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A UNIFIED APPROACH TO SPDES DRIVEN BY SEMIMARTINGALE FIELDS

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ABSTRACT

We establish an original framework referred to as "unified approach" that seeks to generalize the existing theory of integration. For this purpose, we introduce the concept of semimartingale fields and develop the related stochastic integration theory. Meanwhile, we prove that our framework includes the Walsh stochastic integral, integrals with respect to an infinite dimensional Wiener process and the Poisson random measure. As an application, we study the infinite dimensional stochastic partial differential equations (SPDEs) driven by random fields and investigate the stability and regularity properties of their solutions. In particular, we are interested in the L^p-existence of mild and weak solutions to SPDEs with general C_0-semigroups, where the driving noise is a continuous martingale field with independent increments.

Keywords: Unified Approach, Random Fields, Stochastic Partial Differential Equations, Mild and Weak Solutions, Stochastic Integration

ZUSAMMENFASSUNG

Wir werden einen “universellen Ansatz” zur Verallgemeinerung der bestehenden Integrationstheorie ent-wickeln. Dazu führen wir das Konzept von Semimartingalfeldern und eine zugehörige stochastische Integrationstheorie ein. Wir beweisen, dass unser mathematischer Rahmen das Walsh’sche stochastische Integral, Integrale bezüglich unendlich dimensionaler Wienerprozesse und das Poisson’sche Zufallsmaß umfasst. Als Anwendung untersuchen wir unendlich dimensionale stochastische partielle Differentialgleichungen (SPDEs), die durch Zufallsfelder angetrieben werden, und bestimmen die Stabilitäts- und Regularitätseigenschaften der Lösungen. Insbesondere sind wir an der L^p-Existenz milder und schwache Lösungen von SPDEs mit allgemeinen C_0-Halbgruppen interessiert, bei denen der treibende Störterm ein stetiges Martingalfeld mit unabhängigen Zuwächsen ist.

Schlüsselwörter: Einheitlicher Ansatz, Zufälliges Feld, Stochastische partielle Differentialgleichung, Milde und schwache Lösungen, Stochastische Integration

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INHALTSVERZEICHNIS

LISTINGS

a). We start by giving all different abbreviations for the word writing:

- **i.e. :** stands for *that is*
- **w.r.t. :** stands for *with respect to*
- **e.g. :** stands for *for example*
- **a.s. :** stands for *almost surely*
- **resp. :** stands for *respectively*
- **SDE :** stands for *stochastic differential equations*
- **SPDE :** stands for *stochastic partial differential equations*

b) We establish basic notations that will be used through the dissertation.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a stochastic basis that satisfies the usual hypothesises of completeness and right-continuity. Let $(H, \|\cdot\|)$ and $(U, \|\cdot\|_U)$ be separable Hilbert spaces, $(X, \mathcal{X})$ be a measurable space and (E, Σ) be a Blackwell space. Let $T > 0$ be a fixed horizon time. We denote and define:

1. $\#$: the counting measure; δ_x : the Dirac measure sitting at the point x.
2. λ : the Lebesgue measure on $\mathbb{R}_+$.
3. $\mathcal{L}^p_{loc}(\lambda; H)$: the space of all predictable processes $f : \Omega \times \mathbb{R}_+ \times H \to H$ such that
$$\mathbb{P}\left(\int_0^T \int_E \|f(s,x)\|^p \, ds < \infty\right) = 1.$$
4. $L^p(\Omega; H) = L^p(\Omega, \mathcal{F}, \mathbb{P}; H)$ is a Lebesgue space with the standard norm
$$\|M\|_{L^p(\Omega;U)} := \mathbb{E}\left[\|M\|_U^p\right]^{1/p}.$$
We shall write $L^p = L^p(\Omega; \mathbb{R})$ for all $p \geqslant 0$.

5. $\mathcal{O}$ (resp. $\mathcal{P}$) : the optional (resp. **predictable**) σ-fields on $\Omega \times \mathbb{R}_+$, that is generated by all càdlàg (resp. **càg**) adapted processes on $\Omega \times \mathbb{R}_+$.

6. $\mathcal{V}^+$ (resp. $\mathcal{V}$) the class of all real-valued adapted and càdlàg processes N having a non-decreasing path $t \mapsto N_t(\omega)$ (resp. a finite variation over finite interval of the form $[0, t]$ for $t \in \mathbb{R}_+$) and $N_0 = 0$.

   The variation process V_N of N on $[0, t]$ is

   $$V_N(t)(\omega) = \lim_{n\to\infty} \sum_{k=1}^{n} |N_{tk/n}(\omega) - N_{t(k-1)/n}(\omega)|, \quad \text{for any } \omega \in \Omega.$$

7. $\mathcal{A}^+$(resp. $\mathcal{A}$) the set of all $N \in \mathcal{V}^+$ such that $\mathbb{E}[N_\infty] < \infty$ (resp. $N \in \mathcal{V}$ such that $\mathbb{E}[V_N(\infty)] < \infty$).

8. $\mathcal{M}$ the space of all real-valued integrable martingale processes.

9. $\mathcal{H}^2$ the space of all real-valued square-integrable martingale processes.

10. $\langle M, N\rangle$ the predictable process belonging to $\mathcal{V}$ so that $MN - \langle M, N\rangle \in \mathcal{M}_{loc}$, for any $M, N \in \mathcal{H}^2_{loc}$.

11. $\mathcal{S}$: the class of all real-valued semimartingale processes Y of the form:

    $$Y = M + N, \quad \text{with } M \in \mathcal{M}_{loc}, N \in \mathcal{V}.$$

12. $L_2(U, H)$ the space of Hilbert-Schmidt operators from U and H.

13. $\mathcal{P}_T$ the predictable σ-fields on $\Omega \times [0, T]$, while $\mathcal{P}$ the predictable σ-fields on $\Omega \times \mathbb{R}_+$.

14. $M^2_T(H)$ the space of all square-integrable càdlàg martingales $Y : \Omega \times [0, T] \to H$, where indistinguishable processes are identified.

15. $G_{loc}(\mu)$ the space of all $\mathcal{P} \otimes \Sigma$-measurable real-valued functions h on $\Omega \times \mathbb{R}_+ \times E$ such that the stochastic integral $h * (\mu - \nu)$ exists in the sense of [47] (see Page 72) for any random measure μ on $\mathbb{R}_+ \times E$ with its dual predictable projection ν.

INTRODUCTION

Stochastic partial differential equations (SPDEs) are increasingly playing an important role in applications to finance, numerical analysis, physics, and biology. In literature, the theory of SPDEs developed from the work of Walsh [96] on one hand and the from studies on stochastic evolution equations in Hilbert spaces H (such as [22], [16] and [34]) on the other hand. These two approaches led to the development of two distinct methods of understanding SPDEs, based on different theories of stochastic integration. Walsh theory emphasizes integration with respect to a set of functions called "martingale measures" (see [96]) whereas the theory of integration in a Hilbert space H is with respect to H-valued processes such as Q-Wiener processes (see Da Prato and Zabczyk [22]).

Motivated by the above two arguments, we propose and develop a noval integration theory based on a semimartingale field. Walsh describes a random field as an L^2-valued and σ-finite martingale measure on a Lusin space (see [96]). In contrast, we define a random field as mapping on $\mathbb{R}_+ \times \Sigma$, viewed as a process in the direction of time and a random premeasure on Σ for some Blackwell space (E, Σ). First, we shall introduce the generalized Bochner integral with respect to finite variation fields and then build the Itô stochastic integration with respect to a martingale field. We shall establish the unified approach for stochastic integrations and show that our integrals are related to both stochastic integrations cited above.

According to Da Prato and Zabczyk [22], SPDEs can be viewed as stochastic perturbations of partial differential equations. In other words, some factor of randomness (e.g. a infinite dimensional Wiener process) is added to a (semi-linear) PDE to obtain a stochastic partial differential equation (SPDE) of the form

$$du_t = [Au_t + \alpha(t, u_t)]\, dt + \sigma(t, u_t)\, dW_t, \quad u_0 \in H. \tag{0.1}$$

for some $\sigma : \Omega \times H \to H$ and a separable Hilbert space H. The existence and uniqueness as well as stability and regularity conditions of solutions have been studied in general cases. For instance in the case of cylindrical-Wiener process (see [16] or [34]) or non-Gaussian Lévy noise (see [71]). Further developments led to jump-diffusions SPDEs in Hilbert spaces (see [1] or [31]) of the form

$$du_t = [Au_t + \alpha(t, u_t)]\, dt + \sigma(t, u_t)\, dW_t + \int_B \gamma(t, u_t, x)\overline{\mu}(dt, dx),\ u_0 = \xi, \tag{0.2}$$

for some infinite dimensional Wiener process W and compensated Poisson random measure $\overline{\mu}$. The type of SPDE given in (0.2) includes a large class of equations driven by H-valued Lévy noise. According to [31], the existence and uniqueness of mild and weak solutions to SPDE (0.2) can be established through a "moving frame" approach. This consists of a time-dependent transformation of the given SPDE to an SDE that is relatively easier to solve then obtain results to the original SPDE through a pull-back transformation of the results to the SDE.

In this dissertation, we are interested studying stochastic partial differential equations of the form:

$$du_t = Au_t\, dt + \int_E \beta(t, u_{t^-}, x)X(dt, dx), \quad u_0 = \xi, \tag{0.3}$$

for some Blackwell space (E, Σ), generator A of C_0-semigroup $(S_t)_{t\geqslant 0}$ and a semimartingale field X. By means of the moving frame approach, introduced by [31], we prove the existence and uniqueness of mild and weak solutions to the SPDE (0.3) with càdlàg paths. In addition, we investigate the stability and regularity dependence on the initial data for the solutions. Meanwhile, we show that the SPDE (0.2) is a particular case of (0.3). At the end, we examine the case where $(S_t)_{t\geqslant 0}$ is a general C_0-semigroup and X is a continuous martingale field. We derive the L^p-existence and uniqueness of solutions.

The rest of the dissertation is as follows: In Chapter 1, basic definitions and concepts related to the random fields are given, starting with finite variation fields and the definition of the Bochner integral followed by the martingale fields and the related stochastic integration. We present the concept of semimartingale field which is a combination of finite variation and martingale field. Then define the stochastic integration with respect to semimartingale field and introduce the unified approach. For each class of random fields, we also present illustrative examples to aid understanding of the concepts. In Chapter 2, we investigate the SDE induced by the SPDE (0.3) when $A = 0$. In particular, we study the existence and uniqueness of strong solutions, making a fixed point argument on a specific Banach space. The stability and regularity of solutions are also discussed. Chapter 3, starts with the establishment of the concepts of strong, weak and mild solutions followed by the transfer of all results from Chapter 2 to show the existence and uniqueness of mild and weak solutions to the SPDE (0.3), for the case of pseudo-contractive C_0-semigroup, including stability and regularity. Next we show that the SPDE (0.2) is a fundamental example of our framework and then present the full theory of L^p-estimates for solutions of SPDEs, with general C_0-semigroup, driven by continuous martingale.

1

RANDOM FIELDS AND INTEGRATION THEORY

Random fields play a crucial role in the development of stochastic integrals or stochastic differential equations. In this chapter we aim to describe the unification approach to semimartingale fields. For this propose, we introduce the notion of semimartingale fields. Likewise for the classical semimartingale processes, we define a semimartingale field as sum of a finite variation (FV) field and a martingale field. We present the related integral theory for each class of these random fields and finally, discuss the main result of this Chapter arising from our unified approach.

This chapter is organized as follows. In Section 1 we introduce the basic notation and terminology concerning random fields. In Section 2 we describe the concept of finite variation fields and present some fundamental examples. This is followed by the definition of Bochner integrals w.r.t. FV fields. Section 3 is an introductory section focusing on martingale fields including several examples under different settings. The notion of bracket process for martingale fields and the integration theory shall be revisited. Section 4 is devoted to describing the semimartingale field and its basic properties. We then derive the integrability condition of a stochastic integral w.r.t. a semimartingale field. We present the unification framework which consists of combining two stochastic integrals w.r.t. different semimartingale fields to derive a single integral process.

1.1 PRELIMINARIES

In the existing literature, there are several definitions of random fields but we shall adopt the one where a random field is viewed as a set function, precisely a premeasure.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a (continuous-time) stochastic basis satisfying the usual hypothesis of completeness and right-continuity. In the sequel, let (E, Σ) be a Blackwell space (e.g. $\mathbb{N}$ or a Polish space), i.e., there is a countable semi-ring $\mathcal{E}$ on E such that it contains an exhausting sequence $(E_n)_{n \in \mathbb{N}}$ (i.e. $E_n \nearrow E$) and $\Sigma = \sigma(\mathcal{E})$.

Definition 1.1.1 A mapping $\Phi : \Omega \times \mathbb{R}_+ \times \mathcal{E} \to \mathbb{R}$ is said to be *random field* on $\mathbb{R}_+ \times \mathcal{E}$ if

1. $\Phi(\omega; 0, A) = \Phi(\omega; t, \emptyset) = 0$ for each $\omega \in \Omega, (t, A) \in \mathbb{R}_+ \times \mathcal{E}$.
2. $(\omega, t) \mapsto \Phi(\omega; t, A)$ is an adapted stochastic process for each $A \in \mathcal{E}$.
3. $A \mapsto \Phi(\omega; t, A)$ is a signed σ-finite premeasure on $(E, \mathcal{E})$ for ω, t fixed.

Definition 1.1.2 A random field Φ is continuous (resp. càdlàg) if the mapping $t \mapsto \Phi(\omega; t, A)$ is continuous (resp. càdlàg) for any fixed $A \in \mathcal{E}$ and $\omega \in \Omega$.

Throughout this work, we set $\mathcal{A}_0$ is the set of countable disjoint unions of semi-open intervals $(a, b]$ with $0 \leqslant a \leqslant b$. It can be verified that $\mathcal{A}_0$ is an algebra and $\sigma(\mathcal{A}_0) = \mathcal{B}(\mathbb{R}_+)$ (see e.g. [85]). Let Φ be a random field on $\mathbb{R}_+ \times \mathcal{E}$. We are now ready to define the increment process associated to Φ.

Definition 1.1.3 The increment process of Φ is a random signed premeasure I_Φ on $(\mathbb{R}_+ \times E, \mathcal{A}_0 \times \mathcal{E})$ such that

$$I_\Phi((s,t] \times A) := \Phi(t, A) - \Phi(s, A), \qquad \text{for all } A \in \mathcal{E}, (s,t] \in \mathcal{A}_0.$$

Furthermore, for each $A \in \mathcal{E}$, we say that Φ has

1. **independent increments:** if $I_\Phi((s,t] \times A)$ is independent of $\mathcal{F}_s$, for every $(s,t] \times A \in \mathcal{A}_0 \times \mathcal{E}$.
2. **stationary increments:** $I_\Phi((s,t] \times A)$ has the same distribution as $\Phi(t - s, A)$, for every $(s,t] \times A \in \mathcal{A}_0 \times \mathcal{E}$.

Remark 1 *As far as stochastic integration w.r.t a random field is concerned later on and since $\mathcal{E}$ is not a σ-algebra, it is important to point out that in order to define stochastic integrals one has to extend the premeasure $A \mapsto \phi(\cdot, A)$ to a measure* *on* (E, Σ). *First, if* ϕ *takes positive value, then by Lemma A.1.2 there exists a unique positive random field* $\widehat{\Phi}$ *on* $\mathbb{R}_+ \times \Sigma$ *such that the mapping* $\widehat{\Phi}(\omega; t, \cdot)$ *defines a measure on* (E, Σ) *and the restriction of* $\widehat{\Phi}|_{\mathbb{R}_+ \times \mathcal{E}} = \Phi$. *In the case where* $\phi(\omega; t, \cdot)$ *is a signed σ-finite premeasure for each* $(\omega, t) \in \Omega \times \mathbb{R}_+$ *then one needs some extra conditions. In other words, if* ϕ *is a signed random field, one first take* $\phi = \phi^+ - \phi^-$ *where* ϕ^- *and* ϕ^+ *are positive random fields. To avoid the problem of having* $\infty - \infty$, *one should ensure for instance that* $\phi^+ + \phi^- < \infty$ *and then for each* $(\omega, t) \in \Omega \times \mathbb{R}_+$ *the extension procedure for the signed premeasure* $\phi(\omega; t, \cdot)$ *makes sense after extending respectively both premeasures* $\phi^-(\omega; t, \cdot)$ *and* $\phi^+(\omega; t, \cdot)$ *to measures* $\widehat{\phi}^-(\omega; t, \cdot)$ *and* $\widehat{\phi}^+(\omega; t, \cdot)$ *on* (E, Σ). *Therefore, we also obtain the extended signed measure* $A \mapsto \widehat{\Phi}(\cdot, A) = \widehat{\phi}^+(\cdot, A) - \widehat{\phi}^-(\cdot, A)$ *on* (E, Σ) *so that the mapping* $\widehat{\Phi}$ *defines a random field on* $\mathbb{R}_+ \times \Sigma$ *and the restriction of* $\widehat{\Phi}|_{\mathbb{R}_+ \times \mathcal{E}} = \Phi$.

Moreover, note that the construction of integral processes is not affected by the choice of countable generator $\mathcal{E}$. *Indeed, this is due to the fact that if two different countable semi-rings* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are both generators of* Σ *(i.e.* $\Sigma = \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$*). Then they should lead to the same unique mapping* $\widehat{\Phi}$ *by the extension argument.*

1.2 FINITE VARIATION FIELDS

Let N be a càdlàg random field on $\mathbb{R}_+ \times \mathcal{E}$. To begin, we briefly present the definition of variation process of N. Let $\Pi_t = \Big\{\vartheta = \{t_0 < t_1 < \cdots\} \mid \vartheta \text{ is a partion of } [0,t]\Big\}$ for $t \geqslant 0$. For any fixed $A \in \Sigma$, denote by $V_N(\omega, t, A) := V_{N(\cdot, A)}(\omega, t)$ the total variation process of $N(\cdot, A)$ up to time t and for $\omega \in \Omega$. Then define

$$V_N(\omega; t, A) = \sup_{\pi \in \Pi_t} \Bigg\{ \sum_{\substack{t_i, t_{i+1} \in \pi \\ t_i < t \leqslant t_{i+1}}} \big| N(\omega; t_{i+1}, A) - N(\omega; t_i, A) \big| \Bigg\}. \quad (1.1)$$

1.2.1 *Basic Properties*

We consecutively introduce few notions.

Definition 1.2.1 A random field $N : \Omega \times \mathbb{R}_+ \times \mathcal{E} \to \mathbb{R}_+$ is called an *increasing field* if the following are satisfied

1. For each $A \in \mathcal{E}$, the process $(N(t, A))_{t \geqslant 0}$ belongs to $\mathcal{V}^+$.

2. For any $(\omega, t) \in \Omega \times \mathbb{R}_+$, $N(\omega; t, \cdot)$ is a σ-finite premeasure on $(E, \mathcal{E})$.

Denote by $\mathcal{V}^+_{\mathcal{E}}$ the set of all increasing fields on $\mathbb{R}_+ \times \mathcal{E}$. Note that, since $N(\cdot, A) \in \mathcal{V}^+$ for any $A \in \mathcal{E}$, then the process $N(\cdot, A)$ coincides with its variation process up to evanescent set, i.e., $V_N(t, A) = N(t, A)$ for all $t \in \mathbb{R}_+$ almost surely.

Example 1 *Recall that* (E, Σ) *is a Blackwell space such that* $\Sigma = \sigma(\mathcal{G})$ *with* $\mathcal{G}$ *is a countable ring. Let* F *be a positive σ-finite measure on* (E, Σ). *Then there is* $(A_n)_{n \in \mathbb{N}}$ *be an increasing sequence of sets in* Σ *such that* $E = \bigcup_{n \in \mathbb{N}} A_n$ *and* $F(A_n) < \infty$ *for all* $n \in \mathbb{N}$. *Construct the countable*[1] *ring:*

$$\mathcal{E} := \bigcup_{n \in \mathbb{N}} \{B \cap A_n : B \in \mathcal{G}\},$$

so that Σ *is countably generated by* $\mathcal{E}$, *i.e.* $\sigma(\mathcal{E}) = \Sigma$. *Define the field*

$$N(t, A) = \lambda([0, t])F(A), \quad \textit{for all } t \in \mathbb{R}_+.$$

Hence N *is an increasing field. Indeed, since* λ *is a Lebesgue measure for each* $A \in \mathcal{E}$, *then* $N(\cdot, A) \in \mathcal{V}^+$. *One the other hand, we also have* $N(\omega; t, \cdot)$ *is a σ-finite premeasure on* (E, Σ) *for every* $(\omega, t) \in \Omega \times \mathbb{R}_+$ *as* F *is a σ-finite measure on* (E, Σ).

Definition 1.2.2 A random field $N : \Omega \times \mathbb{R}_+ \times \mathcal{E} \to \mathbb{R}$ is called a *finite variation (FV) field* if

1. For each $A \in \mathcal{E}$, the process $(N(t, A))_{t \geqslant 0}$ belongs to $\mathcal{V}$.

2. For any $(\omega, t) \in \Omega \times \mathbb{R}_+$, $V_N(\omega; t, \cdot)$ is a σ-finite premeasure on $(E, \mathcal{E})$.

Denote by $\mathcal{V}_{\mathcal{E}}$ the set of all finite variation fields on $\mathbb{R}_+ \times \mathcal{E}$.

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1 By construction, $\mathcal{E}$ is a ring. Since the set $\{B \cap A_n : B \in \mathcal{G}\}$ is countable for any $n \in \mathbb{N}$, then its countable union is also countable. Moreover, $\sigma(\mathcal{E})$ is separable and $\sigma(\mathcal{E}) \subset \Sigma$ thus it follows $\Sigma = \sigma(\mathcal{E})$.

Since, by definition, any increasing stochastic process is obviously a finite variation process. Then the same holds for increasing and finite variation fields. The following result shows the decomposition of finite variation field.

Lemma 1.2.1 *For every countable semi-ring $\mathcal{E}$ on E with $\Sigma = \sigma(\mathcal{E})$, then we have*

$$\mathcal{V}_{\mathcal{E}} = \mathcal{V}_{\mathcal{E}}^{+} \ominus \mathcal{V}_{\mathcal{E}}^{+}, \qquad \mathbb{P}\text{-almost surely},$$

i.e. for each $N \in \mathcal{V}_{\mathcal{E}}$, there exist unique $N_1, N_2 \in \mathcal{V}_{\mathcal{E}}^{+}$ such that $N = N_1 - N_2$ and $V_N = N_1 + N_2$.

Proof Let $N \in \mathcal{V}_{\mathcal{E}}$.To prove the next point it is sufficient to prove that N can be written as $N = N_1 - N_2$ with a unique pair (N_1, N_2) of increasing fields up to a $\mathbb{P}$-null set. For any fixed $A \in \mathcal{E}$, by a path-wise argument, we obtain the existence of two unique pair $(N_1(\cdot, A), N_2(\cdot, A))$ of processes which are càdlàg, with $N_1(0, A) = N_2(0, A) = 0$ and non-decreasing paths, such that $N(\cdot, A) = N_1(\cdot, A) - N_2(\cdot, A)$ and $V_N(\cdot, A) = N_1(\cdot, A) + N_2(\cdot, A)$ almost surely (see [47], Proposition 3.3). Namely, one can construct both processes by the following way

$$N_1(\cdot, A) = \frac{V_N(\cdot, A) + N(\cdot, A)}{2} \quad \text{and} \quad N_2(\cdot, A) = \frac{V_N(\cdot, A) - N(\cdot, A)}{2}$$

for any $A \in \mathcal{E}$. Lastly, since $\mathcal{E}$ is countable, then we deduce that

$$\bigcup_{A \in \mathcal{E}} \Bigg(\Big\{ \omega \in \Omega : N(\omega; t, A) \neq N_1(\omega; t, A) - N_2(\omega; t, A), t \in \mathbb{R}_+ \Big\} \\ \bigcup \Big\{ \omega \in \Omega : V_N(\omega; t, A) \neq N_1(\omega; t, A) + N_2(\omega; t, A), t \in \mathbb{R}_+ \Big\} \Bigg)$$

is a $\mathbb{P}$-null set. This completes the proof. ∎

Remark 2 *For every FV field $N \in \mathcal{V}_{\mathcal{E}}$, we define the jump process associated to $N(\cdot, A)$ for any fixed A as follows*

$$\Delta N(t, A) = N(t, A) - N(t^{-}, A), \quad \textit{for all } t \geqslant 0,$$

where

$$N(t^{-}, A) = \lim_{s \uparrow t} N(s, A).$$

We now present a canonical example of FV fields in the following lemma.

Lemma 1.2.2 *Assume that* $E = \mathbb{N}$ *and set* $\Sigma = \sigma(\mathcal{E})$ *where*

$$\mathcal{E} = \{A \subset E : A \text{ is a finite set}\}.$$

Let $(N_i)_{i \in \mathbb{N}}$ *be a family of processes from* $\mathcal{V}^+$ *and define the random field as*

$$N(t, A) = \sum_{i \in A} N_i(t), \quad \text{for } A \in \mathcal{E}, t \in [0, T],$$

which is a finite variation field on $\mathbb{R}_+ \times \mathcal{E}$.

Proof To prove this lemma, one need to verify that Definition 1.2.2 is fulfilled for the field N. But before proceeding, it is important to say that $\mathcal{E}$ is a countable ring by construction. Next we have by construction:

(1) without of lose of generality, let $n \in \mathbb{N}$ and take $A = \{1, \ldots, n\}$. Then, by formula 1.1, we compute

$$\begin{aligned} V_N(t, A) &= \sup_{\pi \in \Pi_t} \left\{ \sum_{\substack{t_i, t_{i+1} \in \pi \\ t_i < t}} \left| \sum_{k \in A} N_k(t_{i+1}) - N_k(t_i) \right| \right\}, \quad \text{for each } t \geqslant 0. \\ &= \sum_{k \in A} \sup_{\pi \in \Pi_t} \left\{ \sum_{\substack{t_i, t_{i+1} \in \pi \\ t_i < t}} [N_k(t_{i+1}) - N_k(t_i)] \right\} \\ &= \sum_{k \in A} V_{N_k}(t) = \sum_{k \in A} N_k(t) < \infty, \quad \text{for each } t \geqslant 0. \end{aligned} \tag{1.2}$$

This implies that $N(\cdot, A) \in \mathcal{V}$ for any fixed set $A \in \mathcal{E}$, i.e. each path $t \mapsto N(\omega; t, A)$ has a finite variation over each interval $[0, t]$ for every $\omega \in \Omega$.

(2) Next we show that $N(\omega; t, \cdot)$ is a premeasure on $(E, \mathcal{E})$ for any $(\omega, t) \in \Omega \times \mathbb{R}_+$. Indeed, we have:

- by construction, $V_N(t, \emptyset) = 0$ almost surely;

- for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{E}$ with $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$ (i.e. there is $n_0 \in \mathbb{N}$ such that $A_n = \emptyset$ for $n \geqslant n_0$) and by equation (1.2) we get

$$\begin{aligned} V_N(t, \cup_{n\in\mathbb{N}} A_n) &= \sum_{i\in\bigcup_{n\in\mathbb{N}} A_n} N_i(t) = \sum_{i\in A_1} N_i(t) + \cdots + \sum_{i\in A_{n_0-1}} N_i(t) \\ &= \sum_{n\in\mathbb{N}} \sum_{i\in A_n} N_i(t) = \sum_{n\in\mathbb{N}} V_N(t, A_n), \quad \text{a.s.} \end{aligned} \tag{1.3}$$

∎

Example 2 *Let $\lambda > 0$ and $(N_i)_{i\in\mathbb{N}}$ be a family of Poisson processes with intensity λ (see e.g. [47]). Let us take $E = \mathbb{N}$. As $N_i \in \mathcal{V}^+$ for any $i \in \mathbb{N}$, thus by Lemma 1.2.2, the field associated to the collection $(N_i)_{i\in\mathbb{N}}$ defined as follows*

$$N(t, A) = \sum_{i\in A} N_i(t), \quad \textit{for } A \subset \mathbb{N} \textit{ is finite}, t \in [0, T],$$

is a finite variation field on $\mathbb{R}_+ \times \mathcal{E}$ where $\mathcal{E} = \{A \subset \mathbb{N} : A \text{ is a finite set}\}$.

1.2.2 *Bochner Integral*

This subsection aims to define the integral process with respect to a finite variation field. To this end, we first start with the construction of the so-called Bochner integral with respect to a non-decreasing field. Then we extend it to the class of finite variation fields.

1. Let assume that $N \in \mathcal{V}^+_{\mathcal{E}}$. In order to perform the construction of the integral process we first need to show that, for any fixed ω, the mapping $(t, A) \mapsto N(\omega; t, A)$ induces a random premeasure on $(\mathbb{R}_+ \times E, \mathcal{A}_0 \times \mathcal{E})$. Then we use the Carathéodory extension theorem (see Lemma A.1.2) to get a random measure on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \Sigma)$.

Lemma 1.2.3 *For every increasing field N, there exists an unique optional and σ-finite random measure ζ on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \Sigma)$ such that*

$$\zeta\big((s, t] \times A\big) = N(t, A) - N(s, A), \quad \textit{for each } (s, t] \times A \in \mathcal{A}_0 \times \mathcal{E},$$

Conversely, every such random measure defines an increasing field in $\mathcal{V}^+_{\mathcal{E}}$.

Proof We want to show that there is one-to-one correspondence between the class of finite premeasure on $\mathcal{A}_0 \times \mathcal{E}$ and the class of increasing fields $\mathcal{V}^+_{\mathcal{E}}$. The proof is done by two steps.

(a). Let $N \in \mathcal{V}^+_{\mathcal{E}}$. For any $(s,t] \times A \in \mathcal{A}_0 \times \mathcal{E}$, let us define a set function η as follows

$$\eta\left((s,t] \times A\right) = \begin{cases} \sum_{i=1}^{n} N(t_i, A_i) - N(s_i, A_i) & \text{if } (s,t] \times A = \dot{\bigcup}_{i=1}^{n} (s_i, t_i] \times A_i \\ N(t,A) - N(s,A) & \text{otherwise} \end{cases} \tag{1.4}$$

for some sequence $\left((s_i, t_i] \times A_i\right)_{i=1,\dots,n}$ of pairwise disjoint rectangles in $\mathcal{A}_0 \times \mathcal{E}$. Note that η is well-defined and additive. For the proof of countable additivity of η, it suffices to observe that if $\{(s_n, t_n] \times A_n\}_{n \in \mathbb{N}}$ is an increasing sequence of rectangles in $\mathcal{A}_0 \times \mathcal{E}$ whose union $(s,t] \times A$ is an element of $\mathcal{A}_0 \times \mathcal{E}$ as well. Then we have

$$\eta((s,t] \times A) = \lim_{n \to \infty} \eta((s_n, t_n] \times A_n),$$

that is, by Lemma A.1.1, this is equivalent to say that

$$\eta\left(\dot{\bigcup_{n \in \mathbb{N}}} (a_n, b_n] \times B_n \right) = \sum_{n=1}^{\infty} \eta((a_n, b_n] \times B_n),$$

whenever sequence of pairwise disjoint rectangles $\{(a_n, b_n] \times B_n\} \subset \mathcal{A}_0 \times \mathcal{E}$. Indeed, by the continuity of the process $N(\cdot, A)$ for any $A \in \mathcal{E}$ and the continuity of the premeasure $N(\omega; t, \cdot)$ for $(\omega, t) \in \Omega \times \mathbb{R}_+$, we can write

$$\begin{aligned} \lim_{n \to \infty} \eta((s_n, t_n] \times A_n) &= \lim_{n \to \infty} N(t_n, A_n) - N(s_n, A_n) \\ &= N(t,A) - N(s,A) = \eta((s,t] \times A), \end{aligned}$$

where $t_n \uparrow t$, $s_n \downarrow s$, $A_n \uparrow A$, $(s_n, t_n] \times A_n \uparrow (s,t] \times A \in \mathcal{A}_0 \times \mathcal{E}$. Next, fix $A \in \mathcal{E}$, by definition $N(\cdot, A)$ is an increasing adapted process then follows directly $\eta((u,v] \times A)$ is $\mathcal{F}_t$-measurable for any $t \geqslant 0$ and $(u,v] \subset (0,t]$. That is, η is an adapted[2] premeasure. Moreover, it is optional because $N(\cdot, A)$ is $\mathcal{O}$-measurable for each $A \in \mathcal{E}$. Lastly, it remains to show that η is σ-finite.

2 A random measure ζ is said to be "adapted" if $\zeta(\cdot, B)$ is $\mathcal{F}_t$-measurable for any $B \subset [0,t] \times E$ for all $t \geqslant 0$.

By definition, for each ω, t, the premeasure $N(\omega; t, \cdot)$ is σ-finite then there exists a sequence A_n increasing to E such that $\mathbb{E}\left[|N(\omega; t, A_n)|\right] < \infty$. That implies that $\mathbb{E}\left[|\eta(\omega; (0,t] \times A_n)|\right] < \infty$ and thus by Definition A.2.4 we get η is σ-finite premeasure.

Conversely, let ν be an optional and σ-finite random premeasure on $(\mathbb{R}_+ \times E, \mathcal{A}_0 \times \mathcal{E})$ with $\nu(\{0\} \times A) = 0$ for any $A \in \mathcal{E}$. Define a random field N as follows

$$N(t, A) := \nu((0,t] \times A), \quad \text{for all } t \in \mathbb{R}_+, A \in \mathcal{E}. \tag{1.5}$$

We want to show that $N \in \mathcal{V}^+_{\mathcal{E}}$. Fix $A \in \mathcal{E}$, we have from its definition that $N(\cdot, A)$ is an increasing process and $N(0, A) = 0$. Moreover, by construction it follows that $N(\cdot, A)$ is adapted for each $A \in \mathcal{E}$. Next, by the countable additivity of ν, if $\{t_n\}$ is a sequence in $\mathbb{R}_+$ such that $t_n \uparrow t \in \mathbb{R}_+$ and $(0, t_n] \times A \in \mathcal{A}_0 \times \mathcal{E}$ then we obtain

$$\lim_{n\to\infty} N(t_n, A) = \lim_{n\to\infty} \nu((0,t_n] \times A) = N(t, A) \quad \text{and} \lim_{t\to 0} N(t, A) = 0,$$

that is, $N(\cdot, A)$ is càdlàg process and belongs to $\mathcal{V}^+$. One the other hand, fix $(\omega, t) \in \Omega \times \mathbb{R}_+$, again by the σ-additivity of ν if $\{A_n\}$ is a sequence in $\mathcal{E}$ such that $A_n \uparrow A \in \mathcal{E}$ and $(0,t] \times A_n \uparrow (0,t] \times A \in \mathcal{A}_0 \times \mathcal{E}$. Hence

$$\lim_{n\to\infty} N(t, A_n) = \lim_{n\to\infty} \nu((0,t] \times A_n) = \nu((0,t] \times A) = N(t, A),$$

with $N(t, \emptyset) = 0$. This means that the mapping $A \mapsto N(\omega; t, A)$ is a σ-finite premeasure on $(E, \mathcal{E})$, in particular N defines an increasing field on $\mathbb{R}_+ \times \mathcal{E}$.

At the end, we get the uniqueness due to the fact that one can define

$$\nu((s,t] \times A) = N(t, A) - N(s, A),$$

whenever $(s,t] \times A \in \mathcal{A}_0 \times \mathcal{E}$. If N satisfies equations (1.4) and (1.5) then it follows directly $\eta = \nu$.

(b). Since $\mathcal{B}(\mathbb{R}_+) \otimes \Sigma = \sigma(\mathcal{A}_0 \times \mathcal{E})$ (see Lemma A.1.4), then by Lemma A.1.2, we can uniquely extend the random σ-finite premeasure η to a random σ-finite measure, denoted by ζ, on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \Sigma)$ such that

$$\zeta((s,t] \times A) = N(t, A) - N(s, A), \quad \text{for any } (s,t] \times A \in \mathcal{A}_0 \times \mathcal{E}.$$

This completes the proof. ∎

As result, we can now define the Bochner integral w.r.t an increasing field that we denote by $f \cdot N$ or $\int_0^{\bullet}\int_E f(s,x)N(ds,dx)$ for some H-valued functions f.

Definition 1.2.3 For each $(\omega,t) \in \Omega \times \mathbb{R}_+$ and for every H-valued optional process f such that

$$\int_0^t \int_E \|f(\omega,s,x)\| V_N(\omega;ds,dx) < \infty, \tag{1.6}$$

we define the integral process $f \cdot N_t$ as

$$f \cdot N_t(\omega) := \int_0^t \int_E f(\omega,s,x)N(\omega,ds,dx) = \int_0^t \int_E f(\omega,s,x)\zeta(\omega,ds,dx), \tag{1.7}$$

and if f is an elementary function of the form

$$f = f_a \mathbb{1}_{(a,b]\times A}, \quad (a,b] \times A \subset \mathbb{R}_+ \times E,$$

with f_a is an H-valued bounded and $\mathcal{F}_a$-measurable random variable, then

$$f \cdot N_t = f_a \left(N(t \wedge b, A) - N(t \wedge a, A) \right), \quad t \geqslant 0.$$

Next, we give some properties of the integral process $f \cdot N$.

Proposition 1.2.1 *Let* N *be an increasing field and* h *be a* H*-valued optional process satisfying condition (1.6). Then the following properties are satisfied:*

i. $(f \cdot N(t))_{t\geqslant 0}$ *is a* H*-valued, adapted and càdlàg process with finite variation path, namely, for* $\omega \in \Omega$

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \left\| f \cdot N(\omega, tk/n) - f \cdot N(\omega, t(k-1)/n) \right\| < \infty, \quad \textit{for all } t \geqslant 0.$$

ii. $\Delta(f \cdot N) = f \cdot \Delta N$.

iii. If f *and* N *are predictable, then* $f \cdot N$ *is predictable.*

iv. The mapping $h \mapsto f \cdot N$ *is a linear.*

v. $(f \cdot X)^{\tau} = \left(f \mathbb{1}_{[\![0,\tau]\!]}\right) \cdot X$*, for each stopping time* τ *with* $\mathbb{P}(\tau \leqslant T) = 1$.

Proof **(i).** Despite the fact that f is an optional function and $N(\cdot, A) \in \mathcal{V}^+$ for any $A \in \mathcal{E}$. It follows if $f \cdot N$ is well-defined then it is also càdlàg with

$f \cdot N(0) = 0$ almost surely. On the other hand, fix $t \geqslant 0$, the set function $\gamma(\omega; ds \times dx) := N(\omega; ds, dx)\mathbb{1}_{\{s \leqslant t\}}$ for each ω defines a premeasure on $[0, t] \times E$, such that $\gamma(\cdot; A \times B)$ is $\mathcal{F}_t$-measurable for each Cartesian product $A \times B \subset [0, t] \times E$. Moreover, $f(\omega, s, x)$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t]) \otimes \Sigma$-measurable on $\Omega \times [0, t] \times E$, and thus by Fubini's Theorem for transition measures we deduce that $f \cdot N_t$ is $\mathcal{F}_t$-measurable for any $t \geqslant 0$.

Last, the finite variation path comes from relations (1.1) and (1.6) where

$$\sum_{k=1}^{n} \int_{t\frac{k-1}{n}}^{t\frac{k}{n}} \int_E \|f(\omega, s, x)\| N(\omega; ds, dx) \leqslant \int_0^t \int_E \|f(\omega, s, x)\| V_N(\omega; ds, dx),$$

for all $n \in \mathbb{N}$, and which leads to

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{n} \int_{t\frac{k-1}{n}}^{t\frac{k}{n}} \int_E \|f(\omega, s, x)\| N(\omega; ds, dx) < \infty.$$

(ii). To prove $\Delta(f \cdot N) = f \cdot \Delta N$, we first consider f is an elementary function of the form

$$f = f_0 \mathbb{1}_{[0,a] \times A}, \quad [0, a] \times A \subset \mathbb{R}_+ \times E,$$

where f_0 is a H-valued bounded and $\mathcal{F}_0$-measurable. Then it follows

$$f \cdot N_t = f_0 N(t \wedge a, A)$$

with f_0 is an H-valued bounded, $\mathcal{F}_0$-measurable random variable and we compute

$$\Delta(f \cdot N)_t = f \cdot N_t - f \cdot N_{t^-} = f_0[N(t \wedge a, A) - N(t^- \wedge a, A)] = f \cdot \Delta N_t.$$

Thus, the result is also true if f is a simple function. To complete the proof, we use the limit approximation argument. More precisely, let f be an optional function satisfying (1.6) and let $\{f_n\}$ be a sequence of simple functions such that $\Delta(f_n \cdot N) = f_n \cdot \Delta N$ and $\|f_n - f\| \to 0$ as $n \to \infty$. Noting that if $\|f_n - f\| \to 0$ then follows $f_n \cdot N \to f \cdot N$ as, by Definition 1.2.3, we have

$$\left\| \int_0^t \int_E [f_n(s, x) - f(s, x)] N(ds, dx) \right\| \leqslant \int_0^t \int_E \|f_n(s, x) - f(s, x)\| V_N(ds, dx).$$

Therefore, we deduce that

$$\Delta(f \cdot N) = \Delta(\lim_{n \to \infty} f_n \cdot N) = \lim_{n \to \infty} \Delta(f_n \cdot N) = \lim_{n \to \infty} f_n \cdot \Delta N = f \cdot \Delta N.$$

(iii). By [47, Proposition 2.6, p.17] and (i), $f \cdot N_-$ is predictable as $f \cdot N$ is an H-valued càdlàg and adapted process. Similarly, by (ii), we also have $\Delta(f \cdot N) = f \cdot \Delta N$ is predictable as f and N are predictable. Therefore, we deduce that $f \cdot N = f \cdot N_- + \Delta(f \cdot N)$ is predictable.

(iv). Let f, g be H-valued optional processes satisfying respectively condition (1.6) and c be a real constant. Since $f + cg$ is also optional, so it holds that

$$\int_0^t \int_E \|f(s,x) + cg(s,x)\| V_N(ds, dx) < \infty,$$

as $\|f(s,x) + cg(s,x)\| \leqslant \|f(s,x)\| + |c|\|g(s,x)\|$ for $(s,x) \in \mathbb{R}_+ \times E$. Then the integral process of $f + cg$ w.r.t N is well-defined and we have

$$\int_0^t \int_E [f + cg](s,x) N(ds, dx) = \int_0^t \int_E f(s,x) N(ds, dx) + c \int_0^t \int_E g(s,x) N(ds, dx).$$

(v). Let $t \in [0, T]$, $\omega \in \Omega$ be arbitrary. If f is an elementary function of the form

$$f = f_0 \mathbb{1}_{[0,a] \times A}, \quad [0,a] \times A \subset \mathbb{R}_+ \times E.$$

then, ω-by-ω, we have

$$\begin{aligned} (f \cdot N)_t^\tau(\omega) &= \int_0^{t \wedge \tau(\omega)} \int_E f(\omega, s, x) N(\omega; ds, dx) \\ &= f_0(\omega) \left[N(\omega; t \wedge T(\omega) \wedge a, A) - N(\omega; 0, A) \right], \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} (f \mathbb{1}_{[\![0,\tau]\!]}) \cdot N_t(\omega) &= \int_0^t \int_E f(\omega, s, x) \mathbb{1}_{[\![0,\tau]\!]}(\omega, s) N(\omega; ds, dx) \\ &= \int_0^t \int_E f_0(\omega) \mathbb{1}_A(x) \mathbb{1}_{[\![0,\tau \wedge a]\!]}(\omega, s) N(\omega; ds, dx) \\ &= f_0(\omega) \left[N(\omega; t \wedge T(\omega) \wedge a, A) - N(\omega; 0, A) \right], \end{aligned} \tag{1.9}$$

This implies that $(f \cdot N)^\tau = (f \mathbb{1}_{[\![0,\tau]\!]}) \cdot N$ for f an elementary function. Then by the limit approximation (as in **(ii)**), we obtain

$$(f \cdot N)^\tau = \lim_{n \to \infty} (f_n \cdot N)^\tau = \lim_{n \to \infty} (f_n \mathbb{1}_{[\![0,\tau]\!]}) \cdot N = (f \mathbb{1}_{[\![0,\tau]\!]}) \cdot N.$$

This completes the proof ∎

Remark 3 *For any* $a \in H$ *such that* $\langle f, a\rangle_H \geqslant 0$, $\langle f \cdot N, a\rangle_H$ *belongs to* $\mathcal{V}^+$ *and* $\langle f \cdot N, a\rangle_H(t) := (\langle f, a\rangle_H \cdot N)_t = \langle f, a\rangle_H \cdot N_t$ *is the real-valued Bochner integral. Indeed, we compute that*

$$\begin{aligned}\langle f \cdot N, a\rangle_H(t) &= \int_0^t \int_E (f(\omega, s, x)N(\omega, ds, dx), a)_H \\ &= \int_0^t \int_E \langle f, a\rangle_H(\omega, s, x)N(\omega, ds, dx) = \langle f, a\rangle_H \cdot N(t).\end{aligned}$$

By Proposition 1.2.1, we obtain that $\langle f \cdot N, a\rangle_H$ *is càdlàg and adapted with* $\langle f \cdot N, a\rangle_H(\omega; 0) = 0$ *for* $\omega \in \Omega$. *Since the process* $\langle f, a\rangle_H$ *is optional with* $\langle f, a\rangle_H \geqslant 0$ *and* N *is an increasing field, thus it follows that* $\langle f \cdot N, a\rangle_H$ *still remains an increasing process. This implies that* $\langle f \cdot N, a\rangle_H \in \mathcal{V}^+$.

2. At end, we now extend this definition of the Bochner integral to the class of FV fields. Let $N \in \mathcal{V}_\mathcal{E}$ such that we can uniquely write $N = N^+ - N^-$ (by Lemma 1.2.1). We next give the definition of the Bochner integral $\int_0^\bullet \int_E f(s, x)N(ds, dx)$.

Definition 1.2.4 Let f be H-valued optional process and N is a finite variation field such that

$$\int_0^t \int_E \|f(\omega, s, x)\| V_N(\omega, ds, dx) < \infty.$$

Thus, for all $(\omega, t) \in \Omega \times \mathbb{R}_+$, the Bochner integral $f \cdot N$ exists and is defined by

$$f \cdot N(\omega; t) := \int_0^t \int_E f(\omega, s, x)N^+(\omega; ds, dx) - \int_0^t \int_E f(\omega, s, x)N^-(\omega; ds, dx), \tag{1.10}$$

where both integrals in right side are defined as in Definition 1.2.3.

Similarly to the increasing field, all results in Proposition 1.2.1 can be extended to the class of finite variation fields. In other words, the integral process $f \cdot N$ is a H-valued, adapted and càdlàg process with finite variation path and satisfies all properties in Proposition 1.2.1.

Remark 4 *For any arbitrary* $p \geqslant 1$. *For* $T > 0$, *we define* $L^p_T(N; H)$ *as the space of all predictable processes* $f : \Omega \times \mathbb{R}_+ \times H \to H$ *such that*

$$\mathbb{E}\left[\int_0^T \int_E \|f(s, x)\|^p V_N(ds, dx)\right] < \infty.$$

Furthermore, we extend the space $L^p_T(N;H)$ *to* $\mathcal{L}^p_{loc}(N;H)$ *the space of all predictable processes* $f : \Omega \times \mathbb{R}_+ \times H \to H$ *which satisfies*

$$\mathbb{P}\left(\int_0^T \int_E \|f(s,x)\|^p V_N(ds,dx) < \infty\right) = 1, \quad \textit{for all } T > 0.$$

Note that this can be performed by the so-called ***localization procedure****, (see Section 1.3.4), and the stopping time result in Proposition 1.2.1.*

Proposition 1.2.2 *Let* $a : \Omega \times \mathbb{R}_+ \to H$ *be an optional and* λ*-integrable mapping. Then there exists a Borel space* (E, Σ) *and a function* $f \in \mathcal{L}^p_{loc}(N;H)$ *so that*

$$\int_0^t a(s)\, ds = \int_0^t \int_{\mathbb{E}} f(s,x) N(ds,dx), \quad \textit{for all } t \in \mathbb{R}_+.$$

Proof For the existence of space E, we take (E, Σ) as any arbitrary Blackwell space such that $E \neq \emptyset$. Let η be a finite measure on (E, Σ). For $t \in [0, T]$, $x \in E$ and $A \in \Sigma$, we define

$$f(t,x) := a(t) \quad \text{and} \quad N(t,A) := \lambda([0,t])\frac{\eta(A)}{\eta(E)},$$

Then N is a finite variation field by construction. The Bochner integral $f \cdot N$ is well-defined and it holds

$$\int_0^t \int_{\mathbb{E}} f(s,x) N(ds,dx) = \int_0^t \int_{\mathbb{E}} a(s)\, ds \frac{\eta(dx)}{\eta(E)} = \int_0^t a(s)\, ds.$$

∎

1.3 MARTINGALE FIELDS

In this section, we describe the class of martingale fields and then we perform the associated integration theory.

1.3.1 *Definitions and Properties*

Let M is a random field on $\mathbb{R}_+ \times \mathcal{E}$ (or $\mathbb{R}_+ \times \Sigma$) such that $(M(t,A))_{t \geqslant 0}$ is a locally square-integrable martingale process for all $A \in \mathcal{E}$ (resp. Σ). Let us introduce the concept of quadratic variation or bracket process associated to

$M(\cdot, A)$. Fix $A \in \mathcal{E}$ (resp. Σ), by the standard argument (see [47], Theorem 4.2), there exists an unique predictable process $\langle M\rangle(t, A) := \langle M(\cdot, A)\rangle_t$ with $\langle M\rangle(0, A) = 0$ a.s that makes $M(\cdot, A)^2 - \langle M(\cdot, A)\rangle$ a local martingale.

Remark 5 *Later on, we perform the stochastic integration w.r.t martingale fields by following the Itô construction, i.e. we use the Itô isometry to construct the stochastic integrals and then we formulate the integrability condition in terms of the quadratic variation. As $\mathcal{E}$ is not a σ-algebra, so again we shall need to extend the premeasure induced by the increasing random field $A \mapsto \langle M\rangle(\cdot, A)$ to a measure on (E, Σ) (see Remark 1).*

Before we start, it is important to point out that there are two different types of martingale fields, namely, cylindrical martingale field and true-martingale field. The difference between both fields lies on the space where they are defined.

Definition 1.3.1 a) A cylindrical martingale field is a random field M on $\mathbb{R}_+ \times \mathcal{E}$ which satisfies

1. For each $A \in \mathcal{E}$, the stochastic process $(M(t, A))_{t \geqslant 0}$ belongs to $\mathcal{H}^2_{loc}$.
2. there is a predictable non-decreasing field B on $\mathbb{R}_+ \times \mathcal{E}$ such that $\langle M\rangle(\cdot, A) = B(\cdot, A)$, up to an evanescence set, for all $A \in \mathcal{E}$.

Then $\langle M\rangle(\cdot, A)$ is called the quadratic variation process of $M(\cdot, A)$ for each $A \in \mathcal{E}$, while the covariance functional of $M(t, \cdot)$ is a premeasure on $(E, \mathcal{E})$ with

$$\langle M\rangle(t, A \cap B) := \langle M(\cdot, A), M(\cdot, B)\rangle_t,$$

for any fixed $t \geqslant 0$ and for all $A, B \in \mathcal{E}$. By polarization, we define the co-variance process of two martingale fields M, N by the following process:

$$\langle M, N\rangle(t, A) = \frac{1}{4}\left[\langle M + N\rangle(t, A) - \langle M - N\rangle(t, A)\right], \quad \text{for any } A \in \mathcal{E}, t \geqslant 0.$$

Denote by $\mathcal{M}_{\mathcal{E}}$ the class of cylindrical martingale fields defined on $\mathbb{R}_+ \times \mathcal{E}$.

b) A true martingale field is a cylindrical martingale field on $\mathbb{R}_+ \times \Sigma$, i.e. $\mathcal{E} = \Sigma$, and the mapping $A \mapsto B(\cdot, A)$ is a measure on (E, Σ). We denote by $\mathcal{M}_\Sigma$ the class of true martingale fields.

It is clear that $\mathcal{M}_\Sigma \subset \mathcal{M}_\mathcal{E}$. In the sequel, without lose of generality, for simplicity we use the name "martingale field" to refer a cylindrical martingale field.

Proposition 1.3.1 *For every $M \in \mathcal{M}_\mathcal{E}$, the field B is unique up to a $\mathbb{P}$-null set.*

Proof Fix $A \in \mathcal{E}$ and let B and B' two predictable increasing fields on $\mathbb{R}_+ \times \mathcal{E}$ such that respectively $\langle M\rangle(\cdot, A) = B(\cdot, A)$ and $\langle M\rangle(\cdot, A) = B'(\cdot, A)$ up to evanescent sets (denote respectively by $\mathcal{N}_A$ and $\mathcal{N}_A^{'}$). By Theorem 4.2 [47, p38-39], for all the bracket process $\langle M\rangle(\cdot, A)$ is unique up to an evanescent set (denote by $\mathcal{N}_A^{''}$). This implies that both processes $B(\cdot, A)$ and $B'(\cdot, A)$ coincide in the sense that

$$\left\{\omega \in \Omega : \; B(\omega; t, A) \neq B'(\omega; t, A) \text{ for some } t \in \mathbb{R}_+\right\} = \mathcal{N}_A \cup \mathcal{N}_A^{'} \cup \mathcal{N}_A^{''}$$

is a $\mathbb{P}$-null set.

It follows from the countability of $\mathcal{E}$ that the set, defined as

$$\mathcal{N} = \bigcup_{A \in \mathcal{E}} \left\{\omega \in \Omega : B(\omega; t, A) \neq B'(\omega; t, A) \text{ for some } t \in \mathbb{R}_+\right\}, \quad (1.11)$$

is also a $\mathbb{P}$-null set. This completes the proof. ∎

Let us introduce new definitions.

Definition 1.3.2 Let $M, N \in \mathcal{M}_\mathcal{E}$. All equalities are up to evanescence.

a) M and N are called orthogonal if, for each $A \in \mathcal{E}$, their product $M(\cdot, A)N(\cdot, A)$ is a local martingale or equivalently $\langle M, N\rangle(t, A) = 0$.

b) M is orthogonal to itself if, for any two disjoint sets A and B in $\mathcal{E}$, $\langle M\rangle(t, A \cap B) = \langle M(t, A), M(t, B)\rangle = 0$.

c) A martingale field M is called a purely discontinuous martingale field if $M(0, \cdot) = 0$ and $\langle M, N\rangle(\cdot, A) = 0$, for any continuous martingale fields N and $A \in \mathcal{E}$.

Lemma 1.3.1 *Suppose that M has independent and stationary increments. Then, for $A \in \mathcal{E}$ fixed, the quadratic variation process $\langle M\rangle(\cdot, A)$ is deterministic up to evanescent set, i.e. $\langle M\rangle(\cdot, A)$ is uniquely determined by*

$$\langle M\rangle(t, A) = \mathbb{E}\left[I_M((0, t] \times A)^2\right], \quad \textit{for } t \geqslant 0.$$

Proof Fix $A \in \mathcal{E}$. Since $M(\cdot, A) \in \mathcal{H}^2_{loc}$, we have

$$\mathbb{E}\left[M(t,A)^2 - M(0,A)^2\right] = \mathbb{E}\left[I_M((0,t] \times A)^2\right].$$

As $M(\cdot, A)^2 - \langle M \rangle(\cdot, A)$ is a martingale, then it follows

$$\mathbb{E}\left[I_M((s,t] \times A)^2 | \mathcal{F}_s\right] = \mathbb{E}\left[\langle M \rangle(t,A) - \langle M \rangle(s,A) | \mathcal{F}_s\right], \quad \text{for any } 0 \leqslant s < t.$$

If M has independent increments, then we obtain

$$\mathbb{E}\left[I_M((s,t] \times A)^2\right] = \mathbb{E}\left[\langle M \rangle(t,A) - \langle M \rangle(s,A) | \mathcal{F}_s\right],$$

this means that $\langle M \rangle(\cdot, A)$ is deterministic and it holds , by the tower property of conditional expectation, that

$$\langle M \rangle(t,A) = \mathbb{E}\left[I_M((0,t] \times A)^2\right] = \mathbb{E}\left[M(t,A)^2 - M(0,A)^2\right].$$

∎

Since we are more interested in martingale fields having independent and stationary increments. Then we assume all martingale fields, that shall be used throughout the thesis, have independent and stationary increments.

1.3.2 *Fundamental Examples*

We next provide divers example of martingale fields.

Definition 1.3.3 An (extended) Poisson field on $\mathbb{R}_+ \times \mathcal{E}$, relative to the Filtration $\mathbb{F}$, is an integer-valued martingale field M which satisfies

1. $A \mapsto m(\cdot, A) = \mathbb{E}[M(\cdot, A)]$ is σ-finite premeasure on $(E, \mathcal{E})$.
2. $I_M((s,t] \times A)$ is independent of $\mathcal{F}_s$, for any $(s,t] \times A \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{E}$.

The premeasure m is called the intensity of M.

Example 3 (Compensated Poisson field) *Let* (E, Σ) *be a Blackwell space such that* $\Sigma = \sigma(\mathcal{G})$ *with* $\mathcal{G}$ *is a countable semi-ring. Let* F *be a positive* σ*-finite measure on* (E, Σ)*, namely, there is* $(A_n)_{n \in \mathbb{N}}$ *be an increasing sequence of sets in* Σ *such* *that* $E = \bigcup_{n\in\mathbb{N}} A_n$ *and* $F(A_n) < \infty$ *for all* $n \in \mathbb{N}$. *Construct the countable semi-ring:*

$$\mathcal{E} := \bigcup_{n\in\mathbb{N}} \{B \cap A_n : B \in \mathcal{G}\}.$$

Let μ *be a homogeneous Poisson random measure on* $\mathbb{R}_+ \times E$ *with intensity measure* $\nu(dt \times dx) = dt \otimes F(dx)$. *We define a Poisson field on* $\mathbb{R}_+ \times \mathcal{E}$ *as*

$$M_\mu(t, A) = \int_0^t \int_E \mathbb{1}_A(x) \left[\mu(ds \times dx) - dsF(dx)\right], \quad \textit{for all } A \in \mathcal{E}, t \geqslant 0. \tag{1.12}$$

Note that the integral process in (1.12) is well-defined as it is a stochastic integral of the form $\mathbb{1}_A * (\mu - \nu)$ *(see [47, p.71]).*

To prove the above claim, we make the following two steps:

(a). Fix $A \in \mathcal{E}$. *The compensated random measure* $\mu - \nu$ *is by definition an integer-valued random measure, so* M_μ *also takes its values in* $\overline{\mathbb{N}}$. *We also have* $\mu - \nu$ *is integrable (see [47, I.1.6]) and* $\mathbb{1}_A * (\mu - \nu)$ *is a local martingale as* $\mathbb{1}_A \in G_{loc}(\mu)$. *Define* $C(\mathbb{1}_A)$ *as in [47, I.1.31] and compute:*

$$C(\mathbb{1}_A)_t = \mathbb{1}_A^2 * \nu = \int_0^t \int_E \mathbb{1}_A(x)^2 \, dsF(dx) = tF(A) < \infty.$$

Therefore, we have $C(\mathbb{1}_A) \in \mathcal{A}^+$ *and then by [47, I.1.33.(b)] we obtain* $\mathbb{1}_A * (\mu - \nu) \in \mathcal{H}^2$ *and* $\langle M_\mu \rangle(t, A) = \langle \mathbb{1}_A * (\mu - \nu), \mathbb{1}_A * (\mu - \nu) \rangle_t = C(\mathbb{1}_A)_t$. *Namely, the process* $M_\mu(\cdot, A) \in \mathcal{H}^2$ *and there is a non-decreasing field B such that* $B(t, A) := tF(A)$. *This shows that* M_μ *is a martingale field[3] on* $\mathbb{R}_+ \times \mathcal{E}$.

(b). Next, we prove that (1) and (2) in Definition (1.3.3) are fulfilled. Since F *be a* σ*-finite premeasure on* $(E, \mathcal{E})$ *and by the integrability of* $\mu - \nu$, *so we obtain* $A \mapsto \mathbb{E}[M_\mu(t, A)] = \mathbb{E}[\mu((0, t] \times A) - tF(A)]$ *is a premeasure on* $(E, \mathcal{E})$. *Moreover, as* $\mu(\cdot; (s, t] \times A)$ *is independent of* $\mathcal{F}_s$, *for any* $s \geqslant 0$ *and* $(s, t] \times A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$ *with* $(s, t] \times A \subset \mathbb{R}_+ \times E$, *so also is* $I_{M_\mu}((s, t] \times A)$ *independent of* $\mathcal{F}_s$. *This means that* M_μ *is a Poisson field.*

There is very little to say about discrete-space martingale fields. We consider the case where $E \subseteq \mathbb{N}$ and $\Sigma = \mathscr{P}(E)$.

3 If F is finite measure then M_μ becomes a true martingale field as we can take $\mathcal{E} = \Sigma$.

Lemma 1.3.2 *Let* $I \subset \mathbb{N}$ *finite and* $(M_i)_{i \in I}$ *be a family of local square integrable martingale processes with* $\langle M_i, M_j \rangle_t = 0$ *for* $i \neq j$. *Then the field defined as*

$$M(t, A) = \sum_{i \in A} M_i(t), \quad \textit{for } A \subset \mathbb{N} \textit{ finite}, t \in [0, T],$$

is a martingale field on $\mathbb{R}_+ \times \mathcal{E}$ *where* $\mathcal{E}$ *is a collection of all finite subsets of* $\mathbb{N}$.

Proof First, consider $\mathcal{E} = \{A \subset \mathbb{N} : A \text{ is a finite set}\}$. Fix $A \in \mathcal{E}$, by construction, $M(\cdot, A)$ defines a square-integrable local martingale process. Moreover, we obtain that there is a non-decreasing field B on $\mathbb{N} \times \mathcal{E}$ such that

$$B(\cdot, A) := \langle M \rangle(\cdot, A) = \sum_{i \in A} \langle M_i \rangle + \sum_{\substack{i,j \in A \\ i \neq j}} \langle M_i, M_j \rangle = \sum_{i \in A} \langle M_i \rangle.$$

It is clear that $B(\cdot, \emptyset) = 0$. Next we shall show that $A \mapsto B(\cdot, A)$ is a premeasure on $(\mathbb{N}, \mathcal{E})$. Indeed, if $(A_n)_{n \in \mathbb{N}}$ a sequence of disjoint sets in $\mathcal{E}$ with $\cup_{n \in \mathbb{N}} A_n \in \mathcal{E}$ (i.e. there is $n_0 \in \mathbb{N}$ such that $A_n = \emptyset$, for any $n \geqslant n_0$). Then, we have

$$B(\cdot, \cup_{n \in \mathbb{N}} A_n) = \sum_{i \in \cup_{n \in \mathbb{N}} A_n} \langle M_i \rangle = \sum_{i \in A_1} \langle M_i \rangle + \cdots + \sum_{i \in A_{n_0}} \langle M_i \rangle = \sum_{n=1}^{n_0} B(\cdot, A_n),$$

this implies the σ-additivity of $A \mapsto B(\cdot, A)$ as

$$\sum_{n=1}^{n_0} B(\cdot, A_n) = \sum_{n \in \mathbb{N}} B(\cdot, A_n).$$

This completes the proof. ∎

Definition 1.3.4 A Gaussian martingale field is a martingale field G on $\mathbb{R}_+ \times \mathcal{E}$ such that:

a) $G(0, \cdot) = 0$ up to evanescence;

b) for each $A \in \mathcal{E}$, $G(\cdot, A)$ is a Gaussian process[4].

Example 4 (Finite dimensional Wiener processes) *Let* d *be a positive integer number with* $d \geqslant 1$ *and* $W = (W_1, \ldots, W_d)$ *be a* d*-dimensional Wiener*

4 That is the distribution of any finite family, $(G(t_1, A), \cdots, G(t_n, A))$ is Gaussian.

process. Consider $E = \{1, \dots, d\}$ *and* $\Sigma = \mathscr{P}(E)$*. Then, by Lemma 1.3.2, we can construct a true Gaussian field* [5] G *associated to* W *as follows*

$$G(t, A) = \sum_{i \in A} W_i(t), \quad \textit{for } A \in \Sigma \textit{ and } t \in \mathbb{R}_+,$$

with the quadratic variation

$$\langle G \rangle (t, A) = \sum_{i \in A} \langle W_i \rangle_t = t\#(A), \quad \textit{for } A \in \Sigma \textit{ and } t \in \mathbb{R}_+.$$

Example 5 (Cylindrical Wiener processes) *Let W be a cylindrical Wiener process on* U *on some separable Hilbert space* U*, and* $\{e_i\}_{i \in \mathbb{N}}$ *be an orthonormal basis of* U*. Namely, the sequence* $(W(e_i))_{i \in \mathbb{N}}$ *is a sequence of independent standard Wiener process.*

By taking $E = \mathbb{N}$ *with* $\Sigma = \mathscr{P}(\mathbb{N})$*, and setting* $\mathcal{E} = \{A \subset \mathbb{N} : A \textit{ is a finite set}\}$*. Then we can associate a martingale field* M *to the family* $(W(e_i))_{i \in \mathbb{N}}$ *by the following way:*

$$M(t, A) = \sum_{i \in A} W_t(e_i), \quad \textit{for all } t \in \mathbb{R}_+,\ A \in \mathcal{E}. \tag{1.13}$$

Moreover, M *is a Gaussian martingale field with*

$$\langle M \rangle (t, A) = \sum_{i \in A} \langle W(e_i) \rangle_t = t\#(A), \quad \textit{for } A \in \mathcal{E} \textit{ and } t \in \mathbb{R}_+.$$

Indeed, it follows from Lemma 1.3.2 that M *is a Gaussian martingale field on* $\mathbb{N} \times \mathcal{E}$ *as* $(W(e_i))_{i \in A}$ *is a family of Gaussian martingale processes.*

Last, we provide the definition of Lévy random field.

Definition 1.3.5 A martingale field L on $\mathbb{R}_+ \times \mathcal{E}$ is said to be a Lévy field if the following conditions are fulfilled:

a) $L(0, \cdot) = 0$;

b) L has independent and stationary increments.

Example 6 *Let* $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ *be a sequence with* $\sum_{n \in \mathbb{N}} \lambda_n < \infty$ *and let* $(l^2_\lambda, \langle \cdot, \cdot \rangle_{l^2_\lambda})$ *be a Hilbert space consisting of all weighted sequences*

$$l^2_\lambda := \left\{ (v^n)_{n \in \mathbb{N}} \subset \mathbb{R} : \sum_{n \in \mathbb{N}} \lambda_n |v^n|^2 < \infty \right\} \quad \textit{and} \quad \langle u, v \rangle_{l^2_\lambda} = \sum_{n \in \mathbb{N}} \lambda_n u^n v^n.$$

5 The Gaussian field G is characterised by: $G(0, A) = 0$; G has independent and stationary increments; $G(t, A) \sim \mathcal{N}(0, t\#A)$.

Let $Q \in L(l^2_\lambda)$ *be a self-adjoint, positive definite trace class operator. Let* $\{e^\lambda_n\}_{n\in\mathbb{N}}$ *be an orthonormal basis of* l^2_λ *that consists of eigenvectors of* Q *with corresponding eigenvalues* $(\lambda_n)_{n\in\mathbb{N}}$ *such that*

$$Qe^\lambda_n = \lambda_n e^\lambda_n, \quad \textit{for all } n \in \mathbb{N}.$$

Let X *be an* l^2_λ*-valued square integrable Lévy martingale with covariance operator* Q *(see Tappe [92]). Namely,* X *is a Lévy martingale with* $\mathbb{E}\big[\|X(t)\|^2_{l^2_\lambda}\big] < \infty$ *and* $\mathbb{E}\big[X(t)\big] = 0$*, and we have*

$$\mathbb{E}\left[\langle X(s), x\rangle_{l^2_\lambda}\langle X(t), y\rangle_{l^2_\lambda}\right] = (s \wedge t)\langle Qx, y\rangle_{l^2_\lambda}, \quad x, y \in l^2_\lambda, t, s \in \mathbb{R}_+.$$

According to [92, Proposition 5.4], $\langle X, e^\lambda_n\rangle_{l^2_\lambda}$ *is a sequence of standard Lévy process for every* $n \in \mathbb{N}$*. That is, it consists of square-integrable martingales with* $\langle\langle X, e^\lambda_i\rangle_{l^2_\lambda}, \langle X, e^\lambda_j\rangle_{l^2_\lambda}\rangle_t = \delta_{ij} \cdot t$ *for all* $i, j \in \mathbb{N}, t \geqslant 0$*. Here* δ_{ij} *denotes the Kronecker delta.*

Again let $E = \mathbb{N}$ *with* $\Sigma = \mathscr{P}(E)$*. In the following, we derive two different examples of martingale fields from the process* X*.*

a) Set $\mathcal{E} = \{A \subset E : A \text{ is finite}\}$ *and define*

$$L_1(t, A) = \sum_{n\in A} \frac{1}{\sqrt{\lambda_n}}\langle X(t), e^\lambda_n\rangle_{l^2_\lambda}, \quad \textit{for } A \in \mathcal{E}, t \in \mathbb{R}_+. \tag{1.14}$$

Note that L_1 *is well-defined and is a Lévy (cylindrical) martingale field on* $\mathbb{R}_+ \times \mathcal{E}$*.*

Indeed, since $\{\langle X, e^\lambda_n\rangle_{l^2_\lambda}\}_{n\in\mathbb{N}}$ *defines a standard Lévy process, then the sum in (1.14) converges for all* $A \in \mathcal{E}$*. Moreover, it holds:*

1. *as* $\langle X, e^\lambda_n\rangle_{l^2_\lambda}$ *has independent and stationary increments for any* $n \in \mathbb{N}$*, so also is the field* L_1 *given in equation (1.14) and* $L_1(0, E) = 0$*;*

2. *by definition,* $\langle X, e^\lambda_n\rangle_{l^2_\lambda} \in \mathcal{H}^2_{loc}$ *for each* $n \in \mathbb{N}$*, then any finite sum*

$$\sum_{n\in A} \frac{1}{\sqrt{\lambda_n}}\langle X, e^\lambda_n\rangle_{l^2_\lambda}, \quad \textit{for any fixed } A \in \mathcal{E},$$

   belongs to $\mathcal{H}^2_{loc}$*.*

3. *Fix* $t \geqslant 0$ *and* $A \in \mathcal{E}$. *Since* $\langle\langle X, e_i^\lambda\rangle_{l^2_\lambda}, \langle X, e_j^\lambda\rangle_{l^2_\lambda}\rangle_t = 0$ *for* $i \neq j$, *then we compute*

$$\begin{aligned}\langle L_1\rangle(t,A) &= \sum_{n\in A}\frac{1}{\lambda_n}\langle\langle X, e_n^\lambda\rangle_{l^2_\lambda}, \langle X, e_n^\lambda\rangle_{l^2_\lambda}\rangle_t \\ &\quad + \sum_{\substack{n,k\in A\\ n\neq k}}\frac{2}{\sqrt{\lambda_n\lambda_k}}\langle\langle X, e_n^\lambda\rangle_{l^2_\lambda}, \langle X, e_k^\lambda\rangle_{l^2_\lambda}\rangle_t \\ &= \sum_{n\in A}\frac{t}{\lambda_n}\langle Qe_n^\lambda, e_n^\lambda\rangle_{l^2_\lambda} = t\#(A) < \infty.\end{aligned} \tag{1.15}$$

This shows that $(t,A)\mapsto\langle L_1\rangle(t,A)$ *is a non-decreasing field on* $\mathbb{R}_+\times\mathcal{E}$. *In other words,* L_1 *is a cylindrical martingale field.*

b) Now set $\mathcal{E} = \Sigma = \mathscr{P}(E)$ *and define a Lévy field* $L_2 : \Omega\times\mathbb{R}_+\times\Sigma\to\mathbb{R}$,

$$L_2(t,A) = \sum_{n\in A}\langle X(t), e_n^\lambda\rangle_{l^2_\lambda}, \quad \text{for } A\in\Sigma \text{ and } t\in\mathbb{R}_+, \tag{1.16}$$

which is a true martingale field. Likewise to example (a), we indeed have:

1. *by construction,* $L_2(\cdot,A)$ *has independent and stationary increments;*

2. *for each* $A\in\Sigma$, $L_2(\cdot,A)$ *belongs to* $\mathcal{H}^2_{loc}$;

3. *Fix* $t > 0$ *and* $A\in\Sigma$. *We compute and obtain*

$$\begin{aligned}\langle L_2\rangle(t,A) &= \sum_{n\in A} t\langle Qe_n^\lambda, e_n^\lambda\rangle_{l^2_\lambda} + \sum_{\substack{n,k\in A\\ n\neq k}} 2t\langle Qe_n^\lambda, e_k^\lambda\rangle_{l^2_\lambda} \\ &= t\sum_{n\in A}\lambda_n = t\alpha_A < \infty,\end{aligned}$$

where α_A *defines a measure on* (E,Σ). *This yields that* $(t,A)\mapsto\langle L_2\rangle(t,A)$ *is a non-decreasing field on* $\mathbb{R}_+\times\Sigma$.

1.3.3 *Stochastic Integration*

In this section, we present the Itô stochastic integral with respect to martingale fields. Then we shall investigate the some special cases in which we show the possible connection of our stochastic integral with the classic integration theory in the standard textbooks.

In what follows, let H be a real-separable Hilbert space and M be a martingale field on $\mathbb{R}_+\times\mathcal{E}$. Denote by $f\cdot M$ the integral process $\int_0^\bullet\int_E f(s,x)M(ds,dx)$.

1.3.3.1 *Definition of the Stochastic Integral*

Let us fix the time horizon $T \geqslant 0$ and define

$$L^2_T(M;H) := L^2(\Omega \times [0,T], \mathcal{P}_T \otimes \Sigma, \mathbb{P} \otimes \langle M \rangle; H),$$

the space of all predictable mappings $f : \Omega \times [0,T] \times E \to H$ for which:

$$\mathbb{E}\left(\int_0^T \int_E \|f(s,x)\|^2 \langle M \rangle (ds,dx)\right) < \infty. \qquad (1.17)$$

Notice that the space $L^2_T(M;H)$ is a real Hilbert space equipped with the norm $\|\cdot\|_T$ defined by:

$$\|\phi\|_T = \mathbb{E}\left(\int_0^T \int_E \|\phi(s,x)\|^2 \langle M \rangle (ds,dx)\right)^{1/2}, \quad \text{for } \phi \in L^2_T(M;H).$$

Definition 1.3.6 For each $t \in [0,T]$, a predictable process f is M-integrable if $f \in L^2_T(M;H)$ and we write

$$f \cdot M(\omega;t) := \int_0^t \int_E f(\omega,s,x) M(\omega,ds,dx). \qquad (1.18)$$

Moreover, if M is a continuous then $f \cdot M$ is continuous .

Let us give here a short explanation regarding to the construction of the integral.

1. Let $\mathscr{E}_T(M;H)$ denote the space of H-valued elementary functions adapted to the filtration $(\mathcal{F}_t)_{t \leqslant T}$ that are of the form

$$\phi = \sum_{i=0}^n \phi_i \mathbb{1}_{(t_i,t_{i+1}] \times A_i}, \quad \text{for some } n \in \mathbb{N}, \qquad (1.19)$$

where $\{t_0,\dots,t_n\}$ is a partition of $[0,T]$, $A_0,\dots,A_n$ are pairwise disjoints sets in $\mathcal{E}$ with $(t_i,t_{i+1}] \times A_i \subset \mathbb{R}_+ \times E$ for $i = 0,\dots,n$, and ϕ_i is a H-valued, bounded and $\mathcal{F}_{t_i}$-measurable random variable with $\langle \phi_i, \phi_j \rangle_H = 0$ for $i \neq j$.

2. The stochastic integral of ϕ with respect to M is canonically defined as

$$\phi \cdot M_t := \sum_{i=0}^n \phi_i M\left((t \wedge t_i, t \wedge t_{i+1}], A_i\right), \quad \text{for all } t \in (0,T]. \qquad (1.20)$$

Next, we shall find the properties of the stochastic integral.

Theorem 1.3.1 *Let* M *be a martingale field and* $f, g \in \mathscr{E}_T(M; H)$. *The following proprieties are satisfied:*

(i) *The mapping* $f \mapsto f \cdot M$ *is a linear isometry from* $\mathscr{E}_T(M; H)$ *into* $M_T^2(H)$.

(ii) *For any stopping time* τ *with* $\mathbb{P}(\tau \leqslant T) = 1$, $(f \cdot M)^T = (f \mathbb{1}_{[\![0,\tau]\!]}) \cdot M$.

(iii) *For any* $u, v \in H$, *the covariance process of* $f \cdot M$ *is determined by*

$$\langle (f \cdot M, u)_H, (g \cdot M, v)_H \rangle_t = \int_0^t \int_E (f(s,x), u)_H (g(s,x), v)_H \langle M \rangle (ds, dx). \tag{1.21}$$

Proof Let $f, g \in \mathscr{E}_T(M; H)$ be arbitrary such that

$$f = f_0 \mathbb{1}_{(0,a] \times A} \quad \text{and} \quad g = g_0 \mathbb{1}_{(0,b] \times B}, \quad \text{for } (0,a] \times A, (0,b] \times B \subset [0,T] \times E$$

where f_0 and g_0 are respectively H-valued, bounded and $\mathcal{F}_0$-measurable random variables.

(i) First, we show that $f \cdot M$ is a H-valued square-integrable martingale. Let $h \in H$ be arbitrary. Then, for $0 \leqslant s < t$, we compute

$$\begin{aligned} \mathbb{E}[\langle f \cdot M_t, h \rangle_H | \mathcal{F}_s] &= \mathbb{E}\big[\langle f_0, h \rangle_H I_M \left((0, t \wedge a], A \right) | \mathcal{F}_s \big] \\ &= \langle f_0, h \rangle_H I_M \left((0, s \wedge a], A \right) = \langle f \cdot M_s, h \rangle_H , \end{aligned}$$

proving that the process $f \cdot M$ is an H-valued martingale.

Next, using the independent increment property of M, we compute

$$\begin{aligned} \mathbb{E}\left[\|f \cdot M_t\|^2 \right] &= \mathbb{E}\left[\|f_0 I_M \left((0, t \wedge a], A \right) \|^2 \right] \\ &= \mathbb{E}\left[\left(M(t \wedge a, A) - M(0, A) \right)^2 \|f_0\|^2 \right] \\ &= \mathbb{E}\Big[\mathbb{E}\left[M(t \wedge a, A)^2 - M(0,A)^2 | \mathcal{F}_0 \right] \|f_0\|^2 \Big] \\ &= \mathbb{E}\Big[\left(\langle M \rangle (t \wedge a, A) - \langle M \rangle (0, A) \right) \|f_0\|^2 \Big] \\ &= \mathbb{E}\left[\int_0^t \int_E \|f(s,x)\|^2 \langle M \rangle (ds, dx) \right] = \|f\|_t^2, \end{aligned}$$

which shows the isometry property and if f satisfies (1.17) then we obtain $f \cdot M \in M_T^2(H)$. Moreover, if $\alpha \in \mathbb{R}$ is an arbitrary constant then we have

$$\|f + \alpha g\|_t \leqslant \|f\|_t + |\alpha| \|g\|_t < \infty, \quad \text{for all } t \geqslant 0.$$

This means that $f+\alpha g$ is M-integrable and we obtain

$$\begin{aligned}(f+\alpha g)\cdot M_t &= \int_0^t\int_E (f_0\mathbb{1}_{(0,a]\times A}(s,x)+\alpha g_0\mathbb{1}_{(0,b]\times B}(s,x))M(ds,dx)\\ &= f\cdot M_t+\alpha g\cdot M_t, \quad \text{for all } t\geqslant 0,\end{aligned}$$

and the linearity of stochastic integral follows.

(ii) Let $t\in[0,T]$, $\omega\in\Omega$ be arbitrary. By definition, ω-by-ω, we write:

$$\begin{aligned}(f\cdot M)_t^\tau(\omega) &= \int_0^{t\wedge\tau(\omega)}\int_E f(\omega,s,x)M(\omega;ds,dx)\\ &= f_0(\omega)\left[M(\omega;t\wedge T(\omega)\wedge a,A)-M(\omega;0,A)\right],\end{aligned} \tag{1.22}$$

and

$$\begin{aligned}(f\mathbb{1}_{[\![0,\tau]\!]})\cdot M_t(\omega) &= \int_0^t\int_E f(\omega,s,x)\mathbb{1}_{[\![0,\tau]\!]}(\omega,s)M(\omega;ds,dx)\\ &= \int_0^t\int_E f_0(\omega)\mathbb{1}_A(x)\mathbb{1}_{[\![0,\tau\wedge a]\!]}(\omega,s)M(\omega;ds,dx)\\ &= f_0(\omega)\left[M(\omega;t\wedge T(\omega)\wedge a,A)-M(\omega;0,A)\right],\end{aligned} \tag{1.23}$$

This implies that $(f\cdot M)^\tau=(f\mathbb{1}_{[\![0,\tau]\!]})\cdot M$.

(iii) First, we compute [6]

$$\begin{aligned}&\int_0^t\int_E\left\langle f(s,x),u\right\rangle_H\left\langle g(s,x),v\right\rangle_H\langle M\rangle(ds,dx)\\ =&\langle f_0,u\rangle_H\langle g_0,v\rangle_H\left[\langle M\rangle(t\wedge a\wedge b,A\cap B)-\langle M\rangle(0,A\cap B)\right]\\ =&\langle f_0,u\rangle_H\langle g_0,v\rangle_H\langle M(\cdot\wedge a,A),M(\cdot\wedge b,B)\rangle_t.\end{aligned} \tag{1.24}$$

Equation (1.24) yields

$$\begin{aligned}&\langle f\cdot M_t,u\rangle_H\langle g\cdot M_t,v\rangle_H-\int_0^t\int_E\left\langle f(s,x),u\right\rangle_H\left\langle g(s,x),v\right\rangle_H\langle M\rangle(ds,dx)\\ =&\langle f_0,u\rangle_H\langle g_0,v\rangle_H\big([M(t\wedge a,A)M(t\wedge b,B)-\langle M(\cdot\wedge a,A),M(\cdot\wedge b,B)\rangle_t\\ &-M(0,A)M(t\wedge b,B)-M(0,B)M(t\wedge a,A)-M(0,A)M(0,B)\big),\end{aligned} \tag{1.25}$$

which is a martingale as M is a martingale field. Therefore, by definition of quadratic covariation of real-valued process, we must have

$$\int_0^t\int_E\left\langle f(s,x),u\right\rangle_H\left\langle g(s,x),v\right\rangle_H\langle M\rangle(ds,dx)=\langle\langle f\cdot M,u\rangle_H,\langle g\cdot M,v\rangle_H\rangle_t.$$

6 $\langle M\rangle(\cdot\wedge a\wedge b,A\cap B)$ is the bracket process of both processes $M(\cdot\wedge a,A)$ and $M(\cdot\wedge b,B)$.

∎

In order to extend this definition to the larger space $L^2_T(M;H)$, one may use the fact that the class of bounded elementary processes $\mathscr{E}_T(M;H)$ is dense in $L^2_T(M;H)$ and then apply the limit approximation argument.

Lemma 1.3.3 *Let* $f \in L^2_T(M;H)$, *then there exists a sequence* f_n *in* $\mathscr{E}_T(M;H)$ *approximating* f *in* $L^2_T(M;H)$. *That is,*

$$\|f_n - f\|_T^2 = \mathbb{E}\left(\int_0^T \int_E \|f_n(s,x) - f(s,x)\|^2 \langle M\rangle(ds,dx)\right) \to 0,$$

as $n \to \infty$.

Proof We follow the idea in [96]. We aim to show that $\mathscr{E}_T(M;H)$ is dense in $L^2_T(M;H)$.

(1) Let $f \in L^2_T(M;H)$ and define a sequence of bounded functions

$$f_n(t,x) = \begin{cases} f(t,x) & \text{if } \|f(t,x)\| < n \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|f_n - f\|_T^2 = \mathbb{E}\left(\int_0^T \int_E \|f_n(s,x) - f(s,x)\|^2 \langle M\rangle(ds,dx)\right) \to 0$$

by monotone convergence theorem, as $n \to \infty$. This implies that bounded functions are dense in $L^2_T(M;H)$.

(2) We assume now that f is bounded and predictable. We construct ω-by-ω

$$f_n(t,x,\omega) = \frac{1}{2^n}\int_{(k-1)/2^n}^{k/2^n} f(s,x,\omega)\,ds, \quad \text{if } t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right).$$

Then $\|f_n - f\|_T^2 \to 0$ by Lebesgue dominated convergence theorem. This shows that the space of simple functions dense in the bounded functions.

(3) By definition, it is clear to see that $\mathscr{E}_T(M;H)$ is dense in the space of simple functions. This implies that $\mathscr{E}_T(M;H)$ is dense in $L^2_T(M;H)$.

(4) Next, we use the isometry property of $f \mapsto f \cdot M$ to obtain the existence of approximating sequence. By Lemma A.4.2, the map $f \mapsto f \cdot M$ has a

further extension to the space $L^2_T(M;H)$. Namely, there exists a sequence $\{f_n\} \subset \mathscr{E}_T(M;H)$ and

$$\mathbb{E}\left(\int_0^T \int_E \|f_n(s,x) - f(s,x)\|^2 \langle M\rangle(ds,dx)\right) \to 0, \quad \text{as } n \to \infty. \tag{1.26}$$

To conclude the proof, by analogous reasoning if $\{f'_n\} \subset \mathscr{E}_T(M;H)$ is another sequence satisfying (1.26) then both sequences lead to the same function f. That is the definition of stochastic integral $f \cdot M$ does not depend on the choice of approximating sequence. ∎

At the end, for every $f \in L^2_T(M;H)$, the stochastic integral $f \cdot M$ or $\int_0^\bullet \int_E f(s,x)M(ds,dx)$ is thus well-defined and we write

$$f \cdot M_t = \int_0^t \int_E f(s,x)M(ds,dx) = \lim_{n\to\infty} \int_0^t \int_E f_n(s,x)M(ds,dx), \quad t \in [0.T],$$

where $\{f_n\}$ is a sequence of functions in $\mathscr{E}_T(M;H)$ and by Lemma 1.3.3, all properties in Theorem 1.4.1 hold true for any function $f \in L^2_T(M;H)$.

Remark 6 *Note that for any predictable function* $f \in L^2_T(M;H)$*, the integral process* $\int_0^\bullet \int_E f(s,x)M(ds,dx)$ *has càdlàg sample paths.*

1.3.3.2 *Special Cases*

In the following, we provide the connection between stochastic integration w.r.t martingale fields developed previously and those already existing in literature that is used to study SPDE's: (1) stochastic integral w.r.t Hilbert-space-valued processes; (2) stochastic integral w.r.t Poisson random measures.

1) Fix $T > 0$ and suppose that E is countable and $\Sigma = \mathscr{P}(E)$. Define $\mathcal{E} = \{A \subset \mathbb{N} : A \text{ is finite}\}$ such that $\Sigma = \sigma(\mathcal{E})$. Let M be a martingale field on $\mathbb{R}_+ \times \mathcal{E}$ and H be a separable Hilbert space. For $k \in E$, we set $M_k(t) := M(t,\{k\})$ and according to Definition 1.3.1, $M_k(t)$ is a square-integrable martingale process. Moreover, we have:

$$M(\cdot,A) := \sum_{k\in A} M_k(\cdot) \text{ and } \langle M\rangle(\cdot,A) = \sum_{k\in A} \langle M_k\rangle, \quad \text{for all } A \in \mathcal{E}. \tag{1.27}$$

Before going further, we consider the following result.

Proposition 1.3.2 *Let* $f : \Omega \times [0,T] \times E \to H$ *be such that* $f \in L^2_T(M;H)$. *Then* f *is* M_k*-integrable for each* $k \in E$ *and it holds*

$$\int_0^t \int_E f(s,x)M(ds,dx) = \sum_{k\in E} \int_0^t f(s,k)M^k(ds).$$

Proof Let $k \in E$. By definition $\langle M_k \rangle$ is an increasing process, then we can write

$$\mathbb{E}\left[\int_0^t \|f(s,k)\|^2 \langle M_k \rangle(ds)\right] < \sum_{n\in E} \mathbb{E}\left[\int_0^t \|f(s,n)\|^2 \langle M_n \rangle(ds)\right]. \tag{1.28}$$

Since $f \in L^2_T(M;H)$, and by equation (1.27), we have

$$\sum_{n\in E} \mathbb{E}\left[\int_0^t \|f(s,n)\|^2 \langle M_n \rangle(ds)\right] = \mathbb{E}\left[\int_0^t \int_E \|f(s,x)\|^2 \langle M \rangle(ds,dx)\right] < \infty.$$

This yields that the Itô integral $\int_0^\bullet f(s,k)M_k(ds)$ is well-defined since

$$\mathbb{E}\left[\int_0^t \|f(s,k)\|^2 \langle M_k \rangle(ds)\right] < \infty,$$

Now let f be an elementary function of the form

$$f = f_a \mathbb{1}_{(a,b]\times A}, \quad (a,b] \times A \subset [0,T] \times E,$$

where f_a is a H-valued and $\mathcal{F}_a$-measurable random variable. On one hand, the stochastic integral $f \cdot M_t$ is given by

$$f \cdot M_t = f_a[M(t\wedge b, A) - M(t \wedge a, A)] = \sum_{n\in A} f_a[M_n(t\wedge b) - M_n(t\wedge a)]. \tag{1.29}$$

One the other hand, we also have f is M_k-integrable for each $k \in E$, with

$$\int_0^t f(s,k)M_n(ds) = f_a[M_n(t\wedge b) - M_n(t\wedge a)]. \tag{1.30}$$

Thus combining equations (1.29) and (1.30) yields

$$f \cdot M_t = \int_0^t \int_E f(s,x)M(ds,dx) = \sum_{n\in E} \int_0^t f(s,k)M_n(ds).$$

Finally, by Lemma 1.3.3, the result holds true for any predictable process $f \in L^2_T(M;H)$. ∎

Thanks to the above result, we next present the equivalence between stochastic integrals w.r.t a cylindrical Wiener process on some separable Hilbert space and w.r.t a martingale field.

Let U be a separable Hilbert space and W be a cylindrical Wiener process on U. Let $\{f_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of U. We recall that for any predictable $\Phi \in L_2(U,H)$, i.e.

$$\mathbb{E}\left[\int_0^t \|\Phi(s)\|^2_{L_2(U,H)}\,ds\right] < \infty, \quad \text{for all } t \geqslant 0,$$

we denote $\Phi \cdot W = \int_0^\bullet \Phi(s)W(ds)$ and we define the H-valued stochastic integral $\int_0^t \Phi_s\,dW_s$ as in [22] and [34]. That is, we can define $\Phi \cdot W$ by the following serie representation (see e.g. [34, Lemma 2.8]):

$$\int_0^\bullet \Phi(s)W(ds) = \sum_{i=1}^\infty \int_0^\bullet (\Phi(s)f_i)\,dW_s(f_i). \tag{1.31}$$

Note that the sum in (1.31) does not depend on the chosen orthonormal basis.

Proposition 1.3.3 *If* $\Phi \in L_2(U,H)$, *then there exists* $f \in L^2_T(M;H)$ *such that*

$$\int_0^t \Phi_s\,dW_s = \int_0^t \int_{\mathbb{N}} f(s,x)M(ds,dx), \tag{1.32}$$

where

$$f(t,i) = \Phi(t)e_i, \quad \textit{for } t \in [0,T],\ i \in \mathbb{N}.$$

Proof By equation (1.13), we can construct a sequence of standard independent Wiener processes $\{M_i\}_{i\in\mathbb{N}}$ such that

$$M_i(t) := W_t(f_i), \quad \text{for all } t \geqslant 0, i \in \mathbb{N},$$

and we define a martingale field $M(t,\{i\}) := M_i(t)$. Then we obtain

$$\int_0^\bullet \Phi(s)\,dW_s = \sum_{i=1}^\infty \int_0^\bullet [\Phi(s)f_i]\,M_i(ds). \tag{1.33}$$

According to Proposition 1.3.2, there is a mapping $f : \Omega \times \mathbb{R}_+ \times \Sigma \to H$ associated to Φ with

$$f(t,i) = \Phi(t)f_i, \quad \text{for any } t \in [0,T], i \in \mathbb{N},$$

which is M_i-integrable and satisfies

$$\sum_{i=1}^\infty \int_0^t [\Phi(s)f_i]\,dW_s(f_i) = \sum_{i=1}^\infty \int_0^t f(t,i)M_i(ds) = \int_0^t \int_{\mathbb{N}} f(s,x)M(ds,dx).$$

This concludes the proof. ∎

As a consequence, we also compare our random field integral to the infinite-dimensional stochastic integral as established in [22]. Let $Q \in L(U)$ be a positive, definite, symmetric, linear trace-class operator. If $\mathcal{W}$ is a Q-Wiener process on U, there is a natural way to associate to it a cylindrical Wiener process on U (for more details see [22] and [34]). Namely, we set

$$\mathcal{W}_t = W_t \circ Q^{1/2} := \sum_{i=1}^{\infty} \sqrt{\lambda_i} W_t(e_i) e_i, \quad \text{for all } t \geqslant 0. \tag{1.34}$$

where the serie converges in $L^2(\Omega; \mathcal{C}([0,T]; U))$ and the λ_i are the eigenvalues of Q, each e_i is an eigenvector corresponding to λ_i (i.e. $Qe_i = \lambda_i e_i$). We denote by $L_2(U_Q, H)$ the space of all predictable processes ϕ satisfying

$$\mathbb{E}\left[\int_0^t \|\phi(s)\|_{L_2(U_Q,H)}\, ds\right] < \infty, \quad \text{for all } t \geqslant 0,$$

here $U_Q = Q^{1/2}U$ is a separable Hilbert space with an orthonormal basis $\{\sqrt{\lambda_i} ei\}_{i \in \mathbb{N}}$ and is equipped with the scalar product

$$\langle u, v\rangle_{U_Q} = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} \langle u, e_i\rangle_U \langle v, e_i\rangle_U.$$

We recall that the serie representation of the stochastic integral $\phi \cdot \mathcal{W}$ in terms of ordinary Itô integrals of real-valued processes as in [22] and [34]. The integral process can be written in the form:

$$\int_0^{\bullet} \phi(s)\mathcal{W}(ds) = \sum_{i=1}^{\infty} \int_0^{\bullet} (\phi(s)e_i)\, d\langle \mathcal{W}_s, (e_i)\rangle_U. \tag{1.35}$$

Corollary 1.3.1 *If $\phi \in L_2(U_Q, H)$, then there exists $g \in L_T^2(M; H)$ such that*

$$\int_0^{\bullet} \phi_s\, d\mathcal{W}_s = \int_0^{\bullet} \int_{\mathbb{N}} g(s,x) M(ds, dx), \tag{1.36}$$

where

$$g(t,i) = \phi(t)e_i, \quad \textit{for } t \in [0,T],\ i \in \mathbb{N}.$$

Proof The proof follows directly from Proposition 1.3.3. Indeed, we first obverse that $\langle \mathcal{W}_s, (e_i)\rangle_U$ defines a sequence of standard independent Wiener processes. Since $k \mapsto W_t(k) = \langle \mathcal{W}_t, k\rangle_U$ is a Cylindrical Wiener process on U and $W_s(e_i) = \langle \mathcal{W}_s, (e_i)\rangle_U$ for all $i \in \mathbb{N}$. Therefore, by Proposition 1.3.3

and equation (1.35), we have the existence a mapping $g \in L^2_T(M; H)$ such that

$$g(t, i) = \phi(t)e_i, \quad \text{for } t \in [0, T], \ i \in \mathbb{N},$$

and

$$\phi \cdot \mathcal{W} = \sum_{i=1}^{\infty} \int_0^{\bullet} [\phi(s)e_i] \, d\langle \mathcal{W}_s, (e_i) \rangle_U = \int_0^{\bullet} \int_{\mathbb{N}} g(s, x) M(ds, dx).$$

This completes the proof. ∎

2) Now, we assume that (E, Σ) is a Blackwell space such that there is a countable ring $\mathcal{E}$ with $\Sigma = \sigma(\mathcal{E})$. We recall that M_μ is a compensated Poisson field $\mathbb{R}_+ \times \mathcal{E}$ as in Example 3, that is

$$M_\mu(t, A) = \int_0^t \int_E \mathbb{1}_A(x)[\mu(ds \times dx) - \nu(ds \times dx)], \quad \text{for } A \in \mathcal{E}, \ t \geqslant 0,$$

where $\nu(ds \times dx) = ds F(dx)$ is the intensity of homogeneous Poisson random measure μ. Fix $T \geqslant 0$ and for every predictable functions f that satisfies

$$\mathbb{E}\Big[\int_0^T \int_E \|f(s, x)\|^2 F(dx) \, ds\Big] < \infty \tag{1.37}$$

both stochastic integrals $f * (\mu - \nu)$ and $f \cdot M_\mu$ are well-defined. Note that first $f * (\mu - \nu)$ is defined as in [93] (i.e. the extension of stochastic integral w.r.t random measures developed in [47] to Hilbert-space-valued functions). The aim is to prove that stochastic integral w.r.t Poisson random measures is in fact captured by the stochastic integration w.r.t martingale fields. Namely, we have the following equivalence

$$\int_0^T \int_E f(s, x) M_\mu(ds, dx) = \int_0^T \int_E f(s, x)[\mu(ds \times dx) - F(dx) \, ds].$$

Proposition 1.3.4 *Let f be a predictable function satisfying condition (1.37). Then it holds $f * (\mu - \nu) = f \cdot M_\mu$ (up to indistinguishability).*

Proof By extending the stochastic integration in [47], we first denote by $G^H_{loc}(\mu)$ the space of all H-valued predictable functions f such that the process $C(f) = \|f\|^2 * \nu \in \mathcal{A}^+$, that is

$$\mathbb{E}\Big[\int_0^T \int_E \|f(s, x)\|^2 F(dx) \, ds\Big] < \infty.$$

One the other hand, we have seen in Example 3 that M_μ is a martingale field with

$$\langle M_\mu \rangle(t, A) = tF(A), \quad \text{for all } t \geqslant 0, A \in \mathcal{E}.$$

Hence, by Defintion 1.3.6, f is M_μ-integrable if and only if the following condition is fulfilled

$$\mathbb{E}\Big[\int_0^T \int_E \|f(s,x)\|^2 \langle M_\mu \rangle(ds, dx)\Big] = \mathbb{E}\Big[\int_0^T \int_E \|f(s,x)\|^2 F(dx)\, ds\Big] < \infty.$$

This implies that both spaces $G^H_{loc}(\mu)$ and $L^2_T(M_\mu; H)$ coincides and we have $f * (\mu - \nu) = f \cdot M_\mu$. ∎

1.3.4 *Extension of the Stochastic Integral by Localisation*

In this section, we conclude the construction of the stochastic integral by extending Definition 1.3.6 to the class of integrands satisfying a less restrictive assumption on the integrability condition.

Furthermore, we denote $\mathcal{L}^2_{loc}(M; H)$ the space of all predictable processes $f : \Omega \times \mathbb{R}_+ \times H \to H$ such that

$$\mathbb{P}\left(\int_0^T \int_E \|f(s,x)\|^2 \langle M \rangle (ds, dx) < \infty\right) = 1, \quad \text{for all } T \geqslant 0.$$

We can relax the integrability condition (1.3.6) by showing that $L^2_T(M; H)$ is dense in $\mathcal{L}^2_{loc}(M; H)$.

Lemma 1.3.4 *Let $f \in \mathcal{L}^2_{loc}(M; H)$ be arbitrary. Then there exists an approximation sequence $f_n \in L^2_T(M; H)$ such that*

$$f \cdot M_t = \lim_{n \to \infty} \int_0^t \int_E f_n(t, x) M(ds, dx),$$

Proof Let $f \in \mathcal{L}^2_{loc}(M; H)$ and $T > 0$ be arbitrary. We define

$$\tau_n := n \wedge \inf\left\{t \in [0, T] : \int_0^t \int_E \|h(s,x)\|^2 \langle M \rangle(ds, dx) \geqslant n\right\}, \tag{1.38}$$

where $\tau_n \uparrow T$ as $n \to \infty$ and wet set

$$f_n(\omega, t, x) = f(\omega, t, x) \mathbb{1}_{[\![0, \tau_n]\!]}(t, \omega), \quad \text{for } t \in \mathbb{R}_+, x \in E, \omega \in \Omega. \tag{1.39}$$

Then τ_n is a sequence such that

$$\mathbb{E}\left[\int_0^t\int_E \|f_n(s,x)\|^2\langle M\rangle(ds,dx)\right] = \mathbb{E}\left[\int_0^t\int_E \|f(s,x)\mathbb{1}_{[\![0,\tau_n]\!]}\|^2\langle M\rangle(ds,dx)\right] \leqslant n, \tag{1.40}$$

This implies that the stochastic integral $(f\mathbb{1}_{[\![0,\tau_n]\!]})\cdot M$ is well-defined and $f_n = f\mathbb{1}_{[\![0,\tau_n]\!]} \in L^2_T(M;H)$ for all $n \in \mathbb{N}$.

Now define

$$f\cdot M_t := (f\mathbb{1}_{[\![0,\tau_n]\!]})\cdot M_t, \quad t\in[0,T], \tag{1.41}$$

where $n \in \mathbb{N}$ is arbitrary such that $\tau_n \geqslant t$. Observe that if we take arbitrary $\tau_m \geqslant t$ and $m \geqslant n$ then by Proposition 1.4.2 we have $\mathbb{P}$-*a.s*

$$(f\mathbb{1}_{[\![0,\tau_m]\!]})\cdot M_t = (f\mathbb{1}_{[\![0,\tau_m]\!]})\cdot M_t^{\tau_n} = (f\mathbb{1}_{[\![0,\tau_n]\!]})\cdot M_t.$$

On the other hand, if $\tau'_n \uparrow T$ another sequence of stopping time satisfying 1.40 thus both $(f\mathbb{1}_{[\![0,\tau'_n]\!]})\cdot M$ and $(f\mathbb{1}_{[\![0,\tau_n]\!]})\cdot M$ lead to the same process $f\cdot M$ $\mathbb{P}$-*a.s*. This means that the definition in (1.41) is consistent and does not depend on the choice of sequence of stopping times.

As result, we conclude that the stochastic integral of any function $f \in \mathcal{L}^2_{loc}(M;H)$ with respect to M is determined by

$$f\cdot M_t = \lim_{n\to\infty}\int_0^t\int_E f_n(t,x)M(ds,dx) = \lim_{n\to\infty}\int_0^t\int_E f\mathbb{1}_{\{t\leqslant\tau_n\}}M(ds,dx).$$

∎

Remark 7 *We conclude this section by pointing out that analogous approach has been developed in [96]. In fact, let (E,Σ) be a Lusin space (see [91] or [19]), on the one hand the Walsh's terminology assumes that there is a ring $\mathcal{A}$ and an increasing sequence E_n such that*

- $E = \bigcup_{n\in\mathbb{N}} E_n$
- $\Sigma_n := \Sigma|_{E_n} \subseteq \mathcal{A}$

Walsh defines his set function M on $\Omega\times\mathbb{R}_+\times\mathbb{A}$ as a martingale measure, namely, $\mathbb{P}$-almost surely

1. *$M_t(A)$ is square-integrable martingale with $M_0(A) = 0$, for every $A \in \mathcal{A}$.*

2. $M_t(A \cup B) = M_t(A) + M_t(B)$, *for all* $t \geqslant 0$ *and all disjoint* $A, B \in \mathcal{A}$.

3. $M(A), M(B)$ *are orthogonal martingales for all disjoint* $A, B \in \mathcal{A}$.

4. $\sup\left\{\mathbb{E}[\|M_t(A)\|^2], A \in \Sigma_n\right\} < \infty$ *for all* $t \geqslant 0$.

Moreover, Walsh [96], if M *is a martingale measure then there exists a random* σ*-finite positive measure* ν *on* $\mathbb{R}_+ \times E$ *such that*

$$\left\langle M(A)\right\rangle_t := \nu((0,t] \times A), \quad t \geqslant 0, A \in \Sigma.$$

Since the Lusin set E *has the Blackwell property (see [86]) and if we consider a countable ring*

$$\mathcal{E} = \bigcup_{n \in \mathbb{N}} \{A \cap E_n : A \in \Sigma\},$$

then the martingale measure M *meets Definition 1.3.1. More precisely,* M *is a martingale field on* $\mathbb{R}_+ \times \mathcal{E}$ *with* $B(t, A) = \nu((0,t] \times A)$ *for all* $t \geqslant 0, A \in \mathcal{E}$.

On the other hand, Walsh follows the Itô spirit to construct his integration theory. However, in order to perform the stochastic integration, one need to assume that the martingale measure is worthy, i.e., the existence of a dominating measure (see [96, p.291]). In contrast, our approach do not require such condition and we only works with fairly easy integrability conditions. This shows that by choosing the measure induced by field B *as a dominating measure we obtain that Walsh integral shall coincide with our Itô stochastic integral.*

Moreover, as shown in [26], stochastic integrals w.r.t cylindrical and Q*-Wiener processes coincide with the Walsh's stochastic integral. However, Proposition 1.3.3 and Corollary 1.3.1 also prove that the stochastic integral constructed in Subsection 1.3.3.1 captures both integral processes w.r.t infinite dimensional Wiener processes. This means that again Walsh's stochastic integral meets our stochastic integral.*

1.4 SEMIMARTINGALE FIELDS

In this section, we begin with new concepts and notions. That is the description of the semimartingale field, that combines both class of fields seen previously, as well its properties. The stochastic integration w.r.t. semimartingale fields shall be the main focus, including the Fubini's theorem and the unification result of two stochastic integrals.

1.4.1 *Definitions*

Definition 1.4.1 A semimartingale field on $\mathbb{R}_+ \times \mathcal{E}$ is a random field X of the form $X = M + N$ where $M \in \mathcal{M}_{\mathcal{E}}$ and $N \in \mathcal{V}_{\mathcal{E}}$. If the set A is fixed, we have $S(\cdot, A)$ defines a semimartingale process which admits a decomposition $S(\cdot, A) = M(\cdot, A) + N(\cdot, A)$. We denote by $\mathcal{S}_{\mathcal{E}}$ the class of all semimartingale fields on $\mathbb{R}_+ \times \mathcal{E}$.

It is clear that the decomposition in Definition 1.4.1 is not unique. However, under some conditions, there is at most one decomposition if the field N is in addition predictable as the following result shows.

Theorem 1.4.1 *Any semimartingale field* X *admits a unique decomposition up to an evanescence set if its finite variation part is a predictable finite variation field.*

Proof Let $(M, N), (M', N') \in \mathcal{M}_{\mathcal{E}} \times \mathcal{V}_{\mathcal{E}}$ such that they satisfy respectively the decomposition $X = M + N$ in Definition 1.4.1. On the one hand, we define

$$\mathcal{N}_1 = \bigcup_{A_1 \in \mathcal{E}} \Big\{ \omega \in \Omega : \ X(\omega; t, A_1) \neq M(\omega; t, A_1) + N(\omega; t, A_1), \ t \in \mathbb{R}_+ \Big\},$$

and

$$\mathcal{N}_2 = \bigcup_{A_2 \in \mathcal{E}} \Big\{ \omega \in \Omega : \ X(\omega; t, A_2) \neq M'(\omega; t, A_2) + N'(\omega; t, A_2), \ t \in \mathbb{R}_+ \Big\},$$

are respectively $\mathbb{P}$-null sets. One the other hand, we set

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2,$$

which is a $\mathbb{P}$-null set. Indeed, for any fixed $A \in \mathcal{E}$, one can check that $N'(\cdot, A) - N(\cdot, A) = M(\cdot, A) - M'(\cdot, A) = 0$ (up to an evanescent set) due to the fact that this difference process is a predictable local martingale belonging to $\mathcal{V}$ (see [47, Corollary 3.16]). Now let $\omega \in \mathcal{N}^c$. It follows that for any $A \in \mathcal{E}$

$$M(\omega; t, A) - M'(\omega; t, A) = N'(\omega; t, A) - N(\omega; t, A) = 0, \quad \text{for each } t \geqslant 0.$$

This shows the uniqueness of the decomposition.

∎

This result leads to the following definition.

Definition 1.4.2 A special semimartingale field is a semimartingale field X which admits an unique decomposition $X = M + N$ where M is a martingale field and N is a predictable finite variation field. We denote by $\mathcal{S}^p_{\mathcal{E}}$ the set of all special semimartingale fields.

Remark 8 *Here are some other properties of both spaces $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{S}^p_{\mathcal{E}}$:*

1. *$\mathcal{S}_{\mathcal{E}}$ and $\mathcal{S}^p_{\mathcal{E}}$ are stable under stopping. That is, if $X \in \mathcal{S}_{\mathcal{E}}$ (resp. $X \in \mathcal{S}^p_{\mathcal{E}}$) (T_n) is a localizing sequence of stopping times, then the stopped semimartingale field $X^{T_n} \in \mathcal{S}_{\mathcal{E}}$ (resp. $X^{T_n} \in \mathcal{S}^p_{\mathcal{E}}$).*

2. *$(\mathcal{S}_{\mathcal{E}})_{loc} = \mathcal{S}_{\mathcal{E}}$ and $(\mathcal{S}^p_{\mathcal{E}})_{loc} = \mathcal{S}^p_{\mathcal{E}}$.*

Example 7 *In the following, we provide diverse examples of semimartingale fields.*

a. *Since $\mathcal{V}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ are in $\mathcal{S}_{\mathcal{E}}$, there are many examples of semimartingale fields.*

b. *Recall respectively both finite variation and martingale fields in examples 1 and 5. Let $A \subset \mathbb{N}$ be a finite set and define*

$$S(t, A) = \sum_{i \in A} N_i(t) + \sum_{i \in A} W_t(e_i), \quad \textit{for } t \geqslant 0.$$

If we denote $\mathcal{E} = \{A : A \subset \mathbb{N}$ and is finite$\}$, the field S is a (special) semimartingale field on $\mathbb{R}_+ \times \mathcal{E}$ as all Poisson processes $(N_i)_{i \in \mathbb{N}}$ are predictable.

c. *Take E be a countable set. It is also interesting to recognize any family of semimartingale processes of the form $S = S_0 + M + N$ (as in [47, Definition 4.21]) defines a semimaringale field. Namely, let $t \geqslant 0$ and $(S_i)_{i \in \mathbb{E}}$ be a family of semimartingale processes such that by Corollary 4.16 in [47] we have a decomposition*

$$S^i = S^i_0 + M^i + N^i,$$

where $M_i \in \mathcal{M}_{loc}$, $N_i \in \mathcal{V}$ *and* S_0^i *is a real-valued* $\mathcal{F}_0$*-measurable random variable for each* $i \in \mathbb{N}$. *Let* $A \subset \mathbb{E}$ *be a finite set and set*

$$\begin{aligned} S(t,A) &= \sum_{i \in \mathbb{A}} S_t^i = \sum_{i \in \mathbb{A}} M_t^i + \sum_{i \in \mathbb{A}} [S_0^i + N_t^i] \\ &= M(t,A) + N(t,A), \end{aligned}$$

where M *and* N *define respectively martingale and finite variation fields on* $\mathbb{R}_+ \times \mathcal{E}$ *with* $\mathcal{E} = \{A : A \subset E, \text{finite set}\}$. *Thus, we obtain* S *is a semimartingale field on* $\mathbb{R}_+ \times \mathcal{E}$.

1.4.2 *Stochastic Integration*

In this subsection, we proceed to constructing the stochastic integral of predictable processes with respect to a semimartingale field.

Let $f : \Omega \times \mathbb{R}_+ \times E \to H$ predictable mapping and denote by $f \cdot X$ the integral process defined as

$$f \cdot X_t = \int_0^t \int_E f(\omega, s, x) X(\omega; ds, dx).$$

1. Note that if X is a semimartingale field on $\mathbb{R}_+ \times \mathcal{E}$ and when f is simple enough, then the integral process $f \cdot X$ has only one definition (even if $X(ds, dx)$ is not well-defined), namely:

$$f \cdot X_t = f_a(X(t \wedge b, A) - X(t \wedge a, A)), \quad \text{if } f = f_a \mathbb{1}_{(a,b] \times A}, \quad t \in [0,T], \tag{1.42}$$

where $(a,b] \times A \subset [0,T] \times E$, f_a is an H-valued bounded and $\mathcal{F}_a$-measurable random variable. To show the existence of the stochastic integral $f \cdot X$, for the general case, we use the density property of the space of simple functions. Therefore, we get

$$f \cdot X_t = \lim_{n \to \infty} \int_0^t \int_E f_n(s,x) X(ds, dx), \quad \text{for } t > 0.$$

for some approximation function f_n from the space of simple functions such that $f_n \to f$ and the integral process $f_n \cdot X$ converges (in sense of equation (1.42)).

2. As we have seen in the two previous sections, the stochastic integral $f \cdot X$ can be defined when X belongs to one of the following two classes of fields:

a. if $X \in \mathcal{V}_{\mathcal{E}}$ and for every optional process f satisfying the following:

$$\int_0^t \int_E \|f(s,x)\| V_X(ds,dx) < \infty.$$

b. if $X \in \mathcal{M}_{\mathcal{E}}$ and for every predictable process f satisfying the following:

$$\mathbb{E}\left[\int_0^t \int_E \|f(s,x)\|^2 \langle X\rangle(ds,dx)\right] < \infty.$$

Putting these two classes together we obtain the following definition.

Definition 1.4.3 An H-valued process f is said to be locally integrable with respect to a semimartingale field X, if there exists a decomposition $X = M + N$ such that both integrals $f \cdot M$ and $f \cdot N$ are defined as above. Namely, f satisfies

$$\mathbb{E}\left[\int_0^t \int_E \|f(s,x)\|^2 \langle M\rangle(ds,dx) + \left(\int_0^t \int_E \|f(s,x)\| V_N(ds,dx)\right)^2\right] < \infty, \tag{1.43}$$

In this case, we define the stochastic integral $f \cdot X$ by

$$f \cdot X = f \cdot M + f \cdot N. \tag{1.44}$$

where $f \in L_T(N;H)$ and $f \in L^2_T(M;H)$. Denote by $L^2_T(X;H)$ the space of all predictable processes that satisfied condition (1.43).

Remark 9 *Note that the definition of stochastic integral in (1.44) is independent of the decomposition of* X*. Moreover, the stochastic integral w.r.t to semimartingale field does not preserve the isometry property but instead one has the following*

$$\mathbb{E}\left[\left\|\int_0^t \int_E f(s,x) X(ds,dx)\right\|^2\right] \leqslant \mathbb{E}\left[\int_0^t \int_E \|f(s,x)\|^2 \langle M\rangle(ds,dx)\right] + \mathbb{E}\left[\left(\int_0^t \int_E \|f(s,x)\| V_N(ds,dx)\right)^2\right]. \tag{1.45}$$

Remark 10 *If* $f \in L^2_T(N;H)$ *and* $f \in L^2_T(M;H)$*, then by Cauchy-Schwartz inequality the following condition is sufficient to make* f *integrable w.r.t.* X*:*

$$\mathbb{E}\left[\int_0^t \int_E \|f(s,x)\|^2 \left(\langle M\rangle(ds,dx) + V_N(ds,dx)\right)\right] < \infty, \quad t \in [0,T]. \tag{1.46}$$

It is clear that (1.46) implies (1.43). In this case, we can define a non-decreasing field $[X] := \langle M\rangle + V_N$ *up to indistinguishability for every semimartingale field* $X = M + N$. *Then we shall use* $L^2_T([X];H)$ *instead* $L^2_T(X;H)$.

3. We now can extend the space $L^2_T(X;H)$ to the space $\mathcal{L}^2_{loc}(X;H)$ consisting of all predictable processes $f : \Omega \times \mathbb{R}_+ \times E \to H$ such that for all $T \in \mathbb{R}_+$

$$\mathbb{P}\left(\int_0^T \int_E \|f(s,x)\|^2 \langle M\rangle(ds,dx) + \left(\int_0^T \int_E \|f(s,x)\| V_N(ds,dx)\right)^2 < \infty\right) = 1.$$

This is always possible because by condition (1.43) we can define

$$L^2_T(X;H) = L_T(N;H) \cap L^2_T(M;H),$$

and by Remark (4) and Lemma (1.3.4) we can respectively extended $L_T(N;H)$ to $\mathcal{L}_{loc}(N;H)$ and $L^2_T(M;H)$ to $\mathcal{L}^2_{loc}(M;H)$. Therefore, we extend $L^2_T(X;H)$ to $\mathcal{L}^2_{loc}(X;H)$ where

$$\mathcal{L}^2_{loc}(X;H) = \mathcal{L}_{loc}(N;H) \cap \mathcal{L}^2_{loc}(M;H).$$

4. Last, we state various properties of the stochastic integrals.

Proposition 1.4.1 *Let* X *be a semimartingale field and* f *be an* H*-valued predictable process satisfying 1.43. Then the following statements hold:*

1. $f \mapsto f \cdot X$ *is linear.*

2. $f \cdot X$ *is an* H*-valued semimartingale process* [7].

3. $(f \cdot X)^\tau = \left(f \mathbb{1}_{[\![0,\tau]\!]}\right) \cdot X$, *for each stopping time* τ *with* $\mathbb{P}(\tau \leqslant T) = 1$.

Proof We proceed according to the following steps:

1. The linearity follows directly Propositions 1.2.1 and 1.4.2.

2. By Decomposition (1.42), together with Propositions 1.2.1 and 1.4.2, we obtain $f \cdot X = f \cdot M + f \cdot N$ is a sum of H-valued martingale and finite variation processes.

7 A H-valued semimartingale process is any process of the form $S = M + A$ where M is an H-valued martingale process while A is an H-valued finite variation process

3. By Propositions 1.2.1 and 1.4.2, we compute

$$(f\cdot X)^{\tau} = (f\cdot M)^{\tau} + (f\cdot N)^{\tau} = \left(f\mathbb{1}_{[\![0,\tau]\!]}\right)\cdot M + \left(f\mathbb{1}_{[\![0,\tau]\!]}\right)\cdot N = \left(f\mathbb{1}_{[\![0,\tau]\!]}\right)\cdot X.$$

This completes the proof. ∎

1.4.3 *Stochastic Fubini Theorem*

Here we shall present a form of stochastic Fubini's theorem which can be adequate for our later needs. Let $X = M + N$ be a semimartingale field on $\mathbb{R}_+ \times \mathcal{E}$ and $(U, \mathcal{U}, \eta)$ be a finite measure space.

Let us fix $T > 0$ and let $L^2_T(X, \eta; H)$ be the real Hilbert space of all $\mathcal{P}\otimes\Sigma\otimes\mathcal{U}$-measurable functions $f : \Omega\times[0,\infty)\times E\times U \to H$ for which $\|f\|_{T,\eta} < \infty$, where

$$\begin{aligned}\|f\|_{T,\eta} =& \mathbb{E}\left[\int_U \left(\int_0^T\int_E \|f(s,x)\|^2 \langle M\rangle(ds,dx)\right)\eta(du)\right] \\ &+ \mathbb{E}\left[\int_U \left(\int_0^T\int_E \|f(s,x)\| V_N(ds,dx)\right)^2 \eta(du)\right].\end{aligned} \tag{1.47}$$

Theorem 1.4.2 *Let* f *be in* $L^2_T(X,\eta;H)$. *Then for each* $t\in[0,T]$,

$$\int_U \left[\int_0^t\int_E f(s,x,u)X(ds,dx)\right]\eta(du) = \int_0^t\int_E \left[\int_U f(s,x,u)\eta(du)\right] X(ds,dx). \tag{1.48}$$

Proof First it is important to note that if condition (1.47) is fulfilled then both integrals in equation (1.48) are well-defined.

1) Assume f is an elementary function, i.e.,

$$f = f_0\mathbb{1}_{(0,a]\times A\times B}, \quad (0,a]\times A\times B \subset [0,T]\times E\times U,$$

where f_0 is a bounded $\mathcal{F}_0$-measurable random variable. Note that by computing both integrals in (1.48), lead to the same process

$$f_0 I_X((0,t\wedge a],A)\eta(B),$$

that is the result holds true.

2) In a second step, we use the limit approximation argument to show the result for $L^2_T(X,\eta;H)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple function converging to some function $f \in L^2_T(X,\eta;H)$, i.e., $\|f_n - f\|_{T,\eta} \to 0$ as $n \to +\infty$.

a. To do so, we first assume $X \in \mathcal{V}_{\mathcal{E}}$, then

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_E \left(\int_U \left[f_n(s,x,u) - f(s,x,u)\right]\eta(du)\right) X(ds,dx)\right\|^2\right] \\
&\leqslant \mathbb{E}\left(\int_0^t \int_E \left\| \int_U \left[f_n(s,x,u) - f(s,x,u)\right]\eta(du)\right\| V_X(ds,dx)\right)^2 \\
&\leqslant \mathbb{E}\left(\int_0^t \int_E \int_U \left\|f_n(s,x,u) - f(s,x,u)\right\|\eta(du) V_X(ds,dx)\right)^2 \\
&\leqslant \|f_n - f\|_{T,\eta} \to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_U \left(\int_0^t \int_E \left[f_n(s,x,u) - f(s,x,u)\right] X(ds,dx)\right)\eta(du)\right\|^2\right] \\
&\leqslant \mathbb{E}\left[\int_U \left\| \int_0^t \int_E \left[f_n(s,x,u) - f(s,x,u)\right] X(ds,dx)\right)\right\|^2 \eta(du)\right] \\
&\leqslant \mathbb{E}\left(\int_U \int_0^t \int_E \left\|f_n(s,x,u) - f(s,x,u)\right\| V_X(ds,dx)\eta(du)\right)^2 \\
&\leqslant \|f_n - f\|_{T,\eta} \to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
&\int_U \left[\int_0^t \int_E f(s,x,u) X(ds,dx)\right]\eta(du) \\
&= \lim_{n\to\infty} \int_U \left[\int_0^t \int_E f_n(s,x,u) X(ds,dx)\right]\eta(du) \\
&= \lim_{n\to\infty} \int_0^t \int_E \left[\int_U f_n(s,x,u)\eta(du)\right] X(ds,dx) \\
&= \int_0^t \int_E \left[\int_U f(s,x,u)\eta(du)\right] X(ds,dx).
\end{aligned} \tag{1.49}$$

b. A similar argument to prove that the result also holds true if $X \in \mathcal{M}_{\mathcal{E}}$. By Itô isometry (1.4.2), we estimate

$$
\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_E \left(\int_U \left[f_n(s,x,u) - f(s,x,u)\right]\eta(du)\right) X(ds,dx)\right\|^2\right] \\
&= \mathbb{E}\left[\int_0^t \int_E \left\| \int_U \left[f_n(s,x,u) - f(s,x,u)\right]\eta(du)\right\|^2 \langle X\rangle (ds,dx)\right] \\
&\leqslant \mathbb{E}\left[\int_0^t \int_E \int_U \left\| \left[f_n(s,x,u) - f(s,x,u)\right]\right\|^2 \eta(du) \langle X\rangle (ds,dx)\right] \\
&\leqslant \|f_n - f\|_{T,\eta} \to 0 \quad \text{as } n \to +\infty.
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left\| \int_U \left(\int_0^t \int_E \left[f_n(s,x,u) - f(s,x,u)\right] X(ds,dx)\right)\eta(du)\right\|^2\right] \\
&\leqslant \mathbb{E}\left[\int_U \left\| \int_0^t \int_E \left[f_n(s,x,u) - f(s,x,u)\right] X(ds,dx)\right)\right\|^2 \eta(du)\right] \\
&\leqslant \mathbb{E}\left(\int_U \int_0^t \int_E \left\| f_n(s,x,u) - f(s,x,u)\right\|^2 \langle X\rangle (ds,dx)\eta(du)\right) \\
&\leqslant \|f_n - f\|_{T,\eta} \to 0 \quad \text{as } n \to +\infty.
\end{aligned}
$$

Then using the limit approximation yields

$$
\int_U \left[\int_0^t \int_E f(s,x,u)X(ds,dx)\right]\eta(du) = \int_0^t \int_E \left[\int_U f(s,x,u)\eta(du)\right] X(ds,dx). \tag{1.50}
$$

Finally, we use the decomposition $X = M + N$ together with equations (1.49) and (1.50) to conclude that for any semimartingale field X and $f \in L^2_T(X,\eta;H)$ it holds that

$$
\int_U \left[\int_0^t \int_E f(s,x,u)X(ds,dx)\right]\eta(du) = \int_0^t \int_E \left[\int_U f(s,x,u)\eta(du)\right] X(ds,dx). \tag{1.51}
$$

∎

1.4.4 *Unification Framework*

Recall H is an Hilbert space and let (E_1, Σ_1), (E_2, Σ_2) be two Blackwell spaces such that there are respectively two countable semi-rings $\mathcal{E}_1, \mathcal{E}_2$ with

$\Sigma_1 = \sigma(\mathcal{E}_1)$ and $\Sigma_2 = \sigma(\mathcal{E}_2)$. We consider two semimartingale fields X_1 and X_2 based respectively on $\mathbb{R}_+ \times \mathcal{E}_1$ and $\mathbb{R}_+ \times \mathcal{E}_2$.

Theorem 1.4.3 *Let ϕ and φ be respectively two mappings belong to $\mathcal{L}^2_{\mathrm{loc}}(X_1; H)$ and $\mathcal{L}^2_{\mathrm{loc}}(X_2; H)$. There exist a Blackwell space (E, Σ) with $\Sigma = \sigma(\mathcal{E})$, a semimartingale field X on $\mathbb{R}_+ \times \mathcal{E}$ and a mapping $\Psi : \mathcal{L}^2_{\mathrm{loc}}(X_1; H) \times \mathcal{L}^2_{\mathrm{loc}}(X_2; H) \to \mathcal{L}^2_{\mathrm{loc}}(X; H)$ such that*

$$\int_0^t \int_{E_1} \phi(s,x) X_1(ds,dx) + \int_0^t \int_{E_2} \varphi(s,y) X_2(ds,dy) = \int_0^t \int_E \Psi^{\varphi}_{\phi}(s,z) X(ds,dz), \tag{1.52}$$

with $\Psi^{\varphi}_{\phi} = \phi \mathbb{1}_{E_1} + \varphi \mathbb{1}_{E_2}$.

Proof The proof is done in three steps:

1. Let τ be an abstract point that is not respectively in both sets E_1, E_2. We extend respectively E_1 to $\widetilde{E}_1 := E_1 \cup \{\tau\}$ and E_1 to $\widetilde{E}_2 := E_2 \cup \{\tau\}$ by adjoining the cemetery point τ. Both semi-rings $\mathcal{E}_1, \mathcal{E}_2$ are respectively extended analogously to $\widetilde{\mathcal{E}}_1 = \mathcal{E}_1 \cup \{\{\tau\}\}, \widetilde{\mathcal{E}}_2 = \mathcal{E}_2 \cup \{\{\tau\}\}$ so that $(\widetilde{E}_1, \sigma(\widetilde{\mathcal{E}}_1))$, $(\widetilde{E}_2, \sigma(\widetilde{\mathcal{E}}_2))$ are Blackwell spaces as well. We can now define a new set $E = \widetilde{E}_1 \times \widetilde{E}_2$ and a countable semi-ring $\mathcal{E} := \widetilde{\mathcal{E}}_1 \times \widetilde{\mathcal{E}}_2$ on E such that (E, Σ) is Blackwell space with $\Sigma := \sigma(\widetilde{\mathcal{E}}_1 \times \widetilde{\mathcal{E}}_2)$ (see Lemma A.2.1).

2. Next, we construct a semimartingale field X on $\mathbb{R}_+ \times \mathcal{E}$ from X_1 and X_2. First, we denote by δ^2_τ the Dirac premeasure on $(\widetilde{E}_2, \widetilde{\mathcal{E}}_2)$ and we extend the premeasure $A \mapsto X_1(t, A)$ to a premeasure $\overline{X}_1$ on $(\widetilde{E}_1, \widetilde{\mathcal{E}}_1)$, that is:

$$\overline{X}_1(\cdot, A) = \begin{cases} X_1(\cdot, A) & \text{if } A \in \mathcal{E}_1 \\ 0 & \text{else.} \end{cases}$$

Since both δ^2_τ and X_1 are σ-finites, therefore for each $t \geqslant 0$ the set function

$$\widetilde{X}_1(t, \cdot) : \mathcal{E} \to [0, \infty], \quad \widetilde{X}_1(t, C \times D) = \overline{X}_1(t, C)\delta^2_\tau(D),$$

defines (uniquely) a premeasure $(E, \mathcal{E})$ (see A.1.5). Note that for any decomposition $X_1 = M_1 + N_1$, we get

$$\widetilde{X}_1(t, C \times D) = M_1(t, C)\delta^2_\tau(D) + N_1(t, C)\delta^2_\tau(D), \quad \text{for all } t \geqslant 0, C \times D \in \mathcal{E},$$

and here the field $(t, C \times D) \mapsto M_1(t, C)\delta_\tau^2(D)$ belongs to $\mathcal{M}_\mathcal{E}$ as if $M_1 \in \mathcal{M}_{\mathcal{E}_1}$ then it follows that:

a) $M_1(\cdot, C)\delta_\tau^2(D) \in \mathcal{H}^2$, for each $C \times D \in \mathcal{E}$.

b) $(t, C \times D) \mapsto \langle M_1\delta_\tau^2\rangle(t, C \times D) = \delta_\tau^2(D)\langle M_1\rangle(t, C)$ exists and belongs to $\mathcal{V}_\mathcal{E}^+$.

On the other hand, we also obtain that $(t, C \times D) \mapsto N_1(t, C)\delta_\tau^2(D)$ belongs to $\mathcal{V}_\mathcal{E}$ since δ_τ^2 is a finite measure and $N_1 \in \mathcal{V}_{\mathcal{E}_1}$. This implies that $\widetilde{X}_1$ is a semimartingale field on $\mathbb{R}_+ \times \mathcal{E}$.

Analogously, by symmetry and repeating the same arguments, if δ_τ^1 is the Dirac measure on $(\widetilde{E}_1, \widetilde{\mathcal{E}}_1)$ and $\overline{X}_2(t, \cdot)$ is the extended premeasure of $X_2(t, \cdot) = M_2(t, \cdot) + N_2(t, \cdot)$ on $(\widetilde{E}_2, \widetilde{\mathcal{E}}_2)$ for any $t \geqslant 0$. Then it can be verified that the field defined by

$$\widetilde{X}_2(t, C \times D) = \delta_\tau^1(C)M_2(t, D) + \delta_\tau^1(C)N_2(t, D), \quad \text{for all } t \geqslant 0, C \times D \in \mathcal{E},$$

is actually a semimartingale field on $\mathbb{R}_+ \times \mathcal{E}$ as respectively $\delta_\tau^1(\cdot)M_2(\cdot) \in \mathcal{M}_\mathcal{E}$ and $\delta_\tau^1(\cdot)N_2(\cdot) \in \mathcal{V}_\mathcal{E}$.

Finally, combining these two semimartingales fields, we can define the field X as follows: $X = \widetilde{X}_1 + \widetilde{X}_2$.

3. After constructing the semimartingale field X, now we shall show that there is a mapping $\Psi : \mathcal{L}^2_{loc}(X_1; H) \times \mathcal{L}^2_{loc}(X_2; H) \to \mathcal{L}^2_{loc}(X; H)$ such that for any $\phi \in \mathcal{L}^2_{loc}(X_1; H)$, $\varphi \in \mathcal{L}^2_{loc}(X_2; H)$ it holds that

$$\int_0^t \int_{E_1} \phi(s, x)X_1(ds, dx) + \int_0^t \int_{E_2} \varphi(s, y)X_2(ds, dy) = \int_0^t \int_E \Psi_\phi^\varphi(s, z)X(ds, dz). \tag{1.53}$$

Indeed, let $\Psi : (\phi, \varphi) \mapsto \Psi_\phi^\varphi$ be a mapping from $\mathcal{L}^2_{loc}(X_1; H) \times \mathcal{L}^2_{loc}(X_2; H)$ into $\mathcal{L}^2_{loc}(X; H)$. This implies that for all $t \geqslant 0$ the stochastic integral $\int_0^t \int_E \Psi_\phi^\varphi(s, z)X(ds, dz)$ is well-defined and we have

$$\int_0^t \int_E \Psi_\phi^\varphi(s, z)X(ds, dz) = \int_0^t \int_E \Psi_\phi^\varphi(s, z)\widetilde{X}_1(ds, dz) + \int_0^t \int_E \Psi_\phi^\varphi(s, z)\widetilde{X}_2(ds, dz). \tag{1.54}$$

In order to have the equivalence between relations (1.53) and (1.54), the following conditions must be satisfied

i- $\int_0^t \int_{E_1} \phi(s,x_1) X_1(ds,dx) = \int_0^t \int_E \Psi^{\varphi}_{\phi}(s,z) \widetilde{X}_1(ds,dz) = \Psi^{\varphi}_{\phi} \cdot \widetilde{X}_1(t)$,

ii- $\int_0^t \int_{E_2} \varphi(s,x_2) X_2(ds,dy) = \int_0^t \int_E \Psi_{\phi,\varphi}(s,z) \widetilde{X}_2(ds,dz) = \Psi^{\varphi}_{\phi} \cdot \widetilde{X}_2(t)$.

Since $\overline{X}_1(t,\{\tau\}) = \overline{X}_2(t,\{\tau\}) = 0$, so by Fubini's theorem we respectively compute

$$\begin{aligned} \Psi^{\varphi}_{\phi} \cdot \widetilde{X}_1 &= \int_0^{\bullet} \int_{\widetilde{E}_1} \int_{\widetilde{E}_2} \Psi^{\varphi}_{\phi}(s,x,y) \delta^2_{\tau}(dy) \overline{X}_1(ds,dx) = \int_0^{\bullet} \int_{E_1} \Psi^{\varphi}_{\phi}(s,x,\tau) X_1(ds,dx) \\ \Psi^{\varphi}_{\phi} \cdot \widetilde{X}_2 &= \int_0^{\bullet} \int_{\widetilde{E}_2} \int_{\widetilde{E}_1} \Psi^{\varphi}_{\phi}(s,x,y) \delta^1_{\tau}(dx) \overline{X}_2(ds,dy) = \int_0^{\bullet} \int_{E_2} \Psi^{\varphi}_{\phi}(s,\tau,y) X_2(ds,dy) \end{aligned} \tag{1.55}$$

and combining both expressions in (1.55) with conditions **(i)** and **(ii)** yields

$$\Psi^{\varphi}_{\phi}(s,x,\tau) = \phi(s,x) \quad \text{and} \quad \Psi^{\varphi}_{\phi}(s,\tau,y) = \varphi(s,y), \quad \text{for any } s \geqslant 0. \tag{1.56}$$

Now to construct Ψ, intuitively, we suppose that Ψ is defined as,

$$\Psi^{\varphi}_{\phi}(t,x,y) = \phi(t,x) + \varphi(t,y), \quad \text{if } x \in E_1, y \in E_2. \tag{1.57}$$

Furthermore, if we combine the expression of Ψ^{φ}_{ϕ} in (1.57) with equation (1.53) and then we identify the result with (1.54). So we obtain $\phi \cdot \widetilde{X}_2 = \varphi \cdot \widetilde{X}_1 = 0$, namely,

$$\phi \cdot \widetilde{X}_2 = \int_0^{\bullet} \int_{\widetilde{E}_2} \left[\int_{\widetilde{E}_1} \phi(s,x) \delta^1_{\tau}(dx) \right] \overline{X}_2(ds,dy) = \int_0^{\bullet} \int_{E_2} \phi(s,\tau) X_2(ds,dy) = 0$$

and

$$\varphi \cdot \widetilde{X}_1 = \int_0^{\bullet} \int_{\widetilde{E}_1} \left[\int_{\widetilde{E}_2} \varphi(s,y) \delta^2_{\tau}(dy) \right] \overline{X}_1(ds,dx) = \int_0^{\bullet} \int_{E_1} \varphi(s,\tau) X_1(ds,dx) = 0.$$

This justifies the convention that any function takes value 0 at the cemetery point because the natural way to extend ϕ (resp. φ) to $\widetilde{E}_1$ (resp. $\widetilde{E}_2$) is by taking $\phi(t,\tau) = 0$ (resp. $\varphi(t,\tau) = 0$). This means that we must have $\Psi^{\varphi}_{\phi}(t,\tau,\tau) = 0$ which shows that Ψ^{φ}_{ϕ} is well-defined on $\mathbb{R}_+ \times E$ and the mapping Ψ is defined as

$$\Psi^{\varphi}_{\phi} = \phi \mathbb{1}_{E_1} + \varphi \mathbb{1}_{E_2}.$$

∎

As a consequence, we examine its real application in the existing integration theory. Let Q be a self-adjoint, positive, symmetric, definite trace class operator on U and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in U diagonalizing Q. Let $\mathcal{W}$ be a Q-Wiener process taking value in U. Let $\mu - \lambda \otimes F$ be a compensated Poisson random measure defined on $\mathbb{R}_+ \times E$ with intensity $dt \times F(dx)$.

Corollary 1.4.1 *Let $f \in L_2(U_Q, H)$ and $g \in G^H_{loc}(\mu)$. Then there exists a space $\Lambda := \overline{\mathbb{N}} \times \overline{E}$, a countable semi-ring $\mathcal{E}_\Lambda$ on Λ, a Lévy martingale field L on $\mathbb{R}_+ \times \mathcal{E}_\Lambda$ and a mapping Ψ taking value in $\mathcal{S}(L)$ such that*

$$\int_0^\bullet f(s)\mathcal{W}(ds) + \int_0^\bullet \int_E g(s,x)[\mu(ds\times dx) - dsF(dx)] = \int_0^\bullet \int_\Lambda \Psi_{f,g}(s,z)L(ds,dz), \tag{1.58}$$

where $L = \overline{M}_\mathcal{W} + \overline{M}_\mu$ for some martingale fields $\overline{M}_\mathcal{W}, \overline{M}_\mu$ on $\mathbb{R}_+ \times \mathcal{E}_\Lambda$,
with $\Psi_{f,g}(\cdot,x,y) = \phi(\cdot,x)\mathbb{1}_\mathbb{N}(x) + g(\cdot,y)\mathbb{1}_\mathbb{E}(y)$ such that

$$\phi(t,x) = f(t)e_x, \quad \textit{for any } t \geqslant 0, x \in \mathbb{N},$$

and $\overline{\mathbb{N}}$ (resp. $\overline{E}$) is the completion of $\mathbb{N}$ (resp. E) by adjoining an arbitrary cemetery point.

Proof This follows directly from Proposition 1.4.3 combined with Proposition 1.3.4 and Corollary 1.3.1. Indeed, Proposition 1.3.4 and Corollary 1.3.1 lead respectively to the existence of two martingales fields $M_\mathcal{W}, M_\mu$, where $M_\mathcal{W}(t,\{x\}) = \langle \mathcal{W}_t, e_x\rangle_U$ and $M_\mu(t,A) = \mathbb{1}_A * (\mu - \nu)_t$. Then we define the set $\Lambda := \overline{\mathbb{N}} \times \overline{E}$ and a countable semi-ring $\mathcal{E}_\Lambda = \widetilde{\mathcal{E}}_\mathbb{N} \times \widetilde{\mathcal{E}}_E$ as in Proposition 1.4.3. Simultaneously, we also extend respectively both fields $M_\mathcal{W}, M_\mu$ to $\overline{M}_\mathcal{W}, \overline{M}_\mu$ on $\mathbb{R}_+ \times \mathcal{E}_\Lambda$ so that we can define a Lévy random field L with $L = \overline{M}_\mathcal{W} + \overline{M}_\mu$ by Proposition 1.4.3. To complete the proof, we just apply Proposition 1.4.3. ∎

To conclude this chapter, we highlight that the main purpose is to establish the unified approach for multiple stochastic integrals with respect to semimartingale fields. For this end, we introduced the concept of finite variation and martingale fields. We developed integration theory related to both class of fields. From then, we introduce the definiton of semimartingal field and the related stochastic integrations. Concurrently, we also examined the relationship between our approach and the existing ones in literature. For instance we showed that we can derive a unified integral from both stochastic integrals with respect to a Q-Wiener and a Poisson random measure.

2

GENERALIZED HILBERT SPACE-VALUED SDES

In this chapter, we study a generalized SDE version of SPDE of the kind (0.3) by taking $A \equiv 0$. Typically, we consider a SDE problem in infinite dimension on a Hilbert space. The study of Hilbert space-valued SDEs is not something new in the related literature. Moreover, several approaches and results already exist on the existence and uniqueness of solutions. However, the idea of introducing a Hilbert-space valued SDE driven by random fields is not yet well-established and it is worth exploring. This motivated us to implement our random field framework to study Hilbert-space valued SDE apart the fact it is essential later on.

The Chapter is organized as follows: In Section 1 we introduce the preliminaries and notations. In Section 2 we prove existence and uniqueness results for strong solutions to Hilbert Space-Valued SDEs driven semimartingale fields. In Section 3 we discuss the stability and regularity of solutions.

2.1 PRELIMINARIES

This section provides the required preliminaries and notations. We assume that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ are given.

Let $T > 0$ be arbitrary the time horizon and $(H, \|\cdot\|)$ be a separable Hilbert space. We denote by $\mathcal{B}(H)$ the Borel σ-fields on the separable Hilbert space H. Let (E, Σ) be a Blackwell space such that there is a countable semi-ring $\mathcal{E}$ with $\Sigma = \sigma(\mathcal{E})$. We recall that $\mathcal{S}_{\mathcal{E}}$ the space of all semimartingale fields X of the form $X = M + N$.

For any fixed $\xi \in L^2(\mathcal{F}_0; H) = L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, we consider the stochastic differential equations, on $[0, T]$ in H, of the kind:

$$dy_t = \int_E b(t, y_t, x) X(dt, dx) \quad \text{and} \quad y_0 = \xi, \tag{2.1}$$

where $b : \Omega \times \mathbb{R}_+ \times H \times E \longrightarrow H$ and $X \in \mathcal{S}_\mathcal{E}$.

Now we define the space where we want to find the solutions. We define $\mathbb{H}^2_T$ as the Banach space of all H-valued predictable processes $(\phi_t)_{t\in[0,T]}$ for which

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|\phi_t\|^2\right] < \infty.$$

The typical norm of $\mathbb{H}^2_T$ is

$$\|\phi\|_{\mathbb{H}^2_T} = \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|\phi_t\|^2\right]\right)^{\frac{1}{2}}.$$

To deal with SDE (2.1), we proceed as follow: we show directly, using the extended Banach fixed point theorem, the existence and uniqueness of strong solution on the Banach space $\mathbb{H}^2_T$. In a second step, we shall prove that the solution has a càdlàg modification which solves the stochastic convolution equation

$$y_t = \xi + \int_0^t \int_E b(s, y_{s^-}, x) X(ds, dx), \quad \text{for all } t \in [0,T], \mathbb{P}\text{-}a.s.$$

Remark 11 *For simplicity, we study our SDE on $[0,T]$. Nevertheless it is always possible to extend all results on a larger space $\mathbb{H}^2$ consisting of H-valued adapted processes ϕ such that for each $T \in \mathbb{R}_+$ the restriction of ϕ to $\Omega \times [0,T]$ belongs to $\mathbb{H}^2_T$. Indeed, if $y \in \mathbb{H}^2_T$ is a solution for SDE (2.1) then one can always construct $Y \in \mathbb{H}^2$ such that $y = Y|_{[0,T]}$. For instance, consider the stopped process $Y := y^T$.*

Remark 12 *In our framework, we allow space dependent coefficients which may depend on the randomness ω, the time t and the state of the path of the solution in order to capture a wide class of SDEs. In addition, the continuous drift term does not appear in our SDE (2.1) because it is already incorporated in the driving noise X (see the proof of Theorem 3.4.1). In fact, this is always possible by means of unified approach developed in Chapter 1.*

2.2 EXISTENCE AND UNIQUENESS OF SOLUTIONS TO HILBERT SPACE-VALUED SDES

In this section, we next establish existence and uniqueness of strong solutions to Hilbert space-valued SDEs of the type (2.1).

Definition 2.2.1 A process $y \in \mathbb{H}_T^2$ with lifetime T is called a strong solution for SDE (2.1) if $y_0 = \xi \in \mathcal{L}^2(\Omega, \mathcal{F}_0; H)$ and $b \in \mathcal{L}^2_{loc}(X; H)$ such that

$$y_t = \xi + \int_0^t \int_E b(s, y_s, x) X(ds, dx), \quad \text{for all } t \in [0, T], \mathbb{P}\text{-}a.s. \tag{2.2}$$

Remark 13 *Note that the stochastic integrals at the right-side of SDE (2.2) is well-defined up to indistinguishability. That implies that the uniqueness of solutions for SDE (2.1) is also meant up indistinguishability on $\mathbb{R}_+$ as explained in the following definition.*

Definition 2.2.2 We say that uniqueness of strong solutions to SDE (2.2) holds, if Z, Z' are respectively two strong solutions to SDE (2.2) with initial conditions z_0, z_0' and lifetime τ then we have up to indistinguishablity

$$Z^\tau \mathbb{1}_{\{y_0 = z_0\}} = (Z')^\tau \mathbb{1}_{\{y_0 = z_0'\}}. \tag{2.3}$$

We shall study the existence and uniqueness problem under standard regularity assumptions on the coefficients of SDE (2.1) that include:

Assumption 1 - b *is a* $\mathcal{P} \otimes \mathcal{B}(H) \otimes \Sigma-$ *measurable.*

Assumption 2 - *Lipschtiz continuity:*

a- b *is Lipschtiz function, i.e there is a non-decreasing function* $L : \mathbb{R}_+ \to \mathbb{R}_+$, *such that for all* $h_1, h_2 \in H$ *and* $t \in [0, T]$, $\mathbb{P}$-*a.s,*

$$\begin{aligned} \int_E \|b(t, h_1, x) - b(t, h_2, x)\| V_N(t, dx) &\leqslant L(t) \|h_1 - h_2\|. \\ \int_E \|b(t, h_1, x) - b(t, h_2, x)\|^2 \langle M \rangle (t, dx) &\leqslant L(t)^2 \|h_1 - h_2\|^2. \end{aligned} \tag{2.4}$$

b- $L \in \mathcal{L}^2_{loc}(\lambda, H)$ *and denote by* $L_T = \sup_{t \in [0,T]} L(t)^2$.

Assumption 3 - $b(\cdot, 0, \cdot) \in L_T^2(X; H)$, *i.e., for* $t \in [0, T]$ *we have*

$$\mathbb{E}\left[\left(\int_0^t \int_E \|b(s, 0, x)\| V_N(ds, dx) \right)^2 + \int_0^t \int_E \|b(s, 0, x)\|^2 \langle M \rangle (ds, dx) \right] < \infty. \tag{2.5}$$

Before, we state the main result for the existence and uniqueness of solutions we need the following lemmas.

Lemma 2.2.1 *For every $y \in \mathbb{H}^2_T$, if Assumptions (2) and (3) are fulfilled. Then the functions*

$$t \mapsto \mathbb{E}\left[\left(\int_0^t \int_E \|b(s, y_s, x)\| V_N(ds, dx)\right)^2\right] \tag{2.6}$$

$$t \mapsto \mathbb{E}\left[\int_0^t \int_E \|b(s, y_s, x)\|^2 \langle M\rangle(ds, dx)\right] \tag{2.7}$$

are well-defined and continuous on $\mathbb{R}_+$.

Proof First, let us fix $y \in \mathbb{H}^2_T$. For simplicity, we define respectively the mappings $q_1, q_2 : \Omega \times \mathbb{R}_+ \to H$,

$$t \mapsto q_1(t) = \left(\int_0^t \int_E \|b(s, y_s, x)\| V_N(ds, dx)\right)^2 \tag{2.8}$$

$$t \mapsto q_2(t) = \int_0^t \int_E \|b(s, y_s, x)\|^2 \langle M\rangle(ds, dx) \tag{2.9}$$

1. By the growth estimate (2.4) and Assumption (2-b), for every càdlàg process $y \in \mathbb{H}^2_T$ and $t \in [0, T]$, we have:

$$\mathbb{E}\left[q_1(t)\right] \leqslant 2\int_0^t L(s)^2 \sup_{r\in[0,s]} \mathbb{E}\left[\|y_r\|^2\right] ds + 2\mathbb{E}\left[\left(\int_0^t \int_E \|b(s, 0, x)\| V_N(ds, dx)\right)^2\right]$$

$$\mathbb{E}\left[q_2(t)\right] \leqslant 2\int_0^t L(s)^2 \sup_{r\in[0,s]} \mathbb{E}\left[\|y_r\|^2\right] ds + 2\mathbb{E}\left[\int_0^t \int_E \|b(s, 0, x)\|^2 \langle M\rangle(ds, dx)\right]$$

yielding, by condition (2.5),

$$\begin{aligned} \mathbb{E}\left[q_1(t)\right] &\leqslant 2 \sup_{r\in[0,t]} \mathbb{E}\left[\|y_r\|^2\right] \int_0^t L(s)^2\, ds + 2C_1 < \infty \\ \mathbb{E}\left[q_2(t)\right] &\leqslant 2 \sup_{r\in[0,t]} \mathbb{E}\left[\|y_r\|^2\right] \int_0^t L(s)^2\, ds + 2C_2 < \infty. \end{aligned} \tag{2.10}$$

for some positive real constants C_1 and C_2. This implies that both functions (2.6) and (2.7) are well-defined.

2. The proof of continuity is done by applying the Lebesgue dominated convergence on the mapping $(\omega, t) \mapsto q_i(\omega, t)$, for any $i \in \{1, 2\}$. Likewise for equation (2.10), we estimate

$$q_1(t) \leqslant 2 \sup_{r\in[0,T]} \|y_r\|^2 \int_0^T L(s)^2\, ds + 2 \sup_{r\in[0,T]} \left(\int_0^r \int_E \|b(s, 0, x)\| V_N(ds, dx)\right)^2$$

$$q_2(t) \leqslant 2 \sup_{r\in[0,T]} \|y_r\|^2 \int_0^T L(s)^2\, ds + 2 \sup_{r\in[0,T]} \int_0^r \int_E \|b(s, 0, x)\|^2 \langle M\rangle(ds, dx).$$

(2.11)

Hence by the dominated convergence theorem, if $(t_n)_{n\in\mathbb{N}}$ is an arbitrary sequence with $t_n \to t$ as $n \to \infty$. Then it follows $q_i(t_n) \to q_i(t)$, i.e.

$$\lim_{n\to\infty} \mathbb{E}\left[q_i(t_n,)\right] = \mathbb{E}\left[q_i(t)\right], \quad \text{for each } i \in \{1,2\}.$$

This implies the continuity of both functions (2.6) and (2.7). ∎

Next, let us fix $\xi \in L^2(\mathcal{F}_0; H)$, $t \in [0,T]$ and $y \in \mathbb{H}^2_T$. Then we define the process $I_\xi(y)$ by:

$$(I_\xi(y))_t = \xi + \int_0^t \int_E b(s, y_s, x) X(ds, dx), \tag{2.12}$$

and by Lemma 2.2.1, this process is well-defined, mean-square continuous and $I_\xi(y) \in \mathbb{H}^2_T$. Therefore, it induces a mapping $I : L^2(\mathcal{F}_0; H) \times \mathbb{H}^2_T \to \mathbb{H}^2_T$.

Lemma 2.2.2 *For any $\xi \in L^2(\mathcal{F}_0; H)$, if b is a Lipschitz function then there is $n_0 \in \mathbb{N}$ such that the mapping $I^{n_0}_\xi$ is a contraction on $\mathbb{H}^2_T$.*

Proof Let $Y, Z \in \mathbb{H}^2_T$ and $t \in [0,T]$ be arbitrary. For any $t \in [0,T]$, we denote by

$$q_N(t) = \left(\int_0^t \int_E \|b(s, Y_s, x) - b(s, Z_s, x)\| V_N(ds, dx)\right)^2 \tag{2.13}$$

$$q_M(t) = \int_0^t \int_E \|b(s, Y_s, x) - b(s, Z_s, x)\|^2 \langle M\rangle(ds, dx) \tag{2.14}$$

Combining Hölder's inequality, Assumption (2) and equation (1.45) yields

$$\begin{aligned}
\mathbb{E}\left[\|I_\xi(Y)_t - I_\xi(Z)_t\|^2\right] &= \mathbb{E}\left[\left\|\int_0^t \int_E [b(s, Y_s, x) - b(s, Z_s, x)] X(ds, dx)\right\|^2\right] \\
&\leqslant 2\mathbb{E}\left[q_N(t)\right] + 2\mathbb{E}\left[q_M(t)\right] \\
&\leqslant 4\mathbb{E}\left[\int_0^t L^2(s)\|Y_s - Z_s\|^2\, ds\right] \\
&\leqslant 4 \sup_{r\in[0,T]} L^2(r)\mathbb{E}\left[\int_0^t \|Y_s - Z_s\|^2\, ds\right] \\
&\leqslant 4L_T\mathbb{E}\left[\int_0^t \|Y_s - Z_s\|^2\, ds\right] \\
&\leqslant 4L_T \int_0^t \mathbb{E}\left[\|Y_s - Z_s\|^2\right] ds.
\end{aligned}$$

(2.15)

Thus, altogether, we obtain that for a certain constant C, we have

$$\sup_{t\in[0,T]}\left(\mathbb{E}\left[\|I_\xi(Y)_t - I_\xi(Z)_t\|^2\right]\right)^{\frac{1}{2}} \leqslant C\left(\int_0^T \mathbb{E}\left[\|Y_s - Z_s\|^2\right] ds\right)^{\frac{1}{2}}. \quad (2.16)$$

Next, by induction for every $n \in \mathbb{N}$ and using inequality (2.16), we iterate:

$$\begin{aligned}
\|I_\xi^n(Y) - I_\xi^n(Z)\|_{\mathbb{H}_T^2} &\leqslant \left(C\int_0^T \mathbb{E}\left[\|I_\xi^{n-1}(Y)_{t_1} - I_\xi^{n-1}(Z)_{t_1}\|^2\right] dt_1\right)^{\frac{1}{2}} \\
&\leqslant \left(C^2\int_0^T\left(\int_0^T \mathbb{E}\left[\|I_\xi^{n-2}(Y)_{t_2} - I_\xi^{n-2}(Z)_{t_2}\|^2\right] dt_2\right) dt_1\right)^{\frac{1}{2}} \leqslant \cdots \\
&\leqslant \left[C^{n-1}\int_0^T\int_0^T\cdots\int_0^T\int_0^T \mathbb{E}\left[\|I_\xi(Y)_{t_{n-1}} - I_\xi(Z)_{t_{n-1}}\|^2\right] dt_{n-1}\, dt_{n-2}\dots dt_1\right]^{\frac{1}{2}} \\
&\leqslant \left[C^n\int_0^T\int_0^T\cdots\int_0^T\int_0^T\int_0^T \mathbb{E}\left[\|Y_s - Z_s\|^2\right] ds\, dt_{n-1}\, dt_{n-2}\dots dt_1\right]^{\frac{1}{2}} \\
&\leqslant \left[C^n\int_0^T\int_0^T\cdots\int_0^T\int_0^T\int_0^T ds\, dt_{n-1}\, dt_{n-2}\dots dt_1\right]^{\frac{1}{2}} \\
&\leqslant \left(C^n\frac{T^n}{n!}\sup_{t\in[0,T]} \mathbb{E}\left[\|Y_t - Z_t\|^2\right]\right)^{1/2} \\
&\leqslant \left(C^n\frac{T^n}{n!}\right)^{\frac{1}{2}}\|Y - Z\|_{\mathbb{H}_T^2},
\end{aligned}$$

(2.17)

leading to, $\lim_{n\to\infty}\|I_\xi^n(Y) - I_\xi^n(Z)\|_{\mathbb{H}_T^2} = 0$. More precisely, there exists an index $n_0 \in \mathbb{N}$ such that $I_\xi^{n_0}$ is a contraction on $\mathbb{H}_T^2$. ∎

Theorem 2.2.1 *Suppose that Assumptions (1), (2) and (3) are fulfilled. Then for each $\xi \in L^2(\mathcal{F}_0; H)$ there exists a unique càdlàg strong solution $y \in \mathbb{H}_T^2$ for SDE (2.1) on $[0,T]$ and*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|y(t)\|^2\right] < \infty, \qquad \textit{for all } T > 0. \quad (2.18)$$

Moreover, the mapping $I(\cdot, y) : L^2(\mathcal{F}_0; H) \to \mathbb{H}_T^2$ is Lipschitz continuous for all $y \in \mathbb{H}_T^2$. In this case, the Lipschitz constant does not depend on y.

Proof The proof of theorem is done in three steps:

a) Using Lemma 2.2.2, the existence and uniqueness of strong solution $y(\xi) \in \mathbb{H}_T^2$ of SDE (2.1) with initial condition $\xi \in L^2(\mathcal{F}_0; H)$ follows by the extension of the Banach fixed point theorem (see Corollary A.4.1) on the mapping $I(\xi, \cdot) = I_\xi(\cdot)$. Namely, we get the unique fix point $y(\xi) \in \mathbb{H}_T^2$ such that

$$I(\xi, y(\xi)) = y(\xi),$$

for every $\xi \in L^2(\mathcal{F}_0; H)$ fixed, $y := (y(\xi)_t)_{t\in[0,T]}$ is the solution of (2.1). Next, we discuss the path regularity of the solution. By Remark 6 and Proposition 1.2.1, both stochastic integral processes $\int_0^\bullet \int_E b(s, y_s, x)M(ds, dx)$ and $\int_0^\bullet \int_E b(s, y_s, x)N(ds, dx)$ have respectively càdlàg paths on $[0, T]$. Moreover, the integral process

$$\int_0^t \int_E b(s, y_s, x)M(ds, dx) + \int_0^t \int_E b(s, y_s, x)N(ds, dx),$$

is adapted and mean-square continuous (by Lemma 2.2.1), then by Lemma A.4.3 the stochastic integral $\int_0^t \int_E b(s, y_s, x)X(ds, dx)$ has a predictable version on $[0, T]$ which is càdlàg. This implies that the solution process $(y_t)_{t\in[0,T]}$ admits a predictable modification with càdlàg paths on $[0, T]$ that solves the stochastic convolution equation

$$y_t = \xi + \int_0^t \int_E b(s, y_{s^-}, x)X(ds, dx).$$

b) Fix $y \in \mathbb{H}_T^2$. For any $t \in [0, T]$, we estimate

$$\begin{aligned} \|y(t)\|^2 &\leqslant 2\|\xi\|^2 + 2\left\| \int_0^t \int_E b(s, y_s, x)X(ds, dx)\right\|^2 \\ \sup_{t\in[0,T]} \|y(t)\|^2 &\leqslant 2\|\xi\|^2 + 2 \sup_{t\in[0,T]} \left\| \int_0^t \int_E b(s, y_s, x)X(ds, dx)\right\|^2 \end{aligned} \tag{2.19}$$

Noting that, for any $t \in [0, T]$, we can estimate

$$\begin{aligned}\left\|\int_0^t\int_E b(s,y_s,x)X(ds,dx)\right\|^2 &\leqslant 2\left\|\int_0^t\int_E b(s,y_s,x)M(ds,dx)\right\|^2\\&\quad+2\left\|\int_0^t\int_E b(s,y_s,x)N(ds,dx)\right\|^2\\&\leqslant 2\left\|\int_0^t\int_E b(s,y_s,x)M(ds,dx)\right\|^2\\&\quad+2\left[\int_0^T\int_E \|b(s,y_s,x)\|V_N(ds,dx)\right]^2\end{aligned} \tag{2.20}$$

and by Doob's martingale inequality, this leads to

$$\begin{aligned}\sup_{t\in[0,T]}\left\|\int_0^t\int_E b(s,y_s,x)X(ds,dx)\right\|^2 &\leqslant 4\left\|\int_0^T\int_E b(s,y_s,x)M(ds,dx)\right\|^2\\&\quad+2\left[\int_0^T\int_E \|b(s,y_s,x)\|V_N(ds,dx)\right]^2\end{aligned} \tag{2.21}$$

Combining Lemma 2.2.1 and Proposition 1.4.1 with equations (2.19) and (2.21) yields

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\|y(t)\|^2\right] &\leqslant 2\|\xi\|^2+8\mathbb{E}\left[\int_0^T\int_E \|b(s,y_s,x)\|^2\langle M\rangle(ds,dx)\right]\\&\quad+4\mathbb{E}\left[\left(\int_0^T\int_E \|b(s,y_s,x)\|V_N(ds,dx)\right)^2\right]<\infty.\end{aligned} \tag{2.22}$$

c) To conclude the proof, we show that $I(\cdot, y) : L^2(\mathcal{F}_0; H) \to \mathbb{H}^2_T$ is Lipschitz for any fixed $y \in \mathbb{H}^2_T$. For any $\xi \in L^2(\mathcal{F}_0; H)$ we compute:

$$\|I(\xi_1, y)_t - I(\xi_2, y)_t\|^2 = \|\xi_1 - \xi_2\|^2, \quad \text{for all } t \in [0, T], \tag{2.23}$$

which leads to $\|I(\xi_1, y) - I(\xi_2, y)\|_{\mathbb{H}^2_T} = \|\xi_1 - \xi_2\|_{L^2(\mathcal{F}_0;H)}$, i.e. $I(\cdot, y)$ is Lipschitz function for any $y \in \mathbb{H}^2_T$.

∎

2.3 STABILITY AND REGULARITY OF HILBERT-SPACE VALUED SDES

In this section, we deal with the stability theory of stochastic differential equations (2.1). This followed by the regular dependence on initial data for SDE (2.1).

2.3.1 Stability of Solutions for SDE

Stability of a system is the ability of the system to resist a small influence or perturbation unknown beforehand. In practice, one talks about stability of the solution y_t^0, $t \geqslant 0$, $\|y_t - y_t^0\|$ could be made small enough if some reasonable conditions are imposed, for instance, that the initial disturbance scale $\|y_0 - y_0^0\|$ is very small. Indeed, we shall be interested with H-valued SDE of the form

$$dy_t^n = \int_{E_n} b_n(t, y_{t-}^n, x) X(dt, dx), \quad \text{with } y_0^n = \xi_0, \quad t \in [0, T], \tag{2.24}$$

for each $n \in \mathbb{N}$, and we want to establish a stability result for solutions of the SDE (2.1) under appropriate regularity conditions.

We assume by Theorem 2.2.1, there is a unique (particular) solution $(y_t)_{t \geqslant 0}$ of the SDE (2.1). One the other hand, for each $n \in \mathbb{N}$, there exists a unique solution $y^n \in \mathbb{H}_T^2$ for the SPD (2.24). However, in order to assume the existence and uniqueness result (as in Theorem 2.2.1) for equation (2.24), we need to make the following regularity assumptions on the coefficients of (2.24) for each $n \in \mathbb{N}$:

Assumption 4 - *For each $n \in \mathbb{N}$, b_n is a $\mathcal{P} \otimes \mathcal{B}(H) \otimes \Sigma-$ measurable.*

Assumption 5 - *For each $n \in \mathbb{N}$, there is a non-decreasing function $L : \mathbb{R}_+ \to \mathbb{R}_+$, such that for all $h_1, h_2 \in H$ and $t \in [0, T]$, $\mathbb{P}$-a.s,*

$$\int_E \|b_n(t, h_1, x) - b_n(t, h_2, x)\| V_N(t, dx) \leqslant L(t)\|h_1 - h_2\|.$$
$$\int_E \|b_n(t, h_1, x) - b_n(t, h_2, x)\|^2 \langle M \rangle (t, dx) \leqslant L(t)^2 \|h_1 - h_2\|^2.$$

Assumption 6 - *For each $n \in \mathbb{N}$, $b_n(\cdot, 0, \cdot) \in L_T^2(X; H)$, i.e., for each $t \in [0, T]$ we have*

$$\mathbb{E}\left[\left(\int_0^t \int_E \|b_n(s, 0, x)\| V_N(ds, dx)\right)^2 + \int_0^t \int_E \|b_n(s, 0, x)\|^2 \langle M \rangle (ds, dx)\right] < \infty.$$

In addition, we also consider the following assumption to obtain the convergence of solutions.

Assumption 7 - *For any $Y \in \mathbb{H}_T^2$, we assume that, when $n \to \infty$,* *a-* $E_n \to E$;

b- *we have*

$$\mathbb{E}\left[\int_0^t \int_{E_n} \|b_n(s, Y_{s^-}, x) - b(s, Y_{s^-}, x)\|^2 \langle M \rangle (ds, dx)\right] \to 0$$
$$\mathbb{E}\left[\left(\int_0^t \int_{E_n} \|b_n(s, Y_{s^-}, x) - b(s, Y_{s^-}, x)\| V_N(ds, dx)\right)^2\right] \to 0.$$

Under these assumptions, we now investigate the stability problem for the solution $(y_t)_{t\in[0,T]}$ of the SDE (2.1). More precisely, one says that a particular solution y for SDE (2.1) is stable if, for every sequence $\{y_n\}_{n\in\mathbb{N}}$ of (unique) solutions for SDE (2.24), $\|\xi_n - \xi\| \to 0$ as $n \to \infty$ then $\|y_n - y\|_{\mathbb{H}^2_T} \to 0$ as well.

Theorem 2.3.1 *Suppose that Assumptions (1), (2), (3) and (4), (5), (6), (7) are fulfilled. Then there exists a positive constant* K *such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|y_t^n - y_t\|^2\right] < K(T)\left[\|\xi^n - \xi\|^2_{\mathbb{H}^2_0} + c_n(T)\right], \quad \textit{for all } n \in \mathbb{N}, T > 0, \tag{2.25}$$

where $c_n(T) \to 0$ *as* $n \to \infty$. *Note that here* K *and* c_n *only depend on* T *and the Lipschitz function* L.

Proof The proof is done by the two following steps:

1. First, we shall show the existence of constant c_n which converges to 0. Computing $y^n - y$, for any $n \in \mathbb{N}$ and $t \in [0, T]$, yields

$$y_t^n - y_t = \xi_n - \xi + \int_0^t \int_{E_n} b_n(s, y^n_{s^-}, x) X(ds, dx) - \int_0^t \int_E b(s, y_{s^-}, x) X(ds, dx), \tag{2.26}$$

and we compute

$$\begin{aligned}
&\int_0^t \int_{E_n} b_n(s, y^n_{s^-}, x) X(ds, dx) - \int_0^t \int_E b(s, y_{s^-}, x) X(ds, dx) \\
&= \int_0^t \int_{E_n} b_n(s, y^n_{s^-}, x) X(ds, dx) - \int_0^t \int_{E_n} b_n(s, y_{s^-}, x) X(ds, dx) \\
&\quad + \int_0^t \int_{E_n} b_n(s, y_{s^-}, x) X(ds, dx) - \int_0^t \int_{E_n} b(s, y_{s^-}, x) X(ds, dx) \\
&\quad + \int_0^t \int_{E_n} b(s, y_{s^-}, x) X(ds, dx) - \int_0^t \int_E b(s, y_{s^-}, x) X(ds, dx)
\end{aligned}$$

which leads to

$$
\begin{aligned}
&\int_0^t \int_{E_n} b_n(s, y^n_{s^-}, x) X(ds, dx) - \int_0^t \int_E b(s, y_{s^-}, x) X(ds, dx) \\
&= \int_0^t \int_{E_n} [b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x)] X(ds, dx) \\
&\quad + \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] X(ds, dx) \\
&\quad + \int_0^t \int_{E_n \setminus E} b(s, y_{s^-}, x) X(ds, dx).
\end{aligned}
\tag{2.27}
$$

Then we obtain the following inequality

$$
\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_{E_n} b_n(s, y^n_{s^-}, x) X(ds, dx) - \int_0^t \int_E b(s, y_{s^-}, x) X(ds, dx) \right\|^2\right] \\
&\leqslant 2\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x)] X(ds, dx) \right\|^2\right] + 2c_n(t)
\end{aligned}
\tag{2.28}
$$

where we define

$$
\begin{aligned}
c_n(t) :=& \mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] X(ds, dx) \right\|^2\right] \\
&+ \mathbb{E}\left[\left\| \int_0^t \int_{E_n \setminus E} b(s, y_{s^-}, x) X(ds, dx) \right\|^2\right]
\end{aligned}
\tag{2.29}
$$

Next, we shall successively estimate each term of (2.28).

A) By the growth estimate in Assumption (5) and equation 1.45 we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x)] X(ds, dx) \right\|^2\right] \\
&\leqslant 2\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x)] M(ds, dx) \right\|^2\right] \\
&\quad + 2\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x)] N(ds, dx) \right\|^2\right] \\
&\leqslant 2\mathbb{E}\left[\int_0^t \int_{E_n} \left\| b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x) \right\|^2 \langle M \rangle (ds, dx)\right] \\
&\quad + 2\mathbb{E}\left[\left(\int_0^t \int_{E_n} \left\| b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x) \right\| V_N(ds, dx) \right)^2\right] \\
&\leqslant 4\mathbb{E}\left[\int_0^t L(s)^2 \|y^n_{s^-} - y_{s^-}\|^2 \, ds\right] \leqslant 4 \int_0^t L(s)^2 \sup_{r \in [0,s]} \mathbb{E}\left[\|y^n_r - y_r\|^2\right] ds \\
&\leqslant 4 \|y^n - y\|_{\mathbb{H}^2_T} \int_0^t L(s)^2 \, ds.
\end{aligned}
\tag{2.30}
$$

Therefore, if $\|y^n - y\|_{\mathbb{H}^2_T} \to 0$ as $n \to \infty$, so we deduce

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y^n_{s^-}, x) - b_n(s, y_{s^-}, x)] X(ds, dx)\right\|^2\right] \to 0, \quad \text{as } n \to \infty. \tag{2.31}$$

B) We shall show that $c_n(T) \to 0$ when $n \to \infty$. Indeed, we first estimate:

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] X(ds, dx)\right\|^2\right] \\
&\leqslant \mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] M(ds, dx)\right\|^2\right] \\
&\quad + \mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] N(ds, dx)\right\|^2\right]
\end{aligned} \tag{2.32}$$

where, by triangle inequality and Itô isometry, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] M(ds, dx)\right\|^2\right] \\
&\leqslant \mathbb{E}\left[\int_0^t \int_{E_n} \|b_n(s, y_{s^-}, x) - b_n(s, 0, x)\|^2 \langle M\rangle (ds, dx)\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{E_n} \|b_n(s, 0, x) - b(s, 0, x)\|^2 \langle M\rangle (ds, dx)\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{E_n} \|b(s, 0, x) - b(s, y_{s^-}, x)\|^2 \langle M\rangle (ds, dx)\right].
\end{aligned} \tag{2.33}$$

Noting that, by the uniform convergence in Assumption (7-b), there is a constant $C^M_t > 0$ such that

$$\mathbb{E}\left[\int_0^t \int_{E_n} \|b_n(s, 0, x) - b(s, 0, x)\|^2 \langle M\rangle (ds, dx)\right] \leqslant C^M_t, \tag{2.34}$$

Moreover, by Lipschitz continuity of b_n and b combined with relation 2.33, for all $t \in [0, T]$ we estimate

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] M(ds, dx)\right\|^2\right] \\
&\leqslant 2\mathbb{E}\left[\int_0^t L^2(s) \|y_{s^-}\|^2 ds\right] + C^M_t \leqslant 2 \sup_{r\in[0,T]} \mathbb{E}\left[\|y_r\|^2\right] \int_0^t L^2(s)\, ds + C^M_t
\end{aligned} \tag{2.35}$$

Since the right-hand side of this last inequality does not depend on $n \in \mathbb{N}$, so we deduce from Lesbesgue's dominated convergence theorem

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left\| \int_0^t \int_{E_n} [b_n(s, y_{s^-}, x) - b(s, y_{s^-}, x)] M(ds, dx)\right\|^2\right] \to 0, \quad \text{as } n \to \infty.$$

(2.36)

Analogously, we proceed with the same manner with

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t\int_{E_n}[b_n(s,y_{s^-},x)-b(s,y_{s^-},x)]N(ds,dx)\right\|^2\right]\\
&\leqslant \mathbb{E}\left[\left(\int_0^t\int_{E_n}\|b_n(s,y_{s^-},x)-b_n(s,0,x)\|V_N(ds,dx)\right)^2\right]\\
&+\mathbb{E}\left[\left(\int_0^t\int_{E_n}\|b_n(s,0,x)-b(s,0,x)\|V_N(ds,dx)\right)^2\right]\\
&+\mathbb{E}\left[\left(\int_0^t\int_{E_n}\|b(s,0,x)-b(s,y_{s^-},x)\|V_N(ds,dx)\right)^2\right].
\end{aligned} \tag{2.37}$$

where under the Assumptions (7-b), (2) and (5), there is a constant $C_t^N > 0$ and we also obtain

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t\int_{E_n}[b_n(s,y_{s^-},x)-b(s,y_{s^-},x)]N(ds,dx)\right\|^2\right]\\
&\leqslant 2\sup_{r\in[0,T]}\mathbb{E}\left[\|y_r\|^2\right]\int_0^t L^2(s)\,ds + C_t^N
\end{aligned}$$

and by Lesbesgue's dominated convergence theorem, it follows

$$\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\int_0^t\int_{E_n}[b_n(s,y_{s^-},x)-b(s,y_{s^-},x)]N(ds,dx)\right\|^2\right]\to 0, \quad \text{as } n\to\infty. \tag{2.38}$$

This shows that, by equations (2.29) and (2.32), $c_n(T)\to 0$ when $n\to\infty$, for all $T\geqslant 0$. This is because by Assumption (7-a) the integral

$$\int_0^t\int_{E_n\setminus E} b(s,y_{s^-},x)X(ds,dx)\to 0.$$

2. By growth estimates (2.28),(2.30) and equation (2.26), we can write

$$\begin{aligned}
\sup_{s\in[0,t]}\mathbb{E}\left[\|y_s^n-y_s\|^2\right] \leqslant\, & 2\|\xi^n-\xi\|_{\mathbb{H}_0^2}+2c_n^2(T)\\
&+16\int_0^t L^2(s)\sup_{r\in[0,s]}\mathbb{E}\left[\|y_r^n-y_r\|^2\right]ds,
\end{aligned} \tag{2.39}$$

To get explicitly the constant K, we use the Gronwall Lemma (see A.4.1) to the function $t\mapsto \sup_{s\in[0,t]}\mathbb{E}\left[\|y_s^n-y_s\|^2\right]$. Namely, by equation (2.39), we have

$$\sup_{s\in[0,t]}\mathbb{E}\left[\|y_s^n-y_s\|^2\right]\leqslant 2\left(\|\xi^n-\xi\|_{\mathbb{H}_0^2}+c_n^2(T)\right)e^{16\int_0^t L^2(s)\,ds}, \tag{2.40}$$

Therefore, we deduce that

$$\|y^n - y\|_{\mathbb{H}^2_T} \leqslant K\left(\|\xi^n - \xi\|_{\mathbb{H}^2_0} + c_n^2(T)\right), \quad \text{with } K = 2e^{16\int_0^T L^2(s)\,ds}.$$

∎

As result of Theorem 2.3.1, if $\|\xi^n - \xi\|_{\mathbb{H}^2_0} \to 0$ as $n \to \infty$, thus follows $\|y^n - y\|_{\mathbb{H}^2_T} \to 0$ as well. Then we obtain the stability of solution y for SDE (2.1) relative to any (perturbed) solution y^n, $n \in \mathbb{N}$.

2.3.2 *Regularity of Solutions for SDE*

In this subsection, we study regular dependence on initial data for SDEs. More precisely, in sequel with the stability problem in Subsection 2.3.1, we shall prove the differential and continuity dependence of the solution of SDE (2.1) with respect to the initial data. Motivated by ideas from [31], we extend the regularity approach in [31] to our regularity problem of solutions.

1) We begin by fixing the curve of initial data $\varepsilon \mapsto c(\varepsilon)$ which is smooth enough (i.e. differentiable everywhere) such that its derivative $c'(\varepsilon) \in L^2(\mathbb{F}_0, H)$. Now we consider two solutions that can solve the SDE of the kind (2.1). Namely, under Assumptions (1), (2), (3), we consider $(y_t)_{t\in[0,T]} \in \mathbb{H}^2_T$ the unique solution for

$$dy_t = \int_E b(t, y_{t-}, x) X(dt, dx) \quad \text{and} \quad y_0 = \xi = c(0), \tag{2.41}$$

While we denote by $(y_t^\varepsilon)_{t\in[0,T]} \in \mathbb{H}^2_T$ the unique solution for SDE of the form

$$dy_t^\varepsilon = \int_E b(t, y_{t-}^\varepsilon, x) X(dt, dx) \quad \text{and} \quad y_0^\varepsilon = c(0), \tag{2.42}$$

under some regularity conditions to ensure the existence and uniqueness.

For $\varepsilon \neq 0$, but belongs to a neighborhood of 0, we define the variation process relative to both solutions (y^ε, y)

$$\Delta_t^\varepsilon = \frac{y_t^\varepsilon - y_t}{\varepsilon}, \quad t \geqslant 0. \tag{2.43}$$

which is indeed the unique solution of the following SDE

$$d\Delta^\varepsilon_t = \int_E \left[b(t, y_{t-} + \varepsilon\Delta^\varepsilon_{t-}, x) - b(t, y_{t-}, x)\right] X(dt, dx), \quad \Delta^\varepsilon_0 = \frac{c(\varepsilon) - c(0)}{\varepsilon}, \tag{2.44}$$

Therefore, the study of regular dependence solution of SDE (2.1) reduces to a problem of stability (as in Theorem 2.3.1) between Δ^ε and the following process determined by

$$dJ[y](v)_t = \int_E D_b(J[y](v))(t, J[y](v)_{t-}, x)X(dt, dx), \quad J[y](v)_0 = v_0, \tag{2.45}$$

where y is the unique solution to SDE (2.41) and $D_b(v)$ is the Fréchet derivative at the point b into direction v. In [31], such process $J[y](v)$ is defined as the first variation process at point y (the unique solution to SDE (2.41)) in direction v with the following properties:

1. $y^\varepsilon - y = \int_0^\varepsilon J[y](c'(\eta))\, d\eta.$
2. $v \mapsto J[y](v)$ is a linear map from $L^2(\mathbb{F}_0; H)$ into $\mathbb{H}^2_T$ and additionally it is continuously depending [1] on v.

Note that those results follow from the following theorem.

Theorem 2.3.2 *Let $\varepsilon \mapsto c(\varepsilon)$ be a curve of initial values. Suppose that Assumptions (4), (5),(6) are respectively satisfied for SDE (2.44). We assume furthermore that*

- *$b(\cdot, h, \cdot)$ is Fréchet differentiable in H.*
- *when $\varepsilon \to 0$, then*

$$\frac{b(\cdot, [y + \varepsilon J[y](v)]_-, \cdot) - b(\cdot, y_-, \cdot)}{\varepsilon} \to D_{\alpha_b}(J[y](v))(\cdot, J[y](v)_-, \cdot)$$

  in $L^2([X]; H)$. Here we have $J[y](v)_- := (J[y](v)_{t-})_{t \geqslant 0}$.

1 Continuously depend means: if for any sequence $\{v_n\}_{n\in\mathbb{N}}$ with $\|v^n - v\| \to 0$ then

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|J[y](v) - J[y](v)\|^2\right] \to 0.$$

If we also assume conditions (1), (2),(3) for SDE (2.45) with $\nu = c'(0)$*, then in a neighborhood of* 0*, we have*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|J[y](\nu)_t - \Delta_t^\varepsilon\|^2\right] < \mathcal{K}\left(\left\|c'(0) - \frac{c(\varepsilon)-c(0)}{\varepsilon}\right\|^2 + c_\varepsilon(T)^2\right), \quad T > 0, \tag{2.46}$$

where

$$\left\|c'(0) - \frac{c(\varepsilon)-c(0)}{\varepsilon}\right\|^2 + c_\varepsilon(T)^2 \to 0, \quad \textit{as } \varepsilon \to 0.$$

Proof First, as mentioned above, the regular dependence on initial data for SDE (2.24) is equivalent to a stability problem relative to both processes $(J[y](\nu), \Delta^\varepsilon)$. Therefore, in an analogously fashion, the proof can be thought as a corollary of Theorem 2.3.1. Here as we are allowed use the analogy because we do not deal with *almost-sure* convergence and all necessary conditions in Theorem 2.3.1 are fulfilled. ∎

3

HILBERT SPACE-VALUED SPDES DRIVEN BY SEMIMARTINGALE FIELDS

In sequel with Chapter 2, we aim to study the SPDE of the kind (0.3) in the case of pseudo-semigroup. By mean of the "moving frame" approach we prove successively the existence, uniqueness, stability and regularity of solutions for (0.3). Later on, we consider an infinite dimensional SPDE with a general C_0-semigroup and driven by continuous martingale.

This Chapter is organized as follows. In Section 1, we provide all basic notations. In Section 2, we introduce the different concepts of solutions. In Section 3, we prove with the existence and uniqueness of solutions for SPDE (0.3). In Section 4, we deal with stability of solutions. In Section 5, we present some fundamental examples in which we discuss the advantage of our framework compared to the existing literature. Last Section is devoted for the real application of our approach in interest rate modeling.

3.1 NOTATIONS

In this section, we recall the SPDE type that we are going to deal with. Let $T > 0$ be arbitrary the time horizon and $(H, \|\cdot\|)$ be a separable Hilbert space. We assume that a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ is given.

Let $(S_t)_{t\geqslant 0}$ be a C_0-semigroup on the Hilbert space H with infinitesimal generator $A : \mathcal{D}(A) \subset H \to H$ such that there are constants $M \geqslant 1$ and $c \in \mathbb{R}$ and we have

$$\|S_t\| \leqslant Me^{ct}, \quad \text{for } t \geqslant 0.$$

Note that both domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense in H (see Appendix A.3), where A^* is the adjoint operator of A. Recall that $\mathcal{S}_{\mathcal{E}}$ the space of all semimartingales fields on $\mathbb{R}_+ \times \mathcal{E}$.

Let ξ be a H-valued and $\mathcal{F}_0$-random variable. Given an initial data $u_0 = \xi$, we consider the following SPDE on $[0, T]$ in H:

$$du_t = Au_t\,dt + \int_E \beta(t, u_{t^-}, x) X(dt, dx), \tag{3.1}$$

where $\beta : \Omega \times \mathbb{R}_+ \times H \times E \longrightarrow H$ and $X \in \mathcal{S}_{\mathcal{E}}$ of the form $X = M + N$.

3.2 CONCEPT OF SOLUTIONS

This section is devoted to study and review all concepts solutions (i.e. strong, mild and weak solutions) related to SPDE (3.1). We then establish their possible connections.

Before we define all the three solutions concepts, we point out that uniqueness of solutions for (3.1) up to indistinguishability on the interval $t \in [0, \infty)$.

Definition 3.2.1 A process $u \in \mathbb{H}^2_T$ is called a *strong solution* to (3.1), if $u_0 = \xi$ and for any $t \in [0, \infty)$ we have:

1. $\mathbb{P}\left(u_t \in \mathcal{D}(A)\right) = 1$;
2. $Au \in \mathcal{L}_{loc}(\lambda; H)$ and $\beta \in \mathcal{L}^2_{loc}(X; H)$;
3. $u_t = \xi + \int_0^t Au_s\,ds + \int_0^t \int_E \beta(s, u_{s^-}, x) X(ds, dx), \quad \mathbb{P}\text{-}a.s.$

Definition 3.2.2 A process $u \in \mathbb{H}^2_T$ is called a *weak solution* to (3.1), if $u_0 = \xi$ and for any $\phi \in \mathcal{D}(A^*)$, $t \in [0, \infty)$ it holds that:

1. $Au \in \mathcal{L}_{loc}(\lambda; H)$ and $\beta \in \mathcal{L}^2_{loc}(X; H)$;
2. and we have, $\mathbb{P}$-*a.s*,

$$\langle \phi, u_t \rangle_H = \langle \phi, u_0 \rangle_H + \int_0^t [\langle A^*\phi, u_s \rangle_H]\,ds + \int_0^t \int_E \langle \phi, \beta(s, u_{s^-}, x) \rangle_H X(ds, dx).$$

Definition 3.2.3 A process $u \in \mathbb{H}^2_T$ is called a *mild solution* to (3.1), if

1. $Au \in \mathcal{L}_{loc}(\lambda; H)$ and $\beta \in \mathcal{L}^2_{loc}(X; H)$;
2. $u_t = S_t u_0 + \int_0^t \int_E S_{t-s} \beta(s, u_{s^-}, x) X(ds, dx)$.

Next, we give the connection between these solutions.

Lemma 3.2.1 *Let $u \in \mathbb{H}_T^2$ with $u_0 = \xi$. Then if u is a strong solution (resp. a weak solution) of SPDE (3.1), so u is also a weak solution (resp. a mild solution) of SPDE (3.1).*

Proof 1) Consider u is a strong solution of equation (3.1) and let $\phi \in \mathcal{D}(A^*)$. Since $u_t \in \mathcal{D}(A)$ for any $t \geqslant 0$, then $\langle A^*\phi, u_t\rangle_H = \langle \phi, Au_t\rangle_H$ and it follows that ($\mathbb{P}$-*a.s*):

$$\begin{aligned}\langle \phi, u_t\rangle_H =& \langle \phi, u_0\rangle_H + \int_0^t \langle \phi, Au_s\rangle_H \, ds + \int_0^t \int_E \langle \phi, \beta(s, u_{s^-}, x)\rangle_H X(ds, dx)\\ =& \langle \phi, u_0\rangle_H + \int_0^t \langle A^*\phi, u_s\rangle_H \, ds + \int_0^t \int_E \langle \phi, \beta(s, u_{s^-}, x)\rangle_H X(ds, dx).\end{aligned}$$

This implies that u is a weak solution.

2) We assume that u is a weak solution of equation (3.1). For any $t \geqslant 0$ and $\phi \in \mathcal{D}(A^*)$ we obtain $S^*_{t-s}\phi \in \mathcal{D}(A^*)$ and by Definition 3.2.2 we compute:

$$\begin{aligned}\langle S^*_{t-s}\phi, u_t\rangle_H =& \langle S^*_{t-s}\phi, u_0\rangle_H + \int_0^t \langle A^* S^*_{t-s}\phi, u_s\rangle_H \, ds\\ &+ \int_0^t \int_E \langle S^*_{t-s}\phi, \beta(s, u_{s^-}, x)\rangle_H X(ds, dx)\\ =& \Big\langle S^*_{t-s}\phi, u_0 + \int_0^t \int_E \beta(s, u_{s^-}, x) X(ds, dx)\Big\rangle_H\end{aligned}$$

and as the set $\mathcal{D}(A^*)$ dense in H we deduce that

$$u_t = S_t u_0 + \int_0^t \int_E S_{t-s}\beta(s, u_{s^-}, x) X(ds, dx), \qquad \mathbb{P}\text{-}a.s,$$

namely u is a mild solution . This completes the proof. ∎

Remark 14 *Under some regularity conditions, the converse of above statements hold true as well. In other words, if u is a mild solution (resp. a weak solution and $u \in \mathcal{D}(A)$) of SPDE (3.1) with*

$$\mathbb{E}\left[\left(\int_0^t \int_E \|\beta(s, u_{s^-}, x)\| V_N(ds, dx)\right)^2 + \int_0^t \int_E \|\beta(s, u_{s^-}, x)\|^2 \langle M\rangle(ds, dx)\right] < \infty, \tag{3.2}$$

then u is a weak solution (resp. a strong solution). We make the proof in two steps.

a) Let u be a mild solution of SPDE (3.1). To prove that u is also a weak solution, we proceed analogously with the same technique used in [72, Theorem 9.15]. We recall the mild solution that

$$u_t = S_t u_0 + \int_0^t \int_E S_{t-s}\beta(s, u_{s^-}, x) X(ds, dx),$$

and for simplicity we denote by the Bochner integral $\Lambda_t = \int_0^t \langle A^*\phi, u_s\rangle_H \, ds$.

Thanks to condition (3.2) we can use the Fubini Theorem in 1.4.2 so that one can perform the change of order on integration for the stochastic integral w.r.t. the field X *such that*

$$\begin{aligned}
\Lambda_t =& \int_0^t \left\langle A^*\phi, S_s u_0 + \int_0^s \int_E S_{s-r}\beta(u_{r-}, x)X(dr, dx)\right\rangle_H ds \\
=& \left\langle A^*\phi, \int_0^t S_s u_0 \, ds + \int_0^t \int_0^t \int_E \mathbb{1}_{[0,s]}(r) S_{s-r}\beta(r, u_{r-}, x)X(dr, dx) \, ds\right\rangle_H \\
=& \left\langle A^*\phi, \int_0^t S_s u_0 \, ds + \int_0^t \int_E \left[\int_0^t \mathbb{1}_{[0,s]}(r) S_{s-r} \, ds\right] \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H \\
=& \left\langle A^*\phi, \int_0^t S_s u_0 \, ds\right\rangle_H + \int_0^t \int_E \left\langle A^*\phi, \left[\int_r^t S_{s-r} \, ds\right] \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H \\
=& \left\langle A^*\phi, \int_0^t S_s u_0 \, ds\right\rangle_H + \int_0^t \int_E \left\langle \int_r^t S^*_{s-r} A^*\phi \, ds, \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H
\end{aligned}$$

yielding, as $\dfrac{d}{dt} S_t x = S_t A x$ *and* $A\left(\int_0^t S_s x \, ds\right) = S_t x - x$,

$$\begin{aligned}
\Lambda_t =& \langle \phi, S_t u_0 - u_0\rangle_H + \int_0^t \int_E \left\langle \int_r^t \left[\frac{d}{ds} S^*_{s-r}\phi\right] ds, \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H \\
=& \langle \phi, S_t u_0 - u_0\rangle_H + \int_0^t \int_E \left\langle S^*_{t-r}\phi - \phi, \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H \\
=& \langle \phi, S_t u_0\rangle_H - \langle \phi, u_0\rangle_H + \left\langle \phi, \int_0^t \int_E S_{t-r}\beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H \\
& - \int_0^t \int_E \left\langle \phi, \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H
\end{aligned}$$

and then

$$\begin{aligned}
\int_0^t \langle A^*\phi, u_s\rangle_H \, ds =& \left\langle \phi, S_t u_0 + \int_0^t \int_E S_{t-r}\beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H - \langle \phi, u_0\rangle_H \\
& - \int_0^t \int_E \left\langle \phi, \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H
\end{aligned}$$

We conclude that, $\mathbb{P}$*-a.s,*

$$\langle \phi, u_t\rangle_H = \langle \phi, u_0\rangle_H + \int_0^t \langle A^*\phi, u_s\rangle_H \, ds + \int_0^t \int_E \left\langle \phi, \beta(r, u_{r-}, x)X(dr, dx)\right\rangle_H .$$

That is u *is a weak solution because it holds for any* ϕ *from the dense set* $\in \mathcal{D}(A^*)$.

b) Let $\phi \in \mathcal{D}(A^*)$ *be arbitrary. Again since* $u_t \in \mathcal{D}(A)$*, so* $\langle A^*\phi, u_t\rangle_H = \langle \phi, Au_t\rangle_H$ *and we compute:*

$$\begin{aligned}\langle \phi, u_t\rangle_H =& \langle \phi, u_0\rangle_H + \int_0^t \langle A^*\phi, u_s\rangle_H \, ds + \int_0^t \int_E \langle \phi, \beta(s, u_{s^-}, x)\rangle_H X(ds, dx) \\ =& \langle \phi, u_0\rangle_H + \int_0^t [\langle \phi, Au_s\rangle_H] \, ds + \int_0^t \int_E \langle \phi, \beta(s, u_{s^-}, x)\rangle_H X(ds, dx) \\ =& \left\langle \phi, u_0 + \int_0^t Au_s \, ds + \int_0^t \int_E \beta(s, u_{s^-}, x) X(ds, dx)\right\rangle_H.\end{aligned}$$

As $\mathcal{D}(A^*)$ *is dense in* H*, and therefore we obtain*

$$u_t = u_0 + \int_0^t Au_s \, ds + \int_0^t \int_E \beta(s, u_{s^-}, x) X(ds, dx), \quad \mathbb{P}\text{-a.s.}$$

Remark 15 *For the sake of completeness, it is also worth it to mention (without a detailed proof) that if the semigroup* S *is norm continuous, i.e.* $S_t = e^{tA}$*, for all* $t \geqslant 0$*. Then we have the equivalence between those three concept of solutions. Namely, we obtain:*

$$\text{Strong solution} \iff \text{Weak solution} \iff \text{Mild solution.}$$

Indeed, by Lemma 3.2.1, it is already clear that

$$\text{Strong solution} \implies \text{Weak solution} \implies \text{Mild solution.}$$

It remains to prove that a mild solution is also a strong solution. First, we consider the mild solution

$$u_t = e^{tA} u_0 + \int_0^t \int_E e^{(t-s)A} \beta(s, u_{s^-}, x) X(ds, dx), \quad t \geqslant 0,$$

and the process

$$Z_t = \int_0^t \int_E e^{-sA} \beta(s, u_{s^-}, x) X(ds, dx), \quad t \geqslant 0,$$

then it follows that $\mathbb{P}$*-a.s,*

$$u_t = e^{tA}(u_0 + Z_t), \quad t \geqslant 0,$$

The proof can be done by using Itô-formula on the function $g : \mathbb{R}_+ \times H \to H$ *with* $g(t, x) = e^{tA}x$*, and also the fact that*

$$e^{tA}x - x = \int_0^t Ae^{tA} x \, ds,$$

in order to get

$$u_t = u_0 + \int_0^t Au_s \, ds + \int_0^t \int_E \beta(s, u_{s^-}, x) X(ds, dx), \quad t \geqslant 0.$$

3.3 HILBERT SPACE-VALUED SPDES

In this section, we establish the existence and uniqueness of mild solutions to Hilbert space-valued SPDEs of the type (3.1) by using the moving frame approach. To this end, we firstly introduce the moving framework and then establish existence and uniqueness of strong solutions to the transformed SDEs. Secondly, we proceed with the existence and uniqueness results for mild solutions to SPDEs by using the pull-back method from the moving frame.

3.3.1 *Moving Framework*

We first introduce the concept of the moving frame. For an SPDE, the moving framework is described as the time-dependent transformation from the original differential equation to a SDE in which the non-continuous drift term disappears. The methodology of the moving frame consisted of three steps:

1. **Jump to the moving frame:**
   Apply the time-dependent transformation $u_t \mapsto y_t = S_{-t}u_t = U_{-t}\iota u_t$ to reduce the SPDE (3.1) to SDE problems

2. **Solve the transformed SDE:**
   Use the framework developed in Chapter 2 to solve the derived SDE

3. **Leave the moving frame:**
   Deduce a mild solution of the original SPDE by pulling-back the solution process for the transformed SDE by $y_t \mapsto u_t = S_t y_t = \pi U_t \iota y_t$, as the following diagram shows,

$$\begin{tikzcd} \mathcal{H}_0 \arrow[r, "U_t"] & \mathcal{H}_0 \arrow[d, "\pi"] \\ H \arrow[u, "l"] \arrow[r, "S_t"'] & H \end{tikzcd}$$

Abbildung 3.1: Leaving frame diagram

for some C_0-group $(U_t)_{t\in\mathbb{R}}$ on separable Hilbert space $\mathcal{H}_0$, and continuous linear operators $l \in \mathcal{L}(H, \mathcal{H}_0)$, $\pi \in \mathcal{L}(\mathcal{H}_0, H)$.

Remark 16 *As argued and showed in [31], such diagram is always possible if the semigroup* S *is assumed to be pseudo-contractive.*

To get the existence and uniqueness of a mild solution on $[0,T]$ we make the following assumptions.

Assumption 8 - β *is* $\mathcal{P} \otimes \mathcal{B}(H) \otimes \Sigma -$ *measurable.*

Assumption 9 - *Lipschtiz continuity:*

a- *There is a non-decreasing function* $L : \mathbb{R}_+ \to \mathbb{R}_+$, *such that for all* $h_1, h_2 \in H$ *and* $t \in [0,T]$, $\mathbb{P}$-*a.s,*

$$\begin{aligned} \int_E \|\beta(t,h_1,x) - \beta(t,h_2,x)\| V_N(t,dx) \leqslant L(t)\|h_1 - h_2\|. \\ \int_E \|\beta(t,h_1,x) - \beta(t,h_2,x)\|^2 \langle M\rangle(t,dx) \leqslant L(t)^2\|h_1 - h_2\|^2. \end{aligned} \tag{3.3}$$

b- $L \in \mathcal{L}^2_{loc}(\lambda; H)$.

Assumption 10 - $\beta(\cdot, 0, \cdot) \in L^2_T(X; H)$, *i.e., for* $t \in [0,T]$, $\mathbb{P}$-*a.s we have*

$$\mathbb{E}\left[\left(\int_0^t \int_E \|\beta(s,0,x)\| V_N(ds,dx)\right)^2 + \int_0^t \int_E \|\beta(s,0,x)\|^2 \langle M\rangle(ds,dx)\right] < \infty. \tag{3.4}$$

Assumption 11 *Assume* S *is pseudo-contractive semigroup. Then there exists always another separable Hilbert space* $\mathcal{H}_0$, *a* C_0-*group* $(U_t)_{t\in\mathbb{R}}$ *on* $\mathcal{H}_0$, *and* *continuous linear operators* $\iota \in L(H, \mathcal{H}_0)$, $\pi \in L(\mathcal{H}_0, H)$ *such that we have the leaving frame diagram 3.1, i.e.*

$$\pi U_t \iota h = S_t h \quad \textit{and} \quad S_{-t} h = U_{-t} \iota h, \quad \textit{for all } t \in \mathbb{R}_+, h \in H,$$

where $\pi = \iota^*$ *and there is constants* $M \geqslant 1$, $c \in \mathbb{R}$ *such that*

$$\|U_t\| \leqslant M e^{c|t|}, \quad \textit{for all } t \in \mathbb{R}.$$

Remark 17 *Note that the pseudo-contractive property of* S *is sufficient to have the existence of* $\mathcal{H}_0$, U, ι *and* π *as in Diagram 3.1. Indeed, this follows from the Szökefalvi-Nagy Theorem on unitary dilations (see [90]). Precisely, if* S *is pseudo-contractive then there is* $c \geqslant 0$ *such that* $e^{-ct} S_t, t \geqslant 0$ *is contractive and by Szökefalvi-Nagy Theorem there are another separable Hilbert space* $\mathcal{H}_0$ *and a unitary* C_0*-group* U *on* $\mathcal{H}_0$ *such that*

$$\pi U_t \iota = e^{-ct} S_t, \quad \textit{for all } t \geqslant 0,$$

with $\iota \in L(H, \mathcal{H}_0)$ *is an isometric embedding,* $\pi = \iota^* \in L(\mathcal{H}_0, H)$ *is the orthogonal projection from* $\mathcal{H}_0$ *into* H.

Let ξ be a H-valued and $\mathcal{F}_0$-random variable. Since we are interested in finding mild solutions then we recall the stochastic convolution equation

$$u_t = S_t \xi + \int_0^t \int_E S_{t-s} \beta(s, u_{s^-}, x) X(ds, dx) \quad \text{with} \quad u_0 = \xi.$$

Then applying the time-dependent transformation $y_t = S_{-t} u_t$ to the above SPDE leads to the dynamics of the transformed SDE, namely,

$$y_t = y_0 + \int_E b(t, y_{t^-}, x) X(dt, dx) \quad \text{and} \quad y_0 = u_0, \tag{3.5}$$

or equivalently

$$dy_t = \int_E b(t, y_{t^-}, x) X(dt, dx), \tag{3.6}$$

where $b : \Omega \times \mathbb{R}_+ \times H \times E \longrightarrow H$ such that

$$b(t, h, x) := U_{-t} \iota \beta((t, \pi U_t h, x)). \tag{3.7}$$

Indeed, if we have

$$u_t = S_t \xi + \int_0^t \int_E S_{t-s} \beta(s, u_{s^-}, x) X(ds, dx) \quad \text{with} \quad u_0 = \xi.$$

By replacing $u = \pi U_\bullet l y$ and $S = \pi U_\bullet l$, we obtain

$$\pi U_t l y_t = \pi U_t l \xi + \int_0^t \int_E \pi U_{t-s} l \beta(s, \pi U_{s^-} l y_{s^-}, x) X(ds, dx)$$

and then we deduce that

$$\begin{aligned} y_t &= \xi + \int_0^t \int_E \pi U_{-s} l \beta(s, \pi U_{s^-} l y_{s^-}, x) X(ds, dx) \\ &= \xi + \int_0^t \int_E U_{-s} l \beta(s, \pi U_s l y_{s^-}, x) X(ds, dx) \end{aligned}$$

Therefore, solving SPDE (3.1) is equivalent to solve the following SDE

$$dy_t = \int_E U_{-t} l \beta(s, \pi U_t l y_{t^-}, x) X(dt, dx) \quad \text{with} \quad y_0 = \xi.$$

Remark 18 *If β is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \Sigma$-measurable, then it follows that b is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \Sigma$-measurable because the mapping $x \mapsto U_t x$ is continuous. Moreover, it is important to point out that the structure $u = \pi U_\bullet l y$ preserves the path regularity between both processes u and y. In other words, if the process y has a càdlàg path then the same for u due to the continuity property of the mapping $(t, x) \mapsto U_t x$.*

The main concern is now to solve the SDE in fashion way as in Chapter 2.

Lemma 3.3.1 *If β is Lipschtz as in Assumption (9) and assume that Assumption (11) is fulfilled. Then b is Lipschitz function and Assmuption (2) is satisfied.*

Proof For any $h_1, h_2 \in H$, $t \in [0, T]$, we compute

$$\begin{aligned} \|b(t, h_1, x) - b(t, h_2, x)\| &= \|U_{-t} l \beta(t, \pi U_t h_1, x) - U_{-t} l \beta(t, \pi U_t h_2, x)\| \\ &\leqslant \|U_{-t} l\| \|\beta(t, \pi U_t h_1, x) - \beta(t, \pi U_t h_2, x)\|. \end{aligned} \tag{3.8}$$

1. On one hand, we start with V_N and we use equation (3.8) and Assumption (9) to estimate

$$\begin{aligned} &\int_E \|b(t, h_1, x) - b(t, h_2, x)\| V_N(t, dx) \\ &\leqslant \|U_{-t} l\| \int_E \|\beta(t, \pi U_t h_1, x) - \beta(t, \pi U_t h_2, x)\| V_N(t, dx) \\ &\leqslant \|U_{-t} l\| L(t) \|\pi U_t h_1 - \pi U_t h_2\| \leqslant \|U_{-t} l\| \|\pi U_t\| L(t) \|h_1 - h_2\| \\ &\leqslant \|\pi\| \|U_t\|^2 \|l\| L(t) \|h_1 - h_2\|. \end{aligned} \tag{3.9}$$

2. On the other hand, with $\langle M\rangle$, by equation (3.8) we estimate

$$
\begin{aligned}
&\int_E \|b(t,h_1,x)-b(t,h_2,x)\|^2\,\langle M\rangle\,(t,dx)\\
&\leqslant \|U_{-t}l\|^2\int_E \|\beta(t,\pi U_t l h_1,x)-\beta(t,\pi U_t l h_2,x)\|^2\,\langle M\rangle\,(t,dx)\\
&\leqslant \|U_{-t}l\|^2 L(t)^2\|\pi U_t h_1-\pi U_t h_2\|^2 \leqslant \|U_{-t}l\|\|\pi U_t\|^2 L(t)^2\|h_1-h_2\|^2\\
&\leqslant \|\pi\|^2\|U_t\|^4\|l\|^2 L(t)^2\|h_1-h_2\|^2.
\end{aligned}
\tag{3.10}
$$

By Assumption (11) and taking into account both equations (3.9) and (3.10), we deduce that there exists a non-decreasing function $L_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $h_1, h_2 \in H$ and $t \in [0,T]$,

$$
\begin{aligned}
\int_E \|b(t,h_1,x)-b(t,h_2,x)\|V_N(t,dx) \leqslant L_1(t)\|h_1-h_2\|.\\
\int_E \|b(t,h_1,x)-b(t,h_2,x)\|^2\langle M\rangle(t,dx) \leqslant L_1(t)^2\|h_1-h_2\|^2.
\end{aligned}
\tag{3.11}
$$

where the new Lipschitz function is given by

$$L_1(t) = M^2 e^{2ct}\|\pi\|\|l\|L(t), \quad \text{for all } t \geqslant 0.$$

This completes the proof. ∎

3.3.2 *Existence and Uniqueness of Solutions for SPDEs*

We shall deal with the existence and uniqueness of solutions for SPDE (3.1) using the moving frame framework.

Theorem 3.3.1 *Suppose that Assumptions (8), (9), (10) and (11) are fulfilled for SPDE (3.1). Then, for each $\xi \in \mathcal{L}^2(\mathcal{F}_0; H)$ there existe a unique mild and weak solution $u \in \mathbb{H}^2_T$ for SPDE (3.1) with càdlàg paths on $[0,T]$ and satisfies*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t\|^2\right] < \infty, \quad \textit{for all } T > 0.$$

Proof The proof is done in four steps:

1. We show that if Assumptions (8), (9), (10) and (11) are satisfied for (3.1) then also Assumptions (1), (2), (3) are fulfilled for SDE (3.6). First, by Lemma 3.3.1, we have

$$\text{Assumption } (9) \implies \text{Assumption } (2).$$

Next, by equations (3.7) and (3.8), we obtain

$$\text{Assumptions } (8), (10) \implies \text{Assumptions } (1), (3).$$

Therefore, by Theorem 2.2.1, there exists a unique strong solution for SDE (3.6) with càdlàg paths such that, $\mathbb{P}$-*almost surely*,

$$y_t = l\xi + \int_0^t \int_E b(s, y_{s-}, x) X(ds, dx), \quad \text{for all } t \in [0, T] \tag{3.12}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|y_t\|^2\right] < \infty, \quad \text{for all } T > 0. \tag{3.13}$$

2. We prove, using the leaving frame procedure, that we shall get the existence of mild solution for (3.1), that is, the solution process $u = \pi U y$ is well-defined in $\mathbb{H}_T^2$ and has càdlàg paths on $[0, T]$. Indeed, by Assumption (11) and equation (3.12), for each $t \in [0, T]$ we compute

$$\begin{aligned}
u_t &= (\pi U y)_t = \pi U_t y_t \\
&= \pi U_t \left(l\xi + \int_0^t \int_E b(s, y_{s-}, x) X(ds, dx) \right) \\
&= \pi U_t \left(l\xi + \int_0^t \int_E U_{-s} l \beta(s, \pi U_s y_{s-}, x) X(ds, dx) \right) \\
&= \pi U_t l\xi + \int_0^t \int_E \pi U_{t-s} l \beta(s, u_{s-}, x) X(ds, dx) \\
&= S_t \xi + \int_0^t \int_E S_{t-s} \beta(s, u_{s-}, x) X(ds, dx).
\end{aligned}$$

Noting that, by Definition 3.2.3, the process $(u_t)_{t\in[0,T]}$ is a mild solution for (3.1) on $[0, T]$. Moreover, since the mapping $t \mapsto U_t h$ is continuous for any $h \in \mathcal{H}$, so Uy has càdlàg paths. In other words, u also has càdlàg paths.

3. Last, it remains to show that u is also a weak solution for SPDE (3.1). Indeed, by Assumption (11) and Theorem 2.2.1, for each $T > 0$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t\int_E S_{t-s}\beta(s,u_{s^-},x)X(ds,dx)\right\|^2\right]\\
&\leqslant \sup_{t\in[0,T]}\|\pi U_t\|\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t\int_E b(s,y_{s^-},x)X(ds,dx)\right\|^2\right].
\end{aligned}$$

Showing that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t\|^2\right] < \infty.$$

and noting that it follows from Remark 14 that if u is a mild solution for SPDE (3.1) then it is also a weak solution. ∎

Remark 19 *Let $\xi \in \mathcal{L}^2(\mathcal{F}_0; H)$ be an initial data for SPDE (3.1). If u is a mild solution to SPDE (3.1) with initial condition ξ, then SDE (3.6) admits a unique strong solution with the inital condition $l\xi$. Indeed, this can be proven by using the jumping to the moving frame procedure, i.e., one can consider the transformation $t \mapsto y_t := U_{-t} l u_t$, $t \geqslant 0$, leading to the solution process*

$$\begin{aligned}
y_t &= (U_{-}lu)_t = U_{-t}lu_t\\
&= U_{-t}l\left(S_t\xi + \int_0^t\int_E S_{t-s}\beta(s,u_{s^-},x)X(ds,dx)\right)\\
&= U_{-t}l\left(\pi U_t l\xi + \int_0^t\int_E \pi U_{t-s}l\beta(s,\pi U_s y_{s^-},x)X(ds,dx)\right)\\
&= l\xi + \int_0^t\int_E b(s,y_{s^-},x)X(ds,dx), \quad \mathbb{P}\text{-}a.s.
\end{aligned}$$

which is well-defined and has càdlàg paths on $[0,T]$.

3.3.3 *Stability for Solutions*

We now establish the stability and regularity of solutions for SPDE (3.1). Again, by the leaving the moving frame procedure, we transfer respectively the results on stability and regularity from Subsections 2.3.1 and 2.3.2.

We start with the stability problems. By Theorem 3.3.1, we assume there exists a unique solution $u \in \mathbb{H}_T^2$ for SPDE (3.1) (without perturbations). For each $n \in \mathbb{N}$, we also assume the existence of a unique solution $u^n \in \mathbb{H}^2_T$ for the (perturbed) SPDE

$$du^n_t = Au^n_t\,dt + \int_{E_n} \beta_n(t, u^n_{t-}, x)X(dt, dx), \quad u^n_0 = \xi^n \in \mathcal{L}^2(\mathcal{F}_0; H). \tag{3.14}$$

where, by jumping to moving frame procedure, its transformed SDE is

$$dy^n_t = \int_E b_n(t, y^n_{t-}, x)X(dt, dx) \quad \text{and} \quad y^n_0 = l\xi^n. \tag{3.15}$$

Therefore, the stability problem between (u, u^n) is now reduced to stability problem between (y, y^n).

Proposition 3.3.1 *Suppose that Assumptions (11) and (8), (9), (10) are analogously fulfilled for both SPDEs (3.1) and (3.14). We assume Assumption (7) is also fulfilled for both coefficients b_n and b. Then there exists a positive constant K_1 such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|u^n_t - u_t\|^2\right] < K(T)\left[\|\xi^n - \xi\|^2_{\mathbb{H}^2_0} + c_n(T)\right], \quad \textit{for all } n \in \mathbb{N}, T > 0, \tag{3.16}$$

where $c_n(T) \to 0$ as $n \to \infty$. Note that here K only depends on T and the Lipschitz function L.

Proof First of all, by Assumption (11), it is important to recall that we have respectively the implication

$$\text{Assumptions } (8), (9), (10) \implies \text{Assumption } (1), (2), (3).$$

This allows us to transfer all results from Theorem 2.3.1 to get the stability for SPDE (3.1). Next, we compute and then estimate

$$\|u_t - u^n_t\|^2 = \|\pi U_t y - \pi U_t y^n\|^2 \leqslant \|\pi\|^2\|U_t\|^2\|y - y^n\|^2 \tag{3.17}$$

$$\tag{3.18}$$

Since Assumptions (1), (2), (3) are analogously satisfied for SDEs (3.6) and (3.15). Then, by Assumption (7), we get existence of sequence

$$\begin{aligned} C_n(t) :=& \mathbb{E}\left[\left\|\int_0^t \int_{E_n} [b_n(s, y_{s-}, x) - b(s, y_{s-}, x)]X(ds, dx)\right\|^2\right] \\ &+ \mathbb{E}\left[\int_0^t \int_{E_n\setminus E} b(s, y_{s-}, x)X(ds, dx)\right\|^2\right] \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

(3.19)

If we consider the transformation $t \mapsto u_t = \pi U_t y_t$, (3.19) can be written as

$$
\begin{aligned}
C_n(t) =& \mathbb{E}\left[\left\| \int_0^t \int_{E_n} U_{-t} l \left[\beta_n(s, u_{s^-}, x) - \beta(s, u_{s^-}, x)\right] X(ds, dx) \right\|^2\right] \\
&+ \mathbb{E}\left[\left\| \int_0^t \int_{E_n \setminus E} U_{-t} l \beta(s, u_{s^-}, x) X(ds, dx) \right\|^2\right] \\
\leqslant & \|U_{-t}\|^2 \|l\|^2 \mathbb{E}\left[\left\| \int_0^t \int_{E_n} \left[\beta_n(s, u_{s^-}, x) - \beta(s, u_{s^-}, x)\right] X(ds, dx) \right\|^2\right] \\
&+ \|U_{-t}\|^2 \|l\|^2 \mathbb{E}\left[\left\| \int_0^t \int_{E_n \setminus E} \beta(s, u_{s^-}, x) X(ds, dx) \right\|^2\right] \\
\leqslant & \|l\|^2 c_n(t),
\end{aligned}
\tag{3.20}
$$

where $c_n(T)$ is a sequence of the form (2.36), i.e.,

$$
\begin{aligned}
c_n(t) =& M^2 e^{2ct} \Bigg(\mathbb{E}\left[\left\| \int_0^t \int_{E_n} \left[\beta_n(s, u_{s^-}, x) - \beta(s, u_{s^-}, x)\right] X(ds, dx) \right\|^2\right] \\
&+ \mathbb{E}\left[\left\| \int_0^t \int_{E_n \setminus E} \beta(s, u_{s^-}, x) X(ds, dx) \right\|^2\right] \Bigg) \to 0,
\end{aligned}
\tag{3.21}
$$

when $n \to \infty$. One the other hand, by Theorem 2.3.1, we have the existence of a constant

$$
K_1(T) = 2e^{16 \int_0^T L_1(s)^2 ds} \leqslant 2e^{16 M^4 \|\pi\|^2 \|l\|^2 e^{4cT} \int_0^T L(t)^2 ds}
$$

By Theorem 2.3.1 and the growth estimate (3.21), equation (3.17) can be expressed as

$$
\begin{aligned}
\sup_{t \in [0,T]} \mathbb{E}\left[\|u_t - u_t^n\|^2\right] &\leqslant \|\pi\|^2 \|U_T\|^2 \sup_{t \in [0,T]} \mathbb{E}\left[\|y - y^n\|^2\right] \\
&\leqslant \|\pi\|^2 M^2 e^{2cT} K_1(T) \left[\|y_0^n - y_0\|^2_{\mathbb{H}_0^2} + C_n(T)\right] \\
&\leqslant K(T) \left[\|\xi^n - \xi\|^2_{\mathbb{H}_0^2} + c_n(T)\right]
\end{aligned}
\tag{3.22}
$$

where $K(T) = 2M^2 \|\pi\|^2 \|l\|^2 e^{2cT + 16 M^4 \|\pi\|^2 \|l\|^2 e^{4cT} \int_0^T L(t)^2 ds}$. This shows that if $\|\xi^n - \xi\|^2_{\mathbb{H}_0^2} \to 0$, as $n \to \infty$, then $\|u_t - u_t^n\|^2_{\mathbb{H}_T^2} \to 0$. In other words, the unique solution u for SDPE (3.1) is stable. ∎

3.4 SPECIAL CASE: MARTINGALE FIELDS

In this section we study SPDE (3.1) where X is simply a martingale field. First, we show that our approach includes the SPDE (0.2) in the case where S is a pseudo-contractive semigroup. Next, we investigate the case where driving noise is a continuous martingale field and $(S_t)_{t\geqslant 0}$ is just a general C_0-semigroup. We establish the L^p-existence and uniqueness of solutions.

Let H be a separable Hilbert space. We aim to study SPDE of the form:

$$du_t = [Au_t + a(t,u_t)]\,dt + \int_E \beta(t,u_{t^-},x)M(dt,dx) \quad \text{and} \quad u_0 = \xi, \tag{3.23}$$

where $a : \Omega \times \mathbb{R}_+ \times H \to H$, $b : \Omega \times \mathbb{R}_+ \times H \times E \to H$ are $\mathcal{P} \otimes \mathcal{B}(H) \otimes \Sigma-$ measurables, $M \in \mathcal{M}_{\mathcal{E}}$ is a martingale field.

Corollary 3.4.1 *Suppose that a and b are Lipschitz functions, i.e. there is function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $h_1, h_2 \in H$ and $t \in [0,T]$, $\mathbb{P}$-a.s,*

$$\begin{aligned} \|a(t,h_1) - a(t,h_2)\| &\leqslant L(t)\|h_1 - h_2\|. \\ \|\beta(t,h_1,x) - \beta(t,h_2,x)\|^2\langle M\rangle(t,dx) &\leqslant L(t)^2\|h_1 - h_2\|^2. \end{aligned} \tag{3.24}$$

and we assume that Assumption (11) is fulfilled and

a- $L \in \mathcal{L}^2_{loc}(\mathbb{R}_+)$.

b- $\beta(\cdot,0,\cdot) \in L^2_T(M;H)$ *and* $a(\cdot,0) \in L^2_T(\lambda;H)$.

Then, for $\xi \in \mathcal{L}^2(\mathcal{F}_0;H)$, there exists a unique mild and weak solution $u \in \mathbb{H}^2_T$ for

$$u_t = S_t\xi + \int_0^t S_{t-s}a(s,u_s)\,ds + \int_0^t\int_E S_{t-s}\beta(s,u_{s^-},x)M(ds,dx), \tag{3.25}$$

satisfying

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t\|^2\right] < \infty.$$

Proof The proof follows directly from Theorem 3.3.1 and Proposition 1.2.2. By Proposition 1.2.2, stochastic convolution equation (3.25) can be rewritten as

$$u_t = S_t\xi + \int_0^t\int_{\mathbb{R}} S_{t-s}\alpha(s,u_s,x)N(ds,dx) + \int_0^t\int_E S_{t-s}\beta(s,u_{s^-},x)M(ds,dx),$$

(3.26)

where the FV field N and the coefficient α are respectively determined as follows

$$\alpha(s,h,x) := a(s,h)\mathbb{1}_{[0,1]}(x) \quad \text{and} \quad N(t,A) := \lambda([0,t])\lambda(A),$$

for any $t \geqslant 0$, $h \in H$ and $A \subset \mathbb{R}$. Note that the mapping $A \mapsto N(t,A)$ is a σ-finite premeasure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all $t \geqslant 0$. Then the Bochner integral $\alpha \cdot N$ is well-defined and it holds

$$\int_0^t S_{t-s}a(s,u_s)\,ds = \int_0^t \int_{\mathbb{R}} S_{t-s}\alpha(s,u_s,x)N(ds,dx).$$

Now, by the unification approach in Theorem 1.4.3, there exists a Blackwell space (E', Σ') such that there is a countable semi-ring $\mathcal{E}'$ with $\Sigma' = \sigma(\mathcal{E}')$, a mapping $\psi := \alpha\mathbb{1}_{\mathbb{R}} + \beta\mathbb{1}_E$ and a semimartingale field X on $\mathbb{R}_+ \times \mathcal{E}'$ such that

$$\begin{aligned}&\int_0^t \int_{E'} \psi(s,u_{s-},x)X(ds,dx)\\ &:= \int_0^t \int_{\mathbb{R}} \alpha(s,u_s,x)N(ds,dx) + \int_0^t \int_E \beta(s,u_{s-},x)M(ds,dx)\end{aligned} \tag{3.27}$$

Now the equation (3.23) becomes as a SPDE problem in view of equation (3.1), i.e., SPDE (3.23) is reduced to

$$du_t = Au_t\,dt + \int_{E'} \psi(t,u_{t-},x)X(dt,dx), \tag{3.28}$$

Noting that if a and β are Lipschitz functions then Ψ satisfies Assumption (9). Moreover, Assumptions (8) and (10) are also fulfilled for SPDE (3.28). Finally, by Theorem 3.3.1, we get the existence and uniqueness of mild and weak solution $u \in \mathbb{H}^2_T$ for SPDE (3.28) (equivalent to SPDE (3.23)). Namely, u is also mild and weak solution for SPDE (3.23) and all results follow as well. ∎

3.4.1 *Fundamental Example*

Let Q be a self-adjoint, positive, symmetric, definite trace class operator on U and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in U diagonalizing Q. Let $\mathcal{W}$

be a Q-Wiener process taking value in U. Let $\mu - \lambda \otimes F$ be a compensated Poisson random measure defined on $\mathbb{R}_+ \times E$ with intensity $dt \times F(dx)$.

As a consequence of the above result, we consider the following Markovian SPDE

$$
\begin{aligned}
dr_t &= [Ar_t + a(t, r_t)]\,dt + \sigma(t, r_t)\,dW_t + \int_E \gamma(t, r_{t^-}, x)(\mu(dt, dx) - F(dx)\,dt) \\
r_0 &= h_0,
\end{aligned}
\tag{3.29}
$$

where $a : \mathbb{R}_+ \times H \to H$, $\sigma : \mathbb{R}_+ \times H \to H$, $\gamma : \mathbb{R}_+ \times H \times E \to H$ are predictable mappings, and $A : \mathcal{D}(A) \subset H \to H$ is an infinitesimal generator of a C_0-semigroup $(S_t)_{t \geqslant 0}$ on H.

Noting that such type of infinite dimension stochastic differential equation has been very well studied in the area of SPDEs for the last two decades. It has recently received a lot of attention among researches and practitioners as it is not only more realistic for modeling purpose but also it covers wide class of SPDEs. Motivated by these reasons, it is worth to show that equation (3.29) is a particular case of SPDE (3.23).

Corollary 3.4.2 *We assume that Assumptions (11) is fulfilled and there is a function* $L \in \mathcal{L}^2_{loc}(\mathbb{R}_+)$ *such that for all* $h_1, h_2 \in H$ *and* $t \in [0, T]$, $\mathbb{P}$*-a.s,*

$$
\begin{aligned}
\|a(t, h_1) - a(t, h_2)\| &\leqslant L(t)\|h_1 - h_2\|. \\
\|\sigma(t, h_1) - \sigma(t, h_2)\|_{L_2^0} &\leqslant L(t)\|h_1 - h_2\| \\
\int_E \|\gamma(t, h_1, x) - \gamma(t, h_2, x)\|^2 F(dx) &\leqslant L(t)^2 \|h_1 - h_2\|^2.
\end{aligned}
\tag{3.30}
$$

and we suppose that $\|a(\cdot, 0)\|, \|\sigma(\cdot, 0)\|_{L_2^0} \in \mathcal{L}^2_{loc}(\mathbb{R}_+)$ *and* $\int_E \|\gamma(\cdot, 0, x)\|^2 F(dx) \in \mathcal{L}_{loc}(\mathbb{R}_+)$. *Then, for* $h_0 \in \mathcal{L}^2(\mathcal{F}_0; H)$, *there exists a unique mild and weak solution* $r \in \mathbb{H}^2_T$ *for*

$$
\begin{aligned}
r_t = &S_t h_0 + \int_0^t S_{t-s} a(s, r_s)\,ds + \int_0^t S_{t-s} \sigma(s, r_s)\,dW_s \\
&+ \int_0^t \int_E S_{t-s} \gamma(s, r_{s^-}, x)(\mu(ds, dx) - F(dx)\,ds),
\end{aligned}
\tag{3.31}
$$

satisfying

$$
\mathbb{E}\left[\sup_{t \in [0,T]} \|r_t\|^2\right] < \infty.
$$

Proof We use the unification procedure two times, first with both martingale parts to get a unified martingale field, and then combine this field with the FV derived from the continuous drift term. Namely, we proceed as follows:

1. By Corollary (1.4.1), we obtain the existence of martingale field M such that under Assumptions (11) we have the stochastic convolution:

$$\begin{aligned}&\int_0^t S_{t-s}\sigma(s,h)\mathcal{W}(ds)+\int_0^t\int_E S_{t-s}\gamma(s,h,x)\big[\mu(ds,dx)-dsF(dx)\big]\\&=\int_0^t\int_B S_{t-s}\beta(s,h,z)M(ds,dz),\quad \text{for all } h\in H, t\in[0,T],\end{aligned}\tag{3.32}$$

where $B := \overline{\mathbb{N}}\times\overline{E}$ and $\beta(t,h,x,y) := \phi(t,h,x)\mathbb{1}_{\mathbb{N}}(x)+g(t,h,y)\mathbb{1}_{\mathbb{E}}(y)$ with $\phi(t,h,x)=\sigma(t,h)e_x$, for all $h\in H$ and $t\in[0,T]$.

This implies that equation (3.29) is reduced to SPDE problem (3.23) and by Corollary 3.4.1 we have the existence and uniqueness of mild and weak solution for SPDE (3.29) such that

$$r_t = S_t h_0+\int_0^t S_{t-s}a(s,r_s)\,ds+\int_0^t\int_B S_{t-s}\beta(s,r_{s^-},x)M(ds,dx)\tag{3.33}$$

or equivalently

$$\begin{aligned}r_t =&S_t h_0+\int_0^t S_{t-s}a(s,r_s)\,ds+\int_0^t S_{t-s}\sigma(s,r_s)\,dW_s\\&+\int_0^t\int_E S_{t-s}\gamma(s,r_{s^-},x)(\mu(ds,dx)-F(dx)\,ds),\end{aligned}\tag{3.34}$$

Moreover, it follows

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|r_t\|^2\Big]<\infty,$$

which concludes the proof. ∎

3.4.2 *Continuous Martingale*

In this subsection, we shall deal with the existence and uniqueness problem for SPDE (3.23) but driven by continuous martingale field. We extend our framework in order to work with L^p-setting and we also relax the pseudo-contractive assumption to a general C_0-semigroup.

Let H be a separable Hilbert space. We aim to study SPDE of the form:

$$du_t = [Au_t+a(t,u_t)]\,dt+\int_E\beta(t,u_t,x)M(dt,dx)\quad\text{and}\quad u_0=\xi,\tag{3.35}$$

where $a : \Omega \times \mathbb{R}_+ \times H \to H$, $b : \Omega \times \mathbb{R}_+ \times H \times E \to H$, $M \in \mathcal{M}_\mathcal{E}$ is a continuous martingale field, and $A : \mathcal{D}(A) \subset H \to H$ is an infinitesimal generator of a general C_0-semigroup $(S_t)_{t \geqslant 0}$ on H.

Now, for $p \geqslant 1$, we introduce $\mathbb{H}_T^{2p}$ as a Banach space of all H-valued adapted processes $(\phi_t)_{t\in[0,T]}$ for which

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|\phi_t\|^{2p}\right] < \infty.$$

We define the norm of $\mathbb{H}_T^{2p}$

$$\|\phi\|_{\mathbb{H}_T^{2p}} = \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|\phi_t\|^{2p}\right]\right)^{\frac{1}{2}}.$$

3.4.2.1 *Stochastic Convolution integrals*

Before we state the main result for the existence and uniqueness of solutions. We need to consider the following auxiliary results that allows to estimate the stochastic convolution integral.

Lemma 3.4.1 *Let* $h \in H$, $T > 0$, $\phi_h \in L_T^2(M; H)$ *and* $p \geqslant 1$ *be arbitrary. Then there exists two positive constants* $C_{p,T}$ *and* c_p *such that*

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^t\int_E \phi_h(s,x)M(ds,dx)\right\|^{2p}\right] &\leqslant c_p\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p \\ &\leqslant C_{p,T}\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right].\end{aligned} \tag{3.36}$$

Proof (1) To prove the first inequality we shall apply the Itô's formula on the function

$$F : H \to \mathbb{R}_+, \quad x \mapsto F(x) := \|x\|^{2p},$$

which is continuous and twice Fréchet differentiable with derivatives, for all $y, z \in H$,

$$\begin{aligned}D_x F[x](y) &= 2p\|x\|^{2(p-1)}\langle x, y\rangle_H \\ D_{xx} F[x](y,z) &= 4p(p-1)\|x\|^{2(p-2)}\langle x, y\rangle_H\langle x, z\rangle_H + 2p\|x\|^{2(p-1)}\langle y, z\rangle_H\end{aligned} \tag{3.37}$$

are also continuous and bounded on bounded subsets of H with

$$\|D_{xx}F[x]\|_{\mathcal{L}(H\times H,\mathbb{R}_+)} \leqslant 2p(2p-1)\|x\|^{2(p-1)}. \tag{3.38}$$

Let us denote by $m_t = \int_0^t \int_E \phi_h(s,x)M(ds,dx)$ with $m_0 = 0$. By Itô's formula (see e.g. Métivier [65]), we obtain

$$\|m_t\|^{2p} \leqslant \int_0^t D_xF[m_s](dm_s) + \frac{1}{2}\int_0^t\int_0^t D_{xx}F[m_s](dm_s,dm_s) \tag{3.39}$$

We estimate

$$\|m_t\|^{2p} \leqslant 2p\int_0^t \|m_s\|^{2p-2}\langle m_s, dm_s\rangle_H + p(2p-2)\int_0^t\int_0^t \|m_s\|^{2p-2}\|\,dm_s\|^2 \tag{3.40}$$

Noting that we can estimate

$$\int_0^t \langle m_s, dm_s\rangle_H \leqslant \int_0^t\int_0^t \left\langle \mathbb{1}_{[0,s]}\,dm_r, dm_s\right\rangle_H \leqslant \int_0^t\int_0^t \|\,dm_s\|^2 \leqslant \|m_t\|^2. \tag{3.41}$$

Taking the expectation of (3.40) and using Hölder's, Doob's maxmimal inequalities, Itô isometry and equation (3.41), yield

$$\begin{aligned}
\mathbb{E}\left[\|m_t\|^{2p}\right] \leqslant 2p\mathbb{E}&\left[\sup_{0\leqslant s\leqslant t}\|m_s\|^{2p-2}\int_0^t\int_0^t\|\,dm_s\|^2\right]\\
&+p(2p-1)\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}\|m_s\|^{2p-2}\int_0^t\int_0^t\|\,dm_s\|^2\right]\\
&\leqslant p(2p+1)\mathbb{E}\left[\|m_t\|^2\sup_{0\leqslant s\leqslant t}\|m_s\|^{2p-2}\right]\\
&\leqslant p(2p+1)\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}\|m_s\|^{2p-2}\int_0^t\int_E\|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]\\
&\leqslant p(2p+1)\left(\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}\|m_s\|^{2p}\right]\right)^{\frac{p-1}{p}}\\
&\qquad\times\left(\mathbb{E}\left[\int_0^t\int_E\|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p\right)^{\frac{1}{p}}\\
&\leqslant p(2p+1)\left(\left(\frac{2p}{2p-1}\right)^{2p}\mathbb{E}\left[\|m_t\|^{2p}\right]\right)^{\frac{p-1}{p}}\\
&\qquad\times\left(\mathbb{E}\left[\int_0^t\int_E\|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p\right)^{\frac{1}{p}}
\end{aligned} \tag{3.42}$$

Taking the p-power of both sides of (3.42), we deduce that

$$\mathbb{E}\left[\left\|\int_0^t\int_E \phi_h(s,x)M(ds,dx)\right\|^{2p}\right] \leqslant c_p\mathbb{E}\left[\int_0^t\int_E \|\phi_h(s,x)\|^2 \langle M\rangle (ds,dx)\right]^p \tag{3.43}$$

where

$$c_p = [p(2p+1)]^p\left[\frac{2p}{2p-1}\right]^{2p(p-1)}.$$

(2) Finally, to get the second inequality we again use the Hölder's inequality to

$$\left[\int_0^T\int_E \|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p.$$

Namely, applying Hölder's inequality to $\|\phi_h\|^2$ and $\mathbb{1}_{\Omega\times E}$, leads to

$$\begin{aligned}\int_0^T\int_E \|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx) \leqslant &\left[\int_0^T\int_E \|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right]^{\frac{1}{p}}\\ &\times\left[\int_0^T\int_E \|\langle M\rangle(ds,dx)\right]^{\frac{p-1}{p}}\end{aligned} \tag{3.44}$$

Therefore, taking the p-power to (3.44) and then the expectation, we deduce

$$\mathbb{E}\left[\left\|\int_0^t\int_E \phi_h(s,x)M(ds,dx)\right\|^{2p}\right] \leqslant C_{p,T}\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right],$$

where $C_{p,T} := c_p\langle M\rangle(T,E)^{p-1}$ and this shows the second inequality. ∎

Corollary 3.4.3 *Let* $(S_t)_{t\geqslant 0}$ *be a* C_0*-semigroup on* H *and* $p \geqslant 1$. *For* $h \in H$, $\phi_h \in L^2_T(M;H)$ *and* $t \in [0,T]$

$$\begin{aligned}&\mathbb{E}\left[\left\|\int_0^t\int_E S_{t-s}\phi_h(s,x)M(ds,dx)\right\|^{2p}\right]\\ &\leqslant c^{M,T}_{p,c}\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p\\ &\leqslant C^{M,T}_{p,c}\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right].\end{aligned} \tag{3.45}$$

where $\|S_s\| \leqslant Me^{cs}$, $M \geqslant 1$ *and* $c \in \mathbb{R}$ *for all* $s \geqslant 0$.

Proof The proof is just consequence of the above result.

(1) First, note that if ϕ_h is predictable then so also is $S_{t-s}\phi_h(s,\cdot)$ for all $t \geqslant 0, s \in [0,t]$. Then, for $t \in (0,T]$, by Lemma 3.4.1, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t\int_E S_{t-s}\phi_h(s,x)M(ds,dx)\right\|^{2p}\right] \\
&\leqslant c_p,\mathbb{E}\left[\int_0^T\int_E \|S_{t-s}\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p \\
&\leqslant c_p\mathbb{E}\left[\int_0^T\int_E Me^{c(t-s)}\|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p \\
&\leqslant c_pM^2e^{2cT}\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^2\langle M\rangle(ds,dx)\right]^p
\end{aligned} \tag{3.46}$$

we take $c_{p,c}^{M,T} = C_pM^2e^{2cT}$.

(2) Likewise, by Lemma 3.4.1, we estimate

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t\int_E S_{t-s}\phi_h(s,x)M(ds,dx)\right\|^{2p}\right] \\
&\leqslant C_{p,T}\mathbb{E}\left[\int_0^T\int_E \|S_{t-s}\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right] \\
&\leqslant C_{p,T}\mathbb{E}\left[\int_0^T\int_E Me^{c(t-s)}\|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right] \\
&\leqslant C_pM^2e^{2cT}\mathbb{E}\left[\int_0^T\int_E \|\phi_h(s,x)\|^{2p}\langle M\rangle(ds,dx)\right]
\end{aligned} \tag{3.47}$$

so we have $C_{p,c}^{M,T} = C_pM^2e^{2cT}$. This concludes the proof. ∎

Lemma 3.4.2 *Let M be a continuous martingale field and $(S_t)_{t\geqslant 0}$ be a general C_0-semigroup. Let $y \in \mathbb{H}_T^{2p}$ be a continuous process, $\phi : \Omega \times \mathbb{R}_+ \times H \times E \to H$ be a predictable mapping and there is a constant $p > 1$ such that $\phi(\cdot, y_s, \cdot) \in L_T^2(M; H)$ for all $s \in [0,T]$ and*

$$\mathbb{E}\left[\int_0^T\int_E \|\phi(s,y_s,x)\|^{2p}\langle M\rangle(ds,dx)\right] < \infty, \qquad \textit{for all } T > 0, h \in H.$$

Then the stochastic convolution $S\phi * M$ is well-defined and has a continuous version.

Proof To deal with the existence of the continuous modification, we use the factorization method performed in [22] (or see [73]). The proof consists of the following steps:

(1) We find a good candidate for a modification version of the stochastic convolution $\int_0^t \int_E S_{t-s}\phi(s, y_s, x)M(ds, dx)$. In fact, we have the following identity:

$$\int_v^t (t-s)^{\theta-1}(s-v)^{-\theta}\, ds = \frac{\pi}{\sin(\pi\theta)}, \quad \text{for } 0 < \theta < 1,\ v < t. \tag{3.48}$$

It follows that for $t \in (0, T]$

$$\begin{aligned} &\int_0^t \int_E S_{t-s}\phi(s, y_s, x)M(ds, dx) \\ &= \frac{\sin(\pi\theta)}{\pi} \int_0^t \int_E \left(\int_r^t (t-s)^{\beta-1}(s-r)^{-\beta}\, ds \right) S_{t-r}\phi(r, y_r, x)M(dr, dx), \end{aligned} \tag{3.49}$$

and applying the Fubini theorem 1.4.2 and using the fact that $S_{t-u} = S_{t-s}S_{s-u}$, we obtain

$$\begin{aligned} &\frac{\pi}{\sin(\pi\theta)} \int_0^t \int_E S_{t-s}\phi(s, y_s, x)M(ds, dx) \\ &= \int_0^t \int_E \left(\int_r^t (t-s)^{\theta-1}(s-r)^{-\theta}\, ds \right) S_{t-r}\phi(r, y_r, x)M(dr, dx) \qquad (3.50) \\ &= \int_0^t (t-s)^{\theta-1} S_{t-s} \left[\int_0^s \int_E (s-r)^{-\theta} S_{s-r}\phi(r, y_r, x)M(dr, dx) \right] ds \end{aligned}$$

Let us set

$$Z_t = \int_0^t \int_E (t-r)^{-\theta} S_{t-r}\phi(r, y_r, x)M(dr, dx), \quad \text{for } t \in [0, T],$$

and we can consider the following process as the modification of our stochastic convolution

$$S\phi * M_t := \frac{\sin(\pi\theta)}{\pi} \int_0^t (t-s)^{\theta-1} S_{t-s} Z_s\, ds, \quad \text{for } t \in [0, T]. \tag{3.51}$$

(2) Let us fix $\frac{1}{2p} < \theta < \frac{1}{2}$. We shall show that $S\phi * M_t$ is well-defined. For this, by Hölder's inequality and Lemma 3.4.1, we estimate

$$\begin{aligned}
&\mathbb{E}\left[\left\|\frac{\sin(\pi\theta)}{\pi}\int_0^t (t-s)^{\theta-1} S_{t-s} Z_s \, ds\right\|^{2p}\right] \\
&\leqslant \left(\int_0^T (T-s)^{(\theta-1)\frac{2p}{2p-1}} \, ds\right)^{2p-1} \left(\frac{Me^{cT}\sin(\pi\theta)}{\pi}\right)^{2p} \int_0^T \mathbb{E}\left[\|Z_s\|^{2p}\right] ds \\
&\leqslant C_1 \int_0^T \mathbb{E}\left[\|Z_s\|^{2p}\right] ds \\
&\leqslant C_2 \int_0^T \mathbb{E}\left[\int_0^s \int_E (s-r)^{-2p\theta} \|\phi(r, y_r, x)\|^{2p} \langle M\rangle(dr, dx)\right] ds \\
&\leqslant C_2 \int_0^T \int_0^s \int_E (s-r)^{-2p\theta} \mathbb{E}\left[\|\phi(r, y_r, x)\|^{2p} \langle M\rangle(dr, dx)\right] ds \\
&\leqslant C_2 \left[\int_0^T s^{-2p\theta} \, ds\right] \mathbb{E}\left[\int_0^T \int_E \|\phi(r, y_r, x)\|^{2p} \langle M\rangle(dr, dx)\right] \\
&\leqslant C_3 \mathbb{E}\left[\int_0^T \int_E \|\phi(r, y_r, x)\|^{2p} \langle M\rangle(dr, dx)\right] < \infty
\end{aligned} \tag{3.52}$$

where $C_3 > 0$ and it depends only on p, M, θ, c. This shows that the process $(Z_t)_{t\in[0,T]}$ has almost surely $2p$-integrable paths.

(3) Last, it remains to prove that for any fixed $\theta \in (0, 1]$ and $q \geqslant 1$ the mapping

$$t \mapsto F_\theta(f)_t = \int_0^t (t-s)^{\theta-1} S_{t-s} f(s) \, ds$$

is a continuous mapping and there is a constant C such that $\|F_\theta(f)_t\| \leqslant C\|f\|_{L^q_T(\lambda;H)}$ for all $t \in [0, T], f \in L^q_T(\lambda; H)$.

Indeed, fix $t \in [0, T]$ and et $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \to t$ as $n \to \infty$. We compute

$$\begin{aligned}
\|F_\theta(f)_{t_n} - F_\theta(f)_t\| &= \left\|\int_0^{t_n} (t_n - s)^{\theta-1} S_{t_n-s} f(s) \, ds - \int_0^t (t-s)^{\theta-1} S_{t-s} f(s) \, ds\right\| \\
&\leqslant \left\|\int_0^{t_n} (t_n - s)^{\theta-1} S_{t_n-s} f(s) \, ds - \int_0^{t_n} (t-s)^{\theta-1} S_{t_n-s} f(s) \, ds\right\| \\
&\quad + \left\|\int_0^{t_n} (t-s)^{\theta-1} S_{t_n-s} f(s) \, ds - \int_0^t (t-s)^{\theta-1} S_{t-s} f(s) \, ds\right\| \\
&\leqslant \left\|\int_0^{t_n} \left[(t_n - s)^{\theta-1} - (t-s)^{\theta-1}\right] S_{t_n-s} f(s) \, ds\right\| \\
&\quad + \left\|\int_0^{t_n\wedge t} (t-s)^{\theta-1} \left[S_{t_n-s} - S_{t-s}\right] f(s) \, ds\right\| \\
&\quad + \left\|\int_{t_n\wedge t}^{t_n\vee t} (t-s)^{\theta-1} S_{t_n\vee t-s} f(s) \, ds\right\|
\end{aligned}$$

(3.53)

Noting that $S_{t_n - s}x \to S_{t-s}x$, for all $x \in H$, due to the continuity of the mapping $t \mapsto S_t x$. So when $n \to \infty$ then we obtain

$$\begin{aligned}\left\| \int_0^{t_n} \left[(t_n - s)^{\theta-1} - (t-s)^{\theta-1} \right] S_{t_n - s} f(s)\, ds \right\| &\to 0 \\ \left\| \int_0^{t_n \wedge t} (t-s)^{\theta-1} \left[S_{t_n - s} - S_{t-s} \right] f(s)\, ds \right\| &\to 0 \\ \left\| \int_{t_n \wedge t}^{t_n \vee t} (t-s)^{\theta-1} S_{t_n \vee t - s} f(s)\, ds \right\| &\to 0\end{aligned} \tag{3.54}$$

That imply that $\|F_\theta(f)_{t_n} - F_\theta(f)_t\| \to 0$, namely, $F_\theta(f)$ is continuous. It follows

$$\begin{aligned}\|F_\theta(f)_t\|^p &= \left\| \int_0^t (t-s)^{\theta-1} S_{t-s} f(s)\, ds \right\|^p \\ &\leqslant T^{\theta-1} M e^{cT} \int_0^T \|f(s)\|^p\, ds\end{aligned} \tag{3.55}$$

which leads to

$$\|F_\theta(f)_t\| \leqslant T^{\frac{\theta-1}{p}} M^{\frac{1}{p}} e^{\frac{cT}{p}} \left(\int_0^T \|f(s)\|^p\, ds \right)^{\frac{1}{p}} \leqslant C\|f\|_{L_T^q(\lambda;H)}$$

Finally, the existence of the continuous modification of $\int_0^t \int_E S_{t-s}\phi(s, y_s, x) M(ds, dx)$ follows from the continuity of $F_\theta(Z)_t$, i.e,

$$S\phi * M_t = \frac{\sin(\pi\theta)}{\pi} F_\theta(Z)_t,$$

has a continuous version. ∎

First, we start with the deterministic stochastic convolution $\int_0^t S_{t-s} f(s,x)\, ds$ which can be viewed as a standard Bochner integral.

Lemma 3.4.3 *Let* $a : \Omega \times \mathbb{R}_+ \times H \to H$ *be a progressively measurable process such that*

$$\mathbb{P}\left(\int_0^t \|a(s,h)\|^{2p}\, ds < \infty \right) = 1, \quad \textit{for } t > 0, p \geqslant 1.$$

Then the mapping $Y : \Omega \times \mathbb{R}_+ \times H \to H$ *with*

$$Y(\omega, t, h) := \int_0^t S_{t-s} a(\omega, s, h)\, ds, \quad \textit{for } t > 0 \textit{ and } \omega \in \Omega,$$

is well-defined and continuous in L^{2p}.

Proof (1) Concerning the existence, by Hölder inequality, one compute for $t > 0$:

$$\mathbb{E}\left[\|Y(t,h)\|^{2p}\right] = \mathbb{E}\left[\left\|\int_0^t S_{t-s}a(s,h)\,ds\right\|^{2p}\right]$$
$$\leqslant TM^{2p}e^{2pcT}\mathbb{E}\left[\int_0^t \|a(s,h)\|^{2p}\,ds\right] < \infty.$$

(2) To obtain the continuity, one may use the Lebesgue's theorem. Indeed, we consider an arbitrary sequence $(t_n)_{n\in\mathbb{R}_+}$ sucht that $t_n \to t$ and we compute

$$\begin{aligned}\|Y(t_n,h) - Y(t,h)\|^{2p} &= \left\|\int_0^t S_{t-s}a(s,h)\,ds - \int_0^{t_n} S_{t_n-s}a(s,h)\,ds\right\|^{2p} \\ &\leqslant \int_0^{t\wedge t_n} \left\|S_{t-s}a(s,h) - S_{t_n-s}a(s,h)\right\|^{2p} ds \\ &\quad + \int_{t\wedge t_n}^{t\vee t_n} \left\|S(t\vee t_n - s)a(s,h)\right\|^{2p} ds.\end{aligned} \tag{3.56}$$

We have $S_{t_n-s}h \to S_{t-s}h$, for $h \in H$, as mapping $t \mapsto S_t h$ is continuous. Then, one can apply the Lebesgue's dominated convergence theorem here and obtain

$$\mathbb{E}\left[\int_0^{t\wedge t_n} \|S_{t-s}a(s,h) - S_{t_n-s}a(s,h)\|^{2p}\,ds\right] \to 0, \quad \text{as } n \to \infty.$$

Likewise, when n goes to ∞ therefore we have

$$\mathbb{E}\left[\int_{t\wedge t_n}^{t\vee t_n} \|S(t\vee t_n - s)a(s,h)\|^{2p}\,ds\right] \to 0.$$

Then the continuity follows. ∎

3.4.2.2 *Existence and Uniqueness of Mild Solutions*

We now proceed with the proof of the existence and uniqueness of mild and weak solutions for SPDE (3.35) using the Banach fixed point argument.

For any $p \geqslant 1$, we replace our previous assumptions by the following conditions:

Assumption 12 - α *and* β *are predictable mappings.*

Assumption 13 - *There is a non-decreasing function* $L \in \mathcal{L}^{2p}_{loc}(\lambda; \mathbb{R}_+)$ *such that for all* $h_1, h_2 \in H$ *and* $t \in [0,T]$, $\mathbb{P}$*-a.s,*

$$\|a(t,h_1) - a(t,h_2)\|^{2p} \leqslant L(t)^{2p}\|h_1 - h_2\|^{2p}.$$

$$\int_E \|\beta(t,h_1,x) - \beta(t,h_2,x)\|^{2p}\langle M\rangle(t,dx) \leqslant L(t)^{2p}\|h_1 - h_2\|^{2p}.$$

Assumption 14 - $b(\cdot,0,\cdot) \in L^{2p}_T(M;H)$ *and* $a(\cdot,0) \in L^{2p}_T(\lambda;H)$.

Lemma 3.4.4 *Suppose Assumptions (12), (13) and (14) are fulfilled. For every* $t \in [0,T]$ *and continuous processes* $Y, Z \in \mathbb{H}^{2p}_T$, *there exist two constants* $C_1, C_2 > 0$ *such that we have*

$$\mathbb{E}\left[\left\|\int_0^t S_{t-s}\left[a(s,Y_s) - a(s,Z_s)\right]ds\right\|^{2p}\right] \leqslant C_1\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right]ds;$$

$$\mathbb{E}\left[\left\|\int_0^t\int_E S_{t-s}\left[\beta(s,Y_s,x) - \beta(s,Z_s,x)\right]M(ds,dx)\right\|^{2p}\right]$$
$$\leqslant C_2\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right]ds.$$

Proof By lemmas 3.4.2 and 3.4.3, we have the existence of both constants $C_1 = C_1(p,c,T,M)$ and $C_2 = C_2(p,c,T,M,\theta)$ such that

$$\mathbb{E}\left[\left\|\int_0^t S_{t-s}\left[a(s,Y_s) - a(s,Z_s)\right]ds\right\|^{2p}\right] \leqslant C_1\mathbb{E}\left[\int_0^T \|a(s,Y_s) - a(s,Z_s)\|^{2p}ds\right]$$

and

$$\mathbb{E}\left[\left\|\int_0^t\int_E S_{t-s}\left[\beta(s,Y_s,x) - \beta(s,Z_s,x)\right]M(ds,dx)\right\|^{2p}\right]$$
$$\leqslant C_2\mathbb{E}\left[\int_0^T\int_E \|\beta(s,Y_s,x) - \beta(s,Z_s,x)\|^{2p}\langle M\rangle(ds,dx)\right].$$

Under condition (13), we shall obtain

$$\mathbb{E}\left[\left\|\int_0^t S_{t-s}\left[a(s,Y_s) - a(s,Z_s)\right]ds\right\|^{2p}\right] \leqslant C_1\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right]ds$$

$$\mathbb{E}\left[\left\|\int_0^t\int_E S_{t-s}\left[\beta(s,Y_s,x) - \beta(s,Z_s,x)\right]M(ds,dx)\right\|^{2p}\right]$$
$$\leqslant C_2\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right]ds.$$

This completes the proof. ∎

For any fixed $\xi \in L^2(\mathcal{F}_0; H)$, $t \in [0,T]$ and a continuous process $u \in \mathbb{H}^{2p}_T$, let us define the process $I_\xi(u)$ by:

$$I_\xi(u)_t = \xi + \int_0^t S_{t-s} a(s, u_s)\, ds + \int_0^t \int_E S_{t-s} \beta(s, u_s, x) M(ds, dx), \quad (3.57)$$

note that, by Lemma 3.4.4, this process is well-defined as we can estimate

$$\mathbb{E}\left[\left\| \int_0^t S_{t-s} a(s, u_s)\, ds \right\|^{2p}\right] \leqslant \sup_{r \in [0,t]} \mathbb{E}\left[\|u_r\|^{2p}\right] \int_0^T L(s)^{2p}\, ds + K_1$$

$$\mathbb{E}\left[\left\| \int_0^t \int_E S_{t-s} \beta(s, u_s, x) M(ds, dx) \right\|^{2p}\right] \leqslant \sup_{r \in [0,t]} \mathbb{E}\left[\|u_r\|^{2p}\right] \int_0^T L(s)^{2p}\, ds + K_2$$

where

$$K_1 = \mathbb{E}\left[\int_0^T \|a(s,0)\|^{2p}\, ds\right] \quad \text{and} \quad K_2 = \mathbb{E}\left[\int_0^T \int_E \|\phi(s,0,x)\|^{2p} \langle M \rangle (ds, dx)\right],$$

Note that both terms in right side of the above inequality are finite under Assumptions (13) and (14), and we deduce that $I_\xi(u) \in \mathbb{H}^{2p}_T$ as

$$\sup_{t \in [0,T]} \mathbb{E}\left[\|I_\xi(u)_t\|^{2p}\right] < \infty.$$

Therefore, it induces a mapping $I : L^2(\mathcal{F}_0; H) \times \mathbb{H}^{2p}_T \to \mathbb{H}^{2p}_T$.

Theorem 3.4.1 *Assume that $\xi \in L^2(\mathcal{F}_0; H)$ and conditions (12), (13) and (14) are satisfied. Then there exists a unique mild and weak solution $u \in \mathbb{H}^{2p}_T$ for SPDE (3.35) with*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u(t)\|^{2p}\right] < \infty, \qquad \textit{for all } T > 0. \quad (3.58)$$

Moreover, it has a continuous modification and the mapping $\xi \mapsto u_\xi$ is Lipschitz continuous for all $u \in \mathbb{H}^{2p}_T$.

Proof We show the existence and uniqueness of mild solution $u(\xi) \in \mathbb{H}^{2p}_T$ of (3.35) with initial condition $\xi \in L^2(\mathcal{F}_0; H)$ by using the extension of the Banach fixed point theorem on the mapping $I(\xi, \cdot)$. The proof of the theorem is done in four steps:

a) Let $Y, Z \in \mathbb{H}^{2p}_T$ be continuous processes. First, we compute

$$\begin{aligned} &\mathbb{E}\left[\|I_\xi(Y)_t - I_\xi(Z)_t\|^{2p}\right] \\ &\leqslant 2^{2p-1}\mathbb{E}\left[\left\| \int_0^t S_{t-s}\left[a(s, Y_s) - a(s, Z_s)\right] ds \right\|^{2p}\right] \\ &\quad + 2^{2p-1}\mathbb{E}\left[\left\| \int_0^t \int_E S_{t-s}\left[\beta(s, Y_s, x) - \beta(s, Z_s, x)\right] M(ds, dx) \right\|^{2p}\right]. \end{aligned} \quad (3.59)$$

By Lemma 3.4.4, we estimate

$$
\begin{aligned}
&\mathbb{E}\left[\|I_\xi(Y)_t - I_\xi(Z)_t\|^{2p}\right] = \\
&\leqslant 2^{2p-1}\left[C_1\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right] ds + C_2\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right] ds\right] \\
&\leqslant 2^{2p-1}(C_1 + C_2)\int_0^t L(s)^{2p}\mathbb{E}\left[\|Y_s - Z_s\|^{2p}\right] ds
\end{aligned}
\tag{3.60}
$$

Thus, altogether, we obtain that for a certain constant C, we have

$$
\sup_{t\in[0,T]}\left(\mathbb{E}\left[\|I_\xi(Y)_t - I_\xi(Z)_t\|^{2p}\right]\right)^{\frac{1}{2}} \leqslant C\left(\int_0^T \mathbb{E}\left[\|Y_s - Z_s\|^2\right] ds\right)^{\frac{1}{2}}. \tag{3.61}
$$

b) Next, by induction for every $n \in \mathbb{N}$ and using (2.16), we iterate:

$$
\begin{aligned}
&\|I_\xi^n(Y) - I_\xi^n(Z)\|_{\mathbb{H}_T^{2p}} \leqslant \left(C\int_0^T \mathbb{E}\left[\|I_\xi^{n-1}(Y)_{t_1} - I_\xi^{n-1}(Z)_{t_1}\|^2\right] dt_1\right)^{\frac{1}{2}} \\
&\leqslant \left(C^2\int_0^T\left(\int_0^T \mathbb{E}\left[\|I_\xi^{n-2}(Y)_{t_2} - I_\xi^{n-2}(Z)_{t_2}\|^{2p}\right] dt_2\right) dt_1\right)^{\frac{1}{2}} \\
&\leqslant \cdots \\
&\leqslant \left[C^{n-1}\int_0^T\int_0^T\cdots\int_0^T\int_0^T \mathbb{E}\left[\|I_\xi(Y)_{t_{n-1}} - I_\xi(Z)_{t_{n-1}}\|^{2p}\right] dt_{n-1}\, dt_{n-2}\ldots dt_1\right]^{\frac{1}{2}} \\
&\leqslant \left[C^n\int_0^T\int_0^T\cdots\int_0^T\int_0^T\int_0^T \mathbb{E}\left[\|Y_{s^-} - Z_{s^-}\|^{2p}\right] ds\, dt_{n-1}\, dt_{n-2}\ldots dt_1\right]^{\frac{1}{2}} \\
&\leqslant \left[C^n \sup_{t\in[0,T]}\mathbb{E}\left[\|Y_t - Z_t\|^{2p}\right]\int_0^T\int_0^T\cdots\int_0^T\int_0^T\int_0^T ds\, dt_{n-1}\, dt_{n-2}\ldots dt_1\right]^{\frac{1}{2}} \\
&\leqslant \left(C^n\frac{T^n}{n!}\sup_{t\in[0,T]}\mathbb{E}\left[\|Y_t - Z_t\|^{2p}\right]\right)^{1/2} \\
&\leqslant \left(C^n\frac{T^n}{n!}\right)^{\frac{1}{2}}\|Y - Z\|_{\mathbb{H}_T^{2p}},
\end{aligned}
\tag{3.62}
$$

leading to, $\lim_{n\to\infty}\|I_\xi^n(Y) - I_\xi^n(Z)\|_{\mathbb{H}_T^{2p}} = 0$. More precisely, there exists an index $n_0 \in \mathbb{N}$ such that $I_\xi^{n_0}$ is a contraction on $\mathbb{H}_T^{2p}$. It follows that, by the extension of Banach fixed point theorem (A.4.1), the mapping $I(\xi,\cdot)$ has a unique fixed point $u \in \mathbb{H}_T^{2p}$.

c) The existence of continuous version follows from Lemmas 3.4.2 and 3.4.3, denoted by

$$\xi + \int_0^t S_{t-s} a(s,h)\, ds + \frac{\sin(\pi\theta)}{\pi} F_\theta(Z)_t$$

By Doob's martingale maximal inequality, we obtain (3.63), i.e.,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u_t\|^{2p}\right] \leqslant \left(\frac{2p}{2p-1}\right)^{2p} \mathbb{E}\left[\|u_T\|^{2p}\right] < \infty, \quad \text{for all } T > 0. \tag{3.63}$$

d) Last, we show that $I(\cdot, u) : L^2(\mathcal{F}_0; H) \to \mathbb{H}_T^{2p}$ is Lipschitz for any fixed $u \in \mathbb{H}_T^2$. For any $\xi \in L^2(\mathcal{F}_0; H)$ we compute:

$$\|I(\xi_1, u)_t - I(\xi_2, u)_t\|^2 = \|\xi_1 - \xi_2\|^2, \quad \text{for all } t \in [0, T], \tag{3.64}$$

which leads to $\|I(\xi_1, u) - I(\xi_2, u)\|_{\mathbb{H}_T^{2p}} = \|\xi_1 - \xi_2\|_{L^2(\mathcal{F}_0;H)}$, i.e., $I(\cdot, u)$ is Lipschitz function for any $u \in \mathbb{H}_T^{2p}$. ∎

3.5 APPLICATIONS TO INTEREST RATE THEORY

In this section we give an overview of the evolution of interest rate modeling from the point of view of stochastic differential equations. Then motivate why the approach developed in this dissertation is relevant and useful in interest rate modeling.

Starting with models assigned constant coefficients, for instance Ho and Lee [42] which assume the following dynamics:

$$dr_t = \alpha\, dt + \sigma\, dW_s,$$

where W is standard Brownian motion, α and σ are real constants. Approaches to modeling term structure dynamics have grown tremendously in sophistication over the last two decades. The most popular was the model introduced by Heath, Jarrow and Morton (1992). They approached the problem by specifying the dynamics of all instantaneous forward rates $(f(t,T))_{0\leqslant t\leqslant T}$ with

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\, du\right)$$

where $P(t,T)$ is the price at time t of a bond paying one unit at time $T \geqslant t$. They assume that, for all $T > 0$, the dynamic of forward rates are governed by the Itô process of the form:

$$df(t,T) = \alpha(t,T)\,dt + \sum_{n=1}^{N} \sigma^n(t,T)\,dW_t^n, \quad t \in [0,T], \tag{3.65}$$

where $W = (W^1, \dots, W^N)$ is a standard Brownian motion in $\mathbb{R}^N$. The advantages of this model among others is that they do not admit negative interest rates after some calibrations and they also include a wide class of interest rate models. Moreover, they do not require drift estimation for no-arbitrage principle which asserts that the drift coefficients should be functions of their volatilities and the correlations among themselves.

However, the HJM framework presents some drawbacks. For instance, the time-dependence coefficients are deterministic and need to be constantly updated in order to fit the new term structure. Indeed, this problem comes form the fact that in general there does not exist a possible realization of the N-dimensional Brownian motions. In practice, it seems any model that is only driven by Gaussian processes does not provide good fitting to observed marked data since empirically observed log returns of zero-coupon bonds are not normally distributed. In many cases observed empirically, as argued in Bjrök et al [11, 10], the dynamics of interest rates do not look like only diffusion processes, but rather as diffusions and jumps. To address these issues, two approaches have been introduced to model term structure dynamics.

1. The first approach extends the HJM framework to jumps-diffusions model consistent with term structure innovation. [32] have introduced an extended of HJM term structure model which follows the SDE of the form

$$df(t,T) = \alpha(t,T)\,dt + \sum_{n} \sigma^n(t,T)\,dW_t^n + \int_E \gamma(t,y,T)(\mu(dt,dx) - F(dy)\,dt), \tag{3.66}$$

where $\{W^n\}$ denotes an infinite or finite sequence of real-valued independent Brownian motions, μ is a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$ with compensator $dt \otimes F(dy)$. The term $\int_E \gamma(t,y,T)(\mu(dt,dy) - F(dy)\,dt)$ represents jumps of forward rates.

Denoting by H a Hilbert space of forward curves $h : \mathbb{R}_+ \to \mathbb{R}$ and $(S_t)_{t\geqslant 0}$ the shift semigroup on H with $S_t h = h(t+\cdot)$, Musiela parametrization gives the short rate as

$$r_t = f(t, t+x), \quad x \geqslant 0,$$

Consequently, equation 3.66 leads to a short-rate stochastic partial differential equation

$$\begin{aligned} dr_t &= \left[\frac{d}{dx} r_t + \alpha(r_t)\right] dt + \sum_n \sigma^n(r_t)\, dW_t^n + \int_E \gamma(r_{t-}, y)(\mu(dt, dy) - F(dy)\, dt), \\ r_0 &= h_0. \end{aligned} \tag{3.67}$$

with $h_0 \in H$ represents the initial forward curves. Note that, as we pointed out previously, the drift and volatility coefficients should not be deterministic but instead should be functions of prevailing forward curves. Namely, we have $\alpha : H \to H$, $\sigma^n : H \to H$ and $\gamma : H \times E \to H$.

Under various regularity assumptions, the HJM no-arbitrage condition for the drift coefficient for (3.67) is derived in [32], precisely the map $\alpha = \alpha_{HJM} : H \to H$ is determined as follows

$$h \mapsto \alpha_{HJM}(h) = \sum_n \sigma^n(h)\Upsilon^n(h) - \int_E \gamma(h, y)(e^{\Phi(h,y)} - 1)F(dx),$$

where both Υ^n and Φ are given by

$$\Upsilon^n(h)(x) := \int_0^x \sigma^n(h)(z)\, dz \quad \text{and} \quad \Phi(h, y)(x) := -\int_0^x \gamma(h, y)(z)\, dz.$$

The existence and uniqueness of solutions to the SPDE (3.67) are well-studied in [32], including the positivity preserving forward curves. One says that a short-rate process $r = (r_t)_{t\geqslant 0}$ is a mild solution for (3.67) with a given initial forward curve h_0 if it follows an Itô process of the form:

$$\begin{aligned} r_t =& S_t h_0 + \int_0^t S_{t-s}\alpha_a(r_s)\, ds + \int_0^t S_{t-s}\sigma(r_s)\, dW_s \\ &+ \int_0^t \int_E S_{t-s}\gamma(r_{s-}, y)(\mu(ds, dy) - F(dy)\, ds), \end{aligned} \tag{3.68}$$

2. An alternative approach proposes a random field model for the interest rate term structure. The main advantages of random field models are: they do not require re-calibration and they do accommodate both finite and infinite factor models. Kennedy [51, 52] was the first to develop a model of forward rates as a continuous Gaussian random field which has independent increments. More precisely, the dynamics of the forward rates is given by: $f(s,t) = \mu_{s,t} + X_{s,t}$, with $\mu_{s,t}$ being deterministic and $X_{s,t}$ a Brownian sheet (see, e.g. [96] for this concept). Goldstein [35, 36] and Kimmel [54] generalized the Kennedy model by introducing the conditional volatility, which have resulted in non-Gaussian random fields. Later on, Lijun et al. [60] proposed an extended HJM term structure model driven by a Lévy random field. They assume the forward rate admits the following dynamics

$$\begin{aligned} df(t,T) =& \mu(t,T-t)\,dt + \int_{\mathbb{R}^d} \sigma(x,t,T-t) Y^G(ds,dx) \\ &+ \int_{\mathbb{R}^d} \gamma(x,t^-,T-t^-) Y^P(ds,dx), \end{aligned} \tag{3.69}$$

where Y^G is a martingale measure (see [Walsh]), Y^P is a compensated Poisson random measure. $Y = Y^G + Y^P$ defined as a Lévy random field $\mathbb{R}^d \times (0,T]$.

In conclusion, taking into account all these challenges, we are inspired to build a tractable forward interest rates model driven by a martingale field which already incorporates the continuous diffusive risk and the jump risk. By unification approach, we can generalize the short-rate model 3.67 to the following SPDE

$$\begin{aligned} dr_t &= \left[Ar_t + \alpha(r_t)\right] dt + \int_E \sigma(t, r_{t^-}, x) M(dt,dx), \\ r_0 &= h_0. \end{aligned} \tag{3.70}$$

for some infinitesimal generator A of strongly continuous semigroup of shifts $(S_t)_{t\geqslant 0}$. Under the HJM framework, the goal is to derive an arbitrage-free term structure models of the form

$$df(t,T) = \alpha(t,T)\,dt + \int_E \sigma(t,x) M(dt,dx), \tag{3.71}$$

that preserves the positivity of forward rates and can address all issues mentioned previously. Note that model (3.71) is more general as it includes wide classes of term structure models (even both models (3.66) and (3.69)). Practically, such type of term structure models may be beneficial for both researchers and practitioners. Indeed, model (3.71) is easy to work with and the fitting procedure shall be much more simpler. Solving the stochastic evolution (3.70) leads to a new open problem for further researches and development.

Teil I

APPENDIX

A

BACKGROUNDS AND PRELIMINAIRES

In this appendix, we present all useful concepts and results (without proofs) that we will use through in this dissertation. All materials are taken from standard textbooks (see [6],[47], [85],[14],[87],[4],[82],[97],[38], [98]).

A.1 PREMEASURE THEORY

Assuming that the reader is familiar with notions of σ-algebra and measure, within a few definitions, the reader will get to be familiar with pre-measure.

Let X be a set where $\mathscr{P}(X)$ denotes its power set.

Definition A.1.1

1. A family $\mathcal{R} \subset \mathscr{P}(X)$ is called a ring if it has the following properties:
   - $\emptyset \in \mathcal{R}$
   - $A \cup B \in \mathcal{R}$ if $A, B \in \mathcal{R}$.
   - $A \setminus B \in \mathcal{R}$ if $A, B \in \mathcal{R}$.

   If $X \in \mathcal{R}$, then $\mathcal{R}$ is called an algebra.

2. A semi-ring is a family $\mathcal{S}$ of subsets of a set X with the following properties:
   - $\emptyset \in \mathcal{S}$
   - $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$.
   - $A, B \in \mathcal{S} \Rightarrow$ there exist finitely many disjoint $A_1, \ldots, A_n \in \mathcal{S}$ such that $A \setminus B = \bigcup_{i=1}^{n} A_i$.

   If $X \in \mathcal{R}$, then $\mathcal{R}$ is called a semi-algebra.

Remark 20 *Let $\mathcal{R}$ be a ring of subsets of a set X. Note that if $\mathcal{R}$ is closed with respect to countable unions then it is called a σ-ring. Moreover, if $X \in \mathcal{R}$ then $\mathcal{R}$ becomes a σ-algebra.*

It is also useful to consider the property of σ-additivity.

Lemma A.1.1 *Let $\upsilon : \mathcal{R} \to [0, \infty)$ be an additive[1] set function on a semi-ring $\mathcal{R}$ (or an algebra) on X. Then the following conditions are equivalent:*

(i) the function υ is countably additive, i.e. if $(A_n)_{n\in\mathbb{N}} \subset \mathcal{R}$ is a family of pairwise disjoint sets, then

$$\upsilon\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \upsilon(A_n).$$

(ii) the function υ is continuous at zero, i.e. if $A_n \in \mathcal{R}$, $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and $\cap_{n=1}^{\infty} A_n = \emptyset$, then

$$\lim_{n\to\infty} \upsilon(A_n) = 0.$$

(iii) the function υ is continuous from below, i.e. if $A_n \in \mathcal{R}$, $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$, then

$$\lim_{n\to\infty} \upsilon(A_n) = \upsilon\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Proof For the proof, the reader may refer to [85] or [14]. ∎

We are now ready to define pre-measure. Let $\mathcal{R}$ be a semi-ring on X.

Definition A.1.2 A pre-measure defined on a semi-ring $\mathcal{R}$ is a positive set function $\upsilon : \mathcal{R} \to [0, \infty]$ which satisfies the following:

- $\upsilon(\emptyset) = 0$.
- For any sequence of pairwise disjoint sets $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}$ with $\bigcup_n B_n \in \mathcal{R}$, then
$$\upsilon\left(\bigcup_n B_n\right) = \sum_n \upsilon(B_n).$$

1 Additive in the sense: if $A, B \in \mathcal{R}$ with $A \cap B = \emptyset$, then $\upsilon(A \cup B) = \upsilon(A) + \upsilon(B)$.

Moreover, a pre-measure μ is said to be σ-*finite* and $(X, \mathcal{R}, \mu)$ is called a σ-finite pre-measure space, if it contains an increasing sequence $(A_i)_{i \in \mathcal{N}} \subset \mathcal{R}$ with $X = \bigcup\limits_{i \in \mathbb{N}} A_i$ such that $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$.

In general, it is not an easy task to assign explicitly a μ-value to every set $\mathcal{X}$ from a σ-algebra $\mathcal{X}$ for any measure space $(X, \mathcal{X}, \mu)$. Rather than doing this it is often more natural to assign μ-values to sets from some generator $\mathcal{G}$ of $\mathcal{X}$. To address this issue one can use the Carathéodory extension theorem.

Lemma A.1.2 *If* μ *is* σ*-finite pre-measure on* $\mathcal{R}$ *then there is an unique measure* $\widehat{\mu} : \sigma(\mathcal{R}) \to [0, \infty]$ *such that* $\widehat{\mu}$ *is an extension of* μ*, i.e.* $\mu = \widehat{\mu}|_{\mathcal{R}}$.

Proof This follows from [6, Theorem 5.6]. ∎

Next, we will recall some useful results regarding the notion of product spaces and product σ-algebras [2]. Let $(Y, \mathcal{Y})$ be a measurable space. The first problem which we are faced is that the family

$$\mathcal{X} \times \mathcal{Y} = \{A \times B \ : \ A \in \mathcal{X} \text{ and } B \in \mathcal{Y}\} \tag{A.1}$$

is, in general, not a σ-algebra but rather it is a semi-ring as the following shows.

Lemma A.1.3 *If* $\mathcal{G} \subset \mathscr{P}(X)$ *and* $\mathcal{H} \subset \mathscr{P}(Y)$ *are respectively semi-rings on* X *and* Y*. Then* $\mathcal{G} \times \mathcal{H}$ *is a semi-ring on* $X \times Y$.

Proof The reader may refer to [85, Lemma 13.1]. ∎

Definition A.1.3 Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. Then $\mathcal{X} \otimes \mathcal{Y} := \sigma(\mathcal{X} \times \mathcal{Y})$ is a σ-algebra, and $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ is called the product of measurable spaces.

The following lemma allows to reduce considerations for $\mathcal{X} \otimes \mathcal{Y}$ to respective generators of $\mathcal{X}$ and $\mathcal{Y}$. Let $\mathcal{G}$ and $\mathcal{H}$ be respectively two semi-rings on X and Y.

2 For the proofs of all results we will refer the reader to the reference.

Lemma A.1.4 *If* $\mathcal{X} = \sigma(\mathcal{G})$ *and* $\mathcal{Y} = \sigma(\mathcal{H})$ *and if* $\mathcal{G}, \mathcal{H}$ *contain respectively increasing sequences* $(X_i)_{i\in\mathbb{N}} \subset \mathcal{G}$ *with* $X_i \uparrow X$ *and* $(Y_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ *with* $Y_i \uparrow Y$. *Then we have*

$$\mathcal{X} \otimes \mathcal{Y} := \sigma(\mathcal{X} \times \mathcal{Y}) = \sigma(\mathcal{G} \times \mathcal{H}).$$

Proof The proof can be found in [85, Lemma 13.3]. ∎

Next, we discuss about the product pre-measure on a product space. Let $(X, \mathcal{G}, \mu)$ and $(Y, \mathcal{H}, \nu)$ be σ-finite pre-measure spaces with $\mathcal{G}, \mathcal{H}$ are respectively semi-rings on X, Y. We want to define a product pre-measure ρ on rectangles of the form $A \times B$.

Lemma A.1.5 *If* $(X, \mathcal{G}, \mu)$ *and* $(Y, \mathcal{H}, \nu)$ *are σ-finite pre-measure spaces. Then there exists a unique σ-finite pre-measure* ρ *on* $(X \times Y, \mathcal{G} \times \mathcal{H})$ *such that*

$$\rho(A \times B) = \mu(A) \cdot \nu(B), \quad \textit{for all } A \times B \in \mathcal{G} \times \mathcal{H}.$$

In this case, $(X \times Y, \mathcal{G} \times \mathcal{H}, \mu \times \nu)$ *is called the product pre-measure space.*

Proof The proof follows from [85, Theorem 13.5]. ∎

Remark 21 *Observe that, by Lemma A.1.2, the product pre-measure* ρ *can always be extended to a measure on the product σ-algebra* $\sigma(\mathcal{G} \times \mathcal{H})$.

A.2 RANDOM MEASURES AND STOCHASTIC INTEGRATION

In this section, we review the notion of random measures and focus on describing so some special type of random measures that are used in the dissertation. We do not go deeper into details but for more exposure on this topic the reader may refer to [47] which gives complete and more detailed results, including integration theory and its applications.

A.2.1 *Integer-Valued Random Measures*

Let us consider a measurable space (E, Σ).The following definitions and notions are based on [86],[62] and [27].

Definition A.2.1 A σ-algebra Σ said to be *separable* if

1. Σ is countably generated, i.e., there is a countable semi-ring $\mathcal{E}$ (or a ring or an algebra) such that $\Sigma = \sigma(\mathcal{E})$.
2. $\{x\} \in \Sigma$ for all $x \in E$.

Definition A.2.2 We say that (E, Σ) is a *Blackwell space* if Σ is separable and for every separable σ-algebra $\mathcal{A} \subset \Sigma$, then $\mathcal{A} = \Sigma$.

Lemma A.2.1 *Let* (E_1, Σ_1) *and* (E_2, Σ_2) *be two Blackwell spaces. Then the product space* $(E_1 \times E_2, \sigma(\Sigma_1 \times \Sigma_2))$ *is a Blackwell space.*

Proof First, since both measurable spaces (E_1, Σ_1) and (E_2, Σ_2) are Blackwell spaces then there are respectively countable semi-ring $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\Sigma_1 = \sigma(\mathcal{E}_1)$ and $\Sigma_2 = \sigma(\mathcal{E}_2)$. By Lemma (A.1.3), the product $\mathcal{E}_1 \times \mathcal{E}_2$ defines a semi-ring on $E_1 \times E_2$. Moreover, one can define a σ-algebra $\Sigma = \sigma(\Sigma_1 \times \Sigma_2) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$ (see LemmaA.1.4).

We next need to prove that the semi-ring $\mathcal{E}_1 \times \mathcal{E}_2$ is countable. Indeed, $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) is a countable so there is a bijection ψ_1 (resp. ψ_2) : $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) $\to \mathbb{N}$. So for the existence of a bijection for the product space, one can set $\psi(A, B) := \psi_1(A)\psi_2(B)$ for all pairs $(A, B) \in \mathcal{E}_1 \times \mathcal{E}_2$. Thus, ψ is a bijection from $\in \mathcal{E}_1 \times \mathcal{E}_2$ into $\mathbb{N}$. Moreover, it is straightforward to show that for all $x \in E$ then $\{x\} \in \Sigma$. Let $x \in E$, this means that $\{x\} = \{x_1\} \times \{x_2\}$ where $x_1 \in E_1, x_2 \in E_2$. But observer that $\{x_1\} \in \Sigma_1$ and $\{x_2\} \in \Sigma_2$, hence $\{x_1\} \times \{x_2\} \in \Sigma_1 \times \Sigma_2$. It follows $\{x_1\} \times \{x_2\} \in \Sigma$.

Last, it remains to prove that if $\mathcal{A} \subset \Sigma$ is a separable σ-algebra on E, then $\mathcal{A} = \Sigma$. By definition, if $\mathcal{A}$ is a separable σ-algebra, so there is a countable semi-ring $\mathcal{R}$ with $\mathcal{A} = \sigma(\mathcal{R})$. Then there exist respectively $\mathcal{R}_1$ and $\mathcal{R}_2$ two countable semi-rings on E_1 and E_2 such that $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ and $\sigma(\mathcal{R}) = \sigma(\mathcal{R}_1 \times \mathcal{R}_2) = \sigma(\sigma(\mathcal{R}_1) \times \sigma(\mathcal{R}_2))$. Then $\sigma(\mathcal{R}_1) \subset \Sigma_1$ and $\sigma(\mathcal{R}_1)$ is separable as is $\sigma(\mathcal{R})$. But (E_1, Σ_1) is a Blackwell space so it must follows $\sigma(\mathcal{R}_1) = \Sigma_1$. The same argument holds true for $\sigma(\mathcal{R}_2) = \Sigma_2$. We conclude that $\sigma(\mathcal{R}) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$, i.e., $\mathcal{A} = \Sigma$. ∎

Now, let $(E, \mathscr{E})$ to be a Blackwell space such that $\mathscr{E}$ is generated by a countable algebra. On the other side, by [27], the space $(E, \mathscr{E})$ satisfies the disintegration property, namely for any measurable space $(G, \mathscr{G})$ if γ is a positive finite measure on $(E \times G, \mathscr{E} \otimes \mathscr{G})$ then we shall write $\gamma(dx, dy) = \alpha(y, dx)m(dy)$ on $\mathscr{E} \otimes \mathscr{G}$ with $m(A) = \gamma(E \times A)$.

Definition A.2.3 A random measure on $\mathbb{R}_+ \times E$ is a mapping $\mu : \mathcal{B}(\mathbb{R}) \otimes \mathscr{E} \to L^1(\Omega; \mathbb{R}_+)$ satisfying for any $\omega \in \Omega$:

1. $\mu(\omega; \{0\} \times E) = 0$ $\mathbb{P}$-almost surely.
2. $A \mapsto \mu(\omega; A)$ is a measure on defined on a σ-algebra $\mathcal{B}(\mathbb{R}) \otimes \mathscr{E}$.

Next, we present the other properties of random measures.

Definition A.2.4 A random measure μ on $\mathbb{R}_+ \times E$ is said :

1. σ-finite if there exists a sequence E_n increasing to E such that, for any $t \in \mathbb{R}_+$, $\mathbb{E}\left[|\mu([0,t] \times E_n)|\right] < \infty$.
2. adapted if $\mu(\omega; A)$ is $\mathcal{F}_t$-measurable for any $A \subset [0,t] \times E$, for each $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

Here we give a special type of random measure that we are interested.

Definition A.2.5 An integer-valued random measure is a random measure with the following properties:

1. $\mu(\omega; \{t\} \times E) \leqslant 1 \quad \mathbb{P}$-almost surely, for any $t \geqslant 0$.
2. $\mu : \Omega \times \mathcal{B}(\mathbb{R}) \otimes \mathscr{E} \to \overline{\mathbb{N}}$.
3. μ is optional and $\mathcal{P}$-σ-finite.

We present the following result in order to characterize explicitly any integer-valued random measure by a thin random set.

The most useful example of integer-valued measure is the following:

Definition A.2.6 An extended Poisson measure on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \times \mathscr{E})$, relative to the filtration $\mathbb{F}$, is an integer-valued random measure μ which satisfies the following conditions:

1. the positive measure, also called intensity measure of μ, $A \mapsto m(A) = \mathbb{E}[\mu(A)]$ defined on $\mathbb{R}_+ \times E$ is σ-finite.

2. the random variable $\mu(\cdot, A)$ is independent of $\mathcal{F}_s$ for all $s \in \mathcal{R}_+$ and $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathscr{E}$ with $m(A)$.

Example 8 *There are two fundamental example of extended Poisson measures, namely:*

a) If for each $t \in \mathbb{R}_+$ we have $m(\{t\} \times E) = 0$ then we call μ a Poisson measure.

b) If m has the form $m(dt, dx) = dt \times F(dx)$ on $\mathbb{R}_+ \times E$, where F is a positive σ-finite measure on $(E, \mathscr{E})$, then we shall call μ a homogeneous Poisson measure. The measure $\overline{\mu} := \mu - \lambda \otimes F$ is called compensated Poisson measure.

Lemma A.2.2 *If μ is an extended Poisson measure on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \times \mathscr{E})$, relative to the filtration $\mathbb{F}$, with intensity measure m. Then its compensator defined on $\mathcal{B}(\mathbb{R}_+) \times \mathscr{E}$ is determined by $A \mapsto \mu^p(\omega; A) = m(A)$ for all $\omega \in \Omega$. Moreover, if $\{A_n\}$ is an increasing sequence with $\cup_{n \in \mathbb{N}} A_n = E$ with $m(A_n) < \infty$ and for any fixed $A \in \mathcal{E}$,*

$$\mathcal{E} := \bigcup_{n \in \mathbb{N}} \{B \cap A_n : B \in \mathscr{E}\},$$

the process $Z_A(t) := (\mu - m)[(0, t] \times A]$ is a martingale with respect to $\mathbb{F}$.

Proof (1) For the first claim, the reader may refer to [[47], Proposition 1.21, p.71].

(2) For the martingale property, we use the independent increments and the distribution of Poisson random measure. Let $A \subset E$ such that $(0, t] \times A \in \mathcal{B}(\mathbb{R}_+) \times \mathscr{E}$ for any $t \geqslant 0$. Set $X(t) = (\mu - m)[(0, t] \times A]$. Since μ has independent increments (see Definition A.2.4) then we compute

$$\begin{aligned}
\mathbb{E}[Z_A(t) - Z_A(s) | \mathcal{F}_s] &= \mathbb{E}\left((\mu - m)[(0, t] \times A] - (\mu - m)[(0, s] \times A] \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left((\mu - m)[(s, t] \times A] \middle| \mathcal{F}_s\right) = \mathbb{E}\left((\mu - m)[(s, t] \times A]\right) \\
&= \mathbb{E}\left((\mu[(s, t] \times A]\right) - m[(s, t] \times A] = 0.
\end{aligned}$$

∎

Next we describe the integration theory with respect to random measure, in particular with respect to Poisson random measure, with less details.

A.2.2 *Stochastic integration*

We denote:

$$\widetilde{\Omega} = \Omega \times E, \text{ with the } \sigma\text{-fields } \widetilde{\mathcal{O}} = \mathcal{O} \otimes \mathscr{E} \text{ and } \widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathscr{E}$$

Definition A.2.7 Let μ be a random measure and h an optional function on $\tilde{\Omega}$, i.e. h is $\tilde{\mathcal{O}}$-measurable.

1. Since h is $\mathcal{B}(\mathbb{R}_+) \otimes \mathscr{E}$-measurable and if
$$\iint_{[0,t]\times E} |h(\omega, s, x)|\mu(\omega; ds, dx) < \infty, \quad \text{for all } (\omega, t) \times \mathbb{R}_+ \times E,$$
then for any $\omega \in \Omega$ we define the stochastic integral, $h \star \mu$, as
$$h \star \mu_t(\omega) = \iint_{[0,t]\times E} h(\omega, s, x)\mu(\omega; ds, dx).$$
2. μ is said to be optional measure if the integral process $f \star \mu$ is optional for every optional function f.
3. μ is called integrable if μ is option measure and the integral process $\mathbb{1} \star \mu \in \mathcal{A}^+$ where $\mathbb{1} \star \mu_t = \mu(\cdot; [0, t] \times E)$ for any $t \geqslant 0$.
4. mu is $\tilde{\mathcal{P}}$-σ-finite if there exists a $\tilde{\mathcal{P}}$-measurable partition $(A_n)_{n\in\mathbb{N}}$ of $\tilde{\Omega}$ such that each $(\mathbb{1}_{A_n} \star \mu)_\infty$ is integrable.

Now we provide the characterization of the compensator of random measure through the integral process.

Lemma A.2.3 *If μ is an optional $\tilde{\mathcal{P}}$-σ-finite random measure. Then there is an unique predictable random measure μ^p up to a $\mathbb{P}$-null set, which satisfies either one of the two following equivalent properties:*

a) $\mathbb{E}[h \star \mu^p_\infty] = \mathbb{E}[h \star \mu_\infty]$ *for any positive $\tilde{\mathcal{P}}$-measurable function h on $\tilde{\Omega}$.*

b) If $|h| \star \mu \in \mathcal{A}^+_{loc}$, then $|h| \star \mu^p \in \mathcal{A}^+_{loc}$ or $h \star \mu - |h| \star \mu^p \in \mathcal{M}_{loc}$.

Moreover, $h \star \mu^p$ is the compensator of $h \star \mu$ as μ^p is the compensator of μ.

Proof The reader may refer to [47, Theorem II.1.8, p.67]. ∎

A.3 SEMIGROUP THEORY

The aim of this section is to introduce the notion of C_0-semigroups and their generators, including divers properties. Concerning the proofs of the upcoming results, the reader is referred to [82] or [97].

C_0-semigroups, also known as a strongly continuous one-parameter semigroup, is a generalization of the exponential function that serve to describe the time evolution of autonomous linear systems of type

$$\partial x_t = Tx, \quad x(0) = x_0 \in X, \tag{A.2}$$

where x takes values in some Banach space X and T is a possibly unbounded operator on X. If such a solution exists, one expects the existence of a linear operator S_t that maps the initial condition $x(0)$ onto the solution $x(t)$ of equation (A.2) at time t. Moreover, if such a solution is unique, then the family of operators S_t, $t \geqslant 0$, should satisfy $S_0 = 1$ and $S_t \cdot S_s = S_{t+s}$.

Let $(X, \|\cdot\|_X)$ be Banach space and denote by $\mathcal{L}(X)$ the Banach space of bounded linear operators on X equipped with the norm:

$$\|T\|_{\mathcal{L}(X)} = \sup_{x \in X, \|x\|_X = 1} \|Tx\|_X, \quad T \in \mathcal{L}(X).$$

Definition A.3.1 From a formal point of view, a family $S_t \in \mathcal{L}(X)$, $t \geqslant 0$, of bounded linear operators on a Banach space X is called a strongly continuous semigroup if

1. $S_0 = I$,
2. $S_{t+s} = S_t S_s$ for every $t, s \geqslant 0$,
3. $\lim_{t \to 0^+} S_t x = x$ for every $x \in X$.

For any C_0-semigroup S_t and $x \in X$, the mapping $t \mapsto S_t x$ is continuous. Moreover, there exists constants $\alpha \geqslant 0$ and $M \geqslant 1$ such that

$$\|S_t\|_{\mathcal{L}(X)} \leqslant Me^{\alpha t}, \quad t \geqslant 0.$$

Definition A.3.2 A semigroup S_t is called uniformly continuous if

$$\lim_{t \to 0^+} \|S_t - I\|_{\mathcal{L}(X)} = 0.$$

We now give all terminologies in semigroup theory.

Definition A.3.3 Let S_t, $t \geqslant 0$, be a C_0-semigroup on a Banach space X. Then

1. If $M = 1$, then S_t is a pseudo-contraction semigroup.
2. If $\alpha = 0$, then S_t is uniformly bounded.
3. If $\alpha = 0$ and $M = 1$, then S_t is a semigroup of contractions.
4. If for every $x \in X$, the mapping $t \mapsto S_t x$ is differentiable for $t > 0$, then S_t is called a differentiable semigroup.
5. If the operators S_t, $t > 0$, are compact then S_t is compact semigroup.

Next we introduce the notion of generator of a semigroup.

Definition A.3.4 Let S_t be a C_0-semigroup on a Banach space X. The linear operator A on the Banach space X with domain

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{S_t x - x}{t} \text{ exists} \right\}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S_t x - x}{t}$$

is called the infinitesimal generator of the semigroup S_t.

Recall that the graph of a linear operator T on a Banach space X with a domain $\mathcal{D}(T)$ is defined as the subset of $X \times X$ consisting of all elements of the form (x, Tx) where $x \in \mathcal{D}(T)$. Moreover, the operator T is closed if its graph is a closed subspace of $X \times X$. The operator T is closed operator if and only if the fact that if $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(T)$ is Cauchy in X and $\{Tx_n\}_{n \in \mathbb{N}}$ is also Cauchy.

Lemma A.3.1 *$\mathcal{D}(A)$ is dense in X, and A is a closed linear operator.*

We denote by $\mathcal{D}(A^*)$ the domain of the adjoint A^* which is defined as the set of all elements $\varphi \in X^*$ such that there exists an element $A^*\varphi \in X^*$ with the property that $(A^*\varphi)(x) = \varphi(Ax)$ for every $x \in \mathcal{D}(A)$.

Lemma A.3.2 *A linear operator* A *is the infinitesimal generator of a uniformly continuous semigroup* S_t *on a Banach space* X *if and only if* $A \in \mathcal{L}(X)$. *For each* $t \geqslant 0$, *we have the series*

$$S_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

converges in norm.

Remark 22 *In some cases, it may happen that* $A \notin \mathcal{L}(X)$ *so the series representation is no longer possible, for instance, as in the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t), \quad u(0) = x \in X. \tag{A.3}$$

The following proposition provides useful facts about semigroups.

Proposition A.3.1 *Let* A *be an infinitesimal generator of a* C_0*-semigroup* $S_t, t \geqslant 0$, *on a Banach space* X. *Then, the following properties hold*

1. *For* $x \in X$,
$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S_t x \, ds = S_t x.$$

2. *For* $x \in \mathcal{D}(A)$,
$$S_t x \in \mathcal{D}(A), \quad \textit{and} \quad \frac{d}{dt} S_t x = AS_t x = S_t A x.$$

3. *For* $x \in X$,
$$\int_0^t S_s x \, ds \in \mathcal{D}(A), \quad \textit{and} \quad A\left(\int_0^t S_s x \, ds\right) = S_t x - x.$$

4. *If* S_t *is differentiable then for* $n \in \mathbb{N}$
$$S_t : X \to \mathcal{D}(A^n) \quad \textit{and} \quad S^{(n)}(t) = A^n S_t \in \mathcal{L}(X).$$

5. *For* $x \in \mathcal{D}(A)$,
$$S_t x - S_s x = \int_s^t S(u) A x \, du = \int_s^t A S(u) x \, du.$$

6. $\bigcap_{n \in \mathbb{N}} \mathcal{D}(A^n)$ *is dense in* X.

The following result shows that if A is the generator of a C_0 -semigroup S_t, then $u(t) = S_t u(0)$ is indeed the solution to (A.2) in a weak sense.

Lemma A.3.3 *If a function* $u : [0,\infty) \to \mathcal{D}(A)$ *satisfies* $\frac{d}{dt}u(t) = Au(t)$ *for every* $t \geqslant 0$, *then* $u(t) = S_t u(0)$ *. In particular, no two distinct* C_0*-semigroups can have the same generator.*

Conditions under which an operator A can be an infinitesimal generator of a C_0 -semigroup involve the resolvent of A.

Definition A.3.5 The resolvent set $\rho(A)$ of closed linear operator A on a Banach space X is the set defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : \ \text{range}(\lambda - A) \text{ is dense in } X \text{ and } \lambda - A \text{ has a continuous inverse}\}.$$

The family of bounded linear operators

$$R(\lambda, A) = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A)$$

is called the resolvent of A.

Lemma A.3.4 *Let* S_t *be a* C_0*-semigroup with infinitesimal generator* A *on a Banach space* X. *If* $n_0 = \lim_{t\to\infty} \frac{\ln \|S_t\|_{\mathcal{L}(X)}}{t}$ *, then any real number* $\mathrm{Re}(\lambda) > n_0$ *belongs to the resolvent set* $\rho(A)$, *and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S_t x \, ds, \quad x \in X.$$

Theorem A.3.1 *Let* $A : \mathcal{D}(A) \subset X \to X$ *be a linear operator on a Banach space* X. *In order to have* A *as the generator of a* C_0*-semigroup* S_t, *it is sufficient and necessary that the following are fulfilled*

1. *A is closed and $\overline{\mathcal{D}(A)} = X$.*
2. *There exist real numbers M and α such that for every $\mathrm{Re}(\lambda) > \alpha$, $\lambda \in \rho(A)$ and*

$$\|[R(\lambda, A)]^n\|_{\mathcal{L}(X)} \leqslant M(Re(\lambda) - \alpha)^{-n}, \quad \textit{for } n \geqslant 1.$$

For a special case, consider $X = H$ where H is a real separable Hilbert space. Let A be a closed linear operator on H and define the graph norm

$$\|h\|_{\mathcal{D}(A)} = (\|h\|_H + \|Ah\|_H)^{1/2}.$$

Then $(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)})$ is a real separable Hilbert space.

Theorem A.3.2 *Let* $g : [0, \infty) \to \mathcal{D}(A)$ *be measurable, and let* $\int_0^\infty \|g(s)\|_{\mathcal{D}(A)} < \infty$. *Then*

$$\int_0^t g(s)\,ds \in \mathcal{D}(A), \quad \textit{and} \quad A\int_0^t g(s)\,ds = \int_0^t Ag(s)\,ds.$$

A.4 BANACH'S FIXED POINT THEOREM

This section is concerned with an optimal application of Banach's fixed point theorem. It applies to "contractive" mappings between complete metric spaces, yielding the existence of unique fixed-point to operator involved.

Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping.

Definition A.4.1 The mapping f is called a *contraction*, if there exists a constant $0 \leqslant K < 1$ such that

$$d(f(x), f(y)) \leqslant Kd(x, y), \quad \text{for all } x, y \in X.$$

Moreover, $x \in X$ is said to be a *fixed point* of f, if we have $f(x) = x$.

The following result is the well-known Banach fixed point theorem.

Theorem A.4.1 *Let* (X, d) *be a complete metric space and let* $f : X \to X$ *be a contraction. Then the mapping* f *has a unique fixed point.*

Proof For the proof, reader may refer to [4, Theorem 3.48]. ∎

Corollary A.4.1 (Extended Banach fixed point theorem) *Let* (X, d) *be a complete metric space and let* $f : X \to X$ *such that for some* $n \in \mathbb{N}$ *the mapping* f^n *is a contraction. Then the mapping* f *has a unique fixed point and it holds*

$$f(x) = f(f^n(x)) = f^n(f(x)), \quad n \in \mathbb{N}.$$

Proof By Theorem A.4.1, the mapping f^n has a unique fixed point then there is $x \in X$ such that $f^n(x) = x$. Namely, we get $f(x) = f(f^n(x)) = f^n(f(x))$.

Next, we iterate

$$f^n(x) = f^{n-1}(f(x)) = f^{n-1}(x) = \cdots = f(x) = x.$$

This implies that x is a unique fixed point of f. ∎

Next, we give an useful tool for proving existence and uniqueness of solutions for SDEs and SPDEs.

Lemma A.4.1 (Gronwall's inequality) *Let $T \geqslant 0$ be fixed, let $f : [0, T] \to \mathbb{R}_+$ be a continuous mapping, and let $C \geqslant 0$ be a constant such that*

$$f(t) \leqslant C \int_0^t f(s)\,ds, \quad \textit{for all } t \in [0, T].$$

Then we have $f \equiv 0$.

Moreover, if $C : [0, T] \to \mathbb{R}_+$ integrable function and $c \geqslant 0$ is a constant with the property that

$$f(t) \leqslant c + \int_0^t C(s) f(s)\,ds, \quad \textit{for all } t \in [0, T].$$

Then, it holds

$$f(t) \leqslant c \exp\left(\int_s^t C(r)\,dr\right), \quad \textit{for all } t \in [0, T].$$

Proof The reader may refer to [38]. ∎

Last, we present the extension theorem for continuous linear Operators and recall the existence of predictable version for any adapted and stochastically continuous process.

Lemma A.4.2 (Hahn-Banach theorem) *Let $(X, \|\cdot\|)$ be a normed space, Y be a Banach space, $D \subset X$ be a dense subspace and $\Phi : D \to Y$ be a continuous linear Operator. Then there exists an unique continuous extension $\hat{\Phi} : X \to Y$ such that $\hat{\Phi}|_D = \Phi$ and $\|\hat{\Phi}\| = \|\Phi\|$.*

Proof See [[98], Satz II.1.5]. ∎

Lemma A.4.3 *Let Z be an adapted and stochastically continuous process on a closed interval $[0, T]$. Then the process Z admits a predictable version on $[0, T]$.*

Proof See [[22], Proposition 3.6, page 77]. ∎

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