Lines on K3 quartic surfaces

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This thesis has been a beautiful voyage and is dedicated to my grandmothers, but also to my sister, who courageously travelled with me from home to Hannover by car, and to Andrés, with whom I wish to make an even longer journey.

Zusammenfassung

K3-Quartikflächen sind Flächen vierten Grades im projektiven Raum, die rationale Doppelpunkte als Singularitäten zulassen. Die vorliegende Dissertation befasst sich mit der maximalen Anzahl von Geraden auf einer K3-Quartikfläche.

Unsere Untersuchung ist in drei Abschnitte gegliedert, abhängig von der Charakteristik des Grundfeldes, auf dem die Quartik definiert ist.

In Charakteristik ungleich 2 und 3 verallgemeinern wir den Satz von Segre–Rams–Schütt und beweisen, dass höchstens 64 Geraden auf einer K3-Quartikfläche liegen können. Es wird ein Beweis präsentiert, der die Anwendung der Fleknodal-Kurve vermeidet. Stattdessen wird die Geometrie spezieller Geradenkonfigurationen untersucht, beispielsweise der sogenannten Zwillingsgeraden. Wir stellen verschiedene konkrete glatte und nicht-glatte Quartiken mit einer hohen Anzahl von Geraden vor.

Des Weiteren werden K3-Quartikflächen in Charakteristik 2 und 3 analysiert. Von höchstem Interesse sind Geraden, die eine quasi-elliptische Faserung induzieren. In Charakteristik 3 zeigen wir, dass die Rams–Schütt Schranke von 112 Geraden für glatte Quartiken auch für K3-Quartiken gilt. Außerdem beweisen wir, dass es höchstens 67 Geraden geben kann, wenn diese Schranke nicht erreicht wird. Enthält die Fläche zusätzlich einen Stern, kann eine scharfe obere Grenze von 58 Geraden angegeben werden.

Der Fall Charakteristik 2 weist als Besonderheit das neuartige Phänomen auf, dass nicht-glatte K3-Quartikflächen eine höhere Anzahl von Geraden enthalten können als glatte Flächen. Dies ist eine Konsequenz aus der Anwesenheit quasi-elliptischer Geraden. Wir beweisen die scharfe Schranke von 68 Geraden für die Anzahl von Geraden, die auf einer K3-Quartikfläche liegen können. Zudem zeigen wir, dass alle Flächen, die diese Schranke erreichen, projektiv äquivalent zu einem Mitglied der Rams–Schütt Familie sind.

Schlüsselwörter: K3-Fläche, Quartikfläche, Gerade, rationaler Doppelpunkt, positive Charakteristik, elliptische Faserung, quasi-elliptische Faserung.

Abstract

K3 quartic surfaces are surfaces of degree 4 in projective space which admit rational double points as singularities. This thesis is concerned with the maximum number of lines that can lie on such a surface.

Our analysis is divided into three parts, according to the characteristic of the ground field over which the K3 quartic is defined.

In characteristic different from 2 and 3 we generalize Segre–Rams–Schütt's theorem and prove that at most 64 lines can lie on a K3 quartic surface. We present a proof that avoids the use of the flecnodal curve. Instead, we take advantage of special configurations of lines, such as twin lines. We also provide several examples of smooth and non-smooth K3 quartic surfaces containing many lines.

Furthermore, we investigate K3 quartic surfaces defined over fields of characteristic 2 and 3. Lines that induce quasi-elliptic fibrations play a major role. In characteristic 3, Rams–Schütt's bound of 112 lines for smooth quartics is also valid for K3 quartics. In addition, we show that there can be at most 67 lines if the surface has less than 112. In the case that the surface contains a star, we can prove a sharp bound of 58 lines.

Characteristic 2 features a new phenomenon, namely that non-smooth K3 surfaces contain more lines than smooth surfaces. This is due to the presence of quasi-elliptic lines. We prove the sharp bound of 68 for the number of lines that can be contained in a K3 quartic surface. Moreover, we show that each surface attaining this bound is projectively equivalent to a member of Rams–Schütt's family.

Keywords: K3 surface, quartic surface, line, rational double point, positive characteristic, elliptic fibration, quasi-elliptic fibration.

Contents

Chapter 1

Introduction

> Vous vous trompez, cher, le bateau file à bonne allure. Mais le Zuyderzee est une mer morte, ou presque. Avec ses bords plats, perdus dans la brume, on ne sait où elle commence, où elle finit. Alors, nous marchons sans aucun repère, nous ne pouvons évaluer notre vitesse. Nous avançons, et rien ne change. Ce n'est pas de la navigation, mais du rêve.
>
> Albert Camus, *La chute*

The problem of finding a bound for the number of lines on quartic surfaces is part of a broader topic, namely the enumerative geometry of lines on surfaces of degree d in projective space. This topic, which has gathered momentum in the last years (see, for example, Boissière–Sarti [5], Kollár [19], Rams–Schütt [33], Degtyarev–Itenberg–Sertöz [11], Shimada–Shioda [40]), has a long history that dates back to the 19th century.

The case of smooth cubic surfaces was already thoroughly studied by classical geometers such as Cayley, Clebsch, Salmon, Steiner, Schläfli, Cremona and Sturm. Every smooth cubic surface contains exactly 27 lines which are organized in a highly symmetric way related to the Weyl group of E_6. Several presentations of this result (which holds in any characteristic) exist in the modern literature; for instance, we refer the reader to [4], [13], [17], [24].

The general smooth surface of degree $d \geq 4$ contains no lines at all. The first one to state the correct optimal bound of 64 lines for smooth (complex) quartic surfaces was B. Segre in 1943 [39]; nonetheless, his proof contained some major gaps that have been corrected only 70 years later by Rams and Schütt [33]. Rams and Schütt used some techniques which were unknown to Segre, most notably the theory of elliptic fibrations developed by Kodaira in the 1950's. Segre stated that each line on a smooth quartic surface could meet at most 18 other lines. This was the crucial estimate that Rams and Schütt proved to be false, finding an explicit family of quartics $\mathcal{Z}$ containing surfaces with a line intersecting 19 or even 20 other lines, which prompted them to work out a new proof. They further examined this family in a follow-up article [32].

The problem is still open for smooth surfaces of higher degrees. There are some general bounds for $d \geq 5$, but none of them is known to be optimal. Some special cases with particular symmetries have been investigated by Boissière and Sarti [5], where they also find several surfaces with a high number of lines. We refer to their article also for an account of the known bounds.

Although non-smooth cubics had already been classified by Schläfli in the 1860's, it was not until 1979 that the number of lines lying on them was exactly determined by Bruce and Wall [8]. Non-smooth cubic surfaces always contain less than 27 lines, but one can count the lines with multiplicity – depending on the number and type of singular points lying on them – so that the total number is always 27.

Smooth quartic surfaces in $\mathbb{P}^3$ are K3 surfaces. In this thesis we deal with singular quartic surfaces whose minimal desingularization is a smooth K3 surface; these are precisely the quartic surfaces admitting at most rational double points as singularities. We call them 'K3 quartic surfaces'. González Alonso and Rams deal with the case of quartic surfaces with worse singularities in [14].

The behaviour of K3 quartic surfaces varies greatly according to the characteristic p of the ground field. With respect to the number of lines contained in them, the case $p = 0$ and the cases $p \geq 5$ can be studied together to a great extent. The cases $p = 2$ and $p = 3$, though, present such peculiarities that they demand a separate analysis. Being interested in an upper bound, we will always make the assumption that the ground field is algebraically closed.

1.1 Characteristic different from 2 and 3

The case of K3 quartic surfaces defined over an algebraically closed field of characteristic $p = 0$ or $p \geq 5$ can be approached in a similar way to Rams and Schütt's [33]. Our main result is the following theorem.

Theorem (Theorem 4.0.1)**.** *If X is a K3 quartic surface defined over an algebraically closed field of characteristic $p \neq 2, 3$, then X contains at most 64 lines.*

The main difficulty still lies in providing a bound for the valency of a line, i.e., the number of lines on the surface intersecting it. We study the elliptic fibration induced by the given line on the minimal desingularization of the K3 quartic surface; this fibration restricts to a morphism from the strict transform of the line to $\mathbb{P}^1$. There are two features which make the study of such fibrations much more involved in the K3 case, compared to the smooth one: first, the morphism from the line to $\mathbb{P}^1$ has always degree 3 if the quartic is smooth, but it has smaller degree as soon as there is a singular point on the line; second, more complicated Kodaira fiber types may appear. Thus, adapting Rams and Schütt's proof to the K3 quartic case forces us to study several new configurations; our results are summarized in Table 4.1.1. In addition to the family $\mathcal{Z}$ found by Rams and Schütt, we discover two new configurations on some non-smooth K3 quartic surfaces in which a line meets more than 18 lines. Explicit examples of such configurations are given in Section 4.5.

The bound of 64 lines is sharp and is reached by Schur's quartic, which is smooth. An optimal bound for K3 quartic surfaces with at least one singular point is not known; to our knowledge, the current explicit records are 39 lines over a field of characteristic zero (Example 4.5.3 due to González Alonso and Rams, which is a Delsarte surface) and 48 lines over a field of positive characteristic $\neq 2, 3$ (more precisely, over a field of characteristic 5, see Example 4.5.7). A non-smooth K3 quartic surface over $\mathbb{C}$ with 40 lines exists, but an explicit equation is not known (Example 4.5.4).

A new feature of our proof is that – unlike Segre and Rams–Schütt – we do *not* employ a technical tool called 'flecnodal curve'. Our approach offers a deeper insight into the geometry of the surfaces, and is based on two main ingredients: first, we discover a new, geometrically rich configuration of particular pairs of lines, which we call "twin lines"; second, thanks to seminal ideas by A. Degtyarev, we take advantage of the well-known lattice theory of K3 surface to tackle the so-called "triangle-free" surfaces, i.e., surfaces not containing three lines intersecting pairwise at smooth points of the surface.

The methods presented here stem from a fruitful synergy between the different points of view of two teams, the one – S. Rams, M. Schütt and the author – based in Hannover, Germany, the other – A. Degtyarev, I. Itenberg and A. S. Sertöz – mostly in Ankara, Turkey, which for some time worked on the same problem unaware of each other. The Ankara team, whose results can be found in [11], took a powerful lattice-theoretical approach, which is key to the proof of Lemma 4.2.12.

1.2 Characteristic 2 and 3

In characteristic 2 and 3, the picture changes drastically; notably, the Fermat quartic surface, considered over an algebraically closed field of characteristic 3, contains exactly 112 lines. Rams and Schütt proved that this is the maximum number that can be achieved in the smooth case [30].

In this thesis we prove a stronger result for K3 quartic surfaces.

Theorem (Theorem 5.0.1)**.** *If X is a K3 quartic surface defined over an algebraically closed field of characteristic 3, then X contains at most 112 lines. If X contains exactly 112 lines, then X is projectively equivalent to the Fermat quartic surface; otherwise, X contains at most 67 lines.*

There exist examples of smooth K3 quartic surfaces in characteristic 3 with 58 lines. Degtyarev [10] has proven that for smooth supersingular quartic surfaces this is indeed the maximal number of lines that can be attained after 112, and that there are exactly 3 admissible configurations with 58 lines. There are no known smooth (non-supersingular) quartic surfaces with 60 lines.

We provide the equations of three families of surfaces admitting the three configurations of 58 lines. Two of them have already been found independently by Degtyarev, whereas the third one is to our knowledge new (see Section 5.4).

We do not know whether there exist K3 quartic surfaces with $58 < n \leq 67$ lines. The highest number of lines that we could observe in a non-smooth K3 quartic surface is 48 (Example 5.4.5), attained by the reduction of a surface found by Shimada and Shioda [40]. We conjecture that there are no K3 quartic surfaces with more than 58 lines; in one particular case we are able to prove this stronger bound, namely when the surface X contains a star, i.e., four lines intersecting in a single smooth point (Proposition 5.3.13).

In characteristic 2, a smooth quartic can contain at most 60 lines. This bound was shown in several cases by Rams–Schütt [31], who also provided an explicit example of a quartic attaining the bound; a complete proof has been given by Degtyarev [10].

The case of K3 quartic surfaces shows a new surprising phenomenon when the characteristic of the ground field is 2. In fact, this is the only case in which non-smooth K3 quartic surfaces can contain more lines than smooth ones. More precisely, we prove the following theorem.

Theorem (Theorem 6.0.1)**.** *If X is a K3 quartic surface defined over an algebraically closed field of characteristic 2, then X contains at most 68 lines.*

The bound is attained by a family $\mathcal{X}_{68}$ of examples found by Rams and Schütt. We are able to prove a uniqueness result.

Theorem (Theorem 6.4.1)**.** *If X contains 68 lines, then X is projectively equivalent to a member of family $\mathcal{X}_{68}$.*

The characteristic 2 and 3 cases are special for the same reason: quasi-elliptic fibrations can only exist in these characteristics. The existence of such fibrations forces the surface to be Shioda-supersingular. Lines inducing quasi-elliptic fibrations are called quasi-elliptic lines and play an essential role because they can feature very high valencies: 30 in characteristic 3, and 19 in characteristic 2. Note that in characteristic 2, quasi-elliptic lines only occur if the quartic surface is non-smooth, thus explaining the higher number of lines. Such high valencies are attained by cuspidal lines (see Definitions 5.2.1 and 6.2.3).

Like in characteristic $p \neq 2, 3$, we provide several bounds for the valency of a line, both for elliptic and quasi-elliptic lines. We are able to show the sharpness of some of them (Tables 5.1.1, 5.2.1, 6.1.1, 6.2.1).

Since the flecnodal curve may degenerate in characteristic 2 and 3, it is all the more important to be able to study K3 quartic surfaces without referring to it.

1.3 Structure of the thesis

The thesis is divided into the following chapters.

Chapter 2 We briefly review well-known results about K3 surfaces, rational double points and genus 1 fibrations.

Chapter 3 We present basic results about lines on K3 quartic surfaces, setting up the notation. We give the definition of several key concepts, such as lines of the first and second kind, and triangle free surfaces.

Chapter 4 We deal with K3 quartic surfaces in characteristic different from 2 and 3, and we prove the bound of 64 lines. Particular attention is devoted to the construction of twin lines, special lines, and to the case of triangle free surfaces.

Chapter 5 We deal with K3 quartic surfaces in characteristic 3. We provide sharp bounds for the valency of quasi-elliptic lines and the case where the surface contains a star.

Chapter 6 We deal with K3 quartic surfaces in characteristic 2. We prove the bound of 68 lines and the uniqueness of Rams–Schütt's family.

Chapter 2

Preliminaries

In this chapter we present some well-known definitions and results that will be used throughout this thesis. We work over a fixed algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$.

2.1 K3 surfaces

The term *variety* will denote a separated, geometrically integral scheme of finite type over $\mathbb{K}$.

Definition 2.1.1. A *K3 surface* over $\mathbb{K}$ is a complete non-singular variety X of dimension 2 such that $\Omega_X^2 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

K3 surfaces form one of the four classes of minimal surfaces with Kodaira dimension 0 in the Enriques–Kodaira classification of surfaces (extended to surfaces defined over fields of positive characteristic by Bombieri and Mumford [6], [7], [26]). The name was chosen by André Weil in a famous quote from 1958 in honour of the three mathematicians Kummer, Kähler and Kodaira, and "the beautiful mountain K2 in Kashmir". To our knowledge, the most comprehensive treatise on K3 surfaces appeared until now is [18]. For a good overview on K3 surfaces over positive characteristic fields, see [23].

Example 2.1.2. Examples of K3 surfaces include

- smooth quartic surfaces in $\mathbb{P}^3$;
- smooth complete intersections of quadric and cubic hypersurfaces in $\mathbb{P}^4$;
- smooth complete intersections of three quadric hypersurfaces in $\mathbb{P}^5$;
- double covers of $\mathbb{P}^2$ branched over a smooth sextic curve.

Given a K3 surface X, it is a consequence of the definition and Serre duality that $h^{0,1}(X) = 0$ and $h^{0,0}(X) = h^{0,2}(X) = 1$, where $h^{p,q}(X) = h^q(X, \Omega_X^p)$ denote the Hodge numbers, so that $\chi(\mathcal{O}_X) = 2$. The *Euler–Poincaré characteristic* or *Euler number* of $e(X)$, defined as the alternating sum of the (étale cohomology) Betti numbers, coincides with the second Chern class $c_2(X)$. Noether's formula then yields $c_2(X) = 24$. Since the first and third Betti numbers vanish (in characteristic 0 this is a consequence of the Hodge decomposition; in positive characteristic see for example [23, Proposition 2.3]), $b_2(X) = 22$.

It follows that the Picard number $\rho(X)$, i.e., the rank of the Néron–Severi group of X, is not greater than 22. If $\operatorname{char} \mathbb{K} = 0$, then by the Lefschetz principle $\rho(X) \leq 20$: this is a consequence of the fact that for any complex projective variety $\rho(X) \leq h^{1,1}(X) = b_2(X) - 2h^{0,2}(X)$, because of the Hodge decomposition and Lefschetz theorem on (1,1)-classes.

2.2 Rational double points

In this section we want to collect a few well-known facts about rational double points. Such singularities occur with different names in the literature: simple surface singularities, du Val singularities, Kleinian singularities. For an overview of the techniques involved, we refer the reader to [9, Chapter 0, § 2], while the proofs are contained in [2], [28].

Let X be a surface. A closed point $x \in X$ is a *singularity* of X if its local ring $\mathcal{O}_{X,x}$ is not regular. A *resolution* of $x \in X$ is a birational morphism $\rho : Y \to X$ such that Y is smooth and $x \in \rho(Y)$; a resolution is *minimal* if it cannot be factored through another resolution. If x is normal, then there exists an open neighbourhood U of x such that ρ is an isomorphism over $U \setminus \{x\}$. The reduced (connected) curve $\rho^{-1}(x)$ is called the *exceptional curve* of the resolution.

A singularity $x \in X$ is called *rational* if it is normal and $R^1\pi_*\mathcal{O}_Y$ (which is a coherent sheaf concentrated on x) is zero. If x is rational, then all irreducible components E_i of the exceptional curve are smooth rational curves (the converse is not true, see [20]).

The *multiplicity* of a singularity $x \in X$ is the multiplicity of the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Recall that if A is a local ring of Krull dimension d with maximal ideal $\mathfrak{m}$, then the multiplicity of $\mathfrak{m}$ is by definition $d!$ times the leading coefficient of the Hilbert–Samuel polynomial of A.

Definition 2.2.1. A rational singularity of multiplicity 2 is called a *rational double point.*

A point $x \in X$ is a rational double point if and only if the irreducible components E_i of the exceptional curve are rational smooth curves of self-intersection -2. If $E_i \cdot E_j \neq 0$ for $i \neq j$, then $E_i \cdot E_j = 1$, and no three components intersect pairwise. The dual graph of the E_i's, i.e., the graph whose vertex set is $\{E_i\}$ such that two vertices E_i, E_j are connected by an edge if and only if $E_i \cdot E_j = 1$, is a Dynkin diagram of type **A**, **D** or **E** (see Figure 2.2.1).

If the characteristic of the ground field is equal to 0, then up to (formal) isomorphism a rational double point is given by the following equations:

type	equation
$\mathbf{A}_n$	$xy + z^{n+1}$
$\mathbf{D}_n$	$z^2 + x(y^2 + x^n)$
$\mathbf{E}_6$	$z^2 + x^3 + y^4$
$\mathbf{E}_7$	$z^2 + xy^3 + x^3$
$\mathbf{E}_8$	$z^2 + x^3 + y^5$

If the characteristic of the field is positive (and at most 5), then more isomorphism classes arise (see, for instance, [9]). Nonetheless, the following observations still hold:

- if the tangent cone of x is irreducible, then x is of type $\mathbf{A}_1$;

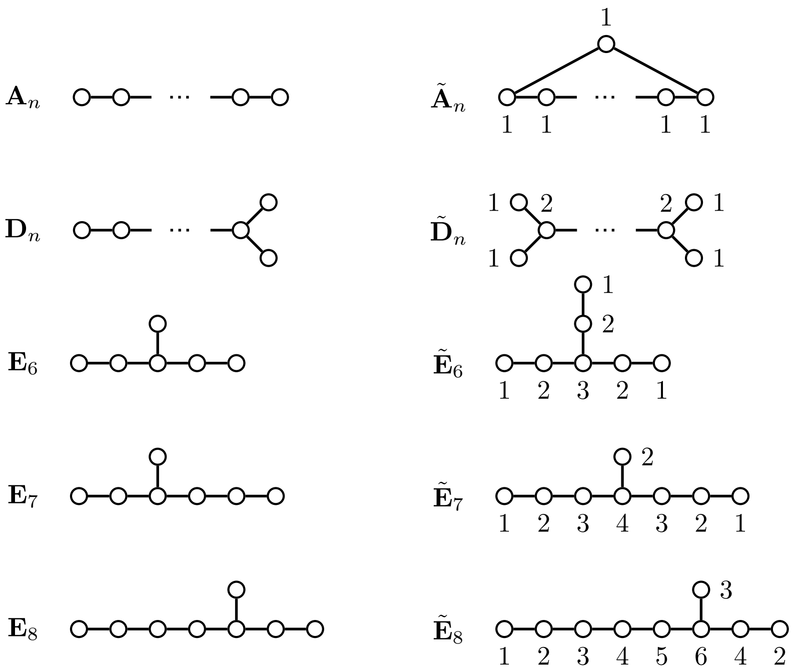


Figure 2.2.1: Dynkin diagrams (left) and extended Dynkin diagrams (right).

- if the tangent cone of x splits into two distinct planes, then x is of type $\mathbf{A}_n$, $n > 1$.

Being projective, the surface X has a dualizing sheaf ω_X [15, Proposition III.7.5]. Rational double points are *canonical* singularities, i.e., if X contains only rational double points, then ω_X is an invertible sheaf and $\rho^*(\omega_X) \cong \omega_Y$.

2.3 Genus 1 fibrations

Throughout this section, X is a smooth projective surface and B is a smooth projective curve with generic point η, both defined over an algebraically closed field $\mathbb{K}$.

Definition 2.3.1. A *genus 1 fibration* is a surjective morphism $X \to B$ such that the generic fiber X_η is a geometrically integral regular algebraic curve of arithmetic genus 1.

We will always assume genus 1 fibrations to be *relatively minimal*, i.e., not containing rational smooth curve of self-intersection -1 as fiber components. A genus 1 fibration is called an *elliptic fibration* if the generic fiber is smooth; otherwise, it is called a *quasi-elliptic fibration* (only possible if $\operatorname{char}\mathbb{K} = 2$ or 3 by a theorem of Tate [42]). Surfaces endowed with an elliptic or quasi-elliptic fibration are called elliptic or quasi-elliptic surfaces, respectively.

Genus 1 fibrations play an important role in the classification of surfaces, both in characteristic 0 and in positive characteristic. In this section we review some of their properties, on which we will base most of our arguments. For more references and proofs, see [3], [9], [18], [22], [25], [38].

2.3.1 Singular fibers

If $f : X \to B$ is a genus 1 fibration, then f is flat, which implies that the arithmetic genus of all fibers is equal to 1. If f is an elliptic fibration, there exists a finite subset $D \subset B$ of closed points such that the fiber $X_t = f^{-1}(t)$ is non-smooth if and only if $t \in D$. If f is a quasi-elliptic fibration, there exists a finite subset $D \subset C$ of closed points such that the fiber X_t is reducible if and only if $t \in D$. The set D is usually called the *degeneracy set* and a fiber X_t, $t \in D$, is called a degenerate fiber. In other words, a degenerate fiber is a singular fiber if the fibration is elliptic, or a reducible fiber if the fibration is quasi-elliptic.

Given a degenerate fiber X_t, we will denote by

$$X_t = \sum_{i \in I} m_i C_i$$

its decomposition into irreducible components; the numbers m_i are called the *multiplicities* of the components C_i and their greatest common divisor $m(t)$ is called the *multiplicity* of X_t.

If X_t is irreducible, then X_t is either a smooth curve of genus 1, a rational curve with an ordinary double point or a rational curve with a cusp. If X_t is reducible, then each component is a smooth rational curve of self-intersection -2 and the dual graph of X_t is an extended Dynkin diagram (see Table 2.3.1). Up to the common factor $m(t)$, the multiplicities m_i are given in Figure 2.2.1.

Table 2.3.1: Classification of fiber types for a genus 1 fibration.

fiber type	dual graph	Euler–Poincaré characteristic	curve
I_0	$\tilde{\mathbf{A}}_0$	0	smooth elliptic curve
I_1	$\tilde{\mathbf{A}}_0$	1	nodal rational curve
I_n, $n \geq 2$	$\tilde{\mathbf{A}}_{n-1}$	n	
II	$\tilde{\mathbf{A}}_0$	2	cuspidal rational curve
III	$\tilde{\mathbf{A}}_1$	3	two tangent rational curves
IV	$\tilde{\mathbf{A}}_2$	4	three concurrent rational curves
I_n^*, $n \geq 0$	$\tilde{\mathbf{D}}_{n+4}$	$n+6$	
IV^*	$\tilde{\mathbf{E}}_6$	8	
III^*	$\tilde{\mathbf{E}}_7$	9	
II^*	$\tilde{\mathbf{E}}_8$	10	

2.3.2 Euler–Poincaré characteristic

The following formula relating the Euler–Poincaré characteristic of X to the Euler–Poincaré characteristics of the fibers will be central to our work (see [9, Proposition 5.1.6]):

$$e(X) = e(B)e(X_{\bar{\eta}}) + \sum_{t \in B} \big(e(X_t) - e(X_{\bar{\eta}}) + \delta_t\big), \tag{2.1}$$

where $X_{\bar{\eta}}$ is the geometric generic fiber. The numbers δ_t are called *wild ramification indices*; they are non-negative integers which vanish if $\operatorname{char} \mathbb{K} \neq 2, 3$ or if f is a quasi-elliptic fibration. Non-degenerate fibers or fibers of type I_n never have wild ramification, so the sum in (2.1) is actually a finite sum over $t \in D$. Schütt and Schweizer [37] provide lower bounds for the wild ramification indices of the other fiber types in the case of an elliptic fibration: these are collected in Table 2.3.2 on page 12.

If f is an elliptic fibration, then the Euler–Poincaré characteristic of the geometric generic curve is equal to 0, so (2.1) takes the following form:

$$e(X) = \sum_{t \in B} \big(e(X_t) + \delta_t\big). \tag{2.2}$$

2.3.3 Base change

A fundamental operation in the theory of elliptic surfaces is the *base change*. Given a morphism $\psi : B' \to B$, the base change of an elliptic surface $X \to B$ is defined as the fiber product $X' = X \times_B B'$. After desingularization, the surface $X' \to B'$ is again an elliptic fibration.

It is important to keep track of how singular fibers behave. If wild ramification does not occur, the fiber type of the fiber X'_s over the point $s \in B'$ will be determined by the ramification index d of s for the morphism ψ. For example, if s is not a point of ramification, then the fiber X'_s will have the same fiber type as $X_{\psi(s)}$. This behavior is described in Table 2.3.3 on page 12 (see [25], [38]).

2.3.4 Mordell–Weil group

Let $f : X \to B$ be an elliptic fibration with a section $\sigma_0 : B \to X$ and at least one singular fiber. Given a smooth fiber X_t, the choice of the point $\sigma_0(t)$ as the origin endows X_t with a group structure. Thanks to the theory of Néron models (for a modern treatment see [41]), this is still true for the open set of smooth points $X_t^\prime$ of a singular fiber X_t; the connected component $(X_t^\prime)^o$ containing $\sigma_0(t)$ has group structure $\mathbb{G}_m$ (if X_t is of type I_n) or $\mathbb{G}_a$ (if X_t is not of type I_n), where $\mathbb{G}_m$ and $\mathbb{G}_a$ denote the multiplicative and additive group structures of $\mathbb{K}$, respectively. The structure of the group of components $X_t^\prime/(X_t^\prime)^o$ is given by Table 2.3.4 on page 12.

Let $K = \mathbb{K}(B)$ be the function field of B, and let $X(B)$ be the set of sections of f. There is a bijective correspondence between $X(B)$ and the set of K-rational points of the generic fiber X_η: the restriction of a section determines a rational point, and the closure of a rational point determines a section. The pair (X_η, O), where $O \in X_\eta$ is the point corresponding to σ_0, is an elliptic curve and thus induces a group structure on $X(B)$.

The group $X(B)$ is called the *Mordell–Weil group* of f (or of X) and it is a finitely generated abelian group (see for instance [38, Theorem 6.1] and references therein).

Torsion sections of elliptic fibrations will play a major role. We will often use the fact (see, for instance [38]) that there exists an injection

$$X(B) \hookrightarrow \prod_t X_t'/(X_t')^o.$$

The Mordell-Weil group of an elliptic surface acts on each smooth fiber by translation, and this action extends to the singular fibers.

Lemma 2.3.2. *Let $X \to B$ be an elliptic fibration endowed with an n-torsion section σ such that $p \nmid n$, where $p = \operatorname{char} \mathbb{K}$. Then, the minimal desingularization of the quotient X/G by the group G generated by the action of σ is an elliptic surface $X' \to B$ such that*

$$e(X) = e(X').$$

Sketch of proof. Let us denote by X' the minimal desingularization of X/G. Since σ acts fiberwise, X' is also an elliptic surface over B. Let $f : X \dashrightarrow X'$ be the rational map induced by composition; it corresponds to a morphism $f : X \setminus F \to X'$, where F is a finite set.

If ω is a regular 1-form, its pullback $f^*\omega$ is a rational 1-form and is regular on $S \setminus F$. The pullback $f^*\omega$ is not the zero form, since $p \nmid n$. Since the poles of a non-zero differential form are divisors, $f^*\omega$ is regular on all X, hence there is an injective map $f^* : \Gamma(X', \Omega_{X'}) \to \Gamma(X, \Omega_X)$. In particular we have $q(X') \leq q(X)$. The same applies to 2-forms, so $p_g(X') \leq p_g(X)$.

Then f induces an isogeny on the generic fibers and the dual isogeny induces a rational map $X' \dashrightarrow X$. The same argument works, so we have the equalities $q(X) = q(X')$ and $p_g(X) = p_g(X')$. Since the surfaces are elliptic, they both have $K^2 = 0$. We conclude by applying Noether's formula. □

2.3.5 Quasi-elliptic fibrations

Bombieri and Mumford introduced the notion of a quasi-elliptic surface in one of their articles on the extension of Enriques's classification to surfaces defined over positive characteristic fields [7]. Nonetheless, it was already known to Tate [42] that if the geometric generic fiber of a genus 1 fibration is not smooth, then it is a rational curve with a cusp; furthermore, this can only happen if the characteristic p of $\mathbb{K}$ is equal to 2 or 3.

In particular, given a quasi-elliptic fibration $f : X \to B$ the Euler–Poincaré characteristic of the geometric generic fiber is equal to 2. Since wild ramification indices vanish, formula (2.1) takes on the following form:

$$e(X) = 2\, e(B) + \sum_{t \in B} \big(e(X_t) - 2\big). \tag{2.3}$$

Definition 2.3.3. The curve formed by the closure of the locus of singular points on the irreducible fibers is called *curve of cusps* or *cuspidal curve*.

The cuspidal curve K is a smooth curve on X such that $K \cdot X_t = p$ for every $t \in B$. Moreover, the restriction of the quasi-elliptic fibration f to K is a purely inseparable morphism of degree p (see [9, Proposition 5.1.7]).

The fiber types of reducible fibers that can appear in a quasi-elliptic fibration are limited. Due to the action of the Mordell–Weil group, the cuspidal curve can intersect reducible fibers only in very special ways. We will describe these phenomena separately for $p = 3$ and $p = 2$, in their respective chapters (see Sections 5.2 and 6.2).

Table 2.3.2: Lower bounds and precise values for wild ramification indices δ_t of an elliptic fibration according to the fiber type of F_t; a single entry indicates equality.

fiber type	$\mathrm{I}_{n\geq 0}$	II	III	IV	$\mathrm{I}^*_{n\neq 1}$	I^*_1	IV^*	III^*	II^*
$p=2$	0	≥ 2	≥ 1	0	≥ 2	1	0	≥ 1	≥ 1
$p=3$	0	≥ 1	0	≥ 1	0	0	≥ 1	0	≥ 1

Table 2.3.3: Fibers after a base change of ramification index d.

before	d	after	before	d	after
I_n	$d\geq 1$	I_{dn}	I^*_n	0 mod 2	I_{dn}
				1 mod 2	I^*_{dn}
II	0 mod 6	I_0	II^*	0 mod 6	I_0
	1 mod 6	II		1 mod 6	II^*
	2 mod 6	IV		2 mod 6	IV
	3 mod 6	I^*_0		3 mod 6	I^*_0
	4 mod 6	IV^*		4 mod 6	IV^*
	5 mod 6	II^*		5 mod 6	II
III	0 mod 4	I_0	III^*	0 mod 4	I_0
	1 mod 4	III		1 mod 4	III^*
	2 mod 4	I^*_0		2 mod 4	I^*_0
	3 mod 4	III^*		3 mod 4	III
IV	0 mod 3	I_0	IV^*	0 mod 3	I_0
	1 mod 3	IV		1 mod 3	IV^*
	2 mod 3	IV^*		2 mod 3	IV

Table 2.3.4: Group of components.

fiber type	$\mathrm{I}_{n\geq 1}$	I^*_{2n}	I^*_{2n+1}	II, II^*	III, III^*	IV, IV^*
$X'_t/(X'_t)^o$	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$

Chapter 3

K3 quartic surfaces

In this chapter we set the main framework of our thesis and present the notation which will be used throughout. We work over a fixed algebraically closed ground field $\mathbb{K}$ of characteristic $p \geq 0$.

Definition 3.0.1. A *K3 quartic surface* is a surface in $\mathbb{P}^3$ of degree 4 admitting only rational double points as singularities.

Rational double points are in particular normal singularities; therefore, a K3 quartic surface only admits isolated singularities. The name stems from the fact that the minimal desingularization of a K3 quartic surface is a K3 surface.

The chapter is structured as follows.

Section 3.1 We describe the birational geometry of K3 quartic surfaces and of the lines lying on them; in particular, we prove that a line on a K3 quartic surface induces a genus 1 fibration on its desingularization.

Section 3.2 We present basic results about the interplay of lines and singularities of the surface.

Section 3.3 Some classical techniques are introduced, such as the distinction between lines of the first and second kind.

Section 3.4 We collect some basic results about triangle free surfaces that are valid in all characteristics and we set the nomenclature for completely reducible planes.

3.1 Lines

Let X be a fixed K3 quartic surface, and let $\operatorname{Sing}(X)$ be the set of singular points of X. Let $\rho : Z \to X$ be the minimal desingularization of X.

Proposition 3.1.1. *The surface Z is a K3 surface and $H := \rho^*(\mathcal{O}_X(1))$ is a nef line bundle such that $H^2 = 4$.*

Proof. We have to show that $\omega_Z \cong \mathcal{O}_Z$ and $h^1(Z, \mathcal{O}_Z) = 0$. By the adjunction formula, $\omega_X \cong \omega_{\mathbb{P}^3}(X)|_X \cong \mathcal{O}_X$. Since X only contains rational double points, $\omega_Z = \rho^*(\omega_X) \cong \mathcal{O}_Z$. Given that X is a hypersurface of dimension 2, $h^1(X, \mathcal{O}_X) = 0$; using the Leray spectral sequence and the definition of rational double point, $h^1(Z, \mathcal{O}_Z) = h^1(X, \mathcal{O}_X)$.

The other two assertions are obvious. □

From now on, ℓ will denote a line lying on a K3 quartic surface X with minimal desingularization $\rho : Z \to X$. Any divisor in the complete linear system defined by $\mathcal{O}_Z(H)$ will be called a *hyperplane divisor* (and often denoted by H, too). The strict transform of ℓ will be denoted by L.

Lemma 3.1.2. *The pencil of planes $\{\Pi_t\}_{t\in\mathbb{P}^1}$ containing the line ℓ induces a genus 1 fibration $\pi : Z \to \mathbb{P}^1$.*

Proof. The pullbacks by the morphism ρ of the curves $\Pi_t \cap X$, obtained by intersecting X with the pencil of planes containing ℓ, define a pencil Σ on Z. Let $F + \Delta$ be a general member of Σ, where Δ is the fixed part; the curve L must be contained in Δ. Given an arbitrary $F' \in |F|$, we have $F' + \Delta \in |H|$, because the linear system $|H|$ is complete; therefore, there is a plane Π' with $\varphi^*\Pi' = F' + \Delta$. Since

$$\rho(\operatorname{supp}(F' + \Delta)) \supset \varphi(L) = \ell,$$

it must be $\ell \subset \Pi'$, so $F' + \Delta \in \Sigma$; hence, $|F| + \Delta = \Sigma$ and, in particular, $\dim |F| = 1$. The Riemann-Roch formula implies that $F^2 = 0$ and $h^1(\mathcal{O}_Z(F)) = 0$; moreover, F is an irreducible curve of arithmetic genus 1 [35, Proposition 2.6.]. It follows that the morphism $\pi := \pi_{|F|} : Z \to \mathbb{P}^1$ induced by the base-point-free complete linear system $|F|$ is a genus 1 fibration. □

The genus 1 fibration π induced by the pencil of planes containing ℓ – or, for short, induced by ℓ – is always elliptic if $\operatorname{char} \mathbb{K} \neq 2, 3$, but in general we have to distinguish the two cases.

Definition 3.1.3. A line ℓ is said to be *elliptic* (respectively *quasi-elliptic*) if it induces an elliptic (respectively *quasi-elliptic*) fibration.

Let x_0, x_1, x_2, x_3 be the coordinates of $\mathbb{P}^3$. Up to projective equivalence, we can suppose that the line ℓ is given by the vanishing of x_0 and x_1, so that the quartic X is defined by

$$X : \sum_{i_0+i_1+i_2+i_3=4} a_{i_0i_1i_2i_3} x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} = 0, \tag{3.1}$$

where $i_0, \dots, i_4$ are non-negative integers, $a_{i_0i_1i_2i_3} \in \mathbb{K}$ and $a_{i_0i_1i_2i_3} = 0$ if $i_2 = i_3 = 0$. It will often be convenient to rewrite the equation of X in this way:

$$X : x_0\alpha(x_2, x_3) + x_1\beta(x_2, x_3) + \text{terms containing } x_0^2,\ x_0x_1 \text{ or } x_1^2 = 0.$$

The forms α and β have degree 3; explicitly:

$$\begin{aligned} \alpha(x_2, x_3) &= a_{1030}x_2^3 + a_{1021}x_2^2x_3 + a_{1012}x_2x_3^2 + a_{1003}x_3^3, \\ \beta(x_2, x_3) &= a_{0130}x_2^3 + a_{0121}x_2^2x_3 + a_{0112}x_2x_3^2 + a_{0103}x_3^3. \end{aligned} \tag{3.2}$$

Remark 3.1.4. We will usually parametrize the planes containing ℓ by $\Pi_t : x_0 = tx_1$, $t \in \mathbb{P}^1$, where of course $t = \infty$ denotes the plane $x_1 = 0$. Two equations which define the *residual cubic* E_t contained in Π_t are the equation of Π_t itself and the equation $g \in \mathbb{K}[t][x_1, x_2, x_3]_{(3)}$ obtained by substituting x_0 with tx_1 in (3.1) and factoring out x_1. An explicit computation shows that the intersection of ℓ with E_t is given by the points $[0 : 0 : x_2 : x_3]$ satisfying

$$g_t(0, x_2, x_3) = t\alpha(x_2, x_3) + \beta(x_2, x_3) = 0. \tag{3.3}$$

Note that we will always consider a residual cubic as a planar curve (of degree 3) in Π_t. In particular, it can be irreducible (in which case either it is smooth, or it contains a node, or it contains a cusp), or reducible (in which case it can split either into the union of a line and an irreducible conic, or into the union of three lines).

The proof of Lemma 3.1.2 shows that a fiber F_t of π ($t \in \mathbb{P}^1$) is the pullback through ρ of the residual cubic E_t. A general fiber will be usually called F, while a general residual cubic will be usually called E. We denote the restriction of π to L again by π.

Definition 3.1.5. If the morphism $\pi : L \to \mathbb{P}^1$ is constant, we say that L has degree 0; otherwise, the *degree* of ℓ is the degree of the morphism $\pi : L \to \mathbb{P}^1$.

Definition 3.1.6. The *singularity* of a line ℓ is the number of singular points of X lying on ℓ.

Proposition 3.1.7. *If ℓ is a line of degree d and singularity s, then,*

$$d = 3 - \sum_{P \in \ell \cap \mathrm{Sing}(X)} I_P(E, \ell), \tag{3.4}$$

where E is a general residual cubic, and $I_P(E, \ell)$ is the intersection multiplicity of E and ℓ at P as planar curves; in particular, $d \leq 3 - s$, and $d = 3$ if and only if $s = 0$.

Proof. The degree of $\pi : L \to \mathbb{P}^1$ is given by 3 minus the common roots of α and β counted with multiplicity (note that α and β cannot be identically zero at the same time, otherwise ℓ would be a line of singular points). Observe that ℓ contains a singularity at the point $[0 : 0 : x_2 : x_3]$ if and only if $[x_2 : x_3]$ is a common root of α and β; moreover, the multiplicity of this common root is exactly $I_P(E_t, \ell)$ for a general $t \in \mathbb{P}^1$. This proves formula (3.4); in particular, if α and β have no roots in common or, equivalently, if there are no singularities on ℓ – and only in that case – the degree of the associated morphism $\pi : L \to \mathbb{P}^1$ is 3. □

Remark 3.1.8. If α and β have all roots in common, i.e., they are multiple of each other, then there is exactly one plane whose intersection with the quartic contains ℓ as a non-reduced component (if there were more, the surface would have worse singularities than rational double points): this plane is the only plane tangent to the surface along the line ℓ. This means precisely that the degree of the morphism $\pi : L \to \mathbb{P}^1$ is zero, that is to say, L is a fiber component for the morphism $\pi : Z \to \mathbb{P}^1$. Conversely, if $\pi : L \to \mathbb{P}^1$ has degree 0, then α and β have all roots in common.

By Proposition 3.1.7, the degree of a line is never greater than 3; therefore, the morphism $\pi : L \to \mathbb{P}^1$ is always separable if $\operatorname{char} \mathbb{K} \neq 2, 3$. Again, in general we have to make a distinction.

Definition 3.1.9. A line ℓ is said to be *separable* (respectively *inseparable*) if the induced morphism $\pi : L \to \mathbb{P}^1$ is separable (respectively inseparable).

Given a separable line ℓ, we will say that a point P on ℓ is a point of ramification n_m if the corresponding point on L has ramification index n and $\text{length}(\Omega_{L/\mathbb{P}^1}) = m$. We recall that if $\operatorname{char}\mathbb{K}$ does not divide n, then $m = n-1$ and can be omitted, whereas if $\operatorname{char}\mathbb{K}$ divides n, then $m \geq n$.

3.2 Singularities

Let P be a singular point on a K3 quartic surface X (not necessarily containing a line). If we choose coordinates so that the point P is given by $[0:0:0:1]$, the defining equation of X becomes

$$X : x_3^2 f_2(x_0,x_1,x_2) + x_3 f_3(x_0,x_1,x_2) + f_4(x_0,x_1,x_2) = 0, \tag{3.5}$$

where the f_i's are homogeneous forms of degree i.

Definition 3.2.1. We call these forms the *(second, third, fourth) Taylor coefficients* of X at P.

Since we are considering only rational double points, the form f_2 is not identically zero and the equation $f_2 = 0$ defines the *tangent cone* of X at P.

Lemma 3.2.2. *If P is a singular point on a K3 quartic surface X, then there are at most 8 lines lying on X and passing through P. Moreover, if there are more than 6, then the second and the third Taylor coefficients of X at P share a common factor; if there are 8, then the second Taylor coefficient of X at P must divide the third.*

Proof. Consider equation (3.5). A line parametrized by $t \mapsto [at : bt : ct : 1]$ is contained in X if and only if $[a : b : c]$ is a point of intersection of the three plane curves of degree $i = 2,3,4$ defined by $f_i = 0$. Recall that by Bézout's theorem two plane curves of degree d and e without irreducible components in common have at most $d \cdot e$ distinct points in common.

Note first that f_2, f_3 and f_4 cannot all have a common irreducible component, otherwise the surface X would be reducible.

- Suppose first that f_2 is irreducible. Then, by Bézout's theorem, it intersects f_3 in at most 6 points, unless it is an irreducible component f_3; in this case f_2 divides f_3 and, since f_2 is not a component of f_4, f_2 and f_4 have at most 8 common points, which is what we claimed.
- Suppose now that $f_2 = gh$ is the union of two lines $g = 0$, $h = 0$, which may be identical or different. If none of these lines is a component of f_3, then the number of common solutions of f_2 and f_3 is at most 6.

Hence, if the number of common solutions is bigger than 6, the curves defined by f_2 and f_3 have at least one common irreducible component, say g. Since each common component of f_2 and f_3 is not a component of f_4, g gives at most 4 solutions with f_4. If h is not a component of f_3, then they intersect in at most 3 distinct point, so the number of intersection points is at most $3 + 4 = 7$.

Therefore, in order to have 8 distinct solutions the two lines g, h must be different and also h must be a component of f_3, which implies that the polynomial f_2 divides f_3. □

Lemma 3.2.3. *If P is a singular point on a line ℓ, a general residual cubic relative to ℓ is smooth at P.*

Proof. If this is not the case, then P is a triple point, as can be checked by an explicit computation. □

Lemma 3.2.4. *Let P be a singular point on a line ℓ. Then, exactly one of the exceptional divisors on Z coming from P is a section of the fibration induced by ℓ, and all others are fiber components.*

Proof. Take a general residual cubic E relative to ℓ. Since E is smooth at P by Lemma 3.2.3, its strict transform F hits exactly one exceptional divisor. On the other hand, all other exceptional divisors have intersection 0 with the fiber F, so they must be fiber components. □

3.3 Valency

Definition 3.3.1. Given a K3 quartic surface X, we will denote by $\Phi(X)$ the number of lines lying on X.

The letter Φ is reminiscent of the name Fano, as the Hilbert scheme of lines on X is usually called the Fano variety of lines on X. The aim of this thesis is to find a bound for $\Phi(X)$. The first observation is that, given any plane Π in $\mathbb{P}^3$, a line ℓ contained in X either lies on Π or meets Π in exactly one point.

We will usually be interested in finding a *completely reducible plane*, i.e., a plane Π such that the intersection $X \cap \Pi$ splits into the highest possible number of irreducible components, namely four lines $\ell_1, \dots, \ell_4$ (not necessarily distinct). If a line ℓ' not lying on Π meets two or more distinct lines ℓ_i, then their point of intersection must be a singular point of the surface. It follows that $\Phi(X)$ is bounded by

$$\begin{aligned}\Phi(X) \leq \#&\{\text{lines in } \Pi\} \\ &+ \#\{\text{lines not in } \Pi \text{ going through } \Pi \cap \operatorname{Sing}(X)\} \\ &+ \sum_{i=1}^{4} \#\{\text{lines not in } \Pi \text{ meeting } \ell_i \text{ in a smooth point}\}.\end{aligned} \tag{3.6}$$

It will then be a matter of finding a bound for the second and third contribution. The former will be usually dealt with using Lemma 3.2.2. As for the latter, it is natural to introduce the following definition.

Definition 3.3.2. The *valency* of ℓ, denoted by $v(\ell)$, is the number of lines on X distinct from ℓ which intersect ℓ in smooth points.

Most of the time we will express the latter contribution in terms of $v(\ell_i)$, and much of the work will be dedicated to finding a bound for these quantities. Of course, not all K3 quartic surfaces admit a completely reducible plane, in which case we will turn to other techniques, such as the ones presented in Section 3.4.

In this section we collect some general facts about the valency of a line that are valid in all characteristics.

Definition 3.3.3. A *3-fiber* is a fiber whose residual cubic splits into three lines, whereas a *1-fiber* is a fiber whose residual cubic splits into a line and an irreducible conic. A line ℓ is said to be *of type* (p, q), $p, q \geq 0$, if in its fibration there are p fibers of the former kind and q fibers of the latter kind.

Definition 3.3.4. The (local) valency of a fiber F, denoted by $v_\ell(F)$, is the number of lines distinct from ℓ contained in the plane corresponding to F that meet ℓ in a smooth point. When it is clear from the context, we will simply write $v(F)$.

Obviously,

$$v(\ell) = \sum_{t \in \mathbb{P}^1} v_\ell(F_t), \tag{3.7}$$

and the sum is actually a finite sum.

Let ℓ be a line of positive degree d. If F is a 3-fiber, then at most d of the 3 lines contained in the corresponding residual cubic can meet ℓ in a smooth point, so $v(F) \leq d$; if F is a 1-fiber, then $v(F) \leq 1$. It follows that if ℓ has type (p, q) formula (3.7) becomes

$$v(\ell) \leq dp + q. \tag{3.8}$$

On the other hand, lines of degree 0 behave in a very special way, as the following lemma shows.

Lemma 3.3.5. *If ℓ is a line of degree 0, then $v(\ell) \leq 2$.*

Proof. As we observed in Remark 3.1.8, a line ℓ is of degree 0 if and only if there exists a plane Π which is tangent to X along ℓ, i.e., ℓ appears with multiplicity at least 2 in the intersection of Π with X. The residual conic on this plane might split into two lines. On the other hand, all other lines meeting ℓ must pass through one of the singular points of ℓ, thus not contributing to the valency of ℓ. □

Lemma 3.3.6. *If ℓ is an elliptic line of type (p, q), then*

$$3\,p + 2\,q \leq 24. \tag{3.9}$$

Proof. A 3-fiber F contains at least 3 components; hence, its Euler–Poincaré characteristic $e(F)$ is at least 3; similarly, if G is a 1-fiber, then $e(G) \geq 2$. The formula follows then from (2.2). □

Lemma 3.3.7. *If ℓ is an elliptic line without 3-fibers, then $v(\ell) \leq 12$.*

Proof. This is a straightforward application of formulas (3.8) and (3.9) with $p = 0$. □

The assumption that ℓ is elliptic in Lemmas 3.3.6 and 3.3.7 is essential, see Sections 5.2 and 6.2.

3.3.1 Lines of the first and second kind

Given a line ℓ of positive degree, a crucial technique to find bounds for $v(\ell)$ is to count the points of intersection of the residual cubics E_t and ℓ which are inflection points for E_t. By *inflection point* we mean here a point which is also a zero of the hessian of the cubic (i.e., the determinant of its hessian matrix – see formula (3.11) when $\operatorname{char} \mathbb{K} = 2$). In fact, if a residual cubic E_t contains a line as a component, all the points of the line will be inflection points of E_t.

Writing out the equation of a cubic in $\mathbb{P}^2$ explicitly and computing the determinant of its hessian matrix, one can also check the following lemma.

Lemma 3.3.8. *Let E be a reducible cubic in $\mathbb{P}^2$ that is the union of an irreducible conic and a line ℓ'. Then, the locus of inflection points of E is exactly ℓ'.*

Supposing that the surface X is defined as in equation (3.1), the hessian of the equation g defining the residual cubic E_t (see Remark 3.1.4) restricted on the line ℓ is given by

$$h := \det\left(\frac{\partial^2 g}{\partial x_i x_j}\right)_{1\leq i,j,\leq 3}\Bigg|_{x_1=0} \in \mathbb{K}[t][x_2,x_3]_{(3)}, \tag{3.10}$$

which is a polynomial of degree 5 in t, with forms of degree 3 in (x_2, x_3) as coefficients. If $\operatorname{char}\mathbb{K} = 2$, we need to modify the definition of the hessian slightly, as suggested by Rams and Schütt [30]. If m is the coefficient of the monomial $x_1x_2x_3$ in g, then one defines

$$\tilde{h} = \frac{1}{4}\left(\frac{1}{2}\det\left(\frac{\partial^2 g}{\partial x_i x_j}\right)_{1\leq i,j,\leq 3} - m^2 g\right)\Bigg|_{x_1=0} \in \mathbb{K}[t][x_2,x_3]_{(3)}, \tag{3.11}$$

which is to be understood first as an algebraic expression over $\mathbb{Z}$ in terms of the generic coefficients of g, then interpreted over $\mathbb{K}$ by reducing modulo 2 and substituting.

We want now to find the number of lines intersecting ℓ by studying the common solutions of (3.3) and (3.10) (or (3.11)) on the line ℓ. It is convenient to extend Segre's nomenclature [39].

Definition 3.3.9. The resultant $R(\ell)$ with respect to the variable t of the polynomials (3.3) and (3.10) (or (3.11) if $\operatorname{char}\mathbb{K} = 2$) is called the *resultant* of the line ℓ.

Definition 3.3.10. We say that a line ℓ of positive degree is a line of the *second kind* if its resultant is identically equal to zero. Otherwise, we say that ℓ is a line of the *first kind*.

A root $[\bar{x}_2 : \bar{x}_3]$ of $R(\ell)$ corresponds to a point $P = [0 : 0 : \bar{x}_2 : \bar{x}_3]$ on ℓ; if P is a smooth surface point, then it is an inflection point for the residual cubic passing through it. A local computation yields the following lemma, which holds in any characteristic.

Lemma 3.3.11. *Let ℓ be a line of the first kind.*

(a) If one line intersects ℓ at a smooth point P, then P is a root of $R(\ell)$.
(b) If two lines or one double line intersect ℓ at a smooth point P, then P is a root of $R(\ell)$ of order at least 2.
(c) If three lines, one double line and a simple line, or one triple line intersect ℓ at a smooth point P, then P is a root of $R(\ell)$ of order at least 5.

Proposition 3.3.12. *If ℓ is a line of the first kind, then $v(\ell) \leq 3 + 5\,d$.*

Proof. Since equation (3.3) is linear in t and – once one has got rid of the common factors of α and β – it has degree d in (x_2, x_3), the resultant $R(\ell)$ of a line ℓ of the first kind is a form in (x_2, x_3) of degree $3 + 5\,d$. The claim follows from Lemma 3.3.11. □

Corollary 3.3.13. *If ℓ is a line of the first kind of valency 18, then ℓ has type $(p, q) = (6, 0)$, $(5, 3)$ or $(4, 6)$.*

Proof. From Proposition 3.3.12 we deduce that $d = 3$. By Lemma 3.3.11, the multiplicity of each root of $R(\ell)$ must be equal to the number of lines different from ℓ passing through the corresponding point (and there can be at most double roots). It follows that (3.8) is indeed an equality, so $q = 18 - 3p$. Since $q \geq 0$, it must be $p \leq 6$. On the other hand, substituting into (3.9) one finds $p \geq 4$. □

3.4 Triangle free surfaces

Definition 3.4.1. Let X be a K3 quartic surface with minimal desingularization Z. The *line graph* of X is the graph whose vertex set is the set of lines on X such that two vertices ℓ, ℓ' are connected by an edge if and only if $L \cdot L' = 1$, where L and L' are the strict transforms of the lines ℓ and ℓ', respectively.

The line graph $\Gamma = \Gamma(X)$ of a K3 quartic surface X is a graph without loops or multiple edges. By definition, the number of its vertices is equal to $\Phi(X)$.

Definition 3.4.2. A K3 quartic surface X is called *triangle free* if its line graph contains no triangles, i.e., cycles of length 3.

In other words, a K3 quartic surface X is triangle free if there are no triples of distinct lines on X forming a *triangle*, i.e., intersecting pairwise in *smooth* points. The next definition has an analogous geometric interpretation.

Definition 3.4.3. A K3 quartic surface X is called *square free* if it is triangle free and if its line graph contains no squares, i.e., cycles of length 4.

Recall that a graph induces a symmetric bilinear form on the lattice generated by its vertices, in the following way:

$$v^2 = v \cdot v := -2 + 2 \cdot \#\{\text{loops around } v\}$$
$$v \cdot w := \#\{\text{edges joining } v \text{ and } w\}$$

Note that the symmetric form on the line graph of a K3 quartic coincides with the intersection form on the lines contained in its minimal desingularization.

Definition 3.4.4. A connected graph is called *elliptic* if its associated form is negative definite; *parabolic* if its associated form is negative semidefinite, with kernel of dimension 1. In other words, elliptic graphs are Dynkin diagrams and parabolic graphs are extended Dynkin diagrams.

In what follows, by 'subgraph' we will always mean an 'induced subgraph'. We will denote by $|G|$ the cardinality of the set of vertices of a graph G. The Milnor number $\mu(G)$ of a graph G is the rank of its associated form.

Let now Γ be the line graph of a K3 quartic surface X. Given a subgraph $G \subset \Gamma$, the *span* of G will be the subgraph

$$\operatorname{span} G = G \cup \{m \in \Gamma : m \cdot l = 1 \text{ for some } l \in G\};$$

the *valency* of G will be

$$v(G) := |(\operatorname{span} G) \smallsetminus G|.$$

Note that this definition extends naturally the notion of 'valency of a line'.

Two subgraphs G, G' of Γ are said to be *disjoint* if $\operatorname{span} G \cap G' = \varnothing$.

Lemma 3.4.5. *If ℓ is an elliptic line on a triangle free K3 quartic surface, then* $v(\ell) \leq 12$.

Proof. For lines of degree 0, we have $v(\ell) \leq 2$ by Lemma 3.3.5, so we can suppose that ℓ has positive degree. It is sufficient to show that

$$2\,v(F) \leq e(F) \tag{3.12}$$

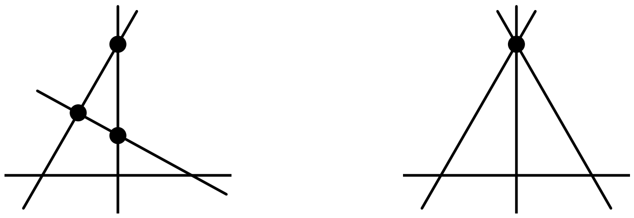
for all fibers F of the fibration induced by ℓ. In fact, if this holds, then by formula (3.7)

$$2\,v(\ell) = \sum_{t\in\mathbb{P}^1} 2\,v(F_t) \leq \sum_{t\in\mathbb{P}^1} e(F_t) \leq 24$$

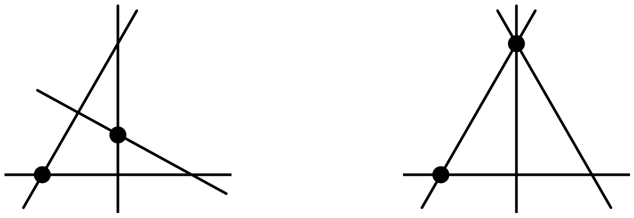
and we conclude.

Formula (3.12) is clear for a 1-fiber F, because $v(F) \leq 1$ and $e(F) \geq 2$. Observe that if a 3-fiber F contains a double line, then $e(F) \geq 6$. Since $v(F)$ can never be greater than 3, we get (3.12). Hence, we can suppose that F is a 3-fiber composed of three distinct lines ℓ_1, ℓ_2, ℓ_3. Moreover, no three lines among ℓ and the ℓ_i's can meet in a smooth point, since there are no triangles.

If $v(F) = 3$, then two configurations may arise, as pictured below: either the ℓ_i's meet in different points or they are concurrent. If they meet in different points, then all points must be singular, because of the triangle free assumption, giving rise to a fiber of type I_n with $n \geq 6$. If they are concurrent, then the intersection point must be singular and the corresponding fiber must have at least 4 components, three of which of multiplicity 1, and no cycle, i.e., it must be of type I_n^* or IV^*. In any case, $e(F_s) \geq 6$ and again we obtain (3.12).



If $v(F) = 2$, then there is a singular point on ℓ and one of the ℓ_i's passes through it, while the other two lines meet ℓ in two other smooth points. Again, the lines ℓ_i can meet in different points or in the same point (see picture below). In the former case, it is not possible that all the intersection points of the ℓ_i's are smooth; thus, the fiber is of type I_n with $n \geq 4$ and (3.12) holds. In the latter case, we can argue as before.



In case $v(F) = 1$ formula (3.12) is automatically satisfied, since for a 3-fiber $e(F) \geq 3$. ☐

In the last part of this section, we would like to classify the possible configurations of lines and singular points on a completely reducible plane. Note that if three lines form a triangle, then they are necessarily coplanar and the plane containing them is completely reducible.

Lemma 3.4.6. *If three lines on X form a triangle, then they are contained in plane Π such that the intersection of Π and X has one of the configurations pictured in Figure 3.4.1.*

Proof. Let ℓ_1, ℓ_2, ℓ_3 be the lines forming a triangle and ℓ_4 the fourth line on the plane. If ℓ_4 coincides with one of the former, then we get configurations $\mathcal{D}_0$ or $\mathcal{E}_0$. Suppose the four lines are pairwise distinct. A priori the following three configurations are possible:

- either the lines meet in pairwise distinct points (configurations $\mathcal{A}$),
- or exactly three of them are concurrent (configurations $\mathcal{B}$),
- or four of them are concurrent (configuration $\mathcal{C}_0$).

Note that a singular point of the surface contained in the plane must be the intersection point of two or more lines. By hypothesis, ℓ_1, ℓ_2 and ℓ_3 meet in smooth points. Up to symmetry, the only possible configurations are those in the picture. □

Lemma 3.4.7. *If X admits a completely reducible plane Π without a triangle, then the intersection of Π and X has one of the configurations in Figure 3.4.2, if the lines on Π are pairwise distinct, or Figure 3.4.3, if there is at least one multiple component.*

Proof. The proof employs the same combinatorial arguments as in Lemma 3.4.6 and we omit it. □

The nomenclature for completely reducible planes introduced in Figures 3.4.1, 3.4.2 and 3.4.3 will be used also in the next chapters.

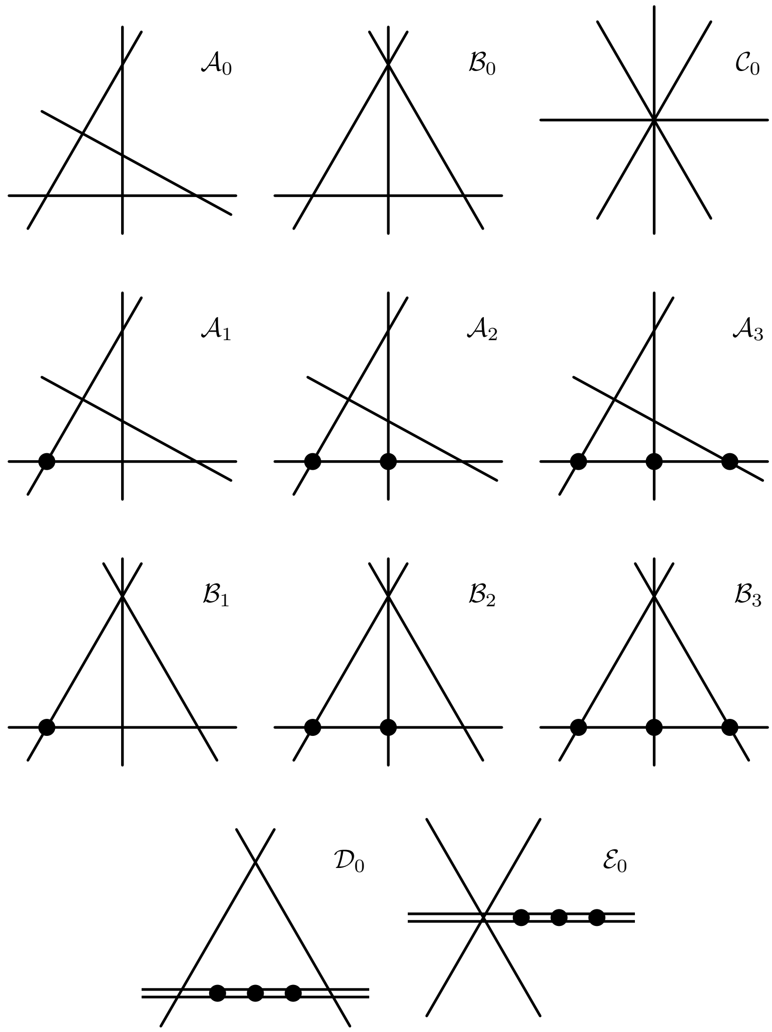


Figure 3.4.1: Possible configurations of lines on a plane with a triangle. Singular points are marked with a bullet. In configurations $\mathcal{D}_0$ and $\mathcal{E}_0$ the singular points might coincide.

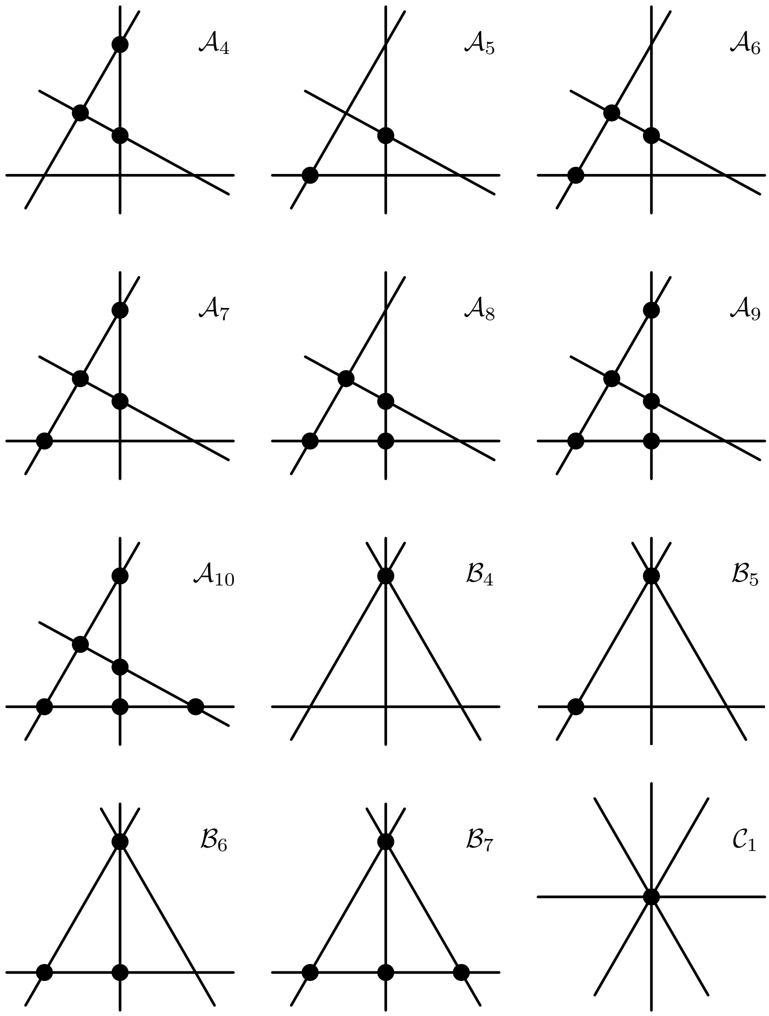


Figure 3.4.2: Possible configurations of lines on a completely reducible plane with four distinct lines and without a triangle. Singular points are marked with a bullet.

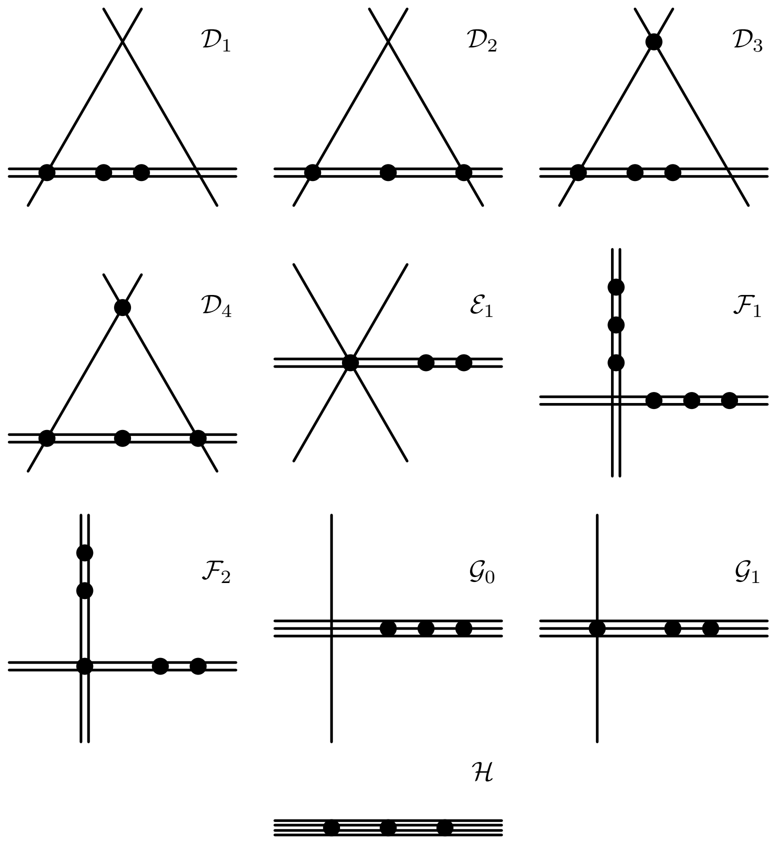


Figure 3.4.3: Possible configurations of lines on a completely reducible plane with a multiple component and without a triangle. Singular points are marked with a bullet. Bullets on the same component, but not belonging also to another component can coincide.

Chapter 4

Characteristic different from 2 and 3

Throughout this chapter we will suppose that the characteristic $p \geq 0$ of the ground field $\mathbb{K}$ is different from 2 and 3.

This chapter is dedicated to the proof of the following theorem, which is a direct generalization of Segre–Rams–Schütt's Theorem [33], [39].

Theorem 4.0.1. *If X is a K3 quartic surface, then $\Phi(X) \leq 64$.*

The assumption on the characteristic of the field rules out several pathologies that can only take place in characteristic 2 and 3. First and foremost, all lines will be separable elliptic lines.

The chapter is structured as follows.

Section 4.1 We provide upper bounds for the valencies of lines according to their kind, degree and singularity, building on a method introduced by Rams and Schütt [33].

Section 4.2 We study two special constructions, namely special lines and twin lines, both of which are to be found on Schur's quartic.

Section 4.3 We prove Theorem 4.0.1 for triangle free surfaces.

Section 4.4 We carry out the rest of the proof.

Section 4.5 We describe several examples of surfaces with a particularly high number of lines.

4.1 Lines of the second kind

Let ℓ be a line of positive degree on a K3 quartic surface X. A consequence of inequalities (3.8) and (3.9) is that $v(\ell)$ cannot be greater than 24. This bound is not sharp. Table 4.1.1, which summarizes the results of this section, offers an overview of the best estimates on $v(\ell)$ known to us.

The bounds for lines of degree 0 and lines of the first kind have already been treated in Lemma 3.3.5 and Proposition 3.3.12, respectively. Finding a bound for lines of the second kind is more involved: this will be the main subject of this section.

The bound 18 for lines of the first kind of degree 3 is reached for example by the lines of type $(p, q) = (4, 6)$ in Schur's quartic (see Section 4.2), while the

Table 4.1.1: Known bounds for the valency of a line according to its kind, degree and singularity. Sharp bounds are marked with an asterisk *.

kind	degree	singularity	valency
first kind	3	0	$\leq 18^*$
	2	1	≤ 13
	1	2 or 1	≤ 8
second kind	3	0	$\leq 20^*$
	2	1	≤ 10
	1	2	≤ 9
	1	1	≤ 11
–	0	3, 2 or 1	$\leq 2^*$

bound 20 for lines of the second kind of degree 3 is reached by some surfaces in family $\mathcal{Z}$ (see Lemma 4.2.2). It is also not difficult to construct surfaces exhibiting configuration $\mathcal{D}_0$ or $\mathcal{E}_0$ of Figure 3.4.1, thus reaching the bound 2 for lines of degree 0. Nonetheless, we do not know whether the other bounds are sharp.

4.1.1 Lines of degree 3

The Riemann–Hurwitz formula (see [15]) applied to the morphism $\pi : L \to \mathbb{P}^1$ yields the following lemma, where we use the notation introduced after Definition 3.1.9.

Lemma 4.1.1. *A line of degree* 3 *can have ramification* 2_1^4, $2_1^2 3_2$ *or* 3_2^2.

The last ramification type will play an important role in the sequel, so we give it a name.

Definition 4.1.2. A line ℓ of degree 3 is said to be *special* if it is of the second kind and has ramification 3_2^2.

Suppose now that ℓ is a line of the second kind of degree 3. The morphism $\pi : L \to \mathbb{P}^1$ corresponds to a (separable) field extension $\mathbb{K}(\mathbb{P}^1) \subset \mathbb{K}(L)$ of degree 3. This extension can be Galois or not, according to the ramification type of π. More precisely, the extension is Galois if and only if ℓ is special. In fact, the index of ramification at a point $P \in L$ is equal to the order of the inertia group of the corresponding place in $\mathbb{K}(L)$. Since the inertia group is a subgroup of the Galois group, its size must divide the size of the Galois group; hence, this extension can never be a Galois extension if there is a point of ramification index 2.

In any case, we can consider the Galois closure of $\mathbb{K}(L)$, which corresponds to a morphism $\psi : \Gamma \to L$, where Γ is a smooth algebraic curve. We set $\eta := \pi \circ \psi$ and $d = \deg \eta$. We perform two consecutive base changes, obtaining the following commuting diagram:

$$\begin{array}{ccccc} Y & \dashrightarrow & W & \dashrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ \Gamma & \xrightarrow{\psi} & L & \xrightarrow{\pi} & \mathbb{P}^1 \end{array}$$

where W is the minimal desingularization of the surface $Z \times_{\mathbb{P}^1} L$, Y is the minimal desingularization of the surface $W \times_L \Gamma$, and dashed arrows represent rational dominant maps. Of course, if the map $\pi : L \to \mathbb{P}^1$ is already Galois, the second base change is trivial, since $\Gamma = L$. Hence, if ℓ is special, then $d = 3$, otherwise $d = 6$.

The inclusion $L \hookrightarrow Z$ lifts to a section $L \to W$, which in turn lifts to a section $s_0 : \Gamma \to Y$. The Galois action provides us with two more sections $s_1, s_2 : \Gamma \to Y$.

Definition 4.1.3. We will call these three sections *Galois sections*, and we choose s_0 as the 0-section in the Mordell-Weil group of the elliptic surface $Y \to \Gamma$.

Since ℓ is a line of the second kind and since the three inflection points that ℓ cuts on the generic cubic are obviously aligned, we obtain the following lemma.

Lemma 4.1.4. *The sections s_1 and s_2 are torsion sections of order 3 inverse to each other.*

The action of the Galois sections induces a rational map $Y \dashrightarrow Y'$, where Y and Y' are two elliptic surfaces over Γ with the same Euler–Poincaré characteristic, by Lemma 2.3.2. To each singular fiber G of Y there corresponds a singular fiber G' of Y' and the type of G' is determined univocally by the type of G and by which components of G are met by the torsion sections.

Let $F = \pi^{-1}(t)$ be a singular fiber of $\pi : Z \to \mathbb{P}^1$ and let $\Pi \supset \ell$ be the corresponding plane in $\mathbb{P}^3$. Let E be the residual cubic in the plane Π. We will say that F is *unramified*, *ramified of index 2* or *ramified of index 3*, according to whether the configuration of $\eta^{-1}(t)$ consists of

(a) d distinct points (t is not a branch point);
(b) 3 points of ramification index 2;
(c) $d/3$ points of ramification index 3.

Observe that the ramified fibers are in one-to-one correspondence with the ramification points of $\pi : L \to \mathbb{P}^1$ and have the same ramification indices. We will now study in detail how the fiber F is modified under the consecutive base changes that we have just described.

Remark 4.1.5. The results of the following three lemmas are summarized in tables. The column "difference" represents the difference of the Euler–Poincaré characteristics of the fibers on Y and Y' obtained from the fiber F by base change.

Note that fibers of type I_{3n} or I_{3n}^* can exhibit two different behaviors.

Lemma 4.1.6. *If F is an unramified fiber, then F has type I_n, IV or IV^* and behaves according to the following table.*

fiber F	fibers on Y	fibers on Y'	difference	$v(F)$
I_n	$d \times \mathrm{I}_n$	$d \times \mathrm{I}_{3n}$	$+2\,dn$	0
I_{3n}	$d \times \mathrm{I}_{3n}$	$d \times \mathrm{I}_n$	$-2\,dn$	3
IV	$d \times \mathrm{IV}$	$d \times \mathrm{IV}$	0	3
IV^*	$d \times \mathrm{IV}^*$	$d \times \mathrm{IV}^*$	0	3

Proof. In this case $\eta^{-1}(t)$ consists of d distinct points and the fiber F is replaced by d fibers on Y. Choose one of them and call it G. Note that F and G are of the same type. Since G accommodates 3-torsion sections, G (hence also F) must be of type I_n ($n \geq 0$), IV or IV^* in Kodaira's notation (see Table 2.3.3).

Suppose that F and G are of type I_n and suppose that the sections s_0, s_1 and s_2 meet the same component of G. Recalling that the sections s_i are induced by the strict transform of ℓ in Z, the former case happens if and only if the line ℓ meets only one component of E, i.e., E is irreducible and gives no contribution to the number of lines meeting ℓ). G must correspond to a fiber G' of type I_{3n} on Y'.

If F and G are of type I_n, but the sections s_i intersect different irreducible components, then n must be a multiple of 3 and G corresponds to a fiber G' of type I_n on Y'. The residual cubic E splits into three lines.

If F and G are of type IV and IV^* then the sections s_i must meet different components of G, hence the residual cubic must fully split and the fiber G' on Y' corresponding to G has the same type of F and G. □

Lemma 4.1.7. *If F is a ramified fiber of index 2, then F has type I_n^*, II, II^* or IV^* and behaves according to the following table.*

fiber F	fibers on Y	fibers on Y'	difference	$v(F)$
I_n^*	$3 \times \mathrm{I}_{2n}$	$3 \times \mathrm{I}_{6n}$	$+12\,n$	2
I_{3n}^*	$3 \times \mathrm{I}_{6n}$	$3 \times \mathrm{I}_{2n}$	$-12\,n$	2
II	$3 \times \mathrm{IV}$	$3 \times \mathrm{IV}$	0	0
II^*	$3 \times \mathrm{IV}^*$	$3 \times \mathrm{IV}^*$	0	2
IV^*	$3 \times \mathrm{IV}$	$3 \times \mathrm{IV}$	0	2

Proof. Suppose $\eta^{-1}(t)$ consists of 3 points of ramification index 2. This is only possible in case ℓ is not special, i.e., when $d = 6$. The fiber F is then replaced by three fibers on Y, whose type can be read off from Table 2.3.3. A priori, the fiber F can be of type I_n, I_n^*, II, IV, II^* or IV^*, yielding three fibers on Y of type I_{2n}, I_{2n}, IV, IV^*, IV^* or IV respectively, all of which could accommodate 3-torsion sections.

We can exclude fibers of type I_n, though. Indeed, since the line ℓ meets the residual cubic in Π at inflection points, ℓ cannot be tangent to the residual cubic, otherwise it would have intersection of order 3 and Π would not correspond to a ramification point of π of index 2. Hence, ℓ cannot be tangent to the fiber F (since all blowups of the desingularization happen outside of ℓ) and, therefore, ℓ meets F in a node. However, on each new fiber on Y two of the Galois sections, say s_0 and s_1, meet the same component, and the third one s_2 meets a different component: this is impossible since we could choose s_0 to be the 0-section, but s_1 and s_2 could not be the inverse of each other.

We also deduce that the residual cubic E in the plane Π corresponding to F cannot split into three different lines. Indeed, these lines could not be concurrent since ℓ would pass through their intersection point and ramification of index 3 would occur. But if they were not concurrent, then they would form a 'triangle'; after blowing up the singular points, the fiber F would still contain a 'cycle', hence it should be of type I_n $(n \geq 3)$, which we have just ruled out.

The residual cubic E cannot split into a line and a conic, because if the two where secant, then they would form a 'cycle' (of length 2) and this cycle would lead to a fiber of type I_n, while if they where tangent then ℓ would pass through the point of tangency (since it cannot be tangent to the conic) and we would get a fiber of type III, which is also excluded; moreover, the residual cubic E cannot be an irreducible cubic with a node, since ℓ should pass through the node and we would have a fiber of type I_1 (recall that all points on ℓ are smooth because $\pi : Z \to \mathbb{P}^1$ has degree 3); finally, E cannot be a triple line, otherwise ramification of index 3 would occur.

Hence, we are left with very few possibilities: either E is an irreducible cubic with a cusp, ℓ passes through the cusp and we have a fiber of type II, or E splits into a double line and another line, hence the fiber F contains a component of multiplicity 2. This rules out a fiber F of type IV, too. □

Lemma 4.1.8. *If F is a ramified fiber of index 3, then F has type I_n, IV or IV* and behaves according to the following table, where $d' = d/3$.*

fiber F	fibers on Y	fibers on Y'	difference	$v(F)$
I_1	$d' \times \mathrm{I}_3$	$d' \times \mathrm{I}_9$	$+6\,d'$	0
I_n $(n \geq 2)$	$d' \times \mathrm{I}_{3n}$	$d' \times \mathrm{I}_{9n}$	$+6\,d'n$	1
IV	$d' \times \mathrm{I}_0$	$d' \times \mathrm{I}_0$	0	3
IV*	$d' \times \mathrm{I}_0$	$d' \times \mathrm{I}_0$	0	≤ 2

Proof. Suppose $\eta^{-1}(t)$ consists of $d/3$ points of ramification index 3. In this case, F is replaced by $d/3$ fibers on Y. As before, we be read off their fiber type from Table 2.3.3: the fiber F can be of type I_n, IV or IV*.

If the residual cubic in Π has a non-reduced component, then it must lead to a fiber of type IV*. If the residual cubic is composed of three distinct lines, then, in order to have ramification of type 3, they must be concurrent and the line ℓ must pass through their intersection point; thus, there cannot be singular points of the surface on the three lines (since this would result in a fiber outside Kodaira's classification) and the fiber must be of type IV.

If the fiber F is of type I_1, then the residual cubic must be irreducible; hence, it gives no contribution to the lines meeting ℓ. If the fiber F is of type I_n, $n \geq 2$, then the residual cubic splits into a line plus a conic (it cannot split into three lines, otherwise we could not have ramification of index 3). In each case, the three Galois sections must meet the same component on each of the two fibers of type I_{3n} on Y (this component comes from the node of the residual cubic through which ℓ passes); therefore, we get two fibers of type I_{9n} on Y'.

Three concurrent lines correspond to a fiber F of type IV. A double or a triple line must lead to a fiber F of type IV*. □

Proposition 4.1.9. *If ℓ is a line of the second kind of degree 3, then $v(\ell) \leq 20$, moreover, if $v(\ell) > 16$, then ℓ is special. If $v(\ell) = 19$, then the line has type $(p, q) = (6, 1)$ and the 1-fiber is a ramified fiber of type I_n, $n \geq 2$; if $v(\ell) = 20$, then the line has type $(p, q) = (6, 2)$ and both 1-fibers are ramified fibers of type I_{n_1}, I_{n_2}, $n_1, n_2 \geq 2$.*

Proof. We compute the valency of ℓ using formula (3.7) and the tables of Lemmas 4.1.6, 4.1.7 and 4.1.8. According to Lemma 2.3.2 the Euler–Poincaré characteristics of Y and Y' must balance out.

- If ℓ has ramification 2_1^4, there are four ramified fibers, so their contribution to the Euler number is always at least 8. Considering the possible combinations of fibers, one can see that each time we get $3n$ lines we must pay with a contribution of at least $4n$ to the Euler number, so the number of lines intersecting ℓ is not greater than 12.
- If ℓ has ramification $2_1^2 3_2$, the contribution to the Euler number coming from the ramified fibers of index 2 is at least 4, without any contribution to the number of lines. Again, looking at the possible combinations, one can see that we need a further contribution of at least $4n$ to the Euler number each time we get $3n$ lines, except when we have a ramified fiber of type I_n, $n \geq 2$, (there can be at most one) paired with n unramified fibers of type I_3, in which case we get $3n + 1$ lines for a loss of $4n$ in the Euler number. Hence, the maximal number of lines meeting ℓ is 16.
- Finally, if ℓ has ramification 3_2^2, i.e., if ℓ is a special line, a direct inspection of the possible combinations yields a bound of 20 lines meeting ℓ. Furthermore, the line ℓ can meet 19 or 20 lines only if there are one or two ramified fibers of type I_{n_1}, I_{n_2}, $n_1, n_2 \geq 2$. □

4.1.2 Lines of degree 2

The following lemma is also a straightforward consequence of the Riemann-Hurwitz formula.

Lemma 4.1.10. *A line of degree 2 has ramification 2_1^2.*

Suppose that ℓ is a line of the second kind of degree 2. By Proposition 3.1.7, the line ℓ contains exactly one singular point of the surface, which we call P, and the general residual cubic intersects ℓ at P with multiplicity 1.

Lemma 4.1.11. *The point P is of type $\mathbf{A}_n$, $n \geq 2$.*

Proof. Let us parametrize the surface X as in (3.1). The point P corresponds to a simple common root of the forms α and β defined in (3.2). Up to projective equivalence, we can suppose that P is $[0 : 0 : 0 : 1]$, which is the same as requiring x_2 to be the common root of α and β; this means that in equation (3.1) we have

$$a_{0103} = a_{1003} = 0. \tag{4.1}$$

In addition, after a suitable change of coordinates we can suppose that

$$a_{0112} = a_{1030} = a_{1021} = 0 \quad \text{and} \quad a_{1012} = a_{0130} = 1. \tag{4.2}$$

In fact, we can assume that $Q = [0 : 0 : 1 : 0]$ is a point of ramification relative to the plane $\Pi_1 : x_1 = 0$ and that the residual cubic in $\Pi_0 : x_0 = 0$ has at least double intersection with ℓ in P. Note that P is a point of ramification if and only if $a_{0121} = 0$. The coefficients a_{1012} and a_{0130} must be different from 0, so we can normalize them to 1.

The resultant $R(\ell)$ must vanish, since ℓ is of the second kind. By looking at the coefficients of x_3^{13} and $x_2 x_3^{12}$ one finds that a_{0202} has to be equal to 0; hence, the tangent cone at P, which is given by

$$f_2 : (a_{2002}x_0 + a_{1102}x_1 + x_2)x_0 = 0, \tag{4.3}$$

is the union of two distinct planes, which means that P is of type $\mathbf{A}_n$, $n \geq 2$. □

From the resolution of the point P we get n smooth rational exceptional divisors $\Delta_1, \dots, \Delta_n$ on Z, such that $\Delta_i.\Delta_{i+1} = 1$ for $i = 1, \dots, n-1$, and $\Delta_i.\Delta_j = 0$ otherwise (as long as $i \neq j$). The intersection line of the two planes in the tangent cone of P (4.3) is a line different from ℓ. This tells us that the strict transform L of ℓ meets one 'extremal' exceptional component, say Δ_1, while the strict transform of a general residual cubic meets the other 'extremal' exceptional component Δ_n (in fact, the two 'extremal' components parametrize the tangent directions in the two planes of the tangent cone).

Consider now the following commuting diagram

$$\begin{array}{ccc} W & \overset{\psi}{\dashrightarrow} & Z \\ \downarrow & & \downarrow \pi \\ L & \xrightarrow{\pi} & \mathbb{P}^1 \end{array} \tag{4.4}$$

where W is the minimal desingularization of $Z \times_{\mathbb{P}^1} L$.

Note that the field extension $\mathbb{K}(\mathbb{P}^1) \subset \mathbb{K}(L)$ corresponding to $\pi : L \to \mathbb{P}^1$ has degree 2; hence, it is always Galois. Therefore, we have three sections s_0, s_1, $s_2 : L \to W$, which we call *Galois sections* as in the degree 3 case of the previous section. We can choose one of them to be the zero section; since ℓ is of the second kind, the other two are 3-torsion sections. Observe that two sections map one-to-one onto ℓ through ψ, whereas the third section maps two-to-one onto Δ_n. The Galois sections induce a rational map $W \dashrightarrow W'$, where $e(W) = e(W')$, as explained in Lemma 2.3.2.

Let $F := \pi^{-1}(t)$ be a singular fiber of the elliptic fibration $\pi : Z \to \mathbb{P}^1$ induced by ℓ corresponding to a plane Π.

Lemma 4.1.12. *If F is an unramified fiber, then F has type* I_n, IV *or* IV^* *and behaves according to the following table.*

fiber F	fibers on W	fibers on W'	difference	$v(F)$
I_n	$2 \times \mathrm{I}_n$	$2 \times \mathrm{I}_{3n}$	$+4\,n$	0
I_{3n}	$2 \times \mathrm{I}_{3n}$	$2 \times \mathrm{I}_n$	$-4\,n$	≤ 2
IV	$2 \times \mathrm{IV}$	$2 \times \mathrm{IV}^*$	0	≤ 2
IV^*	$2 \times \mathrm{IV}^*$	$2 \times \mathrm{IV}^*$	0	≤ 2

Proof. If t is not a branch point of $\pi : L \to \mathbb{P}^1$, then F has type I_n ($n \geq 1$), IV or IV^*, since on W it is substituted by two fibers of the same type and these must accommodate 3-torsion.

Suppose F is a fiber of type I_n. If the residual cubic in Π is irreducible, then the three Galois sections meet the same component; hence, we get two fibers of type I_{3n} on W' and these fibers do not contribute to the valency of ℓ. On the other hand, if the residual cubic in Π is reducible, then n must be divisible by 3 and we get two fibers of type I_{3m} on W and two of type I_m on W', where $n = 3\,m$. ☐

Lemma 4.1.13. *If F is a ramified fiber, then F has type* II, I_n^*, IV^* *or* II^* *and behaves according to the following table.*

fiber F	fibers on W	fibers on W'	difference	$v(F)$
II	IV	IV	0	0
I_n^*	I_{2n}	I_{6n}	$+4\,n$	≤ 1
I_{3n}^*	I_{6n}	I_{2n}	$-4\,n$	≤ 1
IV^*	IV	IV	0	≤ 1
II^*	IV^*	IV^*	0	≤ 1

Proof. If t is a branch point of $\pi : L \to \mathbb{P}^1$, then a priori F can have type I_n $(n \geq 1)$, I_n^* $(n \geq 1)$, II, IV, II^* or IV^* (again, see Table 2.3.3). We can exclude type I_n and IV, though.

We call $\hat{P}$ the point of intersection of Δ_1 with L. There exists exactly one fiber F_0 containing $\hat{P}$; let us denote by Π_0 the corresponding plane (parametrizing X as in the proof of Lemma 4.1.11, Π_0 is given by $x_0 = 0$). Note that the fiber F_0 must contain $\Delta_1, \ldots, \Delta_{n-1}$ as irreducible components plus the strict transform of the components of the residual cubic E_0 in Π_0.

If F_0 is a ramified fiber, one can see by a local computation that the residual cubic E_0 must split into three lines passing through P: in fact, setting $a_{0121} = 0$ (which was the condition for ramification in P) the residual cubic in $x_0 = 0$ has no term containing x_3. The three lines can be all distinct or they might coincide. In any case, we have no cycles and more than three components; hence, we can exclude both type I_n and IV.

Suppose now that F is a ramified fiber different from F_0. The corresponding residual cubic E has thus intersection multiplicity 1 with ℓ at P and 2 at another point $Q \in \ell$. P is the only singular point of X on ℓ by Proposition 3.1.7, so Q must be a smooth point of X. Moreover, since Q is an inflection point of E because ℓ is of the second type, E and ℓ cannot meet tangentially in Q, otherwise the intersection multiplicity would be 3.

Hence, if the cubic E is irreducible, then Q must be a cusp, and F is of type II. In fact, if Q were a node, then two of the Galois sections on Z_1 would meet the same component of the resulting I_2-fiber on Z_1 and the third would meet a different one, which is impossible.

The cubic E cannot split into a line and a conic, because in this case Q would be a point of intersection of the line and the conic, giving rise either to a fiber of type III (which we excluded a priori) or to a fiber of type I_n (I_2 if there are no surface singularities in the plane relative to E, otherwise I_n with $n > 2$) with an impossible configurations of torsion sections as before.

If the cubic E splits into three distinct lines, they could not be concurrent because F is a ramified fiber different from F_0, so again this would lead to an impossible configuration of torsion sections. Finally, if E splits into three lines not all distinct, then F contains a non-reduced component, so fiber types I_n and IV are impossible. □

Proposition 4.1.14. *If ℓ is a line of the second kind of degree 2, then $v(\ell) \leq 10$.*

Proof. According to Lemma 4.1.13, the two ramified fibers have both Euler number ≥ 2, so the remaining local contribution is ≤ 20. Looking at the possible combinations with Lemma 4.1.12, one can see that we we get a maximum of 10 lines intersecting ℓ. □

4.1.3 Lines of degree 1

According to Proposition 3.1.7, a line of degree 1 can have either singularity 2, in which case the general cubic has intersection 1 with ℓ at both singular points, or singularity 1, in which case the general cubic has intersection 2 with ℓ at the only singular point. We study the two cases separately.

Lemma 4.1.15. *If ℓ is a line of the second kind of degree 1 and singularity 2, then the two singular points on ℓ are of type $\mathbf{A}_{n_1}$ and $\mathbf{A}_{n_2}$, with n_1, $n_2 \geq 3$.*

Proof. By hypothesis, the forms α and β defined in (3.2) have two distinct simple roots in common. Up to projective equivalence, we can suppose that the surface is given by equation (3.1) with

$$a_{0130} = a_{0112} = a_{1030} = a_{1021} = a_{1003} = a_{0103} = 0 \quad \text{and} \quad a_{0121} = a_{1012} = 1,$$

so that the two singular points on ℓ are $P = [0 : 0 : 0 : 1]$ and $Q = [0 : 0 : 1 : 0]$. We have chosen coordinates so that the residual cubic in $\Pi_0 : x_0 = 0$ has a double intersection with ℓ at P and the residual cubic in $\Pi_1 : x_1 = 0$ has a double intersection with ℓ at Q.

One can spell out the conditions for ℓ to be a line of the second kind explicitly; in particular, one finds that $a_{0202} = 0$, so the tangent cone at P splits into two planes (one of them isΠ_0) whose intersection is different from ℓ:

$$f_2 = (a_{2002}x_0 + a_{1102}x_1 + x_2)\, x_0. \tag{4.5}$$

This already implies that P is a point of type $\mathbf{A}_n$, $n \geq 2$; moreover, the further condition

$$a_{0301} = a_{0211}a_{1102} - a_{1102}^2$$

allows us to use Bruce and Wall's 'recognition principle' [8, Corollary, p. 246], ruling out the case P of type $\mathbf{A}_2$. The same argument applies symmetrically to Q. □

Proposition 4.1.16. *If ℓ is a line of the second kind of degree 1 and singularity 2, then $v(\ell) \leq 9$.*

Proof. By Lemma 4.1.15, from the resolution of P we get a chain of n exceptional divisors $\Delta_1, \ldots, \Delta_n$, with $\Delta_i.\Delta_{i+1} = 1$ for $i = 1, \ldots n-1$.

Since ℓ is not the intersection of the planes Π_0 and Π_2 making up the tangent cone (4.5), the general residual cubic of the pencil meets one extremal component of the chain of exceptional divisors, say Δ_n, hence if the residual cubic in Π_0 has n_0 components, then the corresponding singular fiber has at least n_0+2 components (because it must contain the strict transforms of the n_0 components of the residual cubic plus $\Delta_1, \dots, \Delta_{n-1}$); in particular, it has Euler number $e_0 \geq n_0 + 2 \geq 3$. The same applies symmetrically to Q: the singular fiber corresponding to the plane Π_1 has Euler number $e_1 \geq 3$.

Let us denote by p' and q' respectively the number of 3- and 1-fibers different from Π_0 and Π_1. Since neither Π_0 nor Π_1 contribute to the valency of ℓ, formula (3.8) becomes $v(\ell) \leq p' + q'$. On the other hand, we must have

$$3\,p' + 2\,q' \leq 24 - e_0 - e_1 \leq 18.$$

Therefore, we infer that $v(\ell) \leq 9$. □

Proposition 4.1.17. *If ℓ is a line of the second kind of degree 1 and singularity 1, then $v(\ell) \leq 11$.*

Proof. In this case α and β have one single double root in common. Suppose the common root is x_2, corresponding to the singular point $P = [0 : 0 : 0 : 1]$. Up to projective equivalence, we can choose coordinates so that the residual cubic in $\Pi_0 : x_0 = 0$ has triple intersection with ℓ at P; hence we can suppose that the surface X is given by equation (3.1) with

$$a_{0103} = a_{1003} = a_{0112} = a_{1012} = a_{0121} = 0.$$

Note that both a_{0130} and a_{1021} must be different from zero, or else the line ℓ would have degree 0.

A necessary condition for ℓ to be of the second kind is $a_{0202} = 0$. The tangent cone at P splits then into two planes: both contain ℓ and one of them is Π_0; in particular, the point P is not of type $\mathbf{A}_1$ and, since the residual cubic in Π_0 has a singular point in P, the corresponding fiber – which does not contribute to the valency of ℓ – has Euler number at least 2.

Denote by p' and q' respectively the number of 3- and 1-fibers different from Π_0. The valency of ℓ is not greater than $p' + q'$ and we have the following bound on the Euler number:

$$3\,p' + 2\,q' \leq 24 - 2 = 22$$

Therefore, $v(\ell) \leq 11$. □

4.2 Schur's quartic

Schur's quartic, defined by the equation

$$X_{64} : x_0^4 - x_0 x_3^3 = x_1^4 - x_1 x_2^3, \tag{4.6}$$

is smooth and contains 64 lines [36]. Although all 64 lines have valency 18, by explicit computation one can check that 16 of them are of the second kind of type $(6, 0)$, and 48 of them are of the first kind of type $(4, 6)$. Both subsets present quite peculiar symmetries and features.

Studying them we were led to examine two particular configurations: special lines and twin lines. The first configuration was already known to Rams and Schütt [33], whereas the second one is – to our knowledge – new.

Both constructions are related to the notion of torsion sections of the Mordell–Weil group and are crucial to the new proof of Segre–Rams–Schütt theorem and to its extension to the K3 quartic case.

We point out a mistake in Rams and Schütt's article [33]: Proposition 7.1, which claims that in a quartic containing 64 lines all lines are of type $(6, 0)$, is false; the flaw lies in the proof of Lemma 7.3 [*ibidem*].

4.2.1 Special lines

The following corollary can be deduced from inspection of Table 4.1.1 and from Proposition 4.1.9.

Corollary 4.2.1. *If ℓ is a line with $v(\ell) > 18$, then ℓ is special.*

We can parametrize surfaces with special lines in the same way as Rams and Schütt did [33, Lemma 4.5].

Lemma 4.2.2. *If X contains a special line ℓ, then X is projectively equivalent to a quartic in the family*

$$\mathcal{Z} : x_0x_3^3 + x_1x_2^3 + x_2x_3q_2(x_0, x_1) + q_4(x_0, x_1) = 0, \tag{4.7}$$

where $q_i \in k[x_0, x_1]$ are homogeneous polynomials of degree i ($i = 2, 4$), and ℓ is given by $x_0 = x_1 = 0$.

Proof. Knowing that there are no singular points on the line ℓ, the proof can be copied word by word from [33, Lemma 4.5]. In the proof one uses the fact that the characteristic of the ground field is different from 3. □

Remark 4.2.3. The parametrization given by equation (4.7) reveals that there exists a (symplectic) automorphism $\sigma : X \to X$ of order 3 which is given by

$$\sigma : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : \zeta x_2 : \zeta^2 x_3],$$

with ζ a primitive third root of unity. In what follows we will refer to this automorphisms as 'the' automorphism of order 3 induced by ℓ. Note that σ permutes the components of the 3-fibers of ℓ.

The following proposition is a generalization of [33, Lemma 6.2] to the K3 quartic case.

Proposition 4.2.4. *If X contains two special lines ℓ and ℓ' intersecting each other, then X is projectively equivalent to Schur's quartic.*

Proof. Let P be the point of intersection of ℓ and ℓ'; Q one of the two ramification points on ℓ, corresponding to the plane $\Pi \supset \ell$; R one of the ramification points of ℓ', corresponding to the plane $\Sigma \supset \ell'$; S one of the points of intersection of the line $\Pi \cap \Sigma$ with X different from P.

Up to projective equivalence, we can suppose that P, Q, R and S are respectively the points $[0 : 0 : 1 : 0]$, $[0 : 0 : 0 : 1]$, $[0 : 1 : 0 : 0]$ and $[1 : 0 : 0 : 0]$. Thus, the line ℓ is given by $x_0 = x_1 = 0$ and the line ℓ' by $x_0 = x_3 = 0$.

This amounts to setting the following coefficients equal to zero in equation (3.1):

$$a_{0400}, a_{0310}, a_{0220}, a_{0130}, a_{1003}, a_{1012}, a_{1021}, a_{1300}, a_{1210}, a_{1120}, a_{4000}.$$

Furthermore, since ℓ and ℓ' do not contain singular points, the following coefficients must be different from zero and we can set them to 1:

$$a_{0103},\ a_{0301},\ a_{1030}.$$

Recall that a necessary condition for a cubic polynomial

$$p(t) = at^3 + bt^2 + ct + d$$

to have a triple root is

$$b^2 - 3\,ac = 0.$$

Therefore, in order for ℓ and ℓ' to have exactly two points of ramification, one sees that the following equations must be satisfied:

$$3\,a_{0121} = a_{0211}^2 = a_{0112}^2.$$

Spelling out the conditions for ℓ and ℓ' to be of the second kind, one sees that these coefficients must be actually zero. Indeed, one obtains a surface which is immediately seen to be projectively equivalent to Schur's quartic. □

Corollary 4.2.5. *A K3 quartic surface X cannot contain two intersecting lines of valency greater than 18.*

Proof. By Corollary 4.2.1, both lines are special. Proposition 4.2.4 implies then that X is projectively equivalent to Schur's quartic, but all lines on Schur's quartic have valency 18. □

4.2.2 Twin lines

According to Corollary 3.3.13, a line of the first kind with valency 18 must be either of type $(p, q) = (6, 0)$, $(5, 3)$ or $(4, 6)$. The lines of the first kind contained in Schur's quartic are all of type $(4, 6)$. Such lines fall into a broader construction which we will describe presently.

Definition 4.2.6. Let ℓ be a line on a K3 quartic surface. A line ℓ' is called an *inflective section* of ℓ if ℓ' meets the general residual cubic relative to ℓ in an inflection point.

Proposition 4.2.7. *Let X be a K3 quartic surface containing two disjoint lines ℓ and ℓ' of degree* 3. *Then, the following conditions are equivalent:*

- *(a) There are at least 9 lines $b_1, \ldots, b_9$ meeting ℓ and ℓ'.*
- *(b) There are exactly 10 lines $b_1, \ldots, b_{10}$ meeting ℓ and ℓ'.*
- *(c) The line ℓ' is an inflective section of ℓ and, vice versa, the line ℓ is an inflective section of ℓ'.*
- *(d) The tangents to the general residual cubic E relative to ℓ at the points of intersection of E with ℓ meet in the point of intersection of E with ℓ'.*
- *(e) The tangents to the general residual cubic E relative to ℓ' at the points of intersection of E with ℓ' meet in the point of intersection of E with ℓ.*
- *(f) The quartic X is projectively equivalent to a quartic in the following family $\mathcal{A}$, where the lines ℓ and ℓ' are given, respectively, by $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$, and $p_0, \ldots, p_3$ are forms of degree 3:*

$$\mathcal{A} := x_0p_0(x_2, x_3) + x_1p_1(x_2, x_3) + x_2p_2(x_0, x_1) + x_3p_3(x_0, x_1) \qquad (4.8)$$

If these conditions are satisfied, then the lines b_i are pairwise disjoint. Moreover, the base change along ℓ induces three 2-torsion sections on Z, if one chooses the 0-section to be the one induced by ℓ', and, vice versa, the base change along ℓ' also induces three 2-torsion sections on Z, if one chooses the 0-section to be the one induced by ℓ.

Proof. Up to coordinate change, we can always suppose that ℓ and ℓ' are respectively given by $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$.

(a) ⇒ (c). The condition of being an inflective section can be computed explicitly and is given by a polynomial of degree 8. The fact that there are at least 9 roots of this polynomial means that it must vanish identically.

(b) ⇒ (a) is obvious.

(c) ⇒ (f). One computes explicitly the conditions for the lines ℓ and ℓ' to be inflective sections of each other, and sees that the following coefficients must be equal to zero:

$$a_{2020},\ a_{2011},\ a_{2002},\ a_{1120},\ a_{1111},\ a_{1102},\ a_{0220},\ a_{0211},\ a_{0202}, \tag{4.9}$$

thus obtaining family $\mathcal{A}$.

(d) ⇒ (f). Let $E_t = E_t(x_1, x_2, x_3)$, $t \in \mathbb{P}^1$, be the residual cubic relative to ℓ, obtained by substituting $x_0 = tx_1$ in the equation of X. Consider the polynomial $\partial E_t/\partial x_1$ restricted on the line $l : x_0 = x_1 = 0$: this is a polynomial of degree 2 in (x_2, x_3) with polynomials of degree 2 in t as coefficients. Since generically it must have three distinct roots, namely the points of intersection of E_t with ℓ, it must be the zero polynomial; hence, the coefficients must be the zero polynomial in t. The result is that the same coefficients listed in (4.9) must be equal to zero.

(e) ⇒ (f) is proven analogously.

(f) ⇒ (c), (d), (e) is immediate.

(f) ⇒ (b) can be proven explicitly by considering the discriminant of the fibration induced by one of the two lines. □

Definition 4.2.8. If ℓ and ℓ' satisfy one of the equivalent conditions of Proposition 4.2.7, we say that ℓ and ℓ' are *twin lines*.

Remark 4.2.9. The family $\mathcal{A}$ has dimension 8; in fact, knowing that there are 10 disjoint lines meeting both ℓ and ℓ', we can assume – up to projective equivalence – that two of them are given respectively by $x_1 = x_2 = 0$ and $x_0 = x_3 = 0$; we are left with 12 parameters, 4 of which can be normalized to 1. Indeed, the lattice generated by the twelve lines and the hyperplane section has rank 12, as expected.

Remark 4.2.10. The explicit parametrization (4.8) of family $\mathcal{A}$ shows the existence of a non-symplectic automorphism $\tau : X \to X$ of degree 2, given by

$$\tau : [x_0 : x_1 : x_2 : x_3] \mapsto [-x_0 : -x_1 : x_2 : x_3].$$

This automorphism fixes ℓ and ℓ' pointwise; it also respects their fibers as sets.

Corollary 4.2.11. *Let ℓ be a line admitting a twin ℓ'. If ℓ has a 3-fiber, then this fiber is ramified.*

Proof. Suppose X, ℓ and ℓ' are given as in family $\mathcal{A}$. Exactly one of the lines in the 3-fiber of ℓ must meet the line ℓ': let us call it ℓ_0 and the other two ℓ_1 and ℓ_2. The points of intersection of ℓ_0 with ℓ and ℓ' are fixed by the automorphism τ (see Remark 4.2.10); hence, ℓ_0 is mapped to itself.

Necessarily, the point P of intersection of ℓ_1 and ℓ_2 is also fixed by τ. Since P does not lie on ℓ', one of its last two coordinates must be different from zero; this implies that its first two coordinates must be zero; therefore, it must lie on ℓ and ramification must occur (of index 3 or 2, according to whether the lines ℓ_i meet at the same point or not). □

Lemma 4.2.12 (Degtyarev–Itenberg–Sertöz). *If ℓ is a line of degree 3 inducing a fibration of type $(p, q) = (4, 6)$, then X is smooth and the line ℓ has a twin ℓ'.*

Sketch of proof. There cannot be singular points outside ℓ, otherwise the Euler number of X would exceed 24; since ℓ has no singular points, the surface is smooth. The surface X is therefore a K3 surface. Let us call $m_{i,j}$, $i = 1, \dots, 4$, $j = 1, 2, 3$, the lines in the 3-fibers and n_k, $k = 1, \dots, 6$, the lines in the 1-fibers.

Suppose first that the base field has characteristic 0. The lattice P generated by the lines and the hyperplane section must admit an embedding into the K3 lattice $\Lambda = U^3 \oplus E_8(-1)^2$. By results of Nikulin [27], this embedding cannot be primitive, due to a condition on the 3-primary part of the discriminant group of P. A careful analysis of the admissible isotropic vectors reveals that – up to symmetry – the following class must also be contained in the Picard lattice of X:

$$\omega := \frac{1}{3}\left(\ell + \sum_{i=1}^{4}(m_{i,1} + m_{i,2}) - \sum_{k=1}^{6} n_k\right).$$

One can check that this is exactly the class of the sought line ℓ'. We refer the reader to [11, Proposition 5.28] for more details.

If the base field has positive characteristic $p > 3$, one has to distinguish two cases.

- If the surface is not Shioda-supersingular, then one can lift it – together with the whole Picard group – to characteristic 0 (see, for instance, Lieblich–Maulik [21] or Esnault–Srinivas [12]), so that one can apply the same arguments.
- If the surface is Shioda-supersingular, then the lattice P must embed in a p-elementary lattice. Since $p > 3$, one obtains the same condition on the 3-primary part of the discriminant group of ℓ which prevents it from embedding primitively. Again, one concludes that ω must be contained in the Picard lattice. □

4.3 Triangle free surfaces

Most of the ideas contained in this section are due to A. Degtyarev.

Proposition 4.3.1. *Let Γ be the line graph of a K3 quartic surface X. If Γ contains a parabolic subgraph D, then*

$$|\Gamma| \leq v(D) + 24.$$

Proof. A parabolic subgraph induces an elliptic fibration [29, §3, Theorem 1]. The vertices in $D \cup (\Gamma \smallsetminus \operatorname{span} D)$ are fiber components of this fibration; hence, on account of the Euler number, they cannot be more than 24 in number. □

We now set

$$\delta := \begin{cases} 20 & \text{if } \operatorname{char} \mathbb{K} = 0 \\ 22 & \text{if } \operatorname{char} \mathbb{K} > 0. \end{cases}$$

The number δ is a well-known bound for the rank of the Néron–Severi group of a K3 surface (see Section 2.1). Since the signature of the Néron–Severi lattice is $(1, \rho - 1)$, the Milnor number of any negative semidefinite subgraph of Γ cannot be greater than $\delta - 1$. In particular, since Γ has neither loops nor multiple edges, it can only contain the following parabolic subgraphs: $\tilde{\mathbf{A}}_2, \ldots, \tilde{\mathbf{A}}_{\delta-1}$, $\tilde{\mathbf{D}}_4, \ldots, \tilde{\mathbf{D}}_{\delta-1}$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, $\tilde{\mathbf{E}}_8$.

Lemma 4.3.2. *If Γ does not contain any parabolic subgraph, then $|\Gamma| \leq \delta - 1$.*

Proof. The associated form of Γ must be negative definite; hence, Γ is the disjoint union of elliptic graphs and its Milnor number is equal to $|\Gamma|$. □

Lemma 4.3.3. *If $v(\ell) \leq 3$ for every line $l \subset X$, then $|\Gamma| \leq \delta + 26$.*

Proof. By the previous lemma, we can assume that there is a parabolic subgraph $D \subset \Gamma$. Under the hypothesis $v(\ell) \leq 3$ for every vertex $l \in \Gamma$, we deduce that $v(\tilde{\mathbf{A}}_n) \leq n + 1$, $v(\tilde{\mathbf{D}}_n) \leq n + 3$, $v(\tilde{\mathbf{E}}_n) \leq n + 3$; hence, by virtue of Proposition 4.3.1, we obtain

$$|\Gamma| \leq v(D) + 24 \leq (n + 3) + 24 \leq (\delta - 1 + 3) + 24 = \delta + 26. \qquad \square$$

Proposition 4.3.4. *A square free K3 quartic surface contains at most 54 lines (51 over a field of characteristic 0).*

Proof. Let ℓ be a vertex of maximal valency. On account of Lemma 4.3.3, we can assume that the valency w of ℓ is at least 4. Let $m_1, \ldots, m_4$ be four vertices adjacent to ℓ and suppose that m_i has valency v_i $(i = 1, \ldots, 4)$.

Since the surface is square free, all vertices adjacent to ℓ are disjoint from the vertices adjacent to m_i; moreover, a vertex adjacent to m_i can be joined to at most one vertex adjacent to m_j, for $i \neq j$. Hence, we can assume that there are a lines meeting ℓ different from m_i, m_j; b lines meeting m_i not intersecting any line meeting m_j; c lines meeting m_j not intersecting any line meeting m_i; d pairs of lines forming a pentagon with m_i, m_j and ℓ. Note that

$$s := w + v_i + v_j = a + b + c + 2\,d + 4.$$

A simple computer-aided computation – which amounts to constructing all possible intersection matrices of $\operatorname{span}\{l, m_i, m_j\}$ for values of a, b, c, d such that $s = \delta + 2$ and computing their ranks – shows that any configuration with $s \geq \delta + 2$ gives rise to a lattice of rank greater than δ. Thus, we can assume that

$$w + v_i + v_j \leq \delta + 1 \quad \text{for all } i \neq j. \tag{4.10}$$

Taking the sum of (4.10) with $(i, j) = (1, 2)$, $(3, 4)$, one finds that

$$w + \sum_{i=1}^{4} v_i \leq 2\,(\delta + 1) - w. \tag{4.11}$$

Hence, if $w \leq \lfloor(\delta+1)/3\rfloor$, by the maximality condition one has

$$w + \sum_{i=1}^{4} v_i \leq 5\,w \leq \frac{5}{3}(\delta+1);$$

on the other hand, if $w \geq \lceil(\delta+1)/3\rceil$, then one obtains the same relation from inequality (4.11). Applying Proposition 4.3.1 to the $\tilde{\mathbf{D}}_4$-subgraph formed by ℓ, $m_1, \ldots, m_4$, we find that

$$|\Gamma| \leq (w-4) + \sum_{i=1}^{4}(v_i - 1) + 24 \leq 16 + \frac{5}{3}(\delta+1),$$

which yields the claim. □

Proposition 4.3.5. *A triangle free K3 quartic surface contains at most 64 lines.*

Proof. By virtue of Proposition 4.3.4, we can assume that Γ contains a square D, formed by the lines ℓ_i, $i = 1, \ldots, 4$. According to Lemma 3.4.5, they must have valency at most 12. Applying Proposition 4.3.1 to the $\tilde{\mathbf{A}}_3$-subgraph D, we infer that

$$|\Gamma| \leq 4 \cdot (12-2) + 24 = 64.$$

□

Remark 4.3.6. The bounds presented in this section are most probably not sharp. Degtyarev [10] has found a triangle free smooth surface over $\mathbb{C}$ with 33 lines and a triangle free smooth surface over an algebraically closed field of characteristic 7 with 47 lines.

4.4 Proof of Theorem 4.0.1

Having dealt with the triangle free case, we can now turn to the proof of the main theorem.

Proposition 4.4.1. *Let X be a K3 quartic surface with a singular point P. Suppose that X admits a completely reducible plane Π containing P. Then, the surface X contains at most 63 lines.*

Proof. Thanks to Lemmas 3.4.6 and 3.4.7, the proof can be carried out on a case-by-case analysis over the possible configurations listed in Figures 3.4.1, 3.4.2 and 3.4.3, except configurations $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{C}_0$, since they do not contain a singular point.

Let us call ℓ_i the lines on Π. In each case, we use formula (3.6) to estimate $\Phi(X)$. We employ Lemma 3.2.2 to control the second contribution, and the following bounds on $v(\ell_i)$ to control the third, depending on the singularity s of ℓ_i (see Table 4.1.1):

- if $s = 0$, then $v(\ell_i) \leq 18$ except for configurations $\mathcal{A}_4$, $\mathcal{B}_4$ and $\mathcal{G}_0$, where $v(\ell_i) \leq 20$ (in fact, in all other cases we can exclude that ℓ_i is special, because of the automorphism σ of order 3 – see Remark 4.2.3);
- if $s = 1$, then $v(\ell_i) \leq 13$;
- if $s = 2$, then $v(\ell_i) \leq 9$;
- if $s = 3$ or ℓ_i appears as a multiple component, then $v(\ell_i) \leq 2$.

The case resulting in the worst bound is $\mathcal{B}_4$. In fact, formula (3.6) yields

$$\Phi(X) \leq 4 + 3 \cdot (8-2) + \big((20-3) + 3 \cdot (9-1)\big) = 63. \qquad \square$$

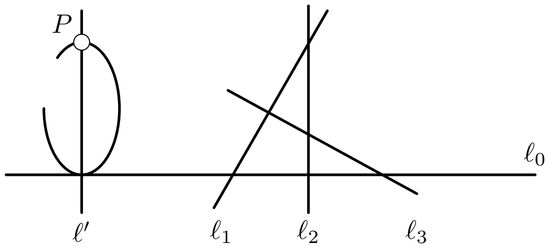
Remark 4.4.2. The bound of Proposition 4.4.1 can be improved, but most of the time the arguments get significantly more involved. With this simple-minded approach one gets a bound greater than 60 for the following configurations: $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$; $\mathcal{A}_4$, $\mathcal{B}_4$.

Proof of Theorem 4.0.1. Assume that X contains more than 64 lines. By virtue of Proposition 4.3.5, we can suppose that X is not triangle free; hence, there are three lines ℓ_1, ℓ_2 and ℓ_3 on X that meet at smooth points of X. These three lines are all contained in a plane Π; let ℓ_0 be the fourth line on this plane. On account of Proposition 4.4.1, we can assume that all points on Π are smooth.

If all four lines on Π have valency less than or equal to 18, then by formula (3.6) the total number of lines lying on X can be at most

$$4 + 4 \cdot (18-3) = 64.$$

Hence, we can assume that one of the lines, say ℓ_0, has valency 19 or 20. By virtue of Table 4.1.1 and Proposition 4.1.9, the line ℓ_0 has a ramified 1-fiber. Let us call ℓ' and C respectively the line and the conic in the 1-fiber, and P the point of intersection of ℓ' and C not lying on ℓ_0. Note that P may or may not be a singular point of X.



Claim 4.4.3. $v(\ell') \leq 10$.

Proof of the claim. If P is not singular, then the line ℓ' is of the first kind, because the point P is certainly not an inflection point of the corresponding residual cubic, whence $v(\ell') \leq 18$. Since the automorphism σ of order 3 induced by ℓ fixes only two points on ℓ' (see Remark 4.2.3), the valency of ℓ' has the form

$$v(\ell') = 1 + 3\,a,$$

for some integer $a \geq 0$; thus, $v(\ell') \leq 16$. On the other hand, ℓ' has no 3-fibers: in fact, since $v(\ell') \leq 16$, if ℓ' had a 3-fiber (with no singular points – see Proposition 4.4.1), then at least one line ℓ'' in the 3-fiber should have valency 19 or 20; the automorphism induced by ℓ'' would force the other two residual lines to have the same valency as ℓ', that is to say, not greater than 16: all in all, the lines on X would be less than 64, which is absurd. Hence, by Lemma 3.3.7, it follows that $v(\ell') \leq 12$, i.e., $a \leq 3$.

On the other hand, if P is a singular point, it follows from Proposition 4.4.1 that ℓ' cannot have 3-fibers, since we are assuming that X contains more than

64 lines. Given that in this case, too, the valency of ℓ' has the form $1 + 3\,a$, we can conclude as before. □

Claim 4.4.4. The lines ℓ_i ($i = 1, 2, 3$) are of type $(4, 6)$.

Proof of the claim. Because of the presence of the automorphism σ induced by ℓ_0, the lines in question have the same valency v and are of the same (p, q)-type. On the one hand, v cannot be greater than 18, by Corollary 4.2.5; on the other hand, if $v \leq 17$, then the total number of lines on X would be at most

$$4 + 3 \cdot (v - 3) + (v(\ell_0) - 3) \leq 4 + 3 \cdot (17 - 3) + (20 - 3) = 63.$$

Therefore, v is exactly 18, whence the ℓ_i's must have type $(p, q) = (6, 0)$, $(5, 3)$ or $(4, 6)$; in fact, $p \leq 3$ is not possible, by a simple Euler number argument.

Observe now that if P is singular, then the plane containing ℓ_i and P cannot be a 3-fiber for ℓ_i, by Proposition 4.4.1. Regardless of P being smooth or singular, it follows that all 3-fibers of ℓ_1, ℓ_2 and ℓ_3 contain a line meeting ℓ' in a point different from P. Since by Claim 4.4.3 the valency of ℓ' is at most 10, the ℓ_i's can only have type $(4, 6)$. □

By virtue of Corollary 4.2.11, the plane Π is a ramified fiber of ramification index 2 for the lines ℓ_i; hence, the three lines must meet in a point. In particular, the plane Π is a fiber of type IV for the line ℓ_0. Recall that, by Proposition 4.1.9, the line ℓ_0 has 6 3-fibers; considering that we could repeat the same argument for any 3-fiber of ℓ_0, we deduce that ℓ_0 has 6 fibers of type IV. But since the line ℓ_0 has also at least one 1-fiber, we deduce that the Euler number of the minimal desingularization of X must be at least

$$6 \cdot 4 + 2 = 26,$$

which is impossible. □

4.5 Examples

In this section we present some examples of K3 quartic surfaces with many lines. Most of them attain some kind of record, as we shall explain. The discriminants of the elliptic fibrations induced by lines are computed using the formulas provided in [1].

4.5.1 Non-smooth K3 quartic surfaces

There is another notion strictly related to the notion of valency of a line, and perhaps more natural.

Definition 4.5.1. The *extended valency* of a line ℓ, denoted by $\tilde{v}(\ell)$, is the number of lines on X that intersect ℓ.

In Section 3.3 we have devoted ourselves to finding bounds for the valency of a line ℓ contained in a K3 quartic surface X; Table 4.1.1 shows that $v(\ell)$ can never be greater than 20 and only in very special configurations it can be greater than 18. Another natural question is the following: what is the maximal *extended* valency that ℓ can have? A rough answer to this question is that, since there cannot be more than 8 lines through a singular point by Lemma 3.2.2,

$$\tilde{v}(\ell) \leq v(\ell) + 7\,s,$$

where s denotes the singularity of the line ℓ.

A much more careful analysis reveals that also

$$\tilde{v}(\ell) \leq 20,$$

for any line $\ell \subset X$. Apart from the obvious case of ℓ as in Proposition 4.1.9, where $\tilde{v}(\ell) = v(\ell)$ can be greater than 18, there are two other configurations in which $\tilde{v}(\ell) > 18$. We do not present a proof of these assertions, as we were only interested in giving a (as sleek as possible) proof of Theorem 4.0.1, but we provide examples of surfaces which exhibit these new behaviors. These examples were found by examining lines of the first kind and lines of degree 0 more closely, for example taking advantage of the restrictions of Lemma 3.2.2.

Example 4.5.2. Consider the surface defined by

$$\begin{aligned} 3\,x_0^4 - 9\,x_0^3x_1 + 6\,x_0^2x_1^2 - 12\,x_0x_1^3 + 8\,x_1^4 - 9\,x_1^3x_2 \\ = 27\,x_0^2x_2^2 + 27\,x_1^2x_2^2 + 27\,x_1x_2^3 + 27\,x_1^2x_3^2 + 27\,x_0x_2x_3^2. \end{aligned}$$

It has one singular point P of type $\mathbf{A}_1$. The line ℓ given by $x_0 = x_1 = 0$ is a line of the first kind of singularity 1 and degree 2; it has valency 12 and extended valency 19.

The fibration induced on the minimal desingularization Z by ℓ has six fibers F_i of type I_3 and one fiber G of type I_2. There are no other lines on the surface; hence, the surface contains exactly 20 lines. This holds true over any field of characteristic $p \neq 2, 3$ such that this fibration does not degenerate, and one can check it in the following way.

The fibers F_i come from residual cubics composed of three lines: we call $m_{i,0}$ the line passing through P, and $m_{i,1}, m_{i,2}$ the other two lines ($i = 0, \ldots, 5$). We call n the line in the residual cubic corresponding to G, which passes through P. One can check explicitly that the lines $m_{i,0}$ and n do not meet other lines. The intersection matrix of the strict transforms of all these lines on Z and of the exceptional divisor resulting from the blowup of P has signature $(1, 14)$. If a section s existed, then it would meet exactly one line between $m_{i,1}, m_{i,2}$, for $i = 0, \ldots, 5$, and no other line; up to symmetry, we can suppose that s would meet $m_{i,1}$ for $i = 0, \ldots, 5$. However, the resulting intersection matrix would have signature $(1, 16)$, which is impossible, since adding one divisor the signature must either stay the same or become $(1, 15)$.

Example 4.5.3 (due to González Alonso and Rams). The surface over $\mathbb{C}$ given by

$$x_0^4 + x_0x_2^3 + x_1^2x_2x_3 + x_0x_3^3 = 0$$

has one singular point of type $\mathbf{A}_3$ and 3 singular points of type $\mathbf{A}_1$. The line given by $x_0 = x_1 = 0$ has singularity 3, valency 2 and extended valency 20.

This surface contains exactly 39 lines. To our knowledge, this is the example of an explicit non-smooth K3 quartic surface with the highest number of lines over a field of characteristic zero that has been found so far. González Alonso and Rams came to this example by checking all Delsarte surfaces in Heijne's list [16].

By a careful inspection of the fibrations induced by the lines lying in the plane $x_0 = 0$, one can conclude that there exists no prime p such that the reduction of this surface modulo p contains more than 39 lines.

Example 4.5.4. A non-smooth complex K3 quartic surface with 40 lines exists and has been found with Degtyarev–Itenberg–Sertöz's lattice-theoretical methods [11]. It contains one singular point of type $\mathbf{A}_1$. An explicit equation of the surface is not known.

The bound expressed by Theorem 4.0.1 is sharp, since Schur's quartic (4.6) – which is smooth – contains exactly 64 lines. It is still an open question what the maximum number of lines on non-smooth K3 quartic surfaces is.

Apart from the Examples 4.5.3 and 4.5.4 with 39 and 40 lines over $\mathbb{C}$, we list here some notable surfaces with many lines defined over fields of positive characteristic, with 42, 45 and 48 lines. The following examples were found either by inspecting the family $\mathcal{Z}$ (4.7) or by imposing a lot of symmetries on the surface. It is worth mentioning here that González Alonso and Rams proved that a complex non-ruled quartic surface with worse singularities than isolated rational double points (i.e., a complex non-K3 quartic surfaces not containing an infinite number of lines) can contain at most 48 lines and their best example (due to Rohn) contains 31 lines and a triple point [14].

Example 4.5.5. The surface over $\mathbb{C}$ given by

$$x_0^2x_1^2 + x_1x_2^3 - x_0^2x_2x_3 - x_0x_1x_2x_3 - x_1^2x_2x_3 + x_0x_3^3 = 0$$

belongs to the family $\mathcal{Z}$, has 5 singular points of type $\mathbf{A}_1$ and contains exactly 33 lines. It has Picard number 20. Its reduction modulo 5 contains 42 lines and has Picard number 22.

Example 4.5.6. The surface over $\mathbb{C}$ given by

$$x_0^3x_1 - 2\,x_0^2x_1^2 + x_0x_1^3 + x_1x_2^3 + x_0^2x_2x_3 - x_0x_1x_2x_3 + x_1^2x_2x_3 + x_0x_3^3 = 0$$

belongs to the family $\mathcal{Z}$, has 1 singular point of type $\mathbf{A}_1$ and contains exactly 36 lines. It has Picard number 20. Its reduction modulo 11 contains 45 lines and has Picard number 22.

Example 4.5.7. The surface over $\mathbb{C}$ given by

$$x_0^2x_1x_2 + x_1^2x_2^2 + x_0x_1^2x_3 + x_0x_2^2x_3 + x_0^2x_3^2 + x_1x_2x_3^2 = 0$$

has 4 singular points of type $\mathbf{A}_1$ and contains exactly 36 lines. Its reduction modulo 5 contains exactly 48 lines. To our knowledge, this is the example of a non-smooth K3 quartic surface with the highest number of lines over a field of positive characteristic $p \neq 2, 3$ that has been found so far.

4.5.2 Smooth quartic surfaces

We conclude with some notable smooth quartic surfaces. The following examples have been found by taking advantage of the parametrizations (4.7) and (4.8). For most of them we were helped by the explicit configurations of lines found by Degtyarev, Itenberg and Sertöz. We follow their nomenclature [11, Table 1].

Example 4.5.8. The surface defined by

$$X_{54} : x_1^3x_2+x_1x_2^3+x_0^3x_3+x_0x_3^3 = \zeta\left(x_0x_1\left(x_2^2-x_3^2\right)+x_2x_3\left(x_0^2-x_1^2\right)\right), \quad (4.12)$$

where ζ satisfies $\zeta^2 = 3$, contains exactly 54 lines forming configuration $\mathbf{X}_{54}$ and has Picard number 20. The lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ are special lines of type $(6, 2)$, while the lines $x_0 = x_2 = 0$ and $x_1 = x_3 = 0$ are twin lines of type $(0, 10)$. Remarkably, this model is real.

According to Degtyarev–Itenberg–Sertöz, there are 8 line configurations with more than 52 lines on smooth quartic surfaces. Explicit examples of surfaces are known with the following configurations of lines:

configuration	reference
$\mathbf{X}_{64}$	Schur [36]
$\mathbf{X}'_{60}$	Rams–Schütt [33]
$\mathbf{X}''_{60}$	Schütt (unpublished)
$\mathbf{X}_{56}$	Shimada–Shioda [40]
$\mathbf{Y}_{56}$	Degtyarev–Itenberg–Sertöz [11]
$\mathbf{X}_{54}$	(4.12)

The missing configurations are $\mathbf{Q}_{56}$ and $\mathbf{Q}_{54}$, which contain neither special lines nor twin lines.

Example 4.5.9. The following surface contains exactly 56 lines forming configuration $\mathbf{X}_{56}$ and has Picard number 20. This model is slightly simpler than the one provided by Shimada and Shioda, involving 8 monomials instead of 12. Here ζ denotes a primitive 8th root of unity.

$$X_{56} : x_1x_2\left(\left(2\,\zeta^3+\zeta^2-2\right)\left(x_1^2+x_2^2\right)+3\,x_0^2+3\,x_3^2\right) = \zeta^2x_0x_3\left(3\,x_1^2+3\,x_2^2-\left(\zeta^2-2\,\zeta+2\right)\left(x_0^2+x_3^2\right)\right).$$

According to [40, Theorem 5.6], the surace X_{56} has good reduction modulo $p \geq 5$ and still contains 56 lines.

Example 4.5.10. The surface defined by the following polynomial contains exactly 52 lines forming configuration $\mathbf{X}'''_{52}$ and has Picard number 20:

$$\begin{aligned}50\,\xi x_1^3x_2 + 20\,\xi x_1x_2^3 - 2\,\xi x_0^3x_3 + 30\,\xi x_0x_2^2x_3 - 150\,\xi x_1x_2x_3^2 \\ + 20\,\xi x_0x_3^3 + 75\,x_0^2x_1x_2 + 625\,x_1^3x_2 + 125\,x_1x_2^3 - 25\,x_0^3x_3 \\ + 375\,x_0x_1^2x_3 + 225\,x_0x_2^2x_3 - 1125\,x_1x_2x_3^2 + 125\,x_0x_3^3 = 0,\end{aligned}$$

where ξ is a root of

$$t^2 + 25\,t + 125.$$

The lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ are twin lines of type $(4, 6)$, while the lines $x_0 = x_2 = 0$ and $x_1 = x_3 = 0$ are twin lines of type $(0, 10)$.

Example 4.5.11. The general member of the following rational family contains 52 lines forming configuration $\mathbf{Z}_{52}$ and has Picard number 19:

$$\begin{aligned}Z_{52} : a^2x_1x_2(ax_0 + ax_3 - 2\,x_1 + 2\,x_2)(ax_0 - ax_3 - 2\,x_1 - 2\,x_2) \\ = -4\,x_0x_3(ax_0 + ax_3 - 6\,x_1 + 6\,x_2)(ax_0 - ax_3 - 6\,x_1 - 6\,x_2). \quad (4.13)\end{aligned}$$

Generically, the lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ are twin lines of type $(2, 8)$, while the lines $x_0 = x_2 = 0$ and $x_1 = x_3 = 0$ are special lines of type $(6, 0)$. All surfaces of the family admit the symmetry

$$[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [-x_3 : x_2 : -x_1 : x_0].$$

We obtain models containing configurations $\mathbf{X}_{64}$ and $\mathbf{X}'_{60}$ when a is a root of the polynomials

$$t^4 + 144 \quad \text{and} \quad t^4 - 12\, t^2 + 144,$$

respectively. On the other hand, if a is a root of

$$t^8 + 224\, t^4 + 20736,$$

then the surface defined by (4.13) contains 6 points of type $\mathbf{A}_1$ and 34 lines (42 in characteristic 5).

Chapter 5

Characteristic 3

Throughout this chapter we will suppose that the ground field $\mathbb{K}$ has characteristic 3. This chapter is dedicated to the proof of the following theorem.

Theorem 5.0.1. *If X is a K3 quartic surface, then $\Phi(X) \leq 112$. Moreover, if $\Phi(X) = 112$, then X is projectively equivalent to the Fermat quartic surface; otherwise, $\Phi(X) \leq 67$.*

The chapter is structured as follows.

Section 5.1 We study elliptic lines. Not all arguments in characteristic 0 carry over to characteristic 3, mainly because they were concerned with 3-torsion sections, which are now not so well behaved.

Section 5.2 We examine a new phenomenon that does not appear in characteristic 0, namely quasi-elliptic lines: these are very important because they exhibit particularly high valencies.

Section 5.3 We carry out the proof of Theorem 5.0.1. We do not know whether the bound of 67 lines is sharp; nonetheless, we are able to prove a far better estimate – 58 lines – under the hypothesis that X contains a star (four lines meeting at the same smooth point) and is not projectively equivalent to the Fermat quartic surface, see Proposition 5.3.13. Improving the bound of 67 lines without the assumption on the existence of a star and without employing lattice-theoretical methods seems quite difficult.

Section 5.4 We discuss some examples of K3 quartic surfaces with many lines. In particular, we present three 1-dimensional families of smooth surfaces with 58 lines and a surface with 8 singular points and 48 lines.

5.1 Elliptic lines

In this section we study elliptic lines, especially separable elliptic lines. Inseparable lines (both elliptic and quasi-elliptic) will be analyzed in Section 5.2. The results of this section are summarized in Table 5.1.1.

The bounds for lines of degree 0 and lines of the first kind have already been treated in Lemma 3.3.5 and Proposition 3.3.12, respectively. In this section we will therefore concentrate on lines of the second kind.

The following two lemmas will also be useful in the study of quasi-elliptic lines.

Table 5.1.1: Known bounds for the valency of a separable elliptic line according to its kind, degree and singularity. Sharp bounds are marked with an asterisk.

kind	degree	singularity	valency
first kind	3	0	$\leq 18^*$
	2	1	≤ 13
	1	2 or 1	≤ 8
second kind	3	0	$\leq 21^*$
	2	1	$\leq 14^*$
	1	2	≤ 9
	1	1	≤ 11
–	0	3, 2 or 1	$\leq 2^*$

Lemma 5.1.1. *Let ℓ be a separable line of the second kind and $P \in \ell$ a smooth point of ramification 2. Then, either the corresponding fiber is of type* II *with a cusp in P, or the corresponding residual cubic splits into a double line plus a simple line.*

Proof. Note that only lines of degree 3 and 2 can have a point P of ramification 2. We choose coordinates so that P is given by $[0:0:0:1]$. This means that

$$a_{0103} = 0 \quad \text{and} \quad a_{0112} = 0.$$

Since P is of ramification index 2 and it is nonsingular, by rescaling variables we can normalize

$$a_{0121} = 1 \quad \text{and} \quad a_{1003} = 1.$$

Since ℓ is of the second kind, the following relations must be satisfied:

$$a_{0202} = 0, \quad a_{0301} = a_{0211}^2 \quad \text{and} \quad a_{0310} = a_{0211}a_{0220}.$$

This means that the residual cubic in $x_0 = 0$ corresponding to P is given by

$$(a_{0211}x_1 - x_2)^2 x_3 + f_3(x_1, x_2),$$

where f_3 is a form of degree 3. Either this cubic is irreducible and gives rise to a fiber of type II, or the polynomial $m = a_{0211}x_1 - x_2$ divides f_3; in the latter case it is immediate to compute that also m^2 divides f_3. □

Lemma 5.1.2. *Let ℓ be a separable line of the second kind and $P \in \ell$ a point of ramification 3_4. Then, either the corresponding fiber is of type* II *with a cusp in P, or the corresponding residual cubic splits into three concurrent lines (not necessarily distinct).*

Proof. Note that ℓ has necessarily degree 3. We choose coordinates so that P is given by $[0:0:0:1]$ and the fiber corresponds to the plane $\Pi_0 : x_0 = 0$. This means that

$$a_{0103} = 0, \quad a_{0112} = 0 \quad \text{and} \quad a_{0121} = 0.$$

A calculation with local parameters shows that $\text{length}(\Omega_{L/\mathbb{P}^1}) = 4$ if and only if $a_{1012} = 0$. Moreover, the following three coefficients must be different from 0:

a_{0130}, a_{1003} and a_{1021}; the first two because otherwise there would be singular points on ℓ (implying that the degree of ℓ is less than 3), the third because otherwise ℓ would be inseparable. We can normalize them to 1, rescaling coordinates. Two necessary conditions for the line ℓ to be of the second kind are

$$a_{0202} = 0 \quad \text{and} \quad a_{0211} = 0.$$

Hence, the residual cubic in Π_0 is given by

$$a_{0301}x_1^2x_3 + x_2^3 + x_1 f_2(x_1, x_2).$$

It is then clear that either the fiber is irreducible and has a cusp in P ($a_{0301} \neq 0$), or it splits into three concurrent lines ($a_{0301} = 0$). ☐

Remark 5.1.3. Suppose that ℓ is an elliptic line of degree 3 with fibration $\pi : Z \to \mathbb{P}^1$. Observe that

$$v_\ell(F_t) \le e(F_t) \le e(F_t) + \delta_t,$$

for any fiber of π since a reducible fiber has $e(F_t) \ge 2$ and a fiber with at least three components has $e(F_t) \ge 3$. From equations (3.7) and (2.2) we infer that

$$v(\ell) = \sum_{t\in\mathbb{P}^1} v_\ell(F_t) \le \sum_{t\in\mathbb{P}^1} \big(e(F_t) + \delta_t\big) = e(Z) = 24. \tag{5.1}$$

The only fiber type whose Euler–Poincaré characteristic is equal to its contribution to the valency of ℓ is type I_3. Hence, if for any subset $S \subset \mathbb{P}^1$ one has

$$\sum_{s\in S} e(F_s) = \sum_{s\in S} v_\ell(F_s) = N,$$

then all fibers F_s must be of type I_3 and, in particular, N must be divisible by 3.

An application of the Riemann–Hurwitz formula yields the following lemma.

Lemma 5.1.4. *If ℓ is a separable line of degree 3, then ℓ has ramification 2_1^4, $2_1 3_3$ or 3_4.*

Proposition 5.1.5. *Let ℓ be a separable elliptic line of the second kind of degree 3. Then, the valency of ℓ is bounded according to the following table, where sharp bounds are marked with an asterisk:*

ramification	valency
2_1^4	≤ 12
$2_1 3_3$	$\le 21^*$
3_4	$\le 21^*$

Proof. Suppose first that ℓ has a point P of ramification index 2. According to Lemma 5.1.1, the corresponding fiber F_P is either of type II, so that $e(F_P)+\delta_P \ge 2 + 1 = 3$ (type II has wild ramification – see Table 2.3.2) and $v(F_P) = 0$, or it contains a double component, so that $e(F_P) \ge 6$ and $v(F_P) = 2$; in any case, the difference $e(F_P)+\delta_P-v(F_P)$ is always at least 3. Therefore, if there are 4 points of ramification 2, then by formula (5.1) $v(\ell)$ is not greater than $24 - 4 \cdot 3 = 12$, while if there is just one, $v(\ell)$ is not greater than $24 - 3 = 21$.

Suppose now that ℓ has no point of ramification index 2, i.e., ℓ has ramification 3_4. If the ramified fiber F_0 is of type II, then there can be at most $24-3 = 21$ lines meeting ℓ. If F_0 splits into three concurrent lines, then $e(F_0)+\delta_0 \geq 5$ (type IV has wild ramification, too), which means that the contribution to $v(\ell)$ of the other fibers is not greater than $24 - 5 = 19$. Nonetheless, by Remark 5.1.3 this contribution cannot be exactly 19, since 19 is not divisible by 3; hence, again, we can have at most 18 lines meeting ℓ. □

Example 5.1.6. The following surface contains a separable line ($x_0 = x_1 = 0$) of ramification $2_1 3_3$ with valency 21:

$$x_0^4 + x_0^2 x_1 x_2 - x_1^3 x_2 + x_0 x_1 x_2^2 + x_1 x_2^3 + x_0^2 x_1 x_3 + x_1^2 x_3^2 + x_0 x_2 x_3^2 + x_0 x_3^3 = 0.$$

Example 5.1.7. The following surface contains a separable line $x_0 = x_1 = 0$ of ramification 3_4 with valency 21:

$$\begin{aligned} i x_0^3 x_1 + i x_1^3 x_2 + i x_1 x_2^3 - i x_0^3 x_3 + i x_0 x_1 x_2 x_3 + i x_0 x_3^3 \\ = x_0^2 x_1 x_2 + x_1^2 x_2^2 + x_0 x_2^2 x_3 - x_0^2 x_3^2, \end{aligned}$$

where i is a square root of -1.

Proposition 5.1.8. *If ℓ is an elliptic line of degree 2, then, $v(\ell) \leq 14$.*

Proof. By Proposition 3.3.12, we can assume that ℓ is of the second kind. Since ℓ has degree 2, it must have singularity 1: let P be the singular point on ℓ. The morphism $\pi : L \to \mathbb{P}^1$, being of degree 2, is separable and has two points of ramification index 2. At least one of the point of ramification must be different from P: let us call it Q. By Lemma 5.1.1 either the fiber corresponding to Q is of type II or the residual cubic splits into a double line and a simple line.

- Suppose the fiber F_Q is of type II. If ℓ is of type (p, q), then $3\,p + 2\,q \leq 24 - 3 = 21$. Applying formula (3.8), we have
$$v(\ell) \leq 2\,p + q = 14.$$
- If the residual cubic corresponding F_Q splits into a double line and a simple line, then it contributes 1 to the valency and at least 6 to the Euler number. Applying formula (3.8) again yields $v(\ell) \leq 13$. □

Example 5.1.9. The following surface contains an elliptic line $\ell : x_0 = x_1 = 0$ of degree 2 with valency 14, thus attaining the bound in Proposition 5.1.8. The surface contains one point $P = [0 : 0 : 0 : 1]$ of type $\mathbf{A}_1$. The line ℓ has 7 fibers of type I_3 and one ramified fiber of type II. The other ramified fiber corresponds to the plane $x_0 = 0$ and is smooth.

$$x_0^4 + x_0^2 x_1 x_2 - x_1^3 x_2 + x_0 x_1 x_2^2 + x_1 x_2^3 + x_1^2 x_3^2 + x_0 x_2 x_3^2 = 0.$$

Proposition 5.1.10. *Let ℓ be an elliptic line of degree 1. Then, $v(\ell) \leq 9$ if ℓ has singularity 2, and $v(\ell) \leq 11$ if ℓ has singularity 1.*

Proof. The proof can be carried over word by word from the characteristic 0 case (see Propositions 4.1.16 and 4.1.17). □

5.2 Quasi-elliptic lines

The phenomenon of quasi-elliptic lines is arguably the main difference with the characteristic 0 case. We will first recall some general facts about quasi-elliptic fibrations in characteristic 3 (see [6], [9], [34]).

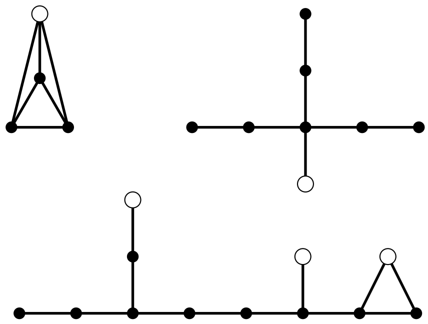
In characterstic 3 only the following fiber types can arise in a quasi-elliptic fibration (for simplicity, we call them *quasi-elliptic fibers*):

$$\mathrm{II},\ \mathrm{IV},\ \mathrm{IV}^*,\ \mathrm{II}^*.$$

We will denote by iv, iv^* and ii^* the number of fibers of type IV, IV* and II*, respectively. On a K3 surface, formula (2.3) takes the following form:

$$iv + 3\,iv^* + 4\,ii^* = 10. \tag{5.2}$$

Recall from Section 2.3 that the cuspidal curve of a quasi-elliptic fibration is a smooth curve K such that $K \cdot F = 3$. The restriction of the fibration to K is an inseparable morphism of degree 3. The cuspidal curve meets a degenerate fiber in the following ways (multiple empty dots represent different possibilities):



We note that

- K intersects a fiber of type IV at the intersection point of the three components.
- The way K intersects F is uniquely determined unless F is of type II*.

5.2.1 Quasi-elliptic lines of degree 3

Quasi-elliptic lines of degree 3 play a crucial role, mainly because it is the only case where the strict transform L of the line itself can serve as the cuspidal curve.

Definition 5.2.1. A line ℓ is said to be *cuspidal* if it is quasi-elliptic and the cuspidal curve K of the induced fibration coincides with the strict transform L.

Table 5.2.1 summarizes the known bounds for the valency of a quasi-elliptic line, which will be proven in this section.

Since the restriction of the fibration on K is an inseparable morphism $K \to \mathbb{P}^1$, a cuspidal line is necessarily inseparable. The following lemma gives a bound on the valency for inseparable lines which are not cuspidal.

Lemma 5.2.2. *If ℓ is an inseparable line and $v(\ell) > 12$, then ℓ is cuspidal.*

Table 5.2.1: Known bounds for the valency of a quasi-elliptic line. Sharp bounds are marked with an asterisk.

degree		valency
3	cuspidal	$\leq 30^*$
	not cuspidal	$\leq 21^*$
2		$\leq 14^*$
1		≤ 10
0		≤ 2

Proof. Up to coordinate change, we can suppose that the residual cubic contained in $x_0 = tx_1$ intersects the line $\ell : x_0 = x_1 = 0$ in $[0 : 0 : 0 : 1]$ for $t = 0$ and in $[0 : 0 : 1 : 0]$ for $t = \infty$. This means that the following coefficients vanish:

$$a_{0103},\ a_{0112},\ a_{0121};\ a_{1012},\ a_{1021},\ a_{1030}.$$

Moreover, a_{1003} and a_{0130} must be different from 0, and can be normalized to 1 and -1, respectively, by rescaling coordinates. Up to a Frobenius change of parameter $t = s^3$, we can explicitly write the intersection point P_s of the residual cubic with ℓ, which is given by

$$P_s = [0 : 0 : s : 1].$$

If a residual cubic E_s is reducible, then all components must pass through P_s; in particular, P_s must be a singular point of E_s. One can see explicitly that P_s is a singular point of E_s if and only if s is a root of the following degree 8 polynomial:

$$\begin{aligned} \varphi(s) := {} & a_{2020}s^8 + a_{2011}s^7 + a_{2002}s^6 + a_{1120}s^5 \\ & + a_{1111}s^4 + a_{1102}s^3 + a_{0220}s^2 + a_{0211}s + a_{0202}. \end{aligned} \tag{5.3}$$

Furthermore, it can be checked by a local computation that if E_s splits into three (not necessarily distinct) lines, then s is a double root of φ. This implies that the valency of ℓ is not greater than $3 \cdot 8/2 = 12$, unless the polynomial φ vanishes identically, but $\varphi \equiv 0$ implies that all points P_s are singular for E_s, i.e., the line ℓ is cuspidal. □

Corollary 5.2.3. *If $\ell \subset X$ is cuspidal, then X is projectively equivalent to a member of the family $\mathcal{C}$ defined by*

$$\mathcal{C} := x_0x_3^3 - x_1x_2^3 + x_2q_3(x_0, x_1) + x_3q_3'(x_0, x_1) + q_4(x_0, x_1),$$

where q_3, q_3' and q_4 are forms of degree 3, 3 and 4, respectively.

Proof. The family can be found imposing that φ vanishes identically. □

Corollary 5.2.4. *If ℓ is cuspidal, then a residual cubic corresponding to a reducible fiber of ℓ is either the union of three distinct concurrent lines, or a triple line.*

Proof. The intersection of a residual cubic with ℓ is always one single point.

A residual cubic of ℓ cannot be the union of a line and an irreducible conic, because the line and the conic would result in a fiber of type I_n (because the conic has to be tangent to ℓ), which is not quasi-elliptic.

Therefore, a residual cubic relative to a degenerate fiber must split into three (not necessarily distinct) lines. If at least two of them coincide, the plane on which they lie contains at least a singular point P of the surface (which is not on ℓ, since ℓ has degree 3). An explicit inspection of this configuration in the family $\mathcal{C}$ (for instance, supposing up to change of coordinates that P is given by $[0:1:0:0]$) shows that the residual cubic degenerates to a triple line. □

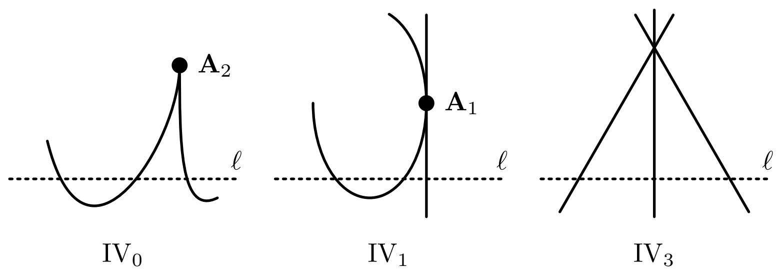
Lemma 5.2.5. *If ℓ is a quasi-elliptic line of degree* 3*, then $v(\ell) \leq 30$.*

Proof. The fibration induced by the line ℓ has at most 10 reducible fibers, each of which can contribute at most 3 to its valency. □

Remark 5.2.6. The bound of Lemma 5.2.5 is sharp. As soon as a K3 quartic surface X is smooth, the valency of a quasi-elliptic line of degree 3 on X is automatically 30, because the fibration induced by ℓ can only have 10 reducible fibers of type IV, whose residual cubics are the union of three concurrent lines. Notably, this happens for all 112 lines on the Fermat surface.

We will now prove that a quasi-elliptic line needs to be cuspidal in order to have valency greater than 21. We will do so by taking advantage of the fact that the fiber types in a quasi-elliptic fibrations are quite rigid and so are the possible residual cubics. We will need to study carefully only fibers of type IV and IV*.

Lemma 5.2.7. *Let ℓ be any line of degree* 3*. A fiber of type* IV *must have one of the following residual cubics, with the only restriction that ℓ cannot pass through a singular point:*



Proof. A fiber of type IV contains three simple components; hence, the corresponding residual cubic can also have only simple components. Since it cannot contain cycles, it must be one of the following, as in the picture:

- a cusp;
- a conic and a line meeting tangentially in one point;
- three distinct lines meeting in one point.

The remaining components must come from the resolution of the singular points on the surface. The types of the singular points can be immediately deduced from the respective Dynkin diagrams. □

Lemma 5.2.8. *Let ℓ be any line of degree* 3*. A fiber of type* IV^* *must have one of the residual cubics pictured in Figure 5.2.1, with the only restriction that ℓ cannot pass through a singular point.*

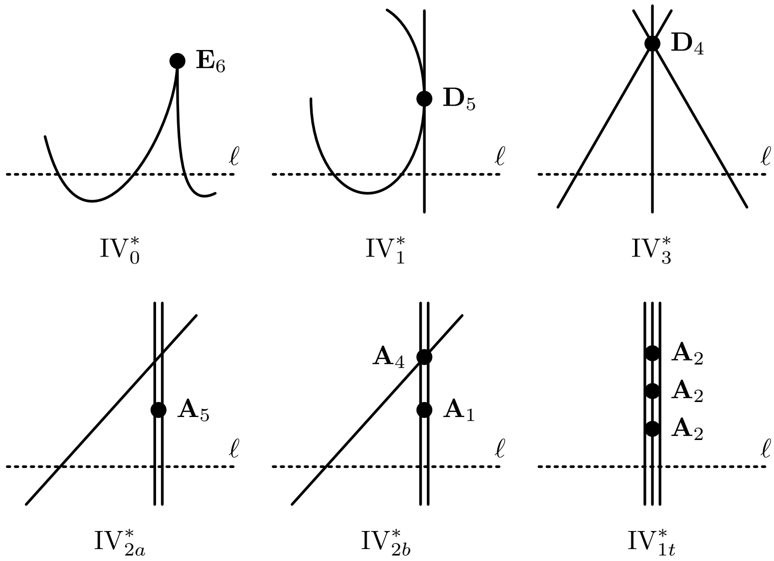


Figure 5.2.1: Possible residual cubics corresponding to a fiber of type IV^*.

Proof. Besides the residual cubics with only simple components described in the previous lemma, we can also have multiple components, namely

- a double line and a simple line;
- a triple line.

In the former case, the strict transforms of the lines can intersect (if their intersection point is smooth) or not (if their intersection point is singular), giving rise to two different configurations, which we distinguish by the letters a and b. In the latter case, there is no ambiguity, since a fiber of type IV^* contains only one triple component. ☐

From now on we will denote by iv_0, iv_1,... the number of fibers of type IV_0, IV_1, and so on. Note that the subscript indicates the local valency of the fiber.

As a last ingredient, we need to find a bound for the degree of the cuspidal curve K, which by definition is given by the intersection number of K with a hyperplane section H.

Lemma 5.2.9. *If ℓ is a separable quasi-elliptic line of degree* 3*, then the degree of its cuspidal curve is at least* 3 *and at most* 7*.*

Proof. Writing $H = F + L$, one gets $k := K \cdot H = 3 + K \cdot L$. The cuspidal curve K and the line L are distinct because ℓ is separable. The curve K can meet L only in points of ramification; moreover, a local computation shows that if K is tangent to L, then ramification 3_4 occurs, and that higher order tangency cannot happen. We thus obtain the following bounds according to the ramification type of ℓ:

- 2_1^4: $K \cdot L \leq 4$.

- $3_2 2_1$: $K \cdot L \leq 2$.
- 3_4: $K \cdot L \leq 2$. □

Proposition 5.2.10. *If ℓ is a separable quasi-elliptic line of degree 3, then $v(\ell) \leq 21$.*

Proof. A fiber of type II* can have local valency at most 2, because it contains only one simple components and three distinct lines would give rise to three distinct simple components. Hence, recalling equation (5.2),

$$\begin{aligned} v(\ell) &\leq 3\,iv + 3\,iv^* + 2\,ii^* \\ &= 3\,(10 - 3\,iv^* - 4\,ii^*) + 3\,iv^* + 2\,ii^* \\ &= 30 - 6\,iv^* - 10\,ii^*. \end{aligned}$$

In particular, if $ii^* > 0$, then $v(\ell) \leq 20$, so we can suppose that ℓ has no II*-fibers. Similarly, we can suppose that ℓ has at most one fiber of type IV*.

If ℓ has no IV*-fiber, then it must have 10 fibers of type IV. Using the classification of Lemma 5.2.7, we list the possible configurations with $v(\ell) > 21$ (16 cases) in the following table.

case	iv^*	iv_3	iv_1	iv_0	valency
1	–	10	0	0	30
2	–	9	1	0	28
3	–	9	0	1	27
4	–	8	2	0	26
5	–	8	1	1	25
6	–	8	0	2	24
7	–	7	3	0	24
8	–	7	2	1	23
9	–	7	1	2	22
10	–	6	4	0	22
11	iv_3^*	7	0	0	24
12	iv_3^*	6	1	0	22
13	iv_{2a}^*	7	0	0	23
14	iv_{2b}^*	7	0	0	23
15	iv_{1a}^*	7	0	0	22
16	iv_{1t}^*	7	0	0	22

For each case, we consider the lattice generated by L, a general fiber F, the fiber components of the degenerate fibers and the cuspidal curve K (which must be different from L, since ℓ is separable). All intersection numbers are univocally determined ($L \cdot F = 3$ because ℓ has degree 3), except for

$$K \cdot L = K \cdot (H - F) = k - 3,$$

but k can only take up the values $3, \ldots, 7$ on account of Lemma 5.2.9. We check that this lattice has rank bigger than 22 in all cases, except for case 6 with $k = 3$ (i.e., $K \cdot L = 0$).

On the other hand, this case does not exist. In fact, suppose that ℓ is as in case 6 with $K \cdot L = 0$; in particular, ℓ has no ramified fibers with multiple components and, since $v(\ell) = 24$, ℓ is of the second kind. It follows that

- if ℓ has a point of ramification 2, then by Lemma 5.1.1, the ramified fiber must be a cusp, i.e., K intersects L so $K \cdot L > 0$;
- if ℓ has ramification 3_4, then by Lemma 5.1.2 the ramified fiber must be either a cusp or the union of three distinct lines; in both cases, $K \cdot L > 0$. □

Example 5.2.11. The following surface contains a separable quasi-elliptic line $\ell : x_0 = x_1 = 0$ of degree 3 with valency 21, thus attaining the bound of Proposition 5.2.10:

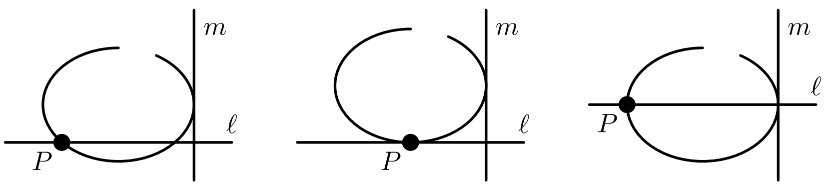
$$X : x_1^4 + x_0^2x_2^2 - x_1^2x_2^2 - x_1x_2^3 + x_0x_2^2x_3 + x_0x_3^3.$$

It contains only one singular point $[1 : 0 : 0 : 0]$ of type $\mathbf{E}_6$.

5.2.2 Quasi-elliptic lines of lower degree

Proposition 5.2.12. *If ℓ is a quasi-elliptic line of degree 2, then $v(\ell) \leq 14$.*

Proof. By Proposition 3.3.12, we can assume that ℓ is of the second kind. Let P be the singular point on ℓ and let F be a fiber of ℓ with $v_\ell(F) > 0$ and C its corresponding residual cubic. The cubic C is reducible, because it contains at least a line. Suppose that C splits into a line m and an irreducible conic (which must be tangent to each other because fibers of type I_n are not admitted in a quasi-elliptic fibration). Since $v_\ell(F) > 0$, the line m meets ℓ in a smooth point; hence, the following three configurations may arise:



All three configurations are impossible for the following reasons:

- in the first configuration, the conic meets ℓ in a non-inflection point (the smooth surface point), by Lemma 3.3.8;
- the second configuration can be ruled out by an explicit parametrization (in a line of the second kind, either the point P is a ramification point, or the cubic passing twice through P is singular at P);
- the third configuration gives rise to a fiber of type III, which is not a quasi-elliptic fiber.

Thus, C must split into three (not necessarily distinct) lines and at least one of them should pass through P. Since there can be at most 8 lines through a singular point (Lemma 3.2.2), there can be at most 7 such reducible fibers, each of them contributing at most 2 to the valency of ℓ, whence $v(\ell) \leq 14$. □

Example 5.2.13. The bound given by Proposition 5.2.12 is sharp. In fact, the following quartic surface contains a quasi-elliptic line $\ell : x_0 = x_1 = 0$ of degree 2 and valency 14:

$$X : x_0^4 + x_0^3x_1 + x_0x_1^3 + x_1x_2^3 + x_0x_1x_3^2 + x_1^2x_3^2 + x_0x_2x_3^2 = 0.$$

The quartic contains two singular points, $P = [0 : 0 : 0 : 1]$ of type $\mathbf{A}_1$ and $Q = [-1 : 1 : 1 : 0]$ of type $\mathbf{E}_6$. The line ℓ has 7 fibers of type IV and one fiber of type IV* corresponding to the plane containing Q.

Lemma 5.2.14. *If ℓ is a quasi-elliptic line of degree 1, then $v(\ell) \leq 10$.*

Proof. The fibration induced by the line ℓ has at most 10 reducible fibers, each of which contributes at most 1 to its valency. □

5.3 Proof of Theorem 5.0.1

5.3.1 Triangle free case

In this section we employ the notation and the ideas of Sections 3.4 and 4.3.

Proposition 5.3.1. *Let Γ be the line graph of a triangle free K3 quartic surface. If Γ contains a parabolic subgraph D, then*

$$|\Gamma| \leq v(D) + 24$$

Proof. The subgraph Γ induces a genus 1 fibration, which can be elliptic or quasi-elliptic [29, §3, Theorem 1]. The vertices in $D \cup (\Gamma \smallsetminus \operatorname{span} D)$ are fiber components of this fibration. If the fibration is elliptic, there cannot be more than 24 components, on account of the Euler number. If the fibration is quasi-elliptic, we obtain from formula (5.2) that

$$iv^* + ii^* \leq 3 \tag{5.4}$$

A fiber of type IV can contain at most 2 lines, since there are no triangles. Hence, from (5.2) and (5.4) we deduce

$$\begin{aligned} |\Gamma| &\leq v(D) + 2\, iv + 7\, iv^* + 9\, ii^* \\ &= v(D) + 20 + iv^* + ii^* \\ &\leq v(D) + 23. \end{aligned}$$
□

Lemma 5.3.2. *If ℓ is a line on a triangle free K3 quartic surface, then $v(\ell) \leq 12$.*

Proof. Thanks to Lemma 3.4.5, we can suppose that ℓ is quasi-elliptic. We can prove that

$$2\, v_\ell(F) \leq e(F) - 2, \tag{5.5}$$

for any degenerate fiber F, which together with formulas (2.1) and (3.7) yields $v(\ell) \leq 10$. To see this, note that a reducible fiber has $e(F) \geq 4$, so (5.5) is obvious for a 1-fiber. On the other hand, a 3-fiber cannot be of type IV, in virtue of the triangle free hypothesis; hence, it must induce a fiber of type IV* or II*, for which $e(F) \geq 8$ and (5.5) holds. □

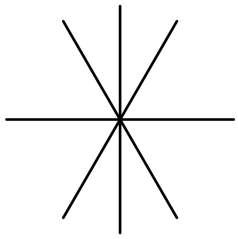
Proposition 5.3.3. *A triangle free K3 quartic surface can contain at most 64 lines.*

Proof. One can adapt the proof in Section 4.3 using Proposition 5.3.1 and Lemma 5.3.2. □

5.3.2 Star case

We will suppose from now on that X has a triangle formed by the lines ℓ_1, ℓ_2, ℓ_3, which are necessarily coplanar. The plane on which they lie must contain a fourth line ℓ_4 (which might coincide with one of the former). We will start our analysis with the following special configuration.

Definition 5.3.4. A *star* on a quartic surface X is the union of four distinct lines meeting in a smooth point.



Since the four lines in a star are necessarily coplanar, a star is the same as a configuration $\mathcal{C}_0$ (Figure 3.4.1). The lines have necessarily degree 3 because there are no singular points on the plane containing them.

We will be able to prove in Proposition 5.3.13 that if X contains a star and is not projectively equivalent to the Fermat surface, then $\Phi(X) \leq 58$, which is a sharp bound.

We will first need a series of lemmas. In all of them, we will parametrize the surface X as in (3.1) in such a way that the star is contained in the plane $x_0 = 0$ and the lines meet at $[0:0:0:1]$, i.e., setting the following coefficients equal to 0:

$$a_{0301}, a_{0211}, a_{0121}, a_{0202}, a_{0112}, a_{0103}.$$

If necessary, we will parametrize a second line in the star ℓ' as $x_0 = x_2 = 0$, by further assuming $a_{0400} = 0$.

Lemma 5.3.5. *If ℓ is a line of the first kind in a star, then $v(\ell) \leq 15$.*

Proof. It can be checked by an explicit computation that the resultant of ℓ has a root of order 6 at the center of the star; this implies that there are at most $18 - 6 = 12$ lines meeting ℓ not contained in the star. □

Lemma 5.3.6. *If ℓ is a separable line of ramification $2_1 3_3$ contained in a star, then it is of the first kind.*

Proof. By a change of coordinates, we can assume that the point of ramification index 2 is $[0:0:1:0]$, and that ramification occurs at $x_1 = 0$. Imposing that ℓ is of the second kind leads to a contradiction (ℓ cannot be separable). □

Lemma 5.3.7. *If three lines in a star are separable and at least two of them have ramification 3_4, then the third one also has ramification 3_4.*

Proof. Beside $\ell : x_0 = x_1 = 0$ and $\ell' : x_0 = x_2 = 0$, we can suppose without loss of generality that a third line is given by $\ell'' : x_0 = x_1 + x_2 = 0$, setting $a_{0220} = a_{0130} + a_{0310}$. The conditions for ℓ, ℓ' or ℓ'' to be of ramification 3_4 are

$a_{1012} = 0$, $a_{1102} = 0$ and $a_{1012} = a_{1102}$, respectively. Clearly, two of them imply the third one. □

Lemma 5.3.8. *If three lines in a star are separable, then at most two of them can be of the second kind.*

Proof. We parametrize ℓ, ℓ' and ℓ'' as in the previous Lemma. Imposing that all three of them are of the second kind leads to a contradiction (at least one of them must be inseparable). □

Lemma 5.3.9. *Let ℓ and ℓ' be two lines in a star; if ℓ is a separable line of the second kind, and ℓ' is a line of the first kind of ramification 3_4, then $v(\ell') \leq 12$.*

Proof. This can be checked again by an explicit computation of the resultant of ℓ', which has now a root of order 9 at the center of the star. □

Lemma 5.3.10. *Let ℓ and ℓ' be two lines in a star; if ℓ is a cuspidal line, and ℓ' is not cuspidal, then $v(\ell') \leq 12$.*

Proof. We parametrize $\ell : x_0 = x_1 = 0$ as in Corollary 5.2.3. By virtue of Lemma 5.2.2, we can suppose that $\ell' : x_0 = x_2 = 0$ is separable. An explicit computation shows that ℓ' cannot be of the second kind, and that its resultant has a root of order 9 in $x_2 = 0$. □

Lemma 5.3.11. *Let ℓ, ℓ' and ℓ'' be three lines in a star; if ℓ and ℓ' are cuspidal, and ℓ'' is not cuspidal, then $v(\ell'') = 3$.*

Proof. We parametrize $\ell : x_0 = x_1 = 0$ and $\ell' : x_0 = x_2 = 0$ as in Corollary 5.2.3, i.e., we suppose that X is given by the family $\mathcal{C}$ where the following coefficients are set to zero:

$$a_{0400},\, a_{0301};\, a_{1201},\, a_{1300};\, a_{2200},\, a_{2101},\, a_{1210}.$$

By a further rescaling we put $a_{0310} = 1$ and we consider $\ell'' : x_0 = x_1 - x_2 = 0$. The line ℓ'' is inseparable and we can compute its polynomial φ as in formula (5.3) in the proof of Lemma 5.2.2 (by parametrizing the pencil with $x_0 = s^3(x_1 - x_2)$), which turns out to be

$$\varphi(s) = a_{2110}s^8.$$

This means that ℓ'' has only one singular fiber in $s = 0$ (namely a fiber of type IV with the maximum possible index of wild ramification), unless $a_{2110} = 0$ and $\varphi \equiv 0$, in which case ℓ'' is cuspidal. □

Lemma 5.3.12. *If ℓ is a cuspidal line which is not contained in at least two stars, then $v(\ell) \leq 6$.*

Proof. On account of Lemma 5.2.4, the number of stars in which ℓ is contained is exactly equal to the number of fibers of type IV in its fibration; moreover, $v_\ell(F) = 1$ if F is of type IV* or II*, yielding

$$v(\ell) = 3\, iv + iv^* + ii^*.$$

Recalling formula (5.2), we deduce that if $iv < 2$ then $v(\ell) \leq 6$. □

Proposition 5.3.13. *If X contains a star and is not projectively equivalent to the Fermat surface, then X contains at most 58 lines.*

Proof. Let ℓ_1, ℓ_2, ℓ_3, ℓ_4 be the lines contained in the star. We will always use the bound (3.6), which takes the form

$$\Phi(X) \leq 4 + \sum_{i=1}^{4} (v(\ell_i) - 3) = \sum_{i=1}^{4} v(\ell_i) - 8.$$

(1) Suppose first that all lines ℓ_i are not cuspidal.
- If $v(\ell_i) \leq 15$ for $i = 1, 2, 3, 4$, then
$$\Phi(X) \leq 4 \cdot 15 - 8 = 52.$$
- If $v(\ell_1) > 15$, then by Lemmas 5.3.5 and 5.2.2, ℓ_1 must be separable of the second kind; hence $v(\ell_1) \leq 21$; if $v(l_i) \leq 15$ for $i = 2, 3, 4$, then
$$\Phi(X) \leq (21 + 3 \cdot 15) - 8 = 58.$$
- If $v(\ell_1) > 15$ and $v(\ell_2) > 15$, then by the same token both ℓ_1 and ℓ_2 are separable lines of the second kind. On account of Lemma 5.3.6, they both have ramification 3_4. We claim that both $v(\ell_3)$ and $v(\ell_4)$ are not greater than 12. In fact, if ℓ_3 is separable, then by Lemmas 5.3.7 and 5.3.8 it must be of the first kind and have ramification 3_4, which in turn implies that $v(\ell_3) \leq 12$, because of Lemma 5.3.9; if ℓ_3 is inseparable, then $v(\ell) \leq 12$ by Lemma 5.2.2. The same applies to ℓ_4. Hence, we conclude that
$$\Phi(X) \leq (2 \cdot 21 + 2 \cdot 12) - 8 = 58.$$

(2) Assume now that exactly one of the lines, say ℓ_1, is cuspidal, so that $v(\ell_1) \leq 30$. On account of Lemma 5.3.10 we have
$$\Phi(X) \leq (30 + 3 \cdot 12) - 8 = 58.$$

(3) Suppose then that both ℓ_1 and ℓ_2 are cuspidal. If ℓ_3 and ℓ_4 are not cuspidal, then by Lemma 5.3.11
$$\Phi(X) \leq (2 \cdot 30 + 2 \cdot 3) - 8 = 58.$$

(4) Finally, suppose that ℓ_1, ℓ_2 and ℓ_3 are cuspidal.
- By a local computation it can be seen that ℓ_4 is also necessarily cuspidal.
- Thanks to the bound of Lemma 5.3.12, we can suppose that at least two lines, say ℓ_1 and ℓ_2, are part of another star.
- Pick two lines ℓ_1' and ℓ_2', each of them in another star containing ℓ_1 respectively ℓ_2, which intersect each other (necessarily in a smooth point).
- Perform a change of coordinates so that ℓ_1, ℓ_2, ℓ_1' and ℓ_2' are given respectively by $x_0 = x_1 = 0$, $x_0 = x_2 = 0$, $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$.
- Impose that ℓ_1, ℓ_2 and ℓ_3 are cuspidal lines: the resulting surface is projectively equivalent to Fermat surface. □

5.3.3 Triangle case

In this section we study the case in which X admits a triangle. The three lines forming the triangle need to be coplanar, and we will denote by Π the plane on which they lie. Obviously, the plane Π intersects X also in a fourth line, which might coincide with one of the first three. We first consider this degenerate case.

Proposition 5.3.14. *If X admits a completely reducible plane Π with a triangle and a multiple component, then X contains at most 60 lines.*

Proof. By Lemma 3.4.6, X admits configuration $\mathcal{D}_0$ or $\mathcal{E}_0$. In order to bound $\Phi(X)$, we use as usual formula (3.6).

Let ℓ_0 be the double line in the plane Π containing one of the two configurations, and let ℓ_1 and ℓ_2 be the two simple lines. Lines meeting ℓ_0 different from ℓ_1 and ℓ_2 must pass through the singular points; hence, by Lemma 3.2.2 there can be at most $3 \cdot (8-1) = 21$ of them.

Note that ℓ_1 and ℓ_2 cannot be cuspidal because of Corollary 5.2.4.

In the fibrations induced by ℓ_1 and ℓ_2 the plane Π corresponds to a fiber with a multiple component, hence with Euler number at least 6; therefore, if ℓ_1 and ℓ_2 are both elliptic, there can be at most 18 more lines meeting them, yielding

$$\Phi(X) \leq 3 \cdot (8-1) + (18+18) + 3 = 60.$$

Suppose that ℓ_1 is quasi-elliptic. The plane Π corresponds to a fiber of type IV^* or II^*; hence, there can be only one singular point on ℓ_0: in fact, by inspection of the Dynkin diagrams, a component of multiplicity 2 in these fiber types meets at most 2 other components (and one of them is the strict transform of ℓ_2). The lines ℓ_1 and ℓ_2 not being cuspidal, we know that they have valency at most 21. It follows that

$$\Phi(X) \leq (8-1) + 2 \cdot (21-2) + 3 = 48. \qquad \square$$

Lemma 5.3.15. *Let ℓ and ℓ' be two lines of degree 3 in configuration $\mathcal{A}_0$ or $\mathcal{A}_1$. If $v(\ell) > 18$, then $v(\ell') \leq 18$.*

Proof. Let Π be the plane containing ℓ and ℓ'. Both lines are separable, since otherwise the respective residual cubics would intersect them in one point. We suppose that also $v(\ell') > 18$ and look for a contradiction.

Since both lines have valency greater than 18, they must be lines of the second kind with ramification $(3(3), 2)$ or 3_4. In particular, they must have a point of ramification 3 (let us call it $P \in \ell$ and $P' \in \ell'$), which does not lie on Π. Up to change of coordinates, we can assume the following:

- Π is the plane $x_0 = 0$;
- ℓ and ℓ' are given respectively by $x_0 = x_1 = 0$ and $x_0 = x_2 = 0$;
- P is given by $[0:0:1:0]$ and P' by $[0:1:0:0]$;
- ramification in P (resp. P') occurs in $x_1 = 0$ (resp. $x_2 = 0$).

This amounts to setting the following coefficients equal to 0:

$$a_{0400}, a_{0301}, a_{0202}, a_{0103}; a_{1030}, a_{1021}, a_{1012}; a_{1300}, a_{1201}, a_{1102}.$$

Furthermore, $a_{0112} \neq 0$, since the two residual lines in Π do not contain $[0:0:0:1]$, the intersection point of ℓ and ℓ'; we set $a_{0112} = 1$ after rescaling one variable.

Two necessary condition for ℓ and ℓ' to be lines of the second kind are

$$a_{0310} = a_{0211}^2 \quad \text{and} \quad a_{0130} = a_{0121}^2.$$

This means that the residual conic in $\Pi : x_0 = 0$ is given explicitly by

$$a_{0211}^2 x_1^2 + a_{0121}^2 x_2^2 + a_{0220} x_1 x_2 + a_{0211} x_1 x_3 + a_{0121} x_2 x_3 + x_3^2 = 0. \tag{5.6}$$

This conic splits into two lines by hypothesis; hence, it has a singular point. Computing the derivatives, one finds that the following condition must be satisfied:

$$a_{0220} = -a_{0121} a_{0211}.$$

Substituting into (5.6), one finds that the conic degenerates to a double line:

$$(a_{0211} x_1 + a_{0121} x_2 - x_3)^2 = 0;$$

thus, we have neither configuration $\mathcal{A}_0$ nor $\mathcal{A}_1$. □

Proposition 5.3.16. *If X admits a triangle but not a star, then X contains at most 67 lines.*

Proof. The proof is a case-by-case analysis on the configurations that are given by Lemma 3.4.6, except configurations $\mathcal{C}$ (a star, treated in Proposition 5.3.13), $\mathcal{D}_0$ and $\mathcal{E}_0$ (treated in Proposition 5.3.14).

Beside the fact that there are at most 8 lines through a singular point and the bounds on the valency of Sections 5.1 and 5.2, one should observe that in configurations of type $\mathcal{B}$, the three lines meeting at the same (smooth) point must be of the first kind by Lemma 5.1.1. For configurations $\mathcal{A}_0$ and $\mathcal{A}_1$, one uses Lemma 5.3.15.

For instance, let us prove the proposition for configuration $\mathcal{A}_1$. Let ℓ_1 and ℓ_2 be the lines through the singular point, and ℓ_3 and ℓ_4 the other two lines. We know that $v(\ell_i) \leq 14$, $i = 1, 2$, whereas Lemma 5.3.15 applies to ℓ_3 and ℓ_4, yielding

$$v(\ell_3) + v(\ell_4) \leq 18 + 21.$$

It follows from (3.6) that

$$\Phi(X) \leq (8-2) + 2 \cdot (14-2) + (18-3+21-3) + 4 = 67.$$

We leave the remaining cases to the reader. □

Proof of Theorem 5.0.1. We have treated the case of triangle free surfaces in Proposition 5.3.3, the star case in Proposition 5.3.13 and the star free triangle case in Proposition 5.3.16, so the proof is now complete. □

5.4 Examples

In this last section, we present examples of K3 quartic surfaces with many lines. In particular, we provide explicit equations for three 1-dimensional families of surfaces with 58 lines. Most of the examples – including the first two families (Examples 5.4.1 and 5.4.2) – were found during the proof of the theorem, especially of Proposition 5.3.13. Note that the first two families had already been discovered independently by A. Degtyarev, who was also aware of the existence of a third configuration with 58 lines. We found the third family (Example 5.4.3) after we were informed of his work; as far as we know, this family is new.

Example 5.4.1. A general member of the 1-dimensional family defined by

$$x_1^3x_2 - x_1x_2^3 + x_0^3x_3 - x_0x_3^3 = ax_0^2x_1x_2$$

is smooth and contains 58 lines.

More precisely, for $a = 0$ we obtain a surface which is projectively equivalent over $\mathbb{F}_9$ to the Fermat surface and thus contains 112 lines.

If $a \neq 0, \infty$, the surface contains a star (in $x_0 = 0$) formed by two cuspidal lines (of valency 30) and two elliptic lines with no other singular fibers than the star itself (hence, of valency 3). The remaining 54 lines are of type $(p, q) = (1, 9)$. The surface contains exactly 19 stars.

For $a = \infty$ we obtain the union of three planes.

Example 5.4.2. A general member of the 1-dimensional family defined by

$$x_1^3x_2 - x_1x_2^3 + x_0^3x_3 - x_0x_3^3 = ax_0x_1(ax_0x_2 + ax_1x_3 + x_1x_2 + x_0x_3)$$

is smooth and contains exactly 58 lines.

More precisely, as long as $a \neq 0,\ 1,\ -1,\ \infty$, the surface contains one cuspidal line (given by $x_0 = x_1 = 0$) which intersects 12 lines of type $(4, 0)$, and 18 lines of type $(1, 9)$; the remaining 27 lines are of type $(4, 6)$ (for instance, $x_2 = x_3 = 0$). The surface contains exactly 10 stars.

For $a = 0$ we find again a model of the Fermat surface, whereas for $a = \pm 1$ the surface contains 20 lines and a triple point. For $a = \infty$ we obtain the union of two planes and a quadric surface.

All surfaces of the family are endowed with the symmetries $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_1 : x_0 : x_3 : x_2]$ and $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_0 : x_1 : -x_2 : -x_3]$.

Example 5.4.3. A general member of the 1-dimensional family defined by

$$\begin{aligned}\left(a^3 + a^2 + a + 1\right)&\left(x_1^3x_2 + x_1x_2^3 - x_0^3x_3 - x_0x_3^3\right) = \\ &(a-1)\left(x_0^2x_1x_2 - x_0^2x_3^2 + x_1x_2x_3^2\right) + (a+1)\left(x_1^2x_2^2 - x_0x_1^2x_3 - x_0x_2^2x_3\right) \\ &\quad + \left(a^2 - 1\right)(x_1x_2 + x_0x_3)(x_0 + x_3)(x_1 + x_2) - \left(a^2 + 1\right)x_0x_1x_2x_3\end{aligned}$$

is smooth and contains exactly 58 lines.

More precisely, if $a \neq 0, 1, -1, \infty$ and $a^2 \neq -1$, then the surface contains exactly one star in the plane

$$x_0 + x_3 = x_1 + x_2. \tag{5.7}$$

The star is formed by two lines of type $(7, 0)$ and two lines of type $(1, 9)$, whose equations can be explicitly written after a change of parameter $a = d/(d^2 + 1)$. Each line of type $(7, 0)$ meets 18 lines of type $(3, 6)$, and each line of type $(1, 9)$ meets 9 lines of type $(4, 6)$. All lines are elliptic.

If $a = 0$, 1 or -1 the surface is the union of a double plane and a quadric surface. If $a = \infty$ the surface is projectively equivalent to the Fermat surface.

If $a^2 = -1$, then the surface contains 9 points of type $\mathbf{A}_1$ and 40 lines. The star in the plane (5.7) is formed by two elliptic lines of type $(4, 0)$ and two quasi-elliptic lines of type $(1, 9)$. Each line of type $(4, 0)$ intersects 9 lines of singularity 2 and valency 6, while each line of type $(1, 9)$ intersects 9 lines of singularity 1 and valency 9.

All surfaces of the family are endowed with the symmetries $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_1 : x_0 : x_2 : x_3]$ and $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_0 : x_1 : x_3 : x_2]$.

Example 5.4.4. The surface defined by

$$\begin{aligned}x_0^3x_1 + x_0^2x_1^2 + x_0x_1^2x_2 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_0^3x_3 \\ - x_0^2x_1x_3 + x_0x_1^2x_3 + x_0^2x_2x_3 + x_0x_1x_2x_3 + x_0x_2^2x_3 + x_0x_3^3 = 0\end{aligned}$$

contains one singular point of type $\mathbf{E}_7$ and 39 lines.

Example 5.4.5. The reduction modulo 3 of Shimada–Shioda's surface X_{56} [40] can be written

$$\Psi(x_0, x_1, x_2, x_3) = \Psi(-x_1, x_0, -x_3, x_2)$$

where

$$\Psi(w, x, y, z) = wz\big(w^2 + wx + x^2 + y^2 + yz + z^2\big)$$

It contains 8 singular points of type $\mathbf{A}_1$ and 48 lines. So far, this is the example known to us with highest number of lines and at least one singular point.

Chapter 6

Characteristic 2

In this chapter we will always work over a fixed algebraically closed field $\mathbb{K}$ of characteristic 2. The main result will be the following theorem.

Theorem 6.0.1. *If X is a K3 quartic surface, then $\Phi(X) \leq 68$.*

The chapter is structured as follows.

Section 6.1 We list the main results about elliptic lines, whose study is essentially the same as in characteristic 0.

Section 6.2 Although a line on a smooth surface never induces a quasi-elliptic fibration (see Remark 6.2.8), the phenomenon of quasi-elliptic lines does indeed take place once we allow for rational double points. We study them extensively in this section.

Section 6.3 We carry out the proof of Theorem 6.0.1.

Section 6.4 The bound of the theorem is indeed sharp. A 1-dimensional family of K3 quartic surfaces with 68 lines was already known to Rams and Schütt [31]. We are able to prove a uniqueness result, namely that if a surface contains 68 lines, then it is projectively equivalent to a member of Rams–Schütt's family (Theorem 6.4.1).

6.1 Elliptic lines

In this section we list some results on separable elliptic lines, especially the bounds on their valency, which are summarized in Table 6.1.1. We postpone the study of inseparable lines (both elliptic and quasi-elliptic) to Lemma 6.2.12.

The analysis of lines of the second kind in characteristic 2 is essentially the same as in characteristic 0. In fact, our arguments dealt with torsion sections of order 3 and fail only in characteristic 3. The only discrepancy is that lines of degree 3 and 2 have different ramification types, but this is compensated by wild ramification. We omit the proofs, but state the main facts that we will use later on.

Lemma 6.1.1. *If ℓ has degree 3, then it is separable and has ramification 2_4, 2_2^2, $2_2 3_2$ or 3_2^2.*

As in characteristic 0, the last ramification type in Lemma 6.1.1 deserves to be given a name.

Table 6.1.1: Known bounds for the valency of a separable elliptic line according to its kind, degree and singularity. Sharp bounds are marked with an asterisk *.

kind	degree	singularity	valency
first kind	3	0	≤ 18
	2	1	≤ 13
	1	2 or 1	≤ 8
second kind	3	0	$\leq 20^*$
	2	1	≤ 10
	1	2	≤ 9
	1	1	≤ 11
–	0	3, 2 or 1	$\leq 2^*$

Table 6.2.1: Known bounds for the valency of a quasi-elliptic line according to its degree and singularity. Sharp bounds are marked with an asterisk.

degree	singularity		valency
3	0		$\leq 16^*$
2	1	cuspidal	$\leq 19^*$
		not cuspidal	≤ 13
1	2		≤ 8
1	1		≤ 12
0	3, 2 or 1		≤ 2

Definition 6.1.2. A line ℓ of degree 3 is said to be *special* if it is of the second kind and has ramification 3_2^2.

Proposition 6.1.3. *If ℓ is a line of the second kind of degree* 3*, then $v(\ell) \leq 20$; moreover, if $v(\ell) > 16$, then ℓ is special. If $v(\ell) = 19$, then the line has type $(p, q) = (6, 1)$ and the 1-fiber is a ramified fiber of type I_n, $n \geq 2$; if $v(\ell) = 20$, then the line has type $(p, q) = (6, 2)$ and both 1-fibers are ramified fibers of type I_n, $n \geq 2$.*

We can also parametrize special lines as in Lemma 4.2.2 (see also [31, Lemma 4.4]). Therefore, special lines induce a symplectic automorphism of degree 3 which permutes the lines in their 3-fibers (Remark 4.2.3).

6.2 Quasi-elliptic lines

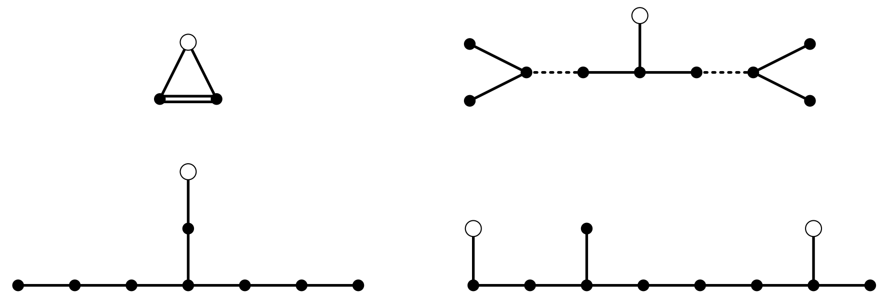
In this section we study quasi-elliptic lines and find bounds on their valency. We summarize our results in Table 6.2.1.

We will first recall some general facts about quasi-elliptic fibrations in characteristic 2 (see [6], [34]).

As we saw in Section 2.3, the cuspidal curve of a quasi-elliptic fibration on a K3 surface $Z \to \mathbb{P}^1$ (in particular, of a fibration induced by a quasi-elliptic line) is a smooth rational curve K such that $K \cdot F = 2$. Only the fiber types

$$\mathrm{II},\ \mathrm{III},\ \mathrm{I}_{2n}^*,\ \mathrm{III}^*,\ \mathrm{II}^*$$

can appear in a quasi-elliptic fibration. We call such fibers *quasi-elliptic fibers*. The restriction of the fibration $Z \to \mathbb{P}^1$ to K is an inseparable morphism of degree 2. The cuspidal curve meets a reducible fiber in the following ways (multiple empty dots represent different possibilities):



In particular, we observe that

- the way K meets a reducible fiber is uniquely determined apart from type II^*;
- K meets a component of a reducible fiber always transversally;
- K always meets only one component of multiplicity 2, with the exception of type III, where it meets two components of multiplicity 1.

Lemma 6.2.1. *A section of a quasi-elliptic fibration does not intersect the cuspidal curve.*

Proof. We suppose the section and the cuspidal curve meet at a point on a fiber F. Since the cuspidal curve intersects F in a singular point or on a double component, the section would have intersection at least 2 with F. □

Lemma 6.2.2. *The cuspidal curve of the fibration induced by a quasi-elliptic line ℓ cannot be an exceptional divisor.*

Proof. Suppose the cuspidal curve of the fibration induced by ℓ coincides with an exceptional irreducible divisor E; then E must come from the resolution of a singular point P on ℓ. Since the general residual cubic is the image of a curve with a cusp on E, it should be singular in P, but this is ruled out by Lemma 3.2.3. □

Quasi-elliptic lines of degree 2 play a special role. In fact, this is the only case where the strict transform L of the line ℓ can be the cuspidal curve itself, since $L \cdot F = 2$. We give a name to these particular lines.

Definition 6.2.3. A line ℓ is said to be *cuspidal* if it is quasi-elliptic of degree 2 and the cuspidal curve on Z coincides with the strict transform of ℓ itself.

Assume that ℓ is a quasi-elliptic line which is *not* cuspidal. By virtue of Lemma 6.2.2, the cuspidal curve K of the fibration induced by ℓ is a smooth rational curve in Z of positive degree $k = K \cdot H > 0$, where H a hyperplane divisor in Z.

We introduce now a way to parametrize such cuspidal curves. Let us first choose a parameter s for K so that the restriction $\pi|_K : K \to \mathbb{P}^1$ is given by

$$s \mapsto t = s^2, \tag{6.1}$$

where t parametrizes the planes containing ℓ: $x_0 = tx_1$. Let $\bar{K} := \rho(K)$ be the image of K in $\mathbb{P}^3$ through the resolution $\rho : Z \to X$. Then, we obtain a morphism from K to $\bar{K} \subset \mathbb{P}^3$ given by

$$\psi : s \mapsto \psi(s) := [\psi_0(s) : \psi_1(s) : \psi_2(s) : \psi_3(s)],$$

where $\psi_i(s)$ is a polynomial of degree $k = K \cdot H$, $i = 0, \dots, 3$.

Let now $\bar{\pi} : X \dashrightarrow \mathbb{P}^1$ be the rational map $\bar{\pi} = \pi \circ \rho^{-1}$. Clearly, on the chart $x_1 \neq 0$, the map $\bar{\pi}$ can be written as

$$[x_0 : x_1 : x_2 : x_3] \mapsto t = \frac{x_0}{x_1},$$

since ℓ is not cuspidal and because of Lemma 6.2.2, $\bar{K}$ has non-trivial intersection with the domain of definition of $\bar{\pi}$; hence, the restriction of $\pi = \bar{\pi} \circ \psi$ to (an open subset of) K is given by

$$s \mapsto t = \frac{\psi_0(s)}{\psi_1(s)},$$

Comparing with (6.1), we see that $\psi_0(s) = s^2\psi_1(s)$, i.e., we can parametrize $\bar{K}$ as

$$s \mapsto \psi(s) = \left[\sum_{i=0}^{k-2} a_i s^{i+2} : \sum_{i=0}^{k-2} a_i s^i : \sum_{i=0}^{k} b_i s^i : \sum_{i=0}^{k} c_i s^i \right], \tag{6.2}$$

where $a_i, b_i, c_i \in \mathbb{K}$, and k is the degree of K. Note that the point $\psi(s)$ lies in the plane $x_0 = s^2 x_1$.

6.2.1 Quasi-elliptic lines of degree 3

Since a quasi-elliptic line of degree 3 is never cuspidal, in the following lemma we will suppose that the image of K in $\mathbb{P}^3$ is parametrized by (6.2).

Lemma 6.2.4. *If ℓ is a quasi-elliptic line of degree 3, then the cuspidal curve of its fibration can have degree at most 4.*

Proof. The pullback of a plane containing ℓ is a divisor $H = F + L$ on the minimal resolution of X. The degree of K is by definition $k := K \cdot H = K \cdot F + K \cdot L = 2 + K \cdot L$.

We claim that $K \cdot L \leq 2$. Indeed, K cannot be the divisor L itself, since $L \cdot F = 3$. If K and L meet at a point P, then the image of P in X (which we will call P again) is a ramification point for ℓ. Since there are at most two ramification points, $K \cdot L \leq 2$, unless K is tangent to L in at least one of them. Suppose therefore that K is tangent to L at P.

Claim 6.2.5. The point P cannot be of ramification 3.

Proof of the claim. The fiber F passing through P must be singular in P, since K is going through P. On the other hand, the corresponding residual cubic C cannot be reducible, because P is a point of ramification index 3. In fact, if C is the union of a line m and an irreducible conic Q, then Q is tangent to ℓ, and m intersects Q in two different points, giving rise to a fiber of type I_n: since ℓ is quasi-elliptic, this is not possible. If C is the union of three lines, possibly not all distinct, then K would have triple intersection with F, also impossible. Therefore the cubic F is irreducible, with a cusp in P and tangent to ℓ. Since K is also tangent to L, we have $K \cdot F \geq 3$. □

We can thus assume that P has ramification index 2. Up to a change of coordinates, P can be given by $[0 : 0 : 0 : 1]$. This means choosing $a_{0112} = 0$, $a_{0103} = 0$, while a_{0121} and a_{1003} must be non-zero, and can be normalized to 1.

Claim 6.2.6. The point P is of ramification 2_4.

Proof of the claim. Consider the parametrization (6.2). Since K goes through P, we can set $a_0 = 0$, $b_0 = 0$ and normalize $c_0 = 1$. Imposing that $\psi(s)$ is a singular point of the cubic in the plane $x_0 = s^2 x_1$ for all s, one finds – beside other relations – that

$$a_{1012} = a_{0130},$$

which is the condition for P to be a point of ramification 2_4. □

It follows that ℓ has only one ramification point, so that $K \cdot L \leq 2$ unless K is tangent to L of order at least 3. Hence, we further set $a_2 = 0$, but this condition leads to ℓ containing a singular point, which contradicts the fact that ℓ has degree 3. □

Proposition 6.2.7. *If ℓ is a quasi-elliptic line of degree 3, then $v(\ell) \leq 16$.*

Proof. Let iii, i_n^*, iii^*, ii^* be the numbers of reducible fibers of type III, I_n^*, III^*, II^*. Formula (2.3) yields

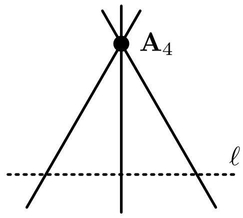
$$iii + \sum (4+n)\, i_n^* + 7\, iii^* + 8\, ii^* = 20. \tag{6.3}$$

A fiber of type III has local valency at most 1 (because if it contains a line, the other component must be an irreducible conic), while other reducible fibers have valency at most 3. Moreover, fibers of type III^* and II^* have valency ≤ 2 because they do not contain three simple components. Hence, using equation (6.3) we obtain a first rough estimate

$$v(\ell) \leq iii + 3 \sum i_n^* + 2\, iii^* + 2\, ii^* \leq 20. \tag{6.4}$$

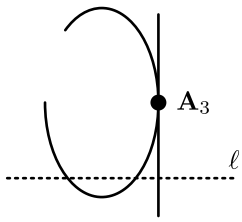
We want to rule out all possible configurations that lead to $v(\ell) > 16$. We first note that a fiber of type III has local valency equal to either 1 or 0, according to whether the residual cubic is the union of a line and a conic, or an irreducible cubic with a cusp at a singular point of type $\mathbf{A}_1$. We denote the number of the former III-fibers with iii' and of the latter with iii''.

First of all, if $v(\ell) > 16$, then $iii^* = ii^* = 0$ and $i_n^* = 0$ for all $n > 2$, by formulas (6.3) and (6.4). If $i_2^* > 0$, then $i_2^* = 1$, $iii = iii' = 14$, and the I_2^*-fiber must have valency 3; the corresponding cubic must then have the following shape:

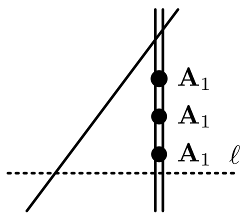


In fact, the lines must meet at the same point (otherwise ℓ would have a fiber of type I_n), and this point must be singular of type $\mathbf{A}_4$ (the strict transforms of the three lines are three simple components of the fiber I_2^*; the dual graph of the remaining components is an $\mathbf{A}_4$-diagram).

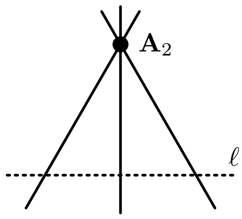
If $i_2^* = 0$ and $i_0^* > 0$, then $i_0^* = 1$ or 2. Observe that there cannot be a fiber of type I_0^* not contributing to $v(\ell)$, otherwise $v(\ell) \leq 16$. Arguing as with type I_2^*, it follows that the residual cubic of a fiber of type I_0^* is one of the types $\mathrm{I}_{0,n}^*$, $n = 1, 2, 3$, as pictured:



$\mathrm{I}_{0,1}^*$



$\mathrm{I}_{0,2}^*$



$\mathrm{I}_{0,3}^*$

If $i_0^* = 2$, then $iii = 12$, one of the I_0^* must have valency 3 (i.e., be of type $\mathrm{I}_{0,3}^*$) and the other must have valency 3 or 2 (i.e., be of type $\mathrm{I}_{0,3}^*$ or $\mathrm{I}_{0,2}^*$). Once the numbers i_2^* and i_0^* are fixed, the sum $iii = iii' + iii''$ is uniquely determined by the number of the other fiber types and it is then a matter of listing all possibilities for iii' and iii'' that lead to $v(\ell) > 16$. There are 14 cases in total, as displayed in Table 6.2.2.

For each case, one can check that the lattice generated by L, the components of its fibers, the cuspidal curve K (which has degree k, $2 \leq k \leq 4$, according to Lemma 6.2.4), and a general fiber has rank 23. □

Remark 6.2.8. An immediate corollary of the last proposition is the fact that on a smooth surface all lines are elliptic, which has already been proven by Rams and Schütt [31, Proposition 2.1] using a different approach. In fact, a quasi-elliptic line on a smooth surface could only have 20 fibers of type III, falling into case 11 of Table 6.2.2 (which would be the only case to be ruled out).

Example 6.2.9. The bound of Proposition 6.2.7 is sharp and is reached, for example, by the line $\ell : x_0 = x_1 = 0$ in the surface

$$X : x_0^3 x_1 + x_0 x_1^3 + x_0 x_2^3 + x_0^2 x_1 x_3 + x_1 x_2^2 x_3 + x_0^2 x_3^2 + x_1^2 x_3^2 + x_0 x_3^3 = 0.$$

The 16 lines meeting ℓ are given by $x_0 = x_3 = 0$ and by $x_1 = s^4 x_0$, $x_0 = a x_2 + b x_3$, where $a = s/(s^4 + s + 1)$, $b = 1/s^2$ and s is a root of

$$s^{15} + s^{12} + s^9 + s^8 + s^7 + s^6 + s^4 + s^3 + s^2 + s + 1.$$

Table 6.2.2: Fiber configurations for a quasi-elliptic line ℓ of degree 3 with $v(\ell) > 16$.

case	i_2^*	$i_{0,3}^*$	$i_{0,2}^*$	$i_{0,1}^*$	iii'	iii''	$v(\ell)$
1	1				14		17
2		2			12		18
3		2			11	1	17
4		1	1		12		17
5		1			16		19
6		1			15	1	18
7		1			14	2	17
8			1		16		18
9			1		15	1	17
10				1	16		17
11					20		20
12					19	1	19
13					18	2	18
14					17	3	17

6.2.2 Quasi-elliptic lines of degree 2

Proposition 6.2.10. *If ℓ is a quasi-elliptic line of degree 2, then $v(\ell) \leq 19$.*

Proof. Consider the quasi-elliptic fibration induced by ℓ; let i be the number of fibers of type III and j the number of reducible fibers not of type III. The residual cubic of a fiber of type III cannot contain more than one line, while the other fibers contribute at most 2 to the valency of ℓ, since ℓ has degree 2. It follows that

$$v(\ell) \leq i + 2\,j.$$

Computing the Euler–Poincaré characteristic yields $i + 4\,j \leq 20$, so $v(\ell) \leq 20$.

We claim that $v(\ell)$ cannot be exactly 20. Indeed, if $v(\ell) = 20$, then ℓ has exactly 20 fibers whose components are a line and an irreducible conic. Moreover, by Lemma 3.2.4 the singular point on ℓ must be of type $\mathbf{A}_1$, giving one divisor E_0 in the resolution which is a section of the fibration, or else the other divisors would form an extra fiber. The line L, the 20 lines meeting L, the divisor E_0 and the general fiber F generate a lattice of rank 23, which is impossible. ☐

Example 6.2.11. The bound of Proposition 6.2.10 is sharp and is reached, for example, by the cuspidal line $\ell : x_0 = x_1 = 0$ in the surface

$$X : x_0^4 + x_1x_2^3 + x_1^3x_3 + x_0x_2x_3^2 = 0.$$

The surface contains exactly one singular point $P = [0 : 0 : 0 : 1]$ of type $\mathbf{A}_2$. The 19 lines meeting ℓ are given by $x_0 = tx_1$, $x_1 = t^5x_2 + t^{15}x_3$, with t any 19th root of unity. The surface contains exactly 20 lines. The line ℓ being quasi-elliptic, the surface X has Picard number 22.

Such high valencies can indeed be reached only by cuspidal lines. To prove this fact we need to find a bound on the degree k of the cuspidal curve K. Up to projective equivalence, we can assume that the singular point on the line is $P = [0:0:0:1]$ and that the cubic in $x_0 = s^2 x_1$ passes through P twice for $s = 0$, and through $[0:0:1:0]$ for $s = \infty$. This means setting the following coefficients equal to zero:

$$a_{1003},\ a_{0103};\ a_{0112},\ a_{1030}.$$

On the other hand, a_{0130} and a_{1012} must be non-zero in order to prevent ℓ from having degree 1; hence, we can normalize both of them to 1. In what follows, this will be the standard parametrization for lines of degree 2.

The following lemma holds also for elliptic lines.

Lemma 6.2.12. *If ℓ is an inseparable line of valency $v(\ell) > 12$, then ℓ is cuspidal.*

Proof. The line ℓ is inseparable if and only if $a_{1021} = a_{0121} = 0$. The smooth point of intersection of the residual cubic E_t in $x_0 = s^2 x_1$ with ℓ is given by

$$P_t = [0:0:s:1].$$

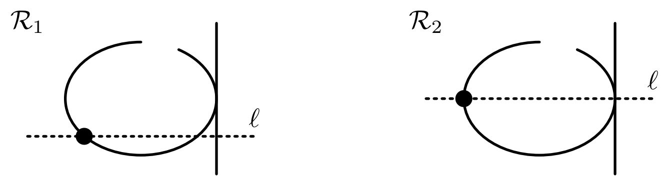
One can see explicitly that P_s is a singular point of E_s if and only if s is a root of the following degree 6 polynomial:

$$\begin{aligned}\varphi(s) := a_{2020}s^6 + a_{2011}s^5 + a_{1120}s^4 + a_{2002}s^4 \\ + a_{1111}s^3 + a_{0220}s^2 + a_{1102}s^2 + a_{0211}s + a_{0202}.\end{aligned} \tag{6.5}$$

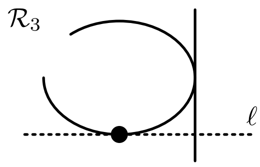
Furthermore, it can be checked by a local computation that if E_s splits off a line, then s is a root of $\varphi(s)$. Since there can be at most 2 lines through P_s, this implies that the valency of ℓ is not greater than $2 \cdot 6 = 12$, unless the polynomial φ vanishes identically, but $\varphi \equiv 0$ implies that all points P_s are singular for E_s, i.e., the line ℓ is cuspidal. □

Let ℓ be a quasi-elliptic line of degree 2, K the cuspidal curve of the fibration induced by ℓ, and $k = K \cdot H$ the degree of K. Considering the pullback of a plane containing ℓ, we can write $H = F + L + E$, where $E = \sum n_i E_i$ has support on the exceptional divisors coming from the singular point on ℓ. We will denote by P the point of intersection of E and ℓ. The following lemma will help us determine the coefficients n_i.

Lemma 6.2.13. *If ℓ is a quasi-elliptic line of degree $d = 2$ and valency $v > 10$, then it is contained in a plane with a configuration $\mathcal{R}_1$ or a configuration $\mathcal{R}_2$.*



Proof. Suppose there are no residual cubics as in the picture. Suppose that F is a fiber of ℓ of type III with $v(F) > 0$. Its residual cubic must split into a line and an irreducible conic; since its residual cubic cannot be of type $\mathcal{R}_1$ or $\mathcal{R}_2$, it



must have configuration $\mathcal{R}_3$ (which can appear only once, since the intersection number of the general residual cubic with ℓ at the singular point is equal to 1).

On the other hand, reducible fibers F not of type III with $v(F) > 0$ must have $e(F) \geq 6$ (because ℓ is quasi-elliptic) and $v(F) \leq 2$ (because ℓ has degree 2). Therefore, there can be at most 5 of them (4, if configuration $\mathcal{R}_3$ appears) and $v(\ell)$ can be at most 10. □

Lemma 6.2.14. *Let ℓ be a separable quasi-elliptic line of degree 2 contained in a plane with one of the residual cubics as in Lemma 6.2.13. If Q is the point of ramification on ℓ, then only the following cases are possible:*

(i) $k = 2$, $K \cdot E = 0$, $Q \notin K$, $Q \neq P$;
(ii) $k = 2$, $K \cdot E = 0$, $Q \notin K$, $Q = P$;
(iii) $k = 3$, $K \cdot E = 0$, $Q \in K$, $Q \neq P$;
(iv) $k = 3$, $K \cdot E = 1$, $Q \notin K$, $Q \neq P$;
(v) $k = 4$, $K \cdot E = 1$, $Q \in K$, $Q \neq P$;
(vi) $k = 4$, $K \cdot E = 1$, $Q \in K$, $Q = P$.

Proof. Since K is not a component of H, $K \cdot H \geq K \cdot F = 2$. Considering a residual cubic as in Lemma 6.2.13, it is clear that the coefficients n_i in E must be equal to 1, since the plane must correspond to a fiber of type I_n for the other line. Therefore we can write $E = E_0 + E_1 + \ldots + E_{n-1}$, where E_0 is a section and the other E_i's are fiber components (necessarily of the same fiber). Therefore, $E \cdot K \leq 1$.

Moreover, by a local computation one can see that $K \cdot L \leq 1$. In fact, K can intersect ℓ only in the point of ramification Q. The local computation is needed to rule out that K might be tangent to ℓ in Q. It follows that $K \cdot L \leq 1$ and the only possible cases are those listed. In fact, if P coincides with Q, then $K \cdot E = 1$. □

Proposition 6.2.15. *If ℓ is a quasi-elliptic line of degree 2 and valency $v(\ell) > 13$, then ℓ is cuspidal.*

Proof. Thanks to Lemma 6.2.12, we can assume that ℓ is separable. We claim that a separable quasi-elliptic line of degree 2 is always of the first kind, whence the bound on the valency follows.

In order to prove this, we parametrize such lines according to the possible values of the degree k of the cuspidal curve K. Assume that the image of K in $\mathbb{P}^3$ is parametrized as in (6.2). By Lemma 6.2.13, we can apply Lemma 6.2.14. According to the cases described there, we have the following conditions, after normalization:

(i) $k = 2$, $a_{1021} = b_0 = c_0 = b_2 = c_2 = 0$, $a_0 = 1$;
(ii) $k = 2$, $a_{0121} = b_0 = c_0 = b_2 = c_2 = 0$, $a_0 = 1$;
(iii) $k = 3$, $a_{1021} = b_0 = c_0 = a_1 = c_3 = 0$, $a_0 = 1$, $b_3 \neq 0$;
(iv) $k = 3$, $a_{0121} = a_0 = b_0 = b_3 = c_3 = 0$, $a_1 = 1$, $c_0 \neq 0$;

(v) $k = 4$, $a_{1021} = a_0 = b_0 = a_2 = c_4 = 0$, $a_1 = 1$, $c_0 \neq 0$, $b_4 \neq 0$;
(vi) $k = 4$, $a_{0121} = a_0 = a_1 = b_0 = b_4 = c_4 = 0$, $a_2 = 1$, $c_0 \neq 0$.

In fact, we can always choose $\psi(0)$ to be either $[0 : 0 : 0 : 1]$ or $[0 : 1 : 0 : 0]$, and $\psi(\infty)$ to be either $[0 : 0 : 1 : 0]$ or $[1 : 0 : 0 : 0]$.

We impose that $\psi(s)$ is indeed a singular point of the residual cubic in the plane $x_0 = s^2 x_1$ for every $s \in \mathbb{P}^1$. It turns out that we can always express the following coefficients in terms of the others:

$$a_{0211}, a_{1111}, a_{2011}; a_{0400}, a_{1300}, a_{2200}, a_{3100}, a_{4000};$$
$$a_{0310}, a_{1210}, a_{2110}, a_{3010}; a_{0301}, a_{1201}, a_{2101}, a_{3001}.$$

(The first three turn out to be always equal to zero). In all cases one can verify that if in addition the conditions for being a line of the second kind are also satisfied, then all points on K are singular, which contradicts the fact that X only admits isolated singularities. □

In other words, separable quasi-elliptic lines of degree 2 have valency at most 13. We do not know if this bound is sharp, but the following example shows that we are very close to it.

Example 6.2.16. The following surface contains a separable quasi-elliptic line $\ell : x_0 = x_1 = 0$ of degree 2 and valency 12:

$$X : x_0^3 x_1 + x_0^2 x_1^2 + x_0 x_1^3 + x_0^2 x_1 x_2 + x_0 x_1^2 x_2 + x_0^2 x_2^2 + x_0 x_1 x_2^2 + x_1 x_2^3 + x_0 x_1^2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_0 x_2 x_3^2 = 0.$$

Two lines meeting ℓ are contained in the plane $x_1 = x_0$, while other 10 lines are contained in $x_1 = t x_0$, where t is a root of

$$t^{10} + t^8 + t^5 + t^4 + 1.$$

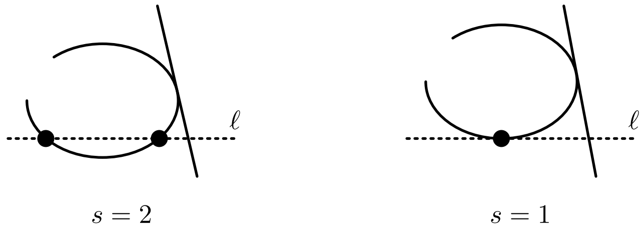
6.2.3 Quasi-elliptic lines of degree 1

In order to study quasi-elliptic lines of degree $d = 1$, we will need to find a bound on the degree $k = K \cdot H$ of the cuspidal curve K. Considering the pullback of a general plane containing ℓ, one can write the hyperplane divisor as

$$H = F + L + \sum n_i E_i$$

where F is a general fiber, L is the strict transform of ℓ and the sum goes over the exceptional divisors coming from the singular points on ℓ. Note that ℓ is a section of the quasi-elliptic fibration, so $K \cdot L = 0$. According to Lemma 3.2.4, for each singular point there is an exceptional divisor E_0 which is a section (hence $K \cdot E_0 = 0$), while the others are fiber components (hence $K \cdot E_i \leq 1$ and equality holds for at most two E_i's per fiber). The following lemma will be useful to determine the coefficients n_i.

Lemma 6.2.17. *If ℓ is a line of degree $d = 1$ and valency $v \geq 5$ and singularity s, then it is contained in a plane with one of the following residual cubics:*



Proof. If F is a fiber of type III with $v_\ell(F) = 1$ (the maximum possible since $d = 1$), then its residual cubic contains a line meeting ℓ in a smooth point and an irreducible conic, i.e., it is one of the residual cubics as in the figure. Without fibers of type III and $v(F) = 1$, ℓ cannot have valency greater than 5: in fact, all other fibers either do not contribute to the valency of ℓ or have Euler–Poincaré characteristic ≥ 6 and local valency 1. □

Lemma 6.2.18. *If a quasi-elliptic line ℓ is contained in a plane with a residual cubic as in Lemma 6.2.17, then its cuspidal curve has degree at least 2 and at most 4.*

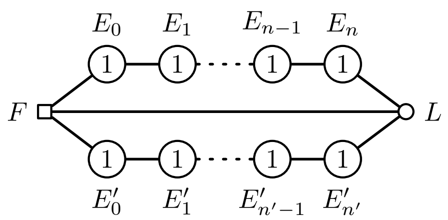
Proof. Let us call ℓ' and Q the line in the residual cubic on the plane Π given by Lemma 6.2.17.

Suppose first that ℓ has two singular points P and P'. Let $H = F + L + \sum n_i E_i + \sum n_i' E_i'$ be the pullback of a general plane containing ℓ, where F is a general fiber of the fibration induced by ℓ and the E_i's (resp. E_i''s) come from the resolution of P (resp. P'). We let E_0 and E_0' be the sections of π (Lemma 3.2.4).

Since the plane Π corresponds to a fiber of type I_N for the line ℓ', the points P and P' must be of type $\mathbf{A}_n$ and $\mathbf{A}_{n'}$ respectively (with $N = n + n' + 2$); moreover, pulling back Π we see that the coefficients of the E_i's and E_i''s must be equal to 1, so

$$H = F + L + \sum_{i=0}^{n} E_i + \sum_{i=0}^{n'} E_i',$$

and we have the following diagram (curves of genus 0 are marked with a circle, curves of genus 1 with a square).

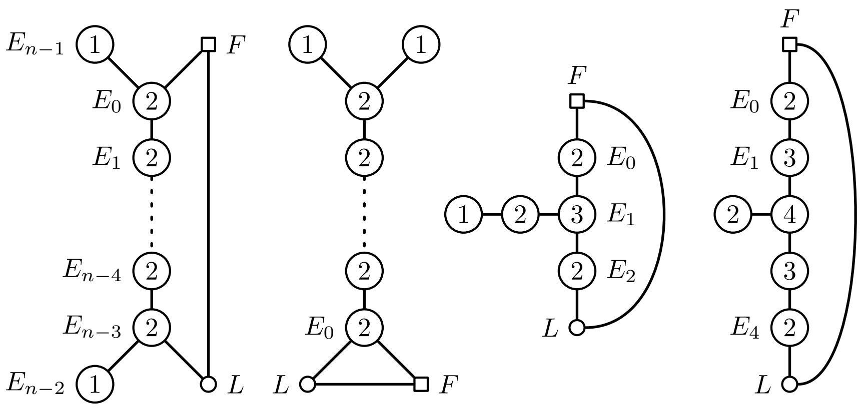


The divisors $E_1, \dots, E_{n-1}$ and $E_1', \dots, E_{n-1}'$ are components of two distinct fibers for ℓ, and the cuspidal curve K can meet at most one of them in each fiber, so

$$K \cdot H = K \cdot F + K \cdot L + K \cdot \sum E_i + K \cdot \sum E_i' \leq 2 + 0 + 1 + 1 = 4.$$

Suppose now that ℓ has only one singular point P (necessarily not of type $\mathbf{A}_1$, by explicit computation of the tangent cone).

The pullback of Π gives $H = F + L + \sum n_i E_i$. The strict transforms L and $\hat{Q}$ and the divisors E_i's make up a fiber of the fibration induced by ℓ' which has at least two simple components (L and $\hat{Q}$) and at least four components; hence, it must be of type I^*_{n+1}, IV^* or III^*. In the former case, we can have two different configurations, according to whether P is of type $\mathbf{A}_n$ or $\mathbf{D}_n$, while in the latter two cases P is of type $\mathbf{D}_5$ resp. $\mathbf{E}_6$. We have the following diagrams, where E_0 always denote the exceptional divisor which is a section for the fibration of ℓ (Lemma 3.2.4).



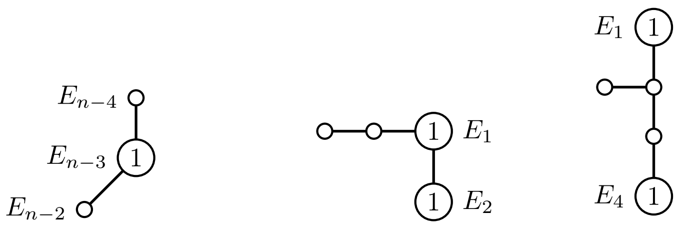
(a) (b) (c) (d)

Claim 6.2.19. The cases (a) with $n \geq 5$, (c) and (d) are not possible.

Proof of the claim. In these cases the exceptional divisors $E_1, \ldots, E_m$, with $m = n-2$, 4 resp. 5, are components of a fiber for ℓ; we write this fiber as

$$F = \sum_{i=1}^{m} r_i E_i + F',$$

where F' denotes the pullback of the residual cubic. Since E_0 and L are sections, the E_i's intersecting them must have multiplity $r_i = 1$. But then we find a contradiction, since no quasi-elliptic fiber can have the following sub-configurations of divisors with multiplicities.



(a), $n \geq 5$ (c) (d)

□

In the few cases left, the multiplicities n_i are not greater than 2.

(i) In case P of type $\mathbf{D}_n$, the E_i's are part of the same fiber for ℓ, so

$$K \cdot \sum n_i E_i \leq 2.$$

(ii) In case P of type $\mathbf{A}_3$, $n_1 = n_2 = 1$, so $K \cdot (E_1 + E_2) \leq 2$.

(iii) Finally, in case P of type $\mathbf{A}_4$, let

$$F = n_1 E_1 + n_2 E_2 + F'$$

be the fiber containing E_1; since E_1 intersects the section E_0, $n_1 = 1$ and since F has more than 3 components, K does not intersect simple components, so $K \cdot E_1 = 0$. Thus,

$$K \cdot \sum n_i E_i = K \cdot (E_2 + E_3) \leq 2. \qquad \square$$

Proposition 6.2.20. *If ℓ is a quasi-elliptic line of degree* 1 *and singularity* 2*, then $v(\ell) \leq 8$.*

Proof. Up to coordinate change, we can suppose that the two singular points on the line ℓ are $P = [0:0:0:1]$ and $P' = [0:0:1:0]$. Moreover, we can assume that the residual cubic in $x_0 = t^2 x_1$ intersects ℓ twice in P for $t = 0$, and twice in P' for $t = \infty$. This means that the following coefficients can be set equal to zero:

$$a_{1003},\ a_{0103};\ a_{1030},\ a_{0130};\ a_{0112},\ a_{1021};$$

whereas a_{0121} and a_{1012} must be non-zero and can be set equal to 1.

Suppose that the image of the cuspidal curve K in $\mathbb{P}^3$ is parametrized by ψ as in (6.2). By Lemmas 6.2.17 and 6.2.18, K can have degree (i) $k = 2$, (ii) $k = 3$, or (iii) $k = 4$. It follows from the proof of the latter lemma that these cases happen exactly when the image of K goes (i) neither through P nor P', (ii) through exactly one of them (say, P), or (iii) through both of them. When $\psi(s)$ is not equal to P or P' for $s = 0$ or $s = \infty$, then up to a further coordinate change we can suppose that $\psi(0) = [0:1:0:0]$ or $\psi(\infty) = [1:0:0:0]$. Thus, we have the following conditions, after normalization:

(i) $k = 2$, $b_0 = c_0 = b_2 = c_2 = 0$, $a_0 = 1$;

(ii) $k = 3$, $a_0 = b_0 = b_3 = c_3 = 0$, $a_1 = 1$;

(iii) $k = 4$, $a_0 = b_0 = a_2 = c_4 = 0$, $a_1 = 1$.

We then impose that the point $\psi(s)$ is in the zero locus of the derivatives of the residual cubics (parametrized by $x_0 = s^2 x_1$) for all $t \in \mathbb{P}^1$, finding conditions on the coefficients $a_{i_0 i_1 i_2 i_3}$.

We observe that each residual cubic which contributes to the valency of ℓ must split off a line m' passing through $\psi(s)$ and the point $[0:0:s^2:1]$ on the line ℓ. The equations of m' can be explicitly found (one of them is $x_0 = s^2 x_1$ and the other is of the form $ax_1 + bx_2 + cx_3 = 0$) and imposing that X contains m' yields a polynomial in s, which is generically of degree $12 - 2\,k$ by an explicit computation. This polynomial cannot be the zero polynomial, otherwise all points on the cuspidal curve would be singular, contrary to the fact that there are only isolated singularities on X. Therefore,

$$v(\ell) \leq 12 - 2\,k \leq 8. \qquad \square$$

Proposition 6.2.21. *If ℓ is a quasi-elliptic line of degree* 1 *and singularity* 1*, then* $v(\ell) \leq 12$.

Proof. By Lemma 6.2.17, we can suppose that there exists a residual cubic splitting into a line m and an irreducible conic tangent to m and tangent to ℓ in its singular point P. Up to a change of coordinates, we can suppose that $P = [0 : 0 : 0 : 1]$, the line m is given by $x_1 = x_3 = 0$, and the point of intersection of m and Q is $[1 : 0 : 0 : 0]$ (so that the plane Π is given by $x_1 = 0$). Moreover, we impose that ℓ is of degree 1. All in all, this amounts to setting the following coefficients equal to zero in (3.1):

$$a_{0103},\, a_{0112},\, a_{1003},\, a_{1012};\, a_{4000},\, a_{3010},\, a_{2020},\, a_{1030};\, a_{3001},\, a_{2011}.$$

On the other hand, to avoid contradictions such as ℓ having degree 0 or the point $[1 : 0 : 0 : 0]$ being singular, the following coefficients must be non-zero:

$$a_{1021},\, a_{0130};\, a_{3100}.$$

We then suppose that the degree of the cuspidal curve K is k, so that the image of K in $\mathbb{P}^3$ is parametrized by (6.2), and we impose that the coordinates of $\psi(s)$ satisfy the equations of the derivatives of the residual cubics relative to ℓ (parametrised by $x_0 = s^2 x_1$). We divide our analysis according to the value of k which, by virtue of Lemma 6.2.18, can be 2, 3 or 4. When $k > 2$ the curve K must necessarily meet at least one exceptional divisor E_i coming from P, so $\psi(t_0) = [0 : 0 : 0 : 1]$ for some t_0, and up to change of coordinates we can suppose that $t_0 = 0$. We can also normalize one of the a_i's, once we know that it is non-zero. We divide the computations according to the following cases:

(i) $k = 2$, $a_0 = 1$;
(ii) $k = 3$, $a_0 = 0$, $a_1 = 1$;
(iii) $k = 4$, $a_0 = 0$, $a_1 = 1$;
(iv) $k = 4$, $a_0 = 0$, $a_1 = 0$, $a_2 = 1$.

In each case, the choice of the following coefficients is unique (the first three turn out to be always equal to zero):

$$a_{0211}, a_{1111}, a_{2011}; a_{0400}, a_{1300}, a_{2200}, a_{3100}, a_{4000};$$
$$a_{0310}, a_{1210}, a_{2110}, a_{3010}; a_{0301}, a_{1201}, a_{2101}, a_{3001}.$$

We observe that if the residual cubic of a fiber contributing to the valency splits into a line m' and an irreducible conic, then m' passes through $\psi(s)$ and the point $[0 : 0 : a_{1021}s^2 : a_{0130}]$ on the line ℓ. The equations of m' can be explicitly found (one of them is $x_0 = s^2 x_1$ and the other is of the form $ax_1 + bx_2 + cx_3 = 0$) and imposing that X contains m' yields a polynomial in s, which is generically of degree $k + 5$. This polynomial cannot be the zero polynomial, otherwise all points on the cuspidal curve would be singular, but there are only isolated singularities on X. Other residual cubics either do not contribute to the valency of ℓ or are singular in P, in which case they must be contained in the tangent cone of P (since P is not of type $\mathbf{A}_1$, there can be at most two of them). Taking into account also the line m, we get that

$$v(\ell) \leq (k + 5) + 2 + 1 \leq 12.$$ □

6.3 Proof of Theorem 6.0.1

6.3.1 Triangle case

Proposition 6.3.1. *If X contains a plane with a triangle and a singular point, then $\Phi(X) \leq 63$.*

Proof. By Lemma 3.4.6, we can suppose that X contains one of the configurations listed in Figure 3.4.1, except $\mathcal{A}_0$, $\mathcal{B}_0$ or $\mathcal{C}_0$ because they do not contain a singular point. In the configurations $\mathcal{A}_i$ and $\mathcal{B}_i$, $i = 1, 2, 3$, all lines must be elliptic (because the corresponding fiber is of type I_n or IV) and those of singularity 0 cannot have valency higher than 18 (since there cannot be an automorphism of degree 3 exchanging the other three lines). Hence, for each line of the configuration we consider the following bounds on its valency v according to its singularity s: if $s = 0$, $v \leq 18$; if $s = 1$, $v \leq 13$; if $s = 2$, $v \leq 11$; if $s = 3$, $v \leq 2$. Moreover, we use the fact that there can be at most 8 lines going through each singular point (Lemma 3.2.2). For configurations $\mathcal{D}_0$ and $\mathcal{E}_0$ we use the bound $v \leq 19$ for the simple lines, which might be quasi-elliptic. We obtain the following bounds on the total number of lines N:

- configurations $\mathcal{A}_1$ and $\mathcal{B}_1$: $N \leq 62$;
- configurations $\mathcal{A}_2$ and $\mathcal{B}_2$: $N \leq 63$;
- configurations $\mathcal{A}_3$ and $\mathcal{B}_3$: $N \leq 57$;
- configurations $\mathcal{D}_0$ and $\mathcal{E}_0$: $N \leq 58$. □

We can now prove Theorem 6.0.1 in the triangle case.

Proposition 6.3.2. *If X has a triangle, then $\Phi(X) \leq 68$.*

Proof. By Lemma 3.4.6 and Proposition 6.3.1 we can assume that X contains a plane Π with four distinct lines and no singular point (configurations $\mathcal{A}_0$, $\mathcal{B}_0$ or $\mathcal{C}_0$). If all lines have valency less or equal than 19, then X has at most $4 \cdot (19 - 3) + 4 = 68$ lines.

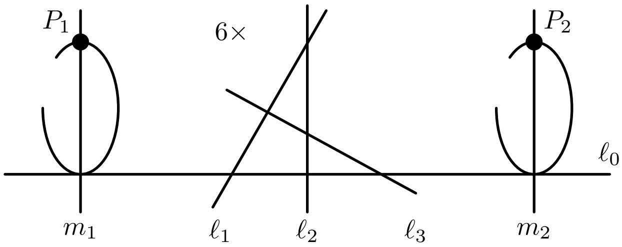
Suppose that X has a line ℓ_0 of valency 20. Then ℓ_0 is a special line of type $(6, 2)$, by Tables 6.1.1 and 6.2.1 and Proposition 6.1.3, and the 1-fibers are the only ramified fibers. Moreover, the six 3-fibers do not contain any singular point, since if one of them did, then on account of the automorphism σ induced by ℓ_0 it would have Euler–Poincaré characteristic at least 6 and $e(Z)$ would exceed 24.

Let ℓ_i, $i = 1, 2, 3$, be the three other lines on Π (which must be one of the unramified 3-fibers), let m_1 and m_2 be the lines in the 1-fibers, and P_i the points on m_i which sit on the residual conic but not on ℓ_0 ($i = 1, 2$).

The lines ℓ_i have the same valency, which must be greater than 18, or else X has at most $(20-3)+3\cdot(18-3)+4 = 66$ lines; moreover, they cannot be quasi-elliptic because they have an I_3-fiber. Therefore, they induce an automorphism of degree 3, whence they must have the same fibration of ℓ_0; in particular, their 3-fibers do not contain singular points and they have valency 20.

Claim 6.3.3. The points P_i are singular and the lines m_i are cuspidal.

Proof of the claim. The lines ℓ_i also have fibration $(6, 2)$. Let n be a line in a 3-fiber of one of the ℓ_i which meets m_1: there are 15 of them. By the same argument as for the ℓ_i's, n must also be special lines of valency 20. Note that,



regardless of P_1 being singular or not, they must meet m_1 in a point different from P_1.

It follows that m_1 must belong to a 1-fiber of n, because it cannot have valency 20: therefore, m_1 has 16 fibers of type III with ramification of order 2. By the Riemann–Hurwitz formula, m_1 must be inseparable, so P_1 must be singular. Recalling Lemma 6.2.12, we conclude that m_1 is cuspidal. The same reasoning applies to P_2 and m_2. □

The points P_i must be of type $\mathbf{A}_1$, otherwise the Euler–Poincaré characteristic of X would be greater than 24. Let now C be the residual cubic contained in the plane on which both m_1 and P_2 lie. We claim that C can be neither irreducible nor reducible, thus finding a contradiction.

In fact, if C is irreducible, then C must have a cusp in P_2, but this is impossible, since the cuspidal curve of the fibration induced by m_1 is the strict transform of m_1. On the other hand, if C is reducible, then it has 2 or 3 components and, since P_2 is of type $\mathbf{A}_1$, the corresponding fiber has 3 or 4 components, but there do not exist quasi-elliptic fibers with 3 or 4 components. □

6.3.2 Square case

We employ here the technique of the dual graph of lines Γ as in Section 4.3. Here, as $\operatorname{char} \mathbb{K} = 2$, a parabolic subgraph $D \subset \Gamma$ might induce an elliptic or a quasi-elliptic fibration.

Lemma 6.3.4. *If D induces an elliptic fibration, then*

$$\Phi(X) \leq v(D) + 24.$$

Proof. The same proof as in Proposition 4.3.1 applies. □

Lemma 6.3.5. *Let X be a triangle-free K3 quartic surface X with a square. If all lines have valency at most 13, then X contains at most 68 lines.*

Proof. The square D induces an elliptic fibration because quasi-elliptic fibration cannot have fibers of type I_4. Hence,

$$\Phi(X) \leq v(D) + 24 \leq 4 \cdot (13 - 2) + 24 = 68. \qquad \square$$

It is therefore important to classify all lines of valency greater than 13 on triangle-free surfaces. By Lemma 3.4.5, all such lines must be quasi-elliptic.

Proposition 6.3.6. *If X is a triangle free surface and admits a completely reducible plane, then $\Phi(X) \leq 68$.*

Proof. The proof is a case-by-case analysis on the configurations of Figures 3.4.2 and 3.4.3 given by Lemma 3.4.7. We use the bound of Lemma 3.4.5 for elliptic lines and those of Table 6.2.1. Moreover, we use the fact that there are at most 8 lines through a singular point (Lemma 3.2.2).

We have to refine our argument only for configurations $\mathcal{A}_8$ and $\mathcal{C}_1$.

- In configuration $\mathcal{A}_8$, all lines are elliptic, have degree 1 and singularity 2. Since the plane corresponds to an I_n-fiber, with $n \geq 5$, they can be met by at most 9 other lines in other planes. It follows that X contains at most $4 \cdot 9 + 4 \cdot (8-2) + 4 = 64$ lines.
- In configuration $\mathcal{C}_1$, if one of the lines is quasi-elliptic, then the plane corresponds to a fiber of type I_0^*; by an Euler–Poincaré characteristic argument, the valency of the lines is not greater than 16, so X contains at most $4 \cdot 16 + 4 = 68$ lines.

Note that $\mathcal{C}_1$ is the only configurations where 68 can be reached. ☐

Now that we have ruled out completely reducible planes, it will be easier to classify lines with valency greater than 13. Since X is triangle-free, such lines must be quasi-elliptic of degree 3 or 2.

Lemma 6.3.7. *If $\ell \subset X$ is a quasi-elliptic line of degree 3 and valency $v > 13$ on a surface without completely reducible planes, then it has one of the fibrations listed in Table 6.3.2.*

Proof. Since X does not have completely reducible planes, a residual cubic of ℓ can be either irreducible or split into a line and a conic. In the former case, it must have a cusp, which might be a singular point of the surface; in the latter case, the line and the conic must be tangent, and their intersection point might be a singular point of the surface. We can have the following possibilities for reducible fibers (the extra subscript number denotes the local valency, while a and b distinguish the two possibilities for $\mathrm{I}_{2n,1}^*$):

III_0 : cusp with a point of type $\mathbf{A}_1$;
III_1 : line and conic with a smooth intersection point;
$\mathrm{I}_{2n,0}^*$: cusp with a point of type $\mathbf{D}_{2n+4}$;
$\mathrm{I}_{2n,1}^{*a}$: line and conic with a point of type $\mathbf{A}_{2n+3}$, $n \geq 0$;
$\mathrm{I}_{2n,1}^{*b}$: line and conic with a point of type $\mathbf{D}_{2n+3}$, $n \geq 1$;
III_0^* : cusp with a point of type $\mathbf{E}_7$;
III_1^* : line and conic with a point of type $\mathbf{E}_6$;
II_0^* : cusp with a point of type $\mathbf{E}_8$.

We then make a list of the fibrations that lead to $v(\ell) > 13$, imitating the arguments of Proposition 6.2.7. The results are shown in Table 6.3.1. There are 14 cases; in two of them (cases 4 and 9) one has to distinguish the type of the fibers $\mathrm{I}_{2n,1}^{*a}$ and $\mathrm{I}_{2n,1}^{*b}$.

We then explicitly compute the rank the intersection matrix of these 14 cases, taking into account the general fiber F, the line L, the fiber components and the cuspidal curve K. The only unknown intersection number is $K \cdot L$, but by Lemma 6.2.4 this can only be 0, 1 or 2. If the rank is always greater than 22, the fibration is discarded. The cases that pass this test are those listed in Table 6.3.2 (all of them with $K \cdot L = 0$). ☐

Table 6.3.1: Candidates to the fiber configuration of a quasi-elliptic line ℓ of degree 3 on a surface without completely reducible planes with $13 < v(\ell) \leq 16$.

case	iii_1^*	$i_{2,1}^{*a,b}$	$i_{2,0}^*$	$i_{0,1}^*$	$i_{0,0}^*$	iii_1	iii_0	$v(\ell)$
1				1		15	1	16
2					1	16	1	16
3						16	4	16
$4^{a,b}$		1				14		15
5				1		14	2	15
6					1	15	1	15
7						15	5	15
8	1					13		14
$9^{a,b}$		1				13	1	14
10			1			14		14
11				2		12		14
12				1		13	3	14
13					1	14	2	14
14						14	6	14

Table 6.3.2: Fiber configurations from Table 6.3.1 for a quasi-elliptic line ℓ of degree 3 with $13 < v(\ell) \leq 16$ which generate a lattice of rank ≤ 22.

case	iii_1^*	$i_{2,1}^*$	$i_{0,1}^*$	$i_{0,0}^*$	iii_1	iii_0	$v(\ell)$	$\mathrm{Sing}(X)$
2				1	16	1	16	$\mathbf{D}_4, \mathbf{A}_1$
3					16	4	16	$4\mathbf{A}_1$
4^b		1			14		15	$\mathbf{D}_5$
5			1		14	2	15	$\mathbf{A}_3, 2\mathbf{A}_1$
8	1				13		14	$\mathbf{E}_6$
9^a		1			13	1	14	$\mathbf{A}_5, \mathbf{A}_1$
11			2		12		14	$2\mathbf{A}_3$

Lemma 6.3.8. *If ℓ is a quasi-elliptic line of degree 2 and valency $v > 13$ on a surface without completely reducible planes, then it has exactly 19 fibers of type* III *with valency 1 and the surface contains only one singular point of type $\mathbf{A}_2$.*

Proof. By Lemma 6.2.12, ℓ is cuspidal and we can assume that it admits the second configuration of Lemma 6.2.13. Therefore, the singular point P on ℓ is of type $\mathbf{A}_n$. Let F_0 be the only fiber whose residual cubic intersects ℓ only in P, and let F be a reducible fiber different from F_0. The residual cubic of F cannot be irreducible (because its cusp is on ℓ, which does not have other singular points, and thus F would be of type II), and it cannot split into three lines by hypothesis. It follows that F is the union of a line and an irreducible conic, necessarily meeting tangentially at a point of ℓ; hence, F is of type III and has valency 1.

On the other hand, the fiber F_0 is a reducible fiber: in fact, if it were not, then ℓ would have 20 fibers of type III and valency 20, but this is excluded by Proposition 6.2.10. The residual cubic C_0 of F_0 is irreducible (necessarily, with a cusp in P): indeed, it can never be the union of an irreducible conic and a line, because it would result in a fiber of type I_n, and by hypothesis it cannot split into three lines. It follows that the strict transform of C_0 is a simple component of F_0, and the remaining components are supplied by the exceptional divisors coming from P. Since P is of type $\mathbf{A}_n$, this is only possible if F is of type III. By Lemma 3.2.4, P is of type $\mathbf{A}_2$ (one exceptional irreducible divisor is a fiber component, the other one a section) and an Euler–Poincaré characteristic argument yields that there are 19 other III-fibers. □

Proposition 6.3.9. *Let X be a K3 quartic surface without completely reducible planes. If X contains a line ℓ of valency $v > 13$, then $\Phi(X) \leq 64$.*

Proof. The proof is to be done case by case according to the possible fibrations of ℓ given by Lemma 6.3.7 and Lemma 6.3.8. The fibrations univocally determine the singular locus of X (for $d = 3$, we refer to Table 6.3.2) and are mutually exclusive, in the sense that the same surface cannot have two lines of valency greater than 13 with different fibrations. We do one case as example; one can argue analogously for the other cases.

Suppose that X contains a line falling in case 2 of Table 6.3.2. The surface X has then two singular points of type $\mathbf{D}_4$ and $\mathbf{A}_1$, ℓ is of degree 3 and $v(\ell) = 16$. Consider the set $S(\ell)$ of lines that do not meet ℓ (and are therefore sections of its fibration). The number of lines on X is not greater than

$$\#S(\ell) + v(\ell) + 1 = \#S(\ell) + 17.$$

If all lines in $S(\ell)$ pass through the singular points, then there can be at most 16 of them. Suppose then that $s \in S(\ell)$ does not go through the singular points. Then, by inspection of the intersection matrix, s must meet exactly 8 lines contained in the III_1-fibers of ℓ. We choose 2 of these 8 lines, say m_1 and m_2, such that the corresponding fibers are not ramified for ℓ (this is possible since ramification occurs in at most two fibers).

Now, ℓ, m_1, m_2 and s form a square D. The lines m_1 and m_2 must be elliptic, since they have a fiber of type I_2. Since the valency of s cannot be greater than 16, the number of lines on X is not greater than

$$\begin{aligned} v(D) + 24 &= (v(m_1) - 2) + (v(m_2) - 2) + 8 + \\ &\quad + (v(s) - 2 - 8) + (v(\ell) - 2 - 8) + 24 \\ &\leq 10 + 10 + 8 + 6 + 6 + 24 = 64. \end{aligned}$$

□

Corollary 6.3.10. *If X is a triangle free surface containing a square, then $\Phi(X) \leq 68$.*

Proof. If X admits a completely reducible plane, then we can use Proposition 6.3.6. Otherwise, we conclude by Proposition 6.3.9 and Lemma 6.3.5. □

6.3.3 Square-free case

We prove an analog of Lemma 6.3.4 for quasi-elliptic fibration.

Lemma 6.3.11. *If $D \subset \Gamma$ induces a quasi-elliptic fibration, then*

$$\Phi(X) \leq v(D) + 25.$$

Proof. We observe that a fiber of type III cannot contain two lines, since two lines never intersect tangentially. Therefore, applying formula (6.3) twice, one gets

$$\begin{aligned}\Phi(X) &\leq v(D) + iii + \sum (5+n)\, i_n^* + 8\, iii^* + 9\, ii^* \\ &\leq v(D) + 20 + \left(\sum i_n^* + iii^* + ii^*\right) \\ &\leq v(D) + 25.\end{aligned}$$

□

Proposition 6.3.12. *If X is a square free K3 quartic surface, then $\Phi(X) \leq 55$.*

Proof. The proof can be copied word by word from Proposition 4.3.5, using both Lemma 6.3.4 and Lemma 6.3.11. □

Proof of Theorem 6.0.1. Having treated the triangle case (Proposition 6.3.2), the square case (Corollary 6.3.10) and the square free case (Proposition 6.3.12), the proof is complete. □

6.4 Rams–Schütt's family

The bound of Theorem 6.0.1 is sharp and is attained by all surfaces of the following family, as long as $\lambda \neq 0$:

$$\mathcal{X}_{68} : \lambda x_0 x_1^2 x_2 + x_1^4 + x_1 x_2^3 + x_0^3 x_3 + x_0 x_2 x_3^2 = 0$$

This family was found by Rams and Schütt [31] and differs by theirs only up to a change of coordinates.

A member X of family $\mathcal{X}_{68}$, for $\lambda \neq 0$, contains one singular point $P = [0 : 0 : 0 : 1]$ of type $\mathbf{A}_3$. The point P sits in a configuration $\mathcal{C}_1$ lying on the plane $x_0 = 0$. The four lines making up this configuration are cuspidal lines. The remaining 64 lines – including, for instance, the line $x_1 = x_3 = 0$ – are special lines of valency 19. The minimal resolution of X is a Shioda supersingular K3 surface of Artin invariant 2.

Theorem 6.4.1. *If $\Phi(X) = 68$, then X is projectively equivalent to a member of family $\mathcal{X}_{68}$.*

The rest of the section is dedicated to the proof of this theorem.

Lemma 6.4.2. *If $\Phi(X) = 68$ and X admits a configuration $\mathcal{C}_1$, then X is projectively equivalent to a member of family $\mathcal{X}_{68}$.*

Proof. We parametrize the surface in such a way that the configuration $\mathcal{C}_1$ sits in the plane $\Pi : x_0 = 0$ and that two lines in Π are given by $\ell_1 : x_0 = x_1 = 0$ and $\ell_2 : x_0 = x_2 = 0$. Let us call the other two lines ℓ_3 and ℓ_4.

Necessarily at least one of the lines in Π, say, ℓ_1, has valency greater than 16. Since ℓ_1 has degree lower than 3, it must be a cuspidal line. It follows that the plane Π represents a fiber of type IV* for ℓ_1, there are no other singular points on the surface and $v(\ell_1)$ is exactly 16. The same must hold for the other three lines in Π. Up to coordinate change, we can suppose that one of the lines meeting ℓ_1 is given by $\ell' : x_1 = x_3 = 0$.

Imposing that the lines $\ell_1, \dots, \ell_4$ are cuspidal, we obtain a quartic which is projectively equivalent to a member of $\mathcal{X}_{68}$. The following coefficients are different from 0 and can be normalized to 1:

$$a_{0220}, a_{1012}, a_{1102}, a_{3001}.$$

We first impose that ℓ_1 and ℓ_2 are cuspidal setting their polynomial φ identically equal to 0, as in Lemma 6.2.12, obtaining the following relations:

$$a_{1111} = a_{2011} = a_{2020} = a_{2101} = a_{2200} = 0,$$
$$a_{0130} = a_{0310} = a_{0220},$$
$$a_{1120} = a_{0130}a_{2002}, \quad a_{1210} = a_{0310}a_{2002}.$$

At this point, ℓ_3 and ℓ_4 are given by $x_0 = x_1^2 + x_1x_2 + x_2^2 = 0$ and imposing that they are cuspidal yields the following equation:

$$a_{2110} = a_{0220}a_{2002}^2.$$

Changing coordinates

$$[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_0 : x_1 : a_{2002}x_0 + x_2 : (a_{2002}^3 + a_{3100})x_1 + x_3],$$

we recover family $\mathcal{X}_{68}$. □

Proposition 6.4.3. *If $\Phi(X) = 68$ and X contains a triangle, then X is projectively equivalent to a member of family $\mathcal{X}_{68}$.*

Proof. Let Π be the plane containing a triangle. By virtue of Proposition 6.3.1, we can suppose that Π has no singular points. Let us call $\ell_0, \dots, \ell_3$ the lines on Π. At least two of them are special; since each special line induces an automorphism of order 3 which exchanges the other three, it follows that all four of them induce fibrations with the same singular fiber types and, in particular, the same valency, which must be equal to 19.

Claim 6.4.4. A 3-fiber of the lines ℓ_i, $i = 0, \dots, 3$, cannot contain singular points.

Proof of the claim. By the presence of the automorphism, the residual cubic of the fiber should be as in configuration $\mathcal{A}_4$ or $\mathcal{B}_4$. We can exclude both of them using formula (3.6) and the known bounds of Table 6.1.1 (the other three lines must necessarily be elliptic since they admit a fiber of type I_n). □

Let ℓ' be the line in the (ramified) 1-fiber of ℓ_0 and let P be the point of intersection of ℓ' with the residual conic C not on ℓ_0. Consider the 15 3-fibers of ℓ_1, ℓ_2 and ℓ_3 other than Π. In each of them, there is a line meeting ℓ': let us call these lines m_i, $i = 1, \dots, 15$. By the same argument as before, the lines m_i are special lines of valency 19.

We now distinguish the two cases, according to whether P is smooth or singular. Suppose first that P is smooth.

Claim 6.4.5. If P is smooth, then $v(\ell') = 16$.

Proof of the claim. Because of the automorphism induced by ℓ_0, the valency of ℓ' has the form $v(\ell') = 1 + 3\,a$; moreover, $v(\ell') \geq 16$ because ℓ' meets ℓ_0, $m_1, \ldots, m_{15}$, but $v(\ell') \leq 18$ because ℓ' is clearly of the first kind. The only possibility is then $v(\ell') = 16$. □

It follows that ℓ' does not belong to a 3-fiber of m_i because otherwise ℓ' should have valency 19; hence, ℓ' sits in the 1-fiber of each m_i (and of ℓ_0), and thus ℓ' has exactly 16 fibers of type III. Necessarily, ℓ' is quasi-elliptic, and all other reducible fibers must have an irreducible residual cubic. By an Euler–Poincaré characteristic argument, there can be two cases:

(i) either ℓ' has 4 more fibers of type III, whose residual cubics have a cusp which is a singular point of type $\mathbf{A}_1$;

(ii) or ℓ' has one fiber of type I_0^*, whose residual cubic has a cusp which is a singular point of type $\mathbf{D}_4$.

- In case (i), the line ℓ_0 has a ramified 1-fiber of type I_2, six 3-fibers without singular point (Claim 6.4.4), and no other 1- or 3-fiber. It follows that ℓ_0 has four more fibers $F_1, \ldots, F_4$ with irreducible residual cubics, each of them containing one of the four points of type $\mathbf{A}_1$ (in fact, an irreducible residual cubic can have at most one singular point of the surface). This means that $e(F_i) \geq 2$, so that

$$e(X) \geq 2 + 6 \cdot 3 + \sum_{i=1}^{4} e(F_i) \geq 28,$$

  which is impossible.
- Similarly, in case (ii), the line ℓ_0 would have one more fiber F containing the point of type $\mathbf{D}_4$, with $e(F) \geq 6$ (since F contains more than 5 components); we would then obtain

$$e(X) \geq 2 + 6 \cdot 3 + e(F) \geq 26,$$

  also impossible.

We can therefore suppose that P is a singular point of X. Consider now the lines ℓ_1', ℓ_2', ℓ_3' in the 1-fibers of ℓ_1, ℓ_2 and ℓ_3. By the same token, also the line ℓ_i' contains a singular point P_i, $i = 1, 2, 3$. If the four points P, P_1, P_2, P_3 are distinct, then ℓ_0 has a 1-fiber of type I_n, $n \geq 3$, six 3-fibers, and three fibers F_i containing P_i with $e(F_i) \geq 2$, $i = 1, 2, 3$, but this cannot be:

$$e(X) \geq 3 + 6 \cdot 3 + 3 \cdot 2 = 27.$$

Hence, the points P_i coincide with P, and $v(\ell') = 16$. Arguing as before, ℓ' has 16 fibers of type III and is quasi-elliptic. By Proposition 6.2.15, ℓ' is cuspidal, and so are ℓ_1', ℓ_2' and ℓ_3'; necessarily, they must lie in the same plane, forming a configuration $\mathcal{C}_1$. We can then conclude applying Lemma 6.4.2. □

Corollary 6.4.6. *If $\Phi(X) = 68$ and X admits a completely reducible plane, then X is projectively equivalent to a member of family $\mathcal{X}_{68}$.*

Proof. By Proposition 6.4.3, we can assume that X is triangle free. By inspection of the proof of Proposition 6.3.6, we see that X must admit a configuration $\mathcal{C}_1$, so we can apply Lemma 6.4.2. □

Proof of Theorem 6.4.1. By Corollary 6.4.6, we can suppose that X does not admit any completely reducible plane. We claim that this assumption leads to a contradiction.

By virtue of Proposition 6.3.12, we can suppose that X admits a square D formed, say, by the lines $\ell_1, \dots, \ell_4$. The square D induces an elliptic fibration $\pi : X \to \mathbb{P}^1$ because I_4 is not a quasi-elliptic fiber. On account of Proposition 6.3.9, all lines on X have valency ≤ 13; hence, by Lemma 6.3.4 we have

$$68 = \Phi(X) \leq 4 \cdot (13 - 2) + 24 = 68.$$

Since equality holds, we deduce two facts:
(i) $v(\ell_i) = 13$, $i = 1, 2, 3, 4$;
(ii) all the components of the singular fibers of the fibration π must be lines.
Let F be a general fiber of the fibration π. Since F is linearly equivalent to $L_1 + L_2 + L_3 + L_4$, we have $F \cdot H = 4$, where H is the hyperplane divisor. It follows from (ii) that all singular fibers are composed by 4 lines. Since the only fiber type with 4 components is I_4, the fibration π has necessarily 6 fibers of type I_4, i.e., 6 squares. Let us put $\ell_i^1 := \ell_i$ and call the other 20 lines ℓ_i^j, $i = 1, \dots, 4$, $j = 2, \dots, 6$. Arguing as before, we deduce that $v(\ell_i^j) = 13$ for every i, j.

Since there are no completely reducible planes, the lines ℓ_i^j have no 3-fibers; by Lemma 3.3.7 and Table 6.2.1, they must be quasi-elliptic of degree 3 or 2.

Claim 6.4.7. The surface X is not smooth.

Proof of the claim. If X were smooth, then ℓ_1 would induce a fibration with 20 fibers of type III formed by a line and an irreducible conic, so $v(\ell_1) = 20$, which contradicts $v(\ell_1) = 13$. □

Let E be an (irreducible) exceptional divisor coming from the resolution of a singular point P. Since E is not a component of a singular fiber of π, E is a multisection of π. On the other hand, E can only have positive intersection with one of the L_i, $i = 1, 2, 3, 4$, since the point P can sit only on one line ℓ_i, so E is actually a section. Up to index permutation, we can suppose then that $E \cdot L_1^j = 1$ for $j = 1, \dots, 6$, i.e., the point P belongs to the lines ℓ_1^j, $j = 1, \dots, 6$ (which have then degree 2).

Claim 6.4.8. The point P is of type $\mathbf{A}_1$.

Proof of the claim. If P is not of type $\mathbf{A}_1$, then there is another exceptional divisor E' coming from P, and we can suppose $E \cdot E' = 1$. Arguing as before, E' is also a section and since P is contained in ℓ_1^j, we have $E' \cdot L_1^j = 1$, too. But E and E' can intersect each L_1^j only in one point, namely the one mapping to P through the resolution, because ℓ_1^j has degree 2 for each j. Hence, E and E' have six different points in common, which is impossible since $E \cdot E' = 1$. □

Finally, let us consider a line in the square D different from ℓ_1 intersecting ℓ_1; up to renaming, we can suppose it is ℓ_2. Let Π be the plane containing both ℓ_1 and ℓ_2 and let F be the fiber corresponding to Π in the fibration induced by ℓ_2. The residual conic C in Π is irreducible since there are no completely reducible planes. The fiber F is composed of the exceptional divisors coming from P, and the strict transforms of ℓ_1 and C, that is to say, three components in total, since P is of type $\mathbf{A}_1$. On the other hand, ℓ_2 is a quasi-elliptic line and in characteristic 2 there are no quasi-elliptic fibers with three components: contradiction. □

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Curriculum vitae

Davide Cesare Veniani was born on 21 February 1988 in Como, Italy. He grew up in Asso and attended the *liceo scientifico* "Galileo Galilei" in Erba.

At the age of 19 he was admitted to the Collegio Ghislieri in Pavia as a member of the Scuola Universitaria Superiore IUSS. For the next five years, the Collegio provided a very stimulating environment for his studies of Mathematics at the Università degli Studi di Pavia.

In 2012 he participated in the Erasmus program at the Universitetet i Bergen, Norway. He earned his *laurea magistrale* under the supervision of Gian Pietro Pirola and Andreas Leopold Knutsen.

In 2013 Davide began to work as a PhD student under the supervision of Matthias Schütt at the Leibniz Universität Hannover, Germany, in the frame of the GRK 1463 "Analysis, Geometry and String Theory".

Thanks to this *Graduiertenkolleg*, he also had the great opportunity to spend two periods of time abroad in 2015, one with Alex Degtyarev at Bilkent Üniversitesi in Ankara, Turkey, and the other with Alessandra Sarti and Samuel Boissière at the Université de Poitiers, France.