Nonholonomic ricci flows and Finsler–Lagrange f(R,F,L)–modified gravity and dark matter effects

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We review the theory of geometric flows on nonholonomic manifolds and tangent bundles and self-similar configurations resulting in generalized Ricci solitons and Einstein-Finsler equations. There are provided new classes of exact solutions on Finsler-Lagrange f(R,F,L)-modifications of general relativity and discussed possible implications in acceleration cosmology.

Keywords: Nonholonomic Ricci flows; Finsler–Lagrange geometry and modified gravity; locally anisotropic Finsler cosmolgy.

Current important and fascinating problems in modern accelerating cosmology and dark energy and dark matter physics involve the finding of canonical (optimal) metric and connection spacetime structures, search for possible topological configurations, and to find the relevant physical applications, see[5,6,8] and references therein. There are strong observational cosmological data and theoretical arguments (e.g. the fundamental unsolved problem of constructing a self-consistent model of quantum gravity) that the standard general relativity, GR, theory of gravity should be modified in a non–Riemannian geometric form and/or as modified gravity theory,

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MGT. In a more conservative approach, cosmological scenarios have to be at least extended for new classes of solutions of nonlinear systems of partial differential equations, PDEs, with generic off-diagonal metrics determined by generating and integration functions depending on all spacetime coordinates, nontrivially polarized vacuum, non-minimal coupling with matter fields etc.[7].

The first our goal is to summarize and develop the results on the geometric evolution theory of Einstein metrics, and possible generalized metric and connection structures, under nonintegrable (equivalently, nonholomic/ anholonomic) constaints resulting in nonholonomic Finsler–Lagrange configurations[1,9]. We argue that self–consistent and physically motivated "minimal" Finsler modifications of the standard Ricci flow and gravitational field equations can be elaborated using the so–called Cartan and canonical distinguished connections (d–connection) structures. The approaches with metric noncompatible Finsler connections, or without linear connections, do not have limits to standard theories of particle physics and do not allow formulations of certain analogs of the axiomatic formalism as GR[2,3]. The second goal of our works is to analyze possible implications of the geometric unifications of nonholonomic Ricci flow evolution and modified gravity theories, MGTs, in modern acceleration and study of dark energy and dark matter problems[4,6].

Intuitively, generic locally anisotropic spacetime constructions with fundamental geometric/physical objects (for instance, metric $\mathbf{g}$ and/or almost symplectic structure, θ; see details in[1,3] and references therein) depend additionally to spacetime coordinates $x = \{x^i\}$ on velocity/momentum type coordinates, $y = \{y^a \simeq v^a$ [velocity], or $\simeq p_a$ [momentum]$\}$, for instance, in the form $g(x,y)$ and/or $\theta(x,y)$. For different geometric model of mechanical, statistical and classical and quantum field theories with nonlinear dispersion relations , there are considered corresponding types of space, and spacetime, manifolds, their (co) tangent bundles, phase spaces etc., endowed with classical and quantum variables. In a similar manner, we can consider certain nonholonomic conventional splitting with local or generalized coordinates $u = (x,y) = \{u^\alpha = (x^i, y^a)\}$, where indices $i, j, ... = 1, 2, ...n$ and $a, b, ... = n+1, n+2, ..., n+m$, for $n \geq 2$ and $m \geq 1$, for a fibered $(n+m)$–structure on a generalized nonholonomic manifold. We use the boldface symbol $\mathbf{V}$, $\dim \mathbf{V} = n+m$, for a conventional fibered manifold, or bundle space, enabled with certain classes of nonholonomic distributions, in particular, nonholonomic frames.

Let V be a real Lorentz C^∞–manifold, $\dim V = n \geq 2$, of signature $(+,+,...,-)$, and denote by TV its tangent bundle. A regular Lagrangian L, i.e. a fundamental/ generating Lagrange function (or Lagrange metric) is a function $L : TM \rightarrow [0, \infty)$ with nondegenerate Hessian $\tilde{g}_{ab} = \frac{1}{2}\partial^2 L/\partial y^a \partial y^b$.

In particular, we can take $L = F^2(x,y)$ for a fundamental Finsler function subjected to the conditions[a]:

[a]The main difference of our approach from that with standard Finsler geometries is that we consider for the base manifolds certain Lorentz signatures instead of Euclidean ones; this is necessary

(1) $F(x,y)$ is C^∞ on $\mathbf{V} \simeq \widetilde{TV} := TV\backslash\{0\}$, where $\{0\}$ denotes the set of zero sections of TV on V;
(2) it is imposed the homogeneity condition $F(x,\beta y) = \beta F(x,y), \forall\beta > 0$, and
(3) $\forall u \in \widetilde{T_x V}$, the vertical, v, induced $\tilde{g}_{ab}$ is non-degenerate and positive definite for a Riemannian structure, but such conditions are relaxed for relativistic systems.[b]

Contrary to the (pseudo) Riemannian geometry which is completely defined by a metric tensor $g_{\alpha\beta}(u)$, a (pseudo) Lagrange, or Finsler, geometry model is not completely defined only by $L(u)$, or $F(u)$, fundamental generating element without assumptions on two other fundamental geometric objects: the nonlinear connection, N–connection, and the distinguished connection, d-connection, see details in[2,3]. Certain models of Finsler geometry are elaborated with the so-called Akbar-Zaedh definition of Ricci type tensor, in term of the semi-spray function

$$\tilde{G}^k = \frac{1}{4}\tilde{g}^{kj}(y^i\frac{\partial^2 L}{\partial y^j \partial x^i} - \frac{\partial L}{\partial x^j}),$$

not involving the concept of linear connection. This is a nice geometric construction but it is not enough for a physical viable theory. In order to include interactions with matter fields and elaborate a variational calculus adapted to the nonholonomic structures in Largange-Finsler spaces, we must consider a physically motivated f covariant derivative (linear connection) and corresponding assumptions on physical frames (determined by a N–connection structure).

A Lagrange/Finsler geometry model is completely defined by the data $(\tilde{\mathbf{g}}, \tilde{\mathbf{N}}, \tilde{\mathbf{D}}, L)$, where L generates for some stated geometric/physical principles a Sasaki type lift of $\tilde{g}_{ab}$ to total space metric $\tilde{\mathbf{g}} = [\tilde{g}_{ij}, \tilde{g}_{ab}, \tilde{N}_i^a := \partial\tilde{G}^a/\partial y^i]$, for a canonical N–connection

$$\tilde{\mathbf{N}} : T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V},\ \tilde{\mathbf{N}} = \tilde{N}_i^a dx^i \otimes \partial/\partial y^i$$

(a nonholonomic distribution with conventional horizontal, h, and vertical, v, splitting).

The Cartan (Finsler like) d–connection $\tilde{\mathbf{D}}$ is uniquely defined by the properties that it is metric compatible, $\tilde{\mathbf{D}}\tilde{\mathbf{g}} = 0$, and compatible with the almost symplectic structure $\tilde{\theta} := \tilde{\mathbf{g}}(x, \tilde{\mathbf{J}}y), \tilde{\mathbf{D}}\tilde{\theta} = 0$, where the almost complex structure $\tilde{\mathbf{J}}$ is naturally determined by $\tilde{\mathbf{N}}$. It is important to note that $\tilde{\mathbf{D}}$ is with nontrivial torsion $\tilde{\mathbf{T}}$ (with zero pure h- and v-components, but non-zero $h-v$ mixed components), completely determined by generic off-diagonal terms of $\tilde{\mathbf{g}}$ determined by a nonintegrable $\tilde{\mathbf{N}}$ (in general, with nonzero Neijenhuis tensor). Another important property is that

for elaborating relativistic like generalized gravity theories with well defined limits to GR and special relativity theories.

[b]The symbols L and F is taken respectively from the Lagrange and Finsler nonlinear quadratic elements, $ds^2 = L(x,dx)$ and $= F^2(x,dx)$, which generalize the quadratic element in (pseudo) Riemannian geometry, $ds^2 = g_{ij}(x)dx^i dx^j, dx^i \sim y^i$.

there is a canonical distortion relation $\tilde{\mathbf{D}}[\tilde{\mathbf{g}}] = \tilde{\nabla}[\tilde{\mathbf{g}}] + \tilde{\mathbf{Z}}[\tilde{\mathbf{g}}]$, where the Levi-Civita, LC, connection $\tilde{\nabla}$ and the Cartan distortion tensor $\tilde{\mathbf{Z}}[\tilde{\mathbf{T}}] = \tilde{\mathbf{Z}}[\tilde{\mathbf{g}}]$ are completely determined by L, or $\tilde{\mathbf{g}}$, or $\tilde{\theta}$.

Using frame transforms, $\tilde{\mathbf{g}} \rightarrow \hat{g} = (hg, vg)$ and $\tilde{\mathbf{N}} \rightarrow \mathbf{N}$, we can consider the so-called canonical d–connection, $\widehat{\mathbf{D}} = (h\widehat{D}, v\widehat{D})$, for which $\widehat{\mathbf{D}}\hat{\mathbf{g}} = 0$ if and only if $h\widehat{D}(hg) = 0$ and $v\widehat{D}(vg) = 0$. This induces another canonical distortion relation, $\hat{\mathbf{D}}[\mathbf{g}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g}]$ for any metric $\mathbf{g}$ on $\mathbf{V}$, which can be always expressed as $\tilde{\mathbf{g}}$ and/or $\hat{\mathbf{g}}$ by corresponding frame transforms and nonholonomic deformations of the linear connection structure.

As a matter of principle, any Lagrange-Finsler and (pseudo) Riemannian geometry can be described in equivalent Finsler-Cartan like, $(\tilde{\mathbf{g}}, \tilde{\mathbf{N}}, \tilde{\mathbf{D}})$, almost Kaehler Finsler-Cartan, $(\tilde{\theta}, \tilde{\mathbf{N}}, \tilde{\mathbf{D}})$ and canonical d–connection, $(\mathbf{g}, \mathbf{N}, \hat{\mathbf{D}})$, variables. For instance, in the first case, we keep an explicit analogy between the Lagrange and Finsler geometry; in the second case, we introduce almost symplectic variables with allow to perform a rigorous deformation quantization of such geometries; in the third case, it is possible to decouple certain generalize Einstein-Finsler equations for $\hat{\mathbf{D}}$ and solve such equations in very general forms with generic off-diagonal metrics determined by generating and integration functions, correspondingly depending on all spacetime coordinates (such variables can be introduced also in GR); see discussion, examples and references in[2–4,7,8]

Finally, we consider generalized Grisha Perelman's functionals [c],

$$\mathcal{F}(\mathbf{g}, \widehat{\mathbf{D}}, f) = \int_{\widehat{\mathcal{V}}} (\widehat{R} + |\widehat{\mathbf{D}} f|^2) e^{-f} \; vol,$$
$$\mathcal{W}(\mathbf{g}, \widehat{\mathbf{D}}, f, \tau) = \int_{\widehat{\mathcal{V}}} [\tau(\widehat{R} + |h\widehat{D}| + |v\widehat{D}|)^2 + f - 2n] \widehat{\mu} \; vol,$$

where $\widehat{R}$ is the scalar curvature of $\widehat{\mathbf{D}}, vol$ is the volume form defined by $\mathbf{g}(\mathbf{u}, \tau)$ generated by a families of $L(x, y, \tau)$ parameterized by real parameter τ; the integration is taken over $\widehat{\mathcal{V}} \subset \mathbf{V}$ as a $2n$ dimensional volume determined as a time like evolution of some initial data on a 3-d compact hypersurface on V (we consider a $(n-1)+1$ spitting of V, naturally imbedded in TV); $\widehat{\mu} = (4\pi\tau)^{-n} e^{-f}$ for a scaling function $f(u, \tau)$ and $\int_{\widehat{\mathcal{V}}} \widehat{\mu} \; vol = 1$ and $\tau > 0$. Self–similar and stationary point configurations of $\mathcal{F}$– and $\mathcal{W}$–functionals result in generalized Ricci soliton eqs and Einstein eqs for MGTs and various Finsler modifications.

New classes of generic off–diagonal solutions, Raychaudhuri eqs on Finsler spaces[6,7] and inhomogeneous and locally anisotropic Finsler cosmology scenarios have been recently studied in Refs.[4,5,8,10]. Mathematically, the approach was recently developed for almost Kähler Ricci flows and Lagrange-Finsler structures on Lie algebroids[11].

[c]Originally, he postulated them for the geometric evoluton of three dimensional Riemannian metrics in the R. Hamilton Ricci flow theory; on nonholonomic constructions, see[1,3,9].

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