Stability of equilibria for a nonlinear population model with age- and spatial-structure

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Abstract

In this thesis we study a nonlinear partial differential equation which models the time evolution of a population with age- and spatial-structure. In an abstract setting, this model reads

$$\begin{aligned} \partial_t u + \partial_a u + A(a)u &= -\mu(u,a)u, \qquad t>0,\ a\in(0,a_m), \\ u(t,0) &= \int_0^{a_m} b(u,a)u(t,a)\,\mathrm{d}a, \qquad t>0, \\ u(0,a) &= u_0(a), \qquad a\in(0,a_m), \end{aligned} \tag{0.1}$$

where $u:[0,T)\to\mathbb{E}_0$ is interpreted as the density function of the population, taking values in an appropriate function space $\mathbb{E}_0$, $b=b(u,a)\geq 0$ and $\mu=\mu(u,a)\geq 0$ are the birth and mortality rates, and $A(a):E_1\subset E_0\to E_0$ is a closed operator on the real Banach lattice E_0, for each $a\in J:=[0,a_m)$. In our considerations to follow we fix $p\in[1,\infty)$ and set $\mathbb{E}_0:=L_p(J,E_0)$.

In the first part we consider the semilinear model of age- and spatial-structured population dynamics, which is obtained when the birth law is assumed to be linear. Put differently, the birth rate in problem (0.1) is supposed to be a function of the age-parameter only, i.e. $b=b(a)$. Assuming A generates a parabolic evolution operator, it is then shown that this semilinear structure allows to formulate problem (0.1) as a semilinear Cauchy problem in the Banach space $\mathbb{E}_0$. In particular, we can study mild solutions, their asymptotic behaviour, and convergence to equilibria, and we will see that the stability analysis can be reduced to the linearised problem. In a subsequent step, the spectral theory of positive compact operators is applied to this linear problem, and as a result we will see that the stability behaviour is completely determined by a single quantity, namely the spectral radius of an associated operator. It should be noted that essential ingredients for this result are the assumptions of maximal L_p-regularity of the spatial diffusion operator A, and the positivity of the parabolic evolution operator generated by A.

In a subsequent part, we introduce a weak solution concept for problem (0.1). Assuming A generates a parabolic evolution operator, these so called *integral solutions* are constructed for the linearised problem in a first step. In the second step we apply a fixed-point argument in order to establish existence of integral solutions for the nonlinear problem. Furthermore we carry out a detailed analysis of the linear inhomogeneous problem, which serves as a preparation for the last part.

In the final part we study the stability behaviour of equilibria to problem (0.1). It has to be pointed out that the birth rate is now allowed to depend on the density, i.e. $b=b(u,.)$, and consequently we lose the semilinear structure, as considered in the first part. In particular, mild solutions are not available any longer, and this is where the integral solutions of the foregoing part come into play. More precisely, we will prove that problem (0.1) is well-posed within this framework. Finally, it is shown that a principle of linearised stability is available within this setting.

Evolution operator - Stability - Well-posedness

Contents

1 Introduction

An interesting problem which arises in biology is the study of the dynamical behaviour of a given population. If one can find a model which is capable of describing this process and at the same time is accessible to a thorough mathematical analysis, it is possible to draw conclusions about the dynamical evolution of the population.

Historically, the first structured population models were introduced during the first quarter of the 20th century. They focused on age-structured populations and did not yet incorporate spatial distribution. Amongst them, the models of Sharpe and Lotka [30] in 1911, and of McKendrick [22] in 1926 have to be mentioned. These models assumed linear mortality- and birth-processes for the underlying population, which made them more accessible from a mathematical point of view, but less appropriate for the description of biological populations. In 1974, Gurtin and MacCamy [14], and Hoppensteadt [17] introduced the first nonlinear models for age-structured populations. These models allow for effects like crowding or limitation of resources, and admit nontrivial equilibrium states, in contrast to the linear models. Around the same time, Gurtin [13] proposed a linear model for age-structured populations with spatial distribution, which in turn lead to research on corresponding nonlinear models, see e.g. [8], [15], [20]. Subsequently, the particular and challenging mathematical structure of these models caused increased interest and activity within the research community, and the development of different approaches, cf. [4], [18], [26], [27], [31] [40], though this list is far from exhaustive.

Let us turn to the model and explain the underlying mechanisms. For simplicity, we neglect spatial structure for the moment, i.e. we consider a population where individuals are distinguished solely by age. The time-evolution of a single-species population is then described by a density function

$$\begin{aligned} u : [0,T) \times [0,a_m) &\to \mathbb{R}_+ \\ (t,a) &\mapsto u(t,a), \end{aligned}$$

where t is the time-parameter, a denotes the age-parameter, and $a_m \in \mathbb{R}_+ \cup \{\infty\}$ is the maximal age. Accordingly, the total population P at time t is then given by

$$P(t) = \int_0^{a_m} u(t,a)\,\mathrm{d}a.$$

In order to obtain an equation for the density function, we consider all individuals of some fixed age $a \in [0,\infty)$. The number of those individuals at time $t \in [0,T)$ is given by $u(t,a) \in \mathbb{R}_+$. After some time-increment $h > 0$, their number amounts to $u(t+h, a+h)$, and therefore the difference

$$u(t+h, a+h) - u(t,a)$$

tells us how the size of this cohort has changed during the time-increment. The *balance-law* of population dynamics says

$$u(t+h, a+h) - u(t,a) \approx -h\,\mu(a)\,u(t,a), \qquad \text{for } 0 < h \ll 1,$$

where $\mu = \mu(a) \geq 0$ denotes the *mortality rate.* Assuming the density is sufficiently regular, this leads to the relation

$$\partial_t u\,(t,a) + \partial_a u\,(t,a) = -\mu(a)\,u(t,a), \quad t > 0,\ a \in (0,a_m), \tag{1.1}$$

also known as the *McKendrick-von Foerster equation.* On the other hand, one would like to allow for reproductory processes, which is achieved by introduction of the so called *birth law*

$$u(t,0) = \int_0^{a_m} b(a) u(t,a)\,\mathrm{d}a, \quad t > 0, \tag{1.2}$$

where $b = b(a) \geq 0$ denotes the *birth rate*. Lastly, one prescribes an inital age-distribution,

$$u(0,a) = u_0(a), \quad a \in (0,a_m). \tag{1.3}$$

The system of equations (1.1) – (1.3) constitute the classical model of linear age-structured population dynamics of Sharpe-Lotka-McKendrick.

In a next step, this model can by extended by allowing for spatial structure of the population, which is important in the description of, for instance, tumor growth or epidemiology. To be more precise, we assume that spatial movement of individuals is governed by a linear diffusion process. Considering furthermore the nonlinear nature of mortality- and birth-processes, the prototypical model of age- and spatial-structured population dynamics takes the form

$$\begin{aligned}
\partial_t u + \partial_a u - \mathrm{div}_x(d(a,x)\nabla_x u) &= -\mu(u,a,x)u, \quad t > 0,\, a \in (0,a_m),\, x \in \Omega, \\
u(t,0,x) &= \int_0^{a_m} b(u,a,x)u(t,a)\,\mathrm{d}a, \quad t > 0,\, x \in \Omega, \\
\partial_\nu u(t,a,x) &= 0, \quad t > 0,\, a \in (0,a_m),\, x \in \partial\Omega, \\
u(0,a,x) &= \phi(a,x), \quad a \in (0,a_m),\, x \in \Omega,
\end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $a_m \in \mathbb{R}_+ \cup \{\infty\}$ denotes the maximal age, and

$$\begin{aligned}
u : [0,T) \times [0,a_m) \times \Omega &\to \mathbb{R}_+ \\
(t,a,x) &\mapsto u(t,a,x),
\end{aligned}$$

is the density function of the population. Accordingly, the number of individuals at time t, with age between a_1 and a_2, in the area $\tilde{\Omega} \subseteq \Omega$, is given by the integral

$$\int_{a_1}^{a_2} \int_{\tilde{\Omega}} u(t,a,x)\,\mathrm{d}x\mathrm{d}a.$$

Subsequently, we want to formulate this model in a more general framework. The corresponding analysis will then also be carried out in an abstract setting, and it will be convenient to have the above prototype model in mind. More precisely, we consider the abstract problem

$$\begin{aligned}
\partial_t u + \partial_a u + A(a)u &= -\mu(u,a)u, \qquad t > 0,\, a \in (0,a_m), \\
u(t,0) &= \int_0^{a_m} b(u,a)u(t,a)\,\mathrm{d}a, \qquad t > 0, \\
u(0,a) &= u_0(a), \qquad a \in (0,a_m),
\end{aligned} \tag{1.4}$$

where $u : [0,T) \to \mathbb{E}_0$ is interpreted as the density function of the population, taking values in an appropriate function space $\mathbb{E}_0$, $b = b(u,a) \geq 0$ and $\mu = \mu(u,a) \geq 0$ are the birth and mortality rates, and $A(a) : E_1 \subset E_0 \to E_0$ is a closed operator on the real Banach lattice E_0, for each $a \in J := [0,a_m)$. In our considerations to follow we fix $p \in [1,\infty)$ and set $\mathbb{E}_0 := L_p(J,E_0)$.

After having declared the problem of interest, let us give an overview of the forthcoming studies. In section 2 we begin with a compilation of the required theoretical background.

In section 3 we consider the semilinear model of age- and spatial-structured population dynamics, which is obtained when the birth law is assumed to be linear. Put differently, the birth rate in problem (1.4) is supposed to be a function of the age-parameter only, i.e. $b = b(a)$. Assuming A generates a parabolic evolution operator, it is then shown that this semilinear structure allows to formulate problem (1.4) as a semilinear Cauchy problem in the Banach space

$\mathbb{E}_0$. In particular, we can study mild solutions, their asymptotic behaviour, and convergence to equilibria, which are determined by the equations

$$\begin{aligned} \partial_a \phi + A(a)\phi &= -\mu(\phi, a)\phi, \qquad a \in (0, a_m), \\ \phi(0) &= \int_0^{a_m} b(a)\phi(a)da. \end{aligned}$$

The existence of nontrivial equilibria poses a separate problem and was considered in [34], [37]. As regards their qualitative behaviour, we will see that the stability analysis can be reduced to the linearised problem, which was studied in [36]. In a subsequent step, the spectral theory of positive compact operators is applied to this linear problem, and as a result we will see that the stability behaviour is completely determined by a single quantity, namely the spectral radius of an associated operator, cf. Theorem 3.16. It should be noted that essential ingredients for this approach are the assumptions of maximal L_p-regularity of the spatial diffusion operator A, and the positivity of the parabolic evolution operator generated by A.

In section 4 we introduce a weak solution concept for problem (1.4). Assuming A generates a parabolic evolution operator, these so called *integral solutions* are constructed for the linearised problem in a first step. In the second step we apply a fixed-point argument in order to establish existence of integral solutions for the nonlinear problem. The remaining part of this section is devoted to a detailed analysis of the linear inhomogeneous problem and serves as a preparation for section 5.

In section 5 we study the stability behaviour of equilibria to problem (1.4). Observe that the birth rate is now allowed to depend on the density, i.e. $b = b(u, .)$, and consequently we lose the semilinear structure, as considered in section 3. In particular, mild solutions are not available any longer, and this is where the integral solutions of section 4 come into play. More precisely, we will prove that problem (1.4) is well-posed within this framework. Subsequently it is shown that a principle of linearised stability is available within this setting, cf. Theorem 5.11.

1.1 Method of characteristics

In the following we formally derive a formula for the solution of equation (1.4), which will be fundamental for our subsequent studies. To this end we integrate the equation along the characteristics $a = c + t$, where $c \geq 0$ is some constant. Let us define

$$w(t) := u(t, c + t), \quad t \geq 0$$

and denote by $\Pi(a, \sigma)$, $0 \leq \sigma \leq a < a_m$, the parabolic evolution operator generated by A, i.e.

$$\begin{aligned} \frac{d}{da}\Pi(a, \sigma)x_0 &= -A(a)\Pi(a, \sigma)x_0, \quad 0 \leq \sigma \leq a, \\ \Pi(\sigma, \sigma)x_0 &= x_0, \quad x_0 \in E_1, \end{aligned}$$

see section 2.6 for a precise definition. Furthermore, we set

$$F : D(F) \subset \mathbb{E}_0 \to \mathbb{E}_0, \qquad F(u) := -\mu(u, .)u. \tag{1.5}$$

Using the first equation in (1.4), we obtain the differential equation

$$\frac{d}{dt}w(t) = -A(c + t)w(t) + F(u(t, .))(c + t) \tag{1.6}$$

in the Banach space E_0. This equation leads us to the Ansatz

$$w(t) = \Pi(c + t, c)x(t),$$

where $x(t)$ is to be determined. With this Ansatz we have

$$\frac{d}{dt}w(t) = -A(c+t)w(t) + \Pi(c+t,c)\dot{x}(t).$$

Now let us set $h(t) := F(u(t,.))(c+t)$ for the moment. Comparing the two previous equations we conclude $\Pi(c+t,c)\dot{x}(t) = h(t)$, and the fundamental theorem of calculus therefore yields

$$\begin{aligned}\Pi(c+t,c)x(t) &= \Pi(c+t,c)x(0) + \int_0^t \Pi(c+t,c)\dot{x}(s)\,\mathrm{d}s \\ &= \Pi(c+t,c)x(0) + \int_0^t \Pi(c+t,c+s)h(s)\,\mathrm{d}s.\end{aligned}$$

Thus we conclude that

$$w(t) = \Pi(c+t,c)w(0) + \int_0^t \Pi(c+t,c+s)h(s)\,\mathrm{d}s.$$

Now let $a \geq t$ and choose $c = a - t$ to obtain

$$u(t,a) = \Pi(a,a-t)u_0(a-t) + \int_0^t \Pi(a,a-t+s)F(u(s,.))(a-t+s)\,\mathrm{d}s, \quad 0 \leq t \leq a. \tag{1.7}$$

In an analogous way we can proceed to obtain a formula for the case $0 \leq a \leq t$. To this end let

$$v(a) := u(c+a,a), \quad a \geq 0.$$

As before we obtain

$$\frac{d}{da}v(a) = -A(a)v(a) + F(u(c+a,.))(a),$$

and consequently

$$v(a) = \Pi(a,0)v(0) + \int_0^a \Pi(a,s)F(u(c+s,.))(s)\,\mathrm{d}s.$$

Now choose $c = t - a$ to obtain

$$u(t,a) = \Pi(a,0)u(t-a,0) + \int_0^a \Pi(a,s)F(u(t-a+s,.))(s)\,\mathrm{d}s, \qquad 0 \leq a \leq t. \tag{1.8}$$

Combining (1.7) and (1.8) leads us to the following definition:

Definition 1.1. Let $u : [0,T] \to \mathbb{E}_0$ be continuous. We say that the function u is an *integral solution* to (1.4) on $[0,T]$, if for all $t \in [0,T]$

$$\begin{aligned} u(t,a) &= \begin{cases} \Pi(a,a-t)u_0(a-t) \\ \Pi(a,0)B(t-a) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi(a,a-t+s)F(u(s,.))(a-t+s)\,\mathrm{d}s, & \text{for a.a. } a \in (t,a_m) \\ \int_0^a \Pi(a,s)F(u(t-a+s,.))(s)\,\mathrm{d}s, & \text{for a.a. } a \in (0,t),\ a < a_m, \end{cases}\end{aligned}$$

where $B(t) = u(t,0)$ satisfies an associated integral equation, which is induced by the age-boundary condition in (1.4):

$$B(t) = \int_0^t b(u,a)u(t,a)\,\mathrm{d}a + \int_t^{a_m} b(u,a)u(t,a)\,\mathrm{d}a.$$

After having declared a solution concept it is natural to ask whether the problem is well posed. This question will be addressed in section 5.2.

2 Definitions and general results

In this section we review some definitions and general results, which will be required in the course of the forthcoming investigations.

2.1 Linear operators

Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, we define

$$\mathcal{L}(E, F) := \{T : E \to F \,|\, T \text{ is } \mathbb{K}\text{-linear and continuous}\},$$

and equip it with the uniform operator topology. In the special case $F = \mathbb{K}$, one obtains the dual space of E, also denoted by $E' = \mathcal{L}(E, \mathbb{K})$. We write $\mathcal{L}_s(E, F)$ if $\mathcal{L}(E, F)$ is given the simple convergence topology, induced by the family of seminorms

$$\{T \mapsto \|Tx\|_F \,:\, x \in E\}.$$

2.2 Banach lattices

Recall that we have formulated the population model from section 1 in the more general framework (1.4). Since we are interested in positive population densities, we have to generalise the concept of positivity to the abstract setting. To this end, we recall the notion of a Banach lattice and state the basic results which will be needed. We folllow the exposition in [29, chapter II].

Definition 2.1. A vector space E over $\mathbb{R}$, endowed with an order relation $\leq$, is called an *ordered vector space* if the following axioms are satisfied:

$$x \leq y \quad \Rightarrow \quad x + z \leq y + z, \qquad \text{for all } x, y, z \in E,$$
$$x \leq y \quad \Rightarrow \quad \alpha x \leq \alpha y, \qquad \text{for all } x, y \in E \text{ and } \alpha \in \mathbb{R}_+.$$

If $(E, \leq)$ is an ordered vector space, the subset $E_+ := \{x \in E : 0 \leq x\}$ is called the *positive cone* of E; elements $x \in E_+$ are called *positive*. If $x \in E_+ \setminus \{0\}$, we write $0 < x$.
For a subset $M \subseteq E$, the *supremum* $\sup M \in E$ is defined as the smallest upper bound of M. More precisely, $\sup M = s \in E$ if and only if $m \leq s$ for all $m \in M$, and if $\tilde{s}$ is such that $m \leq \tilde{s}$ for all $m \in M$, then $s \leq \tilde{s}$. Analogously, one defines the *infimum* $\inf M$ as the largest lower bound of M.
A *vector lattice* is an ordered vector space such that $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in E$.
Let $(E, \leq_E), (F, \leq_F)$ be ordered vector spaces. A linear operator $A : E \to F$ is called *positive*, if it is compatible with the order structure, i.e. if $0 \leq_E x$ implies $0 \leq_F Ax$. The set of positive continuous linear operators from E to F is denoted by $\mathcal{L}_+(E, F)$, the set of positive continuous functionals on E by E'_+.

Let E be a vector lattice. For all $x \in E$, we define $x_+ := x \vee 0$, $x_- := (-x) \vee 0$, $|x| := x \vee (-x)$. x_+, x_- and $|x|$ are called the positive part, the negative part, and the modulus (or absolute value) of x, respectively.

Now, the following identities hold (see [29] for proofs):

$$\begin{aligned} x &= x_+ - x_-. \\ |x| &= x_+ + x_-. \\ |x| = 0 \Leftrightarrow x &= 0, \quad |\alpha x| = |\alpha||x|, \ |x + y| \leq |x| + |y|. \\ x + y &= x \vee y + x \wedge y. \end{aligned}$$

Definition 2.2. Let $(E, \leq)$ be a vector lattice. A norm $\|.\|$ on E is called a *lattice norm* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in E$. If $\|.\|$ is a lattice norm on E, the pair $(E, \|.\|)$ is called a *normed (vector) lattice*; if, in addition, $(E, \|.\|)$ is norm complete it is called a *Banach lattice.*

Remark 2.3. The positive cone E_+ is called *total*, if $\overline{E_+ - E_+} = E$. Let us introduce the relation

$$x' \leq y' :\iff y' - x' \in E'_+, \qquad x', y' \in E',$$

on the dual space E', then it is not difficult to prove that $(E', \leq)$ becomes an ordered vector space if E_+ is total.
Furthermore, if E is a Banach lattice, with E_+ total, then E' becomes a Banach lattice, cf. [29, Proposition II.5.5].
Finally we remark that on a normed vector lattice E, the map $x \mapsto x_+$ is continuous, which implies the closedness of the positive cone E_+, cf. [29, Proposition II.5.2]. □

Definition 2.4. Let E be a Banach lattice with positive cone E_+. An element $x \in E_+$ is called *quasi-interior* if $0 < \langle x', x\rangle$ for all $x' \in E'$ with $0 < x'$.
A linear operator $A : E \to E$ is called *strongly positive* if $0 < x \in E$ implies that Ax is quasi-interior.

Proposition 2.5. *Let E be a Banach lattice with total positive cone E_+, and $S, T : E \to E$ positive, linear operators. Then the implication*

$$Sx \leq Tx, \, \forall x \geq 0 \quad \Rightarrow \quad \|S\| \leq 4 \, \|T\|$$

holds.

Proof. We argue as in the proof of [6, Lemma 12.2]. It is well known, that S, T are continuous, cf. [29, Theorem II.5.3]. By the Hahn-Banach theorem,

$$\|S\| = \sup_{\|x\|\leq 1, \|x'\|\leq 1} |\langle x', Sx\rangle|.$$

Since E' is a Banach lattice, cf. Remark 2.3, we can split the vectors x, x' in their positive and negative parts, and estimate

$$\begin{aligned} \sup_{\|x\|\leq 1, \|x'\|\leq 1} |\langle x', Sx\rangle| &\leq 4 \sup_{\|x\|\leq 1, x\in E_+, \|x'\|\leq 1, x'\in E'_+} \langle x', Sx\rangle \\ &\leq 4 \sup_{\|x\|\leq 1, x\in E_+, \|x'\|\leq 1, x'\in E'_+} \langle x', Tx\rangle \\ &\leq 4\|T\|, \end{aligned}$$

and the claim follows. □

2.3 Spectral theory

Let A be a closed linear operator on a complex Banach space E (if E is a real Banach space, consider its complexification). We denote by $\sigma(A)$ and $\sigma_p(A)$ the spectrum and point spectrum of A, respectively. The *essential spectrum* $\sigma_e(A)$ consists of of those spectral points λ of A such that the image $\mathrm{im}(\lambda - A)$ is not closed, or λ is a limit point of $\sigma(A)$, or the dimension of the kernel $\ker(\lambda - A)$ is infinite. The *peripheral spectrum* $\sigma_0(A)$ is defined as $\sigma_0(A) := \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda = s(A)\}$, where $s(A) := \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$ denotes the *spectral bound* of A. The resolvent set $\mathbb{C} \setminus \sigma(A)$ is denoted by $\rho(A)$. For a more detailed exposition, we refer e.g. to [38, Section 4.3].

2.4 Semigroups

Many evolution equations can be formulated as an autonomous initial-value problem in an appropriate Banach space:

$$\begin{aligned} \frac{d}{dt}u &= -Au + F(u), \quad t > 0, \\ u(0) &= u_0, \end{aligned}$$

where $-A : D(-A) \subset E \to E$ is a closed operator on a Banach space E, and $F : D(F) \subset E \to E$ is continuous. In order to solve this equation, one would like to introduce the formal exponential of the unbounded operator $-A$, which leads to the mathematical concept of strongly continuous semigroups:

Definition 2.6. A *strongly continuous semigroup* $S(t), t \geq 0$, on the Banach space E is a one-parameter family of continuous operators $S(t) \in \mathcal{L}(E)$, such that

1. $S(0) = \mathrm{Id}_E$,
2. $S(t)S(s) = S(t+s), \quad$ for all $t, s \geq 0$,
3. $\lim_{t\to 0_+} S(t)x = x, \quad$ for all $x \in E$.

Definition 2.7. Let $S(t), t \geq 0$, be a strongly continuous semigroup on the Banach space E. The *generator* of the semigroup is defined as

$$-Ax := \lim_{t\to 0_+} \frac{1}{t}(S(t)x - x),$$

with $D(-A) := \{x \in E : \lim_{t\to 0_+} \frac{1}{t}(S(t)x - x) \text{ exists in } E\}$.

Remark 2.8. Every strongly continuous semigroup obviously induces a uniquely determined generator. Conversely, it is well known that for a given generator, its associated semigroup is uniquely determined. Hence, the notation $e^{-tA} = S(t), t \geq 0$, is justified and used to stress the generator under consideration. □

Given a semigroup $S(t), t \geq 0$, on the Banach space E, its growth bound is defined by

$$\omega_0 := \lim_{t\to\infty} \frac{1}{t} \log \|S(t)\|,$$

and its α-growth bound by

$$\omega_1 := \lim_{t\to\infty} \frac{1}{t} \log\left(\alpha(S(t))\right),$$

where α denotes Kuratowski's measure of non-compactness (cf. [38, Section 4.3]). We also write $\omega_i = \omega_i(-A)$, $i = 0, 1$, to stress the generator under consideration.

Proposition 2.9 (Bounded perturbation of semigroups)**.** *Let $(-A, D(-A))$ be the generator of a strongly continuous semigroup $e^{-tA}, t \geq 0$, on the Banach space E, satisfying*

$$\|e^{-tA}\| \leq Me^{\omega t}, \qquad t \geq 0,$$

for some $\omega \in \mathbb{R}$, $M \geq 1$. If $B \in \mathcal{L}(E)$, then the operator

$$-(A+B) \quad \textit{with} \quad D(-(A+B)) := D(-A)$$

generates a strongly continuous semigroup $e^{-t(A+B)}, t \geq 0$, satisfying

$$\|e^{-t(A+B)}\| \leq Me^{(\omega + M\|B\|)t}, \qquad t \geq 0.$$

For a proof of Proposition 2.9 we refer to [11, Thm III.1.3].

In our considerations we will need the following result, a proof of which can be found in [39, Prop. 2.5]:

Proposition 2.10. *Let $S(t), t \geq 0$, be a strongly continuous semigroup of positive bounded linear operators with infinitesimal generator $-A$ on the Banach lattice E. If $s(-A) > \omega_1(-A)$, then $\sigma_0(-A) = \{s(-A)\}$*

2.5 Differential equations in Banach spaces

In the following we collect some results about abstract semilinear Cauchy Problems, i.e. differential equations of the form

$$\begin{aligned} \frac{d}{dt}u &= -Au + F(u), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where $-A$ generates a strongly continuous semigroup $S(t), t \geq 0$, on E and $F : D(F) \subset E \to E$ is continuous.

Definition 2.11. A function $u \in C([0,T),E) \cap C^1((0,T),E)$ with values $u(t) \in D(-A) \cap D(F)$ for all $t \in (0,T)$ and satisfying (2.1) pointwise is called a *strong solution* on $[0,T)$.
A function $u \in C([0,T),E)$, with $u(t) \in D(F)$ for all $t \in [0,T)$, is called a *mild solution* to (2.1) on $[0,T)$, if it satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds, \qquad t \in [0,T). \tag{2.2}$$

In order to gain a better understanding of strong solutions, it is reasonable to consider mild solutions first. Every strong solution is a mild solution, but the converse is not true in general. The existence of mild solutions is well known and can be obtained by an application of Banach's fixed point theorem:

Proposition 2.12 (Existence of solutions)**.** *Let $-A$ be the generator of a strongly continuous semigroup $S(t), t \geq 0$, on E, and $F : D(F) \subset E \to E$ be Lipschitz continuous on bounded sets, with $D(F)$ open in E.*

Then for every $u_0 \in D(F)$ there exists a maximal $T = T(u_0) > 0$ such that there is a unique mild solution $u = u(\cdot, u_0)$ on $[0,T)$. Furthermore, if $D(F) = E$ and $T < \infty$, then $\lim_{t \nearrow T} \|u(t,u_0)\| = \infty$.

The introduction of mild solutions rises the question of their asymptotic behaviour, and one defines:

Definition 2.13. An element $\phi \in D(-A) \cap D(F) \subset E$ is called an *equilibrium* of equation (2.1) if it satisfies

$$-A\phi + F(\phi) = 0. \tag{2.3}$$

Definition 2.14 (Stability)**.** An equilibrium $\phi \in E$ of equation (2.1) is called *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every $u_0 \in E$ with $\|u_0 - \phi\| < \delta$, the mild solution (2.2) exists for all $t \geq 0$ and satisfies $\|u(t) - \phi\| < \epsilon$, for $t \geq 0$.
An equilibrium $\phi \in E$ is called *asymptotically stable* if it is stable and there exists $\delta > 0$ such

that for every $u_0 \in E$ with $\|u_0 - \phi\| < \delta$, the mild solution (2.2) satisfies $\lim_{t\to\infty} \|u(t) - \phi\| = 0$. An equilibrium $\phi \in E$ is called *exponentially asymptotically stable* if it is asymptotically stable, and there exists $\delta > 0$, $\omega > 0$ and $K > 0$ such that for every $u_0 \in E$ with $\|u_0 - \phi\| < \delta$, the mild solution (2.2) satisfies $\|u(t) - \phi\| \le K e^{-\omega t} \|u_0 - \phi\|$, for $t \ge 0$.
If δ can be chosen arbitrarily large in any of these last two definitions, then the corresponding property is said to be *global*.

The following well known result is essential for the principle of linearised stability and will be used frequently.

Lemma 2.15 (Linear shift of the generator). *Let $u_0 \in E$ and $B \in \mathcal{L}(E)$, assume $f : [0,T] \to E$ is continuous and $u : [0,T] \to E$ satisfies*

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\left(Bu(s) + f(s)\right) ds, \qquad t \in [0,T]. \tag{2.4}$$

Then

$$u(t) = e^{-t(A-B)}u_0 + \int_0^t e^{-(t-s)(A-B)} f(s)\, ds, \qquad t \in [0,T].$$

Proof. We only give a sketch of the argument, which relies on the observation that, due to (2.4), u is a mild solution to

$$\begin{aligned} \frac{d}{dt}u &= -Au + Bu + f, \quad t \in (0,T), \\ u(0) &= u_0. \end{aligned}$$

In a next step the assertion is verified for regular data, i.e. $u_0 \in D(-A)$, f continuously differentiable. The general case is proven by a density argument and Gronwall estimate (cf. proof of [38, Prop. 4.17] for details). □

Theorem 2.16 (Principle of linearised stability). *Let $-A$ be the generator of a strongly continuous semigroup on the Banach space E, let $F : B_R(\phi) \subset E \to E$ be Lipschitz continuous and Fréchet-differentiable at ϕ, and suppose ϕ is an equilibrium of equation* (2.1)*. Setting $-\hat{A} = -A + F'(\phi)$, the following hold:*

1. *if $\omega_0(-\hat{A}) < 0$, then ϕ is exponentially asymptotically stable.*
2. *if $\omega_0(-\hat{A}) > 0$ and $\omega_1(-\hat{A}) \le 0$, then ϕ is unstable.*

Proof. The result is classical in the theory of ordinary differential equations, cf. [5], and the corresponding argument can be adopted in the semigroup context, cf. [24, Theorem E]. □

Proposition 2.17 (Positivity of solutions). *Let $S(t), t \ge 0$, be a strongly continuous semigroup of positive bounded linear operators on the Banach lattice E, with infinitesimal generator $-A$. Assume that the nonlinearity $F : D(F) \subset E \to E$ is Lipschitz continuous on bounded sets and satisfies*

$$F(u) + c\,u \ge 0, \qquad \forall\, u \in D(F) \cap E_+,$$

where $c \in \mathbb{R}$ is some constant. Then, for positive initial data $u_0 \in int(D(F))$, the corresponding mild solution remains positive.

Proof. The result is well known, for convenience of the reader, we provide the argument. Given our assumptions, short time existence of mild solutions follows from an application of Banach's fixed point theorem. Regarding the positivity assertion, we first observe that the problem

$$\begin{aligned}\frac{d}{dt}u + Au &= F(u),\\ u(0) &= u_0\end{aligned}$$

is equivalent to

$$\begin{aligned}\frac{d}{dt}u + (A+c)u &= F(u) + c\,u,\\ u(0) &= u_0.\end{aligned}$$

This equivalence also holds in the category of mild solutions, which can be seen as follows: Let us denote by $\tilde{S}(t), t \geq 0$, the semigroup generated by the operator $-(A+c)$, then we have the relation $\tilde{S}(t) = e^{-ct}S(t), t \geq 0$.
Let u_1 be the unique mild solution to the first equation and u_2 the unique mild solution to the equation with the shifted operator. By definition, this means

$$u_2(t) = \tilde{S}(t)u_0 + \int_0^t \tilde{S}(t-s)\left(F(u_2(s)) + c\,u_2(s)\right) ds.$$

By Lemma 2.15, we can shift the linear part $c\,u_2$ into the generator, yielding

$$u_2(t) = S(t)u_0 + \int_0^t S(t-s)F(u_2(s))ds.$$

Due to uniqueness of u_1, we therefore obtain $u_1 = u_2$, which proves the claim.
Let us now continue with the positivity statement. As already pointed out, we have the relation $\tilde{S}(t) = e^{-ct}S(t), t \geq 0$. In particular, the semigroup $\tilde{S}(t), t \geq 0$, is positive as well. Recall that the mild solution u satisfies

$$\begin{aligned}u(t) &= S(t)u_0 + \int_0^t S(t-s)F(u(s))ds\\ &= \tilde{S}(t)u_0 + \int_0^t \tilde{S}(t-s)\left(F(u(s)) + c\,u(s)\right) ds,\end{aligned}$$

and is obtained by a fixed point iteration. We observe that the assumed lower bound on F guarantees that for any initial value $u_0 \in \operatorname{int}(D(F)) \cap E_+$, the corresponding approximate solutions of the iteration procedure are positive as well, and the claim follows. □

Remark 2.18. Even though the assumption concerning the lower bound in Proposition 2.17 can be weakened, it cannot be dropped in general, even for linear, continuous $F : E \to E$ with arbitrarily small operator norm, cf. Example 2.19. □

Example 2.19. We consider the space $E = C([0,1],\mathbb{R})$, equipped with the supremum norm and the natural ordering induced by $(\mathbb{R}, \leq)$. It is not difficult to see that E is a Banach lattice. Furthermore we introduce the operator

$$-A := \alpha \operatorname{Id}_E \in \mathcal{L}(E),$$

where $\alpha \in \mathbb{R}$ is fixed. Then $-A$ generates the positive semigroup $S(t) = e^{t\alpha}\,\mathrm{Id}_E, t \geq 0$, on E. Next we choose $\phi \in E_+$ and define

$$F : E \to E, \quad F(u) := -\int_0^1 u(a)\,da\,\phi.$$

Obviously F is linear and continuous, with $\|F\|_{\mathcal{L}(E)} = \|\phi\|_E$. In the following we consider the Cauchy problem

$$\begin{aligned} \frac{d}{dt}u + Au &= F(u), \\ u(0) &= u_0, \end{aligned}$$

and investigate positivity of the corresponding mild solutions. By the variation of constants formula we have

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\,F(u(s))\,ds,$$

which easily implies

$$e^{-t\alpha}u(t) = u_0 - \int_0^t \int_0^1 e^{-\alpha s}\,u(s,a)\,da\,ds\,\phi. \tag{2.5}$$

Integrating both sides with respect to a then leads to

$$\int_0^1 e^{-\alpha s}u(s,a)\,da = \beta e^{-\gamma s},$$

where $\beta = \int_0^1 u_0(a)\,da$, $\gamma = \int_0^1 \phi(a)\,da$. Inserting this identity into equation (2.5) we conclude

$$e^{-t\alpha}u(t) = u_0 + \frac{\beta}{\gamma}(e^{-\gamma t} - 1)\,\phi,$$

and from this formula it is easy to see that positive initial data u_0 does not necessarily lead to a positive solution u. □

Proposition 2.20 (Growth rate comparison)**.** *Let $e^{-tA}, t \geq 0$, be a strongly continuous semigroup of positive bounded linear operators on the Banach lattice E, with total postive cone E_+. Furthermore let $B \in \mathcal{L}(E)$ such that*

$$0 \leq Bx \leq c\,x, \quad \forall x \geq 0,$$

for some constant $c > 0$. Then the semigroup $e^{-t(A+B)}, t \geq 0$, is positive as well and we have

$$\omega_0\left(-(A+B)\right) \leq \omega_0(-A).$$

Proof. It is well known that for any bounded operator B the relation

$$e^{-t(A+B)}x = e^{-tA}x - \int_0^t e^{-(t-s)A}Be^{-s(A+B)}x\,ds, \quad t \geq 0$$

holds, for all $x \in E$ (cf. [11, Corollary III.1.7]).

Let us assume for the moment, that the semigroup $e^{-t(A+B)}, t \geq 0$, is positive as well. Then the relation above, in connection with the positivity of B and the semigroups, leads to

$$e^{-t(A+B)}x \leq e^{-tA}x, \qquad \forall t \geq 0,\, x \geq 0.$$

Consequently, Proposition 2.5 implies $\|e^{-t(A+B)}\|_{\mathcal{L}(E)} \leq 4\,\|e^{-tA}\|_{\mathcal{L}(E)}$, $\forall t \geq 0$. The claim now follows from the definition of the growth bound ω_0.

It remains to prove the positivity of the semigroup $e^{-t(A+B)}, t \geq 0$. To this end observe that by assumption we have $-Bx + cx \geq 0$, $\forall x \geq 0$, hence the assertion follows from Proposition 2.17. □

Remark 2.21. From the proof of Proposition 2.20 we see that the condition $Bx \leq c\,x$, $\forall x \geq 0$, can be replaced by the weaker assumption that the semigroup $e^{-t(A+B)}, t \geq 0$, be positive. □

2.6 Evolution operators

In the case of non-autonomous intial-value problems, strongly continuous semigroups are not applicable anymore. Instead, one has to work with evolution operators, which will be introduced in the following (note that the independent variable is replaced by $t \mapsto a$ in this section, which is due to the structure of problem (1.4)).

Let $(E_0, \|\cdot\|_{E_0})$ be a Banach space, $J \subset \mathbb{R}$ a closed interval of the form

$$J = \begin{cases} [a_0, a_m], & \text{if } 0 \leq a_0 < a_m < \infty, \\ [a_0, \infty), & \text{if } 0 \leq a_0 < a_m = \infty. \end{cases}$$

In the following we consider a family of operators $\{A(a) : a \in J\}$ in E_0 such that

$$D(A(a)) = E_1, \quad \text{for all } a \in J, \tag{2.6}$$

where E_1 is a linear subspace of E_0, equipped with a norm $\|\cdot\|_{E_1}$ such that

$$(E_1, \|\cdot\|_{E_1}) \text{ is complete and } E_1 \overset{d}{\hookrightarrow} E_0. \tag{2.7}$$

Furthermore, we suppose

$$A \in C(J, \mathcal{L}(E_1, E_0)). \tag{2.8}$$

Remark 2.22. Recall that for a closed linear operator $A_0 : D(A_0) \subset E_0 \to E_0$, we can equip its domain with the graph norm

$$\|x\|_{A_0} := \|x\|_{E_0} + \|A_0 x\|_{E_0}, \quad x \in D(A_0),$$

and by the closed graph theorem, $(D(A_0), \|\cdot\|_{A_0})$ is a Banach space.

Let us consider a family of operators $\{A(a) : a \in J\}$ in E_0, such that (2.6) holds, define

$$(E_1, \|\cdot\|_{E_1}) := \left(E_1, \|\cdot\|_{A(a_0)}\right), \tag{2.9}$$

and assume the operator $A(a_0)$ to be closed. Then $(E_1, \|\cdot\|_{E_1})$ is complete, and by construction $E_1 \hookrightarrow E_0$.

Suppose the induced graph norms on E_1 satisfy

$$\|\cdot\|_{A(a)} \leq c\|\cdot\|_{A(a_0)},$$

for $a \in J$, where $c = c(a) > 0$, then we immediately obtain

$$A(a) \in \mathcal{L}(E_1, E_0).$$

Furthermore, if the operators $\{A(a) : a \in J\}$ are closed in E_0, then the norm $\|\cdot\|_{A(a)}$ is equivalent to the norm $\|\cdot\|_{A(a_0)}$ by the open mapping theorem.
Conversely, suppose $A(a) \in \mathcal{L}(E_1, E_0)$, then by definition

$$\|A(a)x\|_{E_0} \leq \|A(a)\|_{\mathcal{L}(E_1,E_0)} \left(\|x\|_{E_0} + \|A(a_0)x\|_{E_0}\right), \quad x \in E_1,$$

and consequently

$$\|\cdot\|_{A(a)} \leq \left(1 + \|A(a)\|_{\mathcal{L}(E_1,E_0)}\right) \|\cdot\|_{A(a_0)}.$$

□

Remark 2.23. In case we have to distinguish between two operators, it is reasonable to stress the domain under consideration. More precisely, if $\{B(a) : a \in J\}$ is another one-parameter family of linear operators in E_0, satisfying the corresponding conditions (2.6) – (2.8), we denote the associated Banach space in (2.7) by $(E_B, \|\cdot\|_B)$, and write

$$A \in C(J, \mathcal{L}(E_1, E_0)), \qquad B \in C(J, \mathcal{L}(E_B, E_0)).$$

□

After introducing the notation

$$\Delta_J := \{(a,\sigma) \in J \times J : \sigma \leq a\}, \qquad \dot{\Delta}_J := \{(a,\sigma) \in J \times J : \sigma < a\},$$

we define

Definition 2.24. A two-paramter family of linear operators

$$\Pi : \Delta_J \to \mathcal{L}(E_0), \qquad (a,\sigma) \mapsto \Pi(a,\sigma)$$

is called an *evolution operator* on E_0, if the following two properties are satisfied:

1. $\Pi \in C(\Delta_J, \mathcal{L}_s(E_0))$
2. $\Pi(a,a) = \mathrm{Id}_{E_0}$, $\Pi(a,\sigma) = \Pi(a,\tau)\Pi(\tau,\sigma)$ for all $\sigma \leq \tau \leq a$ with $(a,\sigma) \in \Delta_J$.

Two important subclasses of evolution operators are the so called *hyperbolic* and *parabolic evolution operators*, see e.g. [23]. The latter class will be of particular importance for our purposes:

Definition 2.25. A *parabolic evolution operator* for the operator $A \in C(J, \mathcal{L}(E_1, E_0))$ is a map

$$\Pi : \Delta_J \to \mathcal{L}(E_0)$$

satisfying the following properties:

1. $\Pi \in C(\Delta_J, \mathcal{L}_s(E_0)) \cap C(\dot{\Delta}_J, \mathcal{L}(E_0, E_1))$
2. $\Pi(a,a) = \mathrm{Id}_{E_0}$, $\Pi(a,\sigma) = \Pi(a,\tau)\Pi(\tau,\sigma)$ for all $\sigma \leq \tau \leq a$ with $(a,\sigma) \in \Delta_J$.
3. $[(a,\sigma) \mapsto A(a)\Pi(a,\sigma)] \in C(\dot{\Delta}_J, \mathcal{L}(E_0))$ and

$$\sup_{(a,\sigma)\in\dot{\Delta}_J} (a-\sigma)\|A(a)\Pi(a,\sigma)\|_{\mathcal{L}(E_0)} < \infty.$$

4. $\Pi(\cdot,\sigma) \in C^1\left(J\cap(\sigma,\infty),\mathcal{L}(E_0)\right)$ for each $\sigma\in J,$ and for all $a\in J\cap(\sigma,\infty):$

$$\partial_a\Pi(a,\sigma) = -A(a)\Pi(a,\sigma),$$

$\Pi(a,\cdot) \in C^1\left(J\cap[0,a),\mathcal{L}_s(E_1,E_0)\right)$ for each $a\in J,$ and for all $\sigma\in J\cap[0,a):$

$$\partial_\sigma\Pi(a,\sigma)x = \Pi(a,\sigma)A(\sigma)x$$

for all $x\in E_1$.

Remark 2.26. At this point it is natural to ask for sufficient conditions which guarantee that an operator A generates a parabolic evolution operator.
Suppose A satisfies (2.6) – (2.7), and assume

$$A\in L_\infty(J,\mathcal{L}(E_1,E_0)), \quad \sigma + A\in C^\rho(J,\mathcal{H}(E_1,E_0;\kappa,\nu)),$$

for some $\rho,\,\nu>0$, $\kappa\geq 1$, $\sigma\in\mathbb{R}$. Here $\mathcal{H}(E_1,E_0;\kappa,\nu)\subset\mathcal{L}(E_1,E_0)$ consists of all negative generators $-\mathcal{A}$ of analytic semigroups on E_0 with domain E_1 such that $\nu+\mathcal{A}$ is an isomorphism from E_1 to E_0 and

$$\kappa^{-1}\leq\frac{\|(\lambda+\mathcal{A})x\|_{E_0}}{|\lambda|\|x\|_{E_0}+\|x\|_{E_1}}\leq\kappa, \quad x\in E_1\setminus\{0\},\ \mathrm{Re}\,\lambda\geq\nu.$$

Then A generates a parabolic evolution operator $\Pi_A(a,\sigma)$, $0\leq\sigma\leq a<a_m$, on E_0 with regularity subspace E_1 according to e.g. [2, II.Cor.4.4.2], and there are constants $M\geq 1$ and $\bar{\omega}\in\mathbb{R}$ such that

$$\begin{aligned}&\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_\alpha)}+(a-\sigma)^{\alpha-\beta_1}\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_\beta,E_\alpha)}\leq Me^{-\bar{\omega}(a-\sigma)},\\ &0\leq\sigma\leq a<a_m,\end{aligned}$$

for $0\leq\beta_1\leq\beta<\alpha\leq 1$ with $\beta_1<\beta$ if $\beta>0$, see [2, II.Lem.5.1.3]. Here we have set

$$E_\theta := (E_0,E_1)_\theta,$$

for $\theta\in[0,1]\setminus\{1-1/p\}$, with $(\cdot,\cdot)_\theta$ being any admissible interpolation functor, cf. [2, Section I.2.11]. □

2.7 Maximal regularity

In the following, we introduce the concept of maximal L_p-regularity. To this end we consider a fixed operator

$$A\in C(J,\mathcal{L}(E_1,E_0)),$$

which is such that conditions (2.6) – (2.7) are satisfied. Let $p\in(1,\infty)$ be fixed, set $\zeta:=\zeta(p):=1-1/p$ and

$$E_\zeta := (E_0,E_1)_{\zeta,p},$$

$(\cdot,\cdot)_{\zeta,p}$ being the real interpolation functor. Define the Banach spaces

$$\mathbb{E}_0 := L_p(J,E_0), \qquad \mathbb{E}_1 := L_p(J,E_1)\cap W_p^1(J,E_0),$$

and recall that

$$\mathbb{E}_1 \hookrightarrow BUC(J,E_\zeta),$$

see e.g. [2, Theorem III.4.10.2], where BUC denotes the space of bounded and uniformly continuous functions. In particular, the trace

$$\begin{aligned}\gamma_0 : \mathbb{E}_1 &\to E_\zeta \\ u &\mapsto u(a_0)\end{aligned}$$

is well defined and $\gamma_0 \in \mathcal{L}(\mathbb{E}_1, E_\zeta)$.

Definition 2.27. An operator $A \in C(J, \mathcal{L}(E_1, E_0))$ is said to have *maximal L_p-regularity*, if

$$(\partial_a + A, \gamma_0) : \mathbb{E}_1 \to \mathbb{E}_0 \times E_\zeta \text{ is an isomorphism.}$$

An operator $A \in C(J, \mathcal{L}(E_1, E_0))$ is said to have *inhomogeneous maximal L_p-regularity*, if

$$(\partial_a + A, \gamma_0) : \mathbb{E}_1 \cap \ker(\gamma_0) \to \mathbb{E}_0 \times \{0\} \text{ is an isomorphism.}$$

The class of operators having maximal L_p-regularity is denoted by $\mathrm{MR}_p(J, E_1, E_0)$, the inhomogeneous class is denoted by $\mathrm{MR}_p^0(J, E_1, E_0)$.

Definition 2.28. Consider an operator $A \in C(J, \mathcal{L}(E_1, E_0))$, and let $(f, x) \in \mathbb{E}_0 \times E_\zeta$ be given. An element $u = u(\cdot, A, f, x) \in \mathbb{E}_1$ is called an *L_p-solution*, if it satisfies

$$\begin{aligned}\partial_a u + A(a)u(a) &= f(a), \qquad \text{for a.a. } a \in J, \\ u(a_0) &= x.\end{aligned}$$

Remark 2.29. In order to show that an operator A has maximal L_p-regularity, it suffices to verify that there exists a constant $C > 0$, such that for every pair $(f, x) \in \mathbb{E}_0 \times E_\zeta$ there exists a unique L_p-solution $u = u(\cdot, A, f, x) \in \mathbb{E}_1$ satisfying

$$\|u\|_{L_p(J,E_1)} + \|u\|_{W_p^1(J,E_0)} \leq C \left(\|x\|_{E_\zeta} + \|f\|_{L_p(J,E_0)} \right).$$

This is an immediate consequence of the open mapping theorem. □

Remark 2.30. Suppose $A_0 : D(A_0) \subset E_0 \to E_0$ generates an analytic semigroup on E_0. Let $\theta \in (0, 1)$, $p \in (1, \infty)$ be fixed, and consider the real interpolation space $(E_0, E_1)_{\theta,p}$, where $(E_1, \|\cdot\|_{E_1})$ is the Banach space defined in (2.9). Then it can be shown that

$$\|x\|_{\theta,p} := \|x\|_{E_0} + \left(\int_0^\infty \|t^{1-\theta} A_0 e^{tA_0} x\|^p \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}$$

is an equivalent norm on $(E_0, E_1)_{\theta,p}$, cf. [32, 1.14.5]. In particular, if $\theta = \zeta := 1 - 1/p$, then

$$\left(\int_0^\infty \|A_0 e^{tA_0} x\|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \leq \|x\|_{\zeta,p}, \qquad \text{for } x \in E_\zeta.$$

If, in addition, the function $t \mapsto e^{tA_0}x$ is p-integrable, we obtain

$$[t \mapsto e^{tA_0}x] \in L_p((0,\infty), E_1) \cap W_p^1((0,\infty), E_0), \qquad \text{for } x \in E_\zeta.$$

□

In the following it is shown how the property of maximal L_p-regularity can be formulated as an operator theoretic problem. This formulation has the advantage that it allows for perturbation arguments wich preserve the property of maximal L_p-regularity. For the autonomous case, this approach is carried out in [9], and we will adopt this approach in order to establish a perturbation result for non-autonomous operators, cf. Proposition 2.32 below.

So let us consider an operator $A \in C(J, \mathcal{L}(E_1, E_0))$, then we can introduce an unbounded operator in the space $L_p(J, E_0)$:

$$\begin{aligned} \mathcal{A} : D(\mathcal{A}) \subset L_p(J, E_0) &\to L_p(J, E_0), \\ u &\mapsto Au, \end{aligned} \tag{2.10}$$

with $D(\mathcal{A}) = \{u \in L_p(J, E_1) : Au \in L_p(J, E_0)\}$, where we have set $(Au)(a) := A(a)u(a), a \in J$. Furthermore, define

$$\begin{aligned} D_a : D(D_a) \subset L_p(J, E_0) &\to L_p(J, E_0) \\ u &\mapsto \partial_a u, \end{aligned}$$

with $D(D_a) := \{u \in W_p^1(J, E_0) : u(a_0) = 0\}$.

Let $(f, 0) \in \mathbb{E}_0 \times E_\zeta$ be given, then $u = u(\cdot, A, f, 0)$ is an L_p-solution (cf. Definition 2.28) if and only if $u \in D(D_a) \cap D(\mathcal{A})$ and

$$D_a u + \mathcal{A}u = f.$$

If the operator $D_a + \mathcal{A}$ is boundedly invertible on $L_p(J, E_0)$, we denote this inverse by

$$\begin{aligned} \mathcal{M} : L_p(J, E_0) &\to L_p(J, E_0), \\ f &\mapsto u = u(\cdot, A, f, 0). \end{aligned} \tag{2.11}$$

Observe that for every operator $A \in \mathrm{MR}_p^0(J, E_1, E_0)$, its inverse $\mathcal{M} \in \mathcal{L}(L_p(J, E_0))$ is well defined. It turns out, that also the converse is true:

Lemma 2.31. *Let $A \in C(J, \mathcal{L}(E_1, E_0))$ satisfy* (2.6) – (2.7)*, then:*
$A \in MR_p^0(J, E_1, E_0)$ if and only if

$$D_a + \mathcal{A} : D(D_a) \cap D(\mathcal{A}) \to L_p(J, E_0)$$

is invertible with inverse $\mathcal{M} \in \mathcal{L}(L_p(J, E_0))$.

Proof. Notice that we have the inclusion

$$(D(D_a) \cap D(\mathcal{A})) \subset (\mathbb{E}_1 \cap \ker(\gamma_0)).$$

If $A \in \mathrm{MR}_p^0(J, E_1, E_0)$, then by definition there exists $C > 0$, such that for every $f \in L_p(J, E_0)$ there is a unique L_p-solution $u = u(\cdot, A, f, 0) \in \mathbb{E}_1 \cap \ker(\gamma_0)$ satisfying

$$\|u\|_{L_p(J,E_1)} + \|u\|_{W_p^1(J,E_0)} \leq C \|f\|_{L_p(J,E_0)}.$$

Furthermore, since $A(\cdot)u = f - \partial_a u \in L_p(J, E_0)$, it follows $u \in D(D_a) \cap D(\mathcal{A})$ and

$$D_a u + \mathcal{A}u = f, \qquad \|u\|_{L_p(J,E_0)} \leq C \|f\|_{L_p(J,E_0)},$$

which proves the claim.

Conversely, suppose

$$D_a + \mathcal{A} : D(D_a) \cap D(\mathcal{A}) \to L_p(J, E_0)$$

is invertible with inverse $\mathcal{M} \in \mathcal{L}(L_p(J, E_0))$, i.e. there exists $C > 0$, such that for every $f \in L_p(J, E_0)$ there is a unique $u \in D(D_a) \cap D(\mathcal{A})$ satisfying

$$D_a u + \mathcal{A}u = f, \qquad \|u\|_{L_p(J,E_0)} \leq C \|f\|_{L_p(J,E_0)}.$$

It remains to show that there exists $\tilde{C} > 0$, independent of f, u, such that

$$\|u\|_{L_p(J,E_1)} + \|\partial_a u\|_{L_p(J,E_0)} \leq \tilde{C} \|f\|_{L_p(J,E_0)}. \tag{2.12}$$

To this end, we set

$$X_0 = L_p(J, E_0), \qquad \|u\|_0 = \|u\|_{L_p(J,E_0)},$$

and introduce the normed space

$$X_1 := D(\mathcal{A}), \qquad \|u\|_1 := \|u\|_{L_p(J,E_1)} + \|A(\cdot)u\|_{L_p(J,E_0)}.$$

Let us assume for the moment that $(X_1, \|\cdot\|_1)$ is complete, then we are in the following situation:
By assumption, $\mathcal{M} : X_0 \to X_0$ is bounded and $\mathcal{M}(X_0) \subset X_1$, which gives rise to a map

$$\mathcal{M}_1 : X_0 \to X_1.$$

Since we have the inclusion $X_1 \hookrightarrow X_0$ (because $E_1 \hookrightarrow E_0$), it is an easy consequence of the closed graph theorem (for which the completeness of X_1 is required) that $\mathcal{M}_1$ is bounded. Therefore, there exists a constant $\bar{C} > 0$, such that for every $f \in L_p(J, E_0)$ there is a unique $u \in D(D_a) \cap D(\mathcal{A})$ satisfying

$$D_a u + \mathcal{A}u = f, \qquad \|u\|_{L_p(J,E_1)} + \|A(\cdot)u\|_{L_p(J,E_0)} \leq \bar{C} \|f\|_{L_p(J,E_0)}.$$

Now use the identity $\partial_a u = f - A(\cdot)u$, and estimate (2.12) follows.

In order to conclude the proof, we have to verify the completeness of $(X_1, \|\cdot\|_1)$. Consider the map

$$\mathcal{A} : D(\mathcal{A}) \subset L_p(J, E_1) \to L_p(J, E_0), \qquad u \mapsto A(\cdot)u,$$

and observe that its graph is closed in $L_p(J, E_1) \times L_p(J, E_0)$: indeed let $x_n \in D(\mathcal{A})$ be a sequence such that

$$x_n \to x \text{ in } L_p(J, E_1), \qquad \mathcal{A}x_n \to y \text{ in } L_p(J, E_0).$$

Then we can extract a subsequence, which we again denote by x_n, such that

$$x_n(a) \to x(a) \text{ in } E_1, \text{ for a.a. } a \in J, \qquad A(a)x_n(a) \to y(a) \text{ in } E_0, \text{ for a.a. } a \in J.$$

Since $A(a) \in \mathcal{L}(E_1, E_0)$, for $a \in J$, it follows

$$A(a)x_n(a) \to A(a)x(a) \text{ in } E_0, \text{ for a.a. } a \in J,$$

and consequently, $x \in D(\mathcal{A})$, with $\mathcal{A}x = y$. The closedness of the graph then immediately implies the completeness of $(X_1, \|\cdot\|_1)$.

□

After these considerations, let us recall Remark 2.23 and formulate the perturbation result:

Proposition 2.32. *Let $A \in C(J, \mathcal{L}(E_1, E_0))$ satisfy* (2.6)−(2.7)*, and suppose $A \in MR_p^0(J, E_1, E_0)$, with $p \in [1, \infty)$. Let $B \in C(J, \mathcal{L}(E_B, E_0))$ be a one-parameter family of operators in E_0 such that*

1. $E_1 \hookrightarrow E_B$
2. $\exists \alpha, \beta \in \mathbb{R}_+ : \quad \|B(a)x\|_{E_0} \leq \alpha\|x\|_{E_0} + \beta\|A(a)x\|_{E_0}, \quad x \in E_1, \forall a \in J$
3. $\alpha\|\mathcal{M}\|_{\mathcal{L}(L_p(J,E_0))} + \beta\|\mathcal{AM}\|_{\mathcal{L}(L_p(J,E_0))} < 1,$

where $\mathcal{M}, \mathcal{AM}$ are defined as in (2.10) − (2.11)*. Then*

$$A + B \in MR_p^0(J, E_1, E_0).$$

More precisely, there exists a constant $C > 0$, such that for every $f \in L_p(J, E_0)$ there is a unique L_p-solution $u = u(\cdot, A + B, f, 0) \in L_p(J, E_1) \cap W_p^1(J, E_0)$ satisfying

$$\|u\|_{L_p(J,E_1)} + \|u\|_{W_p^1(J,E_0)} \leq C\, \|f\|_{L_p(J,E_0)}.$$

Proof. First we observe that the first condition implies the operator

$$(A + B) \in C(J, \mathcal{L}(E_1, E_0))$$

to be well defined, and by Lemma 2.31 it suffices to show that

$$D_a + \mathcal{A} + \mathcal{B} : D(D_a) \cap D(\mathcal{A}) \to L_p(J, E_0)$$

is well defined and invertible with inverse in $\mathcal{L}(L_p(J, E_0))$.

The remaining part of the argument is analogous to [9, Theorem 6.1], for the sake of completeness we present it here. Let us recall the definition of the operator $\mathcal{A}$ in (2.10),

$$\begin{array}{rcl} \mathcal{A} : D(\mathcal{A}) \subset L_p(J, E_0) & \to & L_p(J, E_0) \\ u & \mapsto & Au, \end{array}$$

with $D(\mathcal{A}) = \{u \in L_p(J, E_1) :\ Au \in L_p(J, E_0)\}$, and introduce

$$\begin{array}{rcl} \mathcal{B} : D(\mathcal{B}) \subset L_p(J, E_0) & \to & L_p(J, E_0) \\ u & \mapsto & Bu, \end{array}$$

with $D(\mathcal{B}) = \{u \in L_p(J, E_B) :\ Bu \in L_p(J, E_0)\}$.
Since $D(D_a+\mathcal{A}) \subset D(\mathcal{A})$, the first and second condition imply $D(D_a+\mathcal{A}) \subset D(\mathcal{B})$. Furthermore, for all $u \in D(D_a + \mathcal{A})$ we can estimate

$$\begin{array}{rcl} \|\mathcal{B}u\|_{L_p(J,E_0)} & \leq & \alpha\|u\|_{L_p(J,E_0)} + \beta\|\mathcal{A}u\|_{L_p(J,E_0)} \\ & = & \alpha\|u\|_{L_p(J,E_0)} + \beta\|\mathcal{AM}(D_a + \mathcal{A})u\|_{L_p(J,E_0)} \\ & \leq & \alpha\|u\|_{L_p(J,E_0)} + \beta\|\mathcal{AM}\|_{\mathcal{L}(L_p(J,E_0))}\|(D_a + \mathcal{A})u\|_{L_p(J,E_0)}. \end{array}$$

Since also the third condition is satisfied, we are in the situation of Theorem A.1, with $R = D_a+\mathcal{A}$ and $S = \mathcal{B}$, so that

$$D_a + \mathcal{A} + \mathcal{B} : D(D_a) \cap D(\mathcal{A}) \to L_p(J, E_0)$$

is invertible with bounded inverse.

□

Definition 2.33. For a closed suberinterval $I \subset J$, we can consider the restriction of an operator $A \in C(J, \mathcal{L}(E_1, E_0))$, which is denoted by

$$A_I := A|_I \in C(I, \mathcal{L}(E_1, E_0)).$$

Lemma 2.34 (Maximal regularity on subintervals)**.** *Suppose there exists a parabolic evolution operator for $A \in C(J, \mathcal{L}(E_1, E_0))$, let (2.6) – (2.7) hold, and $A \in MR_p^0(J, E_1, E_0)$. If $I \subset J$ is any closed subinterval, then*

$$A_I \in MR_p^0(I, E_1, E_0).$$

Proof. Let us consider the case $a_0 = 0, a_m = \infty$, i.e. $J = [0, \infty)$, and note that the other cases can be treated analogously. Without loss of generality, let the subinterval have the form $I = [a_1, \infty)$, with $a_1 > 0$. Consider the Cauchy problem

$$\begin{aligned} \partial_a u + A_I(a)u &= f_I(a), \qquad a \in I, \\ u(a_1) &= x, \end{aligned}$$

where $f_I \in L_p(I, E_0)$, $x \in E_\zeta$. Suppose $u = u(\cdot, a_1, A_I, f_I, x) \in L_p(I, E_1) \cap W_p^1(I, E_0)$ is an L_p-solution, then it has to coincide with the mild solution, i.e.

$$u(a, a_1, A, f_I, x) = \Pi_A(a, a_1)x + \int_{a_1}^a \Pi_A(a, s) f_I(s)\, \mathrm{d}s, \quad a \in I, \tag{2.13}$$

cf. [2, Proposition III.1.3.1], in particular it is unique. Thus, it remains to show existence of an L_p-solution.
To this end, set $x = 0$, fix an element $f_I \in L_p(I, E_0)$ as above and let $f_J \in L_p(J, E_0)$ denote its extension to the interval J, i.e.

$$f_J(a) = \begin{cases} 0, & a \in J \setminus I \\ f_I(a), & a \in I \end{cases}.$$

Since $\in \mathrm{MR}_p^0(J, E_1, E_0)$ by assumption, the Cauchy problem

$$\begin{aligned} \partial_a u + A_J(a)u &= f_J(a), \qquad a \in J, \\ u(0) &= 0, \end{aligned}$$

has a unique solution $u = u(\cdot, 0, A_J, f_J, 0) \in L_p(J, E_1) \cap W_p^1(J, E_0)$. Furthermore, by [2, Proposition III.1.3.1], this solution is given by

$$\begin{aligned} u(a, 0, A_J, f_J, 0) &= \int_0^a \Pi_A(a, s) f_J(s)\, \mathrm{d}s \\ &= \int_{a_1}^a \Pi_A(a, s) f_I(s)\, \mathrm{d}s. \end{aligned}$$

Consequently,

$$u(a, a_1, A_I, f_I, 0) := \int_{a_1}^a \Pi_A(a, s) f_I(s), \quad a \in I,$$

is the desired L_p-solution.
Furthermore, as a direct consequence of this construction we obtain

$$\begin{aligned} &\|u(\cdot, a_1, A_I, f_I, 0)\|_{L_p(I, E_1)} + \|u(\cdot, a_1, A_I, f_I, 0)\|_{W_p^1(I, E_0)} \\ &= \|u(\cdot, 0, A_J, f_J, 0)\|_{L_p(J, E_1)} + \|u(\cdot, 0, A_J, f_J, 0)\|_{W_p^1(J, E_0)} \\ &\le C_J \|f_J\|_{L_p(J, E_0)} \\ &= C_J \|f_I\|_{L_p(I, E_0)}, \end{aligned}$$

where the inequality is an immediate consequence of the assumption
$A \in \mathrm{MR}_p^0(J, E_1, E_0)$. Together with Remark 2.29, this completes the proof. □

Remark 2.35. Let the assumptions of Lemma 2.34 hold, then we conclude that $A_I \in \mathrm{MR}_p^0(I, E_1, E_0)$, for any closed subinterval $I \subset J$. Let $\mathcal{M}_I \in \mathcal{L}(L_p(I, E_0))$ denote the inverse associated with the operator A_I (cf. (2.11)), then we see from the proof of Lemma 2.34 that there exists a positive constant C_J, which depends on J but not on I, such that we can estimate

$$\|\mathcal{M}_I\|_{\mathcal{L}(L_p(I,E_0))} \leq C_J.$$

Furthermore, let $\mathcal{A}_I$ denote the operator in $L_p(I, E_0)$ induced by A_I (cf. (2.10)). Given $f \in L_p(I, E_0)$, let $u = u(\cdot, A_I, f, 0) \in L_p(I, E_1) \cap W_p^1(I, E_0)$ denote the corresponding L_p-solution. In particular, we have the identity $A_I u = f - \partial_a u \in L_p(I, E_0)$, and from the proof of Lemma 2.34 we obtain

$$\begin{aligned} \|A_I u\|_{L_p(I,E_0)} &\leq \|f\|_{L_p(I,E_0)} + \|\partial_a u\|_{L_p(I,E_0)} \\ &\leq (1 + C_J)\|f\|_{L_p(I,E_0)}, \end{aligned}$$

which leads us to

$$\|\mathcal{A}_I \mathcal{M}_I\|_{\mathcal{L}(L_p(I,E_0))} \leq 1 + C_J.$$

□

The following result shows that the property of maximal regularity can be reduced to the inhomogeneous case (in the sense of Definition 2.27).

Proposition 2.36. *Suppose there exists a parabolic evolution operator for $A \in BC(J, \mathcal{L}(E_1, E_0))$, let* (2.6) − (2.7) *hold, and $A \in MR_p^0(J, E_1, E_0)$. Then $A \in MR_p(J, E_1, E_0)$.*

Proof. Consider the Cauchy problem

$$\begin{aligned} \partial_a u + A(a)u &= f(a), \qquad a \in J, \\ u(a_0) &= x, \end{aligned}$$

then we have to show that for all $f \in L_p(J, E_0)$, $x \in E_\zeta$, there exists a unique solution $u = u(\cdot, a_0, f, x) \in L_p(J, E_1) \cap W_p^1(J, E_0)$, which depends continuously on f and x.
Recall from (2.13) that this solution necessarily coincides with the mild solution, consequently is uniquely determined. Furthermore, we have $A \in \mathrm{MR}_p^0(J, E_1, E_0)$ by assumption, hence, by the superposition principle, it remains to verify the claim for the case $f = 0$.
Without loss of generality assume $a_m = \infty$, i.e. $J = [a_0, \infty)$, with $a_0 \geq 0$, then we have to solve

$$\begin{aligned} \partial_a u + A(a)u &= 0, \qquad a \in (a_0, \infty), \\ u(a_0) &= x. \end{aligned} \tag{2.14}$$

To this end, we proceed as in the proof of [3, Lemma 2.2]: choose $w \in L_p(J, E_1) \cap W_p^1(J, E_0)$ such that $w(a_0) = x$ (such an element w exists due to the trace method characterisation of the interpolation space E_ζ, see e.g. [21, Corollary 1.14]), and set

$$f(a) := -\partial_a w - A(a)w, \qquad a \in (a_0, \infty).$$

Since $A : J \to \mathcal{L}(E_1, E_0)$ is bounded by assumption, it follows $f \in L_p(J, E_0)$. Therefore, there exists a unique solution $v \in L_p(J, E_1) \cap W_p^1(J, E_0)$ to

$$\begin{aligned} \partial_a v + A(a)v &= f, \qquad a \in (a_0, \infty), \\ v(a_0) &= 0, \end{aligned}$$

where we used that $A \in \mathrm{MR}_p^0(J, E_1, E_0)$. Setting $u := v + w$ yields the desired solution. As a result, we obtain a mapping

$$\begin{array}{rcl} E_\zeta & \to & L_p(J, E_1) \cap W_p^1(J, E_0) \\ x & \mapsto & u(\cdot, a_0, 0, x), \end{array}$$

which associates to each initial value $x \in E_\zeta$ the unique solution $u = u(\cdot, a_0, 0, x)$ to (2.14). In the remaining part we will verify that this mapping is bounded. Let us consider sequences

$$\begin{array}{rcl} x_n & \to & x \quad \text{in } E_\zeta, \\ u(\cdot, a_0, 0, x_n) & \to & u \quad \text{in } L_p(J, E_1) \cap W_p^1(J, E_0), \end{array}$$

then there exists a subsequence of $u(\cdot, a_0, 0, x_n)$, which converges pointwise almost everywhere on J, without loss of generality

$$u(a, a_0, 0, x_n) \to u(a) \quad \text{in } E_0, \text{ for almost all } a \in J.$$

On the other hand, since Π is a parabolic evolution operator, we have

$$\Pi(a, a_0)x_n \to \Pi(a, a_0)x \quad \text{in } E_0, \text{ for all } a \in J.$$

Together with (2.13), this implies $u = u(\cdot, a_0, 0, x)$, and the closed graph theorem yields the claim. □

The following result is an applier-friendly instance of [12] and was established in [28]:

Proposition 2.37 ([28] Local to global regularity)**.** *Let E_0 be a Banach space, $1 < p < \infty$, $a_m \in (0, \infty]$, and $\{A(a) : a \in [0, a_m]\}$ a family of boundedly invertible sectorial operators in E_0 satisfying*

1. *$D(A(a)) = D(A(0))$, $a \in [0, a_m]$.*
2. *The mapping $A : [0, a_m) \to \mathcal{L}(D(A(0)), E_0))$ is continuous, where $D(A(0))$ is endowed with the graph norm.*
3. *$A(a) \to A(a_m)$ in $\mathcal{L}(D(A(0)), E_0))$, as $a \to a_m$.*
4. *For each $a \in [0, a_m]$, the operator $A(a)$ has maximal L_p-regularity.*

Then the operator A has maximal L_p-regularity.

Proposition 2.38 (Global to local regularity)**.** *Suppose there exists a parabolic evolution operator for $A \in C(J, \mathcal{H}(E_1, E_0; \kappa, \nu))$, let (2.6) − (2.7) hold, and $A \in MR_p^0(J, E_1, E_0)$. In case J is unbounded, i.e. $a_m = \infty$, assume $\omega_0(-A(a)) < 0$ for all $a \in [0, a_m)$. Then $A(a) \in MR_p^0(J, E_1, E_0)$ for every $a \in [0, a_m)$.*

Proof. Since $A \in \mathrm{MR}_p^0(J, E_1, E_0)$ by assumption, it follows that for each $\tilde{a} \in [0, a_m)$ there exists $\delta = \delta(\tilde{a}) > 0$ such that we have

$$A(\tilde{a}) \in \mathrm{MR}_p([0, \delta], E_1, E_0).$$

To be more precise, let $\tilde{a} \in [0, a_m)$ be fixed, and $\delta > 0$ such that $I := [\tilde{a}, \tilde{a} + \delta] \subset J$. Recall that the operator $A_I \in C(I, \mathcal{L}(E_1, E_0))$ induces an operator $\mathcal{A}_I$ in $L_p(I, E_0)$, cf. (2.10). From Lemma

2.34 it follows that $A_I \in \mathrm{MR}_p^0(I, E_1, E_0)$, with corresponding inverse $\mathcal{M}_I \in \mathcal{L}(L_p(I, E_0))$ (see (2.11)). Setting $B_I(a) := A(\tilde{a}) - A_I(a)$, for $a \in I$, we can write

$$A(\tilde{a}) = A_I + B_I \in C(I, \mathcal{L}(E_1, E_0)),$$

and estimate

$$\begin{aligned}
\|B_I(a)x\|_{E_0} &\leq \|A(\tilde{a}) - A_I(a)\|_{\mathcal{L}(E_1,E_0)} \|x\|_{E_1} \\
&\leq \|A(\tilde{a}) - A_I(a)\|_{\mathcal{L}(E_1,E_0)}\, \kappa \, \|(\nu + \sigma + A_I(a))x\|_{E_0} \\
&\leq \|A(\tilde{a}) - A_I(a)\|_{\mathcal{L}(E_1,E_0)}\, \kappa \, |\nu + \sigma| \, \|x\|_{E_0} \\
&+ \|A(\tilde{a}) - A_I(a)\|_{\mathcal{L}(E_1,E_0)}\, \kappa \, \|A_I(a)x\|_{E_0},
\end{aligned}$$

for $a \in I$, $x \in E_1$, where we used the resolvent estimate for the second inequality, cf. Remark 2.26. Since $A_I : [\tilde{a}, \tilde{a} + \delta] \to \mathcal{L}(E_1, E_0)$ is continuous by assumption, we see that the conditions of Proposition 2.32 are fulfilled, if we choose $\delta = \delta(\tilde{a}) > 0$ sufficiently small (observe that the norms $\|\mathcal{M}_I\|, \|\mathcal{A}_I\mathcal{M}_I\|$ cannot blow up as $\delta \to 0$, see Remark 2.35), and consequently $A(\tilde{a}) \in \mathrm{MR}_p^0(I, E_1, E_0)$. A simple shift argument (the operator $A(\tilde{a})$ is autonomous) then yields

$$A(\tilde{a}) \in \mathrm{MR}_p^0([0, \delta(\tilde{a})], E_1, E_0).$$

In the case of finite a_m, i.e. $J = [0, a_m]$, we therefore obtain

$$A(a) \in \mathrm{MR}_p^0(J, E_1, E_0), \qquad \text{for every } a \in [0, a_m),$$

by [9, Cor 5.4]. In the case $a_m = \infty$, this property remains true, since the semigroup $e^{-tA(a)}, t \geq 0$, is assumed to decay exponentially, cf. [9, Thm 5.2]. □

3 Semilinear stability

In this section we consider a simplified instance of problem (1.4), where the birth rate b is assumed to be independent of the density u, i.e.

$$\begin{aligned}
\partial_t u + \partial_a u + A(a)u &= -\mu(u, a)u, \qquad t > 0,\ a \in (0, a_m), \\
u(t, 0) &= \int_0^{a_m} b(a)u(t, a)da, \qquad t > 0, \\
u(0, a) &= u_0(a), \qquad a \in (0, a_m).
\end{aligned} \tag{3.1}$$

If also the mortality rate μ was independent of u, we would recover the linear problem, which was studied in [36], [37] and serves as a basis for the forthcoming considerations of this section. As a first crucial observation we remark that the linear structure of the birth rate allows to interpret (3.1) as a semilinear Cauchy problem in an appropriate Banach space. The question of stability then reduces to the analysis of the corresponding linearised problem. Assuming maximal L_p-regularity of the diffusion operator A, this will enable us to obtain a characterisation of the stability behaviour. Note that in particular we assume $p \in (1, \infty)$ throughout this section.

To be more precise, we would like to think of (3.1) as a problem of the form

$$\begin{aligned}
\frac{d}{dt}u &= -\mathbb{A}u + F(u), \quad t > 0, \\
u(0) &= u_0,
\end{aligned} \tag{3.2}$$

with a suitable linear operator $-\mathbb{A}$ that incorporates the age boundary condition $u(t,0) = \int_0^{a_m} b(a)u(a)\,\mathrm{d}a$ in its domain $D(-\mathbb{A}) \subset \mathbb{E}_0$. This operator should act as

$$-\mathbb{A}\phi = -\partial_a \phi - A\phi, \quad \phi \in D(-\mathbb{A}).$$

As regards the existence of a solution to (3.2), it would furthermore be desirable that $-\mathbb{A}$ generates a strongly continuous semigroup on the Banach space $\mathbb{E}_0$. To this end, an obvious approach would be trying to apply classical results like the Hille-Yosida criterion, with the disadvantage that, in general, one does not know the semigroup. In order to circumvent this issue, we will take a different approach, carried out e.g. in [40], [33], which can be summarised as follows: analogous to section 1.1, one integrates the linear instance of problem (3.1). It can be shown that this gives rise to a strongly continuous semigroup on $\mathbb{E}_0$, and it is natural to conjecture that the corresponding generator agrees with the operator $-\mathbb{A}$ sketched above. In some sense, we have shifted the difficulties: instead of verifying the conditions for the Hille-Yosida (or some related) criterion, we now have to determine the domain of the generator of a semigroup. It turns out that the latter difficulty can be overcome if one imposes maximal L_p-regularity on the diffusion operator A (this assumption is not too restrictive and satisfied in many relevant cases). Observe that in contrast to the alternative approach we now also have an explicit formula for the semigroup, which is crucial for the subsequent considerations of section 3.

Let us start with the analysis outlined above and take a look at the linear instance of problem (3.1):

$$\begin{aligned} \partial_t u + \partial_a u + A(a)u &= 0, \qquad t>0,\ a \in (0,a_m) \\ u(t,0) &= \int_0^{a_m} b(a)u(t,a)da, \qquad t>0 \\ u(0,a) &= u_0(a), \qquad a \in (0,a_m). \end{aligned} \tag{3.3}$$

Here we assumed without loss of generality a vanishing mortality rate $\mu = \mu(a) \equiv 0$, since otherwise it can be absorbed by the linear operator A. Equation (3.3) was studied in [36], in the following we collect some of the results obtained therein. To this end, we first clarify the conditions that are imposed.

3.1 Assumptions

Throughout this section, E_0 denotes a real Banach lattice with closed, total cone E_0^+. Note that we do not distinguish E_0 from its complexification in our notation as no confusion seems likely. Let E_1 be a densely and compactly embedded subspace of E_0. We fix $p \in (1,\infty)$, set $\zeta := \zeta(p) := 1 - 1/p$, and introduce

$$E_\zeta := (E_0, E_1)_{\zeta,p}, \qquad E_\theta := (E_0,E_1)_\theta,$$

for $\theta \in [0,1] \setminus \{1-1/p\}$, with $(\cdot,\cdot)_{\zeta,p}$ being the real interpolation functor and $(\cdot,\cdot)_\theta$ being any admissible interpolation functor, cf. [2, Section I.2.11]. Analogous to sections 2.6-2.7 set

$$J = \begin{cases} [0,a_m], & \text{if } a_m < \infty \\ [0,\infty), & \text{if } a_m = \infty \end{cases} \tag{3.4}$$

and define

$$\mathbb{E}_0 = L_p(J,E_0), \qquad \mathbb{E}_1 = L_p(J,E_1) \cap W_p^1(J,E_0).$$

Furthermore, we denote by $\mathbb{E}_0^+$ those functions in $\mathbb{E}_0$ which take values in E_0^+ almost everywhere and remark that $\mathbb{E}_0$ becomes a Banach lattice.

As regards the operator A, we suppose that $\{A(a) : a \in J\}$ is a one-parameter family of linear closed operators in E_0 such that

$$D(A(a)) = E_1 \subset E_0, \quad \text{for all } a \in J.$$

In accordance with [36], we suppose

$$A \in L_\infty(J, \mathcal{L}(E_1, E_0)), \qquad \sigma + A \in C^\rho(J, \mathcal{H}(E_1, E_0; \kappa, \nu)), \tag{3.5}$$

for some ρ, $\nu > 0$, $\kappa \geq 1$, $\sigma \in \mathbb{R}$. Recalling Remark 2.26, this assumption implies that A generates a parabolic evolution operator $\Pi_A(a,\sigma)$, $0 \leq \sigma \leq a < a_m$, on E_0 with regularity subspace E_1 and there are constants $M \geq 1$ and $\bar{\omega} \in \mathbb{R}$ such that

$$\begin{aligned} &\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_\alpha)} + (a-\sigma)^{\alpha-\beta_1}\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_\beta,E_\alpha)} \leq Me^{-\bar{\omega}(a-\sigma)}, \\ &0 \leq \sigma \leq a < a_m, \end{aligned} \tag{3.6}$$

for $0 \leq \beta_1 \leq \beta < \alpha \leq 1$ with $\beta_1 < \beta$ if $\beta > 0$. We further assume that $\Pi_A(a,\sigma)$ is positive for $0 \leq \sigma \leq a < a_m$, and

$$\bar{\omega} > 0 \quad \text{if } a_m = \infty. \tag{3.7}$$

Moreover, we suppose that

$$\left.\begin{array}{l} \text{for each } \lambda > -\bar{\omega},\ a \in J, \text{ the operator} \\ A_\lambda(a) := \lambda + A(a) \text{ has maximal } L_p\text{-regularity,} \end{array}\right\} \tag{3.8}$$

and in the case $a_m = \infty$, we additionally assume

$$\left.\begin{array}{l} A(\infty) := \lim_{a\to a_m} A(a) \text{ exists in } \mathcal{L}(E_1, E_0), \\ \text{for each } \lambda > -\bar{\omega}, \text{ the operator} \\ A_\lambda(\infty) := \lambda + A(\infty) \text{ has maximal } L_p\text{-regularity.} \end{array}\right\} \tag{3.9}$$

Finally, we require

$$\left.\begin{array}{l} b \in BC(J, \mathcal{L}(E_\theta)) \cap L_{p'}(J, \mathcal{L}(E_\theta)),\ \theta \in [0,1], \\ b(a) \in \mathcal{L}_+(E_0) \text{ a.e. in } J, \\ \lim_{a\to a_m} b(a) = 0, \text{ if } a_m < \infty, \end{array}\right\} \tag{3.10}$$

where $p' = p/(p-1)$ is the dual exponent of p, and that

$$\left.\begin{array}{l} b(a)\Pi_A(a,0) \in \mathcal{L}_+(E_0) \text{ is strongly positive,} \\ \text{for } a \text{ in a subset of } J \text{ of positive measure.} \end{array}\right\} \tag{3.11}$$

We remark that condition (3.11) does not hold, if $b \equiv 0$. In particular, Theorem 3.16 below is not applicable in this case.

Remark 3.1. In [36], condition (3.8) is assumed to hold for all $\lambda \in \mathbb{C}$, such that $\text{Re}\ \lambda > -\bar{\omega}$. At a later stage, we will apply the stability result [36, Theorem 3.5]; it is straightforward to check that it remains true under the weaker assumption (3.8). □

3.2 Auxiliary results

As a first consequence of these assumptions we obtain

Corollary 3.2. *Let $A \in C(J, \mathcal{L}(E_1, E_0))$ satisfy (3.5) – (3.9). Then for each $\lambda > -\bar{\omega}$, the operator*

$$A_\lambda = \lambda + A$$

has maximal L_p-regularity.

Proof. Let us start with the case $a_m = \infty$. Fix $\lambda > -\bar{\omega}$ and let $a \in J \cup \{a_m\}$, then assumptions (3.8) – (3.9) imply

$$\omega_0(A_\lambda(a)) < 0,$$

see [9, Corollary 4.2]. In particular, these operators are boundedly invertible and sectorial, Proposition 2.37 then yields the claim.

Now let $a_m < \infty$. Fix $\lambda > -\bar{\omega}$, then the uniform resolvent estimate in assumption (3.5) (cf. Remark 2.26) implies the existence of a constant $c > 0$, such that

$$\omega_0\left(-(c + A_\lambda(a))\right) < 0, \quad \forall a \in J.$$

In particular, these operators are boundedly invertible and sectorial. Furthermore, assumption (3.8) in combination with a simple shift argument (cf. [9, Theorem 3.3]) leads to

$$c + A_\lambda(a) \in \mathrm{MR}_p^0(J, E_1, E_0), \quad \forall a \in J.$$

Proposition 2.36 and Proposition 2.37 then imply

$$c + A_\lambda \in \mathrm{MR}_p(J, E_1, E_0).$$

With another application of the shift argument we conclude

$$A_\lambda \in \mathrm{MR}_p^0(J, E_1, E_0),$$

and Proposition 2.36 then yields the claim. □

Remark 3.3. Observe that instead of conditions (3.8) – (3.9), we could have imposed

for each $\lambda > -\bar{\omega}$, the operator
$A_\lambda = \lambda + A$ has maximal L_p-regularity.

Proposition 2.38 (together with some additional technical assumptions) then shows that (3.8) – (3.9) is necessary. □

Analogous to section 1.1 one can formally integrate equation (3.3) in order to obtain a formula for the solution:

$$u(t,a) = \begin{cases} \Pi_A(a, a-t)u_0(a-t), & 0 \le t \le a \\ \Pi_A(a,0)B_{u_0}(t-a), & 0 \le a < t, \end{cases} \tag{3.12}$$

where $B_{u_0}(t) = u(t,0)$ satisfies, due to the age-boundary condition in (3.3), the Volterra equation

$$\begin{aligned} B_{u_0}(t) &= \int_0^t b(a)\Pi_A(a,0)B_{u_0}(t-a)\,\mathrm{d}a \\ &+ \int_t^{a_m} b(a)\Pi_A(a,a-t)u_0(a-t)\,\mathrm{d}a, \quad t \ge 0. \end{aligned}$$

Since we are assuming (3.10), there is a unique solution $B_{u_0} \in C([0,\infty), E_0)$ (see Lemma 4.5 below), hence we set

$$\left(S(t)u_0\right)(a) := u(t,a), \tag{3.13}$$

where u is the solution to (3.12). As has already been pointed out in [36, Theorem 2.2], one can argue along the lines of [40, Theorem 4] to obtain:

Theorem 3.4. *Assume the operator A satisfies conditions* (3.5) − (3.7), (3.10)*. Then $S(t), t \geq 0$, given by* (3.13) *is a strongly continuous semigroup on $\mathbb{E}_0$ with*

$$\sup_{t\geq 0} e^{t(\bar{\omega}-\zeta)}\|S(t)\|_{\mathcal{L}(\mathbb{E}_0)} < \infty,$$

where $\zeta = M\|b\|_{L_\infty(J,\mathcal{L}(E_0))}$. If the evolution operator Π_A is positive, then the semigroup $S(t), t \geq 0$, is positive.

In Corollary 3.2 we have shown that the operator $\lambda + A$ has maximal L_p-regularity. This property is used in [36, Theorem 2.8] to characterise the generator

$$-\mathbb{A} : D(-\mathbb{A}) \subset \mathbb{E}_0 \to \mathbb{E}_0$$

of the semigroup $S(t), t \geq 0$:

Theorem 3.5 ([36])**.** *Assume the operator A satisfies conditions* (3.5) − (3.11)*. Then $\phi \in \mathbb{E}_0$ belongs to the domain $D(-\mathbb{A})$ if and only if $\phi \in \mathbb{E}_1$ with*

$$\phi(0) = \int_0^{a_m} b(a)\phi(a)\, da.$$

Furthermore, $\mathbb{A}\phi = \partial_a\phi + A\phi$ for $\phi \in D(-\mathbb{A})$.

In light of Theorem 3.5, we can formulate problem (3.1) as a semilinear Cauchy problem:

$$\begin{aligned} \frac{d}{dt}u &= -\mathbb{A}u + F(u), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{3.14}$$

where we have set

$$F : D(F) \subset \mathbb{E}_0 \to \mathbb{E}_0, \qquad F(u) := -\mu(u,\cdot)u. \tag{3.15}$$

Remark 3.6. Observe that in (3.15), we implicitly assumed the element $F(u) = \mu(u,\cdot)u \in \mathbb{E}_0$ to be well defined. In general this cannot be expected, rather one has to impose proper conditions on the mortality rate μ, see (3.19) − (3.21) below. □

Recalling Definition 2.11, a function $u \in C([0,T),\mathbb{E}_0)$ is said to be a *mild solution* to (3.14) on $[0,T)$, if $u(t) \in D(F)$ for all $t \in [0,T)$ and

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds, \qquad t \in [0,T). \tag{3.16}$$

If the nonlinearity is sufficiently regular, namely

$$F : D(F) \subset \mathbb{E}_0 \to \mathbb{E}_0 \text{ is Lipschitz on bounded sets, with } D(F) \text{ open}, \tag{3.17}$$

then Proposition 2.12 guarantees the existence of mild solutions:

Proposition 3.7 (Existence of solutions)**.** *Assume the operator A satisfies conditions* (3.5) − (3.7), (3.10)*, and the nonlinearity F in* (3.15) *fulfills condition* (3.17)*. Then for every $u_0 \in D(F)$ there exists a maximal $T = T(u_0) > 0$ such that there is a unique mild solution $u = u(\cdot, u_0)$ to* (3.14) *on $[0,T)$. Furthermore, if $D(F) = \mathbb{E}_0$ and $T < \infty$, then $\lim_{t\nearrow T}\|u(t,u_0)\| = \infty$.*

Proof. Assumptions (3.5) – (3.7), (3.10) on the operator A guarantee that the semigroup $S(t)$, $t \geq 0$, is well defined, cf. Theorem 3.4. Since the nonlinearity F satisfies condition (3.17), we can apply Proposition 2.12 and the claim follows. □

After having assured the existence of mild solutions, we can study their asymptotic behaviour, in particular the convergence to equilibrium states:

Definition 3.8. An element $\phi \in D(-\mathbb{A}) \cap D(F) \subset \mathbb{E}_0$ is called an *equilibrium* of equation (3.14) if it satisfies

$$-\mathbb{A}\phi + F(\phi) = 0. \tag{3.18}$$

3.3 Stability of the trivial equilibrium

In the following we study the semilinear Cauchy problem (3.14), in particular the stability behaviour of the trivial equilibrium $\phi = 0$, cf. Definition 2.14. In order to employ the method of linearisation, some regularity conditions have to be imposed on the nonlinearity (3.15), which we clarify below.

We consider a nonlinear mortality rate

$$\begin{aligned} \mu : D \times J &\to \mathcal{L}_+(E_0) \\ (u, a) &\mapsto \mu(u, a), \end{aligned} \tag{3.19}$$

where $D \subset \mathbb{E}_0$ is an open neighbourhood of $\phi = 0$, assume a continuity condition

$$a \mapsto \mu(u, a)v \in C(J, E_0), \qquad \forall u \in D, v \in E_0, \tag{3.20}$$

and existence of a constant $\bar{\mu} \in \mathbb{R}_+$, such that for arbitrary $(u, a) \in D \times J$ we have

$$\mu(u, a)v \leq \bar{\mu}\, v, \qquad \forall\, v \in E_0^+. \tag{3.21}$$

Furthermore, we impose that

$$(a \mapsto \mu(0, a)) \in C^\rho(J, \mathcal{L}(E_1, E_0)), \tag{3.22}$$

with Hölder exponent ρ from (3.5).

Remark 3.9. The growth assumption (3.21) is justified if we recall the single-species model from the introductory section: in that model we have a spatial variable $x \in \Omega \subset \mathbb{R}^n$ and the mortality modulus $\mu = \mu(u, a, x)$ is a real nonnegative scalar. The condition $0 \leq \mu \leq 1$ then means that the portion of deceasing individuals cannot be greater than the actual population. Furthermore, if we want to allow for a spatial dependence of the mortality modulus or a population consisting of several subspecies (e.g. predator-prey models), we have to interpret μ as a bounded positive operator rather than a scalar in the abstract setting. □

Given assumptions (3.19) – (3.21), we see that the nonlinearity gives rise to a well defined map

$$\begin{aligned} f : D \subset \mathbb{E}_0 &\to \mathcal{L}_+(\mathbb{E}_0) \\ u &\mapsto \mu(u, .), \end{aligned}$$

where $(\mu(u, .)v)(a) := \mu(u, a)v(a)$, $a \in J$, $v \in \mathbb{E}_0$, and with Proposition 2.5 we conclude $\|f(u)\|_{\mathcal{L}(\mathbb{E}_0)} \leq 4\,\bar{\mu}$. Observe that condition (3.20) guarantees the strong measurability of the function

$$\begin{aligned} J &\to E_0 \\ a &\mapsto \mu(u, a)v(a), \end{aligned}$$

for $u \in D, v \in \mathbb{E}_0$ (cf. proof of Lemma A.4). We assume

$$f : D \subset \mathbb{E}_0 \to \mathcal{L}(\mathbb{E}_0) \text{ is Lipschitz continuous} \tag{3.23}$$

and define

$$\begin{aligned} F : D \subset \mathbb{E}_0 &\to \mathbb{E}_0 \\ u &\mapsto -f(u)\,u. \end{aligned}$$

Then we have

$$\begin{array}{l} F : D \subset \mathbb{E}_0 \to \mathbb{E}_0 \text{ is Fréchet differentiable at } \phi = 0, \\ \text{with derivative } F'(0)u = -f(0)u,\ u \in \mathbb{E}_0, \\ \text{and Lipschitz continuous on bounded subsets of } D, \end{array} \tag{3.24}$$

which is an immediate consequence of the estimates

$$\begin{aligned} \|F(u) - F(0) + f(0)u\|_{\mathbb{E}_0} &= \| - f(u)u + f(0)u\|_{\mathbb{E}_0} \\ &\leq \|f(u) - f(0)\|_{\mathcal{L}(\mathbb{E}_0)}\, \|u\|_{\mathbb{E}_0} \\ &= o(\|u\|_{\mathbb{E}_0}), \end{aligned}$$

and

$$\begin{aligned} \|F(u) - F(v)\|_{\mathbb{E}_0} &\leq \|f(u) - f(v)\|_{\mathcal{L}(\mathbb{E}_0)}\, \|u\|_{\mathbb{E}_0} \\ &+ \|f(v)\|_{\mathcal{L}(\mathbb{E}_0)}\, \|u - v\|_{\mathbb{E}_0}. \end{aligned}$$

Remark 3.10. Observe that property (3.24) allows to apply Proposition 3.7, and therefore yields existence of mild solutions to equation (3.14). Furthermore, the growth assumption (3.21) implies

$$F(u) + \bar{\mu}\, u \geq 0 \text{ in } \mathbb{E}_0^+, \qquad \forall\, u \in \mathbb{E}_0^+ \cap D,$$

Proposition 2.17 therefore guarantees that a positive initial datum $u_0 \in D$ leads to a positive mild solution. □

After having declared the main assumptions, we can start the analysis of the stability behaviour for the equilibrium $\phi = 0$. According to the principle of linearised stability from Theorem 2.16, it suffices to analyse the semigroup generated by the operator $-\mathbb{A} + F'(0)$. This generator acts as

$$(-\mathbb{A} + F'(0))\, \psi = -\mathbb{A}\psi - \mu(0, .)\psi, \qquad \psi \in D(-\mathbb{A}).$$

To this end, we define

$$\hat{A}(a)v := A(a)v + \mu(0, a)v, \qquad v \in E_1, \tag{3.25}$$

and denote by $\Pi_{\hat{A}}(a, \sigma)$, $0 \leq \sigma \leq a$, the parabolic evolution operator generated by $\hat{A}$, the existence of which will be verified in Lemma 3.11. Observe that in the special case of commutativity of A with $\mu(0, .)$, the evolution operator is given by the formula

$$\Pi_{\hat{A}}(a, \sigma) = e^{-\int_\sigma^a \mu(0,s)ds}\Pi_A(a, \sigma), \qquad 0 \leq \sigma \leq a.$$

Before we inspect the perturbed operator $\hat{A}$ in Lemma 3.11, we have to impose a final assumption. Namely, in the case $a_m = \infty$, we assume

$$\mu(0, a_m) := \lim_{a \to a_m} \mu(0, a) \text{ exists in } \mathcal{L}(E_1, E_0). \tag{3.26}$$

Lemma 3.11 (Perturbation lemma). *Assume the operator A fulfills conditions* (3.5) − (3.7)*, and the mortality rate μ satisfies* (3.19) − (3.22)*. Then, conditions* (3.5) − (3.7) *remain true for the perturbed operator $\hat{A}$ in* (3.25)*.*
Assume, in addition to the previous assumptions, the operator A fulfills conditions (3.8) − (3.9)*, and in the case $a_m = \infty$, let also* (3.26) *hold. Then, conditions* (3.8) − (3.9) *remain true for the perturbed operator $\hat{A}$.*

Proof. Let us start with condition (3.5). As already pointed out, we have $\|\mu(0,a)\|_{\mathcal{L}(E_0)} \leq 4\bar{\mu}$, due to assumption (3.21), and since $\iota : E_1 \hookrightarrow E_0$ is bounded, we obtain $\|\mu(0,a)\|_{\mathcal{L}(E_1,E_0)} \leq 4\,\bar{\mu}\,\|\iota\|_{\mathcal{L}(E_1,E_0)}$. Thus, we see that

$$\hat{A} \in L_\infty(J, \mathcal{L}(E_1, E_0)).$$

Next we consider some fixed $a \in J$ and recall that $-A(a)$ generates an analytic semigroup on E_0. Since $\mu(0,a)$ is a bounded operator on E_0 by (3.19), we conclude that

$$\hat{A}(a) \in \mathcal{H}(E_1, E_0), \quad \forall\, a \in J$$

(cf. [11, Prop III.1.12], alternatively we can argue that $\mu(0,a) \circ \iota : E_1 \to E_0$ is compact, hence $\hat{A}(a)$ generates an analytic semigroup by [7, Thm 1]).
Together with (3.22) this implies that condition (3.5) remains true for the operator $\hat{A}$.

In the next step we verify (3.6) and (3.7), i.e. we have to control the growth bound of the evolution operator corresponding to $\hat{A}$. Essentially this means that we have to establish proper resolvent estimates for the perturbation. However, classical "bounded perturbation" type estimates in the sense of e.g. Proposition 2.9 are not sufficient for our purpose, since they do not guarantee that the growth bound does not increase (or remains negative) in the perturbation process, as required in (3.7). To resolve this issue we will apply maximum principle techniques, for which the positivity of the perturbation term $\mu(0,.)$ will be crucial. To this end we observe the relation

$$\Pi_{\hat{A}}(a,\sigma)v = \Pi_A(a,\sigma)v - \int_\sigma^a \Pi_A(a,s)\mu(0,s)\Pi_{\hat{A}}(s,\sigma)v\,ds, \tag{3.27}$$

(cf. proof of [2, Lem. II.5.1.4]). Now let us assume for the moment that with Π_A also $\Pi_{\hat{A}}$ is positive. Since $\mu(0,.)$ is a positive operator by assumption, the relation above implies,

$$\Pi_{\hat{A}}(a,\sigma)v \leq \Pi_A(a,\sigma)v, \qquad \forall v \geq 0,$$

and Proposition 2.5 then yields $\|\Pi_{\hat{A}}(a,\sigma)\| \leq 4\,\|\Pi_A(a,\sigma)\|$. Together with assumption (3.6), this implies

$$\begin{aligned}&\|\Pi_{\hat{A}}(a,\sigma)\|_{\mathcal{L}(E_\alpha)} + (a-\sigma)^{\alpha-\beta_1}\|\Pi_{\hat{A}}(a,\sigma)\|_{\mathcal{L}(E_\beta,E_\alpha)} \leq 4Me^{-\bar{\omega}(a-\sigma)},\\ &0 \leq \sigma \leq a < a_m,\end{aligned} \tag{3.28}$$

Thus, we see that (3.6) and (3.7) hold and it remains to show the positivity of $\Pi_{\hat{A}}$. This can be achieved with an argument like in the proof of Proposition 2.17, for the sake of completeness we present it here:
Essentially, we want to show that the solution to the problem

$$\begin{aligned}\frac{d}{da}u &= -\hat{A}(a)u,\\ u(\sigma) &= v \in E_0^+,\end{aligned}$$

is positive. To this end, we rewrite the problem in two successive steps. First, we interpret the zero order term in $\hat{A}$ as an inhomogeneity:

$$\begin{aligned} \frac{d}{da}u &= -A(a)u - \mu(0,a)u, \\ u(\sigma) &= v \in E_0^+. \end{aligned}$$

In the second step we add and substract the zero order term $\bar{\mu}\, u$, writing $A_{\bar{\mu}} = A + \bar{\mu}\,\iota$, then yields:

$$\begin{aligned} \frac{d}{da}u &= -A_{\bar{\mu}}(a)u + (-\mu(0,a) + \bar{\mu}\,\iota)\,u, \\ u(\sigma) &= v \in E_0^+. \end{aligned}$$

Now we observe that $\Pi_{A_{\bar{\mu}}}(a,\sigma) = e^{-\bar{\mu}(a-\sigma)}\Pi_A(a,\sigma)$ is positive and the solution u can be obtained by a Banach fixed point iteration. The growth assumption (3.21) guarantees that the iterate solutions are positive, hence the claim follows.

Next, we examine the maximal regularity conditions (3.8) – (3.9). More precisely, we assume the operator A satisfies (3.8) – (3.9), then we have to verify

for each $\lambda > -\bar{\omega}$, $a \in J$, the operator
$\hat{A}_\lambda(a) := \lambda + \hat{A}(a)$ has maximal L_p-regularity,

and in the case $a_m = \infty$,

$\hat{A}(\infty) := \lim_{a \to a_m} \hat{A}(a)$ exists in $\mathcal{L}(E_1, E_0)$,
for each $\lambda > -\bar{\omega}$, the operator
$\hat{A}_\lambda(\infty) := \lambda + \hat{A}(\infty)$ has maximal L_p-regularity.

To this end, we fix $\lambda > -\bar{\omega}$ and consider the case $a_m < \infty$ first. Recall that for each $a \in [0, a_m]$, we have $\mu(0,a) \in \mathcal{L}(E_0)$, in combination with maximal L_p-regularity of the operator $\lambda + A(a)$ this implies

$$\lambda + \hat{A}(a) \in \mathrm{MR}_p^0(J, E_1, E_0), \qquad \forall a \in [0, a_m],$$

see e.g. [9, Theorem 6.2]. Proposition 2.36 then yields the claim.

Now let $a_m = \infty$. Observe that assumption (3.9) together with (3.26) implies existence of the limit

$$\hat{A}(\infty) := \lim_{a \to a_m} \hat{A}(a) \text{ in } \mathcal{L}(E_1, E_0).$$

Let us fix $a \in J \cup \{a_m\}$, then maximal L_p-regularity of the operator $\lambda + A(a)$ implies that the growth bound of the associated semigroup is negative, i.e. $\omega_0(-(\lambda + A(a))) < 0$. Since $\mu(0,a) \in \mathcal{L}_+(E_0)$ by assumption (3.19), we conclude as in the first part of the proof

$$\omega_0(-(\lambda + \hat{A}(a))) < 0.$$

Consequently, $\lambda + \hat{A}(a) \in \mathrm{MR}_p^0(J, E_1, E_0)$ by [9, Theorem 5.2]. Proposition 2.36 then yields the claim. □

Let us suppose, the operator A fulfills conditions (3.5) – (3.7), and the mortality rate μ is such that (3.19) – (3.22) hold.
Then it was shown in Lemma 3.11 that conditions (3.5) – (3.7) remain true for the perturbed operator $\hat{A}$ in (3.25). If, in addition, (3.10) is satisfied, this perturbed operator induces a one-parameter family of operators $\hat{S}(t), t \geq 0$, via formula (3.13) (where the evolution operator Π_A is replaced by $\Pi_{\hat{A}}$). Furthermore, Theorem 3.4 is applicable, and we conclude that $\hat{S}(t), t \geq 0$, is a strongly continuous semigroup on $\mathbb{E}_0$. To sum up, in analogy to Theorem 3.4 we have:

Corollary 3.12 (Perturbed semigroup)**.** *Assume the operator A fulfills conditions* (3.5) – (3.7)*,* (3.10)*, and the mortality rate μ satisfies* (3.19) – (3.22)*. Then the operator $\hat{A}$ in* (3.25) *induces a semigroup $\hat{S}(t), t \geq 0$, on $\mathbb{E}_0$, with*

$$\sup_{t\geq 0} e^{t(\bar{\omega}-\hat{\zeta})} \|\hat{S}(t)\|_{\mathcal{L}(\mathbb{E}_0)} < \infty,$$

where $\hat{\zeta} = 4\,M\|b\|_{L_\infty}$. If the evolution operator Π_A is positive, then the semigroup $\hat{S}(t), t \geq 0$, is positive.

Proof. The existence of the semigroup $\hat{S}(t), t \geq 0$, has already been clarified in the motivation preceding the statement. As regards the estimate, we observe that the assumed conditions were used to establish estimate (3.28) in the proof of Lemma 3.11. An application of Theorem 3.4 to the operator $\hat{A}$ then yields the claim.
In the proof of Lemma 3.11 it was shown that the positivity of the evolution operator Π_A, in combination with assumption (3.19), implies the positivity of $\Pi_{\hat{A}}$. The positivity of the semigroup $\hat{S}(t), t \geq 0$, is then an immediate consequence. ☐

Subsequently, we introduce a one-parameter family of operators $\hat{A}_\lambda := \lambda + \hat{A}$, with $\lambda \in \mathbb{C}$, and their corresponding parabolic evolution operators

$$\Pi_{\hat{A}_\lambda}(a, \sigma) = e^{-\lambda(a-\sigma)} \Pi_{\hat{A}}(a, \sigma), \quad 0 \leq \sigma \leq a < a_m.$$

Finally, for $\lambda \in \mathbb{C}$, with $\mathrm{Re}\,\lambda > -\bar{\omega}$ if $a_m = \infty$, we define linear operators $Q_\lambda, \hat{Q}_\lambda \in \mathcal{L}_+(E_0)$ by

$$\begin{aligned} Q_\lambda &:= \int_0^{a_m} b(a) \Pi_{A_\lambda}(a, 0)da, \\ \hat{Q}_\lambda &:= \int_0^{a_m} b(a) \Pi_{\hat{A}_\lambda}(a, 0)da, \end{aligned} \tag{3.29}$$

and denote by $r(Q_\lambda), r(\hat{Q}_\lambda) \in \mathbb{R}_+$ the corresponding spectral radii. Observe that the boundedness of these operators is a direct consequence of assumptions (3.7), (3.10). Furthermore, the parabolicity of the evolution operator Π_A (in the sense of (3.6)), together with the compact embedding $E_1 \hookrightarrow E_0$, implies the compactness of $Q_\lambda \in \mathcal{L}_+(E_0)$, cf. [36, Lemma 2.4]. This observation, in combination with the crucial assumption (3.11), was used in [36] to employ the Krein-Rutman Theorem and conclude:

Lemma 3.13 ([36])**.** *Assume the operator A fulfills conditions* (3.5) – (3.11)*. Let $\lambda \in \mathbb{R}$, with $\lambda > -\bar{\omega}$ if $a_m = \infty$. Then the spectral radius $r(Q_\lambda)$ is positive and a simple eigenvalue of $Q_\lambda \in \mathcal{L}(E_0)$ with an eigenvector in E_1 that is quasi-interior in E_0^+. It is the only eigenvalue of Q_λ with a positive eigenvector.*

Lemma 3.14 ([36])**.** *Assume the operator A fulfills conditions* (3.5) – (3.11)*. Then the mapping*

$$[\lambda \mapsto r(Q_\lambda)] : [0, \infty) \to (0, \infty)$$

is continuous, strictly decreasing, and $\lim_{\lambda\to\infty} r(Q_\lambda) = 0$.

Lemma 3.15 ([36])**.** *Assume the operator A fulfills conditions* (3.5) – (3.11)*. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\bar{\omega}$ if $a_m = \infty$ and let $m \in \mathbb{N} \setminus \{0\}$. Then $\lambda \in \sigma_p(-\mathbb{A})$ with geometric multiplicity m if and only if $1 \in \sigma_p(Q_\lambda)$ with geometric multiplicity m.*

Lemmata 3.13-3.15 allow us to show that the operator $\hat{Q}_0$ encodes the stability behaviour of the trivial equilibrium $\phi = 0$ (recall formulation (3.14) and Definition 3.8):

Theorem 3.16. *Assume the operator A fulfills conditions* (3.5) − (3.11)*, the nonlinearity F in* (3.15) *satisfies* (3.19) − (3.23)*, and in the case $a_m = \infty$, let also* (3.26) *hold. Lastly, suppose $\hat{A}$ fulfills condition* (3.11)*. Then the asymptotic behaviour of equation* (3.14) *is characterised as follows:*

1. *If the spectral radius $r(\hat{Q}_0)$ is strictly smaller than 1, then the growth bound $\omega_0(-\mathbb{A}+F'(0))$ is negative. In particular, the equilibrium $\phi = 0$ is exponentially asymptotically stable.*
2. *If the spectral radius $r(\hat{Q}_0)$ is strictly greater than 1, then the equilibrium $\phi = 0$ is unstable.*

Proof. Observe that the operator $\hat{A}$ in (3.25) fulfills conditions (3.5) − (3.9) by Lemma 3.11, condition (3.10) obviously remains true and condition (3.11) holds by assumption. Therefore, we can apply the results obtained in [36] to the induced semigroup $\hat{S}(t), t \geq 0$, from Corollary 3.12, and its generator $-\hat{\mathbb{A}}$. By Theorem 3.5, this generator is given by $-\hat{\mathbb{A}} = -(\partial_a + \hat{A}) = -\mathbb{A} + F'(0)$, where we used (3.24) for the last equality. In particular, if $r(\hat{Q}_0) < 1$ holds, then [36, Theorem 3.5] implies that the growth bound $\omega_0(-\hat{\mathbb{A}})$ is negative. Theorem 2.16 then implies the first assertion.

In [36, Lemma 3.2] a negative upper bound for the α-growth bound $\omega_1(-\mathbb{A})$ is established. The perturbation $\hat{A}$ satisfies (3.5) − (3.7), (3.10), so we can apply the aforementioned lemma to the operator $-\hat{\mathbb{A}}$ and see, that also $\omega_1(-\hat{\mathbb{A}})$ is negative. In the next step we will show that $\omega_0(-\hat{\mathbb{A}})$ is positive, Theorem 2.16 then implies that the equilibrium $\phi = 0$ is unstable. In order to see that the growth bound of $-\hat{\mathbb{A}}$ is positive, we will apply some results of [36]. To this end, we consider the one-parameter family of operators $\hat{Q}_\lambda \in \mathcal{L}(E_0)$, for $\lambda \geq 0$, introduced in (3.29). By assumption, $r(\hat{Q}_0) > 1$ holds, so Lemma 3.14 implies that there is some $\lambda > 0$ such that $r(\hat{Q}_\lambda) = 1$. By Lemma 3.13, $r(\hat{Q}_\lambda) = 1$ is an eigenvalue of $\hat{Q}_\lambda$. Now we can use Lemma 3.15 to conclude that λ is an eigenvalue of $-\hat{\mathbb{A}}$. Since λ is positive, we see that the growth bound has to be positive as well. This finishes the proof. □

Remark 3.17 (Interpretation)**.** By Theorem 3.16, stability of the equilibrium $\phi = 0$ is encoded by the compact operator

$$\hat{Q}_0 = \int_0^{a_m} b(a)\Pi_{\hat{A}}(a,0)da \in \mathcal{L}_+(E_0),$$

which allows an intuitive interpretation: If the operator $\hat{Q}_0$ is small, e.g. if the birth rate b is small or the diffusion operator $-\hat{A} = -(A + \mu(0,\cdot))$ has large sinks (meaning there is a spatial drain of population), then a small population will become extinct. □

Example 3.18. In the following we will work out a more concrete example and demonstrate an application of the theory we have developed so far. We consider the example introduced in section 1:

$$\begin{aligned}
\partial_t u + \partial_a u - \operatorname{div}_x(d(a,x)\nabla_x u) &= -\mu(u,a)u, \quad t > 0,\, a \in (0,a_m),\, x \in \Omega,\\
u(t,0,x) &= \int_0^{a_m} b(a,x)u(a)\mathrm{d}a, \quad t > 0,\, x \in \Omega,\\
\partial_\nu u(t,a,x) &= 0, \quad t > 0,\, a \in (0,a_m),\, x \in \partial\Omega,\\
u(0,a,x) &= \phi(a,x), \quad a \in (0,a_m),\, x \in \Omega,
\end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^2 (cf. [1]).

We want to formulate this problem in the abstract framework (1.4). To this end, we fix $q \in (1, \infty)$, and set

$$E_0 := L_q(\Omega), \qquad E_1 := W^2_{q,\mathcal{B}} := \{\phi \in W^2_q(\Omega) : \partial_\nu \phi = 0\},$$

with E_0 ordered by its positive cone of functions that are nonnegative almost everywhere. Furthermore, we introduce for each $a \in [0, a_m)$ the operator

$$\begin{aligned} A(a) : E_1 &\to E_0, \\ (A(a)v)(x) &:= -\operatorname{div}_x(d(a,x)\nabla_x v), \qquad \text{for } v \in E_1, x \in \Omega. \end{aligned}$$

At this point we have to make some assumptions on the coefficient of the above differential operator. Recalling definition (3.4) , we require the ellipticity condition

$$d(a,x) \geq c > 0, \quad \forall (a,x) \in J \times \overline{\Omega},$$

where c is some positive constant, and a regularity condition, for simplicity e.g.

$$d \in BC^2(J \times \overline{\Omega}).$$

In the following we verify that the required assumptions of Theorem 3.16 hold.

Let us start with the family of operators $A(a), a \in J$. Note that our ellipticity and regularity assumptions on the coefficient imply that for $a \in J$ fixed, $(A(a), \mathcal{B})$ constitutes a regular elliptic boundary value problem as studied in [1] (with the symbol $\mathcal{B}$ denoting Neumann boundary conditions). As in [1, Sect. 7] we therefore conclude that $-A(a)$ generates an analytic semigroup on E_0, together with the prescribed regularity of the coefficient this implies that condition (3.5) is satisfied.

As has already been pointed out in (3.6), there exist constants $M \geq 1$, $\bar{\omega} \in \mathbb{R}$ such that

$$\begin{aligned} &\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_\alpha)} + (a-\sigma)^{\alpha-\beta_1}\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_\beta,E_\alpha)} \leq Me^{-\bar{\omega}(a-\sigma)}, \\ &0 \leq \sigma \leq a < a_m, \end{aligned}$$

for $0 \leq \beta_1 \leq \beta < \alpha \leq 1$ with $\beta_1 < \beta$ if $\beta > 0$. Note that condition (3.7) will not hold in general (consider the constant family of operators $A(a) = -\Delta, a \geq 0$, which admit the eigenvalue $\lambda = 0$). For this reason, we will translate the differential operator $A(a)$ by a zero order term, i.e. for $\tau < 0$ fixed, we introduce the operator

$$-A_\tau := -A + \tau\iota,$$

where $\iota : E_1 \hookrightarrow E_0$, and observe that our initial problem can be formulated as

$$\begin{aligned} \partial_t u + \partial_a u + A_\tau(a)u &= -(\tau + \mu(u,a))\,u, \qquad t > 0,\ a \in (0, a_m) \\ u(t,0) &= \int_0^{a_m} b(a)u(a)da, \qquad t > 0 \\ u(0,a) &= u_0(a), \qquad a \in (0, a_m). \end{aligned} \tag{3.30}$$

The idea is that we continue our analysis with the A_τ-formulation instead of the natural A-formulation. It is easy to see, that condition (3.5) is still satisfied. Let Π_A, Π_{A_τ} denote the evolution operators associated with A and A_τ, then we have the identity

$$\Pi_{A_\tau}(a,\sigma) = e^{\tau(a-\sigma)}\Pi_A(a,\sigma), \qquad 0 \leq \sigma \leq a < a_m.$$

Now we multiply estimate (3.6) for the evolution operator Π_A with the factor $e^{\tau(a-\sigma)}$ to obtain

$$\|\Pi_{A_\tau}(a,\sigma)\|_{\mathcal{L}(E_\alpha)} + (a-\sigma)^{\alpha-\beta_1}\|\Pi_{A_\tau}(a,\sigma)\|_{\mathcal{L}(E_\beta,E_\alpha)} \leq Me^{(\tau-\bar{\omega})(a-\sigma)},$$
$$0 \leq \sigma \leq a < a_m,$$

and for τ sufficiently small we can achieve $\tau - \bar{\omega} < 0$ if $a_m = \infty$, hence the operator A_τ satisfies condition (3.7).

Let us now turn to the verification of condition (3.8) for A_τ, i.e. we want to show that for each $\lambda > \tau - \bar{\omega}$, $a \in J$, the operator $\lambda + A_\tau(a)$ has maximal L_p-regularity. So let us fix some $\lambda > \tau - \bar{\omega}$, $a \in J$, and note that [1, Thm 11.1] implies that $-A(a)$ is the generator of a contraction semigroup on E_0 (observe that the scalar structure of the diffusion term d implies $\beta_0^{\#} = 0$ in [1, Thm 11.1]). In particular we see that for $\lambda > \tau - \bar{\omega}$ we have

$$\omega_0\left(-(\lambda + A_\tau(a))\right) = \tau - \lambda + \omega_0\left(-A(a)\right) < \bar{\omega} < 0,$$

where the last inequality is a consequence of [2, Thm II.5.1.1] (alternatively, after inspecting estimate (3.6), we see that we can decrease $\bar{\omega}$ until we have a negative sign). Thus, $-\left(\lambda + A_\tau(a)\right)$ generates for each $a \in J$ a contraction semigroup on E_0 with negative growth bound. Furthermore, the operator is selfadjoint on $L_2(\Omega)$ and by [1, Sect. 7] resolvent positive. Hence we are in the situation of [2, Ex. III.4.7.3(d)], and conclude

$$\lambda + A_\tau(a) \in \mathcal{BIP}(E_0, \theta), \qquad \text{with } \theta = \theta(q) \in [0, \pi/2).$$

Now we can apply [2, Thm III.4.10.7] to obtain that

$$\lambda + A_\tau(a) \in \mathcal{L}(E_1, E_0) \text{ has maximal } L_p\text{-regularity.}$$

For the case $a_m = \infty$ let us assume that the coefficients

$$d(a,.), \nabla_x d(a,.) \text{ converge in } \left(C(\overline{\Omega}), \|.\|_\infty\right) \text{ as } a \to \infty.$$

Then also the limit, denoted by $(A(\infty), \mathcal{B})$, constitutes a regular elliptic boundary value problem in the sense of [1], and as before we conclude

$$\lambda + A_\tau(\infty) \in \mathcal{L}(E_1, E_0) \text{ has maximal } L_p\text{-regularity.}$$

Hence we see that conditions (3.8) – (3.9) are fulfilled.

Condition (3.10) is an assumption on the asymptotic behaviour (and the sign) of the birth rate b, and is assumed to be satisfied in the following.

Finally, let us assume

$$q > n + 2,$$

so that [6, Cor. 13.6] implies that $\Pi_{A_\tau}(a, 0) \in \mathcal{L}_+(E_0)$ is strongly positive for $a > 0$. Therefore, condition (3.11) is satisfied if $b(a) > 0 \in \mathbb{R}$ for a in a subset of J of positive measure, what is true unless $b \equiv 0$, since b is continuous by (3.10).

At last it remains to scrutinise the nonlinearity. Observe that in formulation (3.30), the corresponding nonlinearity is given by

$$\mu_\tau(u, \cdot) := \mu(u, \cdot) + \tau, \tag{3.31}$$

and we have to show that conditions (3.19) – (3.23), (3.26) of Theorem 3.16 hold. This is for instance the case, if μ is of the form

$$\mu(\cdot) : \mathbb{E}_0 \to \mathbb{R}_+, \qquad \mu(u) = g(\int_0^{a_m} k(a)\, u(a)\, da),$$

with $g \in BC^1(\mathbb{R}, \mathbb{R}_+)$ such that $\inf_{\lambda\in\mathbb{R}} g(\lambda) \geq -\tau$, and $k \in L_{p'}(J, E_0{}')$, where p' is the dual exponent of p.

In light of the foregoing considerations, the linearised term takes the form $\mu_\tau(0,.) = \mu(0,.)+\tau$, and the perturbed operator $\hat{A}_\tau$ is given by

$$\begin{aligned}\hat{A}_\tau(a)v &:= (A_\tau(a) + \mu_\tau(0,a))\, v \\ &= (A(a) + \mu(0,a))\, v \\ &= (A(a) + g(0))\, v,\end{aligned}$$

for $v \in E_1, a \in J$. Consequently, also the operator $\hat{A}_\tau$ satisfies condition (3.11), and we see that all assumptions of Theorem 3.16 are fulfilled. □

In the remaining part of this section we study the stability behaviour of the equilibrium $\phi = 0$ for the treshold value $r(\hat{Q}_0) = 1$, cf. Theorem 3.16. To this end we will need the following general result (cf. (24*b*) in [24]):

Proposition 3.19. *Let E be a Banach space and $-A$ the generator of a strongly continuous semigroup $S(t), t \geq 0$, on E. Assume the following conditions are fulfilled:*

1. *$\omega_1(-A) < \omega_0(-A)$.*
2. *Every eigenvalue λ of $-A$ with Re $\lambda = s(-A)$ is simple.*

Then there is a constant $\tilde{M} \geq 1$ such that the estimate

$$\|S(t)\|_{\mathcal{L}(E)} \leq \tilde{M} \exp(s(-A)t), \qquad t \geq 0$$

holds.

The positivity of the semigroup $S(t), t \geq 0$, together with Proposition 3.19 allow us to derive:

Proposition 3.20. *Assume the operator A fulfills conditions* (3.5) – (3.11)*, the nonlinearity F in* (3.15) *satisfies* (3.19) – (3.21) *and is such that* (3.17) *holds. Let $u_0 \in \mathbb{E}_0^+$ be an arbitrary initial value and $u(\,.\,; u_0) : [0, T_{u_0}) \to \mathbb{E}_0$ the corresponding mild solution to equation* (3.14)*. Suppose the spectral radius of the operator $Q_0 \in \mathcal{L}(E_0)$ is equal to* 1*. Then there is a constant $\tilde{M} \geq 1$ such that the estimate*

$$\|u(t; u_0)\|_{\mathbb{E}_0} \leq \tilde{M}\|u_0\|_{\mathbb{E}_0}, \qquad t \in [0, T_{u_0})$$

holds. In particular, if the nonlinearity is such that $D(F) = \mathbb{E}_0$, then the trivial equilibrium is stable within the cone $\mathbb{E}_0^+$.

Proof. Assumptions (3.19) – (3.21) guarantee that $F : D(F) \subset \mathbb{E}_0 \to \mathbb{E}_0$ is well defined, cf. Remark 3.6. Furthermore, condition (3.17) implies that Proposition 3.7 is applicable, and we obtain the existence of a unique mild solution $u = u(\cdot, u_0)$, defined on a maximal interval $[0, T_{u_0})$.

Since the semigroup $S(t), t \geq 0$, and the solution $t \mapsto u(t; u_0)$ are positive (here we used assumption (3.21), cf. Remark 3.10), we can combine the formulae for the mild solution (3.16) and the nonlinearity (3.15) to conclude

$$0 \leq u(t; u_0) \leq S(t)u_0, \qquad 0 \leq t < T_{u_0}.$$

Because $\mathbb{E}_0$ is a Banach lattice, this in turn implies

$$\|u(t; u_0)\|_{\mathbb{E}_0} \leq \|S(t)u_0\|_{\mathbb{E}_0}, \qquad 0 \leq t < T_{u_0}. \tag{3.32}$$

Let us recall the definition of the operator Q_0 in (3.29), then we know from Lemma 3.13 that the spectral radius $r(Q_0)$ is an eigenvalue of Q_0. Furthermore, we have $r(Q_0) = 1$ by assumption. Thus we can apply Lemma 3.15 to conclude that $0 \in \sigma_p(-\mathbb{A})$.

In particular, we must have $\omega_0(-\mathbb{A}) \geq 0$. Since the α-growth bound $\omega_1(-\mathbb{A})$ is strictly negative by [36, Lemma 3.2], we see that the first condition of Proposition 3.19 is fulfilled.

In the following we verify the second condition of Proposition 3.19. To this end we remark that for the essential spectrum, the estimate

$$\sup_{\lambda \in \sigma_e(-\mathbb{A})} \operatorname{Re} \lambda \leq \omega_1(-\mathbb{A}) \tag{3.33}$$

holds (cf. [38, Proposition 4.13]). Now let us assume there exists $\lambda \in \sigma(-\mathbb{A})$ with $\operatorname{Re} \lambda > 0$. In particular we have the strict inequality

$$s(-\mathbb{A}) > \omega_1(-\mathbb{A}),$$

where $s(-\mathbb{A}) = \sup \operatorname{Re} \sigma(-\mathbb{A})$ denotes the spectral bound. Applying Proposition 2.10, we obtain that the peripheral spectrum contains exactly one element, namely the spectral bound:

$$\sigma_0(-\mathbb{A}) = \{s(-\mathbb{A})\}. \tag{3.34}$$

Estimate (3.33) implies, that $s(-\mathbb{A})$ cannot lie in the essential spectrum and therefore is an eigenvalue of $-\mathbb{A}$ (cf. [38, Proposition 4.11]). This in turn implies that $r(Q_{s(-\mathbb{A})}) \geq 1$, by Lemma 3.15. But, since $r(Q_0) = 1$ by assumption, this yields a contradiction to the monotonicity of the spectral radii in Lemma 3.14.

To sum up, we have shown

$$s(-\mathbb{A}) = 0,$$

and it is easy to see that identity (3.34) remains valid. Furthermore, [36, Lemma 3.6] tells us that 0 is a simple eigenvalue of $-\mathbb{A}$ (here we again used the assumption $r(Q_0) = 1$).

Thus we see that Proposition 3.19 is applicable and obtain

$$\|S(t)\|_{\mathcal{L}(\mathbb{E}_0)} \leq \tilde{M}, \qquad t \geq 0.$$

This estimate, together with (3.32), leads to

$$\|u(t; u_0)\|_{\mathbb{E}_0} \leq \tilde{M} \|u_0\|_{\mathbb{E}_0}, \qquad t \in [0, T_{u_0}).$$

Furthermore, if $D(F) = \mathbb{E}_0$, then Proposition 3.7 implies $T_{u_0} = \infty$, and the proof is complete. □

The following result gives, in combination with Theorem 3.16, a rather complete characterisation of the stability behaviour of the equilibrium $\phi = 0$ in terms of the spectral radius of the operator $\hat{Q}_0 \in \mathcal{L}(E_0)$ defined in (3.29):

Theorem 3.21. *Assume the operator A fulfills conditions* (3.5) − (3.10)*, the nonlinearity F in* (3.15) *satisfies* (3.19) − (3.22)*, and is such that* (3.17) *holds. In the case $a_m = \infty$, let* (3.26) *hold. Lastly, suppose $\hat{A}$ fulfillls condition* (3.11)*.*
Let $u_0 \in \mathbb{E}_0^+$ be an arbitrary initial value and $u(\,.\,; u_0) : [0, T_{u_0}) \to \mathbb{E}_0$ the corresponding mild solution to equation (3.14)*. Suppose the spectral radius of the operator $\hat{Q}_0 \in \mathcal{L}(E_0)$ is equal to* 1 *and $\mu(v, .)v \geq \mu(0, .)v$, $\forall v \in D(F) \cap \mathbb{E}_0^+$. Then there is a constant $\tilde{M} \geq 1$ such that the estimate*

$$\|u(t; u_0)\|_{\mathbb{E}_0} \leq \tilde{M} \|u_0\|_{\mathbb{E}_0}, \qquad t \in [0, T_{u_0})$$

holds. In particular, if the nonlinearity is such that $D(F) = \mathbb{E}_0$, then the trivial equilibrium is stable within the cone $\mathbb{E}_0^+$.

Proof. Assumptions (3.19) – (3.21) guarantee that $F : D(F) \subset \mathbb{E}_0 \to \mathbb{E}_0$ is well defined, cf. Remark 3.6. Furthermore, condition (3.17) implies that Proposition 3.7 is applicable, and we obtain the existence a unique mild solution $u = u(\cdot, u_0)$, defined on a maximal interval $[0, T_{u_0})$.

Next we observe that the Cauchy problem (3.14) is equivalent to

$$\begin{aligned}\frac{d}{dt}u &= -\hat{\mathbb{A}}u - (\mu(u,.) - \mu(0,.))\,u, \quad t > 0,\\ u(0) &= u_0,\end{aligned}$$

where $-\hat{\mathbb{A}} := -\mathbb{A} - \mu(0,.) = -(\partial_a + \hat{A})$. We would like to argue as in the proof of Proposition 3.20, so we first verify that the reformulated problem satisfies the corresponding prerequisites: Our assumptions guarantee that Lemma 3.11 is applicable, and we conclude that the operator $\hat{A}$ satisfies conditions (3.5) – (3.9). Observe that the remaining conditions (3.10) – (3.11) hold by assumption.

Next we recall from Corollary 3.12 that $\hat{A}$ induces the semigroup $\hat{S}(t), t \geq 0$, on $\mathbb{E}_0$, and by Theorem 3.5 its generator is $-(\partial_a + \hat{A}) = -\hat{\mathbb{A}}$. Since we have $r(\hat{Q}_0) = 1$ by assumption, we obtain as in the proof of Proposition 3.20 the estimate

$$\|\hat{S}(t)\|_{\mathcal{L}(\mathbb{E}_0)} \leq \tilde{M}, \qquad t \geq 0.$$

Furthermore, we have

$$u(t) = \hat{S}(t)u_0 - \int_0^t \hat{S}(t-s)\,(\mu(u) - \mu(0))\,u(s)\,\mathrm{d}s,$$

cf. Lemma 2.15. Together with the assumed monotonicity of μ, this leads as in the proof of Proposition 3.20 to the estimate

$$\|u(t)\|_{\mathbb{E}_0} \leq \|\hat{S}(t)u_0\|_{\mathbb{E}_0} \leq \tilde{M}\|u_0\|_{\mathbb{E}_0}, \qquad t \in [0, T_{u_0}).$$

Finally, if $D(F) = \mathbb{E}_0$, then Proposition 3.7 implies $T_{u_0} = \infty$, and the assertion follows. □

3.4 Stability of nontrivial equilibria

In the following we investigate the stability behaviour of nontrivial equilibria of equation (3.14). More precisely, we consider a fixed element

$$\phi \in D(-\mathbb{A}) \cap D(F) \subset \mathbb{E}_0, \quad \phi \neq 0,$$

which is such that

$$-\mathbb{A}\phi + F(\phi) = 0, \tag{3.35}$$

cf. Definition 3.8. In order to employ the method of linearisation, we have to impose some conditions on the nonlinearity F in (3.15), which are specified below.

First of all, in analogy to (3.19), we suppose that it is of the form

$$\begin{aligned}\mu : D \times J &\to \mathcal{L}_+(E_0)\\ (u,a) &\mapsto \mu(u,a),\end{aligned} \tag{3.36}$$

where $D \subset \mathbb{E}_0$ an open neighbourhood of ϕ, assume a continuity condition

$$a \mapsto \mu(u,a)v \in C(J, E_0), \qquad \forall u \in D, v \in E_0, \tag{3.37}$$

and existence of a constant $\bar{\mu} \in \mathbb{R}_+$, such that for arbitrary $(u,a) \in D \times J$ we have

$$\mu(u,a)v \leq \bar{\mu}\, v, \qquad \forall v \in E_0^+. \tag{3.38}$$

Furthermore, we impose that

$$(a \mapsto \mu(\phi,a)) \in C^\rho(J, \mathcal{L}(E_1, E_0)), \tag{3.39}$$

with Hölder exponent ρ from (3.5).

Recall from section 3.3 that assumptions (3.36) – (3.38) give rise to a well defined map

$$\begin{array}{rcl} f : D \subset \mathbb{E}_0 & \to & \mathcal{L}_+(\mathbb{E}_0) \\ u & \mapsto & \mu(u,.), \end{array}$$

with $\|f(u)\|_{\mathcal{L}(\mathbb{E}_0)} \leq 4\,\bar{\mu}$. We assume

$$\begin{array}{l} f : D \subset \mathbb{E}_0 \to \mathcal{L}(\mathbb{E}_0) \text{ is Lipschitz continuous} \\ \text{and Fréchet differentiable at } \phi, \end{array} \tag{3.40}$$

and define

$$\begin{array}{rcl} F : D \subset \mathbb{E}_0 & \to & \mathbb{E}_0 \\ u & \mapsto & -f(u)\,u. \end{array}$$

Then we have

$$\begin{array}{l} F : D \subset \mathbb{E}_0 \to \mathbb{E}_0 \text{ is Fréchet differentiable at } \phi, \\ \text{with derivative } F'(\phi)u = -\left(f'(\phi)u\right)\phi - f(\phi)u,\ u \in \mathbb{E}_0, \\ \text{and Lipschitz continuous on bounded subsets of } D, \end{array} \tag{3.41}$$

which is an immediate consequence of the estimates

$$\begin{aligned} &\|F(u) - F(\phi) + \big(f'(\phi)(u-\phi)\big)\phi + f(\phi)(u-\phi)\|_{\mathbb{E}_0} \\ &= \|\big(-f(u) + f(\phi)\big)\phi + \big(f'(\phi)(u-\phi)\big)\phi + f(\phi)(u-\phi) - f(u)(u-\phi)\|_{\mathbb{E}_0} \\ &\leq \|f(u) - f(\phi) - f'(\phi)(u-\phi)\|_{\mathcal{L}(\mathbb{E}_0)}\,\|\phi\|_{\mathbb{E}_0} + \|f(u) - f(\phi)\|_{\mathcal{L}(\mathbb{E}_0}\,\|u-\phi\|_{\mathbb{E}_0} \\ &= o(\|u-\phi\|_{\mathbb{E}_0}), \end{aligned}$$

and

$$\begin{array}{rcl} \|F(u) - F(v)\|_{\mathbb{E}_0} & \leq & \|f(u) - f(v)\|_{\mathcal{L}(\mathbb{E}_0)}\,\|u\|_{\mathbb{E}_0} \\ & + & \|f(v)\|_{\mathcal{L}(\mathbb{E}_0)}\,\|u-v\|_{\mathbb{E}_0}. \end{array}$$

Remark 3.22. Observe that property (3.41) allows to apply Proposition 3.7, and therefore yields existence of mild solutions to equation (3.14). Furthermore, as already pointed out in Remark 3.10, a positive initial datum $u_0 \in D$ leads to a positive solution of (3.14). □

Remark 3.23. In many concrete applications, the nonlinearity $u \mapsto \mu(u,.)$ depends in a nonlocal manner on u, and the corresponding linearisation therefore gives rise to an integral operator. Hence, the first term of $F'(\phi)$ in (3.41) can be expected to define a compact operator on $\mathbb{E}_0$, cf. Example 3.24, Theorem 3.31. □

Example 3.24 (Nonlocal dependence)**.** Recall Example 3.18, where we considered

$$\mu(.) : \mathbb{E}_0 \to \mathbb{R}_+, \qquad \mu(u) = g(\int_0^{a_m} k(a)\, u(a)\, da),$$

with $g \in BC^1(\mathbb{R}, \mathbb{R}_+)$, and $k \in L_{p'}(J, {E_0}')$, with p' the dual exponent of p. It is not hard to verify that

$$\begin{array}{rcl} F : \mathbb{E}_0 & \to & \mathbb{E}_0 \\ u & \mapsto & -\mu(u)u \end{array}$$

is well defined and conditions (3.36) – (3.40) are satisfied. For instance, for the Fréchet derivative we have

$$\begin{aligned} F'(\phi)\psi \;&=\; -g'\left(\int_0^{a_m} k(a)\,\phi(a)\,da\right)\int_0^{a_m} k(a)\,\psi(a)\,da\,\phi \\ &\quad -g\left(\int_0^{a_m} k(a)\,\phi(a)\,da\right)\psi, \end{aligned}$$

with operator norm $\|F'(\phi)\|_{\mathcal{L}(\mathbb{E}_0)} \leq \|g\|_{BC^1}\left(1 + \|k\|_{L_{p'}(J,{E_0}')}\|\phi\|_{\mathbb{E}_0}\right)$. □

Example 3.25 (Local dependence)**.** In the following we consider a nonlinearity which depends in a local way on $u \in \mathbb{E}_0$. More precisely, let $g \in BC^1(\mathbb{R}, \mathbb{R}_+), v \in {E_0}'$ and set

$$\mu(u, a) = g(\langle v, u(a)\rangle_{E_0}), \qquad \mu(u, .) : (0, a_m) \to \mathbb{R}_+.$$

Since g is continuous and bounded by assumption, the map

$$\begin{array}{rcl} F : \mathbb{E}_0 & \to & \mathbb{E}_0 \\ u & \mapsto & -\mu(u, .)u, \end{array}$$

is well defined. It turns out that the differentiability condition in (3.40) is problematic, we do not know if it is satisfied at all. However, we claim that F is Gâteaux-differentiable at $\phi \in \mathbb{E}_1$, with Gâteaux derivative

$$\delta F(\phi)\psi = -g'(\langle v, \phi(.)\rangle_{E_0})\,\langle v, \psi(.)\rangle_{E_0}\,\phi - g(\langle v, \phi(.)\rangle_{E_0})\,\psi, \qquad \psi \in \mathbb{E}_0.$$

In order to verify this claim, introduce an auxiliary function,

$$G : \mathbb{E}_0 \to \mathbb{E}_0, \qquad G(u) := g(\langle v, u(\cdot)\rangle)\,\phi.$$

By the dominated convergence theorem, G is Gâteaux-differentiable, with derivative

$$\delta G(u)\psi = \lim_{t\to 0}\frac{1}{t}\,(G(u + t\psi) - G(u)) = g'(\langle v, u(\cdot)\rangle)\langle\, v, \psi(\cdot)\rangle\,\phi, \quad \psi \in \mathbb{E}_0,$$

and $\|\delta G(u)\|_{\mathcal{L}(\mathbb{E}_0)} \leq \|g\|_{BC^1}\,\|\phi\|_\infty\,\|v\|_{{E_0}'}$, where we used $\phi \in \mathbb{E}_1 \hookrightarrow L_\infty(J, E_0)$, cf. [2, Theorem III.4.10.2]. Now we can write

$$\frac{1}{t}\,(F(\phi + t\psi) - F(\phi)) = -\frac{1}{t}\,(G(\phi + t\psi) - G(\phi) + g(\langle v, \phi + t\psi\rangle)\;t\,\psi)\,,$$

and consequently

$$\delta F(\phi)\psi = -g'(\langle v, \phi(\cdot)\rangle)\langle\, v, \psi(\cdot)\rangle\,\phi - g(\langle v, \phi(\cdot)\rangle)\,\psi, \qquad \psi \in \mathbb{E}_0,$$

with $\|\delta F(\phi)\|_{\mathcal{L}(\mathbb{E}_0)} \leq \|g\|_{BC^1}\,(\|\phi\|_\infty\,\|v\|_{{E_0}'} + 1)$. □

Returning to the general case, let us set

$$-\hat{\mathbb{A}}\psi := -\mathbb{A}\psi + F'(\phi)\psi, \qquad \psi \in D(-\mathbb{A}), \tag{3.42}$$

where $-\mathbb{A}$ denotes the generator of the semigroup $S(t), t \geq 0$, cf. Theorem 3.4 & 3.5. As regards the stability behaviour of equation (3.14), Theorem 2.16 tells us that it suffices to analyse the semigroup generated by the operator $-\hat{\mathbb{A}}$. This generator acts as

$$-\hat{\mathbb{A}}\psi = -\mathbb{A}\psi - (f'(\phi)\psi)\,\phi - \mu(\phi,.)\psi, \qquad \psi \in D(-\mathbb{A}),$$

due to (3.41).

So far, we have proceeded analogous to the trivial equilibrium case, but in the next step a new difficulty arises. To be more precise, one would like to continue the analogy and define an operator $\hat{A}(a) \in \mathcal{L}(E_1, E_0)$, $a \in J$, pointwise, i.e. for every $v \in E_1$. However, the $(f'(\phi)\,.\,)\,\phi$ part of the linearsation $-\hat{\mathbb{A}}$ cannot be defined pointwise in general, since it might depend in a nonlocal way on $\psi \in \mathbb{E}_0$, cf. Example 3.24.

One way around this difficulty is to separate the local part from the (potentially) nonlocal one: set

$$\begin{aligned} \hat{\mathbb{A}}_{loc}\psi &:= \mathbb{A}\psi + \mu(\phi,.)\psi, \qquad \psi \in D(-\mathbb{A}), \\ \mathbb{B}\psi &:= (f'(\phi)\psi)\,\phi, \qquad \psi \in \mathbb{E}_0, \end{aligned}$$

then, by definition,

$$-\hat{\mathbb{A}} = -\hat{\mathbb{A}}_{loc} - \mathbb{B} \qquad \text{on } D(-\mathbb{A}), \tag{3.43}$$

with $\mathbb{B} \in \mathcal{L}(\mathbb{E}_0)$ due to assumption (3.40).

For the local part we can continue the analogy and define, cf. (3.25),

$$\hat{A}(a)v := A(a)v + \mu(\phi, a)v, \qquad v \in E_1, \tag{3.44}$$

and denote by $\Pi_{\hat{A}}(a,\sigma)$, $0 \leq \sigma \leq a < a_m$ the parabolic evolution operator generated by $\hat{A}$, the existence of which can be shown as in the trivial equilibrium case (cf. Lemma 3.11):

Lemma 3.26 (Perturbation lemma)**.** *Assume the operator A fulfills conditions* (3.5) − (3.7)*, and the mortality rate μ satisfies* (3.36) − (3.39)*. Then, conditions* (3.5) − (3.7) *remain true for the perturbed operator $\hat{A}$.*
Assume, in addition to the previous assumptions, the operator A fulfills conditions (3.8) − (3.9)*, and in the case $a_m = \infty$, let the analogue of* (3.26) *(replace* 0 *with ϕ) hold. Then, conditions* (3.8) − (3.9) *remain true for the perturbed operator $\hat{A}$.*

In analogy to Corollary 3.12 we can conclude:

Corollary 3.27 (Perturbed semigroup)**.** *Assume the operator A fulfills conditions* (3.5)−(3.7), (3.10)*, and the mortality rate μ satisfies* (3.36) − (3.39)*. Then the operator $\hat{A}$ in* (3.44) *induces a semigroup $\hat{S}(t), t \geq 0$, on $\mathbb{E}_0$, with*

$$\sup_{t \geq 0} e^{t(\bar{\omega} - \hat{\zeta})} \|\hat{S}(t)\|_{\mathcal{L}(\mathbb{E}_0)} < \infty,$$

where $\hat{\zeta} = 4\,M\|b\|_{L_\infty}$. If the evolution operator Π_A is positive, then the semigroup $\hat{S}(t), t \geq 0$, is positive.

Finally, we define the operator $\hat{Q}_0 \in \mathcal{L}(E_0)$,

$$\hat{Q}_0 := \int_0^{a_m} b(a)\Pi_{\hat{A}}(a,0)da,$$

and formulate a first stability result (cf. Theorem 3.16):

Theorem 3.28 (Stability of small equilibria)**.** *Assume the operator A fulfills conditions* (3.5) – (3.11)*, the nonlinearity F in* (3.15) *satisfies* (3.36) – (3.40)*, and in the case $a_m = \infty$, let the analogue of* (3.26) *(replace* 0 *with ϕ) hold. Lastly, suppose $\hat{A}$ fulfills condition* (3.11)*. Then we have:*
If the spectral radius $r(\hat{Q}_0)$ is strictly smaller than 1, then the growth bound $\omega_0(-\hat{\mathbb{A}}_{loc})$ is negative. Furthermore, if the operator $\mathbb{B} \in \mathcal{L}(\mathbb{E}_0)$ in (3.43) *is "sufficiently small", then the equilibrium ϕ from* (3.35) *is exponentially asymptotically stable.*

Proof. Observe that the operator $\hat{A}$ in (3.44) fulfills conditions (3.5) – (3.9) by Lemma 3.26, condition (3.10) obviously remains true and condition (3.11) holds by assumption. Therefore, we can apply the results obtained in [36] to the induced semigroup $\hat{S}(t), t \geq 0$, from Corollary 3.27, and its generator. By Theorem 3.5, this generator is given by $-(\partial_a + \hat{A}) = -\hat{\mathbb{A}}_{loc}$. Furthermore, if $r(\hat{Q}_0) < 1$ holds, then [36, Theorem 3.5] implies that the growth bound $\omega_0(-\hat{\mathbb{A}}_{loc})$ is negative, and the first claim is proved.

Let us write $e^{-t\hat{\mathbb{A}}_{loc}} = \hat{S}(t), t \geq 0$, cf. Remark 2.8, and $\omega_0 := \omega_0(-\hat{\mathbb{A}}_{loc})$. By definition of the growth bound, for every $\epsilon > 0$ there is a constant $C(\epsilon) \geq 1$ such that

$$\|e^{-t\hat{\mathbb{A}}_{loc}}\| \leq C(\epsilon)\, e^{t(\omega_0+\epsilon)}. \tag{3.45}$$

An application of Proposition 2.9 then yields

$$\|e^{-t(\hat{\mathbb{A}}_{loc}+\mathbb{B})}\| \leq C(\epsilon)\, e^{t(\omega_0+\epsilon+C(\epsilon)\,\|\mathbb{B}\|)}, \qquad t \geq 0.$$

In particular, if $\mathbb{B} \in \mathcal{L}(\mathbb{E}_0)$ is such that

$$\|\mathbb{B}\| < \frac{-(\omega_0+\epsilon)}{C(\epsilon)},$$

then $\omega_0(-(\hat{\mathbb{A}}_{loc}+\mathbb{B})) < 0$ (recall that ω_0 is negative, so we can choose $\epsilon > 0$ such that $-(\omega_0+\epsilon) > 0$). Theorem 2.16 then implies the assertion. ☐

Remark 3.29. In the stability statement of Theorem 3.28, the equilibrium is assumed to be "sufficiently small". If a more quantitative description of this smallness assumption is desired, it is necessary to refine estimate (3.45). Depending on the depth of the corresponding analysis, this endeavour can become more or less involved. One of the simpler approaches can be carried out as follows:
Let us write $e^{-t\hat{\mathbb{A}}_{loc}} = \hat{S}(t), t \geq 0$, cf. Remark 2.8. As in Corollary 3.12 one can show that there exists a constant $C \geq 1$ such that

$$\|e^{-t\hat{\mathbb{A}}_{loc}}\| \leq C\, e^{t(\hat{\zeta}-\bar{\omega})},$$

where $\hat{\zeta} = 4M\|b\|_{L_\infty}$ (in particular, $\omega_0(-\hat{\mathbb{A}}_{loc}) \leq \hat{\zeta} - \bar{\omega}$). An application of Proposition 2.9 then yields

$$\|e^{-t(\hat{\mathbb{A}}_{loc}+\mathbb{B})}\| \leq C\, e^{t(\hat{\zeta}-\bar{\omega}+C\,\|\mathbb{B}\|)}, \qquad t \geq 0.$$

In particular, if $\mathbb{B} \in \mathcal{L}(\mathbb{E}_0)$ is such that

$$\|\mathbb{B}\| < \frac{\bar{\omega} - \hat{\zeta}}{C},$$

then $\omega_0(-(\hat{\mathbb{A}}_{loc} + \mathbb{B})) < 0$, and Theorem 2.16 yields stability of the equilibrium. Note that this approch only makes sense if $\bar{\omega} - \hat{\zeta} > 0$, i.e. $4M \, \|b\|_{L_\infty} < \bar{\omega}$. □

Remark 3.30. In [34] the existence of a smooth branch of nontrivial equilibria, emanating from the trivial equilibrium $\phi = 0$, is shown. In light of Theorem 3.28 it is reasonable to conjecture stability of these equilibria. □

Let us consider a nonlinearity $\mu = \mu(u, a)$, which depends in a nonlocal way on u, cf. Example 3.24. As already pointed out in Remark 3.23, the nonlocal part of linearisation (3.43) can be expected to be compact in this case, i.e.

$$[\psi \mapsto \mathbb{B}\psi = (f'(\phi)\psi)\,\phi] \in \mathcal{K}(\mathbb{E}_0), \tag{3.46}$$

leading to the following result:

Theorem 3.31 (Stability for nonlocal dependence)**.** *Assume the operator A fulfills conditions* (3.5) – (3.11)*, the nonlinearity F in* (3.15) *is of the form described in Example 3.24, with $k \in L_{p'}(J, \mathcal{L}_+(E_0, \mathbb{R}))$, and in the case $a_m = \infty$, let the analogue of* (3.26) *(replace* 0 *with ϕ) hold. Lastly, suppose $\hat{A}$ fulfills condition* (3.11)*.*
Consider the linearised operator $-\hat{\mathbb{A}}$ in (3.42)*, and assume the following conditions hold:*

1. $s(-\hat{\mathbb{A}}) \in \sigma_p(-\hat{\mathbb{A}}) \setminus \{0\}$,
2. $g'\left(\int_0^{a_m} k(a)\,\phi(a)\,da\right) \geq 0$,
3. $\phi \in \mathbb{E}_0^+$.

Then $\omega_0(-\hat{\mathbb{A}})$ is negative. In particular, the equilibrium ϕ from (3.35) *is exponentially asymptotically stable.*

Proof. Due to the assumed form of the nonlinearity, it is not hard to verify that conditions (3.36) – (3.40), (3.46) hold, with

$$\begin{aligned} \hat{\mathbb{A}}_{loc}\psi &= \mathbb{A}\psi + g\Big(\int_0^{a_m} k(a)\phi(a)\,da\Big)\,\psi, \qquad \psi \in D(-\mathbb{A}), \\ \mathbb{B}\psi &= \alpha_\phi \int_0^{a_m} k(a)\,\psi(a)\,da\,\phi, \qquad \psi \in \mathbb{E}_0, \end{aligned}$$

where we have set $\alpha_\phi := g'\left(\int_0^{a_m} k(a)\,\phi(a)\,da\right)$.

From Lemma 3.26 it follows that the operator $\hat{A}$ in (3.44) fulfills conditions (3.5) – (3.9), condition (3.10) obviously remains true and condition (3.11) holds by assumption. Therefore we can apply the results obtained in [36] to the induced semigroup $\hat{S}(t), t \geq 0$, from Corollary 3.27, and its generator. By Theorem 3.5, this generator is given by $-(\partial_a + \hat{A}) = -\hat{\mathbb{A}}_{loc}$, and [36, Lemma 3.2] implies that the α-growth bound $\omega_1(-\hat{\mathbb{A}}_{loc})$ is negative.

Recalling decomposition (3.43), we arrive at

$$\omega_1(-\hat{\mathbb{A}}) = \omega_1(-\hat{\mathbb{A}}_{loc}) < 0,$$

where the equality holds since the operator $\mathbb{B} \in \mathcal{L}(\mathbb{E}_0)$ is obviously compact (cf. [38, Proposition 4.14]). Hence, in order to finish the proof it suffices to establish a negative upper bound for the point spectrum of $-\hat{\mathbb{A}}$.

To this end we suppose there is an eigenvalue $\lambda > 0$ with a corresponding eigenvector $\psi \in D(-\mathbb{A}) \subset \mathbb{E}_1$, i.e.

$$-\hat{\mathbb{A}}\psi = \lambda\psi,$$

and since $\lambda \in \mathbb{R}$, we can assume w.l.o.g. that ψ is real (in particular we can ask whether $\psi(0)$ is positive). Then Theorem 3.5 and the variation of constants formula imply

$$\psi(a) = \Pi_{\hat{A}+\lambda}(a,0)\psi(0) - \int_0^a \Pi_{\hat{A}+\lambda}(a,\sigma)(\mathbb{B}\psi)(\sigma)\,d\sigma, \tag{3.47}$$

cf. (2.13). In the next step we use that $\phi \in D(-\mathbb{A})$ is an equilibrium, i.e. $(\mathbb{A}+\mu(\phi))\phi = 0$. More precisely, by another application of Theorem 3.5 we obtain

$$\begin{aligned} \phi(a) &= \Pi_{\hat{A}}(a,0)\phi(0), && (3.48)\\ \phi(0) &= \int_0^{a_m} b(a)\Pi_{\hat{A}}(a,0)\,da\,\phi(0). && (3.49) \end{aligned}$$

Consider the operator

$$\hat{Q}_\lambda := \int_0^{a_m} b(a)\Pi_{\hat{A}+\lambda}(a,0)\,da \in \mathcal{L}_+(E_0),$$

then due to (3.49) (and $\phi(0) \neq 0$, by (3.48)) we have $r(\hat{Q}_0) \geq 1$, and since $\phi(0) \in E_0^+$ by assumption, equality must hold by Lemma 3.13, i.e. $r(\hat{Q}_0) = 1$. The monotonicity of the spectral radii, see Lemma 3.14, implies

$$r(\hat{Q}_\lambda) < 1.$$

Our next aim is to simplify (3.47). To this end combine (3.48) and Definition 2.25.2 to conclude

$$\int_0^a \Pi_{\hat{A}+\lambda}(a,\sigma)\phi(\sigma)\,d\sigma = \frac{1-e^{-\lambda a}}{\lambda}\Pi_{\hat{A}}(a,0)\phi(0),$$

plugging this into (3.47) then yields

$$\begin{aligned} \psi(a) &= \Pi_{\hat{A}+\lambda}(a,0)\psi(0)\\ &- \alpha_\phi\frac{1-e^{-\lambda a}}{\lambda}\int_0^{a_m} k(a)\psi(a)\,da\,\Pi_{\hat{A}}(a,0)\phi(0). && (3.50) \end{aligned}$$

In the following we will use this formula to obtain additional information about ψ. First, combining the age boundary condition ($\psi \in D(-\mathbb{A})$) with formula (3.50) results in

$$\begin{aligned} \psi(0) &= \int_0^{a_m} b(a)\Pi_{\hat{A}+\lambda}(a,0)\,da\,\psi(0)\\ &- \frac{\alpha_\phi}{\lambda}\int_0^{a_m} k(a)\psi(a)\,da\int_0^{a_m} b(a)(1-e^{-\lambda a})\Pi_{\hat{A}}(a,0)\,da\,\phi(0), \end{aligned}$$

or equivalently

$$\psi(0) = -\frac{\alpha_\phi}{\lambda}\int_0^{a_m} k(a)\psi(a)\,da\,\left(1-\hat{Q}_\lambda\right)^{-1}\int_0^{a_m} b(a)(1-e^{-\lambda a})\Pi_{\hat{A}}(a,0)\,da\,\phi(0).$$

Withous loss of generality we may assume

$$\int_0^{a_m} k(a)\psi(a)\,da \geq 0,$$

since otherwise we can pass to the eigenvector $-\psi$ (recall that $\int_0^{a_m} k(a)\psi(a)\,da \in \mathbb{R}$). Then the previous equation implies

$$\psi(0) \in \begin{cases} E_0^+ & \text{if } \alpha_\phi < 0, \\ -E_0^+ & \text{if } \alpha_\phi \geq 0. \end{cases}$$

Next, multiply (3.50) with $k(a)$ and integrate with respect to a to obtain

$$\int_0^{a_m} k(a)\psi(a)\,da \left(1 + \frac{\alpha_\phi}{\lambda}\int_0^{a_m}(1 - e^{-\lambda a}k(a)\Pi_{\hat{A}}(a,0)\,da\,\phi(0)\right)$$
$$= \int_0^{a_m} k(a)e^{-\lambda a}\Pi_{\hat{A}}(a,0)\,da\,\psi(0).$$

Since $\alpha_\phi \geq 0$ by assumption, we have $\psi(0) \leq 0$. Therefore, the right hand side in the foregoing equation is negative, on the other hand the left hand side is obviously positive, consequently

$$\int_0^{a_m} k(a)\psi(a)\,da = 0,$$

and (3.50) in turn implies $\psi(a) = \Pi_{\hat{A}+\lambda}\psi(0)$, with $\psi(0) \neq 0$. This means that λ is an eigenvalue of $-\mathbb{A}_{loc}$ (with corresponding eigenvector ψ), hence $1 \in \sigma_p(\hat{Q}_\lambda)$ by Lemma 3.15, contradicting $r(\hat{Q}_\lambda) < 1$, for $\lambda > 0$.

This shows that all real eigenvalues of $-\hat{\mathbb{A}}$ are non-positive. Since the spectral bound is an eigenvalue and distinct from zero by assumption, we conclude that it is strictly negative. □

Remark 3.32. 1. The essential step in the proof of Theorem 3.31 is to show that the (nonlocal) operator $-\hat{\mathbb{A}}$ admits no positive eigenvalues. Condition (1) on the spectral bound $s(-\hat{\mathbb{A}})$ is then used to complete the proof.
2. If the semigroup $e^{-t(\hat{\mathbb{A}}_{loc}+\mathbb{B})}, t \geq 0$, is positive and $0 \neq s(-\hat{\mathbb{A}}) > \omega_1(-\hat{\mathbb{A}})$, then condition (1) holds by Proposition 2.10. □

Theorem 3.33 (Instability for nonlocal dependence)**.** *Assume the operator A fulfills conditions* (3.5) – (3.11)*, the nonlinearity F in* (3.15) *satisfies* (3.36) – (3.40) *and compactness condition* (3.46)*, and in the case $a_m = \infty$, let the analogue of* (3.26) *(replace* 0 *with ϕ) hold. Lastly, suppose $\hat{A}$ fulfills condition* (3.11).
Consider the linearised operator $-\hat{\mathbb{A}}$ in (3.42)*, then we have:*

If $\omega_0(-\hat{\mathbb{A}}) > 0$, then the equilibrium ϕ from (3.35) *is unstable.*

Proof. Recall decomposition (3.43) and observe that

$$\omega_1(-\hat{\mathbb{A}}) = \omega_1(-\hat{\mathbb{A}}_{loc}) < 0,$$

where the equality holds because of compactness assumption (3.46) (cf. [38, Proposition 4.14]), and the inequality can be shown as in the proof of Theorem 3.31.

Thus, by an application of Theorem 2.16 the assertion follows. □

In the following we establish another type of stability result. To this end we impose an additional condition, namely we assume existence of a constant $c > 0$ such that the first term in linearisation (3.41) satisfies

$$0 \leq (f'(\phi)\psi)\,\phi \leq c\psi, \quad \forall \psi \in \mathbb{E}_0^+, \tag{3.51}$$

and emphasize that this condition cannot be expected to hold for nonlocal nonlinearities:

Remark 3.34. Condition (3.51) is generally not satisfied if the nonlinearity $\mu = \mu(u, a)$ depends in a nonlocal way on u. To see this, we continue with Example 3.24, i.e.

$$\mu(.) : \mathbb{E}_0 \to \mathbb{R}_+, \qquad \mu(u) = g(\int_0^{a_m} k(a)\,u(a)\,da),$$

with $g \in BC^1(\mathbb{R}, \mathbb{R}_+)$, $k \in L_{p'}(J, {E_0}')$, and p' the dual exponent of p. Recall that we have set $f(u) = \mu(u)$ and calculated

$$(f'(\phi)\psi)\phi = g'\left(\int_0^{a_m} k(a)\,\phi(a)\,da\right)\int_0^{a_m} k(a)\,\psi(a)\,da\,\phi, \qquad \psi \in \mathbb{E}_0.$$

Now it is not difficult to see that condition (3.51) would imply $\phi = 0$, which has been excluded for this section. □

As for the previous stability results, define the operator $\hat{Q}_0 \in \mathcal{L}(E_0)$ by

$$\hat{Q}_0 := \int_0^{a_m} b(a)\Pi_{\hat{A}}(a, 0)da,$$

then we have:

Theorem 3.35. *Assume the operator A fulfills conditions* (3.5) − (3.11)*, the nonlinearity F in* (3.15) *satisfies* (3.36) − (3.40), (3.51)*, and in the case $a_m = \infty$, let the analogue of* (3.26) *(replace 0 with ϕ) hold. Lastly, suppose $\hat{A}$ fulfills condition* (3.11)*. Then we have:*
If the spectral radius $r(\hat{Q}_0)$ is strictly smaller than 1, then the growth bound $\omega_0(-\hat{\mathbb{A}})$ is negative. Furthermore, the equilibrium ϕ from (3.35) *is exponentially asymptotically stable.*

Proof. Observe that the operator $\hat{A}$ in (3.44) fulfills conditions (3.5) − (3.9) by Lemma 3.26, condition (3.10) obviously remains true and condition (3.11) holds by assumption. Therefore, we can apply the results obtained in [36] to the induced semigroup $\hat{S}(t), t \geq 0$,, and its generator. By Theorem 3.5, this generator is given by $-(\partial_a + \hat{A}) = -\hat{\mathbb{A}}_{loc}$. In particular, if $r(\hat{Q}_0) < 1$ holds, then [36, Theorem 3.5] implies that the growth bound $\omega_0(-\hat{\mathbb{A}}_{loc})$ is negative. With assumption (3.51) we are in the situation of Proposition 2.20 and conclude $\omega_0(-(\hat{\mathbb{A}}_{loc} + \mathbb{B})) \leq \omega_0(-\hat{\mathbb{A}}_{loc}) < 0$. Theorem 2.16 then implies the assertion. □

4 The integral equation

In this section we investigate the solvability of problem (1.4). Observe that in contrast to the simplified problem (3.1) in section 3, we do not have a semilinear structure anymore, due to the additional nonlinearity occurring in the age-boundary condition. In particular, semigroup theory and mild solutions are not available any longer. It is therefore necessary to introduce a new solution concept, an undertaking which will constitute a big portion of the present section. To this end, we resume the approach from section 1.1, where we formally derived a formula for the solution by the method of characteristics. More precisely, we want to construct solutions in the sense of Definition 1.1, with $p = 1$, i.e. we set

$$\mathbb{E}_0 = L_1(J, E_0)$$

throughout this section, with $J \subset \mathbb{R}_+$ as in (4.1) below. A quasilinear, homogeneous version of this problem was studied in [35]. For the non-diffusive case, an exposition can be found in [38, Chapter 2].

Another purpose of the present section is to lay the basis for the upcoming section 5, where we study the stability of equilibria to problem (1.4). The corresponding results are established in subsection 4.4.

4.1 Assumptions

Throughout this section, E_0 denotes a real Banach space. Note that we do not distinguish E_0 from its complexification in our notation as no confusion seems likely. We set

$$J = \begin{cases} [0, a_m], & \text{if } a_m < \infty \\ [0, \infty), & \text{if } a_m = \infty \end{cases} \tag{4.1}$$

and define

$$\mathbb{E}_0 = L_1(J, E_0).$$

As regards the operator A, we assume

$$A \in L_\infty(J, \mathcal{L}(E_1, E_0)), \qquad \sigma + A \in C^\rho(J, \mathcal{H}(E_1, E_0; \kappa, \nu)), \tag{4.2}$$

for some ρ, $\nu > 0$, $\kappa \geq 1$, $\sigma \in \mathbb{R}$, where E_1 is a densely and compactly embedded subspace of E_0. Recalling Remark 2.26, this assumption implies that A generates a parabolic evolution operator $\Pi_A(a, \sigma)$, $0 \leq \sigma \leq a < a_m$, on E_0 with regularity subspace E_1 and there are constants $M \geq 1$ and $\bar{\omega} \in \mathbb{R}$ such that

$$\|\Pi_A(a, \sigma)\|_{\mathcal{L}(E_0)} \leq M e^{-\bar{\omega}(a-\sigma)}, \quad 0 \leq \sigma \leq a < a_m. \tag{4.3}$$

Remark 4.1. Let us point out that the results of sections 4 and 5 remain true under the weaker set of assumptions

$$\left.\begin{array}{l} A \in C(J, \mathcal{L}(E_1, E_0)) \text{ generates an evolution} \\ \text{operator } \Pi_A(a, \sigma), 0 \leq \sigma \leq a < a_m, \text{ on } E_0, \end{array}\right\}$$

there are constants $M \geq 1$ and $\bar{\omega} \in \mathbb{R}$ such that

$$\|\Pi_A(a, \sigma)\|_{\mathcal{L}(E_0)} \leq M e^{-\bar{\omega}(a-\sigma)}, \qquad 0 \leq \sigma \leq a < a_m,$$

and for arbitrary $\mu_L \in BC(J, \mathcal{L}(E_0))$, the operator $\hat{A} := A + \mu_L$ induces an evolution operator $\Pi_{\hat{A}}(a, \sigma), 0 \leq \sigma \leq a < a_m$, on E_0, via the formula

$$\Pi_{\hat{A}}(a, \sigma)v = \Pi_A(a, \sigma)v - \int_\sigma^a \Pi_A(a, s)\mu_L(s)\Pi_{\hat{A}}(s, \sigma)v \, ds, \quad v \in E_0,$$

cf. (3.27). To be more precise, the smoothing property of a parabolic evolution operator is not required. In particular we can allow hyperbolic evolution operators, cf. [23], or autonomous operators which generate a strongly continuous semigroup (not necessarily analytic). □

4.2 The linear inhomogeneous equation

Let us consider the linear inhomogeneous version of (1.4),

$$\begin{aligned} \partial_t w + \partial_a w + A(a)w + \gamma w &= -\mu_L(a)w + f(t,a), \quad t>0,\, a\in J, \\ w(t,0) &= \int_0^{a_m} b_L(t,a)w(t,a)\,\mathrm{d}a + h(t), \quad t>0, \\ w(0,a) &= w_0(a), \quad a\in J, \end{aligned} \tag{4.4}$$

with $w:[0,T]\times J\to E_0$ and $\gamma\in\mathbb{R}$.

In section 4.2 we assume the operator A fulfills conditions (4.2) – (4.3), we consider a birth rate

$$b_L \in BC([0,T]\times[0,\infty),\mathcal{L}(E_0)), \quad \mathrm{supp}(b_L)\subset[0,T]\times J, \tag{4.5}$$

and a mortality rate

$$\mu_L \in BC(J,\mathcal{L}(E_0)), \tag{4.6}$$

which is such that

$$(a\mapsto \mu(a)) \in C^\rho(J,\mathcal{L}(E_1,E_0)), \tag{4.7}$$

with Hölder exponent ρ from (4.2). We introduce the notation

$$\begin{aligned} \hat{A}(a)v &:= A(a)v + \mu_L(a)v, \quad v\in E_1, \\ A_\gamma(a)v &:= A(a)v + \gamma v, \quad v\in E_1, \\ \hat{A}_\gamma(a)v &:= A(a)v + \mu_L(a)v + \gamma v, \quad v\in E_1, \end{aligned} \tag{4.8}$$

and recall from the proof of Lemma 3.11 that conditions (4.6) – (4.7) imply existence of the parabolic evolution operator $\Pi_{\hat{A}}(a,\sigma), 0\le\sigma\le a<a_m$, which satisfies

$$\Pi_{\hat{A}}(a,\sigma)v = \Pi_A(a,\sigma)v - \int_\sigma^a \Pi_A(a,s)\mu_L(s)\Pi_{\hat{A}}(s,\sigma)v\,ds, \quad v\in E_0.$$

By assumption (4.3) there are constants $M\ge 1$, $\bar{\omega}\in\mathbb{R}$, such that

$$\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_0)} \le Me^{-\bar{\omega}(a-\sigma)}, \qquad 0\le\sigma\le a<a_m,$$

and without loss of generality we can assume

$$\|\Pi_{\hat{A}}(a,\sigma)\|_{\mathcal{L}(E_0)} \le Me^{-\bar{\omega}(a-\sigma)}, \qquad 0\le\sigma\le a<a_m. \tag{4.9}$$

In contrast to section 3, we do not require any sign condition on $\bar{\omega}$, cf. (3.7).

Analogously to section 1 we can formally integrate (4.4) along characteristics to obtain a formula for the solution:

$$\begin{aligned} w(t,a) &= \begin{cases} \Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t) \\ \Pi_{\hat{A}_\gamma}(a,0)B(t-a) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0\le t\le a<a_m \\ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0\le a<t,\, a<a_m, \end{cases} \end{aligned} \tag{4.10}$$

where $B(t) = w(t,0)$ solves the integral equation

$$\begin{aligned}
B(t) &= \int_0^{a_m} b_L(t,a)w(t,a)\,\mathrm{d}a + h(t) \\
&= \int_0^t b_L(t,a)\Pi_{\hat{A}_\gamma}(a,0)B(t-a)\,\mathrm{d}a \\
&+ \int_0^t b_L(t,a)\int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\
&+ \int_t^{a_m} b_L(t,a)\Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t)\,\mathrm{d}a \qquad (4.11)\\
&+ \int_t^{a_m} b_L(t,a)\int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\
&+ h(t).
\end{aligned}$$

Remark 4.2. 1. The integrals appearing in (4.10) & (4.11) have to be interpreted as Bochner integrals. This will be made more precise in the following.
2. In the case $a_m < \infty$, we have $\int_t^{a_m} b_L(t,a)\ldots\,\mathrm{d}a = 0$ for $t \in [a_m, T]$, due to assumption (4.5). □

As already pointed out in the introductory section, we hide the age variable a in the function space $\mathbb{E}_0 = L_1(J, E_0)$. Thus, we interpret the function w in (4.4) as a curve/trajectory in the phase space $\mathbb{E}_0$:

$$w : [0,T] \to \mathbb{E}_0, \quad t \mapsto w(t).$$

Requiring furthermore that this curve be continuous, it can be identified with an element in the space $L_1((0,T)\times(0,a_m), E_0)$ by Lemma A.3, and we write $w(t)(a) = w(t,a)$. These observations lead to

Definition 4.3. Let $w : [0,T] \to \mathbb{E}_0$ be continuous, with $w(\cdot,0) \in C([0,T], E_0)$. We say that w is an *integral solution* to (4.4) on $[0,T]$ if

I. For all $t \in [0,T]$: (4.11) holds
II. For all $t \in [0,T]$: (4.10) holds for a.a. $a \in (0, a_m)$.

We then write

$$w = w_{w_0}^{\gamma,f,h}, \qquad B = B_{w_0}^{\gamma,f,h}$$

to stress the dependence on the data w_0, γ, f, h.

At this point it is not clear, whether an integral solution exists. Imposing proper conditions on the data, this situation is resolved by the following

Proposition 4.4. *Assume the operator A satisfies* (4.2)−(4.3)*, and conditions* (4.5)−(4.7), (4.9) *hold. Let $w_0 \in \mathbb{E}_0$, $\gamma \in \mathbb{R}$, and suppose that*

$$f \in C([0,T], \mathbb{E}_0), \quad h \in C([0,T], E_0).$$

Then there exists a unique integral solution $w_{w_0}^{\gamma,f,h}$ to (4.4) *on* $[0,T]$.

For the proof of Proposition 4.4, the following lemma is essential:

Lemma 4.5. *Assume the operator A satisfies* (4.2) − (4.3)*, and conditions* (4.5) − (4.7), (4.9) *hold. Let* $w_0 \in \mathbb{E}_0$*,* $\gamma \in \mathbb{R}$*, and suppose that*

$$f \in C([0,T], \mathbb{E}_0), \quad h \in C([0,T], E_0).$$

Then there exists a unique solution $B^{\gamma,f,h}_{w_0} \in C([0,T], E_0)$ *to* (4.11)*. Furthermore we have*

$$\begin{aligned} e^{(\bar{\omega}+\gamma)t}\|B^{\gamma,f,h}_{w_0}(t)\| &\leq \Big(M\|b\|_\infty T e^{|\bar{\omega}+\gamma|T}\|f\|_\infty \\ &+ M\|b\|_\infty\|w_0\|_{\mathbb{E}_0} + e^{|\bar{\omega}+\gamma|T}\|h\|_\infty\Big) e^{tM\|b\|_\infty}. \end{aligned}$$

Proof. Let us consider equation (4.11):

$$B(t) = \int_0^t b_L(t,a)\Pi_{\hat{A}_\gamma}(a,0)B(t-a)\,\mathrm{d}a + g(t) + h(t),$$

where we have set

$$\begin{aligned} g(t) &:= \int_0^t b_L(t,a)\int_{t-a}^t \Pi_{\hat{A}_\gamma}(a, s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_t^{a_m} b_L(t,a)\Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t)\,\mathrm{d}a \\ &+ \int_t^{a_m} b_L(t,a)\int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a. \\ &=: g_1(t) + g_2(t) + g_3(t). \end{aligned}$$

Define $J_\Delta := \{(t,a) \in J \times J : a \leq t\}$ and introduce the integral kernel

$$\begin{aligned} k_\gamma : J_\Delta &\to \mathcal{L}(E_0) \\ (t,a) &\mapsto b_L(t,t-a)\Pi_{\hat{A}_\gamma}(t-a,0), \end{aligned}$$

then we see that the integral equation can be rewritten as

$$B(t) = \int_0^t k_\gamma(t,a)B(a)\,\mathrm{d}a + g(t) + h(t), \qquad t \in [0,T].$$

In the following we verify that the functions $g_1, g_2, g_3 : [0,T] \to E_0$ are well defined and continuous, Proposition 4.22 then implies existence of a unique solution $B = B^{\gamma,f,h}_{w_0} \in C([0,T], E_0)$.

Recall that $f \in L_1((0,T)\times(0,a_m), E_0)$, by Lemma A.3. We formally calculate

$$\begin{aligned} g_1(t) &= \int_0^t \int_{t-a}^t b_L(t,a)\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &= \int_0^t \int_{t-s}^t b_L(t,a)\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}a\mathrm{d}s \\ &= \int_0^t \int_0^s b_L(t,a+t-s)\Pi_{\hat{A}_\gamma}(a+t-s,a)f(s,a)\,\mathrm{d}a\mathrm{d}s. \end{aligned}$$

Observe that the function

$$(0,t)\times(0,a_m) \to E_0, \qquad (s,a) \mapsto b_L(t,a+t-s)\Pi_{\hat{A}_\gamma}(a+t-s,a)f(s,a)$$

is strongly measureable by Lemma A.4. Furthermore, it is integrable, since

$$\|b_L(t,a+t-s)\Pi_{\hat{A}_\gamma}(a+t-s,a)f(s,a)\|_{E_0} \leq \|b_L\|_{L_\infty} M e^{-(\bar{\omega}+\gamma)(t-s)}\|f(s,a)\|_{E_0},$$

where we used (4.5) & (4.9). Hence, by Fubini's theorem, the corresponding iterated integrals exist, and we see that the formal calculation above becomes rigorous (in particular, $g_1(t)$ is well defined). Consequently, the function

$$[0,T] \to E_0, \qquad t \mapsto g_1(t)$$

is continuous by the dominated convergence theorem.

As regards the term g_3, a formal calculation gives

$$\begin{aligned} g_3(t) &= \int_t^{a_m}\int_0^t b_L(t,a)\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &= \int_0^t\int_t^{a_m} b_L(t,a)\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}a\mathrm{d}s \\ &= \int_0^t\int_s^{a_m+s-t} b_L(t,a+t-s)\Pi_{\hat{A}_\gamma}(a+t-s,a)f(s,a)\,\mathrm{d}a\mathrm{d}s, \end{aligned}$$

and as before we conclude that

$$[0,T] \to E_0, \qquad t \mapsto g_3(t)$$

is well defined and continuous (in the case $a_m < \infty$, set $g_3(t) = 0$ for $t \geq a_m$).

Concerning the remaining term, we have

$$\begin{aligned} g_2(t) &= \int_t^{a_m} b_L(t,a)\Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t)\,\mathrm{d}a \\ &= \int_0^{a_m-t} b_L(t,a+t)\Pi_{\hat{A}_\gamma}(a+t,a)w_0(a)\,\mathrm{d}a, \end{aligned}$$

and analogous arguments show that

$$[0,T] \to E_0, \qquad t \mapsto g_2(t)$$

is well defined and continuous (in the case $a_m < \infty$, set $g_2(t) = 0$ for $t \geq a_m$).

Let us turn to the claimed estimate. By (4.11), we have

$$\begin{aligned} \|B_{w_0}^{\gamma,f,h}(t)\| &\leq M\|b\|_\infty \int_0^t e^{-(\bar{\omega}+\gamma)(t-a)}\|B_{w_0}^{\gamma,f,h}(a)\|\,\mathrm{d}a \\ &+ M\|b\|_\infty \int_0^t\int_{t-s}^t e^{-(\bar{\omega}+\gamma)(t-s)}\|f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\ &+ M\|b\|_\infty \int_t^{a_m} e^{-(\bar{\omega}+\gamma)t}\|w_0(a-t)\|\,\mathrm{d}a \\ &+ M\|b\|_\infty \int_0^t\int_t^{a_m} e^{-(\bar{\omega}+\gamma)(t-s)}\|f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\ &+ \|h(t)\|, \end{aligned}$$

or equivalently

$$\begin{aligned}
e^{(\bar\omega+\gamma)t}\|B^{\gamma,f,h}_{w_0}(t)\| &\leq M\|b\|_\infty\int_0^t e^{(\bar\omega+\gamma)a}\|B^{\gamma,f,h}_{w_0}(a)\|\,\mathrm{d}a\\
&+ M\|b\|_\infty\int_0^t\int_{t-s}^t e^{(\bar\omega+\gamma)s}\|f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s\\
&+ M\|b\|_\infty\int_t^{a_m}\|w_0(a-t)\|\,\mathrm{d}a\\
&+ M\|b\|_\infty\int_0^t\int_t^{a_m} e^{(\bar\omega+\gamma)s}\|f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s\\
&+ e^{(\bar\omega+\gamma)t}\|h(t)\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
e^{(\bar\omega+\gamma)t}\|B^{\gamma,f,h}_{w_0}(t)\| &\leq M\|b\|_\infty\int_0^t e^{(\bar\omega+\gamma)a}\|B^{\gamma,f,h}_{w_0}(a)\|\,\mathrm{d}a\\
&+ M\|b\|_\infty\int_0^t e^{(\bar\omega+\gamma)s}\|f(s)\|_{\mathbb{E}_0}\,\mathrm{d}s\\
&+ M\|b\|_\infty\|w_0\|_{\mathbb{E}_0}\\
&+ e^{(\bar\omega+\gamma)t}\|h(t)\|\\
&\leq M\|b\|_\infty\int_0^t e^{(\bar\omega+\gamma)a}\|B^{\gamma,f,h}_{w_0}(a)\|\,\mathrm{d}a\\
&+ M\|b\|_\infty Te^{|\bar\omega+\gamma|T}\|f\|_\infty\\
&+ M\|b\|_\infty\|w_0\|_{\mathbb{E}_0}+e^{|\bar\omega+\gamma|T}\|h\|_\infty,
\end{aligned}$$

and the estimate is a consequence of Gronwall's inequality. □

Proof of Proposition 4.4. By Lemma 4.5 there exists a unique solution $B^{\gamma,f,h}_{w_0}\in C([0,T],E_0)$ to (4.11). For $t\in[0,T]$, define $w^{\gamma,f,h}_{w_0}(t)$ via formula (4.10), i.e.

$$\begin{aligned}
w(t)(a) &:= \begin{cases}\Pi_{\hat A_\gamma}(a,a-t)w_0(a-t)\\ \Pi_{\hat A_\gamma}(a,0)B^{\gamma,f,h}_{w_0}(t-a)\end{cases}\\
&+ \begin{cases}\int_0^t\Pi_{\hat A_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0\leq t\leq a<a_m\\ \int_{t-a}^t\Pi_{\hat A_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0\leq a<t,\ a<a_m.\end{cases}
\end{aligned}\tag{4.12}$$

In the following we will show that

$$w:[0,T]\to\mathbb{E}_0,\quad t\mapsto w(t)$$

is well defined and continuous. This yields the existence claim, uniqueness is a consequence of Lemma 4.5.

First we observe that the integrals appearing in (4.12) are well defined (in the measure-theoretic sense, i.e. for almost all $a\in(0,a_m)$). To see this, fix $t\in[0,T)$, and consider the case $0\leq t\leq a$. Let us set $c:=a-t$, then one shows as in Lemma A.4 that

$$(0,t)\times(0,\infty),\quad (s,c)\mapsto\Pi_{\hat A_\gamma}(t+c,s+c)f(s,s+c)\tag{4.13}$$

is strongly measureable (in case $a_m < \infty$, set the function value to zero for $s + c > a_m$), furthermore we have

$$\|\Pi_{\hat{A}_\gamma}(t+c, s+c) f(s, s+c)\|_{E_0} \leq M e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s, s+c)\|_{E_0}.$$

With Lemma A.3 (and a substitution of variable) we see, that the function on the right is integrable. Therefore, the function (4.13) is integrable, and by Fubini's theorem, the function

$$(0, \infty) \to E_0, \quad c \mapsto \int_0^t \Pi_{\hat{A}_\gamma}(t+c, s+c) f(s, s+c) \,\mathrm{d}s$$

is integrable. Now resubstitute $c = a - t$, and the assertion follows. The case $0 \leq a < t$ can be treated analogously.

In the next step we show that $w(t) \in \mathbb{E}_0$. Let $t \in [0, T]$ be fixed, then

$$\begin{aligned} \|w(t)(a)\|_{E_0} \quad &\leq \quad \begin{cases} M e^{-(\bar{\omega}+\gamma)t} \|w_0(a-t)\|_{E_0} \\ M e^{-(\bar{\omega}+\gamma)a} \|B_{w_0}^{\gamma,f,h}(t-a)\|_{E_0} \end{cases} \\ &+ \quad \begin{cases} \int_0^t M e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s, s+a-t)\|_{E_0} \,\mathrm{d}s, & 0 \leq t \leq a \\ \int_{t-a}^t M e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s, s+a-t)\|_{E_0} \,\mathrm{d}s, & 0 \leq a < t, \end{cases} \end{aligned}$$

where $a < a_m$, and consequently

$$\begin{aligned} \|w(t)\|_{\mathbb{E}_0} \quad &= \quad \int_0^t \|w(t)(a)\|_{E_0} \,\mathrm{d}a + \int_t^{a_m} \|w(t)(a)\|_{E_0} \,\mathrm{d}a \\ &\leq \quad \int_0^t M e^{-(\bar{\omega}+\gamma)a} \|B_{w_0}^{\gamma,f,h}(t-a)\|_{E_0} \,\mathrm{d}a \\ &+ \quad \int_0^t \int_{t-a}^t M e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s, s+a-t)\|_{E_0} \,\mathrm{d}s\mathrm{d}a \\ &+ \quad \int_t^{a_m} M e^{-(\bar{\omega}+\gamma)t} \|w_0(a-t)\|_{E_0} \,\mathrm{d}a \\ &+ \quad \int_t^{a_m} \int_0^t M e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s, s+a-t)\|_{E_0} \,\mathrm{d}s\mathrm{d}a. \end{aligned}$$

The first term in the estimate above is finite, since the integrand is continuous by Lemma 4.5. For the next three terms we observe that they have the same structure as the terms g_i in the proof of Lemma 4.5, hence an analogous argument shows that they are well defined and finite.

It remains to verify the continuity of $w : [0, T] \to \mathbb{E}_0$. Let $t \in [0, T]$ and $\hat{t} \in (0, T)$ such that $0 \leq t < \hat{t}$. Then

$$\begin{aligned} \|w(t) - w(\hat{t})\|_{\mathbb{E}_0} \quad &= \quad \int_0^{a_m} \|w(t)(a) - w(\hat{t})(a)\|_{E_0} \,\mathrm{d}a \\ &= \quad \int_0^t \|w(t)(a) - w(\hat{t})(a)\|_{E_0} \,\mathrm{d}a \\ &+ \quad \int_t^{\hat{t}} \|w(t)(a) - w(\hat{t})(a)\|_{E_0} \,\mathrm{d}a \\ &+ \quad \int_{\hat{t}}^{a_m} \|w(t)(a) - w(\hat{t})(a)\|_{E_0} \,\mathrm{d}a \\ &=: \quad J_1 + J_2 + J_3. \end{aligned}$$

Let us consider the term J_1:

$$\begin{aligned} J_1 &= \int_0^t \big\|\Pi_{\hat{A}_\gamma}(a,0)B_{w_0}^{\gamma,f,h}(t-a) + \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s \\ &- \Pi_{\hat{A}_\gamma}(a,0)B_{w_0}^{\gamma,f,h}(\hat{t}-a) - \int_{\hat{t}-a}^{\hat{t}} \Pi_{\hat{A}_\gamma}(a,s+a-\hat{t})f(s,s+a-\hat{t})\,\mathrm{d}s\big\|_{E_0}\,\mathrm{d}a. \end{aligned}$$

Observe that

$$\lim_{\hat{t}\to t}\int_0^t \big\|\Pi_{\hat{A}_\gamma}(a,0)B_{w_0}^{\gamma,f,h}(t-a) - \Pi_{\hat{A}_\gamma}(a,0)B_{w_0}^{\gamma,f,h}(\hat{t}-a)\big\|_{E_0}\,\mathrm{d}a = 0,$$

since $B_{w_0}^{\gamma,f,h} : [0,T] \to E_0$ is continuous. Now assume $0 < \hat{t} - t < t$, then an estimate analogous to [38, Prop. 2.2] gives (we write $\|.\|$ instead of $\|.\|_{E_0}$ for readability):

$$\begin{aligned} &\int_0^t \| \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s \\ &- \int_{\hat{t}-a}^{\hat{t}} \Pi_{\hat{A}_\gamma}(a,s+a-\hat{t})f(s,s+a-\hat{t})\,\mathrm{d}s\big\|_{E_0}\,\mathrm{d}a \\ &\le \int_0^{\hat{t}-t} \Big(\int_{t-a}^t \|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\|\mathrm{d}s \\ &+ \int_{\hat{t}-a}^{\hat{t}} \|\Pi_{\hat{A}_\gamma}(a,s+a-\hat{t})f(s,s+a-\hat{t})\|\,\mathrm{d}s\Big)\,\mathrm{d}a \\ &+ \int_{\hat{t}-t}^t \Big(\int_{t-a}^{\hat{t}-a} \|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\|\,\mathrm{d}s \\ &+ \int_{\hat{t}-a}^t \big\|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t) \\ &- \Pi_{\hat{A}_\gamma}(a,s+a-\hat{t})f(s,s+a-\hat{t})\big\|\,\mathrm{d}s \\ &+ \int_t^{\hat{t}} \|\Pi_{\hat{A}_\gamma}(a,s+a-\hat{t})f(s,s+a-\hat{t})\|\,\mathrm{d}s\Big)\,\mathrm{d}a \\ &= \int_{2t-\hat{t}}^t \int_{t-s}^{\hat{t}-t} \|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\ &+ \int_t^{\hat{t}} \int_{\hat{t}-s}^{\hat{t}-t} \|\Pi_{\hat{A}_\gamma}(a,s+a-\hat{t})f(s,s+a-\hat{t})\|\,\mathrm{d}a\mathrm{d}s \\ &+ \int_{2t-\hat{t}}^t \int_{\hat{t}-t}^{\hat{t}-s} \|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\ &+ \int_{\hat{t}-t}^{2t-\hat{t}} \int_{t-s}^{\hat{t}-s} \|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\ &+ \int_0^{\hat{t}-t} \int_{t-s}^t \|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\ &+ \int_{\hat{t}-t}^t \int_{\hat{t}-s}^t \big\|\Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t) \end{aligned}$$

$$
\begin{aligned}
&- \Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\big\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_t^{\hat{t}} \int_{\hat{t}-t}^{t} \|\Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\| \,\mathrm{d}a\mathrm{d}s \\
&= \int_{\hat{t}-t}^{t} \int_{t-s}^{\hat{t}-s} \|\Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t)\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_0^{\hat{t}-t} \int_{t-s}^{t} \|\Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t)\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_t^{\hat{t}} \int_{\hat{t}-s}^{t} \|\Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_{\hat{t}-t}^{t} \int_{\hat{t}-s}^{t} \big\|\Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t) \\
&- \Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\big\| \,\mathrm{d}a\mathrm{d}s \\
&\leq \int_{\hat{t}-t}^{t} \int_0^{\hat{t}-t} \|\Pi_{\hat{A}_\gamma}(a+t-s, a) f(s, a)\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_0^{\hat{t}-t} \int_0^{s} \|\Pi_{\hat{A}_\gamma}(a+t-s, a) f(s, a)\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_t^{\hat{t}} \int_0^{s+t-\hat{t}} \|\Pi_{\hat{A}_\gamma}(a+\hat{t}-s, a) f(s, a)\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_{\hat{t}-t}^{t} \int_{\hat{t}-t}^{s} \big\|\Pi_{\hat{A}_\gamma}(a+t-s, a) f(s, a) \\
&- \Pi_{\hat{A}_\gamma}(a+t-s, a+t-\hat{t}) f(s, a+t-\hat{t})\big\| \,\mathrm{d}a\mathrm{d}s.
\end{aligned}
$$

The first integral tends to zero as $\hat{t} \to t$ by the dominated convergence theorem, since the integrand converges to zero pointwise and is dominated by

$$
\int_0^{\hat{t}-t} \|\Pi_{\hat{A}_\gamma}(a+t-s, a) f(s, a)\| \,\mathrm{d}a \leq M e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s)\|_{\mathbb{E}_0}.
$$

An analogous estimate shows that the second and third integral converge to zero by continuity of $f : [0, T] \to \mathbb{E}_0$. The fourth integral can be estimated by

$$
\begin{aligned}
&\int_{\hat{t}-t}^{t} \int_{\hat{t}-t}^{s} \big\|\Pi_{\hat{A}_\gamma}(a+t-s, a) \big(f(s, a) - f(s, a+t-\hat{t})\big) \\
&+ \Big(\Pi_{\hat{A}_\gamma}(a+t-s, a) - \Pi_{\hat{A}_\gamma}(a+t-s, a+t-\hat{t})\Big) f(s, a+t-\hat{t})\big\| \,\mathrm{d}a\mathrm{d}s \\
&\leq \int_{\hat{t}-t}^{t} M e^{-(\bar{\omega}+\gamma)(t-s)} \int_{\hat{t}-t}^{s} \|f(s, a) - f(s, a+t-\hat{t})\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_{\hat{t}-t}^{t} \int_0^{s+t-\hat{t}} \big\|\big(\Pi_{\hat{A}_\gamma}(a+\hat{t}-s, a+\hat{t}-t) \\
&- \Pi_{\hat{A}_\gamma}(a+\hat{t}-s, a)\big) f(s, a)\big\| \,\mathrm{d}a\mathrm{d}s,
\end{aligned}
$$

and the first term converges to zero as $\hat{t} \to t$ by Kolmogorov's compactness criterion in $\mathbb{E}_0 = L_1(J, E_0)$ (here we used that $\{f(s) : s \in [0, T]\}$ is compact in $\mathbb{E}_0$), cf. [16, Thm A.1], the second term tends to zero by the dominated convergence theorem (applied on the product measure space). Thus, we see that J_1 approaches zero as $|\hat{t}-t| \to 0$, if $0 < \hat{t}-t < t$, and a similar argument holds for $t=0$. Next, we consider the term J_2:

$$
\begin{aligned}
J_2 &= \int_t^{\hat{t}} \Big\| \Pi_{\hat{A}_\gamma}(a, a-t) w_0(a-t) + \int_0^t \Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t)\, \mathrm{d}s \\
&- \Pi_{\hat{A}_\gamma}(a, 0) B_{w_0}^{\gamma, f, h}(\hat{t}-a) - \int_{\hat{t}-a}^{\hat{t}} \Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\, \mathrm{d}s \Big\| \,\mathrm{d}a \\
&\leq M e^{-(\bar{\omega}+\gamma)t} \int_0^{\hat{t}-t} \|w_0(a)\| \,\mathrm{d}a \\
&+ M \int_t^{\hat{t}} \int_0^t e^{-(\bar{\omega}+\gamma)(t-s)} \|f(s, s+a-t)\| \,\mathrm{d}s\mathrm{d}a \\
&+ M \int_t^{\hat{t}} e^{-(\bar{\omega}+\gamma)a} \|B_{w_0}^{\gamma, f, h}(\hat{t}-a)\| \,\mathrm{d}a \\
&+ M \int_t^{\hat{t}} \int_{\hat{t}-a}^{\hat{t}} e^{-(\bar{\omega}+\gamma)(\hat{t}-s)} \|f(s, s+a-\hat{t})\| \,\mathrm{d}s\mathrm{d}a,
\end{aligned}
$$

and as $|t-\hat{t}| \to 0$, the first term tends to zero, since w_0 is integrable, and the third term goes to zero due to continuity of $B_{w_0}^{\gamma, f, h} : [0, T] \to E_0$. Regarding the second and fourth term, we remark that the integrands are integrable on the product measure space (see the lines following (4.13)), hence those integrals converge to zero as $|t-\hat{t}| \to 0$ by the dominated convergence theorem. Finally, we turn to the term J_3:

$$
\begin{aligned}
J_3 &= \int_{\hat{t}}^{a_m} \Big\| \Pi_{\hat{A}_\gamma}(a, a-t) w_0(a-t) + \int_0^t \Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t)\, \mathrm{d}s \\
&- \Pi_{\hat{A}_\gamma}(a, a-\hat{t}) w_0(a-\hat{t}) - \int_0^{\hat{t}} \Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\, \mathrm{d}s \Big\| \,\mathrm{d}a \\
&\leq \int_{\hat{t}}^{a_m} \|\Pi_{\hat{A}_\gamma}(a, a-t) \left(w_0(a-t) - w_0(a-\hat{t}) \right) \| \,\mathrm{d}a \\
&+ \int_{\hat{t}}^{a_m} \| \Big(\Pi_{\hat{A}_\gamma}(a, a-t) - \Pi_{\hat{A}_\gamma}(a, a-\hat{t}) \Big) w_0(a-\hat{t}) \| \,\mathrm{d}a \\
&+ \int_{\hat{t}}^{a_m} \int_0^t \Big\| \Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t) \\
&- \Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t}) \Big\| \,\mathrm{d}s\mathrm{d}a \\
&+ \int_{\hat{t}}^{a_m} \int_t^{\hat{t}} \|\Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t})\| \,\mathrm{d}s\mathrm{d}a \\
&\leq M e^{-(\omega+\gamma)t} \int_0^{a_m-\hat{t}} \|w_0(a+\hat{t}-t) - w_0(a)\| \,\mathrm{d}a \\
&+ \int_0^{a_m-\hat{t}} \| \Big(\Pi_{\hat{A}_\gamma}(a+\hat{t}, a+\hat{t}-t) - \Pi_{\hat{A}_\gamma}(a+\hat{t}, a) \Big) w_0(a)\| \,\mathrm{d}a
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^t \int_{\hat{t}}^{a_m} \big\| \Pi_{\hat{A}_\gamma}(a, s+a-t) f(s, s+a-t) \\
&- \Pi_{\hat{A}_\gamma}(a, s+a-\hat{t}) f(s, s+a-\hat{t}) \big\| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_t^{\hat{t}} M e^{-(\omega+\gamma)(\hat{t}-s)} \int_s^{a_m+s-\hat{t}} \|f(s,a)\| \,\mathrm{d}a\mathrm{d}s.
\end{aligned}
$$

As $|t-\hat{t}| \to 0$, the first term tends to zero, since translation is continuous on $\mathbb{E}_0$. The second term converges to zero by the dominated convergence theorem (use (4.9) to obtain an integrable majorant), the fourth term vanishes because $f : [0,T] \to \mathbb{E}_0$ is continuous. The third term can be estimated by

$$
\begin{aligned}
&\int_0^t \int_{s+\hat{t}-t}^{s+a_m-t} \big\| \Pi_{\hat{A}_\gamma}(a+t-s, a) f(s,a) \\
&- \Pi_{\hat{A}_\gamma}(a+t-s, a+t-\hat{t}) f(s, a+t-\hat{t}) \big\| \,\mathrm{d}a\mathrm{d}s \\
&\leq \int_0^t \int_{s+\hat{t}-t}^{s+a_m-t} \|\Pi_{\hat{A}_\gamma}(a+t-s, a) \big(f(s,a) - f(s, a+t-\hat{t}) \big) \| \,\mathrm{d}a\mathrm{d}s \\
&+ \int_0^t \int_{s+\hat{t}-t}^{s+a_m-t} \big\| \big(\Pi_{\hat{A}_\gamma}(a+t-s, a) \\
&- \Pi_{\hat{A}_\gamma}(a+t-s, a+t-\hat{t}) \big) f(s, a+t-\hat{t}) \big\| \,\mathrm{d}a\mathrm{d}s,
\end{aligned}
$$

and, as before, for the term J_1 we conclude that the first integral tends to zero by Kolmogorov's compactness criterion, the second integral converges to zero by the dominated convergence theorem (after a substitution of variable $\tilde{a} = a+t-\hat{t}$). $\square$

4.3 The nonlinear equation

In the following we consider a generalised version of problem (1.4), namely

$$
\begin{aligned}
\partial_t u + \partial_a u + A(a)u + \gamma u &= F(t, u(t))(a), \qquad t>0,\ a \in J \\
u(t,0) &= \int_0^{a_m} b(u(t),a) u(t,a) \,\mathrm{d}a + h(t), \qquad t>0 \\
u(0,a) &= u_0(a) \qquad a \in J.
\end{aligned}
\tag{4.14}
$$

We assume the operator A fulfills conditions (4.2) – (4.3),

$$
\begin{aligned}
&F : [0,T_F] \times \mathbb{E}_0 \to \mathbb{E}_0 \text{ is continuous, there is an} \\
&\text{increasing function } c_F : [0,\infty) \to [0,\infty), \text{ such that} \\
&\|F(t,\phi_1) - F(t,\phi_2)\|_{\mathbb{E}_0} \leq c_F(r) \|\phi_1 - \phi_2\|_{\mathbb{E}_0}, \\
&\text{for all } t \in [0,T_F], \phi_1, \phi_2 \in \mathbb{E}_0 \text{ with } \|\phi_1\|_{\mathbb{E}_0}, \|\phi_2\|_{\mathbb{E}_0} \leq r,
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
&b : \mathbb{E}_0 \times [0,\infty) \to \mathcal{L}(E_0) \text{ is bounded, continuous,} \\
&\text{with } \mathrm{supp}(b) \subset \mathbb{E}_0 \times J, \text{ there is an} \\
&\text{increasing function } c_b : [0,\infty) \to [0,\infty), \text{ such that} \\
&\|b(\phi_1,a) - b(\phi_2,a)\|_{\mathcal{L}(E_0)} \leq c_b(r) \|\phi_1 - \phi_2\|_{\mathbb{E}_0}, \\
&\text{for all } a \in [0,\infty), \phi_1, \phi_2 \in \mathbb{E}_0 \text{ with } \|\phi_1\|_{\mathbb{E}_0}, \|\phi_2\|_{\mathbb{E}_0} \leq r.
\end{aligned}
\tag{4.16}
$$

Recall (4.8), where we have set

$$A_\gamma(a)v = A(a)v + \gamma v, \quad v \in E_1,$$

and let Π_{A_γ} denote the corresponding parabolic evolution operator.

Remark 4.6. We point out that the nonlinearity $F : [0, T_F] \times \mathbb{E}_0 \to \mathbb{E}_0$ is given, and therefore equation (4.14) only makes sense if

$$0 < t \leq T_F,$$

which will be assumed throughout section 4.3 without further notice. □

Definition 4.7. Let $u : [0,T] \to \mathbb{E}_0$ be continuous, with $u(\cdot, 0) \in C([0,T], E_0)$. We say that the function u is an *integral solution* to (4.14) on $[0,T]$, if for all $t \in [0,T]$

$$\begin{aligned} u(t,a) &= \begin{cases} \Pi_{A_\gamma}(a, a-t)u_0(a-t) \\ \Pi_{A_\gamma}(a,0)B(t-a) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi_{A_\gamma}(a, s+a-t)F(s,u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (t, a_m) \\ \int_{t-a}^t \Pi_{A_\gamma}(a, s+a-t)F(s,u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (0,t), \end{cases} \end{aligned}$$

where $a < a_m$, and $B(t) = u(t,0)$ satisfies, for $t \in [0,T]$, the associated integral equation

$$\begin{aligned} B(t) &= \int_0^t b(u(t),a)\Pi_{A_\gamma}(a,0)B(t-a)\,\mathrm{d}a \\ &+ \int_0^t b(u(t),a)\int_{t-a}^t \Pi_{A_\gamma}(a, s+a-t)F(s,u(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_t^{a_m} b(u(t),a)\Pi_{A_\gamma}(a,a-t)u_0(a-t)\,\mathrm{d}a \\ &+ \int_t^{a_m} b(u(t),a)\int_0^t \Pi_{A_\gamma}(a, s+a-t)F(s,u(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ h(t). \end{aligned} \tag{4.17}$$

Remark 4.8. For the sake of brevity, we introduce the notation

$$\|u\|_\infty := \sup_{s\in[0,T]} \|u(s)\|_{\mathbb{E}_0}, \qquad \text{for } u \in C([0,T], \mathbb{E}_0),$$

and analogously for $b \in BC(\mathbb{E}_0 \times [0,\infty), \mathcal{L}(E_0))$. Furthermore, we write

$$(F \circ u)(t) := F(u(t)) := F(t, u(t)).$$

□

Let $u \in C([0,T], \mathbb{E}_0)$ be fixed, then the function

$$[0,T] \to \mathbb{E}_0, \qquad t \mapsto F(t, u(t))$$

is continuous by assumption (4.15) (cf. Remark 4.6), and

$$b_L : [0,T] \times [0,\infty) \to \mathcal{L}(E_0), \qquad (t,a) \mapsto b(u(t),a)$$

is continuous by assumption (4.16), with $\operatorname{supp}(b) \subset [0,T] \times J$. Let

$$u_0 \in \mathbb{E}_0, \qquad \gamma \in \mathbb{R}, \qquad h \in C([0,T], E_0)$$

be given, then by Lemma 4.5 there exists a unique solution

$$B = B_{u_0}^{\gamma, F\circ u, h} \in C([0,T], E_0)$$

to equation (4.17).

After these considerations we can establish the following technical lemma, which will be required in the proof of Theorem 4.10:

Lemma 4.9. *Assume the operator A satisfies* (4.2) – (4.3), *and conditions* (4.15) – (4.16) *hold. Let $u_0 \in \mathbb{E}_0$, $\gamma \in \mathbb{R}$, $h \in C([0,T], E_0)$ be given. Furthermore, let $r > 0$ be fixed and $u_1, u_2 \in C([0,T], \mathbb{E}_0)$, such that $\|u_1\|_\infty, \|u_2\|_\infty \le r$.*
Then there exists a positive constant $c_B = c_B(r,T)$, such that the solutions $B_{u_0}^{\gamma,F\circ u_1,h}$, $B_{u_0}^{\gamma,F\circ u_2,h}$ to (4.17) *satisfy the estimate*

$$\|B_{u_0}^{\gamma,F\circ u_1,h}(t) - B_{u_0}^{\gamma,F\circ u_2,h}(t)\|_{E_0} \le c_B \sup_{s\in[0,T]} \|u_1(s) - u_2(s)\|_{\mathbb{E}_0}, \qquad t \in [0,T].$$

Proof. In the case $b = b(a)$, the claimed estimate is a straightforward consequence of (4.15) and Gronwall's inequality. If $b = b(u,a)$, the situation becomes more involved and we provide the details in the following.

In a first step we use (4.17) to estimate

$$\begin{aligned}
\|B_{u_0}^{\gamma,F\circ u_2,h}(t)\| &\le M\|b\|_\infty \int_0^t e^{-(\bar\omega+\gamma)(t-a)} \|B_{u_0}^{\gamma,F\circ u_2,h}(a)\| \,\mathrm{d}a \\
&+ M\|b\|_\infty \int_0^t \int_{t-s}^t e^{-(\bar\omega+\gamma)(t-s)} \|F(u(s))(s+a-t)\| \,\mathrm{d}a\mathrm{d}s \\
&+ M\|b\|_\infty \int_t^{a_m} e^{-(\bar\omega+\gamma)t} \|u_0(a-t)\| \,\mathrm{d}a \\
&+ M\|b\|_\infty \int_0^t \int_t^{a_m} e^{-(\bar\omega+\gamma)(t-s)} \|F(u(s))(s+a-t)\| \,\mathrm{d}a\mathrm{d}s \\
&+ \|h(t)\| \\
&\le M\|b\|_\infty e^{|\bar\omega+\gamma|T} \int_0^t \|B_{u_0}^{\gamma,F\circ u_2,h}(a)\| \,\mathrm{d}a \\
&+ M\|b\|_\infty e^{|\bar\omega+\gamma|T} \left(\int_0^t \|F(u(s))\|_{\mathbb{E}_0} \,\mathrm{d}s + \|u_0\|_{\mathbb{E}_0} \right) + \|h(t)\|,
\end{aligned}$$

and by (4.15) we have $\|F(u(s))\| \le c_F(r)\|u(s)\| + \|F(0)\|$. Consequently, we can apply Gronwall's inequality to conclude the existence of a constant $c = c(r,T)$, independent of u_2, such that

$$\|B_{u_0}^{\gamma,F\circ u_2,h}(t)\| \le c(r,T), \qquad \text{for } t \in [0,T]. \tag{4.18}$$

Next observe that by (4.17)

$$
\begin{aligned}
&B_{u_0}^{\gamma,F\circ u_1,h}(t) - B_{u_0}^{\gamma,F\circ u_2,h}(t)\\
&= \Big(\int_0^t b(u_1(t),a)\Pi_{A_\gamma}(a,0)B_{u_0}^{\gamma,F\circ u_1,h}(t-a)\,\mathrm{d}a\\
&- \int_0^t b(u_2(t),a)\Pi_{A_\gamma}(a,0)B_{u_0}^{\gamma,F\circ u_2,h}(t-a)\,\mathrm{d}a\Big)\\
&+\Big(\int_0^t b(u_1(t),a)\int_{t-a}^t \Pi_{A_\gamma}(a,s+a-t)F(u_1(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a\\
&- \int_0^t b(u_2(t),a)\int_{t-a}^t \Pi_{A_\gamma}(a,s+a-t)F(u_2(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a\Big)\\
&+ \int_t^{a_m} \big(b(u_1(t),a)-b(u_2(t),a)\big)\Pi_{A_\gamma}(a,a-t)u_0(a-t)\,\mathrm{d}a\\
&+\Big(\int_t^{a_m} b(u_1(t),a)\int_0^t \Pi_{A_\gamma}(a,s+a-t)F(u_1(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a\\
&- \int_t^{a_m} b(u_2(t),a)\int_0^t \Pi_{A_\gamma}(a,s+a-t)F(u_2(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a\Big)\\
&=: I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned}
$$

For the first term we estimate

$$
\begin{aligned}
\|I_1(t)\| \quad &\le \quad \int_0^t \|b(u_1(t),a)\Pi_{A_\gamma}(a,0)\big(B_{u_0}^{\gamma,F\circ u_1,h}(t-a) - B_{u_0}^{\gamma,F\circ u_2,h}(t-a)\big)\|\,\mathrm{d}a\\
&+ \quad \int_0^t \|\big(b(u_1(t),a)-b(u_2(t),a)\big)\Pi_{A_\gamma}(a,0)B_{u_0}^{\gamma,F\circ u_2,h}(t-a)\|\,\mathrm{d}a\\
&\le \quad M\|b\|_\infty \int_0^t e^{-(\bar\omega+\gamma)(t-a)}\|B_{u_0}^{\gamma,F\circ u_1,h}(a) - B_{u_0}^{\gamma,F\circ u_2,h}(a)\|\,\mathrm{d}a\\
&+ \quad Mc_b(r)\|u_1(t)-u_2(t)\| \int_0^t e^{-(\bar\omega+\gamma)(t-a)}\|B_{u_0}^{\gamma,F\circ u_2,h}(a)\|\,\mathrm{d}a\\
&\le \quad M\|b\|_\infty e^{|\bar\omega+\gamma|T} \int_0^t \|B_{u_0}^{\gamma,F\circ u_1,h}(a) - B_{u_0}^{\gamma,F\circ u_2,h}(a)\|\,\mathrm{d}a\\
&+ \quad MTe^{|\bar\omega+\gamma|T}c(r,T)c_b(r)\|u_1(t)-u_2(t)\|
\end{aligned}
$$

where we used (4.16) in the second inequality and (4.18) in the third one.

For the second term one has

$$\begin{aligned}
\|I_2(t)\| &\le \Big\| \int_0^t b(u_1(t),a) \int_{t-a}^t \Pi_{A_\gamma}(a,s+a-t)\big(F(u_1(s)) \\
&- F(u_2(s))\big)(s+a-t)\,\mathrm{d}s\mathrm{d}a \Big\| \\
&+ \Big\| \int_0^t \big(b(u_1(t),a) \\
&- b(u_2(t),a)\big) \int_{t-a}^t \Pi_{A_\gamma}(a,s+a-t)F(u_2(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a \Big\| \\
&\le Me^{|\bar\omega+\gamma|T}\|b\|_\infty \int_0^t \int_{t-s}^t \big\|\big(F(u_1(s)) - F(u_2(s))\big)(s+a-t)\big\|\,\mathrm{d}a\mathrm{d}s \\
&+ Me^{|\bar\omega+\gamma|T} c_b(r)\|u_1(t)-u_2(t)\| \int_0^t \int_{t-s}^t \|F(u_2(s))(s+a-t)\|\,\mathrm{d}a\mathrm{d}s \\
&\le Me^{|\bar\omega+\gamma|T}\|b\|_\infty \int_0^t \|F(u_1(s)) - F(u_2(s))\|_{\mathbb{E}_0}\,\mathrm{d}s \\
&+ Me^{|\bar\omega+\gamma|T} c_b(r)\|u_1(t)-u_2(t)\| \int_0^t \|F(u_2(s))\|\,\mathrm{d}s \\
&\le MTe^{|\bar\omega+\gamma|T}\|b\|_\infty c_F(r)\|u_1-u_2\|_\infty \\
&+ MTe^{|\bar\omega+\gamma|T} c_b(r)\big(c_F(r)r + \|F(0)\|\big)\|u_1(t)-u_2(t)\|.
\end{aligned}$$

It is not hard to see that the term I_4 can be estimated completely analogous. For the remaining term we have

$$\begin{aligned}
\|I_3(t)\| &\le Me^{|\bar\omega+\gamma|T} \int_t^{a_m} \|b(u_1(t),a) - b(u_2(t),a)\|_{\mathcal{L}(E_0)}\,\|u_0(a-t)\|_{E_0}\,\mathrm{d}a \\
&\le Me^{|\bar\omega+\gamma|T}\|u_0\|_{\mathbb{E}_0}\, c_b(r)\,\|u_1(t)-u_2(t)\|.
\end{aligned}$$

Combining the previous estimates, we see that there is a constant $\tilde c = \tilde c(r,T)$ such that

$$\begin{aligned}
&\|B_{u_0}^{\gamma,F\circ u_1,h}(t) - B_{u_0}^{\gamma,F\circ u_2,h}(t)\| \\
&\le M\|b\|_\infty e^{|\bar\omega+\gamma|T} \int_0^t \|B_{u_0}^{\gamma,F\circ u_1,h}(a) - B_{u_0}^{\gamma,F\circ u_2,h}(a)\|\,\mathrm{d}a \\
&+ \tilde c\|u_1-u_2\|_\infty,
\end{aligned}$$

hence the claim follows from Gronwall's inequality. □

Theorem 4.10 (Local existence)**.** *Assume the operator A satisfies* (4.2) − (4.3)*, and conditions* (4.15) − (4.16) *hold. Let $\gamma \in \mathbb{R}$, $h \in C([0,T],E_0)$ be given, and $r > 0$ be fixed. There exists $\tilde T = \tilde T(r) \in (0,T]$ such that if $u_0 \in \mathbb{E}_0$ and $\|u_0\|_{\mathbb{E}_0} \le r$, then there is a unique integral solution to* (4.14) *on $[0,\tilde T]$.*

Proof. The argument is based on Banach's fixed point theorem. To this end recall that for given $u \in C([0,T],\mathbb{E}_0)$, there exists a unique solution $B = B_{u_0}^{\gamma,F\circ u,h} \in C([0,T],E_0)$ to (4.17) by Lemma

4.5. Subsequently, we use this solution to introduce the map

$$
\begin{aligned}
&(Ku)(t,a) \qquad\qquad (4.19)\\
:=\ &\begin{cases} \Pi_{A_\gamma}(a,a-t)u_0(a-t) \\ \Pi_{A_\gamma}(a,0)B(t-a) \end{cases}\\
+\ &\begin{cases} \int_0^t \Pi_{A_\gamma}(a,s+a-t)F(s,u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a\in(t,a_m) \\ \int_{t-a}^t \Pi_{A_\gamma}(a,s+a-t)F(s,u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a\in(0,t), \end{cases}
\end{aligned}
$$

where $a < a_m$, and observe that

$$K : C([0,T],\mathbb{E}_0) \to C([0,T],\mathbb{E}_0),$$

is well defined by Proposition 4.4. The integral solution we are looking for corresponds to a fixed point of K. Let $u_0 \in \mathbb{E}_0$, with $\|u_0\|_{\mathbb{E}_0} \le r$, and introduce the set

$$S := \{u \in C([0,T],\mathbb{E}_0) \mid \|u\|_\infty \le 2\,M\,r\}.$$

In the following we will show that for $T > 0$ sufficiently small, K is a contraction from S into S.

For $u \in S$, we estimate

$$
\begin{aligned}
&\int_0^{a_m} \|(Ku)(t,a)\|\,\mathrm{d}a\\
&\le \int_0^t \|\Pi_{A_\gamma}(t-a,0)B_{u_0}^{\gamma,F\circ u,h}(a)\|\,\mathrm{d}a\\
&+\int_0^t\int_{t-a}^t \|\Pi_{A_\gamma}(a,s+a-t)F(s,u(s))(s+a-t)\|\,\mathrm{d}s\mathrm{d}a\\
&+\int_0^{a_m-t} \|\Pi_{A_\gamma}(a+t,a)u_0(a)\|\,\mathrm{d}a\\
&+\int_t^{a_m}\int_0^t \|\Pi_{A_\gamma}(a,s+a-t)F(s,u(s))(s+a-t)\|\,\mathrm{d}s\mathrm{d}a
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t Me^{-(\bar\omega+\gamma)(t-a)}\|B_{u_0}^{\gamma,F\circ u,h}(a)\|\,\mathrm{d}a\\
&+\int_0^t\int_{t-s}^t Me^{-(\bar\omega+\gamma)(t-s)}\|F(s,u(s))(s+a-t)\|\,\mathrm{d}a\mathrm{d}s\\
&+\int_0^{a_m-t} Me^{-(\bar\omega+\gamma)t}\|u_0(a)\|\,\mathrm{d}a\\
&+\int_0^t\int_t^{a_m} Me^{-(\bar\omega+\gamma)(t-s)}\|F(s,u(s))(s+a-t)\|\,\mathrm{d}a\mathrm{d}s\\
&\le Me^{|\bar\omega+\gamma|T}\left(\int_0^t \|B_{u_0}^{\gamma,F\circ u,h}(a)\|\,\mathrm{d}a + \int_0^t \|F(s,u(s))\|_{\mathbb{E}_0}\,\mathrm{d}s + \|u_0\|_{\mathbb{E}_0}\right)\\
&\le Me^{|\bar\omega+\gamma|T}\big(T\,c(2Mr,T) + T(c_F(2Mr)2Mr + \|F(0)\|) + r\big),
\end{aligned}
$$

where we used (4.18) in the last inequality. Consequently, K maps the set S into itself, if we choose $T > 0$ sufficiently small (observe that the factor $c(2Mr,T)$ from (4.18) remains bounded as T tends to zero).

Consider $u_1, u_2 \in S$, and let $t \in [0,T]$ then we have

$$
\begin{aligned}
&\|(Ku_1)(t) - (Ku_2)(t)\|_{\mathbb{E}_0} \\
&= \int_0^{a_m} \| \left((Ku_1)(t) - (Ku_2)(t)\right)(a)\|_{E_0}\, \mathrm{d}a \\
&\leq \int_0^t \|\Pi_{A_\gamma}(a,0)\left(B_{u_0}^{\gamma,F\circ u_1,h}(t-a) - B_{u_0}^{\gamma,F\circ u_2,h}(t-a)\right)\|\, \mathrm{d}a \\
&+ \int_0^t \int_{t-a}^t \|\Pi_{A_\gamma}(a,s+a-t)\left(F(s,u_1(s)) - F(s,u_2(s))\right)(s+a-t)\|\, \mathrm{d}s\mathrm{d}a \\
&+ \int_t^{a_m} \int_0^t \|\Pi_{A_\gamma}(a,s+a-t)\left(F(s,u_1(s)) - F(s,u_2(s))\right)(s+a-t)\|\, \mathrm{d}s\mathrm{d}a \\
&\leq \int_0^t Me^{-(\bar{\omega}+\gamma)(t-a)}\|B_{u_0}^{\gamma,F\circ u_1,h}(a) - B_{u_0}^{\gamma,F\circ u_2,h}(a)\|\, \mathrm{d}a \\
&+ \int_0^t \int_{t-s}^t Me^{-(\bar{\omega}+\gamma)(t-s)}\|\left(F(s,u_1(s)) - F(s,u_2(s))\right)(s+a-t)\|\, \mathrm{d}a\mathrm{d}s \\
&+ \int_0^t \int_t^{a_m} Me^{-(\bar{\omega}+\gamma)(t-s)}\|\left(F(s,u_1(s)) - F(s,u_2(s))\right)(s+a-t)\|\, \mathrm{d}a\mathrm{d}s \\
&\leq Me^{|\bar{\omega}+\gamma|T}\int_0^t \|B_{u_0}^{\gamma,F\circ u_1,h}(a) - B_{u_0}^{\gamma,F\circ u_2,h}(a)\|\, \mathrm{d}a \\
&+ Me^{|\bar{\omega}+\gamma|T}\int_0^t \|F(s,u_1(s)) - F(s,u_2(s))\|\, \mathrm{d}s \\
&\leq MTe^{|\bar{\omega}+\gamma|T}\big(c_B(2Mr,T)\|u_1-u_2\|_\infty + c_F(2Mr)\|u_1-u_2\|_\infty\big),
\end{aligned}
$$

where we used Lemma 4.9 in the last inequality. Hence, K is a contraction on S, if we choose $T > 0$ sufficiently small (observe that the factor $c_B(2Mr,T)$ from Lemma 4.9 remains bounded as T tends to zero). □

Proposition 4.11 (Global uniqueness)**.** *Assume the operator A satisfies* (4.2) – (4.3)*, and conditions* (4.15) – (4.16) *hold. Let $u_0 \in \mathbb{E}_0$, $\gamma \in \mathbb{R}$, $h \in C([0,T],E_0)$ be given. Then there exists at most one integral solution to* (4.14) *on* $[0,T]$*.*

Proof. Suppose we have two integral solutions, say

$$(u_1,B_1),(u_2,B_2) \in C([0,T],\mathbb{E}_0) \times C([0,T],E_0),$$

where we have set $B_i(t) = u_i(t,0), t \in [0,T]$, for $i = 1,2$. Let us recall the map

$$K : C([0,T],\mathbb{E}_0) \to C([0,T],\mathbb{E}_0)$$

from the proof of Theorem 4.10, then, by assumption, $Ku_i = u_i$, for $i = 1,2$. Furthermore, resuming the notation of Theorem 4.10, we have $B_i = B_{u_0}^{\gamma,F\circ u_i,h}$, for $i = 1,2$. We set

$$z(t) := \|B_1(t) - B_2(t)\|_{E_0}, \quad t \in [0,T],$$

then we have as in the proof of Lemma 4.9

$$z(t) \leq \|I_1(t)\|_{E_0} + \|I_2(t)\|_{E_0} + \|I_3(t)\|_{E_0} + \|I_4(t)\|_{E_0}, \quad t \in [0,T].$$

From the proof of Lemma 4.9 we see that there is a positive constant $c(T)$, such that we can estimate

$$\|I_1(t)\|_{E_0} \le c(T)\,\Big(\int_0^t \|z(a)\|_{E_0}\,\mathrm{d}a + \|u_1(t)-u_2(t)\|_{\mathbb{E}_0}\Big),$$
$$\|I_2(t)\|_{E_0} \le c(T)\,\Big(\int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s + \|u_1(t)-u_2(t)\|_{\mathbb{E}_0}\Big),$$
$$\|I_3(t)\|_{E_0} \le c(T)\,\|u_1(t)-u_2(t)\|_{\mathbb{E}_0},$$
$$\|I_4(t)\|_{E_0} \le c(T)\,\Big(\int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s + \|u_1(t)-u_2(t)\|_{\mathbb{E}_0}\Big),$$

and consequently there is a positive constant, again denoted by $c(T)$, such that

$$\begin{aligned} z(t) \quad &\le \quad c(T)\Big(\int_0^t z(a)\,\mathrm{d}a + \|u_1(t)-u_2(t)\|_{\mathbb{E}_0} \\ &+ \quad \int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s\Big), \quad t\in[0,T]. \end{aligned}$$

Gronwall's inequality then implies

$$\begin{aligned} z(t) \quad &\le \quad c(T)\left(\|u_1(t)-u_2(t)\|_{\mathbb{E}_0} + \int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s\right) \\ &+ \quad c(T)\int_0^t \left(\|u_1(s)-u_2(s)\|_{\mathbb{E}_0} + \int_0^s \|u_1(\tau)-u_2(\tau)\|_{\mathbb{E}_0}\,\mathrm{d}\tau\right)\mathrm{d}s, \end{aligned}$$

and therefore we conclude that there is a positive constant, again denoted by $c(T)$, such that

$$z(t) \le c(T)\left(\|u_1(t)-u_2(t)\|_{\mathbb{E}_0} + \int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s\right), \quad t\in[0,T]. \tag{4.20}$$

In the proof of Theorem 4.10 we obtained the estimate

$$\begin{aligned} \|(Ku_1)(t)-(Ku_2)(t)\|_{\mathbb{E}_0} \quad &\le \quad c(T)\int_0^t z(a)\,\mathrm{d}a \\ &+ \quad c(T)\int_0^t \|F(s,u_1(s))-F(s,u_2(s))\|_{\mathbb{E}_0}\,\mathrm{d}s, \end{aligned}$$

estimate (4.20) and assumption (4.15) then lead to (with $c(T)$ adapted in every step)

$$\begin{aligned} &\|(Ku_1)(t)-(Ku_2)(t)\|_{\mathbb{E}_0} \\ &\le c(T)\int_0^t \left(\|u_1(a)-u_2(a)\|_{\mathbb{E}_0} + \int_0^a \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s\right)\mathrm{d}a \\ &+ c(T)\int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s \\ &\le c(T)\int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s, \quad t\in[0,T]. \end{aligned}$$

Since u_1, u_2 are fixed points of the map K, we conclude

$$\|u_1(t)-u_2(t)\|_{\mathbb{E}_0} \le c(T)\int_0^t \|u_1(s)-u_2(s)\|_{\mathbb{E}_0}\,\mathrm{d}s,$$

and Gronwall's inequality implies $u_1 = u_2$. □

4.4 Further properties

In the following we continue the analysis of equation (4.4) and establish some results which will be needed in section 5. Observe that we allowed the birth rate b_L to be time-dependent, cf. (4.5), which was crucial to establish existence of solutions to the nonlinear equation, see Theorem 4.10. In the present section however, the birth rate must not depend on the time parameter, since we want to translate equation (4.4) into the language of semigroup theory (observe that Theorem 3.4 need not hold anymore, if the birth rate depends on time).

After these considerations, let us formulate the assumptions for the present section: we assume the operator A fulfills conditions (4.2) – (4.3), consider a birth rate

$$b_L \in BC([0,\infty), \mathcal{L}(E_0)), \quad \text{supp}(b_L) \subset J, \tag{4.21}$$

and a mortality rate

$$\mu_L \in BC(J, \mathcal{L}(E_0)), \tag{4.22}$$

which is such that

$$(a \mapsto \mu(a)) \in C^\rho(J, \mathcal{L}(E_1, E_0)), \tag{4.23}$$

with Hölder exponent ρ from (4.2). Recalling definition (4.8), we see as in section 4.2 that the parabolic evolution operator $\Pi_{\hat{A}}(a,\sigma), 0 \le \sigma \le a < a_m$, is well defined and satisfies

$$\Pi_{\hat{A}}(a,\sigma)v = \Pi_A(a,\sigma)v - \int_\sigma^a \Pi_A(a,s)\mu_L(s)\Pi_{\hat{A}}(s,\sigma)v\, ds, \quad v \in E_0. \tag{4.24}$$

By assumption (4.3) we have constants $M \ge 1$, $\bar{\omega} \in \mathbb{R}$, such that

$$\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_0)} \le Me^{-\bar{\omega}(a-\sigma)}, \qquad 0 \le \sigma \le a < a_m,$$

and as in section 4.2 we can assume without loss of generality

$$\|\Pi_{\hat{A}}(a,\sigma)\|_{\mathcal{L}(E_0)} \le Me^{-\bar{\omega}(a-\sigma)}, \qquad 0 \le \sigma \le a < a_m. \tag{4.25}$$

Finally, as in Lemma 4.5, we require

$$w_0 \in \mathbb{E}_0, \quad \gamma \in \mathbb{R}, \quad f \in C([0,T], \mathbb{E}_0), \quad h \in C([0,T], E_0). \tag{4.26}$$

To start with, let us recall Definition 4.3 and observe that the linear structure of equation (4.11), together with Lemma 4.5, implies

$$B_{w_0}^{\gamma,f,h} = B_{w_0}^{\gamma,0,0} + B_0^{\gamma,f,0} + B_0^{\gamma,0,h}, \quad \text{in } C([0,T], E_0),$$

consequently the integral solution (4.10) decomposes as

$$w_{w_0}^{\gamma,f,h} = w_{w_0}^{\gamma,0,0} + w_0^{\gamma,f,0} + w_0^{\gamma,0,h}, \quad \text{in } C([0,T], \mathbb{E}_0). \tag{4.27}$$

In the next step we will show how each term on the right side of (4.27) can be reformulated in the context of semigroup theory.

Consider the integral solution

$$w_{w_0}^{\gamma,0,0}(t,a) = \begin{cases} \Pi_{\hat{A}_\gamma}(a, a-t)w_0(a-t), & 0 \le t \le a \\ \Pi_{\hat{A}_\gamma}(a,0)B_{w_0}^{\gamma,0,0}(t-a), & 0 \le a < t, \end{cases}$$

where $B^{\gamma,0,0}_{w_0}(t)$ solves the corresponding Volterra equation (4.11). Setting

$$(S_\gamma(t)w_0)(a) := w^{\gamma,0,0}_{w_0}(t,a), \tag{4.28}$$

we can argue along the lines of Theorem 3.4, and conclude that $S_\gamma(t), t \geq 0$, is a strongly continuous semigroup on $\mathbb{E}_0$ with

$$\sup_{t\geq 0} e^{t(\bar{\omega}+\gamma-\zeta)} \|S_\gamma(t)\|_{\mathcal{L}(\mathbb{E}_0)} < \infty,$$

where $\zeta = M\|b\|_\infty$.

For the second term in (4.27) we have

Proposition 4.12. *Assume the operator A satisfies* (4.2) − (4.3)*, and conditions* (4.21) − (4.23)*,* (4.25) − (4.26) *hold. Then for all $t \in [0,T]$:*

$$w^{\gamma,f,0}_0(t,a) = \left(\int_0^t S_\gamma(t-s)f(s)\, ds\right)(a), \qquad \text{for a.a. } a \in (0,a_m). \tag{4.29}$$

Proof. By definition,

$$\begin{aligned} w^{\gamma,f,0}_0(t,a) &= \begin{cases} 0 \\ \Pi_{\hat{A}_\gamma}(a,0)B^{\gamma,f,0}_0(t-a) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0 \leq t \leq a < a_m \\ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0 \leq a < t,\ a < a_m. \end{cases} \end{aligned}$$

On the other hand, by defintion of S_γ,

$$\big(S_\gamma(t-s)f(s)\big)(a) = \begin{cases} \Pi_{\hat{A}_\gamma}(a,a-(t-s))f(s,a-(t-s)), & 0 \leq t-s \leq a \\ \Pi_{\hat{A}_\gamma}(a,0)B^{\gamma,0,0}_{f(s)}((t-s)-a), & 0 \leq a < t-s, \end{cases}$$

where $a < a_m$. Let us consider the case $t \leq a$. In particular $t - s \leq a - s \leq a$, and therefore

$$\int_0^t (S_\gamma(t-s)f(s))(a)\,\mathrm{d}s = \int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s,$$

which proves one half of the claim (here we also use the last assertion in Theorem A.2).
In the case $a < t$ we have

$$\begin{aligned} \int_0^t \big(S_\gamma(t-s)f(s)\big)(a)\,\mathrm{d}s &= \int_0^{t-a} \Pi_{\hat{A}_\gamma}(a,0)B^{\gamma,0,0}_{f(s)}((t-s)-a)\,\mathrm{d}s \\ &+ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, \end{aligned}$$

and we see that the second term on the right agrees with the formula for $w^{\gamma,f,0}_0(t,a)$. It remains to show the identity

$$\int_0^{t-a} B^{\gamma,0,0}_{f(s)}(t-s-a)\,\mathrm{d}s = B^{\gamma,f,0}_0(t-a), \quad \text{for } a < t,$$

which is proven in Lemma 4.14. □

Finally, for the third term in (4.27), we have

$$w_0^{\gamma,0,h}(t,a) \quad = \quad \begin{cases} 0, & 0 \le t \le a < a_m \\ \Pi_{\hat{A}_\gamma}(a,0) B_0^{\gamma,0,h}(t-a), & 0 \le a < t,\ a < a_m, \end{cases}$$

where $B_0^{\gamma,0,h}$ satisfies the integral equation

$$B_0^{\gamma,0,h}(t) = \int_0^t b_L(t-a) \Pi_{\hat{A}_\gamma}(t-a,0) B_0^{\gamma,0,h}(a)\, \mathrm{d}a + h(t).$$

In section 5, we want to express the h-dependence of $w_0^{\gamma,0,h}$. To this end, let us introduce the integral kernel

$$k_\gamma(t,a) := b_L(t-a)\Pi_{\hat{A}_\gamma}(t-a,0), \tag{4.30}$$

then the integral equation can be written as

$$\begin{aligned} B_0^{\gamma,0,h}(t) \quad &= \quad \int_0^t k_\gamma(t,a) B_0^{\gamma,0,h}(a)\, \mathrm{d}a + h(t) \\ &=: \quad \left(k_\gamma * B_0^{\gamma,0,h}\right)(t) + h(t). \end{aligned}$$

In order to solve this equation, we require the resolvent kernel corresponding to the integral kernel k_γ. Introduce $J_\Delta := \{(t,a) \in J \times J : a \le t\}$, then by Lemma 4.21 there exists a unique resolvent kernel $r_\gamma \in C(J_\Delta, \mathcal{L}_s(E_0))$ (cf. section 2 for notation), and consequently

$$B_0^{\gamma,0,h}(t) = (r_\gamma * h)(t) + h(t).$$

Putting everything together leads us to

$$w_0^{\gamma,0,h}(t,a) = \begin{cases} 0, & 0 \le t \le a \\ \Pi_{\hat{A}_\gamma}(a,0)\big((r_\gamma * h)(t-a) + h(t-a)\big), & 0 \le a < t, \end{cases} \tag{4.31}$$

where $a < a_m$.

Remark 4.13. In the case $a_m < \infty$, the kernel k has to be cut off at $t - a = a_m$. Therefore, it will generally not be strongly continuous anymore, hence Lemma 4.21 cannot be applied. Assumption (4.5) guarantees that this technical difficulty can be excluded. □

Subsequently, we introduce for $t \in [0,T]$ the linear operator

$$\begin{aligned} V_\gamma(t) : C([0,T], E_0) \quad &\to \quad \mathbb{E}_0 \\ h \quad &\mapsto \quad w_0^{\gamma,0,h}(t,\cdot), \end{aligned}$$

which is well defined by Proposition 4.4. Furthermore, by (4.25) and (4.31) we have

$$\|V_\gamma(t)h\|_{\mathbb{E}_0} \le M \int_0^t e^{-(\bar{\omega}+\gamma)(t-a)} \|(r_\gamma * h)(a) + h(a)\|_{E_0}\, \mathrm{d}a,$$

using Lemma 4.21, we can estimate the first term on the right by

$$
\begin{aligned}
&M \int_0^t e^{-(\bar{\omega}+\gamma)(t-a)} \int_0^a \|r_\gamma(a,s)h(s)\|_{E_0} \, \mathrm{d}s\mathrm{d}a \\
&\leq M \int_0^t e^{-(\bar{\omega}+\gamma)(t-a)} \int_0^a M\|b_L\|_\infty e^{(M\|b_L\|_\infty - \overline{\omega}-\gamma)(a-s)} \|h(s)\|_{E_0} \, \mathrm{d}s\mathrm{d}a \\
&= M^2 \|b_L\|_\infty \int_0^t e^{-(\bar{\omega}+\gamma)t} e^{-(M\|b_L\|_\infty - \overline{\omega}-\gamma)s} \|h(s)\|_{E_0} \int_s^t e^{M\|b_L\|_\infty a} \, \mathrm{d}a\mathrm{d}s \\
&\leq M \int_0^t e^{-(\bar{\omega}+\gamma)t} e^{-(M\|b_L\|_\infty - \overline{\omega}-\gamma)s} e^{M\|b_L\|_\infty t} \|h(s)\|_{E_0} \, \mathrm{d}s,
\end{aligned}
$$

and therefore we obtain

$$
\|V_\gamma(t)h\|_{\mathbb{E}_0} \leq 2M \int_0^t e^{(M\|b_L\|_\infty - \overline{\omega}-\gamma)(t-s)} \|h(s)\|_{E_0} \, \mathrm{d}s. \tag{4.32}
$$

Combining (4.27) – (4.29) and (4.31) leads to

$$
w_{w_0}^{\gamma,f,h}(t,\cdot) = S_\gamma(t)w_0 + \int_0^t S_\gamma(t-s)f(s) \, \mathrm{d}s + V_\gamma(t)h. \tag{4.33}
$$

Lemma 4.14. *Assume the operator A satisfies* (4.2) – (4.3)*, and conditions* (4.21) – (4.23)*,* (4.25) – (4.26) *hold. Then for all $t \in [0,T]$:*

1. $B_{w_0}^{\gamma,f,h}(t) = e^{-\gamma t} B_{w_0}^{0,e^{\gamma \cdot} f, e^{\gamma \cdot} h}(t)$,
2. $\int_0^t B_{f(s)}^{\gamma,0,0}(t-s) \, ds = B_0^{\gamma,f,0}(t)$.

Proof. Ad 1: Observe that (4.8) implies

$$
\Pi_{\hat{A}_\gamma}(a,\sigma) = e^{-\gamma(a-\sigma)} \Pi_{\hat{A}}(a,\sigma), \quad 0 \leq \sigma \leq a,
$$

combining this with (4.11) yields

$$
\begin{aligned}
B_{w_0}^{\gamma,f,h}(t) \;&=\; e^{-\gamma t} \int_0^t b_L(t-a) \Pi_{\hat{A}}(t-a,0) e^{\gamma a} B_{w_0}^{\gamma,f,h}(a) \, \mathrm{d}a \\
&+\; e^{-\gamma t} \int_0^t b_L(a) \int_{t-a}^t \Pi_{\hat{A}}(a, s+a-t) e^{\gamma s} f(s, s+a-t) \, \mathrm{d}s\mathrm{d}a \\
&+\; e^{-\gamma t} \int_t^{a_m} b_L(a) \Pi_{\hat{A}}(a, a-t) w_0(a-t) \, \mathrm{d}a \\
&+\; e^{-\gamma t} \int_t^{a_m} b_L(a) \int_0^t \Pi_{\hat{A}}(a, s+a-t) e^{\gamma s} f(s, s+a-t) \, \mathrm{d}s\mathrm{d}a \\
&+\; h(t).
\end{aligned}
$$

The claim follows from the unique solvability of integral equation (4.11).

Ad 2: The integral is well defined, since the integrand $s \mapsto B_{f(s)}^{\gamma,0,0}(t-s)$ is continuous. To see this, consider two convergent sequences

$$
s_n \to s \quad \text{in } [0,T], \qquad w_n \to w_0 \quad \text{in } \mathbb{E}_0,
$$

then one has

$$\|B^{\gamma,0,0}_{w_0}(s) - B^{\gamma,0,0}_{w_n}(s_n)\|_{E_0} \le \|B^{\gamma,0,0}_{w_0}(s) - B^{\gamma,0,0}_{w_0}(s_n)\|_{E_0} + \|B^{\gamma,0,0}_{w_0} - B^{\gamma,0,0}_{w_n}\|_\infty,$$

and by Lemma 4.5, the first term on the right tends to zero. For the second term, we observe $B^{\gamma,0,0}_{w_0} - B^{\gamma,0,0}_{w_n} = B^{\gamma,0,0}_{w_0-w_n}$ by linearity and uniqueness, hence the estimate of Lemma 4.5 implies that this term converges to zero as well. This proves the claimed continuity.
By definition, $B^{\gamma,f,0}_0$ satisfies

$$\begin{aligned} B^{\gamma,f,0}_0(t) &= \int_0^t b_L(a)\Pi_{\hat{A}_\gamma}(a,0)B^{\gamma,f,0}_0(t-a)\,\mathrm{d}a \\ &+ \int_0^t b_L(a)\int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_t^{a_m} b_L(a)\int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a. \end{aligned} \tag{4.34}$$

On the other hand,

$$\begin{aligned} B^{\gamma,0,0}_{f(s)}(t-s) &= \int_0^{t-s} b_L(a)\Pi_{\hat{A}_\gamma}(a,0)B^{\gamma,0,0}_{f(s)}(t-s-a)\,\mathrm{d}a \\ &+ \int_{t-s}^{a_m} b_L(a)\Pi_{\hat{A}_\gamma}(a,a-(t-s))f(s,a-(t-s))\,\mathrm{d}a, \end{aligned}$$

and consequently

$$\begin{aligned} \int_0^t B^{\gamma,0,0}_{f(s)}(t-s)\,\mathrm{d}s &= \int_0^t\int_0^{t-s} b_L(a)\Pi_{\hat{A}_\gamma}(a,0)B^{\gamma,0,0}_{f(s)}(t-s-a)\,\mathrm{d}a\mathrm{d}s \\ &+ \int_0^t\int_{t-s}^{a_m} b_L(a)\Pi_{\hat{A}_\gamma}(a,a-(t-s))f(s,a-(t-s))\,\mathrm{d}a\mathrm{d}s \\ &= \int_0^t b_L(a)\Pi_{\hat{A}_\gamma}(a,0)\int_0^{t-a} B^{\gamma,0,0}_{f(s)}(t-a-s)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_0^t b_L(a)\int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_t^{a_m} b_L(a)\int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\mathrm{d}a. \end{aligned}$$

Hence, $t \mapsto \int_0^t B^{\gamma,0,0}_{f(s)}(t-s)\,\mathrm{d}s$ satisfies equation (4.34), and by uniqueness the claim follows. □

Proposition 4.15. *Assume the operator A satisfies* (4.2) − (4.3), *and conditions* (4.21) − (4.23), (4.25) − (4.26) *hold. Then for all $t \in [0,T]$:*

$$w^{\gamma,f,h}_{w_0}(t) = e^{-\gamma t} w^{0,e^{\gamma\cdot}f,e^{\gamma\cdot}h}_{w_0}(t)$$

Proof. By definition of $w_{w_0}^{\gamma,f,h}$, and Lemma 4.14, we have

$$
\begin{aligned}
w_{w_0}^{\gamma,f,h}(t) &= \begin{cases} \Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t) \\ \Pi_{\hat{A}_\gamma}(a,0)B_{w_0}^{\gamma,f,h}(t-a) \end{cases} \\
&+ \begin{cases} \int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0 \le t \le a \\ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s, & 0 \le a < t \end{cases} \\
&= \begin{cases} e^{-\gamma t}\Pi_{\hat{A}}(a,a-t)w_0(a-t) \\ e^{-\gamma a}\Pi_{\hat{A}}(a,0)e^{-\gamma(t-a)}B_{w_0}^{0,e^{\gamma\cdot}f,e^{\gamma\cdot}h}(t-a) \end{cases} \\
&+ \begin{cases} e^{-\gamma t}\int_0^t \Pi_{\hat{A}}(a,s+a-t)e^{\gamma s}f(s,s+a-t)\,\mathrm{d}s, & 0 \le t \le a \\ e^{-\gamma t}\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)e^{\gamma s}f(s,s+a-t)\,\mathrm{d}s, & 0 \le a < t, \end{cases}
\end{aligned}
$$

where $a < a_m$, and the claim follows. □

Remark 4.16. Let us recall the definition of the semigroup $S_\gamma(t), t \ge 0$, in (4.28). As an immediate consequence of Proposition 4.15 we have

$$S_\gamma(t)w_0 = e^{-\gamma t}S_0(t)w_0, \quad \forall t \ge 0,\ w_0 \in \mathbb{E}_0.$$

□

Proposition 4.17 (γ-shift)**.** *Assume the operator A satisfies* (4.2)−(4.3)*, and conditions* (4.21)−(4.23)*,* (4.25) − (4.26) *hold. Let $w \in C([0,T],\mathbb{E}_0)$, then the following equivalence holds:*

$$
\begin{aligned}
w(t,a) &= \begin{cases} \Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t) \\ \Pi_{\hat{A}_\gamma}(a,0)B(t-a) \end{cases} \\
&+ \begin{cases} \int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s)(s+a-t)\,ds, & \text{for a.a. } a \in (t,a_m) \\ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f(s)(s+a-t)\,ds, & \text{for a.a. } a \in (0,t), \end{cases}
\end{aligned}
$$

where $a < a_m$, with $B(t) = \int_0^{a_m} b_L(a)w(t,a)\,da + h(t)$, $t \in [0,T]$, if and only if

$$
\begin{aligned}
&w(t,a) \\
&= \begin{cases} \Pi_{\hat{A}}(a,a-t)w_0(a-t) \\ \Pi_{\hat{A}}(a,0)B(t-a) \end{cases} \\
&+ \begin{cases} \int_0^t \Pi_{\hat{A}}(a,s+a-t)\left(-\gamma\,w(s)+f(s)\right)(s+a-t)\,ds, & \text{f.a.a. } a \in (t,a_m) \\ \int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)\left(-\gamma\,w(s)+f(s)\right)(s+a-t)\,ds, & \text{f.a.a. } a \in (0,t), \end{cases}
\end{aligned}
$$

where $a < a_m$, with $B(t) = \int_0^{a_m} b_L(a)w(t,a)\,da + h(t)$, $t \in [0,T]$.

Remark 4.18. Recalling the notation of Definition 4.3, Proposition 4.17 says that for $w \in C([0,T],\mathbb{E}_0)$, we have

$$w = w_{w_0}^{\gamma,f,h} \iff w = w_{w_0}^{0,-\gamma w+f,h}.$$

□

Proof of Proposition 4.17. Let us consider the equation

$$w = w_{w_0}^{0,-\gamma w+f,h}, \tag{4.35}$$

and observe that this is a coupled integral equation for $(w, B) \in C([0,T], \mathbb{E}_0) \times C([0,T], E_0)$, cf. Definition 4.3. Suppose there exist two integral solutions $(w_1, B_1), (w_2, B_2) \in C([0,T], \mathbb{E}_0) \times C([0,T], E_0)$, then we can consider the difference

$$(\tilde{w}, \tilde{B}) := (w_1 - w_2, B_1 - B_2) \in C([0,T], \mathbb{E}_0) \times C([0,T], E_0),$$

and the uniqueness in Lemma 4.5 implies that this difference satisfies

$$\tilde{w} = w_0^{0,-\gamma\tilde{w},0}.$$

Proposition 4.12 then yields

$$\tilde{w}(t) = -\gamma \int_0^t S_0(t-s)\tilde{w}(s)\,\mathrm{d}s, \qquad t \in [0,T],$$

which in turn leads to

$$\|\tilde{w}(t)\|_{\mathbb{E}_0} \leq |\gamma|\, C\, e^{\omega T} \int_0^t \|\tilde{w}(s)\|_{\mathbb{E}_0}\,\mathrm{d}s, \qquad t \in [0,T],$$

where $C, \omega \in \mathbb{R}$ are chosen such that $\|S_0(t)\|_{\mathcal{L}(\mathbb{E}_0)} \leq C\, e^{\omega t}$, $t \geq 0$. Gronwall's inequality then implies $\tilde{w} = 0$, and consequently $\tilde{B} = 0$. Hence, there exists at most one integral solution $(w, B) \in C([0,T], \mathbb{E}_0) \times C([0,T], E_0)$ to equation (4.35).

On the other hand, by Proposition 4.4 and Lemma 4.5, $(w_{w_0}^{\gamma,f,h}, B_{w_0}^{\gamma,f,h}) \in C([0,T], \mathbb{E}_0) \times C([0,T], E_0)$ is well defined. In the following we show that $(w_{w_0}^{\gamma,f,h}, B_{w_0}^{\gamma,f,h})$ satisfies (4.35), by uniqueness the claim then follows.

In order to simplify notation, we write $w^\gamma := w_{w_0}^{\gamma,f,h}$. In the case $t \leq a$, use formula (4.10) to conclude:

$$\begin{aligned}
& w_{w_0}^{0,-\gamma w^\gamma + f,h}(t)(a) \\
&= \Pi_{\hat{A}}(a, a-t) w_0(a-t) \\
&+ \int_0^t \Pi_{\hat{A}}(a, s+a-t)\big(-\gamma\, w^\gamma(s) + f(s)\big)(s+a-t)\,\mathrm{d}s \\
&= \Pi_{\hat{A}}(a, a-t) w_0(a-t) \\
&+ \int_0^t (-\gamma)\Pi_{\hat{A}}(a, s+a-t)\Pi_{\hat{A}_\gamma}(s+a-t, a-t) w_0(a-t)\,\mathrm{d}s \\
&+ \int_0^t (-\gamma)\Pi_{\hat{A}}(a, s+a-t)\left(\int_0^s \Pi_{\hat{A}_\gamma}(s+a-t, r+a-t) f(r, r+a-t)\,\mathrm{d}r\right)\mathrm{d}s \\
&+ \int_0^t \Pi_{\hat{A}}(a, s+a-t) f(s, s+a-t)\,\mathrm{d}s \\
&= \Pi_{\hat{A}}(a, a-t) w_0(a-t) \\
&+ \int_0^t (-\gamma) e^{-\gamma s}\Pi_{\hat{A}}(a, a-t) w_0(a-t)\,\mathrm{d}s \\
&+ \int_0^t \int_0^s (-\gamma) e^{-\gamma(s-r)}\Pi_{\hat{A}}(a, r+a-t) f(r, r+a-t)\,\mathrm{d}r\mathrm{d}s \\
&+ \int_0^t \Pi_{\hat{A}}(a, s+a-t) f(s, s+a-t)\,\mathrm{d}s.
\end{aligned}$$

Observe that the first two terms after the last equality can be simplified to

$$e^{-\gamma t}\Pi_{\hat{A}}(a,a-t)w_0(a-t),$$

whereas for the third term, we can change the order of integration, yielding

$$\begin{aligned}&\int_0^t e^{\gamma r}\Pi_{\hat{A}}(a,r+a-t)f(r,r+a-t)\left(\int_r^t (-\gamma)e^{-\gamma s}\,\mathrm{d}s\right)\mathrm{d}r\\&=\int_0^t \left(e^{-\gamma(t-r)}-1\right)\Pi_{\hat{A}}(a,r+a-t)f(r,r+a-t)\,\mathrm{d}r.\end{aligned}$$

Putting everything together, we arrive at

$$\begin{aligned}&w_{w_0}^{0,-\gamma w^\gamma+f,h}(t)(a)\\&=\Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t)+\int_0^t \Pi_{\hat{A}_\gamma}(a,r+a-t)f(r,r+a-t)\,\mathrm{d}r\\&=w^\gamma(t)(a),\quad t\le a,\end{aligned}\tag{4.36}$$

where we used formula (4.10) in the last step.
Let us turn to the case $t>a$, where we have:

$$\begin{aligned}&w_{w_0}^{0,-\gamma\, w^\gamma+f,h}(t)(a)\\&=\Pi_{\hat{A}}(a,0)B_{w_0}^{0,-\gamma\, w^\gamma+f,h}(t-a)\\&+\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)\left(-\gamma w^\gamma(s,s+a-t)+f(s,s+a-t)\right)\,\mathrm{d}s.\end{aligned}$$

Using formula (4.10), we obtain for the second term

$$\begin{aligned}&\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)\left(-\gamma w^\gamma(s,s+a-t)+f(s,s+a-t)\right)\,\mathrm{d}s\\&=\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)(-\gamma)\Pi_{\hat{A}_\gamma}(s+a-t,0)B_{w_0}^{\gamma,f,h}(t-a)\,\mathrm{d}s\\&+\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)\left((-\gamma)\int_{t-a}^s \Pi_{\hat{A}_\gamma}(s+a-t,r+a-t)f(r,r+a-t)\,\mathrm{d}r\right)\mathrm{d}s\\&+\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\\&=\int_{t-a}^t (-\gamma)e^{-\gamma(s+a-t)}\Pi_{\hat{A}}(a,0)B_{w_0}^{\gamma,f,h}(t-a)\,\mathrm{d}s\\&+\int_{t-a}^t\int_{t-a}^s (-\gamma)e^{-\gamma(s-r)}\Pi_{\hat{A}}(a,r+a-t)f(r,r+a-t)\,\mathrm{d}r\,\mathrm{d}s\\&+\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s\\&=\int_{t-a}^t (-\gamma)e^{-\gamma s}\,\mathrm{d}s\,e^{-\gamma(a-t)}\Pi_{\hat{A}}(a,0)B_{w_0}^{\gamma,f,h}(t-a)\\&+\int_{t-a}^t\int_r^t (-\gamma)e^{-\gamma s}\,\mathrm{d}s\,e^{\gamma r}\Pi_{\hat{A}}(a,r+a-t)f(r,r+a-t)\,\mathrm{d}r\\&+\int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)f(s,s+a-t)\,\mathrm{d}s,\end{aligned}$$

and after integration with respect to s, we arrive at

$$\begin{aligned}
&\int_{t-a}^{t} \Pi_{\hat{A}}(a, s+a-t)\left(-\gamma w^{\gamma}(s, s+a-t) + f(s, s+a-t)\right) \mathrm{d}s \\
&= \Pi_{\hat{A}_\gamma}(a,0) B_{w_0}^{\gamma,f,h}(t-a) - \Pi_{\hat{A}}(a,0) B_{w_0}^{\gamma,f,h}(t-a) \\
&+ \int_{t-a}^{t} \Pi_{\hat{A}_\gamma}(a, r+a-t) f(r, r+a-t)\, \mathrm{d}r, \quad t > a
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\Pi_{\hat{A}}(a,0) B_{w_0}^{\gamma,f,h}(t-a) + \int_{t-a}^{t} \Pi_{\hat{A}}(a, s+a-t)\big(-\gamma w^{\gamma}(s) + f(s)\big)(s+a-t)\, \mathrm{d}s \\
&= \Pi_{\hat{A}_\gamma}(a,0) B_{w_0}^{\gamma,f,h}(t-a) + \int_{t-a}^{t} \Pi_{\hat{A}_\gamma}(a, r+a-t) f(r, r+a-t)\, \mathrm{d}r \\
&= w^{\gamma}(t)(a), \quad t > a. \qquad (4.37)
\end{aligned}$$

Identities (4.36) and (4.37) lead to

$$\begin{aligned}
&B_{w_0}^{\gamma,f,h}(t) \\
&= \int_0^t b_L(a) w^{\gamma}(t,a)\, \mathrm{d}a + \int_t^{a_m} b_L(a) w^{\gamma}(t,a)\, \mathrm{d}a + h(t) \\
&= \int_0^t b_L(a) \Pi_{\hat{A}}(a,0) B_{w_0}^{\gamma,f,h}(t-a)\, \mathrm{d}a \\
&+ \int_0^t b_L(a) \left(\int_{t-a}^{t} \Pi_{\hat{A}}(a, s+a-t)\big(-\gamma\, w^{\gamma}(s) + f(s)\big)(s+a-t)\, \mathrm{d}s \right) \mathrm{d}a \\
&+ \int_t^{a_m} b_L(a) \Pi_{\hat{A}}(a, a-t) w_0(a-t) \mathrm{d}a \\
&+ \int_t^{a_m} b_L(a) \left(\int_0^t \Pi_{\hat{A}}(a, s+a-t)\big(-\gamma\, w^{\gamma}(s) + f(s)\big)(s+a-t)\, \mathrm{d}s \right) \mathrm{d}a \\
&+ h(t), \quad t \in [0,T],
\end{aligned}$$

that is, $B_{w_0}^{\gamma,f,h}$ satisfies the integral equation corresponding to $B_{w_0}^{0,-\gamma w^{\gamma}+f,h}$. Therefore, Lemma 4.5 implies

$$B_{w_0}^{\gamma,f,h} = B_{w_0}^{0,-\gamma w^{\gamma}+f,h},$$

using this equality in (4.37), we conclude

$$w_{w_0}^{0,-\gamma w^{\gamma}+f,h}(t,a) = w^{\gamma}(t,a), \quad \text{for } 0 \le a < t.$$

This, together with (4.36), proves the claim. □

Proposition 4.19 (μ-shift)**.** *Assume the operator A satisfies* (4.2)−(4.3)*, and conditions* (4.21)−(4.23)*,* (4.25) − (4.26) *hold. Let $(w, B) \in C([0,T], \mathbb{E}_0) \times C([0,T], E_0)$, then*

$$\begin{aligned}
w(t,a) \quad &= \quad \begin{cases} \Pi_{\hat{A}}(a, a-t) w_0(a-t) \\ \Pi_{\hat{A}}(a,0) B(t-a) \end{cases} \\
&+ \quad \begin{cases} \int_0^t \Pi_{\hat{A}}(a, s+a-t) f(s)(s+a-t)\, ds, & \text{for a.a. } a \in (t, a_m) \\ \int_{t-a}^t \Pi_{\hat{A}}(a, s+a-t) f(s)(s+a-t)\, ds, & \text{for a.a. } a \in (0,t), \end{cases}
\end{aligned}$$

where $a < a_m$, *with* $B(t) = \int_0^{a_m} b_L(a)w(t,a)\,da + h(t)$, $t \in [0,T]$, *if and only if*

$$
\begin{aligned}
&w(t,a)\\
&= \begin{cases} \Pi_A(a,a-t)w_0(a-t) \\ \Pi_A(a,0)B(t-a) \end{cases}\\
&+ \begin{cases} \int_0^t \Pi_A(a,s+a-t)\,(-\mu_L(\cdot)w(s)+f(s))\,(s+a-t)\,ds, & \textit{f.a.a. } a \in (t,a_m) \\ \int_{t-a}^t \Pi_A(a,s+a-t)\,(-\mu_L(\cdot)w(s)+f(s))\,(s+a-t)\,ds, & \textit{f.a.a. } a \in (0,t), \end{cases}
\end{aligned}
$$

where $a < a_m$, *with* $B(t) = \int_0^{a_m} b_L(a)w(t,a)\,da + h(t)$, $t \in [0,T]$.

Remark 4.20. Observe that due to assumption (4.22), the mortality rate induces an operator $\mu_L(\cdot) \in \mathcal{L}(\mathbb{E}_0)$ by

$$
\big(\mu_L(\cdot)\phi\big)(a) := \mu_L(a)\phi(a), \quad \text{for } a \in J,\ \phi \in \mathbb{E}_0.
$$

In particular, given $w \in C([0,T],\mathbb{E}_0)$, the function

$$
\mu_L(\cdot)w : [0,T] \to \mathbb{E}_0
$$

is well defined and continuous. □

Proof of Proposition 4.19. The argument is analogous to the proof of Proposition 4.17, but instead of the relation

$$
\Pi_{\hat{A}_\gamma}(a,\sigma)v = e^{-\gamma(a-\sigma)}\Pi_{\hat{A}}(a,\sigma)v, \quad v \in E_0,\ 0 \le \sigma \le a,
$$

we require identity (4.24). To be more precise, let us consider the equation

$$
\begin{aligned}
&w(t,a)\\
&= \begin{cases} \Pi_A(a,a-t)w_0(a-t) \\ \Pi_A(a,0)B(t-a) \end{cases}\\
&+ \begin{cases} \int_0^t \Pi_A(a,s+a-t)\,(-\mu_L(\cdot)w(s)+f(s))\,(s+a-t)\,\mathrm{d}s, & \text{f.a.a. } a \in (t,a_m) \\ \int_{t-a}^t \Pi_A(a,s+a-t)\,(-\mu_L(\cdot)w(s)+f(s))\,(s+a-t)\,\mathrm{d}s, & \text{f.a.a. } a \in (0,t), \end{cases}
\end{aligned}
$$

where $a < a_m$, with $B(t) = \int_0^{a_m} b_L(a)w(t,a)\,\mathrm{d}a + h(t)$, $t \in [0,T]$.

As in the proof of Proposition 4.17 we conclude that there exists at most one solution $(w,B) \in C([0,T],\mathbb{E}_0) \times C([0,T],\mathbb{E}_0)$. On the other hand, consider the equation

$$
\begin{aligned}
\hat{w}(t,a) \quad &= \quad \begin{cases} \Pi_{\hat{A}}(a,a-t)w_0(a-t) \\ \Pi_{\hat{A}}(a,0)\hat{B}(t-a) \end{cases}\\
&+ \quad \begin{cases} \int_0^t \Pi_{\hat{A}}(a,s+a-t)f(s)(s+a-t)\,\mathrm{d}s, & \text{f.a.a. } a \in (t,a_m) \\ \int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)f(s)(s+a-t)\,\mathrm{d}s, & \text{f.a.a. } a \in (0,t), \end{cases}
\end{aligned}
$$

where $a < a_m$, with $\hat{B}(t) = \int_0^{a_m} b_L(a)\hat{w}(t,a)\,\mathrm{d}a + h(t)$, $t \in [0,T]$.

By Proposition 4.4 and Lemma 4.5, $(\hat{w},\hat{B}) \in C([0,T],\mathbb{E}_0) \times C([0,T],E_0)$ is well defined. In the following we verify that $(\hat{w},\hat{B})$ also satisfies the first equation, by uniqueness the claim then follows.

Let $0 \leq t \leq a$, then a computation as in the proof of Proposition 4.17, together with formula (4.24), leads to

$$\begin{aligned}
&\Pi_A(a, a-t) w_0(a-t) \\
&+ \int_0^t \Pi_A(a, s+a-t)\big(-\mu_L(.)\hat{w}(s) + f(s)\big)(s+a-t)\,\mathrm{d}s \\
&= \Pi_{\hat{A}}(a, a-t) w_0(a-t) + \int_0^t \Pi_{\hat{A}}(a, r+a-t) f(r, r+a-t)\,\mathrm{d}r \\
&= \hat{w}(t,a), \quad t \leq a, \qquad (4.38)
\end{aligned}$$

cf. (4.36). In the case $t > a$, another lengthy computation and application of formula (4.24) yields

$$\begin{aligned}
&\int_{t-a}^t \Pi_A(a, s+a-t)\big(-\mu_L(.)\hat{w}(s) + f(s)\big)(s+a-t)\,\mathrm{d}s \\
&= -\int_0^a \Pi_A(a,s)\mu_L(s)\Pi_{\hat{A}}(s,0)\hat{B}(t-a)\,\mathrm{d}s \\
&+ \int_{t-a}^t \Pi_{\hat{A}}(a, r+a-t) f(r, r+a-t)\,\mathrm{d}r, \quad t > a.
\end{aligned}$$

Using formula (4.24), this is equivalent to

$$\begin{aligned}
&\Pi_A(a,0)\hat{B}(t-a) \\
&+ \int_{t-a}^t \Pi_A(a, s+a-t)\big(-\mu_L(.)\hat{w}(s) + f(s)\big)(s+a-t)\,\mathrm{d}s \\
&= \Pi_{\hat{A}}(a,0)\hat{B}(t-a) \\
&+ \int_{t-a}^t \Pi_{\hat{A}}(a, r+a-t) f(r, r+a-t)\,\mathrm{d}r \\
&= \hat{w}(t,a), \quad t > a, \qquad (4.39)
\end{aligned}$$

cf. (4.37). Identities (4.38) and (4.39) lead to

$$\begin{aligned}
&\hat{B}(t) \\
&= \int_0^t b_L(a)\hat{w}(t,a)\,\mathrm{d}a + \int_t^{a_m} b_L(a)\hat{w}(t,a)\,\mathrm{d}a + h(t) \\
&= \int_0^t b_L(a)\Pi_A(a,0)\hat{B}(t-a)\,\mathrm{d}a \\
&+ \int_0^t b_L(a)\left(\int_{t-a}^t \Pi_A(a, s+a-t)\big(-\mu_L(.)\,\hat{w}(s) + f(s)\big)(s+a-t)\,\mathrm{d}s \right)\mathrm{d}a \\
&+ \int_t^{a_m} b_L(a)\Pi_A(a, a-t) w_0(a-t)\mathrm{d}a \\
&+ \int_t^{a_m} b_L(a)\left(\int_0^t \Pi_A(a, s+a-t)\big(-\mu_L(.)\,\hat{w}(s) + f(s)\big)(s+a-t)\,\mathrm{d}s \right)\mathrm{d}a \\
&+ h(t), \quad t \in [0,T],
\end{aligned}$$

and we see that $(\hat{w}, \hat{B})$ solves the first equation, as claimed. □

4.5 The resolvent kernel

In this section we establish an existence- and uniqueness result for a Volterra integral equation. More precisely, we set $J_\Delta := \{(t,a) \in J \times J : a \le t\}$, introduce the integral kernel

$$\begin{aligned} k_\gamma : J_\Delta &\to \mathcal{L}(E_0) \\ (t,a) &\mapsto b_L(t,t-a)\Pi_{\hat{A}_\gamma}(t-a,0), \end{aligned} \tag{4.40}$$

and consider the equation

$$\begin{aligned} B(t) &= \int_0^t k_\gamma(t,a)\, B(a)\, \mathrm{d}a + h(t) \\ &=: (k_\gamma * B)(t) + h(t), \qquad t \in [0,T], \end{aligned} \tag{4.41}$$

with $h \in C([0,T], E_0)$ given.
In case $k_\gamma \in C(J_\Delta, \mathcal{L}(E_0))$, it is well known that there exists a unique solution $B \in C([0,T], E_0)$ to (4.41). However, we only have $k_\gamma \in C(J_\Delta, \mathcal{L}_s(E_0))$, since the evolution operator $\Pi_{\hat{A}_\gamma}$ is only strongly continuous. In the following we verify that existence and uniqueness still hold in this setting.

In order to show existence of solutions, we first construct the resolvent kernel:

Lemma 4.21 (Resolvent kernel)**.** *Assume the operator A satisfies* (4.2)*, and conditions* (4.5) − (4.6) *hold. Let the integral kernel k_γ be defined as in* (4.40)*. Then there exists a resolvent kernel $r_\gamma \in C(J_\Delta, \mathcal{L}_s(E_0))$, uniquely determined by the equation*

$$r_\gamma(t,a) = \int_a^t k_\gamma(t,s) r_\gamma(s,a)\, ds + k_\gamma(t,a), \quad (t,a) \in J_\Delta.$$

This resolvent kernel satisfies

$$\|r_\gamma(t,a)\|_{\mathcal{L}(E_0)} \le M \|b_L\|_\infty e^{(M\|b_L\|_\infty - \overline{\omega} - \gamma)(t-a)}, \quad 0 \le a \le t,$$

and the γ-dependence is expressed by

$$r_\gamma(t,a) = e^{-\gamma(t-a)} r_0(t,a).$$

Proof. For $n = 1, 2, \dots$ we recursively define

$$k_{\gamma,1}(t,a) := k_\gamma(t,a), \quad k_{\gamma,n+1}(t,a) := \int_a^t k_\gamma(t,s) k_{\gamma,n}(s,a)\, ds, \quad (t,a) \in J_\Delta.$$

Observe that k_γ is strongly continuous (since b_L is continuous), i.e. $k_\gamma \in C(J_\Delta, \mathcal{L}_s(E_0))$. By an induction and dominated convergence argument we conclude that all products $k_{\gamma,n}$ are strongly continuous as well.

It is well known from the theory of Volterra integral equations that the resolvent kernel is given by the formula

$$r_\gamma = \sum_{n=1}^\infty k_{\gamma,n},$$

hence we have to verify the convergence of the series. We estimate

$$\|k_{\gamma,1}(t,a)\|_{\mathcal{L}(E_0)} \le \|b_L\|_\infty\, M e^{-(\bar{\omega}+\gamma)(t-a)},$$

and inductively

$$\|k_{\gamma,n}(t,a)\|_{\mathcal{L}(E_0)} \le M^n \|b_L\|_\infty^n \frac{(t-a)^{n-1}}{(n-1)!}\, e^{-(\overline{\omega}+\gamma)(t-a)},$$

which yields the claim. The estimate for r_γ is now an easy consequence. Observe that the estimate, together with the Weierstrass M-test, implies the strong continuity of r_γ. As regards uniqueness of the resolvent kernel, fix $a \in J$, $v \in E_0$, and assume $r \in C(J_\Delta, \mathcal{L}_s(E_0))$ satisfies

$$r(t,a)v = \int_a^t k_\gamma(t,s) r(s,a)v\, ds, \qquad t \in [0,T].$$

Then $r(\cdot,a)v \in C([a,T],E_0)$ and

$$\begin{aligned}
\|r(t,a)v\|_{E_0} &\le \int_a^t \|k_\gamma(t,s)\|_{\mathcal{L}(E_0)} \|r(s,a)v\|_{E_0}\, ds \\
&\le \int_a^t \|b_L\|_\infty\, M e^{-(\bar{\omega}+\gamma)(t-s)} \|r(s,a)v\|_{E_0}\, ds \qquad t \in [a,T],
\end{aligned}$$

consequently $r(\cdot,a)v = 0$ in $C([a,T],E_0)$ by Gronwall's inequality. Since $a \in J$, $v \in E_0$ can be chosen arbitrarily, the claim follows.

Regarding the γ-dependence, observe that $k_\gamma(t,a) = e^{-\gamma(t-a)} k_0(t,a)$, hence

$$\begin{aligned}
k_{\gamma,2}(t,a) &= \int_a^t k_\gamma(t,s) k_{\gamma,1}(s,a)\, ds = \int_a^t e^{-\gamma(t-s)} k_0(t,s) e^{-\gamma(s-a)} k_0(s,a)\, ds \\
&= e^{-\gamma(t-a)} k_{0,2}(t,a),
\end{aligned}$$

and inductively

$$k_{\gamma,n}(t,a) = e^{-\gamma(t-a)} k_{0,n}(t,a).$$

The claim now follows from the power series representation of the resolvent kernels. □

Proposition 4.22. *Assume the operator A satisfies* (4.2)*, and conditions* (4.5) − (4.6) *hold. Let the integral kernel k_γ be defined as in* (4.40)*. Then for every $h \in C([0,T],E_0)$ there exists a unique solution $B \in C([0,T],E_0)$ to equation* (4.41).

Proof. Let $r_\gamma \in C(J_\Delta, \mathcal{L}_s(E_0))$ denote the resolvent kernel from Lemma 4.21, and set

$$\begin{aligned}
B(t) &:= \int_0^t r_\gamma(t,a) h(a)\, \mathrm{d}a + h(t) \\
&= (r_\gamma * h)(t) + h(t), \qquad t \in [0,T].
\end{aligned}$$

Then $B \in C([0,T],E_0)$ by the dominated convergence theorem, furthermore

$$\begin{aligned}
(k_\gamma * B)(t) &= \big(k_\gamma * (r_\gamma * h + h)\big)(t) \\
&= \big((k_\gamma * r_\gamma + k_\gamma) * h\big)(t) \\
&= (r_\gamma * h)(t) \\
&= B(t) - h(t),
\end{aligned}$$

for $t \in [0,T]$, which proves the existence claim. Uniqueness is a simple consequence of Gronwall's inequality (cf. proof of Lemma 4.21). □

5 Nonlinear stability

In section 3 we analysed the simplified instance (3.1) of problem (1.4), with the nonlinearity appearing in the mortality process. More presicely, only the mortality rate μ was assumed to depend on the density u, which allowed to formulate (3.1) as a semilinear Cauchy problem. Subsequently the principle of linearised stability was applied, thereby reducing the stability analysis of equilibria to a linear problem.

In the present section we consider the nonlinear problem, where also the birth rate b is allowed to depend on the density u, i.e.

$$\begin{aligned} \partial_t u + \partial_a u + A(a)u &= -\mu(u,a)u, \qquad t > 0,\ a \in J \\ u(t,0) &= \int_0^{a_m} b(u,a)u(a)\,\mathrm{d}a, \qquad t > 0 \\ u(0,a) &= u_0(a), \qquad a \in J, \end{aligned} \tag{5.1}$$

with $J \subset \mathbb{R}_+$ as in (3.4). As a first important observation we note that this additional nonlinearity appearing in the age-boundary condition destroys the semilinear structure. In particular, semigroup theory and mild solutions are not available any longer, and it is therefore necessary to introduce a new solution concept. To this end, we will resort to the results of section 4, where the theory of *integral solutions* was developed. After declaring the assumptions for the present section, we will see that problem (5.1) is well-posed within this framework. In a subsequent step we will study the asymptotic behaviour of this solutions and convergence to equilibria. An equilibrium of problem (5.1), denoted by ϕ in the following, is determined by the equation

$$\begin{aligned} \partial_a \phi + A(a)\phi &= -\mu(\phi,a)\phi, \qquad a \in J \\ \phi(0) &= \int_0^{a_m} b(\phi,a)\phi(a)\,\mathrm{d}a. \end{aligned} \tag{5.2}$$

The question of existence of nontrivial equilibria poses a separate problem and was considered in [34], [37].

In the spatially homogeneous case, a first stability result was obtained by Gurtin & MacCamy [14] by means of Laplace-transform techniques. Another stability result was established by Prüß [25] via a semigroup approach. He showed that the corresponding problem (5.1) induces a semilinear evolution equation and the principle of linearised stability is applicable. A streamlined argument was given by Webb [38, Theorem 4.13]. In the following we will take a similar approach to obtain a stability criterion for (5.1).

5.1 Assumptions

First, we declare the technical assumptions, which are assumed to hold throughout this section. To this end, let $\phi \in \mathbb{E}_0$ denote some fixed equilibrium of (5.1) (a precise definition will be given in Definition 5.6).

Throughout this section, E_0 denotes a real Banach lattice with closed, total cone E_0^+. Note that we do not distinguish E_0 from its complexification in our notation as no confusion seems likely. We recall

$$J = \begin{cases} [0, a_m], & \text{if } a_m < \infty \\ [0, \infty), & \text{if } a_m = \infty \end{cases}$$

and define

$$\mathbb{E}_0 := L_1(J, E_0).$$

Furthermore, we denote by $\mathbb{E}_0^+$ those functions in $\mathbb{E}_0$ which take values in E_0^+ almost everywhere and remark that $\mathbb{E}_0$ becomes a Banach lattice.

Let E_1 be a densely and compactly embedded subspace of E_0. As in section 4, we assume the operator A fulfills conditions (4.2) − (4.3), furthermore we suppose the parabolic evolution operator $\Pi_A(a,\sigma)$ is positive for $0 \le \sigma \le a < a_m$, and

$$\bar{\omega} > 0 \quad \text{if } a_m = \infty. \tag{5.3}$$

We consider a birth rate

$$b \in BC(\mathbb{E}_0 \times [0,\infty), \mathcal{L}_+(E_0)), \quad \operatorname{supp}(b) \subset \mathbb{E}_0 \times J,$$

and a mortality rate

$$\begin{aligned} \mu : \mathbb{E}_0 \times J &\to \mathcal{L}_+(E_0) \\ (u,a) &\mapsto \mu(u,a). \end{aligned} \tag{5.4}$$

For the mortality rate we assume a continuity condition,

$$a \mapsto \mu(u,a)v \in C(J, E_0), \qquad \forall u \in \mathbb{E}_0, v \in E_0, \tag{5.5}$$

and existence of a constant $\bar{\mu} \in \mathbb{R}_+$, such that for arbitrary $(u,a) \in \mathbb{E}_0 \times J$ we have

$$\mu(u,a)v \le \bar{\mu}\, v, \qquad \forall v \in E_0^+. \tag{5.6}$$

Furthermore, we impose

$$(a \mapsto \mu(\phi,a)) \in C^\rho(J, \mathcal{L}(E_1, E_0)), \tag{5.7}$$

with Hölder exponent ρ from (4.2).

In analogy to (4.8), we introduce the notation

$$\begin{aligned} \hat{A}(a)v &:= A(a)v + \mu(\phi,a)v, \quad v \in E_1, \\ A_\gamma(a)v &:= A(a)v + \gamma v, \quad v \in E_1, \\ \hat{A}_\gamma(a)v &:= A(a)v + \mu(\phi,a)v + \gamma v, \quad v \in E_1, \end{aligned} \tag{5.8}$$

and conclude as in section 4.2 that the parabolic evolution operator $\Pi_{\hat{A}}(a,\sigma)$, $0 \le \sigma \le a < a_m$, is well defined and satisfies

$$\Pi_{\hat{A}}(a,\sigma)v = \Pi_A(a,\sigma)v - \int_\sigma^a \Pi_A(a,s)\mu(\phi,s)\Pi_{\hat{A}}(s,\sigma)v\, ds, \quad v \in E_0.$$

By assumption (4.3) we have constants $M \ge 1$, $\bar{\omega} \in \mathbb{R}$, such that

$$\|\Pi_A(a,\sigma)\|_{\mathcal{L}(E_0)} \le Me^{-\bar{\omega}(a-\sigma)}, \qquad 0 \le \sigma \le a < a_m,$$

and since $\mu(\phi,a) \in \mathcal{L}_+(E_0)$ for all $a \in J$, we conclude as in Lemma 3.11

$$\|\Pi_{\hat{A}}(a,\sigma)\|_{\mathcal{L}(E_0)} \le Me^{-\bar{\omega}(a-\sigma)}, \qquad 0 \le \sigma \le a < a_m. \tag{5.9}$$

Recall from section 3 that assumptions (5.4) − (5.6) give rise to a well defined map

$$\begin{aligned} f : \mathbb{E}_0 &\to \mathcal{L}_+(\mathbb{E}_0) \\ u &\mapsto \mu(u,.), \end{aligned}$$

with $\|f(u)\|_{\mathcal{L}(\mathbb{E}_0)} \leq 4\,\bar{\mu}$. We set

$$\begin{array}{rcl} F : \mathbb{E}_0 & \to & \mathbb{E}_0 \\ u & \mapsto & -f(u)u, \end{array}$$

and assume

$$\begin{array}{l} F : \mathbb{E}_0 \to \mathbb{E}_0 \text{ admits an increasing function} \\ c_F : [0,\infty) \to [0,\infty), \text{ such that} \\ \|F(\phi_1) - F(\phi_2)\|_{\mathbb{E}_0} \leq c_F(r)\|\phi_1 - \phi_2\|_{\mathbb{E}_0}, \\ \text{for all } \phi_1, \phi_2 \in \mathbb{E}_0 \text{ with } \|\phi_1\|_{\mathbb{E}_0}, \|\phi_2\|_{\mathbb{E}_0} \leq r, \end{array} \tag{5.10}$$

and

$$\begin{array}{l} b : \mathbb{E}_0 \times [0,\infty) \to \mathcal{L}_+(E_0) \text{ is bounded, continuous,} \\ \text{with } \operatorname{supp}(b) \subset \mathbb{E}_0 \times J, \text{ there is an} \\ \text{increasing function } c_b : [0,\infty) \to [0,\infty), \text{ such that} \\ \|b(\phi_1, a) - b(\phi_2, a)\|_{\mathcal{L}(E_0)} \leq c_b(r)\|\phi_1 - \phi_2\|_{\mathbb{E}_0}, \\ \text{for all } a \in [0,\infty), \phi_1, \phi_2 \in \mathbb{E}_0 \text{ with } \|\phi_1\|_{\mathbb{E}_0}, \|\phi_2\|_{\mathbb{E}_0} \leq r. \end{array} \tag{5.11}$$

Furthermore, we suppose that the nonlinear terms in (5.1) are Fréchet differentiable at the equilibrium $\phi \in \mathbb{E}_0$. More precisely,

$$F(u) = F(\phi) + F'(\phi)(u - \phi) + R_F(u - \phi), \quad u \in \mathbb{E}_0, \tag{5.12}$$

where $F'(\phi) \in \mathcal{L}(\mathbb{E}_0)$, and $R_F : \mathbb{E}_0 \to \mathbb{E}_0$ is continuous such that

$$\|R_F(u)\|_{\mathbb{E}_0} \leq d_F(r)\|u\|_{\mathbb{E}_0}, \quad \text{for } \|u\|_{\mathbb{E}_0} \leq r,$$

and $d_F : [0,\infty) \to [0,\infty)$ is an increasing continuous function with $d_F(0) = 0$ (cf. [38, Proof of Theorem 4.13]). In addition, we assume

$$F'(\phi)\psi = -\mathbb{B}_\phi \psi - \mu(\phi, .)\psi, \qquad \psi \in \mathbb{E}_0, \tag{5.13}$$

such that $\mathbb{B}_\phi \in \mathcal{L}(\mathbb{E}_0)$.

Regarding the birth rate, we assume $b : \mathbb{E}_0 \times [0,\infty) \to \mathcal{L}(E_0)$ is Fréchet differentiable with respect to the first variable at $\phi \in \mathbb{E}_0$. More precisely,

$$\begin{array}{rcl} b(u,.)u & = & b(\phi,.)\phi + \big(D_u b(\phi,.)(u-\phi)\big)\phi \\ & + & b(\phi,.)(u-\phi) + R_b(u-\phi), \qquad u \in \mathbb{E}_0, \end{array} \tag{5.14}$$

where $D_u b(\phi, a) \in \mathcal{L}(\mathbb{E}_0, \mathcal{L}(E_0))$ for all $a \in [0,\infty)$, and $R_b : \mathbb{E}_0 \to \mathbb{E}_0$ is continuous such that

$$\|R_b(u)\|_{\mathbb{E}_0} \leq d_b(r)\|u\|_{\mathbb{E}_0}, \quad \text{for } \|u\|_{\mathbb{E}_0} \leq r,$$

and $d_b : [0,\infty) \to [0,\infty)$ is an increasing continuous function with $d_b(0) = 0$. Finally, we suppose the existence of $\tilde{b}_\phi \in BC([0,\infty), \mathcal{L}(E_0))$, with $\sup(\tilde{b}_\phi) \subset J$, such that

$$\int_0^{a_m} \big(D_u b(\phi, a)w\big)\phi(a)\,\mathrm{d}a = \int_0^{a_m} \tilde{b}_\phi(a)w(a)\,\mathrm{d}a, \quad w \in \mathbb{E}_0. \tag{5.15}$$

Remark 5.1. Since the dependence $u \mapsto b(u,\cdot)$ is nonlocal in general, the same is true for the derivative $w \mapsto D_u b(\phi,\cdot)w$, cf. Expample 5.2. In particular, it cannot be expected to be of the form

$$\big(D_u b(\phi, a)w\big)\phi(a) = \tilde{b}_\phi(a)w(a), \quad a \in [0, a_m),$$

with $\tilde{b}_\phi(a) \in \mathcal{L}(E_0)$ for $a \in [0, a_m)$. However, upon passing to the integrals it is reasonable to assume the existence of $\tilde{b}_\phi \in BC([0, a_m), \mathcal{L}(E_0))$, such that (5.15) holds, cf. (5.16) in Expample 5.2. This formulation will allow us to apply the theory from section 4. □

Example 5.2 (Birth rate). Let $\beta \in BC^2(\mathbb{R}^2, \mathbb{R}_+)$, and consider

$$b : \mathbb{E}_0 \times J \to \mathbb{R}_+, \quad b(u,a) := \beta(Ku, a),$$

where $Ku := \int_0^{a_m} \kappa(a)\, u(a)\, da$, with $\kappa \in BC(J, {E_0}')$. We compute

$$D_u b(\phi, a)w = \frac{\partial \beta}{\partial x}(K\phi, a)\, Kw, \quad w \in \mathbb{E}_0,$$

and, for $a \in [0, a_m)$ fixed,

$$\begin{aligned}
&R_a(u-\phi)\\
&:= b(u,a) - b(\phi,a) - D_u b(\phi,a)(u-\phi)\\
&= \int_0^1 D_u b(\phi + s(u-\phi), a)\,\mathrm{d}s\,(u-\phi) - D_u b(\phi,a)(u-\phi)\\
&= \int_0^1 \big(D_u b(\phi + s(u-\phi), a) - D_u b(\phi,a)\big)\,\mathrm{d}s\,(u-\phi)\\
&= \int_0^1 \left(\int_0^1 \frac{d}{dr}\frac{\partial \beta}{\partial x}(K(\phi + rs(u-\phi)), a)\mathrm{d}r\right)\mathrm{d}s\, K(u-\phi)\\
&= \int_0^1 \left(\int_0^1 \frac{\partial^2 \beta}{\partial x^2}(K(\phi + rs(u-\phi)), a)\, s\, K(u-\phi)\mathrm{d}r\right)\mathrm{d}s\, K(u-\phi),
\end{aligned}$$

hence, $R_a : \mathbb{E}_0 \to \mathbb{R}$ satisfies

$$|R_a(u-\phi)| \le \|\beta\|_{BC^2}\, \|K\|^2 \|u-\phi\|_{\mathbb{E}_0}^2.$$

Furthermore, we can estimate

$$\begin{aligned}
|b(u,a) - b(\phi,a)| &= \Big|\int_0^1 D_u b(\phi + s(u-\phi), a)(u-\phi)\,\mathrm{d}s\Big|\\
&\le \|\beta\|_{BC^1}\, \|K\| \|u-\phi\|_{\mathbb{E}_0},
\end{aligned}$$

and consequently

$$\begin{aligned}
&\|R_b(u-\phi)\|_{\mathbb{E}_0}\\
&= \|b(u,.)u - b(\phi,.)\phi - \big(D_u b(\phi,.)(u-\phi)\big)\phi - b(\phi,.)(u-\phi)\|_{\mathbb{E}_0}\\
&\le \int_0^{a_m} \|\big(b(u,a) - b(\phi,a)\big)(u(a) - \phi(a))\|_{E_0}\,\mathrm{d}a\\
&+ \int_0^{a_m} \|\big(b(u,a) - b(\phi,a) - D_u b(\phi,a)(u-\phi)\big)\phi(a)\|\,\mathrm{d}a\\
&\le \sup_{a\in[0,a_m)} |b(u,a) - b(\phi,a)|\, \|u-\phi\|_{\mathbb{E}_0} + \sup_{a\in[0,a_m)} |R_a(u-\phi)|\, \|\phi\|_{\mathbb{E}_0}\\
&\le \|\beta\|_{BC^1} \|K\|\, \|u-\phi\|_{\mathbb{E}_0}^2 + \|\beta\|_{BC^2}\, \|K\|^2\, \|u-\phi\|_{\mathbb{E}_0}^2\, \|\phi\|_{\mathbb{E}_0}
\end{aligned}$$

which means that

$$\|R_b(u)\| \le d_b(r)\|u\|, \quad \text{for } \|u\| \le r,$$

with $d_b(r) = C\,r$, for some postive constant C, i.e. condition (5.14) is fulfilled . Furthermore, setting $v := \int_0^{a_m} \frac{\partial \beta}{\partial x}(K\phi, a)\,\phi(a)\,\mathrm{d}a \in E_0$, we have for $w \in \mathbb{E}_0$:

$$\begin{aligned} \int_0^{a_m} \big(D_u b(\phi, a) w\big)\phi(a)\,\mathrm{d}a &= \int_0^{a_m} \frac{\partial \beta}{\partial x}(K\phi, a)\,Kw\,\phi(a)\,\mathrm{d}a \\ &= v\,Kw \\ &= \int_0^{a_m} \tilde{b}_\phi(a) w(a)\,\mathrm{d}a, \end{aligned} \tag{5.16}$$

where $\tilde{b}_\phi(a) := \big(v\,\kappa(a)\big) \in \mathcal{L}(E_0)$. ☐

5.2 Well-posedness

After declaring the technical assumptions, we introduce a solution concept for equation (5.1), cf. Definition 4.7:

Definition 5.3. Let $I \subset \mathbb{R}_+$ be an interval, with $0 \in I$. Let $u : I \to \mathbb{E}_0$ be continuous, with $u(\cdot, 0) \in C(I, E_0)$. We say that the function u is an *integral solution* to (5.1) on I, if for all $t \in I$

$$\begin{aligned} u(t,a) &= \begin{cases} \Pi_A(a, a-t) u_0(a-t) \\ \Pi_A(a,0) B(t-a) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi_A(a, s+a-t) F(u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (t, a_m) \\ \int_{t-a}^t \Pi_A(a, s+a-t) F(u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (0, t), \end{cases} \end{aligned}$$

where $a < a_m$, and $B(t) = u(t,0)$ satisfies, for $t \in I$, the associated integral equation

$$\begin{aligned} B(t) &= \int_0^t b(u(t), a) \Pi_A(a, 0) B(t-a)\,\mathrm{d}a \\ &+ \int_0^t b(u(t), a) \int_{t-a}^t \Pi_A(a, s+a-t) F(u(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_t^{a_m} b(u(t), a) \Pi_A(a, a-t) u_0(a-t)\,\mathrm{d}a \\ &+ \int_t^{a_m} b(u(t), a) \int_0^t \Pi_A(a, s+a-t) F(u(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a. \end{aligned}$$

Theorem 5.4. *Assume the operator A satisfies* $(4.2) - (4.3)$*, and conditions* $(5.10), (5.11)$ *hold. Let $u_0 \in \mathbb{E}_0$ be given, then there exists a unique maximal integral solution $u = u(.; u_0) : [0, T_{u_0}) \to \mathbb{E}_0$ to* (5.1)*. Furthermore, if $T_{u_0} < \infty$, then* $\limsup_{t \to T_0} \|u(t)\|_{\mathbb{E}_0} = \infty$.

Proof. Let us set

$$T_{u_0} := \sup\{T \in \mathbb{R}_+ \,|\, \text{there exists an integral solution to (5.1) on } [0, T]\},$$

and observe that T_{u_0} is positive by Theorem 4.10. Furthermore, it follows from Proposition 4.11 that the integral solution

$$u = u(.; u_0) : [0, T_{u_0}) \to \mathbb{E}_0$$

is unique. Finally, let $T_{u_0} < \infty$, and suppose u remains bounded. Then we can set $r := \sup_{t < T_{u_0}} \|u(t)\|_{\mathbb{E}_0} + 1$, and an application of Theorem 4.10 shows that u can be extended beyond T_{u_0}, which contradicts the definition of T_{u_0}. ☐

Proposition 5.5 (Positivity). *Assume the operator A satisfies* (4.2) – (4.3)*, the evolution operator Π_A is positive, the mortality rate μ fulfills* (5.4) – (5.6)*, and conditions* (5.10), (5.11) *hold. Then for given initial value $u_0 \in \mathbb{E}_0^+$, the corresponding maximal integral solution $u = u(.; u_0) : [0, T_{u_0}) \to \mathbb{E}_0$ to* (5.1) *is positive.*

Proof. Recall that

$$\begin{aligned} F : \mathbb{E}_0 &\to \mathbb{E}_0 \\ w &\mapsto -f(w)w, \end{aligned}$$

with $f(w) = \mu(w, .) \in \mathcal{L}_+(\mathbb{E}_0)$, then assumption (5.6) yields

$$F_{\bar{\mu}}(w) := F(w) + \bar{\mu}\, w \geq 0, \qquad \forall w \in \mathbb{E}_0^+.$$

Since u is an integral solution to (5.1), we see from Proposition 4.17 that u satisfies

$$\begin{aligned} u(t,a) &= \begin{cases} \Pi_{A_{\bar{\mu}}}(a, a-t)u_0(a-t) \\ \Pi_{A_{\bar{\mu}}}(a, 0)B(t-a) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi_{A_{\bar{\mu}}}(a, s+a-t)F_{\bar{\mu}}(u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (t, a_m) \\ \int_{t-a}^t \Pi_{A_{\bar{\mu}}}(a, s+a-t)F_{\bar{\mu}}(u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (0, t), \end{cases} \end{aligned}$$

where $a < a_m$, and $B(t) = u(t, 0)$ satisfies, for $t \in [0, T]$, the associated integral equation

$$\begin{aligned} B(t) &= \int_0^t b(u(t), a)\Pi_{A_{\bar{\mu}}}(a, 0)B(t-a)\,\mathrm{d}a \\ &+ \int_0^t b(u(t), a) \int_{t-a}^t \Pi_{A_{\bar{\mu}}}(a, s+a-t)F_{\bar{\mu}}(u(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a \\ &+ \int_t^{a_m} b(u(t), a)\Pi_{A_{\bar{\mu}}}(a, a-t)u_0(a-t)\,\mathrm{d}a \\ &+ \int_t^{a_m} b(u(t), a) \int_0^t \Pi_{A_{\bar{\mu}}}(a, s+a-t)F_{\bar{\mu}}(u(s))(s+a-t)\,\mathrm{d}s\mathrm{d}a. \end{aligned}$$

As in the proof of Theorem 4.10, we introduce the map $K : C([0, T], \mathbb{E}_0) \to C([0, T], \mathbb{E}_0)$,

$$\begin{aligned} &(Ku)(t,a) \\ := &\begin{cases} \Pi_{A_{\bar{\mu}}}(a, a-t)u_0(a-t) \\ \Pi_{A_{\bar{\mu}}}(a, 0)B(t-a) \end{cases} \\ + &\begin{cases} \int_0^t \Pi_{A_{\bar{\mu}}}(a, s+a-t)F_{\bar{\mu}}(u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (t, a_m) \\ \int_{t-a}^t \Pi_{A_{\bar{\mu}}}(a, s+a-t)F_{\bar{\mu}}(u(s))(s+a-t)\,\mathrm{d}s, & \text{for a.a. } a \in (0, t), \end{cases} \end{aligned}$$

where $a < a_m$, and we see that u is a fixed point of K. In the proof of Theorem 4.10 it was shown that K gives rise to a contraction mapping, if $T > 0$ is chosen sufficiently small. Furthermore we note that for $u_0 \in \mathbb{E}_0^+$ fixed, K maps the cone $C([0, T], \mathbb{E}_0^+)$ into itself: the integral kernel

$$k_{\bar{\mu}}(t, a) = b(u(t), t-a)\Pi_{A_{\bar{\mu}}}(t-a, 0), \quad 0 \leq a \leq t,$$

is positive, given $u \in C([0, T], \mathbb{E}_0^+)$ we therefore obtain $B \in C([0, T], E_0^+)$, cf. section 4.5, which in turn yields $Ku \in C([0, T], \mathbb{E}_0^+)$.

Thus, it follows from Banach's fixed point theorem that the fixed point $Ku = u$ is positive. A standard argument then shows $u \in C([0, T_{u_0}), \mathbb{E}_0^+)$. □

5.3 Linearised stability

In the following we will show that a principle of linearised stability is available within the framework of integral solutions. To this end we consider some fixed equilibrium $\phi \in \mathbb{E}_0$ of (5.1), which is determined by equation (5.2). Let $\Pi_{\hat{A}}$ denote the evolution operator generated by $\hat{A}(a) = A(a) + \mu(\phi, a)$, then ϕ has to satisfy

$$\begin{aligned} \phi(a) &= \Pi_{\hat{A}}(a,0)\phi(0), \quad a \in J \\ \phi(0) &= Q(\phi)\phi(0), \end{aligned} \tag{5.17}$$

where

$$Q(\phi) = \int_0^{a_m} b(\phi, a)\Pi_{\hat{A}}(a,0)\,\mathrm{d}a \in \mathcal{L}(E_0),$$

cf. Walker [34]. In (5.11) we supposed the boundedness of b, hence assumption (5.3) together with property (5.9) implies that the operator $Q(\phi)$ is well defined.

Definition 5.6. An element $\phi \in \mathbb{E}_0$ is called an *equilibrium* of (5.1), if it satisfies (5.17).

Remark 5.7. Let us recall the relation

$$\Pi_{\hat{A}}(a,\sigma)v = \Pi_A(a,\sigma)v - \int_\sigma^a \Pi_A(a,\tilde{\sigma})\mu(\phi,\tilde{\sigma})\Pi_{\hat{A}}(\tilde{\sigma},\sigma)v\,\mathrm{d}\tilde{\sigma}, \quad v \in E_0,$$

then we see that the first equation in (5.17) is equivalent to

$$\phi(a) = \Pi_A(a,0)\phi(0) - \int_0^a \Pi_A(a,\tilde{\sigma})\mu(\phi,\tilde{\sigma})\Pi_{\hat{A}}(\tilde{\sigma},0)\phi(0)\,\mathrm{d}\tilde{\sigma}.$$

□

Definition 5.8 (Stability)**.** An equilibrium $\phi \in \mathbb{E}_0$ of (5.1) is called *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every $u_0 \in \mathbb{E}_0$ with $\|u_0 - \phi\| < \delta$, the integral solution $u = u(.;u_0)$ to (5.1) exists for all $t \geq 0$ and satisfies $\|u(t) - \phi\| < \epsilon$, for $t \geq 0$.
An equilibrium $\phi \in \mathbb{E}_0$ of (5.1) is called *asymptotically stable* if it is stable and there exists $\delta > 0$ such that for every $u_0 \in \mathbb{E}_0$ with $\|u_0 - \phi\| < \delta$, the integral solution $u = u(.;u_0)$ to (5.1) satisfies $\lim_{t\to\infty} \|u(t) - \phi\| = 0$.
An equilibrium $\phi \in \mathbb{E}_0$ of (5.1) is called *exponentially asymptotically stable* if it is asymptotically stable, and there exists $\delta > 0$, $\omega > 0$ and $K > 0$ such that for every $u_0 \in \mathbb{E}_0$ with $\|u_0 - \phi\| < \delta$, the integral solution $u = u(.;u_0)$ to (5.1) satisfies $\|u(t) - \phi\| \leq Ke^{-\omega t}\|u_0 - \phi\|$, for $t \geq 0$.
If δ can be chosen arbitrarily large in any of these last two definitions, then the corresponding property is said to be *global*.

The following observation will be essential for the linearisation process:

Proposition 5.9. *Let ϕ be an equilibrium of* (5.1)*, in the sense of Definition 5.6, and $T > 0$. Then for all $t \in [0,T]$:*

$$\begin{aligned} \phi(a) &= \begin{cases} \Pi_A(a,a-t)\phi(a-t) \\ \Pi_A(a,0)\phi(0) \end{cases} \\ &+ \begin{cases} \int_0^t \Pi_A(a,s+a-t)F(\phi)(s+a-t)\,ds, & \text{for a.a. } a \in (t,a_m) \\ \int_{t-a}^t \Pi_A(a,s+a-t)F(\phi)(s+a-t)\,ds, & \text{for a.a. } a \in (0,t), \end{cases} \end{aligned}$$

where $a < a_m$.

Proof. Let $a \in (t, a_m)$, then by formula (5.17) and Remark 5.7 it follows

$$\begin{aligned}
&\Pi_A(a, a-t)\phi(a-t) \\
&= \Pi_A(a, a-t)\left(\Pi_A(a-t, 0)\phi(0) + \int_0^{a-t} \Pi_A(a-t, \sigma)F(\phi)(\sigma)\,\mathrm{d}\sigma\right) \\
&= \Pi_A(a, 0)\phi(0) + \int_0^{a-t} \Pi_A(a, \sigma)F(\phi)(\sigma)\,\mathrm{d}\sigma \\
&= \phi(a) - \int_{a-t}^{a} \Pi_A(a, \sigma)F(\phi)(\sigma)\,\mathrm{d}\sigma \\
&= \phi(a) - \int_0^t \Pi_A(a, s+a-t)F(\phi)(s+a-t)\,\mathrm{d}s.
\end{aligned}$$

Now let $a \in (0, t)$, then by (5.17) and Remark 5.7

$$\begin{aligned}
&\Pi_A(a, 0)\phi(0) \\
&= \phi(a) + \int_0^a \Pi_A(a, \tilde{\sigma})\mu(\phi, \tilde{\sigma})\Pi_\phi(\tilde{\sigma}, 0)\phi(0)\,\mathrm{d}\tilde{\sigma} \\
&= \phi(a) - \int_0^a \Pi_A(a, \tilde{\sigma})F(\phi)(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma} \\
&= \phi(a) - \int_{t-a}^t \Pi_A(a, s+a-t)F(\phi)(s+a-t)\,\mathrm{d}s,
\end{aligned}$$

and the claim follows. □

At this point we are prepared to start the stability analysis. To this end, let $u_0 \in \mathbb{E}_0$ be an arbitrary initial value, and $u = u(\cdot; u_0) : [0, T_{u_0}) \to \mathbb{E}_0$ the corresponding maximal integral solution to (5.1). For given equilibrium $\phi \in \mathbb{E}_0$ of (5.1), Proposition 5.9 then leads to

$$\begin{aligned}
&u(t, a) - \phi(a) \\
&= \begin{cases} \Pi_A(a, a-t)\,(u_0 - \phi)\,(a-t) \\ \Pi_A(a, 0)\,(B(t-a) - \phi(0)) \end{cases} \\
&+ \begin{cases} \int_0^t \Pi_A(a, s+a-t)\,(F(u(s)) - F(\phi))\,(s+a-t)\,\mathrm{d}s, & \text{f.a.a } a \in (t, a_m) \\ \int_{t-a}^t \Pi_A(a, s+a-t)\,(F(u(s)) - F(\phi))\,(s+a-t)\,\mathrm{d}s, & \text{f.a.a. } a \in (0, t), \end{cases}
\end{aligned}$$

where $a < a_m$. Let us introduce

$$\begin{aligned}
w(t) &:= u(t) - \phi, \quad t \in [0, T_{u_0}), \\
w_0 &:= u_0 - \phi, \\
f_w(t) &:= R_F(w(t)), \quad t \in [0, T_{u_0}), \\
h_w(t) &:= \int_0^{a_m} R_b(w(t))(a)\,\mathrm{d}a, \quad t \in [0, T_{u_0}), \\
b_\phi(a)v &:= \tilde{b}_\phi(a)v + b(\phi, a)v, \quad v \in E_0,\ a \in [0, a_m),
\end{aligned}$$

with $\tilde{b}_\phi$ from assumption (5.15).

Remark 5.10. Obviously, $w : [0, T_{u_0}) \to \mathbb{E}_0$ is continuous, consequently the functions $f_w : [0, T_{u_0}) \to \mathbb{E}_0$, $h_w : [0, T_{u_0}) \to E_0$ are continuous by assumptions (5.12), (5.14). □

Using the linearisations (5.12), (5.14), we arrive at

$$
\begin{aligned}
&w(t,a)\\
&=\begin{cases}\Pi_A(a,a-t)w_0(a-t)\\ \Pi_A(a,0)B_w(t-a)\end{cases}\\
&+\begin{cases}\int_0^t \Pi_A(a,s+a-t)\left(F'(\phi)w(s)+f_w(s)\right)(s+a-t)\,\mathrm{d}s, & a\in(t,a_m)\\ \int_{t-a}^t \Pi_A(a,s+a-t)\left(F'(\phi)w(s)+f_w(s)\right)(s+a-t)\,\mathrm{d}s, & a\in(0,t),\end{cases}
\end{aligned}
$$

where $a < a_m$, with

$$
\begin{aligned}
B_w(t) &:= B(t)-\phi(0)\\
&= \int_0^{a_m} b(u(t),a)u(t)-b(\phi,a)\phi(a)\,\mathrm{d}a\\
&= \int_0^{a_m} \big(D_u b(\phi,a)w(t)\big)\phi(a)+b(\phi,a)w(t,a)+R_b(w(t))(a)\,\mathrm{d}a\\
&= \int_0^{a_m} b_\phi(a)w(t,a)\,\mathrm{d}a+h_w(t),
\end{aligned}
$$

where we used assumption (5.15) for the last equality.

In the next step, let us recall the definition of $\hat{A}$ in (5.8), then assumption (5.13) and Proposition 4.19 imply

$$
\begin{aligned}
&w(t,a)\\
&=\begin{cases}\Pi_{\hat{A}}(a,a-t)w_0(a-t)\\ \Pi_{\hat{A}}(a,0)B_w(t-a)\end{cases}\\
&+\begin{cases}\int_0^t \Pi_{\hat{A}}(a,s+a-t)\big(-\mathbb{B}_\phi w(s)+f_w(s)\big)(s+a-t)\,\mathrm{d}s, & a\in(t,a_m)\\ \int_{t-a}^t \Pi_{\hat{A}}(a,s+a-t)\big(-\mathbb{B}_\phi w(s)+f_w(s)\big)(s+a-t)\,\mathrm{d}s, & a\in(0,t),\end{cases}
\end{aligned}
$$

with

$$
B_w(t)=\int_0^{a_m} b_\phi(a)w(t,a)\,\mathrm{d}a+h_w(t).
$$

An application of Proposition 4.17 then leads to

$$
\begin{aligned}
&w(t,a)\\
&=\begin{cases}\Pi_{\hat{A}_\gamma}(a,a-t)w_0(a-t)\\ \Pi_{\hat{A}_\gamma}(a,0)B_w(t-a)\end{cases}\\
&+\begin{cases}\int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)\big(\gamma w(s)-\mathbb{B}_\phi w(s)\big)(s+a-t)\,\mathrm{d}s\\ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)\big(\gamma w(s)-\mathbb{B}_\phi w(s)\big)(s+a-t)\,\mathrm{d}s\end{cases}\\
&+\begin{cases}\int_0^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f_w(s)(s+a-t)\,\mathrm{d}s, & a\in(t,a_m)\\ \int_{t-a}^t \Pi_{\hat{A}_\gamma}(a,s+a-t)f_w(s)(s+a-t)\,\mathrm{d}s, & a\in(0,t),\end{cases}
\end{aligned}
$$

with

$$
B_w(t)=\int_0^{a_m} b_\phi(a)w(t,a)\,\mathrm{d}a+h_w(t).
$$

Let us set $\mu_\phi(a) := \mu(\phi, a), a \in J$, and recall that the linearised problem

$$\begin{aligned}
\partial_t w + \partial_a w + A(a)w + \gamma w &= -\mu_\phi(a)w, \quad t > 0,\, a \in J, \\
w(t,0) &= \int_0^{a_m} b_\phi(a) w(t,a)\,\mathrm{d}a, \quad t > 0, \\
w(0,a) &= w_0(a), \quad a \in J,
\end{aligned}$$

induces a semigroup $S_{\phi,\gamma}(t), t \geq 0$, on $\mathbb{E}_0$, cf. (4.28). Consequently, formula (4.33) yields

$$\begin{aligned}
w(t) &= S_{\phi,\gamma}(t)w_0 + \int_0^t S_{\phi,\gamma}(t-s)\big(\gamma w(s) - \mathbb{B}_\phi w(s) + f_w(s)\big)\,\mathrm{d}s \\
&+ V_\gamma(t)h_w,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
& w(t) - V_\gamma(t)h_w \\
&= e^{-\gamma t} S_{\phi,0}(t)w_0 \\
&+ \int_0^t e^{-\gamma(t-s)} S_{\phi,0}(t-s)\big(\gamma\,(w(s) - V_\gamma(s)h_w) - \mathbb{B}_\phi(w(s) - V_\gamma(s)h_w)\big)\,\mathrm{d}s \\
&+ \int_0^t e^{-\gamma(t-s)} S_{\phi,0}(t-s)\big(\gamma\, V_\gamma(s)h_w - \mathbb{B}_\phi V_\gamma(s)h_w + f_w(s)\big)\,\mathrm{d}s,
\end{aligned}$$

where we used Remark 4.16. Let

$$-\mathbb{A}_\phi : D(-\mathbb{A}_\phi) \subset \mathbb{E}_0 \to \mathbb{E}_0$$

denote the generator of the semigroup $S_{\phi,0}(t), t \geq 0$, and write $S_{\phi,0}(t) = e^{-t\mathbb{A}_\phi}$, $t \geq 0$. By Lemma 2.15, we are led to

$$\begin{aligned}
w(t) &= e^{-t(\mathbb{A}_\phi + \mathbb{B}_\phi)} w_0 + V_\gamma(t)h_w \\
&+ \int_0^t e^{-(t-s)(\mathbb{A}_\phi + \mathbb{B}_\phi)} \left(\gamma\, V_\gamma(s)h_w - \mathbb{B}_\phi V_\gamma(s)h_w + f_w(s)\right)\,\mathrm{d}s. \qquad (5.18)
\end{aligned}$$

After these preparations we can adopt the argument in [38, Theorem 4.13] and establish a principle of linearised stability:

Theorem 5.11 (Principle of linearised stability)**.** *Assume the operator A satisfies* (4.2) – (4.3)*, the evolution operator Π_A is positive, and conditions* (5.3) – (5.7)*,* (5.10) – (5.15) *hold. If the equilibrium $\phi \in \mathbb{E}_0$ of* (5.1) *is such that $\omega_0(-(\mathbb{A}_\phi + \mathbb{B}_\phi)) < 0$, then ϕ is exponentially asymptotically stable.*

Proof. We proceed as in the proof of [38, Theorem 4.13]. Choose $\gamma > M\|b_\phi\|_\infty - \bar{\omega}$, and set

$$\alpha := M\|b_\phi\|_\infty - \bar{\omega} - \gamma.$$

By assumption, $\omega_0(-(\mathbb{A}_\phi + \mathbb{B}_\phi)) < 0$, hence there exist constants $\mathbb{M} \geq 1$ and $\omega \in (\alpha, 0)$ such that $\|e^{-t(\mathbb{A}_\phi + \mathbb{B}_\phi)}\|_{\mathcal{L}(\mathbb{E}_0)} \leq \mathbb{M} e^{\omega t}$ for all $t \geq 0$.
Choose $r > 0$ such that

$$\left(2M + 2M\,\mathbb{M}\,\frac{|\gamma| + \|\mathbb{B}_\phi\|}{\omega - \alpha}\right) d_b(r) + \mathbb{M}\, d_F(r) =: \sigma < -\frac{\omega}{2},$$

set $\epsilon = r/\mathbb{M}$, let $u_0 \in \mathbb{E}_0$ with $\|u_0 - \phi\|_{\mathbb{E}_0} < \epsilon$, and let $t_1 \leq T_{u_0}$ be the largest extended real number such that

$$\|u(t) - \phi\|_{\mathbb{E}_0} \leq r, \qquad 0 \leq t < t_1,$$

where $u : [0, T_{u_0}) \to \mathbb{E}_0$ denotes the maximal integral solution to (5.1) with initial value u_0, cf. Theorem 5.4.

Let $0 \leq t < t_1$, and set $w(t) = u(t) - \phi$, $w_0 = u_0 - \phi$. Since $h_w : [0, T_{u_0}) \to E_0$ is continuous, cf. Remark 5.10, we can apply (4.32) to obtain

$$\begin{aligned}
\|V_\gamma(t) h_w\|_{\mathbb{E}_0} &\leq 2M \int_0^t e^{(M\|b_\phi\|_\infty - \overline{\omega} - \gamma)(t-s)} \|h_w(s)\|_{E_0} \, \mathrm{d}s \\
&\leq 2M \int_0^t e^{(M\|b_\phi\|_\infty - \overline{\omega} - \gamma)(t-s)} \int_0^{a_m} \|R_b(w(s))(a)\|_{E_0} \, \mathrm{d}a \, \mathrm{d}s \\
&\leq 2M \int_0^t e^{(M\|b_\phi\|_\infty - \overline{\omega} - \gamma)(t-s)} d_b(r) \|w(s)\|_{\mathbb{E}_0} \, \mathrm{d}s,
\end{aligned}$$

where we used assumption (5.14) for the last inequality.

Combining this estimate with (5.18) leads to

$$\begin{aligned}
e^{-\omega t} \|w(t)\|_{\mathbb{E}_0} &\leq \mathbb{M} \|w_0\|_{\mathbb{E}_0} \\
&+ 2M \, e^{(\alpha - \omega)t} \, d_b(r) \int_0^t e^{-\alpha s} \|w(s)\|_{\mathbb{E}_0} \, \mathrm{d}s \\
&+ 2M \, \mathbb{M} \, (|\gamma| + \|\mathbb{B}_\phi\|) d_b(r) \int_0^t e^{-\omega s} \left(\int_0^s e^{\alpha(s-\tau)} \|w(\tau)\|_{\mathbb{E}_0} \, \mathrm{d}\tau \right) \mathrm{d}s \\
&+ \mathbb{M} \, d_F(r) \int_0^t e^{-\omega s} \|w(s)\|_{\mathbb{E}_0} \, \mathrm{d}s.
\end{aligned}$$

Regarding the third term, observe that

$$\begin{aligned}
\int_0^t \left(\int_0^s e^{-\omega s} e^{\alpha(s-\tau)} \|w(\tau)\|_{\mathbb{E}_0} \, \mathrm{d}\tau \right) \mathrm{d}s &= \int_0^t \left(\int_\tau^t e^{(\alpha-\omega)s} \, \mathrm{d}s \right) e^{-\alpha\tau} \|w(\tau)\|_{\mathbb{E}_0} \, \mathrm{d}\tau \\
&\leq \int_0^t \frac{1}{\omega - \alpha} e^{(\alpha-\omega)\tau} e^{-\alpha\tau} \|w(\tau)\|_{\mathbb{E}_0} \, \mathrm{d}\tau \\
&= \frac{1}{\omega - \alpha} \int_0^t e^{-\omega\tau} \|w(\tau)\|_{\mathbb{E}_0} \, \mathrm{d}\tau,
\end{aligned}$$

hence we arrive at

$$\begin{aligned}
e^{-\omega t} \|w(t)\|_{\mathbb{E}_0} &\leq \mathbb{M} \|w_0\|_{\mathbb{E}_0} + 2M \, d_b(r) \int_0^t e^{(\alpha-\omega)(t-s)} \, e^{-\omega s} \|w(s)\|_{\mathbb{E}_0} \, \mathrm{d}s \\
&+ \left(2M \, \mathbb{M} \, \frac{|\gamma| + \|\mathbb{B}_\phi\|}{\omega - \alpha} d_b(r) + \mathbb{M} \, d_F(r) \right) \int_0^t e^{-\omega s} \|w(s)\|_{\mathbb{E}_0} \, \mathrm{d}s \\
&\leq \mathbb{M} \|w_0\|_{\mathbb{E}_0} + \sigma \int_0^t e^{-\omega s} \|w(s)\|_{\mathbb{E}_0} \, \mathrm{d}s.
\end{aligned}$$

From Gronwall's lemma it follows that

$$e^{-\omega t} \|w(t)\|_{\mathbb{E}_0} \leq \mathbb{M} \|w_0\|_{\mathbb{E}_0} e^{\sigma t}, \qquad 0 \leq t < t_1,$$

and consequently

$$\|u(t) - \phi\|_{\mathbb{E}_0} \le \mathbb{M}\|u_0 - \phi\|_{\mathbb{E}_0} e^{\omega t/2} < re^{\omega t/2}, \qquad 0 \le t < t_1.$$

Therefore we must have $t_1 = T_{u_0}$, Theorem 5.4 then implies $t_1 = T_{u_0} = \infty$. Hence, ϕ is exponentially asymptotically stable. □

5.4 Stability of the trivial equilibrium

In the following we concretise the results of section 5.3 for the trivial equilibrium $\phi = 0 \in \mathbb{E}_0$. To this end recall that the linearised problem

$$\begin{aligned} \partial_t w + \partial_a w + A(a)w &= -\mu(0,a)w, \quad t > 0,\, a \in J, \\ w(t,0) &= \int_0^{a_m} b(0,a)w(t,a)\,\mathrm{d}a, \quad t > 0, \\ w(0,a) &= w_0(a), \quad a \in J, \end{aligned}$$

induces a semigroup $S_0(t), t \ge 0$, on $\mathbb{E}_0$, cf. (4.28). Let

$$-\mathbb{A}_0 : D(-\mathbb{A}_0) \subset \mathbb{E}_0 \to \mathbb{E}_0$$

denote the generator of the semigroup $S_0(t), t \ge 0$, then we have:

Theorem 5.12. *Assume the operator A satisfies* (4.2) − (4.3)*, the evolution operator Π_A is positive, and conditions* (5.3) − (5.7), (5.10) − (5.15) *hold for $\phi = 0 \in \mathbb{E}_0$. If the equilibrium $\phi = 0 \in \mathbb{E}_0$ of* (5.1) *is such that $\omega_0(-\mathbb{A}_0) < 0$, then $\phi = 0 \in \mathbb{E}_0$ is exponentially asymptotically stable.*

Proof. For $\phi = 0 \in \mathbb{E}_0$, assumption (5.13) simplifies to

$$F'(0)\psi = -\mu(0,.)\psi, \qquad \psi \in \mathbb{E}_0,$$

i.e. $\mathbb{B}_0 = 0 \in \mathcal{L}(\mathbb{E}_0)$. Furthermore we observe that assumption (5.15) implies $\tilde{b}_0 = 0 \in BC([0,\infty), \mathcal{L}(E_0))$, and consequently

$$b_0(a) = b(0,a) \in \mathcal{L}(E_0), \qquad a \in J.$$

Thus, the result is an immediate consequence of Theorem 5.11. □

Remark 5.13. Recall that for this section we have set

$$\mathbb{E}_0 := L_p(J, E_0), \text{ with } p = 1.$$

Assuming maximal L_p-regularity of the operator $A + \mu(0,.)$, with $p > 1$, it was shown in [36] that the operator

$$Q_0 := \int_0^{a_m} b(0,a)\Pi_{A+\mu(0,.)}(a,0)\,\mathrm{d}a \quad \in \mathcal{L}_+(E_0)$$

encodes the asymptotic behaviour of the semigroup $e^{-t\mathbb{A}_0}, t \ge 0$, on $L_p(J, E_0)$. More precisely, if the spectral radius of the operator Q_0 is strictly smaller than 1, then $\omega_0(-\mathbb{A}_0) < 0$, cf. [36, Theorem 3.5]. □

A Appendix

Theorem A.1. *Let R and S be linear operators in a Banach space E such that R has bounded inverse and S is R-bounded, i.e. $D(R) \subset D(S)$ and there exist $a, b \in \mathbb{R}_+$ such that*

$$\|Sx\| \leq a\|x\| + b\|Rx\|, \qquad x \in D(R).$$

Furthermore, suppose that $a\|R^{-1}\| + b < 1$. Then $R + S$ is closed and invertible, with bounded inverse.

A proof of Theorem A.1 can be found in [19, Theorem IV.1.16]

Theorem A.2. *Let (S, Σ_S, μ) and (T, Σ_T, λ) be two positive, σ-finite measure spaces, and let (R, Σ_R, ρ) be their product. Let $1 \leq p \leq \infty$ and let F be a μ-integrable function on S to $L_p(T, \Sigma_T, \lambda, E)$, where E is a real or complex Banach-space. Then there is a ρ-measureable function f on R to E, which is uniquely determined except for a set of ρ-measure zero, and such that $f(s, \cdot) = F(s)$ for μ-almost all $s \in S$.*
Moreover, $f(\cdot, t)$ is μ-integrable on S for λ-almost all t, and the integral $\int_S f(s,t)\mu(ds)$, as a function of t, is equal to the element $\int_S F(s)\mu(ds)$ of $L_p(T, \Sigma_T, \lambda, E)$.

A proof of Theorem A.2 can be found in [10, p. 198, Theorem 17].

Lemma A.3. *Let $a_m \in (0, \infty) \cup \{\infty\}, T > 0$, and $f \in C([0,T], L_1((0, a_m), E_0))$. There is a unique element in $L_1((0,T) \times (0, a_m), E_0)$ (which we also denote by f) such that*

$$\text{for each } t \in [0,T], f(t,a) = f(t)(a) \text{ for almost everywhere } a > 0,$$

and

$$\begin{aligned} \int_0^T \|f(t)\|_{L_1}\, dt &= \int_0^T \int_0^\infty \|f(t,a)\|_{E_0}\, da\, dt \\ &= \int_0^\infty \int_0^T \|f(t,a)\|_{E_0}\, dt\, da \\ &= \int_{[0,T]\times(0,a_m)} \|f(t,a)\|_{E_0}\, d(t,a). \end{aligned}$$

Lemma A.3 follows from Theorem A.2 and an application of Fubini's theorem (cf. Webb [38, Lemmata 2.1] for details).

Lemma A.4. *Let $a_m \in (0, \infty) \cup \{\infty\}, T > 0$, and suppose*

$$b_L \in BC([0,T] \times [0, a_m), \mathcal{L}(E_0)), \quad f \in C([0,T], L_1((0, a_m), E_0)).$$

Let $t \in [0,T]$ be fixed and define $\Gamma_t := \{(s,a) : 0 \leq s \leq t \leq s + a\}$.
Then the function

$$\Gamma_t \to E_0, \qquad (s,a) \mapsto b_L(t,a)\Pi_\gamma(a, s+a-t) f(s, s+a-t)$$

is strongly measureable. The same is true for the function

$$(0,t) \times (0, a_m) \to E_0, \qquad (s,a) \mapsto b_L(t, a+t-s)\Pi_\gamma(a+t-s, a) f(s,a).$$

Proof. We give a proof for the first function, the argument for the other function is analogous.

We consider the element $f \in L_1((0,T) \times (0,a_m), E_0)$, provided by Lemma A.3. In particular $f : (0,T) \times (0,a_m) \to E_0$ is strongly measureable, i.e. there exists a sequence of simple functions converging to f pointwise a.e. in $(0,T) \times (0,a_m)$. Consequently, the function

$$\Gamma_t \to E_0, \quad f_t(s,a) := f(s, s+a-t)$$

is strongly measureable as well (for simple functions this is not hard to see, since $(s,a) \mapsto (s, s+a-t)$ is a homeomorphism on $\mathbb{R}^2$, the general case is an immediate consequence). Therefore, there exists a sequence (f_n) of simple functions such that

$$\lim_{n\to\infty} \|f_n(s,a) - f_t(s,a)\|_{E_0} = 0, \qquad \text{a.e. in } \Gamma_t,$$

and by continuity

$$\lim_{n\to\infty} \|b_L(t,a)\Pi_\gamma(a, s+a-t)\left(f_n(s,a) - f_t(s,a)\right)\|_{E_0} = 0, \tag{A.1}$$

a.e. in Γ_t. Now let us consider the simplest simple function $(s,a) \mapsto e \cdot \chi_D(s,a)$, where $e \in E_0$, and $D \subseteq \Gamma_t$ is measureable. Then

$$\begin{aligned} (s,a) \quad &\mapsto \quad b_L(t,a)\Pi_\gamma(a, s+a-t)(e \cdot \chi_D(s,a)) \\ &= \quad (b_L(t,a)\Pi_\gamma(a, s+a-t)e)\, \chi_D(s,a) \end{aligned}$$

is the product of a continuous, E_0-valued function and the characteristic function χ_D, and therefore strongly measureable. Together with $(A.1)$ we conclude that $(s,a) \mapsto b_L(t,a)\Pi_\gamma(a, s+a-t)f_t(s,a)$ is the pointwise limit of a sequence of strongly measureable functions, and therefore strongly measureable. ☐

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