix} \psi^{+1} & -\psi^{+0} \\ -\psi^{-\bar{0}} & \psi^{-\bar{1}} \end{pmatrix} \begin{pmatrix} \partial \bar{Z}^{\bar{1}} & \partial Z^0 \\ \partial \bar{Z}^{\bar{0}} & \partial Z^1 \end{pmatrix} \begin{pmatrix} u_1 & \bar{u}_2 \\ u_2 & \bar{u}_1 \end{pmatrix} . \tag{IV.84}$$

By right-multiplication with $\left(\begin{smallmatrix} u_1^{-1} & 0 \\ 0 & \bar{u}_1^{-1} \end{smallmatrix}\right)$ as in (IV.11) we can express everything in terms of λ,

$$\begin{pmatrix} G^+(\lambda) & \widetilde{G}^+(\lambda) \\ \widetilde{G}^-(\lambda) & G^-(\lambda) \end{pmatrix} = \begin{pmatrix} \psi^{+1} & -\psi^{+0} \\ -\psi^{-\bar{0}} & \psi^{-\bar{1}} \end{pmatrix} \begin{pmatrix} \partial \bar{Z}^{\bar{1}}(\lambda) & \partial Z^0(\lambda) \\ \partial \bar{Z}^{\bar{0}}(\lambda) & \partial Z^1(\lambda) \end{pmatrix} , \tag{IV.85}$$

where the coordinates

$$Z^0(z, \bar{z}, \lambda) := Z^0(z, \bar{z}) + \lambda \bar{Z}^{\bar{1}}(z, \bar{z}) \quad \text{and} \quad Z^1(z, \bar{z}, \lambda) := Z^1(z, \bar{z}) + \lambda \bar{Z}^{\bar{0}}(z, \bar{z}) \tag{IV.86}$$

define a new complex structure on the target space $\mathbb{C}^{1,1}$ [28, 106]. From eq. (B.34a) we derive that $Z^a(z, \bar{z}, \lambda)$ are null coordinates:

$$Z^a(z, \bar{z}, \lambda) Z^b(w, \overline{w}, \lambda) \sim 0 . \tag{IV.87}$$

The derivation of the string field algebra in (IV.75) can be written entirely in terms of $Z^a(z, \bar{z}, \lambda)$,

$$\widetilde{G}^+(z, \lambda) = \widetilde{G}^+(z) + \lambda G^+(z) = -\varepsilon_{ab} \psi^{+a}(z) \, \partial Z^b(z, \lambda) , \tag{IV.88}$$

and from (IV.87) it follows that

$$\oint \frac{dw}{2\pi i} \widetilde{G}^+(w, \lambda) Z^c(z, \bar{z}, \lambda) = 0 \tag{IV.89}$$

with the integration contour running around z. This equation implies that every analytic functional T_k of the new spacetime coordinates $Z^0(z,\bar{z},\bar{\mu}_k^{-1})$ and $Z^1(z,\bar{z},\bar{\mu}_k^{-1})$ solves (IV.81). Indeed, it can be easily checked that for any integer p,q, we have

$$\oint \frac{dw}{2\pi i}\widetilde{G}^+\left(w,\frac{1}{\bar{\mu}_k}\right) :(Z^0)^p(Z^1)^q(z,\bar{z},\bar{\mu}_k^{-1}): = 0\,. \tag{IV.90}$$

The functional T_k may also depend on arbitrary derivatives $\partial^\ell Z^a(z,\bar{\mu}_k^{-1})$ (note that, for $\ell=1$, $\partial Z^a(z,\bar{\mu}_k^{-1})$ is $\widetilde{G}^+(\bar{\mu}_k^{-1})$-exact). Due to (B.34b), it may furthermore depend on $\psi^{+a}(z)$ or its derivatives. Given some analytic functionals $T_k[Z^a(z,\bar{z},\bar{\mu}_k^{-1}),\partial^\ell Z^a(z,\bar{\mu}_k^{-1}),\psi^{+a}(z),\partial^p\psi^{+a}(z)]$ with values in $\mathrm{Mat}(n\times r,\mathbb{C})$ for $k=1,\dots,m$, we can determine a solution of (III.79) with the help of eq. (IV.71). Note that we do not claim to have found all solutions.

N=2 string solutions for $\boldsymbol{\mathcal{Z}_k\neq 0}$. We restrict ourselves to the abelian case $n=1$. In addition to the coordinates $Z^a(z,\bar{z},\bar{\mu}_k^{-1})$ from above, we introduce vertex operators

$$Y^0(z,\bar{z},\bar{\mu}_k^{-1}) := \tfrac{1}{2}\big(\bar{Z}^{\bar{1}}(z,\bar{z}) - \tfrac{1}{\bar{\mu}_k}Z^0(z,\bar{z})\big) \quad\text{and}\quad Y^1(z,\bar{z},\bar{\mu}_k^{-1}) := \tfrac{1}{2}\big(\bar{Z}^{\bar{0}}(z,\bar{z}) - \tfrac{1}{\bar{\mu}_k}Z^1(z,\bar{z})\big). \tag{IV.91}$$

They satisfy the following OPE with $Z^a(z,\bar{z},\bar{\mu}_k^{-1})$:

$$Z^a(z,\bar{z},\bar{\mu}_k^{-1})\,Y^b(w,\overline{w},\bar{\mu}_k^{-1}) \sim -2\varepsilon^{ab}\ln|z-w|^2\,. \tag{IV.92}$$

From this, we immediately derive

$$\oint \frac{dw}{2\pi i}\widetilde{G}^+\left(w,\frac{1}{\bar{\mu}_k}\right) :e^{\alpha_a^k Y^a}(z,\bar{z},\bar{\mu}_k^{-1}): = -2:e^{\alpha_a^k Y^a}(z,\bar{z},\bar{\mu}_k^{-1}):\,\alpha_b^k\psi^{+b}(z)\,, \tag{IV.93}$$

where α_a^k are complex constants. We see that for $k=1,\dots,m$,

$$T_k = :e^{\alpha_a^k Y^a}(z,\bar{z},\bar{\mu}_k^{-1}): \tag{IV.94}$$

satisfy eq. (IV.75) and therefore produce a solution of (III.79) via (IV.71).

Chapter V

Vacuum superstring field theory

V.1 Introduction

In the last chapter, it was shown that Berkovits' string field theory is integrable in the sense that its equation of motion derives from a system of linear equations. In this chapter, we will see that this idea carries over to cubic superstring field theory and the vacuum versions of both superstring field theories. This chapter is based largely upon [III, IV].

In analogy with gauge field theory, we write down a linear system for cubic as well as nonpolynomial open superstring field theory (in the NS sector) by introducing an auxiliary string field $\Psi(\lambda)$ depending on a "spectral" parameter $\lambda \in \mathbb{C}P^1$. A single-pole ansatz for $\Psi(\lambda)$ leads to a hermitian projector, whose building block is merely subject to a linear equation which can be solved in generality. From it all string fields can be reconstructed. Employing dressing transformations analogous to those in noncommutative field theories [108], we shift the background to the tachyon vacuum and propose a linear equation which governs classical vacuum superstring field theory. As a simple example, the supersliver [126, 6] is based on a trivial solution to this equation. Finally, we propose a strategy to reconstruct classical superstring fields from their building blocks in more detail by taking advantage of the Moyal formulation for superstring field theory.

V.2 Zero-curvature and linear equations for string fields

In cubic open bosonic string field theory [186], the equation of motion for the string field A has a zero-curvature form,

$$F(A) = QA + A^2 = (Q + A)^2 = 0\,, \tag{V.1}$$

where Q denotes the BRST operator (a nilpotent derivation) and Witten's star product is implicit in all string field products. For any string field A one may look for solutions of the linear equation

$$(Q + A)\,\Psi = 0 \tag{V.2}$$

on an auxiliary string field Ψ possibly carrying some internal indices. Equation (V.1) is the compatibility condition of the linear equation (V.2). If we let Ψ take values in the Chan-Paton group, then from (V.2) one may obtain solutions of (V.1) via $A = \Psi Q \Psi^{-1}$ which are, however, pure gauge configurations. The cohomology of Q captures all other solutions.

This situation may change when a parametric dependence is introduced: Let $(Q, A, \Psi) \to (Q(\lambda), A(\lambda), \Psi(\lambda))$ with $\lambda \in \mathbb{C}P^1$. We demand $Q(\lambda)$ and $A(\lambda)$ to be linear in λ,[1]

$$A(\lambda) = a + \lambda A \qquad \text{and} \qquad Q(\lambda) = \eta_0 + \lambda Q \qquad \text{with} \qquad \eta_0^2 = Q^2 = \eta_0\, Q + Q\, \eta_0 = 0\,. \quad \text{(V.3)}$$

In other words, we extend the string configuration space, thereby adding a second string field a and a second BRST-like operator η_0. This case arises for a one-parameter family of N=2 superconformal algebras embedded into a small N=4 algebra and their string field realizations [28, 29]. The extended zero-curvature condition

$$F\big(A(\lambda)\big) = \big(Q(\lambda) + A(\lambda)\big)^2 = \big(\eta_0 a + a^2\big) + \lambda\big(\eta_0 A + Qa + \{A, a\}\big) + \lambda^2\big(QA + A^2\big) = 0 \quad \text{(V.4)}$$

is the compatibility condition of the associated linear equation

$$\big(Q(\lambda) + A(\lambda)\big)\, \Psi(\lambda) = 0\,. \qquad \text{(V.5)}$$

If $\Psi(\lambda)$ is group-valued, it follows that $a + \lambda A = \Psi(\lambda)\big(\eta_0 + \lambda Q\big)\Psi(\lambda)^{-1}$. As was shown in [107] and in the last chapter, this equation yields nontrivial solutions to the equations of motion for a and A.

Exploiting the gauge freedom in (V.4) allows one to gauge away a. Then, the ensuing equations,

$$\eta_0\, A = 0 \qquad \text{and} \qquad QA + A^2 = 0 \qquad \text{(V.6)}$$

are the (NS-sector) equations of motion in Witten's cubic open superstring field theory in the zero picture [150, 9]: Bosonizing the fermionic reparametrization ghosts as in eq. (B.18), $\gamma = \eta e^{\phi}$ and $\beta = e^{-\phi}\partial\xi$, we take η_0 above to be the zero mode of η, which indeed is nilpotent and anticommutes with Q. Then, the first equation in (V.6) simply denies any ξ_0 content in A (originally defined in the large Hilbert space), and the second one is the field equation in the small Hilbert space.[2] Of course, all fields are now NS-sector open superstring fields.

The system (V.6) may be reduced further.[3] Since both η_0 and Q have trivial cohomology in the large Hilbert space $\mathcal{H}$ (cf. appendix D), we may either solve the first equation or alternatively the second one:

$$A = \eta_0\, \Upsilon \qquad \Longrightarrow \qquad Q\, \eta_0\, \Upsilon + (\eta_0\, \Upsilon)^2 = 0\,, \qquad \text{(V.7)}$$

[1]Formally $A(\lambda)$ is a section of the bundle $\mathcal{O}(1)$ over $\mathbb{C}P^1$ with values in the string field Hilbert space $\mathcal{H}$, and $Q(\lambda)$ can be considered as an End($\mathcal{H}$)-valued section of this bundle.

[2]Strictly speaking, it is even better than the field equation for the modified cubic superstring field theory which inludes an unwanted Y_{-2}. – Note that Q and η_0 act via (anti)commutator on world-sheet fields, or, equivalently, via contour integration of the respective currents.

[3]For gauge theory the following goes back to Leznov and to Yang, respectively.

$$A = e^{-\Phi} Q e^{\Phi} \qquad \Longrightarrow \qquad \eta_0\big(e^{-\Phi} Q e^{\Phi}\big) = 0\,. \tag{V.8}$$

Despite appearance, A is *not* pure gauge (in the small Hilbert space) unless $\eta_0 e^{\Phi} = 0$ [140]. The second equation in (V.8) is precisely Berkovits' nonpolynomial equation of motion for the NS string field Φ.

All nonlinear superstring field equations, i. e. (V.6), (V.7) and (V.8), follow from the zero-curvature equation (V.4) (with a=0). Because both Q and η_0 have empty cohomology in the large Hilbert space we can in fact construct all solutions from the associated linear system

$$\big(Q + \tfrac{1}{\lambda}\eta_0 + A\big)\,\Psi(\lambda) = 0 \tag{V.9}$$

for the string fields A and $\Psi(\lambda)$.[4] This equation is the key to generating classical superstring configurations.

Of course, one always has the "trivial" λ-independent solution

$$\Psi = e^{-\Lambda} \quad \text{with} \quad \partial_\lambda \Lambda = 0 \qquad \Longrightarrow \qquad \eta_0\, e^{-\Lambda} = 0 = (Q + A)\, e^{-\Lambda} \tag{V.10}$$

which leads to a pure gauge configuration $A_0 = e^{-\Lambda} Q e^{\Lambda}$. Since $\mathbb{C}P^1$ is compact, the λ dependence of a nontrivial $\Psi(\lambda)$ cannot be holomorphic. Hence, we consider a meromorphic $\Psi(\lambda)$. If we require its regularity for $\lambda \to 0$ and for $\lambda \to \infty$, then one may choose such a gauge that the asymptotics will relate Ψ with the prepotentials Φ and Υ as follows:[5]

$$\Psi(\lambda) \;\longrightarrow\; \begin{cases} \mathcal{I} - \lambda\,\Upsilon + O(\lambda^2) & \text{for} \quad \lambda \to 0 \\ e^{-\Phi} + O(\frac{1}{\lambda}) & \text{for} \quad \lambda \to \infty \end{cases}. \tag{V.11}$$

Clearly, $e^{-\Phi}$, Υ, and $A = \Psi(\infty) Q \Psi(\infty)^{-1} = -\eta_0\, \partial_\lambda \Psi(0)$ are computable once an appropriate $\Psi(\lambda)$ has been found.

V.3 Single-pole ansatz and solutions

Let us employ the linear system (V.9) to solve Witten's or Berkovits' superstring field equations (in the NS sector). We briefly recall the procedure from chapter IV. In contrast to the nonparametric linear equation (V.2), the λ dependence of (V.9) imposes two constraints on $\Psi(\lambda)$. Firstly, isolating A in (V.9),

$$A = \Psi(\lambda)\big(Q + \tfrac{1}{\lambda}\eta_0\big)\Psi(\lambda)^{-1}\,, \tag{V.12}$$

we notice that the right-hand side must not depend on λ, hence all its poles must have vanishing residues. Although the above expression is pure gauge from the point of view of the λ-extended string configuration space, the string field A is nontrivial on the small Hilbert space. A second condition follows from the reality of the string fields. To formulate it one must extend star conjugation (III.30) to an antilinear mapping (which we denote by a bar) on the $\mathbb{C}P^1$ family of

[4] Formally $\Psi(\lambda)$ can be seen as an element of the space $\mathcal{H} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ carrying Chan-Paton labels.

[5] $\mathcal{I}$ denotes the identity string field.

N=2 superconformal algebras where it sends $Q \mapsto -\eta_0$ and $\eta_0 \mapsto Q$ but $\lambda \mapsto \bar{\lambda}$ (cf. eq. (IV.6)). It can be shown that the reality condition requires

$$e^{-\Phi} = \Psi(\lambda)\,\overline{\Psi(-1/\bar{\lambda})}\,. \tag{V.13}$$

Again, the poles on the right-hand side must be removable.

The simplest nontrivial solution displays a single pole in λ,[6]

$$\Psi(\lambda) = \mathcal{I} - \frac{\lambda(1+\mu\bar{\mu})}{\lambda-\mu}\,P\,, \tag{V.14}$$

whose location μ is a moduli parameter. P is a λ-independent string field to be determined. Let us investigate for our ansatz (V.14) the consequences of (V.13) and (V.12), in that order. The residues of the λ-poles of $\Psi\overline{\Psi}$ at $\lambda{=}\mu$ and $\lambda{=}-1/\bar{\mu}$ are proportional to $P(\mathcal{I}{-}\overline{P})$ and $(\mathcal{I}{-}P)\overline{P}$ (for $\mu \in \mathbb{C}P^1$ arbitrary and fixed), respectively, implying the projector property

$$P^2 = P = \overline{P}\,. \tag{V.15}$$

This is achieved by parametrizing

$$P = T\,(\overline{T}T)^{-1}\overline{T} \tag{V.16}$$

with some string field T. Similarly, the absence of poles in (V.12) yields

$$P\big(\mu Q + \eta_0\big)P = 0 \qquad \text{and} \qquad (\mathcal{I}-P)\big(Q-\bar{\mu}\eta_0\big)P = 0 \tag{V.17}$$

which are conjugate to one another. Since $PT = T$ by construction these equations are satisfied if

$$\big(Q-\bar{\mu}\eta_0\big)T = 0\,. \tag{V.18}$$

It is important to note that T is only subject to a *linear* equation and otherwise unconstrained. An obvious solution to (V.18) is

$$T = \big(Q-\bar{\mu}\eta_0\big)W \tag{V.19}$$

for an arbitrary string field W. Every choice of W or solution to (V.18) yields a classical Berkovits string field,

$$e^{-\Phi} = \mathcal{I} - (1{+}\mu\bar{\mu})\,P \quad , \qquad e^{\Phi} = \mathcal{I} - (1{+}\tfrac{1}{\mu\bar{\mu}})\,P \tag{V.20}$$

and, from $\lambda \to 0$,[7]

$$A = -\tfrac{1+\mu\bar{\mu}}{\mu}\,\eta_0\,P\,. \tag{V.21}$$

[6] For more general multi-pole ansätze see chapter IV.
[7] An alternative representation is $A = -\frac{1+\mu\bar{\mu}}{\mu\bar{\mu}}\,\big[QP - PZ(\mathcal{I}{-}P)\big]$ where Z is defined by $\big(Q-\bar{\mu}\eta_0\big)\overline{T} =: \overline{T}\,Z$.

V.4 Shifting the background

The form of the string field equations does not depend on the choice of background (termed “vacuum”). However, the explicit structure of the kinetic operator Q is determined by this choice. For the open-string vacuum,

$$A_0 = 0\,, \qquad P_0 = 0\,, \qquad \Psi_0 = \mathcal{I}\,, \tag{V.22}$$

one has the familiar BRST operator, $Q = Q_{\mathrm{B}}$. Now, one may think of the solution (Ψ, A) to (V.9) as the result of a dressing map [8]

$$\Psi_0 = \mathcal{I} \quad \longmapsto \quad \Psi = \Psi(\lambda)\,\Psi_0 \qquad \text{and} \qquad A_0 = 0 \quad \longmapsto \quad A = \mathrm{Ad}_\Psi A_0 \tag{V.23}$$

applied to a “seed solution” (Ψ_0, A_0). This process can be iterated. Since any two classical superstring configurations are related by such a dressing transformation, a shift of the background (Ψ_0, A_0) to a new reference configuration (Ψ_1, A_1) is exactly of the same nature. The difference is only semantical.

We study the result of shifting the background by a dressing transformation according to

$$\begin{tikzcd}
\text{background:} & \Psi_0{=}\mathcal{I} \arrow[r, "\Psi_1"] \arrow[d, "\Psi"] & \Psi_1 \arrow[d, "\Psi'"] & A_0{=}0 \arrow[r, "\mathrm{Ad}_{\Psi_1}"] \arrow[d, "\mathrm{Ad}_\Psi"] & A_1 \arrow[d, "\mathrm{Ad}_{\Psi'}"] \\
\text{deviation:} & \Psi \arrow[r] & \widetilde{\Psi} & A_0{+}A \arrow[r] & \widetilde{A}
\end{tikzcd} \tag{V.24}$$

where horizontal arrows represent the dressing map to the new background and vertical arrows turn on a deviation via dressing. Composing the two dressing transformations, the linear equation becomes ($\widetilde{\Psi} = \Psi'\Psi_1$ and $\widetilde{A} = A_1{+}A'$)

$$\begin{aligned}
0 \;&=\; \big(Q + \tfrac{1}{\lambda}\eta_0 + \widetilde{A}\big)\,\widetilde{\Psi} \\
&=\; \big[Q\Psi' + A_1\Psi' - \Psi' A_1 + \tfrac{1}{\lambda}\eta_0\Psi' + (\widetilde{A} - A_1)\Psi'\big]\,\Psi_1 \\
&=\; \big[\big(Q' + \tfrac{1}{\lambda}\eta_0 + A'\big)\,\Psi'\big]\,\Psi_1\,,
\end{aligned} \tag{V.25}$$

where we used $(Q + \frac{1}{\lambda}\eta_0)\Psi_1 = -A_1\Psi_1$ and defined $Q'\Psi' := Q\,\Psi' + A_1\Psi' - \Psi' A_1$. Hence, measuring our string fields from the new vacuum A_1, the relevant linear system,

$$\big(Q' + \tfrac{1}{\lambda}\eta_0 + A'\big)\,\Psi' = 0\,, \tag{V.26}$$

has the same form as the original (V.9), but Q has changed into Q'. For the nonlinear string field equations the corresponding form invariance has been observed in [126], a fact almost trivial in our framework.

[8] We abbreviate $\mathrm{Ad}_\Psi A_0 := \Psi(Q + \frac{1}{\lambda}\eta_0 + A_0)\Psi^{-1}$.

V.5 Tachyon vacuum superstring fields

Of special interest is the form of the theory around the tachyonic vacuum (for a general discussion, we refer to section III.4). Deviations from the tachyon vacuum are governed by (V.26), and all equations pertaining to the open-string vacuum simply carry over (with primes added). However, this is not the whole story. As discussed in section III.4.2, a new kinetic operator built entirely from ghosts can be "derived" via a redefinition of the new (tachyon vacuum) superstring fields,

$$A' \;\mapsto\; \mathcal{U}_{\varepsilon_r}\, A' \;=:\; \widehat{\mathcal{A}} \qquad \text{and} \qquad \Psi' \;\mapsto\; \mathcal{U}_{\varepsilon_r}\, \Psi' \;=:\; \widehat{\Psi}\,, \tag{V.27}$$

such that

$$Q' \;\mapsto\; \mathcal{U}_{\varepsilon_r}\, Q' \,\mathcal{U}_{\varepsilon_r}^{-1} \;=:\; \widehat{\mathcal{Q}} \tag{V.28}$$

yields the proper zero-cohomology "vacuum" kinetic operator. The field redefinition (V.27) is induced by a world-sheet reparametrization which is singular for $\varepsilon_r \to 0$. As η has conformal spin one, its zero mode η_0 is inert under the reparametrization. From now on, a hat indicates the presence of internal $2{\times}2$ Chan-Paton matrices distinguishing the GSO($\pm$) sectors, e. g.,

$$\widehat{\mathcal{A}} = \mathcal{A}_+ \otimes \sigma_3 + \mathcal{A}_- \otimes \mathrm{i}\sigma_2 \qquad \text{(odd ghost number)}\,, \tag{V.29}$$

$$\widehat{\Phi} = \Phi_+ \otimes \mathbf{1} + \Phi_- \otimes \sigma_1 \qquad \text{(even ghost number)}\,. \tag{V.30}$$

The kinetic operator of this vacuum superstring field theory (VSSFT) is conjectured to have the form III.105 [2, 141],

$$\widehat{\mathcal{Q}} = \mathcal{Q}_{\text{odd}} \otimes \sigma_3 + \mathcal{Q}_{\text{even}} \otimes \mathrm{i}\sigma_2\,, \tag{V.31}$$

where the subscript refers to the Grassmann parity and

$$\mathcal{Q}_{\text{odd}} \;=\; \tfrac{1}{2\mathrm{i}\varepsilon_r^2}\big[c(\mathrm{i}) - c(-\mathrm{i})\big] \;+\; \tfrac{1}{2}\oint\tfrac{dz}{2\pi\mathrm{i}} b\gamma^2(z)\,, \tag{V.32}$$

$$\mathcal{Q}_{\text{even}} \;=\; \tfrac{1}{2\mathrm{i}\varepsilon_r}\big[\gamma(\mathrm{i}) - \gamma(-\mathrm{i})\big]\,\Pi_+ \;+\; \tfrac{1}{2\mathrm{i}\varepsilon_r}\big[\gamma(\mathrm{i}) + \gamma(-\mathrm{i})\big]\,\Pi_- \tag{V.33}$$

with projectors Π_+ and Π_- onto the GSO(+) and GSO($-$) sectors, respectively. These terms prevail in the limit $\varepsilon_r \to 0$. Consequently, the linear system for VSSFT reads

$$\big(\widehat{\mathcal{Q}} + \tfrac{1}{\lambda}\widehat{\eta}_0 + \widehat{\mathcal{A}}\big)\,\widehat{\Psi}(\lambda) = 0\,, \tag{V.34}$$

where $\widehat{\eta}_0 = \eta_0 \otimes \sigma_3$ and $\widehat{\Psi} = \Psi_+ \otimes \mathbf{1} + \Psi_- \otimes \sigma_1$. Again, solutions to Berkovits' VSSFT or to the cubic VSSFT are obtained from (V.20) or (V.21) by firstly solving the linear equation (V.18) after replacing $Q \to \widehat{\mathcal{Q}}$ and secondly composing the projector via (V.16).

It is usually assumed that the D-brane solutions of VSSFT factorize into a ghost and a matter part, $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_g \otimes \mathcal{A}_m$. Then, the cubic VSSFT equation,

$$\widehat{\mathcal{Q}}\,\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^2 = 0 \qquad \text{with} \qquad \widehat{\eta}_0\,\widehat{\mathcal{A}} = 0\,, \tag{V.35}$$

splits into

$$\mathcal{A}_m^2 = \mathcal{A}_m \qquad \text{and} \qquad \widehat{\mathcal{Q}}\,\widehat{\mathcal{A}}_g + \widehat{\mathcal{A}}_g^2 = 0 \quad \text{with} \quad \widehat{\eta}_0\,\widehat{\mathcal{A}}_g = 0 \tag{V.36}$$

which turns $\mathcal{A}_m$ into a projector. Within our single-pole ansatz (V.14), the full $\widehat{\mathcal{A}}$ is already proportional to a projector $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}_g \otimes \mathcal{P}_m$, hence we must simply factorize (V.21) and have

$$\mathcal{A}_m = \mathcal{P}_m \quad \text{and} \quad \widehat{\mathcal{A}}_g = -\tfrac{1+\mu\bar{\mu}}{\mu}\,\widehat{\eta}_0\,\widehat{\mathcal{P}}_g \qquad \text{with} \qquad \mathcal{P}_m^2 = \mathcal{P}_m \quad \text{and} \quad \widehat{\mathcal{P}}_g^2 = \widehat{\mathcal{P}}_g\,. \tag{V.37}$$

Since $\widehat{\mathcal{Q}}$ is pure ghost the projector equation (V.17) factorizes, and (V.36) reduces to (V.37) plus

$$\big(\widehat{\mathcal{I}}_g - \widehat{\mathcal{P}}_g\big)\big(\widehat{\mathcal{Q}} - \bar{\mu}\widehat{\eta}_0\big)\widehat{\mathcal{P}}_g = 0\,, \tag{V.38}$$

which is solved by (we omit hats over $\mathcal{T}_g$)

$$\widehat{\mathcal{P}}_g = \mathcal{T}_g\,(\overline{\mathcal{T}}_g\mathcal{T}_g)^{-1}\overline{\mathcal{T}}_g \qquad \text{and} \qquad \big(\widehat{\mathcal{Q}} - \bar{\mu}\widehat{\eta}_0\big)\mathcal{T}_g = 0\,. \tag{V.39}$$

In the nonpolynomial formulation, a different ansatz, $\widehat{\Phi} = \widehat{\Phi}_g \otimes \Phi_m$ with $\Phi_m^2 = \Phi_m$, was advocated by Mariño and Schiappa [126]. It allows one to factorize Berkovits' equation (V.8) since one gets

$$e^{\pm\widehat{\Phi}} = \widehat{\mathcal{I}} - \big(\widehat{\mathcal{I}}_g - e^{\pm\widehat{\Phi}_g}\big) \otimes \Phi_m = \widehat{\mathcal{I}}_g \otimes \big(\mathcal{I}_m - \Phi_m\big) + e^{\pm\widehat{\Phi}_g} \otimes \Phi_m\,. \tag{V.40}$$

However, comparison with our solution (V.20),

$$e^{\pm\widehat{\Phi}} = \widehat{\mathcal{I}} - \big(1+(\mu\bar{\mu})^{\mp 1}\big)\,\widehat{\mathcal{P}}_g \otimes \mathcal{P}_m\,, \tag{V.41}$$

implies $\Phi_m = \mathcal{P}_m$ and $\widehat{\Phi}_g = -(\ln \mu\bar{\mu} + \mathrm{i}\pi)\,\widehat{\mathcal{P}}_g$ which is not compatible with the reality of Φ. Hence, our ansatz differs from the one of [126].

A more important distinction of our single-pole ansatz (V.14) from previous work is visible from (V.39): The cohomology problem for $\mathcal{T}_g$ is not based on $\widehat{\mathcal{Q}}$ but on $\widehat{\mathcal{Q}} - \bar{\mu}\widehat{\eta}_0$. Motivated by the freedom to choose a particular embedding of an N=2 superconformal algebra into a small N=4 superconformal algebra, such a coboundary operator (in the case of the open string vacuum) was proposed initially in [28, 29].

V.6 Ghost picture modification

As it stands, the linear equations (V.34) and (V.39) face a problem due to the ghost picture degeneracy of the NSR superstring. If our string fields are to carry a definite picture charge, they must reside in the zero-picture sector. Since η_0 lowers the picture charge by one unit, the above-mentioned coboundary operator is not homogeneous in picture. Therefore, from (V.34) or (V.39) one concludes that any string field, including $\widehat{\mathcal{A}}$ and $\mathcal{T}_g$, must in general be an infinite sum over all picture sectors. Obviously, any such field may be expanded into a formal series $\mathcal{T}_g = \sum_{n\in\mathbb{Z}}(-\bar{\mu})^{-n}\mathcal{T}_n$, where $\mathcal{T}_n$ carries picture number n. From (V.39) we then obtain the recursion relations $\widehat{\eta}_0\mathcal{T}_{n+1} = -\widehat{\mathcal{Q}}\mathcal{T}_n$. If we want to maintain Berkovits' original proposal that all string fields have picture number zero (e. g., $\mathcal{T}_{n\neq 0} = 0$) then only the trivial solutions of (V.34) with $\widehat{\mathcal{Q}}\mathcal{T}_0 = 0 = \widehat{\eta}_0\mathcal{T}_0$ emerge. Clearly, this implies $\widehat{\mathcal{Q}}\,\widehat{\mathcal{P}}_g = 0 = \widehat{\eta}_0\,\widehat{\mathcal{P}}_g$ and therefore $\widehat{\mathcal{A}} = 0$. The supersliver [126, 6] is gauge equivalent to this vacuum [140].

To obtain nontrivial solutions, we have two possibilites: Either we admit string fields inhomogeneous in picture, or we modify our linear equation. In the following we shall pursue the second option and restrict all string fields to the zero picture. The obvious cure then is to introduce a picture-raising multiplier, $\widehat{\eta}_0 \to \widehat{\mathcal{X}}(\mathrm{i})\widehat{\eta}_0$. This is admissible as long as $\widehat{\mathcal{X}}(\mathrm{i})$ commutes with both $\widehat{\eta}_0$ and $\widehat{\mathcal{Q}}$ and can be pulled through the star product.[9] We propose to take $\widehat{\mathcal{X}}(\mathrm{i}) := \{\widehat{\mathcal{Q}}, \widehat{\xi}(\mathrm{i})\}$, i. e. the picture-raising operator $\widehat{\mathcal{X}}$ of VSSFT evaluated at the string midpoint.[10] With this modification, our master linear equation becomes

$$\big(\widehat{\mathcal{Q}} + \tfrac{1}{\lambda}\widehat{\mathcal{X}}(\mathrm{i})\widehat{\eta}_0 + \widehat{\mathcal{A}}\big)\,\widehat{\Psi}(\lambda) = 0\,, \tag{V.42}$$

and all subsequent equations continue to hold after the obvious insertions of $\widehat{\mathcal{X}}(\mathrm{i})$. In particular, the ghost picture modification changes Berkovits' string field equation (V.8) to

$$\widehat{\mathcal{X}}(\mathrm{i})\,\widehat{\eta}_0\big(e^{-\widehat{\Phi}}\widehat{\mathcal{Q}}\,e^{\widehat{\Phi}}\big) = 0\,. \tag{V.43}$$

Any solution $\widehat{\mathcal{A}}$ in the form of (V.21) will, however, automatically be annihilated by $\widehat{\eta}_0$ so that it fulfills also Berkovits' equation of motion without $\widehat{\mathcal{X}}(\mathrm{i})$. Note that the action will remain unchanged; we use $\widehat{\mathcal{X}}(\mathrm{i})$ only as a means to solve our linear equations.

V.7 Towards explicit solutions

In order to extract the physical properties of classical VSSFT configurations, e. g., a D-brane interpretation or the role of our moduli parameter μ, it is desirable to construct solutions to the field equations in a more explicit manner. In keeping with the paradigm of matter-ghost factorization (see (V.36)) we are asked to solve eq. (V.39) with $\widehat{\mathcal{X}}(\mathrm{i})$ inserted. Because $\widehat{\mathcal{Q}} - \bar{\mu}\widehat{\mathcal{X}}(\mathrm{i})\widehat{\eta}_0$ can be "inverted" the general solution of VSSFT may be constructed from

$$\mathcal{T}_g = \big(\widehat{\mathcal{Q}} - \bar{\mu}\widehat{\mathcal{X}}(\mathrm{i})\widehat{\eta}_0\big)\,\widehat{\mathcal{W}}_g \tag{V.44}$$

for an arbitrary ghost string field $\widehat{\mathcal{W}}_g$.

For cubic VSSFT, the ε_r expansion of [141] can be reproduced in this framework.[11] In particular, since the leading term of $\widehat{\mathcal{Q}}{-}\bar{\mu}\widehat{\mathcal{X}}(\mathrm{i})\widehat{\eta}_0$ is identical to $\mathcal{Q}_{\mathrm{GRSZ}}{\otimes}\sigma_3$ [59], the lowest order in ε_r involves only the "natural" Grassmann assignments of all quantities.

Certain special solutions can be seen directly. When $\bar{\mu} = 1$, for instance, one may employ the picture-lowering operator $\widehat{Y}(\mathrm{i})$ to write $\mathcal{T}_g = \widehat{Y}(\mathrm{i})\,\widehat{\xi}(\mathrm{i})\,\widehat{\Xi}_g$ where $\widehat{\mathcal{Q}}\,\widehat{\Xi}_g = 0 = \widehat{\eta}_0\,\widehat{\Xi}_g$. At leading order in ε_r we may identify $\widehat{\Xi}_g$ with the ghost supersliver $\Xi_g{\otimes}\mathbf{1}$.

[9] Any midpoint insertion of conformal spin zero commutes with Witten's star product, as can be seen by its definition in terms of correlation functions of the disk.

[10] Due to the explicit form (V.31)–(V.33) of the kinetic operator, $\widehat{\mathcal{X}}(\mathrm{i})$ consists of Grassmann-even and -odd parts. The Grassmann-even part simply reads $-\partial(b\eta e^{2\phi})(\mathrm{i}) - b\partial\eta e^{2\phi}(\mathrm{i})$; the Grassmann-odd part has to be regularized due to the pole in the OPE of γ with ξ. Around the open-string vacuum, we may simply take $\mathcal{X}(\mathrm{i}) = \{Q, \xi(\mathrm{i})\}$. Note that $\widehat{\mathcal{X}}$ is not simply $\mathcal{X}$ with internal Chan-Paton labels.

[11] Our $\mathcal{Q}_{\mathrm{odd}}$ in (V.32) has a different ε_r-dependence (coinciding with that of [2]), but this is irrelevant here.

In any case, the main difficulty arises in the composition of $\widehat{\mathcal{P}}_g$ from a given $\mathcal{T}_g$ since Witten's star product is implicit in (V.39). In order to circumvent this technical obstacle we propose to make use of the (discrete [13] or continuous [45]) Moyal formulation of Witten's star product. In such a situation, the Moyal-Weyl map can be inferred to encode the non(anti)commutativity into Heisenberg or Clifford algebras, which are represented in auxiliary Fock spaces.[12] The advantage of this (auxiliary) operator formulation is its calculational ease. As an example, the basic projector for a single Moyal pair can be expressed as follows:

$$[a,a^\dagger]=1 \qquad \Longrightarrow \qquad |0\rangle\langle 0| =: e^{-a^\dagger a} : = 1 - a^\dagger(aa^\dagger)^{-1}a\,, \tag{V.45}$$

$$\{c,c^\dagger\}=1 \qquad \Longrightarrow \qquad |0\rangle\langle 0| =: e^{-c^\dagger c} : = 1 - c^\dagger c = 1 - c^\dagger(c\,c^\dagger)^{-1}c\,, \tag{V.46}$$

displaying a simple connection between the Gaussian form and the "fractional" form (cf. (V.16)) of a projector. Of course, for the application to VSSFT infinite tensor products of Heisenberg and Clifford algebras have to be considered [13, 45, 4, 49]. However, (V.45) and (V.46) suggest the possibility to take $\mathcal{T}_g$ not to be an operator but a state $|\mathcal{T}_g\rangle$ in the auxiliary Fock space. This would be in tune with the construction of noncommutative abelian solitons [108]. Finally, a direct comparison with results in the conventional string oscillator basis requires the reverse basis transformation to be applied to the string field configurations constructed in the Moyal basis.

In closing, we should like to stress that we have reduced the problem of solving the superstring field equations (in cubic or nonpolynomial form) to the easier task of considering a linear equation, whose solution T then serves as a building block for the string field configuration. Although demonstrated here with the simplest (single-pole) ansatz for the auxiliary string field $\Psi(\lambda)$, this strategy generalizes to the universal (multi-pole) case. Projectors emerge naturally only in the single-pole setup while T (rather a collection of such) continues to play the decisive role. The formalism is ideally suited to handle the superposition of solitonic objects in integrable systems. We therefore expect it to yield multi-brane configurations automatically.

[12] Such Fock spaces are not to be confused with the string oscillator Fock space.

Chapter VI

The fermionic vertex in N=2 string field theory

VI.1 Introduction

In the last two chapters solution-generating techniques known from integrable field theories were transferred to (open as well as vacuum) superstring field theories. This reduces the search for solutions to the string field equations to finding string fields T which satisfy a certain linear equation. Equivalently, one can try to solve a (generalization of a) linear differential equation on a hermitean projector string field P. The computation of P from T requires the evaluation of star products which is facilitated considerably by going to the Moyal basis in the star algebra, cf. appendix E.1. Alternatively, one can try to classify all projectors in the star algebra and test which of them lead to solutions of the string field equations. Again, this task is rather straightforward in the Moyal basis. Such a basis is known for all N=1 world-sheet fields.

String field theory for N=2 strings shares many characteristic properties with nonpolynomial superstring field theory around the open string vacuum, see chapter III. One of these properties is the mixing of world-sheet fields by the kinetic operators – this was a major problem for solving the string field equations in open string field theory and ultimately lead to the proposal of vacuum string field theory. In the latter, one gets rid of this complication by considering a singular world-sheet reparametrization which changes the BRST charge into a pure ghost operator. However, this cannot be the final answer; it should be possible to find solutions with the full kinetic operator, i. e., without a singular world-sheet reparametrization. Due to this problem, it seems worthwhile to take an apparent sidestep and to consider alternative approaches to this problem. In order to study the properties of string field theory solutions with such mixing properties, one may resort to string field theory for N=2 strings. The main advantage of this model is its simplicity; no ghosts are needed in addition to the matter fields. The equality of the structure and the simplicity of the field realization of the BRST-like operators turn this theory into a viable candidate for studying the intricacies which general solutions to the equation of motion for nonpolynomial string field theory bring about.

Hence, we are finally interested in the classification of the projectors in the 1-string Fock space of N=2 string field theory. Whereas the Moyal formulation for the bosonic sector carries over unchanged from bosonic string field theory, such a formulation is hitherto unknown for the world-sheet fermions in string field theory for self-dual strings. Namely, after twisting (see section III.3.3), these fermions are conformal fields of weights 0 and 1, respectively (and therefore different from world-sheet fermions in superstring (field) theories). It is the goal of this chapter to commence the study of the fermionic world-sheet sector. Since the $\psi^{\pm}$ system is a first order system with weights 0 and 1, the results of this chapter find their application in the computation of $\eta\xi$ vertices in superstring field theories[1], Berkovits' hybrid formalism for covariant N=1 superstring field theory (see, e. g., [28, 19, 12]), and the twisted bc system which is reviewed in section III.5.3.

In this chapter we construct the vertices needed to formulate N=2 string field theory in the twisted fermionic sector from scratch using the operator language. In particular we pay attention to the anomaly of the $U(1)$ current J contained in the N=2 superconformal algebra. Together with the overlap equations for the zero-modes this fixes the choice of vacuum for the vertices when one avoids midpoint insertions. For the identity vertex and the reflector the construction is accomplished using δ-function overlap conditions. The reflector is shown to implement BPZ conjugation as a graded antihomomorphism on the algebra of modes. To obtain the explicit form of the interaction vertex we have to invoke the Neumann function method. Supplemented with the above-mentioned conditions on the vacuum the vertex is fixed. The Neumann coefficients are expressed in terms of coefficients of generating functions. We find an intimate relationship between the coefficients for the fermions and those for bosons allowing us to employ known identities from the boson Neumann matrices. Resorting to this relationship we show that the contribution of the $(0,1)$ system to the reparametrization anomaly cancels this of two real bosons. This is in accordance with their contribution $c = -2$ to the central charge. Finally, we explicitly check that the overlap equations for the interaction vertex are fulfilled.

The chapter is organized as follows. In the next section we review the construction of bosonic matter vertices. In section VI.3 the identity vertex for the world-sheet fermions is constructed. As a starting point δ-function overlap conditions for arbitrary N-string vertices are considered. After deducing the form of the identity vertex from the corresponding overlap equations its symmetries are discussed in detail with particular emphasis on the anomaly of the $U(1)$ current. In section VI.4 the 2-string vertex is considered. Starting from general N-string overlap equations formulated in terms of $\mathbb{Z}_N$-Fourier-transformed fields we discuss constraints on the vacua arising from the zero-mode overlap conditions. Avoiding midpoint insertions these conditions fix the vacuum on which the N-string vertex is built. The reflector is constructed as an application of the tools described in this section. A detailed discussion of BPZ conjugation as implemented by the 2-vertex completes this section. We define BPZ conjugation and its inverse via the the bra-reflector and the ket-reflector, respectively, and their compatibility is shown. The interaction vertex is constructed in section VI.5. The Neumann coefficients of the

[1]Recall that the $\eta\xi$ system plays a distinguished role in these theories. Cubic superstring field theory features insertions of picture changing operators whilst nonpolynomial superstring field theory is formulated in large Hilbert space.

3-string vertex are expressed in terms of generating functions constructed out of the conformal transformations which map unit upper half-disks into the scattering geometry of the vertex. The so obtained Neumann matrices are shown to be closely related to the Neumann matrices for bosons. Therefore, identities for the bosonic Neumann matrices entail corresponding identities for the fermionic ones. In this way, the anomaly of midpoint preserving reparametrizations is shown to cancel the contribution of two real bosons, which is in agreement with conformal field theory arguments. Finally, the overlap conditions for the interaction vertex are checked explicitly. Parts of this calculation are relegated into appendix F where also formulas for the bosonic vertices and Neumann coefficients are collected. This chapter concludes with a short summary and a discussion of possible applications and further developments. The main reference for this chapter is [V].

VI.2 Bosonic matter vertices

In this section, we briefly review the computation of the matter vertices in the operator formulation. The graphical prescription for the product between string fields [186] was originally translated into an operator formalism based on a 3-vertex $\langle V_3|$ in [66, 67, 159] for bosonic strings and in [68] for superstrings. These results relied on the Neumann function method, which was developed before in the framework of light-cone string field theory in [122] and presented in the light of conformal field theory in [114, 115].

The results for real bosons in this section are valid for all string field theories based on the Witten star product, in particular also for the real and imaginary parts of the spacetime coordinate fields Z^a in N=2 string field theory. They will turn out to be useful in section VI.5. We will (very briefly) expound upon overlap equations and their solution via the Neumann function method. Additional details will be explained in the subsequent and more involved case of the fermionic vertex in N=2 string field theory. The discussion is restricted to one real boson X and can be easily generalized to D spacetime dimensions.

Integration vertex. The integration vertex glues the left and right halves of one string, resulting in a number. As explained in section III.2, it may be expressed as an element $\langle\mathcal{I}| \in \mathcal{H}^*$. This dual vector is determined by the overlap equation

$$\langle\mathcal{I}|X(\sigma) = \langle\mathcal{I}|X(\pi-\sigma) \tag{VI.1}$$

at the interaction time $\tau = 0$. Assuming the mode expansion

$$X(\sigma) = x_0 + \sqrt{2}\sum_{m=1}^{\infty} x_n \cos n\sigma \quad \text{with} \quad x_n = \mathrm{i}\sqrt{\tfrac{\alpha'}{n}}(a_n - a_n^\dagger)\,, \quad x_0 = \mathrm{i}\sqrt{\tfrac{\alpha'}{2}}(a_0 - a_0^\dagger)\,, \tag{VI.2}$$

a squeezed state ansatz for the integration vertex leads to the solution

$$\langle\mathcal{I}| = \langle 0|\exp\Big[-\tfrac{1}{2}\sum_{k,l\geq 0}^{\infty} a_k C_{kl}\, a_k\Big] \tag{VI.3}$$

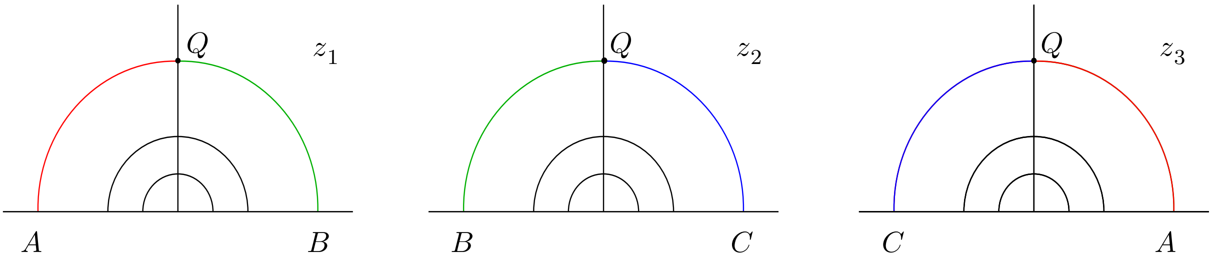

Figure VI.1. *The three local half-disks are parametrized by z_r coordinates. Local operators are inserted at the midpoints; at the interaction time $\tau_r = 0$ (i. e., $|z_r| = 1$), the three half-disks are glued together.*

with $C_{kl} = (-1)^k \delta_{kl}$. The bra-vacuum used in this expression is the oscillator vacuum which is annihilated by all $a_k^\dagger$.

Reflector state. The reflector state glues two strings with opposite orientation together; it is implemented by an element $\langle V_2| \in \mathcal{H}^* \otimes \mathcal{H}^*$. The overlap equations

$$\langle V_2|X^{(1)}(\sigma) = \langle V_2|X^{(2)}(\pi - \sigma) \tag{VI.4}$$

for $\sigma \in [0, \pi]$ are solved by

$$\langle V_2| = {}_1\langle 0| \otimes {}_2\langle 0| \exp\Big[-\tfrac{1}{2} \sum_{k,l\geq 0}^{\infty} a_k^{(1)} C_{kl}\, a_l^{(2)} \Big] \tag{VI.5}$$

with the same matrix C. This state is Grassmann-even and implements BPZ conjugation.

Interaction vertex. The overlap conditions for $\langle \mathcal{I}|$ and $\langle V_2|$ can be solved directly. This will no longer be possible for all higher vertices, although one can decouple the equations by making $_N$ Fourier transforms. Inserting an appropriate ansatz for the N-vertex into the overlap equations, the infinite-dimensional matrices appearing in this ansatz have to be inverted, which is in general very complicated. Therefore one resorts to the Neumann function method; it relies on the fact that the contraction of $\langle V_N|$ with N 1-string states $|\Phi\rangle_r$ can be interpreted as a correlation function of the corresponding vertex operators Φ_r in the N-string scattering geometry. For our purposes, we restrict to $N = 3$ and $\Phi_r = \mathrm{i}\partial X^{(r)}$ for $r = 1, 2$, $\Phi_3 =$ (i. e. the unit operator).

To this aim, consider the vertex operator Φ_r inserted at $z_r = 0$ on its local upper half-disk $\{|z_r| \leq 1,\ \mathrm{Im}\ z_r \geq 0\}$. The Witten-vertex can then be implemented as follows: The local upper half-disks are mapped via

$$h_r(z_r) = \frac{1 + \mathrm{i}z_r}{1 - \mathrm{i}z_r} \tag{VI.6}$$

to vertical half-disks in the $h_r(z_r)$-plane, lying in the region $\mathrm{Re}\ h_r(z_r) \geq 0$, $|h_r(z_r)| \leq 1$. The puncture $z = 0$ is mapped to $h(0) = 1$ at the curved side of the half-disk. This half-disk is now squeezed by the map $w_r = (h_r(z_r))^{2/3}$ into a wedge of 120°; the three wedges are subsequently rotated by phase factors and glued along their radial edges to a full unit disk. This exactly

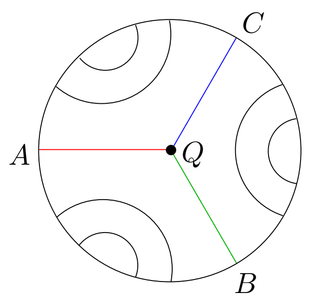

Figure VI.2. *The maps f_r glue the three local half-disks together into the scattering geometry, a three-punctured unit disk.*

matches the definition of the interaction vertex by Witten: The left half of the first string is glued to the right half of the second string[2]. Hence, the full map implementing the 3-vertex scattering geometry is

$$f_r(z) = -e^{\mathrm{i}\pi\frac{1-2r}{3}} f(z)\,, \qquad f(z) = \left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right)^{2/3}\,. \tag{VI.7}$$

If we now insert the mode expansion (cf. appendix B.1)

$$\mathrm{i}\partial X(z) = \sqrt{\frac{\alpha'}{2}}\sum_n \frac{\alpha_n}{z^{n+1}} \tag{VI.8}$$

and the ansatz

$$\langle V_3| = {}_1\langle 0| \otimes {}_2\langle 0| \otimes {}_3\langle 0| \exp\Big(-\frac{1}{2}\sum_{r,s}\sum_{n,m\geq 1} a_m^{(r)} V_{mn}^{rs} a_n^{(s)}\Big) \tag{VI.9}$$

into the expression

$$M = \langle V_3|\mathrm{i}\partial X^{(r)}(z)\,\mathrm{i}\partial X^{(s)}(w)|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3\,, \tag{VI.10}$$

we obtain

$$M = -\frac{\alpha'}{2}\sum_{m,n\geq 1} z^{m-1}w^{n-1}\sqrt{mn}V_{mn}^{rs}\,. \tag{VI.11}$$

Note that this expression holds barring complications from the zero-modes. Reinterpreting M as a correlation function on the disk,

$$M = \big\langle f_r \circ \big(\mathrm{i}\partial X(0)\big) f_s \circ \big(\mathrm{i}\partial X(0)\big)\big\rangle\,, \tag{VI.12}$$

this yields

$$-\frac{\alpha'}{2}\sum_{m,n} z^{m-1}w^{n-1}\sqrt{mn}V_{mn}^{rs} = \frac{\alpha'}{2}\frac{f_r'(z)f_s(w)}{(f_r(z)-f_s(w))^2}\,. \tag{VI.13}$$

The Neumann coefficients are therefore given by

$$V_{mn}^{rs} = -\frac{1}{\sqrt{mn}}\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{1}{z^n\,w^m}\frac{f_r'(w)f_s'(z)}{(f_s(z)-f_r(w))^2} \tag{VI.14}$$

[2]For the nomenclature, cf. section III.5.1.

in momentum basis. The contour integrals can be evaluated to give the values in appendix F.1.

Overlap equations. It is a technically demanding but feasible task to show that the Neumann coefficients given in appendix F.1 satisfy the overlap equations

$$X^{(r)}(\sigma) = X^{(r-1)}(\pi - \sigma) \qquad \text{for } 0 \leq \sigma \leq \frac{\pi}{2}\,, \quad r \text{ cyclic}\,, \tag{VI.15}$$

for the 3-string vertex. The latter are most conveniently formulated in terms of $\mathbb{Z}_3$-transformed string oscillators $A_k^{(a)}$ (cf. section (VI.4), not to be confused with the coefficients A_k defined in (F.3)):

$$A_k^{(a)} = \frac{1}{\sqrt{3}} \sum_{r=1}^{3} a_k^{(r)} e^{\frac{2\pi i a}{3}}\,. \tag{VI.16}$$

Using

$$\langle V_3| = \langle 0| \exp\Big[- \sum_{k,l\geq 0}^{\infty} \Big(\tfrac{1}{2} A_k^{(3)} C_{kl}\, A_l^{(3)} + A_k^{(1)} U_{kl}\, A_l^{(2)} \Big) \Big]\,, \tag{VI.17}$$

as an ansatz for the vertex, we can write the overlap equations in matrix form as

$$(\quad - Y)E(\quad + U) = 0\,, \quad (\quad + Y)E^{-1}(\quad - U) = 0\,. \tag{VI.18}$$

Here the matrix E has components $E_{mn} = \delta_{m,0}\delta_{n,0} + \sqrt{\frac{2}{n}}\delta_{mn}$, and the matrix Y is given by the Fourier components of the operator

$$Y(\sigma,\sigma') = (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}[\mathrm{i}\,\Theta(\tfrac{\pi}{2} - \sigma) - \mathrm{i}\,\Theta(\sigma - \tfrac{\pi}{2})])\delta(\sigma + \sigma' - \pi) =: -\tfrac{1}{2}C(\sigma,\sigma') + \tfrac{\sqrt{3}}{2}X(\sigma,\sigma')\,. \tag{VI.19}$$

Rewritten in terms of the original one-string oscillators the vertex takes the form[3]

$$\langle V_3| = \langle 0| \exp\Big[- \tfrac{1}{2} \sum_{r,s} \sum_{m,n\geq 0}^{\infty} a_n^{(r)} V'^{rs}_{nm}\, a_m^{(s)} \Big] \tag{VI.20}$$

with the matrices ($r = 1, 2, 3$)

$$V'^{r\,r}_{nm} = \tfrac{1}{3}(C + U + \bar{U})\,, \tag{VI.21a}$$
$$V'^{r\,r+1}_{nm} = \tfrac{1}{3}(C - \tfrac{1}{2}(U + \bar{U}) + \tfrac{\sqrt{3}}{2}(U - \bar{U}))\,, \tag{VI.21b}$$
$$V'^{r\,r-1}_{nm} = \tfrac{1}{3}(C - \tfrac{1}{2}(U + \bar{U}) - \tfrac{\sqrt{3}}{2}(U - \bar{U}))\,. \tag{VI.21c}$$

After transformation to momentum basis, these matrices can be identified with the Neumann coefficients.

Reparametrization anomaly. Reparametrizations generated by $K_n = L_n - (-1)^n L_{-n}$ leave the string midpoint invariant and are classical symmetries of string field theory. However, these reparametrizations are potentially anomalous. This usually puts a restriction on the critical dimension; the anomaly of the full system (e. g., ghost plus matter) has to vanish. In the

[3]We stick to the notation of [152] and denote the matrices in the oscillator basis with a prime while those in momentum basis are unprimed.

operator formulation the anomaly arises from operator orderings when two creation operators act on the vertex, i.e., from terms such as

$$-(-1)^m \tfrac{1}{2} \sum_{k=1}^{m-1} \sqrt{k(m-k)} a_k^\dagger \cdot a_{m-k}^\dagger \tag{VI.22}$$

contained in K_m. In [67] it was shown that a single boson contributes (see also [162])

$$\langle V_3|(K_{2n}^{(1)} + K_{2n}^{(2)} + K_{2n}^{(3)}) = -\tfrac{5}{18} n(-1)^n \langle V_3| \,. \tag{VI.23}$$

However, due to a nontrivial relation between the Neumann coefficients for the bosons and the fermions in the twisted theory we will be able show that in the full theory this anomaly is canceled in any even dimension. To this end let us derive the contribution of D bosons to the anomaly in terms of the Neumann coefficients. The application of $\sum_{t=1}^{3} K_m^{(t)}$ to the 3-vertex yields

$$\begin{aligned}
\sum_{t=1}^{3} \langle V_3| K_m^{(t)} &= -(-1)^m \frac{1}{2} \sum_{t=1}^{3} \sum_{k=1}^{m-1} \sqrt{k(m-k)} \langle V_3| a_k^{\dagger(t)} \cdot a_{m-k}^{\dagger(t)} + \ldots \\
&= (-1)^m \frac{1}{4} \sum_{t=1}^{3} \sum_{k=1}^{m-1} \sqrt{k(m-k)} \langle V_3| a_n^{(r)} (V_{nk}^{rt} + V_{kn}^{tr}) \cdot a_{m-k}^{\dagger(t)} + \ldots \\
&= (-1)^m \frac{D}{4} \sum_{t=1}^{3} \sum_{k=1}^{m-1} \sqrt{k(m-k)} \langle V_3| (V_{m-k,k}^{tt} + V_{k,m-k}^{tt}) \\
&= (-1)^m \frac{3\,D}{2} \sum_{k=1}^{m-1} \sqrt{k(m-k)}\, V_{k,m-k}^{tt} \langle V_3| \,.
\end{aligned} \tag{VI.24}$$

VI.3 Fermionic identity vertex

The identity vertex defines the integration in eq. (III.77) in the N=2 case; it is an element $|\mathcal{I}\rangle$ of the one-string Hilbert space corresponding to the identity string field $\mathcal{I}$. The identity vertex glues the left and right halves of a string together; therefore it can be defined via the corresponding overlap equations.

Overlap equations. In general, the overlap equations for an N-vertex can be determined from conformal field theory arguments [68]: On the world-sheet of the r-th string ($r \in \{1, \ldots, N\}$), a strip, we introduce coordinates $\xi_r = \tau_r + \mathrm{i}\sigma_r$. The strip can be mapped into an upper half-disk with coordinates $z_r = e^{\xi_r}$; the upper half-disks are then glued together in the scattering geometry in a such a way that

$$z_r z_{r-1} = -1 \qquad \text{for } |z_r| = 1\,,\ \mathrm{Re}(z_r) \geq 0\,, \quad \text{i.e.,} \quad 0 \leq \sigma_r \leq \frac{\pi}{2}\,,\ \ \tau_r = 0\,. \tag{VI.25}$$

This is achieved by the conformal map

$$f_r(z) = -e^{\mathrm{i}\pi \frac{1-2r}{N}} f(z)\,, \qquad f(z) = \left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right)^{2/N}\,, \tag{VI.26}$$

where the phases have been chosen so as to give a symmetric configuration when mapping back to the upper half-plane.

A primary field $\phi^{(r)}$ of conformal weight h in the boundary conformal field theory on the strip is glued according to

$$\begin{aligned}\phi^{(r)}(\sigma_r,\tau_r=0) &\equiv \phi^{(r)}(\xi_r) = z_r^h \phi^{(r)}(z_r)\\ &= \left(z_r\frac{\partial z_{r-1}}{\partial z_r}\right)^h \phi^{(r-1)}(z_{r-1}) = \left(\frac{z_r}{z_{r-1}}\frac{\partial z_{r-1}}{\partial z_r}\right)^h \phi^{(r-1)}(\sigma_{r-1},\tau_{r-1}=0)\\ &= (-1)^h\phi^{(r-1)}(\pi-\sigma_r,\tau_r=0) \qquad \text{for } 0\le\sigma_r\le\frac{\pi}{2}\,.\end{aligned} \tag{VI.27}$$

In the last two lines we have used (VI.25). This equality is required to hold when applied to the N-string vertex $\langle V_N|$. If we insert the open string mode expansion for $\tau=0$, $\phi^{(r)}(\sigma)=\phi_0^{(r)}+\sum_n(\phi_n^{(r)}+\phi_{-n}^{(r)})\cos n\sigma$, we obtain a condition on the modes. For $N\le 2$, the above condition extends to $0\le\sigma\le\pi$, so that one can take advantage of the orthogonality of the cosine to obtain the diagonal condition $\langle V_N|\big(\phi_n^{(r)}+\phi_{-n}^{(r)}+(-1)^{n+h}(\phi_{-n}^{(r-1)}+\phi_n^{(r-1)})\big)=0$. Instead, we will impose the stricter condition $\langle V_N|\big(\phi_n^{(r)}+(-1)^{n+h}\phi_{-n}^{(r-1)})\big)=0$. For $N>2$, the overlap equations in general mix all modes.

Construction of the identity vertex. For the $\psi^\pm$-system[4], we demand the stricter conditions

$$\langle\mathcal{I}|\big[\psi_n^+-(-1)^n\psi_{-n}^+\big]=0 \qquad\Longrightarrow\qquad \langle\mathcal{I}|\psi^+(\sigma)=\langle\mathcal{I}|\psi^+(\pi-\sigma)\,, \tag{VI.28a}$$
$$\langle\mathcal{I}|\big[\psi_n^-+(-1)^n\psi_{-n}^-\big]=0 \qquad\Longrightarrow\qquad \langle\mathcal{I}|\psi^-(\sigma)=-\langle\mathcal{I}|\psi^-(\pi-\sigma)\,, \tag{VI.28b}$$

from which the gluing conditions (VI.27) follow. The conditions on $\langle\mathcal{I}|$ are compatible since $\{\psi_n^+-(-1)^n\psi_{-n}^+,\psi_n^-+(-1)^n\psi_{-n}^-\}=0$. The obvious solution to eqs. (VI.28) reads

$$\begin{aligned}\langle\mathcal{I}| &= \langle\downarrow|\prod_{n=1}^{\infty}\frac{1}{2}\big[\psi_n^+-(-1)^n\psi_{-n}^+\big]\big[(-1)^n\psi_n^-+\psi_{-n}^-\big]\\ &= \langle\downarrow|\prod_{n=1}^{\infty}\big[1+\frac{1}{2}(-1)^n\psi_n^+\psi_n^-\big] = \langle\downarrow|\exp\Big[\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\psi_n^+\psi_n^-\Big]\,,\end{aligned} \tag{VI.29}$$

where the $SL(2,\mathbb{R})$-invariant vacuum $\langle\downarrow|$ is annihilated by ψ_0^-.

Symmetries of the vertex. Applying the gluing conditions (VI.27) to the complex spin 1 fields

$$\partial Z = -\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_k \alpha_k z^{-k-1} \qquad\text{and}\qquad \partial\bar{Z} = -\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_k \bar{\alpha}_k z^{-k-1}\,, \tag{VI.30}$$

we obtain

$$\langle\mathcal{I}|(\alpha_n+(-1)^n\alpha_{-n})=0\,, \qquad \langle\mathcal{I}|(\bar{\alpha}_n+(-1)^n\bar{\alpha}_{-n})=0\,. \tag{VI.31}$$

[4] Here and in the following, we sometimes omit spacetime labels on $\psi^\pm$ if the statement refers to any of the ψ^{+a}, $\psi^{-\bar{a}}$.

Together with (VI.28), this entails that the gluing conditions for the BRST-like spin 1 currents G^+ and $\widetilde{G}^+$,

$$\langle\mathcal{I}|(G_n^+ + (-1)^n G_{-n}^+) = 0\,, \qquad \langle\mathcal{I}|(\tilde{G}_n^+ + (-1)^n \tilde{G}_{-n}^+) = 0\,, \tag{VI.32}$$

are satisfied. In general, anomalies can only appear if the current contains pairs of conjugate oscillators. Thus, it is clear that the spin 2 currents J^{--}, G^- and $\widetilde{G}^-$ are anomaly-free, just like the spin 0 current J^{++}. More interesting are the (twisted) energy-momentum tensor and the $U(1)$ current J (when treated as primary fields).

The modes of the twisted energy-momentum tensor $T' = -\frac{1}{\alpha'}\partial Z\cdot\partial\bar{Z} - \frac{1}{2}\psi^-\cdot\partial\psi^+$ can be written as

$$L_n = \frac{1}{2}\sum_m \alpha_m\cdot\bar{\alpha}_{n-m} + \frac{1}{2}\sum_m (n-m)\psi_m^-\cdot\psi_{n-m}^+\,. \tag{VI.33}$$

According to (VI.27) these modes have to satisfy[5]

$$\langle\mathcal{I}|K_n = \langle\mathcal{I}|(L_n - (-1)^n L_{-n}) = 0 \tag{VI.34}$$

for the vertex to be reparametrization invariant. In $D/2$ complex dimensions, the contribution of the bosons to the left hand side of eq. (VI.34) can be easily shown to be

$$\langle\mathcal{I}|K_{2n}^\alpha = \frac{D}{2}(-1)^n n\,\langle\mathcal{I}|\,, \tag{VI.35}$$

which is canceled by the fermionic contribution

$$\langle\mathcal{I}|K_{2n}^\psi = -\frac{D}{2}(-1)^n n\,\langle\mathcal{I}|\,. \tag{VI.36}$$

These contributions arise from terms $\frac{1}{2}\alpha_n\cdot\bar{\alpha}_n$ and $\frac{n}{2}\psi_n^-\cdot\psi_n^+$ in K_{2n}^α and K_{2n}^ψ, respectively. Due to the absence of such terms, the K_{2n+1} are automatically anomaly-free.

Before considering the $U(1)$ current J, let us first recall the discussion in [186] of the $U(1)$-anomaly of N-vertices: If the current J is bosonized as $J = \partial\varphi$, the action for this boson reads

$$S = -\frac{1}{4\pi}\int dz\wedge d\bar{z}\,(\partial\varphi\bar{\partial}\varphi + QR\varphi)\,. \tag{VI.37}$$

The operator product expansion is that of the free action, $\varphi(z)\varphi(w)\sim\ln(z-w)$. The energy-momentum tensor for φ reads $T_\varphi = \frac{1}{2}J^2 - Q\partial J$, where Q is the background charge, i. e., the coefficient of the third order pole in the operator product expansion $T(z)J(w)$. For the $\psi^+\psi^-$-system in $D/2$ complex dimensions, $Q = -D/2$.

In a general gluing geometry the curvature is concentrated in one point, namely the midpoint of the string ($\sigma = \pi/2$). On such surfaces the term linear in φ contributes an anomalous factor of

$$\exp\Big(\frac{Q}{2\pi}\varphi(\pi/2)\int d^2\sigma R\Big) \tag{VI.38}$$

[5] Treating the energy-momentum tensor as a primary field is justified iff the central charge vanishes. In this sense, one can understand eq. (VI.34) as a condition on the central charge. Note that the N-string variant $\langle V_N|\sum_{r=1}^N(L_n^{(r)} - (-1)^n L_{-n}^{(r)}) = 0$ of eq. (VI.34) does not follow from eq. (VI.27) for $N > 2$.

in the path integral[6]. This integral measures the deficit angle of this surface when circumnavigating the curvature singularity at the string midpoint and contributes $-(N-2)\pi$ for an N-string vertex. Hence, the factor (VI.38) produces a $U(1)$-anomaly of $(N-2)\frac{D}{4}$ in the path integral. Since the $U(1)$-charge[7] of $\langle\downarrow|$ is $-\frac{D}{4}$, an N-vertex constructed from N $\langle\downarrow|$-vacua requires $(N-2)\frac{D}{4} - N\left(-\frac{D}{4}\right) = \frac{D}{2}(N-1)$ ψ^+-insertions (the exponential factor is neutral).[8] This is consistent with (VI.29) for $N=1$.

Therefore, we do not expect the $U(1)$-current J to be anomaly-free;

$$\langle\mathcal{I}|(J_n + (-1)^n J_{-n}) \neq 0 \tag{VI.39}$$

in general. Since its zero-mode measures the fermion number of the vertex, we instead expect $\langle\mathcal{I}|J_0 = \frac{D}{4}\langle\mathcal{I}|$. This relation holds trivially. For $n \neq 0$ in eq. (VI.39), one obtains $\langle\mathcal{I}|(J_{2n} + J_{-2n}) = (-1)^n\frac{D}{2}\langle\mathcal{I}|$.

VI.4 Reflector

In this section we construct the 2-string vertex for the fermionic $(1,0)$ system (ψ^-, ψ^+). It is convenient to introduce $_N$-Fourier-transformed fields as a tool to diagonalize general N-string overlap equations. The overlap equations fix the zero-mode part of the 2-vertex up to a sign. A discussion of BPZ conjugation motivates our choice for this sign.

$_N$-transforms. Introducing the combinations

$$\sum_n \psi^-_n\, e^{\pm in\sigma} = \pi_{\psi^+}(\sigma) \pm \mathrm{i}\,\psi^-(\sigma)\,, \qquad \sum_n \psi^+_n\, e^{\pm in\sigma} = \psi^+(\sigma) \pm \mathrm{i}\,\pi_{\psi^-}(\sigma) \tag{VI.40}$$

of left and right movers, the conditions imposed on the fermions following from the δ-function overlap of N strings are

$$\psi^{+(r)}(\sigma) = \begin{cases} \psi^{+(r-1)}(\pi-\sigma), & \sigma \in [\,0\,,\frac{\pi}{2}]\,, \\ \psi^{+(r+1)}(\pi-\sigma), & \sigma \in [\frac{\pi}{2},\pi]\,, \end{cases} \tag{VI.41a}$$

$$\pi^{(r)}_{\psi^+}(\sigma) = \begin{cases} -\pi^{(r-1)}_{\psi^+}(\pi-\sigma), & \sigma \in [\,0\,,\frac{\pi}{2}]\,, \\ -\pi^{(r+1)}_{\psi^+}(\pi-\sigma), & \sigma \in [\frac{\pi}{2},\pi]\,. \end{cases} \tag{VI.41b}$$

For $\psi^-(\sigma)$ and $\pi_{\psi^-}(\sigma)$ similar equations have to be fulfilled. The conditions (VI.41) are easily diagonalized if we introduce $_N$-Fourier-transformed fields [66],

$$\Psi^a(\sigma) = \frac{1}{\sqrt{N}}\sum_{r=1}^{N}\psi^{+(r)}(\sigma)e^{\frac{2\pi ira}{N}}\,, \qquad \Pi^a(\sigma) = \frac{1}{\sqrt{N}}\sum_{r=1}^{N}\pi^{(r)}_{\psi^+}(\sigma)e^{\frac{2\pi ira}{N}}\,, \tag{VI.42a}$$

[6] In the integral of the Ricci scalar over this surface we have used $dz \wedge d\bar{z} = 2d\sigma d\tau$. – An analogous argument in the case of the bc system shows that $\#_{bc}(\int) = -\#_{bc}(\star) = -\frac{3}{2}$, cf. eqs. (III.40) and (III.44).

[7] The $U(1)$ charge of a bra vector is measured by $J_0^\dagger$; we use conventions where $J_0 = \frac{1}{2}\sum_{m=1}^{\infty}(\psi^+_{-m}\cdot\psi^-_m - \psi^-_{-m}\cdot\psi^+_m) + \frac{1}{2}\psi^+_0\cdot\psi^-_0 - \frac{D}{4} = -J_0^\dagger$, cf. eq. (C.7).

[8] Alternatively, we could use midpoint insertions to adjust the anomaly of the vertex. However, in our cases, they make life unnecessarily complicated. Since the value of the anomaly, $(N-2)\frac{D}{4}$, never exceeds the maximal charge of $\langle\Omega_N|$, i.e. $N\frac{D}{4}$ for $\langle\Omega_N|_{\max} = \langle\uparrow|^{\otimes N}$, we can avoid midpoint insertions.

where $a \in \{1, \ldots, N\}$. Note that now (Ψ^a, Π^{N-a}) form canonically conjugate pairs (the upper index is taken modulo N). We choose the following ansatz for the N-vertex in terms of $_N$-transformed oscillators

$$\langle V_N| = \langle \Omega_N| \exp\big(\tfrac{1}{2}\sum_{a=1}^{N}\sum_{m,n} \Psi^a_m V^a_{mn} \Pi^{N-a}_n\big) \tag{VI.43}$$

with Neumann matrices V^a and a vacuum state $\langle \Omega_N|$. The vacuum state will be determined below from the zero-mode overlap conditions; the summation range of m, n should then be adjusted in such a way that only creation operators w. r. t. this vacuum appear in the exponential.

In application to $\langle V_N|$, eqs. (VI.41) now read

$$\langle V_N|\Psi^a(\sigma) = \begin{cases} e^{\frac{2\pi i a}{N}} \langle V_N|\Psi^a(\pi - \sigma)\,, \\ e^{\frac{-2\pi i a}{N}} \langle V_N|\Psi^a(\pi - \sigma)\,, \end{cases} \tag{VI.44a}$$

$$\langle V_N|\Pi^a(\sigma) = \begin{cases} -e^{\frac{2\pi i a}{N}} \langle V_N|\Pi^a(\pi - \sigma)\,, \\ -e^{\frac{-2\pi i a}{N}} \langle V_N|\Pi^a(\pi - \sigma)\,. \end{cases} \tag{VI.44b}$$

As already discussed in section VI.3, the overlap conditions will only contain a sum of two oscillators (rather than infinitely many), if after inserting the mode expansions the cosines can be integrated over $[0, \pi]$. This is obviously possible also for $N > 2$ if $\frac{2a}{N} \in$. Therefore, (Ψ^N, Π^N) and, if N is even, $(\Psi^{N/2}, \Pi^{N/2})$ appear in the vertex (VI.43) with Neumann matrices $V^N = -C$ and $V^{N/2} = C$, respectively. Here, C denotes the twist matrix with components $C_{mn} = (-1)^m \delta_{mn}$.

Before we turn to the 2-string vertex, let us briefly discuss the overlap conditions for the zero-modes of the $_N$-transformed oscillators. It is consistent with (VI.44) to demand

$$\langle \Omega_N|\Psi^a_0 = 0 \qquad \text{for } 1 \le a \le N-1\,, \tag{VI.45a}$$

$$\langle \Omega_N|\Pi^N_0 = 0\,. \tag{VI.45b}$$

Note that eqs. (VI.45a) entail that no Ψ^a_0 (for $a \in \{1, \ldots, N-1\}$) may appear in the exponential of the vertex (VI.43). The appearance of Ψ^N_0 is forbidden by eq. (VI.45b) since $V^N = -C$ is diagonal. In terms of the original one-string oscillators, this means that no ψ^+_0 appears in the exponential of the vertex.

It is easy to see that the conditions on the vacuum (VI.45) are solved by[9]

$$\langle \Omega_N| = \pm \sum_{k=1}^{N} {}_1\langle\uparrow| \otimes \cdots {}_{k-1}\langle\uparrow| \otimes {}_k\langle\downarrow| \otimes {}_{k+1}\langle\uparrow| \otimes \cdots {}_N\langle\uparrow|\,. \tag{VI.46}$$

[9]Here one has to use the fact that $\langle\uparrow|$ is Grassmann *even* while $\langle\downarrow|$ is Grassmann *odd*, i.e., the bra-vacua have opposite Grassmannality compared to the corresponding ket-vacua. This is a consequence of the odd background charge.

The subscripts indicate in which string Hilbert space the corresponding vacuum state lives. The vacuum (VI.46) already features the $U(1)$ charge required by the J-anomaly, namely $(N-2)\frac{D}{4}$. This choice allows us to avoid midpoint insertions.

Overlap equations for the reflector. Expressed in terms of $\mathbb{Z}_2$-transforms, the overlap conditions for the reflector simply become

$$\langle V_2|\Psi^1(\sigma) = -\langle V_2|\Psi^1(\pi-\sigma)\,, \qquad \langle V_2|\Psi^2(\sigma) = \langle V_2|\Psi^2(\pi-\sigma)\,, \tag{VI.47a}$$

$$\langle V_2|\Pi^2(\sigma) = -\langle V_2|\Pi^2(\pi-\sigma)\,, \qquad \langle V_2|\Pi^1(\sigma) = \langle V_2|\Pi^1(\pi-\sigma)\,, \tag{VI.47b}$$

which can be rewritten in terms of modes acting on $\langle V_2|$ as

$$(\Psi^1_m+\Psi^1_{-m}) = -(-1)^m(\Psi^1_m+\Psi^1_{-m})\,, \qquad (\Psi^2_m+\Psi^2_{-m}) = (-1)^m(\Psi^2_m+\Psi^2_{-m})\,, \tag{VI.48a}$$

$$(\Pi^1_m+\Pi^1_{-m}) = (-1)^m(\Pi^1_m+\Pi^1_{-m})\,, \qquad (\Pi^2_m+\Pi^2_{-m}) = -(-1)^m(\Pi^2_m+\Pi^2_{-m})\,, \tag{VI.48b}$$

for the nonzero-modes. The conditions for the zero-modes read

$$\langle V_2|\Psi^1_0 = 0\,, \qquad \langle V_2|\Pi^2_0 = 0\,. \tag{VI.49}$$

The zero-modes Ψ^2_0 and Π^1_0 put no restrictions on the vertex. Along the lines of [66], one finds

$$\langle V_2| = \langle\Omega_2|\exp\Big(\tfrac{1}{2}\sum_{m=1}^{\infty}\big[\Psi^2_m(-1)^m\Pi^2_m - \Psi^1_m(-1)^m\Pi^1_m\big]\Big) \tag{VI.50a}$$

$$= \langle\Omega_2|\exp\Big(\tfrac{1}{2}\sum_{m=1}^{\infty}\big[\psi^{+(1)}_m(-1)^m\psi^{-(2)}_m + \psi^{+(2)}_m(-1)^m\psi^{-(1)}_m\big]\Big) \tag{VI.50b}$$

as a solution to eqs. (VI.48). Since no zero-modes appear in the vertex, the vacuum $\langle\Omega_2|$ has to be annihilated by Ψ^1_0 and Π^2_0 in order to satisfy eq. (VI.49). Thus the vacuum is a *symmetric* combination of up- and down-vacua in the two-string Hilbert space,

$$\langle\Omega_2| = \pm({}_1\langle\uparrow|\otimes{}_2\langle\downarrow| + {}_1\langle\downarrow|\otimes{}_2\langle\uparrow|) =: \pm(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|). \tag{VI.51}$$

This is consistent with eq. (VI.46). In the last expression it is understood that the first entry corresponds to string 1, while the second corresponds to string 2. The overall sign is determined by requiring that $\langle V_2|$ implements BPZ conjugation.

BPZ conjugation. On a single field $\phi(z)$ BPZ conjugation (cf. the discussion in section III.2) acts as $I\circ\phi(z)$ with $I(z) = -1/z$; since I inverts the time direction, it is suggestive that on a product of fields, BPZ conjugation should reverse the order of the fields. This statement will be put on a more solid ground below. The action of BPZ on fields induces an action on states: $\mathrm{bpz}(|\phi\rangle)$ defines the out-state $\langle\phi|$ which is created by $\lim_{z\to\infty}\langle 0|I\circ\phi(z)$. In terms of modes this prescription yields

$$\mathrm{bpz}(\phi_n) = (-1)^{n+h}\phi_{-n} \tag{VI.52}$$

for a field of conformal weight h. To fix the choice of vacuum in (VI.50), recall that $\langle V_2|$ is an element of the tensor product of two dual string Hilbert spaces $\langle V_2| \in \mathcal{H}^* \otimes \mathcal{H}^*$ and thus induces an odd linear map from $\mathcal{H}$ to $\mathcal{H}^*$, which is nothing but BPZ conjugation [58]

$$\langle V_2|\phi\rangle_1 = {}_2\langle \mathrm{bpz}(\phi)| \,. \tag{VI.53}$$

In order to be compatible with the usual definitions of BPZ conjugation, we demand in particular that the $SL(2,\mathbb{R})$ invariant vacuum $|\downarrow\rangle$ is mapped into $\langle\downarrow|$ under BPZ conjugation. Therefore we fix the vacuum $\langle\Omega_2|$ to be

$$\langle\Omega_2| = {}_1\langle\uparrow| \otimes {}_2\langle\downarrow| + {}_1\langle\downarrow| \otimes {}_2\langle\uparrow| =: \langle\uparrow\downarrow| + \langle\downarrow\uparrow| \,. \tag{VI.54}$$

Note that choosing this vacuum and using eq. (VI.53), one finds $\mathrm{bpz}(|\downarrow\rangle) = \langle\downarrow|$ and $\mathrm{bpz}(|\uparrow\rangle) = \langle\uparrow|$.

Now consider the corresponding ket state $|V_2\rangle$. Observe that the conformal transformation I maps (τ,σ) to $(-\tau,\pi-\sigma)$. Therefore, the overlap equations for $0 \leq \sigma \leq \pi/2$ for a field ϕ of conformal weight h_ϕ , $\phi^{(r)}(\sigma) = (-1)^{h_\phi}\phi^{(r-1)}(\pi-\sigma)$, transform into $\phi^{(r)}(\sigma) = (-1)^{-h_\phi}\phi^{(r+1)}(\pi-\sigma)$. This implies that the overlap equations for the $N=1$ and $N=2$ vertices are invariant under BPZ conjugation for fields of integral conformal weight. Indeed, this can be verified for (VI.48) using (VI.52) on the level of modes, and we can immediately write down the solution

$$|V_2\rangle = \exp\Big(\tfrac{1}{2}\sum_{m=1}^{\infty}\big[\Pi^2_{-m}(-1)^m\Psi^2_{-m} - \Pi^1_{-m}(-1)^m\Psi^1_{-m}\big]\Big)|\Omega_2\rangle \tag{VI.55a}$$

$$= \exp\Big(\tfrac{1}{2}\sum_{m=1}^{\infty}\big[\psi^{-(1)}_{-m}(-1)^m\psi^{+(2)}_{-m} + \psi^{-(2)}_{-m}(-1)^m\psi^{+(1)}_{-m}\big]\Big)|\Omega_2\rangle \,. \tag{VI.55b}$$

It is easy to see that eqs. (VI.48), now taken to act on the ket vertex, are fulfilled. Eventually we have to fix our choice of vacuum. In order to fulfill the zero-mode overlap equations (VI.49), $|\Omega_2\rangle$ has to be an *antisymmetric* combination of up and down vacua

$$|\Omega_2\rangle = \pm\left(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\right) = \pm\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{VI.56}$$

We fix the overall sign of $|\Omega_2\rangle$ to be a plus sign by requiring

$$\mathrm{bpz}^{-1}(\langle\phi|) := {}_2\langle\phi|V_2\rangle_{12}(-1)^{|\phi|+1} = |\mathrm{bpz}^{-1}(\phi)\rangle_1 \,, \tag{VI.57}$$

where $|\phi|$ denotes the Grassmannality of the state $|\phi\rangle$. Moreover one finds

$$\begin{aligned}
\mathrm{bpz}(\psi^+_{-k}\psi^-_{-l}|\downarrow\rangle) &= {}_{12}\langle V_2|\psi^{+(1)}_{-k}\psi^{-(1)}_{-l}|\downarrow\rangle_1 = {}_{12}\langle V_2|\psi^{+(2)}_{k}(-1)^k\psi^{-(1)}_{-l}|\downarrow\rangle_1 \\
&= {}_{12}\langle V_2|\psi^{-(1)}_{-l}\psi^{+(2)}_{k}(-1)^k(-1)^{|\psi^+||\psi^-|}|\downarrow\rangle_1 \\
&= {}_2\langle\downarrow|\psi^{-(2)}_{l}\psi^{+(2)}_{k}(-1)^{k+l+1}(-1)^{|\psi^+||\psi^-|} \,,
\end{aligned} \tag{VI.58}$$

which can be checked using (VI.50). Eq. (VI.58) is the statement that BPZ conjugation acts as a *graded antihomomorphism* on the algebra of modes. To emphasize the gradation we explicitly kept the sign stemming from the anticommutation of the modes. Note that there is no problem in commuting the modes since after acting on the vertex they belong to different Hilbert spaces, so the only effect is an additional sign. Finally, it is straightforward to check that

$$ {}_{12}\langle V_2|V_2\rangle_{23} = \big(|\!\uparrow\rangle_{3\,1}\langle\downarrow\!| + |\!\downarrow\rangle_{3\,1}\langle\uparrow\!|\big) \exp\Big(\tfrac{1}{2}\sum_{m=1}^{\infty}\big[\psi^{+(3)}_{-m}\psi^{-(1)}_{m} + \psi^{-(3)}_{-m}\psi^{+(1)}_{m}\big]\Big) = {}_{3}\ \ {}_{1}\,, \tag{VI.59} $$

by using standard coherent state techniques (cf. [94, 31, 102]) and eq. (VI.50). One can then check that $\mathrm{bpz}\circ\mathrm{bpz}^{-1} = \mathrm{bpz}^{-1}\circ\mathrm{bpz} = \ \ $. This completes the construction of the reflector state from the overlap equations.

VI.5 Interaction vertex

In this section, we set up the Neumann function method [122, 124, 123, 90, 36, 37, 114, 115] for general N-string vertices, since even in terms of the $\ {}_N$-Fourier-transforms the overlap equations are not directly soluble for $N \geq 3$. In the case of the 3-string vertex, the Neumann coefficients are computed explicitly in terms of generating functions. The observation that they are intimately related to the bosonic Neumann coefficients (VI.14) helps us to show that the K_n-anomaly of the (bosonic and fermionic) 3-vertex vanishes in any even dimension D. Furthermore, it will be shown that the 3-vertex for the $\psi^+\psi^-$ system satisfies its overlap equations.

Neumann function method. The Neumann function method is based on the fact that the large time transition amplitude is given by the Neumann function of the scattering geometry under consideration. To find the Fock space representation of the interaction vertex one makes an ansatz quadratic in the oscillators,

$$ \langle V_N| = \mathcal{N}_N \langle\Omega_N| \exp\big[\tfrac{1}{4}\sum_{r,s}\sum_{k,l}\psi^{+(r)}_{k} N^{rs}_{kl}\psi^{-(s)}_{l}\big]\,, \tag{VI.60} $$

where $\mathcal{N}_N$ is a normalization factor which is determined below.[10] The sum over the string labels r and s runs from 1 to N and the restrictions on the summation range of the oscillator modes has to be determined from the choice of vacuum $\langle\Omega_N|$ (cf. (VI.46)) so that only creation operators appear in the vertex. As derived in section VI.4, ψ^+_0 does not occur in the exponential. The normalization factor $\mathcal{N}_N$ is determined by taking the matrix element $\langle V_N|\tilde{\Omega}_N\rangle$ where $|\tilde{\Omega}_N\rangle$ is the dual vacuum satisfying $\langle\Omega_N|\tilde{\Omega}_N\rangle = 1$. Since this matrix element corresponds to a ψ^+ one-point function and ψ^+ has conformal weight zero this yields $\mathcal{N}_N = \langle V_N|\tilde{\Omega}_N\rangle = \langle\psi^+\rangle = 1$.

To obtain an explicit expression for the coefficients we look at matrix elements of the form

$$ G(z,w) = \langle V_N|\psi^{+(s)}(z)\psi^{-(r)}(w)|\tilde{\Omega}_N\rangle \tag{VI.61} $$

and reinterpret the result as a correlation function on the disk (or, thanks to $PSL(2,\mathbb{R})$ invariance, equivalently on the upper half plane). Note that, in this expression, the J-anomaly has to

[10] The factor of of $\frac{1}{4}$ in the exponent will take care of the nonstandard normalization of the correlation function for the complex fermions.

be saturated in each string separately, i. e., in each Hilbert space we need one ψ_0^+ (which can be attributed to either $\langle\Omega_N|$ or $|\tilde{\Omega}_N\rangle$). Inserting the mode expansions for $\psi^+(z)$ and $\psi^-(w)$ into eq. (VI.61), one obtains by virtue of eq. (VI.60)

$$G(z,w) = \sum_{mn} z^n w^{m-1} N^{rs}_{mn} \,. \tag{VI.62}$$

Following [114], we equate this with

$$G(z,w) = \langle f_s \circ \psi^+(z) f_r \circ \psi^-(w) \tfrac{1}{N} \sum_{i=1}^{N} f_i \circ \psi^+(0)\rangle \,, \tag{VI.63}$$

where the sum on the right hand side was chosen to distribute the background charge symmetrically among the N strings. In principle, any other choice of $\psi^+(0)$-insertions is admissible as long as the J-anomaly on the scattering geometry is saturated, i. e., we need a total $U(1)$ charge of $+1$ in the correlation function. The f_r map the unit upper half-disk into the corresponding wedge of the scattering geometry, as defined in (VI.26). The pole structure of the correlation function (VI.63) is easily evaluated; first order poles arise from $\psi^+\psi^-$-contractions, first order zeros from $\psi^+\psi^+$-contractions. Since the conformal weights of ψ^+ and ψ^- are 0 and 1, respectively, we obtain

$$\langle f_s \circ \psi^+(z) f_r \circ \psi^-(w) \tfrac{1}{N} \sum_{i=1}^{N} f_i \circ \psi^+(0)\rangle = \frac{2f_r'(w)}{f_s(z)-f_r(w)} \frac{1}{N} \sum_{i=1}^{N} \frac{f_s(z)-f_i(0)}{f_r(w)-f_i(0)} \,. \tag{VI.64}$$

Here the unusual factor of 2 appears due to the normalization of the fermionic correlator. From eqs. (VI.61) to (VI.64) one readily finds the expression for the Neumann coefficients in terms of contour integrals,

$$N^{rs}_{mn} = \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1} w^m} \frac{2f_r'(w)}{f_s(z)-f_r(w)} \frac{1}{N} \sum_{i=1}^{N} \frac{f_s(z)-f_i(0)}{f_r(w)-f_i(0)} \,. \tag{VI.65}$$

Neumann coefficients and generating functions. In this paragraph we work out explicitly the integral formula for the Neumann coefficients for the (bra-)interaction vertex and find expressions in terms of the coefficients of generating functions. The vertex will take the form[11]

$$\langle V_3| = \big(\langle\uparrow\uparrow\downarrow| + \langle\uparrow\downarrow\uparrow| + \langle\downarrow\uparrow\uparrow|\big) \exp\big[\tfrac{1}{4} \sum_{r,s} \sum_{k=1,l=0}^{\infty} \psi_k^{+(r)} N^{rs}_{kl} \psi_l^{-(s)}\big] \,. \tag{VI.66}$$

The maps involved in (VI.65) for $N=3$ can be gleaned from (VI.26),

$$f_i(z) = e^{\frac{2\pi \mathrm{i}}{3}(2-i)} \left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right)^{\frac{2}{3}} = \omega^{2-i} f(z) \tag{VI.67}$$

[11] Recall that no $\psi_0^{+(r)}$ appears in the exponent as substantiated in section VI.4.

with $\omega = e^{\frac{2\pi i}{3}}$. Using these maps one can rewrite eq. (VI.65) as

$$\begin{aligned} N^{rs}_{mn} &= 2\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}w^m}\frac{f'_r(w)}{f_s(z)-f_r(w)}\frac{f_s(z)f_r(w)^2-1}{f_r(w)^3-1} \\ &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}w^{m+1}}\frac{2}{3}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\Big[1+U^{rs}(z,w)+U^{sr}(-z,-w)\Big]\,, \end{aligned} \tag{VI.68}$$

where

$$U^{rs}(z,w) = \omega^{(s-r)}\frac{w}{z}\left(\frac{1+iz}{1+iw}\right)^2 f(w)f(-z)\,. \tag{VI.69}$$

Introducing the generating functions

$$G(z) = \frac{f(z)}{(1+iz)^2} = \sum_{n=0}^{\infty} G_n z^n\,, \tag{VI.70a}$$

$$H(z) = (1+iz)^2 f(-z) = \sum_{n=0}^{\infty} H_n z^n\,, \tag{VI.70b}$$

we can write

$$N^{rs}_{mn} = \frac{2}{3}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\Big[C_{mn}(z,w)+\omega^{(s-r)}U_{mn}(z,w)+\bar\omega^{(s-r)}\bar U_{mn}(z,w)\Big]\,, \tag{VI.71}$$

with

$$C_{mn}(z,w) = \frac{1}{z^{n+1}w^{m+1}}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\,, \tag{VI.72a}$$

$$U_{mn}(z,w) = \frac{1}{z^{n+2}w^{m}}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)G(w)H(z)\,, \tag{VI.72b}$$

$$\bar U_{mn}(z,w) = \frac{1}{z^{n+2}w^{m}}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)G(-w)H(-z)\,. \tag{VI.72c}$$

Performing the contour integrals[12], one finds the Neumann coefficients in terms of the coefficients of the generating functions,

$$N^{rs}_{mn} = \frac{2}{3}\left(C_{mn}+\omega^{(s-r)}U_{mn}+\bar\omega^{(s-r)}\bar U_{mn}\right)\,, \tag{VI.73}$$

where

$$U_{mn} = \sum_{k=0}^{n}\Big[(-1)^{n+1-k}G_{m-n-2+k}-G_{m+n-k}\Big]H_k\,, \tag{VI.74a}$$

$$\bar U_{mn} = (-1)^{m+n}\sum_{k=0}^{n}\Big[(-1)^{n+1-k}G_{m-n-2+k}-G_{m+n-k}\Big]H_k\,. \tag{VI.74b}$$

In these formulas, it is implicitly understood that coefficients with negative index are zero, and, as usual, $C_{mn} = (-1)^m\delta_{mn}$ for $m,n>0$. Since ψ^+ does not appear in the exponential of the vertex, we require $N^{rs}_{0m} = 0$ for $m \geq 0$. Note that the Neumann coefficients are real since the

[12]Note that one can choose the contour always so that only the poles at zero contribute.

G_n and H_n are real for n even and purely imaginary for n odd. Obviously, $\bar{U}$ is the complex conjugate of U, and $\bar{U} = CUC$. Eq. (VI.73) makes the cyclic symmetry of the vertex manifest.

Recursion relations. To find recursion relations for the generating functions (VI.70a), we observe that $G(z)$ can be expressed in terms of its derivative:

$$G(z) = -\frac{3}{2}\frac{z^2+1}{3z+\mathrm{i}}G'(z)\,. \tag{VI.75}$$

Inserting the mode expansion, one finds

$$G_k = -\oint \frac{dz}{2\pi\mathrm{i}}\,\frac{1}{z^{k+1}}\,\frac{3}{2}\frac{z^2+1}{3z+\mathrm{i}}\frac{\partial}{\partial z}G(z)\,. \tag{VI.76}$$

Partially integrating and evaluating the resulting contour integral leads to the following recursion formula for G_n:

$$G_{k+2} = -\frac{2\mathrm{i}}{3(k+2)}\,G_{k+1} - G_k\,. \tag{VI.77}$$

Note that this complies with the observation that the G_k are alternatingly real and imaginary. From (VI.77) and the initial condition $G_0 = G(0) = 1$ (and $G_{-1} := 0$), the first coefficients are easily computed to be $G_1 = -\frac{2\mathrm{i}}{3}$, $G_2 = -\frac{11}{9}$, and $G_3 = \frac{76\mathrm{i}}{81}$.

Similarly, we can use

$$H(z) = \left(-\frac{\mathrm{i}}{6} + \frac{z}{2} + \frac{4/3}{3z+\mathrm{i}}\right)H'(z) \tag{VI.78}$$

to find recursion relations for the H_k,

$$(k+2)H_{k+2} = \frac{2\mathrm{i}}{3}H_{k+1} - (k-2)H_k\,, \tag{VI.79}$$

and with the initial condition $H_0 = H(0) = 1$ (and $H_{-1} := 0$), the first coefficients are found to be $H_1 = \frac{2\mathrm{i}}{3}$, $H_2 = \frac{7}{9}$ and $H_3 = \frac{32\mathrm{i}}{81}$. One readily verifies that

$$\sum_{k=0}^{n} G_k H_{n-k} = 0 \qquad \text{for all } n \in \quad , \tag{VI.80}$$

since $G(z) = 1/H(z)$.

Relation to bosonic coefficients. Exemplarily, the first few Neumann coefficients N^{11}_{mn} can be computed via eqs. (VI.73), (VI.74) and the recursion relations (VI.77) and (VI.79):

$$(N^{11})_{mn} = \begin{pmatrix} \frac{10}{27} & 0 & -\frac{64}{729} & 0 & \frac{832}{19683} & \cdots \\ 0 & -\frac{26}{243} & 0 & \frac{1024}{19683} & 0 & \cdots \\ -\frac{64}{243} & 0 & \frac{1786}{19683} & 0 & -\frac{3008}{59049} & \cdots \\ 0 & \frac{2048}{19683} & 0 & -\frac{10250}{177147} & 0 & \cdots \\ \frac{4160}{19683} & 0 & -\frac{15040}{177147} & 0 & \frac{82330}{1594323} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{mn} . \tag{VI.81}$$

The above expression holds for $m, n \geq 1$. This suggests that, for these values of m, n, the Neumann coefficients for the $\psi^+\psi^-$-system agree with those for the bosons in the momentum basis (cf. eq. (VI.14)) up to some factor; the same can be checked for all other r, s:

$$N_{mn}^{rs} = 2\sqrt{\frac{m}{n}} V_{mn}^{rs} \,. \tag{VI.82}$$

A posteriori, one can easily find a proof for this relation. Comparing with (VI.14) and (VI.68), we have to show

$$2\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1} w^m} \frac{f_r'(w)}{f_s(z) - f_r(w)} \frac{f_s(z) f_r(w)^2 - 1}{f_r(w)^3 - 1} = -\frac{2}{n} \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^n w^m} \frac{f_r'(w) f_s'(z)}{(f_s(z) - f_r(w))^2} \,. \tag{VI.83}$$

Since the right hand side can be rewritten as

$$\frac{2}{n} \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^n w^m} \frac{\partial}{\partial z} \frac{f_r'(w)}{(f_s(z) - f_r(w))} = 2 \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1} w^m} \frac{f_r'(w)}{f_s(z) - f_r(w)} \,, \tag{VI.84}$$

the difference of the left hand and the right hand sides of eq. (VI.83) is proportional to

$$\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1} w^m} \frac{f_r'(w) f_r(w)^2}{f_r(w)^3 - 1} \,. \tag{VI.85}$$

This expression vanishes for $n > 0$ due to the absence of poles in the z-contour. This establishes the proof of eq. (VI.82).

Properties of the Neumann matrices. In view of the close relation of the fermionic to the bosonic Neumann matrices one immediately obtains identities for N_{mn}^{rs}, $m, n \geq 1$ from the bosonic ones. Defining $CN^{r\,r} =: N$, $CN^{r\,r+1} =: N_+$ and $CN^{r\,r-1} =: N_-$, one finds that N, N^+ and N^- mutually commute and

$$\begin{gathered} N + N_+ + N_- = 2 \,, \quad N_+ N_- = N(N-2) \,, \quad N^2 + N_+^2 + N_-^2 = 4 \,, \\ N\,N_+ + N_+ N_- + N_- N = 0 \,, \qquad N_\pm^2 - N_\pm = N\,N_\mp \,, \\ C\,N = N\,C \,, \qquad C\,N_+ = N_-\,C \,. \end{gathered} \tag{VI.86}$$

The proof of eq. (VI.82) breaks down for $n = 0$. We have $N_{00}^{rs} = 0$; the Neumann coefficients for the case $n = 0$ and $m > 0$ are given by

$$N_{m0}^{rr} = \begin{cases} -\frac{8\mathrm{i}}{9m} G_{m-1} & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd,} \end{cases} \tag{VI.87}$$

$$N_{m0}^{r\,r+1} = \begin{cases} \frac{4\mathrm{i}}{9m} G_{m-1} & \text{for } m \text{ even,} \\ -\frac{4}{3\sqrt{3}m} G_{m-1} & \text{for } m \text{ odd,} \end{cases} \tag{VI.88}$$

$$N_{m0}^{r\,r-1} = \begin{cases} \frac{4\mathrm{i}}{9m} G_{m-1} & \text{for } m \text{ even,} \\ \frac{4}{3\sqrt{3}m} G_{m-1} & \text{for } m \text{ odd.} \end{cases} \tag{VI.89}$$

The indices r, s are cyclic. From this it is obvious that

$$C_{nm}N^{rt}_{m0} = N^{tr}_{n0}\,, \quad \sum_t\sum_m N^{rt}_{m0} = \sum_t\sum_m N^{tr}_{m0} = 0\,. \tag{VI.90}$$

Exploiting that the generating function $G(z)$ is proportional to the derivative of $((1-\mathrm{i}z)/(1+\mathrm{i}z))^{\frac{1}{3}}$ one sees that the coefficients G_k are related to the coefficients a_n (or equivalently A_n) defined in appendix F.1 [66] via

$$G_{m-1} = \tfrac{3}{2}m(-\mathrm{i})^{(m-1)}a_m\,. \tag{VI.91}$$

Evaluating the generating function for the coefficients A_{2n} (cf. eq. (F.3)),

$$\frac{1}{2}\left[\Big(\frac{1-\mathrm{i}\,z}{1+\mathrm{i}\,z}\Big)^{1/3} + \Big(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\Big)^{1/3}\right], \tag{VI.92}$$

at $z=1$ yields

$$\sum_{m=1}^{\infty} N^{11}_{m0} = \frac{4}{3}\sum_{n=1}^{\infty} A_{2n} = \frac{4}{3}\Big(\frac{2}{\sqrt{3}} - 1\Big)\,. \tag{VI.93}$$

The contour integral around $z=0$ computes

$$\sum_{n=0}^{\infty}(A^2_{2n} - A^2_{2n+1}) = \oint \frac{dz}{2\pi\mathrm{i}}\frac{1}{z}\left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{1/3}\left(\frac{1+\mathrm{i}\,\frac{1}{z}}{1-\mathrm{i}\,\frac{1}{z}}\right)^{1/3} = \frac{1}{2}\,, \tag{VI.94}$$

which establishes that

$$\sum_t\sum_{m=1}^{\infty} N^{1t}_{m0}N^{t1}_{m0} = \frac{8}{3}\Big(\sum_{n=0}^{\infty}(A^2_{2n} - A^2_{2n+1}) - 1\Big) = -\frac{4}{3}\,. \tag{VI.95}$$

Having at hand fermion Neumann coefficients for the nonzero-modes expressed in terms the boson Neumann coefficients puts us in the position to compute the K_n-anomaly of the fermionic 3-vertex in a very simple way. Similarly, the overlap equations can be checked more easily than with the original expression (VI.73). This will be done in the next two paragraphs.

Anomaly of the $\psi^{\pm}$-vertex. We will now demonstrate that the contribution of one $\psi^+\psi^-$-pair to the K_n-anomaly of the 3-vertex cancels the contribution of two real (or one complex) bosons. This agrees with the fact that a $(1,0)$-first order system contributes $c=-2$ to the central charge. Thus, in contrast to bosonic and N=1 strings, no restriction on the critical dimension follows from the K_n-anomaly.

Namely, let $\sum_{r=1}^{3} K^{(r)\psi}_m = \sum_{r=1}^{3}\big(L^{\psi(r)}_m - (-1)^m L^{\psi(r)}_{-m}\big)$ act on the 3-vertex (VI.66). The only contribution to the c-number anomaly comes from the terms in

$$-(-1)^m L^{\psi(r)}_{-m} = -(-1)^m\frac{1}{2}\sum_k (m-k)\psi^{+(r)}_{k-m}\cdot\psi^{-(r)}_{-k} \tag{VI.96}$$

containing two creation operators, i. e., from $\frac{1}{2}\sum_{k=0}^{m-1}(m-k)\psi^{+(r)}_{k-m}\cdot\psi^{-(r)}_{-k}$. The action of $\psi^{+(r)}_{k-m}$ on the bra-vertex pulls down a sum over annihilation operators, and from the interchange of $\psi^{-(r)}_{-k}$ with these creation operators we get a c-number term:

$$\begin{aligned}\langle V_3|K^{(3)\psi}_m &= -(-1)^m\frac{1}{2}\sum_{r=1}^{3}\sum_{k=0}^{m-1}(m-k)\langle V_3|\psi^{+(r)}_{k-m}\cdot\psi^{-(r)}_{-k}+\ldots \\ &= -(-1)^m\frac{1}{4}\sum_{r,s=1}^{3}\sum_{k=0}^{m-1}(m-k)\sum_{l=0}^{\infty}\langle V_3|N^{sr}_{l,m-k}\psi^{+(s)}_{l}\cdot\psi^{-(r)}_{-k}+\ldots \\ &= -(-1)^m\frac{3\,D}{4}\sum_{k=1}^{m-1}(m-k)N^{rr}_{k,m-k}\langle V_3|\,.\end{aligned} \tag{VI.97}$$

In the third equality we have used that $N^{11}=N^{22}=N^{33}$ due to cyclicity and that $N^{rr}_{0,m-k}=0$, i.e., the exponential in the vertex contains no ψ^{+}_{0}. The dots indicate terms which do not contribute to the c-number anomaly. From (VI.24), this equals twice the negative contribution of one real boson; the total anomaly vanishes if we pair each $\psi^{+a}\psi^{-\bar{a}}$-system with a complex boson field $Z^a,\bar{Z}^{\bar{a}}$ in any even dimension.

Overlap conditions. According to the general method outlined in section VI.4 we introduce $\mathbb{Z}_3$-Fourier-transforms

$$\Psi^a = \frac{1}{\sqrt{3}}\sum_{r=1}^{3}\psi^{+(r)}\omega^{ra}\,, \tag{VI.98a}$$

$$\Pi^a = \frac{1}{\sqrt{3}}\sum_{r=1}^{3}\psi^{-(r)}\omega^{ra}\,, \tag{VI.98b}$$

where $\omega=e^{\frac{2\pi\mathrm{i}}{3}}$ and the index a runs from 1 to 3. This diagonalizes the overlap equations which then read

$$\langle V_3|\Psi^1(\sigma) = \begin{cases}\omega\,\langle V_3|\Psi^1(\pi-\sigma)\,, & \sigma\in[0\,,\frac{\pi}{2}]\,,\\ \bar{\omega}\,\langle V_3|\Psi^1(\pi-\sigma)\,, & \sigma\in[\frac{\pi}{2},\pi]\,,\end{cases} \tag{VI.99a}$$

$$\langle V_3|\Psi^2(\sigma) = \begin{cases}\bar{\omega}\,\langle V_3|\Psi^2(\pi-\sigma)\,, & \sigma\in[0\,,\frac{\pi}{2}]\,,\\ \omega\,\langle V_3|\Psi^2(\pi-\sigma)\,, & \sigma\in[\frac{\pi}{2},\pi]\,,\end{cases} \tag{VI.99b}$$

$$\langle V_3|\Psi^3(\sigma) = \langle V_3|\Psi^3(\pi-\sigma)\,, \tag{VI.99c}$$

and

$$\langle V_3|\Pi^1(\sigma) = \begin{cases}-\omega\,\langle V_3|\Pi^1(\pi-\sigma)\,, & \sigma\in[0\,,\frac{\pi}{2}]\,,\\ -\bar{\omega}\,\langle V_3|\Pi^1(\pi-\sigma)\,, & \sigma\in[\frac{\pi}{2},\pi]\,,\end{cases} \tag{VI.100a}$$

$$\langle V_3|\Pi^2(\sigma) = \begin{cases}-\bar{\omega}\,\langle V_3|\Pi^2(\pi-\sigma)\,, & \sigma\in[0\,,\frac{\pi}{2}]\,,\\ -\omega\,\langle V_3|\Pi^2(\pi-\sigma)\,, & \sigma\in[\frac{\pi}{2},\pi]\,,\end{cases} \tag{VI.100b}$$

$$\langle V_3|\Pi^3(\sigma) = -\langle V_3|\Pi^3(\pi-\sigma)\,. \tag{VI.100c}$$

These overlap equations can be written in terms of the Fourier modes of the operator (VI.19) [66] as

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} + \tfrac{1}{2}\tilde{C}_{kl} - \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})\langle V_3|\tilde{\Psi}_l^1 = 0\,, \tag{VI.101a}$$

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} + \tfrac{1}{2}\tilde{C}_{kl} + \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})\langle V_3|\tilde{\Psi}_l^2 = 0\,, \tag{VI.101b}$$

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} - \tfrac{1}{2}\tilde{C}_{kl} + \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})\langle V_3|\tilde{\Pi}_l^1 = 0\,, \tag{VI.101c}$$

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} - \tfrac{1}{2}\tilde{C}_{kl} - \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})\langle V_3|\tilde{\Pi}_l^2 = 0\,, \tag{VI.101d}$$

where here and in the following the indices $k, l, j \in \quad_0$ while $m, n \in \quad$. The matrices $\tilde{E}$ and $\tilde{C}$ are given by

$$\tilde{E}_{kl} = 2\,\delta_{0k}\delta_{0l} + \delta_{kl}\,, \quad \tilde{C}_{kl} = (-1)^k \tilde{E}_{kl}\,. \tag{VI.102}$$

The matrices $\tilde{X}_{kl}$ can be found in appendix F.2. The redefined oscillators are $\tilde{\Psi}_m^a = \Psi_m^a + \Psi_{-m}^a$ and $\tilde{\Pi}_m^a = \Pi_m^a + \Pi_{-m}^a$ for the nonzero-modes and $\tilde{\Psi}_0^a = \Psi_0^a$ and $\tilde{\Pi}_0^a = \Pi_0^a$ for the zero-modes, respectively.

We make the following ansatz for the interaction vertex in terms of the $\mathbb{Z}_3$-transformed oscillators (recall the range of the indices defined above!)

$$\langle V_3| = \langle \Omega_3| \exp\left[\tfrac{1}{2}\sum_{m,n}\Psi_m^3 C_{mn}\Pi_n^3 + \sum_{m,k}(\Psi_m^2 \tilde{U}_{mk}\Pi_k^1 + \Psi_m^1 \tilde{\bar{U}}_{mk}\Pi_k^2)\right]. \tag{VI.103}$$

Note that the exponential does not contain any Ψ_0 modes. The vacuum $\langle\Omega_3|$ is given by

$$\langle \Omega_3| = \left\langle \Pi_0^3 = 0, \Psi_0^1 = 0, \Psi_0^2 = 0\right|\,, \tag{VI.104}$$

which in terms of one string Hilbert space vacua is expressed as[13]

$$\langle\Omega_3| = {}_1\langle\uparrow| \otimes {}_2\langle\uparrow| \otimes {}_3\langle\downarrow| + {}_1\langle\downarrow| \otimes {}_2\langle\uparrow| \otimes {}_3\langle\uparrow| + {}_1\langle\uparrow| \otimes {}_2\langle\downarrow| \otimes {}_3\langle\uparrow|\,. \tag{VI.105}$$

It is straightforward to see that (VI.103) satisfies the overlap equations for the Ψ_m^3's and the Π_m^3's. Comparing the $\mathbb{Z}_3$-transformed version of the interaction vertex with eq. (VI.66) one can identify

$$\tilde{U}_{ml} = U_{ml}\,, \quad \tilde{\bar{U}}_{ml} = \bar{U}_{ml}. \tag{VI.106}$$

[13]The overlap equations fix the vacuum up to an overall sign factor.

After using (VI.103), the overlap equations for the oscillators Π^1, Π^2 and Ψ^1, Ψ^2 become

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} - \tfrac{1}{2}\tilde{C}_{kl} + \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})(\delta_{lj} - U_{lj}) = 0\,, \tag{VI.107a}$$

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} - \tfrac{1}{2}\tilde{C}_{kl} - \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})(\delta_{lj} - \bar{U}_{lj}) = 0\,, \tag{VI.107b}$$

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} + \tfrac{1}{2}\tilde{C}_{kl} - \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})(\delta_{lm} + \bar{U}^T_{lm}) = 0\,, \tag{VI.107c}$$

$$\sum_{l=0}^{\infty}(\tilde{E}_{kl} + \tfrac{1}{2}\tilde{C}_{kl} + \tfrac{\sqrt{3}}{2}\tilde{X}_{kl})(\delta_{lm} + U^T_{lm}) = 0\,. \tag{VI.107d}$$

Let us now exemplify that these overlap conditions are indeed fulfilled by the matrices given in eq. (VI.74). In particular we consider the parts of the overlap equations involving zero-modes.

We start with the $k = 0$ overlap equation for Π^1_0, which is the zero-zero component of eq. (VI.107a):

$$1 - \tfrac{\sqrt{3}}{2}\sum_{m=1}^{\infty}\tilde{X}_{0m}U_{m0} \stackrel{!}{=} 0\,. \tag{VI.108}$$

Inserting the U_{m0} component

$$U_{m0} = -(-\mathrm{i})^m\, a_m \tag{VI.109}$$

into (VI.108) allows us to use known summation formulas for the coefficients [66, 34] to obtain

$$\sum_{m=1}^{\infty}\tilde{X}_{0m}U_{m0} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{a_{2k+1}}{2k+1} = \frac{2}{\sqrt{3}} \tag{VI.110}$$

proving eq. (VI.108). Consider now the overlap equations for $k \neq 0$. Setting $k = 2l$ for k even and $k = 2l+1$ for k odd yields

$$-\tfrac{1}{2}U_{2l,0} - \tfrac{\sqrt{3}}{2}\sum_{m=1}^{\infty}\tilde{X}_{2l,m}U_{m0} \stackrel{!}{=} 0\,, \tag{VI.111a}$$

$$-\tfrac{3}{2}U_{2l+1,0} + \tfrac{\sqrt{3}}{2}\tilde{X}_{2l+1,0} - \tfrac{\sqrt{3}}{2}\sum_{m=1}^{\infty}\tilde{X}_{2l+1,m}U_{m0} \stackrel{!}{=} 0\,. \tag{VI.111b}$$

The first of these equations is proven by

$$\frac{\sqrt{3}}{2}\sum_{m=1}^{\infty}\tilde{X}_{2l,m}U_{m0} = \frac{\sqrt{3}}{\pi}\sum_{k=0}^{\infty}(-1)^l\left(\frac{a_{2k+1}}{2k+1+2l} + \frac{a_{2k+1}}{2k+1-2l}\right) = \frac{1}{2}(-1)^l a_{2l}\,. \tag{VI.112}$$

The second equation in (VI.111) is fulfilled due to

$$\begin{aligned}\frac{\sqrt{3}}{2}\sum_{m=1}^{\infty}\tilde{X}_{2l+1,m}U_{m0} &= -\frac{\sqrt{3}\mathrm{i}}{\pi}\sum_{k=0}^{\infty}(-1)^l\left(\frac{a_{2k}}{2k+2l+1} - \frac{a_{2k}}{2k-2l-1}\right) + \frac{\sqrt{3}\mathrm{i}}{\pi}(-1)^l\frac{2\,a_0}{2k+1}\\ &= -\frac{3\mathrm{i}}{2}(-1)^l a_{2l+1} + \frac{\sqrt{3}\mathrm{i}}{\pi}(-1)^l\frac{2\,a_0}{2k+1}\,.\end{aligned} \tag{VI.113}$$

More involved overlap conditions can be proven using techniques developed in [66, 34]. We postpone their discussion to appendix F.2.

VI.6 Conclusions

In this chapter we explicitly constructed the string field theory vertices for a fermionic first order system $\psi^{\pm}$ with conformal weights $(1, 0)$ in the operator formulation. The technical ingredients needed to construct general N-string vertices were presented in detail. The identity vertex, the reflector and the interaction vertex were discussed with emphasis on their charge under the anomalous $U(1)$ current J and their zero-mode dependence. The identity vertex and the reflector were derived from the corresponding δ-function overlap conditions. The reflector was shown to implement BPZ conjugation as a graded antihomomorphism, and some consistency conditions on the gluing of the reflector were checked. The construction of the interaction vertex was achieved by invoking the Neumann function method. The coefficients of the Neumann matrices are given in terms of coefficients of generating functions; recursion relations for these coefficients were derived. The Neumann coefficients for the $\psi^{\pm}$ system can be neatly expressed in terms of those for the bosons. This allowed us to infer identities for the fermion Neumann matrices directly from those for the bosons. Moreover, the c-number anomaly of midpoint preserving reparametrizations for a $\psi^{\pm}$ pair was straightforwardly shown to cancel the contribution of two real bosons. This agrees with the fact that a $(0, 1)$ first order system contributes $c = -2$ to the central charge. Eventually, it was shown that the overlap equations following from the δ-function overlap conditions are satisfied by the Neumann matrices.

Clearly, the work presented is meant to be a starting point for further investigations. Diagonalizing the vertex is a straightforward task and will be a project in the future. This should pave the way for studying solutions to string field theory in several contexts. Firstly, one might examine how the solution generating techniques proposed in chapters IV and V perform in the more controlled setting of $N = 2$ SFT. This can be expected to give valuable information about how solutions to string field theory can be constructed dropping the factorization assumption of vacuum string field theory. As a direct application to N=1 superstring field theory it appears to be worthwhile to investigate the dependence of solutions on the $\eta\xi$-system more closely. This fermionic $(0, 1)$ first order system emerges in the bosonization of the superconformal ghosts. The related picture changing operation is a delicate subject in string (field) theory and deserves to be examined with minuteness. Finally, we want to point out the similarity of the fermionic first order system considered in this chapter and the twisted bc-system presented in section III.5.3. This auxiliary boundary conformal field theory is used in vacuum string field theory to construct solutions to the ghost part of the equation of motion. The solutions to these equations become projectors in the *twisted* theory. It is tempting to speculate that this similarity can be traced to a deeper interrelation.

Chapter VII

Conclusions

In this work it was shown that superstring field theories are integrable in the sense that their equations of motion can be written as compatibility conditions for certain linear equations. This made it possible to transfer powerful solution generating techniques for integrable field theories to (open and vacuum) superstring field theories.

As a preparation for the string field theory discussion, one particular solution generating technique, the dressing approach, was introduced in a field theory setting, namely, in noncommutative self-dual Yang-Mills theory in 2+2 dimensions. Since this theory arises in the low-energy limit of open N=2 string theory in a B-field background, the connection of these considerations to string (field) theory was apparent. In this part, new soliton-like solutions could be constructed, amongst others, an abelian solution resulting from GMS-like projectors, noncommutative $U(2)$ soliton-like configurations and interacting plane waves.

After a short introduction to the topic of string field theory (including some recent developments), this dressing approach was transferred to Berkovits' WZW-like string field theory. Additionally, a second method for the construction of exact classical solutions was introduced, the splitting technique. In essence, both procedures reduce the nonpolynomial equation of motion to some linear equations. The solutions of these linear equations give us nonperturbative solutions of the original equations of motion. Our discussion was kept general enough to apply to the case of N=1 superstrings as well as to the case of N=2 strings.

In order to demonstrate the power of these methods we explicitly constructed some solutions to the linear equations via the dressing approach. For N=1 superstrings, a quite general class of solutions was presented; for N=2 strings, the same and additional classes of solutions were found. Following the recipe given in section IV.5, one can easily translate all these to classical configurations of Berkovits' (super)string field theory.

With a suitable relation between the Witten and the Berkovits string fields, the condition that the Witten string field is contained in the small Hilbert space (which does not contain the ξ zero-mode) may be reinterpreted as the equation of motion to Berkovits' superstring field theory. Assuming this relation, the equation of motion for Witten's cubic string field theory is trivially satisfied. This observation makes it possible to generalize the above ideas also to cubic superstring field theory. The transition from the description of string field theory around the open string vacuum to a description around the closed string vacuum is embedded in a natural way into the framework of the dressing approach. A strategy for the computation of solutions to the string field equations, based on the Moyal formulation for superstring field theory, was proposed.

As a (in some respects) simplified model for a string field theory with kinetic operators which mix different world-sheet sectors, N=2 string field theory seems to be a viable candidate. It shares many characteristic properties with Berkovits' nonpolynomial superstring field theory. However, little is known in the literature about this theory, in particular, the vertex for the world-sheet fermions of this theory was only graphically defined. As a first step to the concrete computation of solitonic solutions in nonpolynomial string field theory for N=2 strings, the integration and reflector states as well as the 3-string vertex for the world-sheet fermions were determined. This sets the stage for forthcoming investigations, e. g., the Moyal formulation of this sector. Amongst others, our results have applications in Berkovits' hybrid formulation of a covariant superstring field theory in $D = 4$, in the $\eta\xi$ system from the bosonization of N=1 world-sheet ghosts and the twisted bc system used in bosonic vacuum superstring field theory.

A lot of work remains to be done: The Moyal formulation of the world-sheet fermion system should put us into the position to classify all projectors of the star algebra. These were particularly important in the dressing approach – and naturally in vacuum string field theory, where they deliver solutions of the ghost part of the string field equations.

In order to establish which among the proposed solutions represent soliton-like objects within the theory, one has to evaluate their energy. It would be interesting to find criteria on the T_k for the solution to be a soliton, an instanton, or a monopole. In the case of N=2 strings explicit solitonic solutions to the corresponding field theory equations have been constructed earlier [108, 112, 109]; it is plausible that they can be promoted to the string level. An examination of the fluctuations around these nonperturbative solutions should determine what kind of object they represent in string theory. If some of these solutions turn out to describe D-branes, perhaps another check of Sen's conjecture on the relation between the tension of D-branes and the string field theory action is feasible. Due to our choosing the simplest ansätze for the splitting and the dressing methods, we have obtained not the broadest classes of field configurations. However, nothing prevents one from employing more general ansätze and thereby creating more general solutions for N=1 superstring field theory.

Appendix A

Mathematical background for the field theory part

A.1 Self-duality, twistor space and holomorphicity

In this section, we explain the geometric setup underlying the method we use to solve the self-duality equations of Yang-Mills theory on $\mathbb{R}^{2,2}$. We mostly restrict ourselves to the commutative case, comments on the noncommutative generalization are added where appropriate.[1] For our purposes, $U(N)$ Yang-Mills theory is formulated in terms of a $GL(N,\mathbb{C})$ principal bundle $P \cong \mathbb{R}^{2,2} \times GL(N,\mathbb{C})$ over the (pseudo-)Riemannian "space-time" manifold $\mathbb{R}^{2,2}$. This principal bundle should be endowed with an irreducible $GL(N,\mathbb{C})$ connection A and its respective curvature F. We will impose a reality condition on A below. The self-duality equations $F = *F$ are tackled with the help of a Lax pair, whose geometrical meaning will now be described.

A.1.1 Isotropic coordinates

We will see in section A.1.4 that the self-duality equations on $\mathbb{R}^{2,2}$ can be written in real coordinates x^μ as

$$\overline{W}_1^\mu \overline{W}_2^\nu F_{\mu\nu} = 0 \tag{A.1}$$

for certain 4-vectors $\overline{W}_i$. To derive constraints on the $\overline{W}_i$, it turns out to be useful to switch to a spinor notation. Exploiting that $so(2,2) \cong sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$, we can rewrite $\overline{W}_i$ as

$$(\overline{W}_i^{\dot{\alpha}\alpha}) = (\overline{\tau}_\mu^{\dot{\alpha}\alpha} \overline{W}_i^\mu) = \begin{pmatrix} \overline{W}_i^4 + \overline{W}_i^2 & \overline{W}_i^1 - \overline{W}_i^3 \\ \overline{W}_i^1 + \overline{W}_i^3 & \overline{W}_i^4 - \overline{W}_i^2 \end{pmatrix} \tag{A.2}$$

with the help of $SL(2,\mathbb{R})$-generators $\overline{\tau}_a$, $a = 1,2,3$ and $\overline{\tau}_4 = 1$. If we define as for the Pauli matrices $\tau^\mu_{\beta\dot{\beta}} = \eta^{\mu\nu} \overline{\tau}_\mu^{\dot{\alpha}\alpha} \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{\alpha\beta}$ with $\varepsilon_{12} = -1$, eq. (A.1) can be rewritten as

$$\overline{W}_1^{\dot{\alpha}\alpha} \overline{W}_2^{\dot{\beta}\beta} (F_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} + F_{\dot{\alpha}\dot{\beta}} \varepsilon_{\alpha\beta}) = 0. \tag{A.3}$$

[1] For a description of twistors in the noncommutative case, see [91, 175, 110].

These are the self-duality equations $F_{\dot\alpha\dot\beta}=0$ iff we choose

$$\overline{W}_1^{\dot\alpha\alpha}=\xi^\alpha\pi^{\dot\alpha}\quad\text{and}\quad\overline{W}_2^{\dot\beta\beta}=\chi^\beta\pi^{\dot\beta}, \tag{A.4}$$

with arbitrary commutative spinors ξ^α,χ^α, and $\pi^{\dot\alpha}$. That is, $\overline{W}_1$ and $\overline{W}_2$ have to span a null plane in $\mathbb{R}^{2,2}$.

On $\mathbb{R}^{2,2}$, there are two possibilities to satisfy (A.4), related to the existence of Majorana-Weyl spinors in 2+2 dimensions: One can choose complex or real spinors. Since the $\overline{\tau}$-matrices in (A.2) are real, this will lead to complex and real coordinates on $\mathbb{R}^{2,2}$.

A.1.2 Complex coordinates

Almost complex structures on $\mathbb{R}^{2,2}$. To elucidate the meaning of the $\overline{W}_i$ it is necessary to introduce an *almost complex structure* on $\mathbb{R}^{2,2}$. An almost complex structure is a tensor field J of type (1,1) such that $J_\mu{}^\nu J_\nu{}^\lambda=-\delta_\mu{}^\lambda$. We shall consider translationally invariant (constant) and therefore integrable almost complex structures, i.e., complex structures. It is easy to see that complex structures on $\mathbb{R}^{2,2}$ are parametrized by the coset $SO(2,2)/U(1,1)\cong SO(2,1)/SO(2)$.[2] Without loss of generality, we can restrict the discussion to almost complex structures compatible with the metric (so that the metric is hermitean). Then, (anti)holomorphic basis vectors are automatically null vectors.

One can realize [81] this coset space on $so(2,1)$ in the following way [86]: We start from a matrix representation of $so(2,1)$,

$$I_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\qquad I_2=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$
$$I_3=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}, \tag{A.5}$$

satisfying $I_aI_b=g_{ab}+f_{ab}{}^cI_c$ with structure constants $f_{12}{}^3=-f_{23}{}^1=-f_{31}{}^2=1$ and metric $(g_{ab})=\mathrm{diag}(1,1,-1)$ on $so(2,1)$. Then we can write a general complex structure in the form

$$J=-s^aI_a \tag{A.6}$$

[2] For a given almost complex structure J one can choose coordinates x^1, x^2, y^1, y^2 such that in this basis, J as an endomorphism of the tangent bundle maps $J(\frac{\partial}{\partial \mathrm{x}^k})=\frac{\partial}{\partial\mathrm{y}^k}$ and $J(\frac{\partial}{\partial\mathrm{y}^k})=-\frac{\partial}{\partial\mathrm{x}^k}$ for $k=1,2$. A linear combination $\partial_{\mathrm{x}^k}-\mathrm{i}J\partial_{\mathrm{y}^k}=:\partial_{z^k}$ (as a section of the complexified tangent bundle to $\mathbb{R}^{2,2}$) obviously has eigenvalue i, it only rotates homogeneously under rotations M_k^n of the structure group $SO(2,2)$ if J and M commute. This singles out a subgroup $U(1,1)$ of $SO(2,2)$ which leaves the fixed complex structure invariant.

for $s^1, s^2, s^3 \in \mathbb{R}$. We easily read off

$$J^2 = g_{ab} s^a s^b \stackrel{!}{=} -1 \quad \Leftrightarrow \quad (s^1)^2 + (s^2)^2 - (s^3)^2 = -1. \tag{A.7}$$

Obviously, the $\{s^a\}$ parametrize a two-sheeted hyperboloid H^2. We can map the upper half H^2_+ of H^2 onto the interior of the unit disk in the y-plane by a stereographic projection

$$s^1 := \frac{2y^1}{1-r^2}, \quad s^2 := \frac{2y^2}{1-r^2}, \quad s^3 := \frac{1+r^2}{1-r^2}, \quad r^2 := (y^1)^2 + (y^2)^2. \tag{A.8}$$

Simultaneously, the lower half H^2_- is mapped onto the exterior of the unit disk. If we define $\lambda := -(y^1 + \mathrm{i}y^2)$, both regions are related by the map

$$\sigma\colon \lambda \mapsto 1/\bar{\lambda}. \tag{A.9}$$

Note that for $|\lambda| \neq 1$, σ has no fixed points. Recapitulating, we can state that the moduli space of complex structures on $\mathbb{R}^{2,2}$ is $\mathbb{C}P^1 \backslash S^1$ (the S^1 being given by $|\lambda| = 1$).

(Anti)holomorphic vector fields. A given complex structure J on $\mathbb{R}^{2,2}$ as in (A.6) has holomorphic and antiholomorphic eigenvectors with eigenvalues i and $-$i; as a (local) basis for antiholomorphic vector fields, we can choose

$$\overline{W}_1 = \overline{W}{}^\mu_1 \partial_\mu = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2) - \frac{\lambda}{2}(\partial_3 - \mathrm{i}\partial_4) = \partial_{\bar{z}^1} - \lambda \partial_{z^2}, \tag{A.10a}$$

$$\overline{W}_2 = \overline{W}{}^\mu_2 \partial_\mu = \frac{1}{2}(\partial_3 + \mathrm{i}\partial_4) - \frac{\lambda}{2}(\partial_1 - \mathrm{i}\partial_2) = \partial_{\bar{z}^2} - \lambda \partial_{z^1}. \tag{A.10b}$$

Their components, $\overline{W}{}^\mu_1 = (\frac{1}{2}, \frac{\mathrm{i}}{2}, -\frac{\lambda}{2}, \frac{\mathrm{i}\lambda}{2})$ and $\overline{W}{}^\mu_2 = (-\frac{\lambda}{2}, \frac{\mathrm{i}\lambda}{2}, \frac{1}{2}, \frac{\mathrm{i}}{2})$ satisfy $J_\mu{}^\nu \overline{W}{}^\mu_1 = -\mathrm{i}\overline{W}{}^\nu_1$, $J_\mu{}^\nu \overline{W}{}^\mu_2 = -\mathrm{i}\overline{W}{}^\nu_2$, and $\eta_{\mu\nu} \overline{W}{}^\mu_i \overline{W}{}^\nu_j = 0$ for $i, j = 1, 2$, respectively. The vector fields (A.10) will become our Lax operators subsequently (cf. eqs. (II.72)). The definitions of z^1, z^2 coincide with those given in (II.7). We can introduce coordinates $\eta^1 = z^1 + \lambda \bar{z}^2$ and $\eta^2 = z^2 + \lambda \bar{z}^1$ (cf. eq. (II.81)) in the kernel of (A.10).

Almost complex structure on $\mathbf{H^2}$. If we introduce the standard complex structure ϵ on H^2,

$$\epsilon^k_i \epsilon^j_k = -\delta^j_i, \quad \epsilon^2_1 = -\epsilon^1_2 = 1, \quad \epsilon^1_1 = \epsilon^2_2 = 0 \tag{A.11}$$

for $i, j, k = 1, 2$, we can give explicit expressions for the (local) antiholomorphic vector field on $H^2_+ \subset \mathbb{C}P^1$:

$$\overline{W}_3 = -\frac{1}{2}\left(\frac{\partial}{\partial y^1} + \mathrm{i}\frac{\partial}{\partial y^2}\right) = \frac{\partial}{\partial \bar{\lambda}}. \tag{A.12}$$

Noncommutative description. In the noncommutative framework, one has to incorporate some modifications to the above description. All functions now have to be multiplied by a deformed product; alternatively, the Moyal-Weyl map may be used to transform them into operators with the usual operator product. In this interpretation, the space-time manifold $\mathbb{R}^{2,2}$ has to be replaced by the Heisenberg algebra $\mathbb{R}^{2,2}_\theta$ generated by operators $\hat{x}^\mu$ subject to $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}$. The (Lie algebra of) inner derivations of $\mathbb{R}^{2,2}_\theta$ corresponds to the (Lie algebra of) sections of the tangent bundle $T\mathbb{R}^{2,2}$. If we denote by $\mathfrak{R}$ a four-dimensional representation of $SO(2,2)$, the Lie algebra of inner derivations can be understood as a free $\mathfrak{R}$-module. From these arguments it is clear that the construction of the moduli space of complex structures on $\mathbb{R}^{2,2}_\theta$ can be treated analogously to the commutative setup. Note that H^2 remains commutative.

Let us reconsider the noncommutative setup from a different point of view. To this aim, we assume without loss of generality that $\theta = \widetilde{\theta} > 0$. Then, just as z^1 and z^2 are mapped to annihilation operators (II.15a) under the Moyal-Weyl map, η^1 and η^2 are mapped to new annihilation operators

$$c_1 := (1-\lambda\bar{\lambda})^{-1/2}\big(a_1 + \lambda a_2^\dagger\big) \quad \text{and} \quad c_2 := (1-\lambda\bar{\lambda})^{-1/2}\big(a_2 + \lambda a_1^\dagger\big), \tag{A.13}$$

where $|\lambda| < 1$. The $SO(2,2)$ rotation of the commutative discussion above transforming old coordinates z^i to new coordinates η^i after transition to operators takes the form of a Bogoliubov transformation (A.13). In general, transformations $c_1 = Ua_1U^\dagger$, $c_2 = Ua_2U^\dagger$ yield equivalent representations of the Heisenberg algebra $\mathbb{R}^{2,2}_\theta$ if U is unitary. One can easily show that this is the case for $|\lambda| \neq 1$ for (A.13). Obviously, the Bogoliubov transformations leaving a given representation invariant are parametrized by the maximal pseudo-unitary subgroup $U(1,1)$ of $SO(2,2)$ leading again to the same coset space as in the commutative case.

A.1.3 Real isotropic coordinates

Although $|\lambda| = 1$ according to the preceding discussion will not correspond to a complex structure on $\mathbb{R}^{2,2}$, the vector fields (A.10) in this case still span a null plane in $\mathbb{R}^{2,2}$ [106]. Using that now $\lambda = \bar{\lambda}^{-1}$, one readily sees that complex conjugation maps $\overline{W}_1$ to a multiple of $\overline{W}_2$ and vice versa, i.e., the isotropic plane is real. One is free to choose a real basis for this plane, which is most easily accomplished with the help of the map (II.64) sending the unit circle to the real axis in the ζ-plane.

Real isotropic planes, being parametrized by $S^1 = \{\lambda \in \mathbb{C} \big|\, |\lambda| = 1\}$, supplement the moduli space of complex isotropic two-planes (or complex structures) to $\mathbb{C}P^1 \cong H^2 \cup S^1$ [85, 106]. So, $\mathbb{C}P^1$ can be considered as the moduli space of all null two-planes (or extended complex structures) in $\mathbb{R}^{2,2}$.

A.1.4 Extended twistor space for $\mathbb{R}^{2,2}$

Extended twistor space. In this section, Ward's theorem [184] on a one-to-one correspondence between vector bundles E with self-dual connections over euclidean $\mathbb{R}^4$ and holomorphic bundles E' over the so-called twistor space is rephrased for the case of $\mathbb{R}^{2,2}$. The twistor space for $\mathbb{R}^{2,2}$ is the bundle $\mathbb{R}^{2,2} \times H^2 \to \mathbb{R}^{2,2}$ of all constant complex structures on $\mathbb{R}^{2,2}$. It can be endowed with the direct sum $\mathcal{J}$ of the complex structures J and ϵ. The vector fields (A.10) and (A.12) for $|\lambda| \neq 1$ are the $\mathcal{J}$-antiholomorphic vector fields on $\mathbb{R}^{2,2} \times H^2$ with respect to this complex structure. Admitting $|\lambda| = 1$ in (A.10) and (A.12), we can extend these vector fields naturally to $\mathcal{Z} := \mathbb{R}^{2,2} \times \mathbb{C}P^1$.

Vector bundle over $\mathcal{Z}$. Now, we can use the canonical projection $\pi\colon \mathcal{Z} \to \mathbb{R}^{2,2}$ to lift the vector bundle $E := P \times_{GL(N,\mathbb{C})} \mathbb{C}^N$ to a bundle π^*E over the extended twistor space $\mathcal{Z}$. By construction, the connection on π^*E is flat along the fibers $\mathbb{C}P^1$ of $\mathcal{Z}$, so that the lifted connection π^*A on π^*E can be chosen to have only components along $\mathbb{R}^{2,2}$, $\pi^*A = A_\mu dx^\mu$. Thus, the lift takes the covariant derivative $D_\mu = \partial_\mu + A_\mu$ on E to

$$\pi^*D = dx^\mu D_\mu + dy^i \frac{\partial}{\partial y^i} \tag{A.14}$$

on π^*E. Now, the $\mathcal{J}$-antiholomorphic components of (A.14) are the $(0,1)$ components of π^*D along the antiholomorphic vector fields $\overline{W}_i$ on $\mathcal{Z}$:

$$D_1^{(0,1)} \equiv \overline{W}_1^\mu D_\mu = \overline{W}_1 + \frac{1}{2}(A_1 + \mathrm{i}A_2) - \frac{\lambda}{2}(A_3 - \mathrm{i}A_4), \tag{A.15a}$$

$$D_2^{(0,1)} \equiv \overline{W}_2^\mu D_\mu = \overline{W}_1 + \frac{1}{2}(A_3 + \mathrm{i}A_4) - \frac{\lambda}{2}(A_1 - \mathrm{i}A_2), \tag{A.15b}$$

$$D_3^{(0,1)} \equiv \overline{W}_3^i \partial_{y^i} = \overline{W}_3. \tag{A.15c}$$

Holomorphic sections. Local sections φ of the complex vector bundle π^*E are holomorphic if

$$D_1^{(0,1)}\varphi = 0, \tag{A.16a}$$

$$D_2^{(0,1)}\varphi = 0, \tag{A.16b}$$

$$D_3^{(0,1)}\varphi = 0. \tag{A.16c}$$

We can also view this as the local form of meromorphic sections of $E' := \pi^*E$ in a given trivialization of the bundle. One can combine N such sections (as columns) into an $N \times N$ matrix to obtain the matrix-valued function ψ used in (II.65). Using (A.15), a comparison with (II.65) shows that after solving (A.16c) these are exactly the linear equations (Lax pair) for self-dual Yang-Mills theory. In this framework, the self-duality equations (II.66) emerge as the condition that eqs. (A.16) are compatible, i.e., the $(0,2)$ components of the curvature of E' vanish.

A.1.5 Reality condition

So far, we have been working with a complex vector bundle associated to a $GL(N,\mathbb{C})$-principal bundle P to describe $U(N)$ self-dual Yang-Mills theory. Therefore, we have to implement a reality condition on our gauge fields, i.e., impose the additional constraint $A_\mu^\dagger = -A_\mu$.

Let us now scrutinize the action of hermitean conjugation on the linear equations (II.65). Eq. (II.65a) is equivalent to

$$(\partial_{\bar{z}^1} - \lambda\partial_{z^2})\psi^{-1}(\lambda) = \psi^{-1}(\lambda)(A_{\bar{z}^1} - \lambda A_{z^2}), \tag{A.17}$$

where we suppress the additional dependence of ψ of the space-time coordinates $z^i, \bar{z}^i$. Since we demand this to hold for all λ, we can as well first apply σ from (A.9) and then take the hermitean conjugate,

$$(\partial_{\bar{z}^2} - \lambda\partial_{z^1})\left[\psi^{-1}(\bar{\lambda}^{-1})\right]^\dagger = -(A_{\bar{z}^2} - \lambda A_{z^1})\left[\psi^{-1}(\bar{\lambda}^{-1})\right]^\dagger . \tag{A.18}$$

This coincides with (II.65b) if we choose $\psi(\lambda) = \left[\psi(\bar{\lambda}^{-1})^\dagger\right]^{-1}$, i.e., eq. (II.67). With these restrictions, the $gl(N,\mathbb{C})$-curvature F naturally descends to a $u(N)$-valued curvature.

In the noncommutative case, the vector bundle E has to be replaced by a free module over $\mathbb{R}^{2,2}_\theta$. Accordingly, $D = d + A$ is chosen to be a connection on the module E [99]. It is understood that the above discussion can be applied analogously, taking into account that multiplication of A and ψ becomes noncommutative.

A.2 Abelian pseudo-instantons

This appendix concludes our considerations with the discussion of a special class of abelian, i.e., $U(1)$ solutions with finite action (in contrast to their commutative counterparts).[3] We work in the operator formalism. Let us introduce "shifted" operators acting on the two-oscillator Fock space $\mathcal{H}$:

$$X_\mu := A_\mu + \mathrm{i}(\theta^{-1})_{\mu\nu}x^\nu, \tag{A.19}$$

where $(\theta^{-1})_{\mu\sigma}\theta^{\sigma\nu} = \delta^\nu_\mu$. The operator-valued field strength $F_{\mu\nu}$ can be expressed in terms of the shifted operators X_μ as

$$F_{\mu\nu} = [X_\mu, X_\nu] - \mathrm{i}(\theta^{-1})_{\mu\nu}. \tag{A.20}$$

The incarnation of the ncYM equations in this context is

$$[X^\mu, [X_\mu, X_\nu]] = 0. \tag{A.21}$$

They are, of course, automatically satisfied by X_μ subject to the ncSDYM equations (cf. [51])

$$[X_\mu, X_\nu] = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[X^\rho, X^\sigma] + \mathrm{i}(\theta_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}). \tag{A.22}$$

Observe that the last term of the ncSDYM equations vanishes for self-dual $\theta^{\mu\nu}$, i.e., $\theta = \widetilde{\theta}$. Switching to complex coordinates[4] and assuming self-dual $\theta^{\mu\nu}$ from now on, eqs. (A.22) become

$$[X_{z^1}, X_{z^2}] = [X_{\bar{z}^1}, X_{\bar{z}^2}] = 0, \tag{A.23a}$$

$$[X_{z^1}, X_{\bar{z}^1}] - [X_{z^2}, X_{\bar{z}^2}] = 0, \tag{A.23b}$$

where $X_{z^i} := A_{z^i} + \mathrm{i}(\theta^{-1})_{i\bar{j}}\bar{z}^j$ and $X_{\bar{z}^i} := A_{\bar{z}^i} + \mathrm{i}(\theta^{-1})_{\bar{i}j}z^j$, $i \in \{1,2\}$. It is easily checked that

$$X^0_{z^1} = \mathrm{i}(\theta^{-1})_{1\bar{1}}\bar{z}^1, \qquad X^0_{\bar{z}^1} = \mathrm{i}(\theta^{-1})_{\bar{1}1}z^1, \tag{A.24a}$$

$$X^0_{z^2} = \mathrm{i}(\theta^{-1})_{2\bar{2}}\bar{z}^2, \qquad X^0_{\bar{z}^2} = \mathrm{i}(\theta^{-1})_{\bar{2}2}z^2, \tag{A.24b}$$

[3] Noncommutative instantons in euclidean space were introduced in [136].

[4] We denote $[z^i, \bar{z}^j] = \mathrm{i}\theta^{i\bar{j}}$ and $(\theta^{-1})_{\bar{\imath}k}\theta^{k\bar{\jmath}} = \delta^{\bar{\jmath}}_{\bar{\imath}}$.

i.e., $A_{z^i} = A_{\bar{z}^i} = 0$ yields a (trivial) solution of (A.23).

New solutions X^1 may be obtained by shift operator "dressing" of the solutions X^0, namely

$$X^1_{z^1} = SX^0_{z^1}S^\dagger, \quad X^1_{\bar{z}^1} = SX^0_{\bar{z}^1}S^\dagger, \tag{A.25a}$$

$$X^1_{z^2} = SX^0_{z^2}S^\dagger, \quad X^1_{\bar{z}^2} = SX^0_{\bar{z}^2}S^\dagger. \tag{A.25b}$$

In these expressions, S and $S^\dagger$ are shift operators acting on the two-oscillator Fock space $\mathcal{H}$ according to

$$S^\dagger S = 1, \quad SS^\dagger = 1 - P_0, \quad P_0 S = S^\dagger P_0 = 0. \tag{A.26}$$

Apparently, the representation of S on $\mathcal{H}$ is not unique (see, e.g., [1, 110] for various explicit forms of S and $S^\dagger$). Here, P_0 denotes the projector onto the ground state $|0,0\rangle$ of the Fock space $\mathcal{H}$:

$$P_0 = |0,0\rangle\langle 0,0|. \tag{A.27}$$

The field strength for such configurations turns out to be of the form

$$F_{z^i\bar{z}^i} = [X^1_{z^i}, X^1_{\bar{z}^i}] - \mathrm{i}(\theta^{-1})_{i\bar{\imath}} = -\mathrm{i}(\theta^{-1})_{i\bar{\imath}} P_0 = -\frac{1}{2\theta}P_0, \quad i \in \{1,2\}. \tag{A.28}$$

This coincides with the solution first presented in [1] for the euclidean case, namely on $\mathbb{R}^4$. The action for this type of solution is known to be finite; this is also the case here:

$$S_1 = -\frac{1}{2g^2_{YM}}(2\pi\theta)^2 \mathrm{Tr}_{\mathcal{H}}\, F_{z^i\bar{z}^j}F^{z^i\bar{z}^j} = \frac{4\pi^2}{g^2_{YM}}. \tag{A.29}$$

In the context of D-branes, solutions of type (A.28) have been interpreted as a D-brane of codimension four sitting at the origin of a space-time filling D-brane [1]. This may be transferred to our case.

Appendix B

String theory conventions

B.1 Bosonic and N=1 string theories

In this section, we note the conventions used for bosonic and N=1 strings throughout the text. In general, references [64, 65, 121, 146, 147, 93] provide detailed introductions into the basics of (super)string theory.

Spacetime and world-sheet conventions. We formulate critical string theory on flat $\mathbb{R}^{1,D-1}$ with $D = 26$ in the case of bosonic string theory and $D = 10$ in the case of N=1 superstring theory. The metric on these spacetimes is taken in the "mostly plus" convention,

$$(\eta_{\mu\nu}) = \mathrm{diag}(-1, +1, \ldots, +1)\,, \tag{B.1}$$

the spacetime indices μ, ν running from 0 to $D-1$. Spacetime indices are lowered and raised with the help of $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$.

The Minkowski world-sheet is parametrized locally by coordinates σ and τ; in superconformal gauge, the metric on the world-sheet is $ds^2 \propto -d\tau^2 + d\sigma^2$. The light-cone coordinates are $\sigma^\pm = \tau \pm \sigma$. The Wick rotation $\tau \mapsto -\mathrm{i}\tau$ maps

$$\sigma^+ \mapsto -\mathrm{i}(\tau + \mathrm{i}\sigma) =: -\mathrm{i}z'\,, \tag{B.2a}$$

$$\sigma^- \mapsto -\mathrm{i}(\tau - \mathrm{i}\sigma) = -\mathrm{i}\bar{z}'\,; \tag{B.2b}$$

in coordinates $z := e^{z'}$ on the complex plane the world-sheet of an open string is mapped into the upper half plane (points (σ, τ) with $\sigma \in [0, \pi]$ are mapped to points $z \in \{\zeta \in \mathbb{C} |\, \mathrm{Im}\ \zeta \geq 0\}$).

World-sheet actions. In this thesis, we restrict to open strings. In the case of superstrings, they should be in the Neveu-Schwarz sector. We apply the doubling trick so that all fields are (holomorphically) defined on the double cover $\tilde{\Sigma}$ of the world-sheet Σ. On Euclidean world-sheets, the matter action for N=1 supersymmetric strings in superconformal gauge takes the following form:

$$S_{\mathrm{mat}} = \frac{1}{4\pi} \int\limits_{\tilde{\Sigma}} dz \wedge d\bar{z} \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu \right)\,. \tag{B.3}$$

As usual, X^μ is the bosonic embedding coordinate (formally of weight 0), ψ^μ is its superpartner, a Majorana-Weyl spinor[1] of weight $\frac{1}{2}$. The ψ^μ form a fermionic first order system with $\lambda = \frac{1}{2}$.[2] Each boson X^μ contributes $+1$ to the central charge of the conformal field theory, each fermion ψ^μ contributes $+\frac{1}{2}$. For an anomaly-free theory, the central charge of the matter system has to be compensated by the central charge of the Faddeev-Popov ghosts for the reparametrization symmetry on the world-sheet; their action reads

$$S_{\text{gh}} = \frac{1}{2\pi} \int\limits_{\tilde{\Sigma}} dz \wedge d\bar{z} \left(b\bar{\partial}c + \beta\bar{\partial}\gamma \right) . \tag{B.4}$$

The reparametrization ghosts b and c also make up a fermionic first order system with $\lambda = 2$, i. e., their respective weights are 2 and 1, respectively. Their world-sheet superpartners β and γ similarly make up a bosonic first order system with $\lambda = \frac{3}{2}$, their respective weights are $\frac{3}{2}$ and $-\frac{1}{2}$. The action for bosonic strings in conformal gauge can be obtained from (B.3) and (B.4) by setting ψ^μ, β, and γ formally to zero. Since the central charge of the bc system is -26 and of the $\beta\gamma$ system is $+11$, we obtain the above critical dimensions D from the condition that the total central charge should vanish.

Mode expansions. The Minkowski counterparts of both actions are real, if we choose all fields to be hermitean. The doubling trick implies that all fields are holomorphically (apart from the punctures) defined on the entire world-sheet $\tilde{\Sigma}$. We will need the mode expansions of the following fields in σ, τ:

$$X^\mu(\sigma,\tau) = x_0^\mu - 2\mathrm{i}\alpha' p_0^\mu \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{m\in\mathbb{Z}, m\neq 0} \frac{\alpha_m^\mu}{m} e^{-m\tau} \cos m\sigma \,, \tag{B.5a}$$

$$c^\pm(\sigma,\tau) = \sum_{m\in\mathbb{Z}} c_m e^{-m(\tau\pm\mathrm{i}\sigma)} = c(\sigma,\tau) \pm \mathrm{i}\pi_b(\sigma,\tau) \,, \tag{B.5b}$$

$$b^\pm(\sigma,\tau) = \sum_{m\in\mathbb{Z}} b_m e^{-m(\tau\pm\mathrm{i}\sigma)} = \pi_c(\sigma,\tau) \pm \mathrm{i}b(\sigma,\tau) \,. \tag{B.5c}$$

The $c^\pm$ are left- and right-movers of the c-ghosts, respectively; the same holds true for the $b^\pm$ in the case of the b-antighosts. The π_b and π_c are canonically conjugate momenta to the b and c. The mode expansion for X^μ is chosen such that the X^μ-direction is a Neumann-Neumann direction. As usual, the momentum eigenstate is denoted by $|0,p\rangle$ with $\alpha^\mu_{m>0}|0,p\rangle = 0$. For the formulation of overlap equations in string field theory, eq. (B.5a) is often rewritten as

$$X^\mu(\sigma) = x_0^\mu + \sqrt{2} \sum_{m\in\mathbb{Z}, m\neq 0} x_m^\mu \cos m\sigma \tag{B.6}$$

at $\tau = 0$. Obviously, $x_m^\mu = \frac{\mathrm{i}\sqrt{\alpha'}}{m}\left(\alpha_m^\mu - \alpha_{-m}^\mu\right)$.

[1] Strictly speaking, they are only Majorana-Weyl spinors on a *Minkowskian* world-sheet.

[2] The parameter λ is equal to the conformal weight of the conformal field b in a first order system bc.

Operator product expansions. The actions are normalized in such a way that the fundamental operator product expansions (OPEs) for the fields read

$$X^\mu(z)X^\nu(w) \sim -\frac{\alpha'}{2}\ln(z-w)\,, \tag{B.7a}$$

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}\,, \tag{B.7b}$$

$$c(z)b(w) \sim \frac{1}{z-w}\,, \tag{B.7c}$$

$$\gamma(z)\beta(w) \sim \frac{1}{z-w}\,, \tag{B.7d}$$

all other OPEs are regular. In terms of the oscillators from eq. (B.5), the OPEs (B.7a) and (B.7c) correspond to the (anti-)commutation relations $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}$, $[x^\mu_0, p^\nu_0] = \mathrm{i}\eta^{\mu\nu}$, and $\{b_m, c_n\} = \delta_{m+n,0}$. It is common practice to introduce oscillators $a^\mu_m := \sqrt{m}\alpha^\mu_m$ and $a^{\dagger\mu}_m := \sqrt{m}\alpha^\mu_{-m}$ for $m > 0$ with normalized commutation relations $[a^\mu_m, a^{\dagger\nu}_n] = \eta^{\mu\nu}\delta_{mn}$; the zero-modes are splitted as $x^\mu_0 = \mathrm{i}\sqrt{\frac{\alpha'}{2}}(a^\mu_0 - a^{\mu\dagger}_0)$ and $p^\mu_0 = \frac{1}{\sqrt{2\alpha'}}(a^\mu_0 + a^{\mu\dagger}_0)$.

Ghost currents and vacua. W.r.t. to the $U(1)$ current for the bc system,[3]

$$J_{bc} = -bc\,, \tag{B.8}$$

the field b has charge -1, whereas c has charge $+1$. Similarly, β and γ have charges -1 and $+1$ w.r.t.

$$J_{\beta\gamma} = -\beta\gamma\,. \tag{B.9}$$

The background charge for the fermionic bc system is $Q_{bc} = 1 - 2\times 2 = -3$, the bosonic $\beta\gamma$ system has background charge $Q_{\beta\gamma} = -(1 - 2\times\frac{3}{2}) = 2$. The charges of the operators in a correlation function on a disk have to add up to $-Q$ in both conformal field theories in order to give a nonvanishing result.

The $SL(2,\mathbb{R})$ invariant vacuum $|0\rangle_{bc}$ is annihilated by $b_{m\geq -1}$ and $c_{m\geq 2}$. The vacuum $|{\downarrow}\rangle_{bc} := c_1|0\rangle_{bc}$ is killed by $b_{m\geq 0}$ and $c_{m\geq 1}$; and the vacuum $|{\uparrow}\rangle_{bc} := c_0|{\downarrow}\rangle_{bc}$ is annilihated by $b_{m\geq 1}$ and $c_{m\geq 0}$. The easiest way to deal with the anomalies of interaction vertices in string field theory is by using the following convention for normal ordering the zero-mode of J_{bc}:

$$J_{bc,0} = \sum_{m=1}^{\infty}(c_{-m}b_m - b_{-m}c_m) + c_0 b_0 - \tfrac{1}{2}\,. \tag{B.10}$$

If we denote the $J_{bc,0}$-eigenvalue of ket-states by $\#_{bc}$, this is a "symmetric" choice with $\#_{bc}(|{\uparrow}\rangle_{bc}) = \frac{1}{2}$, $\#_{bc}(|{\downarrow}\rangle_{bc}) = -\frac{1}{2}$ and $\#_{bc}(|0\rangle_{bc}) = -\frac{3}{2}$. Since b and c are Grassmann-odd, the Grassmannality of a state which consists of α^μ_{-m} and b_{-m}, c_{-m} acting onto $|0\rangle_{bc}$ is given by its $\#_{bc}$-grading plus 3/2. Note, however, that the charge of an operator (rather than a state) is measured by the commutator of $\#_{bc}$ with this operator (or, alternatively, by the contour integral of J_{bc} around this operator); therefore, the choice of different normal ordering constants does

[3] We always assume normal ordering for composite operators.

not affect ghost numbers of operators. An operator in the combined X-bc conformal field theory is Grassmann-even if and only if its bc-charge is even.

The "symmetric" choice (B.10) is the unique choice for which the zero-mode of the current J_{bc} is antihermitean. Choosing a particular normal ordering constant in $J_{bc,0}$ basically fixes the ghost number of dual states. Let us briefly review this argument for a general fermionic first order system with $U(1)$ current J, energy-momentum tensor T and background charge Q:

Using the mode expansions

$$T(z) = \sum \frac{L_n}{z^{n+2}}\,, \tag{B.11}$$

$$J(z) = \sum \frac{J_m}{z^{m+1}}\,, \tag{B.12}$$

one can show that the commutation relations between the modes read [121]

$$[L_n, J_m] = \frac{Q}{2} m^2 \delta_{m+n,0} - m J_{m+n}\,. \tag{B.13}$$

The normalization of J_0 here is chosen in such a way that the $SL(2,\mathbb{R})$ invariant vacuum has ghost number $\frac{Q}{2}$. This is the above symmetric choice generalized to arbitrary fermionic first order systems. For such systems, $L_n^\dagger = L_{-n}$ and $J_m^\dagger = -J_{-m}$ for $m \neq 0$. The antihermiticity of J_0 can then be inferred from eq. (B.13):

$$J_0^\dagger = -[L_{-1}, J_1]^\dagger + \frac{Q}{2} = -[L_1, J_{-1}] + \frac{Q}{2} = -J_0\,. \tag{B.14}$$

Now let $\mathcal{O}_p$ be an operator with $U(1)$ charge p, i. e., $[J_0, \mathcal{O}_p] = p\mathcal{O}_p$, and $|q\rangle$ and $|q'\rangle$ two states of charge q and q', respectively. Then from

$$p\,\langle q'|\mathcal{O}_p|q\rangle = \langle q'|[J_0, \mathcal{O}_p]|q\rangle = -(q+q')\langle q'|\mathcal{O}_p|q\rangle \tag{B.15}$$

we can conclude that we have to insert an operator of charge $p = -q - q'$ in order to get a nonvanishing result. Thus, we can normalize the nonvanishing inner products as $\langle -q|q\rangle = 1$, e. g. $\langle\downarrow|\uparrow\rangle = 1$.

Bosonization of superghosts. For many purposes it turns out to be useful to reexpress the $\beta\gamma$ system in terms of a bosonic field ϕ of weight 0 and the OPE

$$\phi(z)\phi(w) \sim -\ln(z-w) \tag{B.16}$$

and a fermionic first order system with $\lambda = 1$ consisting of a weight 0 field ξ and a conjugate field η of weight 1; their OPE is given by

$$\eta(z)\xi(w) \sim \frac{1}{z-w}\,. \tag{B.17}$$

Then it is easy to check that the $\beta\gamma$ sytem can be reproduced by setting

$$\beta = e^{-\phi}\partial\xi\,, \qquad \gamma = \eta e^{\phi}\,. \tag{B.18}$$

The $U(1)$ current associated to the boson ϕ is $J_\phi = -\partial\phi$. The $\eta\xi$ system has its own $U(1)$ current $J_{\eta\xi} = \xi\eta$ under which ξ carries one positive and η one negative unit of charge. It features a background charge $Q_{\eta\xi} = 1 - 2 \times 1 = -1$; on a disk, e.g., one ξ_0-insertion (and no explicit further ξ- and η-insertions) saturates the anomaly of $J_{\eta\xi}$. Vertex operators constructed from β and γ do not contain ξ_0 (cf. eq. (B.18)); therefore, we have effectively enlarged the so-called *small Hilbert space* to the *large Hilbert space* containing ξ_0 [52]. Witten's cubic superstring field theory can be formulated in the small Hilbert space, Berkovits' nonpolynomial superstring field theory, however, requires the large Hilbert space. With the help of these two new currents, we can define linear combinations

$$J_{\mathrm{gh}} = J_{bc} - J_{\eta\xi} = -bc - \xi\eta\,, \tag{B.19}$$

$$J_{\mathrm{pic}} = J_\phi + J_{\eta\xi} = -\partial\phi + \xi\eta\,. \tag{B.20}$$

They measure the total ghost number and the picture charge, respectively. The weights, ghost and picture numbers of the world-sheet fields are collected in table B.1.

The mode expansions for η and ξ read

$$\xi^\pm(\sigma,\tau) = \sum_{m\in\mathbb{Z}} \xi_m e^{-m(\tau\pm \mathrm{i}\sigma)} = \xi(\sigma,\tau) \pm \mathrm{i}\pi_\eta(\sigma,\tau)\,, \tag{B.21a}$$

$$\eta^\pm(\sigma,\tau) = \sum_{m\in\mathbb{Z}} \eta_m e^{-m(\tau\pm \mathrm{i}\sigma)} = \pi_\xi(\sigma,\tau) \pm \mathrm{i}\eta(\sigma,\tau)\,. \tag{B.21b}$$

The interpretation of the $\xi^\pm$, $\eta^\pm$, π_η, and π_ξ are as in the case of the bc system. In terms of the above oscillators, the OPE (B.17) takes the form $\{\xi_m, \eta_n\} = \delta_{m+n,0}$. There are two vacua of the same energy, the $SL(2,\mathbb{R})$ invariant vacuum $|0\rangle_{\eta\xi} =: |{\downarrow}\rangle$ with $\xi_{m>0}|{\downarrow}\rangle = 0$ and $\eta_{m\geq 0}|{\downarrow}\rangle = 0$, and the vacuum $|{\uparrow}\rangle := \xi_0|{\downarrow}\rangle$ with $\xi_{m\geq 0}|{\downarrow}\rangle = 0$ and $\eta_{m>0}|{\downarrow}\rangle = 0$. For the zero-mode of $J_{\eta\xi}$, we choose the following normal ordering:

$$J_{\eta\xi,0} = \sum_{m=1}^{\infty} (\xi_{-m}\eta_m - \eta_{-m}\xi_m) + \xi_0\eta_0 - \tfrac{1}{2}\,. \tag{B.22}$$

We denote its eigenvalue on ket-states by $\#_{\eta\xi}$. Then, the above choice is symmetric in the sense that $\#_{\eta\xi}(|{\downarrow}\rangle) = -\frac{1}{2}$ and $\#_{\eta\xi}(|{\uparrow}\rangle) = \frac{1}{2}$.

N=1 superconformal algebras and picture changing. The energy-momentum tensor for the matter and ghost parts are

$$T_{\mathrm{mat}} = -\tfrac{1}{\alpha'}\partial X^\mu \partial X_\mu - \tfrac{1}{2}\psi^\mu \partial\psi_\mu\,, \tag{B.23}$$

$$T_{\mathrm{gh}} = -(\partial b)c - 2b\partial c - \tfrac{1}{2}(\partial\beta)\gamma - \tfrac{3}{2}\beta\partial\gamma\,. \tag{B.24}$$

Together with

$$G_{\mathrm{mat}} = \mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\psi^\mu \partial X_\mu\,, \tag{B.25}$$

$$G_{\mathrm{gh}} = (\partial\beta)c + \tfrac{3}{2}\beta\partial c - 2b\gamma \tag{B.26}$$

operator	∂X^μ	ψ^μ	b	c	β	γ	$e^{\ell\phi}$	ξ	η	T	G	Q	$\mathcal{X}$	Y
weight h	1	$\frac{1}{2}$	2	-1	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}\ell^2-\ell$	0	1	2	$\frac{3}{2}$	0	0	0
ghost no. $\#_{\mathrm{gh}}$	0	0	0	0	-1	$+1$	0	-1	$+1$	0	0	$+1$	0	0
pict. no. $\#_{\mathrm{pic}}$	0	0	0	0	0	0	ℓ	$+1$	-1	0	0	0	$+1$	-1
ws. statistics	B	F	F	F	B	B	F (ℓ odd) B (ℓ even)	F	F	B	F	F	B	B

Table B.1. *Weights, ghost and picture numbers and world-sheet statistics of fields and operators for N=1 strings.*

they form N=1 superconformal algebras (SCAs). The combined matter-ghost system has a residual BRST symmetry under transformations generated by

$$J_{\mathrm{BRST}} = cT_{\mathrm{mat}} + \gamma G_{\mathrm{mat}} + bc\partial c + \tfrac{3}{4}(\partial c)\beta\gamma + \tfrac{1}{4}c(\partial\beta)\gamma - \tfrac{3}{4}c\beta\partial\gamma - b\gamma^2\,. \tag{B.27}$$

Its zero-mode is the BRST-operator Q. The picture raising operator is defined as

$$\begin{aligned}\mathcal{X}(z) &= \oint \frac{dw}{2\pi \mathrm{i}} J_{\mathrm{BRST}}(w)\xi(z) \\ &= e^{\phi}G_{\mathrm{mat}}(z) + be^{2\phi}\partial\eta(z) + \partial(be^{2\phi}\eta)(z) + c\partial\xi(z)\,.\end{aligned} \tag{B.28}$$

The integration contour in the first line encircles z. The picture raising operator is a conformal field of weight 0. In the large Hilbert space, it has an inverse [133]

$$Y(z) = c\partial\xi e^{-2\phi}\,. \tag{B.29}$$

It satisfies $\lim_{z\to w}\mathcal{X}(z)Y(w) = \lim_{z\to w}Y(z)\mathcal{X}(w) = 1$.

B.2 N=2 string theory

In this section, we note the conventions for string theory with extended N=2 world-sheet supersymmetry used throughout the text. In general, [125, 104, 89, 105, 137] provide detailed introductions to this topic. However, we will mostly deal with nonpolynomial string field theory for N=2 strings in this thesis, for which a twisted version of the N=2 string is needed. Therefore, we omit N=2 superconformal ghosts and many of the intricacies connected to them almost completely from our discussion.

It should, however, be mentioned that just as for bosonic and for superstrings, the critical dimension of untwisted N=2 string theory can be determined from anomaly considerations. Namely, the N=2 world-sheet reparametrization superghost system has a central charge of -6, which has to be compensated by matter fields. It turns out that this is the case for four real bosons X^μ and four $so(1,1)$ Dirac spinors ψ^μ (their four NSR partners, $\mu \in \{1,\dots,4\}$). It was shown that an anomaly-free quantization is only possible in signature $(2,2)$ or $(4,0)$ [38]. The theory in $4+0$ dimensions does not possess any on-shell degrees of freedom. Therefore, we restrict in this thesis to N=2 string theory on $\mathbb{R}^{2,2}$ or, equivalently, on $\mathbb{C}^{1,1}$.

Spacetime conventions. As stated above, we work on flat $\mathbb{C}^{1,1}$ with hermitean metric

$$(\eta_{a\bar{a}}) = \mathrm{diag}(-1,+1)\,, \tag{B.30}$$

for $a = 0,1$ and $\bar{a} = \bar{0},\bar{1}$. We will see, however, that nonpolynomial N=2 string field theory can be formulated in any even dimension, irrespective of the signature. Therefore, we mostly formulate twisted N=2 string field theory in an arbitrary even dimension D. In this case, η denotes an arbitrary Kähler metric on the spacetime manifold, which satisfies $\eta_{a\bar{a}}\eta^{a\bar{a}} = \frac{D}{2}$. All indices now run from 0 to $\frac{D}{2}-1$.

The world-sheet conventions agree with those for N=1 string theory.

World-sheet action. The Kähler spacetime is naturally parametrized by $D/2$ complex bosons Z^a,

$$\begin{aligned} Z^0 &:= X^1 + \mathrm{i}X^2\,, \quad Z^1 := X^3 + \mathrm{i}X^4\,, \quad \dots\,, \\ \bar{Z}^{\bar{0}} &= X^1 - \mathrm{i}X^2\,, \quad \bar{Z}^{\bar{1}} := X^3 - \mathrm{i}X^4\,, \quad \dots\,. \end{aligned} \tag{B.31}$$

Their NSR supersymmetry partners can be combined into

$$\psi^{+0} := \psi^1 + \mathrm{i}\psi^2\,, \quad \psi^{-\bar{0}} := \psi^1 - \mathrm{i}\psi^2\,, \quad \psi^{+1} := \psi^3 + \mathrm{i}\psi^4\,, \quad \psi^{-\bar{1}} := \psi^3 - \mathrm{i}\psi^4\,, \quad \dots\,. \tag{B.32}$$

In superconformal gauge, the world-sheet action for the matter fields reads (on a Euclidean world-sheet Σ with double cover $\tilde{\Sigma}$)

$$S = \frac{1}{4\pi\alpha'}\int_{\tilde{\Sigma}} dz\wedge d\bar{z}\,(\partial Z\cdot\bar{\partial}\bar{Z} + \bar{\partial}Z\cdot\partial\bar{Z}) + \frac{1}{8\pi}\int_{\tilde{\Sigma}} dz\wedge d\bar{z}\,(\psi^+\cdot\bar{\partial}\psi^- + \psi^-\cdot\bar{\partial}\psi^+)\,. \tag{B.33}$$

Here, the dots denote a contraction with $\eta_{a\bar{a}}$, e. g., $\partial Z\cdot\bar{\partial}\bar{Z} = \eta_{a\bar{a}}\partial Z^a\bar{\partial}\bar{Z}^{\bar{a}}$. For convenience, we will often omit the spacetime indices on fields.

Operator product expansions. The action is normalized in such a way that the operator product expansions are the ones which should be expected from the transition from real to complex coordinates:

$$Z^a(z)\bar{Z}^{\bar{a}}(w) \sim -\alpha'\eta^{a\bar{a}}\ln|z-w|^2\,, \tag{B.34a}$$

$$\psi^{+a}(z)\psi^{-\bar{a}}(w) \sim \frac{2\,\eta^{a\bar{a}}}{z-w}\,, \tag{B.34b}$$

cf. eqs. (B.7a) and (B.7b). With their help, it is easy to see that

$$\begin{aligned} T &= -\frac{1}{\alpha'}\partial Z\cdot\partial\bar{Z} - \frac{1}{4}(\psi^+\cdot\partial\psi^- + \psi^-\cdot\partial\psi^+)\,, \\ G^+ &= \frac{\mathrm{i}}{\sqrt{2\alpha'}}\psi^+\cdot\partial\bar{Z}\,, \quad G^- = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\psi^-\cdot\partial Z\,, \\ J &= \frac{1}{2}\psi^+\cdot\psi^- \end{aligned} \tag{B.35}$$

form an N=2 superconformal algebra with central charge $c = 3D/2$ (i. e., $c = 6$ in $D = 4$, as required from the ghosts).[4] The next step will be to embed this algebra into a small N=4 superconformal algebra and subsequently twist it such that the central charge of the twisted algebra vanishes. This will be done in appendix C.

[4]Note that the superscripts $\pm$ on each quantity label the charge under the $U(1)$ current J.

Appendix C

Small N=4 superconformal algebra

C.1 Realization in terms of N=1 multiplets

Generators of the N=2 SCA. For the construction of an anomaly-free N=1 superstring theory, ten massless matter multiplets are needed. A massless N=1 matter multiplet (X, ψ) consists of real bosons X (the ten string coordinates) and $so(1,1)$ Majorana spinors ψ (their superpartners, each splitting up into a left- and a right-handed Majorana-Weyl spinor). Due to the reparametrization invariance of the related supersymmetric sigma model its covariant quantization entails the introduction of world-sheet (anti)ghosts b and c and their superpartners β and γ as described in appendix B.1. The superghosts are bosonized according to (B.18). The realization of the previously mentioned N=2 superconformal algebra in terms of these multiplets is given by [28]

$$\begin{aligned} T &= T_{N=1} + \frac{1}{2}\partial(bc + \xi\eta)\,, \\ G^- = b\,, \quad G^+ &= J_{\mathrm{BRST}} + \partial^2 c + \partial(c\xi\eta)\,, \\ J &= J_{\mathrm{gh}}\,, \end{aligned} \tag{C.1}$$

where J_{gh} is the total ghost number current (B.19) and $T_{N=1} = T_{\mathrm{mat}} + T_{\mathrm{gh}}$ is the energy-momentum tensor given in (B.23) and (B.24). $T_{N=1}$ spans a Virasoro algebra with central charge $c = 0$ and J_{BRST} is the BRST current. These generators make up an N=2 superconformal algebra with $c = 6$, i. e., with the same central charge as the critical N=2 superstring.

Generators of the N=4 SCA. A straightforward method for calculating scattering amplitudes would be to introduce an N=2 superghost system with $c^{\mathrm{gh}} = -6$ compensating the positive central charge [27]. However, there is a more elegant method [28]: One can embed the N=2 algebra into a small N=4 algebra (as described above) and afterwards twist by the $U(1)$ current J. Then,

$$\begin{aligned} \widetilde{G}^- = [Q, b\xi] &= -b\mathcal{X} + \xi T_{N=1}\,, \quad \widetilde{G}^+ = \eta\,, \\ J^{--} = b\xi\,, &\quad J^{++} = c\eta\,, \end{aligned} \tag{C.2}$$

where $\mathcal{X}$ denotes the picture raising operator and Q is the BRST-operator of the original N=1 string theory. These generators together form a small N=4 superconformal algebra with $c = 6$. The twist [29] $T \to T + \frac{1}{2}\partial J$ in this case amounts to removing the term $\frac{1}{2}\partial(bc + \xi\eta)$ from T, thereby reproducing the original $T_{N=1}$ with $c = 0$. It shifts the weight of each conformal field by $-1/2$ of its $U(1)$ charge — in particular, G^+ and $\widetilde{G}^+$ after twisting become fermionic spin 1 generators which subsequently serve as BRST-like currents. Their zero-modes are exactly Q and η_0, respectively.

C.2 Realization in terms of N=2 multiplets

Generators of the N=4 SCA. In the critical dimension, the generators (B.35) form an N=2 superconformal algebra of central charge 6. As above, one way to obtain an anomaly-free theory would be to introduce N=2 superghosts compensating this central charge. Equivalently, the N=2 super Virasoro algebra can be embedded into a small N=4 superconformal algebra [28] which after twisting has vanishing central charge. Note that in this approach, contrary to the N=1 case, we do not need to introduce reparametrization ghosts. The N=4 extension is achieved by adding the currents

$$\begin{aligned} J^{++} &= \frac{1}{4}\varepsilon_{ab}\,\psi^{+a}\psi^{+b}\,, \quad & J^{--} &= \frac{1}{4}\varepsilon_{\bar{a}\bar{b}}\,\psi^{-\bar{a}}\psi^{-\bar{b}}\,, \\ \widetilde{G}^{+} &= \frac{\mathrm{i}}{\sqrt{2\alpha'}}\varepsilon_{ab}\,\psi^{+a}\partial Z^{b}\,, \quad & \widetilde{G}^{-} &= \frac{\mathrm{i}}{\sqrt{2\alpha'}}\varepsilon_{\bar{a}\bar{b}}\,\psi^{-\bar{a}}\partial \bar{Z}^{\bar{b}}\,, \end{aligned} \tag{C.3}$$

where we choose the convention that $\varepsilon_{01} = \varepsilon_{\bar{0}\bar{1}} = -\varepsilon^{01} = -\varepsilon^{\bar{0}\bar{1}} = 1$. In checking the operator product expansions, the identity

$$\varepsilon_{ab}\eta^{b\bar{a}}\varepsilon_{\bar{a}\bar{b}} = \eta_{a\bar{b}} \tag{C.4}$$

turns out to be useful. If the theory is formulated on general hyperkähler manifolds,[1] ε_{ab} and $\varepsilon_{\bar{a}\bar{b}}$ have to be replaced by the components of nondegenerate $(2,0)$- and $(0,2)$-forms in such a way that a similar relation is satisfied. Twisting $T \to T' := T + \frac{1}{2}\partial J$, the new energy-momentum tensor takes the form

$$T' = -\frac{1}{\alpha'}\eta_{a\bar{a}}\partial Z^{a}\partial \bar{Z}^{\bar{a}} - \frac{1}{2}\eta_{a\bar{a}}\psi^{-\bar{a}}\partial\psi^{+a}\,. \tag{C.5}$$

W.r.t. the twisted energy-momentum tensor, ψ^+ now has weight 0, and ψ^- has weight 1. They form a fermionic first order system with $\lambda = 1$, just as ξ and η in the N=1 case. In fact, our vertex constructions for the $\psi^+\psi^-$ system in chapter VI remain valid for the $\eta\xi$ system upon substitution $\psi^+ \mapsto \sqrt{2}\xi$ and $\psi^- \mapsto \sqrt{2}\eta$. The rescaling is necessary because of the different normalizations of the OPEs (B.17) and (B.34b).

[1] Untwisted N=4 world-sheet supersymmetry requires that the spacetime manifold is hyperkähler. It is unclear to me which properties the spacetime manifold has to satisfy for twisted N=4 supersymmetry. For the introduction of the nondegenerate $(2,0)$- and $(0,2)$-forms it is (naturally) necessary that its (real) dimension D is at least 4. String field theory anomalies for N=2 strings do not seem to impose further constraints on the dimension.

In analogy to the bc and the $\xi\eta$ systems, we can introduce momenta $\pi_{\psi^\pm}$ conjugate to $\psi^\pm$. As fields of integral weight, both ψ^+ and ψ^- are now integer moded:

$$\psi^+(\sigma,\tau) \pm \mathrm{i}\pi_{\psi^-}(\sigma,\tau) = \sum_{m\in\mathbb{Z}} \psi^+_m e^{-m(\tau\pm\mathrm{i}\sigma)}\,, \tag{C.6a}$$

$$\pi_{\psi^+}(\sigma,\tau) \pm \mathrm{i}\psi^-(\sigma,\tau) = \sum_{m\in\mathbb{Z}} \eta_m e^{-m(\tau\pm\mathrm{i}\sigma)}\,. \tag{C.6b}$$

The OPE (B.34b) implies $\{\psi^{+a}_m, \psi^{-\bar{a}}_n\} = \delta_{m+n,0}\eta^{a\bar{a}}$. The spin 0 field ψ^+ has a zero-mode on the sphere. Thus, in analogy to the $\eta\xi$ system (and the bc system) there are two vacua at the same energy level: the bosonic $SL(2,\mathbb{R})$-invariant vacuum $|0\rangle =: |{\downarrow}\rangle$ is annilihated by the Virasoro modes $L_{m\geq -1}$ and $\psi^+_{m>0}$, $\psi^-_{m\geq 0}$; its fermionic partner, $|{\uparrow}\rangle := \psi^+_0|{\downarrow}\rangle$, is annihilated by $\psi^+_{m\geq 0}$, $\psi^-_{m>0}$. To get nonvanishing fermionic correlation functions, we need one ψ^+-insertion, i.e., $\langle{\downarrow}|{\downarrow}\rangle = \langle{\uparrow}|{\uparrow}\rangle = 0$, $\langle{\downarrow}|{\uparrow}\rangle = 1$.

For the zero-mode of J in (B.35), we choose the following normal ordering:

$$J_0 = \tfrac{1}{2}\sum_{m=1}^{\infty}(\psi^+_{-m}\psi^-_m - \psi^-_{-m}\psi^+_m) + \tfrac{1}{2}\psi^+_0\psi^-_0 - \tfrac{D}{4}\,. \tag{C.7}$$

We denote its eigenvalue on ket-vectors by $\#_\psi$. Then, the above choice is symmetric in the sense that $\#_\psi(|{\downarrow}\rangle) = -\frac{D}{4}$ and $\#_\psi(|{\uparrow}\rangle) = \frac{D}{4}$. Since the normal ordering constant is immaterial in commutators, operators will still have integral $\#_\psi$-charge. This charge modulo two measures the Grassmannality of the operator.

Twisted action. The world-sheet action for the twisted conformal field theory takes the same form as the untwisted action in superconformal gauge, but now, the field contents is different:

$$S' = \frac{1}{4\pi\alpha'}\int_{\tilde{\Sigma}} dz\wedge d\bar{z}\,(\partial Z\cdot\bar{\partial}\bar{Z} + \bar{\partial}Z\cdot\partial\bar{Z}) + \frac{1}{4\pi}\int_{\tilde{\Sigma}} dz\wedge d\bar{z}\,\psi^+\cdot\bar{\partial}\psi^-\,. \tag{C.8}$$

Now, ψ^+ and ψ^- are fields of conformal weights 0 and 1, respectively. Again, the action is normalized in such a way that the operator product expansions are (B.34). To prove the reality of this action, we consider the fermionic part of its counterpart in Minkowski space,

$$S'_M = -\frac{\mathrm{i}}{2\pi}\int_{\Sigma} d\sigma\wedge d\tau\big(\psi^+(\sigma^-)\partial_+\psi^-(\sigma^-) + \tilde{\psi}^+(\sigma^+)\partial_-\tilde{\psi}^-(\sigma^+)\big)\,. \tag{C.9}$$

This is indeed real if we choose all fields to be hermitean. The fact that ψ^+ and ψ^- are no longer connected via complex conjugation dovetails with the different shifts of their respective weights.

It is easy to check that the action (C.8) is invariant under transformations generated by the currents in the twisted small N=4 SCA. On flat $\mathbb{R}^{2,2}$, they are seen to be:

$$T' = -\frac{1}{\alpha'}\partial Z \cdot \partial \bar{Z} - \frac{1}{2}\psi^- \cdot \partial\psi^+ \; : \qquad \begin{aligned} \delta\psi^{+a} &= \partial\psi^{+a}\,, \\ \delta\psi^{-\bar{a}} &= \partial\psi^{-\bar{a}}\,, \\ \delta Z^a &= \partial Z^a\,, \\ \delta\bar{Z}^{\bar{a}} &= \partial\bar{Z}^{\bar{a}}\,; \end{aligned} \tag{C.10a}$$

$$J = \frac{1}{2}\psi^+ \cdot \psi^- \; : \qquad \begin{aligned} \delta\psi^{+a} &= \psi^{+a}\,, \\ \delta\psi^{-\bar{a}} &= -\psi^{-\bar{a}}\,; \end{aligned} \tag{C.10b}$$

$$J^{++} = \frac{1}{4}\varepsilon_{ab}\,\psi^{+a}\psi^{+b} \; : \qquad \begin{aligned} \delta\psi^{-\bar{0}} &= \psi^{+1}\,, \\ \delta\psi^{-\bar{1}} &= \psi^{+0}\,; \end{aligned} \tag{C.10c}$$

$$J^{--} = \frac{1}{4}\varepsilon_{\bar{a}\bar{b}}\,\psi^{-\bar{a}}\psi^{-\bar{b}} \; : \qquad \begin{aligned} \delta\psi^{+0} &= \psi^{-\bar{1}}\,, \\ \delta\psi^{+1} &= \psi^{-\bar{0}}\,; \end{aligned} \tag{C.10d}$$

$$G^+ = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\psi^+ \cdot \partial\bar{Z} \; : \qquad \begin{aligned} \delta\psi^{-\bar{a}} &= \mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\partial\bar{Z}^{\bar{a}}\,, \\ \delta Z^a &= -\mathrm{i}\sqrt{\tfrac{\alpha'}{2}}\psi^{+a}\,; \end{aligned} \tag{C.10e}$$

$$G^- = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\psi^- \cdot \partial Z \; : \qquad \begin{aligned} \delta\psi^{+a} &= \mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\partial Z^a\,, \\ \delta\bar{Z}^{\bar{a}} &= -\mathrm{i}\sqrt{\tfrac{\alpha'}{2}}\psi^{-\bar{a}}\,; \end{aligned} \tag{C.10f}$$

$$\widetilde{G}^+ = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\varepsilon_{ab}\,\psi^{+a}\partial Z^b \; : \qquad \begin{aligned} \delta\psi^{-\bar{0}} &= -\mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\partial Z^1\,, \\ \delta\psi^{-\bar{1}} &= -\mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\partial Z^0\,, \\ \delta\bar{Z}^{\bar{0}} &= -\mathrm{i}\sqrt{\tfrac{\alpha'}{2}}\psi^{+1}\,, \\ \delta\bar{Z}^{\bar{1}} &= -\mathrm{i}\sqrt{\tfrac{\alpha'}{2}}\psi^{+0}\,; \end{aligned} \tag{C.10g}$$

and

$$\widetilde{G}^- = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\varepsilon_{\bar{a}\bar{b}}\,\psi^{-\bar{a}}\partial\bar{Z}^{\bar{b}} \; : \qquad \begin{aligned} \delta\psi^{+0} &= -\mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\partial Z^1\,, \\ \delta\psi^{+1} &= -\mathrm{i}\sqrt{\tfrac{2}{\alpha'}}\partial Z^0\,, \\ \delta Z^0 &= -\mathrm{i}\sqrt{\tfrac{\alpha'}{2}}\psi^{-\bar{1}}\,, \\ \delta Z^1 &= -\mathrm{i}\sqrt{\tfrac{\alpha'}{2}}\psi^{-\bar{0}}\,. \end{aligned} \tag{C.10h}$$

This might reassure us that the action (C.8) is indeed the correct one.

C.3 Twisted N=4 superconformal algebra

In this section, we give the operator product expansions of a twisted small N=4 superconformal algebra. This algebra is an essential ingredient of Berkovits' superstring field theory. The twist $T \mapsto T' = T + \frac{1}{2}\partial J$ reduces the conformal weights of all operators by half of their $U(1)$ charge, since an operator $\mathcal{O}$ of conformal weight h w.r.t. T and charge q has conformal weight $h - \frac{1}{2}q$

w.r.t. T',

$$\begin{aligned} T'(z)\mathcal{O}(w) &\sim \frac{h}{(z-w)^2}\,\mathcal{O}(w) + \frac{1}{z-w}\partial\mathcal{O}(w) + \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{q}{z-w}\mathcal{O}\right) \\ &= \frac{h-\frac{1}{2}q}{(z-w)^2}\,\mathcal{O}(w) + \frac{1}{z-w}\partial\mathcal{O}(w)\,. \end{aligned} \tag{C.11}$$

Therefore, the operator product expansions with T' read:

$$T'(z)\,T'(w) \sim \frac{2T'(w)}{(z-w)^2} + \frac{\partial T'(w)}{z-w}\,, \tag{C.12a}$$

$$T'(z)\,G^+(w) \sim \frac{G^+(w)}{(z-w)^2} + \frac{\partial G^+(w)}{z-w}\,, \tag{C.12b}$$

$$T'(z)\,\widetilde{G}^+(w) \sim \frac{\widetilde{G}^+(w)}{(z-w)^2} + \frac{\partial \widetilde{G}^+(w)}{z-w}\,, \tag{C.12c}$$

$$T'(z)\,G^-(w) \sim \frac{2G^-(w)}{(z-w)^2} + \frac{\partial G^-(w)}{z-w}\,, \tag{C.12d}$$

$$T'(z)\,\widetilde{G}^-(w) \sim \frac{2\widetilde{G}^-(w)}{(z-w)^2} + \frac{\partial \widetilde{G}^-(w)}{z-w}\,, \tag{C.12e}$$

$$T'(z)\,J^{++}(w) \sim \frac{\partial J^{++}(w)}{z-w}\,, \tag{C.12f}$$

$$T'(z)\,J^{--}(w) \sim \frac{2J^{--}(w)}{(z-w)^2} + \frac{\partial J^{--}(w)}{z-w}\,, \tag{C.12g}$$

Since J is not a conformal field, the OPE between T' and J is anomalous:

$$T'(z)J(w) \sim -\frac{D/2}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}\,. \tag{C.12h}$$

Checking this OPE for N=2 matter multiplets, each $\psi^+\psi^-$ first order system contributes -1 to the coefficient of the anomalous third order pole.[2]

Operator product expansions with the $U(1)$ current J determine the $U(1)$ charge of the generators, which is labeled by the superscripts $\pm$:

$$J(z)\,G^\pm(w) \sim \frac{G^\pm(w)}{z-w}\,, \tag{C.13a}$$

$$J(z)\,\tilde{G}^\pm(w) \sim \frac{\pm\tilde{G}^\pm(w)}{z-w}\,, \tag{C.13b}$$

$$J(z)\,J^{\pm\pm}(w) \sim \frac{\pm 2J^{\pm\pm}(w)}{z-w}\,. \tag{C.13c}$$

[2]We remark that the central charge of the twisted superconformal algebra is indeed zero, since the fourth order pole in the $T'T'$ OPE vanishes. For an untwisted superconformal algebra, the central charge also appears as the third order pole in the TJ, the G^+G^-, and the $\widetilde{G}^+\widetilde{G}^-$ OPEs and the second order pole in the JJ and the $J^{++}J^{--}$ OPEs. The fact that these poles do not vanish in the twisted superconformal algebra is a remnant of its untwisted ancestor. Since the G^+G^- OPE, e. g., does not change under the twist, one can read off the central charge of the original untwisted algebra from the third order pole of this OPE. The same holds true for all other OPEs which do not contain the energy-momentum tensor (as a factor); in fact, all these OPEs are identical to the OPEs in the untwisted algebra, with the additional replacement $\frac{D}{2} \to \frac{c}{3}$.

Eq. (C.13c) together with

$$J(z)\,J(w) \sim \frac{D/2}{(z-w)^2}\,, \tag{C.14a}$$

$$J^{++}(z)\,J^{--}(w) \sim \frac{D/4}{(z-w)^2} + \frac{J(w)}{z-w} \tag{C.14b}$$

is the statement that J, J^{++}, and J^{--} form an affine $su(2)$ Kac-Moody algebra of level 2 (for N=1 strings) or an affine $su(1,1)$ Kac-Moody algebra of level $\frac{D}{2}$ (for N=2 strings), respectively.

Taking the operator product with J^{++} raises the $U(1)$ charge by two units:

$$J^{++}(z)\,G^{-}(w) \sim \frac{\widetilde{G}^{+}(w)}{z-w}\,, \tag{C.15a}$$

$$J^{++}(z)\,\widetilde{G}^{-}(w) \sim \frac{G^{+}(w)}{z-w}\,; \tag{C.15b}$$

likewise, the operator product with J^{--} lowers the $U(1)$ charge by two units:

$$J^{--}(z)\,G^{+}(w) \sim \frac{\widetilde{G}^{-}(w)}{z-w}\,, \tag{C.16a}$$

$$J^{--}(z)\,\widetilde{G}^{+}(w) \sim \frac{G^{-}(w)}{z-w}\,. \tag{C.16b}$$

The operator products of J^{++} with $G^{+}, \widetilde{G}^{+}$ and of J^{--} with $G^{-}, \widetilde{G}^{-}$ are regular.

Finally, we have

$$G^{+}(z)\,G^{-}(w) \sim \frac{D/2}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T'(w)}{z-w}\,, \tag{C.17a}$$

$$\widetilde{G}^{+}(z)\,\widetilde{G}^{-}(w) \sim -\frac{D/2}{(z-w)^3} - \frac{J(w)}{(z-w)^2} - \frac{T'(w)}{z-w}\,, \tag{C.17b}$$

$$G^{+}(z)\,\widetilde{G}^{+}(w) \sim -\frac{2J^{++}(w)}{(z-w)^2} - \frac{\partial J^{++}(w)}{z-w}\,, \tag{C.17c}$$

$$G^{-}(z)\,\widetilde{G}^{-}(w) \sim -\frac{2J^{--}(w)}{(z-w)^2} - \frac{\partial J^{--}(w)}{z-w}\,. \tag{C.17d}$$

The operator products of G^{+} with $\widetilde{G}^{-}$ and of G^{-} with $\widetilde{G}^{+}$ are regular.

Appendix D

A cohomology theorem

In this appendix, we want to prove a theorem on the cohomology of a complex of vector spaces. It assures that the cohomology is trivial iff there exists a contracting homotopy map. The theorem in this thesis applies to the cases of BRST cohomology in the large Hilbert space, the cohomology of η_0 in the large Hilbert space, the (nonvanishing) cohomology of linear combinations of Q and η_0, the vanishing cohomology of the vacuum BRST operator (already in the small Hilbert space), and many more. It is based on ideas in [48].

D.1 Proof of the theorem

Theorem. *Let (V,d) denote the complex*

$$\dots \longrightarrow V_{n-1} \xrightarrow{d^{n-1}} V_n \xrightarrow{d^n} V_{n+1} \xrightarrow{d^{n+1}} V_{n+2} \xrightarrow{d^{n+2}} \dots$$

of vector spaces. Then the cohomology of V is trivial iff there exists a homomorphism $k=(k_n)$ of complexes of vector spaces, $k_n\colon V_n \longrightarrow V_{n-1}$ such that

$$kd + dk = \mathrm{id} \tag{D.1}$$

holds.

The statement (D.1) is the customary abbreviation for

$$k^{n+1}d^n + d^{n-1}k^n = \mathrm{id}_{V_n}\,. \tag{D.2}$$

Proof. Let us denote the vector space of d^n-closed vectors by $Z^n(V)$. The quotient of V_n by $Z^n(V)$ will be abbreviated by $N^n(V)$; it consists of those vectors in V_n which are not d^n-closed and decomposes V_n into $V_n = Z^n(V) \oplus N^n(V)$.

"$\Longleftarrow$": First, let us assume that there exists a homomorphism k with the specified properties. Obviously, this guarantees that each element $v \in Z^n(V)$ is in the image of d^{n-1} since

$$v = (k^{n+1}d^n + d^{n-1}k^n)(v) = d^{n-1}\big(k^n(v)\big)\,.$$

Thus, $Z^n(V) = d^{n-1}V_{n-1}$, and the cohomology $H^n(V)$ is trivial.

"$\Longrightarrow$": Now, let us assume that the cohomology of the complex is trivial. Then we have to construct a map $k_n\colon V_n \longrightarrow V_{n-1}$ satisfying eq. (D.2). By definition of $N^{n-1}(V)$, the restriction $d^{n-1}|_{N^{n-1}(V)}\colon N^{n-1}(V) \longrightarrow Z^n(V)$ has trivial kernel. Furthermore, it is also surjective since $H^n(V) = \{0\}$, i. e., $Z^n(V) = d^{n-1}V_{n-1}$. Thus, it is an isomorphism; we call the inverse map $k^n|_{Z^n(V)}\colon Z^n(V) \longrightarrow N^{n-1}(V)$. The claim is now that on $Z^n(V)$, the homomorphisms $k^n|_{Z^n(V)}$ will satisfy eq. (D.2). Indeed, for an arbitrary $v \in Z^n(V)$, we have

$$(k^{n+1}d^n + d^{n-1}k^n)(v) = d^{n-1}k^n(v) = v\,.$$

Finally, we have to extend this map to $N^n(V)$. There is some freedom in doing this; we can choose any homomorphism $k^n|_{N^n(V)}\colon N^n(V) \longrightarrow Z^{n-1}(V)$. In principle, $k^n|_{N^n(V)} := 0$ will do the job. Namely, for an arbitrary $v' \in N^n(V)$ we have

$$(k^{n+1}d^n + d^{n-1}k^n)(v') = v' + d^{n-1}d^{n-2}(w) = v'\,,$$

where we have used that $k^n(v') \in Z^{n-1}(V)$ and since $H^{n-1}(V) = \{0\}$, there exists a $w \in V_{n-2}$ with $k^n(v') = d^{n-2}(w)$. $\square$

As a side remark, it should be mentioned that this ties in neatly into the mathematical terminology of homological algebra: What we have just proven is the statement that the identity on V is homotopy equivalent to the zero-map: Let (V, d), $(W, \tilde{d})$ be two complexes of objects in an abelian category and $f, g\colon V \longrightarrow W$ two morphisms between these complexes. Then, f and g are called homotopy equivalent if there exists a morphism $k = (k_n)$ with $k_n\colon V_n \longrightarrow V_{n-1}$ such that

$$kd + dk = f - g\,. \tag{D.3}$$

The corresponding diagram reads:

$$\begin{tikzcd}
\cdots \arrow[r, "d^{n-1}"] & V_n \arrow[r, "d^n"] \arrow[d, "f^n"', "g^n"] & V_{n+1} \arrow[r, "d^{n+1}"] \arrow[dl, "k^{n+1}"'] \arrow[d, "f^{n+1}"', "g^{n+1}"] & V_{n+2} \arrow[r, "d^{n+2}"] \arrow[dl, "k^{n+2}"'] \arrow[d, "f^{n+2}"', "g^{n+2}"] & \cdots \\
\cdots \arrow[r, "\tilde{d}^{n-1}"] & W_n \arrow[r, "\tilde{d}^n"] & W_{n+1} \arrow[r, "\tilde{d}^{n+1}"] & W_{n+2} \arrow[r, "\tilde{d}^{n+2}"] & \cdots
\end{tikzcd}$$

Whenever eq. (D.3) holds, f and g induce the same maps on the cohomology since cycles are in the kernel of d, therefore (D.3) reduces to $f - g = dk$.

The above theorem shows that for two identical complexes of vector spaces, $V = W$, the maps induced by $f = \mathrm{id}$ and $g = 0$ on the cohomology are the same: $\mathrm{id}^* = 0^*\colon H^*(V) \longrightarrow H^*(V)$ shows that the cohomology is indeed trivial. This means that f and g in this case are quasi-isomorphisms. In general abelian categories, quasi-isomorphic complexes have the same cohomology, but the converse is in general not true [179].

D.2 Applications

The theorem applies (amongst others) to the following cases:

- Since $\{\mathcal{C}_n, \mathcal{B}_n\} = 1$ (cf. eq. (III.87)), it is guaranteed that the vacuum BRST operator in cubic string field theory has vanishing cohomology. Therefore, there are no open string excitations around the tachyon vacuum.
- Since there exists an operator $K(z) := \xi Y(z)$ in the large Hilbert space with $\{K, Q\} = 1$ [133], the cohomology of the open superstring BRST operator is empty (only in the large Hilbert space). In particular, the cohomology of G^+ in Berkovits' superstring field theory is trivial.
- By virtue of $\{\eta_0, \xi_0\} = 1$ the cohomology of $\widetilde{G}^+$ in Berkovits' superstring field theory is empty.
- Non-trivial linear combinations $\widetilde{G}^+ + \lambda G^+$ of G^+ and $\widetilde{G}^+$ in Berkovits' superstring field theory have non-trivial cohomology since there is no operator $\mathcal{K}$ with $\{\mathcal{K}, G^+ + \lambda \widetilde{G}^+\} = 1$. This can be seen from the following argument: The only possible candidate is a non-trivial linear combination $\mathcal{K} = \alpha K + \beta \xi_0$. This ansatz leads to the requirement $\{\eta_0 + \lambda Q, \alpha K + \beta \xi_0\} = \alpha Y + \lambda \beta \mathcal{X} + \beta + \alpha \lambda \stackrel{!}{=} 1$. Due to the picture gradings of the involved operators, this equation possesses no solution.

Appendix E

Connections between field theory and string field theory

In this appendix, we display some connections between noncommutative field theories and string field theory. We review briefly the results of Douglas, Liu, Moore, and Zwiebach which show that the star algebra is isomorphic to a continuous tensor product of Heisenberg algebras. Therefore, star products may be computed just as ordinary Moyal products in a certain basis of the string field algebra. This facilitates the computation of star products considerably. The projectors introduced in section III.5 take a particularly simple form in this basis which can often be recognized from their field theory counterparts.

Furthermore, we show that the zero-mode part of the string field equation for N=2 strings contains the field theory self-duality equation. To this aim, string field theory in a B-field background is discussed (but without developing this theory much further); it turns out that, in the Seiberg-Witten limit, a discussion of the dressing approach leads to Lax operators acting only on the oscillator (nonzero-mode) part of a string field.

E.1 Witten and Moyal star products

In this section, we briefly review the main ideas of [45], where a Moyal formulation of the Witten vertex was introduced. It was shown that the star algebra is isomorphic to a continuous tensor product of Heisenberg algebras; in a certain basis of the string field algebra, the star product can be computed via ordinary Moyal products (cf. chapter II). These considerations were transferred to the world-sheet fermion, ghost and superghost sectors in [17, 4, 49, 18, 14, 15] and can be applied literally to the real and imaginary parts of the complexified world-sheet bosons in N=2 string field theory. Here, we omit spacetime indices and consider only one real boson for simplicity. To stress their operator properties, position and momentum operators are denoted with hats in this section.

E.1.1 Continuous basis

Discrete Fock space basis. The oscillator modes $\hat{x}_n$ and $\hat{p}_n$ of the open bosonic field operator (B.6) and its conjugate momentum,

$$X(\sigma) = \hat{x}_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{x}_n \cos n\sigma\,, \qquad \pi\widehat{P}(\sigma) = \hat{p}_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{p}_n \cos n\sigma\,, \tag{E.1}$$

can be related to the usual creation and annihilation operators a_n and $a_n^\dagger$ (with $[a_n, a_m^\dagger] = \delta_{nm}$) through

$$\hat{x}_n = \mathrm{i}\sqrt{\frac{\alpha'}{n}}\big(a_n - a_n^\dagger\big)\,, \qquad \hat{p}_n = \frac{1}{2}\sqrt{\frac{n}{\alpha'}}\big(a_n + a_n^\dagger\big) \tag{E.2}$$

for $n \neq 0$, and through

$$\hat{x}_0 = \mathrm{i}\sqrt{\frac{\alpha'}{2}}\big(a_0 - a_0^\dagger\big)\,, \qquad \hat{p}_0 = \frac{1}{\sqrt{2\alpha'}}\big(a_0 + a_0^\dagger\big) \tag{E.3}$$

for the zero-modes. From the vacuum $|0\rangle$ annihilated by all $a_n, n \geq 0$, one can construct a $\hat{p}_0$-eigenstate by

$$|p\rangle = \frac{1}{\pi^{1/4}} \exp\Big[-\frac{\alpha'}{2}p^2 + \sqrt{2\alpha'}a_0^\dagger p - \frac{1}{2}(a_0^\dagger)^2\Big]|0\rangle \tag{E.4}$$

so that $|0\rangle$ is the Bogoliubov transform of the vacuum $|p_0{=}0\rangle$ annihilated by all $a_n, n > 0$, and p_0. Let $\mathcal{H}$ denote the bosonic string theory Fock space created by the $a_n^\dagger$, $n > 0$, on the vacuum $|p_0{=}0\rangle$ endowed with the Witten star product. In terms of these oscillators, the star product of two zero-momentum states can be realized through the three-string vertex

$$|V_3\rangle = \exp\Big[-\frac{1}{2}\sum_{r,s}\sum_{m,n\geq 1} a_m^{r\dagger}(CM^{rs})_{mn}a_n^{s\dagger}\Big]|p{=}0\rangle_1 \otimes |p{=}0\rangle_2 \otimes |p{=}0\rangle_3\,, \tag{E.5}$$

where $r, s = 1, 2, 3$ label the strings, $C_{mn} = (-1)^m\delta_{mn}$, and the explicit form of the matrices $V^{rs} = CM^{rs}$ is given in F.1 [66, 152]. The matrices $(M^{rs})_{mn}$ were diagonalized in [155], the corresponding eigenvectors were used in [45] to define a new basis for $\mathcal{H}$ in which the Witten star product is computable through simple Moyal products:[1]

Eigenvectors of $\boldsymbol{M_{mn}^{rs}}$. The eigenvectors $v(\kappa) = (v_m(\kappa))$ of $(M^{rs})_{mn}$ are labeled by a continuous parameter $-\infty < \kappa < \infty$, i.e.,

$$\sum_{n=1}^{\infty} M_{mn}^{rs} v_n(\kappa) = \mu^{rs}(\kappa) v_m(\kappa)\,, \tag{E.6}$$

and given by the generating function

$$f_\kappa(z) = \sum_{n=1}^{\infty}\frac{v_n(\kappa)}{\sqrt{n}}z^n = \frac{1}{N(\kappa)^{1/2}}\frac{1}{\kappa}\big(1 - e^{-\kappa\tan^{-1} z}\big) \tag{E.7}$$

[1] In [17], this program was carried out for the three-string vertex of nonzero momentum constructed from the $|0\rangle$-vacuum.

with $N(\kappa) = \frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}$ and eigenvalues

$$\begin{aligned}
\mu(\kappa) = \mu^{11}(\kappa) &= -\frac{1}{1+2\cosh\frac{\pi\kappa}{2}}\,, \\
\mu^{12}(\kappa) &= \frac{1+2\cosh\frac{\pi\kappa}{2}+\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}\,, \\
\mu^{21}(\kappa) &= \frac{1+2\cosh\frac{\pi\kappa}{2}-\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}\,.
\end{aligned} \tag{E.8}$$

Under the action of the twist matrix C, the eigenvectors transform as

$$\sum_{n=1}^{\infty} C_{mn} v_n(\kappa) = -v_m(-\kappa)\,, \tag{E.9}$$

which suggests a separation of even and odd components according to

$$v_{2n}(-\kappa) = -v_{2n}(\kappa)\,, \qquad v_{2n+1}(-\kappa) = v_{2n+1}(\kappa)\,. \tag{E.10}$$

The eigenvectors can be shown to be orthogonal and complete,

$$\sum_{n=1}^{\infty} v_n(\kappa_1)v_n(\kappa_2) = \delta(\kappa_1-\kappa_2)\,, \tag{E.11}$$

$$\int_{-\infty}^{\infty} d\kappa\, v_m(\kappa)v_n(\kappa) = \delta_{mn}\,. \tag{E.12}$$

These relations allow for a separation of even and odd modes, respectively,

$$2\sum_{n=1}^{\infty} v_{2n}(\kappa_1)v_{2n}(\kappa_2) = \delta(\kappa_1-\kappa_2)\,, \qquad 2\sum_{n=1}^{\infty} v_{2n-1}(\kappa_1)v_{2n-1}(\kappa_2) = \delta(\kappa_1-\kappa_2)\,, \tag{E.13}$$

$$2\int_0^{\infty} d\kappa\, v_{2m}(\kappa)v_{2n}(\kappa) = \delta_{mn}\,, \qquad 2\int_0^{\infty} d\kappa\, v_{2m-1}(\kappa)v_{2n-1}(\kappa) = \delta_{mn}\,, \tag{E.14}$$

holding for $\kappa_1, \kappa_2 > 0$.

Continuous Fock space basis. In the above-mentioned basis in $\mathcal{H}$, $a_n^\dagger$ and a_n are replaced by

$$e_\kappa^\dagger := \sqrt{2}\sum_{n=1}^{\infty} v_{2n}(\kappa)a_{2n}^\dagger\,, \qquad o_\kappa^\dagger := -\sqrt{2}\mathrm{i}\sum_{n=1}^{\infty} v_{2n-1}(\kappa)a_{2n-1}^\dagger\,, \tag{E.15}$$

and their hermitean adjoints. Either eq. (E.11) or eqs. (E.13) may be used to invert these relations:

$$a_{2n}^\dagger = \sqrt{2}\int_0^{\infty} d\kappa\, v_{2n}(\kappa)e_\kappa^\dagger\,, \tag{E.16}$$

$$a_{2n-1}^\dagger = \sqrt{2}\mathrm{i}\int_0^{\infty} d\kappa\, v_{2n-1}(\kappa)e_\kappa^\dagger\,. \tag{E.17}$$

The new oscillators satisfy the commutation relations

$$[e_\kappa, e^\dagger_{\kappa'}] = [o_\kappa, o^\dagger_{\kappa'}] = \delta(\kappa - \kappa')\,, \qquad [o_\kappa, e_{\kappa'}] = [o_\kappa, e^\dagger_{\kappa'}] = 0\,. \tag{E.18}$$

It should be noted that $\{\kappa \geq 0\}$ is a "fundamental region" for the κ-indices; only in this region we have a pair of one annihilation and one creation operator $e_\kappa, e^\dagger_\kappa$ (or $o_\kappa, o^\dagger_\kappa$), cf. (E.15). The vacua for the e_κ and o_κ oscillators can be identified with the vacua for the a_n. Just as one can combine conjugate position and momentum operators $(\hat{x}_n, \hat{p}_n)$ out of harmonic oscillators $a_n, a^\dagger_n$ (cf. (E.2)), we can form new position and momentum operators $(\hat{x}_\kappa, \hat{q}_\kappa)$, $(\hat{y}_\kappa, \hat{l}_\kappa)$ out of the new oscillators (E.15) and their hermitean adjoints:

$$\hat{x}_\kappa := \frac{\mathrm{i}}{\sqrt{2}}(e_\kappa - e^\dagger_\kappa)\,, \qquad \hat{q}_\kappa := \frac{1}{\sqrt{2}}(e_\kappa + e^\dagger_\kappa)\,, \tag{E.19}$$

$$\hat{y}_\kappa := \frac{\mathrm{i}}{\sqrt{2}}(o_\kappa - o^\dagger_\kappa)\,, \qquad \hat{l}_\kappa := \frac{1}{\sqrt{2}}(o_\kappa + o^\dagger_\kappa)\,. \tag{E.20}$$

The eigenvalues of the new position operators will be denoted by $x(\kappa)$ and $y(\kappa)$, respectively. In analogy to eigenstates of the position operator $\hat{x}_0$ the eigenstates of $\hat{x}_\kappa, \hat{y}_\kappa$ take the following form:

$$\langle x, y| \equiv \langle x(\kappa), y(\kappa)| = \langle p_0{=}0| \exp\Big(- \int\limits_0^\infty d\kappa \, \Big[\frac{1}{2}\vec{X}(\kappa)\cdot\vec{X}(\kappa) - \sqrt{2}\mathrm{i}\vec{A}_\kappa \cdot \vec{X}(\kappa) - \vec{A}_\kappa \cdot \vec{A}_\kappa\Big]\Big) \tag{E.21}$$

with $\vec{X}(\kappa) = (x(\kappa), y(\kappa))$ and $\vec{A}_\kappa = (e_\kappa, o_\kappa)$. Note that these position eigenstates are independent of κ and satisfy

$$\langle x, y|x', y'\rangle = \delta\big[x(\kappa) - x'(\kappa)\big]\delta\big[y(\kappa) - y'(\kappa)\big]\,. \tag{E.22}$$

As mentioned in section III.2, string functionals are associated to states via

$$\Psi[x, y] = \langle x, y|\Psi\rangle\,. \tag{E.23}$$

By a comparison of the three-string vertex (E.5) in terms of the new oscillators with the integral kernel representation of the Moyal product between two (noncommutative) spacetime coordinates the authors of [45] identified the Witten star product with a continuous tensor product (over κ) of Moyal products (between $x(\kappa)$ and $y(\kappa)$) so that the following commutation relation holds:

$$[x(\kappa), y(\kappa')]_\star = 2\mathrm{i}\tanh\frac{\pi\kappa}{4}\delta(\kappa - \kappa') \equiv \mathrm{i}\,\theta(\kappa)\delta(\kappa - \kappa')\,. \tag{E.24}$$

This identification works in the bosonic matter sector only up to an (infinite) proportionality constant.

E.1.2 Projectors in the continuous basis

Sliver state. The sliver state in the matter sector was introduced in section III.5.1. Since eqs. (E.8) and (E.24) yield $\mu(\kappa) = \frac{\theta^2 - 4}{12 + \theta^2}$, the sliver state can be rewritten [35] in the continuous basis (E.15) as

$$|\Xi\rangle = N' \exp\left(-\frac{1}{2}\int_0^\infty d\kappa\, \frac{\theta-2}{\theta+2}(e_\kappa^\dagger e_\kappa^\dagger + o_\kappa^\dagger o_\kappa^\dagger)\right)|p_0{=}0\rangle \tag{E.25}$$

with a (divergent) normalization constant N'. Via (E.23), this corresponds to a functional

$$\Xi[x(\kappa), y(\kappa)] \equiv \langle x(\kappa), y(\kappa)|\Xi\rangle = N^D \exp\left(-\int_0^\infty d\kappa\, \frac{x(\kappa)^2 + y(\kappa)^2}{\theta(\kappa)}\right), \tag{E.26}$$

where the normalization factor N is given by

$$N = \exp\left(\frac{\log L}{2\pi}\int_0^\infty d\kappa\, \log\frac{16}{12+\theta^2}\right) \tag{E.27}$$

in D spacetime directions if one approximates the infinite K_1 matrix in [155] by an $L \times L$-matrix.

Butterfly states. The family of butterfly states was introduced in section III.5.2. The corresponding functional of $\vec{X}(\kappa) = \big(x(\kappa), y(\kappa)\big)$ is given by [53]

$$B_\alpha[x(\kappa), y(\kappa)] \equiv \langle x(\kappa), y(\kappa)|B_\alpha\rangle = \tilde{N} \exp\Big(-\frac{1}{2}\int_0^\infty d\kappa \vec{X}(\kappa) L(\kappa) \vec{X}(\kappa)\Big) \tag{E.28}$$

with some infinite normalization constant $\tilde{N}$ and the matrix

$$L(\kappa) = \coth\left(\frac{\pi\kappa}{4}\right)\begin{pmatrix} \tanh(\frac{\pi\kappa(2-\alpha)}{4\alpha}) & 0 \\ 0 & \coth(\frac{\pi\kappa(2-\alpha)}{4\alpha}) \end{pmatrix}. \tag{E.29}$$

In the limit $\alpha \to 0$ we get

$$L(\kappa) = \coth\left(\frac{\pi\kappa}{4}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.30}$$

in agreement with eqs. (E.24) and (E.26).

Translating the Moyal product for each κ to the operator formalism, cf. chapter II, it should be possible to classify all matter projectors in the star algebra.

E.2 Relation between field theory and string field theory discussions

In this section, we want to clarify the relation of the field theory discussion in chapter II to N=2 string field theory in the presence of a B-field. In the next paragraph, we will show that the zero-mode part of the string field theory equation of motion contains the field theory self-duality equation. After that, it will be argued that, in the Seiberg-Witten limit, an analogous discussion of the dressing approach (see chapter IV) leads to Lax operators acting only on the oscillator (nonzero-mode) part of a string field.[2]

[2]Since no special form of the star product was demanded in chapter IV, the discussion there is equally well valid in the case of nonvanishing B-field.

Field theory content of string field theory. Let us first briefly show that the gauge-fixed self-duality equation (II.74) is contained in the equation of motion of nonpolynomial string field theory for N=2 strings [19, 26, 107] (for finite α'). Its equation of motion is given in eq. (III.79). Recall the implementation of G^+ and $\widetilde{G}^+$ in terms of world-sheet fields $Z, \bar{Z}$ and $\psi^\pm$,

$$G^+ = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\eta_{i\bar{\jmath}}\psi^{+i}\partial\bar{Z}^j \qquad \text{and} \qquad \widetilde{G}^+ = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\varepsilon_{ij}\psi^{+i}\partial Z^j. \tag{E.31}$$

Here, $\eta_{i\bar{\jmath}}$ denotes the (pseudo-)Kähler spacetime metric with non-vanishing components $\eta_{1\bar{1}} = -\eta_{2\bar{2}} = 1$; for the antisymmetric tensor we choose the convention that $\varepsilon_{12} = -1$. For compatibility with our field theory conventions, we have chosen to denote holomorphic spacetime indices by $i, j \in \{1, 2\}$; the 2-direction is the time direction. Taking into account the bosonic operator product expansions

$$Z^i(w,\overline{w})\bar{Z}^j(w',\overline{w}') \sim -2\alpha'\eta^{i\bar{\jmath}}\ln|w-w'|^2, \quad Z^i(w,\overline{w})Z^j(w',\overline{w}') \sim 0, \quad \bar{Z}^i(w,\overline{w})\bar{Z}^j(w',\overline{w}') \sim 0, \tag{E.32}$$

we see that due to (III.71) G^+ and $\widetilde{G}^+$ act as derivatives on string fields containing only world-sheet bosons. Concretely, the equations of motion (III.79) for such string fields can be written as

$$\psi_0^{+1}\psi_0^{+2}\eta^{i\bar{\jmath}}\partial_{\bar{z}^j}\left(e^{-\Phi}\partial_{z^i}e^{\Phi}\right) + \ldots = 0. \tag{E.33}$$

Here, the bosonic zero-modes z^i and $\bar{z}^j$ coincide with the spacetime coordinates used in section II.4.3; ψ_0^{+i} denote the zero-modes of ψ^{+i}. The dots indicate the oscillator-dependent part of the equation of motion. The zero-mode part in (E.33) coincides with the remaining self-duality equation (II.74) in the Yang gauge (II.71) if we identify $h(z^i, \bar{z}^i) = e^{\Phi(z^i,\bar{z}^i)}$ (cf. (II.71b)).

In the same way, the linear equation given in (IV.18) includes the field theory Lax pair (II.72). For $A = e^{-\Phi}G^+e^{\Phi}$, it can be written as

$$\begin{aligned} 0 &= \{\widetilde{G}^+ + \lambda G^+ + \lambda A\}\Psi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(E.34)} \\ &= \frac{1}{2}\left\{\psi_0^{+1}\left(\partial_{\bar{z}^2} - \lambda\partial_{z^1} - \lambda e^{-\Phi}\partial_{z^1}e^{\Phi}\right) + \psi_0^{+2}\left(\partial_{\bar{z}^1} - \lambda\partial_{z^2} - \lambda e^{-\Phi}\partial_{z^2}e^{\Phi}\right) + \ldots\right\}\Psi. \end{aligned}$$

Because ψ_0^{+1} and ψ_0^{+2} are mutually independent, the zero-mode part coincides with (II.72).

Star product in the Seiberg-Witten limit. Now we will scrutinize the Seiberg-Witten limit of string field theory in a B-field background and argue that, in this limit, the above BRST-like operators G^+ and $\widetilde{G}^+$ act only on the oscillator-part of a string field e^{Φ}.[3] In order to avoid clumsy notation, we decompose the bosons Z^i into their real and imaginary parts again. The fermions will be left in complex notation since they cannot easily be decomposed in a similar way (recall that they have different weight after twisting). The metric will be taken either in real or in complex coordinates and denoted by the same symbol. In covariant string field theory, strings are glued with Witten's star product identifying the left half of the first string with the right half of the second string. This product is noncommutative even without a B-field background, but

[3] This argument is along the lines of [188, 160].

in order to make contact with the discussion of ncSDYM in this thesis, we switch on a constant B-field.[4] Witten's star product for our purposes will be computed in an oscillator representation of the three-vertex ${}_{123}\langle V_3|$ joining two string states $|A\rangle_1$ and $|B\rangle_2$ according to

$$\begin{aligned} {}_3\langle C| &= {}_{123}\langle V_3||A\rangle_1|B\rangle_2 && \text{(E.35a)} \\ \text{with}\quad {}_{123}\langle V_3| &= \int d^4p^{(1)}d^4p^{(2)}d^4p^{(3)}\delta(p^{(1)}+p^{(2)}+p^{(3)})(\langle 0,p|\otimes\langle 0,p|\otimes\langle 0,p|)\otimes \\ &\quad \big(\langle\uparrow\uparrow\downarrow| + \langle\uparrow\downarrow\uparrow| + \langle\downarrow\uparrow\uparrow|\big)\exp(-E_{\mathrm{mat}}) && \text{(E.35b)} \end{aligned}$$

with

$$\begin{aligned} E_{\mathrm{mat}} =& \frac{1}{2}\sum_{m,n\geq 1} G_{\mu\nu}a_m^{(r)\mu}V_{mn}^{rs}a_n^{(s)\nu} + \sqrt{\alpha'}\sum_{n\geq 1}G_{\mu\nu}p^{(r)\mu}V_{0n}^{rs}a_n^{(s)\nu} + \frac{\alpha'}{2}p^{(r)\mu}V_{00}^{rr}p^{(r)\nu}G_{\mu\nu} \\ &+ \frac{\mathrm{i}}{2}\theta_{\mu\nu}p^{(1)\mu}p^{(2)\nu} + \frac{1}{4}\sum_{k=1,l=0}^{\infty}\psi_k^{+(r)i}N_{kl}^{rs}\psi_l^{-(s)\bar{\jmath}}G_{i\bar{\jmath}}\,, \end{aligned} \qquad \text{(E.35c)}$$

where V_{nm}^{rs} and N_{nm}^{rs} are the Neumann coefficients for world-sheet bosons and fermions and $a_n^{(r)\mu}$ and $\psi_n^{\pm(r)\mu}$ denote the bosonic and fermionic oscillators of the r-th string in the μ-direction, respectively. The open string metric $G_{\mu\nu}$ was introduced in eq. (II.2). A summation over $r,s=1,2,3$ and over $\mu,\nu=1,\dots,4$ is implicit. This expression is valid for N=2 strings in a B-field background and is constructed analogously to [160, 32, 33].

We will now consider the properties of this vertex in the Seiberg-Witten limit $B\to\infty$ keeping fixed all other closed string parameters. For this purpose, we set $B=tB_0$ and take $t\to\infty$;[5] then, the effective open string parameters scale as [160]

$$G_{\mu\nu}\sim G_{0\mu\nu}t^2, \qquad \theta^{\mu\nu}\sim\theta_0^{\mu\nu}t^{-1}. \qquad \text{(E.36)}$$

In checking the operator product expansions for the N=4 superconformal algebra, the relations (C.4),

$$\varepsilon_{ij}\eta^{j\bar{\jmath}}\varepsilon_{\bar{\jmath}\bar{\imath}} = \eta_{i\bar{\imath}}\,, \qquad \text{(E.37)}$$

are needed. Since $\eta_{i\bar{\jmath}}$ in eq. (E.31) has to be replaced by $G_{i\bar{\jmath}}$ in the case of a nonvanishing B-field, the "(anti)holomorphic part of the volume element" ε_{ij} is changed to ϵ_{ij} with the same scaling behavior as G_{ij} (cf. (E.37)).

For the commutation relations

$$\begin{aligned} [a_m^\mu, a_n^\nu] &= \delta_{m+n,0}G^{\mu\nu}, && \text{(E.38a)} \\ \big[x^\mu, x^\nu\big] &= \mathrm{i}\theta^{\mu\nu}, && \text{(E.38b)} \\ \big[p^\mu, x^\nu\big] &= -\mathrm{i}G^{\mu\nu}, && \text{(E.38c)} \\ \{\psi_m^{+i}, \psi_n^{-\bar{\jmath}}\} &= \delta_{m+n,0}G^{i\bar{\jmath}} && \text{(E.38d)} \end{aligned}$$

[4]For N=2 strings, the B-field must be a (pseudo-)Kähler two-form [111].

[5]This limit is not to be confused with the large time limit in section II.6.

to be invariant in the large B-field limit, we have to introduce rescaled oscillators

$$\tilde{a}^\mu_m = t a^\mu_m \quad \text{for } m \neq 0, \tag{E.39a}$$
$$\tilde{p}^\mu = t^{3/2} p^\mu, \tag{E.39b}$$
$$\tilde{x}^\mu = t^{1/2} x^\mu, \tag{E.39c}$$
$$\tilde{\psi}^{+i}_m = t \psi^{+i}_m, \tag{E.39d}$$
$$\tilde{\psi}^{-\bar{\jmath}}_m = t \psi^{-\bar{\jmath}}_m. \tag{E.39e}$$

In terms of these modes, the matter part of the three-vertex (E.35c) takes the form

$$\begin{aligned} E_{\text{mat}} = & \sum_{m,n=1}^{\infty} \frac{1}{2} \tilde{a}^{(r)\mu}_n V^{rs}_{nm} \tilde{a}^{(s)\nu}_m G_{0\mu\nu} + \frac{\mathrm{i}}{2} \theta_{0\mu\nu} \tilde{p}^{(1)\mu} \tilde{p}^{(2)\nu} + \frac{1}{\sqrt{t}} \sum_{n=1}^{\infty} \sqrt{\alpha'}\ \tilde{p}^{(s)\mu} V^{rs}_{0n} \tilde{a}^{(r)\nu}_n G_{0\mu\nu} \\ & + \frac{\alpha'}{2t} \tilde{p}^{(r)\mu} V^{rr}_{00} \tilde{p}^{(r)\nu} G_{0\mu\nu} + \sum_{m=1,n=0}^{\infty} \frac{1}{4} \tilde{\psi}^{+(r)i}_n N^{rs}_{nm} \tilde{\psi}^{-(s)\bar{\jmath}}_m G_{0i\bar{\jmath}}\,. \end{aligned} \tag{E.40}$$

Observe that, for $t \to \infty$, the terms coupling a-oscillators and momenta p vanish. Thus, the string star algebra $\mathcal{A}$ factorizes into a zero-momentum part $\mathcal{A}_0$ spanned by $\tilde{p}$-, $\tilde{a}$-, and $\tilde{\psi}$-oscillators and a spacetime part $\mathcal{A}_1$ generated by $\tilde{x}^\mu$ [188]. The star product in $\mathcal{A}_1$ “degenerates” to the usual Moyal-Weyl product with constant noncommutativity parameter θ_0.

To read off the scaling behavior of the BRST-like operators G^+ and $\widetilde{G}^+$, we switch back to complex coordinates (labeled by roman spacetime indices) and exemplarily pick two typical terms from G^+ (up to an overall constant):

$$\psi^{+i}_0 p^{\bar{\jmath}} G_{i\bar{\jmath}} = \frac{1}{\sqrt{t}} \tilde{\psi}^{+i}_0 \tilde{p}^{\bar{\jmath}} G_{0i\bar{\jmath}} \tag{E.41}$$

and

$$\psi^{+i}_1 a^{\bar{\jmath}}_{-1} G_{i\bar{\jmath}} = \tilde{\psi}^{+i}_1 \tilde{a}^{\bar{\jmath}}_{-1} G_{0i\bar{\jmath}}. \tag{E.42}$$

Eq. (E.41) is the only term in G^+ acting onto $\mathcal{A}_1$; obviously it is suppressed for large t. Eq. (E.42) exemplifies a term in G^+ acting onto $\mathcal{A}_0$; it is independent of t. This affirms the claim that, in the large B-field limit, G^+ and $\widetilde{G}^+$ act only onto $\mathcal{A}_0$.

As a consequence, all BRST-like operators in the equations of chapter IV in the Seiberg-Witten limit act only onto the oscillator algebra $\mathcal{A}_1$. Thus, if we assume a factorized solution $\Phi = \Phi_0 \otimes \Phi_1$ with $\Phi_0 \in \mathcal{A}_0$ and $\Phi_1 \in \mathcal{A}_1$, the equation of motion can be restricted to $\mathcal{A}_0$ if Φ_1 is chosen to be a projector (i.e., $\Phi_1 \star \Phi_1 = \Phi_1$):

$$0 = \widetilde{G}^+(e^{-\Phi} G^+ e^{\Phi}) = \widetilde{G}^+(e^{-\Phi_0} G^+ e^{\Phi_0}) \otimes \Phi_1. \tag{E.43}$$

Nevertheless, for finite B, the string field theory equation of motion contains the ncSDYM equation of motion. Therefore, the solutions constructed in sections II.5 and II.6 can serve as a guide in the search for nonperturbative solutions to string field theory. Note that some proposals for string functionals T were made in chapters IV and V; indeed, these solutions were motivated by the above ideas.

Appendix F

Fermionic overlap equations

F.1 Bosonic Neumann coefficients

In this appendix we give a list of the boson Neumann coefficients. These are results of [66] but presented in the notation of [152]. The interaction vertex in momentum basis is given by

$$\langle V_3| = \int d^D p^{(1)} d^D p^{(2)} d^D p^{(3)} \delta^D(p^{(1)}+p^{(2)}+p^{(3)})_{123}\langle p,0|\exp\big[-V\big]\,, \tag{F.1}$$

where

$$V = \frac{1}{2}\sum_{r,s}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(r)\mu}V_{mn}^{rs}a_n^{(s)\nu} + \sqrt{\alpha'}\sum_{r,s}\sum_{n\geq 1}\eta_{\mu\nu}p^{(r)\mu}V_{0n}^{rs}a_n^{(s)\nu} + \frac{\alpha'}{2}\sum_{r}\eta_{\mu\nu}p^{(r)\mu}V_{00}^{rr}p^{(r)\nu}\,. \tag{F.2}$$

The Neumann matrices are expressed in terms of coefficients A_n and B_n which are defined as

$$\left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{1/3} = \sum_{n\text{ even}} A_n\,z^n + \mathrm{i}\sum_{n\text{ odd}} A_n\,z^n\,, \qquad \left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{2/3} = \sum_{n\text{ even}} B_n\,z^n + \mathrm{i}\sum_{n\text{ odd}} B_n\,z^n\,. \tag{F.3}$$

The Neumann coefficients read

$$V_{mn}^{r\,r} = -\sqrt{mn}\,\frac{(-1)^n+(-1)^m}{6}\Big(\frac{A_m\,B_n + A_n\,B_m}{m+n} + \frac{A_m\,B_n - A_n\,B_m}{m-n}\Big)\,, \quad m\neq n,\ m,n\neq 0\,, \tag{F.4a}$$

$$\begin{aligned} V_{mn}^{r\,r+1} = {} & \sqrt{mn}\,\frac{(-1)^n+(-1)^m}{12}\Big(\frac{A_m\,B_n + A_n\,B_m}{m+n} + \frac{A_m\,B_n - A_n\,B_m}{m-n}\Big) \\ & - \sqrt{mn}\,\sqrt{3}\,\frac{1-(-1)^{n+m}}{12}\Big(\frac{A_m\,B_n - A_n\,B_m}{m+n} + \frac{A_m\,B_n + A_n\,B_m}{m-n}\Big)\,, \quad m\neq n,\ m,n\neq 0\,, \end{aligned} \tag{F.4b}$$

$$\begin{aligned} V_{mn}^{r\,r-1} = {} & \sqrt{mn}\,\frac{(-1)^n+(-1)^m}{12}\Big(\frac{A_m\,B_n - A_n\,B_m}{m-n} + \frac{A_m\,B_n + A_n\,B_m}{m+n}\Big) \\ & + \sqrt{mn}\,\sqrt{3}\,\frac{1-(-1)^{n+m}}{12}\Big(\frac{A_m\,B_n + A_n\,B_m}{m-n} + \frac{A_m\,B_n - A_n\,B_m}{m+n}\Big)\,, \quad m\neq n,\ m,n\neq 0\,, \end{aligned} \tag{F.4c}$$

The coefficients on the diagonal are given by

$$V^{r\,r}_{nn} = -\frac{1}{3}\big[2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\big]\,, \quad n \neq 0\,, \tag{F.5a}$$

$$V^{r\,r+1}_{nn} = V^{r\,r-1}_{nn} = \frac{1}{2}\big[(-1)^n - V^{rr}_{nn}\big]\,, \quad n \neq 0\,, \tag{F.5b}$$

$$V^{r\,r}_{00} = \ln(27/16)\,. \tag{F.5c}$$

The coefficients with one index zero are obtained as limits of the coefficients V_{mn} above

$$V^{r\,r}_{0n} = -\sqrt{\frac{2}{n}}\,\frac{1+(-1)^n}{3}A_n\,, \tag{F.6a}$$

$$V^{r\,r+1}_{0n} = -\sqrt{\frac{2}{n}}\,\Big[-\frac{1+(-1)^n}{6}A_n - \sqrt{3}\frac{1-(-1)^n}{6}A_n\Big]\,, \tag{F.6b}$$

$$V^{r\,r-1}_{0n} = -\sqrt{\frac{2}{n}}\,\Big[-\frac{1+(-1)^n}{6}A_n + \sqrt{3}\frac{1-(-1)^n}{6}A_n\Big]\,. \tag{F.6c}$$

The value of the coefficients for different conventions for α' are easily obtained absorbing the explicit α' into these coefficients.

F.2 More overlap equations

In this appendix we continue the discussion of the overlap equations started at the end of section VI.5. We adopt the convention about the index range chosen there so that the indices i, j, k and l start from zero, $i, j, k, l = 0, 1, \dots$, while $m, n = 1, 2, \dots$. The matrices $\tilde{X}_{kl}$ are

$$\tilde{X}_{0m} = -\tilde{X}_{m0} = \frac{2\,\mathrm{i}}{\pi\, m}(-1)^{\frac{m-1}{2}}\,[1-(-1)^m]\,, \tag{F.7a}$$

$$\tilde{X}_{nm} = \frac{\mathrm{i}}{\pi}(-1)^{\frac{n-m-1}{2}}\,[1-(-1)^{n+m}]\Big[\frac{1}{n+m} + \frac{(-1)^m}{n-m}\Big]\,. \tag{F.7b}$$

Note that compared to the matrices defined in [66] we have $\tilde{X}_{nm} = X^{\mathrm{GJ}}_{nm}$, but for the parts containing a zero index $-\sqrt{2}\,\tilde{X}_{0m} = X^{\mathrm{GJ}}_{0m}$! Using the relation to the bosonic coefficients given in eq. (VI.82) and the definition of $N^{r\,r+1}_{mn}$ in (VI.73) one finds for $m \neq n$

$$N^{r\,r}_{mn} = \frac{2}{3}(U_{mn} + \bar{U}_{mn}) = 2\sqrt{\frac{m}{n}}V^{r\,r}_{mn}\,, \tag{F.8a}$$

$$N^{r\,r+1}_{mn} = \frac{2}{3}(\omega U_{mn} + \bar{\omega}\bar{U}_{mn}) = -\frac{1}{3}(U_{mn} + \bar{U}_{mn}) + \frac{\mathrm{i}}{\sqrt{3}}(U_{mn} - \bar{U}_{mn}) = 2\sqrt{\frac{m}{n}}V^{r\,r+1}_{mn}\,, \tag{F.8b}$$

and hence

$$\begin{aligned} U_{mn} = &-\frac{m}{4}\big[(-1)^n + (-1)^m\big]\Big[\frac{A_m\,B_n + A_n\,B_m}{m+n} + \frac{A_m\,B_n - A_n\,B_m}{m-n}\Big] \\ &+ \frac{\mathrm{i}\,m}{4}\big[1-(-1)^{m+n}\big]\Big[\frac{A_m\,B_n - A_n\,B_m}{m+n} + \frac{A_m\,B_n + A_n\,B_m}{m-n}\Big]\,, \quad m \neq n\,, \end{aligned} \tag{F.9}$$

from which it is once more apparent that $CUC = \bar{U}$. This prepares the stage to scrutinize the overlap equations for Π^1_m following from eq. (VI.107a). Taking $k = 0$ and $j = 2l$ gives

$$\sum_{m=1}^{\infty}\tilde{X}_{0m}U_{m,2l} \stackrel{!}{=} 0\,. \tag{F.10}$$

Inserting eqs. (F.7a) and (F.9), we see that

$$\sum_{m=1}^{\infty} \tilde{X}_{0m} U_{m,2l} = -\frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \Big[\frac{A_{2k+1}\, B_{2l} - A_{2l}\, B_{2k+1}}{2k+1+2l} + \frac{A_{2k+1}\, B_{2l} + A_{2l}\, B_{2k+1}}{2k+1-2l} \Big] , \tag{F.11}$$

Using the relation between the coefficients A_n and a_n,

$$A_{2k} = (-1)^k a_{2k} , \qquad A_{2k+1} = (-1)^k a_{2k+1} , \tag{F.12}$$

and the summation formulas for the coefficients a_n derived in [66],

$$O^a_k = \sum_{l=0}^{\infty} \frac{a_{2l+1}}{(2l+1)+k} = \frac{\pi\, a_k}{\sqrt{3}} , \quad O^a_{-n} = \sum_{l=0}^{\infty} \frac{a_{2l+1}}{(2l+1)-n} = -\frac{1}{2} \frac{\pi\, a_n}{\sqrt{3}} , \quad \text{for } k, n \text{ even}, \tag{F.13a}$$

$$E^a_k = \sum_{l=0}^{\infty} \frac{a_{2l}}{(2l)+k} = \frac{\pi\, a_k}{\sqrt{3}} , \qquad E^a_{-n} = \sum_{l=0}^{\infty} \frac{a_{2l+1}}{(2l+1)-n} = -\frac{1}{2} \frac{\pi\, a_n}{\sqrt{3}} , \quad \text{for } k, n \text{ odd}, \tag{F.13b}$$

$$O^b_k = \sum_{l=0}^{\infty} \frac{b_{2l+1}}{(2l+1)+k} = \frac{\pi\, a_k}{\sqrt{3}} , \quad O^b_{-n} = \sum_{l=0}^{\infty} \frac{b_{2l+1}}{(2l+1)-n} = \frac{1}{2} \frac{\pi\, b_n}{\sqrt{3}} , \quad \text{for } k, n \text{ even}, \tag{F.13c}$$

$$E^b_k = \sum_{l=0}^{\infty} \frac{b_{2l}}{(2l)+k} = \frac{\pi\, b_k}{\sqrt{3}} , \qquad E^b_{-n} = \sum_{l=0}^{\infty} \frac{b_{2l+1}}{(2l+1)-n} = \frac{1}{2} \frac{\pi\, b_n}{\sqrt{3}} , \quad \text{for } k, n \text{ odd}, \tag{F.13d}$$

one finds

$$\sum_{m=1}^{\infty} \tilde{X}_{0m} U_{m,2l} = -\frac{2}{\pi} \big[O^a_{2l} B_{2l} - O^b_{2l} A_{2l} + O^a_{-2l} B_{2l} + O^b_{-2l} A_{2l} \big] = 0 \tag{F.14}$$

proving (F.10). Now let us look at $k = 2l+1$ and $j = 2n+1$ in eq. (VI.107a):

$$\tfrac{3}{2} U_{2l+1,2n+1} + \tfrac{\sqrt{3}}{2} \sum_{i=0}^{\infty} \tilde{X}_{2l+1,i} U_{i,2n+1} \overset{!}{=} 0 . \tag{F.15}$$

If one inserts eqs. (F.7b) and (F.9), the sum can be written in terms of eqs. (F.13) as

$$\begin{aligned} \sum_{i=0}^{\infty} \tilde{X}_{2l+1,i} U_{i,2n+1} =& (-1)^l \frac{2l+1}{\pi} \Big[\frac{E^a_{2l+1} B_{2n+1} - E^b_{2l+1} A_{2n+1}}{(2n+1)-(2l+1)} + \frac{E^a_{-2l-1} B_{2n+1} - E^b_{-2l-1} A_{2n+1}}{(2n+1)+(2l+1)} \\ & - \frac{E^b_{-2l-1} A_{2n+1} + E^a_{-2l-1} B_{2l+1}}{(2n+1)-(2l+1)} - \frac{E^a_{2l+1} B_{2n+1} + E^b_{2l+1} A_{2n+1}}{(2n+1)+(2l+1)} \Big] \\ =& -\sqrt{3}\, U_{2l+1,2n+1} , \end{aligned} \tag{F.16}$$

proving eq. (F.15). The case $k = 2l$ and $j = 2n$ can be easily treated along the same lines.

Bibliography

[I] M. Ihl and S. Uhlmann, *Noncommutative extended waves and soliton-like configurations in $N = 2$ string theory*, to be published in IJMPA [arXiv:hep-th/0211263].

[II] O. Lechtenfeld, A. D. Popov and S. Uhlmann, *Exact solutions of Berkovits' string field theory*, Nucl. Phys. B **637** (2002) 119 [arXiv:hep-th/0204155].

[III] A. Kling, O. Lechtenfeld, A. D. Popov and S. Uhlmann, *On nonperturbative solutions of superstring field theory*, Phys. Lett. B **551** (2003) 193 [arXiv:hep-th/0209186].

[IV] A. Kling, O. Lechtenfeld, A. D. Popov and S. Uhlmann, *Solving string field equations: New uses for old tools*, Fortschr. Phys. **51** (2003) 775 [arXiv:hep-th/0212335].

[V] A. Kling, S. Uhlmann, *String field theory vertices for fermions of integral weight*, arXiv:hep-th/0306254.

[1] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in non-commutative gauge theory*, JHEP **0104** (2001) 001 [arXiv:hep-th/0009142].

[2] I. Ya. Aref'eva, D. M. Belov and A. A. Giryavets, *Construction of the vacuum string field theory on a non-BPS brane*, JHEP **0209** (2002) 050 [arXiv:hep-th/0201197].

[3] I. Ya. Aref'eva, D. M. Belov, A. A. Giryavets, A. S. Koshelev and P. B. Medvedev, *Non-commutative field theories and (super)string field theories*, arXiv:hep-th/0111208.

[4] I. Ya. Aref'eva and A. A. Giryavets, *Open superstring star as a continuous Moyal product*, JHEP **0212** (2002) 074 [arXiv:hep-th/0204239].

[5] I. Ya. Aref'eva, A. A. Giryavets and A. S. Koshelev, *NS ghost slivers*, Phys. Lett. B **536** (2002) 138 [arXiv:hep-th/0203227].

[6] I. Ya. Aref'eva, A. A. Giryavets and P. B. Medvedev, *NS matter sliver*, Phys. Lett. B **532** (2002) 291 [arXiv:hep-th/0112214].

[7] I. Ya. Aref'eva, A. S. Koshelev, D. M. Belov and P. B. Medvedev, *Tachyon condensation in cubic superstring field theory*, Nucl. Phys. B **638** (2002) 3 [arXiv:hep-th/0011117].

[8] I. Ya. Aref'eva, P. B. Medvedev and A. P. Zubarev, *Background formalism for superstring field theory*, Phys. Lett. B **240** (1990) 356.

[9] I. Ya. Aref'eva, P. B. Medvedev and A. P. Zubarev, *New representation for string field solves the consistence problem for open superstring field*, Nucl. Phys. B **341** (1990) 464.

[10] M. F. Atiyah and R. S. Ward, *Instantons and algebraic geometry*, Commun. Math. Phys. **55** (1977) 177.

[11] D. Bak, *Exact multi-vortex solutions in noncommutative abelian Higgs theory*, Phys. Lett. B **495** (2000) 251 [arXiv:hep-th/0008204].

[12] L. Barosi and C. Tello, *GSO(−) vertex operators and open superstring field theory in hybrid variables*, JHEP **0305** (2003) 004 [arXiv:hep-th/0303246].

[13] I. Bars, *Map of Witten's $\star$ to Moyal's $\star$*, Phys. Lett. B **517** (2001) 436 [arXiv:hep-th/0106157].

[14] I. Bars and Y. Matsuo, *Computing in string field theory using the Moyal star product*, Phys. Rev. D **66** (2002) 066003 [arXiv:hep-th/0204260].

[15] I. Bars, I. Kishimoto and Y. Matsuo, *Fermionic ghosts in Moyal string field theory*, arXiv:hep-th/0304005.

[16] S. Bellucci and A. Galajinsky, *Can one restore Lorentz invariance in quantum $N = 2$ string?*, Nucl. Phys. B **630** (2002) 151 [arXiv:hep-th/0112024].

[17] D. M. Belov, *Diagonal representation of open string star and Moyal product*, arXiv:hep-th/0204164.

[18] D. M. Belov and A. Konechny, *On continuous Moyal product structure in string field theory*, JHEP **0210** (2002) 049 [arXiv:hep-th/0207174].

[19] N. Berkovits, *Super-Poincaré invariant superstring field theory*, Nucl. Phys. B **450** (1995) 90 [Erratum-ibid. B **459** (1996) 439] [arXiv:hep-th/9503099].

[20] N. Berkovits, *The tachyon potential in open Neveu-Schwarz string field theory*, JHEP **0004** (2000) 022 [arXiv:hep-th/0001084].

[21] N. Berkovits, *Review of open superstring field theory*, arXiv:hep-th/0105230.

[22] N. Berkovits, *The Ramond sector of open superstring field theory*, JHEP **0111** (2001) 047 [arXiv:hep-th/0109100].

[23] N. Berkovits and C. T. Echevarria, *Four-point amplitude from open superstring field theory*, Phys. Lett. B **478** (2000) 343 [arXiv:hep-th/9912120].

[24] N. Berkovits, M. T. Hatsuda and W. Siegel, *The big picture*, Nucl. Phys. B **371** (1992) 434 [arXiv:hep-th/9108021].

[25] N. Berkovits, A. Sen and B. Zwiebach, *Tachyon condensation in superstring field theory*, Nucl. Phys. B **587** (2000) 147 [arXiv:hep-th/0002211].

[26] N. Berkovits and W. Siegel, *Covariant field theory for self-dual strings*, Nucl. Phys. B **505** (1997) 139 [arXiv:hep-th/9703154].

[27] N. Berkovits and C. Vafa, *On the uniqueness of string theory*, Mod. Phys. Lett. A **9** (1994) 653 [arXiv:hep-th/9310170].

[28] N. Berkovits and C. Vafa, *$N = 4$ topological strings*, Nucl. Phys. B **433** (1995) 123 [arXiv:hep-th/9407190].

[29] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, JHEP **9903** (1999) 018 [arXiv:hep-th/9902098].

[30] S. Bieling, *Interaction of noncommutative plane waves in 2+1 dimensions*, J. Phys. A **35** (2002) 6281 [arXiv:hep-th/0203269].

[31] L. Bonora, C. Maccaferri, D. Mamone and M. Salizzoni, *Topics in string field theory*, arXiv:hep-th/0304270.

[32] L. Bonora, D. Mamone and M. Salizzoni, *Vacuum string field theory with B field*, JHEP **0204** (2002) 020 [arXiv:hep-th/0203188].

[33] L. Bonora, D. Mamone and M. Salizzoni, *Vacuum string field theory ancestors of the GMS solitons*, JHEP **0301** (2003) 013 [arXiv:hep-th/0207044].

[34] J. Bordes, A. Abdurrahman and F. Anton, *N-string vertices in string field theory*, Phys. Rev. D **49** (1994) 2966 [arXiv:hep-th/9306029].

[35] B. Chen and F. L. Lin, *D-branes as GMS solitons in vacuum string field theory*, Phys. Rev. D **66** (2002) 126001 [arXiv:hep-th/0204233].

[36] E. Cremmer and L. Gervais, *Combining and splitting relativistic strings*, Nucl. Phys. B **76** (1974) 209.

[37] E. Cremmer and L. Gervais, *Infinite component field theory of interacting relativistic strings and dual theory*, Nucl. Phys. B **90** (1975) 410.

[38] A. D'Adda and F. Lizzi, *Space dimensions from supersymmetry for the N=2 spinning string: A four-dimensional model*, Phys. Lett. B **191** (1987) 85.

[39] J. R. David, *Tachyon condensation in the D0/D4 system*, JHEP **0010** (2000) 004 [arXiv:hep-th/0007235].

[40] P.-J. De Smet, *Tachyon condensation: Calculations in string field theory*, arXiv:hep-th/0109182.

[41] P. J. De Smet and J. Raeymaekers, *Level four approximation to the tachyon potential in superstring field theory*, JHEP **0005** (2000) 051 [arXiv:hep-th/0003220].

[42] P. J. De Smet and J. Raeymaekers, *The tachyon potential in Witten's superstring field theory*, JHEP **0008** (2000) 020 [arXiv:hep-th/0004112].

[43] H. J. de Vega, *Nonlinear multi-plane wave solutions of self-dual Yang-Mills theory*, Commun. Math. Phys. **116** (1988) 659.

[44] M. R. Douglas and C. M. Hull, *D-branes and the noncommutative torus*, JHEP **9802** (1998) 008 [arXiv:hep-th/9711165].

[45] M. R. Douglas, H. Liu, G. Moore and B. Zwiebach, *Open string star as a continuous Moyal product*, JHEP **0204** (2002) 022 [arXiv:hep-th/0202087].

[46] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73** (2002) 977 [arXiv:hep-th/0106048].

[47] S. L. Dubovsky, V. A. Rubakov and S. M. Sibiryakov, *Quasi-localized states on noncommutative solitons*, JHEP **0201** (2002) 037 [arXiv:hep-th/0201025].

[48] I. Ellwood, B. Feng, Y. H. He and N. Moeller, *The identity string field and the tachyon vacuum*, JHEP **0107** (2001) 016 [arXiv:hep-th/0105024].

[49] T. G. Erler, *Moyal formulation of Witten's star product in the fermionic ghost sector*, arXiv:hep-th/0205107.

[50] P. Forgacs, Z. Horvath and L. Palla, *Solution generating technique for self-dual monopoles*, Nucl. Phys. B **229** (1983) 77.

[51] F. Franco-Sollova and T. A. Ivanova, *On noncommutative merons and instantons*, J. Phys. A **36** (2003) 4207 [arXiv:hep-th/0209153].

[52] D. Friedan, E. Martinec and S. Shenker, *Conformal invariance, supersymmetry and string theory*, Nucl. Phys. B **271** (1986) 93.

[53] E. Fuchs, M. Kroyter and A. Marcus, *Squeezed state projectors in string field theory*, JHEP **0209** (2002) 022 [arXiv:hep-th/0207001].

[54] K. Furuta, T. Inami, H. Nakajima and M. Yamamoto, *Low-energy dynamics of noncommutative $\mathbb{C}P^1$ solitons in 2+1 dimensions*, Phys. Lett. B **537** (2002) 165 [arXiv:hep-th/0203125];

[55] K. Furuta, T. Inami, H. Nakajima and M. Yamamoto, *Non-BPS solutions of the noncommutative $\mathbb{C}P^1$ model in 2+1 dimensions*, JHEP **0208** (2002) 009 [arXiv:hep-th/0207166].

[56] K. Furuta, T. Inami and M. Yamamoto, *Topics in nonlinear sigma models in $D = 3$*, arXiv:hep-th/0211129.

[57] K. Furuuchi and K. Okuyama, *Comma vertex and string field algebra*, JHEP **0109** (2001) 035 [arXiv:hep-th/0107101].

[58] M. R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories I: Foundations*, Nucl. Phys. B **505** (1997) 569 [arXiv:hep-th/9705038].

[59] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, arXiv:hep-th/0111129.

[60] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Star algebra projectors*, JHEP **0204** (2002) 060 [arXiv:hep-th/0202151].

[61] D. Gaiotto and L. Rastelli, *Experimental string field theory*, arXiv:hep-th/0211012.

[62] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, Commun. Math. Phys. **233** (2003) 355 [arXiv:hep-th/0103256].

[63] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, JHEP **0005** (2000) 020 [arXiv:hep-th/0003160].

[64] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory. Vol. 1: Introduction*, Cambridge University Press (1987).

[65] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology*, Cambridge University Press (1987).

[66] D. J. Gross and A. Jevicki, *Operator formulation of interacting string field theory*, Nucl. Phys. B **283** (1987) 1.

[67] D. J. Gross and A. Jevicki, *Operator formulation of interacting string field theory. 2*, Nucl. Phys. B **287** (1987) 225.

[68] D. J. Gross and A. Jevicki, *Operator formulation of interacting string sield theory. 3. NSR superstring*, Nucl. Phys. B **293** (1987) 29.

[69] D. J. Gross and N. A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, JHEP **0007** (2000) 034 [arXiv:hep-th/0005204];

[70] D. J. Gross and N. A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, JHEP **0010** (2000) 021 [arXiv:hep-th/0007204].

[71] D. J. Gross and N. A. Nekrasov, *Solitons in noncommutative gauge theory*, JHEP **0103** (2001) 044 [arXiv:hep-th/0010090].

[72] L. Hadasz, U. Lindstrom, M. Rocek and R. von Unge, *Noncommutative multi-solitons: Moduli spaces, quantization, finite theta effects and stability*, JHEP **0106** (2001) 040 [arXiv:hep-th/0104017].

[73] M. Hamanaka, Y. Imaizumi and N. Ohta, *Moduli space and scattering of D0-branes in noncommutative super Yang-Mills theory*, Phys. Lett. B **529** (2002) 163 [arXiv:hep-th/0112050].

[74] M. Hamanaka and S. Terashima, *On exact noncommutative BPS solitons*, JHEP **0103** (2001) 034 [arXiv:hep-th/0010221].

[75] M. Hamanaka and K. Toda, *Towards noncommutative integrable systems*, arXiv:hep-th/0211148.

[76] J. A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, arXiv:hep-th/0102076.

[77] J. A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, JHEP **0012** (2000) 024 [arXiv:hep-th/0010060].

[78] K. Hashimoto, *Fluxons and exact BPS solitons in noncommutative gauge theory*, JHEP **0012** (2000) 023 [arXiv:hep-th/0010251].

[79] K. Hashimoto and H. Ooguri, *Seiberg-Witten transforms of noncommutative solitons*, Phys. Rev. D **64** (2001) 106005 [arXiv:hep-th/0105311].

[80] H. Hata and T. Kawano, *Open string states around a classical solution in vacuum string field theory*, JHEP **0111** (2001) 038 [arXiv:hep-th/0108150].

[81] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press (1978).

[82] Z. Horvath, O. Lechtenfeld and M. Wolf, *Noncommutative instantons via dressing and splitting approaches*, JHEP **0212** (2002) 060 [arXiv:hep-th/0211041].

[83] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer (1972).

[84] A. Iqbal and A. Naqvi, *Tachyon condensation on a non-BPS D-brane*, arXiv:hep-th/0004015.

[85] T. A. Ivanova and O. Lechtenfeld, *Hidden symmetries of the open $N = 2$ string*, Int. J. Mod. Phys. A **16** (2001) 303 [arXiv:hep-th/0007049].

[86] T. A. Ivanova and A. D. Popov, *Self-dual Yang-Mills fields in $D = 4$ and integrable systems in $1 \leq D \leq 3$*, Theor. Math. Phys. **102** (1995) 280.

[87] T. A. Ivanova and A. D. Popov, *Some new integrable equations from the selfdual Yang-Mills equations*, Phys. Lett. A **205** (1995) 158 [arXiv:hep-th/9508129].

[88] K. Jünemann, O. Lechtenfeld and A. D. Popov, *Non-local symmetries of the closed $N = 2$ string*, Nucl. Phys. B **548** (1999) 449 [arXiv:hep-th/9901164].

[89] K. Jünemann, *Symmetrien des $N = 2$ Strings*, preprint ITP-UH-02/99, PhD Thesis, Hannover (1999).

[90] M. Kaku and K. Kikkawa, *The field theory of relativistic strings. Pt. 2: Loops and pomerons*, Phys. Rev. D **10** (1974) 1823.

[91] A. Kapustin, A. Kuznetsov and D. Orlov, *Noncommutative instantons and twistor transform*, Commun. Math. Phys. **221** (2001) 385 [arXiv:hep-th/0002193].

[92] T. Kawano and K. Okuyama, *Open string fields as matrices*, JHEP **0106** (2001) 061 [arXiv:hep-th/0105129].

[93] E. Kiritsis, *Introduction to superstring theory*, arXiv:hep-th/9709062.

[94] I. Kishimoto, *Some properties of string field algebra*, JHEP **0112** (2001) 007 [arXiv:hep-th/0110124].

[95] I. Kishimoto and K. Ohmori, *CFT description of identity string field: Toward derivation of the VSFT action*, JHEP **0205** (2002) 036 [arXiv:hep-th/0112169].

[96] J. Klusoň, *Some remarks about Berkovits' superstring field theory*, JHEP **0106** (2001) 045 [arXiv:hep-th/0105319].

[97] J. Klusoň, *Proposal for background independent Berkovits' superstring field theory*, JHEP **0107** (2001) 039 [arXiv:hep-th/0106107].

[98] J. Klusoň, *Some solutions of Berkovits' superstring field theory*, arXiv:hep-th/0201054.

[99] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry. II*, Phys. Rept. **360** (2002) 353 [arXiv:hep-th/0107251].

[100] V. E. Korepin and T. Oota, *Scattering of plane waves in self-dual Yang-Mills theory*, J. Phys. A **29** (1996) L625 [arXiv:hep-th/9608064].

[101] A. Koshelev, *Solutions of vacuum superstring field theory*, arXiv:hep-th/0212055.

[102] V. A. Kostelecký and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, Phys. Rev. D **63** (2001) 046007 [arXiv:hep-th/0008252].

[103] V. A. Kostelecký and S. Samuel, *On a nonperturbative vacuum for the open bosonic string*, Nucl. Phys. B **336** (1990) 263.

[104] O. Lechtenfeld, *The self-dual critical $N = 2$ string*, arXiv:hep-th/9607106.

[105] O. Lechtenfeld, *Mathematics and physics of $N = 2$ strings*, arXiv:hep-th/9912281.

[106] O. Lechtenfeld and A. D. Popov, *Closed $N = 2$ strings: Picture-changing, hidden symmetries and SDG hierarchy*, Int. J. Mod. Phys. A **15** (2000) 4191 [arXiv:hep-th/9912154].

[107] O. Lechtenfeld and A. D. Popov, *On the integrability of covariant field theory for open $N = 2$ strings*, Phys. Lett. B **494** (2000) 148 [arXiv:hep-th/0009144].

[108] O. Lechtenfeld and A. D. Popov, *Noncommutative multi-solitons in 2+1 dimensions*, JHEP **0111** (2001) 040 [arXiv:hep-th/0106213].

[109] O. Lechtenfeld and A. D. Popov, *Scattering of noncommutative solitons in 2+1 dimensions*, Phys. Lett. B **523** (2001) 178 [arXiv:hep-th/0108118].

[110] O. Lechtenfeld and A. D. Popov, *Noncommutative 't Hooft instantons*, JHEP **0203** (2002) 040 [arXiv:hep-th/0109209].

[111] O. Lechtenfeld, A. D. Popov and B. Spendig, *Open $N = 2$ strings in a B-field background and noncommutative self-dual Yang-Mills*, Phys. Lett. B **507** (2001) 317 [arXiv:hep-th/0012200].

[112] O. Lechtenfeld, A. D. Popov and B. Spendig, *Noncommutative solitons in open $N = 2$ string theory*, JHEP **0106** (2001) 011 [arXiv:hep-th/0103196].

[113] O. Lechtenfeld and S. Samuel, *Gauge invariant modification of Witten's open superstring*, Phys. Lett. B **213** (1988) 431.

[114] A. LeClair, M. E. Peskin and C. R. Preitschopf, *String field theory on the conformal plane. 1. Kinematical principles*, Nucl. Phys. B **317** (1989) 411.

[115] A. LeClair, M. E. Peskin and C. R. Preitschopf, *String field theory on the conformal plane. 2. Generalized gluing*, Nucl. Phys. B **317** (1989) 464.

[116] B. H. Lee, K. M. Lee and H. S. Yang, *The $\mathbb{C}P^n$ model on noncommutative plane*, Phys. Lett. B **498** (2001) 277 [arXiv:hep-th/0007140].

[117] R. Leese, *Extended wave solutions in an integrable chiral model in 2+1 dimensions*, J. Math. Phys. **30** (1989) 2072.

[118] M. Legaré, *Reduced systems of (2,2) pseudo-euclidean noncommutative self-dual Yang-Mills theories*, J. Phys. A **35** (2002) 5489.

[119] D. E. Lerner, *The linear system for self-dual gauge fields in a space-time of signature 0*, J. Geom. Phys. **8** (1992) 211.

[120] A. Losev, G. W. Moore, N. Nekrasov and S. Shatashvili, *Four-dimensional avatars of two-dimensional RCFT*, Nucl. Phys. Proc. Suppl. **46** (1996) 130 [arXiv:hep-th/9509151].

[121] D. Lüst and S. Theisen, *Lectures on string theory*, Lect. Notes Phys. **346**, Springer (1989).

[122] S. Mandelstam, *Interacting string picture of dual resonance models*, Nucl. Phys. B **64** (1973) 205.

[123] S. Mandelstam, *Dual-resonance models*, Phys. Rept. **13** (1974) 259.

[124] S. Mandelstam, *Lorentz properties of the three-string vertex*, Nucl. Phys. B **83** (1974) 413.

[125] N. Marcus, *A tour through $N = 2$ strings*, arXiv:hep-th/9211059.

[126] M. Mariño and R. Schiappa, *Towards vacuum superstring field theory: The supersliver*, J. Math. Phys. **44** (2003) 156 [arXiv:hep-th/0112231].

[127] P. Matlock, *Butterfly tachyons in vacuum string field theory*, Phys. Rev. D **67** (2003) 086002 [arXiv:hep-th/0211286].

[128] K. Matsubara, *Restrictions on gauge groups in noncommutative gauge theory*, Phys. Lett. B **482** (2000) 417 [arXiv:hep-th/0003294].

[129] N. Moeller and W. Taylor, *Level truncation and the tachyon in open bosonic string field theory*, Nucl. Phys. B **583** (2000) 105 [arXiv:hep-th/0002237].

[130] G. Moore and W. Taylor, *The singular geometry of the sliver*, JHEP **0201** (2002) 004 [arXiv:hep-th/0111069].

[131] J. Murugan and R. Adams, *Comments on noncommutative sigma models*, JHEP **0212** (2002) 073 [arXiv:hep-th/0211171].

[132] V. P. Nair and J. Schiff, *Kähler Chern-Simons theory and symmetries of anti-self-dual gauge fields*, Nucl. Phys. B **371** (1992) 329.

[133] F. J. Narganes-Quijano, *Picture changing operation and BRST cohomology in superstring field theory*, Phys. Lett. B **212** (1988) 292.

[134] N. A. Nekrasov, *Trieste lectures on solitons in noncommutative gauge theories*, arXiv:hep-th/0011095.

[135] N. A. Nekrasov, *Lectures on open strings, and noncommutative gauge fields*, arXiv:hep-th/0203109.

[136] N. Nekrasov and A. Schwarz, *Instantons on noncommutative $\mathbb{R}^4$ and (2,0) superconformal six-dimensional theory*, Commun. Math. Phys. **198** (1998) 689 [arXiv:hep-th/9802068].

[137] B. Niemeyer, $N = 2$ *Strings und selbstduale Feldtheorien*, preprint ITP-UH-06/01, PhD Thesis, Hannover (2001).

[138] K. Ohmori, *A review on tachyon condensation in open string field theories*, arXiv:hep-th/0102085.

[139] K. Ohmori, *Tachyonic kink and lump-like solutions in superstring field theory*, JHEP **0105** (2001) 035 [arXiv:hep-th/0104230].

[140] K. Ohmori, *Comments on solutions of vacuum superstring field theory*, JHEP **0204** (2002) 059 [arXiv:hep-th/0204138].

[141] K. Ohmori, *On ghost structure of vacuum superstring field theory*, Nucl. Phys. B **648** (2003) 94 [arXiv:hep-th/0208009].

[142] K. Ohmori, *Level-expansion analysis in NS superstring field theory revisited*, arXiv:hep-th/0305103.

[143] K. Okuyama, *Siegel gauge in vacuum string field theory*, JHEP **0201** (2002) 043 [arXiv:hep-th/0111087].

[144] K. Okuyama, *Ghost kinetic operator of vacuum string field theory*, JHEP **0201** (2002) 027 [arXiv:hep-th/0201015].

[145] H. Ooguri and C. Vafa, *Self-duality and* $N = 2$ *string magic*, Mod. Phys. Lett. A **5** (1990) 1389.

[146] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge University Press (1998).

[147] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*, Cambridge University Press (1998).

[148] A. P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, Phys. Lett. B **495** (2000) 407 [arXiv:hep-th/0007043].

[149] A. D. Popov, *Self-dual Yang-Mills: Symmetries and moduli space*, Rev. Math. Phys. **11** (1999) 1091 [arXiv:hep-th/9803183].

[150] C. R. Preitschopf, C. B. Thorn and S. Yost, *Superstring field theory*, Nucl. Phys. B **337** (1990) 363.

[151] L. Rastelli, A. Sen and B. Zwiebach, *String field theory around the tachyon vacuum*, Adv. Theor. Math. Phys. **5** (2002) 353 [arXiv:hep-th/0012251].

[152] L. Rastelli, A. Sen and B. Zwiebach, *Classical solutions in string field theory around the tachyon vacuum*, Adv. Theor. Math. Phys. **5** (2002) 393 [arXiv:hep-th/0102112].

[153] L. Rastelli, A. Sen and B. Zwiebach, *Boundary CFT construction of D-branes in vacuum string field theory*, JHEP **0111** (2001) 045 [arXiv:hep-th/0105168].

[154] L. Rastelli, A. Sen and B. Zwiebach, *Vacuum string field theory*, arXiv:hep-th/0106010.

[155] L. Rastelli, A. Sen and B. Zwiebach, *Star algebra spectroscopy*, JHEP **0203** (2002) 029 [arXiv:hep-th/0111281].

[156] L. Rastelli and B. Zwiebach, *Tachyon potentials, star products and universality*, JHEP **0109** (2001) 038 [arXiv:hep-th/0006240].

[157] A. A. Rosly and K. G. Selivanov, *On amplitudes in self-dual sector of Yang-Mills theory*, Phys. Lett. B **399** (1997) 135 [arXiv:hep-th/9611101].

[158] M. Sakaguchi, *Pregeometrical formulation of Berkovits' open RNS superstring field theories*, arXiv:hep-th/0112135.

[159] S. Samuel, *The ghost vertex in E. Witten's string field theory*, Phys. Lett. B **181** (1986) 255.

[160] M. Schnabl, *String field theory at large B-field and noncommutative geometry*, JHEP **0011** (2000) 031 [arXiv:hep-th/0010034].

[161] M. Schnabl, *Wedge states in string field theory*, JHEP **0301** (2003) 004 [arXiv:hep-th/0201095].

[162] M. Schnabl, *Anomalous reparametrizations and butterfly states in string field theory*, Nucl. Phys. B **649** (2003) 101 [arXiv:hep-th/0202139].

[163] V. Schomerus, *D-branes and deformation quantization*, JHEP **9906** (1999) 030 [arXiv:hep-th/9903205].

[164] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[165] A. Sen, *Stable non-BPS states in string theory*, JHEP **9806** (1998) 007 [arXiv:hep-th/9803194].

[166] A. Sen, *Stable non-BPS bound states of BPS D-branes*, JHEP **9808** (1998) 010 [arXiv:hep-th/9805019].

[167] A. Sen, *Tachyon condensation on the brane antibrane system*, JHEP **9808** (1998) 012 [arXiv:hep-th/9805170].

[168] A. Sen, $SO(32)$ *spinors of type I and other solitons on brane-antibrane pair*, JHEP **9809** (1998) 023 [arXiv:hep-th/9808141].

[169] A. Sen, *Type I D-particle and its interactions*, JHEP **9810** (1998) 021 [arXiv:hep-th/9809111].

[170] A. Sen, *Descent relations among bosonic D-branes*, Int. J. Mod. Phys. A **14** (1999) 4061 [arXiv:hep-th/9902105].

[171] A. Sen, *Non-BPS states and branes in string theory*, arXiv:hep-th/9904207.

[172] A. Sen and B. Zwiebach, *A proof of local background independence of classical closed string field theory*, Nucl. Phys. B **414** (1994) 649 [arXiv:hep-th/9307088].

[173] A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, JHEP **0003** (2000) 002 [arXiv:hep-th/9912249].

[174] H. S. Snyder, *Quantized space-time*, Phys. Rev. **71** (1947) 38.

[175] K. Takasaki, *Anti-self-dual Yang-Mills equations on noncommutative spacetime*, J. Geom. Phys. **37** (2001) 291 [arXiv:hep-th/0005194].

[176] W. I. Taylor, *Lectures on D-branes, gauge theory and M(atrices)*, arXiv:hep-th/9801182.

[177] W. Taylor, *A perturbative analysis of tachyon condensation*, JHEP **0303** (2003) 029 [arXiv:hep-th/0208149].

[178] W. Taylor, *Lectures on D-branes, tachyon condensation, and string field theory*, arXiv:hep-th/0301094.

[179] R. P. Thomas, *Derived categories for the working mathematician*, arXiv: math.AG/0001045.

[180] C. B. Thorn, *String field theory*, Phys. Rept. **175** (1989) 1.

[181] R. S. Ward, *Integrable and solvable systems, and relations among them*, Phil. Trans. Roy. Soc. Lond. A **315** (1985) 451.

[182] R. S. Ward, *Soliton solutions in an integrable chiral model in 2+1 dimensions*, J. Math. Phys. **29** (1988) 386.

[183] R. S. Ward, *Classical solutions of the chiral model, unitons, and holomorphic vector bundles*, Commun. Math. Phys. **128** (1990) 319.

[184] R. S. Ward and R. O. Wells, *Twistor geometry and field theory*, Cambridge University Press (1990).

[185] C. Wendt, *Scattering amplitudes and contact interactions in Witten's superstring field theory*, Nucl. Phys. B **314** (1989) 209.

[186] E. Witten, *Noncommutative geometry and string field theory*, Nucl. Phys. B **268** (1986) 253.

[187] E. Witten, *Interacting field theory of open superstrings*, Nucl. Phys. B **276** (1986) 291.

[188] E. Witten, *Noncommutative tachyons and string field theory*, arXiv:hep-th/0006071.

[189] M. Wolf, *Soliton-antisoliton scattering configurations in a noncommutative sigma model in 2+1 dimensions*, JHEP **0206** (2002) 055 [arXiv:hep-th/0204185].

[190] C. N. Yang, *Condition of self-duality for $SU(2)$ gauge fields on euclidean four-dimensional space*, Phys. Rev. Lett. **38** (1977) 1377.

[191] T. Yoneya, *Spontaneously broken space-time supersymmetry in open string theory without GSO projection*, Nucl. Phys. B **576** (2000) 219 [arXiv:hep-th/9912255].

[192] V. E. Zakharov and A. V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique*, Sov. Phys. JETP **47** (1978) 1017.

[193] V. E. Zakharov and A. B. Shabat, *Integration of non-linear equations of mathematical physics by the method of inverse scattering*, Funct. Anal. Appl. **13** (1979) 166.

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