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Characterization of informational completeness for covariant phase space observables

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In the nonrelativistic setting with finitely many canonical degrees of freedom, a shift-covariant phase space observable is uniquely characterized by a positive operator of trace one and, in turn, by the Fourier-Weyl transform of this operator. We study three properties of such observables, and characterize them in terms of the zero set of this transform. The first is informational completeness, for which it is necessary and sufficient that the zero set has dense complement. The second is a version of informational completeness for the Hilbert-Schmidt class, equivalent to the zero set being of measure zero, and the third, known as regularity, is equivalent to the zero set being empty. We give examples demonstrating that all three conditions are distinct. The three conditions are the special cases for $p = 1, 2, \infty$ of a more general notion of p-regularity defined as the norm density of the span of translates of the operator in the Schatten-p class. We show that the relation between zero sets and p-regularity can be mapped completely to the corresponding relation for functions in classical harmonic analysis.
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I. INTRODUCTION

A basic task of practical quantum mechanics is to determine the density operator, which completely describes the statistical properties of a source. This "tomography" has to be done by a suitable family of measurements, sometimes called a "quorum."[1,2] For example, in a quantum optical setting[3] the family of homodyne measurements will do. However, in some cases it turns out that measuring a single observable is enough. These observables were called "super-observables" in the terminology of Ludwig[4] and "informationally complete" in Ref. 5. Formally, such an observable is a positive operator valued measure F such that $\mathrm{tr}[\rho_1 F(\Delta)] = \mathrm{tr}[\rho_2 F(\Delta)]$ for all measurable sets Δ and some density operators ρ_1, ρ_2 imply $\rho_1 = \rho_2$. Little can be said about the general characterization of such observables beyond the defining property. However, in specific contexts such as observables on homogeneous spaces which are covariant with respect to a projective representation of a Lie group[6] it may become more tractable. We are here taking a fresh look at a very specific instance of such a setting, namely shift covariant observables on the phase space of a non-relativistic quantum system with finitely many degrees of freedom.

The problem, which in this case remained open and somewhat controversial is the characterization of informational completeness in terms of the zero set of a certain Fourier transform, which we describe in detail below. Actually, two distinct conditions have been forwarded as necessary

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and sufficient, although somewhat in passing as a side issue in a broader context. In the notation introduced below these are "condition (Z 1)" in Ref. 7, and "condition (Z 2)" in Refs. 8–11. What we show in the present paper is that the correct condition (Z 3) is different from both of these, and also that the other conditions have interesting consequences in their own right.

It would be interesting to obtain similar results for other groups and for relativistic phase spaces in particular. However, since the Fourier transform is not available in a non-commutative context, it is not clear to us how to even state a conjecturable analog. A number of examples of informationally complete observables in such a context are known; see, e.g., Ref. 12 and the references therein.

II. SETTING AND MAIN RESULTS

We consider a non-relativistic quantum system with $N < \infty$ canonical degrees of freedom; i.e., we have the phase space X, identified with its dual group $\widehat{X}$ via the symplectic form $\{(q, p), (q', p')\} = q'p - qp'$, and a continuous irreducible representation of the Weyl commutation relations. These are unitary operators $W(x)$, with $x = (q, p) \in X$ such that $W(x)W(y) = \exp(i\{x, y\}/2)W(x + y)$. We will normalize the Lebesgue measure dx on X such that $dx = (2\pi)^{-N}dqdp$.

For simplicity, we present the results in the case of one degree of freedom. The generalization to arbitrary N is straightforward, as one can readily see by inspecting the proofs. Indeed, the crucial ingredient to our result is Eq. (4) below, which follows from the properties of the Fourier-Weyl transform, as we describe in Sec. III.

By the Stone-von Neumann uniqueness theorem, the Weyl operators $W(x)$ are determined up to unitary equivalence, and described explicitly in terms of the standard phase space coordinate operators Q and P acting on $L^2(\mathbb{R})$. In fact, $W(q, p) = e^{i\frac{qp}{2}}e^{-iqP}e^{ipQ}$. For concreteness, we can think of position and momentum of a spinless particle confined to move in one dimension, or quadrature components of a one-mode electromagnetic field.

A shift-covariant phase space observable is a normalized positive operator valued measure, which assigns to each Borel set $M \subset X$ a positive operator $\mathsf{G}(M)$ such that G is σ-additive in the weak*-topology, $\mathsf{G}(X) = \mathbb{1}$, and the covariance condition

$$W(x)\mathsf{G}(M)W(x)^* = \mathsf{G}(M + x) \tag{1}$$

holds for all Borel sets M and all $x \in X$. Such observables are necessarily[13,14] of the form $\mathsf{G}(M) = \int_{x\in M} dx\; W(x)TW(x)^*$, with a positive trace class operator T with trace one, which is to say that the measurement outcome probability density f_ρ is given by

$$f_\rho(x) = \mathrm{tr}\left[\rho\, W(x)TW(x)^*\right]. \tag{2}$$

The key practical question is how we can reconstruct ρ from the measured density f_ρ. A necessary condition for the reconstruction is obviously that the observable is informationally complete, in the sense defined in the Introduction. Indeed, this condition is equivalent to the map $\rho \mapsto f_\rho$ being injective. The nature of the reconstruction, a sort of non-commutative deconvolution, becomes clearer if we take the symplectic Fourier transform of (2). For any integrable function $f : X \to \mathbb{C}$, this is defined via

$$\widehat{f}(y) = \int e^{-i\{x,y\}} f(x)\, dx. \tag{3}$$

By direct computation using the properties of the Weyl operators, one obtains

$$\widehat{f_\rho}(y) = \mathrm{tr}\,[\rho W(y)]\; \overline{\mathrm{tr}\,[TW(y)]} = \widehat{\rho}(y)\overline{\widehat{T}(y)}, \tag{4}$$

where the Weyl transform $\hat{S}$ of any trace class operator S is a function $\widehat{S} : X \to \mathbb{C}$ defined via

$$\widehat{S}(y) = \mathrm{tr}\,[SW(y)]\,. \tag{5}$$

The Weyl transform is the operator equivalent to the Fourier transform, and has many analogous properties. In particular, it is well known that $S \mapsto \widehat{S}$ is injective. Hence, in order to determine the state ρ from the density f_ρ, we only need to reconstruct the function $\widehat{\rho}$ from the right-hand side of (4).

Clearly, this will work if $\widehat{T}(x)$ is non-zero everywhere. Given the nature of ρ, some zeros can be tolerated, but how many exactly is not obvious. This is the main motivation of our paper.

Let us introduce the zero set

$$Z(T) = \{x \in X \mid \mathrm{tr}\,[T\,W(x)] = 0\}. \tag{6}$$

There are three obvious possibilities to formalize "smallness" of $Z(T)$, an algebraic, a measure theoretic, and a topological way:

(Z 1) $Z(T)$ is empty,
(Z 2) $Z(T)$ is of Lebesgue measure zero,
(Z 3) $Z(T)$ contains no open set, i.e., has dense complement.

Trivially, (Z 1) $\Rightarrow$ (Z 2) $\Rightarrow$ (Z 3). Since the Weyl transform $\widehat{\rho}$ is a continuous function, one immediately sees that the weakest condition (Z 3) is sufficient to guarantee the informational completeness of the observable. We will show (Proposition 4 below) that this is also a necessary condition. In addition, we demonstrate that neither of the obvious implications can be reversed (Propositions 8 and 9), correcting thus some of the earlier statements, as explained in the Introduction. Since (Z 3) is indeed a necessary condition, this also shows, contrary to some formal state reconstruction formulas,[15] that not all covariant phase space observables can be used in quantum tomography. For instance, a phase space observable generated by a slit state $T = |\varphi\rangle\langle\varphi|$, with a compactly supported φ, is not informationally complete.

With the above three conditions of the smallness of $Z(T)$, the last one being equivalent to injectivity of $\rho \mapsto \mathrm{tr}\,[\rho\,W(x)T\,W(x)^*]$ on the set of states, an obvious question to ask is whether the other two can be characterized in an analogous fashion. The key point in our approach is to understand $\mathrm{tr}\,[\rho\,W(x)T\,W(x)^*]$ as an operator equivalent of convolution between two operators ρ and T, according to the theory systematically developed in Ref. 14. In fact, for a trace class operator S we set

$$S * T = \mathrm{tr}\left[S\,W(x)T_-W(x)^*\right], \tag{7}$$

where $T_- = \Pi T \Pi$ and Π is the parity operator. The usefulness of this notation is apparent from (4), which now just reads $\widehat{\rho * T} = \widehat{\rho}\widehat{T}$, expressing a property one expects of a convolution. In fact, this puts the problem of informational completeness into the general context of quantum harmonic analysis: it turns out that the injectivity of $S \mapsto S * T$ on the set of bounded, Hilbert-Schmidt, and trace class operators characterizes (Z 1), (Z 2), and (Z 3), respectively (Proposition 4). In general, injectivity of $S \mapsto S * T$ on the dual of the Schatten-p class turns out to be equivalent to p-regularity of T, i.e., the property of the span of Weyl-translates of T being norm dense in the Schatten-p class. Using the correspondence theory of Ref. 14, we characterize this property entirely within the framework of classical harmonic analysis (Propositions 1–3).

III. NOTATIONS AND PRELIMINARIES

For the relevant Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, we let $\mathcal{B}(\mathcal{H})$, $\mathcal{T}_2(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$, equipped with the norms $\|\cdot\|_\infty$, $\|\cdot\|_2$, and $\|\cdot\|_1$, denote the spaces of bounded, Hilbert-Schmidt, and trace class operators on $\mathcal{H}$, respectively. As mentioned above, we also consider the general Schatten classes $\mathcal{T}_p(\mathcal{H})$ for $1 \leq p < \infty$ with the norm $\|A\|_p = \mathrm{tr}[|A|^p]^{1/p}$ (Chap. IX.4 of Ref. 16). These contain the Hilbert-Schmidt and trace class as $p = 2$ and $p = 1$. By $\mathcal{K}(\mathcal{H})$ we denote the space of compact operators, and by $\mathcal{C}_0(X)$ the space of continuous complex valued functions on X vanishing at infinity.

The proofs of our main results rest heavily on quantum harmonic analysis on phase space;[14] hence we present here the basic definitions and results. The idea is to extend the definitions of convolution and Fourier transform to combinations of operators and functions. With this convolution $\mathcal{L}^1(X) \oplus \mathcal{T}(\mathcal{H})$ becomes a $\mathbb{Z}_2$-graded commutative Banach algebra, meaning that, for functions f, g and operators A, B, $f * g$, and $A * B$ are functions and $f * A = A * f$ is an operator. The associated Fourier transform, which turns the convolution into a product of functions, is the symplectic Fourier transform (3) on the function part, and the Weyl transform (5) on the operator part.

One of the classic themes of classical harmonic analysis is the mapping properties of function spaces. In the extended structure this becomes a correspondence theory by which spaces of functions on phase space (assumed to be closed under phase space translations) are associated with spaces of operators. The moral is that while quantum-classical correspondences between individual observables are "fuzzy" and generally depend on the choice of some parameters, the correspondence between translation invariant *spaces* of functions and operators is canonical. In this paper we just need the instances:

$$\begin{array}{c} \mathcal{L}^1(X) \leftrightarrow \mathcal{T}_1(\mathcal{H}) \\ \mathcal{L}^p(X) \leftrightarrow \mathcal{T}_p(\mathcal{H}) \\ \mathcal{L}^\infty(X) \leftrightarrow \mathcal{B}(\mathcal{H}) \\ \mathcal{C}_0(X) \leftrightarrow \mathcal{K}(\mathcal{H}) \end{array} . \tag{8}$$

Here the double arrow indicates that the convolution of an element on one side by an arbitrary trace class operator gives an element on the other side. As customary for functions the convolution is extended here from $\mathcal{L}^1(X) \oplus \mathcal{T}(\mathcal{H})$ to allow one factor from $\mathcal{L}^\infty(X)$ or $\mathcal{B}(\mathcal{H})$.

The convolution in some sense was defined above already by describing its Fourier transform. To give a direct definition let us fix some more notations. Let α_x denote the automorphism induced by phase space translations, i.e., $(\alpha_x f)(y) = f(y - x)$ for $f \in \mathcal{L}^\infty(X)$ and $\alpha_x(A) = W(x)AW(x)^*$ for $A \in \mathcal{B}(\mathcal{H})$. The map $x \mapsto \alpha_x$ is strongly continuous on $\mathcal{L}^p(X)$, $\mathcal{T}_p(\mathcal{H})$ for $1 \leq p < \infty$, and on $\mathcal{C}_0(X)$ and $\mathcal{K}(\mathcal{H})$. It is weak*-continuous on $\mathcal{L}^\infty(X)$ and $\mathcal{B}(\mathcal{H})$. The phase space inversion of a function is written by a subscript "$-$," so $(g_-)(x) = g(-x)$. Its operator analog is $S_- = \Pi S \Pi$, where Π is the parity operator. We can then write the usual convolution of integrable functions in two equivalent ways, which suggest the extensions to trace class operators A, B:

$$\begin{aligned} (f * g)(y) &= \int f(x) g(y-x)\, dx = \int f(x)(\alpha_y g_-)(x)\, dx \\ f * A = A * f &= \int f(x)\alpha_x(A)\, dx \\ (A * B)(y) &= \mathrm{tr}\left[A\alpha_y(B_-)\right] \end{aligned} \tag{9}$$

(Note that here the last one is just (7).) It is a crucial fact of the theory, based on the square integrability of the Weyl operators, that $A * B$ is always integrable. In fact, the integral of $A * B$ is given by

$$\int (A * B)(x)\, dx = \mathrm{tr}\,[A]\,\mathrm{tr}\,[B]\,. \tag{10}$$

In particular, this result, together with the basic properties of Weyl operators, gives (4).

For extending the convolution to one merely bounded (but not integrable or trace-class) factor, we use the duality relation

$$\int f_-(x)(A * B)(x)\, dx = \mathrm{tr}\,[A_-(f * B)] = (A * (f * B))(0), \tag{11}$$

which follows immediately from the definitions. This is an identity for integrable/trace class elements. If f is merely bounded, the first expression still makes sense, and thus we define the convolution $f * B$ by the second expression. That is equivalent to take the integral (9) in the weak* sense. For $A \in \mathcal{B}(\mathcal{H})$ we can proceed similarly, but in this case the expressions in (9) can also be taken literally. The correspondences (8) are then associated with the norm estimates

$$\|A * S\|_1 \leq \|A\|_1 \|S\|_1, \qquad \|A * S\|_p \leq \|A\|_p \|S\|_1, \qquad \|A * S\|_\infty \leq \|A\|_\infty \|S\|_1. \tag{12}$$

The Fourier transform of an integrable function is defined as (3), and for operators the Weyl transform (5) for $S \in \mathcal{T}(\mathcal{H})$ has the equivalent role. In particular, in each of the above cases the Fourier transform maps convolutions into products, namely, $\widehat{f * g} = \widehat{f}\,\widehat{g}$, $\widehat{A * S} = \widehat{A}\,\widehat{S}$, and $\widehat{f * S} = \widehat{f}\widehat{S}$. All of the standard results of harmonic analysis also hold. We make explicit use of the Plancherel theorem which states that the maps $f \mapsto \widehat{f}$ and $S \mapsto \widehat{S}$ extend to Hilbert space unitaries $\mathcal{L}^2(X) \to \mathcal{L}^2(X)$ and $\mathcal{T}_2(\mathcal{H}) \to \mathcal{L}^2(X)$.

IV. REGULARITY, DENSITY, AND INJECTIVITY

In his classic paper Wiener[17] connected two conditions on the zero set of the Fourier transform of a function *f*, analogous to (Z 1) and (Z 2), with the property that the translates of *f* should span an appropriate function space. It turns out that such density conditions are precisely what is needed also in the quantum case. This motivates Definition 1 below. Moreover, we will show that ∞-regularity of an operator $T \in \mathcal{T}(\mathcal{H})$ is equivalent to informational completeness of the phase space observable it generates (see Proposition 3, condition (3.3)). The connection with zero sets will be discussed in Sec. V.

Definition 1: For $1 \le p < \infty$, *we say that* $T \in \mathcal{T}_p(\mathcal{H})$ *is* ***p*-regular** *if the linear span of* $\{\alpha_x(T) | x \in X\}$ *is dense in* $\mathcal{T}_p(\mathcal{H})$. *Similarly, a function* $f \in \mathcal{L}^p(X)$ *is called p-regular, if its translates span a norm dense subspace of* $\mathcal{L}^p(X)$. 1*-regular elements are just called* **regular**.

$T \in \mathcal{B}(\mathcal{H})$ *(resp.* $f \in \mathcal{L}^\infty(X)$*) is called* ∞**-regular** *if the span of translates is weak*-dense*.

For $1 \le p \le p' \le \infty$ we have the inclusions

$$\mathcal{T}(\mathcal{H}) \subset \mathcal{T}_p(\mathcal{H}) \subset \mathcal{T}_{p'}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \tag{13}$$

with $\|\cdot\|_\infty \le \|\cdot\|_{p'} \le \|\cdot\|_p \le \|\cdot\|_1$; hence *p*-regularity implies *p*′-regularity.

The following three propositions characterize *p*-regularity for the quantum case. In fact, Proposition 2 covers the open interval $1 < p < \infty$, and the endpoints $p = 1$ and $p = \infty$ are stated separately as Propositions 1 and 3, respectively. The reason is that the $\mathcal{T}_p(\mathcal{H})$ spaces for these endpoints are not reflexive, so there is an ambiguity in what one might understand under the Schatten class $\mathcal{T}_\infty(\mathcal{H})$, a notation we therefore avoid: should it be $\mathcal{B}(\mathcal{H})$, the dual of the other endpoint $\mathcal{T}(\mathcal{H}) = \mathcal{T}_1(\mathcal{H})$, or should it be its predual, the space $\mathcal{K}(\mathcal{H})$ of compact operators, since all other $\mathcal{T}_p(\mathcal{H})$ consist of compact operators? So, for example, condition (2.3) with $p = 1$ is (1.3) with the understanding $\mathcal{T}_\infty(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$, and (2.2) turns into (3.7) for $\mathcal{T}_\infty(\mathcal{H}) \mapsto \mathcal{K}(\mathcal{H})$. Proposition 3 also has additional statements connecting weak*-density in $\mathcal{B}(\mathcal{H})$ with norm-density in $\mathcal{K}(\mathcal{H})$. We remark that this option exists also for the definition of ∞-regularity: it can be stated equivalently as the norm density of the translates in $\mathcal{K}(\mathcal{H})$.

To emphasize the common features we first state the three propositions and then give the proofs, using parallel arguments as much as possible. The spectral characterizations in terms of zero sets are given in Proposition 4.

Proposition 1: Let $T \in \mathcal{T}(\mathcal{H})$. *Then the following conditions are equivalent.*

(1.0) *T is regular*.
(1.1) *If* $f \in \mathcal{L}^\infty(X)$ *and* $f * T = 0$, *then* $f = 0$.
(1.2) *The set* $\mathcal{T}(\mathcal{H}) * T$ *is dense in* $\mathcal{L}^1(X)$.
(1.3) *If* $A \in \mathcal{B}(\mathcal{H})$ *and* $A * T = 0$, *then* $A = 0$.
(1.4) *The set* $\mathcal{L}^1(X) * T$ *is dense in* $\mathcal{T}(\mathcal{H})$.
(1.5) $T * T$ *is regular*.
(1.6) *For some (resp. all) regular* $T_0 \in \mathcal{T}(\mathcal{H})$, $T * T_0$ *is regular*.

Moreover, there exists a regular operator $T \in \mathcal{T}(\mathcal{H})$.

Proposition 2: Let $T \in \mathcal{T}(\mathcal{H})$, $1 < p < \infty$, *and set* $q = (1 - p^{-1})^{-1}$. *Then the following conditions are equivalent.*

(2.0) *T is p-regular*.
(2.1) *If* $f \in \mathcal{L}^q(X)$ *and* $f * T = 0$, *then* $f = 0$.

(2.2) The set $\mathcal{T}_p(\mathcal{H}) * T$ is dense in $\mathcal{L}^p(X)$.
(2.3) If $A \in \mathcal{T}_q(\mathcal{H})$ and $A * T = 0$, then $A = 0$.
(2.4) The set $\mathcal{L}^p(X) * T$ is dense in $\mathcal{T}_p(\mathcal{H})$.
(2.5) $T * T$ is p-regular.
(2.6) For some (resp. all) regular $T_0 \in \mathcal{T}(\mathcal{H})$, $T * T_0$ is p-regular.

Proposition 3: Let $T \in \mathcal{T}(\mathcal{H})$. Then the following conditions are equivalent.

(3.0) T is ∞-regular.
(3.1) If $f \in \mathcal{L}^1(X)$ and $f * T = 0$, then $f = 0$.
(3.2) The set $\mathcal{B}(\mathcal{H}) * T$ is weak-dense in $\mathcal{L}^\infty(X)$.*
(3.3) If $A \in \mathcal{T}(\mathcal{H})$ and $A * T = 0$, then $A = 0$.
(3.4) The set $\mathcal{L}^\infty(X) * T$ is weak-dense in $\mathcal{B}(\mathcal{H})$.*
(3.5) $T * T$ is ∞-regular.
(3.6) For some (resp. all) regular $T_0 \in \mathcal{T}(\mathcal{H})$, $T * T_0$ is ∞-regular.
(3.7) The set $\mathcal{K}(\mathcal{H}) * T$ is dense in $\mathcal{C}_0(X)$.
(3.8) The set $\mathcal{C}_0(X) * T$ is dense in $\mathcal{K}(\mathcal{H})$.

Proof: **(a.1)⇔(a.2) and (a.3)⇔(a.4) for a=1,2,3, and (3.3)⇔(3.8)**
are all based on the same basic fact concerning continuous linear operators between dual pairings of topological vector spaces (Chap. IV.2.3 of Ref. 18), i.e., in the most general setting in which the notion of adjoint makes sense: a continuous linear operator is injective if and only if its adjoint has dense range in the weak topology induced by the pairing. Indeed, the vectors in the kernel of the operator are precisely those vanishing on the range of the adjoint. In the cases at hand we have the canonical dual pairings of the Banach spaces $\langle \mathcal{L}^1(X),\ \mathcal{L}^\infty(X)\rangle$, $\langle \mathcal{L}^p(X),\ \mathcal{L}^q(X)\rangle$, $\langle \mathcal{T}(\mathcal{H}),\ \mathcal{B}(\mathcal{H})\rangle$ and $\langle \mathcal{T}_p(\mathcal{H}),\ \mathcal{T}_q(\mathcal{H})\rangle$. The operator involved is always written as $X \mapsto X * T$ whose adjoint, taken from (11), is $Y \mapsto Y * T_-$. We have omitted the minus subscripts from the statements of the theorem, because all conditions of the propositions are obviously equivalent for T and for T_-. We note that in the general result the natural topology in which the range is taken to be dense is the weak one induced by the pairing. However, since in a Banach space the weak closure is equal to the norm closure, the density in $\mathcal{T}(\mathcal{H})$, $\mathcal{T}_p(\mathcal{H})$, $\mathcal{L}^1(X)$, $\mathcal{L}^p(X)$ is also in norm as stated. In contrast, in Proposition 3, it is the weak* topology of $\mathcal{B}(\mathcal{H})$, $\mathcal{L}^\infty(X)$, i.e., the weak topology coming from the predual. The statement in the norm topology would be false. Indeed, (3.2) is always false with norm density, because all functions $A * T$, and hence their norm limits are uniformly continuous on phase space.

For proving (3.3)⇔(3.8) we take the dualities $\langle \mathcal{T}(\mathcal{H}),\ \mathcal{K}(\mathcal{H})\rangle$ and $\langle \mathcal{L}^1(X),\ \mathcal{C}_0(X)\rangle$. The latter is now not a pair of a Banach space and its dual, but still satisfies the mutual separation conditions for a duality. The operators $T* : \mathcal{T}(\mathcal{H}) \to \mathcal{L}^1(X)$ and $T_-* : \mathcal{C}_0(X) \to \mathcal{K}(\mathcal{H})$ are adjoints with respect to these pairings, therefore T* is injective (⇔ (3.3)) iff the range of T_- * is dense in the weak topology of $\mathcal{K}(\mathcal{H})$, and hence in the norm topology, which is (3.8).

(a.0)⇔(a.3) for a=1,2,3
Since $A * T(x) = \mathrm{tr}[A_- \alpha_{-x}(T)]$, also this follows immediately from the dualities $\langle \mathcal{T}(\mathcal{H}),\ \mathcal{B}(\mathcal{H})\rangle$ and $\langle \mathcal{T}_p(\mathcal{H}),\ \mathcal{T}_q(\mathcal{H})\rangle$.

(a.1) ⇒ (a.3) for a=1,2,3
follows from the associativity of convolution. Assume (a.1), i.e., injectivity on the appropriate function class $\mathcal{L}^1(X)$, $\mathcal{L}^q(X)$, $\mathcal{L}^\infty(X)$, and assume $A * T = 0$ for some operator in the corresponding operator space $\mathcal{T}(\mathcal{H})$, $\mathcal{T}_q(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$. Then, for all trace class operators S, we have $S * A * T = 0$. But $S * A$ is in the appropriate function class, so with (a.1) we get $S * A = 0$. In particular, $S * A(0) = \mathrm{tr}[SA_-] = 0$. Since S was arbitrary, $A_- = A = 0$.

(a.5) ⇒ (a.2) and (a.6) ⇒ (a.2) for a=1,2,3
Given any $T' \in \mathcal{T}(\mathcal{H})$ we have $\alpha_x(T' * T) = \alpha_x(T') * T$ for all $x \in X$, so that $\{\alpha_x(T * T') \mid x \in X\} \subset \mathcal{T}(\mathcal{H}) * T \subset \mathcal{T}_p(\mathcal{H}) * T \subset \mathcal{B}(\mathcal{H}) * T$ for all $1 \le p < \infty$.

Existence of a T_0 satisfying (1.2)
Let T_0 be the one-dimensional projection onto any Gaussian wave function. Indeed, then $\widehat{T_0 * T_0}(x) = \widehat{T}(x)^2$ is of (complex) Gaussian form which thus never vanishes, so we can apply the classic Wiener's approximation theorem (discussed more in Sec. V) to conclude that the translates of $T_0 * T_0$ span $\mathcal{L}^1(X)$. Since we have already proved that (1.5) ⇒ (1.2), T_0 satisfies (1.2).

(a.3) ⇒ (a.1) for a=1,2,3
is analogous to (a.1) ⇒ (a.3), with one additional idea. Assume (a.3) and $f * T = 0$, with f in $\mathcal{L}^1(X)$, $\mathcal{L}^q(X)$, $\mathcal{L}^\infty(X)$. Then as before we get $S * f = 0$ for *all* $S \in \mathcal{T}(\mathcal{H})$. To conclude the proof we choose some T_0 satisfying (1.2). Since $T_0 * S * f = 0$, we have that $g * f = 0$ for the $\mathcal{L}^1$-dense set of functions $g = T_0 * S$. This implies $f = 0$.

(a.2) ⇒ (a.5) and (a.2) ⇒ (a.6) for a=1,2,3
Assuming (a.2), we can approximate any $f \in \mathcal{L}^p(X)$ by $A * T$ with some $A \in \mathcal{T}_p(\mathcal{H})$ ($A \in \mathcal{B}(\mathcal{H})$ in case $p = \infty$). On the other hand, we have already proved that (a.2) ⇒ (a.1) ⇒ (a.3) ⇒ (a.0), so T is p-regular. Hence, we can further approximate A by a linear combination $\sum_{j=1}^n c_j \alpha_{x_j}(T)$; then $\sum_{j=1}^n c_j \alpha_{x_j}(T * T)$ approximates f because of the p-norm (weak* in case $p = \infty$) continuity of $A \mapsto A * T$. This proves (a.5). If $T_0 \in \mathcal{T}(\mathcal{H})$ is regular, it is also p-regular for all $1 \le p \le \infty$, so we can also approximate A by a linear combination of translates of T_0 instead of those of T; this proves (a.6).

(3.7)⇔(3.8)
Now note that both statements (3.7) and (3.8) hold for a regular T_0. Indeed, by (1.2) we can find T_1 so that $T_0 * T_1$ is close in 1-norm to a normalized density concentrated in a small ball around the origin. Hence $\|f - T_0 * T_1 * f\|_\infty$ can be made arbitrarily small for any uniformly continuous $f \in \mathcal{L}^\infty(X)$, and in particular for $f \in \mathcal{C}_0(X)$. Since $T_1 * f \in \mathcal{K}(\mathcal{H})$, we conclude that $T_0 * \mathcal{K}(\mathcal{H}) \subset \mathcal{C}_0(X)$ is dense. Similarly, $T_0 * \mathcal{C}_0(X) \subset \mathcal{K}(\mathcal{H})$ is dense.

Now assume (3.8), for some T. Then the set of all $A * T$ contains those with $A = f * T_0$, $f \in \mathcal{C}_0(X)$. But then in $A * T = (T * f) * T_0$ the first factor ranges over a dense subset of $\mathcal{K}(\mathcal{H})$, which by the density property of T_0 implies (3.7). Again the converse is completely analogous. □

V. REGULARITY AND ZERO SETS

The following proposition establishes the announced equivalence between 1, 2, ∞-regularity of a trace class operator T and the "spectral" conditions (Z 1)–(Z 3). Of course, (Z 1) ⇒ (Z 2) ⇒ (Z 3). These inclusions will be shown to be strict in Sec. VI.

Proposition 4: Let $T \in \mathcal{T}(\mathcal{H})$. Then

(1) T is regular iff $Z(T)$ is empty (*Z 1*).
(2) T is 2-regular iff $Z(T)$ is of measure zero (*Z 2*).
(3) T is ∞-regular iff $Z(T)$ has dense complement (*Z 3*).

Since ∞-regularity is equivalent to informational completeness, Proposition 4 (3) gives the desired spectral characterization of this property. We note that in Propositions 1–4, the positivity of T is not required, so they are a bit more general than needed for the discussion of covariant observables. Of course, when we show later that (Z 2) is not necessary, we have to be careful to construct a counterexample of a positive T, since the reverse implication might be true just under this additional assumption.

Proof: **(Z 1) ⟹ (1.5)**
Here we just refer to Wiener's approximation theorem.[17]

(1.3) ⟹ (Z 1)
Let $A = W(x)$ be a Weyl operator. Then $A * T$ is equal to the Weyl transform multiplied by an exponential. Hence if the Weyl transform of T had a zero at x, we would conclude that $W(x) * T = 0$ and hence, by (1.3), $W(x) = 0$, which is a contradiction.

(Z 2)⇔(2.1) for $p = 2$
Assume that *Z*(*T*) has measure zero, and $T * f = 0$. This convolution is, in general, defined by continuous extension from $\mathcal{L}^1$-functions *f* with respect to the 2-norms. Since the Weyl transform is isometric by the quantum version of the Plancherel theorem, this means that $\widehat{T}(x)\widehat{f}(x) = 0$ for almost all *x*. But since $\widehat{T}(x) \neq 0$ almost everywhere, $\widehat{f}(x) = 0$ almost everywhere, which by definition of $\mathcal{L}^2(X)$ means that $f = 0$.

Conversely, assume that *Z*(*T*) has positive measure. Then we can find a bounded subset *Y*, which still has positive measure. Then let $\widehat{f}$ be the indicator function of *Y*. This is non-zero and in $\mathcal{L}^2(X)$, and hence so is *f*. On the other hand, by construction, $\widehat{T}(x)\widehat{f}(x) = 0$ for all *x*, and hence by Fourier transform $T * f = 0$. Hence (2.1) fails, too.

(Z 3)⇔(3.3) (Characterization of informational completeness)
We have already noted the implication (Z 3) $\Longrightarrow$ (3.3) in the Introduction: When $A * T = 0$, we have $\widehat{A}(x)\widehat{T}(x) = 0$ for all *x*. By assumption $\widehat{T}(x) \neq 0$ on a dense set on which, consequently, $\widehat{A}(x) = 0$. Since $\widehat{A}$ is a continuous function, it vanishes identically.

Now suppose that (Z 3) does not hold, i.e., there exists an open set $\Omega \subset Z(T)$. We will construct a non-zero function $f \in \mathcal{L}^1(X)$ such that $\widehat{f}$ has support in Ω. (This will immediately be a counterexample to (3.1).) Since any regular T_0 satisfies (Z 1) according to what we just proved above, a counterexample to (3.3) can be obtained as $A = f * T_0$.

It remains to construct a non-zero $f \in \mathcal{L}^1(X)$ with $\operatorname{supp}\widehat{f} \subset \Omega$. For this we can take any sufficiently smooth function $\widehat{f}$ with the required support and define *f* by the inverse Fourier transform of $\widehat{f}$. The Fourier transform will thus decrease faster than any desired power, and will therefore be integrable. □

Wiener's approximation theorem was used in a crucial way in this proof. In fact, we could have obtained the whole proposition as a corollary of classical results of classical harmonic analysis. Let us make these connections more explicit, since they will also be crucial for understanding the more subtle cases of *p*-regularity with $p \neq 1, 2, \infty$. Regularity statements of operators can be reduced to those of functions via the equivalences (a.0)⇔(a.5) for $a = 1, 2, 3$ above. Similarly, the properties of zero sets are translated via the relation $Z(T) = Z(T * T)$. Here, analogously to the Introduction, we define the set *Z*(*f*) for $f \in \mathcal{L}^1(X)$ as the zero set of its Fourier transform $\widehat{f}$. With this translation, Wiener's approximation theorem[17] becomes Proposition 4(1). For (2) we can invoke another result from Ref. 17, namely that 2-regularity is equivalent to *Z*(*f*) having zero measure. Finally, the characterization of ∞-regularity of functions by *Z*(*f*) having dense complement is, e.g., in Theorem 2.3 of Ref. 19.

In his classic paper Wiener already raised the question (p. 93 of Ref. 17) about other values of *p*. This has turned out to be a subtle problem, generating a rich literature (see, e.g., Refs. 19–23). The point we wish to make here is that the results obtained in this context can be turned directly into statements about operators using the translation principles sketched above. We begin with a statement that makes this relation more symmetric: results about operator regularity also imply classical results.

Various notions of "smallness" for zero sets have been considered in the literature. In the following proposition, we introduce another one, which we call a *p-slim* set for the sake of discussion. The terminology echoes the stronger notion of *p*-thin sets of Edwards,[19] and a still stronger condition, sets of "type $U^{p/(1-p)}$" in Ref. 22 (see, Ref. 19 particularly Theorem 2.2 for these comparisons).

Proposition 5: For $f \in \mathcal{L}^p(X) \cap \mathcal{L}^1(X)$, $T \in \mathcal{T}(\mathcal{H})$, a regular $T_0 \in \mathcal{T}(\mathcal{H})$, and $1 \leq p \leq \infty$ we have that f is p-regular iff $f * T_0$ is p-regular, and T is p-regular iff $T_0 * T$ is p-regular.

For a subset $S \subset \mathbb{R}^2$ the following conditions are equivalent:

(1) For any $f \in \mathcal{L}^1(X) \cap \mathcal{L}^p(X)$, $Z(f) \subset S$ implies that f is p-regular.
(2) For any $T \in \mathcal{T}(\mathcal{H})$, $Z(T) \subset S$ implies that T is p-regular.

*We call such sets p-***slim**.

Proof: The equivalence between the regularity of T and of $T * T_0$ is just (a.0)⇔(a.6) for $a = 1, 2, 3$ above. The corresponding statement for functions is proved in a similar fashion: p-regularity of $f * T_0$ implies the density of $\mathcal{T}_p(\mathcal{H}) * f$ in $\mathcal{T}_p(\mathcal{H})$, which by duality implies the injectivity of $A \mapsto f * A$ on $\mathcal{T}_q(\mathcal{H})$, which by associativity of the convolution implies the injectivity of $g \mapsto f * g$ on $\mathcal{L}^q(X)$, i.e., p-regularity of f. On the other hand, if f is p-regular, then any $A \in \mathcal{T}_p(\mathcal{H})$ can be approximated by $g * T_0$, where g is a linear combination of translates of f, so $f * T_0$ is p-regular. The equivalence of (1) and (2) now follows immediately, because $Z(T_0 * f) = Z(f)$ and $Z(T_0 * T) = Z(T)$. □

Clearly, $Z(T)$ being p-slim is a natural sufficient condition for p-regularity of T. There are several sufficient conditions for the p-slimness of a set S. As an example we give the following result, which uses Hausdorff dimension[24] as a measure of smallness. Intuitively, this describes the scaling of the number of balls needed to cover the set as a function of the radius of the balls, and is one of the standard characteristics of fractal sets. The connection with Hausdorff dimension and the closure of translates problem has been first noted by Beurling,[21] and extended to any dimension in Ref. 25. Since $0 \leq h \leq 2$ in our two-dimensional phase space, the bound ranges from 1 to 2. The proof follows by combining (Theorem 4.(ii) of Ref. 22) with the fact that any set of type U^q, $p^{-1} + q^{-1} = 1$ is also p-slim.

Proposition 6: Let $2 \geq p > 4/(4 - h)$, *and let* $S \subset X$ *be a set of Hausdorff dimension* h. *Then* S *is* p-*slim.*

It seems that a necessary and sufficient characterization of p-slim sets is a hard problem. More importantly for our context, the whole research program initiated by Wiener's remark, namely to extend the clean characterizations of Proposition 4 to values $p \neq 1, 2, \infty$ has been resolved in the negative:[23] information about zero sets is not in general sufficient to decide regularity. The following proposition rephrases this result in the operator context.

Proposition 7: Let $1 < p < 2$. *Then there exists* $T, T' \in \mathcal{T}(\mathcal{H})$ *such that* T *is* p-*regular and* T' *is not, but* $Z(T) = Z(T')$.

Proof: We need to extend the example established for functions of one variable in Corollary 2 of Ref. 23, to functions on phase space X. Clearly, if $h, g \in L^1(\mathbb{R}) \cap \mathcal{L}^p(\mathbb{R})$, then $(q, p) \mapsto h(q)g(p)$ is p-regular iff the spans of translates of h and g are both dense in $\mathcal{L}^p(\mathbb{R})$. This follows easily by using the fact that $f \in \mathcal{L}^1 \cap \mathcal{L}^p$ is p-regular iff $g * f = 0$ implies $g = 0$ for all $g \in \mathcal{L}^q$. Hence, we can use Corollary 2 of Ref. 23 to conclude that there exist two functions $f, f' \in \mathcal{L}^1(X) \cap \mathcal{L}^p(X)$, such that f is p-regular and f' is not, but $Z(f) = Z(f')$. Then $T = f * T_0$ and $T' = f' * T_0$ have the stated properties if $T_0 \in \mathcal{T}(\mathcal{H})$ is any regular operator. □

VI. STRICT IMPLICATIONS

Here we show that the implications (Z 1) ⇒ (Z 2) and (Z 2) ⇒ (Z 3) are in fact strict.

Proposition 8: There is a positive trace class operator T *satisfying* (Z 2) *but not* (Z 1).

Proof: We take $T = T_1 = |\varphi_1\rangle\langle\varphi_1|$, where $\varphi_1(q) = \left(\frac{1}{\pi}\right)^{1/4} q e^{-\frac{q^2}{2}}$ is the first excited state of the harmonic oscillator. The Weyl transform is then

$$\widehat{T}_1(q, p) = \left(\frac{1}{2} - \frac{1}{4}(q^2 + p^2)\right) e^{-\frac{1}{4}(q^2+p^2)}$$

so clearly the zero set $Z(T_1) = \{(q, p) \in X | q^2 + p^2 = 2\}$ is nonempty but of measure zero. □

Proposition 9: There is a positive trace class operator T *satisfying* (Z 3) *but not* (Z 2).

Proof: We have to construct a positive operator T_2 of trace one such that $Z(T_2)$ is of nonzero measure but has dense complement. We choose the form $T_2 = f * T$, with T satisfying (Z 1), $T \geq 0$, and

$\mathrm{tr}[T] = 1$. Thus we reduce this to the construction of a function $f \in \mathcal{L}^1(X)$, which must be positive with integral one, such that the zero set of $\widehat{f}$, which is equal to $Z(T_2)$, satisfies the required conditions. We further specialize this to a one-dimensional construction, by setting $f(x) = f(q, p) = \phi(q)e^{-p^2}$. Now $\phi \in \mathcal{L}^1(\mathbb{R})$ has to be positive, and its zero set has to meet the description. Since the zero set of $\widehat{f}$ now consists of infinite strips, $Z(T_2)$ constructed in this way will even have infinite measure.

As the starting point for our construction we choose a positive $\varphi \in \mathcal{L}^1(\mathbb{R})$ such that also $\widehat{\varphi}$ is positive and $\widehat{\varphi}(q) \neq 0$ if and only if $q \in (-1, 1)$. This is satisfied, e.g., when $\varphi = \widehat{\chi * \chi}$ where χ is the characteristic function of the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. For each $\lambda > 0$ define

$$\widehat{\psi}_\lambda(q) = \sum_{k\in\mathbb{Z}} \alpha_k \widehat{\varphi}(\lambda(q+k)),$$

where $(\alpha_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ and $\alpha_k > 0$ for all $k \in \mathbb{Z}$. Then $\widehat{\psi}_\lambda \in \mathcal{L}^1(\mathbb{R})$ and the inverse Fourier transform gives

$$\psi_\lambda(q) = \tfrac{1}{\lambda}\varphi\left(\tfrac{q}{\lambda}\right) \sum_{k\in\mathbb{Z}} e^{-iqk}\alpha(k)$$

so clearly $\psi_\lambda \in \mathcal{L}^1(\mathbb{R})$. To ensure the positivity of ψ_λ we need to require that the sum is positive for all $q \in \mathbb{R}$. The choice $\alpha_k = 2^{-|k|}$ will work here. Finally, let $\lambda_n, \mu_n > 0$ for all $n \in \mathbb{N}$ and define

$$\widehat{\phi}(q) = \sum_{n=1}^{\infty} \beta_n \widehat{\psi}_{\lambda_n}(\mu_n q),$$

where $(\beta_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$ and $\beta_n > 0$ for all $n \in \mathbb{N}$. In order to ensure that $\widehat{\phi} \in \mathcal{L}^1(\mathbb{R})$ we further assume that $\sup_{n\in\mathbb{N}}(\mu_n\lambda_n)^{-1} < \infty$. By construction, $\widehat{\phi}$ and ϕ are non-negative, integrable functions and $\widehat{\phi}(q) = 0$ if and only if $\lambda_n(\mu_n q + k) \in (-1, 1)^c$ for all $n \in \mathbb{N}, k \in \mathbb{Z}$. The zero set $Z(\phi)$ of $\widehat{\phi}$ is thus

$$Z(\phi) = \bigcap_{n\in\mathbb{N}} \left(\tfrac{1}{\mu_n}\mathbb{Z} + \left(-\tfrac{1}{\mu_n\lambda_n}, \tfrac{1}{\mu_n\lambda_n}\right)\right)^c. \tag{14}$$

Clearly, we can normalize ϕ such that f has norm one.

What remains is to show that the scaling parameters can be chosen so that $Z(\phi)$ has positive measure and dense complement. A convenient choice is now $\mu_n = 2^n$, $\lambda_n = 2^{n+2}$. Then $Z(\phi)^c$ is clearly dense since $\bigcup_{n\in\mathbb{N}} \frac{1}{2^n}\mathbb{Z} \subset Z(\phi)^c$. To show that $Z(\phi)$ is of positive measure it is sufficient to show that the measure of $Z(\phi)^c \cap [0, 1]$ is strictly less than 1. For that purpose, note that $Z(\phi)^c \cap [0, 1] = \bigcup_{n\in\mathbb{N}} I_n$ where

$$I_n = \left(\tfrac{1}{2^n}\mathbb{Z} + \left(-\tfrac{1}{2^{2(n+1)}}, \tfrac{1}{2^{2(n+1)}}\right)\right) \cap [0, 1]$$

and the measure of I_n is $\frac{1}{2^{n+1}}$. It follows from the subadditivity of the Lebesgue measure that the measure of $Z(\phi)^c \cap [0, 1]$ is less than $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$. In other words, $Z(\phi) \cap [0, 1]$ is of positive measure. □

VII. EXTENSIONS TO MORE GENERAL PHASE SPACES

A more general phase space X can be defined as a locally compact abelian group equipped with an antisymmetric "symplectic" bi-character. More commonly one considers pairings $X = G \times \widehat{G}$, where G is a locally compact abelian group, and $\widehat{G}$ is its dual. The work[14] was written for $G = \mathbb{R}^n$, but the extension to general G is work in progress (J.S.). We do not wish to enter subtleties here which are better discussed separately. Therefore, we only give a simple extension of our propositions, which is easily proved and still covers many practical cases.

Proposition 10: Suppose that $X = G \times \widehat{G}$ is a phase space such that G is a finite product of copies of $\mathbb{R}$, $\mathbb{Z}$, the 1-torus group $\mathbb{T}$, and finite abelian groups.
Then Propositions 1–4 hold mutatis mutandis, and (Z 1) ⇒ (Z 2) ⇒ (Z 3).
Suppose that one of the reverse implications also holds. Then G is finite.

Proof: For a finite Cartesian product of groups written additively as $G = \bigoplus_i G_i$, we get Weyl operators which are tensor products with respect to $\mathcal{H} = \bigotimes_i \mathcal{H}_i$. The existence of a regular trace class operator can therefore be shown by tensoring such elements for each factor. We have seen this already for $G_i = \mathbb{R}$. For $G_i = \mathbb{Z}$, which is equivalent to $G_i = \mathbb{T} = \widehat{\mathbb{Z}}$, we can take a vector $\psi \in \ell^2(\mathbb{Z})$ with $\psi(n) = a^n$ for $n \geq 0$ and $\psi(n) = 0$ for $n < 0$. Finally, on a finite group the Hilbert space of the regular representation is also finite dimensional. For fixed x the equation $\langle\psi|W(x)\psi\rangle = 0$ holds only on a manifold of vectors of smaller dimension, and since there are only finitely many x, we have that almost all pure states are regular. This was the only specific property of G needed in the proofs of the first three propositions, and the rest of the proofs is entirely parallel to the ones given above.

Suppose now that one of the implications (Z 1) $\Rightarrow$ (Z 2) $\Rightarrow$ (Z 3) is strict for any one of the factors G_i. By tensoring the appropriate counterexample T_i with regular elements T_j we get an element T whose zero set is empty/measure zero/without open sets if and only if $Z(T_i)$ has these properties. Hence in order to exclude all but finite factors, we only need to show that the inclusions are strict for $G = \mathbb{Z}$. By taking $T = T_0 * f$ with T_0 regular, and f depending only on the $\mathbb{Z}$ coordinate, we can reduce this to finding appropriate functions on $\mathbb{T}$, exactly as in the proof of Proposition 9. Finding $f \in \ell^1(\mathbb{Z})$, whose Fourier transform has only some isolated zeros is easy. For a f such that the zero set of $\widehat{f}$ has positive measure, but contains no open sets, we can take the same example as in the proof of Proposition 9. □

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