H^∞-Calculus for Cone Pseudodifferential Operators and the Dirichlet to Neumann Map

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Abstract

We study the Dirichlet to Neumann operator $\mathcal{N}$ acting on distributions over a manifold $\mathbb{B}$ with conical singularities. $\mathbb{B}$ itself is the boundary of a singular manifold $\mathbb{D}$. $\mathcal{N}$ assigns to a given boundary datum f in the weighted cone Besov space $\mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\mathbb{B})$, the exterior normal derivative of the solution to the associated Dirichlet problem:

$$\Delta_c u = 0 \text{ in } \mathbb{D}, \qquad \gamma_0(u) = f \text{ on } \mathbb{B}. \tag{1}$$

Here, Δ_c denotes the Laplace Beltrami operator with respect to a conical metric g on $\mathbb{D}$. Hence, denoting by γ_0 the restriction to the boundary $\partial\mathbb{D} = \mathbb{B}$ of $\mathbb{D}$, and by D_n the derivative in the exterior normal direction on $\mathbb{D}$, $\mathcal{N}$ is defined by:

$$\begin{aligned} \mathcal{N} : \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\partial\mathbb{D}) &\to \mathcal{B}_p^{s-1-\frac{1}{p},-\frac{1}{2}}(\partial\mathbb{D}), \\ f &\mapsto \gamma_0(x^{-1}D_n u). \end{aligned} \tag{2}$$

We show that the solution to the Dirichlet problem exists, which shows that $\mathcal{N}$ is well defined. Further, we use the entries of the Calderon projector to construct $\mathcal{N}$ as an operator which we show to be contained in Schulze's cone calculus.

We show that $\mathcal{N}$ is parameter elliptic, which guarantees the existence of the resolvent $(\mathcal{N} - \lambda)^{-1}$.

We can identify operators in Schulze's cone algebra with operators of the b-calculus, which is developed by Melrose. Building on a resolvent construction for operators in the small b-calculus, we outline a generalization of this construction to operators of the full b-calculus in which $\mathcal{N}$ is contained.

This construction allows us to analyze the asymptotic structure of resolvents for cone pseudodifferential operators, which we use to to prove the existence of a bounded H^∞ functional calculus for this class.

Therefore, we can conclude that Dirichlet to Neumann map on manifolds with conical singularities admits a bounded H^∞ calculus.

Keywords: Dirichlet to Neumann Operator, Functional Calculus, Conical Singularities.

Zusammenfassung

Wir untersuchen die Wirkung des Dirichlet zu Neumann Operators $\mathcal{N}$ auf Distributionen über einer Mannigfaltigkeit $\mathbb{B}$ mit konischen Singularitäten. Hierbei ist $\mathbb{B}$ der Rand einer singulären Mannigfaltigkeit $\mathbb{D}$. $\mathcal{N}$ weist einem gegebenen Randwert f in einem gewichteten Konus Besov Raum $\mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\mathbb{B})$ die äußere Normalenableitung der Lösung des zugehörigen Dirichletproblems zu:

$$\Delta_c u = 0 \text{ in } \mathbb{D}, \qquad \gamma_0(u) = f \text{ auf } \mathbb{B}. \tag{3}$$

Hierbei bezeichnet Δ_c den Laplace Beltrami Operator unter der konischen Metrik g auf $\mathbb{D}$. Ist also γ_0 die Einschränkungsabbildung auf den Rand $\partial\mathbb{D} = \mathbb{B}$ von $\mathbb{D}$ und D_n die Ableitung in die äußere Normalenrichtung, so ist $\mathcal{N}$ definiert durch:

$$\begin{aligned} \mathcal{N} : \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\partial\mathbb{D}) &\rightarrow \mathcal{B}_p^{s-1-\frac{1}{p},-\frac{1}{2}}(\partial\mathbb{D}), \\ f &\mapsto \gamma_0(x^{-1}D_n u). \end{aligned} \tag{4}$$

Wir zeigen dass eine Lösung des Dirichlet Problems existiert was zeigt, dass $\mathcal{N}$ wohldefiniert ist. Ferner benutzen wir die Einträge des Calderonprojektors um $\mathcal{N}$ als einen Operator innerhalb von Schulze's Konusalgebra zu konstruieren.

Ferner zeigen wir dass $\mathcal{N}$ parameter elliptisch ist, was die Existenz einer Resolvente, also von $(\mathcal{N} - \lambda)^{-1}$, garantiert.

Wir können Operatoren aus Schulze's Konusalgebra mit Operatoren des b Kalküls identifizieren das von Melrose entwickelt wurde. Wir skizzieren eine Erweiterung einer existierenden Resolventenkonstruktion für das kleine b Kalkül auf das volle B Kalkül, in welchem $\mathcal{N}$ enthalten ist.

Diese Konstruktion erlaubt es uns die Struktur der Resolvente von Konuspseudodifferentialoperatoren zu untersuchen, was es uns erlaubt die Existenz eines beschränkten H^∞ Funktionalkalküls für diese Klasse zu beweisen.

Damit können wir folgern dass der Dirichlet zu Neumann Operator auf Mannigfaltigkeiten mit konischen Singularitäten ein beschränktes H^∞ Kalkül besitzt.

Schlüsselwörter: Dirichlet zu Neumann Operator, Funktionalkalkül, Konische Singularitäten.

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Contents

Chapter 1

Introduction

The Dirichlet problem on a Riemannian manifold M with boundary ∂M for a given boundary datum f, which is defined on ∂M, consists in finding a solution u to:

$$\Delta_g u = 0, \qquad \gamma_0(u) = f, \tag{1.1}$$

where Δ_g is the Laplace Beltrami operator on M with respect to the Riemannian metric g on M.

Having established the existence of solutions for the Dirichlet problem for a class of boundary data f of a certain regularity, one can define the Dirichlet to Neumann map $\mathcal{N}$ as the mapping which takes the boundary data f and assigns to it the Neumann data $\mathcal{N}f = (\partial_\nu u)|_{\partial M}$. Here ∂_ν denotes the exterior normal derivative on M.

For the case that M is a Riemannian manifold admitting a smooth boundary, it is a well known result that the Dirichlet to Neumann map can be expressed as a pseudodifferential operator, see e.g. [38].

Having an $(n+1)$ dimensional manifold D with boundary and conical singularity, we blow up D near the conical point. By this, we obtain a smooth manifold $\mathbb{D}_0$ with boundary to which a cylinder $C \cong [0,1) \times Y$ is glued. Here Y is an n dimensional manifold with boundary and the blow-down of $\{0\} \times Y$ is the conical point. On $\mathbb{D}$ the conical singularity is modelled by a singular Riemannian metric, which near C takes the form: $g = dx^2 + x^2 h_Y$, where h_Y denotes a Riemannian metric on Y.

$\mathbb{B} = \partial\mathbb{D}$ is a manifold with a conical singularity without boundary. Our interest lies in the situation where the Dirichlet to Neumann operator $\mathcal{N}$ is defined for the data on such a manifold $\mathbb{B}$. In this context, the Laplace Beltrami operator to the associated Dirichlet problem on $\mathbb{D}$ assumes, due to the conical metric, the form of a Fuchs type operator.

That is an operator which, in local coordinates on an n+1 dimensional manifold $\mathbb{D}$, can be written in the form $x^{-\nu}\sum_{k=0}^{\mu} A_k(x)(x\partial_x)^k$ for a smooth family of differential operators $A_k(x)$ of order $\mu - k$ acting on the cross section Y. These type of operators have been introduced independently by Melrose [24] and Schulze [32], [33].

The natural domains for Fuchs type operators are weighted Sobolev spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$. The restriction of the weighted Sobolev spaces from $\mathbb{D}$ to its boundary $\mathbb{B}$ gives weighted Besov spaces $\mathcal{B}_p^{s,\gamma}(\mathbb{B})$, the natural domains for the boundary data of the Dirichlet problem. In fact, we have that $\gamma_0 : \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \to \mathcal{B}_p^{s-1/p,\gamma-1/2}(\mathbb{B})$. For details see [8].

While there exists an extensive treatment of the Dirichlet problem in bounded domains with point singularities, see e.g. [21], [16], [15], the specific type of conical manifold and weighted Sobolev respective Besov spaces which is of interest in our case, has so far not been treated in the literature. Therefore, a first problem which arises in the framework of the analysis of the Dirichlet to Neumann operator on $\mathbb{B}$ which is solved in this thesis, is to guarantee the existence of solutions to the associated Dirichlet problem on $\mathbb{D}$.

In the case of manifolds with conical singularities there exists a pseudodifferential calculus which is due to Schulze, [32], [33]. The majority of problems which are treated in the context of singular analysis consider the case of Fuchs type differential operators. Imposing certain ellipticity conditions on those operators allows for the construction of parametrices which are pseudodifferential operators in Schulze's cone calculus. In our case, the underlying operator $\mathcal{N}$ is no longer a Fuchs type differential operator, but already a pseudodifferential operator which turns out to be an operator contained in the cone algebra.

Having established the existence of solutions to the Dirichlet problem on $\mathbb{D}$ for data in weighted Sobolev spaces $\mathcal{B}_p^{s,\gamma}(\mathbb{B})$ over $\mathbb{B}$, allows us to define the Dirichlet to Neumann map $\mathcal{N}$ as a mapping between weighted Besov spaces.

Let $A : \mathcal{D}(A) \subset \mathcal{Y} \to \mathcal{Y}$ be a closed and densely defined operator on a Banach space $\mathcal{Y}$ and $\Lambda = \Lambda(\theta)$ a sector in the complex plane:

$$\Lambda(\theta) = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \geq \theta\},$$

with $0 < \theta < \pi$. Further we assume that A has no spectrum in Λ.

Then, if $\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{Y})}$ is uniformly bounded for large $\lambda \in \Lambda$, and for functions $f \in H$, where we denote by $H = H(\theta)$ the space of all holomorphic functions $\mathbb{C}\backslash\Lambda \to \mathbb{C}$ for which $\|f(\lambda)\| \leq c(|\lambda|^{\delta} + |\lambda|^{-\delta})^{-1}$ for some $\delta > 0$ and $c > 0$, we can define $f(A)$ by:

$$f(A) := \frac{1}{2\pi i}\int_{\mathcal{C}} f(\lambda)(A - \lambda)^{-1} d\lambda,$$

with $\mathcal{C} = \partial\Lambda$.

Having an sectorial operator A, we say that A admits a bounded H^∞ calculus, if $f(A)$ defines a bounded operator, satisfying the following estimate for its operator norm:

$$\|f(A)\|_{\mathcal{L}(\mathcal{Y})} \leq c_p \|f\|_\infty \quad \forall f \in H. \tag{1.2}$$

We establish in this thesis a bounded $\mathcal{H}^\infty$ calculus for a class of cone pseudodifferential operators on manifolds with conical singularities which includes the Dirichlet to Neumann operator $\mathcal{N}$.

The proof of the H^∞ calculus requires a good understanding of the asymptotic properties of the resolvents of cone pseudodifferential operators. For cone differential operators a resolvent construction exists [6], as well as a proof of H^∞ calculus, even for operators on conical manifolds with boundary [7].

It is only in the language of b-calculus, which is due to Melrose [24], in which a resolvent has been constructed for cone pseudodifferential operators by Gil and Loya, see [10]. It has been shown by Lauter and Seiler in [18], that we can identify certain elements of Schulze's cone algebra with elements of the b-calculus and vise versa. However, the resolvent construction of Gil and Loya is limited to operators of the "small b-calculus", which is not enough to cover the case of the Dirichlet to Neumann operator. This is why we generalize the existing resolvent construction from Gil and Loya to sectorial operators which are contained in Schulze's cone algebra, making use of the identifications from Lauter and Seiler.

The motivation to study the bounded $\mathcal{H}^\infty$ calculus of operators is due to its strong applications in parabolic evolution equations, see [17] for an extensive treatise. In particular, the choice of $f(\lambda) = \lambda^{it}$ for $t \in \mathbb{R}$, implies the boundedness of imaginary powers for A, that is $A^{it} \in \mathcal{L}(\mathcal{Y})$ and $\|A^{it}\| \leq Me^{t\theta}$.

Having Banach spaces $\mathcal{D}(A) \hookrightarrow \mathcal{Y}$, and $A : \mathcal{D}(A) \to \mathcal{Y}$ a closed densely defined operator. Assume that $-A$ generates an analytic semigroup. Then the operator A is said to have maximal regularity for the pair $(\mathcal{D}(A), \mathcal{Y})$ and $1 \leq q \leq \infty$, if for every v_0 in the interpolation space $\mathcal{Y}_q = (\mathcal{D}(A), \mathcal{Y})_{1-1/q,q}$, and for every $g \in L^q(0,T;\mathcal{Y})$ there exists a unique solution $v \in L^q(0,T;\mathcal{D}(A)) \cap W^1(0,T;\mathcal{Y}) \cap C(0,T;\mathcal{Y}_q)$ of the equation

$$\dot{v} + Av = g, \quad t \in (0,T), \quad v(0) = v_0,$$

depending continuously on the data v_0 and g.

It is due to a theorem of Dore and Venni, see Theorem 3.2. of [9], that the property of bounded imaginary powers of angle $> \frac{\pi}{2}$ of an operator implies maximal regularity.

The maximal regularity of A allows to establish short time existence of solutions quasi-linear equations of the form:

$$\begin{cases} \partial_t u(t) + A(u(t))u(t) = f(t, u(t)) + g(t) \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Theorem 1.0.1. *(Clement and Li,[5], Theorem 2.1) Assume that there exists an open neighborhood U of u_0 in X_q, such that $A(u_0)$ has maximal regularity for $(\mathcal{D}(A), \mathcal{Y})$ and q, and that*

1. $A \in C^{1-}(U, \mathcal{L}(\mathcal{D}(A), \mathcal{Y}))$,
2. $f \in C^{1-,1-}([0, T_0] \times U, \mathcal{Y})$,
3. $g \in L^q([0, T_0], \mathcal{Y})$.

Then there exists a $T > 0$ and a unique $u \in L^q(0, T; \mathcal{D}(A)) \cap W^1_q(0, T; \mathcal{Y}) \cap X([0, 1]; \mathcal{Y}_q)$ solving the equation (1.3) on $]0, T[$.

The thesis is organized as follows:

- In **Chapter 2** we introduce the precise notion of a manifold with boundary and conical singularities. Further, we introduce certain weighted function spaces on which our cone operators turn out to operate continuously.

- In **Chapter 3** we give a short introduction of the operators of Schulze's cone algebra, which is a pseudodifferential calculus for manifolds with conical singularities. We explain basic notions like cone ellipticity, which is the ellipticity condition on cone pseudodifferential operators which allows for the construction of Fredholm inverses of cone pseudodifferential operators.

- In **Chapter 4** we prove the solvability of the Dirichlet problem for spaces of distributions on which the calculus of cone pseudodifferential operators is established. We do this by establishing solutions in the setup of weighted L^p-Sobolev spaces $\mathcal{H}_p^{s,\frac{1}{2}}(\mathbb{D})$, first for the case of $p = 2$. Then we prove an imbedding theorem on weighted spaces which allows to generalize the solvability for arbitrary $1 < p < \infty$.

- Having established the solvability of the Dirichlet problem in Chapter 4, we can introduce in **Chapter 5** the Dirichlet to Neumann map $\mathcal{N}$ on weighted Besov spaces, that is
$$\mathcal{N} : \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\mathbb{B}) \to \mathcal{B}_p^{s-1-\frac{1}{p},-\frac{1}{2}}(\mathbb{B}).$$

We show that the Calderón projector is well defined on conical spaces and further that the entries of the Calderón projector are pseudodifferential operators which are contained in Schulze's cone calculus. Further, we use the mapping properties of the Calderón projector to construct, up to regularizing terms, the Dirichlet to Neumann operator $\mathcal{N}$ out of the entries of the Calderón projector. Finally we prove cone ellipticity of $\mathcal{N}$.

- We introduce in **Chapter 6** the notion of parameter ellipticity and show that $\mathcal{N}$ meets the stated requirements of parameter ellipticity.

- In **Chapter 7** we give a short introduction to the b-calculus and an extended resolvent calculus which is due to Loya [19].

- Gil and Loya construct in [10] the resolvents for cone pseudodifferential operators in the small b-calculus. In **Chapter 8** we give an (incomplete) sketch about the generalization of resolvents to the case of the full b-calculus. Out motivation to do this is to obtain a resolvent for the Dirichlet to Neumann operator $\mathcal{N}$, which can be identified with an operator in the full b-calculus.

- Finally, in **Chapter 9**, we use the asymptotic structure of the resolvent of cone pseudodifferential operators to show that they admit an H^∞ functional calculus.

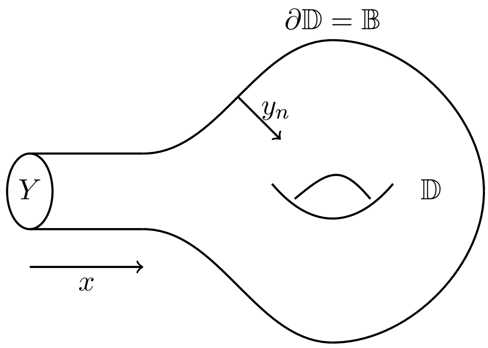
Chapter 2

Preliminaries

All material from this chapter is well known and can be found e.g. in [8]

2.1 Notation

Definition 2.1.1. *Let D be an $(n+1)$-dimensional manifold with boundary and conical singularities. Here we assume that $n > 1$. As usual, we blow up D near the conical points. We obtain an object $\mathbb{D}$ consisting of a smooth manifold $\mathbb{D}_0$ with boundary to which finitely many cylinders $C_j \cong [0,1) \times Y_j$ are glued. Here, Y_j is a smooth – not necessarily connected – n-dimensional manifold with boundary and the blow-down of $\{0\} \times Y_j$ is the j-th conical point. In order to simplify the notation, we will assume that there is only one conical point and correspondingly only one cylinder $C \cong [0,1) \times Y$.*



It is on $\mathbb{D}$ that we will perform the analysis. We model the conical singularity by endowing $\mathbb{D}$ with a Riemannian metric which, near C takes the form

$$g = dx^2 + x^2 h_Y, \tag{2.1}$$

where h_Y is a (fixed) Riemannian metric on Y. This corresponds to the case of a straight conical singularity. More generally, we can consider the situation, where

$$g = dx^2 + x^2 h_Y(x), \tag{2.2}$$

where $x \mapsto h_Y(x)$ is a smooth family of metrics on Y, $0 \leq x < 1$, resulting in the structure of a warped cone.

Choosing an atlas $(U_\alpha, \kappa_\alpha)$ of Y, with $\kappa_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ homeomorphisms, we use local coordinates $y = \kappa_\alpha(p)$ for $p \in Y$, with $y = (y_1, ..., y_n) \in \mathbb{R}^n$ for a fixed chart κ_α.

Further, we will assume that $y \in \mathbb{R}^n$ is of the form $y = (y', y^n)$ with $y' = (y^1, \ldots, y^{n-1}) \in \mathbb{R}^{n-1}$ and $y^n \in \mathbb{R}$ such that $y^n \geq 0$ is a boundary defining function on Y.

Definition 2.1.2. *We denote by Ω a smooth n-dimensional manifold without boundary in which Y is contained. The above structures induce a topology on $\mathbb{D}$; we assume $\mathbb{D}$ to be compact. By Σ we denote a smooth $(n+1)$-dimensional manifold without boundary, in which $\mathbb{D}$ is contained.*

The boundary $B = \partial D$ of D is a (boundaryless) manifold with conical singularities. It is modeled by the boundary $\mathbb{B} = \partial\mathbb{D}$ of $\mathbb{D}$; here the boundary is defined to be of the form $[0,1) \times \partial Y \cup \partial\mathbb{D}_0$ along the blow-up of the conical singularity.

2.2 Function Spaces

We begin with the definition of Sobolev spaces on a compact n-dimensional manifold Ω without boundary.

First we define the cylinder $\Omega^\wedge$ over Ω as the Cartesian product with $\mathbb{R}_+$:

$$\Omega^\wedge = \mathbb{R}_+ \times \Omega. \tag{2.3}$$

Now we consider the space of smooth functions with compact support on $\Omega^\wedge$, namely $C_c^\infty(\Omega^\wedge)$.

We introduce a norm, first on $C_c^\infty(\Omega^\wedge)$:

Definition 2.2.1. *We use the following mapping:*

$$\mathcal{S}_\gamma : C_c^\infty(\mathbb{R}^{1+n}) \to C_c^\infty(\mathbb{R}^{1+n}), \quad v(s,x) \mapsto e^{(\frac{1+n}{2}-\gamma)s} v(e^{-s}, x). \tag{2.4}$$

Moreover, let $\kappa_j : U_j \subseteq \Omega \to \mathbb{R}^n, j = 1, ..., n$, *be a covering of* Ω *by coordinate charts and* $\{\varphi_j\}$ *a subordinate partition of unity. Then we define* $\|v\|_{\mathcal{H}_p^{s,\gamma}(\Omega^\wedge)}$ *for* $v \in C_c^\infty(\Omega^\wedge)$ *by:*

$$\|v\|_{\mathcal{H}_p^{s,\gamma}(\Omega^\wedge)} = \sum_{j=1}^{n} \|\mathcal{S}_\gamma(1 \otimes \kappa_j)_*(\varphi_j v)\|_{H_p^s(\mathbb{R}^{1+n})}. \tag{2.5}$$

This allows us to define the space $\mathcal{H}_p^{s,\gamma}(\Omega^\wedge)$ *as the closure of* $C_c^\infty(\Omega^\wedge)$ *with respect to the norm* $\|\cdot\|_{\mathcal{H}_p^{s,\gamma}(\Omega^\wedge)}$

We remember that the manifold $\mathbb{D}$ is diffeomorphic to $[0,1) \times Y$ close to the conical singularity, where Y is a manifold with boundary which is assumed to be embedded in a compact manifold Ω without boundary.

Definition 2.2.2. *We define two different weighted Sobolev spaces on* $\mathbb{R}_+ \times Y$*:*

- *Let* r^+ *denote the restriction of* Ω *to* Y*, and let:*

$$\mathcal{H}_p^{s,\gamma}(Y^\wedge) = \{r^+ f \,:\, f \,\in\, \mathcal{H}_p^{s,\gamma}(\Omega^\wedge)\}. \tag{2.6}$$

  The space $\mathcal{H}_p^{s,\gamma}(Y^\wedge)$ *carries the quotient norm:*

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(Y^\wedge)} = \inf\{\|f\|_{\mathcal{H}_p^{s,\gamma}(\Omega^\wedge)} : f \in \mathcal{H}_p^{s,\gamma}(\Omega^\wedge), r^+ f = u\}. \tag{2.7}$$

- *The second space is defined as the closure of* $C_c^\infty(intY^\wedge)$ *in* $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ *with respect to the norm* $\|\cdot\|_{\mathcal{H}_p^{s,\gamma}(\Omega^\wedge)}$ *and is denoted by* $\dot{\mathcal{H}}_p^{s,\gamma}(\Omega^\wedge)$.

To define the space $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$ on $\mathbb{D}$, we first define ordinary Sobolev spaces $H_p^s(X)$ on a manifold X with boundary:

Definition 2.2.3. *Let* X *be a manifold with boundary which imbeds into the double of* X*,* $2X$*, which is a manifold without boundary. Denoting by* r^+ *the restriction* $r^+ : 2X \to X$ *from* $2X$ *to* X *we define:*

$$H_p^s(X) = \{r^+ v | v \in H_p^s(2X)\}. \tag{2.8}$$

Further:

$$\dot{H}_p^s(X) = \overline{C_c^\infty(\text{int}X)}_{H_p^s(2X)}, \tag{2.9}$$

the closure of the smooth functions with compact support in the interior int$X$ *of* X *with respect to the norm on* $H_p^s(2X)$.

Then we can define $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$ with the help of a cut-off function ω:

Definition 2.2.4.

$$\mathcal{H}_p^{s,\gamma}(\mathbb{D}) = \{u \in \mathcal{D}'(\text{int}\mathbb{D}) \,|\, \omega u \in \mathcal{H}_p^{s,\gamma}(Y^\wedge) \; and \, (1-\omega)u \in H_p^s(2\mathbb{D})\}. \tag{2.10}$$

We also define $\dot{\mathcal{H}}_p^{s,\gamma}(\mathbb{D})$ *as:*

$$\dot{\mathcal{H}}_p^{s,\gamma}(\mathbb{D}) = \{u \in \mathcal{D}'(\mathrm{int}\mathbb{D}) \,|\, \omega u \in \dot{\mathcal{H}}_p^{s,\gamma}(Y^\wedge) \; and \, (1-\omega)u \in \dot{H}_p^s(2\mathbb{D})\}. \tag{2.11}$$

We are also interested in function spaces on the boundary $\mathbb{B}$ of $\mathbb{D}$:

Definition 2.2.5. *The boundary* $\mathbb{B}$ *of* $\mathbb{D}$ *is an* n*-dimensional compact manifold with boundary* ∂Y*. Hence, we can define the norm* $\|\cdot\|_{\mathcal{H}_p^{s,\gamma}(Y^\wedge)}$ *by choosing* $\Omega = \partial Y$ *in Definition 2.2.1. Now, we can define* $\mathcal{H}_p^{s,\gamma}(\partial Y^\wedge)$ *simply as the closure of* $C_c^\infty(\partial Y^\wedge)$ *with respect to* $\|\cdot\|_{\mathcal{H}_p^{s,\gamma}(\partial Y^\wedge)}$*. Finally, we define the space* $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ *in analogy to (2.10).*

The restriction of weighted Sobolev spaces on $\mathbb{D}$ to the boundary $\mathbb{B}$ results for arbitrary $1 < p < \infty$ in distributions which are contained in weighted Besov spaces:

Definition 2.2.6. *Using the Besov spaces* $B_p^s(\mathbb{R}\times\partial Y^\wedge) := B_{p,p}^s(\mathbb{R}\times\partial Y^\wedge)$ *on the cylinder and the transformation:*

$$(S'_\gamma u)(x, y') := x^{-\frac{n}{2}+\gamma} u(\log x, y'), \qquad \gamma \in \mathbb{R}, \tag{2.12}$$

we introduce the Banach spaces:

$$\mathcal{B}_p^{s,\gamma}(\mathbb{B}) = \{u \in \mathcal{D}'(int\mathbb{B}) | \omega u \in \mathcal{B}_p^{s,\gamma}(\mathbb{R}_+ \times \partial Y^\wedge) \; and \; (1-\omega)u \in B_p^s(2\mathbb{B})\}. \tag{2.13}$$

We denote by γ_j the usual boundary operator $\gamma_0 \circ \partial_\nu^j$, where γ_0 denotes the restriction to the boundary.

We have:

Lemma 2.2.7. *For any* $1 < p < \infty$ *and* $s > \frac{1}{p} + j$ *the boundary operator induces continuous maps*

$$\gamma_j : \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \to \mathcal{B}_p^{s-j-\frac{1}{p},\gamma-\frac{1}{2}}(\mathbb{B}). \tag{2.14}$$

2.3 Dual Spaces

The space $\mathcal{H}_2^{0,0}(\mathbb{B})$ coincides with the ordinary L^2 space on $\mathbb{B}$. In general we have the following duality result:

Theorem 2.3.1. *The dual space of* $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ *is isomorphic to* $\mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{B})$*. The* L^2 *scalar product gives rise to a dual pairing:*

$$\langle\cdot,\cdot\rangle : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \times \mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{B}) \to \mathbb{C}. \tag{2.15}$$

Regarding the duality of the spaces over $\mathbb{D}$, we have the following result:

Theorem 2.3.2. *The dual* $\dot{\mathcal{H}}_p^{s,\gamma}(\mathbb{D})^*$ *of* $\dot{\mathcal{H}}_p^{s,\gamma}(\mathbb{D})$ *is given by* $\mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{D})$. *Again, the duality between* $\dot{\mathcal{H}}_p^{s,\gamma}(\mathbb{D})$ *and* $\mathcal{H}_{p'}^{-s,-\gamma}(\mathbb{D})$ *is given by the* L^2 *scalar product.*

2.4 Fuchs-type Operators, Definition and Basic Properties, Cone Laplacian

First, we define:

$$D_0 = -i\, x\partial_x, \quad D_j = -i\, \partial_{y^j} \text{ for } j = 1, ..., n. \tag{2.16}$$

The conical structure of the metric g as defined in (2.1) gives rise to a certain class of differential operators, the so called Fuchs-type operators:

Definition 2.4.1. *Diff*$^{\mu,\nu}(\mathbb{D})$ *is the class of operators which can be written in the form:*

$$x^{-\nu} \sum_{k=0}^{\mu} A_k D^k, \tag{2.17}$$

where $A_k \in$ *Diff*$^{\mu-k}(Y)$, *the class of ordinary differential operators of order* $\mu - k$ *acting on* Y.

Example 2.4.2. *In local coordinates near x=0:*

$$g = dx^2 + x^2 \sum_{i,j=1}^{n} h_{ij} dy^i\, dy^j. \tag{2.18}$$

This yields the cone Laplacian:

$$\Delta_c = \frac{1}{x^2}((x\partial_x)^2 + (n-1)(x\partial_x) + \Delta_\partial), \tag{2.19}$$

with the Laplacian on the boundary Δ_∂ *given by*

$$\begin{aligned} \Delta_\partial &= \frac{1}{\sqrt{h}} \sum_{i,j=1}^{n} \partial_{y^i}(\sqrt{h}\, h^{ij})\partial_{y^j}, \\ &\quad \textit{with } h = det(h_{ij}), \quad (h^{ij}) = (h_{ij})^{-1}. \end{aligned} \tag{2.20}$$

Chapter 3

The Cone Algebra

We give in this chapter a short introduction to Schulze's cone algebra. Everything in this chapter is well known material, which can be found e.g. in [36], [8], [34] or [33].

3.1 Mellin Operators

The first thing we will need to treat Mellin symbols is the so called Mellin transform:

Definition 3.1.1. *The Mellin transform $\mathcal{M}u$ of a complex-valued $C_c^\infty(\mathbb{R}_+)$-function u is given by*

$$(\mathcal{M}u)(z) = \int_0^\infty x^{z-1}u(x)\mathrm{d}x, \quad z \in \mathbb{C}. \tag{3.1}$$

We let $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$ and write $\Gamma_\beta, \beta \in \mathbb{R}$, for the vertical line $\{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\}$.

We summarize some properties of the Mellin transform which reveal their analogy to the Fourier transform when ∂_x is replaced by $x\partial_x$:

Lemma 3.1.2. *(a) The Mellin transform of $u \in C_c^\infty(\mathbb{R}_+)$ is holomorphic on $\mathbb{C}$.*

(b) $\mathcal{M}(x^\gamma u)(z) = (\mathcal{M}u)(z+\gamma)$.

(c) $\mathcal{M}(ln(x)u(z)) = (\partial_z \mathcal{M}u)(z)$.

(d) $\mathcal{M}(-x\partial_x u)(z) = z\mathcal{M}u(z)$.

(e) $\mathcal{M}$ extends to an isomorphism $\mathcal{M} : L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}})$.

Proof. (e) simply follows from Plancherel's theorem, using that

$$(\mathcal{M}u)(z) = \mathcal{F}u(e^{-x})(-iz). \tag{3.2}$$

where $\mathcal{F}$ denotes the Fourier transform.

The rest is easy to check. □

Lemma 3.1.3. *For $u \in C_c^\infty(\mathbb{R}_+)$, the Mellin transform $\mathcal{M}u$ is rapidly decreasing on each line Γ_β, uniformly for β in finite intervals.*

Further, we introduce the weighted Mellin transform:

Definition 3.1.4.

$$\mathcal{M}_\gamma u = \mathcal{M}u|_{\frac{\dim \mathbb{B}}{2}-\gamma}, \tag{3.3}$$

which has the inverse:

$$(\mathcal{M}_\gamma^{-1} g)(x) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{\dim \mathbb{B}}{2}-\gamma}} x^{-z} g(z)dz. \tag{3.4}$$

Definition 3.1.5. *By $\Psi_{cl}^\mu(\partial Y, \Gamma_\omega)$ we denote the space of classical pseudodifferential operators acting on the compact manifold ∂Y with parameter dependence in Γ_ω. That means, that we have in local coordinates symbols $p \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n; \Gamma_\omega)$, such that the following estimate holds:*

$$|D_x^\alpha D_\xi^\beta \partial_\lambda^\gamma p(x,\xi;\omega + i\lambda)| \leq C_{\alpha,\beta,\gamma}\langle(\xi,\lambda)\rangle^{\mu-|\beta|-\gamma}, \tag{3.5}$$

for multi-indices α, β, γ and $\lambda \in \mathbb{R}$.

Definition 3.1.6. *Let $\gamma \in \mathbb{R}$ and $h \in C^\infty(\overline{\mathbb{R}}_+, \Psi_{cl}^\mu(\partial Y, \Gamma_{\frac{\dim \mathbb{B}}{2}-\gamma}))$. We define the Mellin operator with Mellin symbol $h(x,z)$ with $x \in \mathbb{R}_+$ and $z \in \Gamma_{\frac{\dim \mathbb{B}}{2}-\gamma}$:*

$$\mathrm{op}_{\mathcal{M}}^\gamma(h) : C_c^\infty(\mathbb{R}_+ \times \partial Y) \to C^\infty(\mathbb{R}_+ \times \partial Y),$$

as:

$$\mathrm{op}_{\mathcal{M}}^\gamma(h)\, u(x) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{\dim \mathbb{B}}{2}-\gamma}} x^{-z} h(x,z)\mathcal{M}_\gamma\, u(z)\, dz. \tag{3.6}$$

Definition 3.1.7. *We call a function $\omega(x) \in C^\infty(\overline{\mathbb{R}}_+)$ a cut-off function if supp ω is bounded and $\omega \equiv 1$ near $x = 0$.*

The following result can be derived from the standard theory of pseudodifferential operators, for details see [1]:

Lemma 3.1.8. *Choosing cut-off functions ω_1, ω_2, $s \in \mathbb{R}$ the operator $\mathrm{op}_{\mathcal{M}}^\gamma(h)$ extends to a bounded operator:*

$$\omega_1\, \mathrm{op}_{\mathcal{M}}^\gamma(h)\, \omega_2 \,:\, \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma}(\mathbb{B}). \tag{3.7}$$

Further we introduce the notion of holomorphic Mellin symbols:

Definition 3.1.9. *By $M^{\mu}_{\mathcal{O}}(\partial Y)$ we denote the space of all holomorphic functions $A : \mathbb{C} \to \Psi^{\mu}_{cl}(\partial Y)$ such that, for each $\beta \in \mathbb{R}$, the restriction $a|_{\Gamma_\beta}$ is a parameter-dependent pseudodifferential operator with parameter-space Γ_β:*

$$a|_{\Gamma_\beta} \in \Psi^{\mu}_{cl}(\partial Y; \Gamma_\beta), \text{ uniformly for } \beta \text{ in compact intervals.} \tag{3.8}$$

3.2 Meromorphic Mellin Symbols

We want to obtain a full calculus of Mellin pseudodifferential operators which involves holomorphic Mellin symbols. It is reasonable that there will appear singularities in the construction of parametrices of holomorphic pseudodifferential operators. Hence, we have to include meromorphic Mellin symbols in our calculus.

Definition 3.2.1. *A set P is called a discrete asymptotic type for Mellin symbols if*

$$P = \{(p_j, n_j, N_j)|\, Re\, p_j \to \pm\infty \, for\, j \to \mp\infty, n_j \in \mathbb{N}_0, j \in \mathbb{Z}\}, \tag{3.9}$$

with finite dimensional subspaces $N_j \subset \Psi^{-\infty}(\partial Y)$ of finite rank operators. We also allow P to be a finite set. Let $\pi_{\mathbb{C}}P = \{p_j | j \in \mathbb{Z}\}$ and $\mathcal{O}$ the empty asymptotic type.

We write $P \in As(\partial Y)$.

The asymptotic types are used to describe the behavior of the meromorphic Mellin symbols close to the singularities. Meromorphic Mellin symbols are defined as:

Definition 3.2.2. *Given a Mellin asymptotic type $P \in As(\partial Y)$ as defined in 3.2.1, we write $M^{\mu}_{P}(\partial Y)$ for the space of all meromorphic functions a on $\mathbb{C} \backslash \pi_{\mathbb{C}} P$ with values in $\Psi^{\mu}_{cl}(\partial Y)$.*

In a neighborhood of each p_j, a is supposed to be of the form:

$$a(z) \equiv \sum_{k=0}^{m_j} R_{jk}(z - p_j)^{-k-1}, \tag{3.10}$$

modulo a function which is holomorphic near p_j, with $R_{jk} \in \Psi^{-\infty}(\partial Y), k = 0, ..., m_j$.

Further, we need a analogous condition as in (3.8). This is formulated with the help of an excision function χ for the poles p_j (i.e. $\chi \in C^{\infty}(\mathbb{C})$, $\chi \equiv 1$ outside a neighborhood of $\pi_{\mathbb{C}}P$, and $\chi \equiv 0$ near each p_j). We need that:

$$\chi a|_{\Gamma_\beta} \in \Psi^{\mu}_{cl}(\partial Y; \Gamma_\beta), \text{ uniformly for } \beta \text{ in compact intervals.} \tag{3.11}$$

A meromorphic Mellin symbol with asymptotic type P of order μ is a function $h \in C^{\infty}(\overline{\mathbb{R}}_+, M^{\mu}_{P}(\partial Y))$.

We define $M_P^{-\infty}(\partial Y) = \cap_\mu M_P^\mu(\partial Y)$.

Theorem 3.2.3. *Each* $a \in M_P^\mu(\partial Y)$ *can be decomposed as* $a = a_0 + a_P$*, with* $a_0 \in M_\mathcal{O}^\mu(\partial Y)$ *and* $a_P \in M_P^{-\infty}(\partial Y)$.

3.3 Green Operators

The Green operators are the regularizing operators in the cone calculus.

There are two weights $\gamma, \gamma' \in \mathbb{R}$ associated with the space of Green operators, as well as a width $\theta > 0$. The weights are used to determine the scale of weighted Sobolev spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and $\mathcal{H}_p^{t,\gamma'}(\mathbb{B})$ between which the operators are acting (for suitable choices of s and t).

The Green operators improve the smoothness and the weight and they also induce a certain asymptotic behavior close to the boundary. The data which is used to specify the asymptotic behavior is collected in the following two types of sets:

Definition 3.3.1. *(a) A weight datum is a collection:*

$$\mathbf{g} = (\gamma, \gamma', \theta), \tag{3.12}$$

with $\gamma, \gamma' \in \mathbb{R}$ *and* $\theta > 0$.

(b) Let $\theta > 0$*,* $\gamma \in \mathbb{R}$*. An asymptotic type with respect to* γ *and* θ *is a set:*

$$Q = \{(p_j, m_j, L_j) : j = 1, ..., N\}, \tag{3.13}$$

of triples (p_j, m_j, L_j)*, where* $q_j \in \mathbb{C}$ *with* $\frac{\dim\mathbb{B}}{2} - \gamma - \theta < Re\, p_j < \frac{\dim\mathbb{B}}{2} - \gamma, m_j \in \mathbb{N}_0$*, and* L_j *is a finite dimensional subspace of* $C^\infty(\partial Y)$*. We denote by* $As(\gamma,\theta)$ *the collection of all these.*

We say that $Q \in As(\gamma, \infty)$*, if* $Q \in As(\gamma,\theta)$ *for all* $\theta \in \mathbb{N}$.

The Green operators are characterized by the property of mapping to the right asymptotic spaces which are specified below:

Definition 3.3.2. *Let* $\theta > 0, \gamma \in \mathbb{R}$*. Let* $Q = ((p_j, m_j, L_j))_{j=1,\dots,N} \in As(\gamma,\theta)$ *be a finite asymptotic type. We denote by* $\mathcal{H}_{p,Q}^{s,\gamma}(\mathbb{B})$ *the space of all* $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ *which can be written in the form:*

$$u(x,y) = u_0(x,y) + \sum_{j=1}^{N} \sum_{k=0}^{m_j} c_{jk}(y)\omega(x)x^{-p_j} ln^k x, \tag{3.14}$$

with $c_{jk} \in L_j$ *and* $u_0 \in \mathcal{H}_p^{s,\gamma+\theta-\varepsilon}(\mathbb{B})\ \forall\, \varepsilon > 0$.

Now we can define a Green operator as an operator with the following mapping properties:

Definition 3.3.3. *Let* $\mathbf{g} = (\gamma, \gamma', \theta)$ *be a weight datum. Let* $Q_1 \in As(-\gamma, \theta)$ *and* $Q_2 \in As(\gamma', \theta)$ *be two asymptotic types.*

We write $G \in C_{\mathbf{G}}(\mathbb{B}, \mathbf{g})$ *and call it a Green operator associated with the weight datum* $\mathbf{g}$*, if we have the following mapping properties for* G *and its formal adjoint* G^**:*

$$G \quad : \quad \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_{p,Q_2}^{\infty,\gamma'}(\mathbb{B}) \quad \text{for all } s \in \mathbb{R}, \tag{3.15}$$

$$G^* \quad : \quad \mathcal{H}_p^{s,-\gamma'}(\mathbb{B}) \to \mathcal{H}_{p,Q_1}^{\infty,-\gamma}(\mathbb{B}) \quad \text{for all } s \in \mathbb{R}. \tag{3.16}$$

To be able to study compositions of operators, we define two additional weight data:

$$\mathbf{g}' = (\gamma', \gamma'', \theta) \text{ and } \mathbf{g}'' = (\gamma, \gamma'', \theta). \tag{3.17}$$

We obtain:

Lemma 3.3.4. *The composition* $G_1 G_2$ *of a Green operator* G_2 *in* $C_{\mathbf{G}}(\mathbb{B}, \mathbf{g})$ *and a Green operator* G_1 *in* $C_{\mathbf{G}}(\mathbb{B}, \mathbf{g}')$*, is an element of* $C_{\mathbf{G}}(\mathbb{B}, \mathbf{g}'')$*.*

3.4 Smoothing Mellin Operators

Definition 3.4.1. *A smoothing Mellin operator associated with the weight datum* $\mathbf{g} = (\gamma, \gamma - \mu, \theta)$ *is an operator of the form:*

$$M = \omega_1(x) \sum_{l=0}^{\theta-1} x^{-\mu+l} op_{\mathcal{M}}^{\gamma_l}(h_l) \omega_2(x), \tag{3.18}$$

for Mellin symbols $h_l \in M_{P_l}^{-\infty}(\partial Y)$*, where the* P_l *are Mellin asymptotic types with* $\pi_{\mathbb{C}}(P_l) \cap \Gamma_{\frac{\dim \mathbb{B}}{2} - \gamma_l} = \emptyset$ *and* $\gamma - l \leq \gamma_l \leq \gamma$*.*

Theorem 3.4.2. *(a) Let* M *be as in Definition 3.4.1. Then* M *furnishes a continuous map:*

$$M : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{\infty,\gamma-\mu}(\mathbb{B}). \tag{3.19}$$

(b) Changing one of the cut-off functions changes the smoothing Mellin operator by a Green operator.

Theorem 3.4.2 motivates the following definition:

Definition 3.4.3. *For a weight datum* $\mathbf{g} = (\gamma, \gamma', \theta)$*, we denote by* $C_{\mathbf{M+G}}(\mathbb{B}, \mathbf{g})$ *the space of all operators of the form:*

$$\omega_1 x^{-\mu} op^{\gamma}_{\mathcal{M}}(h_0)\omega_2 + G, \tag{3.20}$$

with $h_0 \in M^{-\infty}_P(\partial Y)$ *for some Mellin asymptotic type* P*, such that* $\Gamma_{\frac{dim\mathbb{B}}{2}-\gamma} \cap \pi_{\mathbb{C}} P = \emptyset$*, cut-off functions* ω_1, ω_2 *and* $G \in C_{\mathbf{G}}(\partial Y, \mathbf{g})$*.*

Theorem 3.4.4. *Let* $\mathbf{g}, \mathbf{g}', \mathbf{g}''$ *be weight data as defined in (3.17). The composition of elements in* $C_{\mathbf{M+G}}(\mathbb{B}, \mathbf{g}')$ *and* $C_{\mathbf{M+G}}(\mathbb{B}, \mathbf{g})$ *furnishes operators in* $C_{\mathbf{M+G}}(\mathbb{B}, \mathbf{g}'')$*. If either of the factors is a Green operator, so is the composition.*

3.5 Operators in the Cone Calculus.

Definition 3.5.1. *Let* $\mathbf{g} = (\gamma, \gamma', \theta)$*. The space* $C^{\mu}(\mathbb{B}, \mathbf{g})$ *consists of all operators*

$$A = x^{-\mu}\omega_1 op^{\gamma}_{\mathcal{M}}(h)\omega_2 + M + (1-\omega_1)P(1-\omega_3) + G, \tag{3.21}$$

where

(i) $\omega_1, \omega_2, \omega_3$ *are cut-off functions, such that* $\omega_1\omega_2 = \omega_1$ *and* $\omega_1\omega_3 = \omega_3$*,*

(ii) $h \in C^{\infty}(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}}(\partial Y))$*,*

(iii) P *is a pseudodifferential operator of order* μ *on int* $\mathbb{B}$*,*

(iv) M *as in Definition 3.4.1,*

(v) G *is a Green operator in* $C_{\mathbf{G}}(\mathbb{B}, \mathbf{g})$*.*

Theorem 3.5.2. *Given* $h \in C^{\infty}(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}}(\partial Y))$*, there is a pseudodifferential operator* P_M *of order* μ *on int*$\mathbb{B}$ *such that* $op^{\gamma}_{\mathcal{M}}(h) - P_M$ *is regularizing on* $(0,1)\times\partial Y$*.*

Conversely, given a pseudodifferential operator P *of order* μ *on int*$\mathbb{B}$*, there is an element* $h \in C^{\infty}(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}}(\partial Y))$ *such that* $op^{\gamma}_{\mathcal{M}}(h) - P$ *is regularizing on* $(0,1)\times\partial Y$*.*

We associate three different symbols to a cone pseudodifferential operator:

Definition 3.5.3. *Let* $A \in C^{\mu}(\mathbb{B}, \mathbf{g})$*. We associate three symbols with* A*:*

- *Letting* P_M *be the pseudodifferential operator which coincides up to regularizing operators with* A *close to* $\partial\mathbb{B}$ *and exists due to Theorem 3.5.2, we define:*

$$\sigma^{\mu}_{\psi}(A) = \omega_1(x)x^{-\mu}\sigma^{\mu}_{\psi}(P_M) + (1-\omega_1(x))\sigma^{\mu}_{\psi}(P). \tag{3.22}$$

- *Secondly, we have the rescaled symbol $\tilde{\sigma}^{\mu}_{\psi}(A)$. Writing $\sigma^{\mu}_{\psi}(A) = \sigma^{\mu}_{\psi}(A)(x, y, \rho, \zeta)$, we define $\tilde{\sigma}^{\mu}_{\psi}(A)$ in a neighborhood of $x = 0$ by:*

$$\tilde{\sigma}^{\mu}_{\psi}(A) = x^m \sigma^{\mu}_{\psi}(A)(x, y, x^{-1}\rho, \zeta). \tag{3.23}$$

- *Further, we have the conormal symbol $\sigma^{\mu}_{\mathcal{M}}(A)$ defined by:*

$$\sigma^{\mu}_{\mathcal{M}}(A)(z) = h(0, z) + h_0(z) \quad z \in \mathbb{C}. \tag{3.24}$$

  This is a meromorphic function in z taking values in the pseudodifferential operators on $\partial\mathbb{B}$ of order at most μ. In particular $\sigma^{\mu}_{\mathcal{M}}(A) \in M^{\mu}_{\mathcal{P}}(\partial Y)$.

For compositions in the cone algebra, we have the following Theorem:

Theorem 3.5.4. *Let the weight data $\mathbf{g}, \mathbf{g}', \mathbf{g}''$ as defined in (3.17), assume that $A_0 \in C^{\mu}(\mathbb{B}, \mathbf{g})$, $A_1 \in C^{\mu'}(\mathbb{B}, \mathbf{g}')$. Then, the composition $(A_0, A_1) \mapsto A_0 A_1$ induces a continuous map:*

$$C^{\mu}(\mathbb{B}, \mathbf{g}) \times C^{\mu'}(\mathbb{B}, \mathbf{g}') \to C^{\mu+\mu'}(\mathbb{B}, \mathbf{g}''). \tag{3.25}$$

If one of the two factors belongs to the $C_{\mathbf{G}}$ or $C_{\mathbf{M}++\mathbf{G}}$ class, so does the product. The conormal symbol behaves multplicative up to a shift, we have:

$$\sigma^{\mu+\mu'}_{\mathcal{M}}(A_0 A_1)(z) = \sigma^{\mu}_{\mathcal{M}}(A_0)(z + \mu') \cdot \sigma^{\mu'}_{\mathcal{M}}(A_0)(z) \tag{3.26}$$

Using the symbols from Definition 3.5.3, we can define the notion of cone-ellipticity:

Definition 3.5.5. *We say that a Mellin operator $A \in C^{\mu}(\mathbb{B}, \mathbf{g})$ is cone degenerate elliptic (or simply: cone-elliptic) with respect to $\mathbf{g}$, if:*

- *A is elliptic over the interior $\mathbb{B}^{\circ}$ of $\mathbb{B}$, that is, $\sigma^{\mu}_{\psi}(A)$ is invertible and further the rescaled symbol $\tilde{\sigma}^{\mu}_{\psi}(A)$ is uniformly invertible in a neighborhood of $x = 0$.*

- *The restriction of $\sigma^{\mu}_{\mathcal{M}}(z)$ to the line $\Gamma_{\frac{\dim \mathbb{B}}{2}} - \gamma$ gives a parameter dependent family of pseudodifferential operators which is pointwise invertible as a mapping: $H^s(Y) \to H^{s-\mu}(Y)$.*

Theorem 3.5.6. *Let $A \in C^{\mu}(\mathbb{B}, \mathbf{g})$. Then the following are equivalent:*

1. *A is cone-elliptic.*

2. *There exists a parametrix to A in the cone calculus, an operator B in $C^{-\mu}(\mathbb{B}, \mathbf{g}^{-1})$ with $BA - I \in C_{\mathbf{G}}(\mathbb{B}, \mathbf{g}_0)$, and $AB - I \in C_{\mathbf{G}}(\mathbb{B}, \mathbf{g}_1)$. Here $\mathbf{g}^{-1} = (\gamma - \mu, \gamma, \theta)$, $\mathbf{g}_0 = (\gamma, \gamma, \theta)$ and $\mathbf{g}_1 = (\gamma - \mu, \gamma - \mu, \theta)$.*

3. *$A : \mathcal{H}^{s,\gamma}_p(\mathbb{B}) \to \mathcal{H}^{s-\mu,\gamma-\mu}_p(\mathbb{B})$ is a Fredholm operator for all $s \in \mathbb{R}$.*

4. $A : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$ is a Fredholm operator for some $s \in \mathbb{R}$.

Example 3.5.7. *We treat the cone Laplacian on Σ. Therefore, we replace $\mathbb{B}$ by Σ and ∂Y by Ω in the definitions above. We split the cone Laplacian into a part near the boundary and a part away from the boundary, which can be treated as a usual differential operator:*

$$\Delta_c = x^{-2}\omega_1 \operatorname{op}_{\mathcal{M}}^{\gamma}(\sigma_M(\Delta))\omega_2 + (1-\omega_1)\Delta_c(1-\omega_3) \tag{3.27}$$

with cut-off functions ω_i such that $\omega_1\omega_2 = \omega_1$ and $\omega_1\omega_3 = \omega_3$.

The conormal symbol of the cone Laplacian (2.19) is $\sigma_M(\Delta_c)(z) = z^2 - (n-1)z + \Delta_\partial$. Evaluating $\sigma_M(\Delta_c)(z)$ on $\Gamma_{\frac{\dim\mathbb{D}}{2}-\gamma}$, writing $z_\omega = \frac{\dim\mathbb{D}}{2} - \gamma + i\omega$ with $\omega \in \mathbb{R}$ and choosing $\gamma = 1$:

$$\sigma_M(\Delta_c)(z_\omega) = -\frac{(\dim\,\mathbb{D}-2)^2}{4} - \omega^2 + \Delta_\partial, \tag{3.28}$$

which is invertible for $\omega \in \mathbb{R}$ for $\dim\,\mathbb{D} \geq 3$.

We have for the interior symbol of Δ_c in the sense of (3.22):

$$\sigma_\psi^2(\Delta_c) = x^{-2}(-x^2\rho^2 - i\frac{(\dim\,\mathbb{D}-1)}{2}x\rho - \xi^2), \tag{3.29}$$

where ξ^2 is the symbol of Δ_∂. We see that $\sigma_\psi^2(\Delta_c)$ is invertible on int $\mathbb{D}$.

The rescaled symbol of Δ_c as defined in (3.23) is:

$$\tilde{\sigma}_\psi^2 = -\rho^2 - i\frac{(\dim\,\mathbb{D}-1)}{2}\rho - \xi^2 \tag{3.30}$$

which is uniformly invertible up to $x = 0$.

Therefore we can conclude that for all $\theta > 0$, $\gamma \in \mathbb{R}$, $\Delta_c \in C^2(\mathbb{B}, \mathbf{g})$ with $\mathbf{g} = (\gamma, \gamma-2, \theta)$ is cone-elliptic. Hence, we can find a meromorphic Mellin symbol $q \in M_{\mathcal{P}}^{\mu}(\Omega)$, and a parametrix $Q \in C^{-2}(\mathbb{B}, \mathbf{g}')$ with $\mathbf{g}' = (\gamma-2, \gamma, \theta)$ of Δ_c, which admits a representation:

$$Q = x^2\,\omega_1' \operatorname{op}_{\mathcal{M}}^{\gamma}(q)\omega_2' + (1-\omega_1')P(1-\omega_3'), \tag{3.31}$$

where P is a parametrix for $\Delta_c|_{x>0}$.

Here Q acts as parametrix of Δ_c up to smoothing operators and maps:

$$Q : \mathcal{H}_p^{s-2,\gamma-2}(\Sigma) \to \mathcal{H}_p^{s,\gamma}(\Sigma). \tag{3.32}$$

3.6 Parameter Dependent Kernel Cut-off

We focus now on the decomposition of a meromorphic Mellin symbol into the sum of a holomorphic Mellin symbol and a smoothing meromorphic symbol. The technique which is used for this decomposition is called kernel cut-off. We introduce in this section a parameter dependent version of this kernel cut-off which allows to choose the seminorms of the meromorphic symbol arbitrarily small.

Theorem 3.6.1. *Let $\varphi \in C_c^\infty(\mathbb{R}_+)$ with $\psi(\rho) \equiv 1$ near $\rho = 1$. Let $h \in C^\infty(\overline{\mathbb{R}}_+, \Psi_{cl}^\mu(\Omega, \Gamma_{\frac{\dim \mathbb{B}}{2}-\gamma}))$. Then, for each $\varepsilon > 0$:*

1. *The operator-valued function $h_{\mathcal{O},\varepsilon}$ defined by:*

$$h_{\mathcal{O},\varepsilon}(x,z) = \mathcal{M}_{\frac{\dim \mathbb{B}}{2}-\gamma,\rho\to z}\varphi(\rho^\varepsilon)\mathcal{M}^{-1}_{\frac{\dim \mathbb{B}}{2}-\gamma,\zeta\to\rho}h(x,\zeta),$$

   is an element of $C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(\Omega))$.

2. *The operator-valued function $h_{\mathfrak{M},\varepsilon}$ defined by:*

$$h_{\mathfrak{M},\varepsilon}(x,z) = \mathcal{M}_{\frac{\dim \mathbb{B}}{2}-\gamma,\rho\to z}(1-\varphi(\rho^\varepsilon))\mathcal{M}^{-1}_{\frac{\dim \mathbb{B}}{2}-\gamma,\zeta\to\rho}h(x,\zeta),$$

   is an element of $C^\infty(\overline{\mathbb{R}}_+, M_P^{-\infty}(\Omega))$ for a suitable asymptotic type P.

3. *We have that all seminorms for $h_{\mathfrak{M},\varepsilon}$ in $C^\infty(\overline{\mathbb{R}}_+, M_P^{-\infty}(\Omega))$ tend to zero as $\varepsilon \to 0$.*

   This implies, that:

$$\lim_{\varepsilon\to 0} \|x^{-\mu} op^\gamma(h_{\mathfrak{M},\varepsilon})\|_{\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B})\to\mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B}))} = 0. \tag{3.33}$$

Proof. A proof of (1) and (2) can be found e.g. in Section 2.2.2. of [34]. For (3), we have to show that all the seminorms for $h_{\mathfrak{M},\varepsilon}$ tend to zero for $\varepsilon \to 0$. For this, it suffices to show that (details can be found in the proof of Theorem 2.2.8. in [34]):

$$\lim_{\varepsilon\to 0} \pi_k(\log^M(\rho)(\rho\frac{\partial}{\partial\rho})^N(1-\varphi(\rho^\varepsilon))(\mathcal{M}^{-1}_{\frac{\dim \mathbb{B}}{2}-\gamma,\zeta\to\rho}h(x,\zeta))) = 0 \quad \forall\, k, M, N \in \mathbb{N}_0, \tag{3.34}$$

where $\{\pi_k | k \in \mathbb{N}_0\}$ is a system of seminorms on $\Psi^{-\infty}(Y)$. This follows from the fact that $\lim_{\varepsilon\to 0}(1-\varphi(\rho^\varepsilon)) = 0$ for each $\rho \geq 0$, and that for $j \in \mathbb{N}$, $j > 0$ there exist constants c_k, such that: $(\rho\frac{\partial}{\partial\rho})^j(1-\varphi(\rho^\varepsilon)) = \varepsilon^j \sum_{k=1}^j c_k \varphi^{(k)}(\rho^\varepsilon)\rho^{k\varepsilon}$. ☐

Chapter 4

The Dirichlet Problem on $\dot{\mathcal{H}}_p^{1,1}(\mathbb{D})$

We establish in this chapter the existence of solutions for the Dirichlet problem on conical spaces. Our motivation for solving this problem is to define the Dirichlet to Neumann map which takes the boundary value data of the Dirichlet problem and maps it to its corresponding Neumann data. That indeed, the mapping of the Dirichlet data to its Neumann data admits a representation in the form of a certain operator, a cone pseudodifferential operator in Schulze's cone algebra $C^1(\mathbb{B}, \mathbf{g})$, is a result which is treated in the subsequent chapters.

The Dirichlet problem consists in finding solutions $u \in \mathcal{H}_p^{1,1}(\mathbb{D})$ for a given boundary data $f \in \mathcal{B}_p^{1-\frac{1}{p},\frac{1}{2}}(\mathbb{B})$, and a $p \in \mathbb{N}$, such that:

$$\Delta_c u = 0; \qquad \gamma_0(u) = f. \tag{4.1}$$

To prove the existence of solutions for all $1 < p < \infty$, we will first establish the existence of solutions on $\dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$. In a second step we use this existence to prove the existence of solutions for the Dirichlet problem for all $u \in \dot{\mathcal{H}}_p^{1,1}(\mathbb{D})$.

The proof of the problem on $\mathbb{D}$ for $p = 2$ makes use of the fact that $\dot{\mathcal{H}}_2^{s,\gamma}(\mathbb{D}),\ s, \gamma \in \mathbb{R}$ is a Hilbert space which allows us to apply the Theorem of Lax-Milgram to the right bilinear form.

We denote by $\| \cdot \|_{\dot{\mathcal{H}}^{s,\gamma}(\mathbb{D})}$ the Hilbert space norm on $\dot{\mathcal{H}}_2^{s,\gamma}(\mathbb{D})$ which is induced by the scalar product.

Further, we can make use of the fact that the dual space $\dot{\mathcal{H}}_2^{s,\gamma}(\mathbb{D})'$ for $s = \gamma = 1$ is given by $\dot{\mathcal{H}}_2^{1,1}(\mathbb{D})' = \mathcal{H}_2^{-1,-1}(\mathbb{D})$, so that $\Delta_c : \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}) \to \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})'$.

The Lax-Milgram Theorem deals with bilinear forms $a : V \times V \to \mathbb{C}$ on a Hilbert space V, that are V-elliptic:

Definition 4.0.2. *A bilinear form* $a(\cdot,\cdot)$*:*

$$a(\cdot,\cdot) : V \times V \to \mathbb{C}, \tag{4.2}$$

on a Hilbert space V *with norm* $\|\cdot\|$ *is said to be V-elliptic if there exists an* $\alpha > 0$*, such that:*

$$|a(u,u)| \geq \alpha \, \|u\|^2. \tag{4.3}$$

The Lax-Milgram Theorem ([40], Thm 17.9) states:

Theorem 4.0.3. *(Lax-Milgram)*

If $a(\cdot,\cdot)$ *is a bilinear continuous complex valued form on a Hilbert space* V *which is* V*-elliptic, and if further* $l(\cdot) : V \to \mathbb{C}$ *denotes a linear continuous functional on* V*, then there exists a unique element* $u \in V$*, such that:*

$$a(u,v) = l(v) \qquad \forall\, v \in V. \tag{4.4}$$

From here on, until the end of this chapter, we specify as Hilbert space V the space $V = \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$.

If $u, v \in C_c^\infty(\mathbb{D}^\circ)$, then we define $a(u,v)$ as:

$$a(u,v) = (\nabla_{\mathbb{D}} u, \nabla_{\mathbb{D}} v)_g. \tag{4.5}$$

Since $C_c^\infty(\mathbb{D}^\circ)$ is dense in $\dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$, the form $a(\cdot,\cdot)$ extends to a bilinear mapping $a : \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}) \times \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}) \to \mathbb{C}$ by continuity.

We choose for the linear form l_f an arbitrary element f of the dual space $\mathcal{H}_2^{-1,-1}(\mathbb{D})$ of $\dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$ and define:

$$l_f(u) = \langle f, u \rangle \qquad \forall\, u \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}). \tag{4.6}$$

We will show that a satisfies the requirements posed on a for the Lax-Milgram theorem.

To apply Lax-Milgram to $V = \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$, a and l_f as defined above, it remains to prove that a is V-elliptic. To prove this, we will make use of the Poincaré inequality for conical spaces, which is Theorem 2.5 in [4]:

Theorem 4.0.4. *(Poincaré inequality)*

Let $M = (0,1) \times Y$ *with* $Y \subset \mathbb{R}^n$ *open and bounded, and* $1 < p < \infty$, $\gamma \in \mathbb{R}$*. Let* $\nabla_M = (x\partial_x, \partial_{y_1}, ..., \partial_{y_n})$*.*

If $u(x,y) \in \dot{\mathcal{H}}_p^{1,\gamma}(M)$, then there exists $c > 0$, such that:

$$\|\nabla_M u(x,y)\|_{\dot{\mathcal{H}}_p^{0,\gamma}(M)} \geq c\|u(x,y)\|_{\dot{\mathcal{H}}_p^{0,\gamma}(M)}. \tag{4.7}$$

The Poincaré inequality gives us the following estimate: taking into account that we have $\nabla_{\mathbb{D}} = x^{-1}\nabla_M$:

Corollary 4.0.5. *For $u \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$, it holds that there exists a $c > 0$, such that:*

$$a(u,u) \geq c\|u\|^2_{\dot{\mathcal{H}}_2^{0,1}(\mathbb{D})}. \tag{4.8}$$

Lemma 4.0.6. *The bilinear form a as defined above is $V-$elliptic in the sense of Definition 4.0.2.*

Proof. We want to prove that there exists $\alpha > 0$, such that:

$$a(u,u) \geq \alpha(\|u\|_{\dot{\mathcal{H}}_2^{1,1}})^2, \qquad \forall\, u \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}). \tag{4.9}$$

This follows from the Poincaré inequality:

$$\begin{aligned} a(u,u) &= \frac{1}{2}(a(u,u) + a(u,u)) \\ &\geq \frac{1}{2}(c\|u\|^2_{\dot{\mathcal{H}}_2^{0,1}(\mathbb{D})} + \tilde{c}\,\|\nabla_{\mathbb{D}} u\|^2_{\dot{\mathcal{H}}_2^{0,0}(\mathbb{D})}) \\ &\geq \frac{\min\{c,\tilde{c}\}}{2}(\|u\|^2_{\dot{\mathcal{H}}_2^{0,1}(\mathbb{D})} + \|\nabla_{\mathbb{D}} u\|^2_{\dot{\mathcal{H}}_2^{0,0}(\mathbb{D})}) \\ &= \frac{\min\{c,\tilde{c}\}}{2}\|u\|^2_{\dot{\mathcal{H}}_2^{1,1}(\mathbb{D})}. \end{aligned}$$

□

Corollary 4.0.7. *It follows from the theorem of Lax-Milgram, that given $f \in \mathcal{H}_2^{-1,-1}(\mathbb{D})$, we can find $u \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$, such that:*

$$(\nabla_{\mathbb{D}} u, \nabla_{\mathbb{D}} v)_g = \langle f, v\rangle, \qquad \forall\, v \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}). \tag{4.10}$$

The solution u, which solves the problem posed in Corollary 4.0.7, is called a weak solution of the Dirichlet problem. It turns out that every weak solution to the problem is a solution in the following sense:

Theorem 4.0.8. *(Inhomogeneous Dirichlet problem for $p = 2$)*

Given $f \in \mathcal{H}_2^{-1,-1}(\mathbb{D})$, there exists $u \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$ such that:

$$-\Delta_c u = f \qquad \gamma_0(u) = 0, \tag{4.11}$$

where $\gamma_0(u)$ denotes the restriction of u to $\partial\mathbb{D}$.

Proof. Let $\mathcal{D}(\mathbb{D})$ denote functions on $\mathbb{D}$ which are smooth and have compact support in $\operatorname{int}\mathbb{D}$. It is true that $\mathcal{D}(\mathbb{D})$ is dense in $\dot{\mathcal{H}}_2^{s,\gamma}(\mathbb{D})$ for every $s, \gamma \in \mathbb{R}$ and $1 < p < \infty$.

Corollary 4.0.7 gives us the existence of $u \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D})$, such that

$$(\nabla_{\mathbb{D}} u, \nabla_{\mathbb{D}} \varphi)_g = \langle f, \varphi \rangle \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{D}). \tag{4.12}$$

Here it holds in the distributional sense, that:

$$(\nabla_{\mathbb{D}} u, \nabla_{\mathbb{D}} \varphi)_g = \langle -\Delta_c u, \varphi \rangle_{L^2} = \langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{D}). \tag{4.13}$$

Hence, it holds that:

$$-\Delta_c u = f, \tag{4.14}$$

in the distributional sense. □

Next, we want to generalize the inhomogeneous Dirichlet problem from $\mathcal{H}_2^{s,1}(\mathbb{D})$ to $\mathcal{H}_p^{s,1}(\mathbb{D})$ for $1 < p < \infty$. We have the following imbedding result, which we use to generalize the existence of solutions of the Dirichlet problem on $\mathcal{H}_p^{s,\gamma}$ spaces for $p \neq 2$:

Lemma 4.0.9. *We have for an arbitrary $\varepsilon > 0$ and a fixed $p_0 \in \mathbb{R}$ with $p_0 > 1$:*

$$\dot{\mathcal{H}}_{p_0}^{\infty,\gamma+\varepsilon}(\mathbb{D}) \hookrightarrow \bigcap_{p>1} \dot{\mathcal{H}}_p^{\infty,\gamma}(\mathbb{D}). \tag{4.15}$$

Proof. Let us assume that $u_0 \in \dot{\mathcal{H}}_{p_0}^{\infty,\gamma+\varepsilon}(\mathbb{D})$.

We remember that we defined the spaces $\dot{\mathcal{H}}_p^{\infty,\gamma}(\mathbb{D})$ in (2.11) as:

$$u_0 \in \dot{\mathcal{H}}_p^{\infty,\gamma+\varepsilon}(\mathbb{D}) \quad \Leftrightarrow \quad \omega\, u_0 \in \dot{\mathcal{H}}_p^{\infty,\gamma+\varepsilon}(Y^\wedge), \quad \text{and} \quad (1-\omega)\, u_0 \in \dot{H}_p^s(\mathbb{D}),$$

for a cut-off function ω.

Hence, we have $\omega u_0 \in \dot{\mathcal{H}}_{p_0}^{\infty,\gamma+\varepsilon}(Y^\wedge)$.

Let $\kappa_j : U_j \subseteq Y \to \mathbb{R}^n$ be a finite covering by coordinate charts and let φ_j be a subordinate partition of unity.

We take the localizations $\mathcal{S}_{\gamma+\varepsilon}(1 \otimes \kappa_j)_*(\varphi_j \omega u_0)$, of ωu_0. We can use the continuity of the following imbedding, which follows from [39](p.203,2.8.1(c)), since $F_{p,2}^s(\mathbb{R}^{1+n}) = H_p^s(\mathbb{R}^{1+n})$ (see [39], p.169,Def. 2.3.1.c)):

Let $\infty > q \geq p_0 > 1$ and $-\infty < t \leq s < \infty$ with

$$s - \frac{n+1}{p_0} = t - \frac{n+1}{q}, \tag{4.16}$$

then:

$$H^s_{p_0}(\mathbb{R}^{1+n}) \overset{\imath}{\hookrightarrow} H^t_q(\mathbb{R}^{1+n}). \tag{4.17}$$

Choosing $q_0 \geq p_0$ and $t_0 \in \mathbb{R}$, we can choose s_0, such that the condition (4.16) holds. Then, $u \in H^\infty_{p_0}(\mathbb{R}^{1+n}) \Rightarrow u \in H^{s_0}_{p_0}(\mathbb{R}^{1+n})$ and $H^{s_0}_{p_0}(\mathbb{R}^{1+n}) \hookrightarrow H^{t_0}_{q_0}(\mathbb{R}^{1+n})$ continuously. We obtain:

$$\imath(\mathcal{S}_{\gamma+\epsilon}(1 \otimes \kappa_j)_*(\varphi_j \omega u_0)) \in H^t_q(\mathbb{R}^{1+n}), \qquad \forall\, t \in \mathbb{R},\, q \geq p_0, \forall j. \tag{4.18}$$

This gives us:

$$\omega u_0 \in \dot{\mathcal{H}}^{\infty,\gamma+\varepsilon}_q(Y^\wedge), \qquad \forall q \geq p_0, \tag{4.19}$$

and we can conclude:

$$\omega u_0 \in \bigcap_{q \geq p_0} \dot{\mathcal{H}}^{\infty,\gamma+\varepsilon}_q(Y^\wedge). \tag{4.20}$$

A completely analogous argumentation yields the embedding for $(1-\omega)u_0$:

$$u_0 \in \bigcap_{q \geq p_0} \dot{\mathcal{H}}^{\infty,\gamma+\varepsilon}_q(\mathbb{D}). \tag{4.21}$$

Now, for the case $q_0 < p_0$, we use localizations φ_j and a subordinate partition of unity κ_j for Y, this is equivalent to the convergence of $\mathcal{S}_{\gamma+\varepsilon}(1 \otimes \kappa_j)_*(\varphi_j \omega u_0) = e^{(\gamma+\varepsilon-\frac{n+1}{2})x}(1 \otimes \kappa_j)_*(\varphi_j \omega u_0)(e^{-x}, y)$ for all j and $p \geq p_0$ with respect to the standard H^s_p norm $\|\cdot\|_{H^s_p(\mathbb{R}^{1+n})}$ on $\mathbb{R}^{1+n}$.

Since Y is compact, we can assume that the image of U_j under κ_j is contained in a compact subset K_j of $\mathbb{R}^n$. We can define $\chi_j(y)$ as a smooth function on $\mathbb{R}^n$, such that $\chi_j(y) = 1$ for $y \in K_j$, $\chi_j(y) = 0$ for $y \in \mathbb{R}^n \backslash (2 \cdot K_j)$. Then:

$$e^{(\frac{n+1}{2}-\gamma+\varepsilon)x}(1 \otimes \kappa_j)_*(\varphi_j \omega u_0)(e^{-x}, y) = \chi_j(y) e^{(\frac{n+1}{2}-\gamma+\varepsilon)x}(1 \otimes \kappa_j)_*(\varphi_j \omega u_0)(e^{-x}, y).$$

We have $e^{-x} \in (0,1) \Leftrightarrow x \in \mathbb{R}_{>-1}$. Since the support of ω is contained in $[0,1)$ we can define a smooth function $\sigma(x)$ on $\mathbb{R}$ with support in $\mathbb{R}_{>-1}$, such that $\sigma(x)\omega(e^{-x}) = \omega(e^{-x})$.

We can first choose $s = 0$, and since $\|\cdot\|_{H^0_p} = \|\cdot\|_p$, where $\|\cdot\|_p$ denotes the standard L^p-norm on $\mathbb{R}^{1+n}$, we can apply the following generalized Hölder inequality:

$$\|fg\|_r \leq \|f\|_q\|g\|_p, \quad \text{for } r = \frac{1}{\frac{1}{q}+\frac{1}{p}}. \tag{4.22}$$

We choose $f = \sigma(x)\chi_j(y)e^{-\varepsilon x}$, $g = e^{(\gamma+\varepsilon-\frac{n+1}{2})x}(1\otimes\kappa_j)_*(\varphi_j\omega u_0)(e^{-x},y)$, and obtain

$$\begin{aligned}\|\sigma(x)\chi_j(y)e^{(\gamma-\frac{n+1}{2})}(1\otimes\kappa_j)_*(\varphi_j\omega u_0)(e^{-x},y)\|_r \quad \leq \quad & \|\sigma(x)\chi_j(y)e^{-\varepsilon x}\|_q \cdot \\ & \cdot\|e^{(\gamma+\varepsilon-\frac{n+1}{2})x}(1\otimes\kappa_j)_*(\varphi_j\omega u_0)(e^{-x},y)\|_p.\end{aligned}$$

Since $\sigma(x)\chi_j(y)e^{-\varepsilon x} \in L^q(\mathbb{R}^{1+n})$, for all $q \in \mathbb{N}$ and since $e^{(\gamma+\varepsilon-\frac{n+1}{2})x}(1\otimes\kappa_j)_*(\varphi_j\omega u_0)(e^{-x},y) \in L^p(\mathbb{R}^{1+n})$ for $p \geq p_0$, we obtain that $e^{(\gamma+\varepsilon-\frac{n+1}{2})x}(1\otimes\kappa_j)_*(\varphi_j\omega u_0)(e^{-x},y) \in L^r(\mathbb{R}^{1+n})$ for all $1 < r < p_0$.

This implies that:

$$\omega u_0 \in \dot{\mathcal{H}}^{0,\gamma}_p(Y^\wedge), \qquad \forall\, p > 1. \tag{4.23}$$

Now the result for the higher order spaces can be derived from applying the generalized Hölder inequality to derivatives of $\sigma(x)\chi_j(y)e^{(\gamma-\frac{n+1}{2})}(1\otimes\kappa_j)_*(\varphi_j\omega u_0)(e^{-x},y)$, since $\partial^k_x\partial^\alpha_y(\chi_j(y)e^{-\varepsilon x})$ is contained in $L^q(\mathbb{R}^{1+n})$ for each $k \in \mathbb{N}, \alpha \in \mathbb{N}^n$, $q \in \mathbb{N}$.

We obtain the imbedding:

$$\omega u_0 \in \bigcap_{q>1} \dot{\mathcal{H}}^{\infty,\gamma+\varepsilon}_q(Y^\wedge). \tag{4.24}$$

Again, the same considerations yield the imbedding for $(1-\omega)u_0$, which gives the imbedding 4.15. □

Theorem 4.0.10. *(Inhomogeneous Dirichlet Problem for $1 < p < \infty$)*

Given $f \in \mathcal{H}^{-1,-1}_p(\mathbb{D})$, $1 < p < \infty$, there exists a unique $u \in \dot{\mathcal{H}}^{1,1}_p(\partial\mathbb{D})$, such that:

$$\Delta_c u = f. \tag{4.25}$$

Proof. An equivalent formulation of Theorem 4.0.10 is to say, that:

$$\Delta_c : \dot{\mathcal{H}}^{1,1}_p(\mathbb{D}) \to \mathcal{H}^{-1,-1}_p(\mathbb{D}), \tag{4.26}$$

is an isomorphism for $1 < p < \infty$.

Since the Dirichlet problem $\mathcal{A}_D = \begin{pmatrix} \Delta \\ \gamma_0 \end{pmatrix}$ is elliptic on $\mathbb{D}$ for $\gamma = 1$ (see Example 6.1 of [8]), we can conclude that $\Delta_c : \dot{\mathcal{H}}_p^{1,1}(\mathbb{D}) \to \mathcal{H}_p^{-1,-1}(\mathbb{D})$ is Fredholm, hence has closed range.

Therefore, saying that $\Delta_c : \dot{\mathcal{H}}_p^{1,1}(\mathbb{D}) \to \mathcal{H}_p^{-1,-1}(\mathbb{D})$ is an isomorphism for $1 < p < \infty$ is equivalent to:

$$\ker(\Delta_c : \dot{\mathcal{H}}_p^{1,1}(\mathbb{D}) \to \mathcal{H}_p^{-1,-1}(\mathbb{D})) = \{0\}, \quad \operatorname{coker}(\Delta_c : \dot{\mathcal{H}}_p^{1,1}(\mathbb{D}) \to \mathcal{H}_p^{-1,-1}(\mathbb{D})) = \{0\}. \tag{4.27}$$

We prove $\ker(\Delta_c : \dot{\mathcal{H}}_p^{1,1}(\mathbb{D}) \to \mathcal{H}_p^{-1,-1}(\mathbb{D})) = \{0\}$ by contradiction, assuming that there exists $p_0 \in \mathbb{N}$ and a $u_0 \in \dot{\mathcal{H}}_{p_0}^{1,1}$, with $u_0 \neq 0$, such that:

$$\Delta_c u_0 = 0. \tag{4.28}$$

Since $0 \in \mathcal{H}_{p_0}^{\infty,-1}(\mathbb{D})$, we can apply a parametrix P to Δ_c, such that $P\Delta_c = I + S$ for a regularizing Greens operator S. Applying P to (4.28), immediately gives $u_0 \in \dot{\mathcal{H}}_{p_0}^{\infty,1+\varepsilon}(\mathbb{D})$.

Now, since $\dot{\mathcal{H}}_{p_0}^{\infty,1+\varepsilon}(\mathbb{D})$ imbeds into $\bigcap_p \dot{\mathcal{H}}_p^{\infty,1}(\mathbb{D})$ by Lemma 4.0.9, we can conclude that:

$$u_0 \in \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}). \tag{4.29}$$

But this contradicts Theorem 4.0.8, which says that $\Delta_c : \dot{\mathcal{H}}_2^{1,1}(\mathbb{D}) \to \mathcal{H}_2^{-1,-1}(\mathbb{D})$ is an isomorphism.

Proving that $\operatorname{coker}(\Delta_c) = \{0\}$ works completely analogous, using that $\operatorname{coker}(\Delta_c) = \ker(\Delta_c^*)$ and $\Delta_c = \Delta_c^*$ □

We use elliptic regularity to generalize Theorem 4.0.10 to Sobolev spaces of higher regularity:

Theorem 4.0.11. *Let $s \geq -1$ and $\dim(\mathbb{D}) = n + 1 \geq 2$. Given $f \in \mathcal{H}_p^{s,-1}(\mathbb{D})$, there exists a $u \in \mathcal{H}_p^{s+2,1}(\mathbb{D})$, such that:*

$$-\Delta_c u = f \qquad \gamma_0(u) = 0, \tag{4.30}$$

where $\gamma_0(u)$ denotes the restriction of u to $\partial\mathbb{D}$.

Proof. Let $f \in \mathcal{H}_p^{s,-1}(\mathbb{D})$. Now $\mathcal{H}_p^{s,-1}(\mathbb{D})$ imbeds continuously into $\mathcal{H}_p^{-1,-1}(\mathbb{D})$ as long as $s \geq -1$. Therefore, we can apply Theorem 4.0.8 to obtain a $u \in \dot{\mathcal{H}}_p^{1,1}(\mathbb{D})$, such that $\Delta_c u = f$.

Considering the Dirichlet problem $\mathcal{A}_D = \begin{pmatrix} \Delta \\ \gamma_0 \end{pmatrix}$ on $\mathbb{D}$, we can associate a principal conormal symbol $\sigma^2_{\mathcal{M}}(\mathcal{A}_D)(z)$ to $\mathcal{A}_D$. Then $\sigma^2_{\mathcal{M}}(\mathcal{A}_D)(z)$ is invertible on $\Gamma_{\frac{n+1}{2}-1}$, this is Example 6.1 of [8].

Therefore, since $f \in \mathcal{H}^{s,-1}_p(\mathbb{D})$, elliptic regularity for the boundary value problem (see e.g. [8] for details) $\Delta_c(u) = f$, $\gamma_0(u) = 0$ gives us that $u \in \mathcal{H}^{s+2,1}_p(\mathbb{D}) \oplus \mathcal{E}$, where $\mathcal{E}$ denotes a finite dimensional space of asymptotic functions as described in Definition 3.3.2; $\mathcal{E} \subseteq \mathcal{H}^{\infty,-1}_p(\mathbb{D})$. However, we further know that $u \in \dot{\mathcal{H}}^{1,1}_p(\mathbb{D})$ and since $(\mathcal{H}^{s+2,1}_p \oplus \mathcal{E}) \cap \dot{\mathcal{H}}^{1,1}(\mathbb{D}) = \dot{\mathcal{H}}^{s+2,1}_p(\mathbb{D})$, we can conclude that $u \in \dot{\mathcal{H}}^{s+2,1}_p(\mathbb{D})$. ☐

We can use the solution of the inhomogeneous Dirichlet problem to solve the homogeneous problem: for this we will need the following lemma, this is Lemma 3.4 of [8]:

Lemma 4.0.12. *Given $g \in \mathcal{B}^{s-\frac{1}{p},\frac{1}{2}}_p(\partial\mathbb{D})$ with $1 < p < \infty$, there exists $u_g \in \mathcal{H}^{s,1}_p(\mathbb{D})$, such that:*

$$\gamma_0(u_g) = g. \tag{4.31}$$

Theorem 4.0.13. *Let $dim(\mathbb{D}) = n + 1 \geq 2$ and $s \in \mathbb{R}$ with $s \geq 1$.*

Given $g \in \mathcal{B}^{s-\frac{1}{p},\frac{1}{2}}_p(\partial\mathbb{D})$ with $1 < p < \infty$, there exists $u \in \mathcal{H}^{s,1}_p(\mathbb{D})$, such that:

$$\Delta_c u = 0, \qquad \gamma_0(u) = g. \tag{4.32}$$

Proof. Due to Lemma 4.0.12 there exists a $u_g \in \mathcal{H}^{s,1}_p(\mathbb{D})$, such that $\gamma_0(u_g) = g$.

Now, since $\Delta_c u_g \in \mathcal{H}^{s-2,-1}_p(\mathbb{D})$, Theorem 4.0.11 states the existence of a $\tilde{u} \in \mathcal{H}^{s,1}_p(\mathbb{D})$ with $\gamma_0(\tilde{u}) = 0$, such that:

$$\Delta_c\tilde{u} = \Delta_c u_g. \tag{4.33}$$

Now, if we define u as:

$$u = u_g - \tilde{u}, \tag{4.34}$$

we have that $u \in \mathcal{H}^{s,1}_p(\mathbb{D})$, $\gamma_0(u) = \gamma_0(u_g) - \gamma_0(\tilde{u}) = g$ and $\Delta_c u = \Delta_c u_g - \Delta_c\tilde{u} = 0$. ☐

Chapter 5

The DtN Operator

Since we have shown the existence of solutions for the Dirichlet problem in the previous chapter, we can define now for $f \in \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\mathbb{B})$ and $s \geq 1$ the Dirichlet to Neumann map as the mapping which assigns to f the restriction of the exterior normal derivative of the solution to the boundary.

In this chapter we use the mapping properties of the Calderón projector to relate the Dirichlet data of the Dirichlet problem to the Neumann data and to finally prove the existence of the Dirichlet to Neumann operator as an operator in $C^1(\mathbb{B}, \mathbf{g})$ for $\mathbf{g} = (\frac{1}{2}, -\frac{1}{2}, \theta)$ and arbitrary $\theta > 0$.

5.1 Formal Definition of the DtN Map

We have already established in Chapter 4 the existence of solutions for the Dirichlet Problem on $\dot{\mathcal{H}}_p^{s,1}(\mathbb{D})$ for boundary data $f \in \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\mathbb{B})$ and $s \geq 1$. We choose the local coordinates $(y_1, ..., y_n)$ on Y in a way such that the inward pointing normal direction is given by y_n, the inward pointing normal derivative by the Fuchs type operator $x^{-1}\partial y_n$.

By definition of $\mathcal{H}_p^{s,\gamma}(\mathbb{D})$, we have: $x^{-1}\partial_{y_n} : \mathcal{H}_p^{s,\gamma}(\mathbb{D}) \to \mathcal{H}_p^{s-1,\gamma-1}(\mathbb{D})$ for $s, \gamma \in \mathbb{R}$.

The Dirichlet to Neumann map sends Dirichlet boundary data to its corresponding Neumann boundary data.

Lemma 5.1.1. *Let $f \in \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\partial\mathbb{D})$. For $s \geq 1$, Theorem 4.0.13 states that we have a $u_0 \in \mathcal{H}_p^{s,1}(\mathbb{D})$ solving the homogeneous Dirichlet problem such that $\gamma_0(u_0) = f$.*

Then the Dirichlet to Neumann operator $\mathcal{N}$ is well defined as the following mapping:

$$\begin{aligned} \mathcal{N} : \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\partial\mathbb{D}) &\to \mathcal{B}_p^{s-1-\frac{1}{p},-\frac{1}{2}}(\partial\mathbb{D}), \\ f &\mapsto \gamma_0(x^{-1}D_n u_0). \end{aligned} \tag{5.1}$$

Proof. For $s > 1 + \frac{1}{p}$ this result is trivial by the mapping properties of γ_0. For arbitrary $s \geq 1$ we can use that $\Delta_c u_0 = 0$, where Δ_c is elliptic. Therefore the results in [35] apply to our situation, giving well-definedness of the trace. □

5.2 The Calderón Projector on Conical Spaces

We establish in this subsection the existence of the Calderón projector C^+ (A.P. Calderón in [3]) on weighted cone Sobolev spaces.

A crucial ingredient which is used in the construction of C^+ is that the operators which are involved satisfy the *transmission property*. A more detailed exposition of the transmission property can be found in section 1.3.5 in [13].

R.T. Seeley proved in [35] that the Calderón projection for elliptic operators is a pseudodifferential projection.

We show that the entries of the operator valued matrix C^+ in the case of conical manifold are cone pseudodifferential operators which are contained in Schulze's cone algebra $C^\mu(\mathbb{B}, \mathbf{g})$.

Further, we show that the upper right entry C_{01}^+ of C^+ is cone degenerate elliptic. An important consequence of this is that the Dirichlet to Neumann map can be constructed, up to regularizing operators, out of the entries of C^+. Consequently $\mathcal{N} \in C^\mu(\mathbb{B}, \mathbf{g})$.

5.2.1 The Construction of the Calderón Projector

As usual, we denote the double of $\mathbb{D}$ along the $x-$ direction by $2\mathbb{D}$ and we imbed $2\mathbb{D}$ into an open manifold Ω without boundary. We assume $\mathbb{D}$ to be isomorphic to $\mathbb{R}_+ \times Y$ close to the conical singularity and imbed Y into an open manifold Σ without boundary.

Definition 5.2.1. *The following operators map between spaces of distributions on $\mathbb{D}$ and Ω:*

- *Write e^+u for the the extension of distributions $u \in \mathcal{H}_2^{s,\gamma}(\mathbb{D})$ for $s > -\frac{1}{2}$ by zero to a distribution e^+u on Ω.*
- *The restriction of distributions $u \in \mathcal{H}^{s,\gamma}(\Omega)$ to $\mathbb{D}$ is defined by r^+u.*

It follows from L^2 duality that the adjoint $\tilde{\gamma}_0^*$ of $\tilde{\gamma}_0$ acts for a given $s < 0$ as a map $\mathcal{H}^{s,\gamma}(\partial\mathbb{D}) \to \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\Omega)$ and is given by:

$$\tilde{\gamma}_0^*(u) = x^{-1} \cdot (u(x') \otimes \delta(y_n)). \tag{5.2}$$

We know from Example 3.5.7, that $\Delta_c \in C^2(1, -1, \infty)$ is cone-elliptic on $\mathcal{H}_2^{1,1}(\Omega)$ for $\dim(\Omega) \geq 3$. As discussed in Chapter 3, it is known that there exists a parametrix $Q \in C^{-2}(-1, 1, \infty)$ which is contained in the cone calculus and inverts Δ_c up to a regularizing operator.

We collect the trace operators γ_0, γ_1 in a vector, which we define by:

$$\rho(u) = \begin{pmatrix} \gamma_0(u) \\ \gamma_1(u) \end{pmatrix}, \tag{5.3}$$

and in the same way, we define $\tilde{\rho}$ with γ_0, γ_1 replaced by $\tilde{\gamma}_0, \tilde{\gamma}_1$.

We pick the Greens Matrix $\mathfrak{A}$ for Δ_c, which is determined by:

$$\langle \Delta_c u, v \rangle_{\mathcal{H}_2^{0,0}(\mathbb{D})} - \langle u, \Delta_c^* v \rangle_{\mathcal{H}_2^{0,0}(\mathbb{D})} = \langle \mathfrak{A}\rho(u), \rho(v) \rangle_{\mathcal{H}_2^{0,0}(\partial\mathbb{D})}. \tag{5.4}$$

The explicit computations for the entries of $\mathfrak{A}$ can be found in the Appendix in Section A.1. The result is:

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{a}_{00} & \mathfrak{a}_{01} \\ \mathfrak{a}_{10} & \mathfrak{a}_{11} \end{pmatrix} = \begin{pmatrix} \frac{i}{x} L_1 & i\, h_{nn} \\ i\, h_{nn} & 0 \end{pmatrix}, \tag{5.5}$$

with a differential operator L_1 of order 1 with smooth coefficients.

It looks tempting to define the Calderón projector by:

$$C^+ = \tilde{\rho}\, Q\, \tilde{\rho}^* \mathfrak{A}. \tag{5.6}$$

However, a brief look at the required regularity properties for the application of γ_0, γ_1 and their adjoints, and taking into account that Q increases the regularity s by 2 shows that there exists no $s \in \mathbb{R}$, such that the composition $\tilde{\rho}\, Q\, \tilde{\rho}^*$ is a priori well defined.

Our strategy to make sense of the compositions involved in C^+ will be to analyze the mapping properties of the Mellin symbols which appear in the construction of the parametrix Q more carefully and to use an additional property which they obey, the so called transmission property. This will allow us to show that the composition indeed makes sense and gives a matrix with values in cone pseudodifferential operators.

At first we introduce the transmission property, using techniques developed within the Boutet de Monvel calculus to treat pseudodifferential operators on manifolds with boundary. See [27] for a short introduction to Boutet de Monvels algebra and [28] and [29] for a Boutet de Monvel calculus on conical manifolds.

5.2.2 The Transmission Property

Definition 5.2.2. *Given a function f on $\mathbb{R}^n_+$ we take as e^+f its extension by zero to a function on $\mathbb{R}^n$. e^-g is the extension to of a function g on $\mathbb{R}^n_-$ to $\mathbb{R}^n$.*

We let:

$$H^+ = \{(e^+u)\hat{} : u \in \mathcal{S}(\mathbb{R}_+)\}, \tag{5.7}$$

$$H^-_0 = \{(e^-u)\hat{} : u \in \mathcal{S}(\mathbb{R}_-)\}. \tag{5.8}$$

The H^+ and H^-_0 are spaces of smooth functions on $\mathbb{R}$, decaying to first order near infinity. By H'_d we denote the space of polynomials of degree $\leq d-1$. We let:

$$H_d = H^+ \oplus H^-_0 \oplus H'_d.$$

With this space we define the transmission property:

Definition 5.2.3. *Let $\Omega = \Omega' \times \mathbb{R}, \Omega' \subseteq \mathbb{R}^{n-1}$ open. A symbol $p \in S^\mu(\Omega, \mathbb{R}^n)$ has the transmission property at $r = 0$ if for every $k \in \mathbb{N}$:*

$$D^k_r p(x', r, \xi', \langle\xi'\rangle\rho)|_{r=0} \in S^\mu(\Omega'_{x'}, \mathbb{R}^{n-1}_{\xi'})\hat{\otimes}_\pi H_{d,\rho}, \tag{5.9}$$

where $d = entier(\mu) + 1$. We shall also say that p has the transmission property with respect to (r, ρ).

We write $p \in S^\mu_{\mathrm{tr}}(\Omega, \mathbb{R}^n)$ for symbols p with transmission property.

Definition 5.2.4. *Let $\Omega = \Omega' \times \mathbb{R}, \Omega' \subseteq \mathbb{R}^{n-1}$ be open. Identifying Γ_β with $\mathbb{R}$ by writing $z = \beta + i\rho$ for $\rho \in \mathbb{R}$, we say that $p \in S^\mu(\Omega' \times \mathbb{R}_r, \mathbb{R}^{n-1} \times \mathbb{R}_\rho \times \mathbb{R}_\omega)$ has the transmission property (with parameter), if it has the transmission property with respect to (r, ρ).*

Lemma 5.2.5. *We have for the holomorphic Mellin symbol $\sigma^2_{\mathcal{M}}(\Delta_c)$ of Δ_c, that $\sigma^2_{\mathcal{M}}(\Delta_c) \in \Psi^2_{tr}(\Sigma, \Gamma_{\frac{n-1}{2}})$.*

Further, we have that the inverse $q(z) = (\sigma^2_{\mathcal{M}}(\Delta_c))^{-1}$, which is the conormal symbol $\sigma^{-2}_{\mathcal{M}}(Q)$ of the parametrix Q for Δ_c, that $q(z) \in \Psi^{-2}_{tr}(\Sigma, \Gamma_{\frac{n+3}{2}})$.

To introduce a general notion of pseudodifferential operators acting between Banach spaces, we have to define group actions first:

Definition 5.2.6. *A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbb{R}_+\}$ of isomorphisms in $\mathcal{L}(E)$ such that $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}$ and the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous for every $e \in E$. For all the above Sobolev spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$ we shall use the group action defined on functions by:*

$$(\kappa_\lambda u)(x) = \lambda^{q/2} u(\lambda x). \tag{5.10}$$

It extends to distributions by $(\kappa_\lambda u)(\varphi) = u(\kappa_{\lambda^{-1}}\varphi), \varphi \in C_0^\infty$. On $E = \mathbb{C}^l, l \in \mathbb{N}$, we use the trivial group action $\kappa_\lambda \equiv \mathbb{I}$. Sums of spaces of the above kind will be endowed with the sum of the group actions.

We use the group action to define of operator valued symbols which were introduced by Schulze in [30]:

Definition 5.2.7. *Let E, F be Banach spaces with strongly continuous group actions κ and $\tilde{\kappa}$, respectively. Let $a \in C^\infty(\mathbb{R}^n, \mathbb{R}^n, \mathcal{L}(E,F))$ and $\mu \in \mathbb{R}$. We shall write $a \in S^\mu(\mathbb{R}^n, \mathbb{R}^n, \mathcal{L}(E,F))$ provided that, for all multi-indices α, β, γ, there is a constant $C = C(\alpha, \beta, \gamma)$ with*

$$\|\tilde{\kappa}_{\langle\eta\rangle^{-1}} D_\eta^\alpha D_y^\beta a(y,\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \le C \langle\eta\rangle^{\mu-|\alpha|}. \tag{5.11}$$

If a is independent of y or $\tilde{y}$ we shall write $a \in S^\mu(\mathbb{R}^n, \mathbb{R}^n; E, F)$.

Note that we recover the definition of the symbol class $S^\mu(\mathbb{R}^n \times \mathbb{R}^n)$ for the case that $E = F = \mathbb{C}$

The following theorem is a parameter dependent version of Lemma 2.11 in [27]:

Lemma 5.2.8. *Let $n \in S^\mu_{tr}(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma_\beta)$ and $l \in \mathbb{N}$. Define:*

$$k_l(y, \xi, \lambda) = r^+[op_{y_n} q](\delta_0^{(l)}). \tag{5.12}$$

Then $k_l(y, \xi, \lambda)$ yields an operator valued symbol with parameter:

$$k_l \in S^{\mu+l+1/2}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \times \Gamma_\beta; \mathbb{C}, \mathcal{S}(\mathbb{R}_+)). \tag{5.13}$$

Proof. Take a right symbol $p_R = p_R(x', y_n, \xi, \lambda)$ for $\mathrm{op}_{x_n} p$ with $x' \in \mathbb{R}^{n-1}$, $y_n \in \mathbb{R}$ denoting the distance to the boundary ∂Y of Y. Fix a function $\omega \in C_0^\infty(\mathbb{R})$ with

$\omega(t) \equiv 1$ near $t = 0$ and write:

$$\begin{aligned} & p_R(x', y_n, \xi, \lambda) \\ & = \sum_{j=0}^{l} \frac{y_n^j}{j!}\omega(y_n)\partial_{y_n}^j p_R(x', 0, \xi, \lambda) + y_n^{l+1}\omega y_n p_{Rl}(x', y_n, \xi, \lambda) + \\ & + (1 - \omega(y_n))p_R(x', y_n, \xi, \lambda), \end{aligned}$$

with suitable p_{Rl}. $\operatorname{op} x_n$ is applied to a distribution with a singularity only in $y_n = 0$. Since pseudodifferential operators behave pseudo local we obtain a smoothing operator valued symbol away from $y_n = 0$. Because the second and third summand in the above sum vanish in $y_n = 0$, we can focus on the first one.

Since p_R satisfies the transmission condition, this also holds for $\partial_{y_n}^j p_R$. In our case the transmission condition is fulfilled by p_R with respect to (y_n, ξ_n), while λ serves as an additional covariable of the operator valued symbol. Hence we have to deal with a tuple (ξ', λ) of covariables.

So we have $\partial_{y_n}^j p_R(x', 0, \xi', \lambda, \langle(\xi', \lambda)\rangle\xi_n) \in S^\mu(\mathbb{R}^{n-1}, \mathbb{R}_{\xi'}^{n-1} \times \Gamma_\beta)\hat{\otimes}_\pi H_{d,\rho}$.

We can write, using the definition of the direct tensor product:

$$\begin{aligned} & \partial_{y_n}^j p_R(x', 0, \xi', \lambda, \xi_n) \\ & = \sum_{k=0}^{\mu} s_{jk}(x', \xi', \lambda)\xi_n^k + \sum_{k=0}^{\infty} \lambda_{jk} b_{jk}(x', \xi', \lambda) h_{jk}(\xi_n/\langle(\xi', \lambda)\rangle), \end{aligned}$$

with $s_{jk} \in S_{1,0}^{\mu-k}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}\times\Gamma_\beta)$, $\{\lambda_{jk}\}_k \in l^1$, and null sequences $b_{jk} \in S_{1,0}^{\mu}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}\times \Gamma_\beta)$, $h_{jk} \in H_0$. Polynomials in ξ_n convert into Dirac delta distributions and it derivatives under quantization, which have non-zero support only in $y_n = 0$. So they don't contribute to the result due to the application of r^+. Hence we have no contribution from the polynomial part to (5.12).

By this, it suffices to consider a single term $b(x', \xi')h(\xi_n/\langle(\xi', \lambda)\rangle)$ under the summation and to show that its contribution to (5.12) is an element of $S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$, and to check that the semi-norms for this element depend continuously on those for b and h. Since b is of order μ and since $y_n^j\delta_0^{(l)} = \sum_{k=0}^{j} \begin{pmatrix} l \\ k \end{pmatrix} \delta_0^{(l-k)}$, it suffices that, for all $\sigma \in \mathbb{R}^2$,

$$r^+\kappa_{\langle(\xi',\lambda)\rangle^{-1}}[\operatorname{op}_{x_n} D_{(\xi',\lambda)}^\alpha h(\xi_n/\langle(\xi', \lambda)\rangle)]\delta_0^{(l)} : \mathbb{C} \to H^\sigma(\mathbb{R}_+),$$

has norm $O(\langle(\xi', \lambda)\rangle^{-|\alpha|+l+1/2})$. Now $D_{(\xi',\lambda)}^\alpha h(\xi_n/\langle(\xi', \lambda)\rangle)$ is a linear combination of terms of the form:

$$(\xi_n/\langle(\xi', \lambda)\rangle)^k h^{(k')}(\xi_n/\langle(\xi', \lambda)\rangle)s(\xi', \lambda),$$

where $s \in S_{1,0}^{-|\alpha|}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \times \Gamma_\beta)$, and $0 \le l \le k' \le |\alpha|$. If we define $\nu := \xi_n/\langle(\xi', \lambda)\rangle$, the function $\nu^k h^{k'}$ is an element in H_0, so we may focus on the case $\alpha = 0$. We observe that:

$$\begin{aligned}
&\kappa_{\langle(\xi',\lambda)\rangle^{-1}} \mathrm{op}_{x_n} h(\xi_n/\langle(\xi',\lambda)\rangle)\delta_0^{(l)} \\
&= c_l \langle(\xi',\lambda)\rangle^{-1/2} \mathcal{F}^{-1}_{\xi_n \to x_n}[h(\xi_n/\langle(\xi',\lambda)\rangle)\xi_n^l](x_n/\langle(\xi',\lambda)\rangle) \\
&= c_l \langle(\xi',\lambda)\rangle^{1/2+l} \mathcal{F}^{-1}[h(\nu)\nu^l](x_n),
\end{aligned}$$

with $c_l = (2\pi)^{-1/2} i^l$. Since we have that $r^+\mathcal{F}^{-1}(h\nu^l)$ is a function in $\mathcal{S}(\mathbb{R}_+)$, this gives the desired result. □

Lemma 5.2.9. *Provided that $s > j + \frac{1}{2}$, we can regard γ_j as an operator valued symbol independent of the variables y, λ and η, then:*

$$\gamma_j \in S^{j+\frac{1}{2}}(\mathbb{R}^{q-1}, \mathbb{R}^{q-1} \times \Gamma_\beta; H^s(\mathbb{R}_+), \mathbb{C}).$$

Definition 5.2.10. *Let E, κ_λ as in Definition (5.2.6), $n \in \mathbb{N}$, $s \in \mathbb{R}$. The wedge Sobolev space $\mathcal{W}(\mathbb{R}^n, E)$ is the completion of $\mathcal{S}(\mathbb{R}^n, E) = \mathcal{S}(\mathbb{R}^n)\hat{\otimes}_\pi E$ in the norm:*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^n,E)} = \left(\int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta} u(\eta)\|_E^2 d\eta\right)^{\frac{1}{2}}, \tag{5.14}$$

If we consider $op(a)$ for $a \in S^\mu(\mathbb{R}^n, \mathbb{R}^n \times \Gamma_\beta, \mathcal{L}(E, F))$, we obtain the desired mapping properties which are analogous to the mapping property of ordinary pseudodifferential operators if $\mathcal{W}^s$ is replaced by H^s:

Theorem 5.2.11. *Let E, F be Banach spaces, $s, \mu \in \mathbb{R}$, and $a \in S^\mu(\mathbb{R}^n_y, \mathbb{R}^n_\eta \times \mathbb{R}^l_\lambda; E, F)$. Then for every $\lambda \in \mathbb{R}^l$*

$$\mathrm{op}\ a(\lambda) : \mathcal{W}^s_{comp}(\mathbb{R}^n, E) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\mathbb{R}^n, F),$$

is bounded.

The mapping op :″ symbol $\mapsto$ operator″ is continuous in the corresponding topologies.

Proof. A proof can be found in Section 3.2.1. of [32]. □

In the situation considered in Lemma 5.2.8 we have the spaces $E = \mathbb{C}$ and $F = \mathcal{S}(\mathbb{R}_+)$. It follows from the definition of the wedge Sobolev spaces, that $\mathcal{W}^s_{\mathrm{comp}}(\mathbb{R}^n, \mathbb{C}) = H^s(\mathbb{R}^n)$, using the trivial group action $\kappa_\lambda = \mathbb{I}$ on $\mathbb{C}$, and $\mathcal{W}^{s-\mu}_{\mathrm{loc}}(\mathbb{R}^n, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}^{n+1}_+)$. This gives us:

Corollary 5.2.12. *We see that the quantization* $\operatorname{op} k_l(\lambda)$ *of the operator valued symbol defined in* (5.12) *is a parameter dependent operator and maps:*

$$\operatorname{op} k_l(\lambda) : H^s(\mathbb{R}^{n-1}) \longrightarrow H^{s-\mu}(\mathbb{R}^n_+).$$

We can summarize the results of Lemma 5.2.8 and of Lemma 5.2.9 in the following Corollary:

Corollary 5.2.13. *The composition of* γ_k *and* k_l *as defined in* (5.12) *defines a parameter dependent pseudodifferential symbol* $\gamma_k\, k_l \in S^{\mu+k+l+1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Gamma_\beta)$.

5.2.3 The Calderón Projector

Lemma 5.2.13 already points out the right strategy how to make sense of the compositions which are involved in the definition of C^+:

We take the Mellin symbols of Q which are parameter dependent pseudodifferential operators on Ω, and build the compositions $\gamma_k\, k_l$. Then we get from Lemma 5.2.13, that we obtain parameter dependent operators acting on distributions on ∂Y, from which we can compute the Mellin quantization to obtain Mellin pseudodifferential operators acting on $\mathbb{B}$.

We summarize these results in the following lemma:

Lemma 5.2.14. *Now let* Ω *be a* $n+1$ *dimensional manifold without boundary and consider the cone algebra* $C^\mu(\Omega, \mathbf{g})$ *for a weight datum* $\mathbf{g} = (\gamma, \gamma - \mu, \theta)$. *Assume that* $Q \in C^\mu(\Omega, \mathbf{g})$ *and further, that* Q *has the transmission property. Then, for* $k, l \in \mathbb{N}$, *it holds that:*

$$\tilde{\gamma}_k Q \tilde{\gamma}_l^* \in C^{\mu+k+l+1}(\mathbb{B}, (\gamma - \frac{1}{2}, \gamma + \frac{1}{2} - \mu - k - l), \theta).$$

Proof. Assume that we have a Mellin symbol $h(z)$ which is either in $M^\mu_{\mathcal{O}}(\Sigma)$ or in $M^{-\infty}_P(\Sigma)$ for a certain Mellin asymptotic type P and for $\mu \in \mathbb{R}$. Taking local coordinate charts for Σ, it is then straight forward from 5.2.13 above, that the composition $\gamma_k h(z) \gamma_l^*$ is well defined and gives a Mellin symbol which is contained in $M^{\mu+1}_{\mathcal{O}}(\partial Y)$ respective in $M^{-\infty}_P(\partial Y)$.

Taking an operator $Q \in C^\mu(\Omega, \mathbf{g})$, which has the structure $Q = x^{-\mu}\omega_1 \operatorname{op}^\gamma_{\mathcal{M}}(h)\omega_2 + (1 - \omega_1)P(1-\omega_3) + M + G$ as described in Definition 3.5.1, we obtain immediately:

$$\tilde{\gamma}_k\, Q\, \tilde{\gamma}_l^* = x^{-\mu}\omega_1 \tilde{\gamma}_k \operatorname{op}^\gamma_{\mathcal{M}}(h)\, \tilde{\gamma}_l^* \omega_2 + (1-\omega_1)\tilde{\gamma}_k\, P\, \tilde{\gamma}_l^*(1-\omega_3) + \tilde{\gamma}_k\, M\, \tilde{\gamma}_l^* + \tilde{\gamma}_k\, G\, \tilde{\gamma}_l^*. \quad (5.15)$$

Then we obtain by Corollary 5.2.13, that $x^{-\mu}\omega_1\tilde{\gamma}_k \operatorname{op}^\gamma_{\mathcal{M}}(h)\tilde{\gamma}_l^*\omega_2$ and $\tilde{\gamma}_k M \tilde{\gamma}_l^*$ give contributions to the right spaces.

Further, $(1-\omega_1)P(1-\omega_3)$ defines a pseudodifferential operator of the desired order on $\mathbb{B}$ by the standard theory for manifold with smooth boundary as discussed e.g. in [27].

Regarding G, since G defines a smoothing operator which maps to $\mathcal{H}_Q^{\infty,\gamma-\mu}(\Omega)$ for a certain asymptotic type Q, it is clear that the composition $\tilde{\gamma}_k\, G\, \tilde{\gamma}_l^*$ is well defined and gives a Greens operator in $C^{\mu+1+k+l}(\mathbb{B}, (\gamma-\frac{1}{2}, \gamma+\frac{1}{2}-\mu-k-l), \theta)$.

□

Now we can define the Calderón projector:

Definition 5.2.15. *We define:*

$$K^+ = -r^+\, Q(\tilde{\gamma}_0^* \quad \tilde{\gamma}_1^*)\mathfrak{A}_\Delta. \tag{5.16}$$

Such that the Calderón projector C^+ is given by:

$$C^+ = \rho\, K^+. \tag{5.17}$$

Lemma 5.2.16. *The Calderón projector maps: $\mathcal{H}_2^{s-\frac{1}{2},\gamma}(\mathbb{B}) \times \mathcal{H}_2^{s-\frac{3}{2},\gamma}(\mathbb{B})$ into itself.*

Proof. The Greens matrix $\mathfrak{A}$ obeys the following mapping properties:

$$\mathfrak{A}: \mathcal{H}_2^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{B}) \times \mathcal{H}_2^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{B}), \to \mathcal{H}_2^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{B}) \times \mathcal{H}_2^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{B}).$$

Now, the mapping property of C^+ follows from Lemma 5.2.14. □

5.2.4 The Construction of the DtN Operator

We take the Calderón projector C^+ as defined in Equation (5.17) and write:

$$C^+ = \begin{pmatrix} C_{00}^+ & C_{01}^+ \\ C_{10}^+ & C_{11}^+ \end{pmatrix}. \tag{5.18}$$

We have:

$$Q\Delta_c = I + R_G,$$

for a Greens operator $R_G \in C_G(\Sigma, (1, 1, \theta))$.

We obtain from equation (20.1.6), p 235 of [14], that:

$$\rho\, u + \rho R_G e^+ u = \rho Q e^+(\Delta_c u) + \rho\, Q \rho^* \mathcal{A} \rho\, u, \tag{5.19}$$

for Cauchy data

$$\rho\, u = \begin{pmatrix} \gamma_0(u) \\ \gamma_1(u) \end{pmatrix}, \tag{5.20}$$

and $u \in \mathcal{H}^{s,1}(\mathbb{D})$.

We know that we can solve the Dirichlet problem for $f \in \mathcal{H}^{s,\frac{1}{2}}(\mathbb{B})$, using a u which solves $\Delta_c u = 0$, we obtain:

$$\rho\, u + \rho R_G e^+ u = \rho\, Q \rho^* \mathcal{A} \rho\, u. \tag{5.21}$$

Or, with $C^+ = \rho\, Q\rho^* \mathcal{A}$:

$$\rho\, u + \rho R_G e^+ u = C^+ \rho\, u. \tag{5.22}$$

Here, we need to relate the Cauchy data $\rho(u)$ to the solution u of the Dirichlet problem. However, we know, that the solution u of the Dirichlet problem exists and is uniquely determined for $\gamma_0(u) = f \in \mathcal{H}^{s,\frac{1}{2}}(\mathbb{B})$, and $s \geq \frac{1}{2}$. Further, the solution operator K^D : $\mathcal{H}^{s,\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}^{s+\frac{1}{2},1}(\mathbb{D})$ is continuous.

We obtain:

$$\rho\, u + \rho R_G e^+ (K^D \gamma_0(u)) = C^+ \rho\, u. \tag{5.23}$$

or:

$$(C^+ - I)\rho\, u = \rho R_G e^+ (K^D \gamma_0(u)) \tag{5.24}$$

where we obtain that for $\gamma_0(u) = g \in \mathcal{H}^{s,\frac{1}{2}}(\mathbb{B})$, the mapping $g \mapsto \rho R_G e^+(K^D g)$ gives a Greens operator due to the following considerations:

Given $g \in \mathcal{H}^{s,\frac{1}{2}}(\mathbb{B})$ with $s \geq 0$, we have $K^D g \in \mathcal{H}^{s+\frac{1}{2},1}(\mathbb{D})$, therefore $e^+(K^D g) \in \mathcal{H}^{0,1}(\mathbb{D})$. Due to the mapping properties of R_G, we then have $R_G e^+(K^D g) \in \mathcal{H}_P^{\infty,1}(\mathbb{D})$ for some asymptotics type P. Then, the restriction to the boundary yields, for some asymptotic types P' and P'':

$\gamma_0(R_G e^+(K^D g)) \in \mathcal{H}_{P'}^{\infty,\frac{1}{2}}(\mathbb{B})$ and $\gamma_1(R_G e^+(K^D g)) \in \mathcal{H}_{P''}^{\infty,-\frac{1}{2}}(\mathbb{B})$

We extend the mapping properties from $s \geq \frac{1}{2}$ to $s \geq 0$ by conjugation with order reducing operators and their inverses.

To show the mapping properties for the adjoint operators, we compute:

$$(\rho R_G e^+ K_D)^* = (K^D)^* r^+ R_G^* \rho^*. \tag{5.25}$$

First, we observe that $r^+R_G^*\gamma_0^*$ maps $\mathcal{H}^{s,-\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}_Q^{\infty,-1}(\mathbb{D})$ and $r^+R_G^*\gamma_1^* : \mathcal{H}^{s,\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}_{Q'}^{\infty,-1}(\mathbb{D})$ for some asymptotic types Q and Q'.

So we compute $(K_D)^*$:

Let $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$, $s \geq \frac{1}{2}$, $v \in \mathcal{H}^{s,\gamma}(\mathbb{D})$. We can solve $v = \Delta w$ with $\gamma_0(w) = 0$.

We obtain, using Green's formula:

$$\begin{aligned}\int_{\mathbb{D}} K^D u\, v\, dx &= \int K^D u\, \Delta w\, dx \\ &= \int_{\mathbb{D}} K^D u \Delta w\, dx - \int_{\mathbb{D}} \underbrace{(\Delta K^D u)}_{=0}\, w\, dx \\ &= \int_{\mathbb{B}} \gamma_0(u)\gamma_1(w)\, dS - \int_{\mathbb{B}} \gamma_1(u)\underbrace{\gamma_0(w)}_{=0}\, dS \\ &= \int_{\mathbb{B}} \gamma_0(u)\gamma_1(\Delta_D^{-1} v)\, dS \end{aligned} \tag{5.26}$$

With $\Delta_D^{-1} : \mathcal{H}^{s-1,-1}(\mathbb{D}) \to \mathcal{H}^{s+1,1}(\mathbb{D})$ being the solution operator for the Dirichlet problem $v = \Delta w$, $\gamma_0(w) = 0$.

Hence, $(K^D)^* = \gamma_1 \Delta_D^{-1} : \mathcal{H}^{s-1,-1}(\mathbb{D}) \to \mathcal{H}^{s-\frac{1}{2},-\frac{1}{2}}(\mathbb{B})$.

We end up with the desired mapping properties:

$$(\gamma_0 R_G e^+ K_D)^* : \mathcal{H}^{s,-\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}_{\tilde{Q}}^{\infty,-1}(\mathbb{B}) \tag{5.27}$$

$$(\gamma_1 R_G e^+ K_D)^* : \mathcal{H}^{s,\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}_{\tilde{Q}'}^{\infty,-1}(\mathbb{B}), \tag{5.28}$$

for asymptotic types $\tilde{Q}$ and $\tilde{Q}'$.

Consequently the identity:

$$(C^+ - I)\rho(u) = 0, \tag{5.29}$$

is fulfilled up to regularizing Greens operators.

Here the left side of the above equation is a vector with two entries. Defining $G = \gamma_0 R_G e^+ K_D$, the upper enties give:

$$(C_{00}^+ - 1)\gamma_0(u) + C_{01}^+\gamma_1(u) = G\gamma_0(u),$$

with $G \in C_G(\mathbb{B}, (\frac{1}{2}, -\frac{1}{2}, \theta))$.

Therefore:

$$C_{01}^{+}\,\gamma_1(u) = (1 - C_{00}^{+})\gamma_0(u) + G\gamma_0(u). \tag{5.30}$$

At this point, we see that the entries C_{00} and C_{01} of C^+ relate the Dirichlet and Neumann data of the Dirichlet problem to each other. At this point we would like to apply an inverse of C_{01} from the left in equation (5.30), which would give $\mathcal{N} = (C_{01}^{+})^{-1}(1 - C_{00})$, up to regularizing Greens operators.

Since $C_{01} \in C^1(\mathbb{B}, \mathbf{g})$ it is a natural question to ask for the existence of a Fredholm inverse of C_{01}^{+}.

C_{01}^{+} is explicitly given by:

$$C_{01}^{+} = -\gamma_0 r^+ Q \gamma_0^* \mathfrak{a}_{01}, \qquad \text{with } \mathfrak{a}_{01} = i\, h_{nn}. \tag{5.31}$$

The crucial property for the existence of a parametrix for C_{01} is the invertibility of the conormal symbol $\sigma_{\partial\mathcal{M}}^{-1}(C_{01})$ of C_{01}. The following lemma allows us to compute the conormal symbols of the involved cone operators:

Lemma 5.2.17. *We consider the composition $\gamma_0 Q \gamma_l^*$ as described in 5.2.14. We have seen that this defines a Mellin operator acting on distributions on $\mathbb{B}$. We have the conormal symbol $\sigma_{\partial\mathcal{M}}^{\mu+l+1}(\tilde{\gamma}_0 Q \tilde{\gamma}_l^*)$:*

$$\sigma_{\partial\mathcal{M}}^{\mu+l+1}(\tilde{\gamma}_0 Q \tilde{\gamma}_l^*) = \lim_{y_n \to 0^+} (r^+ \sigma_{\mathcal{M}}^{\mu}(Q)(z + 1 + l)(\delta^{(l)}(y_n) \otimes \mathbb{I})). \tag{5.32}$$

Proof. This follows directly from the definition of the conormal symbol together with the Expression (A.27) for γ_0^* and the identity $\mathrm{op}_{\mathcal{M}}^{\gamma}(h(z))x^{-l} = x^{-l}\mathrm{op}_{\mathcal{M}}^{\gamma+l}(h(z+l))$. □

Lemma 5.2.18. *The element $C_{01}^{+} = -\tilde{\gamma}_0 r^+ Q \tilde{\gamma}_0^* i\, h_{nn}$ defines a cone operator which is cone degenerate elliptic in $C^{-1}(\mathbb{B}, \mathbf{g})$ for the weight data $\mathbf{g} = (-\frac{1}{2}, \frac{1}{2}, \infty)$.*

Proof. We have to show that $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})$ is invertible on $\Gamma_{\frac{n+1}{2}}$.

We split the proof in three parts: First, we compute the conormal symbol and show that it is injective. In the second part, we show that also $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})^*$ is injective. In the third part, we show that $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+}) \in \Psi^1(\partial Y, \Gamma_{\frac{n-1}{2}})$ is an elliptic pseudodifferential operator and therefore a Fredholm operator. Consequently it has closed range and the invertibility of $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})(z) : H^s(\partial Y) \to H^{s+1}(\partial Y)$ follows from the closed range theorem.

Step 1

The operator Q which is contained in the definition of C_{01}^{+} is a parametrix for Δ_c which is cone degenerate elliptic as an operator mapping $\mathcal{H}_2^{1,1}(\Omega) \to \mathcal{H}_2^{-1,-1}(\Omega)$.

We have, by Lemma 5.2.17:

$$
\begin{aligned}
\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+}) &= \lim_{y_n\to 0^+}(-r^{+}\sigma_{\mathcal{M}}^{-2}(Q)(z+1)i\,h_{nn}(\mathbb{I}\otimes\delta(y_n)) \\
&= \lim_{y_n\to 0^+}(-r^{+}\mathfrak{q}(z+1)i\,h_{nn}(\mathbb{I}\otimes\delta(y_n)),
\end{aligned} \tag{5.33}
$$

with $\mathfrak{q}(z) = \sigma_{\mathcal{M}}^{-2}(Q)(z) = ((z-2)^2-(n-1)(z-2)+\Delta_\Sigma)^{-1}$.

The operator C_{01}^{+} maps: $\mathcal{H}_2^{s,-\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}_2^{s+1,\frac{1}{2}}(\mathbb{B})$, therefore it is defined on $\mathcal{H}_2^{s,-\frac{1}{2}}(\mathbb{B})$. This means that its conormal symbol $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})(z)$ is evaluated on z with $\Re z = \frac{n}{2}-\gamma = \frac{n+1}{2}$, hence on the line $\Gamma_{\frac{n+1}{2}}$.

We parametrize $\Gamma_{\frac{n+1}{2}}$ by $z_\omega = \frac{n+1}{2}+i\omega$ with $\omega\in\mathbb{R}$.

We evaluate $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})$ on z_ω:

$$
\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})(z_\omega) = -\gamma_0 r^{+}(-(\frac{n-1}{2})^2-\omega^2+\Delta_Y)^{-1}(i\,h_{nn}(\mathbb{I}\otimes\delta(y_n))). \tag{5.34}
$$

We assume that there is a $u\in H^{-\frac{1}{2}}(\partial Y)$, $u\neq 0$, such that $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})(z_\omega)u = 0$ for some $\omega\in\mathbb{R}$.

We define $v := r^{+}(-\frac{(n-1)^2}{4}-\omega^2+\Delta_Y)^{-1}(i\,h_{nn}(\mathbb{I}\otimes\delta(y_n)))u$ and observe, that:

$$
\begin{aligned}
&(-\frac{(n-1)^2}{4}-\omega^2+\Delta_Y)v \\
=&r^{+}(-\frac{(n-1)^2}{4}-\omega^2+\Delta_Y)(-\frac{(n-1)^2}{4}-\omega^2+\Delta_Y)^{-1}(i\,h_{nn}(\mathbb{I}\otimes\delta(y_n)))u \\
=&r^{+}\gamma_0^{*}u = 0.
\end{aligned}
$$

This shows, that v is a solution of the following Dirichlet problem:

$$
(\Delta_Y - \frac{(n-1)^2}{4}-\omega^2)v = 0, \qquad v\in H_2^1(Y). \tag{5.35}
$$

Further, since we assumed that $\gamma_0(v) = \sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})u = 0$, we get by the maximum principle that $v=0$.

Now, since $(z^2-(n-1)z+\Delta_Y)^{-1}$ is invertible, $v=0$ implies $\gamma_0^{*}(u)=0$, and consequently, $u=0$.

Therefore we have shown the injectivity of $\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+}) : H^{\frac{1}{2}}(\partial Y)\to H^{-\frac{1}{2}}(\partial Y)$.

Step 2

With $-\mathfrak{a}_{01} = -i\,h_{nn}$, we have that:

$$\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01}) = \tilde{\gamma}_0\sigma^{-2}_{\mathcal{M}}(Q)(z+1)\gamma_0^*\mathfrak{a}_{01},$$

and it follows that:

$$\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})^* = -\tilde{\gamma}_0(\sigma^{-2}_{\mathcal{M}}(Q)(z+1))^*\gamma_0^*\mathfrak{a}_{01}.$$

Now, Q defines a parametrix for the Laplacian Δ_c, and it follows for $u, v \in \mathcal{H}^{1,1}_2(\Omega)$ by partial integration, that:

$$\langle \Delta_c u, v\rangle = \langle u, \Delta_c v\rangle. \tag{5.36}$$

Hence: $\Delta_c^* = \Delta_c$ on $\mathcal{H}^{1,1}_2(\Omega)$, and consequently $\sigma^2_{\mathcal{M}}(\Delta_c) = \sigma^2_{\mathcal{M}}(\Delta_c^*)$. It follows that:

$$\sigma^{-2}_{\mathcal{M}}(Q) = \sigma^{-2}_{\mathcal{M}}(Q^*) = \sigma^{-2}_{\mathcal{M}}(Q)^*.$$

Therefore, we have $\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})^* = -\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})$, and the injectivity of $\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})^*$ follows from the injectivity of $\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})$.

Step 3

We have, by equation (5.33), with $\mathfrak{q}(z) = \sigma^{-2}_{\mathcal{M}}(Q) = ((z-2)^2 - (n-1)(z-2) + \Delta_\Sigma)^{-1}$:

$$\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})(z) = \lim_{y_n\to 0^+}(-r^+\mathfrak{q}(z+1)i\,h_{nn}(\mathbb{I}\otimes\delta(y_n)) \tag{5.37}$$

Using:

$$\sigma^{-2}_{\psi}(\mathfrak{q}(z+1)) = \frac{1}{-\sum_{i,j=1}^n h_{ij}\xi_i\,\xi_j}, \tag{5.38}$$

we obtain:

$$\begin{aligned}
&\sigma^{-1}_{\psi}\left(\sigma^{-1}_{\partial\mathcal{M}}(C^{+}_{01})\right)\\
&= \left(\lim_{y_n\to 0^+} r^+(2\pi)^{-n}\int\limits_{\mathbb{R}^n} e^{iy\xi}\frac{-i(2\pi)^{-n}\,h_{nn}}{-\sum_{i,j=1}^n h_{ij}\xi_i\xi_j}\cdot\mathcal{F}_{y\to\xi}(\delta(y_n)\otimes\mathbb{I})\mathrm{d}\xi\right)\\
&= \left(\int\limits_{\mathbb{R}^{n-1}} e^{iy'\xi'}\lim_{y_n\to 0}\int\limits_{\xi_n} e^{iy_n\xi_n}\frac{(2\pi)^{-n}i\,h_{nn}}{\sum_{i,j=1}^n h_{ij}\xi_i\xi_j}\cdot\mathcal{F}_{y\to\xi}(\delta(y_n)\otimes\mathbb{I})\mathrm{d}\xi\right).
\end{aligned}$$

We close the integration along ξ_n in the upper complex plane in order to apply the residue theorem and denote the integration along this path by Ω_{ξ_n}:

$$\sigma_\psi^{-1}\left(\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^{+})\right) = \left(\int\limits_{\mathbb{R}^{n-1}} e^{iy'\xi'} \int\limits_{\Omega_{\xi_n}} \frac{(2\pi)^{-n} i\, h_{nn}\,\mathrm{d}\xi_n}{\sum_{i,j=1}^{n} h_{ij}\xi_i\xi_j} \underbrace{\mathcal{F}_{y_n\to\xi_n}(\delta(y_n))}_{=1} \mathcal{F}_{y'\to\xi'}\,\mathbb{I})\mathrm{d}\xi'\right).$$

We regard the denominator of the fraction contained in the above integral as a polynomial in ξ_n.

We define:

$$v_1(\xi') = \sum_{i=1}^{n-1}(h_{in}+h_{ni})\xi_i, \qquad v_2(\xi') = \sum_{i,j=1}^{n-1} h_{ij}\xi_i\xi_j, \tag{5.39}$$

so that we can write the denominator as:

$$(h_{nn}\,\xi_n^2 + v_1(\xi')\,\xi_n + v_2(\xi')), \tag{5.40}$$

which is a polynomial of second order in ξ_n.

We compute the zeros of (5.40), regarded as a function in ξ_n:

$$\kappa_{1/2}(\xi', z-1) = -\frac{v_1(\xi')}{2\cdot h_{nn}} \pm \frac{1}{2}\sqrt{\frac{v_1(\xi')^2}{(h_{nn})^2} - 4\frac{v_2(\xi')}{h_{nn}}}, \tag{5.41}$$

so that (5.40) factorizes in:

$$h_{nn}(\xi_n - \kappa_1(\xi'))(\xi_n - \kappa_2(\xi')),$$

and we have:

$$\sigma_\psi^{-1}\left(\sigma_{\partial\mathcal{M}}^{-1}(C_{00,0}^{+})\right) = \left(\int\limits_{\mathbb{R}^{n-1}} e^{iy'\xi'} \int\limits_{\Omega_{\xi_n}} \frac{(2\pi)^{-n} i\,\mathrm{d}\xi_n}{(\xi_n-\kappa_1(\xi'))(\xi_n-\kappa_2(\xi'))}\,\mathrm{d}\xi'\right). \tag{5.42}$$

Our aim is now to compute the integral:

$$\int\limits_{\Omega_{\xi_n}} \frac{\mathrm{d}\xi_n}{(\xi_n-\kappa_1(\xi'))(\xi_n-\kappa_2(\xi'))}, \tag{5.43}$$

which appears in (5.42).

The integrator contains two singularities in ξ_n. To evaluate the integral with the residue theorem we have to find out which of them are contained in the upper half plane.

We know that the metric (h_{ij}) is positive definite, which gives us that:

$$h_{nn}\xi_n^2 + v_1(\xi')\xi_n + v_2(\xi') \geq 0 \text{ and } h_{nn} > 0. \tag{5.44}$$

Evaluation of the first inequality at $\xi_n = -\frac{v_1(\xi')}{2h_{nn}}$ and multiplication of both sides by $(h_{nn})^{-1}$ gives:

$$\begin{aligned} &-\frac{v_1(\xi')^2}{4(h_{nn})} + v_2(\xi') > 0 \text{ and } h_{nn} > 0, \\ \Rightarrow &\frac{v_1(\xi')^2}{(h_{nn})^2} - 4\frac{v_2(\xi')}{h_{nn}} > 0. \end{aligned} \tag{5.45}$$

Now from (5.45) it is clear, that the term under the square root in (5.41) is positive.

We obtain:

$$\kappa_{1/2} \in \mathbb{C}_{-/+}, \tag{5.46}$$

with $\mathbb{C}_+ = \{z \in \mathbb{C} | \mathrm{im}(z) > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} | \mathrm{im}(z) < 0\}$.

Locating the κ_i in the complex plane allows us to evaluate (5.43) with help of the residue theorem:

$$\int\limits_{\Omega_{\xi_n}} \frac{\mathrm{d}\xi_n}{(\xi_n - \kappa_1(\xi'))(\xi_n - \kappa_2(\xi'))} = \frac{2\pi\, i}{\cdot(\kappa_1 - \kappa_2)(\xi')}.$$

Hence:

$$\begin{aligned} \sigma_\psi^{-1}\left(\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^+)\right) &= \sigma_\psi^{-1}\left(\int\limits_{\mathbb{R}^{n-1}} e^{iy'\xi'}\frac{(2\pi)^{-(n-1)}}{(\kappa_1 - \kappa_2)(\xi')}\mathrm{d}\xi'\right) \\ &= \frac{1}{(\kappa_1 - \kappa_2)(\xi')} = \left(\sqrt{\frac{v_1(\xi')^2}{(h_{nn})^2} - 4\frac{v_2(\xi')}{h_{nn}}}\right)^{-1}. \end{aligned} \tag{5.47}$$

Therefore we see that $\sigma_\psi^{-1}(\sigma_{\partial\mathcal{M}}^{-1}(C_{01}^+))$ is clearly non-zero for all $\xi' \in \mathbb{R}^{n-1}$ and $z \in \Gamma_{\frac{n+1}{2}}$. □

Theorem 5.2.19. *The Dirichlet to Neumann operator $\mathcal{N}$ can be expressed, up to smoothing terms, as a cone pseudodifferential operator of order 1 which is contained in $C^1(\mathbb{B}, \mathbf{g})$ with $\mathbf{g} = (\frac{1}{2}, -\frac{1}{2}, \theta)$ for all $\theta > 0$, as defined in Defintion 3.5.1, and maps continuously:*

$$\mathcal{N} : \mathcal{B}_p^{s-\frac{1}{p},\frac{1}{2}}(\mathbb{B}) \to \mathcal{B}_p^{s-1-\frac{1}{p},-\frac{1}{2}}(\mathbb{B}). \tag{5.48}$$

Proof. We have shown in Lemma 5.2.18 that C_{01} is cone degenerate elliptic, therefore we can choose a parametrix $(C_{01})^{\circ -1}$ for C_{01} and obtain:

$$\gamma_1(u) \sim (C_{01})^{\circ -1}(1 - C_{00}))\gamma_0(u). \tag{5.49}$$

If we denote by $\mathcal{N}$ the operator which maps Dirichlet to Neumann boundary data, we have shown that:

$$\mathcal{N} \sim \underbrace{(C_{01})^{\circ -1}(1 - C_{00}))}_{=:\bar{\mathcal{N}}}. \tag{5.50}$$

□

5.3 Ellipticity of the DtN Operator

We have already seen in 5.2.19, that the Dirichlet to Neumann map $\mathcal{N}$ is given by a pseudodifferential operator which is contained in $C^1(\mathbb{B}, (\frac{1}{2}, -\frac{1}{2}, \infty))$ and is determined up to a regularizing operator by $\bar{\mathcal{N}} = (C^+_{01})^{\circ -1}(1 - C^+_{00})$.

We denote the conormal symbol of $\bar{\mathcal{N}}$ by:

$$\sigma^1_{\partial\mathcal{M}}(\bar{\mathcal{N}}) =: \lambda(z). \tag{5.51}$$

Here $\lambda(z)$ is a parameter dependent pseudodifferential operator which is meromorphic in z with no poles on the line $\Gamma_{\frac{n-1}{2}}$.

Now we compute the principal symbol of $\lambda(z)$. This will be useful to prove the Fredholm property of $\mathcal{N}$:

Lemma 5.3.1. *The principal symbol $\sigma^1_\psi(\lambda(z))$ of $\lambda(z) \in \Psi^1(\partial Y, \Gamma_{\frac{n-1}{2}})$ is given in local coordinates by:*

$$\sigma^1_\psi(\lambda(z)) = \kappa_1(\xi'),$$

where

$$\kappa_1(\xi') = -\frac{v_1(\xi')}{2 \cdot h_{nn}} + \frac{1}{2}\sqrt{\frac{v_1(\xi')^2}{(h_{nn})^2} - 4\frac{v_2(\xi')}{h_{nn}}},$$

with:

$$v_1(\xi') = \sum_{i=1}^{n-1}(h_{in} + h_{ni})\xi_i, \quad v_2(\xi') = \sum_{i,j=1}^{n-1} h_{ij}\xi_i\xi_j.$$

Proof. The entries of C^+ are explicitly given by:

$$C^+ = \begin{pmatrix} -\gamma_0 r^+ Q\gamma_0^* \mathfrak{a}_{00} - \gamma_0 r^+ Q\gamma_1^* \mathfrak{a}_{10} & -\gamma_0 r^+ Q\gamma_0^* \mathfrak{a}_{01} \\ -\gamma_1 r^+ Q\gamma_0^* \mathfrak{a}_{00} - \gamma_1 r^+ Q\gamma_1^* \mathfrak{a}_{10} & -\gamma_1 r^+ Q\gamma_0^* \mathfrak{a}_{01} \end{pmatrix}, \tag{5.52}$$

and we go on to compute $\sigma^1_\psi(\sigma^1_{\mathcal{M}}((C^+_{01})^{-1}))$ and $\sigma^0_\psi(\sigma^0_{\mathcal{M}}((I - C^+_{00})))$ separately.

We write:

$$C^+_{00} = \underbrace{-\tilde{\gamma}_0 r^+ Q \tilde{\gamma}^*_0(\frac{i}{x} L_1 f)}_{=:C^+_{00,0}} + \underbrace{i\tilde{\gamma}_0 r^+ Q(\frac{1}{x}\mathrm{D}_n)^* \tilde{\gamma}^*_0\, i\, h^{nn}}_{=:C^+_{00,1}},$$

such that $C^+_{00} = C^+_{00,0} + C^+_{00,1}$.

Therefore we can split the calculation of $\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C_{00}))$ into two parts:

$$\sigma^0_\psi(\sigma^0_{\mathcal{M}}(C_{00})^+) = \sigma^0_\psi(\sigma^0_{\mathcal{M}} C_{00,0})) + \sigma^0_\psi(\sigma^0_{\mathcal{M}}(C_{00,1})). \tag{5.53}$$

Computation of $\sigma^0_\psi(\sigma^0_{\mathcal{M}}(C_{00,0}))$

We obtain, by Lemma 5.2.17, for the conormal symbol:

$$\sigma^0_{\partial\mathcal{M}}(C^+_{00,0})(z) = \sigma^{-1}_{\partial\mathcal{M}}(-\tilde{\gamma}_0 r^+ Q \tilde{\gamma}^*_0)(z+1)\sigma^1_{\partial\mathcal{M}}(\frac{i}{x} L_1) \tag{5.54}$$

$$= \lim_{y_n \to 0^+}(-r^+ \sigma^{-2}_{\mathcal{M}}(Q)(z+2)\delta(y_n) \otimes \mathbb{I})(i\, L_1). \tag{5.55}$$

Again, we write: $\mathfrak{q}(z) = \sigma^{-2}_{\mathcal{M}}(Q)(z) = ((z-2)^2 - (n-1)(z-2) + \Delta_\Sigma)^{-1}$.

Therefore, we have:

$$\sigma^0_{\partial\mathcal{M}}(C^+_{00,0})(z) = \lim_{y_n \to 0^+}(-r^+ \mathfrak{q}(z+2)(\delta(y_n) \otimes \mathbb{I}))i\, L_1. \tag{5.56}$$

And we obtain for the principal symbol:

$$\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C^+_{00,0})) = \sigma^{-1}_\psi(-(\tilde{\gamma}_0 r^+ \mathfrak{q}(z+2)(\delta(y_n) \otimes \mathbb{I})))\sigma^1_\psi(i\, L_1). \tag{5.57}$$

We have, that:

$$L_1 = \mathrm{op}_{\xi'}(v_1(\xi')) + \sqrt{\underline{h}}^{-1} \sum_{i=1}^{n-1} D_i(\sqrt{\underline{h}}(\underline{h}^{in} + \underline{h}^{ni})).$$

Therefore $\sigma^1_\psi(L_1) = v_1(\xi')$, and we arrive at the result:

$$\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C^+_{00,0})) = \frac{v_1(\xi')}{h_{nn} \cdot (\kappa_1 - \kappa_2)(\xi')}. \tag{5.58}$$

Computation of $\sigma^0(\sigma^0_{\partial\mathcal{M}}(C^+_{00,1}))$

We have:

$$(\sigma^0_{\partial\mathcal{M}}(C^+_{00,1}))(z) = \sigma^{-1}_{\partial\mathcal{M}}(i\tilde{\gamma}_0 r^+(Q))(z+1)\sigma^1_{\partial\mathcal{M}}(\frac{1}{x}\mathrm{D}_n)^*\tilde{\gamma}_0^*\, i\, h^{nn})$$
$$= i\tilde{\gamma}_0 r^+(\sigma^{-2}_{\mathcal{M}}(Q)(z+2))((\mathrm{D}_n)^*\tilde{\gamma}_0^*\, i\, h^{nn}).$$

The calculations for $\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C_{00,1}))$ work analogous to the ones for $\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C_{00,0}))$:

$$\begin{aligned}
&\sigma^0_\psi\left(\sigma^0_{\partial\mathcal{M}}(C_{00,1})\right)\\
&= \int\limits_{\mathbb{R}^{n-1}} e^{iy'\xi'} \int\limits_{\Omega_{\xi_n}} \frac{(2\pi)^{-n} i}{\sum_{i,j=1}^n h_{ij}\xi_i\xi_j} \underbrace{\mathcal{F}_{y_n\to\xi_n}(\delta^{(1)}(y_n))}_{=\xi_n} \mathcal{F}_{y'\to\xi'}(h_{nn})\mathrm{d}\xi_n\,\mathrm{d}\xi'\\
&= \int\limits_{\mathbb{R}^{n-1}} e^{iy'\xi'} \int\limits_{\Omega_{\xi_n}} \frac{(2\pi)^{-n} i\,\xi_n}{h_{nn}(\xi_n-\kappa_1(\xi'))(\xi_n-\kappa_2(\xi'))}\mathrm{d}\xi_n \mathcal{F}_{y'\to\xi'}(h_{nn})\mathrm{d}\xi',
\end{aligned}$$

where the residue theorem gives:

$$\int\limits_{\Omega_{\xi_n}} \frac{\xi_n}{(\xi_n-\kappa_1(\xi'))(\xi_n-\kappa_2(\xi'))}\mathrm{d}\xi_n = 2\pi\, i\, \frac{\kappa_1(\xi')}{(\kappa_1-\kappa_2)(\xi')},$$

and we obtain as expression for $\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C_{00,1}))$:

$$\sigma^0_\psi\left(\sigma^0_{\partial\mathcal{M}}(C_{00,1})\right) = \frac{\kappa_1(\xi')}{(\kappa_1-\kappa_2)(\xi')}. \tag{5.59}$$

Computation of $\sigma^0_\psi(\sigma^0_{\partial\mathcal{M}}(C^+_{01})^{\circ -1})$

For the conormal symbol, it holds that:

$$\sigma^1_{\partial\mathcal{M}}((C^+_{01})^{\circ -1})(z) = \left(\sigma^{-1}_{\partial\mathcal{M}}(C^+_{01}))(z+1)\right)^{-1}. \tag{5.60}$$

Therefore, we have that:

$$\sigma^1_\psi(\sigma^1_{\partial\mathcal{M}}(C^+_{01})^{\circ -1}))(\xi') = \left(\sigma^{-1}_\psi(\sigma^{-1}_{\partial\mathcal{M}}(C^+_{01}))(\xi'))\right)^{-1}. \tag{5.61}$$

We have already computed $\sigma^{-1}_\psi(\sigma^{-1}_{\partial\mathcal{M}}(C^+_{01}))$ in (5.33), the result was:

$$\sigma^{-1}_\psi(\sigma^{-1}_{\partial\mathcal{M}}(C^+_{01}))(\xi') = \frac{1}{(\kappa_1-\kappa_2)(\xi')}.$$

Consequently, we obtain the result:

$$\sigma^1_\psi(\sigma^1_{\partial\mathcal{M}}(C^+_{01})^{\circ -1}))(\xi') = (\kappa_1-\kappa_2)(\xi'). \tag{5.62}$$

Computation of $\sigma^1_\psi(\sigma^1_{\partial\mathcal{M}}(\lambda))$

Since the conormal symbol of operators behaves multiplicative, we have the following relation for $\sigma^1_{\mathcal{M}}(\mathcal{N})$:

$$\begin{aligned}\lambda(z) = \sigma^1_{\mathcal{M}}(\mathcal{N}) &= \sigma^1_{\mathcal{M}}((C^+_{01})^{-1}(I - C^+_{00})) \\ &= \sigma^1_{\mathcal{M}}((C^+_{01})^{-1}) \cdot \sigma^0_{\mathcal{M}}((I - C^+_{00})).\end{aligned}$$

Consequently, we have for the principal symbol of $\lambda(z)$:

$$\sigma^1_\psi(\lambda) = \sigma^1_\psi(\sigma^1_{\mathcal{M}}((C^+_{01})^{-1})) \cdot \sigma^0_\psi(\sigma^0_{\mathcal{M}}((I - C^+_{00}))),$$

which gives us the result:

$$\begin{aligned}\sigma^1_\psi(\lambda) &= (\kappa_1 - \kappa_2)(\xi') \left(1 - \frac{\kappa_1(\xi')}{(\kappa_1 - \kappa_2)(\xi')} - \frac{v_1(\xi')}{h_{nn}(\kappa_1 - \kappa_2)(\xi')}\right) \\ &= (\kappa_1 - \kappa_2)(\xi') - \kappa_1(\xi') - \frac{v_1(\xi')}{h_{nn}} \\ &= -\kappa_2(\xi') - \frac{v_1(\xi')}{h_{nn}} = \kappa_1(\xi'). \end{aligned} \tag{5.63}$$

□

Theorem 5.3.2. *The conormal symbol $\sigma^1_{\mathcal{M}}(\mathcal{N})$ of $\mathcal{N}$ is invertible as an operator:*

$$\sigma^1_{\mathcal{M}}(\mathcal{N}) : H^s(\partial Y) \to H^{s-1}(\partial Y), \quad s \in \mathbb{R}. \tag{5.64}$$

Proof. Again, we divide the proof into three steps, establishing injectivity, injectivity of the adjoint and the Fredholm property:

Step 1 The conormal symbol of Δ_c is $\sigma^2_{\mathcal{M}}(\Delta_c) = z^2 - (n-1)z + \Delta_Y : H^1_2(Y) \to H^{-1}_2(Y)$ with $z \in \Gamma_{\frac{n-1}{2}}$. We parametrize $\Gamma_{\frac{n-1}{2}}$ by $z_\omega = \frac{n-1}{2} + i\omega$ for $\omega \in \mathbb{R}$, and see that:

$$\sigma^2_{\mathcal{M}}(\Delta_c)(z_\omega) = -\frac{(n-1)^2}{4} - \omega^2 + \Delta_Y.$$

Now, if we define $c_\omega = \frac{(n-1)^2}{4} + \omega^2$, it follows from the multiplicity of the conormal symbols and Lemma 5.2.17, that the conormal symbol of $\mathcal{N}$ is given by:

$$\sigma^1_{\partial\mathcal{M}}(\mathcal{N}) = (\tilde{C}^+_{01})^{-1}(I - \tilde{C}^+_{00}),$$

where $\tilde{C}^+_{00}, \tilde{C}^+_{01}$ are the entries of the following Calderón projector:

$$\tilde{C}^+ = \tilde{\rho}(-c_\omega + \Delta_\Omega)^{-1}\tilde{\rho}^*. \tag{5.65}$$

Therefore, it is straight forward to see that $\sigma^1_{\partial\mathcal{M}}(\mathcal{N}) : H_2^{\frac{1}{2}}(\partial Y) \to H_2^{-\frac{1}{2}}(\partial Y)$ coincides with the Dirichlet to Neumann operator for the following "shifted" Dirichlet problem:

$$(\mathbf{D1}) \begin{cases} (\Delta - c_\omega)u = 0 & \text{on } Y, \\ \gamma_0(u) = f & \text{on } \partial Y. \end{cases}$$

Using Greens first identity:

$$\int_Y (\psi \Delta_Y \varphi + \nabla\psi \cdot \nabla\varphi) dV = \int_{\partial Y} \psi \frac{\partial \varphi}{\partial \mathbf{n}}, \tag{5.66}$$

and choosing $\psi = \varphi$ as solutions of (**D1**), we obtain:

$$\langle \sigma^1_{\partial\mathcal{M}}(\mathcal{N})\varphi, \varphi\rangle = \int_Y (c_\omega |\varphi|^2 + |\nabla\varphi|^2). \tag{5.67}$$

Which gives us that $\sigma^1_{\partial\mathcal{M}}(\mathcal{N})$ is injective.

Step 2

We show that the conormal symbol $\sigma^1_{\partial\mathcal{M}}(\mathcal{N})$ is symmetric under the dual pairing:

$$\langle \sigma^1_{\partial\mathcal{M}}(\mathcal{N})(u), v\rangle_{L^2(\partial Y)} = \langle u, \sigma^1_{\partial\mathcal{M}}(\mathcal{N})(v)\rangle_{L^2(\partial Y)}.$$

We have seen in the proof of Lemma 5.3.1, that $\sigma^1_{\partial\mathcal{M}}(\mathcal{N}) : H_2^{\frac{1}{2}}(\partial Y) \to H_2^{-\frac{1}{2}}(\partial Y)$ coincides with the Dirichlet to Neumann operator for the shifted Dirichlet problem (**D1**) on Y. Therefore, we can employ Greens second identity:

$$\int_Y (\psi \Delta_c \varphi - \varphi \Delta_c \psi) dV = \int_{\partial Y} (\psi \frac{\partial \varphi}{\partial \mathbf{n}} - \frac{\partial \psi}{\partial \mathbf{n}} \varphi) dS, \tag{5.68}$$

to solutions ψ, φ of (**DP1**), we obtain with $u = \gamma_0(\psi), v = \gamma_0(\varphi)$:

$$0 = \int_{\partial Y} (\psi \frac{\partial \varphi}{\partial \mathbf{n}} - \frac{\partial \psi}{\partial \mathbf{n}} \varphi) dS = \langle u, \sigma^1_{\partial\mathcal{M}}(\mathcal{N})(v)\rangle - \langle \sigma^1_{\partial\mathcal{M}}(\mathcal{N})(u), v\rangle. \tag{5.69}$$

Step 3

We have already computed $\sigma^1_\psi(\sigma^1_{\partial\mathcal{M}}(\mathcal{N}))$ in Lemma 5.3.1, with the result that $\sigma^1_\psi(\lambda(z)) = \mathrm{op}_{\xi'}(\kappa_1(\xi', z))$. Since $\kappa_1(\xi', z)$ is non vanishing on $\Gamma_{\frac{n-1}{2}}$, we can conclude that $\sigma^1_{\partial\mathcal{M}}(\mathcal{N})$ defines a Fredholm operator, which immediately gives that it has closed range.

Hence, we can conclude that $\sigma^1_{\partial\mathcal{M}}(\mathcal{N}) : H^{\frac{1}{2}}(\partial Y) \to H^{-\frac{1}{2}}(\partial Y)$ is invertible.

□

5.4 Basic Properties of the DtN Operator

Lemma 5.4.1. *For boundary datum $f \in C_c^\infty(int\mathbb{B})$, we have for the solution of the Dirichlet problem $\Delta_c u = 0$, $\gamma_0(u) = f$, that $u \in \mathcal{H}_2^{\infty,1}(\mathbb{D})$.*

Proof. Since $C_c^\infty(\mathrm{int}\,\mathbb{B}) \subset \mathcal{H}_2^{s-\frac{1}{2},\frac{1}{2}}(\mathbb{B})$, we know by Theorem 4.0.13, that a solution u exists for all $s \geq 1$. To show that $u \in \mathcal{H}_2^{\infty,1}(\mathbb{D})$, we apply the left entry of K^+ as defined in 5.16 to f. Then the claim follows from the mapping properties of K^+. □

Remark 5.4.2. *Let $\psi, \varphi \in C^2(\mathbb{D})$. Then:*

$$\int_{\mathbb{D}} (\psi\Delta_c\varphi + \nabla\psi\cdot\nabla\varphi)dV = \int_{\partial\mathbb{D}} \psi\frac{\partial\varphi}{\partial\mathbf{n}}, \tag{5.70}$$

and:

$$\int_{\mathbb{D}} (\psi\Delta_c\varphi - \varphi\Delta_c\psi)dV = \int_{\partial\mathbb{D}} (\psi\frac{\partial\varphi}{\partial\mathbf{n}} - \frac{\partial\psi}{\partial\mathbf{n}}\varphi)dS, \tag{5.71}$$

This follows from Gauss theorem which holds on manifolds with conical singularities, see [26] for a fairly general proof.

We gather a few basic results about the DtN map $\mathcal{N}$:

Theorem 5.4.3. *1. The Dirichlet to Neumann operator $\mathcal{N} : \mathcal{H}_2^{\frac{1}{2},\frac{1}{2}}(\mathbb{B}) \to \mathcal{H}_2^{-\frac{1}{2},-\frac{1}{2}}(\mathbb{B})$ for $p = 2$ is self-adjoint with respect to the L^2 scalar product.*

2. The Dirichlet to Neumann operator is positive in the following sense:

Let $f \in \mathcal{H}_2^{1,1}(\mathbb{B})$, then:

$$\langle\mathcal{N}(f), f\rangle = \langle f, \mathcal{N}(f)\rangle \geq 0. \tag{5.72}$$

Proof. 1. We can first apply Greens second identity (5.71):

We have in (5.71) for $f, g \in C_c^\infty(\partial\mathbb{D})$ due to Lemma 5.4.1 the solutions $\psi, \varphi \in \mathcal{H}^{\infty,\infty}(\mathbb{D})$ to the Dirichlet problem, such that $\gamma_0(\varphi) = f, \gamma_0(\psi) = g$, $\Delta_c\varphi = \Delta_c\psi = 0$. It follow that $\frac{\partial\psi}{\partial\mathbf{n}} = \mathcal{N}(f)$, $\frac{\partial\varphi}{\partial\mathbf{n}} = \mathcal{N}(g)$, and we obtain:

$$0 = \int_{\partial\mathbb{D}} (f\mathcal{N}(g) - \mathcal{N}(f)g)dS. \tag{5.73}$$

We obtain with respect to the L^2 dual pairing on $\mathcal{H}_2^{0,0}(\mathbb{D})$:

$$\langle\mathcal{N}(f), g\rangle_{\mathcal{H}_2^{0,0}(\partial\mathbb{D})} = \langle f, \mathcal{N}(g)\rangle_{\mathcal{H}_2^{0,0}(\partial\mathbb{D})}. \tag{5.74}$$

Since $C_c^{\infty}(\mathbb{B})$ is dense in $\mathcal{H}_2^{\frac{1}{2},\frac{1}{2}}(\mathbb{B})$, the identity (5.74) extends to $f, g \in \mathcal{H}_2^{\frac{1}{2},\frac{1}{2}}(\mathbb{B})$ by continuity.

Considered as a bounded operator which maps $\mathcal{H}_2^{\frac{1}{2},\frac{1}{2}}(\mathbb{B})$ to $\mathcal{H}_2^{-\frac{1}{2},-\frac{1}{2}}(\mathbb{B})$, $\mathcal{N}$ can be considered as a self-adjoint operator.

2. We can prove that $\mathcal{N}$ has a positive spectrum with the help of Green's first identity 5.70:

   We choose $f \in C_c^{\infty}(\mathbb{B})$ such that $\Delta_c = u_0, \gamma_0(u_0) = f$. Again, we have by Lemma 5.4.1, that $u_0 \in \mathcal{H}_2^{\infty,1}(\mathbb{D})$. We obtain by setting $\psi = \varphi = u_0$ in (5.70):

$$\int_{\mathbb{D}} (|\nabla u_0|^2) dV = \int_{\partial\mathbb{D}} f\mathcal{N}(f) dS, \tag{5.75}$$

   which gives:

$$\langle f, \mathcal{N}(f)\rangle \geq 0 \qquad \forall f \in \mathcal{H}_2^{\frac{1}{2},\frac{1}{2}}(\partial\mathbb{D}). \tag{5.76}$$

□

Chapter 6

Parameter Ellipticity

We outlined in the last section the construction of the Dirichlet to Neumann operator $\mathcal{N}$ on conical manifolds. Later on, our goal is to prove H^∞ functional calculus for a class of operators containing $\mathcal{N}$. Defining $f(A)$ is the first step to establish H^∞ calculus, and to do this we need the existence of the resolvent $(\mathcal{N}-\lambda)^{-1}$ for λ being contained in a sector Λ of the complex plane. The conditions on $\mathcal{N}$ which are necessary for the existence of the resolvent are summarized by the notion of parameter ellipticity. Therefore, we introduce in this chapter parameter ellipticity and give a prove that $\mathcal{N}$ is parameter elliptic on the right spaces.

While $\mathcal{N}$ was constructed in the last chapter as an operator acting on weighted Besov spaces, we use in this chapter the constructed operator acting on weighted Sobolev spaces which are more handsome for practical computations.

6.1 Definition of Parameter Ellipticity

6.1.1 Model Cone Operator, Kegel Spaces

We want to show that the Dirichlet to Neumann operator is parameter elliptic. The definition of parameter ellipticity for a cone pseudodifferential operators P depends on the invertibility of the model cone operator $P_\wedge$ which is an operator which is obtained from P as described below. $P_\wedge$ acts on the so called Kegel spaces which have been introduced and extensively studied by Schulze, see e.g. [31] or [32]:

Definition 6.1.1. *Let $\partial Y = U_1 \cup \cdots \cup U_J$ be an open covering of ∂Y; let $\kappa_j : U_j \to V_j$ be coordinate maps and $\{\varphi_1, \cdots, \varphi_J\}$ a subordinate partition of unity.*

Given a function $u = u(x, y)$ on $\mathbb{R} \times \partial Y$, we shall say that $u \in H^s_{p,cone}(\mathbb{R} \times \partial Y)$ provided that, for each j, the function:

$$v(x, \tilde{y}) = \varphi_j(y)u(x, y), \quad y = \kappa_j^{-1}(\tilde{y}/[x]), \tag{6.1}$$

is an element of $H^s_p(\mathbb{R} \times \mathbb{R}^{n-1})$ (we consider the right hand to be zero for $x \notin V_j$). In other words: $\varphi_j u$ is the pull-back of a function in $H^s(\mathbb{R}^n)$ under the composition of the maps

$$\mathrm{id} \times \kappa_j : \mathbb{R} \times X_j \ni (x, y) \mapsto (z, [x]y) \in \mathbb{R}^n, \tag{6.2}$$

and

$$\xi : \mathbb{R} \times U_j \ni (t, \tilde{y}) \mapsto (t, [t]\tilde{y}) \in \mathbb{R}^n, \tag{6.3}$$

so that the definition extends to distributions in the usual way for $s \in \mathbb{R}, 1 < p < \infty$.

$\mathcal{K}^{s,\gamma}_p(\partial Y^\wedge)$ is the space of all distributions $u \in H^s_{p,loc}(\mathbb{R}_+ \times \partial Y)$ such that, for an arbitrary cut-off function ω,

$$\omega u \in \mathcal{H}^{s,\gamma}_p(\partial Y) \quad \textit{and} \quad (1 - \omega)u \in H^s_{p,cone}(\mathbb{R} \times \partial Y). \tag{6.4}$$

Usually, the model cone operator is defined for a certain class of differential operators, the so called Fuchs Type operators, which are operators of the form:

$$A = x^{-\mu} \sum_{j=0}^{\mu} a_j(x)(-x\partial_x)^j. \tag{6.5}$$

For this class of operators, the model cone operator $\hat{A}$ is obtained by freezing the coefficients at the boundary, i.e. $\hat{A}$ is expressed as the differential operator given by:

$$\hat{A} = x^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-x\partial_x)^j, \tag{6.6}$$

on the infinite half-cylinder $\mathbb{B}^\wedge = \mathbb{R}_+ \times \mathbb{B}$.

The parameter ellipticity is described by the properties of the model cone operator $B_\wedge$ of a cone pseudodifferential operator $B \in C(\mathbf{g}, \theta)$:

Definition 6.1.2. *Let $B = x^{-\mu}\omega_1 op^\gamma_{\mathcal{M}}(h)\omega_2 + (1-\omega_1)P(1-\omega_3) + x^{-\mu}\omega_1 op^\gamma_{\mathcal{M}}(h_0)\omega_2 + G$, as defined in 3.5.1, then we can define $B_\wedge$ by:*

$$\begin{aligned} B_\wedge : \mathcal{K}^{s,\gamma}(\partial Y^\wedge) &\to \mathcal{K}^{s-\mu,\gamma-\mu}(\partial Y^\wedge), \\ u &\mapsto B_\wedge u := (x^{-\mu}\mathrm{op}^\gamma(h(0, z) + h_0(z)))u. \end{aligned} \tag{6.7}$$

Parameter ellipticity is formulated with respect to a sector Λ in the complex plane:

$$\Lambda = \Lambda(\theta) = \{\lambda = r\, e^{i\varphi} | r \geq 0, \theta \leq \varphi \leq 2\pi - \theta\}.$$

Remark 6.1.3. *The definition of $B_\wedge$ in equation (6.7) is the natural generalisation of the definition of the model cone operator of cone differential operators.*

However, the cone algebra over over the stretched cone $\partial Y^\wedge$ as defined in Chapter 2.2.4. of [32] consists of operators of the form as defined in Definition 3.5.1.

Using cut-off funtions ω, ω', we can rewrite $B_\wedge$ as:

$$B_\wedge = \omega B_\wedge \omega' + (1-\omega) B_\wedge (1-\omega') + (1-\omega) B_\wedge \omega' + \omega B_\wedge (1-\omega')$$

.

We conjecture that $(1-\omega)B_\wedge(1-\omega')$ can be expressed, up to regularizing Green's operators, as $(1-\omega)B_\wedge(1-\omega') = (1-\omega)P(1-\omega')$, P denoting an ordinary Pseudodifferential operator of order μ.

Further, since their kernels are supported away from the diagonal, both $(1-\omega)B_\wedge\omega'$ and $\omega B_\wedge(1-\omega')$ should contribute as regularizing Green's operators.

Unfortunately, we where not able to prove those two statements.

Therefore, we assume in the preceding considerations that $B_\wedge$ maps continuously $\mathcal{K}_p^{s,\gamma}(\partial Y^\wedge) \to \mathcal{K}_p^{s-\mu,\gamma-\mu}(\partial Y^\wedge)$. Further, we assume that $B_\wedge$ as defined in (6.7) is contained in the cone algebra over $\partial Y^\wedge$.

We define parameter-ellipticity:

Definition 6.1.4. *An operator $B \in C^\mu(\mathbb{B}, (\gamma, \gamma - \mu, \theta))$ is said to be parameter-elliptic on a sector $\Lambda \subset \mathbb{C}$ with respect to the weight γ, if and only if:*

- *Both the homogeneous symbol $\sigma_\psi^\mu(B)$ and the rescaled symbol $\tilde{\sigma}_\psi^\mu(B)$ have no spectrum in Λ,*
- *$B_\wedge - \lambda : \mathcal{K}_p^{s,\gamma}(\partial Y^\wedge) \to \mathcal{K}_p^{s-\mu,\gamma-\mu}(\partial Y^\wedge)$ is invertible for every $\lambda \in \Lambda$ sufficiently large, and for some $s \in \mathbb{R}$, $1 < p < \infty$.*

6.1.2 Symmetry Properties

Theorem 6.1.5. *For $u, v \in \mathcal{K}_2^{\frac{1}{2},\frac{1}{2}}(Y^\wedge)$, it is true that:*

$$\langle \mathcal{N}_\wedge u, v \rangle_{\mathcal{K}_2^{0,0}(Y^\wedge)} = \langle u, \mathcal{N}_\wedge v \rangle_{\mathcal{K}_2^{0,0}(Y^\wedge)}. \tag{6.8}$$

Proof. This is a direct consequence of the symmetry of $\mathcal{N}$. □

6.1.3 Positivity

Theorem 6.1.6. *Let* $u \in \mathcal{K}_2^{\frac{1}{2},\frac{1}{2}}(\partial Y^\wedge)$*. Then:*

$$\langle \mathcal{N}_\wedge u, u \rangle \geq 0. \tag{6.9}$$

Proof. Since the Mellin symbol of $\mathcal{N}_\wedge$ coincides with the conormal symbol of $\mathcal{N}$, we can use the result of Theorem 5.3.2, which says that $\sigma^1_{\mathcal{M}}(\mathcal{N})$ coincides with the Dirichlet to Neumann operator of the following Dirichlet problem:

Given $f \in H^{\frac{1}{2}}(\partial Y)$, find $u \in H^1(Y)$, such that:

$$(\mathbf{D1}) \begin{cases} (\Delta_Y(0) - (-z^2 + (n-1)z))u = 0 & \text{on } Y, \\ \gamma_0(u) = f & \text{on } \partial Y. \end{cases}$$

We evaluate in z on the line $\Gamma_{\frac{n+1}{2}-1}$. Writing for $z \in \Gamma_{\frac{n+1}{2}-1}$, $z_\omega = \frac{n-1}{2} + i\omega$ with $\omega \in \mathbb{R}$, we see that

$$(-(z_\omega)^2 + (n-1)z_\omega) = \frac{(n-1)^2}{4} + \omega^2 > 0. \tag{6.10}$$

Consequently, the Mellin symbol defines a positive operator in the sense that:

$$\langle \lambda_\wedge(z)\, u, u \rangle \geq 0 \qquad \forall\, u \in H^{\frac{1}{2}}(\partial Y),\ z \in \Gamma_{\frac{n-1}{2}}. \tag{6.11}$$

We compute:

$$\begin{aligned} \langle \mathrm{op}_{\mathcal{M}}^{\frac{1}{2}}(\lambda_\wedge)u, u \rangle_{L^2(\partial Y^\wedge)} &= \langle \mathcal{M}^{-1} \lambda_\wedge \mathcal{M} u, u \rangle_{L^2(\partial Y^\wedge)} \\ &= \langle \lambda_\wedge \mathcal{M} u, \mathcal{M} u \rangle_{L^2(\Gamma_{\frac{n-1}{2}} \times \partial Y)} \\ &= \int_{\Gamma_{\frac{n-1}{2}}} \langle \lambda_\wedge \mathcal{M} u(z), \mathcal{M} u(z) \rangle_{L^2(\partial Y^\wedge)} dz. \end{aligned} \tag{6.12}$$

□

Lemma 6.1.7. $\mathcal{N}_\wedge : \mathcal{K}_2^{\frac{1}{2},\frac{1}{2}}(\partial Y^\wedge) \to \mathcal{K}_2^{-\frac{1}{2},-\frac{1}{2}}(\partial Y^\wedge)$ *is a Fredholm operator.*

Proof. We have to check the ellipticity condition of $\mathcal{N}_\wedge$ for the cone algebra over $\partial Y^\wedge$. Here, as in the case for the cone algebra over $\mathbb{B}$, we need the invertibility of $\sigma^1_{\mathcal{M}}(\mathcal{N}_\wedge)$ and $\sigma^1_\psi(\mathcal{N}_\wedge)$. Further, we have to introduce an additional symbol, $\sigma^0_e(\mathcal{N}_\wedge)$. Taking (x, y, τ, ξ) as local variables on the fiber $T^*\partial Y$ and developing the pseudodifferential symbol $\mathcal{N}_\wedge$ in contributions which are homogeneous in (x, y), $\sigma_e^0(\mathcal{N}_\wedge)$ denotes the contribution which is homogeneous of order 0 in y. Then, $\mathcal{N}_\wedge$ is Fredholm, if all three symbols are invertible, this is Chapter 2.2.4., Theorem 14 of [32].

- $\sigma^1_{\mathcal{M}}(\mathcal{N}_\wedge)$ is invertible on $\Gamma_{\frac{n-1}{2}}$:

  This follows since $\sigma^1_{\mathcal{M}}(\mathcal{N})(z) : H^{\frac{1}{2}}(\partial Y) \to H^{-\frac{1}{2}}(\partial Y)$ is the Dirichlet to Neumann operator associated to the Dirichlet problem (**D1**) as described in Theorem 6.1.6. It is well known that the Dirichlet to Neumann operator associated to $(\Delta - c)u = 0$ is invertible for $c > 0$.

- $\sigma^1_{\psi}(\mathcal{N}_\wedge)$ is invertible on $T^*\partial Y^\wedge \backslash \{0\}$: In the interior of the manifold, away from the conical singularity, the ellipticity of the principal symbol away from 0 follows from the standard theory, of the Dirichlet to Neumann operator for the smooth case, since here $\sigma^1_{\psi}(\mathcal{N}_\wedge) = |\xi'|$.

- The invertibility of the exit symbol $\sigma_e^0(\mathcal{N}_\wedge)$ is trivial in this case, since the pseudodifferential symbol of $\mathcal{N}_\wedge$ coincides, up to smoothing operators, with the part which is homogeneous of order 0. This is due to the fact that the Mellin operator is evaluated at $x = 0$, and due to the rescaling of the local coordinate charts in 6.1.

□

Theorem 6.1.8. *Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) < 0$. Then $\mathcal{N}_\wedge - \lambda : \mathcal{K}_2^{\frac{1}{2},\frac{1}{2}}(\partial Y^\wedge) \to \mathcal{K}_2^{-\frac{1}{2},-\frac{1}{2}}(\partial Y^\wedge)$ is invertible.*

Proof. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) < 0$. Take a non-zero $u \in \mathcal{K}^{\frac{1}{2},\frac{1}{2}}(\partial Y^\wedge)$. Then, using the dual pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{K}_2^{\frac{1}{2},\frac{1}{2}}(\partial Y^\wedge)$ and $\mathcal{K}_2^{-\frac{1}{2},-\frac{1}{2}}(\partial Y^\wedge)$, we have that:

$$\langle (\mathcal{N}_\wedge - \lambda)u, u \rangle = \langle \mathcal{N}_\wedge u, u \rangle - \lambda \langle u, u \rangle.$$

Therefore, the injectivity of $\mathcal{N}_\wedge - \lambda$ follows by Theorem 6.1.6, since the above expression has strictly positive real part, therefore $(\mathcal{N}_\wedge - \lambda)u \neq 0$ for all $u \neq 0$.

Passing to the formal adjoint gives the injectivity of $(\mathcal{N}_\wedge - \lambda)^*$, since by Theorem 6.1.5, we have $(\mathcal{N}_\wedge - \lambda)^* = (\mathcal{N}_\wedge - \overline{\lambda})$ and $\Re(\overline{z}) = \Re(z)$.

Now we observe that $\sigma^1_{\mathcal{M}}(\mathcal{N}_\wedge - \lambda) = \sigma^1_{\mathcal{M}}(\mathcal{N}_\wedge)$ and $\sigma_\psi(\mathcal{N}_\wedge - \lambda) = \sigma_\psi(\mathcal{N}_\wedge)$. Therefore, the ellipticity of $\mathcal{N}_\wedge - \lambda$ can be derived from that of $\mathcal{N}_\wedge$. Hence $\mathcal{N}_\wedge - \lambda$ is cone degenerate elliptic, therefore a Fredholm operator, therefore has closed range. This implies the invertibility of $\mathcal{N}_\wedge - \lambda$. □

The following theorem establishes the spectral invariance of operators in $C^{\mu}(\partial Y^{\wedge}, \mathbf{g})$:

Theorem 6.1.9. *If $A \in C^{\mu}(\partial Y^{\wedge}, \mathbf{g})$ is invertible as an operator for $\mu \in \mathbb{R}$, $\mathbf{g} = (\gamma, \delta, \Theta)$, $\gamma, \delta \in \mathbb{R}$ as an operator:*

$$A : \mathcal{K}^{s_0,\gamma}_{p_0}(\partial Y^{\wedge}) \to \mathcal{K}^{s_0-\mu,\delta}_{p_0}(\partial Y^{\wedge}), \quad \textit{for some } s_0 \in \mathbb{R}, p_0 \in \mathbb{N},$$

it is invertible for all $s \in \mathbb{R}$ and $1 < p < \infty$.

Proof. An inspection of the proof of Theorem 2.4.49 in [34] shows that we can set $W = 0$ in the prerequisites to obtain an inverse A^{-1} which is contained in $C^{-\mu}(\partial Y^{\wedge}, \mathbf{g}^{-1})$. □

Finally, summarizing the results of Theorem 5.3.2, Theorem 6.1.8 and Theorem 6.1.9 in the following result:

Theorem 6.1.10. *The Dirichlet to Neumann operator $\mathcal{N}$ is parameter elliptic with respect to the weight $\gamma = \frac{1}{2}$ as an operator:*

$$\mathcal{N} : \mathcal{H}^{\frac{1}{2},\frac{1}{2}}_p(\mathbb{B}) \to \mathcal{H}^{-\frac{1}{2},-\frac{1}{2}}_p(\mathbb{B}). \tag{6.13}$$

Chapter 7

Parameter Dependent b-Calculus

7.1 Why b-Calculus?

Everything which has been done so far was formulated in Schulze's cone calculus. Now our aim is to estimate the norm of $f(A)$ for certain bounded functions f. If A is a cone pseudodifferential operator, an important ingredient for the definition of $f(A)$ is the existence and structure of the resolvent $(A-\lambda)^{-1}$, for λ being contained in a suitable subset of $\mathbb{C}$.

Regarding the case of A being a differential operator on conical manifolds, namely a Fuchs type operator as defined in 2.4.1, the resolvent has been computed in the terms of Schulze's cone calculus by Coriasco, Schrohe and Seiler in [6].

In the case of A being a cone pseudodifferential operator, there exist only computations in language of a different school of singular analysis, namely the b-calculus, as developed by Melrose et al, see [22] or [23]. The resolvent is constructed in [10].

As discussed in [18] by Lauter and Seiler, certain elements of Schulze's cone calculus can be identified with operators in the b-calculus and vise versa. However, Gil and Loya introduce additional classes of parameter dependent operators for the construction of the resolvent which have so far no analogue within the language of Schulze's cone calculus.

This is why we use in this chapter this alternative language of b-calculus in order to estimate $f(A)$, for A being a cone pseudodifferential operator.

We begin with a short introduction to the parts of b-calculus which are needed for our calculations. See e.g [11] for a short introduction to the b-calculus, or also [18] for a comparison between b-calculus and cone-algebra.

The basic object on which we establish our calculus is a manifold with conical singularity like the manifold $\mathbb{B}$ as defined in Chapter 2. Since we restrict our studies to the boundary $\mathbb{B}$ of a manifold $\mathbb{D}$, and to stay within the notation which is used in the b-calculus language, we write here X instead of $\mathbb{B}$.

7.2 Blow-up Spaces

The function spaces in b-calculus are basically the same weighted Sobolev spaces $\mathcal{H}^{s,\gamma}$ as defined in chapter 2.2. Since we are working with the notation used in [10], we define the function spaces here as:

Definition 7.2.1.

$$H_b^{s,p}(X) := \mathcal{H}_p^{s,\frac{n}{2}}(X). \tag{7.1}$$

Further, we make use of weighted Sobolev spaces:

$$x^\alpha H_b^{s,p}(X) := \mathcal{H}_p^{s,\frac{n}{2}+\alpha}(X). \tag{7.2}$$

The objects which are blown up are manifolds with corners:

Definition 7.2.2. *An n dimensional manifold with corners X is a paracompact topological space with local models of the form $\mathbb{R}^{n,k} := [0,\infty)_x^k \times \mathbb{R}_y^{n-k}$, where k can run between 0 and n, such that X has only finitely many boundary hypersurfaces, say $\{H_1, ..., H_r\}$ for some $r \in \mathbb{N}_0$, where each H_i is embedded. The set of boundary hypersurfaces is denoted by $X_1(X)$. A total boundary defining function is a function of the form $\rho = \prod_{i=1}^r \rho_i$, where ρ_i is a boundary defining function for H_i.*

We introduce b-densities on X:

Definition 7.2.3. *The b-density bundle, Ω_b, is the trivial bundle with sections $\mathfrak{m}$ of the form $\mathfrak{m} = \rho^{-1}\mu$, where μ is a smooth density on X, and where ρ is a total boundary defining function on X.*

We introduced the wedge space $\Omega^\wedge$ in chapter 2.2 in definition (2.3) and modeled the function spaces over it. The wedge space $\Omega^\wedge$ can be thought of as a special case of a manifold with corners for $k = 1$. Higher orders in k enter e.g. in the analysis of a kernel of an operator which lives on the Cartesian product $\Omega^\wedge \times \Omega^\wedge$ which, in local coordinates, is a manifold with corners with $k = 2$.

Example 7.2.4. *Let $X = [0,\infty)_x \times [0,\infty)_{x'}$ and $Y = (0,0)$. Then we define "X blown up at Y" as the set $[X;Y] \equiv [0,\infty)_r \times \mathbb{S}_\theta^{1,2}$, where $\mathbb{S}^{1,2} = \mathbb{S}^1 \cap [0,\infty)^2$, and where $r = |(x,x')|$ and $\theta = \tan^{-1}(x'/x)$. Hence, the blow-up corresponds just to the introduction of polar coordinates.*

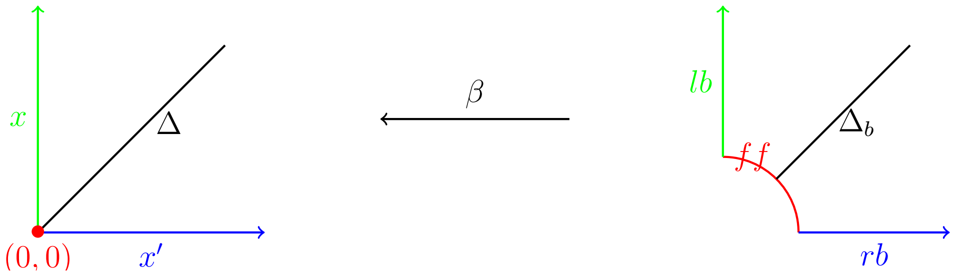


FIGURE 7.1: Blow up of $X = [0, 1) \times [0, 1)$

Further, we define the left boundary, lb, as the set where $\theta = \frac{\pi}{2}$*, the right boundary rb as the set where* $\theta = 0$*, and the front face ff as the face where* $r = 0$*.*

This approach is generalized to a manifold X with corners and an embedded submanifold Y of X, here one can define "X blown-up at Y", $[X; Y]$, by taking polar coordinates about Y. The boundary face created in the blow-up is called the front face, denoted by ff$[X; Y]$, and the polar coordinates map $\beta : [X; Y] \to X$ is called the blow down map.

If $Z \subseteq X$ is a closed subset of X, then we define the lift of Z into $[X; Y]$, $\beta^* Z \subseteq [X; Y]$, as $\beta^* Z := \beta^{-1}(Z)$ if $Z \subseteq Y$, or as $\beta^* Z := \overline{\beta^{-1}(Y \backslash Z)}$ if $Z = Z \backslash Y$

7.3 The Operators of the b-Calculus

7.3.1 The Small b-Calculus

We begin with the definition of $\Psi_b^\mu(X)$, the algebra of b-pseudodifferential operators. They can be identified with the elements of the so called holomorphic cone algebra, which is a subset of all elements which are contained in the cone Algebra $C^\mu(X, \mathbf{g})$ as defined in 3.5.1

Let X be the manifold specified above, such that $\partial X = Y$ and $X \cong [0, 1) \times Y$ close to ∂X.

The kernels of b-pseudodifferential operators are defined on the following blow-up X_b^2 of $X^2 = X \times X$:

Definition 7.3.1. *We define the b-stretched product,* X_b^2*, by* $X_b^2 := [X^2; Y \times Y]$*. If* $\beta : X_b^2 \to X^2$ *is the blow-down map, we set* $lb := \beta^*(Y \times X)$*,* $rb := \beta^*(X \times Y)$*, and* $ff := \beta^*(Y \times Y)$*. The b-diagonal is defined by* $\Delta_b := \beta^*(\Delta)$*, where* Δ *is the diagonal in* X^2*.*

Let $0 < \nu \in C^\infty(X, \Omega_b)$ be any trivialization of Ω_b. Let $\beta : X_b^2 \to X \times X$ be the blowdown map for X_b^2. Let ν' be the lift of ν under $\pi_1 \circ \beta : X_b^2 \to X$ to X_b^2.

Now, the class $\Psi_b^m(X)$ is defined as:

Definition 7.3.2. *The space of b-pseudodifferential operators of order* $m \in \mathbb{R}$, $\Psi_b^m(X)$, *consists of operators* A *on* $C^\infty(X)$ *that have a Schwartz kernel* K_A *satisfying the following two conditions:*

1. *Given* $\varphi \in C_c^\infty(X_b^2 \backslash \Delta_b)$, *the kernel* φK_A *is of the form* $k\nu'$, *where* $k \in C^\infty(X_b^2)$ *and vanishes in Taylor series at the sets lb and rb.*

2. *Given a coordinate patch of* X_b^2 *near* Δ_b *of the form* $\mathcal{U}_y \times \mathbb{R}_z^n$ *such that* $\Delta_b \cong \mathcal{U} \times \{0\}$, *and given* $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, *we have*

$$\varphi K_A = \int_{\mathbb{R}^n} e^{iz\cdot\xi} a(y,\xi)\, đ\xi \cdot \nu', \qquad đ\xi = \frac{1}{(2\pi)^n}\, d\xi, \tag{7.3}$$

   where $a(y,\xi)$ *is a classical symbol of order* m.

The elements of $\Psi_b^\mu(X)$ form an algebra which is closed under compositions. However, elements which are contained in $\Psi_b^{-\infty}(X)$ are not necessarily compact. To be able to construct Fredholm inverses of elements in $\Psi_b^\mu(X)$, we need an enlargement of the small calculus by elements living on X_b^2, which are not longer of rapid decay on the boundary hypersurfaces *lb* and *rb*. To capture the non trivial asymptotic behavior of those terms, we need the notion of asymptotic expansions:

7.3.2 Asymptotic Expansions

The first notion we will need, is that of an index set:

Definition 7.3.3. *Let* $\mathbb{N}$ *be the set of positive integers and let* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *An index set* E *is a discrete subset of* $\mathbb{C} \times \{\mathbb{N}_0\}$, *such that:*

- $(z,k) \in E \Rightarrow (z,l) \in E$ *for all* $0 \leq l \leq k$, *and*
- *given any* $N \in \mathbb{R}$, *the set* $\{(z,k) \in E | \Re z \leq N\}$ *is finite.*
- $(z,k) \in E \Rightarrow (z+l,k) \in E \quad \forall l \in \mathbb{N}$.

For $\alpha \in \mathbb{R}$ *and an index set* E, *we say that* $E > \alpha$ *iff* $(z,k) \in E \Rightarrow \Re z > \alpha$.

The index sets allow us to describe the asymptotic of functions towards the boundary hypersurfaces. Let $\mathcal{U} = [0,1)_x^k \times (-1,1)_y^{n-k}$. Then for $a \in \mathbb{R}^k$ the space of symbols $\Sigma^a(\mathcal{U})$ consists of those smooth functions of the form:

$$u(x,y) = x_1^{a_1} \cdots x_k^{a_k}\, v(x,y),$$

where for each α and β, $(x\partial_x)^\alpha \partial_y^\beta v(x,y)$ is a bounded function.

Definition 7.3.4. *Given any index set E, a function $u \in \Sigma^a(\mathcal{U})$ is said to have asymptotic expansion at $x_1 = 0$ with index set E if, for each $N > 0$:*

$$u(x,y) = \sum_{(z,k)\in E, \Re z \leq N} x_1^z (\log x_1)^k u_{(z,k)}(x',y) + x_1^N u_N(x,y), \tag{7.4}$$

with $u_N(x,y) \in \sum^a(\mathcal{U})$ and $u_{z,k}(x',y) \in \sum^{a'}(\mathcal{U}')$, where $a = (a_1, a'), x = (x_1, x')$, and $\mathcal{U}' = [0,1)_{x'}^{k-1} \times (-1,1)_y^{n-k}$.

On a manifold with corners X one can define asymptotic expansions at a hypersurface H with index set E by reference to local coordinates. First of all, a function $u \in C^\infty(\mathring{X})$ is said to be in $\sum^0(X)$, if for any patch $\mathcal{U}$ on X and for any $\varphi \in C_c^\infty(\mathcal{U})$, the function φu is an element of $\sum^0(\mathcal{U})$. Let $H_1, ..., H_m$ be the hypersurfaces of X with corresponding boundary defining functions $\rho_1, ..., \rho_m$. For $a \in \mathbb{R}^m$ we define:

$$\sum^a(X) = \{\rho_1^{a_1} ... \rho_m^{a_m} v | v \in \sum^0(X)\}. \tag{7.5}$$

A function $u \in \sum^a(X)$ has a partial expansion at H with index set E of order κ, if for any patch $\mathcal{U} = [0,1)_{x_1} \times \mathcal{U}'$ on X with $H \cap \mathcal{U} = \{x_1 = 0\}$, and for any $\varphi \in C_c^\infty(\mathcal{U})$, the function φu has a partial expansion at $x_1 = 0$ with index set E of order κ in the sense described above.

If $\mathcal{E}$ is a collection of index sets $\mathcal{E} = \{E_{H_1}, ..., E_{H_l}\}$ corresponding to some family of hypersurfaces $H_1, ..., H_l$ of X, then we denote by $\mathcal{A}_\kappa^\mathcal{E}(X)$ the space of functions $u \in \sum^a(X)$ for some $a \in \mathrm{R}^m$ such that for each H, u has a partial expansion at H with index set E_H of order κ.

Finally, we define:

$$A^\mathcal{E}(X) = \cap_{\kappa>0} \mathcal{A}_\kappa^\mathcal{E}(X). \tag{7.6}$$

7.3.3 The Full b-Calculus

We define two new classes of operators, the first one corresponds to the smoothing Mellin operators of the cone calculus:

Definition 7.3.5. *The class $\Psi_b^{-\infty,\mathcal{E}}(X)$ is described with the help of an index set $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$. This class is characterized by integral kernels living on the blow up space X_b^2 with an asymptotic behavior which is described by $\mathcal{E}$. We say that:*

$$B \in \Psi_b^{-\infty,\mathcal{E}}(X) \Leftrightarrow K_B = k\nu', \quad k \in \mathcal{A}_{phg}^\mathcal{E}(X_b^2). \tag{7.7}$$

Definition 7.3.6. *For a collection of index sets* (E_{lb}, E_{rb})*, we define* $\Psi^{-\infty,(E_{lb},E_{rb})}(X)$ *as those operators with a Schwartz kernel of the form* $k\nu'$*, where* $k \in \mathcal{A}^{(E_{lb},E_{rb})}_{phg}(X^2)$*. Thus:*

$$C \in \Psi^{-\infty,(E_{lb},E_{rb})}(X) \Leftrightarrow C = k\nu' \quad k \in \mathcal{A}^{(E_{lb},E_{rb})}_{phg}(X^2). \tag{7.8}$$

The following result relates the mapping properties of operators in $\tilde{\Psi}^{m,\mathcal{E}}_b(X)$ to the defining index set $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$.

Theorem 7.3.7. *Let* $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$ *be an index set such that* $E_{lb} > \beta, E_{rb} > -\alpha$ *and* $\alpha + E_{ff} \geq \beta$*. Then any operator* $A \in \tilde{\Psi}^{m,\mathcal{E}}_b(X)$ *defines a continuous mapping:*

$$A : x^\alpha H^s_b(X) \to x^\beta H^{s-m}_b(X). \tag{7.9}$$

Finally, the full b-calculus is defined as the following set of operators:

$$\tilde{\Psi}^{m,\mathcal{E}}_b(X) := \Psi^m_b(X) + \Psi^{-\infty,\mathcal{E}}_b(X) + \Psi^{-\infty,E_{lb},E_{rb}}(X). \tag{7.10}$$

7.3.4 Comparison of b-Calculus and Cone Algebra

A comparison between b-calculus and cone algebra has been established in [18]. Let $C^m(X, \Omega_b)$ be the cone algebra as defined in 3.5.1. Theorem 5.17 there states that:

Theorem 7.3.8. *Let* $\gamma, \mu \in \mathbb{R}$*,* $j \in \mathbb{N}_0$ *be arbitrary,* x *a boundary defining function for* Y*, and suppose that the boundary* ∂X *of* X *is connected. Then we have*

$$C^\mu(X, \Omega_b, (\gamma, \gamma - \mu, \infty)) \quad \subset \quad \cup_{\mathcal{E}} x^{-\mu} \tilde{\Psi}^{\mu,\mathcal{E}}_b(X, \Omega_b)$$

where the union is over all index families $\mathcal{E} = (E_{lb}, E_{rb}, 0)$ *satisfying:*

$$-\inf E_{rb} < \gamma - \frac{n}{2} < \inf E_{lb}.$$

7.3.5 Parameter Ellipticity in the b-Calculus

In the case that $P \in \tilde{\Psi}^{m,\mathcal{E}}_b(X)$, Loya defines the model cone operator in [19] as follows:

Let (x, y) be local coordinates on $Y^\wedge$, define a group action on $C^\infty_c(Y^\wedge)$ by:

$$\kappa_\rho(u(x, y)) := u(\rho x, y). \tag{7.11}$$

The action extends in a natural way to the $\mathcal{K}^{s,\gamma}(Y^\wedge)$ spaces.

Definition 7.3.9. *Let* $P \in x^{-\mu}\tilde{\Psi}^{\mu,\mathcal{E}}_b(X)$ *with an index set* $\mathcal{E}$ *as in Theorem 7.3.7.*

We associate to P the model cone operator $P_\wedge$ by:

$$P_\wedge : \mathcal{K}^{s,\gamma}(Y^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(Y^\wedge)$$
$$u \mapsto \lim_{\rho\to 0} \rho^\mu \kappa_\rho \varphi A(\psi \kappa_\rho^{-1} u),$$

for smooth cut-off functions φ, ψ supported in a collar neighborhood of $\partial Y^\wedge$.

Remark 7.3.10. *For a pseudodifferential operator P which is contained in Schulze's cone algebra, that is $P \in C^\mu(X, \mathbf{g})$ with $\mathbf{g} = (\gamma, \gamma-\mu, \infty)$, we can find by Theorem 7.3.8 a index family $\mathcal{E}$, such that $P \in x^{-\mu}\tilde{\Psi}_b^{\mu,\mathcal{E}}(X, \Omega_b)$. In this case, it is easy to check that the Definition 7.3.9 of $P_\wedge$ coincides with Definition 6.1.2.*

Remark 7.3.11. *If the conditions (E1) and (E2) are satisfied, they automatically hold for a slightly larger keyhole region (by closedness of the spectrum, compactness of X, and the homogeneity of the rescaled symbol, the homogeneous principal symbol, as well as the conormal symbol).*

The following Theorem is Theorem 4.11 from [10], where the construction of the resolvent is carried out in detail.

7.4 b-Pseudodifferential Resolvent Calculus

In this sub chapter we cite the basic definitions and results from [10]. These result contain a resolvent calculus which allows for the construction of resolvents of parameter-elliptic cone pseudodifferential operators which are contained in $x^{-\mu}\Psi_b^\mu(X)$.

The resolvent calculus is based on the following class of parameter dependent symbols:

Definition 7.4.1. *For $\mu, p \in \mathbb{R}$ and $d > 0$ we define $S^{\mu,p,d}(\mathbb{R}^n; \Lambda)$ as the space of functions $a \in C^\infty(\mathbb{R}^n \times \Lambda)$ such that*

$$|\partial_\xi^\alpha \partial_\lambda^\beta a(\xi, \lambda)| \le C_{\alpha\beta}(1 + |\xi|)^{\mu-p-|\alpha|}(1 + |\xi| + |\lambda|^{1/d})^{p-d|\beta|}. \tag{7.12}$$

The space $S_r^{\mu,p,d}(\mathbb{R}^n; \Lambda), p/d \in \mathbb{Z}$, consists of elements $a \in S^{\mu,p,d}(\mathbb{R}^n; \Lambda)$ such that if we set

$$\tilde{a}(\xi, z) := z^{p/d} a(\xi, 1/z),$$

then $\tilde{a}(\xi, z)$ is smooth at $z = 0$, and:

$$|\partial_\xi^\alpha \partial_z^\beta \tilde{a}(\xi, z)| \le C_{\alpha\beta}(1 + |\xi|)^{\mu-p-|\alpha|+d|\beta|}(1 + |z||\xi|^d)^{p/d-|\beta|}, \tag{7.13}$$

uniformly for $|z| \leq 1$. *Further let* $S^{\mu,p,d}_{r,cl}(\mathbb{R}^n; \Lambda)$ *be the space of elements* $a \in S^{\mu,p,d}_r(\mathbb{R}^n; \Lambda)$ *that, for every* $N \in \mathbb{N}$, *admit a decomposition*

$$a(\xi, \lambda) = \sum_{j=0}^{N-1} \chi(\xi) a_{\mu-j}(\xi, \lambda) + r_N(\xi, \lambda), \tag{7.14}$$

where $r_N \in S^{\mu-N,p,d}_r(\mathbb{R}^n; \Lambda), \xi \in C^\infty(\mathbb{R}^n)$ *with* $\chi(\xi) = 0$ *for* $|\xi| \leq \frac{1}{2}$ *and* $\chi(\xi) = 1$ *for* $|\xi| \geq 1$, *and where each* $a_{\mu-j}(\xi, \lambda)$ *has the following properties:*

- $a_{\mu-j}(\delta\xi, \delta^d\lambda) = \delta^{\mu-j} a_{\mu-j}(\xi, \lambda)$ *for every* $\delta > 0$,
- $z^{p/d} a_{\mu-j}(\xi, 1/z)$ *is smooth at* $z = 0$.

The following class of parameter dependent operators is based on the definition of the parameter dependent symbols introduced in 7.4.1:

Definition 7.4.2. *Given* $\mu, p, d \in \mathbb{R}$ *with* $p/d \in \mathbb{Z}$ *and* $d > 0$, *the space* $\Psi^{\mu,p,d}_c(X; \Lambda)$ *consists of parameter-dependent operators* $A(\lambda)$ *that have a Schwartz kernel* $K_{A(\lambda)}$ *satisfying the following two conditions:*

- *Given* $\varphi \in C^\infty_c(X^2_b \backslash \Delta_b)$, *the kernel* $\varphi K_{A(\lambda)}$ *is of the form* $k(\rho^d\lambda, q) \cdot \mathfrak{m}'$, *where* $k(\lambda, q)$ *is a smooth function of* $(\lambda, q) \in \Lambda \times X^2_b$ *that vanishes to infinite order in* q *at the sets lb and rb, and is such that if we define* $\tilde{k}(z, q) = z^{p/d} k(1/z, q)$, *then* $\tilde{k}(z, q)$ *is smooth at* $z = 0$.
- *Given a coordinate patch of* X^2_b *overlapping* Δ_b *of the form* $\mathcal{U}_y \times \mathbb{R}^n_\xi$ *such that* $\Delta_b \equiv \mathcal{U} \times \{0\}$, *and given* $\varphi \in C^\infty_c(\mathcal{U} \times \mathbb{R}^n)$, *we have*

$$\varphi K_{A(\lambda)} = \int e^{i\zeta \cdot \xi} a(y, \xi, \rho^d \lambda) \, d\!\!\bar{}\xi \, \mathfrak{m}', \tag{7.15}$$

  where $y \mapsto a(y, \xi, \lambda)$ *is smooth with values in* $S^{\mu,p,d}_{r,cl}(\mathbb{R}^n; \Lambda)$.

7.4.1 Residual Operators for the Resolvent Calculus

This chapter deals with the residual classes which are needed to construct Fredholm inverses of the parameter dependent operators introduced before. The kernels of these operators are defined on a manifold which results from a two step iterative blow-up of $X \times X \times \overline{\Lambda}_d$, where $\overline{\Lambda}_d$ denotes the one point compactification of the complex parameter space: First, we blow up X^2 along its origin, obtaining X^2_b as usual.

Now, let $[\Lambda; \{0\}]$ be the sector Λ blown up at the origin; that is, Λ with polar coordinates taken at $\lambda = 0$, let $\overline{\Lambda}$ denote the stereographic compactification of $[\Lambda; \{0\}]$ in the Riemann sphere. Coordinates on $\overline{\Lambda}$ near the blown up origin are $\rho_0 = |\lambda|$ and $\theta = \lambda/|\lambda|$; near

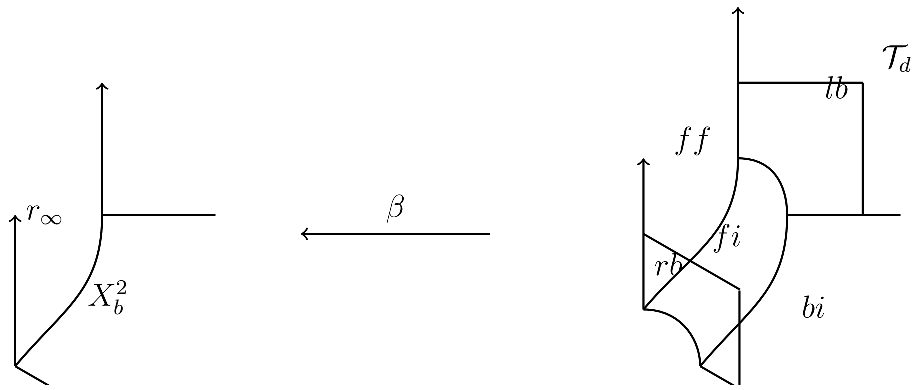


FIGURE 7.2: The blow up of $\overline{\Lambda}_d \times X_b^2$ along $\{r_\infty = 0\} \times \mathrm{ff}_b$.

$\lambda = \infty$ the coordinates are $\rho_\infty = |\lambda|^{-1}$ and $\theta = \lambda/|\lambda|$. Let $d > 0$ and let $\overline{\Lambda}_d = \{\lambda^{1/d}\}$ so that the radial coordinates on $\overline{\Lambda}_d$ are $r_0 = |\lambda|^{1/d}$ near the origin and $r_\infty = |\lambda|^{-1/d}$ near infinity.

We define:

$$\mathcal{T}_d(X) := [\overline{\Lambda}_d \times X_b^2; \{r_\infty = 0\} \times \mathrm{ff}_b], \tag{7.16}$$

the blow up of $\overline{\Lambda} \times X_b^2$ along $\{r_\infty = 0\} \times \mathrm{ff}_b$, where ff_b is the front face of X_b^2.

Then, if $\beta : \mathcal{T}_d(X) \to \overline{\Lambda}_d \times X_b^2$ is the blow-down map, we set $\mathrm{lb} := \beta^*(\overline{\Lambda}_d \times lb(X_b^2))$, left boundary; $\mathrm{rb} := \beta^*(\overline{\Lambda}_d \times rb(X_b^2))$, right boundary; $\mathrm{ff} := \beta^*(\overline{\Lambda}_d \times ff(X_b^2))$, front face; $\mathrm{fi} := \beta^*(\partial_\infty \overline{\Lambda}_d \times ff(X_b^2))$, face at infinity; and $bi := \beta^*(\partial_\infty \overline{\Lambda}_d \times X_b^2)$, boundary at infinity.

We introduce a class of residual operators whose integral kernels are defined as functions on $\mathcal{T}_d(X)$ with certain asymptotic at the boundary hypersurfaces. These operators are needed in the construction of parametrices of parameter dependent cone pseudodifferential operators:

Definition 7.4.3. *Let:*

$$\mathcal{E} = (E_{lb}, E_{rb}, E_{\mathit{ff}}, E_{\mathit{fi}}, \emptyset), \tag{7.17}$$

be an index family for $\mathcal{T}_d(X)$ associated to the faces $(lb, rb, \mathit{ff}, \mathit{fi}, bi)$. We denote by $\Psi_c^{-\infty,d,\mathcal{E}}(X;\Lambda)$ the space of those parameter-dependent operators $A(\lambda)$ that have a Schwartz kernel of the form

$$K_A = k \cdot \mathfrak{m}', \quad \text{with } k \in \mathcal{A}^{\mathcal{E}}(\mathcal{T}_d). \tag{7.18}$$

Thus k defines a function on $\mathcal{T}_d(X)$ that vanishes to infinite order at bi and have asymptotic expansions at the hypersurfaces lb, rb, ff and fi, determined by the index sets E_{lb}, E_{rb}, E_{ff}, and E_{fi}, respectively.

7.4.2 b-Calculus Resolvents

Gil and Loya construct in [10] the resolvent of an operator $A \in x^{-\mu}\Psi_b^{\mu}(X), \mu > 0$, which is assumed to be parameter elliptic. Here, parameter ellipticity is given in Definition 4.5. of [10]. Since the operators in $x^{-\mu}\Psi_b^{\mu}(X)$ can be identified with holomorphic operators in the cone Algebra by Theorem 5.4. of [18], it is easy to see that our Definition 6.1.4 of parameter ellipticity coincides with the one given in Definition 4.5. of [10].

This is Theorem 4.11. of [10]:

Theorem 7.4.4. *Let $A \in x^{-\mu}\Psi_b^{\mu}(X), \mu > 0$, be such that $A - \lambda$ is parameter-elliptic on Λ with respect to some $\alpha \in \mathbb{R}$. Then for $\lambda \in \Lambda$ sufficiently large,*

$$A - \lambda : x^{\alpha - \frac{n}{2}} H_b^s(X) \to x^{\alpha - \frac{n}{2} - \mu} H_b^{s-\mu}(X),$$

is invertible for any $s \in \mathbb{R}$, and:

$$(A - \lambda)^{-1} \in x^{\mu}\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda) + x^{\mu}\Psi_c^{-\infty,\mu,\mathcal{E}(\alpha)}(X;\Lambda), \tag{7.19}$$

where $\mathcal{E}(\alpha) = (E_{lb}, E_{rb}, E_{ff}, E_{fi}, E_{bi})$ is an index family associated to $\mathcal{T}_d$ such that $E_{lb} > \alpha - \frac{n}{2} - \mu, E_{rb} > -(\alpha - \frac{n}{2} - \mu), E_{ff} > 0, E_{fi} = \mathbb{N}$ and $E_{bi} = \emptyset$. Moreover, for $\alpha = \mu = s$ we have that:

$$(A - \lambda)^{-1} : L_b^2(X) \to x^{\mu} H_b^{\mu}(X),$$

is uniformly bounded in λ.

Chapter 8

Resolvents in the Full b-Calculus

8.1 The Full Resolvent

Gil and Loya construct in [10] the resolvent for a cone pseudodifferential operator $P \in x^{-\mu}\Psi_b^{\mu}(X)$. We discuss here a generalization of this result to an operator which is contained in $x^{-\mu}\tilde{\Psi}_b^{\mu,\mathcal{E}}(X)$, the full b-calculus as defined in (7.10).

We begin with a few composition results which are needed to compose operators of the full b-calculus with operators of the parameter dependent resolvent calculus of Gil and Loya as introduced in [10].

Lemma 8.1.1. *Let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$ be an index family for X_b^2. Then:*

$$\Psi_b^{-\infty,\mathcal{E}}(X) \circ \Psi_c^{-\mu,-\mu,\mu}(X;\Lambda) \subset \Psi_c^{-\infty,\mu,\mathcal{F}}(X;\Lambda), \tag{8.1}$$

where $\mathcal{F} = (E_{lb}, E_{rb}, E_{ff}, \mathbb{N}_0)$, as well as:

$$\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda) \circ \Psi_b^{-\infty,\mathcal{E}}(X) \subset \Psi_c^{-\infty,\mu,\mathcal{F}}(X;\Lambda). \tag{8.2}$$

Proof. The composition of the operators is expressed with the help of pullbacks and pushforwards. We let $\pi_F, \pi_S, \pi_C : X^3 \to X^2$ be the maps

$$\pi_F(u,v,w) = (uv), \quad \pi_S(u,v,w) = (v,w), \quad \pi_C(u,v,w) = (u,w).$$

Further, we define the manifold X_b^3 by blowing up Y^3 in X^3 first ("the origin"), and then blowing up the submanifolds coming from the codimension two corners of X^3, see Figure 8.1, which is taken from [10].

Then, $\pi_{F,b}$, $\pi_{S,b}$, $\pi_{C,b}$ denote the maps π_F, π_S, π_C expressed in the polar coordinates of X_b^3 and X_b^2. Then, we can express the composition of M and A with Schwartz kernels

K_M and K_A by:

$$\nu K_{MA} = (\pi_{C,b})_* (\pi^*_{C,b} \nu \pi^*_{F,b} K_M \pi^*_{S,b} K_A) \tag{8.3}$$

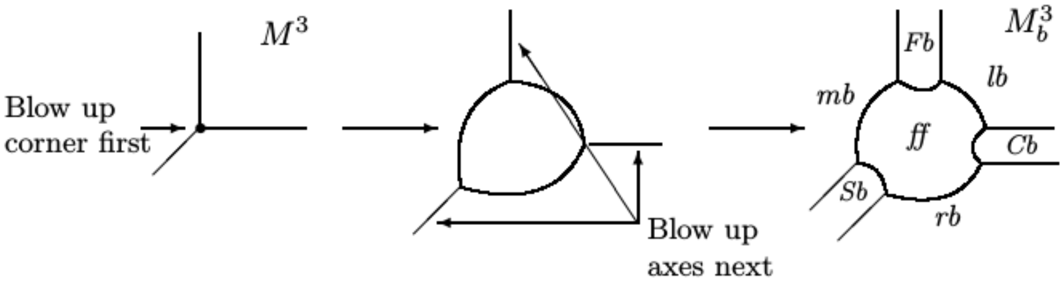


FIGURE 8.1: The blowup space X^3_b

We assume $A \in \Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$, $M \in \Psi_b^{-\infty,\mathcal{E}}(X)$. We decompose A into two parts, $A = A_1 + A_2$ where the kernel of A_1 is supported around the diagonal Δ_b and the kernel of A_2 has support away from Δ_b.

For the composition MA_1, we choose a coordinate neighborhood, such that:

$$X^3_b \cong X^2_b \times \mathbb{R}_u, \quad \pi^{-1}_{F,b}(\Delta_b) \cong X^2_b \times \{0\}_u, \tag{8.4}$$

where $\pi_{C,b}(p,u) = p$ and $\pi_{S,b}(p,u) = p$ for all $(p,u) \in X^2_b \times \mathbb{R}_u$. Further, $\pi^*_{F,b}\rho = \rho\rho_{lb}$, where ρ_{lb} is a boundary defining function for lb of X^2_b. Let $X^2_b \cong [0,1)_\rho \times [-1,1]_y$, where $lb = \{y = -1\}$ and $rb = \{y = 1\}$.

We begin with MA_1. We obtain:

$$\pi^*_{C,b} \nu \pi^*_{F,b} k_M \pi^*_{S,b} k_{A_1} = m(\rho\rho_{lb}, y) \int e^{iu\cdot\xi} a_1(\rho^\mu \lambda, \rho, \xi)\, đ\xi |du| \nu\nu'. \tag{8.5}$$

Now, since $\pi_{C,b}(p,u) = p$, we have:

$$\begin{aligned} \nu k_{MA_1} = (\pi_{C,b})_*(\pi_{C,b})^* \nu \pi^*_{F,b} k_M \pi^*_{S,b} k_{A_1}) &= m(\rho\rho_{lb}, y) a_1(\rho^\mu \lambda, \rho, 0)|du|\nu\nu' \\ &= C(\rho^\mu\lambda, \rho, \rho_{lb}, y)\nu\nu', \end{aligned}$$

where $C(\lambda, \rho, \rho_{lb}, y) = m(\rho\rho_{lb}, y) a_1(x^\mu \lambda, \rho, 0)|du|$. Now the asymptotic properties for $C(\lambda, \rho, \rho_{lb}, y)$ follow, since $m(\rho\rho_{lb}, y)$ has asymptotic expansions at $y = -1$ and $y = 1$ with index sets E_{lb} and E_{rb} and at $\rho = 0$ with index set E_{ff}.

For the composition MA_2 we assume that $\pi^*_{C,b} \nu \pi^*_{F,b} k_M \pi^*_{S_b} k_{A_2}$ is supported near the intersection of mb, ff and Fb of X^3_b. In this region of X^3_b we use the coordinates (s, t, x''), where $s = x/x''$ and $t = x'/x$. In these coordinates, $\pi_{C,b}$ and π_{S_b} map near lb

in X_b^2 and in terms of local coordinates (s, x') near lb of X_b^2, with $s = x/x'$, are given by:

$$\pi_{S,b}(s,t,x'') = (st, x''); \quad \pi_{C,b}(s,t,x'') = (s, x''). \tag{8.6}$$

Now, $\pi_{F,b}$ maps near rb in X_b^2, and in the local coordinates (x, t) with $t = x'/x$ near rb on X_b^2, we have:

$$\pi_{F,b}(s,t,x'') = (sx'', t). \tag{8.7}$$

Near rb in X_b^2, we can write $K_M = M(x,t)|dx'/x'|$, where $M(x,t)$ admits asymptotic expansions at $x = 0$ and $t = 0$ with index sets E_{ff} and E_{rb}. Near lb in X_b^2, we can write $K_{A_2} = A_2(r,s,v')|dx'/x'|$, where $r = \lambda^{-1/d}$ and $v' = x'/r$, and where $A_2(r,s,v')$ has expansions at $r = 0$, $s = 0$, $v' = 0$ and $v' = \infty$, with index sets $F_{fi} = \mathbb{N}_0$, $f_{lb} = \emptyset$, $E_{ff} = \mathbb{N}_0$ and $F_{bi} = \emptyset$. We obtain:

$$\pi_{C,b}^* \nu \pi_{F,b}^* K_M \pi_{S,b}^* K_{A_2} = M(sx'',t) A_2(r, st, x''/r) |\frac{dsdtdx''}{stx''}|. \tag{8.8}$$

Therefore:

$$\begin{aligned} \nu K_{MA_2} = (\pi_{C,b})_* (\pi_{C,b}^* \nu \pi_{F,b}^* K_M \pi_{S,b}^* K_{A_2}) &= \int M(sx',t) A_2(r,st,x'/r) \frac{dt}{t} \cdot |\frac{dsdx'}{sx'}| \\ &= B(r,s,v)|\frac{dsdx'}{sx'}|, \end{aligned}$$

where $B(r,s,v') = \int M(srv',t)A_2(r,st,v')dt/t$ with $v' = x'/r$. Then the asymptotic properties of M and A_2 imply that B has asymptotic expansions at $r = 0$, $s = 0$, $v' = 0$ and $v' = \infty$ with index sets $\mathbb{N}_0$, E_{lb}, E_{ff} and $\emptyset$.

The calculations for the remaining five regions of intersections on X_b^3 can be computed similarly, for further details see e.g. the proof of Proposition 4.2. of [20]. □

For the sake of completeness we give here the definitFion of two classes of smoothing operators with bounds which are needed in the construction of the resolvent, they are Definition 3.9. and Definition 3.11. of [10]:

Definition 8.1.2. *Let $N \in \mathbb{N}$ and $d > 0$. For $m \in \mathbb{N}$ we define $\Psi_{m,N}^{-\infty;\mu}(X;\Lambda)$ as the space of those parameter-dependent operators $A(\lambda)$ whose Schwartz kernel $K_{A(\lambda)}$ is of the form $k(x^\mu \lambda, q) \cdot \mathfrak{m}'$ with $k(\lambda, q)$ satisfying the following properties:*

(a) For some $\varepsilon > 0$, $x_l^{-N\mu-\varepsilon} x_r^{-N\mu-\varepsilon} k$ is a symbol in $\Sigma^0(\overline{\Lambda} \times X_b^2)$ having a partial expansion at the face $\overline{\Lambda} \times ff$ with index set $\mathbb{N}_0$ of order $N\mu + \varepsilon$. Again, ρ_l and ρ_r are boundary defining functions for lb and rb in X_b^2,

(b) For each $N' \leq N$,

$$k(\lambda, q) = \sum_{j=m}^{N'-1} \lambda^{-j} f_j(q) + \lambda^{-N'} k_{N'}(\lambda, q),$$

where $f_j \in \mathcal{A}^{\mathcal{E}}_{2N\mu d - jd}(X^2_b)$, with $\mathcal{E} = (\emptyset, \emptyset, \mathbb{N}_0)$, and $k_{N'}$ satisfies (a) with $N\mu$ replaced by $2N\mu - N'\mu$. If $m \geq N$, then we disregard the summation and require instead $k(\lambda, q) = \lambda^{-N} k_N(\lambda, q)$, where k_N satisfies (a).

Definition 8.1.3. *Let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi})$ be an index family for $\mathcal{T}_\mu$. Then we define $\Psi^{-\infty,\mu,\mathcal{E}}_N(X; \Lambda)$ as those parameter dependent operators $A(\lambda)$ that have a Schwartz kernel of the form $K_{A(\lambda)} = k \cdot \mathfrak{m}'$, where k is a symbol on $\mathcal{T}_\mu$, of order $N\mu$ at bi, that satisfies:*

- *Given $\varphi \in C^\infty(\mathcal{T}_\mu)$ supported near fi, φk is in $\mathcal{A}^{\mathcal{E}}_{N\mu+\varepsilon}$ for some $\varepsilon > 0$.*
- *Given $\psi \in C^\infty(\mathcal{T}_\mu)$ supported away from fi, ψk is the kernel of a parameter dependent operator in $\Psi^{-\infty,\mu}_{N,N}(X; \Lambda)$.*

For the construction of resolvents in the full calculus, we assume that the following composition result of operators in the full b-calculus with operators which are contained in the calculus with bounds is true. However, we leave the proof of this lemma open:

Lemma 8.1.4. *Let $m, N \in \mathbb{N}_0$, and $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$ be an index family for X^2_b.*

$$\Psi^{-\infty,\mathcal{E}}_b(X) \circ \Psi^{-\infty,\mu}_{m,N}(X) \subset \Psi^{-\infty,\mu,\mathcal{E}}_N(X). \tag{8.9}$$

The following Lemma is Proposition 5.5. of [19]:

Lemma 8.1.5. *Let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi})$ and $\mathcal{F} = (F_{lb}, F_{rb}, F_{ff}, F_{fi})$ be two index sets on $\mathcal{T}_d$. Provided that $E_{rb} + F_{lb} > 0$, we have*

$$\Psi^{-\infty,\mu,\mathcal{E}}_c(X; \Lambda) \circ \Psi^{-\infty,\mu,\mathcal{F}}_c(X; \Lambda) \subseteq \Psi^{-\infty,\mu,\mathcal{E}\hat{\circ}\mathcal{F}}_c(X; \Lambda).$$

where the index set $\mathcal{G} = \mathcal{E}\hat{\circ}\mathcal{F}$ is defined by:

$$G_{lb} = E_{lb}\overline{\cup}(E_{ff} + F_{lb}); \quad G_{rb} = (E_{rb} + F_{ff})\overline{\cup}F_{rb};$$
$$G_{ff} = (E_{ff} + F_{ff})\overline{\cup}(E_{lb} + F_{rb}), \quad \text{and } G_{fi} = E_{fi} + F_{fi}.$$

This Lemma follows from Theorem 3.18. of [10]:

Lemma 8.1.6. *Having an index set $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi})$, the space $\Psi^{-\infty,\mu,\mathcal{E}}_c(X; \Lambda)$ is closed under composition with $\Psi^{\mu,p,d}_c(X; \Lambda)$, for instance,*

$$\Psi^{\mu,p,d}_c(X; \Lambda) \circ \Psi^{-\infty,\mu,\mathcal{E}}_c(M; \Lambda) \subset \Psi^{-\infty,\mu,\mathcal{E}}_c(M; \Lambda). \tag{8.10}$$

For the construction of resolvents of operators $P \in x^{-\mu}\tilde{\Psi}_b^{\mu,\mathcal{E}}(X)$, we assume that A is parameter elliptic. From this we know that the resolvent of the model cone $P_\wedge$ of A exists. We make here an additional assumption on the structure of the model cone operator which we were not able to prove:

Lemma 8.1.7. *Let $P \in x^{-\mu}\tilde{\Psi}_b^{\mu,\mathcal{E}}(X)$ be parameter elliptic on a sector Λ. Then for $\lambda \in \Lambda$ sufficiently large, $(P_\wedge - \lambda)^{-1} \in x^{\mu}(\Psi_c^{-\mu,-\mu,\mu}(Y^\wedge;\Lambda) + \Psi_c^{-\infty,\mu,\mathcal{F}}(Y^\wedge;\Lambda))$. Here $\mathcal{F} = (F_{lb}, F_{rb}, F_{ff}, F_{fi})$ is an index family for $\mathcal{T}_d$, such that $F_{lb} > (\alpha-\mu) + \frac{n}{2}$, $F_{rb} > -(\alpha-\mu) + \frac{n}{2}$, $F_{ff} \geq 0$, $F_{fi} = \mathbb{N}_0$.*

Now we come to the central result of this chapter about the existence and structure of resolvents in the full b calculus:

Theorem 8.1.8. *Let $P \in x^{-\mu}\tilde{\Psi}_b^{\mu,\mathcal{E}}(X)$ with $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$ with $E_{lb} > \alpha + \frac{n}{2}$, $E_{rb} > -\alpha - \frac{n}{2}$ and $E_{ff} = \mathbb{N}_0$ be parameter elliptic in the sense of Definition 6.1.4 with respect to α.*

Then, for λ sufficiently large in Λ, $(P-\lambda)^{-1} = F + G$, where $F \in x^{\mu}\Psi_c^{-\mu-\mu,\mu}(X;\Lambda)$, $G \in x^{\mu}\Psi_c^{-\infty,\mu,\mathcal{G}}$, with $\mathcal{G} = (G_{lb}, G_{rb}, G_{ff}, G_{fi})$. Here $\mathcal{G}$ is a index family for $\mathcal{T}_d$, such that $G_{lb} > \alpha - \mu - \frac{n}{2}$, $G_{rb} > -(\alpha - \mu - \frac{n}{2})$, $G_{ff} \geq 0$, $G_{fi} = \mathbb{N}_0$.

Proof. First, by the definition of $\tilde{\Psi}_b^{\mu,\mathcal{E}}(X)$, we can write $P = A+M+G$, here $A \in \Psi_b^{\mu}(X)$, $M \in \Psi_b^{-\infty,\mathcal{E}}(X)$ and $G \in \Psi^{-\infty,(E_{lb},E_{rb})}(X)$. Next, we can summarize $N = M+G$, where $N \in \Psi_b^{-\infty,\mathcal{E}'}$, with $\mathcal{E}' = (E_{lb}, E_{rb}, E_{lb}+E_{rb})$, this follows from the push forward Lemma, see e.g. [12] for details. Therefore, we can assume that we have $P = A+N$, with $N \in x^{-\mu}\Psi_b^{-\infty,\mathcal{E}}(M)$ with $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$ such that $E_{lb} > \alpha + \frac{n}{2}$, $E_{rb} > -\alpha - \frac{n}{2}$, $E_{ff} \geq 0$.

Gil and Loya construct in the proof of Theorem 4.11. in [10] a resolvent $(A-\lambda)^{-1}$ for the case that $A \in x^{-\mu}\Psi_b^{\mu}(M)$. There, they invert $(A-\lambda)$ first up to a regularizing term. Precisely, they construct an inverse $B_3(\lambda) \in \Psi_c^{-\mu-\mu,\mu}(M;\Lambda) + \Psi_{1,N}^{-\infty,\mu}(M;\Lambda)$, such that $(A-\lambda)$ can be inverted up to an error $S_3(\lambda) \in \Psi_N^{-\infty,\mathcal{G}}(M;\Lambda)$, for $\mathcal{G} = (\emptyset, \emptyset, \mathbb{N}_0, \mathbb{N}_0)$:

$$(A-\lambda)x^{\mu}B_3(\lambda) = I - S_3(\lambda).$$

In our case, we want to invert $P-\lambda$. Since $P = A+N$, where N is smoothing on the diagonal, we replace A by $P = A+N$ in the above equation, obtaining:

$$(A+N-\lambda)x^{\mu}B_3(\lambda) = I - S_3'(\lambda).$$

Here, $S_3'(\lambda) = S_3(\lambda) + Nx^{\mu}B_3(\lambda)$.

To describe the composition $Nx^\mu B_3(\lambda)$, we first observe that for $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff})$, it holds that $x^{-\mu}\Psi_b(X)^{-\infty,\mathcal{E}}x^\mu \subset \Psi_b^{-\infty,\mathcal{E}'}(X)$ with $\mathcal{E}' = (E_{lb} - \mu, E_{rb} + \mu, E_{ff})$. Consequently, using the composition results of Lemma 8.1.1 and Lemma 8.1.4, we obtain that $Nx^\mu B_3(\lambda) \in \Psi_N^{-\infty,\mathcal{F}}(X)$, with $\mathcal{F} = (E_{lb} - \mu, E_{rb} + \mu, E_{ff}, F_{fi} = \mathbb{N}_0)$. Therefore we obtain that $S_3'(\lambda) \in \Psi_N^{\mathcal{E}\cup\mathcal{G}}$.

Now, one can use the resolvent of the model cone operator $(P_\wedge - \lambda)^{-1}$, which exists on Λ due to our assumptions on P, to obtain an error term which decays in first order λ and can be inverted within the calculus. See the proof of Theorem 6.3. in [19] for details. In the end one obtains as result:

$$(P - \lambda)^{-1} \in x^\mu \Psi_c^{-\mu,-\mu,\mu}(X;\Lambda) + x^\mu \Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda),$$

where $\mathcal{G} = (G_{lb}, G_{rb}, G_{ff}, G_{fi})$ is an index family for $\mathcal{T}_\mu$, such that $G_{lb} > \alpha - \mu + \frac{n}{2}$, $G_{rb} > -(\alpha - \mu) - \frac{n}{2}$, $G_{ff} \geq 0$ and $G_{fi} = \mathbb{N}_0$.

$\square$

Chapter 9

H^∞ Calculus

Definition 9.0.9. *We denote by* $\Lambda = \Lambda(\theta)$ *the complex sector:*

$$\Lambda(\theta) = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \geq \theta\},$$

with $0 < \theta < \pi$.

Let A be a closed, densely defined operator:

$$A : \mathcal{D}(A) \subset F \to F, \tag{9.1}$$

in a Banach space F.

Definition 9.0.10. *We call such an operator* A *sectorial, if*

- *The spectrum of* A *has empty intersection with* $\Lambda \backslash \{0\}$

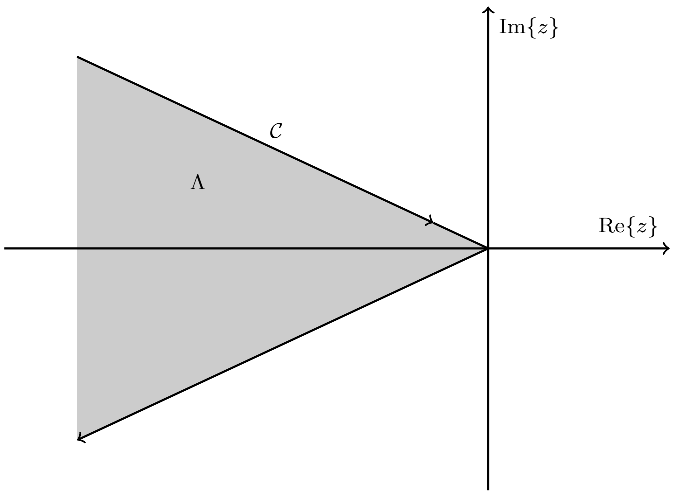


FIGURE 9.1: The path of integration along the complex sector Λ

- *$\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(F)}$ is uniformly bounded for large $\lambda \in \Lambda$.*

By $H = H(\theta)$ we denote the space of all holomorphic functions $\mathbb{C}\backslash\Lambda \to \mathbb{C}$ for which $|f(\lambda)| \leq c(|\lambda|^{\delta} + |\lambda|^{-\delta})^{-1}$ for some $\delta > 0$ and $c > 0$.

If A is sectorial, we can define with $\mathcal{C} = \partial\Lambda$:

$$f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(\lambda)(A - \lambda)^{-1} d\lambda \quad \text{for } f \in H. \tag{9.2}$$

The second condition for sectoriality yields the absolute convergence of (9.2) with respect to the operator norm on $\mathcal{L}(F)$.

We define for an operator A acting as an unbounded operator on $\mathcal{H}_p^{0,\alpha-\mu}(X)$:

Definition 9.0.11. *We say that an unbounded operator A on $\mathcal{H}_p^{s,\alpha-\mu}(X)$ with domain $\mathcal{D}(\mathcal{H}_p^{s,\alpha-\mu}(X))$ admits H^∞ Calculus with respect to θ, if $f(A)$ being defined by equation (9.2) defines a bounded operator on $\mathcal{H}_p^{s,\alpha-\mu}(X)$, and we have, that:*

$$\|f(A)\|_{\mathcal{L}(\mathcal{H}_p^{s,\alpha-\mu}(X))} \leq c_p \|f\|_\infty \quad \forall f \in H. \tag{9.3}$$

9.1 H^∞ Calculus for Cone Pseudodifferential Operators

Theorem 9.1.1. *Let $P \in x^{-\mu}\tilde{\Psi}_b^{\mu,\mathcal{E}}(X)$ be an operator in the full b-Calculus which is parameter elliptic. Due to Theorem 8.1.8, we can use the Resolvent of P to define $f(P)$ using 9.2. Further, P admits H^∞ Calculus in the sense of Definition 9.0.11.*

It holds, that $(A - \lambda)^{-1} \in \mathcal{L}(L_b^2(X), x^\mu H_b^\mu(X))$ is uniformly bounded in λ if and only if $\|(A - \lambda)^{-1}\|_{\mathcal{L}(L_b^2(X))} = \mathcal{O}(|\lambda|^{-1})$ as $|\lambda| \to \infty$.

This shows, that the condition on A to be sectorial is already sufficient to define $f(A)$ of A via the Dunford Integral (9.2).

9.2 The Proof of Theorem 9.1.1

If we want to estimate (9.2), we can use the results of Theorem 8.1.8, to write $(A-\lambda)^{-1}$ as $(A - \lambda)^{-1} = Q(\lambda) + R(\lambda)$ with $Q(\lambda) \in x^\mu \Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$ and $R(\lambda) \in x^\mu \Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda)$. We obtain for $f(A)$:

$$\begin{aligned} f(A) &= \frac{1}{2\pi i}\int\limits_{\mathcal{C}} f(\lambda)(A-\lambda)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int\limits_{\mathcal{C}} f(\lambda)\, Q(\lambda) d\lambda + \frac{1}{2\pi i}\int\limits_{\mathcal{C}} f(\lambda)\, R(\lambda) d\lambda \\ &=: Q_f + R_f. \end{aligned} \tag{9.4}$$

If we want to obtain estimates on $\|f(A)\|_{\mathcal{H}_p^{s,\alpha-\mu}(Y^\wedge)}$ as defined in (9.6), we have to take estimates on $\|(A-\lambda)^{-1}\|_{\mathcal{H}_p^{s,\alpha-\mu}(Y^\wedge)}$. In our case of a cone elliptic pseudodifferential operator, the structure of $(A-\lambda)^{-1}$ is described in Theorem 8.1.8.

The following lemma helps us to simplify the estimates:

Lemma 9.2.1. *Estimating $\|(A-\lambda)^{-1}\|_{\mathcal{H}_p^{0,\alpha-\mu}(Y^\wedge)}$ for a cone elliptic pseudodifferential operator, we can assume without loss of generality, that $\alpha-\mu=0$.*

Proof. Taking coordinates $(x,y), x\in\mathbb{R}_+, y\in Y$, the multiplication with x^α induces an isomorphism between $\mathcal{H}_p^{s,\gamma}(Y^\wedge)$ and $\mathcal{H}_p^{s,\gamma+\alpha}(Y^\wedge)$.

Hence, we have an equivalence of the norms:

$$\|(A-\lambda)\|_{\mathcal{H}_p^{s,\alpha-\mu}(Y^\wedge)} \sim \|x^{(\alpha-\mu)}(A-\lambda)x^{-(\alpha-\mu)}\|_{\mathcal{H}_p^{s,0}(Y^\wedge)}. \tag{9.5}$$

Now we decompose $f(A)$ into Q_f and R_f as described in (9.4). We prove the Lemma for both parts separately:

Estimates on Q_f:

Let $Q\in x^\mu\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$. Then, if we take $\varphi\in C_c^\infty(X_b^2\backslash\Delta_b)$, we have rapid decay of the integral kernel φk_Q towards lb and rb. Taking local coordinates $x', \frac{x'}{x}$ on rb, a look at (9.5) reveals, that a change of the weight corresponds to a multiplication of the integral kernel with powers of $(\frac{x}{x'})$. This leaves the property of rapid decay towards lb and rb invariant and is a multiplication by a bounded function away from rb and lb since φ is compactly supported.

If we take $\varphi\in C_c^\infty(X_b^2)$ with compact support around the diagonal but away from lb and rb, then the multiplication with $\frac{x'}{x}$ corresponds just to the multiplication with a compactly supported function.

Estimates on R_f:

Having $R \in x^{\mu}\Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda)$, we realize that the asymptotic behavior of R is described by the index sets:

$$\mathcal{G} = (G_{lb}, G_{rb}, G_{ff}, G_{fi}, G_{bi}),$$

where $G_{lb} > \alpha - \mu - \frac{n}{2}, G_{rb} > -(\alpha-\mu) + \frac{n}{2}, G_{ff} \geq 0, G_{fi} = \mathbb{N}$ and $G_{bi} = \emptyset$.

As local coordinates near rb, we can choose the set $(x, \frac{x'}{x})$. Then the transition from $\mathcal{H}^{0,\mu-\alpha}(X)$ to $\mathcal{H}^{0,0}(X)$ can be expressed by multiplying the kernel with $(\frac{x'}{x})^{\alpha-\mu}$. While this leaves the bounds for the index sets G_{ff}, G_{fi} and G_{bi} invariant, it changes the bounds on the asymptotic for G_{lb} and G_{rb} from $G_{lb} > \alpha - \mu - \frac{n}{2}, G_{rb} > -(\alpha-\mu) + \frac{n}{2}$ to $G_{lb} > -\frac{n}{2}, G_{rb} > \frac{n}{2}$). □

The Dunford integral (9.2) as well as the corresponding estimates for H^{∞} (9.3) can be computed separately for the integrals Q_f and R_f.

We consider the integral:

$$Q_f = \frac{1}{2\pi i}\int_{\mathcal{C}} f(\lambda)\, Q(\lambda) d\lambda, \tag{9.6}$$

where $Q(\lambda) \in x^{\mu}\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$.

We know that $\partial X = Y$ and $X \cong [0,1) \times Y$ close to ∂X. Now let $\mathcal{U} = [0,1)_x \times \mathbb{R}^{n-1}_y$ be a coordinate patch on X near Y. It follows that $X^2 \cong [0,1)^2_{(x,x')} \times \mathbb{R}^{n-1}_y \times \mathbb{R}^{n-1}_{y'}$, where (x', y') are local coordinates for the right factor of $\mathcal{U}^2$.

It follows that locally:

$$X_b^2 \cong [[0,1)^2; (0,0)] \times \mathbb{R}^{n-1}_y \times \mathbb{R}^{n-1}_{y'}. \tag{9.7}$$

Working through the construction of the resolvent of A in the proof of Theorem 4.11 in [10], it can be seen, that only the leading order contribution of $Q(\lambda)$ is an operator contained in $x^{\mu}\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$, while the lower order contributions can be summarized into an operator which is contained in $x^{\mu}\Psi^{-\mu-1,-2\mu,\mu}(X;\Lambda)$.

Our strategy to prove the H^{∞} Calculus estimate for Q_f will be to write $Q(\lambda)$ as:

$$Q(\lambda) = Q_1(\lambda) + Q_2(\lambda), \tag{9.8}$$

where $Q_1(\lambda) \in x^{\mu}\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$, $Q_2(\lambda) \in x^{\mu}\Psi^{-\mu-1,-2\mu,\mu}(X;\Lambda)$.

This leads to a further decomposition of Q_f being defined for $Q(\lambda)$ into

$$Q_f = Q_{1,f} + Q_{2,f}. \tag{9.9}$$

We will prove the desired H^∞ Calculus estimate for the concrete form of $Q_1(\lambda)$ as defined in the construction of the resolvent in [10], as well as for a general operator of class $\Psi_c^{-\mu-1,-2\mu,\mu}(X;\Lambda)$.

9.2.1 Estimates for the Terms of Lower Order

Recall that $Y^\wedge = \overline{\mathbb{R}}_+ \times Y$. Using a partition of unity on Y, we can assume $Y = \mathbb{R}^{n-1}$, hence $Y^\wedge = \overline{\mathbb{R}} \times \mathbb{R}^{n-1}$. The H^∞ Calculus estimate (9.3) for $Q_{2,f}$ follows from the following lemma:

Lemma 9.2.2. *Let* $P(\lambda) \in x^\mu \Psi_c^{-\mu-1,-2\mu,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1};\Lambda)$.

We define the operator P_f *by:*

$$P_f = \int_{\mathcal{C}} f(\lambda)\, P(\lambda) d\lambda. \tag{9.10}$$

Then P_f *acts as a bounded operator on* $\mathcal{H}_p^{0,0}(X)$ *and its operator norm satisfies the estimate:*

$$\| \int_{\mathcal{C}} f(\lambda)\, P(\lambda) d\lambda \|_{\mathcal{L}(\mathcal{H}_p^{0,0}(X))} \le c_p \|f\|_\infty. \tag{9.11}$$

Proof. Since X is a compact manifold, as well as X_b^2, we can cover X_b^2 with a finite number of coordinate patches and a subordinate partition of unity $\varphi_i,\ i \in \{1,...k\}$.

We simplify the problem in several steps:

- As long as the kernel of $P(\lambda)$ is supported away from ∂X_b^2, there exist cut-off functions ω_1, ω_2, such that $P(\lambda) = (1-\omega_1)P(\lambda)(1-\omega_2)$. Therefore, we have that $P(\lambda)$ is just an ordinary pseudodifferential operator with parameter, the result follows in this case by the standard theory.

- Therefore, we have reduced the proof to the case where the kernel of $P(\lambda)$ is supported in a collar neighborhood of ∂X_b^2.

  We distinguish here between two different cases:

  1. A coordinate patch supported on $X_b^2 \backslash \Delta_b$,

  2. A coordinate patch supported in a neighborhood of $ff(X_b^2) \cap \Delta_b$, but away from rb and lb.

We begin with a remark which applies to both cases:

Let $\mathcal{H}_p^{0,\gamma}(Y^\wedge)$ be the weighted Sobolev spaces modeled over $Y^\wedge$. Since we can assume the kernel of $P(z)$ to be supported close to $ff(X_b^2)$, we can choose cut-off functions $\omega_1, \omega_2 \in C_c^\infty([0,1))$, such that $P(z) = \omega_1 P(\lambda)\omega_2$. Now, the multiplication with any cut-off function $\omega \in C_c^\infty([0,1))$ induces continuous operators $\mathcal{H}_p^{0,\gamma}(X) \to \mathcal{H}_p^{0,\gamma}(Y^\wedge)$ and $\mathcal{H}_p^{0,\gamma}(Y^\wedge) \to \mathcal{H}_p^{0,\gamma}(X)$. Therefore it suffices to prove the corresponding estimates with respect to the $\|\cdot\|_{\mathcal{H}_p^{0,\gamma}(Y^\wedge)}$ norm, which we denote from here on by $\|\cdot\|_p$.

1. **Integral kernel supported away from the diagonal:**

   Choosing $\varphi \in C_c^\infty(X_b^2 \backslash \Delta_b)$, we have by by Definition 7.4.1, that $\varphi P(\lambda)$ can be represented by a Schwartz kernel of the form $k(\rho^\mu \lambda, q)$ with $\lambda \in \Lambda$, $q \in X_b^2$ and ρ being a distance function for ff in X_b^2.

   Let us assume that we have a coordinate patch in a neighborhood of rb with possible intersection of ff but disjoint from lb. In this case, we can choose local coordinates (x, s, y, y') with $s = \frac{x'}{x}$. Then a distance function for ff is given by x. Therefore, in local coordinates, suppressing the coordinate dependence on y, y':

$$m(s,x)\nu' := x^\mu \int_{\mathcal{C}} \varphi K_{P(\lambda)} d\lambda = x^\mu \int_{\mathcal{C}} k(x^\mu \lambda, s, x) d\lambda\, \nu'. \tag{9.12}$$

   We have to show that $m(s,x)$ satisfies the properties of operators in $\Psi_b^0(X)$ as described in 7.3.2.

   By definition of $\Psi_c^{-\mu-1,-2\mu,\mu}$, the kernel $k(z,s,x)$ is of rapid decay on rb, that is for $s \to 0$, is smooth in x up to zero and is such that if we define:

$$\tilde{k}(z,s,x) = z^{-2}\, k(z^{-1}, s, x),$$

   then $\tilde{k}(z,s,x)$ is smooth in $z = 0$. From this and since x varies in a bounded subset, we can conclude that $|k(x^\mu \lambda, s, x)|$ is bounded from above by:

$$(1 + |x^\mu \lambda|)^{-2}\, \tilde{k}(s), \tag{9.13}$$

   where $\tilde{k}(s)$ is a smooth function of s and with rapid decay for $s \to 0$. The estimate (9.13) immediately implies that the defining integral for $m(s,x)$ in (9.12) is absolutely converging and therefore gives smoothness of $m(s,x)$ in the interior of X_b^2 as well as rapid decay for $s \to 0$, that is on rb.

   Here, (9.13) allows us to estimate:

$$|m(s,x)| \leq x^{\mu} \int_{\mathcal{C}} |f(\lambda)|(1+|x^{\mu}\lambda|)^{-2} d\lambda\, \tilde{k}(s).$$

Integrating along $\mathcal{C}$ as in Figure 9.1, $|m(s,x)|$ is trivially bounded along the inner arch, therefore it is enough to estimate the integral along the two rays $\omega e^{\pm i\theta}$, for $\omega \in [1,\infty)$. We treat here the case of $\omega e^{+i\theta}$, the estimate along the second ray works analogous.

We have to estimate:

$$x^{\mu} \int_{1}^{\infty} \|f\|_{\infty}(1+x^{\mu}\omega)^{-2} d\omega\, \tilde{k}(s),$$

which can be evaluated explicitly, giving:

$$|m(s,x)| \leq \frac{\|f\|_{\infty}}{1+x^{\mu}} \tilde{k}(s) \leq \|f\|_{\infty} \tilde{k}(s). \tag{9.14}$$

Therefore we can conclude that $|m(s,x)|$ is bounded up to $x \to 0$ and of rapid decay in s for $s \to 0$.

The calculations for lb work completely analogous as the ones on rb done above, yielding the same type of estimates.

Now we apply the integral kernel (9.12) to a distribution. Here, since the support of φ is a coordinate patch which is supported around rb, it holds that $m(s,x)$ is supported right to the diagonal, that is $x' \leq x$ on $\text{supp}(m(s,x))$ for $s = \frac{x'}{x}$.

Then we obtain with help of the estimate (9.14) with respect to $(\|\cdot\|_{\mathcal{H}_p^{0,0}})^p$, using that m is of rapid decay in $(\frac{x'}{x})$ for each $N \in \mathbb{N}_0$:

$$\begin{aligned}
&| \int_0^{\infty} | \int_0^{\infty} m(\frac{x'}{x}, x)\, u(x')\, \frac{dx'}{x'} |^p\, x^{\frac{n}{2}p} \frac{dx}{x} \\
&\leq | \int_0^{\infty} | \int_0^{x} \|f\|_{\infty} (\frac{x'}{x})^N\, u(x')\, \frac{dx'}{x'} |^p\, x^{\frac{n}{2}p} \frac{dx}{x} \\
&= \int_0^{\infty} | \int_0^{x} \|f\|_{\infty} (x')^{N-1}\, u(x')\, dx' |^p\, x^{-1-p(N-\frac{n}{2})} dx
\end{aligned}$$

We use the following Hardy inequality (cf. [37], Lemma 3.14, page 196), assuming that $N > \frac{n}{2}$:

$$\int_0^\infty \left(\int_0^t g(s)ds \right)^p t^{-1-r} dt \leq (\frac{p}{r})^p \int_0^\infty g(t)^p t^{p-1-r} dt, \tag{9.15}$$

we arrive at:

$$| \int_0^\infty | \int_0^\infty m(\frac{x'}{x}, x)\, u(x')\, \frac{dx'}{x'} |^p\, x^{\frac{n}{2}p} \frac{dx}{x} \leq \|f\|_\infty^p \left(\frac{1}{N - \frac{n}{2}} \right)^p \int_0^\infty |u(x)|^p\, x^{\frac{n}{2}p} \frac{dx}{x}. \tag{9.16}$$

2. **Integral kernel supported around the diagonal:**

We choose for a subset $\mathcal{U} \subset \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}$ an explicit set of local coordinates on X_b^2 around $\Delta_b \cap ff$, but away from lb and rb by:

$$X_b^2 \cong \mathcal{U} \times \mathbb{R}^n_w, \quad \text{where } (x, y) \in \mathcal{U} \text{ and } w = (\log(x'/x), y' - y), \tag{9.17}$$

with $x, x' \in [0, 1)$ and $y, y' \in \mathbb{R}^{n-1}$.

Using this set of local coordinates around $\Delta_b \cap ff$, a possible boundary defining function is given by $\rho = x \in [0, 1)$.

On the chosen coordinate patch $\mathcal{U}$ and set of coordinates, we have that $\Delta_b \cong \mathcal{U} \times \{0\}$ and $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, and there exists a smooth mapping $(x, y) \mapsto a(x, y, \xi, \lambda)$ with values in $S_{r,\text{cl}}^{-\mu-1,-2\mu,\mu}(\mathbb{R}^n; \Lambda)$, such that:

$$\varphi K_{P(\lambda)} = \int e^{iw\cdot\xi} p(x, y, \xi, x^\mu \lambda)\, đ\xi \cdot m'. \tag{9.18}$$

We define:

$$p_f(x, y, \xi) = x^\mu \int_{\mathcal{C}} f(\lambda) p(x, y, \xi, x^\mu \lambda) d\lambda. \tag{9.19}$$

By exchanging the orders of integration in (9.10), we see that if $K_{p_f}(x, y, w)$ defines the integral kernel of the operator (9.10), then it holds, that:

$$\varphi K_{p_f}(x, y, w) = \int e^{i\, w\cdot\xi} p_f(x, y, \xi)\, đ\xi. \tag{9.20}$$

Our aim is now to prove that the symbol $p_f(x, y, \xi)$ is a symbol which is associated to an operator $\Psi_b^0(X)$ as specified in Equation (7.3.2).

For this, we have to show that the mapping $(x, y) \mapsto p_f(x, y, \xi)$ defines a smooth mapping into $S^0(\mathbb{R}^n)$. Further, to prove H^∞ Calculus for $\varphi f(P)$, we compute the boundedness of the corresponding seminorms, i.e. we check, that:

$$|\partial_\xi^\alpha (x\partial_x)^k \partial_y^\beta p_f(x,y,\xi)(\|f\|_\infty)^{-1}\langle\xi\rangle^\alpha, \tag{9.21}$$

is uniformly bounded for $x \in [0,1), y \in \mathbb{R}^n$ and $f \in H(\theta)$ and for all k, β.

Since we have that $x\partial_x\, x^\mu = \mu\, x^\mu$, we obtain for an arbitrary $p \in \Psi_c^{\mu,p,d}$:

$$x\partial_x p(x,y,\xi,x^\mu\lambda) = (x\partial_x p)(x,y,\xi,x^\mu\lambda) + \mu\, x^\mu\,(\lambda\partial_\lambda p)(x,y,\xi,x^\mu\lambda). \tag{9.22}$$

Now the first summand in the above expression for $x\partial_x p(x,y,\xi,x^\mu\lambda)$ is clearly contained in the same symbol class as $p(x,y,\xi,x^\mu\lambda)$, and since $|\lambda|$ is bounded from above by $(1+|\xi|+|\lambda|^{\frac{1}{\mu}})^\mu$, also the second summand satisfies the same symbol estimates as $p(x,y,\xi,x^\mu\lambda)$.

Therefore, by differentiating p_z under the integral, proving the estimate (9.21) reduces to the proof in the case of $\beta = 0$, $k = 0$.

We obtain:

$$\begin{aligned} |\partial_\xi^\alpha p_f(x,y,\xi)| &\leq x^\mu \int_{\mathcal{C}} |f(\lambda)|\, |\partial_\xi^\alpha a(x,y,\xi,x^\mu\lambda)| d\lambda \\ &\leq x^\mu \int_{\mathcal{C}} |f(\lambda)| C_\alpha (1+|\xi|)^{\mu-1-|\alpha|}(1+|\xi|+x\,|\lambda|^{1/\mu})^{-2\mu} d\lambda \\ &= \langle\xi\rangle^{-|\alpha|} C_\alpha x^\mu \int_{\mathcal{C}} |f(\lambda)| \frac{(1+|\xi|)^{\mu-1}}{(1+|\xi|+x\,|\lambda|^{1/\mu})^{2\mu}} d\lambda. \end{aligned}$$

We can use, that on the path $\mathcal{C}$, $|f(\lambda)|$ is bounded from above by $\|f\|_\infty$, hence:

$$|\partial_\xi^\alpha p_f(y,\xi)| \leq \langle\xi\rangle^{-|\alpha|} \tilde{C}_{\alpha n}\, \|f\|_\infty x^\mu \int_{\mathcal{C}} \frac{(1+|\xi|)^{\mu-1}}{(1+|\xi|+x\,|\lambda|^{1/\mu}|)^{2\mu}} d\lambda. \tag{9.23}$$

At this point we substitute $\omega = x^\mu\lambda$, to obtain:

$$\begin{aligned} |\partial_\xi^\alpha p_f(y,\xi)| &\leq \langle\xi\rangle^{-|\alpha|} C_\alpha\, \|f\|_\infty \int_{\mathcal{C}} \frac{(1+|\xi|)^{\mu-1}}{(1+|\xi|+|\omega|^{1/\mu})^{2\mu}} d\omega \\ &\leq \langle\xi\rangle^{-|\alpha|} C_\alpha\, \|f\|_\infty \int_{\mathcal{C}} (1+|\xi|+|\omega|^{1/\mu})^{-\mu-1} d\omega \\ &\leq \langle\xi\rangle^{-|\alpha|} C_\alpha\, \|f\|_\infty \int_{\mathcal{C}} (1+|\omega|^{1/\mu})^{-\mu-1} d\omega. \end{aligned} \tag{9.24}$$

Since $a(x, y, \xi, x^d\lambda)$ is the symbol for the integral kernel $\varphi K_{A_1(\lambda)}$, where φ is compactly supported, we can assume without loss of generality that $x^\mu \leq 1$ and replace $x^\mu \mathcal{C}$ by $\mathcal{C}$:

$$|\partial_\xi^\alpha p_f(y, \xi)| \quad \leq \quad \langle\xi\rangle^{-|\alpha|} C_\alpha \, \|f\|_\infty \int\limits_{\mathcal{C}} (1 + |\omega|^{1/\mu})^{-\mu-1} d\omega. \tag{9.25}$$

Now the desired estimate follows, since $\int\limits_{\mathcal{C}} (1 + |\omega|^{1/\mu})^{-\mu-1} d\omega < \infty$.

□

9.2.2 Estimates on the Terms of Leading Order

In the proof of Theorem 4.11 in [10] it is shown that the highest order contribution to $Q(\lambda)$ arises from an operator $Q_1(\lambda)$, whose integral kernel is defined as follows:

First a_μ is defined as:

$$q_{-\mu}(x, y, \xi, \lambda) = \chi(\xi)(a_\mu(x, y, \xi) - x^\mu \lambda)^{-1}, \tag{9.26}$$

with $(x, y) \in \mathcal{U} = [0, c) \times \mathbb{R}^{n-1}$. Then choose coordinates (x, y, w) with $w = (log(x'/x), y' - y)$ on X_b^2 near Δ_b. Further choose $\varphi \in C_c^\infty(\mathcal{U})$, $\psi(w) \in C_c^\infty(\mathbb{R}^n)$ with $\psi(w) = 1$ on a neighborhood of $w = 0$ and define:

$$K_{Q_1(\lambda)} = \varphi(x, y)\psi(w) \int e^{iw\cdot\xi} q_{-\mu}(x, y, \xi, \lambda) \, đ\xi \cdot \mathfrak{m}', \tag{9.27}$$

where $\mathfrak{m}' = |(dx/x)dy|$.

Then, $K_{Q_1(\lambda)}$ defines the integral kernel of $Q_1(\lambda)$, the highest order contribution for the resolvent part $Q(\lambda)$.

If we assign the Dunford integral to the kernel $K_{Q_1(\lambda)}$ to $Q_1(\lambda)$ to compute $(Q_1)_z$, then we see by exchanging the orders of integration that $(Q_1)_z$ is a conormal distribution with symbol $\frac{x^\mu}{2\pi i} \int\limits_{\mathcal{C}} f(\lambda)\varphi(x', y')\psi(w)\chi(\xi)(a_\mu(x, y, \xi) - x^\mu\lambda)^{-1} d\lambda$.

Lemma 9.2.3. *We define:*

$$(q_1)_f(x, y, \xi) = \frac{x^\mu}{2\pi i} \int\limits_{\mathcal{C}} f(\lambda)\varphi(x, y)\psi(w)\chi(\xi)(a_\mu(x, y, \xi) - x^\mu\lambda)^{-1} d\lambda. \tag{9.28}$$

Then, $(q_1)_f$ defines a classical symbol of order zero and the symbol estimates for $(q_1)_f$ are uniform in $\|f\|_\infty$. In particular:

$$\|Q_{1,f}\|_{\mathcal{L}(\mathcal{H}_p^{0,\alpha}(X))} \leq c_p \|f\|_\infty. \tag{9.29}$$

Proof. First we subsitute $\omega = x^\mu \lambda$ in (9.28), we obtain:

$$(q_1)_f(x, y, \xi) = \frac{1}{2\pi i} \int\limits_{\mathcal{C}} f(x^{-\mu}\omega)\varphi(x, y)\psi(w)\chi(\xi)(a_\mu(x, y, \xi) - \omega)^{-1} d\omega.$$

Now, since $a_\mu(x, y, \xi)$ is the positively homogenous principal symbol of A, we have that $\mathrm{spec}(a_\mu(x, y, \xi)) \subset \Omega_\xi$, where:

$$\Omega_\xi = \{\lambda \in \mathbb{C}\backslash\Lambda \mid c_1|\xi|^\mu \leq |\lambda| \leq c_2|\xi|^\mu\}. \tag{9.30}$$

Further we define $q_{-\mu}(x, y, \xi, \lambda)$ by $q_{-\mu}(x, y, \xi, \lambda) := \chi(\xi)(a_\mu(x, y, \xi) - \lambda)^{-1}$. Then it is easy to see that $q_{-\mu}(x, y, \xi, \lambda) \in S^{-\mu,-\mu,\mu}(\mathbb{R}^n; \Lambda)$. Therefore we have that:

$$|(\partial_\xi)^\alpha q_{-\mu}| \leq c_\alpha \langle\xi\rangle^{-|\alpha|}(1 + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu} \tag{9.31}$$

for each multi index $\alpha \in \mathbb{N}_0^k$.

Now we want to estimate:

$$|(\partial_\xi)^\alpha(q_1)| = |(\partial_\xi)^\alpha \frac{1}{2\pi} \int\limits_{\mathcal{C}} f(x^{-\mu}\omega)\varphi(x, y)\psi(w)\chi(\xi)(a_\mu(x, y, \xi) - \omega)^{-1} d\omega| \tag{9.32}$$

Since the integrand is holomorphic outside of Ω_ξ, we can change by Cauchy's theorem the integral path from $\mathcal{C}$ to $\Upsilon(\xi) = \partial\Omega_\xi$, see figure 9.2. Further, $\varphi(x, y)\psi(w)$ is smooth and compactly supported, hence bounded by a constant and f can be estimated on $\mathcal{C}$ by $\|f\|_\infty$. We obtain:

$$|(\partial_\xi)^\alpha(q_1)| \leq c\,\|f\|_\infty \int\limits_{\Upsilon(\xi)} |(\partial_\xi)^\alpha q_{-\mu}(x, y, \xi)| d\omega \tag{9.33}$$

Now, using the symbol estimates for $S^{-\mu,-\mu,\mu}(\mathbb{R}^n; \Lambda)$:

$$|(\partial_\xi)^\alpha(q_1)| \leq c\,\|f\|_\infty \int\limits_{\Upsilon(\xi)} |C_\alpha \langle\xi\rangle^{-\alpha}(1 + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu}| d\omega. \tag{9.34}$$

On $\Upsilon(\xi)$ we can estimate $(1 + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu} \leq (1 + |\xi| + (c_1)^{\frac{1}{\mu}}\langle\xi\rangle)^{-\mu} \leq c'\langle\xi\rangle^{-\mu}$ for a $c' > 0$.

We obtain:

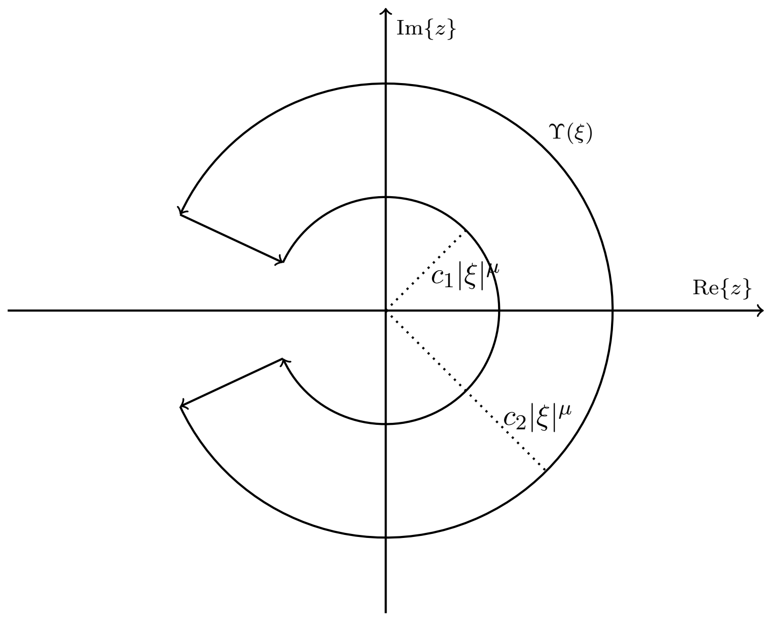


FIGURE 9.2: The path $\Upsilon(\xi)$

$$|(\partial_\xi)^\alpha(q_1)| \leq c\,\|f\|_\infty \int\limits_{\Upsilon(\xi)} |C_\alpha\,\langle\xi\rangle^{-\alpha} c'\langle\xi\rangle^{-\mu}|d\omega \leq c\,C_\alpha\,c'\,\|f\|_\infty\langle\xi\rangle^{-\alpha}\langle\xi\rangle^{-\mu}\,\mathrm{length}(\Upsilon(\xi)). \tag{9.35}$$

Here we can estimate $\mathrm{length}(\Upsilon(\xi)) \leq (2+4\pi)c_2\langle\xi\rangle^\mu$, thus we arrive at the desired symbol estimate for $(q_1)_f$, with $\tilde{c}_\alpha := c\,C_\alpha\,c'\,(2+4\pi)c_2$:

$$|(\partial_\xi)^\alpha(q_1)| \leq \tilde{c}_\alpha\,\|f\|_\infty\langle\xi\rangle^{-\alpha}. \tag{9.36}$$

□

9.2.3 Estimates on the Residual Operators

Let the model cone be:

$$Y^\wedge = [0,\infty) \times Y, \tag{9.37}$$

and choose local coordinates (x, y) on $Y^\wedge$ with $x \in [0,\infty), \quad y \in Y$.

We can introduce weighted Sobolev spaces $\rho^\alpha H_b^{s,p}(Y^\wedge)$ over $Y^\wedge$ in the same way as over $\mathbb{B}$. Choosing functions $\varphi \in C_c^\infty(\mathcal{T}_d)$, we can distinguish between two cases:

1. φ having disjoint support with ff: In this case, $\varphi R(\lambda)$ is of rapid decay for $|\lambda| \to \infty$.

2. φ being supported in a neighborhood of ff.

In this case there exist cut-off functions $\omega, \omega' \in C_0^\infty(X)$ such that $\omega, \omega' \equiv 1$ near $Y = \partial X$ and compactly supported in a neighborhood of Y, such that

$$R(\lambda) = \omega R(\lambda)\omega'. \tag{9.38}$$

Using that a cut off functions ω, ω' with compact support in a neighborhood of $x = 0$, x being a distance function for ∂X on X, multiplication with ω induces continuous maps $\omega : H_b^{s,p}(Y^\wedge) \to H_b^{s,p}(X)$ resp. $\omega' : H_b^{s,p}(X) \to H_b^{s,p}(Y^\wedge)$. Therefore, we can estimate R_f equivalently with respect to $\| \cdot \|_{\mathcal{H}_b^{s,p}(Y^\wedge)}$.

Therefore it suffices to prove H^∞ Calculus with respect to the $\mathcal{H}_p^{0,0}(Y^\wedge)$ norm for a kernel which is supported on $\mathcal{T}_d(Y^\wedge)$ acting on distributions over $Y^\wedge$. Let $(x, y), (x', y') \in \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}$, such that coordinates on $(Y^\wedge)^2$ can be expressed locally by (x, y, x', y'). To shorten notation, we suppress the integration along Y, Y'. Then, we work with residual operators having kernels which are defined on $\mathcal{T}_d$, such that:

$$\tilde{\beta} : \mathcal{T}_d \to \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge. \tag{9.39}$$

We choose a cut-off function χ, such that we have the blowdown-map:

$$\chi(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in (1, \infty) \end{cases}, \tag{9.40}$$

which is explicitly constructed in the Appendix in (B.8).

Our aim is to construct a partition of unity which is defined on $\mathcal{T}_d$ which allows us to treat the singular behavior of the integral kernel of $B(\lambda)$ towards the boundary faces of $\mathcal{T}_d$ separately.

We begin with an explicit construction of a partition of unity which is defined on $\overline{\Lambda}_d$:

$$\begin{aligned} \mathrm{id} &= (\chi(\frac{x}{r}) + \chi(\frac{r}{x}))(\chi(\frac{x'}{x}) + \chi(\frac{x}{x'})) \\ &= \underbrace{\chi(\frac{x}{r})\chi(\frac{x'}{x})}_{=:\kappa_1} + \underbrace{\chi(\frac{x'}{r})\chi(\frac{x}{x'})}_{=:\kappa_2} + \underbrace{\chi(\frac{r}{x})\chi(\frac{x'}{x})}_{=:\kappa_3} + \underbrace{\chi(\frac{r}{x'})\chi(\frac{x}{x'})}_{=:\kappa_4} \\ &= \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4. \end{aligned} \tag{9.41}$$

We lift the four functions $\kappa_i,\ i \in \mathbb{N}$ to $\mathcal{T}_d$:

We consider a blowdown map β, and define the lift of (9.41) to $\mathcal{T}_d$ by:

$$\beta^*(\mathrm{id}) = \beta^*(\kappa_1) + \beta^*(\kappa_2) + \beta^*(\kappa_3) + \beta^*(\kappa_4). \tag{9.42}$$

The index set $\mathcal{G}$ which is associated to $R(\lambda)$ captures the asymptotic behavior of the integral kernel $K_{R(\lambda)}$. Now, since $R(\lambda) \in x^{\mu}\Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda)$, we have an integral kernel of $R(\lambda)$ which is defined on $\mathcal{T}_d$. Hence, the action of $R(\lambda)$ on distributions can be expressed by:

$$R(\lambda)u = \tilde{\pi}_{2b*}(\tilde{\pi}_{1b}^*(u)\tilde{\pi}_{2b}^*(\nu)\, K_{R(\lambda)}), \tag{9.43}$$

where $K_P = k\,\mathfrak{m}'$ with $k \in \mathcal{A}^{\mathcal{E}}(\mathcal{T}_d)$.

In (9.43), the product $\tilde{\pi}_{1b}^*(u)\tilde{\pi}_{2b}^*(\nu)\, K_{R(\lambda)}$ lives on $\mathcal{T}_d$, while the pull-back $\beta^*(\mathrm{id})$ of the identity is the identity on $\mathcal{T}_d$. Hence, we can insert (9.42) into (9.43), to split the action of $R(\lambda)$ on distributions into four disjoint components:

$$R(\lambda)u = \sum_{i=1}^{4} \tilde{\pi}_{2b*}(\beta^*(\eta_i)\, \tilde{\pi}_{1b}^*(u)\tilde{\pi}_{2b}^*(\nu)\, K_{R(\lambda)}). \tag{9.44}$$

Our strategy to prove the estimate (9.3) for R_f will be as follows: We decompose (9.43) by inserting a partition of unity. Then we treat the four components obtained in (9.44) separately and perform the occurring calculations in local coordinates which are expressed by appropriate distance functions to the boundary hypersurfaces which are contained in the regions $\Omega_i,\ i \in \{1, ..., 4\}$.

We denote by $K_{R(\lambda)}$ the integral kernel of $R(\lambda)$. Then $K_R(\lambda) = k\,\mathfrak{m}'$ with $k \in \mathcal{A}^{\mathcal{E}}(\mathcal{T}_d)$, where $\mathcal{E} = (G_{\mathrm{lb}}, G_{\mathrm{rb}}, G_{\mathrm{ff}}, G_{\mathrm{fi}}, \emptyset)$.

Part 1:

Now, if we denote the Dunford integral of $R(\lambda)$ in the sense of (9.2) with respect to f as R_f, we can compute $R_f u$ as:

$$R_f u(x) = \int\limits_{x'}\int\limits_{\mathcal{C}} f(\lambda)\, x^{\mu}\, k(r, \frac{x}{r}, \frac{x'}{x})\chi(\frac{x}{r})\chi(\frac{x'}{x})\mathrm{d}\lambda\, u(x')\frac{dx'}{x'}.$$

We use, that $r = |\lambda|^{-\frac{1}{\mu}}$ and substitute $\lambda = \rho^{\mu}\, e^{i\theta}$:

$$\int\limits_{\mathcal{C}} f(\lambda)\mathrm{d}\lambda = \mu\, \mathrm{e}^{i\theta}\int\limits_{1}^{\infty} f(\rho^{\mu}\mathrm{e}^{i\theta})\rho^{\mu-1}\mathrm{d}\rho. \tag{9.45}$$

Hence,

$$R_f u(x) = \int\limits_{x'} \mu\, \mathrm{e}^{i\theta}\int\limits_{1}^{\infty} f(\rho^{\mu}\mathrm{e}^{i\theta})\, x^{\mu}\, k(\rho^{-1}, \rho x, \frac{x'}{x})\chi(\rho x)\chi(\frac{x'}{x})\rho^{\mu-1}\mathrm{d}\rho\, u(x')\frac{dx'}{x'}.$$

As $\rho \in [1, \infty)$ it is clear, that $\chi(\rho x)$ is nonzero only for $x \in [0,1]$ and further $\rho \in [1, x^{-1})$:

$$R_f u(x) = \chi(x) \int_{x'} \mu\, \mathrm{e}^{i\theta} \int_1^{x^{-1}} f(\rho^\mu \mathrm{e}^{i\theta})\, x^\mu\, k(\rho^{-1}, \rho x, \frac{x'}{x}) \chi(\frac{x'}{x}) \rho^{\mu-1} \mathrm{d}\rho\, u(x') \frac{dx'}{x'}.$$

Evaluating $\chi(\frac{x'}{x})$, we obtain:

$$R_f u(x) = \chi(x) \int_0^x \mu\, \mathrm{e}^{i\theta} \int_1^{x^{-1}} f(\rho^\mu \mathrm{e}^{i\theta})\, x^\mu\, k(\rho^{-1}, \rho x, \frac{x'}{x}) \rho^{\mu-1} \mathrm{d}\rho u(x') \frac{dx'}{x'}. \tag{9.46}$$

We proceed to find an estimate for $(\|R_f u\|_{\mathcal{H}_p^{0,0}})^p$:

$$(\|R_f u\|_{\mathcal{H}_p^{0,0}})^p = \int_0^\infty |R_f u(x)|^p\, x^{\frac{n}{2}p} \frac{\mathrm{d}x}{x}. \tag{9.47}$$

We insert the expression which we obtained for $R_f u$:

$$\begin{aligned}(\|R_f u\|_{\mathcal{H}_p^{0,0}})^p &= \int_0^x |\int_0^x \mu\, \mathrm{e}^{i\theta} \int_1^{x^{-1}} f(\rho^\mu \mathrm{e}^{i\theta}) x^\mu k(\frac{1}{\rho}, \rho x, \frac{x'}{x}) \rho^{\mu-1} \mathrm{d}\rho\, u(x') \frac{\mathrm{d}x'}{x'}|^p\, x^{\frac{n}{2}p} \frac{\mathrm{d}x}{x} \\ &\leq \int_0^1 (\int_0^x \mu \int_1^{x^{-1}} \|f\|_\infty\, \rho^{\mu-1} x^\mu |k(\frac{1}{\rho}, \rho x, \frac{x'}{x})| \mathrm{d}\rho\, |u(x')| \frac{\mathrm{d}x'}{x'})^p\, x^{\frac{n}{2}p} \frac{\mathrm{d}x}{x}.\end{aligned} \tag{9.48}$$

At this point, we note that it follows from the fact that the integral kernel k has asymptotic expansions with index sets satisfying the estimates as stated in the Appendix, that is:

$$|k(\frac{1}{\rho}, \rho x, \frac{x'}{x})| \leq C_1\, (\frac{x'}{x})^{\varepsilon + \frac{n}{2}} \quad \forall\, (\frac{1}{\rho}), (\rho x), (\frac{x'}{x}) \in (0,1]. \tag{9.49}$$

And plug this into (9.48), to arrive at:

$$\|R_f u\|_p^p \leq \int_0^1 (\int_0^x C_1\, \mu \int_1^{x^{-1}} \|f\|_\infty \rho^{\mu-1} \mathrm{d}\rho\, x^{\mu-\varepsilon}\, x'^{\varepsilon} |u(x')|\, (x')^{\frac{n}{2}} \frac{\mathrm{d}x'}{x'})^p \frac{\mathrm{d}x}{x}.$$

Evaluating the integral along ρ, we obtain:

$$\|R_f u\|_p^p \leq \|f\|_\infty^p C_1^p \int_0^1 (\int_0^x (1 - x^\mu) x^{-\varepsilon}\, x'^{\varepsilon-1} |(x')^{\frac{n}{2}} u(x')| \mathrm{d}x')^p \frac{\mathrm{d}x}{x}.$$

Since $(1-x^{\mu}) \leq 1$, we obtain:

$$\|R_f u\|_p^p \leq 2^p \|f\|_\infty^p C_1^p \int_0^\infty (\int_0^x x'^{\varepsilon-1} |u(x')| (x')^{\frac{n}{2}} \mathrm{d}x')^p x^{-1-p\varepsilon} \mathrm{d}x. \tag{9.50}$$

We can use the following Hardy inequality (cf. [37], Lemma 3.14, page 196):

$$\int_0^\infty \left(\int_0^t g(s) ds \right)^p t^{-1-r} dt \leq (\frac{p}{r})^p \int_0^\infty g(t)^p t^{p-1-r} dt, \tag{9.51}$$

to estimate:

$$\begin{aligned} \|R_f u\|_p^p &\leq (\frac{p}{p\,\varepsilon})^p \|f\|_\infty^p C_1^p \int_0^\infty |u(x)|^p x^{p\varepsilon-p} x^{p-1-p\varepsilon} x^{p\frac{n}{2}} \mathrm{d}x \\ &= (\frac{1}{\varepsilon})^p \|f\|_\infty^p C_1^p \int_0^\infty |u(x)|^p x^{p\frac{n}{2}} \frac{\mathrm{d}x}{x}. \end{aligned} \tag{9.52}$$

We obtain:

$$\|R_f u\|_{\mathcal{H}_p^{0,0}}^p \leq \|f\|_\infty^p (\frac{1}{\varepsilon})^p C_1 \|u\|_{\mathcal{H}_p^{0,0}}^p. \tag{9.53}$$

Part 2:

Using the usual procedure of pull-backs and push-forwards, it is straightforward to show, that:

$$R_f u(x) = \int_{x'} \int_{\mathcal{C}} f(\lambda) x^{\mu} k(r, \frac{x'}{r}, \frac{x}{x'}) \chi(\frac{x}{r}) \chi(\frac{x}{x'}) \mathrm{d}\lambda \, u(x') \frac{dx'}{x'}. \tag{9.54}$$

The corresponding estimates can be obtained analogously to the estimates for Part 1.

Part 3:

Using the usual procedures of pull-backs and push-forwards, we obtain as expression for $R_f u$:

$$R_f u(x) = \int_{x'} \int_{\mathcal{C}} f(\lambda) \cdot x^{\mu} \cdot k(x, \frac{r}{x}, \frac{x'}{x}) \chi(\frac{r}{x}) \chi(\frac{x'}{x}) \mathrm{d}\lambda u(x') \frac{dx'}{x'}, \tag{9.55}$$

with $r = |\lambda|^{-\frac{1}{\mu}}$.

We use (9.45) to evaluate (9.55) along $\mathcal{C}$:

$$R_f u(x) = \int_0^\infty \mu\, e^{i\theta} \int_1^\infty f(\rho^\mu\, e^{i\theta}) x^\mu\, k(\frac{1}{\rho\, x}, \frac{x'}{x}, x) \chi(\frac{1}{\rho x}) \chi(\frac{x'}{x}) \rho^{\mu-1} d\rho\, u(x') \frac{dx'}{x'} \tag{9.56}$$

$$= \int_0^x \mu\, e^{i\theta} \int_1^\infty f(\rho^\mu\, e^{i\theta}) x^\mu k(\frac{1}{\rho\, x}, \frac{x'}{x}, x) \chi(\frac{1}{\rho x}) \rho^{\mu-1} d\rho\, u(x') \frac{dx'}{x'}. \tag{9.57}$$

We go on to evaluate $(\|R_f u\|_{\mathcal{H}_p^{0,0}})^p$:

$$(\|R_f u\|_{\mathcal{H}_p^{0,0}})^p \leq \int_0^\infty (\int_1^x \mu \int_0^\infty \|f\|_\infty \rho^{\mu-1} x^\mu\, |k(\frac{1}{\rho\, x}, \frac{x'}{x}, x)| \chi(\frac{1}{\rho x}) d\rho\, u(x') \frac{dx'}{x'})^p\, x^{\frac{n}{2}p}\, \frac{dx}{x}.$$

Defining:

$$I_1 := \int_0^1 (\int_0^x \mu \int_1^\infty \|f\|_\infty \rho^{\mu-1} x^\mu\, |k(\frac{1}{\rho\, x}, \frac{x'}{x}, x)| \chi(\frac{1}{\rho x}) d\rho\, u(x') \frac{dx'}{x'})^p\, x^{\frac{n}{2}p}\, \frac{dx}{x}, \tag{9.58}$$

and

$$I_2 := \int_1^\infty (\int_0^x \mu \int_1^\infty \|f\|_\infty \rho^{\mu-1} x^\mu\, |k(\frac{1}{\rho\, x}, \frac{x'}{x}, x)| \chi(\frac{1}{\rho x}) d\rho\, u(x') \frac{dx'}{x'})^p\, x^{\frac{n}{2}p}\, \frac{dx}{x}, \tag{9.59}$$

we can split the above integral in two parts, which gives us:

$$(\|R_f u\|_{\mathcal{H}_p^{0,0}})^p \leq I_1 + I_2. \tag{9.60}$$

Now, since we have that $x \leq 1$ on I_1 and $x \geq 1$ on I_2, this allows us to evaluate $\chi(\frac{1}{\rho x})$ on both integrals and two estimate them separately: in both integrals, we have the same bound for $|k(\frac{1}{\rho\, x}, \frac{x'}{x}, x)|$:

$$|k(\frac{1}{\rho\, x}, \frac{x'}{x}, x)| \leq (\frac{1}{\rho\, x})^N\, (\frac{x'}{x})^{\varepsilon + \frac{n}{2}}\, C_3 \quad \forall (\frac{1}{\rho\, x}), (\frac{x'}{x}) \in [0, 1],\ N \in \mathbb{N}. \tag{9.61}$$

We choose $N = \mu$ in 9.61.

- Estimate for I_1:

  Evaluating $\chi(\frac{1}{\rho x})$ in I_1, we have:

$$\int_0^1 (\int_0^x \mu \int_{x^{-1}}^\infty \|f\|_\infty \rho^{\mu-1} x^\mu\, |k(\frac{1}{\rho\, x}, \frac{x'}{x}, x)| d\rho\, u(x') \frac{dx'}{x'})^p\, x^{\frac{n}{2}p}\, \frac{dx}{x}. \tag{9.62}$$

Using the kernel estimate 9.61 gives:

$$I_1 \leq \int_0^1 \left(\int_0^x \mu \int_{x^{-1}}^{\infty} \|f\|_\infty \rho^{\mu-1} x^\mu \, (\frac{1}{\rho\, x})^{\mu+\varepsilon} \, (\frac{x'}{x})^\varepsilon \, C_3 \, d\rho \, u(x') \, (x')^{\frac{n}{2}} \, \frac{dx'}{x'} \right)^p \frac{dx}{x}$$
$$= \int_0^1 \left(\int_0^x \mu \int_{x^{-1}}^{\infty} \|f\|_\infty \rho^{-1} d\rho \, x'^{\varepsilon-1} x^{-2\varepsilon} \, C_3 \; u(x') \, (x')^{\frac{n}{2}} \, dx' \right)^p \frac{dx}{x}.$$

We obtain, if we evaluate the above integral along ρ, defining $\kappa_3 := \frac{C_3 \, \mu \|f\|_\infty}{\varepsilon}$:

$$I_1 \leq (\kappa_3)^p \int_0^1 \left(\int_0^x x'^{\varepsilon-1} x^{-2\varepsilon} \, u(x') \, (x')^{\frac{n}{2}} \, dx' \right)^p \frac{dx}{x}$$
$$= (\kappa_3)^p \int_0^1 x^{-1-p\varepsilon} \left(\int_0^x x'^{\varepsilon-1} \, u(x') \, (x')^{\frac{n}{2}} \, dx' \right)^p dx$$
$$\leq (\kappa_3)^p \int_0^\infty x^{-1-p\varepsilon} \left(\int_0^x x'^{\varepsilon-1} \, u(x') \, (x')^{\frac{n}{2}} \, dx' \right)^p dx. \tag{9.63}$$

At this point we can use the modified Hardy inequality (9.51) to obtain:

$$I_1 \leq (\kappa_3)^p (\frac{p}{p\varepsilon})^p \int_0^\infty x^{p\varepsilon - p} u(x)^p x^{p-1-p\varepsilon} \, x^{\frac{n}{2}p} \, dx$$
$$= (\frac{\kappa_3}{\varepsilon})^p \int_0^\infty u(x)^p \, x^{\frac{n}{2}p} \, \frac{dx}{x} = (\frac{\kappa_3}{\varepsilon})^p \cdot \|u\|^p_{\mathcal{H}^{0,0}_p}. \tag{9.64}$$

- Estimate for I_2:

  Evaluating $\chi(\frac{1}{\rho x})$ in I_2, we have:

$$I_2 = \int_1^\infty (\int_0^x \mu \int_1^\infty \|f\|_\infty \rho^{\mu-1} x^\mu \, k(\frac{1}{\rho\, x}, \frac{x'}{x}, x) d\rho \, |u(x')| \frac{dx'}{x'})^p \, x^{\frac{n}{2}p} \, \frac{dx}{x}. \tag{9.65}$$

Using 9.61 gives with $N = \mu$:

$$I_2 \leq \int_1^\infty (\int_0^x \mu \int_1^\infty \|f\|_\infty \rho^{-1-\varepsilon} d\rho \, x^{-2\varepsilon} (x')^{\varepsilon-1} C_3 u(x') \, (x')^{\frac{n}{2}} \, dx')^p \frac{dx}{x}. \tag{9.66}$$

Evaluating the integral along ρ, we obtain:

$$\begin{aligned} I_2 &\leq (\kappa_3)^p \int_1^{\infty} (\int_0^{x} x^{-2\varepsilon}(x')^{\varepsilon-1}|u(x')|(x')^{\frac{n}{2}}dx')^p dx \\ &= (\kappa_3)^p \int_1^{\infty} x^{-1-p\varepsilon}(\int_0^{x} x^{-\varepsilon}(x')^{\varepsilon-1}|u(x')|(x')^{\frac{n}{2}}dx')^p dx. \end{aligned} \tag{9.67}$$

Again we can use the inequality (9.51), we obtain:

$$\begin{aligned} I_2 &\leq (\frac{\kappa_3}{\varepsilon})^p \int_0^{\infty} u(x)^p x^{p\varepsilon-p}x^{p-p\varepsilon}\, x^{\frac{n}{2}p}\frac{dx}{x} \\ &= (\frac{\kappa_3}{\varepsilon})^p \int_0^{\infty} u(x)^p\, x^{\frac{n}{2}p}\frac{dx}{x} = (\frac{\kappa_3}{\varepsilon})^p(\|u\|_{\mathcal{H}_p^{0,0}})^p. \end{aligned} \tag{9.68}$$

Part 4:

Using the usual procedure of pull-backs and push-forwards, we arrive as expression for $R_f u$:

$$R_f u(x) = \int_{x'}\int_{\mathcal{C}} f(\lambda)x^{\mu}\, k(x', \frac{r}{x'}, \frac{x}{x'})\chi(\frac{r}{x'})\chi(\frac{x}{x'})d\lambda u(x')\frac{dx'}{x'},$$

which is evaluated equivalently to the expression in Part 3.

9.3 H^{∞} Calculus for Operators Acting on Higher Order Spaces and Besov Spaces

At first, we generalize the H^{∞} Calculus result of Theorem 9.1.1 to higher order Sobolev spaces:

Theorem 9.3.1. *Let $A_s \in x^{-\mu}\Psi_b^{\mu}(X)$, $\mu > 0$ be an operator mapping:*

$$A_s : \mathcal{H}_p^{s+\mu,\alpha}(X) \to \mathcal{H}_p^{s,\alpha-\mu}(X), \quad \text{for } s \geq 0. \tag{9.69}$$

Assume that A_0 satisfies the requirements of Theorem 9.1.1. Then $(A_s - \lambda)$ is uniformly bounded as an operator $(A_s - \lambda) : \mathcal{H}_p^{s+\mu,\alpha}(X) \to \mathcal{H}_p^{s,\alpha-\mu}(X)$. Therefore $f(A_s)$ is well defined in the sense of Equation 9.2. Further, $f(A_s)$ admits H^{∞} Calculus as defined in 9.0.11 for $s \geq 0$.

The arguments are very similar to Theorem 3.3 in [25], therefore we keep the proof short:

Proof. We begin by proving that $(A_s - \lambda)^{-1}$ is uniformly bounded in λ. For $s = 0$ this is part of Theorem 4.11 in [10]. For higher orders in s, we first restrict to integer s and begin with $s = 1$.

Therefore, it suffices to show uniform boundedness of $\|\lambda\, x\partial_x(A_{s=1} - \lambda)^{-1}u\|_{\mathcal{H}^{0,\gamma}}$ resp. $\|\lambda\, \partial_{y^j}(A_{s=1} - \lambda)^{-1}u\|_{\mathcal{H}^{0,\gamma}}$ by $c\,\|u\|_{\mathcal{H}^{1,\gamma+\mu}}$ for a $c > 0$.

In [10] the uniform boundedness is derived for $s = 0$ from the contributions to $(A-\lambda)^{-1}$ by $(A_\wedge - \lambda)^{-1}$ and $B_0(\lambda)$. There the boundedness of $(A - \lambda)^{-1}$ is derived from the κ homogeneity of $(A_\wedge - \lambda)^{-1}$. Now, if $(A_\wedge - \lambda)^{-1}$ is κ homogeneous, the same is true for $[x\partial_x, (A_\wedge - \lambda)^{-1}]$ as well as $[\partial_{y^j}, (A_\wedge - \lambda)^{-1}]$. Therefore boundedness of $(A_\wedge - \lambda)^{-1}$ follows for $s = 1$ and by iteration of the argument for arbitrary integer s.

The Kernel of $B_0(\lambda)$ from Theorem 4.11 in [10] is given by $K_{Q_1(\lambda)}$ as defined in 9.27. Again, using local coordinates around Δ_b, it can be checked by direct computations similar to the ones in the proof of Theorem 3.3 in [25], that $[x\partial_x, B_0(\lambda)]$ and $[\partial_{y^j}, B_0(\lambda)]$ are of the same type as $B_0(\lambda)$. This together with the uniform boundedness on L^2 yields uniform boundedness for $s = 1$ and again by iteration for integer s. Now the uniform boundedness for $s \in \mathbb{R}$ can be obtained by interpolation theory.

Therefore $(A_s)^z$ is well-defined via 9.2. We remember that $(A_s - \lambda)^{-1} = (A_0 - \lambda)^{-1}$ can be decomposed after equation 7.19 into contributions from $x^\mu\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$ and $x^\mu\Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda)$.

Here it follows from Lemma 9.2.2 and Lemma 9.2.3, that the contribution of $x^\mu\Psi_c^{-\mu,-\mu,\mu}(X;\Lambda)$ to the Dunford integral defines a b pseudodifferential operator of order zero with the desired symbol estimates. This gives the desired H^∞ Calculus bounds for this term on $\mathcal{H}_p^{s,\gamma}(X)$.

Regarding the contribution of $x^\mu\Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda)$ to the Dunford integral, we can again refer to the H^∞ Calculus bounds on $\mathcal{H}^{0,\gamma}$, since an application of $x\partial_x$ resp. ∂_y^j to the Schwartz kernel of an operator in $x^\mu\Psi_c^{-\infty,\mu,\mathcal{G}}(X;\Lambda)$ leaves its asymptotic structure invariant. Iteration gives integer s, arbitrary $s > 0$ follows by interpolation. □

Lemma 9.3.2. *The Besov spaces $\mathcal{B}_p^{s,\gamma}$ are obtained as real interpolation spaces of Sobolev spaces by:*

$$(\mathcal{H}_p^{s_1,\gamma}(X), \mathcal{H}_p^{s_2,\gamma}(X))_{p,\theta} = \mathcal{B}_p^{s,\gamma}(X), \quad s = \theta\, s_1 + (1-\theta)s_2, \tag{9.70}$$

Proof. Using the Definition 2.2.6 of the weighted Besov spaces, we see that the lemma follows from the following interpolation result for the standard Besov spaces, which is part of Theorem 6.2.4 of [2]:

$$(H_p^{s_0}, H_p^{s_1})_{\theta,q} = B_{pq}^s \quad (1 \le p, q \le \infty, 0 < \theta < 1), \tag{9.71}$$

for $s_0 \neq s_1$, $s = (1-\theta)s_0 + \theta s_1$. □

We have the following corollary regarding the H^∞-Calculus of operators acting on the Besov spaces $\mathcal{B}_p^{s,\gamma}(X)$:

Corollary 9.3.3. *We obtain by interpolation theory:*

$$\|f(A)\|_{\mathcal{B}_p^{1-\frac{1}{p},\frac{1}{2}}(X)} \leq C\|f\|_\infty. \tag{9.72}$$

Proof. We choose here $p \in \mathbb{N}$, $s_1 = -1$, $s_2 = 0$, $\gamma = \frac{1}{2}$ and $\theta = \frac{1}{p}$, we see that:

$$(\mathcal{H}_p^{-1,\frac{1}{2}}(X), \mathcal{H}_p^{0,\frac{1}{2}}(X))_{p,\theta} = \mathcal{B}_p^{1-\frac{1}{p},\frac{1}{2}}(X).$$

This can be applied to Lemma A.3.4 from the appendix. We can choose $X_0 = Y_0 = \mathcal{H}_p^{0,-\frac{1}{2}}$, $X_1 = Y_1 = \mathcal{H}_p^{-1,-\frac{1}{2}}$. Since we have that:

$$\|f(A)\|_{\mathcal{H}_p^{0,-\frac{1}{2}}(X)} \leq c_0\|f\|_\infty, \qquad \|f(A)\|_{\mathcal{H}_p^{-1,-\frac{1}{2}}(X)} \leq c_1\|f\|_\infty,$$

we obtain from A.3.4:

$$\begin{aligned}\|f(A)\|_{\mathcal{B}_p^{1-\frac{1}{p},\frac{1}{2}}(X)} &\leq (c_0\|f\|_\infty)^{1-\frac{1}{p}}(c_1\|f\|_\infty)^{\frac{1}{p}} \\ &= \underbrace{(c_0)^{1-\frac{1}{p}}(c_1)^{\frac{1}{p}}}_{=:C}\|f\|_\infty.\end{aligned}$$

□

Appendix A

Appendix A - Cone Calculus

A.1 Greens Formula

We have duality between $\mathcal{H}_2^{s,\gamma}$ and $\mathcal{H}_2^{-s,-\gamma}$ under the L^2 scalar product for the metric $g = dx^2 + x^2 h_{ij} dy^i\, dy^j$, which has the functional determinant $\det(g) = h \cdot x^n$ in case of a manifold of dimension $n+1$, with $h = \det(h_{ij})$.

Because C_c^∞ is dense in $\mathcal{H}^{s,\gamma}$ for all $s, \gamma \in \mathbb{R}$, we assume $u, v \in C_c^\infty(\mathbb{D})$ in the proof of the following identity.

Definition A.1.1. *Let $\mathbb{D}$ be a $n+1$ dimensional manifold with metric g. We consider a general Differential operator $P = \sum_{|\alpha| \leq n} a_\alpha D^\alpha$ of order n with smooth, real valued coefficient functions $a_\alpha \in C^\infty(\mathbb{D}, \mathbb{R})$. Then we define the adjoint P^* of P as:*

$$P^* u = \sum_{|\alpha| \leq n} \sqrt{g}^{-1} D^\alpha (\sqrt{g} a_\alpha u). \tag{A.1}$$

We consider now a second order Differential Operator:

$$P_2 = \sum_{i,j=0}^{n} a_{ij}(x,y) D_i\, D_j + \sum_{i=0}^{n} b_i(x,y) + c(x,y), \tag{A.2}$$

defined on a $n+1$ dimensional manifold with local coordinates $x, y_1, ..., y_n$ and metric g_{ij}. We denote the induced boundary metric by $\hat{g}_{ij}$ and the functional determinants with respect to g_{ij}, respective $\hat{g}_{ij}$ by g and $\hat{g}$.

Due to boundary terms the expression:

$$\langle P_2 u,\, v \rangle_{\mathbb{D}} - \langle u,\, P_2^* v \rangle_{\mathbb{D}}, \tag{A.3}$$

is non-vanishing. However, non vanishing terms appear only by partial integration along the boundary directions, i.e. they enter only by terms of P_2 in which D_n is involved.

This means, that if we define:

$$\hat{P}_2 = a_{nn} D_n D_n + \sum_{i=0}^{n-1} (a_{in} + a_{ni}) D_i D_n + b_n D_n, \tag{A.4}$$

we have, that:

$$\langle P_2 u,\, v\rangle_{\mathbb{D}} - \langle u,\, P_2^* v\rangle_{\mathbb{D}} = \langle \hat{P}_2 u,\, v\rangle_{\mathbb{D}} - \langle u,\, \hat{P}_2^* v\rangle_{\mathbb{D}}. \tag{A.5}$$

We compute now contributions from the relevant terms, i.e. the three summands of $\hat{P}_2$. Performing the partial Integration under the scalar product, we have to evaluate the functional determinant $\sqrt{g}$ of g at the boundary, i.e. at $y_{n=0}$. We define for this restriction:

$$\sqrt{\underline{g}} := \sqrt{g}\big|_{y_n=0} \in \mathcal{C}^\infty(\partial\mathbb{D}). \tag{A.6}$$

Further, we define the restrictions $\underline{a}_{in}$:

$$\underline{a}_{in} := a_{in}(x, y_1, ..., y_n)\big|_{y_n=0} \in \mathcal{C}^\infty(\partial\mathbb{D}). \tag{A.7}$$

The leading term is of second order in D_n:

$$\begin{aligned}
& \int_{\mathbb{D}} (a_{nn} D_n D_n u)\overline{v}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n \\
= & -i \int_{\partial\mathbb{D}} (\underline{a}_{nn} D_n u)\overline{v}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_{n-1} + \int_{\mathbb{D}} (D_n u)\overline{D_n(a_n v\sqrt{g})}\mathrm{d}y_1...\mathrm{d}y_n \\
= & -i \int_{\partial\mathbb{D}} (\sqrt{\hat{g}}^{-1}\sqrt{\underline{g}}\underline{a}_{nn} D_n u)\overline{v}\sqrt{\hat{g}}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_{n-1} + \\
& -i \int_{\partial\mathbb{D}} u\overline{(\sqrt{\hat{g}}^{-1}(vD_n(\underline{a}_{nn}\sqrt{\underline{g}}) + \underline{a}_{nn}\sqrt{\underline{g}}(D_n v))}\sqrt{\hat{g}}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_{n-1} + \\
& + \int_{\mathbb{D}} u\overline{\sqrt{g}^{-1} D_n D_n(a_{nn} v\sqrt{g})}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n.
\end{aligned}$$

So we have:

$$\begin{aligned}
& \langle a_{nn} D_n D_n u,\, v\rangle_{\mathbb{D}} - \langle u,\, (a_{nn} D_n D_n)^* v\rangle_{\mathbb{D}} \\
= & -i\,\langle \sqrt{\hat{g}}^{-1}\sqrt{g}\underline{a}_{nn} D_n u,\, v\rangle_{\partial\mathbb{D}} + \\
& -i\,\langle u,\, \sqrt{\hat{g}}^{-1}(vD_n(\underline{a}_{nn}\sqrt{g}) + \underline{a}_{nn}\sqrt{g}(D_n v))\rangle_{\partial\mathbb{D}}.
\end{aligned} \tag{A.8}$$

The next term one is:

$$\begin{aligned}
&\int\limits_{\mathbb{D}} (\sum_{i=0}^{n-1}(a_{in}+a_{ni})D_iD_nu)\overline{v}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n \\
= \quad & \int\limits_{\mathbb{D}} D_nu\overline{\sum_{i=0}^{n-1}D_i(\sqrt{g}(a_{in}+a_{ni})v)}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n \\
= \quad & -i\int\limits_{\partial\mathbb{D}} u\overline{\sum_{i=0}^{n-1}\sqrt{\hat{g}}^{-1}D_i(\sqrt{\underline{g}}(\underline{a}_{in}+\underline{a}_{ni})v)}\sqrt{\hat{g}}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_{n-1}+ \\
&+\int\limits_{\mathbb{D}} u\overline{\sum_{i=0}^{n-1}\sqrt{g}^{-1}D_iD_n(\sqrt{g}(a_{in}+a_{ni})v)}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n.
\end{aligned}$$

We obtain:

$$\begin{aligned}
&\langle\sum_{i=0}^{n-1}(a_{in}+a_{ni})D_iD_nu,\, v\rangle_{\mathbb{D}} - \langle u,\, (\sum_{i=0}^{n-1}(a_{in}+a_{ni})D_iD_n)^*v\rangle_{\mathbb{D}} \\
= \quad & -i\,\langle u,\, v\sum_{i=0}^{n-1}\sqrt{\hat{\underline{g}}}^{-1}D_i(\sqrt{g}(\underline{a}_{in}+\underline{a}_{ni}))\rangle_{\partial\mathbb{D}} - \\
& -i\,\langle u,\, \sum_{i=0}^{n-1}\sqrt{\hat{g}}^{-1}(\sqrt{\underline{g}}(\underline{a}_{in}+\underline{a}_{ni})D_iv\rangle_{\partial\mathbb{D}}.
\end{aligned} \tag{A.9}$$

The last term is:

$$\begin{aligned}
&\int\limits_{\mathbb{D}} b_nD_nu\overline{v}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n \\
= \quad & -i\int\limits_{\partial\mathbb{D}} u\overline{\underline{b}_nv\sqrt{\underline{g}}\sqrt{\hat{g}}^{-1}}\sqrt{\hat{g}}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_{n-1}+ \\
&+\int\limits_{\mathbb{D}} u\overline{\sqrt{g}^{-1}D_n(\underline{b}_nv\sqrt{g})}\sqrt{g}\mathrm{d}x\mathrm{d}y_1...\mathrm{d}y_n,
\end{aligned}$$

and so:

$$\langle b_nD_nu,\, v\rangle_{\mathbb{D}} - \langle u,\, (b_nD_n)^*v\rangle_{\mathbb{D}} = -i\,\langle u,\, \underline{b}_n\sqrt{\underline{g}}\sqrt{\hat{g}}^{-1}v\rangle_{\partial\mathbb{D}}. \tag{A.10}$$

We summarize our results in the formula:

$$\langle P_2u,\, v\rangle_{\mathbb{D}} - \langle u,\, P_2^*v\rangle_{\mathbb{D}} = \langle\mathcal{A}\rho(u),\, \rho(v)\rangle_{\partial\mathbb{D}}, \tag{A.11}$$

with

$$\mathcal{A} = \begin{pmatrix} \mathfrak{a}_{00} & \mathfrak{a}_{01} \\ \mathfrak{a}_{10} & \mathfrak{a}_{11} \end{pmatrix} \quad \text{and} \quad \rho(u) = \begin{pmatrix} \gamma_0(u) \\ \gamma_0(\frac{1}{x}D_nu) \end{pmatrix}. \tag{A.12}$$

We obtain from (A.8), (A.9) and (A.10):

$$\mathfrak{a}_{00} = -i\sum_{i=0}^{n-1}(\sqrt{\hat{g}}^{-1}\sqrt{\underline{g}}(\underline{a}_{in}+\underline{a}_{ni}))D_i - i\sqrt{\underline{g}}\sqrt{\hat{g}}^{-1}b_n+$$
$$+ i\sqrt{\hat{g}}^{-1}D_n(\underline{a}_{nn}\sqrt{\underline{g}}) - i\sqrt{\hat{g}}^{-1}\sum_{i=0}^{n-1}D_i(\sqrt{\underline{g}}(\underline{a}_{in}+\underline{a}_{ni})), \tag{A.13}$$

and

$$\mathfrak{a}_{01} = \mathfrak{a}_{10} = -ix\,a_{nn}\sqrt{\hat{g}}^{-1}\sqrt{\underline{g}}. \tag{A.14}$$

We note that $\mathfrak{a}_{00}$ is a first order differential operator with respect to $x, y_1, ..., y_{n-1}$with smooth coefficients away from $x=0$, while a_{01} and a_{10} are smooth functions away from $x=0$.

If we choose for the metric $g = \mathrm{d}x^2+x^2h_Y(x,y_1,...,y_n)$, with $h_Y = \sum_{i,j=1}^n h_{ij}(y_1,...,y_n)\mathrm{d}x^i\mathrm{d}x^j$, then the induced metric $\hat{g}$ on the boundary is given by: $\hat{g} = \mathrm{d}x^2+x^2\hat{h}_{\partial Y}(x,y_1,...,y_{n-1})$, with

$$\hat{h}_{\partial Y} = \sum_{i,j=1}^{n-1} h_{ij}(y_1,...,y_{n-1},0)\mathrm{d}x^i\mathrm{d}x^j =: \sum_{i,j=1}^{n-1} \underline{h}_{ij}(y_1,...,y_{n-1})\mathrm{d}x^i\mathrm{d}x^j.$$

Hence, we obtain:

$$\sqrt{\hat{g}} = x^{n-1}\sqrt{\underline{h}}. \tag{A.15}$$

While it holds, that $\sqrt{g} = x^n\sqrt{h}$, and so:

$$\sqrt{\underline{g}} = x^n\sqrt{\underline{h}}. \tag{A.16}$$

We conclude, that $\sqrt{\hat{g}}^{-1}\sqrt{\underline{g}} = x$, and we use this to simplify the expressions for $\mathfrak{a}_{00}$, $\mathfrak{a}_{01}$ and $\mathfrak{a}_{10}$:

$$\mathfrak{a}_{00} = -ix\sum_{i=0}^{n-1}(\underline{a}_{in}+\underline{a}_{ni})D_i + ix\sqrt{\underline{h}}^{-1}D_n(\underline{a}_{nn}\sqrt{\underline{h}})$$
$$- ix\sqrt{\underline{h}}^{-1}\sum_{i=0}^{n-1}D_i(\sqrt{\underline{h}}(\underline{a}_{in}+\underline{a}_{ni})) - i\,x\,\underline{b}_n, \tag{A.17}$$

and

$$\mathfrak{a}_{01} = \mathfrak{a}_{10} = -i\,x^2\,\underline{a}_{nn}. \tag{A.18}$$

We call the matrix $\mathcal{A}$ defined as in (A.12) greens matrix.

We give an explicit computation for the Greens Matrix for Δ_c:

A.1.1 Greens Matrix for the Laplacian

We want to compute in this section the entries of the Green's matrix as defined for a general second order differential operator P_2 in (A.12) for the case of $P_2 = \Delta_c$.

If we take the general form of a second order differential operator in the notation used in section A.1, we have:

$$P_2 = \sum_{i,j=0}^{n} a_{ij}(x,y) D_i \, D_j + \sum_{i=0}^{n} b_i(x,y) + c(x,y). \tag{A.19}$$

To apply the results from A.1 to the Laplacian, we write Δ_c in the notation of (2.16): Using the (2.16) we have for $u \in \mathcal{H}^{s,\gamma}$:

$$\Delta_c f = -\frac{1}{x^2}(D_0^2 f + i(n-1)D_0 f + \frac{1}{\sqrt{h}} \sum_{i,j=1}^{n} D_i(\sqrt{h}\, h^{ij} D_j f). \tag{A.20}$$

It follows, that:

$$a_{00} = -\frac{1}{x^2}, \quad a_{0n} = a_{n0} = 0, \tag{A.21}$$

$$a_{ij} = -\frac{1}{x^2} h^{ij} \quad \text{for } 1 \leq i,j \leq n, \tag{A.22}$$

and:

$$b_0 = -\frac{1}{x^2}(i(n-1)), \quad b_j = -\frac{1}{x^2} \frac{1}{\sqrt{h}} \sum_{i=1}^{n} D_i(\sqrt{h} h^{ij}) \quad \text{for } 1 \leq j \leq n, \tag{A.23}$$

as well as:

$$c = 0. \tag{A.24}$$

While we denote the metric on Y by h_y, the metric on ∂Y will be denoted by $h_{\partial Y}$.

We denote the functional determinants of $g_{\mathbb{D}}$ respective $g_{\partial \mathbb{D}}$ by g and $\hat{g}$.

Lets denote the explicit expression of the Green's matrix $\mathcal{A}$ as defined in (A.12) for $P_2 = \Delta_c$ by $\mathcal{A}_\Delta$.

We obtain for its entries:

$$\begin{aligned}
\mathfrak{a}_{00} &= -i \sum_{i=1}^{n-1} (x(-\frac{1}{x^2}(\underline{h}^{in} + \underline{h}^{ni}))D_i - ix(-\frac{1}{x^2}\frac{1}{\sqrt{\underline{h}}} D_n(\sqrt{\underline{h}}\, \underline{h}_{nn})) + \\
&+ ix\sqrt{\underline{h}}^{-1} D_n(-\frac{1}{x^2} h_{nn}\sqrt{\underline{h}}) - ix\sqrt{\underline{h}}^{-1} \sum_{i=1}^{n-1} D_i(\sqrt{\underline{h}}(-\frac{1}{x^2}(\underline{h}^{in} + \underline{h}^{ni}))) \\
&= \frac{i}{x} \sum_{i=1}^{n-1} ((\underline{h}^{in} + \underline{h}^{ni}))D_i + \frac{i}{x}\sqrt{\underline{h}}^{-1} \sum_{i=1}^{n-1} D_i(\sqrt{\underline{h}}((\underline{h}^{in} + \underline{h}^{ni}))).
\end{aligned} \tag{A.25}$$

Note that we have used the fact $a_{0n} = a_{n0}$ in the expression above, so that the sum starts at $i = 1$ instead of $i = 0$.

We compute for $\mathfrak{a}_{01} = \mathfrak{a}_{10}$:

$$\begin{aligned} \mathfrak{a}_{01} = \mathfrak{a}_{10} &= -i\, x^2(-\frac{1}{x^2}\underline{h}^{nn}) \\ &= i\underline{h}^{nn}. \end{aligned} \tag{A.26}$$

A.2 Adjoint of the Trace Operator on Weighted Sobolev Spaces

We want to compute the adjoint of $\tilde{\gamma}_0$. Again, we us the definition of h, $\hat{h}$, $\underline{h}$, g, $\hat{g}$ and $\underline{g}$ as in section A.1.

We choose $u \in \mathcal{H}^{s,\gamma}(\mathbb{D}),\ v \in \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\partial\mathbb{D})$:

$$\begin{aligned} \langle\tilde{\gamma}_0(u), v\rangle_{\partial\mathbb{D}} &= \int\limits_{\partial\mathbb{D}} (u(x,y',0))v(x,y')\sqrt{\hat{g}}\mathrm{d}x\mathrm{d}y' \\ &= \int\limits_{\partial\mathbb{D}} (u(x,y',0))v(x,y')x^{n-1}\sqrt{\hat{h}}\mathrm{d}x\mathrm{d}y' \\ &= \int\limits_{\partial\mathbb{D}} (u(x,y',0))v(x,y')x^{n-1}\sqrt{\underline{h}}\mathrm{d}x\mathrm{d}y' \\ &= \int\limits_{\mathbb{D}} (u(x,y',y_n))(x^{n-1}\sqrt{\underline{h}}\cdot v(x,y')\otimes\delta(y_n))\mathrm{d}x\mathrm{d}y'\mathrm{d}y^n \\ &= \int\limits_{\mathbb{D}} (u(x,y',y_n))(x^{-1}\cdot v(x,y')\otimes\delta(y_n))\underbrace{x^n\sqrt{h}}_{=\sqrt{g}}\mathrm{d}x\mathrm{d}y'\mathrm{d}y^n. \end{aligned}$$

Hence, if we define the operator γ_0^* as:

$$\tilde{\gamma}_0^* : u(x') \mapsto x^{-1}\cdot(u(x')\otimes\delta(y_n)) \quad \text{for } u \in \mathcal{H}^{s,\gamma}(\partial\mathbb{D}), \tag{A.27}$$

we have:

$$\langle\tilde{\gamma}_0(u), v\rangle_{\partial\mathbb{D}} = \langle u, \tilde{\gamma}_0^*(v)\rangle_{\mathbb{D}}. \tag{A.28}$$

We obtain as mapping properties of γ_0^* by duality:

$$\tilde{\gamma}_0^* : \mathcal{H}^{s,\gamma}(\partial\mathbb{D}) \to \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{D}). \tag{A.29}$$

A.2.1 The Adjoint of Mellin Operators

Let$Y$ be an $(n+1)$ dimensional manifold. Let $h \in M^{\mu}_{\mathcal{O}}(\tilde{Y})$ be a Mellin symbol. It can be checked by direct computation, using a shift in the integration variables, that the adjoint M^* of $M = \mathrm{op}^{\gamma}_{\mathcal{M}}(h)$ with respect to the L^2 scalar product is given by:

$$M^* = \mathrm{op}^{-\gamma}_{\mathcal{M}}(h^*(n+1-\overline{z})). \tag{A.30}$$

Here h^* denotes the adjoint of the pseudodifferential operator $h \in M^{\mu}_{\mathcal{O}}(\tilde{Y})$.

This yields for a Mellin pseudodifferential operator F of the form:

$$F = x^{-\mu}\mathrm{op}^{\gamma}_{\mathcal{M}}(h^*(n+1-\overline{z})), \tag{A.31}$$

the following adjoint:

$$\begin{aligned} F^* &= \mathrm{op}^{-\gamma}_{\mathcal{M}}(h^*(n+1-\overline{z}))x^{-\mu} \\ &= x^{-\mu}\mathrm{op}^{-\gamma+\mu}_{\mathcal{M}}(h^*(n+1-\overline{z+\mu})) \\ &= x^{-\mu}\mathrm{op}^{-\gamma+\mu}_{\mathcal{M}}(h^*(n+1-\mu-\overline{z})). \end{aligned} \tag{A.32}$$

A.3 Interpolation of Weighted Sobolev Spaces

We give a few basic definitions and properties of real interpolation spaces. For details see [39].

A.3.1 Basic Definitions and Results

Definition A.3.1. *Let X_0, X_1 be Banach spaces over $\mathbb{K}$. Then the pair (X_0, X_1) is called admissible, if there is a Hausdorff topological vector space Z such that $X_0, X_1 \hookrightarrow Z$ with continuous embeddings.*

Let (X_0, X_1) be an admissible pair of Banach spaces.

Definition A.3.2. *For $t > 0$, $x \in X_0 + X_1$ let:*

$$K(t,x) \equiv K(t,x;X_0,X_1) = \inf_{x=x_0+x_1, x_0\in X_0, x_1\in X_1} \|x_0\|_{X_0} + t\|x_1\|_{X_1}.$$

Definition A.3.3. *For $\theta \in (0,1), 1 \leq p \leq \infty$ we define the real interpolation space $(X_0, X_1)_{\theta,p}$ as:*

$$(X_0,X_1)_{\theta,p} := \{x \in X_0 + X_1 : \Phi_{\theta,p}(K(\cdot,x)) < \infty\},$$

where

$$\Phi_{\theta,p}(K(\cdot,x)) = \|t^{-\theta}K(t,x)\|_{L^p((0,\infty),\frac{dt}{t})}.$$

We endow $(X_0, X_1)_{\theta,p}$ *with the norm* $\|x\|_{\theta,p} := \Phi_{\theta,p}(K(\cdot,x))$.

The following lemma can be found in 1.3.3. of [39]:

Lemma A.3.4. *Let* $1 \leq p \leq \infty, \theta \in (0,1)$ *and let* (X_0, X_1), (Y_0, Y_1) *be admissible Banach spaces. Let* $X = (X_0, X_1)_{\theta,p}, Y = (Y_0, Y_1)_{\theta,p}$. *Then:*

$$T \in \mathcal{L}(X_j, Y_j), j = 0, 1, \quad \Rightarrow \; T|_X \in \mathcal{L}(X, Y),$$

and

$$\|T\|_{\mathcal{L}(X,Y)} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1,Y_1)},$$

for all $T \in \mathcal{L}(X_j, Y_j), j = 0, 1$.

Appendix B

Appendix B - b Calculus

B.1 b-Pseudodifferential operators

B.1.1 Computations in b-Calculus in Local Coordinates

Definition B.1.1. *Let $f : X \to Z$ be a smooth map between manifolds X and Z. The pull-back f^*u of a function u on Z is then defined as:*

$$f^*u = u \circ f. \tag{B.1}$$

The push-forward on measures is defined as an operation which assigns to a Borel measure μ on X a measure $f_*\mu$ on Z by:

Definition B.1.2. *Let μ be a measure on:*

$$f_*\mu(V) = \mu(f^{-1}(V)) \quad V \in Z. \tag{B.2}$$

In terms of integrals, this can be expressed by duality, using $\Phi \in C_0^\infty$:

$$\int_Z (f_*\mu)\Phi = \int_X \mu f^*\Phi. \tag{B.3}$$

The action of b-Pseudo is defined using pull backs and push forwards by coordinate projections:

Definition B.1.3. *We use the coordinate projections π_i, $i = 1, 2$, which are projections on the first or second component of $X \times X$, to define:*

Let $\beta : X_b^2 \to X^2$ be the blow-down map. Then:

$$\pi_{i\,b} := \pi_i \circ \beta \quad \textit{for}\, i \in \{1, 2\}. \tag{B.4}$$

Now, given a b-pseudodifferential operator $A \in \Psi_b^m(X)$ as defined in (7.3.2), we can use the preceding definitions to define the action of A in terms of its integral kernel K_A:

Definition B.1.4. *We take a auxiliary density, in this case a b-density $\mathfrak{m}$ to obtain:*

$$(Au)(x) = \pi_{1b*}(\pi_{1b}^* \mathfrak{m}\, \pi_{2b}^* u\, K_A). \tag{B.5}$$

The definition (B.5) finally justifies to use integral kernels supported on X_b^2.

Now, if we want to define the action operators contained in $\Psi_c^{-\infty,d,\mathcal{E}}(X;\Lambda)$ to distributions, we start by using the blow down map γ_1 of X_b^2 with:

$$\gamma_1 : X_b^2 \to X \times X. \tag{B.6}$$

Next, we consider the blowdown map γ_2 as:

$$\gamma_2 : \mathcal{T}_d \to \overline{\Lambda}_d \times X_b^2, \tag{B.7}$$

to finally introduce:

$$\begin{aligned} \tilde{\beta} &:= (\mathrm{id} \times \gamma_1) \circ \gamma_2, \\ \tilde{\beta} : \mathcal{T}_d &\to \overline{\Lambda}_d \times X \times X. \end{aligned} \tag{B.8}$$

Additionally to the projections π_1, π_2 introduced before, we use the projections $\tilde{\pi}_i, \quad i \in \{1, 2\}$ acting on $\overline{\Lambda}_d \times X \times X$ and projecting:

$$\begin{aligned} \tilde{\pi}_1 : \overline{\Lambda}_d \times X \times X &\to \overline{\Lambda}_d \times X, \\ (\omega, x, y) &\mapsto (\omega, x), \end{aligned} \tag{B.9}$$

respective:

$$\begin{aligned} \tilde{\pi}_2 : \overline{\Lambda}_d \times X \times X &\to \overline{\Lambda}_d \times X, \\ (\omega, x, y) &\mapsto (\omega, y). \end{aligned} \tag{B.10}$$

Finally, we define:

$$\tilde{\pi}_{ib} := \tilde{\pi} \circ \tilde{\beta}, \tag{B.11}$$

such that:

$$\tilde{\pi}_{ib} : \mathcal{T}_d \to \overline{\Lambda}_d \times X, \tag{B.12}$$

which allows us to define an action of an Integral Kernel living on $\mathcal{T}_d$ on a distribution.

For this, take an auxiliary b-density $d\nu$. Then, using a distribution u and an operator $A(\lambda) \in \Psi_c^{-\infty, d\mathcal{E}(X;\Lambda)}$ with Kernel $k \in \mathcal{A}^{\mathcal{E}}(\mathcal{T}_d)$, we can define:

$$Au := \tilde{\pi}_{2b*}(\tilde{\pi}_{1b}^*(u)\tilde{\pi}_{2b}^*(\nu)\, k). \tag{B.13}$$

B.2 Computations on the Parameter Blow-Up Space in Local Coordinates

We define $\Omega_i = \text{supp}(\alpha_i)$. Then we have:

$$\begin{aligned}
\Omega_1 &= \{(r,\omega,x,x',y,y') \in \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge | x' \leq x \leq r\}, \\
\Omega_2 &= \{(r,\omega,x,x',y,y') \in \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge | x \leq x' \leq r\}, \\
\Omega_3 &= \{(r,\omega,x,x',y,y') \in \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge | \max\{r,x'\} \leq x\}, \\
\Omega_4 &= \{(r,\omega,x,x',y,y') \in \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge | \max\{r,x\} \leq x'\}.
\end{aligned}$$

Further, the $\Omega_i, i \in \{1,...,4\}$ form a disjoint decomposition of $\overline{\Lambda}_d \times Y^\wedge \times Y^\wedge$.

We choose local stereographic coordinates on the regions $\Omega_i,\ i \in \{1,...,4\}$ and construct explicit blowdown maps $\gamma_i : \Omega_i \to \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge$.

We use local coordinates (x,x',y,y',r,ω). The integral Kernels on $\mathcal{T}_d$ which are of interest, are integrated along rays of constant angle in $\overline{\Lambda}_d$. Further, the coordinates $y,y' \in Y$ are compactly supported and remain untouched by the blow-up procedure. Hence, for the sake of a simpler notation, we will neglect the coordinates ω, y, y'.

So, let (r,x,y') be local coordinates on $\overline{\Lambda}_d \times Y^\wedge \times Y^\wedge$. We define:

- On $\beta^{-1}(\Omega_1)$:

  Appropriate local coordinates on $\mathcal{T}_d$ close to the region which is supported by $\tilde{\beta}^*(\eta_1)$ are given by $(\eta_1,\gamma_1,\zeta_1) = (r, \frac{x}{r}, \frac{x'}{x})$.

  Now, a blowdown map is given by:

$$\begin{aligned}
\tilde{\beta}_1 : \mathcal{T}_d &\to \Omega_1 \subset \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge, \\
(\eta_1,\gamma_1,\zeta_1) &\mapsto (\eta_1, \eta_1\cdot\gamma_1, \eta_1\cdot\gamma_1\cdot\zeta_1).
\end{aligned} \tag{B.14}$$

$\beta^{-1}(\Omega_1)$ intersects the hypersurfaces ff, rb and fi. Boundary defining functions for these hypersurfaces are given by η_1 for fi, γ_1 for ff and ζ_1 for rb.

- On $\beta^{-1}(\Omega_2)$:

  Appropriate local coordinates on $\mathcal{T}_d$ close to the region which is supported by $\tilde{\beta}^{-1}(\eta_2)$ are given by $(\eta_2, \gamma_2, \zeta_2) = (r, \frac{x'}{r}, \frac{x}{x'})$. Now, a blowdown map is given by:

$$\begin{aligned} \tilde{\beta}^* : \mathcal{T}_d &\rightarrow \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge, \\ (\eta_2, \gamma_2, \zeta_2) &\mapsto (\eta_2, , \eta_2 \cdot \gamma_2 \cdot \zeta_2, \gamma_2 \cdot \zeta_2). \end{aligned} \tag{B.15}$$

  $\beta^{-1}(\Omega_2)$ intersects regions around the hypersurfaces ff, lb and fi. Boundary defining functions for those hypersurfaces can be expressed by η_2 for fi, γ_2 for ff and ζ_2 for lb.

- On $\beta^{-1}(\Omega_3)$:

  Local coordinates on $\mathcal{T}_d$ along the region on which $\beta^{-1}(\eta_3)$ is supported, are given by $(\eta_3, \gamma_3, \zeta_3) = (\frac{r}{x}, x, \frac{x'}{x})$:

$$\begin{aligned} \beta : \mathcal{T}_d &\rightarrow \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge, \\ (\eta_3, \gamma_3, \zeta_3) &\mapsto (\eta_3 \cdot \gamma_3, \gamma_3, \gamma_3 \cdot \zeta_3). \end{aligned} \tag{B.16}$$

  Now, η_3 serves as a boundary defining function for bi, γ_3 as a boundary defining function for fi, ζ_3 as a boundary defining function for rb.

- On $\beta^{-1}(\Omega_4)$

  Local stereographic coordinates on $\mathcal{T}_d$ close to the region on which $\beta^*(\eta_4)$ is supported are given by $(\eta_4, \gamma_4, \zeta_4) = (\frac{r}{x'}, \frac{x}{x'}, x')$. Hence, the blowdown map β can be expressed in this coordinates by:

$$\begin{aligned} \beta : \mathcal{T}_d &\rightarrow \overline{\Lambda}_d \times Y^\wedge \times Y^\wedge, \\ (\eta_4, \gamma_4, \zeta_4) &\mapsto (\eta_4 \cdot \ \zeta_4, \zeta_4, \gamma_4 \cdot \zeta_4). \end{aligned} \tag{B.17}$$

  Here, η_4 serves as a boundary defining function for bi, γ_4 as a boundary defining function for lb, ζ_4 as a boundary defining function for fi.

B.3 Kernel Estimates

The asymptotic behavior of the Integral Kernel towards the boundary faces is encoded in the collection of index sets $\mathcal{E}(\alpha) = (E_{\mathrm{lb}}, E_{\mathrm{rb}}, E_{\mathrm{ff}}, E_{\mathrm{fi}}, E_{\mathrm{bi}})$. It holds, that $E_{\mathrm{lb}} > \alpha - \mu - \frac{n}{2}$, $E_{\mathrm{rb}} > -(\alpha - \mu) + \frac{n}{2}$, $E_{\mathrm{ff}} > 0$, $E_{\mathrm{fi}} = \mathbb{N}$ and $E_{\mathrm{bi}} = \emptyset$.

- On $\beta^{-1}(\Omega_1)$

  In a local neighborhood of fi, our conical manifold X is locally isomorphic to a subset of $\mathcal{U} = [0,1)_{\eta_1} \times [0,1)_{\gamma_1} \times [0,1)_{\zeta_1} \times \mathcal{U}'$. Since the integral kernels of our residual operators are compactly supported in $\mathcal{U}'$, we suppress the dependence of the Integral kernel $K_{R(\lambda)}$ on $\mathcal{U}'$ in the preceding discussion:

$$K_{R(\lambda)} = \sum_{(z,k)\in E_{fi}, \Re z \le N} (\gamma_1)^z (\log \gamma_1)^k u_{(z,k)}(\eta_1, \zeta_1, y) + (\gamma_1)^N u_N(\eta_1, \gamma_1, \zeta_1), \quad \text{(B.18)}$$

  $u_N(\eta_1, \gamma_1, \zeta_1) \in \Sigma^a(\mathcal{U})$ and $u_{(z,k)}(\eta_1, \zeta_1) \in \Sigma^{a'}([0,1)_{\eta_1} \times [0,1)_{\zeta_1})$.

  Since $E_{\mathrm{rb}} > -(\alpha - \mu) + \frac{n}{2}$, it follows that there exists an $\varepsilon > 0$ with:

$${\zeta_1}^{-(\varepsilon - (\alpha-\mu) + \frac{n}{2})} \cdot K_{R(\lambda)}(\eta_1, \gamma_1, \zeta_1), \quad \text{(B.19)}$$

  is a smooth and bounded function in ζ_1 up to $\zeta_1 = 0$.

  Further,we have that $u_N(\eta_1, \gamma_1, \zeta_1) \in \Sigma^a(\mathcal{U})$ and $u_{(z,k)}(\eta_1, \zeta_1) \in \Sigma^{a'}([0,1)_{\eta_1} \times [0,1)_{\zeta_1})$.

  Summarizing the remaining index sets in $\mathcal{E}' = (E_{\mathrm{lb}}, E_{\mathrm{ff}}, E_{\mathrm{fi}}, \emptyset)$, we have that $u_{(z,k)}(\eta_1, \zeta_1) \in \mathcal{A}^{\mathcal{E}'}$.

  Applying the same argument as above to $u_{z,k}$ with respect to the index sets E_{ff} and E_{rb},we can conclude that the Integral kernel $K_{R(\lambda)}$, expressed in local coordinates is bounded on Ω_1 by:

$$|K_{R(\lambda)}(\eta_1, \gamma_1, \zeta_1)| < C_1 \cdot {\zeta_1}^{\varepsilon - (\alpha-\mu) + \frac{n}{2}}. \quad \text{(B.20)}$$

- On $\beta^{-1}(\Omega_2)$

  We use the local coordinates $(\eta_2, \gamma_2, \zeta_2)$ on $\beta^{-1}(\Omega_2)$ as defined in (B.15).

  First of all, the variable ζ_2 on Ω_2 ranges from zero to infinity. Now, $Y^\wedge$ is not compactified around infinity, hence there is no boundary face there. However, since the residual operators which contribute the resolvent are multiplied with a compactly supported cut-off function, we can assume without loss of generality that K_R is compactly supported in ζ_2. Since $\zeta_2 \to 0$ corresponds to $\rho_{fi} \to 0$ and $E_{fi} = \mathbb{N}$, we can assume $|K_{R(\lambda)}(\eta_2, \gamma_2, \zeta_2)|$ to be bounded in ζ_2.

Similar considerations as for $\beta^{-1}(\Omega_1)$ yield, since $E_{\mathrm{ff}} > 0$, $E_{\mathrm{lb}} > \alpha - \mu - \frac{n}{2}$ and $E_{\mathrm{fi}} = \mathbb{N}$:

$$|K_{R(\lambda)}(\eta_2, \gamma_2, \zeta_2)| \leq C_2 \cdot (\gamma_2)^{\varepsilon} \cdot (\zeta_2)^{\varepsilon + (\alpha-\mu) - \frac{n}{2}}. \tag{B.21}$$

- <u>On $\beta^{-1}(\Omega_3)$</u>

  Since $E_{\mathrm{bi}} = \emptyset$,$E_{\mathrm{fi}} = \mathbb{N}$, and $E_{\mathrm{rb}} > -(\alpha - \mu) + \frac{n}{2}$, the integral Kernel on $\beta^{-1}(\Omega_3)$ satisfies the estimates:

$$|K_{R(\lambda)}(\eta_3, \gamma_3, \zeta_3)| \leq C_3 \cdot (\eta_3)^N \cdot (\zeta_3)^{\varepsilon - (\alpha-\mu) + \frac{n}{2}}, \qquad \forall N \in \mathbb{N}. \tag{B.22}$$

- <u>On $\beta^{-1}(\Omega_4)$</u>

  Since $E_{\mathrm{bi}} = \mathbb{N}$, $E_{\mathrm{lb}} > (\alpha - \mu) - \frac{n}{2}$ $E_{\mathrm{fi}} = \mathbb{N}$:

$$|K_{R(\lambda)}(\eta_4, \gamma_4, \zeta_4)| \leq C_4 \cdot (\eta_4)^N \cdot (\gamma_4)^{\varepsilon + (\alpha-\mu) - \frac{n}{2}}, \qquad \forall N \in \mathbb{N}. \tag{B.23}$$

We summarize the kernel estimates:

i	$(\eta_i, \gamma_i, \zeta_i)$	$\beta_i : (\eta_i, \gamma_i, \zeta_i) \mapsto$	$\lvert K(\eta_i, \gamma_i, \zeta_i)\rvert \leq$	ρ_{lb}	ρ_{rb}	ρ_{ff}	ρ_{fi}	ρ_{bi}
1	$(r, \frac{x}{r}, \frac{x'}{x})$	$(\eta_1, \eta_1\,\gamma_1, \eta_1\,\gamma_1\,\zeta_1)$	$C_1\,{\zeta_1}^{\varepsilon-(\alpha-\mu)+\frac{n}{2}}$	-	ζ_1	γ_1	η_1	-
2	$(r, \frac{x'}{r}, \frac{x}{x'})$	$(\gamma_2\,\zeta_2\,\eta_2, \gamma_2\,\zeta_2, \zeta_2)$	$C_2\,\zeta_2^{\varepsilon+(\alpha-\mu)-\frac{n}{2}}$	ζ_2	-	γ_2	η_2	-
3	$(\frac{r}{x}, x, \frac{x'}{x})$	$(\eta_3\,\gamma_3, \gamma_3, \gamma_3\,\zeta_3)$	$C_3\,\eta_3^{N+\varepsilon}\,\zeta_3^{\varepsilon-(\alpha-\mu)+\frac{n}{2}}$	-	ζ_3	-	γ_3	η_3
4	$(\frac{r}{x'}, \frac{x}{x'}, x')$	$(\eta_4\,\gamma_4\,\zeta_4, \gamma_4\,\zeta_4, \zeta_4)$	$C_4\,\eta_4^{N+\varepsilon}\,\gamma_4^{\varepsilon+(\alpha-\mu)-\frac{n}{2}}$	γ_4	-	-	ζ_4	η_4

B.4 b Calculus Computations in Local Coordinates

We use the local coordinates and blowdown map on $\beta^{-1}(\Omega_1)$ as defined above to give an explicit calculation of the action of a residual operator on a function:

For this, we use that the action of Operators on functions is defined using pull-backs and push-forwards as defined in (B.13):

$$\begin{aligned} R(\lambda)\,u\,\tilde{\mathfrak{m}} &= \pi_{2b*}(K_{R(\lambda)}\tilde{\beta}^*(\kappa_1)\pi_{2b}^*(u)\pi_{1b}^*(\mathfrak{m})) \\ &= \pi_{2b*}(k(\eta_1, \gamma_1, \zeta_1)\chi(\gamma_1)\chi(\zeta_1)u(\eta_1\,\gamma_1\,\zeta_1)\frac{d\gamma_1}{\gamma_1}\frac{d\zeta_1}{\zeta_1}). \end{aligned} \tag{B.24}$$

We use a test function $\Phi \in C^{\infty}(Y^{\wedge};\Lambda)$ to compute the action of the density (B.24) on Φ:

$$\begin{aligned}
&\int\limits_{\gamma_1}\int\limits_{\zeta_1} \pi_{2b*}(k(\eta_1,\zeta_1,\gamma_1)\chi(\gamma_1)\chi(\zeta_1)u(\eta_1\,\gamma_1\,\zeta_1))\Phi\frac{d\gamma_1}{\gamma_1}\frac{d\zeta_1}{\zeta_1}\\
=\;&\int\limits_{\gamma_1}\int\limits_{\zeta_1}(k(\eta_1,\gamma_1,\zeta_1)\chi(\gamma_1)\chi(\zeta_1)u(\eta_1\,\gamma_1\,\zeta_1))\pi_{2b}^{*}(\Phi)\frac{d\gamma_1}{\gamma_1}\frac{d\zeta_1}{\zeta_1}\\
=\;&\int\limits_{\gamma_1}\int\limits_{\zeta_1}(k(\eta_1,\gamma_1,\zeta_1)\chi(\gamma_1)\chi(\zeta_1)u(\eta_1\,\gamma_1\,\zeta_1))(\Phi(\eta_1\,\gamma_1))\frac{d\gamma_1}{\gamma_1}\frac{d\zeta_1}{\zeta_1}.
\end{aligned}$$

Substituting $x=\eta_1\,\gamma_1$ we obtain:

$$\int\limits_{\gamma_1}\int\limits_{\zeta_1}(k(\eta_1,\frac{x}{r},\zeta_1)\chi(\frac{x}{r})\chi(\zeta_1)u(x\,\zeta_1))(\Phi(x))\frac{dx}{x}\frac{d\zeta_1}{\zeta_1}.$$

Now, with $x'=x\,\zeta_1$:

$$\int\limits_{x}\int\limits_{x'}(k(r,\frac{x}{r},\frac{x'}{x})u(x'))\chi(\frac{x}{r})\chi(\frac{x'}{x})(\Phi(x))\frac{dx}{x}\frac{dx'}{x'}. \tag{B.25}$$

Clearly, we can identify the resulting distributional b-density in (B.25) with a distribution, hence we obtain as result:

$$(R(\lambda)u)(r,x)=\int\limits_{x'}(k(r,\frac{x}{r},\frac{x'}{x})\chi(\frac{x}{r})\chi(\frac{x'}{x})u(x'))\frac{dx'}{x'}. \tag{B.26}$$

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