Continuous time limit of repeated quantum observations

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Abstract

In this thesis we study the problem of continuous measurement on an open quantum system.

The starting point of our analysis is the time-evolution of an open quantum system where the evolution of the environment is not explicitly described, i.e. traced out. More explicitly we restrict to time-evolutions having the structure of a semigroup or evolution system. That is, for time-points $0 \leq r \leq s \leq t$ there exists a two-parameter family of completely-positive maps $\mathbb{E}_*(t,s)$, such that $\mathbb{E}_*(t,s)\mathbb{E}_*(s,r) = \mathbb{E}_*(t,r)$ and the systems state evolved according to $\rho(t) = \mathbb{E}_*(t,s)(\rho(s))$.

Physically this means that in our Ansatz the interaction between system and environment is already fixed and can only be weak. The main goal of this thesis is now to describe, mathematically rigorous, all continuous measurements compatible with the given evolution. A measurement is compatible with the given time-evolution, if the evolution does not change when measurement results are ignored. In other words the measurement is indirect and does not further disturb the systems evolution.

Our results can be interpreted as a rigorous construction of continuous matrix product states, and are related to the Hudson-Parthasarathy quantum stochastic calculus.

The analysis in this thesis separates into two parts. In the first we study the class of quantum systems having an evolution system structure as described above. Precisely we study the theory of minimal solutions to a Cauchy equation with a generator of Gorini-Kossakowski-Sudarshan-Lindblad type. We do this for the case of time-dependent and unbounded generators. This analysis culminates in the description of a counterexample due to Holevo of a semigroup which is not the minimal solution to such a standard Lindblad equation.

In the second part we construct a description of all measurements compatible with a given evolution and thus achieve our main goal. The analysis starts with the discrete-time version of the problem, i.e. where the time-points r, s, t in the above description of the evolution are chosen from a finite set. This discrete-time problem is completely solved by Stinespring dilation theory. We then construct our solution of the continuous-time case as a refinement limit over arbitrary discretizations of the evolution.

Hence, our construction gives a continuous-time analogue of the Stinespring dilation for semigroups and evolution systems. It can be interpreted as a complete quantum description of the information the system radiates into the environment, i.e. it yields a quantum state of the environment. We complement the description of the state with a description of the observables for continuous measurements and the calculation expectation values.

Keywords: open quantum systems, delayed-choice quantum measurement, lindblad generators

Zusammenfassung

In der vorliegenden Arbeit untersuchen wir das Problem der Beschreibung von kontinuierlichen Messungen an einem offenen Quantensystem.

Der Startpunkt unserer Analyse ist die Zeitentwicklung eines offenen Quantensystems, bei welcher die Zeitentwicklung der Umgebung nicht Teil der Beschreibung, also ausgespurt, ist. Genauer gesagt, beschränken wir die Klasse an möglichen Zeitentwicklungen auf diejenigen, welche die Stuktur einer Halbgruppe oder eines Propagators haben. Zu Zeitpunkten $0 \leq r \leq s \leq t$ existiert also eine zwei-Parameter-Familie von vollständig positiven Abbildungen, $\mathbb{E}_*(t,s)$, mit der Eigenschaft $\mathbb{E}_*(t,s)\mathbb{E}_*(s,r) = \mathbb{E}_*(t,r)$. Die Zeitentwicklung des Systems ist gegeben durch die Gleichung $\rho(t) = \mathbb{E}_*(t,s)(\rho(s))$.

Auf physikalischer Ebene bedeutet dieser Ansatz, dass die Wechselwirkung zwischen System und Umgebung von vorne herein festgelegt ist und nur schwach sein kann. Das Primärziel dieser Arbeit ist die mathematisch strenge Beschreibung aller Messungen in kontinuierlicher Zeit die mit der gegebenen Zeitentwicklung verträglich sind. Verträglichkeit von Messung und Zeitentwicklung bedeutet hierbei, dass die Zeitentwicklung unverändert bleibt falls die Messergebnisse nicht beachtet werden. Die Messung verläuft also indirekt und stört das System nicht zusätzlich.

Unsere Ergebnisse können als eine kontinuierliche Stinespring Dilatation verstanden werde. Es besteht ein Zusammenhang mit kontinuierlichen Matrixproduktzuständen und dem quantenstochastischen Calculus von Hudson und Parthasarathy.

Diese Arbeit gliedert sich in zwei Abschnitte. Im ersten Abschnitt werden Zeitentwicklungen mit der oben beschriebenen Propagator-Struktur näher beleuchtet. Genauer gesagt, untersuchen wir minimale Lösungen von Cauchy Gleichungen mit einem infinitesimalem Erzeuger von Gorini-Kossakowski-Sudarshan-Lindblad Form. Wir behandeln hier den Fall von zeitabhängigen und unbeschränkten Erzeugern. Dieser Abschnitt der Arbeit endet mit der Beschreibung eines Gegenbeispiels von Holevo, einer Halbgruppe die nicht minimale Lösung einer Lindblad Gleichung ist.

Im zweiten Abschnitt der Arbeit konstruieren wir die Beschreibung aller mit einer vorgegebenen Zeitentwicklung verträglichen Messungen und erfüllen damit unser Primärziel. Unser Lösungsansatz basiert auf der Diskretisierung der Zeitentwicklung, das heißt die möglichen Zeitpunkte r, s, t in der obigen Beschreibung beschränken sich auf eine endliche Menge. Diese diskretisierte Variante des Problems wird vollständig durch die Stinespring Dilatation gelöst. Der Kontinuumsfall ergibt sich daraus als ein Verfeinerungslimes über beliebige Diskretisierungen.

Die Konstruktion kann also als ein kontinuierliches Analogon zur Stinespring Dilatation für Halbgruppen und Propagatoren betrachtet werden. Das Ergebnis ist eine vollständige Beschreibung der Quanteninformation die das System an die Umgebung abgibt. Wir ergänzen diese Beschreibung der Zustände mit einer Theorie der dazu gehörigen Observablen und der Berechnung der Erwartungswerte.

Schlagwörter: offene Quantensysteme,verzögerte Quantenmessung, Lindblad Erzeuger

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$\#\mathbb{A}$ Cardinality of a set $\mathbb{A}$.

$\pm$ Used as an index for Bose(+) or Fermi(-) Fock space valued maps

$\circ$ Iteration or concatenation of maps

φ, ψ Vectors in a Hilbert space.

B,X Usually an operator on a Hilbert space, i.e. in $\mathfrak{B}(\mathcal{H})$

$\mathcal{A}$, $\mathcal{B}$ A Banach space, usually $\mathcal{B} = \mathfrak{B}(\mathcal{H})$

$\mathcal{H}$ Hilbert space, usually the system

$\mathbb{A}$ The set of labels for Kraus operators, i.e. a countable set

$\mathfrak{B}(\mathcal{H})$ The set of bounded operators on the space $\mathcal{H}$

$\mathfrak{CP}(\mathcal{A}, \mathcal{B})$ The set of completely-positive operators from $\mathcal{A}$ to $\mathcal{B}$

$\mathfrak{CP}(\mathcal{B})$ The set of completely-positive operators on the space $\mathcal{B}$

$\mathcal{K}$ Hilbert space, usually the environment/dilation space

L A Lebesque space or a Bochner-Lebesque space

W A Sobolev space

ad The adjunction, i.e. usually $\mathrm{ad}_B(A) = B^*AB$

FCS Finitely correlated state, see section 6.3.

MPS Matrix product state, see section 6.3.

$\mathbb{C}$ Complex numbers.

$|\mathcal{D}\rangle\langle\mathcal{D}|$ The linear span of ketbra operators with $\psi \in \mathcal{D}$

diam Diameter of an Interval decomposition, i.e. length of longest subinterval

$\mathbb{E}$,$\mathbb{F}$ A completely-positive map, $\mathbb{E} \in \mathfrak{CP}(\mathcal{A}, \mathcal{B})$

$\mathbb{E}_*$ Pre-adjoint of a map, i.e. usually a completely-positive map in Schrödinger picture

$\mathcal{E}$ An exit space, for some evolution system $U(t)$

$\widehat{\mathbb{E}}$ A completely positive map, usually $\widehat{\mathbb{E}} \in \mathfrak{CP}(\mathfrak{B}(\mathcal{H})))$

exp	The exponential function, or a semigroup
$\Gamma_\pm$	The Fock space functor, either Bose or Fermi.
$\mathrm{d}\Gamma_\pm$	The differential Fock space functor, either Bose or Fermi.
$\mathrm{id}_{\mathcal{B}}$	Identity map on $\mathcal{B}$.
$\mathbb{1}_{\mathcal{H}}$	Identity operator on $\mathcal{H}$.
$\mathfrak{Im}$	Imaginary part.
$\mathcal{J}$	A reinsertion map like $B \to L^*\mathbb{1} \otimes BL$
L	A Lindblad operator, usually $L : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$
$\underrightarrow{J\text{-}lim}$	Inductive limit of a family of spaces
$\underleftarrow{J\text{-}lim}$	Projective limit of a family of spaces
s-lim	Limit in strong topology
w-lim	Limit in weak topology
w ∗ -lim	Limit in weak-∗ topology
$\mathcal{L}$	Generator of a Semigroup/Evolution-system usually of Lindblad form
$\mathcal{M}$	An arrival time measure
$\mathbb{N}$	Natural numbers.
Φ	A field operator/ Generator of Weyl operator
$\mathbb{R}$	Real numbers.
$\mathcal{R}$	A resolvent operator
$\mathfrak{Re}$	Real part.
span	The linear span of a set
$\overline{\mathrm{span}}$	The closed linear span of a set
$\mathcal{T}$	The transit space for a given reinsertion
Θ,Ξ,Λ	Interval decomposition.
tr	The trace of a linear operator
$U(t)$	A semigroup, usually $U(t) \in \mathfrak{B}(\mathcal{H})$
V	Stinespring isometry, usually: $V : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$
$\mathcal{Z}$	Generator of a dissipative evolution, usually $\mathcal{Z}(B) = K^*B + BK$
$\mathbb{Z}$	Integers.
$\mathfrak{Z}([0,T])$	The set of interval decompositions of $[0,T]$

1. Outline

1.1. Measurements compatible with an evolution

The central theme of this thesis is the continuous observation of an open quantum system.

The starting point of our analysis is the time-evolution of an open quantum system where the evolution of the environment is not explicitly described, i.e. traced out. The interaction between the system and its environment is thus fixed throughout our whole analysis.

We set ourselves the goal of describing all continuous-time measurements compatible with a given continuous time-evolution. A measurement is compatible with a given evolution, when ignoring the measurement results means that the evolution does not change. We always depict time-evolutions of quantum systems as in figure 1.1.

This leads to two main principles in our description of measurements: On the one hand the observation process should not add any additional disturbance to the system. On the other hand we try to extract the maximal amount of information, i.e. we want to describe all measurements.

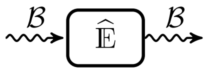


Figure 1.1.: A single time-step

At first it might seem paradoxical that we choose not to add any disturbance, since it is well known that any measurement always has to add perturbation, see e.g. [63]. But since we work with open systems there already exists an environment. And our goal is to extract the maximal amount of knowledge from the already existing interactions with this environment. In some sense we allow ourselves complete access to that environment and simply perform our measurements on the environment of the system alone.

Such a complete access to the environment may however be hard to reconcile with our premise not to add any further disturbance to the system. If we want to reconcile this with the wish to describe all measurements, we have to assume that state changes of the environment due to our measurement apparatus do not have any influence on the systems evolution. Hence the interaction between our measurement apparatus and the system is not allowed to be too strong.

Apart from the obvious solution to this problem, i.e. to restrict the class of measurements on the environment, there is another way. It is well known that one can view every possible evolution of an open system as a unitary evolution on a bigger system. Hence alternatively we can just redefine the “environment” to include only those parts which do interact only weakly with the rest of the system. Practically this just amounts to a restriction of the class of open quantum systems we are interested in.

We call the class of open quantum evolutions, where state changes of the environment

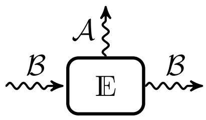


Figure 1.2.: Single time-step and possible measurements compatible with it

due to a measurement do not have any influence on the system evolution, Markovian quantum systems. As we shall see, they form a natural generalization of classical Markov evolutions, since they too have the property that knowledge of the systems state at any time-point provides knowledge of the state for all future times.

This viewpoint now leaves us with the following task: given a Markovian quantum evolution, we want to describe all measurements, which do not disturb this evolution any further. That is, if we ignore our measurement results, the evolution of the quantum system does not change.

On the contrary it is clear that evaluating the measurement results, provides additional knowledge about the system. And hence its quantum state changes, which is a representation of our knowledge about the system.

This approach to continuous measurements has the nice property that it can not run into a quantum Zeno paradox, because the evolution of the observed system does not change at all and hence can not freeze in particular.

Furthermore this scheme is compatible with arbitrary discretizations of a continuous measurement. That is we can just discretize the time-evolution and quantify the information we can obtain during the different time intervals. This means we get less information about the time of the measurement and leads to a simplified description of the possible measurements.

1.2. Thesis outline

Our struggle to reach the goal of describing all measurements compatible with a given Markovian evolution falls into two parts. In the first half of this thesis we focus on getting a solid understanding of Markovian quantum evolutions. We focus on the most important class of such systems, i.e. those generated by a Lindblad equation.

We do not attempt to solve the question if this class comprises actually all systems we are interested in. But rather concentrate on understanding these systems. This leads to a study of a arrival time measures, which serve as good basis for understanding and eventually generalizing the Lindblad equation.

This first part of the thesis culminates in the presentation of a seemingly widely unknown example by Holevo, which tries to fathom the limits of the usual approach to Lindblad equations.

In the second half of the thesis we present a continuous-time limit approach to the problem of measurement in continuous time. These results are extensions of those described in the authors Master Thesis [69]. This part of the thesis contains our main results. It can be seen as a discrete approach to quantum stochastic calculus [72] and quantum input output theory [48].

Our constructions are build around the observation, that repeated measurement scenarios are easily described and understood in a discrete setting. The central notion for a measurement scenario, we want to introduce is that of a delayed-choice iterated measurement. A viewpoint on repeated measurements, which relates them to the theory of finitely correlated states (FCS) [46] or matrix product states (MPS) [83]. We start the second part of the thesis with a description of these discrete constructions in section 6.1.

Our continuous limit construction for the discrete measurements, also yields a discrete approximation scheme for the Fock space of a one dimensional quantum field. The construction arises from a few simple assumptions about the structure of short time-steps and is described in section 6.4.

We then proceed with a presentation of the continuous-time limit of repeated quantum observations in section 7, the eponymous construction of this thesis. The states resulting in the limit were called continuous matrix product states (cMPS) in [82]. Our approach to the limit is compatible with time-dependent preparation and measurement setups. Furthermore it allows for a lot of mathematical generality, e.g. including unbounded generators. The rest of this thesis elucidates this construction.

A special focus lies on their application to the calculation of expectation values of field observables. This is the topic of section 8. We close with the treatment of an example of cMPS originating from quasi free semigroups on the CCR-algebra.

1.3. Central ideas of part I

As the main part of this thesis is very mathematical. And since despite our best efforts to structure the results, may be a bit hard to read. We start with an overview about the physical ideas behind our constructions.

As already mentioned, a central assumption in our approach is that the quantum system we try to measure couples only "weakly" to its environment. That is the knowledge of the systems state at an arbitrary time-point fixes it evolution for all later times. Hence we can describe its time-evolution is in Heisenberg picture by a differential equation of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}B(t) = \mathcal{L}(B(t)) \tag{1.1}$$

We call a system Markovian when such an description exists. The systematic study of these Cauchy equations leads to the theory of semigroups and evolution systems.

The further requirement that a Markov evolution should also be compatible with the statistical interpretation of quantum mechanics, i.e., completely positive, forces the generator to have a special form, the Lindblad form:

$$\mathcal{L}(B) = KB + BK + \sum_{\alpha} L_{\alpha}^{*} B L_{\alpha} \tag{1.2}$$

The generator furthermore has to fulfill the condition $\mathcal{L}(\mathbb{1}) \leq 0$ which ensures that the total probability, i.e. the expectation of the identity, stays bounded by 1.

We do however allow for systems which "loose" probability. In that case we have $\mathcal{L}(\mathbb{1}) < 0$. Such a behavior can easily be interpreted as an "escape" of the system, i.e. the system avoids detection altogether. On the contrary "growing" total probability does not make sense in a purely statistical interpretation. However, if we interpret the expectation of the identity as a particle number rather than the total probability, different normalization conditions would make perfect sense.

One of the main goals of the first part of this thesis is to get a better understanding of the Lindblad form. The various mathematical tools we introduce mostly try to serve this purpose.

A crucial point about the Lindblad form is that it allows to "decompose" the time-evolution into two distinct contributions. On the one hand there the dissipative part $B \to K^*B + BK$. Generators of this form are well known from the Heisenberg equation of motion, where we would have $K = iH$ for a Hamiltonian H. In the present case there is an additional absorptive term, i.e. $K = D + iH$ where D is negative and describes the "loss" of probability.

This loss of probability due to the dissipative part of the evolution is (partially) mitigated by reinsertion of escaped systems. Such reinsertions correspond to the part: $B \to \sum_\alpha L_\alpha^* B L_\alpha$. These reinsertions happen instantaneously. The normalization condition $\mathcal{L}(\mathbb{1}) \leq 0$ ensures that we do not reinsert more probability than what we actually loose.

This division of the Lindblad generator in a dissipative and a reinsertion part explains its relation to arrival time measures. One just has to assume that the arrival time measure describes the times at which we "loose" the system. That is, it describes the distribution of waiting times between "loss-events". Together with a description of reinsertions we can interpret the capture and reinsertion process as a quantum jump event, i.e. an instantaneous state transformation. And in total we obtain an understanding of the Lindblad form, which might actually be more general than the Lindblad form itself.

As we shall see, our understanding of the Lindblad form, greatly helps in describing measurements on systems described by such an equation.

1.4. Central ideas of part II

In the second part of this thesis we try to construct a description of the continuous-time measurement of an open quantum system undergoing an evolution of Lindblad type. As required above, the measurement should satisfy two conditions. Firstly, it should not disturb the evolution any further, i.e. if we ignore our measurement results the evolution does not change. And secondly, we want to simultaneously describe all measurements satisfying this condition. And hence obtain a quantum description of the information the system "emits" into the environment.

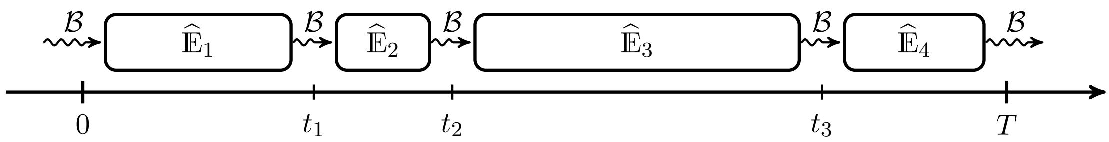


Figure 1.3.: The evolution of a quantum system in discrete time

Our construction of a solution tries to exploit the fact, that the discrete version of the problem is easy to solve. A discrete-time evolution of a quantum system is depicted in figure 1.3. The solution of the discrete time problem rest on two facts. On the one hand there is a general solution for the one-step case. That is a construction which describes the quantum information "released" into the environment in a time-step: the Stinespring dilation. This construction gives us a canonical way to go from figure 1.1 to 1.2.

On the other hand it is straightforward to extend this solution to multiple time-steps, i.e. any discretization of a problem. The idea is to collect and store the information released in each single time-step. This provides us with a description of the information released during the whole process, and hence to a description of all measurements we can perform on it. The result of this process can be seen in figure 1.4.

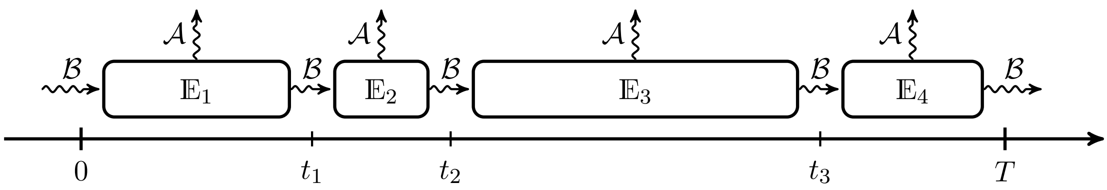


Figure 1.4.: Multiple time-steps in discrete time and the compatible measurements

Actually these two ideas, and the structure they introduce, are sufficient to construct a continuous time limit, see section 6.5.2 or [71]. But if we add one more idea we get a solution, whose structure is much easier to understand. As an added benefit this helps us to put the focus on a structure common to the continuous limit off all such processes.

So far there are two more problems keeping us from writing down a limit construction. The first one is that the existence of the Stinespring dilation, i.e. the description measurements compatible with a single time step, is indeed guaranteed by abstract theory, but the dilation may still be quite hard to calculate explicitly. However if we are only interested in the existence of a limit, rather than in its structure this is not a problem.

The second problem problem we have to solve, is the question how to compare to different discretizations of the same process. From the abstract Stinespring theory, we already know that it is easiest to compare two different discretization of the same time step. Even better with the right limit structure it is sufficient to be able to compare a single time step with all its discretizations. Hence in our construction of the limit we do not compare arbitrary discretizations but rather keep refining a given discretization by further subdividing existing time-steps. This is depicted in figure 1.5.

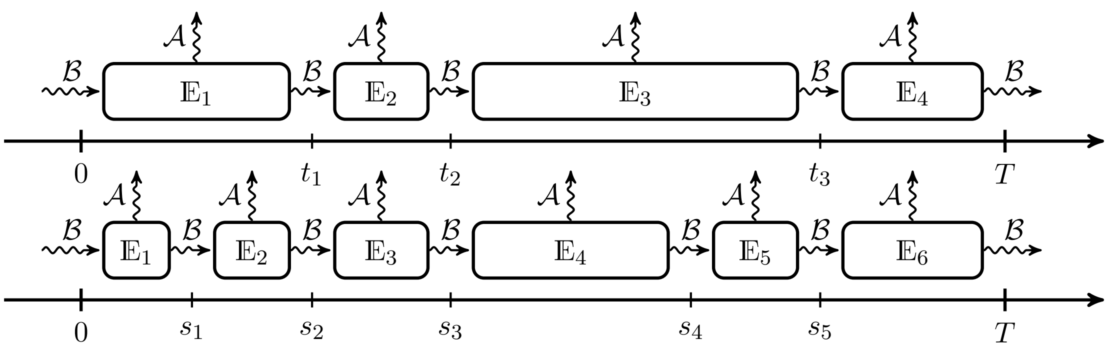


Figure 1.5.: Comparing to discretizations of the same evolution

We get a nice solution to the remaining problems, if we restrict our attention to measurements on Markovian quantum evolutions and capitalize on our understanding of the Lindblad equation. An important lesson about Lindblad evolutions is that they can be interpreted as a continuous evolution, interlaced with instantaneous "quantum jumps".

The moral of this story is that, if we go to a fine enough discretization there should be at most one jump event per subinterval. So if we can approximate short time-steps as having at most one event, we get a simplified description of the time-evolution.

Such a simplified description of short time-steps solves both our problems at the same time. Firstly, the possible measurements compatible with a single short time-step are easy to describe. They are fixed by the information, if there was a jump event during that time step, and if yes of which type. And secondly, when comparing a short time-step with a discretization, we only have to average over the time-point at which the single jump event might have happened.

Thus in total our limit is based on the following three basic principles:

1. Short time-steps are easy to describe approximately.
2. Piecing together solutions for time-steps is easy.
3. Comparing approximate short time-steps is easy.

2. Mathematical and physical basics

2.1. Inductive limits

In the second half of this thesis we are going to construct a continuous process from a collection of discrete processes which approximate the continuous one, i.e. we are going to perform a limit. Here we try to set up the necessary language. There are at least two possible strategies to handle limits. The first one is to find a space in which all the discrete objects and the limit object are defined and do the limit process in this space. This needs a great deal of insight because one needs to know where the limit object "lives" from the very beginning.

The other possibility, the one which we implement, is to work entirely on the basis of "finite" objects and show convergence without knowledge of the explicit structure in the limit. To do this we only have to be able to compare the finite objects with each other. Of course we do not have to be able to compare arbitrary objects, but only according to some order structure on our objects. In other words we construct something like a Cauchy sequence.

Thus we omit two of the basic structures one usually encounters in limits: the explicit knowledge of a limit object and the sequence structure, i.e. the existence of a total order. The concepts which we need to describe such a limit are the inductive or directed limit, and the notion of a net, as a generalization of a sequence.

The limit concepts, we describe here, can be found in many introductory topology textbooks. Since nets are slightly out of fashion, many books only cover this topic very briefly. An introductory text on topology with a slightly longer than usual treatment of nets, is e.g. [37, chapter 10]

2.1.1. Directed sets and interval decompositions

To deviate from the concept of a sequence and arrive at the more flexible concept of a net, one has to substitute the index set $\mathbb{N}$ of the sequence with a more general order structure. The necessary structure is encompassed in the following definition of a *directed set*.

Definition 2.1. Let $\mathfrak{Z}$ be a set, and $\Theta, \Xi, \Lambda \in \mathfrak{Z}$. The set $\mathfrak{Z}$ is called a *directed set* $\mathfrak{Z}$ if it has a pre-order $\leq$, a binary relation which is:

transitiv $\Theta \leq \Xi$ and $\Xi \leq \Lambda$ imply $\Theta \leq \Lambda$

reflexiv $\Theta \leq \Theta$

Furthermore for every two elements Θ,Ξ there exists a common upper bound Λ such that $\Theta \leq \Lambda$ and $\Xi \leq \Lambda$.

The easiest example of a directed set is just $\mathbb{R}$ or $\mathbb{N}$ with the normal order operation. But these are totally ordered sets, i.e. we can compare any two elements via the order relation. Directed sets allow for a significantly richer order structure, where one is not able to compare arbitrary elements. The set of bounded operators with the usual operator-ordering is an example of a more general directed set. In this case the existence of an upper bound is guaranteed by the fact that every operator is bounded by a scalar multiple of the identity, i.e. the identity is an order unit.

Apart from wanting to advertise the versatility of more general order structures, we do not need the full generality of directed sets at all. We are actually only concerned with one particular example, the set of decompositions of a fixed interval:

Figure 2.1.: An interval decomposition

Definition 2.2. Let $[0,T]$ be a fixed interval. An *interval decomposition* Θ is a finite ordered set of points in $[0,T]$, where the endpoints 0 and T are always included, i.e.:

$$\Theta := \{0 = t_0 < t_1 < t_2 \cdots < t_n = T | t_i \in [0,T], n \in \mathbb{N}\} \tag{2.1}$$

We refer to the set of labels $\{1, 2, \ldots, n\}$ of an interval decompositions Θ as $I(\Theta)$. Note that the label 0 is not included. The length of the subintervals is denoted as $\tau_i = t_i - t_{i-1}$ for $i \in I(\Theta)$.

The set of all interval decompositions of a given interval is denoted as $\mathfrak{Z}([0,T])$. Figure 2.1 shows an interval decomposition and figure 2.2 demonstrates the labeling of subintervals.

An interval decomposition $\Theta \in \mathfrak{Z}([0,T])$ segments $[0,T]$ in a family of subintervals $[t_{i-1}, t_i]$ labeled by $i \in I(\Theta)$. The labels $i \in I(\Theta)$ are always thought to correspond to the subintervals defined by Θ. The set of interval decompositions $\mathfrak{Z}([0,T])$ with this order forms an order lattice.

Lemma 2.3. *For an interval* $[0,T]$ *the set of interval decompositions* $\mathfrak{Z}([0,T])$ *is ordered by the set theoretic inclusion* $\subset$. *With this order structure* $(\mathfrak{Z}([0,T]), \subset)$ *is an order lattice. For* $\Theta, \Xi \in \mathfrak{Z}([0,T])$ *the common upper bound is given by* $\Theta \cup \Xi$. *Furthermore* $\Theta \cap \Xi$ *gives a common lower bound for two sets and* $\Theta \subset \Xi$ *and* $\Xi \subset \Theta$ *imply* $\Theta = \Xi$.

Figure 2.2 depicts this order relation.

In particular lemma 2.3 means that the set of interval decompositions of a fixed interval, i.e. $\mathfrak{Z}([0,T])$, is a directed set. We mainly concentrate on this part of the order structure.

From now on we always denote the ordering on $\mathfrak{Z}([0,T])$ by $\subset$ and the set of interval decompositions is always thought of to be equipped with this structure.

Of course $\mathfrak{Z}([0,T])$ has even more structure. For example if $\Theta \subset \Xi \in \mathfrak{Z}([0,T])$ then Ξ defines an interval decomposition on every subinterval $[t_{i-1}, t_i]$ for $i \in I([\Theta])$.

Definition 2.4. For $\Theta, \Xi \in \mathfrak{Z}([0,T])$ and $\Theta \subset \Xi$ we denote the decomposition of $[t_{i-1}, t_i]$ for $i \in I(\Theta)$ given by $\{t_j \in \Xi | t_{i-1} \leq t_j \leq t_i\}$ as $\Xi|_i \in \mathfrak{Z}([t_{i-1}, t_i])$ or as ${}_{\Xi|i}$ when used as an index.

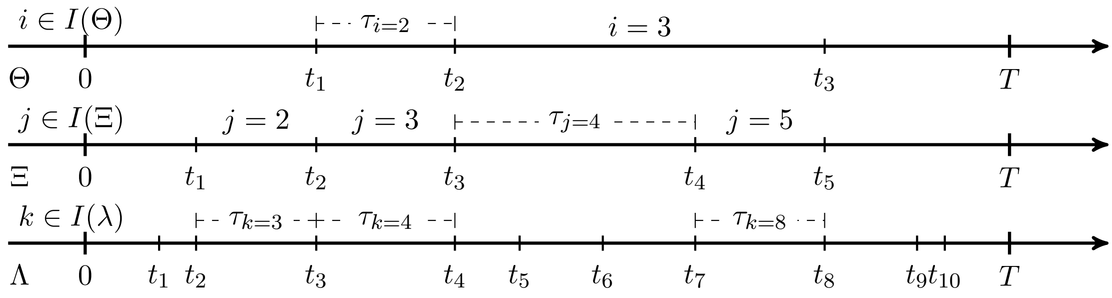


Figure 2.2.: Three interval decompotitions, s.t. $\Theta \subset \Xi \subset \Lambda$

This property later turns out to play an important role in our limit construction. This is because we have the same structure as before when we restrict to a subinterval. Even more it turns out that it is sufficient to be able to compare objects belonging to the generic interval decomposition $\{s, t\} \in \mathfrak{Z}([s, t])$ to an arbitrary interval decomposition $\Theta \in \mathfrak{Z}([s, t])$. This of course greatly simplifies the construction and is owed to the fact that the other main ingredient to our limit construction is an evolution system and hence satisfies a similar property, i.e. Markovianity, see section 2.3.

The scheme of labeling objects by labels of subintervals, as in definition 2.4, is a recurrent theme throughout this thesis. We actually index almost all objects either by an interval decomposition or by the subinterval of an interval decomposition.

2.1.2. Nets

The probably best known application of directed sets is that to a straightforward generalization of sequences to *nets* or Moore-Smith sequences.

Definition 2.5. Given a topological space $\mathcal{B}$ and a directed set $\mathfrak{Z}$ a net in $\mathcal{B}$ is a mapping $\eta : \mathfrak{Z} \to \mathcal{B}$ typically denoted as η_Θ for $\Theta \in \mathfrak{Z}$.

For nets convergence is defined in direct analogy to convergence of sequences. This is, a net η_Θ converges to η, iff for all neighbourhoods of η we can find a $\Theta \in \mathfrak{Z}$, such that η_Ξ is in the neighbourhood for all $\Theta \subset \Xi$.

For our purposes the case of a Banach space valued net is completely sufficient, and we usually use the Cauchy condition to define convergence.

Definition 2.6. For a directed set $\mathfrak{Z}$ let η_Θ for $\Theta \in \mathfrak{Z}$ be a net with values in a Banach space. η_Θ is said to be *Cauchy*, iff:

$$\text{For all } \epsilon \text{ there is a } \Theta \in \mathfrak{Z} \text{ s.t. } \|\eta_\Theta - \eta_\Xi\| \leq \epsilon \text{ for all } \Theta \subset \Xi \in \mathfrak{Z} \tag{2.2}$$

If it exists we denote the limit of a net η_Θ as $\lim_\Theta \eta_\Theta$. The definition of convergence is analogue to the definition of a Cauchy sequence.

A Banach space valued net η_Θ is called *bounded* iff there exists a $C \in \mathbb{R}$ such that $\|\eta_\Theta\| \leq C$ for all $\Theta \in \mathfrak{Z}$.

Corollary 2.7. *Every Cauchy net in a Banach space converges.*

Proof. This is a simple consequence of the completeness for sequences. One has to construct a sequence $\Theta(n) \in \mathfrak{Z}$ such that for every ϵ there is an n such that $\|\eta_\Xi - \eta_\Lambda\| \leq \epsilon$ for all $\Lambda, \Xi \supset \Theta(n)$. The Cauchy net then converges to the limit of this sequence. □

The above definition and corollary have an obvious generalization to (complete) metric spaces. However, there is, as for Cauchy sequences, no generalization for general topological spaces. The most abstract form are Cauchy nets on uniform spaces.

In general it turns out that in terms of Banach or metric spaces the differences between sequences and nets are quite trifling. For example the collection of all converging sequences in a Banach space completely determines its topology. For more general topological spaces nets still determine the topology of a space while sequences do not.

Sadly the theory of nets is a bit out of fashion in modern mathematics, so it is hard to find a good comprehensive introduction. This is mainly the case because the concept of nets is equivalent to that of filters on a topological space and the latter notion seems to be better adapted to modern topology. Filters on a space are collections of subsets and are roughly equivalent to the sets of tails of a net, i.e. sets of the type $\{\eta_\Xi \in \mathcal{B} | \Theta \subset \Xi\}$ for a fixed Θ.

2.1.3. Inductive limits

An important part of this thesis is the construction of a Hilbert space describing the possible measurements compatible with a continuous-time evolution as an inductive limit. That is to say, we construct the Hilbert space corresponding to possible continuous measurements on an open quantum system from the ones describing a discretized measurement procedure.

An inductive limit of spaces is a way of constructing a limit of a "net" of spaces without reference to a common enveloping space. To be able to construct such a limit, one needs to be able to compare different spaces in the "net". That is, one needs a family of structure preserving embeddings, i.e. homomorphisms, between the spaces.

Hence we can construct the limit of a net of spaces without making any additional assumptions on the limit space. This gives us a very flexible and general way to construct spaces. In the case of the continuous-time evolution, we are able to describe the continuous measurements completely in terms of discretized measurements.

The concept of inductive limits and the related projective limit seems to find its natural environment in category theory, since inductive limits conserve most of the structure of the finite spaces, under the condition that one uses homomorphisms to compare spaces. However one has to mention that inductive limits do not exist in every category.

Introductions to inductive limits can be found in most basic topology textbooks. Another common name for this construction is directed limit. Most books from the Bourbaki series feature the inductive limit construction for the category of objects they describe, e.g. topological spaces, vector spaces, etc. The most general construction can be found in [13, chapter III.7]. To see how this fits into the general setup of category theory see [2, chapter VIII].

Definition 2.8. Let $\mathfrak{Z}$ be a directed set and $(\mathcal{K}_\Theta)_{\Theta\in\mathfrak{Z}}$ be a family of sets. Assume for every pair $\Theta \subset \Xi \in \mathfrak{Z}$ we have an embedding: $J_{\Xi,\Theta} : \mathcal{K}_\Theta \to \mathcal{K}_\Xi$ and this family of embeddings has the property:

$$J_{\Lambda,\Xi} J_{\Xi,\Theta} = J_{\Lambda,\Theta} \text{ for } \Theta \subset \Xi \subset \Lambda \in \mathfrak{Z}. \tag{2.3}$$

Then we can define an equivalence relation on $\bigcup_{\Theta\in\mathfrak{Z}} \mathcal{K}_\Theta$ the disjoint union of all $\mathcal{K}_\Theta$ by

$$\begin{aligned} &\varphi \in \mathcal{K}_\Theta \equiv \psi \in \mathcal{K}_\Xi \\ \Leftrightarrow &\text{There exists a } \Lambda \in \mathfrak{Z} \text{ s.t. } J_{\Lambda,\Theta}\varphi. = J_{\Lambda,\Xi}\psi. \end{aligned} \tag{2.4}$$

The *inductive limit* of the family $\mathcal{K}_\Theta$ with respect to the family of embedding J is denoted $\underrightarrow{J\text{-}lim}\, \mathcal{K}_\Theta$ and defined as the disjoint union of all $\mathcal{K}_\Theta$ modulo the equivalence relation $\equiv$:

$$\underrightarrow{J\text{-}lim}\, K_\Theta := \bigcup_{\Theta\in\mathfrak{Z}} \mathcal{K}_\Theta \Big/ \equiv . \tag{2.5}$$

The canonical embedding of K_Θ into the limit is denoted as $J_\Theta : \mathcal{K}_\Theta \to \underrightarrow{J\text{-}lim}\, K_\Theta$. It fulfills the relation:

$$J_\Xi J_{\Xi,\Theta} = J_\Theta \text{ for } \Theta \subset \Xi \in \mathfrak{Z}. \tag{2.6}$$

.

Again this definition serves more as an advertisement for generality in which these concepts can be applied. We shall exclusively use inductive limits of Hilbert spaces. Observe that the inductive limit is defined entirely in terms of the "finite" spaces $\mathcal{K}_\Theta$, so if we want to compare elements in $\underrightarrow{J\text{-}lim}\, \mathcal{K}_\Theta$, we can always do that in some finite space space $\mathcal{K}_\Theta$.

However comparing does not make too much sense for elements in "raw" sets. So we would like to equip the sets with more structure. And see to it that the structure is conserved in the limit. In the end we need only the special case of Hilbert spaces.

Lemma 2.9. *In the situation of definition 2.8, let each $\mathcal{K}_\Theta$ be a Hilbert space and let each member of the family $J_{\Xi,\Theta}$ be an isometry. Then the completion of the inductive limit $\overline{\underrightarrow{J\text{-}lim}\, K_\Theta}$ is again a Hilbert space.*

Note that we need an additional completion to make the limit space into a Hilbert space. In the case of Hilbert or Banach spaces, we are always interested in the completion of the inductive limit, rather than the inductive limit itself. Hence when we talk about the inductive limit of such space, we frequently mean, in abuse of notation, its completion.

For complete spaces there is an alternative construction of the inductive limit, or rather its completion, as a space of nets. This viewpoint is interesting because it allows us a common view on all the different limits we are going to construct. Furthermore it allows some "generalization" of the inductive limit. We introduce this construction on page 15ff.

2.1.4. Projective limits

The categorically dual notion to an inductive limit the *projective* or *inverse limit* does not play an equally prominent role in this work. We shall nonetheless mention it because some parts of the limit we are going to construct naturally have this structure, namely the limit considered as a limit of states. This is an instance of the important duality between effects and states in quantum mechanics.

Furthermore including this structure in our construction is a key insight in generalizing the construction from [69] to unbounded generators.

The main difference between an inductive and an projective limit is, as it is usually the case with categorical duals, that all the arrows, i.e. maps, go into the other direction. This is instead of "including" smaller spaces into bigger ones we project down. For a rigorous definition one can look into [13].

Definition 2.10. Let $\mathfrak{Z}$ be a directed set and $(\mathcal{K}_\Theta)_{\Theta\in\mathfrak{Z}}$ be a family of sets. Assume for every pair $\Theta \subset \Xi \in \mathfrak{Z}$ we have a homomorphism: $J_{\Theta,\Xi} : \mathcal{K}_\Xi \to \mathcal{K}_\Theta$ and this family of embeddings has the property:

$$J_{\Theta,\Xi} J_{\Xi,\Lambda} = J_{\Theta,\Lambda} \text{ for } \Theta \subset \Xi \subset \Lambda \in \mathfrak{Z}. \tag{2.7}$$

The *projective limit of the family* $(\mathcal{K}_\Theta)$ *with respect to the family of mappings* $J_{\Xi,\Theta}$ is the largest subspace of the product of all the spaces $\mathcal{K}_\Theta$:

$$\prod_{\Theta\in\mathfrak{Z}} \mathcal{K}_\Theta \tag{2.8}$$

such that for the canonical projections of the product space construction $\pi_\Theta : \prod_\Theta \mathcal{K}_\Theta \to \mathcal{K}_\Theta$ one has:

$$\pi_\Theta = J_{\Theta,\Xi} \circ \pi_\Xi \text{ for } \Theta \subset \Xi \in \mathfrak{Z} \tag{2.9}$$

We denote the projective limit by $\underleftarrow{J\text{-}lim}\,\mathcal{K}_\Theta$. The canonical projections from the inductive limit to the "finite" spaces are usually called: $J_\Theta : \underleftarrow{J\text{-}lim}\,\mathcal{K}_\Theta \to \mathcal{K}_\Theta$ and fulfill the relation:

$$J_\Theta = J_{\Theta,\Xi} J_\Xi \text{ for } \Theta \subset \Xi \in \mathfrak{Z} \tag{2.10}$$

The differences between this definition and definition 2.8 are quite hard to understand. In both cases one selects from a vastly large set a subset such that elements with the same image under $J_{\Theta,\Xi}$ are identified. Also the differences are not as big as one might think after a first glance on the definition.

If one looks for example at both constructions for a finite collection of vector spaces than the inductive limit as well as the projective limit are naturally subspaces of the direct sum of all the vector spaces, i.e. the vector space spanned by the union of all their bases. Furthermore the two subset constructions are quite similar.

Many subtle differences between the two notions only start playing out when one has an infinite collection of sets and more structure on them, like a topology or some algebraic structures, see section 6.5.1. Both notions are already set up such that they still give the right construction when one imposes these structures.

2.1.5. Operators between inductive limit spaces

When constructing limit spaces as an inductive limit, it is of course a key interest to also understand the operators on this limit space through the operators on the finite descriptions. Here we shall try to do this.

We have not found any detailed study of this or similar situations in the literature. However the subject is so fundamental that it is bound to be described somewhere. In the following we examine a few of the fundamental concepts related to this, that is mostly possible topologies which naturally arise.

We are mainly interested in operators from a fixed Hilbert space $\mathcal{H}$ into an Hilbert space $\mathcal{K}$ defined as an inductive limit, i.e. $O : \mathcal{H} \to \mathcal{K}$ or operators on the limit space, i.e. $O : \mathcal{K} \to \mathcal{K}$.

Since we want to stick to the philosophy of defining limit objects entirely through finite versions, we are interested in conditions on the convergence of a family of operators of the form $O_\Theta : \mathcal{H} \to \mathcal{K}_\Theta$, or $O_{\Xi,\Theta} : \mathcal{K}_\Theta \to \mathcal{K}_\Xi$, with respect to a family of maps $J_{\Xi,\Theta}$ to an operator in or on the limit space, as in definition 2.8. Observe that these are not nets in the sense of definition 2.5 because they do not map into the same space.

Definition 2.11. Let $\mathcal{H}$ be a Hilbert space and $(K_\Theta, J_{\Xi,\Theta})$ for $\Theta \subset \Xi \in \mathfrak{Z}$ family of Hilbert spaces, as in definition 2.8, with inductive limit Hilbert space $\mathcal{K}$. Here $\mathcal{K}$ denotes the closure of the bare inductive limit.

Let $V_\Theta : \mathcal{H} \to \mathcal{K}_\Theta$ for $\Theta \in \mathfrak{Z}$ be a net of operators. Convergence is now defined through the operator $\widetilde{V}_\Theta : \mathcal{H} \to \mathcal{K}$ defined as $\widetilde{V}_\Theta = J_\Theta V_\Theta$ and convergence in sense of definition 2.6. In abuse of language, we say that the net V_Θ is

norm convergent iff the net $\widetilde{V}_\Theta$ is convergent,

strongly convergent iff all the nets $\widetilde{V}_\Theta \varphi$ for $\varphi \in \mathcal{H}$ are convergent,

weakly convertent iff all the nets $\langle \lambda, V_\Theta \varphi \rangle$ for $\varphi \in \mathcal{H}$ and $\lambda \in \mathcal{K}$ are convergent.

Let $O_\Theta : \mathcal{K}_\Theta \to \mathcal{K}_\Theta$ for $\Theta \in \mathfrak{Z}$ be a net of operators. Let $\widetilde{O}_\Theta : \mathcal{K} \to \mathcal{K}$ be defined by $\widetilde{O}_\Theta = J_\Theta O_\Theta J_\Theta^*$. In abuse of language say that the net O_Θ is:

norm convergent iff the net $\widetilde{O}_\Theta$ is convergent,

strongly convergent iff all the nets $\widetilde{O}_\Theta \varphi$ for $\varphi \in \mathcal{K}$ are convergent,

weak-$*$ convertent iff all the nets $\mathrm{tr}(\rho \widetilde{O}_\Theta)$ for ρ in $\mathfrak{T}(\mathcal{K})$ are convergent.

Both strong convergence conditions can be weakened to include unbounded operators. Let $\mathcal{D} \subset \mathcal{K}$ be a subspace. Then the net O_Θ is said to converge on $\mathcal{D}$ or to converge strongly to an unbounded operator $O : \mathcal{D} \to \mathcal{K}$ iff the net $\widetilde{O}_\Theta \psi$ converges for all $\psi \in \mathcal{D}$ and analogously for nets V_Θ.

These are straightforward definitions of convergence conditions. However they are of limited use when we want to work entirely on the level of "finite" spaces, since in all cases the convergence is defined "in the limit". We shall now take a look at proper "finite" convergence conditions.

Such convergence conditions obviously have to be of Cauchy type. The following basic property of the inductive limits helps in the study of analogues of the Cauchy nets from definition 2.6. In the notation of the above definition we have:

$$\widetilde{O}_\Xi - \widetilde{O}_\Theta = O_\Xi - J_{\Xi,\Theta} O_\Theta \tag{2.11}$$

Hence when analyzing Cauchy nets, we encounter doubly indexed nets of real numbers. A typical convergence condition for such doubly indexed nets is the following:

Definition 2.12. Let $\mathfrak{Z}$ be a directed set. For $\Theta \leq \Xi \in \mathfrak{Z}$ let $c_{\Xi,\Theta}$ be a real number. We denote

$$\lim_{\Xi \gg \Theta} c_{\Xi,\Theta} := \lim_{\Theta} \left(\limsup \{ c_{\Xi,\Theta} | \Xi \in \mathfrak{Z}, \Xi \geq \Theta \} \right). \tag{2.12}$$

The next lemma shows that this notion can be used to characterize convergence of Banach space valued nets. Actually the lemma directly generalizes to metric spaces.

Lemma 2.13. *Let $\mathfrak{Z}$ be a directed set and B_Θ for $\Theta \in \mathfrak{Z}$ a Banach space valued net. Then B_Θ is convergent iff $\lim\limits_{\Xi \gg \Theta} \|B_\Xi - B_\Theta\| = 0$.*

Proof. One direction is trivial. For the other assume that $\lim\limits_{\Xi \gg \Theta} \|B_\Xi - B_\Theta\| = 0$ hence for every ϵ there exists Θ such that $\limsup \|B_\Xi - B_\Lambda\| \leq \frac{1}{2}\epsilon$ for all $\Theta \leq \Xi \leq \Lambda$. And hence there exists a Ξ, dependent on Θ and ϵ such that $\|B_\Lambda - B_\Theta\| \leq \frac{1}{2}\epsilon$ for $\Xi \subset \Lambda$. Hence by the subadditivity of the norm for $\Xi \leq \Lambda_1 \leq \Lambda_2$ we have $\|B_{\Lambda_1} - B_{\Lambda_2}\| \leq \|B_{\Lambda_1} - B_\Theta\| + \|B_\Theta - B_{\Lambda_2}\| \leq \epsilon$ ☐

Corollary 2.14. *Let $V_\Theta : \mathcal{H} \to \mathcal{K}_\Theta$ be a net operators as in definition 2.11. The net V_Θ is:*

norm convergent *iff* $\lim\limits_{\Xi \gg \Theta} \|V_\Xi - J_{\Xi,\Theta} V_\Theta\| = 0$

strongly convergent *(on $\mathcal{D} \subset \mathcal{H}$) iff* $\lim\limits_{\Xi \gg \Theta} \|V_\Xi \psi - J_{\Xi,\Theta} V_\Theta \psi\| = 0$ *for all $\psi \in \mathcal{D}$*

weakly convergent *iff* $\lim\limits_{\Xi \gg \Theta} \langle \lambda_\Xi, (V_\Xi - J_{\Xi,\Theta} V_\Theta)\, \varphi \rangle$ *for all $\varphi \in \mathcal{H}$ and all Cauchy nets $\lambda_\Theta \in \mathcal{K}_\Theta$.*

Let $O_\Theta : \mathcal{K}_\Theta \to \mathcal{K}_\Theta$ be a net of operators as in definition 2.11. The net O_Θ is

norm convergent *iff* $\lim\limits_{\Xi \gg \Theta} \|O_\Xi - J_{\Xi,\Theta} O_\Theta J^*_{\Xi,\Theta}\| = 0$

strongly convergent *(on $\mathcal{D} \subset \mathcal{K}$) iff* $\lim\limits_{\Xi \gg \Theta} \|O_\Xi \psi - J_{\Xi,\Theta} O_\Theta J^*_{\Xi,\Theta} \psi\| = 0$ *for all $\psi \in \mathcal{D}$*

weak-$*$ convergent *iff* $\lim\limits_{\Xi \gg \Theta} \operatorname{tr}\left(\rho_\Xi \left(O_\Xi - J_{\Xi,\Theta} O_\Theta J^*_{\Xi,\Theta}\right)\right)$ *for all nets $\rho_\Theta \in \mathfrak{T}(\mathcal{K}_\Theta) = 0$ which are Cauchy w.r.t. trace norm.*

Proof. This is a direct consequence of the preceding lemma 2.13 and basic properties of the inductive limit and closure, i.e. $\|J_\Theta V_\Theta - J_\Xi V_\Xi\| = \|J_{\Xi,\Theta} V_\Theta - V_\Xi\|$ and similarly for the other expressions. It is noteworthy that the preceding expression holds for the completed inductive limit. ☐

In conclusion this gives a simple finite characterization of the operator into an inductive limit space. We mainly need norm and strong convergence for operator V into $\mathcal{K}$ and for the case of operators O on K strong convergence on a subspace, i.e. convergence to an unbounded operator or convergence of bounded nets of operators.

In the case of bounded nets of operators the convergence conditions for the strong and weak cases can be further simplified.

Corollary 2.15. *Let $V_\Theta : \mathcal{H} \to \mathcal{K}_\Theta$ be a net of operators as in definition 2.11. Furthermore let V_Θ be bounded, i.e. there is a $C \in \mathbb{R}$ such that $\|V_\Theta\| \leq C$ for all $\Theta \in \mathfrak{Z}$. The net V_Θ is*

weakly convergent *iff* $\lim\limits_{\Lambda \gg \Xi} \langle J_{\Lambda,\Theta} \lambda, (V_\Lambda - J_{\Lambda,\Xi} V_\Xi)\, \varphi \rangle = 0$ *for all $\varphi \in \mathcal{H}$, all $\Theta \in \mathfrak{Z}$ and all $\lambda_\Theta \in \mathcal{K}_\Theta$.*

Let $O_\Theta : \mathcal{K}_\Theta \to \mathcal{K}_\Theta$ be a net operators as in definition 2.11. Furthermore let O_Θ be bounded, i.e. there is a $C \in \mathbb{R}$ such that $\|O_\Theta\| \leq C$ for all $\Theta \in \mathfrak{Z}$. The net O_Θ is

strongly convergent *iff* $\lim_{\Lambda \gg \Xi} \|O_\Lambda J_{\Lambda,\Theta}\varphi_\Theta - J_{\Lambda,\Xi}O_\Xi J_{\Xi,\Theta}\varphi_\Theta\| = 0$ *for all* $\psi \in \mathcal{D}$

weak-$*$ convergent *iff* $\lim_{\Lambda \gg \Xi} \left\langle \varphi_\Theta, \left(J^*_{\Lambda,\Theta}O_\Lambda J_{\Lambda,\Theta} - J^*_{\Xi,\Theta}O_\Xi J_{\Xi,\Theta}\right)\varphi_\Theta\right\rangle = 0$ *for all* $\Theta \in \mathcal{K}$ *and* $\varphi_\Theta \in \mathcal{K}_\Theta$.

Proof. All reformulations are based on the observation that we can approximate a vector $\varphi \in \mathcal{K}$ by an vector $\varphi_\Theta \in \mathcal{K}_\Theta$. A simple application of the triangle equality and the boundedness of the net gives then the desired estimate.

$$\|(J_\Lambda O_\Lambda J^*_\Lambda - J_\Xi O_\Xi J^*_\Xi)\varphi\| \tag{2.13}$$
$$\leq 2C\|\varphi - J_\Theta\varphi_\Theta\| + \|O_\Lambda J_{\Lambda,\Theta}\varphi_\Theta - J_{\Lambda,\Xi}O_\Xi J_{\Xi,\Theta}\varphi_\Theta\| \tag{2.14}$$

And similarly for the other cases. In the case of weak-$*$ convergence we can restrict to pure states because the set of finite rank states is dense in $\mathfrak{T}(\mathcal{K})$. This is again an application of the triangle inequality. For every $\Theta \in \mathfrak{Z}$, $\rho \in \mathfrak{T}(\mathcal{K}_\Theta)$ and every ϵ there is an $n \in \mathbb{N}$ and a family of vectors $\varphi_i \in \mathcal{K}_\Theta$ for $1 \leq i \leq n$ such that ρ and $\sum_{i=1}^n |\varphi_i\rangle\langle\varphi_i|$ are ϵ close in trace norm. We then get:

$$\lim_{\Lambda \gg \Xi} \operatorname{tr}\left(\rho_\Theta\left(J^*_{\Lambda,\Theta}O_\Lambda J_{\Lambda,\Theta} - J^*_{\Xi,\Theta}O_\Xi J_{\Xi,\Theta}\right)\right) \tag{2.15}$$
$$\leq 2C \operatorname{tr}\left|\rho - \sum_{i=1}^n |\varphi_i\rangle\langle\varphi_i|\right|$$
$$+ \lim_{\Lambda \gg \Xi} \sum_{i=1}^n \left\langle \varphi_i, \left(J^*_{\Lambda,\Theta}O_\Lambda J_{\Lambda,\Theta} - J^*_{\Xi,\Theta}O_\Xi J_{\Xi,\Theta}\right)\varphi_i\right\rangle \leq 2C\epsilon. \tag{2.16}$$

□

Later we work mainly with nets of bounded operators. So this is the most important set of conditions for us.

2.1.6. Generalized inductive limits

There is an alternative construction for the inductive limit spaces, which is interesting because it is more in the spirit of the rest of this section, i.e. in terms of nets, and more adapted to the setting of Banach spaces. Furthermore it is possible to generalize this alternative definition of the inductive limit. The following way to define an inductive limit is similar to the spirit of [10, 9].

Lemma 2.16. *Let $\mathcal{B}_\Theta$, $\Theta \in \mathfrak{Z}$ be a family of Banach spaces indexed by an directed set $\mathfrak{Z}$, and let $J_{\Xi,\Theta}$ be a two-parameter family of isometries as in definition 2.8. Let B_Θ, $\Theta \in \mathfrak{Z}$ be a net such that $B_\Theta \in \mathcal{B}_\Theta$. In abuse of notation we say that the net B_Θ is* Cauchy *or* convergent *iff* $\lim_{\Xi \gg \Theta} \|J_{\Xi,\Theta}B_\Theta - B_\Xi\| = 0$. *A net is said to be a* null net *or* null sequence *iff* $\lim_\Theta \|B_\Theta\| = 0$. *Then the completion of the inductive limit* $\underrightarrow{J\text{-lim}}_\Theta B_\Theta$ *is equal to the space of Cauchy nets modulo the null nets. Addition and other operations can be defined element-wise. Given two nets $(A_\Theta)_{\Theta\in\mathfrak{Z}}$ and $(B_\Theta)_{\Theta\in\mathfrak{Z}}$ we define: $(A_\Theta)_{\Theta\in\mathfrak{Z}} + (B_\Theta)_{\Theta\in\mathfrak{Z}} = (A_\Theta + B_\Theta)_{\Theta\in\mathfrak{Z}}$. The norm is defined through the limit* $\|(B_\Theta)_{\Theta\in\mathfrak{Z}}\| := \lim_\Theta \|B_\Theta\|$

Proof. Let $\mathcal{A}$ denote the "Cauchy net version" of inductive limit as described above and let $\mathcal{B}$ be the uncompleted direct limit space $\mathcal{B} := \underrightarrow{J\text{-}lim}_{\Theta} \mathcal{B}_\Theta$ as defined in definition 2.8 and $\overline{\mathcal{B}}$ its closure. The equality of $\mathcal{A}$ and $\overline{\mathcal{B}}$, as Banach spaces, can be seen when we express completion in terms of nets and not of sequences.

In one direction for every Cauchy net, we can find a Cauchy sequence converging to the same limit, as shown in corollary 2.7. Note that by construction for every $B \in \mathcal{B}$ there is a $B_\Theta \in \mathcal{B}_\Theta$ such that $B = J_\Theta B_\Theta$. If we map $B \in \mathcal{B}_\Theta \in \mathcal{B}$ to the net $B_\Xi := J_\Xi B_\Theta$ and 0 else, hence we get an isometric embedding of $\overline{\mathcal{B}} \to \mathcal{A}$.

Conversely every Cauchy sequence in $\mathcal{B}$ gives rise to a Cauchy net $B_\Theta \in \mathcal{B}_\Theta$ for $\Theta \in \mathfrak{Z}$ in the above sense. To see this start to organize the Cauchy sequence B_i in $\mathcal{B}$, s.t. $B_i = J_{\Theta_i} B_{\Theta_i}$ and $\Theta_i \subset \Theta_j$ if $i \leq j$, which can always be done. Now we extend the sequence to a net over $\mathfrak{Z}$ by setting $B_\Xi := J_{\Xi,\Theta_i} B_{\Theta_i}$ iff $\Xi \supset \Theta_i$ and $\Xi \not\supset \Theta_{i+1}$ or 0 if iff such a condition is not satisfied.

This construction defines an isometric embedding $\mathcal{A} \to \overline{\mathcal{B}}$, which shows equality. $\square$

The above lemma easily extends to the case of inductive limits of Hilbert spaces, i.e. the Hilbert space structure is conserved. The above lemma gives a common interpretation, as Cauchy nets, to all the limit objects we define in this thesis.

Building on the preceding lemma we can now give a "generalized" definition of inductive limits for Banach spaces.

Definition 2.17. Let $\mathcal{B}_\Theta$ for $\Theta \in \mathfrak{Z}$ be a family of Banach spaces indexed by a directed set $\mathfrak{Z}$. Let $\Theta \subset \Xi \subset \Lambda$. We call a two-parameter family of maps $J_{\Xi,\Theta} : \mathcal{B}_\Theta \to \mathcal{B}_\Xi$ asymptotically inductive in:

norm topology iff $\lim_{\Lambda \gg \Xi} \|J_{\Lambda,\Xi} J_{\Xi,\Theta} - J_{\Lambda,\Theta}\| = 0$.

strong topology iff $\lim_{\Lambda \gg \Xi} \|J_{\Lambda,\Xi} J_{\Xi,\Theta}\psi - J_{\Lambda,\Theta}\psi\| = 0$.

In distinction a family such that $J_{\Lambda,\Xi} J_{\Xi,\Theta} = J_{\Lambda,\Theta}$ as in definition 2.8 is called *strictly inductive*. We call the two-parameter family $J_{\Xi,\Theta}$ *asymptotically isometric* iff $\lim_{\Xi} \|J_{\Xi,\Theta}\| = 1$ for all Θ. We define the generalized inductive limit w.r.t. a asymptotically isometric and either norm or strongly asymptotically inductive limit as in lemma 2.16, i.e. as the set of Cauchy nets w.r.t the asymptotically inductive family modulo the null nets.

This general inductive limit construction was used in application to mean field theory in [77], see [36] for an introduction.

The theory of inductive limits of Hilbert spaces and operators between them can be extended to this setting and many of the lemmas from the preceding section still hold.

2.2. Evolution of open quantum systems

We now want to take a closer look at evolutions of general quantum systems. From our point of view those are always described by a quantum channel. That is a completely-positive and unital or trace preserving map between two operator algebras, the algebras of observables or states. If the maps are restricted to be unital or trace preserving depends on the question whether we want to describe them in Heisenberg or Schrödinger picture.

It is well known that the Schrödinger picture describes the evolution of states, while the Heisenberg picture describes the evolution of observables. We shall always describe states in terms of the trace class operators on a Hilbert space $\mathcal{H}$ denoted as $\mathfrak{T}(\mathcal{H})$ and the corresponding observable by the bounded operators on the space Hilbert space, denoted as $\mathfrak{B}(\mathcal{H})$. We prefer the Heisenberg picture and hence unital evolutions.

The requirement of complete positivity is linked to the statistical interpretation of quantum mechanics. It corresponds to the assumption that we can always enlarge our state space to include another quantum system which does not interact at all with the first system. When one requires that the class of channels is closed under this operation, one arrives at the notion of complete positivity. We only consider completely-positive quantum evolutions.

Furthermore we always restrict the observable algebras to be full matrix algebras, i.e. the algebra of bounded operators on a given Hilbert space $\mathcal{H}$. The algebras are usually denoted $\mathcal{B} = \mathfrak{B}(\mathcal{H})$ or $\mathcal{A} = \mathfrak{B}(\mathcal{K})$. Using the full algebra of bounded operators has the advantage that one can describe the evolution in terms of pure states, i.e. vectors in the Hilbert space $\mathcal{H}$.

Occasionally it is helpful to think about the case of classical stochastic evolutions, which can be embedded in the quantum framework by restricting to diagonal trace-class matrices as the class of states and a commutative, i.e. diagonal, algebra of observables.

Of course one does not have to impose the condition of complete-positivity, e.g. because one does not believe in the "extension hypothesis". Another reason to deviate from complete-positivity could be that not all "reduced" quantum evolutions are completely-positive. That is, if we look at a unitary quantum evolution and trace out a subsystem, we do not arrive at a completely-positive map. This point of view is described, e.g. in [75, 35]

The problem with this critique is that it only works in cases where it is hard to describe the "subsystem" as a subsystem. One of the things we show in the following section is that in all cases where we can embed the subsystem into the bigger system as an operator algebra, i.e. through a representation, the restricted evolution is completely-positive.

So our response to a departure from complete-positivity would not be that the resulting evolution is not quantum, or that the resulting evolution is non physical or uninteresting, but that the considered "subsystem" is not a "system", because one can not prepare such "systems" independent of the environment anyway.

2.2.1. The Stinespring dilation

The Stinespring dilation theorem may look a bit abstract at the first sight, but it is a very versatile and important theorem in quantum theory and it has a surprisingly clear physical interpretation: It fully characterizes how an open quantum system interacts with the environment. The Stinespring dilation is actually the strongest result in a whole family of similar theorems, which are all based on the construction of a positive definite kernel [43]. In terms of diagrams the Stinespring dilation implements the operation depicted in figure 2.3.

Our presentation of the Stinespring theorem is based on unpublished expositions by my supervisor Reinhard F. Werner. The Stinespring dilation can be used to characterize all quantum channels, i.e. completely-positive unital maps between operator algebras. As we now see the Stinespring theorem comes in four parts:

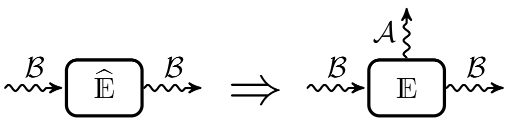


Figure 2.3.: Stinespring dilation: Describes measurements compatible with a given timestep

Firstly, the existence of the Stinespring dilation. Secondly, a minimality clause for minimal Stinespring dilations. Thirdly, a comparison clause showing under which circumstances Stinespring dilations of different maps can be compared. And last, a continuity clause showing that similar channels have similar Stinespring dilations.

The first three properties are generic to the kernel approach from [43] on which most modern proofs of Stinespring dilation [80] are based. The continuity property is not as widely known as the rest of the theorem and can be found in [63, 62]. However, the continuity clauses for some of the related theorems are common knowledge. In the case of the GNS-theorem, which dilates general states to pure states on a bigger Hilbert space, this continuity clause goes back to Bures [16].

Theorem 2.18. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $\mathbb{E}, \mathbb{F} : \mathfrak{B}(\mathcal{H}_2) \to \mathfrak{B}(\mathcal{H}_1)$ be completely-positive normal unital maps. Then there exists a separable Hilbert space $\mathcal{K}$, called the* dilation space, *with identity operator $\mathbb{1}_K : \mathcal{K} \to \mathcal{K}$, and an operator of the form $V : \mathcal{H}_1 \to \mathcal{K} \otimes \mathcal{H}_2$ such that:*

1. *For all $B \in \mathfrak{B}(\mathcal{H}_2)$ we have $\mathbb{E}(B) = V^* \mathbb{1}_{\mathcal{K}} \otimes BV$. And due to the unitality of $\mathbb{E}$ the operator V is an isometry, called* Stinespring isometrie

2. *V and $\mathcal{K}$ can be chosen* minimal. *In this case:*
$$\mathcal{K} \otimes \mathcal{H}_2 = \overline{\operatorname{span}} \left\{ \mathbb{1}_{\mathcal{K}} \otimes XV\varphi | \varphi \in \mathcal{H}_1, X \in \mathfrak{B}(\mathcal{H}_2) \right\}, \tag{2.17}$$
and for any non minimal dilation $\widetilde{V} : \mathcal{H}_1 \to \widetilde{\mathcal{K}} \otimes \mathcal{H}_2$ there exists a unique isometrie $W : \mathcal{K} \to \widetilde{\mathcal{K}}$, such that $\widetilde{V}\varphi = W \otimes \mathbb{1}_{\mathcal{H}_1} V\varphi$ for all $\varphi \in \mathcal{H}_1$.

3. *If $\mathbb{E} - \mathbb{F}$ is completely-positive then there exists a unique positive operator $F \in \mathfrak{B}(\mathcal{K})$ with $0 \leq F \leq \mathbb{1}$, such that $\mathbb{F}(B) = V^* F \otimes BV$. This property is called the Radon-Nikodym property of the Stinespring dilation.*

4. *If the two completely-positive maps are close in complete boundedness norm, i.e. $\|\mathbb{E} - \mathbb{F}\|_{\text{c.b.}} \leq \epsilon$, then there is a separable Hilbert space $\mathcal{K}$ and Stinespring isometries $V_{\mathbb{F}}, V_{\mathbb{E}} : \mathcal{H}_1 \to \mathcal{K} \otimes \mathcal{H}_2$ such that:*
$$\mathbb{E}(B) = V_{\mathbb{E}}^* \mathbb{1}_{\mathcal{K}} \otimes BV_{\mathbb{E}} \qquad \mathbb{F}(B) = V_{\mathbb{F}}^* \mathbb{1}_{\mathcal{K}} \otimes BV_{\mathbb{F}} \tag{2.18}$$
and:
$$\|\mathbb{E} - \mathbb{F}\|_{\text{c.b.}} \leq \|V_{\mathbb{E}} - V_F\| \leq \sqrt{\|\mathbb{E} - \mathbb{F}\|_{\text{c.b.}}}. \tag{2.19}$$

.

The Stinespring dilation theorem can be stated in substantially higher generality. Already the original version [80] is for maps from C^*-algebras into the bounded operators on a Hilbert space, which do not have to be normal. In general the dilation space does not have the structure $\mathcal{K} \otimes \mathcal{H}$ and the map $B \to \mathbb{1}_{\mathcal{K}} \otimes B$ is replaced by a representation of $\mathfrak{B}(\mathcal{H}_2)$.

There is an important alternative form of the Stinespring theorem. Instead of defining the Stinespring dilation as an isometry $V : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$, one can define the dilation in terms of a family of operators on $\mathcal{H}$, the so called Kraus operators. This basically corresponds to introducing a basis on the dilation space $\mathcal{K}$, as one sees in the finite dimensional case where we can write $\mathbb{1} = \sum_\alpha |\alpha\rangle\langle\alpha|$ for α labeling an orthonormal basis of $\mathcal{K}$.

Corollary 2.19. *Under the assumptions from theorem 2.18 there exists a family of operators $V_\alpha : \mathcal{H}_1 \to \mathcal{H}_2$ for a countable set $\alpha \in \mathbb{A}$, such that we can write:*

$$\mathbb{E}(X) = \sum_{\alpha \in \mathbb{A}} V_\alpha^* X V_\alpha. \tag{2.20}$$

The sum converges in weak-$*$ topology.

A proof of this result can for example be found in [42, theorem 4.16]. The connection between these two representations is the following. Let $|\alpha\rangle$ for $\alpha \in \mathbb{A}$ be a basis for an Hilbert space $\mathcal{K}$. Then one gets a Stinespring isometry V from the set of Kraus operators V_α by defining:

$$V : \mathcal{H}_1 \to \mathcal{K} \otimes \mathcal{H}_2 \qquad \varphi \to \sum_{\alpha \in \mathbb{A}} |\alpha\rangle \otimes V_\alpha \varphi. \tag{2.21}$$

We regularly use both forms of the Stinespring dilation theorem and freely switch between these two viewpoints.

One should mention that completely-positive maps are by definition always bounded. Indeed to calculate the norm of a completely-positive map $\mathbb{E} : \mathcal{B}_2 \to \mathcal{B}_1$ one only has to evaluate the norm of $\mathbb{E}(\mathbb{1})$. Moreover for a completely-positive map $\mathbb{E}$ with Stinespring isometry V:

$$\|\mathbb{E}\|_{\text{c.b.}} = \|\mathbb{E}\| = \|\mathbb{E}(\mathbb{1})\| = \|V\|^2 = \left\| \sum_{\alpha \in \mathbb{A}} V_\alpha^* V_\alpha \right\|. \tag{2.22}$$

The nontrivial part of these series of equalities can be found in [73, proposition 3.6].

The Stinespring or Kraus representation of a completely-positive map can be used to partly extend the concept of completely-positive maps to unbounded operators. Occasionally one, in abuse of notation, still calls a map $\widetilde{\mathbb{E}}(X) \to \sum_\alpha V_\alpha^* X V_\alpha$ completely-positive, if all the maps V_α are unbounded operators.

But one has to be careful when talking about this kind of completely-positive map $\widetilde{\mathbb{E}}$, because it is not clear if they extend to an unbounded operator $\widetilde{\mathbb{E}} : \mathrm{dom}(\widetilde{\mathbb{E}}) \subset \mathfrak{B}(\mathcal{H}_2) \to \mathfrak{B}(\mathcal{H}_1)$. If one wants to define such a map rigorously one can do so on the trace-class operators $\mathfrak{T}(\mathcal{H})$ the pre-dual of $\mathfrak{B}(\mathcal{H})$, at least if the domains V_α share a common core. The construction is:

$$\widetilde{\mathbb{E}}_* : \mathrm{dom}(\widetilde{\mathbb{E}}_*) \subset \mathfrak{T}(\mathcal{H}_2) \to \mathfrak{T}(\mathcal{H}_1) \qquad |\psi\rangle\langle\psi| \to \sum_\alpha |V_\alpha \psi\rangle\langle V_\alpha \psi|, \tag{2.23}$$

for vectors ψ in a common subspace of all the domains of the Kraus operators V_α. The above definition might extend to a completely-positive map on a subalgebra of the trace-class operators, or can be interpreted as a completely-positive map between operator spaces. We take a closer look at this construction in section 4.1.4.

We meet a few examples of such "completely-positive" maps. The most prominent example is the "completely-positive" term in an unbounded Lindblad generator (3.11). In some cases it feels more natural to define results in terms of these "generalized" completely-positive terms.

Reminder 1 (Duality in Banach spaces). For any two topological vector spaces $\mathcal{B}$ we can define the continuous dual $\mathcal{B}' = \mathfrak{B}(\mathcal{B}, \mathbb{C})$, as the set of all continuous linear functions from the vector space into the complex numbers. All these functions are bounded in the sense that they map bounded sets to bounded sets [79, theorem 1.32].

On this dual space, we encounter two different topologies. The first one is the weak-$*$ topology, i.e. the coarsest topology making all the functionals $\Psi_B : \mathcal{B}' \to \mathbb{C}$ for $B \in \mathcal{B}$ which are defined by $\Psi_B(\omega) = \omega(\mathcal{B})$ continuous. This topology is important because it has nice compactness properties by the Banach-Alaoglu theorem. Further information can be found in [79, chapter 3].

If $\mathcal{B}$ is a Banach space there is another important topology on $\mathcal{B}'$. This topology is given by a norm and actually a special case of a norm for bounded linear operators between two Banach spaces. An extensive treatment can be found in [79, chapter 4]. Let $\mathcal{B}$ and $\mathcal{A}$ be Banach spaces and $\mathbb{E} : \mathcal{B} \to \mathcal{A}$ be a bounded linear map between $\mathcal{A}$ and $\mathcal{B}$. Then we can define a norm for $\mathbb{E}$ by:

$$\|\mathbb{E}\| := \sup_{B \in \mathcal{B}} \|\mathbb{E}(\mathcal{B})\|. \tag{2.24}$$

It turns out that this norm renders the space of all bounded linear maps between $\mathcal{A}$ and $\mathcal{B}$ into a Banach space. In the case of Banach spaces, we always equip $\mathcal{B}'$ with this norm and write $\mathcal{B}^*$ for this normed dual.

Another important concept in this direction is the notion of a pre-dual space. The pre-dual of a Banach space $\mathcal{B}$ is the Banach space $\mathcal{B}_*$ such that $(B_*)^* = \mathcal{B}$. In contrary to the dual the pre-dual does not always exist.

For quantum mechanics we have the important duality between traceclass operators and bounded operators. Let $\mathcal{H}$ be a Hilbert space then the normed dual of the traceclass operators on $\mathcal{H}$ is the set of bounded operators on $\mathcal{H}$, i.e. $\mathfrak{T}(\mathcal{H})^* = \mathfrak{B}(\mathcal{H})$. The relationship between $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ is given by identifying for $\rho \in \mathfrak{T}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{H})$ the operator B with the functional $\operatorname{tr}(\rho B)$. An nice introduction into this duality can be found in [78, chapter 6.6].

2.2.2. Interpretation of the Stinespring dilation

The usual interpretation of the Stinespring dilation theorem goes via a slightly different statement of the theorem, i.e. we extend the Stinespring isometry $V : \mathcal{H}_1 \to \mathcal{K} \otimes \mathcal{H}_2$ to an unitary $U : \widetilde{\mathcal{K}} \otimes \mathcal{H}_1 \to \mathcal{K} \otimes \mathcal{H}_2$ for another Hilbert space $\widetilde{\mathcal{K}}$. Then we can interpret every channel as consisting of three parts: Adding of an ancilla system $\widetilde{\mathcal{K}}$ in a pure state, followed by a unitary evolution U and the discarding of another ancilla system $\mathcal{K}$.

However, we won't try to further justify this interpretation, but head for a "minimal" interpretation. Our chosen route to the interpretation of the Stinespring theorem goes via the often "overlooked" Radon-Nikodym property, i.e. part 3 of the theorem, it can be found, e.g. in [86].

The basic question in an interpretation of the Stinespring dilation is: How do we interpret the space $\mathcal{K}$. Or more precisely how do we justify to interpret it as an environment. The key to this is the Radon-Nikodym property. It is easy to see that this property gives us an one-to-one correspondence between *resolutions of the identity* on $\mathcal{K}$, i.e. families of positive operators $F_i \in \mathfrak{B}(\mathcal{K})$, such that $\sum_i F_i = \mathbb{1}$, and decompositions of the evolution $\mathbb{E}$ into completely-positive maps $\mathbb{F}_i : \mathfrak{CP}(\mathfrak{B}(\mathcal{H}_2), \mathfrak{B}(\mathcal{H}_1)$ such that $\mathbb{E} = \sum_i \mathbb{F}_i$.

This is a remarkable relation, because in the standard interpretation of quantum mechanics positive operators correspond to effects, i.e. yes-no measurements on the system. Consequently, on the one hand resolutions of the identity on $\mathcal{K}$ describe measurements on $\mathcal{K}$. This concept is known under the name positive-operator-valued measure, which we introduce in definition 2.20.

On the other hand a decomposition of an evolution $\mathbb{E}$ into completely-positive maps similarly corresponds to repeatable measurements. That is an observation where we do not destroy the system upon measurement, but instead keep the quantum system and gain additional classical information about it. In other words such a decomposition is an evolution, which is conditioned on a classical measurement result. The map $\mathbb{E}$ in this case describes the average evolution, i.e. the case where we ignore the measurement result.

In summary the Stinespring dilation of a map $\mathbb{E}$ describes all measurements on a quantum system $\mathcal{K}$, which are compatible with the evolution $\mathbb{E}$. Here compatible means, that when we ignore the measurement results, we get back the original evolution. The system $\mathcal{K}$ thus describes everything we can learn about the evolution without introducing additional disturbance, i.e. the quantum information the system emits to its environment during the time-step $\mathbb{E}$.

Due to the above analysis the Stinespring dilation is naturally interpreted as a *delayed choice* measurement. That is in a time-step we obtain a quantum sytem describing the measurement, which we could store and put in a shelf, and can later decide which property we want to measure on the system.

This picture of the Stinespring dilation can also be nicely understood on the basis of two important examples: the Heisenberg microscope and cavity quantum electro dynamics. We discuss these examples in the two following subsections.

But before we come to that, let us formalize the above discussion. First we need a few definitions:

A *positive operator valued measure*, or short *POVM*, is the most general description of a destructive measurement and a core notion in quantum information theory.

Definition 2.20. Let (Ω, σ) be a measure space with σ-algebra $\mathcal{F}$ and $\mathcal{K}$ a Hilbert space. A *POVM* is a map $F : \mathcal{F} \to \mathfrak{B}(\mathcal{K})$ that satisfies the following requirements:

1. $F(\sigma)$ positive for all $\sigma \in \mathcal{F}$, $F(\emptyset) = 0$
2. The map is σ-additive in the weak-$*$ topology. That is, for a countable collection of mutually disjoint sets σ_i, for $i \in \mathbb{N}$, we have:

$$\operatorname{tr}\left(\rho \sum_i F\left(\sigma_i\right)\right) = \operatorname{tr}\left(\rho F\left(\bigcup \sigma_i\right)\right), \tag{2.25}$$

   for every $\rho \in \mathfrak{T}(\mathcal{K})$

3. F is normalized, i.e. $F(\Omega) = \mathbb{1}_{\mathcal{K}}$.

An *instrument* is the generalization of POVMS to completely-positive-map valued measures. It is a central notion in the theory of repeated or continuous measurements. The notion of an instrument goes back to Davies [27].

Definition 2.21. Let (Ω, σ) be a measure space with σ-algebra $\mathcal{F}$, and $\mathcal{H}_1$, $\mathcal{H}_2$ Hilbert spaces. An *instrument* in Heisenberg picture is a map from $\mathbb{F} : \sigma \to \mathfrak{CP}(\mathfrak{B}(\mathcal{H}_2), \mathfrak{B}(\mathcal{H}_1))$ such that:

1. $\mathbb{F}(\sigma)$ is completely-positive, $\mathbb{F}(\emptyset) = 0$

2. The map is σ-additive in the weak-$*$ topology. That is, for a countable collection of mutually disjoint sets σ_i, with $i \in \mathbb{N}$, we have:

$$\operatorname{tr}\left(\rho \sum_i \mathbb{F}(\sigma_i)\right) = \operatorname{tr}\left(\rho \mathbb{F}\left(\bigcup \sigma_i\right)\right) \tag{2.26}$$

3. $\mathbb{F}(\Omega)$ is a channel, i.e. $\mathbb{F}(\Omega)(\mathbb{1}) = \mathbb{1}$

Analogous one can define an instrument in Schrödinger picture by looking at "measures" with values in $\mathfrak{CP}(\mathfrak{T}(\mathcal{H}_1), \mathfrak{T}(\mathcal{H}_2))$ which are σ-additive in strong topology.

The correspondence between measurements on $\mathcal{K}$ and decompositions of $\mathbb{E}$ then has the following form:

Corollary 2.22. *Let $\mathbb{E} : \mathfrak{B}(\mathcal{H}_2) \to \mathfrak{B}(\mathcal{H}_1)$ be a completely-positive map with $V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathcal{K}$ a dilating Stinespring isometry and $(\Omega, \mathcal{F})$ a measure space. Then there is a one-to-one correspondence between POVM $F : \Omega \to \mathfrak{B}(\mathcal{K})$ and instruments $\mathbb{F} : \omega \to \mathfrak{CP}(\mathfrak{B}(\mathcal{H}_2), \mathfrak{B}(\mathcal{H}_1))$, such that $\mathbb{F}(\Omega) = \mathbb{E}$*

Proof. One direction is given by associating to the POVM F the instrument defined by $B \to V^* F(\sigma) \otimes BV$ and the other direction is the comparison clause from Stinespring theorem, i.e point number 3, associating to each $\mathbb{F}(\sigma)$ a unique operator $F(\sigma)$. The sigma additivity directly passes over and all other properties are obvious. □

2.2.3. The Heisenberg microscope

A very instructive example in understanding the previous discussion of the Stinespring dilation theorem is the Heisenberg microscope. Heisenberg used this example of a microscope in his famous uncertainty paper [52] to discuss some simple classical motivation for the uncertainty principle. The idea is that one tries to "measure" an electron with the help of a microscope.

The setup is as follows and depicted in figure 2.4. We want to measure properties of an electron through the photons it scatters. Hence, the incident light comes from the side and the microscope is set up to detect the photons scattered by the electron. There are, however, at least two quite distinctive ways to do this. We could put the electron in the image plane of the microscope, such that we get a good picture of the electron and hence measure its position.

Or, alternatively we can put the electron in the focal plane of the microscope, such that all light scattered in the same direction by the electron is focused to a point. Hence,

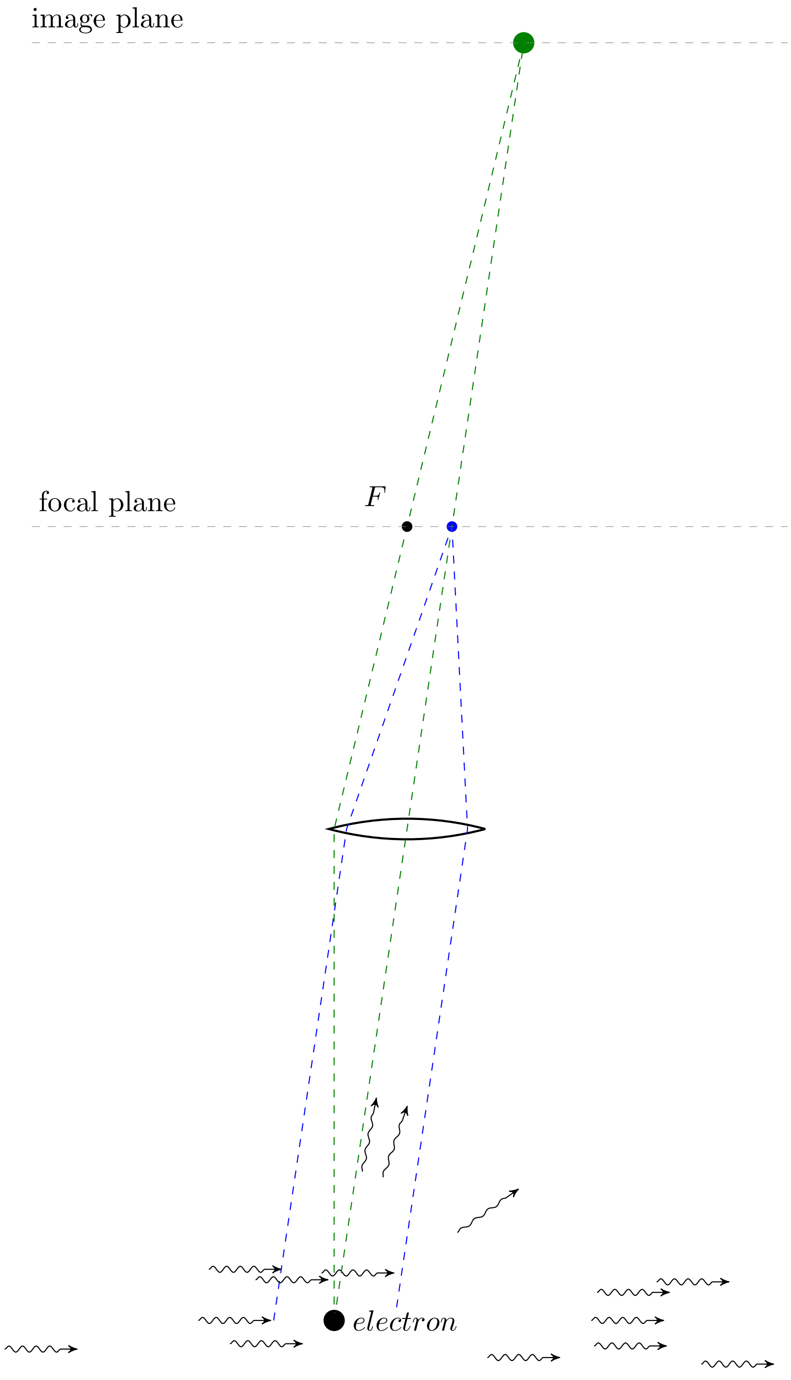


Figure 2.4.: Heisenberg microscope: Depending on our detector setup we can measure incompatible observables through the same interaction. If we measure in the image plane we get information about the position of the electron and if we measure in the focal plane we get information on the momentum.

via the Compton effect we can learn something about the momentum kicks the electron receives and ultimately about the momentum of the electron.

Heisenberg uses this discussion to hint at the problem that the resolution of the position measurement is limited by the wavelength of the light, while on the other hand light of a shorter wavelength means a stronger disturbance of the momentum of the electron.

From our point of view the central message is that through the same measurement interaction we can actually measure quite different observables. Since we know that the disturbed evolution of the electron in the light field $\mathbb{E}$ already characterizes all measurements compatible with it through the Stinespring dilation. We can conclude, that the dilation space $\mathcal{K}$ must contain a minimal description of the scattered light field.

Let us remark that the measurement scheme described above is inherently continuous in time, while on the other hand the description via Stinespring dilation is manifestly discrete, i.e. we can only ever dilate a whole time-step. To adapt the Stinespring dilation theorem to such a continuous setup is one of the main goals of this thesis. The construction can be found in section 7.

2.2.4. Cavity quantum electrodynamics

Apart from the Heisenberg microscope, the other example of a quantum system, which accompanies us throughout this thesis, and guides our intuition, is cavity quantum electodymanics, or short cavity QED. One of the merits of this system is that it has a clear separation between system, environment and measurement mediating system.

The example we talk about is that of a cavity, i.e. an arrangement of two mirrors, s.t. the light field between the mirrors has a resonant frequency. One of the mirrors is slightly transparent such that we can couple light into the cavity, i.e. driving laser, and out of the cavity. To make the setup more interesting we put a small quantum system, like an atom, in the cavity. We occasionally call this system the memory of the cavity.

This setup has a nice separation into three parts: Firstly an open quantum system, this is the cavity and atom system. Secondly a classically controlled environment, i.e. the driving laser field. And thirdly an output channel which we can use to learn about the dynamics inside the cavity without disturbing them, i.e. the output light. It is this output of the cavity which we are most interested in. Especially the relations between the input laser field and the output laser field.

Such cavity systems are a nice example of a quantum Markov system. A class of systems we introduce in section 2.3. Systems of this type are usually described by cavity input output theory. One should, however, mention that in this context the main purpose of the cavity from the viewpoint of a theoretician is to visually separate the system from the environment. That is, cavity input output theory can be and is applied to wide range of models, many of which do not involve cavities.

2.2.5. Beams and rates

We want to use this opportunity to point out a discrepancy between the usual description of quantum measurement in text books and many experiments as they are implemented in laboratories. These ideas are drawn from [86], and where partially fleshed out in [61]. This thesis can be seen as a part of the project layed out in [86].

The discrepancy is the following: text books usually focus on an ideal destructive von Neumann measurement, a scenario where we destroy a quantum system and gain some classical output. Sometimes the state of the quantum system after the measurement is included and one arrives at a description similar to an instrument.

However many quantum measurements in laboratories, especially those including photons are conducted in a beam type scenario, see [61]. That is, one sends a beam of particles through some apparatus consisting of filters and other transformations and then on a detector. In this setup it is, contrary to the classical description of measurement, not clear when the detector reacts or even how many particles it detects.

Most quantum optic experiments are of this type. But even many classic examples, like the Stern-Gerlach experiment or the double slit experiment fit well into this scheme.

The point is that when one is confronted with a beam of quantum systems, he is often not only interested in measuring one particle observables. The result of such an experiment is not a single position, momentum or spin direction, but rather a collection of single particle measurement outcomes together with the time and space coordinates of their occurrence. Hence, it is natural to ask for arrival times of particles, distributions of wait times or other statistics of clicks either over time or in space.

The natural description for such experiments is not a single probability distribution, but rather something like a *point process*. For the theory of point processes we recommend [25, 24]. Let us start with a physicists definition of a point process. The result of a single run of an experiment is a distribution of points in an outcome space, usually this is a time coordinate and the result of a one particle measurement. We don't know which results we get, when we get them and how many there are.

A way to formalize this is to model a point process as a random sum of Dirac measures, i.e. point measures of mass one. Each Dirac measure represents a single "click" event. And hence, if we want to know how many events there are in a given time interval in a single run we integrate over the corresponding region. If we want statistical averages we additionally integrate over the random variable determining the measure.

Definition 2.23. Let $\mathcal{X}$ be a complete separable metric space with Borel σ-algebra $\mathcal{E}$.

A *boundedly-finite* measure on $(\mathcal{X}, \mathcal{E})$ is a measure ν with the property that: $\mu(\epsilon)$ is finite for all bounded $\epsilon \in \mathcal{E}$.

A *counting measure* is a boundedly-finite measure such that $\mu(\epsilon) \in \mathbb{N}$ for all $\epsilon \in \mathcal{E}$.

A *random measure* on the state space $\mathcal{X}$ is a measurable mapping from a probability space $(\Omega, \mathcal{F}, \mu)$ into the boundedly-finite measures ν on $(\mathcal{X}, \mathcal{E})$. Here we equip the space of measures on $\mathcal{X}$ with the initial topology given by the evaluation maps $\pi_\epsilon(\nu) = \nu(\epsilon)$ for $\epsilon \in \mathcal{E}$ and the corresponding Borel σ-algebra.

A *point process* is a counting measure valued random measure.

It turns out that every counting measure can be written as a sum of Dirac measures, this is a consequence of the fact that boundedly-finite, or more generally σ-finite, measures can be written as a weighted sum of Dirac measures and a diffuse measure, i.e. a measure being zero on all sets with only one element. More abstractly, iff ν is a counting measure on $\mathcal{X}$ then there is a family of points $x_i \in \mathcal{X}$ and associated Dirac measures δ_{x_i} such that:

$$\nu = \sum_i \delta_{x_i}. \tag{2.27}$$

A *Dirac measure* is a measure δ_x for $x \in \mathcal{X}$ on $(\mathcal{X}, \mathcal{E})$, such that $\delta_x(\epsilon) = 1$ for $\epsilon \in \mathcal{E}$ iff $x \in \epsilon$ and 0 else.

An alternative definition of random measures and hence point processes goes via the notion of kernels. This is interesting mainly because kernels also play a major role in the study of general stochastic processes and Markov processes. This characterization can, e.g. be found in [60, lemma 1.37]

Lemma 2.24. *Let $(\Omega, \mathcal{F})$ and $(\mathcal{X}, \mathcal{E})$ be two measure spaces as in definition 2.23. A map $k : \Omega \times \mathcal{E} \to [0, \infty]$ such that for every $\epsilon \in \mathcal{E}$, the following function is measurable*

$$k_\epsilon : \Omega \to [0, \infty] \qquad\qquad \omega \mapsto k(\omega, \epsilon), \tag{2.28}$$

and furthermore that for every $\omega \in \Omega$ the function:

$$k_\omega : \mathcal{E} \to [0, \infty] \qquad\qquad \epsilon \mapsto k(\omega, \epsilon) \tag{2.29}$$

is a boundedly-finite measure is called a kernel *from Ω to $\mathcal{X}$. Then the mapping $\omega \to k_\omega$ is a* random measure *and a* point process *if every k_ω is a counting measure. The map k is called the kernel of the random measure.*

The interpretation of the Kernel in this setup is straightforward. The space Ω is just the space of "hidden variables", and hence the concrete choice of Ω is usually quite unimportant, while of course it may have a helpful interpretation. In the case of a point process the induced measure $k_\omega = \sum \delta_{x_i}$ is then just the distribution of points, when the hidden variable takes the value ω.

For the physicist point processes correspond to experimental setups where one waits for an array of detectors to click and then takes note of click times and detector number, or where one notes down space, or space-time, coordinates of events, with the corresponding event type. These are situations in which an experiment has an unknown but finite number of results, which can be classified using a set of discrete and continuous variables.

But of course not all experiments fall under this scheme. There is another class of natural beam-type measurements. These are Brownian motion like setups, i.e. setups where one monitors a "continuous" but random path. An example in the spirit of cavity QED would be the monitoring of an intensity and its fluctuations, i.e. photocurrents, on a detector screen. These experiments are not described well by point processes.

Obviously both these types of processes arise naturally in quantum optics, and other quantum experiments. Later we see how this fits into the abstract theoretical description, e.g. see section 8.3.

2.3. Semigroups and evolution systems

2.3.1. Some physical considerations

The theory of semigroups and evolution systems is very important for the theory of quantum systems in general and especially for open systems. As unitary evolution governed by the Schrödinger equation falls under this approach, one can argue that every continuous evolution of a quantum system is described either by a semigroup or in the explicitly time-dependent case by an evolution system. However, from the point of semigroup theory the unitary one-parameter groups describing this type of evolution are a rather specialized class.

Here we are interested in those semigroups and evolution systems describing the evolution of a certain class of *open quantum systems*. The class of open quantum systems we try to describe are called *Markovian*. They have the property that one can stop and restart the evolution at arbitrary times. This behaviour excludes some types of interaction with the environment.

Measurements on the environment can not have any direct influence on the evolution of the open system, in the sense that state changes of a detector, in the environment, due to the system do not influence the system. This, however, does not exclude the possibility to learn something about the open system by measurements on the environment, only that we can actively steer the system by influencing the environment.

Markovian systems emerge naturally from the weak coupling approach [26]. They are an important class of systems in the research on open quantum systems and there is a rich set of experiments accurately described by this approximation.

The most prominent example of a Markovian quantum system is that of an atom in a cavity driven by a classically controlled laser. Under some standard approximation in quantum optics the driving of the laser can, up to some classical control parameters like intensity and frequency, be absorbed into the equations of motion of the atom. The coupling between field and atom is nonetheless, due to the cavity, quite strong and can lead to interesting effects, like Rabi oscillations.

The system has to be described as open because it leaks information to the environment in form of the output field of the cavity. In this setup the Markovian approximation is easy to justify, because photons leaving the cavity depart with the speed of light, so there is obviously very little back action of the outside light field into the cavity.

The evolution of the atom in the cavity under a constant driving laser field, is now in very good approximation described by a semigroup. If we want to describe a pulsed, or otherwise non-constant, driving we can describe the system by an evolution system.

We now begin with a short introduction to the theory of semigroups and evolution systems. Keep in mind that we want to describe quantum systems and quantum mechanics is a probabilistic theory. Hence we can restrict to contractive evolution systems, reflecting the fact that the total probability is upper bounded by 1. We however admit occasionally for loss of probability and interpret the loss as a run of the experiment with no usable result, i.e. "escape" of the system.

The natural type of continuity one expects from quantum system in an algebraic framework is weak-$*$ continuity which translates to the continuity of expectation values as an argument of time. Due to the nature of evolution systems it is however no problem at all to allow for so called quantum quenches, i.e. instantaneous changes of some control parameters, as it is very natural to impose the continuity restrictions only piece-wise.

Another peculiarity of quantum mechanics is of course that we are mainly interested in completely-positive evolution systems, this is another tribute to the statistical structure of quantum theory. In this case reflecting some natural assumptions of how one expects composed systems to behave. The requirement of complete positivity emerges naturally when one supposes the existence of bystander systems not influencing the evolution at all.

2.3.2. Basic properties

For the general theory of one-parameter semigroups there are many excellent introductions. As an introductory text to one-parameter semigroups with one of the few systematic accounts of the basic theory of evolution systems we recommend [74]. For brevity and to improve readability we mostly drop the prefix and will only call them semigroups. In this section we shortly restate some of the basic results in the theory of semigroups and evolution systems. Partly with proofs or proof ideas because many of the tricks used in these proofs resurface in the main part of this text.

Basic definitions and continuity

Definition 2.25. A *one-parameter semigroup* or shortly semigroup is a one-parameter family of maps $\mathbb{E}(t) \in \mathfrak{B}(\mathcal{B})$ for $t \geq 0$ to the bounded operators on a Banach space $\mathcal{B}$ which satisfy Cauchys functional equation:

$$\begin{aligned} U(t)U(s) &= U(t+s) \\ U(0) &= \mathbb{1} \end{aligned} \tag{2.30}$$

Let id be the identity map in $\mathfrak{B}(\mathcal{B})$. A semigroup $\mathbb{E}(t)$ is called:

contractive: iff $\|\mathbb{E}(t)\| \leq 1$ for all $0 \leq t$. We also call $\mathbb{E}(t)$ a *contraction semigroup.*

unital: iff $\mathbb{E}(t)(\mathbb{1}) = \mathbb{1}$ for all t

norm continuous: iff $\lim_{t\to 0} \|\mathbb{E}(t) - \mathbb{1}\| = 0$

strongly continuous: iff $\lim_{t\to 0} \|(\mathbb{E}(t) - \mathbb{1})B\| = 0$ for all $B \in \mathcal{B}$.

Let $\mathcal{B}$ have a predual space $\mathcal{B}_*$, then $\mathbb{E}(s,t)$ is called:

weak-$*$ continuous: iff $\lim_{t\to 0} \rho(\mathbb{E}(t)(B) - B) = 0$ for all $B \in \mathcal{B}$ and $\rho \in \mathcal{B}_*$ the pre-dual of $\mathcal{B}$.

Strongly continuous one-parameter semigroups are also known as C_0-semigroups.

The first two continuity conditions for one-parameter semigroups, i.e. norm and strong continuity, are by far the most important continuity conditions for semigroups. The main reason for this is that the algebraic property of semigroups, i.e. Cauchys functional equation, is so restrictive that most weaker continuity restrictions are actually equivalent to strong continuity.

The most important result in this direction is the equivalence between weak and strong continuity for semigroups. Here weak continuity means that we have $\lim_{t\to 0} \rho(\mathbb{E}(t)(B) - B) = 0$ for all $B \in \mathcal{B}$ and $\rho \in \mathcal{B}^*$ the dual space of B [74, Theorem 2.1.4]. This result has important consequences for this work since it tells us something about weak-$*$-continuous semigroups via their *preduals.*

Definition 2.26. Let $\mathbb{E}(t)$ be a weak-$*$-continuous semigroup on the Banach space $\mathcal{B}$ with predual $\mathcal{B}_*$. We define the *predual semigroup* $\mathbb{E}_*(t)$ via:

$$\mathbb{E}_*(t) : \mathcal{B}_* \to \mathcal{B}_* \qquad \rho \to \rho \circ \mathbb{E}(t) \tag{2.31}$$

We use this definition exclusively in the case where $\mathcal{B} = \mathfrak{B}(\mathcal{H})$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}$. In this case the predual is given by the trace class operators on that Hilbert space $\mathfrak{B}(\mathcal{H})_*$. Predual semigroups are important because the weak equals strong theorem for semigroups immediately entails that:

Corollary 2.27. *The predual semigroup of a weak-$*$-continuous semigroup is weakly and hence also strongly continuous.*

One-parameter semigroups are a generalization of the exponential function, in the sense that they satisfy the same functional equation (2.30). This manifests itself also in the notation since they are often denoted as an exponential, see definition 2.29.

The search for all solutions to Cauchys functional equation is also what started the theory of semigroups. Under some mild continuity assumptions the exponential function is the only complex valued solution to this equation. However if one drops all assumptions of continuity there is a whole family of solutions to this equation. These solutions are quite pathological. This does not matter to much for our case but since part of our construction can be extended to semigroups without any continuity conditions the reader might be warned that these solutions might be highly irregular.

Reminder 2 (Integral operators). Apart from positive definite kernels, which play a role in Stinespring decomposition, probability kernels and kernels of functions, there is one other definition of kernel of relevance for this thesis. The notion of a kernel of an integral operator, which is at least closely related to the notion of a probability kernel.

If a linear operator on the function space $\mathrm{L}^2(\mathbb{R})$ has, for $\psi \in \mathcal{H}$, the form:

$$B\psi(x) = \int_{\mathbb{R}} k(x,y)\psi(y)\mathrm{d}y \tag{2.32}$$

then we call $B \in \mathfrak{B}(\mathrm{L}^2(\mathbb{R}))$ an *integral operator* and $k(x,y)$ its kernel.

The connection to probability kernels $p : \mathbb{R} \times \Omega \to [0,\infty]$ with σ being the Lebesgue σ-algebra is clear if we associate to a given probability kernel the operator:

$$\int \psi(y)p(x,\mathrm{d}y), \tag{2.33}$$

for $\psi \in \mathcal{H}$. We get an integral operator, iff all of the measures $p(x,\cdot)$ are continuous with respect to the Lebesgue measure, such that we can define them via their Radon-Nikodym derivatives.

Example 1 (Diffusion on $\mathbb{R}$). Let $\mathcal{H}$ be the Hilbert space $\mathrm{L}^2(\mathbb{R})$. The diffusion on $\mathcal{H}$ is defined as an integral operator, namely as the convolution with the heat kernel $h_t(x)$:

$$h_t \in \mathrm{L}^2(\mathbb{R}) \qquad x \to \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t}\right) \tag{2.34}$$

$$U(t) : \mathcal{H} \to \mathcal{H} \qquad U(t)\psi(x) \to \int_{\mathbb{R}} h_t(x-y)\psi(y)\mathrm{d}y \tag{2.35}$$

Hence, h_t is a Gaussian centered at the origin with width $\sigma = \sqrt{2t}$ and furthermore $\|h_t\| = 1$. The kernel of the operator $U(t)$ is $k(x,y) := h_t(x-y)$. To see that this defines a semigroup we note that the convolution is associative, and that Gaussians are invariant under Fourier transformation, i.e. $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_t(x) \exp(-ixp)\mathrm{d}x = \exp(-|p|^2 t)$.

This semigroup is not norm continuous but strongly continuous, colloquially because $\lim_{t\to 0} h_t$ is the δ distribution.

The generator of a semigroup

It turns out that the behavior of the semigroup is completely-described by a single operator on the Banach space $\mathcal{B}$, its generator:

Definition 2.28. Let $\mathbb{E}(t)$ be a strongly continuous semigroup. Whenever the following limit exists we define:

$$\mathcal{L}(B) = \lim_{t \to 0} \frac{1}{t}(\mathbb{E}(t)(B) - B) \tag{2.36}$$

The set of all B for which the preceding expression is defined is called the domain of the generator of the and denoted $\mathrm{dom}(\mathcal{L})$. The operator $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$ is called *generator* of the semigroup.

In the case of a norm continuous semigroup, which is by definition also strongly continuous, the generator is just a bounded operator on $\mathcal{B}$, and hence the domain is the whole of $\mathcal{H}$. In this case the semigroup can actually rigorously be defined via the exponential series, in complete analogy to the exponential function:

$$\mathbb{E} = \exp(t\mathcal{L}) = \sum_{n=0}^{\infty} \frac{1}{n!}(t\mathcal{L})^n. \tag{2.37}$$

The simple proof of this hinges on the fact that the submultiplicativity of operator norm $\|\mathcal{L}^n\| \leq \|\mathcal{L}\|^n$ reduces everything to the real valued case.

In the case of a weak-$*$-continuous semigroup we still regularly talk about its generator, and we could rigorously define one by a weak-$*$-topology version of equation (2.36). However, it seems easier to characterize weak-$*$-continuous semigroups by the generator of their pre-dual semigroups which are strongly continuous anyway. So when we talk about generators of weak-$*$-continuous semigroups we mean the generator of their predual.

Example 2 (Diffusion on $\mathbb{R}$)*.* A straightforward calculation shows that the heat kernel h_t defined in equation (2.34) satisfies the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t = \frac{\mathrm{d}^2}{\mathrm{d}x^2} h_t. \tag{2.38}$$

Integrating by parts twice shows that the generator of the diffusion semigroup $U(t)$ defined in equation (2.35) has as generator the one dimensional Laplacian: $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Definition 2.29. Since semigroups can be considered as a generalization of the exponential function to unbounded operators we extend the notation from equation (2.37) and given a strongly continuous semigroup $\mathbb{E}$ with generator $\mathcal{L}$ denote:

$$\exp(t\mathcal{L}) := \mathbb{E}(t) \tag{2.39}$$

The Cauchy equation

Via their generator the theory of semigroups is intimately linked to the theory of solutions the Cauchy equation. An ordinary differential equation of the following form, with initial condition X. Here $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$ is a unbounded operator:

$$\frac{\mathrm{d}}{\mathrm{d}t} B(t) = \mathcal{L}B(t) \qquad\qquad B(0) = X \tag{2.40}$$

If $\mathcal{L}$ is the generator of a semigroup $\mathbb{E}(t)$ and B is in its domain we can construct solutions to the differential equation as $B(t) = \mathbb{E}(t)(X)$.

When trying to model physical situations with differential equations, and subsequently describing the situation with the resulting semigroups, one can already experience or imagine a few of the problems connected to unbounded generators. The main obstacle in such situations is that properties of unbounded operators depend sensitively on their domains, while on the other hand physical intuition is hard to use when describing domains of operators. We encounter similar problems later, when trying to state an unbounded version of the Lindblad equation.

A notable special case of one-parameter semigroups is the *one-parameter group*, i.e. a semigroup which index range can, conserving the above mentioned properties, be extended to the whole of $\mathbb{R}$. That the time evolution of quantum systems is given by a one-parameter semigroup $\mathbb{E}(t) \in \mathfrak{B}(\mathcal{H})$ of bounded an unitary operators on a Hilbert space $\mathcal{H}$ is a standard postulate in quantum mechanics. The Stone theorem saying that the generators of unitary one-parameter groups are given by self-adjoint operators, then explains the quantum mechanics enthusiasm for Hamiltonians, and shows the conection between unitary one-parameter groups and solutions to the Schrödinger equation.

Generators of strongly continuous (contraction-)semigroups

Compared to the generators of norm-continuous semigroups the description of generators of strongly-continuous semigroups is a lot more interesting. For simplicity we restrict ourselfs to generators of contraction semigroups.

It is fairly easy to see that the domain of a generator is in general dense since one can approximate a vector in B by:

$$\frac{1}{h}\int_0^h \mathbb{E}(t)(B)\mathrm{d}t, \tag{2.41}$$

which is easily seen to be in the domain of the generator $\mathcal{L}$ and also tends to B in the limit $h \to 0$. It is also easy to see that $\mathbb{E}(t)$ conserves the domain of its generator, i.e. $\mathbb{E}(t)(\mathrm{dom}(\mathcal{L})) = \mathbb{E}(t)$. Also a semigroup commutes with its generator $\mathbb{E}(t)\mathcal{L} = \mathcal{L}\mathbb{E}(t)$.

There is an important norm on the domain of a semigroup which has a few very nice properties:

Definition 2.30. Let $\mathcal{L}$ be the generator of a semigroup $\mathbb{E}(t)$ on a Banach space $\mathcal{B}$. On $\mathrm{dom}(\mathcal{L})$ we define the *domain norm* or *graph norm* by $\|B\|_{\mathrm{dom}(\mathcal{L})} := \|B\| + \|\mathcal{L}(\mathcal{B})\|$

With respect to the norm $\|\|_{\mathrm{dom}(\mathcal{L})}$ the domain $\mathrm{dom}(\mathcal{L})$ is closed, also the generator $\mathcal{L} : (\mathrm{dom}(\mathcal{L}), \|\|_{\mathrm{dom}(\mathcal{L})}) \to (\mathcal{B}, \|\|)$ is a bounded operator. Furthermore if we restrict the semigroup $\mathbb{E}(t)$ to $(\mathrm{dom}(\mathcal{L}), \|\|_{\mathrm{dom}(\mathcal{L})})$, which is possible since the domain is invariant under $\mathbb{E}(t)$, then the resulting semigroup is again strongly continuous.

The two main theorems for the description of the generator of a strongly continuous (contraction-)semigroup are the Hille-Yoshida [74, Theorem 1.3.1] and the Lumer-Phillips [74, Theorem 1.4.3] theorems.

Theorem 2.31 (Hille-Yosida and Lumer-Phillips)**.** *An operator $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$ is the generator of a strongly continuous one-parameter semigroup of contractions $\mathbb{E}$ iff (Hille-Yosida),*

- *$\mathcal{L}$ is a closed densely defined operator,*

- $\mathrm{Res}(\mathcal{L})$, *the resolvent set of* $\mathcal{L}$ *contains the positive half axis* $t \geq 0$,
- *for all* λ *we have* $(\lambda\mathbb{1} - \mathcal{L}))^{-1} \leq \frac{1}{\lambda}$.

Or alternatively iff (Lumer-Phillips)

- $\mathcal{L}$ *is a closed densely defined operator,*
- $\mathcal{L}$ *is dissipative, i.e.* $\|(\lambda\mathrm{id} - \mathcal{L})B\| \geq \lambda\|B\|$ *for all* $x \in \mathrm{dom}(\mathcal{L})$ *and* $\lambda > 0$,
- *there exists a* $\lambda_0 > 0$ *such that* $\lambda_0\mathrm{id} - \mathcal{L}$ *is onto.*

For non contraction semigroups the above theorem is in a slightly altered form still true. In the case of an operator acting on a Hilbert space $\mathcal{H}$ the dissipativity can also be characterized in the following way. An operator $K : \mathrm{dom}(K) \to \mathcal{H}$ is dissipative iff $\mathfrak{Re}\,\langle\psi, K\psi\rangle \leq 0$.

Reminder 3 (Closed Operators). Closedness of an operator guarantees us at least some continuity. It means that for every convergent sequence in the domain of the operator, whose image under the operator also converges, the limit of the image sequence and the image of the limit of the sequence coincide. I.e. given a sequence B_n with $\lim B_n = B$ and an operator $\mathcal{L}$, closedness of $\mathcal{L}$ means $\lim \mathcal{L}B_n = \mathcal{L}\lim B_n$.

Given two unbounded operators $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ with the property that $\mathrm{dom}(\widetilde{\mathcal{L}}) \subset \mathrm{dom}(\mathcal{L})$ and $\mathcal{L}(B) = \widetilde{\mathcal{L}}(B)$ for $B \in \mathrm{dom}(\mathcal{L})$. Then $\mathcal{L}$ is said to extend $\widetilde{\mathcal{L}}$. When an operator $\mathcal{L}$ has a closed extension it is said to be closeable and the smallest closed extension is called the closure of $\mathcal{L}$

Core for a generator of a semigroup

The point that generators of semigroups are closed operators is a very important technical point. The other conditions are in both cases mainly needed to ensure that the semigroups do not blow up in norm too fast, or not at all in this case.

Due to their closedness generators of semigroups are often defined only on a dense set $\mathcal{D}$, the actual generator is then defined as the closure of the operator on $\mathcal{D}$. This way of describing generators of semigroups is often immensely useful, because it is often very difficult to explicitly describe the full domain of a generator. Properties of unbounded operators may, however, rely heavily on the choice of the domain, as many physicists know very well from the theory of self-adjoint operators.

Definition 2.32. Let $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$ be the generator of a semigroup. A dense subspace $\mathcal{D}$ with the property that the closure of the operator $\widetilde{\mathcal{L}} : \mathcal{D} \to \mathcal{B}$ is $\mathcal{L}$ is called *core* for $\mathcal{L}$.

Checking if a dense set is a core for a given generator is hence quite important. The following theorem helps at least in cases where the semigroup is explicitly known, it can be found in [31, theorem 6.1.18].

Theorem 2.33. *Let* $\mathbb{E}(t)$ *be a strongly continuous semigroup with generator* $\mathcal{L}$. *Then* $\mathcal{D}$ *is a core for* $\mathcal{L}$ *iff it is dense and invariant under* $\mathbb{E}(t)$, *i.e.* $\mathbb{E}(t)(B) \in \mathcal{D}$ *for all* $t \geq 0$ *and* $B \in \mathcal{D}$.

Resolvents of generators of semigroups

Many interesting properties of unbounded operators can be understood by the study of their resolvents, which, since they are bounded operators, are usually far easier to handle than the operator itself. This is, as theorem 2.31 suggests, even more the case for generators of semigroups. A fact which greatly helps this business is that the resolvent of the generator of a semigroup is identical to the Laplace transformation of the semigroup:

$$(\lambda \mathrm{id} - \mathcal{L})^{-1} B = \int_0^\infty e^{-\lambda t} \mathbb{E}(t) B \mathrm{d}t \tag{2.42}$$

A proof of this result can for example be found in [74, theorem 1.3.1] This equation is obviously most useful when the semigroup is explicitly known. Knowledge of the resolvent could then be used to describe and calculate the domain of the generator.

Example 3 (Diffusion on $\mathbb{R}$). We can now use this trick to calculate the resolvent of the diffusion on $\mathbb{R}$. To do so we first calculate the Laplace transformation of the heat kernel h_t defined in equation (2.34), i.e:

$$\int_0^\infty e^{-\lambda t} h_t \mathrm{d}t \tag{2.43}$$

A substitution reduces the necessary integrals to a well known one:

$$\begin{aligned}
&\int_a^b \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t} - \lambda t\right) \mathrm{d}t \\
=&\int_a^b \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t} - \lambda t\right) \frac{1}{2\sqrt{\lambda}} \left(\left(-\frac{x}{2t} + \sqrt{\lambda}\right) + \left(\frac{x}{2t} + \sqrt{\lambda}\right)\right) \mathrm{d}t \\
=&\frac{1}{\sqrt{4\pi\lambda}} \int_a^b \exp\left(-\left(\frac{x}{2\sqrt{t}} + \sqrt{\lambda t}\right)^2 + x\sqrt{\lambda}\right) \left(-\frac{x}{4t^{3/2}} + \sqrt{\frac{\lambda}{2t}}\right) \mathrm{d}t \\
&- \frac{1}{\sqrt{4\pi\lambda}} \int_a^b \exp\left(-\left(\frac{x}{2\sqrt{t}} - \sqrt{\lambda t}\right)^2 - x\sqrt{\lambda}\right) \left(-\frac{x}{4t^{3/2}} - \sqrt{\frac{\lambda}{2t}}\right) \mathrm{d}t \\
=&\frac{e^{x\sqrt{\lambda}}}{\sqrt{4\pi\lambda}} \int_{\frac{x}{2\sqrt{a}}+\sqrt{\lambda a}}^{\frac{x}{2\sqrt{b}}+\sqrt{\lambda b}} \exp\left(-s^2\right) \mathrm{d}s - \frac{e^{-x\sqrt{\lambda}}}{\sqrt{4\pi\lambda}} \int_{\frac{x}{2\sqrt{a}}-\sqrt{\lambda a}}^{\frac{x}{2\sqrt{b}}-\sqrt{\lambda b}} \exp\left(-s^2\right) \mathrm{d}s
\end{aligned}$$

For the case $x > 0$ in the limit $a \to 0$ and $b \to \infty$ the first integral vanishes. The limit is independent off the order of the two limits, because it integrates in any case the tail of a Gauss distribution. The second integral approaches an integral over the whole line, which is known to be $\sqrt{4\pi}$. The case $x < 0$ is analogous with interchanged roles. We arrive at:

$$\int_0^\infty h_t(x) e^{-\lambda t} \mathrm{d}t = \frac{e^{-|x|\sqrt{\lambda}}}{2\sqrt{\lambda}} \tag{2.44}$$

The calculation of the resolvent of the diffusion generator, i.e. the one-dimensional Laplacian is now a straightforward application of the Fubini-Tonelli theorem and the Hölder inequality, which shows that the integrand is in $L^1(\mathbb{R}\times\mathbb{R}^+)$. Let K be the one dimensional Laplacian.

$$(\lambda \mathbb{1} - K)^{-1}(\varphi)(x) = \int_0^\infty \int_{-\infty}^\infty \exp(-\lambda t) h_t(y-x)\varphi(y)\mathrm{d}y\mathrm{d}t \tag{2.45}$$

$$= \int_{-\infty}^\infty \frac{e^{-|y-x|\sqrt{\lambda}}}{2\sqrt{\lambda}}\varphi(y)\mathrm{d}y. \tag{2.46}$$

Hence, the resolvent is again defined by an integral kernel.

We can also recover the semigroup from the resolvents of its generator by the following formula:

$$\lim_{n\to\infty}\left(\frac{1}{n}\mathrm{id} - \mathcal{L}\right)^{-n} B = \mathbb{E}(t). \tag{2.47}$$

This method goes back to Hille. The preceding approximation is also a key step in proving theorem 2.31. An alternative construction of the semigroup via its generators is the Yosida approximation:

$$\widetilde{\mathcal{L}}_\lambda B := \mathcal{L}\lambda\left(\lambda \mathrm{id} - \mathcal{L}\right)^{-1} B. \tag{2.48}$$

With this definition we have: $\widetilde{\mathcal{L}}_\lambda B$ is a bounded operator and converges strongly to $\mathcal{L}$ in the limit $\lambda \to \infty$. Furthermore the semigroups generated by $\widetilde{\mathcal{L}}_\lambda$ converge to the semigroup generated by $\mathcal{L}$:

$$\mathbb{E}(t)(B) = \lim_{\lambda\to\infty}\exp\left(\widetilde{\mathcal{L}}_\lambda\right) B. \tag{2.49}$$

Lemma 2.34. *Yosida approximations of completely-positive maps are completely-positive.*

Proof. Observe that:

$$\mathcal{L}\lambda(\lambda\mathbb{1} - \mathcal{L})^{-1} = (\lambda\mathbb{1} + (\mathcal{L} - \lambda\mathbb{1}))(\lambda\mathbb{1} - \mathcal{L})^{-1} = \lambda^2(\lambda\mathbb{1} - \mathcal{L})^{-1} - \lambda\mathbb{1} \tag{2.50}$$

This map is conditionally completely-positive and hence $\exp(t\mathcal{L}\lambda(\lambda\mathbb{1} - \mathcal{L})^{-1})$ is a completely-positive semigroup. □

Perturbations of semigroups

Another important question in the theory of semigroups is: under which pertubations does the generator of a semigroup stay the generator of a semigroup? The importance of this question for physical applications is immediate, not only because perturbation theory plays a central role in solving realistic scenarios, but also from a more theoretical perspective, as it tells for example which potentials lead to well defined evolutions for a free particle.

Again we are mostly interested in perturbation results on contraction semigroups, because we want to apply them to the theory of open quantum systems. For unitary quantum mechanics the perturbation theory for hermitian operators plays a central role, and often helps to understand many otherwise quite supprising technical conditions.

The simplest perturbation result is about the perturbation of a generator of a strongly continuous semigroup by a bounded operator. Concisely it states that this type of perturbation is always possible and leads to a semigroup. However, the norm bound we have on the semigroup worsens according to the norm of the perturbation. The theorem can be found in any basic textbook on semigroups, e.g. [74, chapter 3.1], and has the following form:

Theorem 2.35. *Let $\mathbb{E}(t) : \mathcal{B} \to \mathcal{B}$ be a strongly continuous semigroup on a Banach space $\mathcal{B}$ such that $\|\mathbb{E}(t)\| \leq Me^{\omega t}$. Denote the generator of $\mathbb{E}(t)$ by $\mathcal{L}$ and let $\mathcal{P}$ be a bounded operator. Then $\mathcal{L} + \mathcal{P}$ generates a strongly continuous semigroup and $\|\exp(t(\mathcal{L} + \mathcal{P}))\| \leq Me^{(\omega + M\|\mathcal{P}\|)t}$.*

The theorem can be proven for example by a simple Picard-Lindelöf like iteration, i.e. the limit of:

$$\widetilde{\mathbb{E}}^{n+1}(t) = \mathbb{E}(t) + \int_0^t \mathbb{E}(t-s)\mathcal{P}\widetilde{\mathbb{E}}^n(s)\mathrm{d}s \tag{2.51}$$

Since we are later mainly interested in unbounded perturbations, for us theorem 2.35 is mainly interesting as a motivation for the proof technique we later use. A far more interesting result is the following one about generators of contraction semigroups, i.e.:

Theorem 2.36. *Let $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$ be the generator of a contraction semigroup on the Banach space $\mathcal{B}$ and let $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \to \mathcal{B}$ be an unbounded operator. Then $\mathcal{L} + \mathcal{P}$ is the generator of a contraction semigroup if:*

1. $\mathrm{dom}(\mathcal{L}) \subset \mathrm{dom}(\mathcal{P})$
2. *$\mathcal{P}$ is realtively bounded by $\mathcal{L}$ with relative bound < 1*
3. *$\mathcal{L} + \mathcal{P}$ is dissipative.*

This result can be found, e.g., in [31, theorem 11.5.1].

Reminder 4 (relative boundedness). Let $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$ and $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \to \mathcal{B}$ be two unbounded operators such that $\mathrm{dom}(\mathcal{L}) \subset \mathrm{dom}(\mathcal{P})$ and there exists $b, c \in \mathbb{R}$, such that:

$$\|\mathcal{P}(B)\| \leq b\|\mathcal{L}(B)\| + c\|B\| \text{ for all } B \in \mathrm{dom}(\mathcal{L}) \tag{2.52}$$

Then $\mathcal{P}$ is called *relatively bounded* by $\mathcal{L}$ and the smallest possible constant c in equation (2.52) is called the *relative bound.*

Solutions of inhomogenous equations

Another type of "perturbation" of semigroups arises when one tries to solve inhomogenous Cauchy equations. That is solutions to equations like (2.40) with an additional "driving" term. The physical significance of such equations is immediate. We encounter them later when calculating expectation values of cMPS.

More formally the inhomogenous Cauchy problem has the following form: Given a Banach space $\mathcal{B}$, an possibly unbounded operator $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathcal{B}$, a $\mathcal{B}$ valued function $f(t)$ and an initial value X, we want to find a function $t \mapsto B(t)$ such that:

$$\frac{\mathrm{d}}{\mathrm{d}t}B(t) = \mathcal{L}(B(t)) + f(t) \qquad B(0) = X \tag{2.53}$$

In many cases solutions $B(t)$ to the inhomogenous equation can be constructed from the semigroup solving the homogeneous equation $\mathbb{E}(s)$, i.e. the case with $f(t) = 0$, through the formula:

$$B(t) := \mathbb{E}(t)(X) + \int_0^t \mathbb{E}(t-s)f(s)(X)\mathrm{d}s, \tag{2.54}$$

That is, the semigroup solving the homogenous equation is the Greens function for the inhomogenous equation. Exemplary conditions can be found in the following theorem. We adapted it from [74, chapter 4.2].

Theorem 2.37. *Let $\mathcal{L}$ and $\mathcal{B}$ be as above. And let $f(t) \in \mathrm{L}^1([0,T],\mathcal{B})$ be continuous on the interior of $[0,T]$. Then equation (2.53) has a solution if one of the following two sets of conditions is satisfied.*

1a. The function $v(t) : [0,T] \to \mathcal{B}$ given by $v(t) : \int_0^t \mathbb{E}(t-s)f(s)\mathrm{d}s$ maps into $\mathrm{dom}(\mathcal{L})$ for all $t \in [0,T]$

1b. The function $v(t) : [0,T] \to \mathcal{B}$ as above is continuously differentiable on $[0,T]$ and $t \to \mathcal{L}(v(t))$ is continuous in the interior of $[0,T]$

Or alternatively:

2a. $f(t) \in \mathrm{dom}(\mathcal{L})$ for $0 \leq t \leq T$ and $t \to \mathcal{L}(f(t)) \in \mathrm{L}^1([0,T],\mathcal{B})$

2b. $f(t)$ is continuously differentiable on $[0,T]$

2.3.3. The Markov property

Our interest in semigroups arises from the study of continuous-time stochastic processes. The connection between these two topics arises roughly as follows: We have a space $\mathcal{B}$ describing the state of some system. For simplicity this space stays fixed throughout time, i.e. the system can at all times be described by the same properties. Elements in the space $\mathcal{B}$ are used to describe the state of the system.

The semigroup $\mathbb{E}(t) : \mathcal{B} \to \mathcal{B}$ describes the evolution of the system through time. Cauchys functional equation (2.30) reflects in this setup the Markov property of the stochastic system. That is the evolution of the state of the system is completely determined by its state at a certain time, and not by the whole evolution up to this point.

Furthermore the semigroup has to satisfy some positivity restrictions to ensure that it maps probability distributions to probability distributions. Unitality of the semigroup is connected to probability conservation.

In the case of quantum systems the space $\mathcal{B}$ is usually the space of bounded operators, i.e. $\mathcal{B} = \mathfrak{B}(\mathcal{H})$, or trace-class operators, i.e. $\mathcal{B} = \mathfrak{T}(\mathcal{H})$, on a given Hilbert space, depending on if we describe the system in Heisenberg or in Schroedinger picture. The semigroup is then just the normal description of the evolution of the system.

In classic probability theory the appearance of semigroups is not weaved into the basic formalism, but also quite straightforward. The main point is to describe a probability distribution not directly by its measure on a measure space, but rather by the set of all positive real-valued measurable functions on the set. This is equivalent to the usual description because all indicator functions of measurable sets are of this form, i.e. functions being 1 for elements in a measurable set and 0 else. It is obvious how to reconstruct the probability measure from the integrals of these functions.

Now instead of having a time depenent family of measures we can describe the time evolution of a system by the evolution of such functions. For classical Markov processes it is again natural to describe this time-evolution by a semigroup. A very nice description of these relations can be found in [8, chapter 36], or any other textbook on probability theory like [60].

2.3.4. Evolution systems

Reminder 5 (Vector valued integration)*.* Often when one learns functional analysis the theory of vector valued integration is neglected a bit. This is partly due to the reason that one integrates only continuous functions anyway, e.g. semigroups, and there is no need of a careful treatment. In the case of evolution systems a little more caution is needed. For a thorough introduction to this topic we recommend the book by Diestel and Uhl [34], where the most relevant section for us are II.1 and II.2.

We base our discussion mostly on one of the most restrictive notions of vector valued integration, namely the Bochner integral, i.e. integrability “in norm”. Doubtlessly many of our results generalize to the more general Pettis integral, i.e. integrability in evaluation with a functional. However, this weak form of integrability would introduce many technicalities we wish to avoid for the sake of presentation.

A Banach space valued function $B : \mathbb{R} \to \mathcal{B}$ is *Bochner measurable* when there exists a family of simple functions $B_n : \mathbb{R} \to \mathcal{B}$, i.e. measurable functions whose image contains only finitely many points, converging pointwise to B, i.e. $\lim_n \|B(t) - B_n(t)\| = 0$ almost everywhere. The generalization to other measure spaces than $\mathbb{R}$ is obvious.

Furthermore a function $B(t)$ as above is called *Bochner integrable* if there exists a sequence of simple functions B_n, as above, such that:

$$\lim_{n\to\infty} \int \|B(t) - B_n(t)\| \mathrm{d}t = 0, \tag{2.55}$$

again with the obvious generalization to other measure spaces. The Bochner integral of a function $t \to B(t)$ is then defined as the limit of the integrals of the approximating sequence which are defined elementary.

The Bochner integrability of a Bochner measurable function $t \to B(t)$ is equivalent to the integrability of the real valued function $t \to \|B(t)\|$. Moreover the following useful inequality holds for the Bochner integral:

$$\left\| \int B(t) \mathrm{d}t \right\| \leq \int \|B(t)\| \, \mathrm{d}t. \tag{2.56}$$

In general the Bochner integral fulfills all relations one expects from an integral, e.g., that sums of integrals over disjoint regions compose as expected and the integral commutes with bounded and even closed operators. Furthermore straightforward generalizations of the Lebesgue dominated convergence theorem and the fundamental theorem of calculus hold. The last one looks like:

$$\lim_{h\to 0} \int_t^{t+h} \|B(s) - B(t)\| \mathrm{d}s = 0 \text{ for almost all } s. \tag{2.57}$$

We furthermore need the following generalizations of the Lebesgue spaces. The space $L^1(\mathbb{R}, \mathcal{B})$ is just the space of Bochner integrable functions, with the usual equivalence relation. And the space $L^2(\mathbb{R}, \mathcal{B})$ is the space of functions such that the following integral is finite:

$$\sqrt{\int \|B(t)\|^2 \mathrm{d}t}. \tag{2.58}$$

Again this has an obvious generalization to $L^p(\mathbb{R}, \mathcal{B})$ spaces and more general measure spaces.

Another important topic is the integration of scalar valued functions against vector valued measures. Here we also refer to [34, chapter I] as a reference. We later also refer to the Radon-Nikodym statement for vector measures, which is the topic of chapter [34, chapter III]. We shall not give an introduction into vector measures here, but want to mention that we assume vector measures to be countably additive in norm topology. Hence in this convention POVMs and instruments are not vector measures in the definition of a vector measure .

Evolution systems are a straightforward generalization of one-parameter semigroups to two arguments. In difference to semigroups they solve the Cauchy equation (2.40) with a time-dependent generator. That is, they correspond to evolutions where some time-dependence is fixed for the system from the outset. Many properties of semigroups generalize to the slightly more general case of evolution system. However compared to the theory of semigroups there are only very few introductory texts on this topic and there is not even clear consensus on the basic nomenclature, i.e. evolution systems are often called propagator, evolution operator, or fundamental solution.

Basic definition and continuity

Definition 2.38. Let $\mathcal{B}$ be a Banach space and $0 \leq r \leq s \leq t \in \mathbb{R}$. A two parameter family of maps $\mathbb{E}(s,t) : \mathcal{B} \to \mathcal{B}$ is called an *evolution system* iff:

$$\mathbb{E}(r,s)\mathbb{E}(s,t) = \mathbb{E}(r,t) \qquad \text{for all } 0 \leq r \leq s \leq t \in \mathbb{R} \tag{2.59}$$

$$\mathbb{E}(t,t) = \mathrm{id}_{\mathcal{B}} \qquad \text{for all } 0 \leq t \in \mathbb{R} \tag{2.60}$$

An evolution system is said to be:

contractive iff $\|\mathbb{E}(s,t)\| \leq 1$ for all $0 \leq s \leq t$

norm continuous: iff $(s,t) \to \mathbb{E}(s,t)$ is almost-everywhere continuous in the norm topology on $\mathfrak{B}(\mathcal{B})$.

strongly continuous: iff $(s,t) \to \mathbb{E}(s,t)(B)$ is almost-everywhere continuous for all $B \in \mathcal{B}$ in the norm topology on $\mathcal{B}$.

Let $\mathcal{B}$ be an algebra with identity element $\mathbb{1}$. $\mathbb{E}$ is called:

unital: iff $\mathbb{E}(s,t)(\mathbb{1}) = \mathbb{1}$.

Let $\mathcal{B}$ have a predual space $\mathcal{B}_*$, then $\mathbb{E}(s,t)$ is called:

weak-$*$ continuous: iff $(s,t) \to \rho \circ \mathbb{E}(s,t)(B)$ is amost-everywhere continuous for all $\rho \in B_*$ and $B \in \mathcal{B}$.

Analogous to semigroups, which describe the evolution of a system over a time period t, we can understand evolution systems as describing the evolution of a system from a timepoint s to a timepoint t.

We always denote evolution systems on $\mathfrak{B}(\mathcal{H})$, the bounded operators on an Hilbert space $\mathcal{H}$, as above, since they represent evolution in the Heisenberg picture. If on the contrary we want to describe evolution in the Schrödinger picture, i.e. for pure states or the predual semigroup, time flows naturally in the other direction, which we incorporate in our notation.

In general we try to define evolution systems such that the middle argument cancels. Remember that according to the Heisenberg picture operators in $\mathfrak{B}(\mathcal{H})$ evolve backwards in time while according to the Schrödinger picture states in $\mathfrak{T}(\mathcal{H})$ or pure states in $\mathcal{H}$ evolve forwards in time. Hence, for an evolution system $U(t,s) : \mathcal{H} \to \mathcal{H}$ for $0 \leq r \leq s \leq t$, representing pure state evolution, we change the order of the arguments and write $U(t,r) = U(t,s)U(s,r)$.

We can again define a *predual evolution system* for a given evolution system $\mathbb{E}(s,t)$ analogous to definition 2.26. Note the change in the order of the arguments.

Definition 2.39. Let $\mathbb{E}(t)$ be a weak-$*$ continuous evolution system on the Banach space $\mathcal{B}$ with predual $\mathcal{B}_*$, and $0 \leq s \leq t$. We define the *predual evolution system* $\mathbb{E}_*(t,s) : \mathcal{B}_* \to \mathcal{B}_*$:

$$\mathbb{E}_*(t,s) : \mathcal{B}_* \to \mathcal{B}_* \qquad\qquad \rho \to \rho \circ \mathbb{E}(s,t) \tag{2.61}$$

Given an evolution in the Heisenberg picture the predual evolution system would represent the evolution in the Schrödinger picture.

Piecewise definition of an evolution system

The simplest example of an evolution system is any one-parameter semigroup. Setting $\mathbb{E}(s,t) := \widetilde{\mathbb{E}}(t-s)$ associates an evolution system to any given semigroup. Similarly we can "glue together" any finite number of semigroups or evolution systems, by dividing the time-line in subintervals and letting each evolution system govern the evolution over a given time-interval. Let $\Theta \in \mathfrak{Z}([0,T])$ be an interval decomposition of the interval $[0,T]$, and for every $i \in I(\Theta)$ let $\widetilde{\mathbb{E}}_i(s,t)$ be an evolution system, then for s in subinterval i_s and t in subinterval i_t we define:

$$\mathbb{E}(s,t) := \widetilde{\mathbb{E}}_i(s,t_{i_s}) \prod_{\substack{i_s<k<i_t \\ k\in I(\Theta)}} \widetilde{\mathbb{E}}_k(t_{k-1},t_k)\widetilde{\mathbb{E}}(t_{i_t-1},t) \tag{2.62}$$

iff s,t are in different subintervals and $\mathbb{E}(s,t) := \widetilde{\mathbb{E}}_i(s,t)$ iff both s and t are in subinterval i. This constriction can of course be extended to evolutions over the whole of $\mathbb{R}$.

This way of constructing evolution systems from families of evolution systems is very important for us.

Firstly, because it is the basic method to construct any non trivial evolution system. Actually most evolution systems are constructed as limits of evolution systems of the above type.

Secondly, because it shows that it is always sufficient to construct evolution systems only piecewise. This is especially important when one looks at evolutions which radically change in an instant, e.g., because some potential was turned on or some other switch in the laboratory was switched.

2.3.5. $\mathcal{D}$-valued evolution systems

An important assumption, which we demand from almost every evolution system appearing in this thesis, is the existence of a "common core". To not further complicate the statement of our constructions and results, we require the existence of subspace $\mathcal{D}$ such that, among other things, the functions $t \to \mathbb{E}(t,s)(D)$ are differentiable for every $D \in \mathcal{D}$ and $s \in \mathbb{R}$. This amounts to the requirement that $\mathcal{D}$ is invariant under the evolution. In this sense the space $\mathcal{D}$ is something like a common core for the generators at different time-points. We call such evolution systems $\mathcal{D}$-valued.

Definition 2.40 ($\mathcal{D}$-valued evolution system)**.** Let $\mathcal{B}$ be a Banach space and $0 \leq r \leq s \leq t \in \mathbb{R}$. Let $\mathcal{D}$ be a Banach space which can be densely and continuously embedded into $\mathcal{B}$. A two parameter family of maps $\mathbb{E}_*(s,t) : \mathcal{B} \to \mathcal{B}$ is called a *strongly-continuous $\mathcal{D}$-valued contractive evolution system* with *(family of) generator(s) $\mathcal{L}(t)$* iff:

1. $\mathbb{E}(t,s)$ is a contractive strongly-continuous evolution system, i.e.:
    a) $\mathbb{E}_*(t,s)\mathbb{E}_*(s,r) = \mathbb{E}_*(t,r)$ for all $0 \leq r \leq s \leq t \in \mathbb{R}$
    b) $\mathbb{E}_*(t,t) = \mathrm{id}_{\mathcal{B}}$ for all $0 \leq t \in \mathbb{R}$
    c) $\|\mathbb{E}_*(t,s)\| \leq 1$ for all $0 \leq s \leq t \in \mathbb{R}$
    d) For every $\rho \in B$ the map $(s,t) \to \mathbb{E}_*(t,s)\rho$ is continuous.
2. $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_*(t,s)\omega = -\mathbb{E}_*(t,s)\mathcal{L}(s)\omega$
3. $\mathbb{E}_*(t,s)\omega \subset \mathcal{D}$ for $\omega \in \mathcal{D}$
4. For every $\omega \in \mathcal{D}$ the map $(s,t) \to \mathbb{E}_*(t,s)\omega$ is continuous in the norm of $\mathcal{D}$
5. $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_*(t,s)\omega = \mathcal{L}(t)\mathbb{E}_*(t,s)\omega$

In the case that $\mathcal{B}$ is actually a Hilbert space, we choose $\mathcal{D}$ to be a Hilbert space as well.

Such $\mathcal{D}$-valued evolution systems do indeed exists. The following lemma gives conditions on the family of generators $\mathcal{L}(t)$.

Lemma 2.41. *For every $t \in [0,T]$ let $\mathcal{L}(t)$ be the generator of a contraction semigroup. Under the following conditions the family $\mathcal{L}(t)$ creates a $\mathcal{D}$-valued evolution system:*

1. *There exists a family of isomorphisms: $Q(t) : \mathcal{D} \to \mathcal{B}$ such that:*
    a) For every $B \in \mathcal{B}$ the function $t \to Q(t)$ is continuously differentiable.
    b) $Q(t)\mathcal{L}(t)Q(t)^{-1} = \mathcal{L}(t) + B(t)$ where $B(t)$ is bounded.

2. *The function $t \to \mathcal{L}$ is continuous if $\mathcal{L}$ is considered as an operator from $\mathcal{D} \to \mathcal{B}$, in the sense that for all ϵ there exists a δ, such that for $|t-s| \leq \delta$ we have*

$$\frac{\|(\mathcal{L}(t) - \mathcal{L}(s))(\omega)\|}{\|B\|_{\mathcal{D}}} \leq \epsilon \tag{2.63}$$

Proof. This is just an application of [74, theorem 5.4.6]. The assumption that every $\mathcal{L}(t)$ is the generator of a contraction semigroup ensures that the family $\mathcal{L}(t)$ is a stable family. □

We do use the previous lemma almost exclusively in the case where the operator $\mathcal{L}(t)$ acts on some Hilbert space $\mathcal{H}$. In this case we also choose $\mathcal{D}$ to be a Hilbert space. Furthermore we then usually use the symbol $K(t)$ for the generator.

2.3.6. Evolution systems and non autonomous Cauchy equations

In the same way that semigroups correspond to the solutions of time independent Cauchy equations like (2.40), evolution systems correspond to the solution of non-autonomous evolutions. These are evolutions of the same form but determined by a differential equation with an explicit time dependence. That is, we search for a $\mathcal{B}$ valued function $t \to B(t)$ such that, for a given initial value X:

$$\frac{\mathrm{d}}{\mathrm{d}t} B(t) = \mathcal{L}(t)(\mathcal{B}(t)) \qquad\qquad B(0) = X. \tag{2.64}$$

The "generator" of an evolution system is now, however, a whole family of operators. Since the generator can change from one moment to another to a completely different operator as in the previous example, also their domains can change. This is why evolution systems "solving" a given time-dependent Cauchy equation often do not even generate a solution to the given equation. We have the following theorem, which is a simplified version of [74, theorem 5.3.1].

Theorem 2.42. *Let $\mathcal{H}$ be a Banach space and let $\mathcal{D}$ be a Banach space which can be densely and continuously embedded in $\mathcal{H}$, i.e. $\mathcal{D}$ can be considered as a subspace of $\mathcal{H}$ and $\|\psi\| \leq C\|\psi\|_{\mathcal{D}}$. Given that the following assumptions are satisfied for $[0, T]$:*

1. *For all $t \geq 0$ let $K(t)$ be the generator of an contraction semigroup*

2. *For $\psi \in \mathcal{D}$, $\mathcal{D}$ is invariant under the semigroups generated by $\mathcal{K}$, i.e. $\exp(tK)\mathcal{D} \subset \mathcal{D}$ and the restriction to $\mathcal{D}$ is again a strongly continuous contraction semigroup, i.e. $\lim_{h\to h0} \|\exp(hK)\psi - \psi\|_{\mathcal{D}} \to 0$ and $\|\exp(tK)\psi\|_{\mathcal{D}} \leq \|\psi\|_{\mathcal{D}}$*

3. *Every K is bounded, considered as an operator from $(\mathcal{D}, \|\|_{\mathcal{D}}) \to (\mathcal{H}, \|\|)$.*

4. *The function $t \to K(t)$ is continuous when each $K(t)$ is considered as a bounded operator from $(\mathcal{D}, \|\|_{\mathcal{D}}) \to (\mathcal{H}, \|\|)$. This is $\lim_{t\to s} \sup_{\psi\in\mathcal{D}} \|K(t) - K(s)\psi\|/\|\psi\|_{\mathcal{D}} = 0$ for all s.*

There exists a unique evolution system $U(t,s)$ for $0 \leq s \leq t \leq T$ such that for all $\psi \in \mathcal{D}$ we have:

1. $\|U(t,s)\| \leq 1$

2. $\lim_{h\to 0} \frac{1}{h}\left(U(t+h,t)\psi - \psi\right) = K(t)\psi$

3. $\frac{\mathrm{d}}{\mathrm{d}s} U(t,s)\psi = -K(s)\psi$

If the family K is constant, i.e. we are actually looking at a semigroup, an example of a subspace $\mathcal{D}$ satisfying all the above requirements is $\mathrm{dom}(K)$. Furthermore due to the invariance of $\mathcal{D}$ under all $K(t)$ it needs to be a mutual core for all times. For the construction of the evolution system belonging to the family of generators $K(t)$ one uses the trick of piecewise definition, concatenating the semigroups $U_i(t) := \exp(tK(t_i))$. The list of prerequisites in the previous theorem is essentially a wishlist of all properties necessary to get this family of evolution systems converge on $\mathcal{D}$.

The piecewise construction of evolution systems has also found its way into some standard physics notation. Let $\Theta \in \mathfrak{Z}([0,T])$, then one can rewrite equation (2.62) in the following way using the exponential notation for semigroups from definition 2.29. Let $\mathcal{L}_t$ be the generator of a semigroup for all t, then one could approximate the evolution system from theorem (2.64)

$$\mathbb{E}(0,T) = \prod_{i\in I(\Theta)} \exp(\tau_i \mathcal{L}_{t_i}). \tag{2.65}$$

This can be seen as some kind of non commutative Riemann integral approximation to an evolution system. In the limit, this justifies the notation as a time ordered integral:

$$\mathbb{E}(0,T) = \mathcal{T}\exp\left(\int_0^T \mathcal{L}_t \mathrm{d}t\right). \tag{2.66}$$

Note that even under the quite restrictive conditions from theorem 2.42 it is not guaranteed that the evolution system $U(t,s)$ conserves the subspace $\mathcal{D}$. Hence the functions $t \to U(t,s)\psi$ may not be differentiable even for $\psi \in \mathcal{D}$. Such differentiable solutions to the Cauchy equation (2.64) are called classical solutions. Sadly we do not know any simple conditions guaranteeing the existence of such general classical solutions. However under even stronger conditions on the family of generators $K(t)$ we can guarantee the existence of en evolution system with even stronger properties. The following theorem can be found in [74, theorem 5.3.1].

Theorem 2.43. *If we replace assumption number 2 in theorem 2.42 by the following:*

2'. There exists a family of isomorphisms $Q(t) : \mathcal{D} \to \mathcal{H}$, such that the functions $t \to Q(t)\psi$ are continuously differentiable for all $\psi \in \mathcal{D}$ with the property that the function:

$$t \to K(t) - Q(t)K(t)Q(t)^{-1} \tag{2.67}$$

is strongly continuous and maps into the bounded operators.

Then the unique evolution system satisfies the two additional properties:

1. $U(t,s)\mathcal{D} \subset \mathcal{D}$

2. $t \to U(t,s)\psi$ *for* $\psi \in \mathcal{D}$ *is continuous as a* $(\mathcal{D}, \|\|_{\mathcal{D}})$ *valued function*

Later we need evolution systems with the domain conservation property number 1. The above theorem guarantees the existence of non trivial evolution systems with this property.

Part I.

The Lindblad equation

3. Lindblad generators

In this chapter we study the generators of completely-positive evolutions. The focus lies on the theory of semigroups. After a short overview of the norm continuous case we describe some of the problems with weaker continuity conditions.

3.1. Norm continuous semigroups

3.1.1. The Lindblad theorem

It is an important problem to characterize the generators leading to completely-positive semigroups, and hence evolutions describing the aforementioned class of Markovian open quantum systems. It is therefore not surprising that the two papers which independently solved this problem, for the case of norm continuous semigroups, are known to almost every researcher in the open quantum systems community. The two solutions both got published in 1976 and are due to Lindblad [66] on the one hand and Gorini, Kossakowksi and Sudarshan on the other [50].

Throughout this thesis we call the following theorem Lindblad theorem, and refer to the specific form of the operator as Lindblad form. This is not to be understood as any kind of rating of the relative importance of the two contributions, but rather resembles the fact that Lindblad's notation and methodology is better adapted to our purpose.

Theorem 3.1 (Lindblad)**.** *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{E}_t : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ for $t \geq 0$ be a norm-continuous semigroup of completely-positive and unital maps. One can find bounded operators $K : \mathcal{H} \to \mathcal{H}$ and an operator $L : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$, where $\mathcal{K}$ is a separable Hilbert space, such that the generator $\mathcal{L}$ of $\mathbb{E}_t$ is given by:*

$$\mathcal{L}(B) = K^*B + BK + L^*\mathbb{1}_{\mathcal{K}} \otimes BL \tag{3.1}$$

$$\mathcal{L}(\mathbb{1}) = 0 \tag{3.2}$$

Conversely, every operator satisfying these two equations generates a completely-positive unital and norm-continuous semigroup.

Note that Theorem 3.1 characterizes operator on $\mathfrak{B}(\mathcal{H})$ via a few operators on $\mathcal{H}$. It thus greatly reduces the complexity of the generator, especially if the space $\mathcal{K}$ has only small dimension.

To simplify a bit the notation of time-independent generators we deviate slightly from the usual presentation of the generator. Namely we choose not to introduce a basis $\{e_\alpha\}$ for $\alpha \in \mathcal{A}$ on $\mathcal{K}$, which would allow us to write $\mathcal{L}(B) = K^*B + BK + \sum_{\alpha\in\mathcal{A}} L^*_\alpha B L_\alpha$.

Another possible way to express equation (3.1) is to define a completely-positive *jump map* $\mathcal{J}(B) := L^*\mathbb{1}_{\mathcal{K}} \otimes BL$ and the *dissipator* $\mathcal{Z}(B) = K^*B + BK$. This reflects the separation of the Lindblad generator in two parts. We explain the naming of these two parts later in this section.

In this case of a bounded generator, the two equations (3.1) and 3.2 are equivalent to the existence of a self-adjoint operator H and a family of operators L_α such that:

$$\mathcal{L}(B) = i[H, B] + \sum_\alpha (-L_\alpha^* L_\alpha B - B L_\alpha^* L_\alpha + L_\alpha^* B L_\alpha) \tag{3.3}$$

3.1.2. Interpretation of Lindblad generators

Of course for a physicist the greatest merit in theorem 3.1 and especially formula (3.1) is in giving us insight into the interpretation of open quantum systems. So let us look what the form of the generator tells us about the evolution of the open quantum system.

The generator (3.1) is obviously made up of two different parts. The first is the so-called *dissipative* evolution and is generated by $\mathcal{Z}(B) := K^* B + BK$. Generators of this form belong to evolutions of the form $B \to U^*(t) B U(t)$ where $U(t) : \mathcal{B} \to \mathcal{B}$ is the, in general non-unitary, semigroup generated by K, i.e. $U(t) = \exp(tK)$. We usually denote this part of the evolution as $\mathbb{F}_0(t)(B) := U_t^* B U_t$.

From equation (3.3) we can see that the generator K itself contains two parts: On the one hand an unitary Hamiltonian evolution generated by iH, which can be interpreted as the free evolution of the open system undisturbed by the environment. An unitary evolution is in this sense a special case of open system evolution. On the other hand a contractive part generated by the manifestly negative operator $-\sum_\alpha L_\alpha^* L_\alpha$. This part of K "destroys" probability in the sense that there are pure states $\varphi \in \mathcal{H}$ such that $\|U_t \varphi\| < 1$ for $t > 0$.

To get a better understanding of the second part of the generator, $\mathcal{J}(B) := L^* \mathbb{1}_\mathcal{K} \otimes BL$, we first note that this term is manifestly completely-positive, since the map is already in Stinespring form. Observe that we get a semigroup with the generator (3.1) as a perturbation of semigroups of the form $\mathbb{F}_0(B) = U_t^* B U_t$. We just have to construct solutions to the following integral version of the Cauchy equation:

$$\mathbb{E}(t) = \mathbb{F}_0(t) + \int_0^t \mathbb{F}_0(t-s) \circ \mathcal{J} \circ \mathbb{E}_0(s) \mathrm{d}s. \tag{3.4}$$

To see that the solution $\mathbb{E}$ has the right generator we look at the differential of $\frac{\mathrm{d}}{\mathrm{d}s} \mathbb{F}_0(t-s) \circ \mathbb{E}(s)|_{t=s}$. Solutions to the above equations can be constructed through a simple Picard-Lindelöf iteration:

$$\mathbb{E}(t)^{n+1} = \mathbb{F}_0(t) + \int_0^t \mathbb{F}_0(t-s) \circ \mathcal{J} \circ \mathbb{E}^n(s) \mathrm{d}s. \tag{3.5}$$

It is easy to see that this sequence converges since the difference between two terms is completely-positive and all terms are bounded, because all the maps $\mathbb{E}^n$ are contractive by (3.2).

This construction of the generator gives us a clear interpretation of the second part of the generator: it corresponds to quantum events or jumps which are interlaced with the absorptive evolution $\mathbb{F}_0(t)$ given by the dissipative part of the generator.

Proof idea We have now shown that all semigroups with a generator in Lindblad form (3.1) are completely-positive. Since the structure of the Lindblad generators is of imminent importance to us, we also sketch a proof of the other direction, showing that all generators of completely-positive semigroups are of Lindblad form. The outlined proof idea can be found in [53, 57] and has the advantage that it, to some extend, generalizes to the unbounded case.

We start by taking advantage of the complete-positivity of the semigroup $\mathbb{E}_t$, which is equivalent to:

$$\sum_{i,j} \langle \varphi_i, \mathbb{E}(t)(B_i^* B_j)\varphi_j \rangle \geq 0 \tag{3.6}$$

for finite families of operators $B_i \in \mathfrak{B}(H)$ and vectors $\varphi_i \in \mathcal{H}$ for $1 \leq i \leq n$ and arbitrary n. Under the additional constraint $\sum_i B_i \varphi_i = 0$, we get the same inequality for the generator of the semigroup:

$$\sum_{i,j} \langle \varphi_i, \mathcal{L}(B_i^* B_j)\varphi_j \rangle \geq 0 \tag{3.7}$$

This is the case because the constraint entails that $\sum_{i,j} \langle \varphi_i, \mathrm{id}(B_i^* B_j)\varphi_j \rangle = 0$. This property is called *complete dissipativity* and can be shown to be sufficient to guarantee that an operator generates an completely-positive semigroup.

The last step in the proof is to find an operator $K \in \mathfrak{B}(\mathcal{H})$ such that the map $B \mapsto \mathcal{L}(B) - K^* B - BK$ is completely-positive. By the Stinespring theorem this is sufficient to show Lindblad form. If we fix an arbitrary vector φ_0, a straightforward calculation shows that the operator K defined by

$$\langle \varphi, K\psi \rangle := \langle \varphi_0, \mathcal{L}\left(|\varphi_0\rangle\langle\varphi|\right)\psi \rangle - \frac{1}{2}\langle \varphi_0, \mathcal{L}\left(|\varphi_0\rangle\langle\varphi_0|\right)\varphi_0 \rangle \langle \varphi, \psi \rangle \tag{3.8}$$

does the trick.

The freedom of choice for the vector φ_0 in the above proof turns out to play an important role for the form of the Lindblad equation. It leads to a gauge freedom of the Lindblad generator, which is intimately connected with the structure of the continuous time-dilations we are going to construct. We discuss this gauge freedom in section 3.3.

There are numerous directions in which theorem 3.1 has been extended. We want to mention the version for completely-positive norm-continuous semigroups between general C^*-algebras by [23] exploiting a connection between completely-dissipative operators and cohomology of operator algebras [65].

However, of far greater importance to us is the large body of work extending the Lindblad theorem results to semigroups with weaker continuity conditions, i.e. strong continuity. We discuss this body of work in the following section.

3.2. Strongly continuous semigroups

Strongly continuous semigroups of completely-positive maps are arguably of vital importance to the theory of open quantum systems, since they regularly emerge as a natural description in the case of infinite-dimensional open quantum systems. An example is the already mentioned atom in a cavity, see section 2.2.4. If we include the light field in the cavity into the open system the problem obviously gets infinite dimensional, and typically the evolution is only strongly continuous. This is similar for any quantum system with a continuous degree of freedom, like position.

The case of weak-$*$ continuous semigroups however does not add much further generality since, as already mentioned, we can always restrict to its pre-dual semigroup which is again strongly continuous. Surprisingly there does not exist a direct analogue of theorem 3.1 for this case of strongly continuous semigroups. Hence the description of all generators of strongly continuous semigroups remains open.

There are however strong partial results for the generalization of theorem 3.1. For example the results by Holevo and Davies [28, 56], which show how to construct completely-positive semigroups from generators of the form (3.1). Or [29, 53] showing that under some additional assumptions on the generator of completely-positive semigroups is of form (3.1).

The short summary of these results seems to be that for all practical purposes the Lindblad theorem still holds true in the case of unbounded generators, however of course with some modifications. That is, generators of Lindblad type lead to completely-positive evolutions. Conversely every semigroup a physicist was ever interested in, seems to have a Lindblad-type generator.

This situation if however foiled by a counterexample due to Holevo. It shows the existence of a completely-positive semigroup, whose generator can in some sense not be of the standard Lindblad form. In terms of notation yet to be introduced, the semigroup can not be the minimal solution of a Lindblad equation. However the constructed semigroup does not seems to describe a physically interesting situation. Furthermore the example still has the already described structure of absorptive evolution interlaced with events. This might explain why it was, so far, almost never cited. Nevertheless it seems to be far better known than its citation statistics suggest. And at least from a mathematical viewpoint it is an extremely interesting result!

One of the main problems with the Lindblad form for unbounded generators is that to even write it down we have to make some pretty serious assumptions about its domain. Consequently this plays a major role in the construction of Holevo's counterexample [54]. We later return to an in-depth discussion of the counterexample in section 5.

3.2.1. Minimal solutions

A theorem proving the existence of solutions to the Lindblad equation with unbounded Lindbladian, generally looks like the following. We adapted a version from [56].

Theorem 3.2. *With $\mathcal{H}$ a Hilbert space and $\mathcal{K}$ a separable Hilbert space, let $K : \mathcal{H} \supset \operatorname{dom} K \to \mathcal{H}$ be the generator of a one-parameter contraction semigroup and $L : \operatorname{dom} K \subset \operatorname{dom} L \to \mathcal{K} \otimes \mathcal{H}$ be an operator satisfying the* infinitesimal conservativity condition*:*

$$\|L\psi\|^2 \leq -2\,\mathfrak{Re}\,\langle \psi, K\psi \rangle \,. \tag{3.9}$$

Then there is a unique weak-∗-continuous contraction semigroup $\mathbb{E}_{min}$ on $\mathfrak{B}(\mathcal{H})$ called the minimal solution *solving the Cauchy equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi, \mathbb{E}_{min}(t)(B)\psi \rangle = \langle \psi, \mathcal{L}\left(\mathbb{E}_{min}(B)\right) \psi \rangle \,, \tag{3.10}$$

where the generator $\mathcal{L}$ is determined by:

$$\langle \psi, \mathcal{L}(B)\psi \rangle = \langle K\psi, B\psi \rangle + \langle \psi, BK\psi \rangle + \langle L\psi, \mathbb{1}_{\mathcal{K}} \otimes BL\psi \rangle \,. \tag{3.11}$$

This solution satisfies the minimality condition that for any other weak-∗-continuous contraction semigroup $\widetilde{\mathbb{E}}(t)$ on $\mathfrak{B}(\mathcal{H})$ satisfying equation (3.11) we have that $\widetilde{\mathbb{E}}(t) - \mathbb{E}_{min}(t)$ is completely-positive for all $t \geq 0$.

Here we do not discuss proof techniques for theorem 3.2 in detail. However we reproduce a construction in section 4.3.2. It is important to mention that in order to understand the above result, many of the techniques discussed in previous section about bounded generators still are useful. This is especially true for the separation of the evolution into two parts: The dissipative part and the quantum jump part.

There are a few things we should note about the above theorem. Firstly we again deviate a bit from standard notation by not introducing a basis on $\mathcal{K}$. This would lead to the Lindblad form:

$$\langle \psi, \mathcal{L}(B)\psi \rangle = \langle K\psi, B\psi \rangle + \langle \psi, BK\psi \rangle + \sum_\alpha \langle L_\alpha \psi, BL_\alpha \psi \rangle . \tag{3.12}$$

Here $L_\alpha : \mathrm{dom}(K) \subset \mathrm{dom}(L_\alpha) \to \mathcal{H}$ is a family of unbounded operators on $\mathcal{H}$, such that:

$$\sum_\alpha \|L_\alpha \psi\|^2 \leq -2\,\mathfrak{Re}\,\langle \psi, K\psi \rangle . \tag{3.13}$$

Furthermore we note that equations (3.10) and (3.11) are not exactly specifying a generator for the semigroup $\mathbb{E}_{\min}(t)$. Instead they define a generator for the pre-dual evolution $\mathbb{E}_{\min,*}(t) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$, which is strongly continuous and acts on the trace-class operators.

However even that is not entirely true, because the generator of the semigroup is only defined on the span of Ketbras of vectors in the $\mathrm{dom}(K)$:

$$|\mathcal{D}\rangle\langle\mathcal{D}| := \mathrm{span}\,\{|\psi\rangle\langle\psi| | \psi \in \mathrm{dom}\,\mathcal{K}\} . \tag{3.14}$$

The problem is that $|\mathcal{D}\rangle\langle\mathcal{D}|$, while certainly being a dense set, might not be core for $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to \mathfrak{T}$. This means that the generator of $\mathbb{E}_{\min,*}(t)$ is not just the closure of $\mathcal{L}$ defined on $|\mathcal{D}\rangle\langle\mathcal{D}|$.

The set $|\mathcal{D}\rangle\langle\mathcal{D}|$ is nonetheless immensely important, for example because it is a core for the dissipative part of $\mathbb{E}_{\min}(t)$. To see what this means we need the following notation:

Definition 3.3. Let K be as in theorem 3.2. Then $U(t) := \exp(tK)$ denotes the semigroup generated by K, $\mathbb{F}_{0,*}(t)(\rho) := U(t)\rho U^*(t)$ its ampliation to the trace class operators given by the adjunction and $\mathbb{F}_0(t)(B) := U^*(t)BU(t)$ the natural dual acting on $\mathfrak{B}(\mathcal{H})$. That is:

$$U(t) : \mathcal{H} \to \mathcal{H} \qquad \phi \to \exp(tK)\phi, \tag{3.15}$$

$$\mathbb{F}_{0,*}(t) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}) \qquad T \to U(t)TU^*(t)\phi, \tag{3.16}$$

$$\mathbb{F}_0(t) : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \qquad B \to U^*(t)TU(t)\phi. \tag{3.17}$$

We denote the generator of $\mathbb{F}_{0,*}(t)$ by $\mathcal{Z} : \mathrm{dom}(\mathcal{Z}) \to \mathfrak{T}(\mathcal{H})$, i.e.:

$$\mathbb{F}_{0,*}(t) = \exp(t\mathcal{Z}). \tag{3.18}$$

Corollary 3.4. *In the notation of the previous definition* $\mathcal{Z} : \mathrm{dom}(\mathcal{Z}) \to \mathfrak{T}(\mathcal{H})$ *is the closure of the operator defined by:*

$$\widetilde{\mathcal{Z}}(|\psi\rangle\langle\psi|) = |K\psi\rangle\langle\psi| + |\psi\rangle\langle K\psi|. \tag{3.19}$$

Proof. This is just a reformulation of the fact that $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a core for $\mathcal{Z}$, which follows from its invariance under $\mathbb{F}_{0,*}(t)$ by theorem 2.33. □

Especially helpful for the interpretation of the dissipative part of the generator is the following observation. The quadratic form defining the generator of the dissipative evolution $\mathcal{Z}$ is just the norm loss of the semigroup $U(t)$. This observation is the central idea behind the exit space construction, which we describe in section 4.2.

Corollary 3.5. *Let $U(t) : \mathcal{H} \to \mathcal{H}$ be a contraction semigroup with generator K on a Hilbert space $\mathcal{H}$. With $\psi \in \mathrm{dom}(K)$ the following expression extends to a quadratic form on* $\mathrm{dom}(K)$*:*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\psi\|^2\bigg|_{t=0} = -2\,\mathfrak{Re}\langle\psi, K\psi\rangle. \tag{3.20}$$

Polarization shows that the quadratic form is given by:

$$(\varphi, \psi) \to -\left(\langle\varphi, K\psi\rangle + \langle K\varphi, \psi\rangle\right). \tag{3.21}$$

The semigroup $\mathbb{F}_{0,*}(t)$ defined above is exactly one possible regularization of the above quadratic form. This construction of the dissipator gives a rigorous meaning to the interpretation that the part of the Lindblad equation given by $\mathcal{Z}$ describes the norm loss of the system. This viewpoint is for example useful to construct arrival time measures [85].

Theorem 3.2 can actually be slightly optimized using the notation from definition 3.3. That is, equation (3.11) might be extended to $\mathrm{dom}(\mathcal{Z})$. However even $\mathrm{dom}(\mathcal{Z})$ might still not be a core for the generator of $\mathbb{E}_{\mathrm{min},*}$ [28, 56]. An example of this phenomenon is part of Holevo's counterexample 5.

Observe that to write down a straightforward generalization of the Lindblad equation we would have to assume that $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a core for the generator $\mathcal{L}$ of $\mathbb{E}_{\mathrm{min},*}(t)$. On the contrary if there are not "enough" ketbras $|\psi\rangle\langle\psi|$ in the domain, we can not even write down equation (3.11).

The domain problems just described are precisely the reason why one has to be careful when comparing different versions of unbounded Lindblad equations. For example Davies original proof [28] is formulated only for the hermitian subspace of $\mathfrak{T}(\mathcal{H})$. The paper furthermore constructs the strongly continuous semigroup $\mathbb{E}_{\mathrm{min},*}$ instead of its pre-dual. In many other cases equation (3.11) is viewed as a quadratic form on $\mathrm{dom}(K)$, e.g. [53].

Observe that, apart from the domain requirements, the only condition regularizing the operator $L : \mathrm{dom}\, L \to \mathcal{K} \otimes \mathcal{H}$ in theorem 3.2 is the infinitesimal conservativity condition, given in equation (3.9). Apart from that the operator L can be pretty wild, in particular it does not have to be closeable. Nonetheless we get that it is relatively bounded by the generator of the dissipative evolution.

Lemma 3.6. *Under the conditions from theorem 3.2 especially equation (3.9) the operator L is relatively bounded by K in the sense that for every $\alpha \geq 0$ we have:*

$$\|L\psi\| \leq \|(\alpha K - \frac{1}{\alpha}\mathbb{1})\psi\| \leq \alpha\|K\psi\| + \frac{1}{\alpha}\|\psi\| \tag{3.22}$$

Proof. The statement is an immediate consequence of the following calculation:

$$\left\|(\alpha K - \frac{1}{\alpha}\mathbb{1})\psi\right\|^2 = \alpha^2\|K\psi\|^2 + \frac{1}{\alpha}\|\psi\| - 2\,\mathfrak{Re}\langle\psi, K\psi\rangle \geq \|L\psi\|^2. \tag{3.23}$$

□

Definition 3.7. Analogous to $\mathcal{Z}$ in definition 3.3 the completely-positive part of the generator in equation (3.11) can also be used to define an unbounded operator $\mathcal{J}$ on the trace-class operators.

We start with a regularized version, defined on $\mathrm{dom}(\mathcal{Z})$ with the domain norm:

$$\mathcal{J}_{\mathrm{reg.}} : \left(\mathrm{dom}(\mathcal{Z}), \|\cdot\|_{\mathrm{dom}(\mathcal{K})}\right) \to \mathfrak{T}(\mathcal{H}) |\psi\rangle\langle\psi| \to \mathrm{tr}_{\mathcal{K}}(|L\psi\rangle\langle L\psi|). \tag{3.24}$$

Since $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a core for $\mathrm{dom}(\mathcal{Z})$ the above definition uniquely extends to an operator. We can also view equation (3.24) as defining an unbounded operator

$$\mathcal{J} : |\mathcal{D}\rangle\langle\mathcal{D}| \to \mathfrak{T}(\mathcal{H}). \tag{3.25}$$

We take a closer look at this regularization in section 4.1.4. There we analyze the needed notion of complete-positivity.

It turns out that the minimal solution $\mathbb{E}_{\min}(t)$ in theorem 3.2 is actually completely determined by the regularized operator $\mathcal{J}_{\mathrm{reg.}}$ [56]. So for the theory of minimal solutions it does not matter which extension of $\mathcal{J}$ we take to define the "generator" (3.11).

There is an important other way to state theorem 3.2, i.e. its integral form.

Corollary 3.8. *Under the condition of theorem 3.2 and using the notation from definition 3.3, every weak-∗-continuous semigroup fulfilling the Cauchy equation* (3.10) *also fulfills the* integral form of the Cauchy equation. *That is for* $\psi \in \mathrm{dom}(K)$ *we have:*

$$\langle\psi, (\mathbb{E}(t) - \mathbb{F}_0(t))\,(B)\psi\rangle = \int_0^t \langle\psi, \mathbb{F}_0(t-s)\mathcal{J}\mathbb{E}(s)(B)\psi\rangle \,\mathrm{d}s \tag{3.26}$$

Conversely every weak-∗continuous semigroup fulfilling the integral form of the Cauchy equation also solves the Cauchy equation 3.10.

Proof. The equivalence follows by looking at the derivative of the scalar-valued function $s \to \langle\psi, \mathbb{F}_0(t-s) \circ \mathbb{E}(s)\psi\rangle$. □

3.2.2. Unitality of minimal solutions

Another important observation about theorem 3.2 is that the solution $\mathbb{E}_{\min}$ is neither promised to be unique, nor unital. This is true even in the case when we have equality in equation (3.9). The issue of unitality, also called *conservativity*, has of course attracted much attention, since it reflects conservation of probability. Hence unitality is of crucial importance to the physical interpretation of the semigroup [55, 12, 22, 20, 19, 18].

Corollary 3.9. *In the case when the minimal solution is conservative it is the unique conservative weak-∗continuous solution to equation* (3.10)*.*

This is fairly straightforward to see since, due to the minimality of $\mathbb{E}_{\min}(t)$ for any other solution $\widetilde{\mathbb{E}}(t)$ and an effect $0 \le B \le \mathbb{1}$, the two operators:

$$-\left(\left(\widetilde{\mathbb{E}}(t) - \mathbb{E}_{\min}(t)\right)(B)\right) = \left(\widetilde{\mathbb{E}}(t) - \mathbb{E}_{\min}(t)\right)(\mathbb{1} - B) \tag{3.27}$$

would both have to be positive.

To understand the appearance and behavior of non-unital solutions it is instructive to take a look at the case of classical Markov semigroups. These are semigroups describing the infinitely many states of some classical system. The statistical interpretation of classical Markov semigroups is similar to the quantum case. As usual the classical case is included in the quantum one in the form of diagonal matrices and generators $\mathcal{L}$ leaving the diagonal invariant.

Examples of classical non-unital Markov semigroups are for example pure jump processes with a non-zero probability for infinitely many jumps in finite time. Another possibility are birth processes where infinitely many particles are generated in finite time. From a minimal solution of such a system, where some trajectories "terminate" at infinity, one can then construct different unital solutions by reinserting the system, i.e. resetting the system to some fixed state after it "reached infinity". Exactly the same phenomena arises in the fully quantum case.

3.3. Gauge invariance

It is well known that for a given Lindblad generator $\mathcal{L}$ the choice of operators K and L are far from unique. Discussions of this gauge freedom can be found in [72, 53, 30]. The reason for this non-uniqueness is essentially the same as in the construction of the Stinespring dilation, which is clear from [53, theorem 1].

That is, the non-uniqueness of the Lindblad corresponds to the choice of a dilation space for the associated quadratic form. Since $\mathcal{L}$ is not completely-positive but only conditionally-completely-positive the situation becomes a bit more complicated. In the end we mainly have the freedom of choosing the dilation space for the completely-positive part of the Lindblad form, i.e. a unitary on $\mathcal{K}$ and additionally a vector in $\mathcal{K}$. On top of that there is the usual freedom of choice for the phase of pure states.

Definition 3.10. Let $K(t) : \mathrm{dom}(K) \to \mathcal{H}$ and $L(t) : \mathrm{dom}(K) \to \mathcal{K} \otimes \mathcal{H}$ define a Lindbladian $\mathcal{L}(t)$ as in equation (3.11)

For every $t \in [0, T]$, let $x(t) \in \mathbb{R}$, let $\lambda(t) \in \mathcal{K}$ and let $U(t) \in \mathfrak{B}(\mathcal{K})$ be a unitary operator.

The following families of operators define a *gauge shifted Lindbladian* with the gauge triple $(U(t), \lambda(t), x(t))$:

$$\widetilde{L}(t) = U(t) \otimes \mathbb{1}_{\mathcal{H}} L(t) + |\lambda(t)\rangle \otimes \mathbb{1}_{\mathcal{H}} \tag{3.28}$$

$$\widetilde{K}(t) = K(t) - \frac{1}{2}\|\lambda(t)\|^2 - (\langle U(t)^*\lambda| \otimes \mathbb{1})L(t) + ix(t)\mathbb{1}_{\mathcal{H}} \tag{3.29}$$

If the Lindbladian or the gauge is time-independent we use the notation accordingly.

An explicit calculation immediately shows that $\mathcal{L}(t) = \widetilde{\mathcal{L}}(t)$. It requires a bit more work to see that $\mathcal{L}(t)$ is actually a standard Lindblad generator, i.e. $\widetilde{K}(t)$ is a generator of a contraction semigroup. Note that $\mathrm{dom}(K) = \mathrm{dom}(\widetilde{K})$

Lemma 3.11. *In the situation of definition 3.10 we have that for any $\psi, \varphi \in \mathrm{dom}(K)$ and $B \in \mathfrak{B}(\mathcal{H})$:*

$$\langle \varphi, \mathcal{L}(B)\psi \rangle = \left\langle \varphi, \widetilde{\mathcal{L}}(B)\psi \right\rangle \tag{3.30}$$

Furthermore $\widetilde{K} : \mathrm{dom}(K) \to \mathcal{H}$ is the generator of a contraction semigroup.

Proof. First note that $\widetilde{\mathcal{L}} = \mathcal{L}$, which follows by direct calculation.

To see that $\widetilde{\mathcal{L}}$ is again a valid Lindblad generator, we have to show that $\widetilde{K}(t)$ is the generator of a contraction semigroup. Firstly note that the gauged $K(t)$ is again a dissipative operator. To see that use the infinitesimal normalization condition from equation (3.9).

$$2\,\mathfrak{Re}\left\langle \psi, \left(K - \langle\lambda| \otimes \mathbb{1}L - \tfrac{1}{2}\|\lambda\|^2\right)\psi\right\rangle \tag{3.31}$$

$$\leq - \|L\psi\|^2 - 2\,\mathfrak{Re}\left\langle \psi, \langle\lambda| \otimes \mathbb{1}L\psi\right\rangle - \|\lambda \otimes \psi\|^2 \tag{3.32}$$

$$= - \|(L + \lambda \otimes \mathbb{1})\psi\|^2 \leq 0 \tag{3.33}$$

Furthermore note that L is relatively bounded with respect to K with relative bound smaller than one by lemma 3.6. More explicit we have the following bound for any $\alpha > 0$

$$\|(\langle\lambda| \otimes \mathbb{1})L\psi\|^2 \leq \|\lambda\|^2\|L\psi\|^2 = -2\|\lambda\|^2\,\mathfrak{Re}\langle\psi, K\psi\rangle \tag{3.34}$$

$$\leq \alpha^2\|K\psi\|^2 - 2\|\lambda\|^2\,\mathfrak{Re}\langle\psi, K\psi\rangle + \tfrac{\|\lambda\|^4}{\alpha^2}\|\psi\|^2 \tag{3.35}$$

$$= \alpha\left\|(K - \frac{\|\lambda\|^2}{\alpha^2})\psi\right\|^2 \tag{3.36}$$

By [74, Corollary 3.3.3] this guarantees that $\widetilde{K}$ is again the generator of a contraction semigroup. □

The preceding lemma only asserts us that a gauge maps a family of Lindblad generators to another family of Lindblad generators. A priory it is not clear whether the associated new Cauchy equation is solvable or not. The continuity conditions which ensure that $t \to \widetilde{L}(t)$ is still continuous are easy to work out. However it is not that clear when $\widetilde{K}(t)$ still generates an evolution system. Even more we have to show that $K(t)$ generates a $\mathcal{D}$-valued evolution system for some Hilbert space $\mathcal{D}$. In the special case that $\mathrm{dom}(K(t)) = \mathrm{dom}(K(0))$ for all t, we can find easy criteria for this to happen.

Definition 3.12. In the situation of definition 3.10 we define the following three functions:

$$\lambda : [0,T] \to \mathcal{K} \qquad t \mapsto \lambda(t) \tag{3.37}$$

$$U(t) : [0,T] \to \mathfrak{B}(\mathcal{K}) \qquad t \mapsto U(t) \tag{3.38}$$

$$x(t) : [0,T] \to \mathbb{R} \qquad t \mapsto x(t) \tag{3.39}$$

$$\tag{3.40}$$

Let x be continuously differentiable, λ continuously differentiable in norm and $U(t)$ continuously differentiable in strong topology. The triple (U, λ, x) is called a *continuously differentiable gauge.*

Lemma 3.13. *In the situation of definition* (3.28) *let* (U, λ, x) *be a continuously differentiable gauge in the sense of definition 3.12.*

Assume that $\mathrm{dom}(K(t)) = \mathrm{dom}(K(0)$ *for all* t *and set* $\mathcal{D} = \mathrm{dom}(K(0))$.

The family of gauged operators $\widetilde{K}(t)$ *generates an* $\mathcal{D}$*-valued evolution system and the Cauchy equation with Lindblad generator* $\widetilde{\mathcal{L}}$ *possesses a minimal solution.*

Proof. Note that also $\mathrm{dom}(\widetilde{K}(t)) = \mathrm{dom}(K(0))$ for every t. We define the set $\mathcal{D} := (\mathrm{dom}(K(0)), \|\cdot\|_{\mathrm{dom}(K(0))})$. And for some $\mu \geq 0$ we define $Q(t) := \mu\mathbb{1} - \widetilde{K}(t)$. The map $t \to Q(t)\psi$ is clearly continuously differentiable for all $\psi \in \mathrm{dom}(\mathcal{K})$. Furthermore for every μ, $\widetilde{K}(t)$ is the generator of a contraction semigroup. By theorem [74, theorem 5.4.6] this is sufficient to ensure that the family $K(t)$ generates an $\mathcal{D}$-valued evolution system. □

3.4. Stinespring continuity

Another possible approach to the Lindblad form uses the continuity of the Stinespring dilation which we mentioned in section 2.2.1. In particular one can reason that the continuity of the semigroup implies that the Stinespring dilation of the semigroup near the identity can be chosen in such a way that it has a single Kraus operator converging in norm to the identity, while all the others vanish, i.e. a dominant Kraus operator.

The connections between the Lindblad form, Stinespring continuity and the existence of a dominant Kraus operator are far from understood. Hence we can only present partial results. We think that these connections are an interesting topic for further investigations, see 10.

The main obstacle in the analysis it that the Stinespring continuity theorem, i.e. point 4 of theorem 2.18 needs a rather strong form of continuity on the side of the completely-positive maps, i.e. continuity in c.b.-norm. So it looks like we can not even make such an argument for norm-continuous semigroups. Luckily it turns out that near the identity the operator norm and the c.b.-norm are equivalent, as it is shown e.g. in [64].

Let us start with a closer look at the Stinespring dilations of the identity operator $\mathrm{id} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$, which is obviously a completely-positive and normal map. A minimal Stinespring isometry is given by the single Kraus operator $\mathbb{1}$. The minimal dilation space is hence just the space of complex numbers $\mathbb{C}$.

By the uniqueness clause of Stinespring theorem, we get all Stinespring dilations of the identity with dilation space $\mathcal{K}$, via an isometry from $\mathbb{C}$ to $\mathcal{K}$. It is easy to see that all these isometries are given by "Kets" $|\psi\rangle$. In other words all Stinespring dilations $V_\psi : \mathcal{H} \to \mathcal{K}\otimes\mathcal{H}$ have the form $\varphi \to \psi \otimes \varphi$ for a vector $\psi \in \mathcal{K}$.

Hence Stinespring dilations of the identity can always be chosen to have only one Kraus operator. By Stinespring continuity this determines a dominant Kraus operator for maps near the identity.

Corollary 3.14. *Let $\mathbb{E}(t)$ be a unital evolution system continuous in c.b.-norm, i.e. $\lim_{t\to 0} \|\widehat{\mathbb{E}}(t) - \mathrm{id}\|_{\text{c.b.}} = 0$. Then there is a constant k such that for every t one can find a set of Kraus operators V_α, labeled by a countable set $\alpha \in \mathbb{A} \cup \{0\}$, with a distinguished element V_0, s.t.*

$$\| \sum_{\alpha\in\mathbb{A}} V_\alpha^* V_\alpha \| \le tk \qquad \left\| \frac{1}{2}\left(V_{\mathbb{E},0} + V_{\mathbb{E},0}^*\right) - \mathbb{1} \right\| \le \frac{1}{2} kt \tag{3.41}$$

Proof. Fix a time t. By theorem 2.18 one can find a separable Hilbert space $\mathcal{K}$ and two Stinespring isometries:

$$V_{\mathbb{E}} : \mathcal{H} \to \mathcal{K}\otimes\mathcal{H} \text{ such that } V_{\mathbb{E}}^* \mathbb{1}_{\mathcal{K}} \otimes X V_{\mathbb{E}} = \mathbb{E}(t)(X) \tag{3.42}$$

$$\text{and } V_{\text{id}} : \mathcal{H} \to \mathcal{K}\otimes\mathcal{H} \text{ such that } V_{\text{id}}^* \mathbb{1}_{\mathcal{K}} \otimes X V_{\text{id}} = X \tag{3.43}$$

$$\text{such that } \|V_{\mathbb{E}} - V_{\text{id}}\|^2 \le \|\mathbb{E}(t) - \text{id}\|_{\text{c.b.}} . \tag{3.44}$$

By the previous discussion $V_{\mathbb{1}}$ is of the form $\varphi \to \psi \otimes \varphi$ for a unit vector $\psi \in \mathcal{K}$. Let us complement ψ to a countable basis $\{|\alpha\rangle\}$ off $\mathcal{K}$. That is we write $\alpha \in \mathbb{A} \cup \{0\}$ for a countable set $\mathbb{A}$ and $\psi = |0\rangle$. In abuse of notation we can now write $V_{\mathbb{1}} = |0\rangle \otimes \mathbb{1}$. Let $V_{\mathbb{E},\alpha}$ for $\alpha \in \mathbb{A} \cup \{0\}$ denote the corresponding Kraus operators of $\mathbb{E}(t)$. We get:

$$\|\mathbb{E}(t) - \mathrm{id}\|_{\mathrm{c.b.}} \geq \|V_{\mathbb{E}} - V_{\mathrm{id}}\|^2 \tag{3.45}$$
$$= \left\|(V_{\mathbb{E},0} - \mathbb{1})^*(V_{\mathbb{E},0} - \mathbb{1}) + \sum_{\alpha \in \mathcal{A}} V_{\mathbb{E},\alpha}^* V_{\mathbb{E},\alpha}\right\| \tag{3.46}$$
$$= \left\|(V_{\mathbb{E},0} - \mathbb{1})^*(V_{\mathbb{E},0} - \mathbb{1}) + \mathbb{1} - V_{\mathbb{E},0}^* V_{\mathbb{E},0}\right\| \tag{3.47}$$
$$= \left\|V_{\mathbb{E},0} - \mathbb{1} + V_{\mathbb{E},0}^* - \mathbb{1}\right\| \tag{3.48}$$
$$\tag{3.49}$$

In the second equality we used the unitality of $\mathbb{E}(t)$. The bound on $\sum_\alpha V_\alpha^* V_\alpha$ follows by positivity. The linear behavior of $\|\mathbb{E}_t - \mathrm{id}\|_{\mathrm{c.b.}}$ can be seen from

$$\|\mathrm{id} - \mathbb{E}(t)\| = \left\|\int_0^t \mathbb{E}(t)\mathcal{L}\right\| \leq t\|\mathcal{L}\|. \tag{3.50}$$

□

To extend this result at least to norm continuous semigroups, one would need a stronger version of the Stinespring continuity. Luckily it turns out that near the identity the continuity result can be considerably strengthened. That is we can replace the c.b.-norm by the normal operator norm and choose an arbitrary dilation space. This has however the drawback that we have to trade in the order $\sqrt{\tau}$ for a order $\tau^{1/4}$. This has already been noted in [63].

Lemma 3.15. *Let $\mathbb{E} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a completely-positive normal map and let $V_{\mathbb{E}} : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ be a Stinespring isometry for $\mathbb{E}$, i.e. $\mathbb{E}(X) = V_{\mathbb{E}}^* \mathbb{1}_{\mathcal{K}} \otimes X V_{\mathbb{E}}$. Then one can find a Stinespring isometry for the identity $V_{\mathrm{id}} : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ on the same dilation space such that*

$$\|U \otimes \mathbb{1}_{\mathcal{H}} V_{\mathbb{E}} - V_{\mathrm{id}}\|^4 \leq 8\|\mathbb{E} - \mathrm{id}\| \tag{3.51}$$

Proof. We can evaluate the distance on an arbitrary dilation space, because a minimal dilation space for $\mathbb{E}$ is automatically also an dilation space for $\mathbb{1}$. The second part it the norm inequality from [64, proposition 4.3]:

$$\|\mathbb{E} - \mathrm{id}\|_{\mathrm{c.b.}} \leq 8\|\mathbb{E} - \mathrm{id}\|^{\frac{1}{4}}. \tag{3.52}$$

□

Sadly these results do not seem to extend to the case of weak-∗continuous or even strongly continuous semigroups. It is not even clear how such a generalization should look like.

4. Measurement on arrival

In this chapter we want to elucidate the connections between the Lindblad form of the generators of completely-positive evolutions, covariant arrival-time measures, capture by absorption, measurement on arrival, reinsertion of quantum systems and perturbation theory of semigroups.

There are indeed strong connections between all these different topics. Here we are mainly interested in leveraging these connections to get a better understanding of the Lindblad form.

We want to clarify the validity of the fundamental interpretation of the Lindblad form. That is, describing the evolution of an open system as alternation between a dissipative, i.e. "probability loosing", part and instantaneous quantum jump events.

Ultimately we also want to understand the limits of the Lindblad form which have a lot to do with domains of unbounded operators, and whether or not they contain certain algebras of operators [3]. These difficulties culminate in the counterexample to the Lindblad form [54], which is described in section 5.

For this purpose it seems helpful to take a step back and try to understand the general occurrence of these phenomena. This leads us to the idea of capture by absorption combined with reinsertion of the absorbed quantum systems.

The general problem underlying this whole chapter is the following: Given a quantum system with a known free evolution, how does the evolution change when we add boundaries and counters to the description. Furthermore when we have introduced detectors, we would like to describe their click statistics given that we know the initial state of the system. Another question would be what we can learn about the system from the click statistics of the detectors.

We try to break this problem into a series of simpler questions, and then answer as many of them as independently as possible. These questions are:

1. How does the system behave up to be point where it is absorbed by a counter or hits a boundary? That is, we want to describe the no-event evolution.

2. When does the system first encounter a detector or hit a boundary? That is, we want to describe arrival-time measures.

3. In which ways can the system be absorbed?

4. What happens to the system at the time the system is absorbed? Reflection, absorption or reinsertion in a fixed state? Can we classify all possibilities?

5. Which overall evolution do we get if we continue the evolution after the reinsertion?

Of course these problems can't be solved in full generality, i.e. incorporating all possible effects. Instead we have to make some serious restrictions to enable us giving simple answers to any of the questions.

Our main assumption is about question number 1. Here we assume that the detectors do not change the free evolution to much, i.e, we ignore back actions of the change of the detector state to the field. In other words we assume that even with the introduced environment, i.e. the detectors and boundaries, the evolution is still given by a semigroup or an evolution system. Boundaries and detectors hence only appear as part of the generator of the semigroup. More specifically they introduce a dissipative, i.e. absorptive, part in the generator. We refer to semigroup or evolution system describing this as the *no-event evolution.*

Since our analysis focuses mainly on the generators of the evolutions, i.e. the infinitesimal state changes, we carry it out in the Schrödinger picture. This has the advantage that with our usual assumptions the semigroups under question are strongly-continuous and there is a one-to-one correspondence between generators and semigroups.

With this assumption on the structure of the systems the three other questions have simple answers. At least if we want to conserve the Markovianity, i.e. semigroup structure, of the evolution, through all stages of our analysis.

We distinguish two cases of completely-positive semigroups or evolution systems for the description of the disturbed system before absorption.

Firstly the case of an evolution conserving the class of pure quantum states, i.e. evolution systems of the form:

$$\mathbb{F}_*(t,s)(\rho) = U(t,s)\rho U(t,s)^*. \tag{4.1}$$

The evolution is contractive in the sense that $\|U(t,s)\| \leq 1$ for all $0 \leq s \leq t$. We shall see that case is closely related to the Lindblad form. We base our exposition of this part on the work [85] and the ideas for an extension laid out in [86].

The second case is that of a general contractive evolution system, i.e. we only require that $\|\mathbb{F}_*(t,s)\| \leq 1$. This case is needed in the construction of Holevo's counterexample in section 5. An extensive study of this case which is the third main source for this exposition was given by Holevo in [56].

Our main contribution is in working out the details of [86] and adding time dependence for the generator. This time dependence is not a mere five-finger exercise, but later turns out to be important for an understanding of the Lindblad form.

To the work of Holevo we also add time-dependence and clarify the relation to our other two main sources [85] and [86]. Furthermore we improve the arrival-time measures to allow for a measurement on arrival and even a delayed choice measurement where it is possible. We also shift the whole treatment to the Schrödinger picture such that we can focus our attention on the infinitesimal behavior. In the Schrödinger picture almost all integrals can be interpreted in the sense Bochner, rather than as weak integrals, i.e. Pettis integrals.

4.1. Arrival-time measurements

We start with a short description of the general structure we expect of an arrival-time measurement. Here by arrival-time measurement we mean an instrument in the sense of Davies, see definition 2.21. The instrument should describe the distribution of the time of the first absorption event, i.e. a detector click or the system hitting a "wall" the first time together with the posterior state. Since we include a description of the posterior state, the description of the first-arrival-time is sufficient to describe multiple arrivals.

Arrival-time measures in this sense are a common solution to both questions 2 and 4. Furthermore we are not yet interested in the question which detector clicked or how to continue the evolution afterwards. We also so far ignore the possibility of employing a delayed choice measurement scheme, i.e. treating the type of detection as a quantum event. Note that we always treat the time of detection classically.

Let $\mathcal{H}$ be the Hilbert space describing the system. The time evolution disturbed by the detectors is assumed to be given in Schrödinger picture by a strongly-continuous contraction semigroup $\mathbb{F}_*(t) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ of completely-positive maps. This semigroup "looses" norm which we interpret as the probability that the system is captured.

4.1.1. Definition

An arrival-time measurement should be an instrument assigning a completely-positive map on the trace-class operators to every time-interval $[s,t]$. This is for every $0 \leq s \leq t$ we get an operator $\mathcal{M}([s,t]) \in \mathfrak{CP}(\mathfrak{T}(\mathcal{H}))$. To make the assignment of operators to intervals consistent $\mathcal{M}$ needs to be an instrument, i.e. has to be σ-additive in the strong topology.

Based on the physical interpretations we have two more requirements on arrival-time measures. Remember that an arrival-time measure always only describes the distribution of the time of the first detector click. Hence if we prepare our system earlier, or correspondingly switch on our detector later, i.e. start looking at its output later, the distribution of the first-click time should change accordingly. As a formula this reads $\mathcal{M}([s,t]) \circ \mathbb{F}_*(r) = \mathcal{M}([s+r, t+r])$. We call this property *covariance* of $\mathcal{M}([s,t])$ with respect to the semigroup $\mathbb{F}_*(t)$.

The other requirement is that the total probability for the first detection has to be bounded by the norm loss of the semigroup $\mathbb{F}_*(t)$. This is the case because we work in the Schrödinger picture and thus the norm, i.e. the trace, of a state $\rho(t)$ is just the total probability of detection. We thus interpret the norm loss of the semigroup on positive states as the probability of the system to be "lost", i.e. absorbed at a boundary. In other words for positive $0 \leq \rho \in \mathfrak{T}\mathcal{H}$ we get, using the linearity of the trace norm on positive states:

$$\|\mathcal{M}([s,t])(\rho)\|_{\mathrm{tr}} \leq \|\mathbb{F}_*(t)(\rho)\|_{\mathrm{tr}} - \|\mathbb{F}_*(s)(\rho)\|_{\mathrm{tr}} = \|(\mathbb{F}_*(t) - \mathbb{F}_*(r))(\rho)\|_{\mathrm{tr}} . \tag{4.2}$$

Observe that by covariance it is sufficient to require this only for sets of the form $[0,s]$.

Definition 4.1. Let $\mathcal{H}$ be a Hilbert space and $\mathbb{F}_* : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ a strongly-continuous contraction semigroup of completely-positive maps. Let σ be the Lebesgue σ-algebra on $\mathbb{R}^+$. We define an *(first-)arrival-time measure* for the semigroup $\mathbb{F}_*(t)$ to be a $\mathfrak{CP}(\mathfrak{T}(\mathcal{H}), \mathfrak{T}(\mathcal{K}))$ valued instrument on σ such that:

1. The measure is bounded by the norm loss of $\mathbb{F}_*(s)$ in the sense that for $[s,t] \subset \mathbb{R}^+$ we have for any positive $0 \leq \rho \in \mathfrak{T}(\mathcal{H})$:

$$\|\mathcal{M}([s,t])(\rho)\|_{\mathrm{tr}} = \|\mathbb{F}_*(s)(\rho)\|_{\mathrm{tr}} - \|\mathbb{F}_*(t)(\rho)\|_{\mathrm{tr}} \tag{4.3}$$

2. The measure is $\mathbb{F}_*$-*covariant* in the following sense. Let τ be a measurable set $s \in \mathbb{R}^+$ and $\tau + s$ the translation of the measurable set by s. Then we have:

$$\mathcal{M}(\tau) \circ F(s)(\rho) = \mathcal{M}(\tau + s)(\rho) \tag{4.4}$$

For technical reasons we require $\mathcal{M}$ to be inner regular and hence a Radon measure. That is:

$$\mathcal{M}(\tau) = \sup \{M(\kappa) | \kappa \subset \tau, \kappa \text{ compact}\} \tag{4.5}$$

We also we require $\tau \to \mathcal{M}(\tau)(\rho)$ to be an vector measure for all $\rho \in \mathfrak{T}(\mathcal{H})$ that is σ-additive in the norm topology.

The inner regularity ensures that it is sufficient to define $\mathcal{M}$ on intervals $[s,t]$ and then extend to the whole σ-algebra by completion.

Important cases for the space $\mathcal{K}$ are: $\mathcal{K} = \mathbb{C}$ for the case of a destructive arrival-time measurement, i.e. the system is destroyed on arrival. $\mathcal{K} = \mathcal{H}$ for the case of a repeatable first-arrival-time measurement, i.e. we can proceed our observation of the system after the first arrival and the nature of the quantum system does not fundamentally change, i.e. is still described by the same Hilbert space. The last case we shall study in more detail is $\mathcal{K} = \mathcal{T} \otimes \mathcal{H}$, with some other Hilbert space $\mathcal{T}$, for a delayed choice measurement on the first arrival, i.e. we can continue the observation and obtain additional "quantum" information about the arrival event.

Evolution system case

As a next step we turn to the notion of an arrival-time measure for the case where the no-event evolution is given by an evolution system. The generalization is straight forward when one remembers that the case of an evolution system corresponds to experiments which depend explicitly on the lab time, but are otherwise Markovian in the same sense as semigroups. This corresponds to the fact that the "generator" of an evolution system depends on an explicit time. Hence if we describe arrival-time measures in this setup we naturally end up with a whole family of arrival-time measurements, describing measurements indexed by the time t where the measurement was started. We arrive at the definition.

Definition 4.2. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}$ a Banach space which can be densely and continuously embedded into $\mathfrak{T}(\mathcal{H})$ and $\mathbb{F}_*(t,s) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ for $0 \le s \le t$ a strongly-continuous contractive evolution system of completely-positive maps which is $\mathcal{D}$-valued in the sense of definition 2.40. Let σ be the Lebesgue σ-algebra on $\mathbb{R}^+$. We define an *(first-) arrival time measure* for the evolution system $\mathbb{F}_*(t,s)$ to be a family of $\mathfrak{CP}(\mathfrak{T}(\mathcal{H}), \mathfrak{T}(\mathcal{K}))$ valued instruments $\mathcal{M}_t$ on σ for $t \ge 0$ such that:

1. The family $\mathcal{M}_t$ is bounded by the norm loss of $\mathbb{F}_*(t,s)$ in the sense that for $[s,t] \subset \mathbb{R}^+$ we have for any positive $0 \le \rho \in \mathfrak{T}(\mathcal{H})$ and $0 \le r \le s$:

$$\|\mathcal{M}_r([s,t])(\rho)\| = \|\mathbb{F}_*(s,r)(\rho)\| - \|\mathbb{F}_*(t,r)(\rho)\| \tag{4.6}$$

2. The family$\mathcal{M}_t$ is $\mathbb{F}_*$-*covariant*. That is: let τ be a measurable set in $[t,\infty)$ and $0 \le s \le t$. Then we have:

$$\mathcal{M}_t(\tau) \circ F(t,s)(\rho) = \mathcal{M}_s(\tau)(\rho) \tag{4.7}$$

Further every $\mathcal{M}_t$ should be inner regular and hence a Radon measure. And again every $\mathcal{M}_t$ is should be σ-additive in the strong topology.

4.1.2. Outline

The rest of our analysis of arrival-time measures is now divided into three parts. In the first we classify arrival-time measures in terms of their Radon-Nikodym derivatives w.r.t the Lebesgue measure. This characterization turns out to be even more important than the original definition of the measures, because it allows to define the integration of operator valued functions against arrival-time measures. Later we are exclusively interested in arrival-time measures which are of the form discussed there.

Next we look at various generalizations of the notion of an arrival-time measure, e.g. measures which describe not only the distribution of the first click time but simultaneously also the distribution of the corresponding detector output. And furthermore we try to describe how to treat the type of detection as a quantum event.

In the last part we analyze how one can integrate operator valued functions against an first-arrival-time measure. Since the arrival-time measure also tells us what happens on arrival we can we can then perturb the contractive evolution by repeatedly integrating against the arrival-time measure and hence adding the possibility of more and more arrivals. From that technique we can also obtain a description of the click statistics and of the system state post selected on the click distribution.

Many of the techniques used in this section have there counterparts in the fully quantum treatment of the delayed choice measurement, which culminates in the construction of cMPS in section 7.

4.1.3. Infinitesimal description of arrival-time measures

It is helpful for the physical intuition to have an infinitesimal description of first arrivals. That is we want to understand how the state of a system changes instantaneously on arrival, which can be deduced from the first-arrival-time measurement since these are given by an instrument, and hence also describe the post measurement state. It is already clear from the previous discussion that the state change on arrival is finite. We would like to classify which state changes are possible for a fixed no-event semigroup or evolution system.

Radon-Nikodym derivatives

Another viewpoint on the goal of this section is that we try to classify arrival time measures in terms of their Radon-Nikodym derivatives with respect to the Lebesgue measure. Note that for a separable Hilbert space $\mathcal{K}$ the set of trace-class operators $\mathfrak{T}(\mathcal{H})$ has the Radon-Nikodym property, as a separable dual space [34, theorem III.3.1].

The Radon-Nikodym theorem [34, section III] says that for every ω there is a Bochner integrable function $M_\omega : [0,T] \to \mathfrak{T}(\mathcal{K})$ such that

$$\mathcal{M}(\tau)(\omega) = \int_\tau M_\omega(t)\mathrm{d}t \tag{4.8}$$

The measure of integration being the Lebesgue measure. The necessary requirements, i.e that each of the vector measures $\tau \to \mathcal{M}(\tau)(\omega)$ has bounded variation and is continuous w.r.t. Lebesgue measure are imminent from definition 4.1. Of course we later want to interpret $\omega \to M_\omega(t)$ as a linear operator.

It is already clear at this point that to transfer this analysis to the case of evolution systems we need some kind of common dense domain $\mathcal{D}$, i.e. $\mathcal{D}$-valued evolution systems. Otherwise continuity w.r.t the Lebesgue measure can not be deduced from point 1 of definition 4.2.

Let us now try to further characterize the maps $M_\omega(t)$. Of course we want to interpret them as a family of operators $\omega \to M_\omega(t)$. But before we try to do that let us look at the consequences of the covariance of the arrival-time measure with respect to the no-event evolution described in equations (4.4) and (4.7). These two equations ensure that the structure of arrivals is time-independent. Or in the case of evolution systems only dependent on the external/lab time. In other words we expect arrivals to be described by a single map, or a one-parameter family of maps in the case of evolution systems. In the notation of the Radon-Nikodym theorem above this means that $M_\omega(t) = M_{\mathbb{F}_*(s)(\omega)}(t-s)$ and hence fundamentally $M_\omega(t) = \mathcal{M}_{\mathbb{F}_*(t)(\omega)}(0)$. That is, we expect the existence of a single linear operator $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{K})$ such that $M_\omega(t) = \mathcal{J} \circ \mathbb{F}_*(s)\omega$.

Since we are working with strongly-continuous semigroups and evolution systems it is not to be expected that the infinitesimal description is given in terms terms of a bounded operator, i.e. that we can extend $\mathcal{J}$ to an operator $\mathcal{J} : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$. However if we want to treat $\mathcal{J}$ as a bounded operator we can do so by equipping $\mathrm{dom}(\mathcal{Z}_*)$ with the graph norm and considering $\mathcal{J}$ as a map from $(\mathrm{dom}(\mathcal{Z}_*), \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)})$ into $\mathfrak{T}(\mathcal{K})$.

Infinitesimal arrivals

It turns out that the infinitesimal arrivals are completely-positive and not just conditionally-completely-positive. The general idea of an "unbounded completely-positive map" was already mentioned in section 2.2.1 and we discussed a few ideas on their regularization in section 3.2. As promised we avoid this concept and define the complete positivity in the regularized setting, i.e. we describe a tensor norm structure on $\mathrm{dom}(\mathcal{Z}_*)$. This is the topic of section 4.1.4.

Apart from the complete positivity the only requirement on infinitesimal arrivals comes from normalization condition of the arrival-time measure. As the arrival-time measure is bounded by the total norm loss over an interval, the infinitesimal arrival has to be bounded by the infinitesimal norm loss the no-event evolution.

That is for every $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ the map $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{K})$ has to fulfill:

$$\|\mathcal{J}(\omega)\|_{\mathrm{tr}} \leq -\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{F}_*(t)(\omega)\|_{\mathrm{tr}} \tag{4.9}$$

This already reminds us of the exit space approach which we get to know in section 4.2. As we already mentioned the maps $\mathcal{J}$, giving the infinitesimal description of an arrival, are positive, and even completely-positive. It is hence sufficient to evaluate the norm on positive operators where we have with $\mathrm{dom}(\mathcal{Z}_*) \ni \omega \geq 0$:

$$\|\mathcal{J}(\omega)\|_{\mathrm{tr}} = \mathrm{tr}(\mathcal{J}(\omega)) \leq -\,\mathrm{tr}(\mathcal{Z}_*(\omega)) \tag{4.10}$$

Observe that the positivity of the right side of this inequality is just a statement of the dissipativity of the operator $\mathcal{Z}_*$, because $\|\omega\|_{\mathrm{tr}}\mathbb{1}_{\mathcal{H}} \in \mathfrak{B}(\mathcal{H})$ is obviously in the duality set of $\omega \in \mathfrak{T}(\mathcal{H})$

In the case of a time-dependent generator we can perform the above analysis point-wise and get that for $\mathcal{D} \ni \omega \geq 0$:

$$\|\mathcal{J}_s(\omega)\|_{\mathrm{tr}} \leq -\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{F}_*(t,s)(\omega)\|_{\mathrm{tr}}\bigg|_{t=s} = -\operatorname{tr}(\mathcal{Z}_*(s)(\omega)) \tag{4.11}$$

Again the existence of a common dense domain is imminent for the construction.

The following characterizations of arrival-time measures have, at least, one big "deficiency": despite of characterizing all possible arrival-time measures in terms of completely-positive maps, they fail to characterize the possible measurements upon arrival.

To see what this means assume that the domain of $\mathcal{Z}_*$ is a C^*-algebra whose dual is unital, e.g. in the case of finite dimension, then we could apply the Stinespring decomposition theorem to $\mathcal{J}$ to characterize all possible measurements compatible with the arrival-time measurement, i.e. measurements on arrival. This means that apart from being able to describe arrival-time measurements, we would be able to describe the statistics about how the particle has been captured. In other words we would like to have a canonical map from the case $\mathcal{K} = \mathcal{H}$ to the case $\mathcal{K} = \mathcal{T} \otimes \mathcal{H}$. However in the case that the domain of $\mathcal{Z}_*$ is not an algebra, which is generic, we can not perform, such an analysis.

Morally such an characterization should be possible if the domain algebra [3] for $\mathcal{Z}_*$ is dense in $\operatorname{dom}(\mathcal{Z}_*)$. We do not examine this general case but only look at the construction from [85] which can be used to obtain a concise characterization in the case that the semigroup $\mathbb{F}_*(t)$ is pure, in the sense that it maps pure states to pure states.

The central characterization theorem for first-arrival-time measures for given no-event semigroups is now the following:

Theorem 4.3. *Let $\mathbb{F}_*(t) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ be a strongly-continuous semigroup of completely-positive maps with generator $\mathcal{Z}_* : \operatorname{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$.*

Then there is a one-to-one correspondence between completely-positive maps of the form $\mathcal{J} : (\operatorname{dom}(\mathcal{Z}_*), \|\cdot\|_{\operatorname{dom}(\mathcal{Z}_*)}) \to \mathfrak{T}(\mathcal{K})$ and with the condition that for $\operatorname{dom}(\mathcal{Z}_*) \ni \omega \geq 0$:

$$\operatorname{tr}(\mathcal{J}(\omega)) \leq -\operatorname{tr}(\mathcal{Z}_*(\omega)) = -\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{F}_*(t)(\omega)\|\bigg|_{t=0} \tag{4.12}$$

and arrival-time measures $\mathcal{M}(\tau) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$ for Lebesgue measurable sets $\tau \subset \mathbb{R}^+$. Complete positivity for maps with domain $\operatorname{dom}(\mathcal{Z}_*)$ is to be understood as described in section 4.1.4.

Every map $\mathcal{J}$ defines a first-arrival-time measure by setting for $\omega \in \operatorname{dom}(\mathcal{Z}_*)$ and τ in the Lebesgue σ-algebra of $\mathbb{R}^+$:

$$\mathcal{M}(\tau)(\omega) := \int_\tau \mathcal{J} \circ \mathbb{F}_*(t)(\omega)\mathrm{d}t. \tag{4.13}$$

The integral is to be interpreted as a Bochner integral. For each measurable set τ the above definition extends to an bounded operator $\mathcal{M} : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$. The assignment $\tau \to \mathcal{M}(\tau)$ is an arrival-time measure.

Conversely given an arrival-time measurement $\mathcal{M}$ then:

$$\mathcal{J} : \left(\operatorname{dom}(\mathcal{Z}_*), \|\cdot\|_{\operatorname{dom}(\mathcal{Z}_*)}\right) \to \mathfrak{T}(\mathcal{H}) \qquad\qquad \omega \to -\mathcal{M}(\mathbb{R}^+) \circ \mathcal{Z}_* \tag{4.14}$$

defines a completely-positive map as above. $\mathcal{M}$ is the first-arrival-time measure defined by this map. We call the map $\mathcal{J}$ an infinitesimal arrival *for the semigroup $\mathbb{F}_*(t)$.*

We delay the proof of these results to the next section.

Evolution systems

In the case of evolution systems we do not get a one-to-one correspondence but only one direction of the above theorem.

Theorem 4.4. *Let* $(\mathcal{D}, \|\|_{\mathcal{D}})$ *be a Banach space which can be densely and continuously embedded in* $\mathfrak{T}(\mathcal{H})$. *Furthermore for* $0 \leq s \leq t \leq T$ *let* $\mathbb{F}_*(t,s) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ *be an* $\mathcal{D}$*-valued evolution system with family of generators* $\mathcal{Z}_*(t) : \mathcal{D} \subset \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$.

Then every piece-wise strongly-continuous family $\mathcal{J}(t) : \mathcal{D} \to \mathfrak{T}(\mathcal{K})$ *of completely-positive maps, with* $0 \leq t \leq T$ *and such that for* $\mathcal{D} \ni \omega \geq 0$*:*

$$\mathrm{tr}_{\mathcal{K}}\left(\mathcal{J}(s)(\omega)\right) \leq -\,\mathrm{tr}_{\mathcal{H}}(\mathcal{Z}_*(s)(\omega)) = -\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{F}_*(t,s)(\omega)\|\Big|_{t=s} \tag{4.15}$$

defines a first arrival-time measure for the evolution system $\mathbb{F}_*(t,s)$ *by the following construction.*

For $s \geq 0$, $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ *and* τ *in the Lebesgue* σ*-algebra of* $[s,\infty]$ *set:*

$$\mathcal{M}_s(\tau)(\omega) := \int_\tau \mathcal{J}(t) \circ \mathbb{F}_*(t,s)(\omega)\mathrm{d}t. \tag{4.16}$$

For each $s \in \mathbb{R}^+$ *and each measurable set* τ *this definition extends to a bounded operator* $\mathcal{M}_s(\tau) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$. *The assignment* $\tau \to \mathcal{M}_s(\tau)$ *for each* s *defines an arrival-time measure.*

If there is a semigroup $\mathbb{F}(s)$ *such that* $\mathbb{F}_*(t,s) = \mathbb{F}_*(t-s)$ *and the function* $t \to \mathcal{J}(t)$ *is constant then we are in the situation of theorem 4.3 with* $\mathcal{M}_0(\tau) = \mathcal{M}(\tau)$. *We call the family of maps* $\mathcal{J}_s$ *an* infinitesimal arrival *for the evolution system* $\mathbb{F}_*(t,s)$.

Contrary to the case of semigroups the relation between covariant measures and completely-positive maps from $\mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{K})$ is not that simple. To obtain a converse of the above theorem we would have to analyze the Radon-Nikodym derivatives of family of the measures $\mathcal{M}_s$ in the sense described at the beginning of this section.

Similarly to the discussion in the beginning the covariance condition from equation (4.7) suggests the form $\mathcal{J}_t \circ \mathbb{F}(t,s)$ with a time dependent family of completely-positive maps $\mathcal{J}_t$. To proof linearity and complete positivity for each of the operators $\mathcal{J}_t$ would be at least a lot more work as in the semigroup case, where we can use a trick to shift the Radon-Nikodym derivative entirely on $\mathbb{F}_*$.

Let us now briefly mention a standard example of a possible arrival map: The reset to a fixed state. Physically this corresponds to situations where upon an arrival the experimenter intervenes and resets the system to a fixed state. In the mathematical description we choose some $\rho_0 \in \mathfrak{T}(\mathcal{K})$ and set for a given semigroup $\mathbb{F}_* : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ with generator $\mathcal{Z}_*$:

$$\mathcal{J}(\rho) = -\rho_0\,\mathrm{tr}(\mathcal{Z}_*(\rho)) \tag{4.17}$$

This map evidently satisfies the normalization condition. It is a completely-positive map as a concatenation of two completely-positive maps: a positive functional and the reset to a fixed state. This type of arrival is mathematically even easier if one chooses $\rho_0 \in \mathrm{dom}(\mathcal{Z}_*)$, especially when we later want to describe more than just the first arrival. However with the theory described here we do not need to make this restriction. This perturbation has a long history and was considered for example in [56, 28, 11].

In the case of an evolution system we can of course choose ρ_0 time dependent and have to use the generator at the right time-point, i.e:

$$\mathcal{J}(\rho) = -\rho_0(t)\,\mathrm{tr}(\mathcal{Z}_*(t)(\rho)) \tag{4.18}$$

4.1.4. $\mathrm{dom}(\mathcal{Z}_*)$ as an operator space

Before we can turn our attention to the classification of arrival-time measures we have to clarify a few technicalities. We want to find a regularized version of the following generalization of complete positivity to an unbounded operator $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$: A matrix $(\omega_{i,j})$ with $1 \leq i,j \leq n$ and $\omega_{i,j} \in \mathrm{dom}(\mathcal{Z}_*)$ is called positive iff for all finite sequences $(B_i)_{i=1}^n \in \mathfrak{B}(\mathcal{H})$ we have that the following expression is positive:

$$\sum_{i,j} \mathrm{tr}\left(\omega_{i,j} B_i^* B_j\right) \tag{4.19}$$

An operator $\mathcal{J}$ as above is completely-positive iff for all n and all positive $n \times n$ matrices $(\omega_{i,j})$ the matrix $(\mathcal{J}(\omega_{i,j}))$ is positive.

We shall now proof that the unbounded operator $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$ is completely-positive iff the bounded operator $\mathcal{J} : (\mathrm{dom}(\mathcal{Z}_*), \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)}) \to \mathfrak{T}(\mathcal{H})$ is completely-positive. For this to make sense we have to introduce a tensor norm structure on $\mathrm{dom}(\mathcal{Z}_*)$.

We start with a short analysis of the positivity structure of $\mathrm{dom}(\mathcal{Z}_*)$. First observe that the domain of the generator of a positive semigroup is hermitian in the sense that iff $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ it follows that $\omega^* \in \mathrm{dom}(\mathcal{Z}_*)$. This is a simple consequence of the fact that $\mathbb{F}_*(t)$ maps hermitian operators to hermitian operators and hence $\mathbb{F}_*(t)(\omega^*) = \mathbb{F}_*(t)(\omega)^*$ and consequently the same is true for the generator $\mathcal{Z}_*$.

It follows that $\mathrm{dom}(\mathcal{Z}_*)$ can be equipped with the positivity structure it inherits from $\mathfrak{T}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$. That is $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ is positive iff ω is positive as an element of $\mathfrak{B}(\mathcal{H})$.

We want to be able to speak about completely-positive maps from $\mathrm{dom}(\mathcal{Z}_*)$ into $\mathfrak{T}(\mathcal{H})$. This means we have to look at the spaces $\mathrm{dom}(\mathcal{Z}_*) \otimes \mathfrak{B}(\mathbb{C}^n)$ for arbitrary n. If we equip $\mathfrak{B}(\mathbb{C}^n)$ with the standard basis of matrix units $E_{i,j}$, i.e. matrices with one nonzero entry in row j column i, we can identity $\mathrm{dom}(\mathcal{Z}_*) \otimes \mathfrak{B}(\mathbb{C}^n)$ with the space of $n \times n$- matrices with entries in $\mathrm{dom}(\mathcal{Z}_*)$. We shall henceforward always do this. If $\omega_{i,j} \in \mathrm{dom}(\mathcal{Z}_*)$ for $1 \leq i,j \leq n$ then a map $\mathbb{E}_* \otimes \mathbb{1}_n$ acts on the matrix with entries $(\omega_{i,j})$ as $\mathbb{E}_* \otimes \mathbb{1}_n(\omega_{i,j}) = (\mathbb{E}_*\omega_{i,j})$.

We already know that $\mathrm{dom}(\mathcal{Z}_*)$ is a Banach space when equipped with the the graph norm $\|\omega\|_{\mathrm{dom}(\mathcal{Z}_*)} = \|\omega\|_{\mathrm{tr}} + \|\mathcal{Z}_*(\omega)\|_{\mathrm{tr}}$. If we want to speak about completely-positive maps from $\mathrm{dom}(\mathcal{Z}_*)$ into $\mathfrak{B}(\mathcal{H})$ we have to equip the space $\mathrm{dom}(\mathcal{Z}_*) \otimes \mathfrak{B}(\mathbb{C}^n)$ with a norm for each n in $\mathbb{N}$. The canonical choice is:

$$\|\omega\| := \|\omega\|_{(\mathrm{dom}(\mathcal{Z}_* \otimes \mathbb{1}_n)} = \|\omega\|_{\mathrm{tr}} + \|\mathcal{Z}_* \otimes \mathbb{1}_n \omega\|_{\mathrm{tr}} \tag{4.20}$$

To see that this is well defined we have to check that if $\omega_{i,j} \in \mathrm{dom}(\mathcal{Z}_*)$ for $1 \leq i,j \leq n$ then the matrix with entries $(\omega_{i,j})$ is in $\mathrm{dom}(\mathcal{Z}_* \otimes \mathbb{1}_n)$ for all n. This is obviously the case. Furthermore the set of such matrices is a core for the domain of $\mathcal{Z}_* \otimes \mathbb{1}_n$ because it is evidently dense and invariant under the semigroup. It follows that a matrix $(\omega_{i,j})$ is in $\mathrm{dom}(\mathcal{Z}_* \otimes \mathbb{1}_n)$ iff $\omega_{i,j} \in \mathrm{dom}(\mathcal{Z}_*)$ for all $1 \leq i,j \leq n$.

If we from now on speak about a completely-positive map with domain $\mathrm{dom}(\mathcal{Z}_*)$ we have the above tensor structure in mind. Note that we can not apply the Stinespring dilation theorem to a completely-positive map $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$ because the domain $\mathrm{dom}(\mathcal{Z}_*)$ need not be an algebra.

4.1.5. Proof of the characterization theorems

We start with the case of a semigroup. Later the proof for evolution systems is a straight forward generalization. As a first step we proof that every completely-positive map on the domain of the generator semigroup which is bounded by the infinitesimal norm loss defines a unique arrival-time measure.

Proof of theorem 4.3: part 1. For $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ and a measurable set τ we want to define the measure by equation (4.13) and then extend it to the whole of $\mathfrak{T}(\mathcal{H})$ by closure of the operators $\mathcal{M}(\tau) : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{K})$ in $\mathfrak{T}(\mathcal{H})$. To see that this works we have to show, that $\mathcal{M}(\tau)(\omega)$ is well defined and that $\mathcal{M}(\tau)$ is a bounded operator for each τ when restricted to $\mathrm{dom}(\mathcal{Z}_*)$.

To see that the integral (4.13) exists we have to show that the function $t \to \mathcal{J} \circ \mathbb{F}_*(t)(\omega)$ is Bochner integrable. But for $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ this function is actually continuous, since if if we restrict $\mathbb{F}_*$ to $\mathbb{F}_* : (\mathrm{dom}(K), \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)} \to (\mathrm{dom}(K), \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)}$ it stays a strongly-continuous semigroup. Hence the function is Bochner measurable, see page37. Integrability of $t \to \mathcal{J} \circ \mathbb{F}_*(t)(\omega)$ is a straight forward consequence of the normalization bound on $\mathcal{J}$ which entails continuity. This calculation is the same as that for the normalization condition for $\mathcal{M}$:

For $0 \le s \le t \in \mathbb{R}$ and $0 \le \omega \in \mathrm{dom}(\mathcal{Z}_*)$ we have:

$$\|\mathcal{M}([s,t))(\omega)\| = \left\| \int_s^t \mathcal{J} \circ \mathbb{F}_*(r)(\omega) \mathrm{d}r \right\| \le \int_s^t \|\mathcal{J} \circ \mathbb{F}_*(r)(\omega)\| \, \mathrm{d}r \tag{4.21}$$

$$\le - \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} \|\mathbb{F}_*(r)(\omega)\| \, \mathrm{d}r \le \|\mathbb{F}_*(s)(\omega)\| - \|\mathbb{F}_*(t)\omega\| \tag{4.22}$$

We thus have $\|M(\tau)(\rho)\| \le \|M(\mathbb{R}^+)\| \le 1$.

For $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ the Bochner integrability of $t \to \mathcal{J} \circ \mathbb{F}_*(t)(\omega)$ directly implies countable additivity, by [34, theorem II.4]. By positivity of $\mathcal{M}$ and denseness of $\mathrm{dom}(\mathcal{Z}_*)$ this extends to countable additivity for $\rho \in \mathfrak{T}(\mathcal{H})$.

For $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ and $B \in \mathfrak{B}(\mathcal{H})$ the expression $\mathrm{tr}\,(\mathcal{M}(\tau)(\omega)B)$ obviously defines a Radon measure. By boundedness of $\mathcal{M}(\tau)$, the expectations $\mathrm{tr}\,(\mathcal{M}(\tau)(\omega)B)$ fix the operator uniquely. Hence inner regularity of $\mathcal{M}(\tau)$ follows from the corresponding property of $\mathrm{tr}\,(\mathcal{M}(\tau)(\omega)B)$. □

The converse relies heavily on the semigroup structure and resulting covariance of the measure.

Proof of theorem 4.3: part 2. $\mathcal{J}$ is a positive map as a strong limit of positive maps. To see that we write $\mathcal{Z}_*$ as a limit:

$$\mathcal{J}(\omega) = \lim_{t\to 0} -\mathcal{M}(\mathbb{R}^+) \left(\frac{1}{h} \mathbb{F}(h)(\omega) - \omega \right) = \lim_{t\to 0} \frac{1}{h} \mathcal{M}([0,h])(\omega) \tag{4.23}$$

The limit evidently exists and is a positive bounded operator. The tensor norm structure on $\mathrm{dom}(\mathcal{Z}_*)$ is defined such that this calculation directly generalizes to complete positivity. Since the measure is bounded in the sense of equation (4.3) we get for positive elements in the domain $\mathrm{dom}(\mathcal{Z}_*) \ni \omega \geq 0$:

$$\|\mathcal{M}([0,h])(\omega)\|_{\mathrm{tr}} \leq \|\omega\|_{\mathrm{tr}} - \|\mathbb{F}_*(h)(\omega)\|_{\mathrm{tr}} = \mathrm{tr}\left((\mathbb{1} - \mathbb{F}(h))(\omega)\right) \tag{4.24}$$

This inequality gives the desired bound for $\mathcal{J}$ on positive elements ω.

Since the measure $\mathcal{M}$ is a Radon measure it is sufficient to define it on half open intervals and extend it to arbitrary measurable sets. On such sets $\mathcal{M}$ and the measure defined through $\mathcal{J}$ obviously coincide.

□

The proof of the time-dependent case is analogous to the semigroup case. Note that we only want to proof one direction.

Proof of theorem 4.4. First note that the function $s \to \mathcal{J}(s) \circ \mathbb{F}_*(s,r)(\omega)$ is continuous for every $\omega \in \mathcal{D}$ and $r \leq s \leq t \in \mathbb{R}$, hence the function is Bochner measurable. With $\mathcal{D} \ni \omega \geq 0$, we get:

$$\left\| \int_s^t \mathcal{J} \circ \mathbb{F}_*(\tau,r)(\omega)\mathrm{d}\tau \right\|_{\mathrm{tr}} \leq \int_s^t \|\mathcal{J} \circ \mathbb{F}_*(\tau,r)(\omega)\|_{\mathrm{tr}}\, \mathrm{d}\tau \tag{4.25}$$

$$\leq - \int_s^t \mathrm{tr}\left(\mathcal{Z}_*\mathbb{F}_*(\tau,r)\right)\mathrm{d}\tau = \|\mathbb{F}_*(s,r)(\omega)\|_{\mathrm{tr}} - \|\mathbb{F}_*(t,r)(\omega)\|_{\mathrm{tr}}\,. \tag{4.26}$$

This shows the boundedness and hence that the integration in the definition of the measure is well defined. The rest of proof is basically identical to that of theorem 4.3. □

4.2. Exit spaces

We now turn to a situation where we can give a more detailed analysis of the structure of arrivals. That is the situation where the underlying no-event semigroup is a pure map. It is described in detail in the two papers [86, 85]. Our main contribution is to give rigorous proofs of some of the results announced in [86].

4.2.1. General remarks

The exit space construction is of interest to the above description of arrival-time measures, because in this situation we are able to split the arrival into an “absorption” and a reinsertion. Hence we can give a more detailed analysis of the possible types of arrival. To achieve this the first step is to obtain a description of the state space on arrival, which is called the exit space. Later we extract some ideas to generalize the definition of arrival-time measures given above even further.

For simplicity we restrict for the moment to the case of semigroups. Such that we can leverage the results obtained there to get a simplified treatment of the evolution system case. A point-wise treatment as we used so far does indeed lead to well defined theory but introduces additional technical difficulties we wish to avoid.

The central assumption for the exit space theory is the purity of the no-event semigroup. Purity of $\mathbb{F}_*(t)$ means that it conserves the class of pure states. In other words there is a semigroup $U(t) : \mathcal{H} \to \mathcal{H}$ on the Hilbert space level such that $\mathbb{F}_*(t)\rho = U(t)\rho U^*(t)$. One has to remark that because $\mathcal{H}$ is its own dual the dual semigroup $U^*(t)$ is again a strongly-continuous semigroup on $\mathcal{H}$. We denote the generator of $U(t)$ as K. Consequently K^* is the generator of $U^*(t)$.

The generator of $\mathbb{F}_*(t)$ is thus formally given as:

$$\mathcal{Z}_*(\omega) = K\omega + \omega K^*. \tag{4.27}$$

A central object in the coming analysis is a core for this operator. Namely the space of finite rank operators on $\mathrm{dom}(K)$ or in other words the span of ketbra operators in $\mathrm{dom}(K)$, i.e. let $\mathcal{D}$ be a core for $\mathrm{dom}(K)$

$$|\mathcal{D}\rangle\langle\mathcal{D}| := \mathrm{span}\,\{|\psi\rangle\langle\varphi|\,|\psi, \varphi \in \mathcal{D}\} \tag{4.28}$$

This space is important because it gives a connection between a core for K and a core for $\mathcal{Z}_*$. It is indeed easy to see that $|\mathcal{D}\rangle\langle\mathcal{D}|$ is dense and invariant under $\mathbb{F}_*(t)$ and hence a core for the semigroup $\mathbb{F}_*(t)$. Accordingly $\mathcal{Z}_*$ is indeed given by the closure of the expression in equation (4.27) for $\omega \in |\mathcal{D}\rangle\langle\mathcal{D}|$.

The analysis of Holevo in [54, 53] showed that the existence of such Ketbra is intimately connected with the Lindblad form for generators of completely-positive semigroups.

4.2.2. Definition

We finally arrive at the actual subject of this section. The construction of exit spaces. The idea of is to construct a new Hilbert space from $\mathrm{dom}\,K$ whose scalar product describes the norm loss of the semigroup $U(t)$. Such a Hilbert space then has a natural interpretation as describing the states of the system at arrival.

Definition 4.5. Let $\mathcal{H}$ be a Hilbert space and $U(t) : \mathcal{H} \to \mathcal{H}$ a strongly-continuous contraction semigroup with generator $K : \mathrm{dom}(K) \to \mathcal{H}$. The *minimal exit space* $\mathcal{E}$ of the semigroup is the completion of $\mathrm{dom}\,K$ with respect to the following sesquilinear form and the corresponding semi-norm:

$$\langle\psi, \varphi\rangle_K := -\langle K\psi, \varphi\rangle - \langle\psi, K\varphi\rangle \text{ for } \varphi, \psi \in \mathrm{dom}(K) \tag{4.29}$$

$$\|\varphi\|_K^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\varphi\|^2\bigg|_{t=0} \text{ for } \varphi \in \mathrm{dom}(K) \tag{4.30}$$

By completion we mean the space of Cauchy sequences, w.r.t. the above sesquilinear form, equipped with the natural induced Hilbert space structure. Notice that in the process of completion we automatically also go to a quotient of $\mathrm{dom}(K)$. By construction we also get an embedding $j : \mathrm{dom}\,K \to \mathcal{E}$ such that:

$$\langle j\psi, j\varphi\rangle_{\mathcal{E}} = \langle\psi, \varphi\rangle_K = -\langle K\psi, \varphi\rangle - \langle\psi, K\varphi\rangle \tag{4.31}$$

Note that the embedding does not have to be an isometry because the completion process involves also taking a quotient, i.e. we mod out the kernel of K. The exit space construction is of great importance for our analysis because it allows us to split the arrival in two steps: the “absorption” and some kind of “reinsertion”. The exit space evidently describes the absorption part, in the sense that it classifies all possible types of arrival. We shall shortly see that we can classify all possible re-insertions similarly. This corresponds to a slightly more general definition of exit space.

The central equation for the exit space, as defined above, is the embedding j of the domain into the exit space given by equation (4.31). Later we see that it is the only property needed of the exit space for the purpose of constructing arrival-time measures. Consequently we arrive at the following definition:

Definition 4.6. Let $\mathcal{H}$ be a Hilbert space and $U(t) : \mathcal{H} \to \mathcal{H}$ a strongly-continuous contraction semigroup with generator $K : \mathrm{dom}(K) \to \mathcal{H}$. Furthermore let $\mathcal{E}$ be a Hilbert space with a linear map $j : \mathrm{dom}(K) \to \mathcal{E}$, s.t.:

$$\langle j\varphi, j\psi \rangle = -\langle K\varphi, \psi \rangle - \langle \varphi, K\psi \rangle . \tag{4.32}$$

Then the pair $(\mathcal{E}, j)$ is called Exit space for the semigroup $U(t)$.

For a general exit space the minimal exit space can always be identified with the closure of the image of j in $\mathcal{E}$.

4.2.3. Relation to arrival-time measures

To connect the theory of exit spaces to our notion of arrival-time measure in the sense of definition 4.1 we just have to extend the exit space embedding $j : \mathrm{dom}(K) \to \mathcal{E}$ to an embedding on the level of trace class operators.

Lemma 4.7. *Let $\mathcal{H}$ be a Hilbert space and $U(t) : \mathcal{H} \to \mathcal{H}$ a strongly-continuous semigroup with generator K. We denote the associated strongly-continuous semigroup of completely-positive maps on $\mathfrak{T}(\mathcal{H})$ as $\mathbb{F}_*(t)(\rho) := U(t)\rho U(t)^*$ and call its generator $\mathcal{Z}_*$. Furthermore let $(\mathcal{E}, j)$ be a, possibly non-minimal, exit space for $U(t)$.*

Let $|\mathcal{D}\rangle\langle\mathcal{D}|$ be defined by equation (4.28). For $\omega \in |\mathcal{D}\rangle\langle\mathcal{D}|$ we set:

$$\mathrm{ad}_j : |\mathcal{D}\rangle\langle\mathcal{D}| \to \mathfrak{T}(\mathcal{E}) \qquad\qquad \mathrm{ad}_j(\omega) = j\omega j^* \tag{4.33}$$

This extends by linearity and closure to a completely-positive contraction $\mathrm{ad}_j : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{E})$ such that for $\omega \in \mathrm{dom}(\mathcal{Z}_*)$:

$$\| \mathrm{ad}_j(\omega)\|_{\mathrm{tr},\mathcal{E}} = -\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbb{F}_*(t)(\omega)\|_{\mathrm{tr},\mathcal{H}} \Big|_{t=0} \tag{4.34}$$

The norm on the left side of the equation being the trace-norm on $\mathfrak{B}(\mathcal{E})$ and the norm on the right side the trace norm on $\mathfrak{B}(\mathcal{H})$.

The map ad_j is pure, i.e. the only completely-positive maps $\mathbb{F} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{E})$ which are dominated by ad_j, in the sense that that the difference $\mathrm{ad}_j - \mathbb{F}$ is completely-positive, are scalar multiples of ad_j. Furthermore iff $(\mathcal{E}, j)$ is a minimal exit space the map ad_j is onto.

Proof. For $\psi, \varphi \in \mathrm{dom}\, K$ we have:

$$\mathrm{tr}_{\mathcal{E}}\left(\mathrm{ad}_J(|\varphi\rangle\langle\psi|)\right) = \langle j\psi, j\varphi \rangle_{\mathcal{E}} = -\,\mathrm{tr}_{\mathcal{H}}(\mathcal{Z}_*(|\varphi\rangle\langle\psi|)) \tag{4.35}$$

This result is sufficient to get the desired norm bound on ad_j. In particular it follows that ad_J is a contraction since:

$$\| \mathrm{ad}_j(|\varphi\rangle\langle\varphi|)\|_{\mathrm{tr}} = -\operatorname{tr}(\mathcal{Z}_*(|\varphi\rangle\langle\varphi|)) \leq \|\mathcal{Z}_*(|\varphi\rangle\langle\varphi|)\|_{\mathrm{tr}} \leq \||\varphi\rangle\langle\varphi|\|_{\mathrm{dom}(\mathcal{Z}_*)} \tag{4.36}$$

By linearity of the trace norm for positive elements this extends to general positive $\omega \in |\mathcal{D}\rangle\langle\mathcal{D}|$.

We now have to check that the definition uniquely extends to an completely-positive map. As we already mentioned the set $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a core for $\mathcal{Z}_*$. Consequently it is also dense in $(\mathcal{Z}_*, \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)})$. Similarly $|\mathcal{D}\rangle\langle\mathcal{D}|\otimes M_n$ is dense in $\mathrm{dom}(\mathcal{Z}_*)\otimes M_n$ if we equip $|\mathcal{D}\rangle\langle\mathcal{D}|$ with the natural operator space structure it inherits as a subspace from $(\mathrm{dom}(\mathcal{Z}_*), \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)})$.

Furthermore ad_j is a completely-positive map from $\mathcal{D} \to \mathfrak{T}(\mathcal{E})$ by construction. Hence the closure of ad_j is a completely-positive map on $\mathrm{dom}(\mathcal{Z}_*)$. It is clear that ad_j is onto for a minimal exit space by the open mapping theorem [79, theorem 2.11].

W.l.o.g. let $(\mathcal{E}, j)$ be a minimal exit space. Otherwise we can restrict the following discussion to the closure of the image of j. Now assume that we have a completely-positive map $\mathbb{F} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$ such that $\mathrm{ad}_j - \mathbb{F}(\mathcal{Z}_*)$ is completely-positive. We can define an completely-positive map $\mathbb{G} : \mathfrak{B}(\mathfrak{B}(\mathcal{E})) \to \mathfrak{B}(\mathfrak{B}(\mathcal{E}))$ by setting for $E \in \mathfrak{B}(\mathcal{E})$:

$$\langle j\psi, \mathbb{G}(E) j\varphi\rangle := \operatorname{tr}(\mathbb{F}(|\varphi\rangle\langle\psi|)E) \tag{4.37}$$

By construction $j(\mathrm{dom}(K))$ is dense in $\mathcal{E}$ furthermore the operator is obviously positive and bounded and hence extends uniquely to a positive bounded operator on $\mathfrak{B}(\mathcal{E})$. Complete positivity follows as above.

It follows that $\mathbb{G}_* \circ \mathrm{ad}_j = \mathbb{F}$. The purity of ad_j is now a consequence of the Stinespring dilation theorem, the complete positivity of $\mathrm{id}_{\mathfrak{B}(\mathcal{E})} - \mathbb{G}_*$ which follows from the complete positivity of $(\mathrm{id} - \mathbb{G}_*) \circ \mathrm{ad}_j = \mathrm{ad}_j - \mathbb{F}$ and the purity of the identity. □

Corollary 4.8. *In the situation of the previous lemma* ad_j *is an infinitesimal arrival, in the sense of theorem 4.3, for the semigroup* $\mathbb{F}_*(t)$, *i.e. it corresponds to a first-arrival-time measure* $\mathcal{M}(\tau) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathbb{E})$.

The above lemma is of fundamental technical importance, but seemingly only allows to characterize a quite narrow class of arrival-time measures, i.e. we are restricted to measures where the Hilbert space after arrival is the exit space $\mathcal{E}$.

4.2.4. Re-insertions

Luckily there is a one-to-one correspondence between general first-arrival-time measures $\mathcal{M}(\tau) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$ and exit-space embeddings mediated by completely-positive maps from $\mathfrak{T}(\mathcal{E}_{\min})$ to $\mathfrak{T}(\mathcal{K})$.

The improvement of the following characterization theorem over the previous theorem 4.3 lies in the already mentioned fact that we are able to classify arrival-time measures by completely-positive maps between the bounded operators on Hilbert spaces, i.e. we can apply the Stinespring dilation theorem to further classify these maps.

The result relies mainly on the denseness of the ketbra domain $|\mathcal{D}\rangle\langle\mathcal{D}|$ in $\mathrm{dom}(\mathcal{Z}_*)$ also for the tensor stabilized versions.

Theorem 4.9. *In the notation of the previous lemma and with $(\mathcal{E}_{min}, j)$ being the minimal Exit space: There is a one to one correspondence between completely-positive contractions $\mathbb{J}_* : \mathfrak{T}(\mathcal{E}_{min}) \to \mathfrak{T}(\mathcal{K})$ and completely-positive maps $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{K})$ such that $\mathrm{tr}_{\mathcal{K}}(\mathcal{J}(\omega)) \leq -\mathrm{tr}_{\mathcal{H}}(\mathcal{Z}_*(\omega))$*

Proof. Given a completely-positive contraction $\mathbb{J}$ we define:

$$\mathcal{J}(\omega) := \mathbb{J} \circ \mathrm{ad}_j(\omega) \tag{4.38}$$

For positive $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ we have:

$$\|\mathcal{J}(\omega)\|_{\mathrm{tr}} = \|\mathbb{J} \circ \mathrm{ad}_j(\rho)\|_{\mathrm{tr}} \leq \|\mathbb{J}\| \| \mathrm{ad}_j(\rho)\|_{\mathrm{tr}} \leq -\mathrm{tr}(\mathcal{Z}_*(\rho)) \tag{4.39}$$

And given a completely-positive map $\mathcal{J}(\omega)$, using that ad_j is onto we define:

$$\mathbb{J}(\mathrm{ad}_j(\omega)) := \mathcal{J}(\omega) \tag{4.40}$$

Contractivity follows from :

$$\|\mathbb{J}(\mathrm{ad}_j(|\psi\rangle\langle\psi|))\|_{\mathrm{tr},\mathcal{K}} = \mathrm{tr}_{\mathcal{K}}(\mathcal{J}(|\psi\rangle\langle\psi|) \leq -\mathrm{tr}_{\mathcal{H}}(\mathcal{Z}_*(|\psi\rangle\langle\psi|)) = \||\psi\rangle\langle\psi|\|_{\mathrm{tr},\mathcal{E}_{\mathrm{min}}} \tag{4.41}$$

□

Colloquially for a given exit space $(\mathcal{E}, j)$ the embedding j takes care of the normalization by the norm loss, while the family of maps $\mathbb{J}$ describes all the possible arrival types of arrival.

The completely-positive maps $\mathbb{J}_* : \mathfrak{T}(\mathcal{E}) \to \mathfrak{T}(\mathcal{K})$ have a natural interpretation as some reinsertion of the system. This is clearest in the case $\mathcal{K} = \mathcal{H}$. In any other cases the nature of the quantum system, i.e. its Hilbert space, changes upon reinsertion. Note that the dual map of a reinsertion is a completely-positive map from $\mathfrak{B}(\mathcal{H})$ to $\mathfrak{B}(\mathcal{E})$. This is remarkable, because we are now in a situation where we could apply the Stinespring dilation theorem.

4.2.5. Re-insertions and Lindblad generators

By the Stinespring decomposition theorem we can find for every map $\mathbb{J}_* : \mathfrak{T}(\mathcal{E}_{\mathrm{min}}) \to \mathfrak{T}(\mathcal{H})$ a Hilbert space $\mathcal{T}$ and a linear operator $V : \mathcal{E} \to \mathcal{T} \otimes \mathcal{H}$ such that $\mathbb{J}(B) = V^* \mathbb{1}_{\mathcal{T}} \otimes BV$. We arrive at the following corollary.

Corollary 4.10. *In the notation of the previous theorem there is a one-to-one correspondence between completely-positive maps $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{K})$ and non-minimal exit spaces of the form $(\mathcal{T} \otimes \mathcal{K}, j)$ given by:*

$$\mathcal{J}(\omega) = \mathrm{tr}_{\mathcal{T}}(\mathrm{ad}_j(\omega)) \tag{4.42}$$

Hence equivalently to the description through a completely-positive map $\mathbb{J}_*$ the re-insertions can be described through the choice of a non minimal exit space of the form $\mathcal{E} = \mathcal{T} \otimes \mathcal{H}$.

The above corollary is especially notable because it includes the case of Lindblad generators. They correspond to the case $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$. In theorem 4.16 we later show that from such a map one can construct a minimal solution to the Lindblad equation with generator $\mathcal{L}(\omega) := \mathcal{Z}_*(\omega) + \mathrm{tr}_{\mathcal{T}}(\mathrm{ad}_j(\omega))$. The above discussion can thus be sen as a classification of all Lindblad equations with a fixed dissipative part $\mathcal{Z}_*$.

4.2.6. Measurements on arrival

The key idea in this section is to apply the Radon-Nikodym statement of the Stinespring dilation, to describe all measurements compatible with the measurement of the first-arrival-time, i.e. the possible measurements on arrival. This analysis relies heavily on the purity of the map ad_j.

The following theorem states that measurements on arrival are in correspondence to POVMs on the transit space $\mathcal{T}$. These are measurements which give us additional information about the first arrival, apart from the arrival-time. This could be e.g. the number of the counter at which the arrival happened. The theorem is a direct consequence of the preceding analysis.

Definition 4.11. Let σ be the Lebesgue σ-algebra of $\mathbb{R}^+$ and let (Ω, χ) be some other measure space.

A *measurement on arrival* $\mathcal{M} : \sigma \otimes \chi \mathfrak{CP}(\mathfrak{T}(\mathcal{H}), \mathfrak{T}(\mathcal{K})$ for the first-arrival-time measurement $\mathcal{M}_\Omega(\tau) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$ an instruments such that for every $\xi \in \chi$ the map $\tau \to \mathcal{M}(\tau \times \xi)$ is a first-arrival-time measure in the sense of definition 4.1 for $\mathbb{F}_*(t)$ and $\mathcal{M}_\Omega(\tau) = \mathcal{M}(\tau \times \Omega)$.

Theorem 4.12. *Let $\mathcal{T} \otimes \mathcal{K}, j$ be an exit space for the semigroup $U(t)$, σ be the Lebesgue σ-algebra of $\mathbb{R}^+$ and (Ω, χ) be some other measure space.*

Measurements on arrival as in the preceding definition are in one to one correspondence with POVMs $F : \chi \to \mathfrak{B}(\mathcal{T})$ in the following sense: let $\mathcal{M}_j$ be the $\mathfrak{CP}(\mathfrak{T}(\mathcal{H}), \mathfrak{T}(\mathcal{T} \otimes \mathcal{K})$ valued arrival-time measure defined through

$$\mathcal{M}_j(\tau)(\omega) := \int_\tau \mathrm{ad}_j \circ \mathbb{F}(t) \mathrm{d}t. \tag{4.43}$$

Then for every measurement on arrival $\mathcal{M}$ for the first arrival time measure $\mathrm{tr}_\mathcal{T}(\mathcal{M}(\tau)(\rho))$ there exists a unique POVM $F : \chi \to \mathfrak{B}(\mathcal{T})$ such that for every $K \in \mathfrak{B}(\mathcal{K})$:

$$\mathrm{tr}(\mathcal{M}(\tau \times \xi)(\rho) K) = \mathrm{tr}_\mathcal{T}(\mathcal{M}_j(\tau) F(\xi) \otimes K) \tag{4.44}$$

4.2.7. Exit spaces for evolution systems

The idea of an exit space can be carried over point-wise to the case of evolution systems in exact analogy to the above case. We would just complete $\mathrm{dom}(\mathcal{K}(t)$ for each t with respect to the following scalar product:

$$\langle \varphi, \psi \rangle_{K(t)} := -\langle K(t)\psi, \varphi \rangle - \langle \psi, K(t)\varphi \rangle \text{ for } \varphi, \psi \in \mathrm{dom}(K(t)) \tag{4.45}$$

This leads to a t dependent family of Hilbert spaces, which could again be called exit space. We however try to avoid having to work with an direct integral of Hilbert spaces and furthermore later would like to be able to integrate against the exit space embeddings.

This goal is easily achieved if we skip the introduction of minimal exit spaces and directly proceed to the case of non-minimal exit spaces:

Definition 4.13. Let $\mathcal{D}$ be a Hilbert space which can be densely and continuously embedded into $\mathcal{H}$ and let $U(t, s) : \mathcal{H} \to \mathcal{H}$ be a $\mathcal{D}$-valued evolution system with family of generators $K(t)$. An exit space for $U(t,s)$ is a pair $(\mathcal{E}, j_t)$ of a Hilbert space $\mathcal{E}$ and a family of embeddings $j_t : \mathrm{dom}(K(t)) \to \mathcal{E}$ such that:

$$\langle j_t\varphi, j_t\psi\rangle = -\langle K(t)\varphi, \psi\rangle - \langle \varphi, K(t)\psi\rangle \tag{4.46}$$

The family j_t is strongly-continuous in the sense that for all $\psi \in \mathcal{D}$ the function $j_t\psi$ is continuous.

Of course we are again mainly interested in exit spaces of the form $\mathcal{E} = \mathcal{T} \otimes \mathcal{H}$ for some Hilbert space $\mathcal{T}$.

If we define $|\mathcal{D}\rangle\langle\mathcal{D}|$ as in equation (4.28) but with $\mathcal{D}$ being the core of $K(t)$ we can generalize the results from the previous section by point-wise application. Note that $|\mathcal{D}\rangle\langle\mathcal{D}|$ is still a core for $\mathcal{Z}_*(t)$

Lemma 4.14. *There is a family of embeddings* $\mathrm{ad}_{j,t} : \mathrm{dom}(\mathcal{Z}_*(t)) \to \mathfrak{T}(\mathcal{E})$ *such that for* $\omega \in \mathrm{dom}(\mathcal{Z}_*(t))$*:*

$$\|\mathrm{ad}_{j,s}(\omega)\|_{\mathrm{tr},\mathcal{E}} = -\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{F}_*(t,s)(\omega)\|_{\mathrm{tr},\mathcal{H}}\Big|_{t=s} \tag{4.47}$$

The family $\mathrm{ad}_{j,t}$ *is strongly-continuous in the sense that for all* $\omega \in |\mathcal{D}\rangle\langle\mathcal{D}|$ *the function* $\mathrm{ad}_{j,t}(\omega)$ *is continuous.*

If the exit space has the form $\mathcal{E} = \mathcal{T} \otimes \mathcal{K}$ *then* $\mathrm{ad}_{j,t}$ *and the map*

$$\omega \to \mathrm{tr}_{\mathcal{T}}(\mathrm{ad}_j(\omega)) \tag{4.48}$$

are infinitesimal arrivals in the sense of theorem 4.4.

Proof. After point-wise application of lemma 4.7, we only have to check that the continuity condition for j_t is stronger than the continuity condition for ad_j. This is the “easy” half of the continuity theorem for the Stinespring dilation, for $\psi \in \mathcal{D}$ we have:

$$\|\mathrm{ad}_{j,t}(|\psi\rangle\langle\psi|) - \mathrm{ad}_{j,s}(|\psi\rangle\langle\psi|\omega)\|_{\mathrm{tr}} = \||j_t\psi\rangle\langle j_t\psi| - |j_s\psi\rangle\langle j_s\psi|\| \tag{4.49}$$

$$\leq \|(j_t - j_s)\psi\|(\|j_t\psi\| + \|j_s\psi\|) \tag{4.50}$$

This extends to strong continuity on $|\mathcal{D}\rangle\langle\mathcal{D}|$, because this set only consists of finite dimensional matrices. ☐

We conjecture that with a more thorough analysis, one can proof a one-to-one correspondence between exit spaces and the families of maps $\mathcal{J}(t)$ from theorem 4.4, under the natural identification of Bochner-Lebesgue functions. Furthermore we conjecture that if one replaces the continuity condition on $\mathcal{J}(t)$ by some natural measurability assumption we also get a one-to-one correspondence with arrival-time measures.

4.2.8. Exit spaces for general contractive semigroups

It remains to answer the question why we did not carry over the notion of an exit space to the case of general contractive semigroups. This is even more surprising since the basic notion of an exit space carries over almost identically to this case. We would just close $\mathrm{dom}(\mathcal{Z}_*)$ with respect to the semi-norm:

$$\|\omega\|_{\mathcal{Z}_*} = -\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{F}_*(t)(\omega)\| \tag{4.51}$$

The resulting space is obviously a Banach space. The canonical "embedding" would be completely-positive by construction, since we equip the exit space with the positivity structure it inherits from $\mathrm{dom}(\mathcal{Z}_*)$. It satisfies the equality $\| \mathrm{ad}_j(\omega)\| = \|\omega\|_{\mathcal{Z}_*}$, where the norm on the left side is the norm of the exit space

However it is not at all clear if the exit space has any algebra structure at all. And without that structure it seems hard to capitalize on the notion of an exit space. This is the case because for our most useful characterizations in this section we needed the Stinespring dilation theorem, which relies heavily on the algebra structure.

A detailed analysis of these generalized exit spaces and the way in which they fail to be an algebra could nonetheless be insightful for an analysis of the limits of the Lindblad form.

In the case of a pure contractive semigroup as above the two definitions of exit space "coincide", i.e. the exit space as defined here is $(\mathfrak{T}(\mathcal{E}_{\min}), \mathrm{ad}_j)$, where $(\mathcal{E}_{\min}, j)$ is the minimal exit space for the semigroup $U(t)$. This is clear because the set $|\mathcal{D}\rangle\langle\mathcal{D}|$ is dense in $(\mathrm{dom}(\mathcal{Z}_*), \|\cdot\|_{\mathrm{dom}(\mathcal{Z}_*)})$ and the defining property of the embedding is satisfied on positive elements, and hence on the whole of $|\mathcal{D}\rangle\langle\mathcal{D}|$:

$$\left\|\sum_i |\psi\rangle\langle\psi|\right\|_{\mathcal{Z}_*} = -\sum_i \mathrm{tr}(\mathcal{Z}_*(|\psi\rangle\langle\psi|) = \left\|\mathrm{ad}_j\left(\sum_i |\psi\rangle\langle\psi|\right)\right\|_{\mathrm{tr},\mathcal{E}} \tag{4.52}$$

Consequently the general definition of Exit spaces would add nothing to the analysis of the exit space construction on Hilbert space level.

4.3. Completely-positive perturbations of semigroups

We haven't yet discussed or even appreciated one of the most interesting properties of first-arrival-time measures as defined in definitions 4.1 and 4.2: the possibility to integrate against an arrival-time measure. Which is not very surprising except that we can integrate not only scalar valued functions but $\mathfrak{CP}(\mathfrak{T}(\mathcal{K}), \mathfrak{T}(\mathcal{H}_2))$ valued functions. This allows us to describe settings in which the evolution of the system continues after the first arrival. On this idea we then build a perturbation theory for semigroups and evolution systems.

4.3.1. Integration against arrival-time measures

The following lemma is the key ingredient for the integration theory to come. The integral is defined via its Radon-Nikodym derivative w.r.t. the Lebesgue measure and hence reduced to "ordinary" Bochner integration theory. For simplicity we restrict to integration of continuous functions. The class of integrable functions could probably be further extended, through the Egoroff theorem, to include functions which are Bochner integrable in some strong sense.

Lemma 4.15. *Let $\mathcal{D}$ be a Banach space which can be densely and continuously embedded into $\mathfrak{T}(\mathcal{H}_1)$ and $\mathbb{F}(t,s) \in (\mathfrak{CP}(\mathfrak{T}(\mathcal{H}_1))$ a strongly-continuous $\mathcal{D}$-valued evolution system of completely-positive maps.*

Let $\mathcal{M}_s$ be a family of $\mathfrak{CP}(\mathfrak{T}(\mathcal{H}_1), \mathfrak{T}(\mathcal{K}))$ valued first-arrival-time measures for $\mathbb{F}(t,s)$, as defined in definition 4.1 and given through the family of maps $\mathcal{J}(t)$ as in theorem (4.13).

Furthermore let $\mathbb{E}_* : [0,T] \to \mathfrak{CP}(\mathfrak{T}(\mathcal{K}), \mathfrak{T}(\mathcal{H}_2))$ *be a piece-wise strongly-continuous function, such that* $\sup_{t\in[0,T]} \|\mathbb{E}_*(t)\| = C < \infty$*.*

Then for all $0 \leq s \leq T$ *and measurable* $\tau \in [s,T]$ *the integrals*

$$\int_\tau \mathbb{E}_*(t) \circ \mathcal{M}_s(\mathrm{d}t)(\rho) \tag{4.53}$$

exists for all $\rho \in \mathfrak{T}(\mathcal{H}_1)$ *and define a completely-positive operator such that:*

$$\left\| \int_\tau \mathbb{E}_* \circ \mathcal{M}(\mathrm{d}t)(\rho) \right\| \leq \sup_{t\in\tau} \|\mathbb{E}_*(t)\| \, \|\mathcal{M}_s(\tau)(\rho)\| \leq C \tag{4.54}$$

In abuse of notation we allow $T = \infty$

Proof. Let $\omega \in \mathcal{D}$ then the function $t \to \mathbb{E}_*(t) \circ \mathcal{J} \circ \mathbb{F}_*(t,s)(\omega)$ is piece-wise continuous for all $s \in \mathbb{R}$ since the operator product commutes with strong limits as long as we have a uniform bound on $\|\mathbb{E}(t)\|$ which was assumed. Hence for all $s \in \mathbb{R}^+$ and measurable sets $\tau \subset [s,\infty]$ the following integrals exist in the sense of Bochner.

$$\int_\tau \mathbb{E}_*(t) \circ \mathcal{M}_s(\mathrm{d}t)(\omega) = \int_\tau \mathbb{E}_*(t) \circ \mathcal{J}(t) \circ \mathbb{F}_*(t,s)(\omega)\mathrm{d}t \tag{4.55}$$

The integral is defined piece-wise. The norm bound also follows directly from the Bochner integrability of the functions $t \to \mathbb{E}_*(t) \circ \mathcal{J} \circ \mathbb{F}_*(t)(\omega)$ for $\omega \in \mathrm{dom}(\mathcal{Z}_*)$:

$$\|\mathcal{M}(\tau)(\omega)\| = \left\| \int_\tau \mathbb{E}_*(t) \circ \mathcal{J} \circ \mathbb{F}_*(t,s)(\omega)\mathrm{d}t \right\|_{\mathrm{tr}} \tag{4.56}$$

$$\leq \int_\tau \|\mathbb{E}_*(t) \circ \mathcal{J} \circ \mathbb{F}_*(t)(\omega)\|_{\mathrm{tr}} \, \mathrm{d}t. \tag{4.57}$$

Hence we can extend the operators $\int \mathbb{E}_*(t)\mathcal{M}(\mathrm{d}T)$ to the whole of $\mathfrak{T}(\mathcal{H})$ by closure . The integrals are completely-positive as limits of completely-positive functions.

□

We want to remark that this lemma of course also allows to define the integration against arrival-time measures for semigroups $\mathbb{F}_*(t)$. Just set $\mathbb{F}_*(t,s) := \mathbb{F}_*(t-s)$ and use the evolution system case. In the case of semigroups we of course do not have to assume the existence of a Radon-Nikodym derivative $\mathcal{J}(t)$ but can just use theorem 4.3 to construct one.

Also note that in the situation of lemma 4.15 for every s and $s \leq t$ the function $t \to \int_s^t \mathbb{E}_*(t) \circ \mathcal{M}(\mathrm{d}t)$ is strongly continuous and takes values in the completely-positive maps.

4.3.2. Repeated integration and perturbation of semigroups

The most interesting application, from the viewpoint of a physicist, of the above integration theory is probably in the case of where the arrival-time measure is of the form $\mathcal{M}(\tau) \in \mathfrak{CP}(\mathfrak{T}(\mathcal{H}))$. In this case we can integrate the no-event semigroup or evolution system against the arrival-time measure:

$$\int_r^t \mathbb{F}(t,s) \circ \mathcal{M}_r(\mathrm{d}t) \tag{4.58}$$

The resulting operator has a natural interpretation as the Evolution of the system under $\mathbb{F}_*(t,s)$ with the condition of exactly one arrival. The integral just averages over all possible times of arrival. Remember that we interpret $\mathbb{F}_*(t,s)$ as describing the evolution of system in the vicinity of the counter, before an arrival.

This integration procedure can be iterated to obtain descriptions of the evolution with an arbitrary but fixed number of arrivals.

$$\int_r^t \int_{t_1}^t \cdots \int_{t_{n-1}}^t \mathbb{F}(t,s) \circ \mathcal{M}_r(\mathrm{d}t_n) \cdots \circ \mathcal{M}_r(\mathrm{d}t_1) \tag{4.59}$$

The sum of all these possible evolutions then corresponds to an average over all possible distributions of arrivals and hence describes the disturbed evolution of the system in vicinity of the counters. We hence expect it to be a semigroup again. The following theorem shows that this is true.

Theorem 4.16 (Perturbation of Evolution Systems)**.** *Let $\mathbb{F}(t,s)$ be a contractive $\mathcal{D}$-valued evolution system. The following iteration converges and defines a strongly-continuous contractive evolution system $\mathbb{E}_*(t,r)$ of completely-positive maps.*

$$\mathbb{E}_*^{n+1}(t,r) := \mathbb{F}_*(t,r) + \int_r^t \mathbb{E}_*^n(t,s) \circ \mathcal{M}_r(\mathrm{d}s) \tag{4.60}$$

The series converges in the sense that, $\lim_{n\to\infty} \mathrm{tr}\left(\left(\mathbb{E}_*^n(t,r)(\rho) - \mathbb{E}_*(t,r)(\rho)\right) B\right) = 0$ *for all $B \in \mathfrak{B}(\mathcal{H})$ and $\rho \in \mathfrak{T}(\mathcal{H})$.*

The dual of the evolution system $\mathbb{E}_*$, i.e. $\mathbb{E}$ is a minimal solution to the Cauchy equation (3.10) *with generator $\mathcal{L}(t) = \mathcal{Z}_*(t) + \mathcal{J}(t)$ on $\mathcal{D}$.*

Proof. The proof of [56] directly generalizes to the case of evolution systems. The only part which requires some work is the evolution system property of the limit map.

We have that $\|\mathbb{E}^n\|_{\mathrm{tr}} \leq 1$ for all n since we have by contractivity of $\mathbb{F}_*(t,s)$ and the boundedness of the measures $\mathcal{M}_s$:

$$\|\mathbb{E}_*^{n+1}(t,r)(\rho)\|_{\mathrm{tr}} \leq \|\mathbb{F}_*(t,r)\|_{\mathrm{tr}} + \left\| \int_r^t \mathbb{E}_*^n(t,s) \circ \mathcal{M}_r(\mathrm{d}s)(\rho) \right\| \tag{4.61}$$

$$\leq \|\mathbb{F}_*(t,r)(\rho)\|_{\mathrm{tr}} + \|\mathcal{M}_r([t,r])(\rho))\| \leq \|\rho\|_{\mathrm{tr}}. \tag{4.62}$$

Hence as a monotone bounded sequence of completely-positive operators $\mathbb{E}_*^n(t,r)$ converges for every $t, r \in \mathbb{R}$ to an completely-positive operator.

A similar calculation yields $\|\mathbb{E}_*^n(t,r)(\rho) - \rho\| \leq 2\|\mathbb{F}_*(t,r)(\rho) - \rho\|$. That is strong continuity of the family $\mathbb{E}_*(t,r)$ follows from strong continuity of the evolution system $\mathbb{F}_*(t,r)$.

The minimality of $\mathbb{E}(t,r)$ follows from the observation that for every other solution of the Cauchy equation $\widetilde{\mathbb{E}}(t,r)$ the expression $\widetilde{\mathbb{E}}(t,r) - \mathbb{E}^n(t,r)$ is completely-positive for every n this can be seen from the Cauchy equation in integral form and an straight forward induction.

Define: $\mathbb{E}_*(t,s)^{(n)} := \mathbb{E}_*(t,s)^{n+1} - \mathbb{E}_*(t,s)^n$ with $\mathbb{E}_*^{(0)}(t,s) = \mathbb{E}_*^1(t,s) = \mathbb{F}_*(t,s)$. These are evolutions with exactly n events. It is easy to see that:

$$\mathbb{E}(t,r)^{(k)} \tag{4.63}$$

$$= \int_r^t \cdots \int_{s_{k-2}}^t \int_{s_{k-1}}^t \mathbb{F}_*(t,s_k) \circ \mathcal{M}_{s_{k-1}}(\mathrm{d}s_k) \circ \mathcal{M}_{s_{k-2}}(\mathrm{d}s_{k-1}) \cdots \circ \mathcal{M}_r(\mathrm{d}s_1) \tag{4.64}$$

For every n the family of operators $\mathbb{E}_*(t,s)^{(n)}$ is evidently completely-positive. To show the evolution system property of the family $\mathbb{E}(t,r)$ we follow [11, proposition 3.9 - 3.11]. It is easy to see that if $n+m=k$ with $0 \leq n, m \leq k$ then every $\mathbb{E}^{(n)}(t,s)\mathbb{E}^{(m)}(s,r)$ gives an identical expression with different integral borders, i.e.:

$$\int_r^s \cdots \int_{s_{m-1}}^s \int_s^t \cdots \int_{s_{k-2}}^t \mathbb{F}_*(t,s_k) \circ \mathcal{M}_{s_{k-1}}(\mathrm{d}s_k) \circ \cdots \circ \mathcal{M}_r(\mathrm{d}s_1) \tag{4.65}$$

We observe that:

$$\mathbb{E}^1(t,s)\mathbb{E}^{k-1}(s,r) + \mathbb{E}^0(t,s)\mathbb{E}^k(s,r) \tag{4.66}$$

$$= \int_r^s \cdots \int_{s_{k-1}}^s \int_s^t (\cdots) + \int_r^s \cdots \int_{s_{k-1}}^s \int_{s_k}^s (\cdots) = \int_r^s \cdots \int_{s_{k-1}}^s \int_{s_k}^t (\cdots) \tag{4.67}$$

A straight forward induction then shows that: $\sum_{n+m=k} \mathbb{E}^{(n)}(t,r)\mathbb{E}^{(m)}(r,s) = \mathbb{E}^{(k)}(t,r)$. □

Corollary 4.17. *In the situation of the above theorem. Let $\mathbb{F}_*(t,s) = \mathbb{F}_*(t-s)$ for an semigroup $\mathbb{F}_*(t)$ and $\mathcal{J}(t) = \mathcal{J}$ constant, then $\mathbb{E}_*(t,s) = \mathbb{E}_*(t-s)$ for a semigroup $\mathbb{E}_*(t)$. Let $\mathcal{L}$ be the generator of $\mathbb{E}_*(t)$ and $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ then:*

$$\mathcal{L}(\omega) = \mathcal{Z}_*(\omega) + \mathcal{J}(\omega) \tag{4.68}$$

But beware that $\mathrm{dom}(\mathcal{Z}_*)$ might not be a a core for $\mathcal{L}$. Accordingly $\mathcal{L}$ might look different on $\mathrm{dom}(\mathcal{L})$. One can indeed find examples where $\mathcal{L} = \mathcal{Z}_*$ on $\mathrm{dom}(\mathcal{Z}_*)$ and nonetheless $\mathbb{E}_*(t)$ and $\mathbb{F}_*(t)$ are clearly different semigroups. See for example section 5.

That $\mathcal{Z}_*$ need not be a domain for $\mathcal{L}$ is a well known phenomenon which can be observed already in the case of classical Markov chains. An example was already given together with the first construction of a minimal solution for a Lindblad equation in [28, theorem 3.2]. In such a case there are numerous different non-minimal solutions to the same Cauchy equation.

4.3.3. Resolvents for perturbed semigroups

Since $\mathrm{dom}(\mathcal{Z}_*)$ is not necessarily a core for the perturbed semigroup $\mathbb{E}(t)$. A characterization of the domain $\mathbb{E}(t)$ seems highly desirable. In the case of semigroups we can at least construct the resolvent of the generator of the perturbed semigroup in terms of the perturbation and the resolvent of the unperturbed generator. This result is a sufficient to characterize the resolvent.

The characterization of the resolvent is a simple consequence of the equality between the Laplace transform of a semigroup and the resolvent for its generator and the following lemma about the Laplace transform of the convolution of a function with an arrival-time measure.

Lemma 4.18. *Let $\mathbb{F}_*(t)$ be a semigroup with generator $\mathcal{Z}_*$. Let $\mathcal{M}$ be a $\mathfrak{CP}(\mathfrak{T}(\mathcal{H})$ valued arrival-time measure as defined in 4.1 with Radon-Nikodym derivative $\mathcal{J} : \mathrm{dom}(\mathcal{Z}_*) \to \mathfrak{T}(\mathcal{H})$. And let $\mathbb{E}_*(t) \in \mathfrak{CP}(\mathfrak{T}(\mathcal{H}))$ be a strongly-continuous function which is uniformly bounded on $\mathbb{R}^+$. Then the Laplace transform*

$$\mathcal{R}_{\mathbb{E}_*}(\lambda) := \int_0^\infty e^{-\lambda t}\mathbb{E}_*(t)\mathrm{d}s \tag{4.69}$$

exists. Furthermore we get for the Laplace transform of the integral of $\mathbb{E}_*$ against $\mathcal{M}$ that:

$$\int_0^\infty e^{-\lambda t}\left(\int_0^t \mathbb{E}_*(t-s)\circ\mathcal{J}\circ\mathbb{F}_*(s)\mathrm{d}s\right)\mathrm{d}t = \mathcal{R}_{\mathbb{E}_*}(\lambda)\circ\mathcal{J}\circ(\lambda\mathbb{1}-\mathcal{Z}_*)^{-1} \tag{4.70}$$

Proof. We want to use the Fubini-Tonelli theorem to be able to view the two one-dimensional integrals in the preceding expression as a single two-dimensional one. For $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ and $B \in \mathfrak{B}(\mathcal{H})$ we have that the function :

$$(s,t) \to \mathrm{tr}\left(e^{-\lambda t}\mathbb{E}_*(t-s)\circ\mathcal{J}\circ\mathbb{F}_*(s)(\omega)B\right) \in \mathrm{L}^1(\mathbb{R}^+\oplus\mathbb{R}^+,\mathbb{C}). \tag{4.71}$$

It is integrable against the product measure. Since this is the case for arbitrary $B \in \mathfrak{B}(\mathcal{H})$, the same holds true for the above integral. Viewing the above integral as an integral over all points $(s,t) \in \mathbb{R}^+ \oplus \mathbb{R}^+$ such that $s \leq t$ and doing the coordinate transform $(s,t) \to (s,t+s)$ we get:

$$\int_0^\infty e^{-\lambda(t+s)}\int_0^\infty \mathbb{E}_*(t)\circ\mathcal{J}\circ\mathbb{F}_*(s)(\omega)\mathrm{d}s\mathrm{d}t \tag{4.72}$$

$$=\int_0^\infty e^{-\lambda t}\mathbb{E}_*(t)\mathrm{d}t\circ\mathcal{J}\circ\int_0^\infty e^{-\lambda s}\mathbb{F}_*(s)(\omega)\mathrm{d}s \tag{4.73}$$

$$=\mathcal{R}(\lambda,\mathbb{E}_*)\circ\mathcal{J}\circ\mathcal{R}(\lambda,\mathbb{F}_*) \tag{4.74}$$

□

If we iterate the above lemma we end up with the following theorem.

Theorem 4.19. *Let $\mathbb{F}_*(t)$ be a semigroup with generator $\mathcal{Z}_*$, and let $\mathbb{E}_*(t)$ be the semigroup with generator $\mathcal{Z}_* + \mathcal{J}$ as in theorem 4.16. The following sequence of completely-positive operators converges to the resolvent of $\mathcal{Z}_* + \mathcal{J}$*

$$\mathcal{R}^{n+1} := (\lambda\mathbb{1}-\mathcal{Z}_*)^{-1} + \mathcal{R}^n\circ\mathcal{J}\circ(\lambda\mathbb{1}-\mathcal{Z}_*)^{-1} \tag{4.75}$$

The sequence converges in the sense that for all $B \in \mathfrak{B}(\mathcal{H})$ and $\rho \in \mathfrak{T}(\mathcal{H})$:

$$\lim_{n\to\infty}\mathrm{tr}(\left(\mathcal{R}^n(t,r) - (\lambda\mathbb{1}-(\mathcal{Z}_*+\mathcal{J}))^{-1}\right)(\rho)B) = 0.$$

Hence to understand the resolvent of $\mathbb{E}_*(t)$ we have to understand the iterates of the map $(\lambda\mathrm{id} - \mathcal{Z}_*)^{-1} \circ \mathcal{J}$. In some cases this is possible and leads to simple results. An example is the perturbation $\mathcal{J}(\omega) = -\rho_0 \operatorname{tr}(\mathcal{Z}_*(\omega))$ i.e. the reset to a fixed state.

Corollary 4.20. *If we perturb the semigroup $\mathbb{F}_*$ with generator $\mathcal{Z}_*$ with the operator $\mathcal{J}(\omega) = -\rho_0 \operatorname{tr}(\mathcal{Z}_*(\omega))$ we have* $\operatorname{dom}(\mathcal{Z}_*) = \operatorname{dom}(\mathcal{J})$.

Proof. Since we always reset the system the resolvent has the following form:

$$(\lambda\mathrm{id} - \mathcal{L})^{-1} \tag{4.76}$$

$$=\mathcal{R}(\lambda)(\rho) + \mathcal{R}(\lambda)(\rho_0) \operatorname{tr}(\mathcal{Z}_*\mathcal{R}(\lambda)(\rho)) \sum_{n=0}^{\infty} (-1)^{n+1} \operatorname{tr}(\mathcal{Z}_* \circ \mathcal{R}(\lambda)(\rho_0))^n \tag{4.77}$$

That is a sum of two elements in $\operatorname{dom}(\mathcal{Z}_*)$ and hence in $\operatorname{dom}(\mathcal{Z}_*)$. □

4.3.4. Delayed choice measurements

We now turn to a different viewpoint on the perturbation construction in lemma 4.16. In this theorem we describe the evolution of the system disturbed by an array of detectors. That is, the theorem enables us to describe different measurements on the system at a later time.

When we want to describe a system which interacts with its boundaries in some Markovian way this turns out to be very useful. But in the case that it interacts with an array of counters we would rather like to describe the measurement which is already running, i.e. the click statistics of the counters.

This viewpoint fits perfectly in the general theme of this thesis, and is the main reason why we went through all this trouble. If we are able to describe the detector click statistics we are describing possible measurements on the system compatible with its evolution, i.e. the perturbed evolution $\mathbb{E}_*(t)$. And since the perturbation construction is build from a "fine-grained" picture of the interaction with the environment this set of measurements should be fairly complete.

Precisely we are able to describe, the full statistics for, all measurements on the system which fit into the measurement on arrival scheme. These are measurements where the system undergoes a finite change, i.e. a quantum jump, and we are interested in the time and type of the jump.

The natural description for such a setup was already described in the introduction in section 2.2.5. It is the language of point processes, i.e. of random measures.

It is straight forward to extract a description of the point-process from our picture of the physical process and the explicit description of the evolution given in equation (4.60). Colloquially the explicit description of the evolution in this equation can be read as a "density function" for the random measure. And one can just read of the "state" of the quantum system for e given sequence of detector clicks.

A rigorous statement of the relation is equally straight forward, but we have to check that the necessary measures are well defined.

We want to describe point processes modelling sequences of results. These results can always be ordered through the time-point of their occurrence.

Definition 4.21. By $\Delta_n([0,T])$ we denote the set of ordered n-tuples in $[0,T]$, i.e. $\Delta_n := \{(t_1,\ldots,t_n) | t_n \in [0,T], t_i \leq t_j \text{ iff } i \leq j\}$.

We start with a delayed choice description of a fixed number of arrivals. That is, for the moment we retain a quantum description of the possible measurements on arrival. Essentially we view the integral defined in theorem 4.15 as an instrument describing the distribution of exactly one arrival, and the natural generalizations to multiple arrivals. The region we integrate on is just the distribution of arrivals we average over.

Lemma 4.22. *Let $\mathcal{H},\mathcal{T}$ be Hilbert spaces and $\mathcal{D}$ a Banach space which can be densely and continuously embedded into $\mathfrak{T}(\mathcal{H})$. Let $\mathbb{F}_*(t,s)$ be a strongly-continuous $\mathcal{D}$-valued evolution system. Furthermore let $\mathcal{J}(t):\mathcal{D}\to\mathfrak{T}(\mathcal{T}\otimes\mathcal{H})$ be an infinitesimal arrival for $\mathbb{F}_*(t,s)$. Denote by $\mathcal{M}_s(\tau):\mathfrak{T}(\mathcal{H})\to\mathfrak{T}(\mathcal{T}\otimes\mathcal{H})$ the induced first arrival-time measure, as in theorem 4.4.*

For every $0\le s\le t\le T$ every n and measurable sets $\tau\in[0,T]$ and $\sigma\in[0,T]^n$ the following expression extends uniquely to a two-parameter family of $\mathfrak{CP}\left(\mathfrak{T}(\mathcal{H}),\mathfrak{T}(\mathcal{T}^{\otimes n+1}\otimes\mathcal{H})\right)$ valued instruments on $[0,T]^{n+1}$.

$$\mathbb{E}^{(n+1)}(t,r)((\tau,\sigma)) = \int\limits_{\tau\cap[r,t]} \mathrm{id}_{\mathcal{T}}\otimes\mathbb{E}^{(n)}(t,s)(\sigma)\circ\mathcal{M}_r(\mathrm{d}s) \tag{4.78}$$

$$\mathbb{E}^{(1)}(t,r)(\tau) = \int\limits_{\tau\cap[r,t]} \mathrm{id}_{\mathcal{T}}\otimes\mathbb{F}_*(t,s)\circ\mathcal{M}_r(\mathrm{d}s) \tag{4.79}$$

For every $0\le s\le t\le T$, every $\rho\in\mathfrak{T}(\mathcal{H})$ and every $n\in N$ the assignment $\sigma\to\mathbb{E}_*^{(n)}(t,r)(\sigma)(\rho)$ for Lebesgue measurable $\sigma\subset[0,T]^n$ is a vector measure, i.e. σ-additive in norm topology. The measure is supported on $\Delta^n([0,T])$ this is we have $\mathbb{E}^{(n)}(t,r)(\sigma)=\mathbb{E}^{(n)}(t,r)(\sigma\cap\Delta_n([s,t])$.

For every n the family $\mathbb{E}^{(n)}$ is continuous in the sense that for every $\rho\in\mathfrak{T}(\mathcal{H})$ and $\sigma\in[0,\infty]^n$ the map $(s,t)\to\mathbb{E}^{(n)}(t,r)(\rho)$ is continuous.

Proof. The case $n=1$ is clear. Assume that the theorem is true for the case n. It is clear that $\mathbb{E}^{(n+1)}(t,r)$ as defined above vanishes on sets of Lebesgue measure zero, because the measure $\mathcal{M}(\tau)$ and $\mathbb{E}^{(n)}$ have this property. Hence by the Carathéorory-Hahn-Kluvanek extension theorem [34, theorem I.5.2] it extends uniquely to a bounded vector measure on $[0,T]^{n+1}$, which again vanishes on sets of Lebesgue measure zero. The continuity follows follows as in 4.15. It is clear that the measure is non-zero only on $\Delta_{n+1}([s,t])$. □

Since POVMs are in general not assumed to be vector measures, but only satisfy σ-additivity in the weak topology, we can define the resulting measure for a specific choice only "in expectation". It is then a straight forward consequence of the classical Carathéorory-Hahn extension theorem.

Corollary 4.23. *In the situation of the previous lemma, choose a measure space (X,Ω) a POVM $F:\Omega\to\mathfrak{B}(\mathcal{T})$, a state $0\le\rho\in\mathfrak{T}(\mathcal{H})$ and an effect $B\in\mathfrak{B}(\mathcal{H})$ s.t. $0\le B\le\mathbb{1}$.*

Then for every n there exists a unique real valued measure $\mu^{(n)}$ on $\Delta^n([0,T])\otimes X^n$ such that for measurable $\sigma\subset\Delta_n([0,T])$ and $\chi_i\in\Omega$ for $1\le i\le n$:

$$\nu^{(n)}(\sigma\times\chi_1\cdots\times\chi_n) = \mathrm{tr}\left(\mathbb{E}_*^{(n)}(\sigma)(T,0)(\rho)F(\chi_1)\otimes\cdots F(\chi_n)\otimes B\right) \tag{4.80}$$

If we set $\Delta^0([0,T])\otimes X^0$ to be set with one element and $\mathbb{E}_*^{(0)}=\mathbb{F}_*(T,0)$ or zero this introduces a measure on $\bigcup_{n=0}^\infty\Delta_n\otimes X^n$ with total mass $(\mathrm{tr}(\mathbb{E}_*(T,0)(\rho)B)))$. Where $\mathbb{E}_*(t,s)$

denotes the pre-adjoint of the minimal solution of the Cauchy equation with generator $\mathcal{L}(\omega) = \mathcal{Z}_*(\omega) + \mathrm{tr}_{\mathcal{T}}(\mathcal{J}(\omega))$ *defined through theorem 4.16.*

To get a point process we just have to identify points in $\Delta_n([0,T]) \otimes X^n$ with sums of Dirac measures on $[0,T] \times X$. And normalize the above measure on $\bigcup_{n=0}^{\infty} \Delta_n \otimes X^n$ to mass 1.

Theorem 4.24. *The situation of the previous corollary defines a unique point process with state space* $([0,T] \times X, \mathcal{X})$*, where* $\mathcal{X}$ *is the product sigma algebra.*

$$k : \left(\bigcup_{n=0}^{\infty} (\Delta_n([0,T]) \times X^n) \right) \times \mathcal{X} \to \mathbb{R} \tag{4.81}$$

$$((t_1, \ldots, t_n), (x_1, \ldots, x_n)), \xi) \mapsto \#\{i | (t_i, x_i) \in \xi\} \tag{4.82}$$

There is a shortcoming in this approach which we briefly want to mention. Even if we add measurements on arrival to the description the theory still fails to describe certain measurements which one could perform on the open system under investigation. Namely those which are not second quantized versions of one particle observables but genuine measurements of "field"-quantities of the information radiated into the environment. In the example of the cavity these would e.g. be homodyne measurements on the output field.

5. Holevo's counterexample

One of the most intriguing examples of a completely-positive semigroup is the "non commutative diffusion", introduced by Holevo in the end of his study of arrival times for completely-positive semigroups [56]. The peculiar property of this example is, that it can not be represented as the minimal solution of an ordinary Lindblad equation[54].

Nonetheless this example is still of the absorptive-evolution-plus-reinsertion type. That is, there is still a clear distinction between no-event evolution and reinsertion events.

Holevo's counterexample shows that one has to be quite careful when defining what a Lindblad generator is and how one can go "beyond" the usual interpretation of the Lindblad form. It is however not clear if there are simple modifications to the definition of a Lindblad generator which include the example.

Since the presentation of the example in [54] is arguably a bit brief we try to give an expanded and mostly self-contained account. To streamline the presentation in this chapter a bit we embedded some helpful calculations into the introduction, section 2.3.

5.1. Outline

In the following section we construct three different semigroups. The first one, denoted $U(t)$, is a semigroup on Hilbert space level, namely the diffusion on $\mathrm{L}^2(\mathbb{R}^+)$ with boundary condition 0.

The first semigroup constitutes the dissipative part of the second semigroup $\mathbb{E}_{1*}(t)$, which acts on trace class operators and is a completely-positive semigroup with a normal Lindblad generator. The construction is a straightforward application of the exit space and reinsertion theory described in 4.2.

The second semigroup, i.e. $\mathbb{E}_{1*}(t)$, is non-conservative so we can again add reinsertion events. This results in the third semigroup $\mathbb{E}_{2*}(t)$, which is the actual example of a non-standard semigroup. We proof that if we attempt to construct this semigroup as a minimal solution to a Lindblad equation we end up with the semigroup number two.

Loosely speaking, the additional Lindblad term that we add to get the third semigroup, lies perpendicular to the domain of the other Lindblad operators, in the sense that the intersection of their domains is the zero vector.

5.1.1. Setup

In the entire rest of the chapter the Hilbert space $\mathcal{H}$ is the space of square integrable functions on the positive half axis $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^+)$.

We want to analyze the diffusion semigroup on $\mathcal{H}$. Its generator is the Laplace operator. Hence we need the notion of (classical-)differentiation in Lebesgue spaces as well as weak derivatives. The definition of the former is clear and we do not repeat it here. However we want to remark that one has to be a bit cautious about the equivalence relation underlying the Lebesgue spaces.

The set of twice differentiable functions with boundary condition 0 plays an important role. It serves as a core for the generator of the first semigroup.

$$\mathcal{D} = \{\psi \in \mathrm{L}^2(\mathbb{R}^+) | \psi(0) = 0, \psi'' \in \mathrm{L}^2(\mathbb{R}^+)\} \tag{5.1}$$

This set is evidently dense in $\mathrm{L}^2(\mathbb{R}^+)$. Another important set is:

$$|\mathcal{D}\rangle\langle\mathcal{D}| := \operatorname{span}\left\{ |\psi\rangle\langle\varphi| \,\middle|\, \psi, \varphi \in \mathcal{D} \right\} \tag{5.2}$$

It is dense in $\mathfrak{T}(\mathcal{H})$. We need this set in the construction of the minimal solution, as described in 4.3. Such ketbra domains are intimately linked with the definition of the Lindblad form.

The weak derivative is a straightforward generalization of the classical derivative via partial integration. A function ψ is weakly differentiable iff there exists a function ψ' such that the following equation is true for all infinitely (classically)-differentiable functions supported away from 0:

$$\int_0^\infty \psi'(x)\bar{\varphi}(x)\mathrm{d}x = -\int_0^\infty \psi(x)\bar{\varphi}'(x)\mathrm{d}x \tag{5.3}$$

If one interprets the derivatives in the definition of $\mathcal{D}$ as weak derivatives and equips the space with the norm $\|\psi\|_{\mathrm{W}} := \|\psi\| + \|\psi''\|$ we get the Sobolev space $\mathrm{W}^{2,2}(\mathbb{R}^+)$ with the boundary condition 0 which we denote as $\mathrm{W}_0^{2,2}(\mathbb{R}^+)$. The boundary condition is well defined because of the Sobolev embedding theorem which states for this case that every function in $\mathrm{W}^{2,2}(\mathbb{R}^+)$ has a once continuously differentiable representative. Actually the derivative is Hölder continuous with constant $\alpha = \frac{1}{2}$.

The Sobolev space $\mathrm{W}_0^{2,2}$ is the domain of the diffusion semigroup, i.e. the first semigroup in our example.

5.2. Construction of the counterexample

5.2.1. The diffusion on $\mathbb{R}^+$

We start as in the exit space construction, compare 4.2, with a contractive strongly-continuous semigroup on the Hilbert space $\mathcal{H}$. The formal generator restricted to $\mathcal{D}$ is given as:

$$K_1 : \mathcal{D} \to \mathcal{H} \qquad \varphi \to \varphi'' \tag{5.4}$$

The closure of this operator is a maximally dissipative operator and hence generator of a semigroup. Instead of showing this directly we construct a semigroup $U(t) = \exp(tK_1)$ which solves the Cauchy equation $\frac{\mathrm{d}}{\mathrm{d}t}U(t)\psi = K_1 U(t)\psi$ for $\psi \in \mathcal{D}$ and show that $\mathcal{D}$ is a core for its generator. This is sufficient to show that the closure of K_1 is indeed the generator of $U(t)$.

Construction of the semigroup

It is clear that the semigroup $U(t)$ is some kind of diffusion on $\mathbb{R}^+$. The "boundary condition" for $U(t)$ is fixed through the definition of $\mathcal{D}$ to be $\varphi(0) = 0$. The diffusion on

$\mathbb{R}^+$ with zero boundary condition can easily be constructed from the diffusion $\widetilde{U}(t)$ on the whole $\mathbb{R}$ by the reflection trick. Luckily we already explicitly constructed the diffusion on $\mathbb{R}$ on page 29, it is given through convolution with the heat kernel:

$$h_t \in \mathrm{L}^2(\mathbb{R}) \qquad x \to \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t}\right) \tag{5.5}$$

$$\widetilde{U}(t) : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}) \qquad \psi \to (h_t * \psi)(x) \tag{5.6}$$

Because of the reflection symmetry of the heat kernel we have the following identity:

$$\int_{-\infty}^{\infty} h_t(y)\psi(x-y)\mathrm{d}y = \int_{-\infty}^{\infty} h_t(y)\psi(x+y)\mathrm{d}y \tag{5.7}$$

The next step is to restrict the semigroup $\widetilde{U}(t)$ to the positive half axis, s.t., the boundary condition is satisfied. This is done by the reflection trick. That is we make use of the symmetry of the diffusion kernel and do a point inversion at the origin of the distribution on the positive half axis. The distribution and its "mirror charge" then annihilate each other at the origin. Furthermore the distributions conserve their symmetry under the semigroup evolution, so it is possible to later restrict the whole setting to one half axis.

Definition 5.1. Let $\varphi \in \mathrm{L}^2(\mathbb{R}^+)$. We define the following symmetrization map $\varphi \to \widetilde{\varphi}$ by:

$$\mathrm{L}^2(\mathbb{R}) \supset \mathrm{L}^2(\mathbb{R}^+) \to \mathrm{L}^2(\mathbb{R}) \qquad \varphi \mapsto \widetilde{\varphi} \tag{5.8}$$

$$\text{where} \quad \widetilde{\varphi} \in \mathrm{L}^2(\mathbb{R}) \qquad x \mapsto \begin{cases} \varphi(x) & x \geq 0 \\ -\varphi(|x|) & x < 0 \end{cases} \tag{5.9}$$

The operation can either be seen as a map from $\mathrm{L}^2(\mathbb{R}^+) \to \mathrm{L}^2(\mathbb{R})$ or as a map $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$. We use the same notation for both definitions.

We can now use this symmetry to define the diffusion semigroup on $\mathrm{L}^2(\mathbb{R}^+)$ by symmetric extension of functions on $\mathbb{R}^+$ to functions on the whole line. The central idea is that the diffusion on $\mathbb{R}$ conserves the point inversion symmetry. We can thus define a new semigroup $U(t)$ acting on a function φ on the positive half line, such that:

$$\widetilde{U(t)\varphi} = \widetilde{U}(t)\widetilde{\varphi}. \tag{5.10}$$

Lemma 5.2. *Let $\widetilde{U}(t)$ be the diffusion on $\mathbb{R}$. The following assignment defines a strongly-continuous semigroup on $\mathcal{H}$:*

$$U(t) : \mathrm{L}^2(\mathbb{R}^+) \to \mathrm{L}^2(\mathbb{R}^+) \qquad x \to \widetilde{U}(t)\widetilde{\varphi}(x) \tag{5.11}$$

As an alternative form we get:

$$U(t)\varphi(x) = \int_0^{\infty} (h_t(y-x) - h_t(y+x))\varphi(y)\mathrm{d}y \tag{5.12}$$

The generator of U is given by the closure of K_1 as defined in equation (5.4)*. The semigroup is contractive.*

Proof. The only thing we have to check is the Cauchy functional equation (2.30). All other properties directly transfer. This can be done either by straightforward but tedious calculation. Or alternatively we make use of the fact that the semigroup conserves the point inversion symmetry . In other words we want to show that

$$\widetilde{U}(t)\widetilde{\varphi} = \widetilde{\widetilde{U}(t)\widetilde{\varphi}}. \tag{5.13}$$

This can be seen from the following calculation:

$$\widetilde{U}_t(\widetilde{\varphi})(-x) = \int_{-\infty}^{\infty} h_t(y)\widetilde{\varphi}(-x+y)\mathrm{d}y \tag{5.14}$$

$$= -\int_{-\infty}^{\infty} h_t(-y)\widetilde{\varphi}(-x-y)\mathrm{d}y = -\widetilde{U}_t(\widetilde{\varphi})(x) \tag{5.15}$$

This is sufficient to see that $U(t)U(s) = U(t+s)$ since:

$$U(t)U(s)\varphi = U(t)\widetilde{U}(s)\widetilde{\varphi} = \widetilde{U}(t)\widetilde{U}(s)\widetilde{\varphi} \tag{5.16}$$

A few simple integral transformations yield the alternative definition of $U(t)$. From this definition it can be easily seen that $U(t)$ conserves the boundary condition. Furthermore the semigroup obviously conserves the domain $\mathcal{D}$, and hence by the core theorem 2.33 the closure of K_1 is the generator of a semigroup. $U(t)$ is also contractive because it is the restriction of the contractive semigroup $\widetilde{U}(t)$. □

It is clear that the semigroup looses norm. Due to the boundary condition at zero probability "diffuses out of the system".

Resolvent

We do not need the resolvent to characterize the domain of the generator $U(t)$, because we already know a core of the domain. However the resolvent of $U(t)$ is very helpful in calculating the resolvents of the other two semigroups we are going to construct. And for those domain we won't get a core for free.

We already calculated the resolvent for the diffusion on the whole of $\mathbb{R}$ in example 3 on page 33 via the Laplace transform of the semigroup. The resolvent $(\lambda\mathbb{1} - K_1)^{-1}$ is now a straightforward consequence.

$$(\lambda\mathbb{1} - K_1)^{-1}(\varphi)(x) = \int_{-\infty}^{\infty} \frac{e^{-|y|\sqrt{\lambda}}}{2\sqrt{\lambda}}\widetilde{\varphi}(y+x)\mathrm{d}y \tag{5.17}$$

$$= \int_0^{\infty} \left(\frac{e^{-|x-y|\sqrt{\lambda}}}{2\sqrt{\lambda}} - \frac{e^{-|x+y|\sqrt{\lambda}}}{2\sqrt{\lambda}} \right) \varphi(y)\mathrm{d}y. \tag{5.18}$$

As we already mentioned the domain of K_1 is equal to the Sobolev space $W_0^{2,2}$.

5.2.2. Two types of diffusion on $\mathbb{R}^+ \times \mathbb{R}^+$

We now turn to a completely-positive semigroup with dissipative part generated by $U(t)$. That is a minimal solution to Lindblad equations whose dissipative part is generated by K_1.

The action of the resulting semigroup on trace class operators on $\mathcal{H}$ is clear. It is a general result that trace class operators on function spaces can be written as integral operators, e.g. see [78, section VI.6] or the reminder number 2 on page 29.

For $\rho \in \mathfrak{T}(\mathcal{H})$ and $\psi \in \mathcal{H}$ there exists an integral kernel $\rho(x,y)$ such that:

$$(\rho\varphi)(x) := \int_0^\infty \rho(x,y)\phi(y)\mathrm{d}y \tag{5.19}$$

The trace of ρ can be written as:

$$\mathrm{tr}(\rho) = \int_0^\infty \rho(x,x)\mathrm{d}x \tag{5.20}$$

Subsequently we always identify operators with their kernels. An especially important subclass are ketbras, i.e. operators of the form $|\psi\rangle\langle\varphi|$ for $\psi, \varphi \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}^+)$. In this case we have $|\psi\rangle\langle\varphi|(x,y) = \bar{\varphi}(y)\psi(x)$.

No-event evolution

First we turn our attention to the natural ampliation of the semigroup $U(t)$ to the trace class operators:

$$\mathbb{F}_*(t) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}) \qquad \mathbb{F}_*(t) := U(t)\rho U(t)^*. \tag{5.21}$$

To see that this semigroup is strongly-continuous, observe that this is evidently the case on ketbra operators, due to the strong continuity of $U(t)$. Since the span of ketbra operators is dense in $\mathfrak{T}(\mathcal{H})$ the strong continuity extends to the whole of $\mathfrak{T}(\mathcal{H})$.

Its generator is formally given as $\mathcal{Z}_*(\rho) = K_1\rho + \rho K_1$. It can be rigorously defined as the closure of:

$$\mathcal{Z}_* : |\mathcal{D}\rangle\langle\mathcal{D}| \to \mathfrak{T}(\mathcal{H}) \qquad |\psi\rangle\langle\psi| \to |\psi''\rangle\langle\psi| + |\psi\rangle\langle\psi''|, \tag{5.22}$$

where the definition is extended by linearity to $|\mathcal{D}\rangle\langle\mathcal{D}|$. This works because $|\mathcal{D}\rangle\langle\mathcal{D}|$ is by construction a core for $\mathcal{Z}_*$. It is dense and invariant under $\mathbb{F}_*(t)$. We shall later see that although it is a core for $\mathcal{Z}_*$, the domain $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a bit too small to capture the whole norm loss of $\mathbb{F}_*$.

It is clear that $\mathbb{F}_*$ is (two-dimensional) diffusion on the upper right quadrant with boundary condition 0. Precisely we have that with the following symmetry operation and the two dimensional heat kernel h_{2t}:

$$\widehat{\bullet} : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathrm{L}^2(\mathbb{R})) \qquad \widehat{\rho}(x,y) = \operatorname{sign} x \operatorname{sign} y \rho(|x|,|y|) \tag{5.23}$$

$$h_{2t} \in \mathrm{L}^2(\mathbb{R}) \qquad (x,y) \mapsto \frac{1}{4\pi t} \exp\left(\frac{-(x^2+y^2)}{4t}\right), \tag{5.24}$$

we have the following form for the no event semigroup:

$$\mathbb{F}_*(t)(\rho)(x,y) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h_{2t}((v-x, w-y))\widehat{\rho}(x+v, y+w)\mathrm{d}x\mathrm{d}y. \tag{5.25}$$

It is again clear that the $\mathbb{F}_*(t)$ is compatible with the symmetry operation and hence that it is a semigroup. It is clear that the domain of $\mathcal{Z}_*$ consists of kernels that satisfy the boundary condition and have two weak partial derivatives. The resolvent of $\mathcal{Z}_*$ can in principle be calculated by the same method as before, i.e., it is the convolution with the Laplace transform of the heat kernel. Remember that $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a core for $\mathcal{Z}_*$.

Diffusion along the diagonal

As a next step we perturb the semigroup $\mathbb{F}_*(t)$ by adding events. For a characterization of the structure of possible events it is instructive to look at the exit space $\mathcal{E}$ for the semigroup $U(t)$. It is readily calculated from the observation that for $\psi, \varphi \in \mathcal{D}$ the norm-loss scalar product evaluates to:

$$\langle\varphi, \psi\rangle_{K_1} = -2\,\mathfrak{Re}\langle\varphi, K_1\psi\rangle = 2\langle\varphi', \psi'\rangle \tag{5.26}$$

The exit space is therefore just the space of square integrable function, i.e., $\mathcal{E} = \mathrm{L}^2(\mathbb{R}^+)$, with the derivative as the natural embedding $j : \mathrm{dom}(K_1) \to \mathcal{E}$. This notation is well defined because $\mathcal{D}$ consists of twice weakly differentiable and hence at least once classically differentiable functions. The natural reinsertion is therefore just the identity map.

If we do this, we end up with the derivation as the sole Lindblad operator. To summarize, our choice for K_1 and L_1 is:

Definition 5.3. Let $\mathcal{D}$ be defined as in equation (5.1). We set:

$$K_1 : \mathcal{D} \to \mathcal{H} \qquad\qquad \psi \to \psi'' \tag{5.27}$$

$$L_1 : \mathcal{D} \to \mathcal{H} \qquad\qquad \psi \to \sqrt{2}\psi'. \tag{5.28}$$

We denote the completely-positive map associated to L_1 as $\mathcal{J}_1$.

We want to construct a solution for the associated Lindblad equation:

$$\begin{aligned}\langle\psi, \mathcal{L}_1(X)\psi\rangle &= \langle K_1\psi, X\psi\rangle + \langle\psi, XK_1\psi\rangle + \langle L_1\psi, XL_1\psi\rangle \\ &= +\left\langle\psi'', X\psi\right\rangle + \left\langle\psi, X\psi''\right\rangle + 2\left\langle\psi', X\psi'\right\rangle\end{aligned} \tag{5.29}$$

It turns out that the solution is non-unique, i.e. not trace-preserving. According to [54] the solution we are going to construct is actually the minimal solution, i.e. the semigroup promised by theorem 3.2. However the following discussion it is not important, if the solution is minimal, but only that it looses probability. Hence we do not attempt to prove minimality.

Since we choose a trace-preserving re-embedding, i.e. the identity map, the generator fulfills the infinitesimal conservativity condition, i.e. equation (3.9). Partial integration can be used to verify this fact. For $\psi \in \mathcal{D}$ we get:

$$\langle\psi, \mathcal{L}_1(\mathbb{1})\psi\rangle = \frac{1}{2}\langle\psi, \psi''\rangle + \frac{1}{2}\langle\psi, \psi''\rangle + \langle\psi', \psi'\rangle \tag{5.30}$$

$$= \left[\bar{\psi}'(x)\psi(x)\right]_0^\infty - \langle\psi', \psi'\rangle + \langle\psi', \psi'\rangle = 0 \tag{5.31}$$

Hence L_1 is a "full" set of Lindblad operators in the sense that we can not add another Lindblad operator without violating conservativity.

Construction of the diffusion

Now we are going to construct the promised solution to (5.29). As in the case of the first semigroup, we again explicitly construct the semigroup $\mathbb{E}_{1*}(t)$ and then show that it is a solution to the Lindblad equation, and according to [54] actually the minimal one. To motivate the construction we examine how generator $\mathcal{L}_{1*}$ of $\mathbb{E}_{1*}(t) = \exp(t\mathcal{L}_{1*})$ acts on pure states $\Psi := |\psi\rangle\langle\psi|$. Evidently we have that for operators $(x, y) \to \rho(x, y) \in |\mathcal{D}\rangle\langle\mathcal{D}|$ the generator $\mathcal{L}_{1*}$ acts as:

$$\mathcal{L}_{1*}(\rho) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\frac{\partial^2}{\partial x \partial y}\right)\rho = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \rho \tag{5.32}$$

In conclusion the semigroup $\mathbb{E}_{1*}(t)$ should again be a diffusion. This time along diagonal lines in the positive square $\mathbb{R}^+ \oplus \mathbb{R}^+$. The boundary condition is this time that $(x, y) \to \rho(x, y)$ is zero on the x and y-axis, as it follows from the definition of $|\mathcal{D}\rangle\langle\mathcal{D}|$. The solution is constructed in analogy with the diffusion on $\mathbb{R}^+$, i.e. by the reflection trick. This time with a slightly more complicated symmetry.

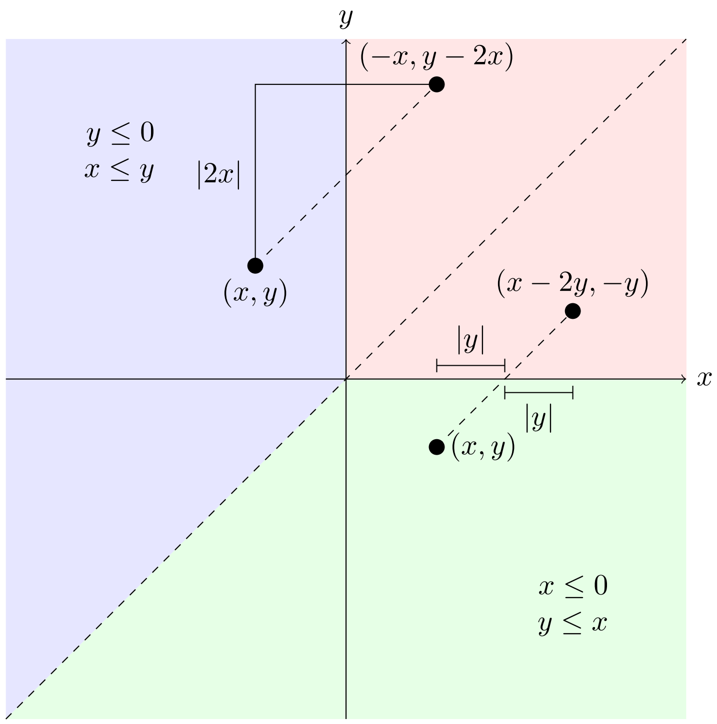


Figure 5.1.: Sketch of the reflection symmetry

The recipe to get the right symmetrization is simple. When we want to extend a kernel $\rho(x, y) \in \mathfrak{T}(\mathrm{L}^2(\mathbb{R}^+))$ to a kernel $\widetilde{\rho}(x, y) \in \mathfrak{T}(\mathrm{L}^2(\mathbb{R})$ we do a point inversion of every diagonal at its intersection with the coordinate system. The symmetry is depicted in figure 5.1. All the coordinate plane is divided into three parts, one of which has $x, y \geq 0$, $x \leq 0$ and $x \leq y$ another and $y \leq 0$, $y < x$ the third. We hence define:

$$\mathfrak{T}(\mathrm{L}^2(\mathbb{R})) \supset \mathfrak{T}(\mathrm{L}^2(\mathbb{R}^+)) \to \mathfrak{T}(\mathrm{L}^2(\mathbb{R})) \qquad \rho \to \widetilde{\rho} \tag{5.33}$$

$$\widetilde{\rho}(x,y) = \begin{cases} \rho(x,y) & x,y \geq 0 \\ -\rho(x-2y,-y) & y \leq 0, x \geq y \\ -\rho(-x,x-2y) & x \leq 0, x \leq y \end{cases} \tag{5.34}$$

$$= \begin{cases} \rho(x,y) & x,y \geq 0 \\ -\rho(x-2\min\{x,y\}, y-2\min\{x,y\}) & \text{else} \end{cases} \tag{5.35}$$

Analogue to the Hilbert space case the diffusion along diagonal lines is easy to define for kernel operators in $\mathfrak{T}(\mathrm{L}^2(\mathbb{R}))$.

Theorem 5.4. *Let $\widetilde{\mathbb{E}}_{1*}$ be the diffusion along the diagonal on $\mathfrak{T}(\mathrm{L}^2(\mathbb{R}))$, i.e.:*

$$\widetilde{\mathbb{E}}_{1*}(t) : \mathfrak{T}(\mathrm{L}^2(\mathbb{R})) \to \mathfrak{T}(\mathrm{L}^2(\mathbb{R})) \tag{5.36}$$

$$\widetilde{\mathbb{E}}_{1*}(t)(\rho)(x,y) = \int_{-\infty}^{\infty} h_t(z)\rho(x+z,y+z)\mathrm{d}z \tag{5.37}$$

The diffusion along the diagonal on the upper right quadrant is defined as:

$$\mathbb{E}_{1*}(t) : \mathfrak{T}(\mathrm{L}^2(\mathbb{R}^+)) \to \mathfrak{T}(\mathrm{L}^2(\mathbb{R}^+)) \qquad \rho \to \widetilde{\mathbb{E}}_{1*}(t)\widetilde{\rho}. \tag{5.38}$$

Alternatively we can write:

$$\mathbb{E}_{1*}(t)(\omega)(x,y) = \int_{-\infty}^{\infty} h_t(z)\widetilde{\omega}(x+z,y+z)\mathrm{d}z \tag{5.39}$$

$$= \int_{-\min\{x,y\}}^{\infty} \left(h_t(z) - h_t(z+2\min\{x,y\})\right)\omega(x+z,y+z) \tag{5.40}$$

The generator of $\mathbb{E}_{1*}$ is the closure of the operator:

$$\mathcal{L}_{1*}(\rho)(x,y) := \frac{\mathrm{d}^2}{\mathrm{d}z^2}\rho(x+z,y+z)\bigg|_{z=0} \tag{5.41}$$

defined on kernels twice differentiable along the diagonals and with boundary condition 0. *For $\omega \in |\mathcal{D}\rangle\langle\mathcal{D}|$ we have*

$$\mathcal{L}_{1*}(\omega)(x,y) := \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \omega(x,y) = \left(\mathcal{Z}_* + \mathcal{J}_1\right)(\omega)(x,y) \tag{5.42}$$

$\mathbb{E}_{1*}(t)$ *is a strongly-continuous semigroup of completely-positive maps.*

Proof. Again the main step in showing that $\mathbb{E}_{1*}(t)$ as defined above is indeed a semigroup, is in showing that $\widetilde{\mathbb{E}}_{1*}(t)$ conserves the symmetry in the sense that:

$$\widetilde{\mathbb{E}}_{1*}(t)(\widetilde{\omega}) = \widetilde{\widetilde{\mathbb{E}}_{1*}(t)(\widetilde{\omega})}. \tag{5.43}$$

This is obvious from the definition, and can be verified by straightforward but tedious calculation.

Since $\mathbb{E}_{1*}(t)$ is a formal solution to a Lindblad equation, it is completely-positive. This statement is part of [56, Proposition 3] and a direct consequence of the integral form of the Cauchy equation, see corollary (3.26). □

According to [54] this semigroup is the minimal solution to the Lindblad equation (5.29). Even though it satisfies the infinitesimal normalization condition (3.9) The semigroup $\mathbb{E}_{1*}(t)$ is not trace-preserving.

Corollary 5.5.

$$\operatorname{tr}\left(\mathbb{E}_{1*}\omega\right) = \int_0^\infty \operatorname{erf}\left(\frac{z}{2\sqrt{t}}\right)\omega(z,z)\mathrm{d}z \tag{5.44}$$

Proof. This follows by explicit evaluation of the trace. A substitution yields:

$$\int_0^\infty h_t(z-x)\mathrm{d}x = \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)$$

And hence by interchanging the order of integration in the definition of the trace we get the result:

$$\operatorname{tr}\left(\mathbb{E}_{1*}\omega\right) = \int_0^\infty\int_0^\infty \left(h_t(z-x)-h_t(z+x)\right)\omega(z,z)\mathrm{d}x\mathrm{d}z \tag{5.45}$$

□

This has a few important consequences. First of all it follows that the minimal solution of the Lindblad equation is not trace-preserving, and hence $\operatorname{dom}(\mathcal{Z}_*)$ is not a core for the domain of the minimal solution and $\mathbb{E}_{1*}$ by [28, theorem 3.2].

This can also be seen directly. We are able to explicitly calculate the resolvent of $\mathcal{L}_{1*}$ in exact analogy with the resolvent of K_1.

Corollary 5.6. *Let $\mathcal{L}_{1*}$ be the generator of the diffusion along the diagonal $\mathbb{E}_{1*}$, then we find the following expression for its resolvent:*

$$\left(\lambda\mathrm{id}-\mathcal{L}_{1*}\right)^{-1}(\rho)(x,y) = \int_{-\infty}^{\infty}\frac{e^{-|z|\sqrt{\lambda}}}{2\sqrt{\lambda}}\widetilde{\rho}(x+z,y+z)\mathrm{d}z. \tag{5.46}$$

$$= \int_{-\min\{x,y\}}^{\infty}\frac{1}{2\sqrt{\lambda}}\left(e^{-|z|\sqrt{\lambda}}-e^{-|z+2\min\{x,y\}|\sqrt{\lambda}}\right)\omega(x+z,y+z)\mathrm{d}z \tag{5.47}$$

This determines the domain of $\mathcal{L}_{1*}$ since it is equal to the image of $\mathfrak{T}(\mathcal{H})$ under the resolvent. The domain is given as all the kernel operators $\rho(x,y)\in\mathfrak{T}(\mathcal{H})$ which satisfy the boundary condition $\rho(x,y)=0$ and for which the second derivative along the diagonals exists in the same distributional sense as for $\operatorname{dom}K_1$.

Hence functions in the domain of $\mathcal{L}_{1*}$ do not have to be differentiable, even in the weak sense, in directions other than the diagonal. Since $\operatorname{dom}(\mathcal{Z}_*)$ is not a core for $\operatorname{dom}(\mathcal{L}_{1*})$ differentiability in other directions is also not "conserved" under the semigroup.

When we restrict the semigroup $\mathbb{E}_{1*}$ to a diagonal, we get again diffusion with boundary condition 0. Observe that this is the diffusion on the Lebesgue space $\mathrm{L}^1(\mathbb{R}^+)$ and not $\mathrm{L}^2(\mathbb{R}^+)$. A simple consequence of the following lemma is also, that the boundary value for elements in $\operatorname{dom}(\mathcal{L}_{1*})$ is well defined.

Lemma 5.7. *For every point (x, y) in the upper half plane the restriction of a density operator to the diagonal through (x, y) is a linear map of the trace-class operators to $L^1(\mathbb{R}^+)$:*

$$\begin{aligned} &D_{x,y} : \mathfrak{T}(\mathcal{H}) \to L^1(\mathbb{R}^+) \\ &D_{x,y}(\omega)(z) = \omega(x + z - \min\{x, y\}, y + z - \min\{x, y\}) \end{aligned} \tag{5.48}$$

Under this map the semigroup $\mathbb{E}_{1*}$is mapped to diffusion on the diagonal, in the sense that:

$$D_{x,y} \left(\mathbb{E}_{1*}(t)(\omega)\right) = U(t)D_{x,y}(\omega) \tag{5.49}$$

$$D_{x,y} \operatorname{dom}(\mathcal{L}_{1*}) \subset \operatorname{dom}(K_1) \tag{5.50}$$

Here we have to interpret $U(t)$ as an operator on on $L^1(\mathbb{R}^+)$

Proof. We only have to prove that $\|D_{x,y}(\omega)\|_{\mathrm{L}^1} \leq \infty$. To see this, observe that the shift on $\mathrm{L}^1(\mathbb{R}^+)$, i.e. $(S_y\psi)(x) := \psi(x + y)$, is a contraction for every $y \in \mathbb{R}$. As always we identify trace-class operators with their kernels. The shift operator acts as $(S(z)\omega)(x, y) = \omega(x + z, y)$. Let w.l.o.g. $x \geq y$ then:

$$\|D_{x,y}\omega\|_{\mathrm{L}^1} = \|U_{x-y}\omega\|_{\mathrm{tr}} \leq \|U_{x-y}\|_{\mathfrak{B}(\mathcal{H})}\|\omega\|_{\mathrm{tr}}. \tag{5.51}$$

Equation (5.49) follows directly by insertion of the definition. The observation about the domains is a direct consequence. □

5.3. A non-standard semigroup

Since $\mathbb{E}_{1*}$ still looses norm, we can apply our perturbation machinery again. But this time we surely end up with a trace-preserving semigroup. To see that we calculate the (generalized) exit space for the semigroup $\mathbb{E}_{1*}$. The relevant norm, evaluates on positive elements in the domain of $\mathcal{L}_{1*}$ to:

$$-\operatorname{tr}(\mathcal{L}_{1*}(\omega))(x, y) \tag{5.52}$$

$$= -\operatorname{tr}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}\omega(x + z, y + z)\bigg|_{z=0}\right) \tag{5.53}$$

$$= -\int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}z}\omega(x + z, y + z)\bigg|_{z=0} \mathrm{d}z \tag{5.54}$$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\omega(z, z)\bigg|_{z=0} \tag{5.55}$$

If we complete $\operatorname{dom}\mathcal{L}_{1*}$ with respect to this norm the resulting Banach space is just $\mathbb{C}$ with absolute value as the norm.

5.3.1. Definition

We know from [28, theorem 3.4] that any strongly-continuous not trace-preserving semigroup can be made into a trace-preserving one, by adding a ricochet to some fixed state $\rho_0 \in \mathfrak{T}(\mathcal{H})$. Even more for the present case of completely-positive semigroups the domain is invariant under this kind of perturbation as we have shown in corollary 4.20. By the above argument, these are also the only possible re-insertions.

Definition 5.8. Fix an $\rho_0 \in \mathfrak{T}(\mathcal{H})$ with $\rho_0 \geq 0$ and $\mathrm{tr}(\rho_0) = 1$. We define the following map:

$$\mathrm{dom}(\mathcal{L}_{1*}) \to \mathfrak{T}(\mathcal{H}) \qquad \mathcal{J}_2(\omega) = -\rho_0 \, \mathrm{tr}(\mathcal{L}_{1*}(\omega)) \tag{5.56}$$

When $\omega \in \mathrm{dom}(\mathcal{L}_{1*})$ is twice regularly differentiable we have:

$$\mathcal{J}_2(\omega) = -\,\rho_0 \, \mathrm{tr}(\mathcal{L}_{1*}(\omega)) = -\rho_0 \int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}x^2}\omega(x,x)\mathrm{d}x \tag{5.57}$$

$$= -\,\rho_0 \left[\frac{\mathrm{d}}{\mathrm{d}x}\omega(x,x)\right]_0^\infty = \rho_0 \left.\frac{\mathrm{d}}{\mathrm{d}x}\omega(x,x)\right|_{x=0} \tag{5.58}$$

The last expression is defined on the whole of $\mathrm{dom}(\mathcal{L}_{1*})$. This map fulfills the requirements to define an $\mathbb{E}_{1*}$-covariant measure in the sense of theorem 4.3 i.e., for positive ω:

$$\|\mathcal{J}_2(\omega)\|_{\mathrm{tr}} = -\,\mathrm{tr}(\mathcal{L}_{1*}(\omega)) \tag{5.59}$$

Observe that for $\omega \in \mathrm{dom}(\mathcal{Z}_*)$ we have that $\mathcal{J}(\omega) = 0$ since for $\psi \in \mathcal{D}$ we have $\mathcal{J}_2\left(|\psi\rangle\langle\psi|\right)(x,y) = \rho(x,y)\left(\bar{\psi}'(0)\psi(0) + \bar{\psi}(0)\psi'(0)\right) = 0$ and $|\mathcal{D}\rangle\langle\mathcal{D}|$ is a core for $\mathrm{dom}(\mathcal{Z}_*)$. Hence the perturbation is "perpendicular" to the perturbation $\mathcal{J}_{1*}$

Definition 5.9. Let $\mathbb{E}_{1*}$ be the semigroup defined in theorem 5.4 and let $\mathcal{J}_2$ be the map defined in equation (5.56). We define $\mathbb{E}_{2*}(t)$ to be the perturbed semigroup we get by applying theorem 4.16. Denote the generator of $\mathbb{E}_{2*}(t)$ as $\mathcal{L}_{2*}$.

We note a few important properties of the semigroup $\mathbb{E}_{2*}(t)$. A trivial but important consequence of them is that $\mathbb{E}_{2*}(t)$ can not be equal to $\mathbb{E}_{1*}$ and that it hence neither can be a minimal solution for the Lindblad equation (5.29).

Corollary 5.10. *The semigroup* $\mathbb{E}_{2*}(t)$ *is completely-positive and trace-preserving. Furthermore we have* $\mathrm{dom}(\mathcal{L}_{2*}) = \mathrm{dom}(\mathcal{L}_{1*})$.

Proof. The invariance of the domain was proven in corollary 4.20. That the semigroup is trace preserving then follows from [28, theorem 3.2]. □

Now we have two semigroups whose generators coincide on $\mathrm{dom}\,\mathcal{Z}_*$. Remember however that $\mathrm{dom}(\mathcal{Z})$ is not a core for the semigroup $\mathbb{E}_{1*}(t)$, and hence neither for the semigroup $\mathbb{E}_{2*}(t)$. The two semigroups do not coincide, because one is conservative and the other is not.

$\mathbb{E}_{2*}(t)$ is non-standard

We are now going to proof that the semigroup $\mathbb{E}_{2*}(t)$ can not be the minimal solution to a Lindblad equation. Our analysis starts with a few simple consequences of the assumption that $\mathbb{E}_{2*}$ is standard.

A central property of minimal solutions of Lindblad equations is that they contain a large set of ketbra operators, i.e., there is always a set like $|\mathcal{D}\rangle\langle\mathcal{D}|$ contained in the domain. On the other hand, if the generator of a semigroup contains a large set of ketbras then it satisfies Lindblad form on this domain [53]. This is the structure we are going to exploit.

A key step in the following analysis is the next lemma. It shows that every ketbra domain contained in $\mathrm{dom}(\mathcal{L}_{1*})$ is made up of pure states in $\mathrm{dom}(K_1)$.

Lemma 5.11. *Let K_1 be the generator of the diffusion on $\mathrm{L}^2(\mathbb{R}_+)$ as defined in equation(5.4). Let $\mathcal{D}_2$ be a dense subspace in $\mathcal{H}$ such that the set of ketbras is contained in* $\mathrm{dom}(\mathcal{L}_{2*})$, *i.e:*

$$|\mathcal{D}_2\rangle\langle\mathcal{D}_2| := \mathrm{span}\left\{|\psi\rangle\langle\varphi| | \psi, \varphi \in \mathcal{D}_2\right\} \subset \mathrm{dom}(\mathcal{L}_{2*}) \tag{5.60}$$

In this situation $\mathcal{D}_2 \subset \mathrm{dom}(K_1)$.

Proof. Let $\psi, \varphi \in \mathcal{D}_2$ and let $\lambda, \mu \subset \mathcal{D}$, where $\mathcal{D}$ is as defined in (5.1). Since $\mathrm{dom}(\mathcal{L}_{2*}) = \mathrm{dom}(\mathcal{L}_{1*})$, the following expression is well defined:

$$\mathrm{tr}(\mathcal{L}_{1*}(|\psi\rangle\langle\varphi|)|\lambda\rangle\langle\mu|) = \mathrm{tr}(|\psi\rangle\langle\varphi|\mathcal{L}_1(|\lambda\rangle\langle\mu|)) \tag{5.61}$$

$$= \int_0^\infty \int_0^\infty \psi(x)\bar{\varphi}(y)(\partial_x + \partial_y)^2\lambda(y)\bar{\mu}(x)\mathrm{d}y\mathrm{d}x \tag{5.62}$$

$$= \int_0^\infty \int_0^\infty \psi(x)\bar{\varphi}(y)\left(\lambda''(y)\bar{\mu}(x) + 2\lambda'(y)\bar{\mu}'(x) + \lambda(y)\bar{\mu}''(x)\right)\mathrm{d}y\mathrm{d}x \tag{5.63}$$

$$= \langle\mu, \psi\rangle\langle\varphi, \lambda''\rangle + 2\langle\mu', \psi\rangle\langle\varphi, \lambda'\rangle + \langle\mu'', \psi\rangle\langle\varphi, \lambda\rangle \tag{5.64}$$

We view this as a family of linear functionals in ψ. We want to show existence of the two weak derivatives of ψ. To get this we have to show that for fixed ψ and μ we can get a system of three independent equations. Which would mean that all three functionals $\langle\mu, \psi\rangle$ $\langle\mu', \psi\rangle$ and $\langle\mu'', \psi\rangle$ can be reconstructed from the system of equations and hence are well defined.

It is clear that we can find a function $\lambda \in \mathcal{D}$ such that λ, λ' and λ'' are linearly independent. The only thing that could go wrong is that φ satisfies the given weak differential equation given. However the space of solutions to this equation is only finite dimensional. Since $\mathcal{D}$ is dense we can certainly choose an φ, which is not a solution.

The boundary condition is satisfied because for vectors $\psi \in \mathcal{D}_2$ the condition $|\psi\rangle\langle\psi| \subset \mathrm{dom}(\mathcal{L}_{2*})$ is fulfilled. Lemma 5.7 shows that $\psi(0) = 0$ □

We get another important result as a direct consequence of the previous lemma. The two generators $\mathcal{L}_{1*}$ and $\mathcal{L}_{2*}$ coincide on Ketbra domains.

Corollary 5.12. *Let $\mathcal{D}_2$ be a set of vectors dense in $\mathcal{H}$ such that the set of ketbras is contained in* $\mathrm{dom}(\mathcal{L}_{1*})$, *i.e:*

$$|\mathcal{D}_2\rangle\langle\mathcal{D}_2| := \mathrm{span}\left\{|\psi\rangle\langle\varphi| | \psi, \varphi \in \mathcal{D}_2\right\} \subset \mathrm{dom}(\mathcal{L}_{2*}) \tag{5.65}$$

We have that for $\omega \in |\mathcal{D}_2\rangle\langle\mathcal{D}_2|$:

$$\mathcal{L}_{1*}(\omega) = \mathcal{L}_{2*}(\omega) \tag{5.66}$$

Proof. We only have to show that $\mathcal{J}_2(\omega) = 0$ for $\omega \in |\mathcal{D}_2\rangle\langle\mathcal{D}_2|$. It is sufficient to show this for operators of the form $\omega = |\psi\rangle\langle\varphi|$ with $\psi, \varphi \in \mathcal{D}_2$. Since by lemma 5.11 ψ and φ are twice weakly differentiable and hence once continuously differentiable by the Sobolev embedding theorem. The result follows by application of the product rule. □

We can now carefully use all the structure results we collected so far and combine them to the main theorem of this section. To be clear, this result is entirely due to Holevo [54]. We merely expanded the proof to a hopefully readable format.

Theorem 5.13. *The semigroup $\mathbb{E}_{2*}$ is not the minimal solution to a Lindblad equation.*

Proof. Assume the contrary, i.e., there exists a generator of a contraction semigroup $K_2 : \mathrm{dom}(K_2) \to \mathcal{H}$ and a countable family of operators $L_{2\alpha}$ for $\alpha \in \mathbb{A}$ such that $\mathbb{E}_{2*}$ is the minimal solution to the Lindblad equation with the following Lindbladian. For $\psi, \varphi \in \mathrm{dom}(K_2)$:

$$\mathcal{L}_{2*}\left(|\psi\rangle\langle\varphi|\right) = |K_2\psi\rangle\langle\varphi| + |\psi\rangle\langle K_2\varphi| + \sum_{\alpha\in\mathbb{A}} |L_{2\alpha}\psi\rangle\langle L_{2\alpha}\varphi| \tag{5.67}$$

From the construction of the semigroup it follows that $|\psi\rangle\langle\varphi| \in \mathrm{dom}(\mathcal{L}_{2*}) = \mathrm{dom}(\mathcal{L}_{1*})$ for $\psi, \varphi \in \mathrm{dom}(K_2)$ and hence by lemma 5.11 we have $\mathrm{dom}(K_2) \subset \mathrm{dom}(K_1)$.

We shall see that from this assumption it follows that $\mathrm{dom}(K_1) = \mathrm{dom}(K_2)$ and furthermore that $\mathcal{L}_{2*} = \mathcal{L}_{1*}$ on ketbra operators over $\mathrm{dom}(K_2)$. Hence $\mathbb{E}_{2*}(t)$ would be a minimal solution of the Lindblad equation (5.29), which is impossible because $\mathbb{E}_{1*}(t)$ is a solution to this Lindblad equation which is clearly dominated by $\mathbb{E}_{2*}(t)$ as a completely-positive map. It follows that $\mathbb{E}_{2*}$ can not be the minimal solution of any Lindblad equation, i.e. the semigroup is non-standard in the terminology of [54].

We are now going to reconstruct the operators K_2 and $L_{2\alpha}$ for all $\alpha \in \mathbb{A}$ from the action of $\mathcal{L}_2$ on ketbra operators over $\mathrm{dom}(K_2)$. On such a domain we have by lemma 5.12 that $\mathcal{L}_{2*} = \mathcal{L}_{1*}$.

As we observed earlier the Lindblad form is far from unique. So it is natural that in order to show equality of two Lindblad generators we have to apply gauge transformations to both of them.

We first fix a unit vector $\varphi_0 \in \mathrm{dom}(K_2)$ and choose a gauge such that for the gauge shifted $\widetilde{L}_\alpha$ we have: $\langle\psi_0, \widetilde{L}_\alpha\psi_0\rangle = 0$ for all $\alpha \in \mathbb{A}$. Denote the gauge shifted K_2 as $\widetilde{K}_2$. Remember that the domain of K_2 is invariant under such gauge operations, i.e. $\mathrm{dom}(K_2) = \mathrm{dom}(\widetilde{K}_2)$.

In this gauge we have that:

$$\mathcal{L}_{2*}\left(|\psi\rangle\langle\varphi_0|\right)\varphi_0 \tag{5.68}$$
$$= \widetilde{K}_2\psi\langle\varphi_0, \varphi_0\rangle + \psi\langle\widetilde{K}_2\varphi_0, \varphi_0\rangle + \sum_\alpha \widetilde{L}_{2\alpha}\psi\langle\widetilde{L}_{2\alpha}\varphi_0, \varphi_0\rangle \tag{5.69}$$
$$= \widetilde{K}_2\psi + \psi\langle\widetilde{K}_2\varphi_0, \varphi_0\rangle \tag{5.70}$$

We hence get for $\widetilde{K}_2$

$$\widetilde{K}_2\psi = -\psi\langle\widetilde{K}_2\varphi_0, \varphi_0\rangle + \mathcal{L}_{1*}\left(|\psi\rangle\langle\varphi_0|\right)\varphi_0 \tag{5.71}$$
$$= -\psi\langle\widetilde{K}_2\varphi_0, \psi_0\rangle + \psi''\langle\psi_0, \varphi_0\rangle + 2\psi'\langle\varphi_0', \varphi_0\rangle + \psi\langle\varphi_0'', \varphi_0\rangle \tag{5.72}$$
$$= \psi'' + 2\langle\varphi_0', \varphi_0\rangle\psi' + \left(\langle\widetilde{K}_2\varphi_0, \varphi_0\rangle + \langle\varphi_0'', \varphi_0\rangle\right)\psi \tag{5.73}$$

Here we used that $\mathcal{L}_{2*} = \mathcal{L}_{1*}$ on such ketbras. As a solution to the Lindblad equation (5.29) $\mathcal{L}_1$ has Lindblad form when restricted to such ketbras.

It is clear that equation (5.71) defines a dissipative operator for all $\psi \in \mathrm{dom}(K_1)$ since in that case $|\psi\rangle\langle\varphi_0|$ lies in $\mathrm{dom}\left(\mathcal{L}_{1*}\right)$ and by the dissipativity of $\mathcal{L}_{1*}$ we get

$$\mathfrak{Re}\,\langle\psi, \mathcal{L}_{1*}\left(|\psi\rangle\langle\varphi_0|\right)\varphi_0\rangle \leq 0. \tag{5.74}$$

Since $\widetilde{K}_2$ is a maximally dissipative operator, i.e. there is no dissipative operator extending $\widetilde{K}_2$ [31, chapter 10.4] , it follows that $\mathrm{dom}(K_1) \subset \mathrm{dom}(K_2)$ and hence $\mathrm{dom}(K_1) =$

dom(K_2). It remains to be shown that we can also bring K_1 from in equation (5.29) into the same form, to see that $K_1 = K_2$ up to gauge transformations. Remember that $K_1 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}$, $L_{1,1} = \sqrt{2}\frac{\mathrm{d}}{\mathrm{d}t}$ and formally we can choose a second Linblad term $L_{1,2} = 0$. We now apply a gauge transformation with vector:

$$\lambda_1 = \sqrt{2}\langle \psi_0', \psi_0 \rangle \tag{5.75}$$

$$\lambda_2 = \sqrt{2\|\psi_0'\|^2 - 2\,\mathfrak{Re}\langle \widetilde{K}_2\psi_0, \psi_0\rangle - 2\,|\langle \psi_0', \psi_0\rangle|^2} \tag{5.76}$$

$$x = \mathfrak{Im}\langle \widetilde{K}_2\psi_0, \psi_0\rangle \tag{5.77}$$

To see that the term under the square root is positive use that K is dissipative and the Cauchy-Schwarz inequality. It is easy to see that the gauged K_1 is equal to $\widetilde{K}_2$.

The equality of K_1 and K_2 immediately shows that the completely-positive maps induced by L_1 and $L_{2,\alpha}$ are equal. By Stinesprings theorem we can hence choose them equal.

□

Part II.

Continuous time limit of repeated observations

6. Delayed choice measurement

The second half of this thesis can be viewn, and is probably best understood, as the study of a specific measurement scheme: the delayed choice iterated measurement. Our main contribution lies in a systematic study of a continuous time limit of this construction. We already gave a short overview about the underlying physics and the essential ideas in the introduction in section 1 and 2.

The ideas described there should be kept in mind as an Ariadne's thread for the coming formalization of the problem. Let us shortly recapitulate the central points.

We are trying to describe all measurements which are compatible with the evolution of a given open quantum system in the sense that, if we ignore the measurement outcomes, the evolution does not change. That is, we do not add any further disturbance to the open system, but perform the measurements on the environment of the system.

In the end, we want to describe measurements which continuously monitor the system. For that notion to be well defined, we restrict to the description of systems which do interact only weakly with their environment, in the sense that their evolution is Markovian. This means that, to describe the state of the system at later times, the knowledge of the state of the system alone is sufficient. Hence in such a scheme, state changes of the environment, due to the continuous measurement, can not have a back action on the evolution of the system. Such evolutions are usually governed by a Lindblad generator as described in the first half of this thesis.

Our Ansatz for a description of continuous-time measurements starts from a description of the same class of measurements in a discrete time setup. There a solution is easy to construct, since the Stinespring dilation describes the possible measurements in a single time-step, or equivalently the quantum information released to the environment. To obtain a description of a sequence of time-steps, we just have to collect and "store" the information of every single time-step, i.e. perform an iterated delayed-choice measurement.

This whole Ansatz is successful because as we shall see:

1. Short time-steps are easy to describe approximately.
2. Piecing together solutions for time-steps is easy.
3. Comparing approximate short time-steps is easy.

The central ideas of the construction are depicted in the two figures 1.4 and 1.5.

Our comparison of time-steps is based on the idea that Lindblad evolutions can be described as alternation of instantaneous quantum jumps and a dissipative evolution. And that in a short enough time-step there is only one quantum jump.

6.1. Iterated measurements

As a first step towards our goals, we formalize the process of a iterated delayed choice measurement compatible with a given discrete time evolution, i.e. measurements as described in the preceding section in a discrete-time setting. The resulting iteration procedure is at the heart of our limit construction and the notation introduced here of central importance.

As always we work with "full" quantum systems, that is the statistics of the system as well as the environment are described by the bounded operators on a Hilbert space, and the ideal of trace class operators. We fix the system Hilbert space to be $\mathcal{H}$ for all times. A time-dependent system Hilbert space, would not pose a problem in the discrete-time setting but leads to notational inconveniences.

Let $\Theta \in \mathfrak{Z}([0,T])$ be an interval decomposition. We use $i \in I(\Theta)$ as labels for the "steps" of our discrete evolution, i.e. we assume the evolution to be the discretization of some continuous process.

Furthermore let $\widehat{\mathbb{E}}_i : \mathfrak{CP}(\mathfrak{B}(\mathcal{H}))$ for $i \in I(\Theta)$ be a family of unital completely-positive maps. The family $\widehat{\mathbb{E}}_i$ naturally describes a discrete time evolution.

It is now easy to describe measurements compatible with each of the time-steps $\widehat{\mathbb{E}}_i$. We know from the introduction 2.2.2 that by applying the Stinespring dilation theorem separately to each step i we can construct another a family $\mathbb{E}_i : \mathbb{E}_i : \mathfrak{CP}(\mathfrak{B}(\mathcal{K}_i \otimes \mathcal{H}), \mathfrak{B}(\mathcal{H}))$, such that $\mathbb{E}_i$ describes the evolution in step i and all possible measurements compatible with this evolution. These maps have the form $\mathbb{E}_i(X) = V_i^* X V_i$ with $V_i : \mathcal{H} \to \mathcal{K}_i \otimes \mathcal{H}$.

It is easy to extend this description to an evolution over several time-steps. We simply "collect" the information of all the single steps, i.e. we perform a delayed-choice measurement. This idea is captured by the following definition:

Definition 6.1. Let $\mathcal{H}$, $\mathcal{K}_1$ and K_2 be Hilbert spaces. For two operators of the form: $\mathbb{E}_1 : \mathfrak{B}(\mathcal{K}_1) \otimes \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$, and $\mathbb{E}_2 : \mathfrak{B}(\mathcal{K}_1) \otimes \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$. We define their iteration as:

$$\mathbb{E}_1 \circ \mathbb{E}_2 : \mathfrak{B}(\mathcal{K}_1) \otimes \mathfrak{B}(K_2) \otimes \mathfrak{B}(\mathcal{H}) \to \mathcal{B} \qquad \mathbb{E}_1 \circ \mathbb{E}_2 := \mathbb{E}_1(\mathrm{id}_{\mathfrak{B}(\mathcal{K}_1)} \otimes \mathbb{E}_2) \tag{6.1}$$

Likewise for two operators: $V_1 : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{H}$ and $V_2 : \mathcal{H} \to \mathcal{K}_2 \otimes \mathcal{H}$ we set:

$$V_2 \circ V_1 : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathcal{H} \qquad V_2 \circ V_1 = \big(\mathbb{1}_{\mathcal{K}_1} \otimes V_2\big) V_1$$

Observe the difference in order which, as we shortly see, renders the two iterations compatible. This type of iteration of maps is essentially what we call a delayed-choice iterated measurement, because in the description of the measurement we delayed the choice of observable in favor of a full quantum description of the information leaked to the environment.

It is clear that if $\mathbb{E}_1$ and $\mathbb{E}_2$ each describe a time-step and all possible measurements compatible with it, then the map $\mathbb{E}_1 \circ \mathbb{E}_2$ describes the evolution over both steps and all measurements compatible with that. Also we still retain the information about the time-point when information is released into the environment, i.e. we can describe measurement only during one of the time-steps. It is clear that in general the description of possible measurements by $\mathbb{E}_1 \circ \mathbb{E}_2$ is highly non-minimal. Precisely because we explicitly include information which might be irrelevant: the time of the measurement.

It is straightforward to extend the iteration to families of of maps through an induction.

Definition 6.2. Let $\mathcal{H}$ be a Hilbert space and for each $1 \leq i \leq n$ let $\mathcal{K}_i$ be a Hilbert space. For a family of maps $(\mathbb{E}_i)_{i=1}^n : \mathfrak{B}(\mathcal{K}_i) \otimes \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ we define inductively the ordered iteration

$$\prod_{i=1}^{n} \mathbb{E}_i : \mathfrak{B}\left(\bigotimes_{i=1}^{n} \mathcal{K}_i \otimes \mathcal{H}\right) \to \mathfrak{B}(\mathcal{H}) \qquad \prod_{i=1}^{n} \mathbb{E}_i := \prod_{i=1}^{n-1} \mathbb{E}_i \circ \mathbb{E}^n. \tag{6.2}$$

where we take the empty product to be the identity id : $\mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ and likewise for a family of maps $(V_i)_{i\in\mathbb{N}} : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ we define:

$$\prod_{i=1}^{n} V_i : \mathcal{H} \to \bigotimes_{i=1}^{n} \mathcal{K}_i \otimes \mathcal{H} \qquad \prod_{i=1}^{n} V_i = V_n \circ \left(\prod_{i=1}^{n-1} V_i\right). \tag{6.3}$$

The iteration product has a few important properties, e.g. it maps Stinespring dilations to Stinespring dilations in the sense of the following lemma.

Lemma 6.3. *The iteration is distributive over the sum of operators and associative. If in the situation of the previous definition we have for each* $0 \le i \le n$ *that* $\mathbb{E}_i(X) = V_i^* X V_i$*, then:*

$$\prod_{i=1}^{n} \mathbb{E}_i(X) = \left(\prod_{i=1}^{n} V_i\right)^* X \left(\prod_{i=1}^{n} V_i\right). \tag{6.4}$$

For $1 \le i \le n$ *let* $\widetilde{V}_i : \mathcal{H} \to \mathcal{K}_i \otimes \mathcal{H}$ *be a second family of operators. Then:*

$$\prod_{i=1}^{n} V_i - \prod_{i=1}^{n} \widetilde{V}_i = \sum_{i=1}^{n} \prod_{k=i+1}^{n} V_i \circ (V_i - \widetilde{V}_i) \circ \prod_{i=1}^{i-1} \widetilde{V}_i. \tag{6.5}$$

And similarly for families of maps $\mathbb{E}_i$ *and* $\widetilde{\mathbb{E}}_i$*.*

Proof. The only property which is not entirely obvious is the telescopation identity, which follows from the distributivity, associativity and a straightforward induction. □

The central construction of this thesis is about a continuous-time limit construction for such iterated delayed-choice measurements. The states describing these measurements are well known: the finitely correlated states or matrix product states [46, 84, 83], we look at them in section 6.3.

6.2. Main results

Using the notation introduced in the previous section we are now able to state the main results of this section. Apart from a short description of the states which described iterated delayed-choice measurement, i.e. finitely correlated states, the main point of this section is a description of the Hilbert spaces which describes such measurements in the limit case.

In the example of a driven cavity system, which is as we described earlier an excellent example of a system satisfying the assumptions necessary for our analysis, it is quite clear that the information "leaked" into the environment is described by a quantum field. Namely the light-field coupled out of the cavity.

It is the central result of this section that this structure carries over to the general case. The Hilbert space describing continuous measurements compatible with a given evolution can be identified with the Bose-Fock space over a one dimensional quantum field, i.e. $\Gamma_+\left(\mathrm{L}^2([0,T])\otimes\mathcal{K}\right)$ where Γ_+ is the Fock space functor mapping a Hilbert space to the corresponding Fock space and $\mathcal{K}$ is a separable Hilbert space.

We shall now outline, how the Fock space structure appears. For complete results and details we refer to section 6.4. In particular for the moment we neglect the construction of the limit Hilbert space as an inductive limit. This construction complements the identification of the space as a Fock space.

From the definition of the iteration in the preceding section it is clear that, if for a given discretization $\Theta\in\mathfrak{Z}([0,T])$ the measurements compatible with the step $i\in I(\Theta)$ are described by the Hilbert space $\mathcal{K}_\Theta$, the measurements compatible with the whole evolution are described by the Hilbert space $\bigotimes_{i\in I(\Theta)}\mathcal{K}_i$.

As we shall see, for the limit we want to perform, it is sufficient to choose $\mathcal{K}_i=\mathbb{C}\oplus\mathcal{K}$ for all $i\in I(\Theta)$ with a fixed separable Hilbert space $\mathcal{K}$. The physical interpretation is that the distinguished vector, $|0\rangle$, stands for the possibility that no quantum jump has happened and no corresponding particle was emitted into the field. Each basis vector of $\mathcal{K}$ stands for a different type of quantum jump event and hence a different kind of particle.

Building on this interpretation it is easy to embed the space $\bigotimes_{i\in I(\Theta)}\mathcal{K}_i$ into the Fock space over $\mathrm{L}([0,T])\otimes\mathcal{K}$. For the mathematical details we refer to section 6.4.1.

6.3. Finitely correlated states

We are now going to take a closer look at the class of states, the iteration procedure induces on the "chain" of environments $\mathcal{A}_i:=\mathfrak{B}(\mathcal{K}_i)$. They are called matrix product states (MPS) or finitely correlated states (FCS). Usually these states are used to describe states of spin chains, rather than iterated measurements. The basic construction is quite simple, in terms of the delayed choice picture we just fix the state of and the measurement performed on the "system". This leaves us with a state on the family of environments alone.

The name finitely correlated state comes from the fact that in such states the correlation between different time-steps is fixed by a (potentially finite) quantum system. In our case the (potentially finite dimensional) "system" fixes the state on the infinite dimensional "environment". The name matrix product state becomes apparent when one tries to calculate the coefficients of the state in a canonical basis.

Definition 6.4. Let $\mathcal{H}$ be a Hilbert space and $\rho\in\mathfrak{T}(\mathcal{H})$ be a state on $\mathfrak{B}(\mathcal{H})$, i.e. positive and of trace one. Furthermore for every $1\le i\le n$ let $\mathcal{K}_i$ be a separable Hilbert space and $\mathbb{E}_i:\mathfrak{B}(\mathcal{K}_i\otimes\mathcal{H})\to\mathfrak{B}(\mathcal{H})$ be a channel, i.e. completely-positive, unital and normal.

We define the *finitely correlated state* ω belonging to the following tuple $\left((\mathbb{E}_i)_{i=1}^n,\rho\right)$ as:

$$\omega:\bigotimes_{i=1}^n\mathfrak{B}(\mathcal{K}_i)\to\mathbb{C}\qquad\qquad X\mapsto\operatorname{tr}\left(\rho\left(\prod_{i=1}^n\mathbb{E}_i\right)(X\otimes\mathbb{1})\right).\tag{6.6}$$

If there exists a family of isometries $V_i:\mathcal{H}\to\mathcal{K}_i\otimes\mathcal{H}$ for $1\le i\le n$ such that:

$$\mathbb{E}_i:\mathfrak{B}(\mathcal{K}_i\otimes\mathcal{H})\to\mathfrak{B}(\mathcal{H})\qquad\qquad X\mapsto V_i^*XV_i\tag{6.7}$$

the state is called *purely-generated*. In this case we also say that it is generated by the family (V_i,ρ).

In the case that the $\mathbb{E}_i$ are actually independent of i, i.e. there is a Hilbert space $\mathcal{K}$, s.t. $\mathcal{K}_i = \mathcal{K}$ for all i and a channel $\mathbb{E} : \mathfrak{B}(\mathcal{K} \otimes \mathcal{H}) \to \mathfrak{B}(\mathcal{H})$, s.t., $\mathbb{E}_i = \mathbb{E}$ for all i, the state is called *translation invariant*.

Of course we also could have chosen the Hilbert space $\mathcal{H}$ "step-dependent". We shall however not need that more general constructions. Actually we do not even exploit the freedom of a i-dependent family $\mathcal{K}_i$.

Another possible degree of freedom, we do not use, is the choice of a $B \in \mathfrak{B}(\mathcal{H})$. In this case one would define a finitely correlated state as:

$$\widetilde{\omega} : \bigotimes_{i=1}^{n} \mathfrak{B}(\mathcal{K}_i) \to \mathbb{C} \qquad\qquad X \mapsto \operatorname{tr}\left(\rho\left(\prod_{i=1}^{n} \mathbb{E}_i\right)(X \otimes B)\right). \tag{6.8}$$

With the additional normalization $\omega = \widetilde{\omega}(\mathbb{1})^{-1}\widetilde{\omega}$. This gives a priori an even wider class of of states. However at least in the case of translation invariant FCS, it was shown that these two classes coincide [46].

Furthermore it is clear that we can directly extend the above definition to families of channels indexed by $\mathbb{N}$ or $\mathbb{Z}$. The above definition then naturally extends to the definition of a state on the quasi local algebra $\bigotimes_{\mathbb{N}/\mathbb{Z}} \mathfrak{B}(\mathcal{K}_i)$ [46]. The only difficulty is that we have to find a family of states ρ_i for $i \in \mathbb{Z}$, such that $\operatorname{tr}(\rho_{i-1}\mathbb{E}_{i-1}(\mathbb{1} \otimes B)) = \operatorname{tr}(\rho_i B)$ for all $B \in \mathfrak{B}(\mathcal{H})$. We then say that the family of states ρ_i is *compatible* with the family of channels $B \to \mathbb{E}_i(\mathbb{1}_{\mathcal{K}} \otimes B)$. For a given family of maps $\mathbb{E}_i$ such a family of states can always be found.

We already presented this result in [69]. For the convenience of the reader let us reproduce the proof:

Lemma 6.5. *For a given family of channels $\mathbb{E}_i : \mathfrak{B}(\mathcal{K}_i \otimes \mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ for $i \in \mathbb{Z}$ there exists at least one family of states $(\rho_i)_{i\in\mathbb{Z}}$ fulfilling* $\operatorname{tr}(\rho_{i-1}\mathbb{E}_{i-1}(\mathbb{1} \otimes B)) = \operatorname{tr}(\rho_i B)$.

Proof. For each $n \in \mathbb{N}$ define:

$$M_n = \{(\rho_i)_{i\in\mathbb{Z}} \,|\, \rho_i \in \mathfrak{T}(\mathcal{H}), \operatorname{tr}(\rho_{i-1}\mathbb{E}_{i-1}(\mathbb{1} \otimes B)) \text{ for all } j \geq -n\}. \tag{6.9}$$

Firstly this set is non empty for every $n \in \mathbb{N}$. To see that we set $\widehat{\mathbb{E}}_{i*}$ to be the pre-adjoint of the map $B \to \mathbb{E}_i(\mathbb{1} \otimes B)$. Now for every $n \in \mathbb{N}$ we can choose a family of states $\rho_k := \prod_{i=-n}^{k} \widehat{\mathbb{E}}_{i,*}(\rho)$ for $k \geq -n$ and $\rho_k = \rho$ else.

Furthermore the family of sets M_n has the finite intersection property. And the set of states $\mathfrak{T}(\mathcal{H})$ is compact in the weak-$*$ topology on $\mathfrak{T}(\mathcal{H})$, i.e. the weak-topology induced by considering $\mathfrak{T}(\mathcal{H})$ as the dual of the compact operators.

One can now see that also all the sets M_n are closed in the weak-$*$ topology and hence the intersection of all M_n is nonempty by the finite intersection property. □

In the case of an translation invariant FCS we only need a single state invariant under $\mathbb{E}$, i.e. $\operatorname{tr}(\rho\mathbb{E}(\mathbb{1} \otimes B)) = \operatorname{tr}(\rho B)$, to be able to define a finitely correlated state on the quasi local algebra. The resulting state is translation invariant.

For our limit construction we later only need the notion of purely-generated finitely-correlated states [47]. For them the connection to the better known class of matrix product states, or MPS, is easily seen.

Corollary 6.6. *Let $\mathcal{H}$ be a Hilbert space and for every $1 \leq i \leq n$ let $\mathcal{K}_i$ be a separable Hilbert space and $V_i : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ an isometry.*

For any $\varphi, \psi \in \mathcal{H}$ the following FCS is a pure state on $\bigotimes_{i=1}^n \mathfrak{B}(\mathcal{K})$, i.e. there exists a $\lambda \in \bigotimes_{i=1}^n \mathcal{K}$:

$$\omega_{MPS}(X) = \operatorname{tr}\left(|\psi\rangle\langle\psi| V^* \left(X \otimes |\varphi\rangle\langle\varphi|\right) V\right) = \langle \lambda, X\lambda \rangle \tag{6.10}$$

Let for every $1 \leq i \leq n$ $|\alpha_i\rangle$ for $\alpha_i \in \mathcal{A}_i$ denote a orthonormal basis of $\mathcal{K}_i$. Then the vector λ satisfies:

$$\left\langle \bigotimes_{i=1}^n |\alpha_i\rangle, \lambda \right\rangle = \operatorname{tr}\left(\prod_{i=1}^n V_{i,\alpha_i} |\psi\rangle\langle\varphi| \right). \tag{6.11}$$

Where V_{i,α_i} is defined as $\langle \varphi, V_{i,\alpha_i}\psi \rangle := \langle |\alpha_i\rangle \otimes \varphi, V_i \psi \rangle$.

Such a pure FCS is called *matrix product states* or *MPS with open boundary conditions*. An interesting special case arises when the initial Hilbert space is finite dimensional, say dimension D^2 and of the form $\mathcal{H} \otimes \mathcal{H}$ and a family of isometries of the form $V_i \otimes \mathbb{1}$. For ψ and φ we choose the maximally entangled state:

$$\psi = \varphi = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle \otimes |i\rangle \tag{6.12}$$

where $|i\rangle$ enumerates a basis of $\mathcal{H}$. The resulting FCS has the form:

$$\left\langle \bigotimes_{i=1}^n |\alpha_i\rangle, \lambda \right\rangle = \operatorname{tr}\left(\prod_{i=1}^n V_{i,\alpha_i} \right). \tag{6.13}$$

Since the trace is invariant under cyclic permutations, such a MPS does not live on the spin chain $\bigotimes_{i=1}^n \mathcal{K}_i$ but on the corresponding torus, i.e. they fulfill periodic boundary conditions.

The main theorem of this thesis can be interpreted as a continuous-time limit of a family of FCS/MPS to a state on Fock space. The search for such a limit was the original motivation for this research. The resulting class of states was described before and called continuous matrix product states or cMPS [82, 70]. Similar constructions also arise in quantum-stochastic calculus [72] and quantum input-output theory [48]. It is worth to be noted that in the sense of the initial definition [46] cMPS are just finitely correlated states on one-dimensional quantum fields.

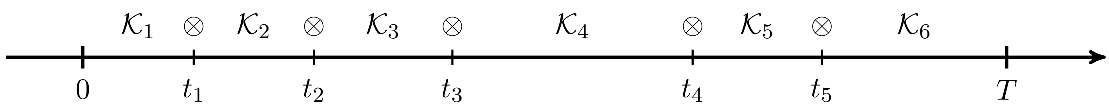
6.4. Limit space

In this section we take a closer look at the Hilbert spaces underlying our limit construction. We construct the space as an inductive limit. The limit is identified as Bose and Fermi Fock space over $\mathrm{L}([0,T],\mathcal{K})$. We also take a short look at some related spaces and topologies.

6.4.1. Hilbert space of cMPS

Due to the structure of our limit construction the limit of the dilation spaces is largely independent from the limit of the dilations. We can hence use the "same" dilation space for every semigroup. Or rather the dilation space only depends on the "label-space" for the Lindblad operators $\mathcal{K}$, i.e. the dilation space for the completely-positive part of the Lindblad generator.

Construction



Let us shortly recapitulate the structure we already fixed for the cMPS Hilbert space, before we capitalize on it and extract a complete description of the limit space. Our assumptions on the label space can be summarized in the few following statements:

1. A single time-step either has a single event or none.
2. The description of possible events in a single step is independent of the actual time-step, in particular of its length.
3. To describe multiple time-steps we just iterate the single step construction.

These three requirements or rather their formalizations completely fix the structure of the space. The assumption number 1 is the main approximation. This approximation is easy to justify since this idea is already at the foundation of the Lindblad form and its minimal solutions. It is then however clear that every time-step can always only be an approximation.

Assumption number 2 could probably be avoided at the cost of a more complicated limit construction. From the viewpoint of a physicist this assumption just means that we are able to track different types of events throughout time. From a mathematical point of view it is similar to the definition of exit spaces of evolution systems in definition 4.13. That is primarily we go to a potentially non-minimal description. The trade-off in mathematical complexity, one pays when avoiding this assumption, seems hard to justify.

In the mathematical framework of quantum mechanics the assumption number 1 just means that, given a discretization $\Theta \in \mathfrak{Z}([0,T])$, for every step $i \in I(\Theta)$ the corresponding Hilbert space $\mathcal{K}_i$ has a distinguished vector representing the "no-event event", i.e. we can write it as $\mathcal{K}_i = \mathbb{C} \oplus \widetilde{\mathcal{K}}_i$, where $\widetilde{\mathcal{K}}_i$ is another Hilbert space.

From the next statement, number 2, we extract that $\widetilde{K}_i$ is actually independent of the step. Hence we can write $\mathcal{K}_i = \mathbb{C} \oplus \mathcal{K}$ for every step $i \in I(\Theta)$.

The collection of events in a delayed choice scheme, i.e. the iteration procedure, just means that we treat every time-step as generating its own "environment". The "global" environment is then naturally described by the tensor product over all steps. This leads to the following definition:

Definition 6.7. Let $\mathfrak{Z}([0,T])$ be the (directed) set of interval decompositions of a fixed interval, see definition 2.2 and let $\mathcal{K}$ be a Hilbert space. For every $\Theta \in \mathfrak{Z}([0,T])$ we define:

$$\mathcal{K}_\Theta := \bigotimes_{i \in I(\Theta)} (\mathbb{C} \oplus \mathcal{K}) \tag{6.14}$$

This defines a net of spaces with index set $\mathfrak{Z}([0,T])$. The tensor structure of the spaces $\mathcal{K}_\Theta$ is compatible with the order structure of interval decompositions noted in definition 2.4, i.e:

Lemma 6.8. *Let $\Theta, \Xi \in \mathfrak{Z}([0,T])$ be a pair of interval decomposition such that $\Theta \subset \Xi$ and for every $i \in I(\Theta)$ denote by $\Xi|_i$ the interval composition of the i-th sub-interval of Θ. We have:*

$$\mathcal{K}_\Xi = \bigotimes_{i \in I(\Theta)} \mathcal{K}_{\Xi|i} \tag{6.15}$$

The last missing ingredient, we need to rigorously set up the limit space, is a way to compare different discretizations of the same process. The way we set up the theory, we only have to be able to compare two spaces $\mathcal{K}_\Theta$ and $\mathcal{K}_\Xi$ if we are able to compare the discretization, i.e. iff $\Theta \leq \Xi$ or the other way around. There are two more points to be considered:

1. Because of the structure of the interval decompositions to compare two different discretizations of the same time-step, we only have to be able to compare a single step with an arbitrary finite sequence amounting to a step of the same length.

2. Such a comparison can be done by assuming that events are instantaneous, and hence the single event in the big step must have happened in one of the subintervals.

The following family of isometries exactly captures these two ideas. Precisely the first one fixes the "tensor structure" of the embedding. We have depicted this idea in figure 6.1. The second idea, i.e. that events happen instantaneously fixes the form of each tensor factor.

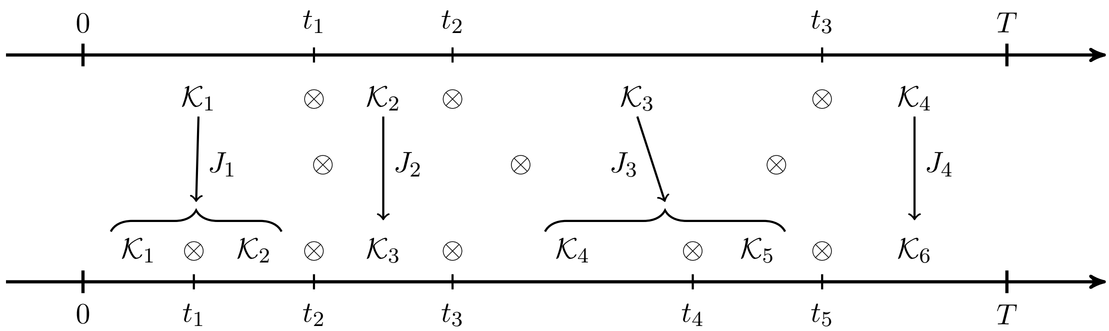


Figure 6.1.: Comparison of two dilation spaces

Definition 6.9. Let $\Theta, \Xi \in \mathfrak{Z}([0,T])$ be a pair of interval decomposition such that $\Theta \subset \Xi$. Let $|0\rangle$ be the distinguished basis vector in $\mathbb{C} \oplus \mathcal{K}$ and let $\varphi \in \mathcal{K}$. We introduce the following notation for vectors with exactly one event:

$$|\varphi@i\rangle := \bigotimes_{i>k\in I(\Theta)} |0\rangle \otimes \phi \otimes \bigotimes_{i<k\in I(\Theta)} |0\rangle \tag{6.16}$$

For every $i \in I(\Theta)$ define $J_i : \mathbb{C} \oplus \mathcal{K} \to K_{\Xi|i}$ to be the extension by linearity of:

$$J_i|0\rangle = \bigotimes_{j\in I(\Xi|i)} |0\rangle; \qquad J_i|\varphi\rangle = \sum_{j\in I(\Xi|i)} \sqrt{\frac{\tau_j}{\tau_i}} |\phi@j\rangle. \tag{6.17}$$

The overall comparison map is then given as:

$$J_{\Xi,\Theta} : \mathcal{K}_\Theta \to \mathcal{K}_\Xi \qquad J_{\Xi,\Theta} := \bigotimes_{i\in I(\Theta)} J_i \tag{6.18}$$

These maps have all the desired properties as a straightforward direct calculation shows. The details can also be found in [69].

Lemma 6.10. *Let $\Xi, \Theta, \Lambda \in \mathfrak{Z}([0,T])$ and $\Theta \subset \Xi \subset \Lambda$. Then $J_{\Theta,\Xi}$ is an isometry and $J_{\Lambda,\Theta} = J_{\Lambda,\Xi} J_{\Xi,\Theta}$, i.e. $J_{\Xi,\Theta}$ is an inductive system of mappings.*

The existence and most properties of the limit dilation space is now merely a consequence of its definition:

Definition 6.11. Let $\Theta \subset \Xi \in \mathfrak{Z}([0,T])$. We define the limit space to be the closure of the inductive limit of the family $(\mathcal{K}_\Theta, J_{\Xi,\Theta})$.

$$\mathcal{K}_{[0,T]} := \overline{\underrightarrow{J\text{-}lim}_{\Theta\in\mathfrak{Z}([0,T])} \mathcal{K}_\Theta} \tag{6.19}$$

Denote by $J_\Theta : \mathcal{K}_\Theta \to \mathcal{K}_{[0,T]}$ the natural embedding. $K_{([0,T])}$ is a Hilbert space.

Exponential property

Notably the limit space is still compatible with the "refinement" of interval decompositions. This property allows us on the one hand to restrict our dilation to arbitrary subintervals and on the other hand to "piece" dilations of subintervals together. In other words the following property of the dilation space is essential for the dilation to be a "true" continuous Stinespring dilation.

Lemma 6.12. *For $r \le s \le t \in \mathbb{R}$ have $\mathcal{K}_{[r,s]} \otimes \mathcal{K}_{[s,t]} = \mathcal{K}_{[r,t]}$ in the sense that there is a canonical unitary equivalence.*

Proof. For $\Theta \in \mathfrak{Z}([r,s])$, $\Xi \in \mathfrak{Z}([s,t])$ and $\Lambda \in \mathfrak{Z}([r,t])$ it is clear that we can naturally view elements $\mathcal{K}_\Theta \otimes \mathcal{K}_\Xi$ as elements in $\mathcal{K}_\Lambda$ iff $s \in \Lambda$ which can always be assumed in the limit case. This unitary equivalence naturally extends to the inductive limit. □

Related with to this the family of spaces $\mathcal{K}_{[s,t]}$ for $0 \le s \le t$a comes with a canonical family of embeddings $\mathcal{K}_{[s,t]} \subset \mathcal{K}_{[r,u]}$ for $[s,t] \subset [r,u]$. For this we just note that the space $\mathcal{K}_{[s,t]}$ has a distinguished vector, i.e. the zero event vector $|0\rangle$. Together with the "exponential property" from the previous lemma this naturally defines a family of embeddings and respective projections. This embedding structure plays a crucial role in quantum stochastic calculus. There it is used to define a filtration of the stochastic process.

Definition 6.13. For $r \leq s \leq t \in \mathbb{R}$ we can define an embedding:

$$I : \mathcal{K}_{[r,s]} \to \mathcal{K}_{[r,t]} = \mathcal{K}_{[r,s]} \otimes \mathcal{K}_{[s,t]} \qquad \lambda \mapsto \lambda \otimes |0\rangle \tag{6.20}$$

The corresponding family of projections is denoted by $P_{[s,t]}$

$$P_{[r,s]} \in \mathfrak{B}\left(K_{[r,t]}\right) = \mathfrak{B}\left(\mathcal{K}_{[r,s]} \otimes \mathcal{K}_{[s,t]}\right) \qquad P_{[r,s]} = \mathbb{1}_{\mathcal{K}_{[r,s]}} \otimes |0\rangle\langle 0| \tag{6.21}$$

Operators on $\mathcal{K}_\Theta$

Later we are of course also interested in operators on $\mathcal{K}_{[s,t]}$. If we want to construct them as inductive limits, as in section 2.1.5, we need a notation for operators on the discrete dilation spaces $\mathcal{K}_\Theta$. We usually construct those from tensor product of operators on $\mathbb{C}\oplus\mathcal{K}$, which have the following structure:

Definition 6.14. Let $\mathcal{K}$ be a Hilbert space. Furthermore let $c \in \mathbb{C}$, $\lambda, \mu \in \mathcal{K}$ and $O_i \in \mathfrak{B}(\mathcal{H})$ and denote:

$$\begin{pmatrix} c & \langle\lambda| \\ |\mu\rangle & O \end{pmatrix} \in \mathfrak{B}(\mathbb{C}\oplus\mathcal{K}) \qquad (a, \mu) \mapsto (ac + \langle\lambda, \mu\rangle, a\mu + O\nu) \tag{6.22}$$

Where $a \in \mathbb{C}$ and $\mu \in \mathcal{K}$.

Since $\mathfrak{B}(\mathcal{K}_\Theta) = \bigotimes_{i\in I(\Theta)} \mathfrak{B}(\mathbb{C}\oplus\mathcal{K})$ we can approximate all operators $X \in \mathfrak{B}(\mathcal{K}_\Theta)$ in the following form:

$$X \approx \sum_{n=1}^{N} \bigotimes_{i\in I(\Theta)} O_{i,n}, \tag{6.23}$$

with $O_{i,n} \in \mathfrak{B}(\mathbb{C}\oplus\mathcal{K})$. Especially operators acting non trivially only on one tensor factor are of crucial importance.

Definition 6.15. For $\Theta \in \mathfrak{Z}([0,T])$ and $A \in \mathfrak{B}(\mathbb{C}\oplus\mathcal{K})$ we define for every $i \in I(\Theta)$ the following two operators:

$$(A@i) \in \mathfrak{B}\left(\mathcal{K}_\Theta\right) \qquad (A@i) = \left(\bigotimes_{k<i\in I(\Theta)} \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right) \otimes A \otimes \left(\bigotimes_{k>i\in I(\Theta)} \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right) \tag{6.24}$$

and:

$$\begin{aligned} &(A@i)_- \in \mathfrak{B}\left(\mathcal{K}_\Theta\right) \\ &(A@i)_- = \left(\bigotimes_{k<i\in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_{\mathcal{K}} \end{pmatrix}\right) \otimes A \otimes \left(\bigotimes_{k>i\in I(\Theta)} \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right). \end{aligned} \tag{6.25}$$

The former operators are of universal importance, while the latter type of operator only seems to be important, if we want to interpret $\mathcal{K}_\Theta$ as an approximation to Fermi Fock space.

Number operators

To get a concrete picture of the limit space $\mathcal{K}_{[0,T]}$ we need to better understand its structure. The exponential property from lemma 6.12 is already a strong hint, that the space $\mathcal{K}_{[0,T]}$ may be related to Bose Fock space, which also has such an exponential property, i.e. $\Gamma_+(\mathcal{K} \oplus \mathcal{H}) = \Gamma + (\mathcal{K}) \otimes \Gamma_+(\mathcal{H})$. Here $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and Γ_+ is the Fock space functor, mapping a Hilbert space to the corresponding (Bose-) Fock space.

Furthermore the spaces $\mathcal{K}_\Theta$ for $\Theta \in \mathfrak{Z}([0,T])$ have a natural notion of event-number or particle-number, since we explicitly distinguish between no-event and event. Simple counting of basis vectors then defines an "event-number" operator.

Definition 6.16. Let $\Theta \in \mathfrak{Z}([0,T])$, $\mathcal{K}$ a Hilbert space and K_Θ the corresponding Hilbert space from definition 6.7. Denote by $|0\rangle$ the distinguished element in $\mathcal{K}_i$ and by $|0\rangle\langle 0|$ the corresponding projector. We define the *number operator* to be:

$$N_\Theta : \mathcal{K}_\Theta \to \mathcal{K}_\Theta \qquad\qquad N_\Theta := \sum_{i \in I(\Theta)} ((\mathbb{1} - |0\rangle\langle 0|)@i) \tag{6.26}$$

Here the $B@i$ denotes an operator acting as B on the i-th tensor factor and as the identity elsewhere.

The number operator N_Θ is clearly self-adjoint. Its eigenvalues and eigenspaces are easy to describe. It has as eigenvalues natural numbers between 0 and to $\#I(\Theta) - 1$, the number of subintervals. For $\varphi \in \mathcal{K}$ the one-event vectors $|\varphi@i\rangle$ defined in definition 6.9 span the eigenspace to eigenvalue 1. We can naturally extend this notation of vectors to multiple events and write down a basis for eigenspaces to eigenvalue k in the form $|(\varphi_1, \ldots, \varphi_k)@(i_1, \ldots, i_k)\rangle$ for $i_j \neq i_l$ when $l \neq j$ and $\varphi_1, \ldots, \varphi_k \in \mathcal{K}$. These vectors are supposed to have tensor factor $|0\rangle$ except at the noted places. In other words this notation is to be read like the definition of a sparse array.

The notion of particle/event-number naturally extends to the inductive limit. This is a consequence of the fact that the inductive family $J_{\Xi\Theta}$ conserves the particle/event-number.

Corollary 6.17. *For $\Theta \leq \Xi \in \mathfrak{Z}([0,T])$ and any $\psi \in \mathcal{K}_\Theta$ we have:*

$$N_\Xi J_{\Xi\Theta}\psi = J_{\Xi,\Theta} N_\Theta \psi. \tag{6.27}$$

The notion of a number operator hence passes naturally to the inductive limit of $(\mathcal{K}_\Theta, J_{\Xi\Theta})$ and on the completed inductive limit $\mathcal{K}_{[0,T]}$ this defines an unbounded self-adjoint operator.

Definition 6.18. Let $\widetilde{N}$ be the natural operator number operator on the inductive limit $\underrightarrow{J\text{-}lim}_{\Theta \in \mathfrak{Z}([0,T])} \mathcal{K}_\Theta$, i.e. the operator acting as N_Θ on $\mathcal{K}_\Theta$. This operator canonically induces an unbounded operator on the completion $\mathcal{K}_{[0,T]}$. The *number operator* on $\mathcal{K}_{[0,T]}$, which we denote as $N_{[0,T]}$ or simply N if the underlying space is clear, is defined to be the closure of $\widetilde{N}$.

Lemma 6.19. *$N_{[0,T]} \in \mathfrak{B}(\mathcal{K}_{[0,T]})$ is well defined and a self-adjoint operator.*

Proof. To see that this is well defined observe that $i\widetilde{N}$ generates a strongly-continuous one-parameter group of operators on the pre-completed inductive limit which extends to a strongly-continuous group of unitary operators on $\mathcal{K}_{[0,T]}$. □

Clearly N has as spectrum the natural numbers including 0. The spectral projections of N naturally decompose $\mathcal{K}_{[0,T]}$ into n-particle spaces.

Equivalence to Fock space

After showing that the limit dilation space behaves like a Fock space for all practical purposes, it remains to be shown that it actually "is" a Fock space. That is, we can find a "canonical" isomorphism on the level of N-particle spaces. The isomorphism conserves n-particle spaces and is compatible with the exponential property.

Before showing equivalence with the Fock spaces, we show equivalence to a space of time ordered functions. To be able to state our results, we repeat definition 4.21 from section 4.3.

Definition 6.20. By $\Delta_n([0,T])$ we denote the set of ordered n-tuples in $[0,T]$.

Lemma 6.21. *Let $\mathcal{K}$ be a Hilbert space. There is a unique unitary equivalence between the eigenspace to eigenvalue n of N in $\mathcal{K}_{[0,T]}$ denoted as $\mathcal{K}_{[0,T]}|_n$ and $\mathrm{L}^2(\Delta_n,\mathcal{K})\otimes\mathcal{K}^{\otimes N}$. s.t. for $\Theta\in\mathfrak{Z}([0,T])$ $\varphi_1,\ldots,\varphi_n\in\mathcal{K}$ and $i_1<i_2<\ldots i_n$ we have*

$$U_nJ_\Theta|(\varphi_1,\ldots,\varphi_n)@(i_1,i_2,\ldots,i_n)\rangle=\prod_{l=1}^{n}1(i_l)\otimes\bigotimes_{l=1}^{n}|\phi_l\rangle. \tag{6.28}$$

Here $1(i_n)$ is the index function of the i_n-th subinterval of Θ, i.e. the function taking the value 1 on the subinterval and 0 elsewhere. We write $U_n:\mathcal{K}_{[0,T]}|_n\to L^2(\Delta_n)\otimes\mathcal{K}^{\otimes N}$ for the unitary.

Proof. We want extend the definition in equation (6.28) by linearity and closure to an unitary operator. We first have to show that for $\Theta\in\mathfrak{Z}([0,T])$ the operator U_nJ_Θ is isometric. This can be seen by direct evaluation of the scalar products. For two families $\varphi_1,\ldots,\varphi_n\in\mathcal{K}$ and $i_1<i_2<\ldots<i_n$ such that $i_k\in I(\Theta)$ for all $1\le k\le n$ and $\psi_1,\ldots,\psi_n\in\mathcal{K}$ and $1\le j_1<j_2<\ldots<j_n\le n$ such that $j_k\in I(\Theta)$ for all $1\le k\le n$ the scalar products of $|(\varphi_1,\ldots,\varphi_n)@(i_1,i_2,\ldots,i_n)\rangle$ and $|(\psi_1,\ldots,\psi_n)@(i_1,i_2,\ldots,i_n)\rangle$ as well as their image under U_nJ_Θ both clearly evaluate to:

$$\prod_{k=1}^{n}\left(\langle\varphi_k,\psi_k\rangle\cdot\delta_{i_k,j_k}\right) \tag{6.29}$$

Hence U_n clearly is an isometric operator. It remains to be shown that U_n is onto. This is a consequence of the uniqueness of the completion and the fact that continuous functions are on the one hand dense in $L_2(\Delta_n)$. On the other hand they are Riemann integrable and hence in the closure of the image of all U_nJ_Θ. We have seen that U_n is isometric and onto, i.e. a unitary operator. □

The desired unitary equivalence to a space of time-ordered functions is now a direct consequence.

Theorem 6.22. *In the sense that there exists a canonical unitary equivalence on the level of n-particle spaces, we have that:*

$$\mathcal{K}_{[0,T]}=\overline{\bigoplus_{n=0}^{\infty}\mathrm{L}^2(\Delta_n)\otimes\mathcal{K}^{\otimes n}}. \tag{6.30}$$

Proof. The result follows because the domain of N is dense in $\mathcal{K}_{[0,T]}$. □

Theorem 6.23. *Let $\mathcal{K}$ be a Hilbert space. In the sense that there exists a canonical unitary equivalence on the level of n-particle spaces, we have that:*

$$\mathcal{K}_{[0,T]} = \Gamma_+(\mathrm{L}^2([0,T],\mathcal{K})) = \Gamma_-(\mathrm{L}^2([0,T],\mathcal{K})). \tag{6.31}$$

Here Γ_+ is the Bose Fock space functor and Γ_- the Fermi Fock space functor.

Proof. To see this, remember how Fock space over $L^2([0,T],\mathcal{K})$ is constructed. For a given Hilbert space $\mathcal{H}$ the Bose/Fermi Fock space $\Gamma_\pm(\mathcal{H})$ is defined via the following family of unitary representations of the permutation group of N elements $\mathcal{P}_N$ on $\mathcal{H}^{\otimes N}$. For $\varphi_1,\dots,\varphi_N \in \mathcal{H}$ $\pi \in \mathcal{P}_N$ we define $U_\pi \varphi_1 \otimes \cdots \otimes \varphi_N := \varphi_{\pi(1)} \otimes \cdots \varphi_{\pi(N)}$. Summing over all permutations defines the projectors on the completely (anti-)symmetric subspace of $\mathcal{H}^{\otimes n}$ as $P_{\pm,N} := \frac{1}{N!}\sum_{\pi\in\mathcal{P}_N}(\pm 1)^\pi U_\pi$. Fock space is then defined as $\Gamma_\pm(\mathcal{H}) := \sum_{N\in\mathbb{N}} P_{\pm,N}\mathcal{H}^{\otimes N}$. The chain of unitary equivalences [78] defined by the following series of equalities also naturally carries over the representations of the permutation group. The elements on the right side are identified.

$$L([0,T],\mathcal{K})^{\otimes N} \qquad \bigotimes_n f_n \cdot \phi_n \tag{6.32}$$

$$L([0,T])^{\otimes N} \otimes \mathcal{K}^{\otimes N} \qquad \bigotimes_n (f_n \otimes \phi_n) \tag{6.33}$$

$$L([0,T]^N) \otimes \mathcal{K}^{\otimes N} \qquad \prod_n f_n \otimes \bigotimes_n \phi_n \tag{6.34}$$

We have $L^2(\Delta^N,\mathcal{K}^{\otimes N})^{\otimes N} \subset (L^2([0,T]^N,\mathcal{K}^{\otimes N})^{\otimes N})$, and hence symmetrization of the time-ordered function gives an unitary equivalence $U_\pm \mathcal{K}_{[0,T]} \to \Gamma_\pm(L^2([0,T],\mathcal{K}))$. □

The theorem shown above entails an equivalence of Bose and Fermi Fock spaces over one dimensional Lebesgue spaces. This equivalence is not as surprising as the main difference between Bose and Fermi spaces in this case is that the functions in the Fermi Fock space have to vanish when the two arguments coincide. Since this only fixes functions on a subspace of measure zero the two variants of Fock space coincide. But beware that continuous functions in the two Fock spaces still look very different. Hence for most applications there is still a huge difference.

6.4.2. Exponential vectors

For the equivalence of $\mathcal{K}_{[0,T]}$ to the Bose Fock space over $\mathrm{L}^2([0,T],\mathcal{K})$ it is helpful to find the analogue of exponential vectors in $\mathcal{K}_{[0,T]}$. The following definition captures their behavior in the discrete setting.

Definition 6.24. Let $\lambda \in \mathrm{L}^2([0,T])$ and $\Theta \in \mathfrak{Z}([0,T])$, then we define:

$$e_\Theta(\lambda) := \bigotimes_{i\in I(\Theta)} (1, \sqrt{\tau_i}\lambda(t_i)) \in \mathcal{K}_\Theta \tag{6.35}$$

Indeed the limit of these vectors exists. Furthermore they are exponential vectors, in the sense that they satisfy their defining property.

Corollary 6.25. *For $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ and $\Theta \in \mathfrak{Z}([0,T])$ the net $J_\Theta e_\Theta(\lambda)$ converges. If we denote $e(\lambda) = \lim_\Theta J_\Theta e_\Theta(\lambda)$, then for $\mu \in \mathrm{L}^2([0,T],\mathcal{K})$ we have the following exponential property:*

$$\langle e(\lambda), e(\mu)\rangle = \exp\left(\langle \lambda, \mu\rangle\right) \tag{6.36}$$

Proof. For a continuous function λ the convergence is easily seen by explicit calculation:

$$\|\exp_\Xi(\lambda) - J_{\Xi\Theta}\exp_\Theta(\lambda)\| \tag{6.37}$$

$$= \prod_{j\in I(\Xi)}\left(1+\tau_j\,\|\lambda(t_j)\|^2\right) + \prod_{i\in I(\Theta)}\left(1+\tau_i\,\|\lambda(t_i)\|^2\right) \tag{6.38}$$

$$-\,2\,\mathfrak{Re}\prod_{i\in I(\Theta)}\left(1+\sum_{j\in I(\Xi|i)}\tau_j\,\langle\lambda(t_j),\lambda(t_i)\rangle\right). \tag{6.39}$$

The following exponential property also follows directly for continuous functions. Here we use that continuous functions on an interval are Riemann integrable.

$$\lim_\theta\,\langle \exp_\theta(\xi), \exp_\theta(\lambda)\rangle = \lim_\theta \prod_{i\in I(\Theta)}\left\langle (1,\sqrt{\tau_j}\lambda_j),(1,\sqrt{\tau_i}\lambda_i)\right\rangle \tag{6.40}$$

$$= \lim_\theta \prod_{i\in I(\Theta)}\left(1+\tau_i\,\langle\xi_i,\lambda_i\rangle\right) = \exp\left(\langle\xi,\lambda\rangle\right) \tag{6.41}$$

Using denseness of continuous functions, we can extend this relation to arbitrary functions in $\mathrm{L}^2([0,T],\mathcal{K})$. □

6.5. Other dilation spaces

We end this section with two construction of "dilation" spaces which are closely related to the above construction. They are of potential use in extensions and generalizations of the limit construction currently under consideration.

6.5.1. A projective limit construction

It turns out that the space $\mathcal{K}_{[0,T]}$ can alternatively be defined as a projective limit. Or precisely, the space can almost be constructed as a projective limit, since in this construction we naturally end up with a different topology. Loosely speaking this alternative topology is the finest one compatible with the locality structure of $\mathcal{K}_{[0,T]}$.

Lemma 6.26. *On the level of vector spaces we have that:*

$$\underleftarrow{J\text{-}lim}\,\mathcal{K}_\Theta = (\underrightarrow{J\text{-}lim}\,\mathcal{K}_\Theta)'. \tag{6.42}$$

Here the $'$ denotes the topological dual space. The natural topologies on the two spaces do not coincide. If we equip the dual space with its natural topology as a Banach space dual, see page 20.

On the level of Hilbert spaces we have:

$$(\underrightarrow{J\text{-}lim}\,\mathcal{K}_\Theta)' = \overline{\underrightarrow{J\text{-}lim}\,\mathcal{K}_\Theta}. \tag{6.43}$$

That is, the natural topologies coincide.

Proof. It is clear that $\underleftarrow{J\text{-}lim}\, K_\Theta \subset \left(\underrightarrow{J\text{-}lim}\, K_\Theta\right)'$ because every element ψ in the projective limit defines naturally a continuous linear functional on the inductive limit by setting $f_\psi(\varphi) = \langle \mathrm{p}_\Theta(\psi), \varphi\rangle$ if $\varphi \in \mathcal{K}_\Theta$. The functional is well defined because of the defining property of the projective limit.

The other way around every linear functional on the inductive limit space can be interpreted as an element in the projective limit. To see this we note that a linear functional f on the inductive limit defines a family of linear functional $f_\Theta \in \mathcal{K}'_\Theta = \mathcal{K}_\Theta$. We can thus identify f with the intersection of the inverse images $\bigcup_{\Theta\in\mathfrak{Z}([0,T])} \mathrm{p}_\Theta^{-1}(f_\Theta)$. The intersection is non-empty because of the finite intersection property and can only contain one element since all projections are fixed.

For the last assertion we note that it is clear that the dual of the inductive limit is complete in its natural topology, since that is true in far greater generality [79, theorem 4.1]. Also it is clear that the norm of $f \in \left(\underrightarrow{J\text{-}lim}\, K_\Theta\right)'$ is just the limit of the norms $\|f_\Theta\|$ so the two topologies coincide. □

6.5.2. A minimal dilation space

There is a second construction of an inductive limit space quite similar to the one which leads to Fock spaces. To get it we stick with the interval decomposition structure, i.e. comparing on the level of subintervals, but all other constructions are then chosen canonically.

However for this construction the limit space is not associated to an abstract limit construction, but to a specific semigroup. The space may thus depend heavily on the chosen semigroup. However it is clear that the space constructed below is in some sense a minimal space.

Given a semigroup $\mathbb{E}(t) : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ of completely-positive normal maps. We do not need any continuity restrictions on the completely-positive map. Furthermore we could also omit the requirement of normality for the maps $\mathbb{E}(t)$. This dilation space is intimately related to the continuous-time Stinespring dilation by Parthasarathy [71].

Fix an interval $[0, T]$ and for every $t \in [0, T]$ choose a minimal Stinespring dilation $(V_t, \mathcal{K}_t \otimes \mathcal{H})$. For $\Theta \in \mathfrak{Z}([0, T])$ we define:

$$\mathcal{K}_{\min,\Theta} = \bigotimes_{i\in I(\Theta)} \mathcal{K}_{\tau_i}. \tag{6.44}$$

Let $\Theta \subset \Xi$ for $\Theta, \Xi \in \mathfrak{Z}([0, T])$ then for every $i \in I(\Theta)$ there exists a unique isometry $J_{\min,i} : \mathcal{K}_{\tau_i} \to \mathcal{K}_{\min \Xi|i}$, such that $J_{\min,i} \otimes \mathbb{1}_\mathcal{H} V_{\tau_i} = \prod_{j\in I(\Xi|i)} V_{\tau_j}$. We define:

$$J_{\min,\Xi\Theta} := \bigotimes_{i\in I(\Theta)} J_{\min,i}. \tag{6.45}$$

The family is inductive because :

$$J_{\min,\Lambda\Xi} J_{\min,\Xi\Theta} \prod_{i\in I(\Theta)} V_{\min,\tau_i} = J_{\min,\Lambda\Xi} \prod_{j\in I(\Xi)} V_{\min,\tau_j} = \tag{6.46}$$

$$J_{\min,\Lambda\Xi} \prod_{j\in I(\Xi)} V_{\min,\tau_j} = \prod_{k\in I(\Lambda)} V_{\min,\tau_k} = J_{\min,\Lambda\Theta} \prod_{i\in I(\Theta)} V_{\min,\tau_i}. \tag{6.47}$$

The uniqueness property of $J_{\min,i}$ ensures that $J_{\min,\Lambda\Xi} J_{\min,\Xi\Theta} = J_{\min,\Lambda\Theta}$. Hence $J_{\min,\Xi\Theta}$ is an inductive family of mappings and the completion of the inductive limit $\overline{\underrightarrow{J\text{-}lim}\,\mathcal{K}_{\min,\Theta}}$ is a well defined Hilbert space.

The space is a product minimal dilation space. To see that remember that every element of $\underrightarrow{J\text{-}lim}\,\mathcal{K}_{\min,\Theta}$ lives in some $\mathcal{K}_{\min,\Theta}$, which can be canonically embedded into any continuous dilation space. The argument that this induces a well defined embedding on the inductive limit is essentially the same as that showing that $J_{\min,\Xi\Theta}$ is an inductive family.

7. Limit of repeated quantum observations

7.1. Main results

Here we shall discuss the main results of this thesis. A Stinespring dilation of a Markovian open quantum evolution compatible with the continuity and evolution-system structure of the time evolution. The dilation describes all possible measurements compatible with a given evolution and the measurements are "time-resolved".

Contrary to the similar construction in section 4, where we mainly worked in Schrödinger picture because we focused on the generators of the evolution, we present the following ideas in Heisenberg picture. The reason is that we use much of the theory of Stinespring dilations, which is naturally understood in this picture.

The proofs of the results we present here can be found in section 7.4 and 7.5.

7.1.1. Results for unbounded generators

Our main assumption is that the evolution we dilate is given as the minimal solution to a Lindblad equation. This mainly restricts the time-dependence of the generator. As always we include as much time-dependence as possible.

We note that Holevo's example in section 5 shows that the assumption that the evolution is a minimal solution might actually restrict the possible types of interaction. For the rest of this thesis we shall assume at least the following.

Assumptions 1. Let $\mathcal{H}, \mathcal{K}$ be a Hilbert spaces and $\mathcal{D}$ a Banach space, which can be densely and continuously embedded into $\mathfrak{B}(\mathcal{H})$. Let $s \leq t \in [0, T]$.

- Let $U(t,s)$ be a contractive evolution system with common core $\mathcal{D}$.
- The family of operators $L(t) : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ fulfills the following two conditions. This makes the pair $(\mathcal{K} \otimes \mathcal{H}, L(t))$ into an exit space for $U(t,s)$.
  - $L(t) : \mathcal{D} \to \mathcal{K} \otimes \mathcal{H}$ is a family of operators, s.t.: For all $\psi \in \mathcal{D}$ the function $t \to L(t)\psi$ is piece-wise continuous.
  - For all $\psi \in \mathcal{D}$ we have that:

$$\lim_{h \to 0} \frac{1}{h} \|U(t+h,t)\psi\|^2 + \|L(t)\psi\|^2 \leq 0. \tag{7.1}$$

The primal example is that $L(t)$ is time independent and that $U(t,s) = \exp((t-s)K)$, i.e. a strongly-continuous semigroup with generator K. With the additional condition that the infinitesimal normalization condition in equation (7.1) is satisfied. In other words:

$$\langle \varphi, \mathcal{L}(B)\psi \rangle = \langle K\varphi, B\psi \rangle + \langle \varphi, BK\psi \rangle + \langle L\varphi, \mathbb{1}_{\mathcal{K}} \otimes BL\psi \rangle \tag{7.2}$$

is a standard Lindblad generator.

Since we are working with evolution systems, it is sufficient that the above assumptions are only satisfied piece-wise, i.e. for each subinterval of an interval decomposition $\Theta \in \mathfrak{Z}([0,T])$. We can of course always solve the problem for each subinterval, and then join the pieces together.

The dilation of single time-step is the central definition for the construction of the limit. The definition only tries to capture the no and one-event behavior of the underlying evolution, i.e. it is not correct to arbitrary order. Furthermore it is not obvious that following definitions are well-defined, i.e. yield bounded operators. A proof of this and a few general properties can be found in corollary 7.14.

Definition 7.1. Under the assumptions 1 and with $\Theta \in \mathfrak{Z}([s,t])$ we define for each subinterval $i \in I(\Theta)$:

$$V_{i,0} : \mathcal{H} \to \mathcal{H} \qquad \varphi \to U(t_i, t_{i-1})\varphi \tag{7.3}$$

$$V_{i,1} : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H} \qquad \mathcal{D} \ni \psi \to \frac{1}{\sqrt{\tau_i}} \int_{t_{i-1}}^{t_i} \mathbb{1} \otimes U(t_i, s)L(s)U(s, t_{i-1})\psi \mathrm{d}s \tag{7.4}$$

$$V_i : \mathcal{H} \to (\mathbb{C} \oplus \mathcal{K}) \otimes \mathcal{H} \qquad \varphi \to (V_0\varphi, V_{i,1}\varphi). \tag{7.5}$$

This definition is inspired by the exit space embedding in [85]. To dilate multiple steps we just iterate the single-step construction as it was described in section 6.1.

Definition 7.2. In the situation of the previous definition 7.1 we set:

$$V_\Theta : \mathcal{H} \to \mathcal{K}_\Theta \otimes \mathcal{H} \qquad V_\Theta = \prod_{i \in I(\Theta)} V_i. \tag{7.6}$$

With these definitions the net of operators $J_\Theta V_\Theta \in \mathfrak{B}(\mathcal{H}, \mathcal{K}_{[0,T]} \otimes \mathcal{H})$ converges strongly.

Theorem 7.3 (convergence)**.** *Under the assumptions 1, and with the operator V_Θ from the two previous definitions number 7.1 and 7.2 with the family of embeddings 6.9 and the inductive limit space $\mathcal{K}_{[0,T]}$ as in definition 6.11, the net $J_\Theta V_\Theta \in \mathfrak{B}(\mathcal{H} \to \mathcal{K}_{[0,T]} \otimes \mathcal{H})$ for $\Theta \in \mathfrak{Z}([s,t])$ converges in the strong topology for every $s \le t \in \mathbb{R}$.*

In other words, we have that for all $s \le t \in \mathbb{R}$, every $\varphi \in \mathcal{H}$ and every ϵ there exists a $\Theta \subset \mathfrak{Z}([s,t])$ such that:

$$\left\| (V_\Lambda - J_{\Lambda\Xi} \otimes \mathbb{1}_\mathcal{H} V_\Xi)\varphi \right\| \le \epsilon \text{ for all } \Theta \subset \Xi \subset \Lambda \in \mathfrak{Z}([s,t]). \tag{7.7}$$

This allows us to define an continuous family of Stinespring dilations and the corresponding family of channels.

Definition 7.4. In the situation of the previous theorem let $s \le t \in \mathbb{R}$. We denote:

$$V_{[s,t]} : \mathcal{H} \to \mathcal{K}_{[s,t]} \otimes \mathcal{H} \qquad \varphi \to \lim_{\Theta \in \mathfrak{Z}([s,t])} J_\Theta V_\Theta \varphi \tag{7.8}$$

$$\mathbb{E}(s,t) : \mathfrak{B}(\mathcal{K}_{[s,t]} \otimes \mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \qquad X \to V_{[s,t]}^* X V_{[s,t]} \tag{7.9}$$

$$\widehat{\mathbb{E}}(s,t) : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \qquad B \to \mathbb{E}(s,t)(\mathbb{1}_{\mathcal{K}_{[s,t]}} \otimes B). \tag{7.10}$$

The thus defined family of Stinespring dilations is a minimal solution to the Cauchy equation with Lindblad generator given by $K(t)$ and $L(t)$ from the assumptions 1.

Theorem 7.5 (infinitesimal behaviour)**.** *The family of maps $\widehat{\mathbb{E}}(s,t)$ for $0 \leq s \leq t \leq T$ is a weak-∗continuous evolution system of completely-positive maps and a minimal solution to the Cauchy equation:*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle \psi, \widehat{\mathbb{E}}(s,T)(B)\varphi \right\rangle = \langle \psi, \mathcal{L}(s)\mathbb{E}(s,T)(B)\varphi\rangle \tag{7.11}$$

with generator:

$$\langle \psi, \mathcal{L}(t)(B)\varphi\rangle = \langle K(s)\psi, B\varphi\rangle + \langle \psi, BK(s)\varphi\rangle + \langle L(s)\psi, \mathbb{1}_{\mathcal{K}} \otimes BL(s)\varphi\rangle . \tag{7.12}$$

$\mathbb{E}(s,t)$ as well as $\widehat{\mathbb{E}}(s,t)$ are normal, i.e. weak-∗ continuous, for all pairs s and t.

The family of maps $V_{[s,t]} : \mathcal{H} \to \mathcal{K}_{[s,t]} \otimes \mathcal{H}$ for $0 \leq s \leq t \leq T$ is a continuous Stinespring dilation for $\widehat{\mathbb{E}}(s,t)$, i.e.:

- *$\widehat{\mathbb{E}}(s,t)(B) = V^*_{[s,t]}\mathbb{1} \otimes BV_{[s,t]}$ for every $B \in \mathfrak{B}(\mathcal{H})$*
- *$V_{[s,t]} \circ V_{[r,s]} = V_{[r,t]}$*
- *The function $t \to V_{[s,t]}\varphi$ is continuous for every $\varphi \in \mathcal{H}$ and every $s \in [0,T]$. To define continuity in the preceding expression we use the canonical embedding I of $\mathcal{K}_{[s,t]}$ into $\mathcal{K}_{[0,T]}$ from definition 6.13.*

We do now continue with the important case of a bounded Lindblad generator, where one can obtain stronger results. Subsequently we shall examine a few simple consequences and applications of these results and compare them to the literature. Last but not least we present the proofs of the above results in section 7.4. The section on proofs contains a few technical results of independent interest, e.g. on the validity of the approximation of short time steps by up to one event.

7.1.2. Results for bounded generators

An important special case is that of a bounded Lindblad generator, i.e.:

Assumptions 2. Let $\mathcal{H}, \mathcal{K}$ be a Hilbert spaces. Let $s \leq t \in [0,T]$. Furthermore let $K(t) \in \mathfrak{B}(\mathcal{H})$ and $L : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ be two families of bounded operators, such that:

- The function $t \to K(t)$ is continuous and $\|K(t)\| \leq C$ for $t \in [0,T]$.
- The function $t \to L(t)$ is continuous.
- For every t we have that:

$$K^*(t) + K(t) + L(t)^*L(t) \leq 0. \tag{7.13}$$

Under these assumptions we can get far stronger results that is, we can replace strong continuity with norm continuity. Also we can get explicit bounds on the speed of convergence.

Theorem 7.6. *Under the assumptions 2 with V_Θ from definitions 7.1 and 7.2, with the family of embeddings 6.9 and the inductive limit space $\mathcal{K}_{[0,T]}$ as in definition 6.11, the net $J_\Theta V_\Theta \in \mathfrak{B}(\mathcal{H} \to \mathcal{K}_{[0,T]} \otimes \mathcal{H})$ for $\Theta \in \mathfrak{Z}([s,t])$ converges in the norm topology for every $s \leq t \in \mathbb{R}$.*

In other words, we have that for all $s \leq t \in \mathbb{R}$ and every ϵ there exists a $\Theta \subset \mathfrak{Z}([s,t])$, such that:

$$\left\| V_\Lambda - J_{\Lambda\Xi} \otimes \mathbb{1}_\mathcal{H} V_\Xi \right\| \leq \epsilon \text{ for all } \Theta \subset \Xi \subset \Lambda \in \mathfrak{Z}([s,t]). \tag{7.14}$$

If the function $t \to L(t)$ is Lipschitz continuous with constant D we have the simple estimate:

$$\|\mathbb{E}_\Xi - \mathbb{E}_\Theta\|_{cb} \leq 2\left(C^2 + 2C(C+1) + D\right) \sum_{i \in I(\Xi)} \tau_i^2. \tag{7.15}$$

where $\mathbb{E}_\Theta(X) := V_\Theta^* \mathbb{1}_{K_\Theta} \otimes X V_\Theta$.

The family of maps $\widehat{\mathbb{E}}(s,t)$ for $0 \leq s \leq t \leq T$ is a norm continuous evolution system of normal and completely-positive maps and a minimal solution to the Cauchy equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\widehat{\mathbb{E}}(s,T)(B) = \mathcal{L}(s)\mathbb{E}(s,T)(B), \tag{7.16}$$

with generator:

$$\mathcal{L}(s)(B) = K(s)^*B + BK(s) + L(s)^*\mathbb{1}_\mathcal{K} \otimes BL(s). \tag{7.17}$$

$\mathbb{E}(s,t)$ as well as $\widehat{\mathbb{E}}(s,t)$ are normal maps, i.e. weak-$*$ continuous.

The family of maps $V_{[s,t]} : \mathcal{H} \to \mathcal{K}_{[s,t]} \otimes \mathcal{H}$ for $0 \leq s \leq t \leq T$ is a continuous Stinespring dilation for $\widehat{\mathbb{E}}(s,t)$, i.e.:

- $\widehat{\mathbb{E}}(s,t)(B) = V_{[s,t]}^* \mathbb{1} \otimes BV_{[s,t]}$ *for every $B \in \mathfrak{B}(\mathcal{H})$*
- $V_{[s,t]} \circ V_{[r,s]} = V_{[r,t]}$
- *The function $t \to V_{[s,t]}$ is continuous in norm for every $s \in [0,T]$. To define continuity use the canonical embedding I of $\mathcal{K}_{[s,t]}$ into $\mathcal{K}_{[0,T]}$ from definition 6.13.*

In the norm continuous case we can use a completely different proof method. That is we can explicitly evaluate and bound the differences between different discretizations.

7.1.3. Discussion of the assumptions

Here we shortly discuss the validity of our assumption and possibilities for improvement.

First of all in most of the relevant literature time-dependence of the Lindblad generator does not play a role, e.g. [59, 58, 45, 11, 68, 67]. One of the few exceptions is [21]. On the other hand because of the gauge freedom of the Lindblad equation, time dependence, is a natural assumption. As we shall see it also helps in understanding its structure.

Our assumptions on time dependence are however still mainly chosen for simplicity and could probably be weakened quite a bit. In particular we do not explicitly need the strong continuity of $L(t)$, but only Bochner integrability of the functions $t \to L(t)U(t,s)\psi$ and $t \to U(T,t)L(t)U(t,s)\psi$ for all $s \leq T$ and $\psi \in \mathcal{D}$. The strong continuity is a sufficient condition for this to happen but by no means necessary.

So we can potentially replace the continuity by some integrability condition on $L(t)$. Similarly we can probably relax the integrability from Bochner integrability to a slightly weaker notion of integral.

Since our theory extends without any work to the case of piece-wise continuous functions, it already contains the important case of a simple, i.e. piece-wise constant, $L(t)$.

Furthermore the requirement of the existence of a common core for the no event evolution system $U(t, s)$ is quite restrictive. It could potentially be weakened, if we follow the limit procedure described here by a second limit on the level of dilations analogue to the usual construction of solutions to hyperbolic non-autonomous Cauchy equations, e.g. [74, section 5.3]. It is clear that in this way one can construct minimal solutions to Lindblad equations where $\mathcal{L}$ does not have a common core. However to prove convergence on the level of dilations, $V_{[s,t]}$ is not clear in this approach.

7.1.4. Discussion of the dilation

Since the dilation only tries to capture the exact behavior up to one event per time-step, it is quite clear that we approach any exact solution to the Lindblad equation from below. We show this rigorously in lemma 7.19. This does ensure that we construct a minimal solution in the end.

The other important property of the single-step dilation is that its extension, by iteration, to multiple time steps is compatible with the family of embeddings $J_{\Xi\Theta}$. That is in lemma 7.15 we show that $J^*_{\Xi\Theta}V_\Xi = V_\Theta$. It means that our dilation acquires the structure of an inverse limit. This property is a central pillar in the construction of the limit. It ensures the convergence of the dilation. See the proof on page 127.

It is worth mentioning that bounded perturbations of an order greater than τ_i in every single-step dilation do not change the limit. This property is very useful in the case of bounded generators.

Lemma 7.7. *For a given $\Theta \in \mathfrak{Z}([s,t])$ let $V_i : \mathcal{H} \to (\mathcal{K} \oplus \mathbb{C}) \otimes \mathcal{H}$ for $i \in I(\Theta)$ be as in definition 7.1 and let $\widetilde{V}_i : \mathcal{H} \to (\mathcal{K} \oplus \mathbb{C}) \otimes \mathcal{H}$ be a family of maps such that $\|V_i - \widetilde{V}_i\| \le C_i \tau_i^\alpha$ with $\alpha > 1$ then:*

$$\lim_{\Theta \in \mathfrak{Z}([s,t])} \left\| V_\Theta - \prod_{i \in I(\Theta)} \widetilde{V}_i \right\| = 0. \tag{7.18}$$

Proof. After telescoping the two iterations over $i \in I(\Theta)$ the lemma becomes obvious. □

In the case that $\mathcal{L}$ is a bounded operator this lemma gives us a lot of freedom in choosing the explicit form dilations $V_{i,0}$ and $V_{i,1}$. For example we could define:

$$\widetilde{V}_{i,0}\varphi = \mathbb{1} + \tau_i K(t_i)\varphi \tag{7.19}$$

$$\widetilde{V}_{i,1}\varphi = \sqrt{\tau_i} L(t_i)\varphi. \tag{7.20}$$

This dilation then leads to an identical limit.

7.2. cMPS

Note that contrary to our announcement we did not construct a limit of matrix product states or finitely correlated states. Instead we constructed a limit on the level of the dilation V_i, which amounts to taking the limit of a whole family of states. The actual limit states are constructed in exact parallel to our definition of finitely correlated states in 6.4.

7.2.1. Basic properties

Definition 7.8 (cMPS)**.** Under the assumptions 1 let $B \in \mathfrak{B}(\mathcal{H})$ and let $\rho \in \mathfrak{T}(\mathcal{H})$ be a state, i.e. positive and with trace 1. Then we define the *cMPS* ω as $\omega := \widetilde{\omega}\|\widetilde{\omega}\|^{-1}$.

$$\widetilde{\omega} : \mathfrak{B}(\Gamma_\pm(\mathrm{L}([0,T],\mathcal{K})) \to \mathbb{C} \qquad\qquad X \mapsto \mathrm{tr}\left(\rho\mathbb{E}_{[0,T]}\left(X \otimes B\right)\right) \tag{7.21}$$

We are again mainly interested in the case where $B = \mathbb{1}_\mathcal{H}$.

Note that even in the case $B = \mathbb{1}_\mathcal{H}$ it is not guaranteed that the evolution $\widehat{\mathbb{E}}(s,t)$ is unital and hence we have to explicitly normalize the state.

As already mentioned, cMPS are best considered to be states over Bose-Fock space, rather than Fermi-Fock space. The reason being that for a generic semigroup, the Fermi states would be highly discontinuous where two arguments coincide. The Bose variant does not suffer from this problem.

One can of course just construct the semigroup/Lindbladian, s.t. events repulse each other, i.e. the chance for two events of the same type vanishes for very short time intervals. A natural condition to ensure such behavior is to choose the Lindblad operators to be point wise nil-potent, i.e. $L_\alpha^2 = 0$. This can be done for example by choosing them to anti-commute at same times. In such a case terms like $L_\alpha(t)U(t,s)L_\alpha(s)$ have to vanish for $t \to s$. Since, when defined, the function is continuous. We do not try to give a rigorous treatment of such situations. It would furthermore be interesting to figure out in how far the results on covariant generators in [53] generalize to our dilation construction.

7.2.2. cMPS over $\mathbb{R}$

As in the case of finitely correlated states we can extend the cMPS to a state on the quasi-local algebra for the one dimensional quantum field, i.e. we can assign numbers to all operators $A \in \mathfrak{B}(\Gamma_\pm(\mathrm{L}(\mathbb{R},\mathcal{K}))$ such that A acts as the identity outside of a finite interval $[s,t]$.

To do this, we need again to switch to a family of states compatible with the evolution system $\widehat{\mathbb{E}}(s,t)$ in the sense that $\widehat{\mathbb{E}}_*(t,s)\rho(s) = \rho(t)$. Furthermore we need to assume that the minimal solution $\widehat{\mathbb{E}}(s,t)$ is unital or has some other eigenvector.

The existence of the compatible family of states is guaranteed by lemma 6.5 since the family of channels $\mathbb{E}_i : \mathfrak{B}(\Gamma_\pm(\mathrm{L}([iT,(i+1)T])))$ for $i \in \mathbb{Z}$ creates a valid FCS. From the compatible family of states ρ_i we can build a continuous family setting $\rho(t) = \widehat{\mathbb{E}}_*(t, iT)(\rho_i)$ for $t \in [iT,(i+1)T]$ and so on.

Note that while we are constructing states in Fock space for any finite interval $[0,T]$, when we change to the whole line $\mathbb{R}$, we do not get states on $\mathfrak{B}(\Gamma_+(\mathrm{L}(\mathbb{R},\mathcal{K}))$, but only on operators "localized" in a finite interval $[S,T]$. It is not entirely clear when the constructed states can be extended to that whole algebra. However it is clear that this is not always the case, since we can easily construct states with finite and constant particle density. This is actually a generic phenomenon for states corresponding to semigroups $\mathbb{E}(s)$, i.e. constant Lindblad generators.

7.2.3. cMPS as a variational class

A major application one has in mind for cMPS, is using them as a variational class to approximate some unknown field state, e.g. for numerical calculation. Such applications are a field where the discrete counterpart MPS/FCS excels at. So the hope is that cMPS could help in numerical treatment of one dimensional quantum fields. And also if we want to study quantum fields by approximation with discrete models, i.e. conventional MPS, the study of the limit is of course of interest.

An important point in this direction is that our discretization of a cMPS always systematically underestimate the exact expectation value for positive operators, because be approach the true cMPS state from below.

When cMPS are used in such a case one would usually try to describe pure states of some quantum field, as one uses MPS usually to describe pure states. Hence in analogy to the case of cMPS, one looks mainly at pure states with either open or circular boundary conditions, i.e. states of the form:

$$X \mapsto \left\langle \psi, \mathbb{E}_{[0,T]} \left(X \otimes |\varphi\rangle\langle\varphi| \right) \psi \right\rangle , \tag{7.22}$$

for $\psi, \varphi \in \mathcal{H}$. Circular boundary conditions can again be implemented as described before. Another possibility to generate periodic states, is to engineer the Lindbladian $\mathcal{L}(t)$ such that its minimal solution $\widehat{\mathbb{E}}(s,t)$ is periodic for some period T, i.e. $\widehat{\mathbb{E}}(0,T) = \mathrm{id}$. The resulting cMPS is clearly periodic.

7.3. Comparison with the literature

Continuous time measurements, are of course not a new idea and the problem of their description has been investigated and solved before. The same thing is true for discrete versions of those constructions and their limits. In essence their is a lot of literature one can compare our approach to.

The two bodies of work which to our knowledge are nearest to our approach are quantum stochastic calculus and quantum input-output theory. The former theory generalizes stochastic-calculus to the quantum regime and thereby solves a wider range of problems. A most important application of stochastic calculus is however exactly the problem we solved, a continuous Stinespring dilation for completely-positive semigroups. The standard text book on this subject is [72].

The latter theory, quantum input output theory, invented similar methods while trying to describe one of the most important models, the output field of a driven cavity [49, 48]. In contrast to our exposition and quantum stochastic calculus this field is aiming less for mathematical generality but tailored to applications.

Compared to these two theories our exposition is tailored more to the problem and tool set of quantum information, i.e. we tried to systematically describe the flow of information from the system to the environment, while on the other hand seeking to minimize assumptions. We also tried to systematically include possible time dependence. Last but not least it is sometimes helpful that our approach is based on a discrete to continuous limit, such that the relationship between discrete and continuous objects is inherently clear.

Yet another approach we want to prominently mention, is the quite recent definition of cMPS [82, 70]. Another approach to similar problems following a completely different motivation. In this case the goal was to generalize the class of matrix product states, a very successful calculational tool in the analysis of spin chains, to the case of one dimensional quantum field. The main motivation here was to get an interesting class of states for variational calculations.

7.3.1. Quantum stochastic calculus

The theory of quantum stochastic calculus is very interesting for this thesis, because it reveals very similar construction in a different light. We try to give a short simplified overview of some key constructions in QSC important for the description of continuous measurements.

For the start of a further study of QSC we recommend the textbook [72]. The central theme of QSC is to generalize classic stochastic analysis to the quantum setting. A main object in this process, setting the basis of "integrable" stochastic processes, are adapted processes. In our language an adapted process is a family of operators $X(t) \in \mathfrak{B}(\Gamma_+(L^2([0,T],\mathcal{K})) \otimes \mathcal{H}$ for $t \in [0,T]$ such that $X(t) = \mathbb{1}_{[t,T]} \otimes X_t$ with an operator $X_t \in \mathfrak{B}(\Gamma_+(L^2([0,t],\mathcal{K})) \otimes \mathcal{H}$, i.e, the process acts non-trivially on larger and larger segments of the total system space.

The usual formulation of QSC is heavily tied to viewing Bose Fock space as an exponential Hilbert space, i.e. "$\Gamma_+(\mathcal{H}) = \exp(\mathcal{H})$". There are however versions for Fermi Fock space, etc. Also in the above definition unbounded operators are allowed. Most objects and properties are defined in application on exponential vectors. So, e.g., the standard continuity condition for adapted processes is continuity of $X(t)\exp(\lambda) \otimes \psi$ as a function of t.

The most important example of an adapted process is the annihilation process $A_\lambda(t)$ for an integrable function λ defined on the domain of exponential vectors by $A_\lambda(t) := a(\lambda|_{[0,t]})$, i.e., the smoothed creation operator of a function λ restricted to $[0,t]$. The point wise adjoint of this process is the creation process.

Together with the conservation process $\Lambda(t)$, defined as differential second quantization of restriction to a time interval of an operator O on $L^2([0,T],\mathcal{K})$, i.e. $\Lambda_O(t) = \mathrm{d}\Gamma_+(O|_{[0,t]})$, these form the three fundamental processes. One then shows how to integrate adapted processes against these fundamental processes.

One of the signature results of quantum stochastic calculus is an dilation of a given completely-positive semigroup $\widehat{\mathbb{E}}(t)$ to an unitary process on Fock space. We can then recover the semigroup as the vacuum (conditional) expectation value of the stochastic process restricted to $[0,t]$, i.e. there exists an unitary operator valued adapted process $U(t)$ such that:

$$\langle \psi, \widehat{\mathbb{E}}(t)(X)\psi \rangle = \langle |0\rangle \otimes \psi, U(t)^* \mathbb{1} \otimes XU(t)|0\rangle \otimes \psi \rangle \tag{7.23}$$

The adapted process $U(t)$ fulfills the following quantum stochastic differential equation:

$$\mathrm{d}U = \left(\sum_\alpha L_\alpha \mathrm{d}A_\alpha^\dagger - L_\alpha^* \mathrm{d}A_\alpha + K\mathrm{d}t \right) U. \tag{7.24}$$

We can compare the stochastic processes to the following objects, which almost form an evolution system.

Corollary 7.9. *Let $V_{[s,t]} : \mathcal{H} \to K_{[s,t]} \otimes \mathcal{H}$ be the family of Maps from definition 7.4 and $0 \leq s \leq t \leq T$. We set:*

$$\widetilde{V}(t,s) : \mathcal{K}_{[0,t]} \otimes \mathcal{H} \to \mathcal{K}_{[0,t]} \otimes \mathcal{H}$$
$$\widetilde{V}(t,s) := \left(\mathbb{1}_{\mathcal{K}_{[0,s]}} \otimes V_{[s,t]}\right) P_{[0,s]} \otimes \mathbb{1}_{\mathcal{H}} \tag{7.25}$$

where $P_{[0,r]}$ is the projection from definition 6.13. And given this we define $V(t,s)$: $\mathcal{K}_{[0,T]} \otimes \mathcal{H} \to \mathcal{K}_{[0,T]} \otimes \mathcal{H}$ as acting trivially on $\mathcal{K}_{[t,T]}$ and as $\widetilde{V}(t,s)$ else.

The resulting two-parameter family of maps $V(s,r)$ is a strongly-continuous evolution system. That is the family $V(t,s)$ is continuous in the strong topology, and we have $V(t,r) = V(t,s)V(s,r)$ and $V(t,t) = \mathbb{1}$.

The family $V(t,s)$ is an example of a adapted process as defined above, at least with one small limitation due to the time dependence of the generator. As in the passage from semigroups to evolution systems it is not anymore guaranteed that we can find a single domain $\mathcal{D}$ on which the process is differentiable and which is conserved under the evolution $V(t,s)$. In other words we do not know if the evolution system $V(t,s)$ is a $\mathcal{D}$-valued evolution system. The existence of such a domain is usually required in quantum stochastic calculus, in contrast we only impose this condition on the "input" evolution system $U(t,s)$.

The process $V(t,s)$ is up to the described domain problems a solution to the quantum stochastic differential equation:

$$\mathrm{d}V(t,0) = \left(\sum_{\alpha} L_\alpha(t)\mathrm{d}A^\dagger_\alpha + K(t)\mathrm{d}t\right) V(t,0). \tag{7.26}$$

In our language this is the consequence of the following lemma. For the comparison with quantum stochastic calculus one should use a differential version of the first fundamental lemma [72, proposition 25.9].

Lemma 7.10. *Let $\mathcal{D}$ be the space from the assumptions 1. Given a $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$, $\psi, \varphi \in \mathcal{D}$ and $\mu \in \mathcal{K}_{[0,T]}$, then:*

$$\lim_{h\to 0} \frac{1}{h}\langle \exp(\lambda) \otimes \varphi, (V(t+h,t) - V(t,t))\mu \otimes \psi\rangle \tag{7.27}$$
$$= \langle \exp(\lambda) \otimes \varphi, P_{[0,t]} \otimes (K + \langle \lambda(s)|L(s)\rangle \, \mu \otimes \psi\rangle. \tag{7.28}$$

This is basically a straight forward consequence of the short time behavior of the dilation as described in corollary 7.20.

In the quantum stochastic calculus literature there are also various discrete to continuous limit constructions like,.e.g. [4, 51]. The constructions show the limit of a large class of repeated interaction models to solutions of quantum stochastic differential equations. However the convergence results hold only for bounded generators. One should also note [14] which is similar in spirit and contains a nice pedagogical writeup of the discrete field model.

The applications of quantum stochastic calculus to quantum optics and continuous measurement has been studied extensively and can be found, e.g., in the works of Belavkin and Barchielli [5, 6]. Instead let us try to present a few important objects of study in the language we build up.

7.3.2. Others

Of the other approaches to continuous measurement the two most important ones for our Ansatz are as mentioned before are quantum input-output theory and continuous matrix product states. In the case of a bounded generator all the approaches are different sides of the same medal. We already explained how to construct Matrix product states from the dilations $V_{[s,t]}$ in section 7.2.

For the comparison quantum input output theory it should be noted, that one of the basic papers of this fields [48] develops a quantum stochastic calculus very similar to that of Hudson and Parthasarathy [59]. This calculus can of course be used to compare our results to quantum input output theory. There is also a very readable explanation on the connection between input-output theory and cMPS as an appendix to [7].

There are two other approaches to similar problems we want to mention, because they motivated and inspired parts of this work. First of all this is the theory of stochastic processes by Davies [27, chapter 5]. Quantum stochastic processes describe continuous measurements, but not as a delayed choice measurement. That is they describe the continuous stochastic processes when we already know which observable we measure on the field.

Furthermore the theory of quantum stochastic processes uses in many places the assumption of a bounded interaction rate. That is essentially supposing the operator $K + K^*$ is bounded. Whereas we do not need this assumption for our main results.

Apart from this restriction the theory of quantum stochastic processes contains many results similar to ours. And it has certainly inspired some of our constructions. Note for examples that it contains many ingredient of exit-space theory, which we heavily exploit and described in section 4.

The last publication important to our work, which we explicitly want to mention here is the construction of a continuous Stinespring dilation by Parthasarathy [71]. This paper introduces an abstract counterpart to our construction. Parthasarathy only treats dilations of semigroups, notably without any continuity restrictions, but the construction extends to the time-dependent case without any additional work.

The dilation constructed in [71] is in the case of semigroups of normal maps equivalent to a minimal Stinespring dilation concatenated with the embedding constructed in section 6.5.2. Let us stress again, that in this construction the dilation space heavily depends on the explicit semigroup.

It would be interesting to find out under which continuity conditions on the semigroup, one can show that the minimal dilation space is in fact equivalent to the space $\mathcal{K}_{[s,t]}$, i.e. when there exists a unitary equivalence conserving the Fock space structure, e.g. the exponential property and the particle number grading. In other words we would like to know under which circumstances "our" dilation space is "minimal". We conjecture this to be the case for strongly continuous semigroups, being the minimal solution to a Lindblad equation.

Of course there are numerous other approaches and solutions to the description of continuous time measurements. Their omission has nothing to do with their importance, or even the authors opinion on their importance, but only with the lack of time to study them.

7.4. Proof: unbounded case

Let us stop looking at consequences of theorems 7.3 and 7.5. Instead we now take a look at the proofs of these results. The proof separates into three parts. First we show that our definitions are well defined. Then we show convergence of the limit constructions. And lastly we examine the "short-time" behavior of our dilation. In all steps of the proof the structure provided by lemma 7.15 can not be underestimated.

7.4.1. Technical preliminaries

Before we dive right into business, we need to introduce the following abbreviations for the single- and no-event part of the completely-positive map induced by the single-step dilation V_i, as they are repeatedly used throughout the following proof. They serve as a connection to the theory of first-arrival time measures and semigroup perturbation theory in section 4.

Definition 7.11. Under the assumption 1 and for $0 \leq s \leq t \leq T$, we define the *no-event evolution* $\mathbb{F}_0(s,t)(B)$ to be $\mathbb{F}_0(s,t)(B) := U(t,s)^* B U(t,s)$. This is obviously a weak-$*$continuous evolution system. Furthermore in the notation of definition 7.1 with $i \in I(\Theta)$, we set for the *no-event* and single-event part of the time-step i:

$$\mathbb{F}_{i,0} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \qquad B \mapsto V_{i,0}^* B V_{i,0} = \mathbb{F}_0(t_{i-1}, t_i) \tag{7.29}$$

$$\mathbb{F}_{i,1} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \qquad B \mapsto V_{i,1}^* \mathbb{1}_{\mathcal{K}} \otimes B V_{i,1}. \tag{7.30}$$

Later in lemma 7.17 we shall see that the single event evolution $\mathbb{F}_{i,1}$ is essentially the adjoint of the measure $\widehat{\mathcal{M}}([t_{i-1}, t_i]) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ belonging to the infinitesimal arrival or Radon-Nikodym derivative $\mathcal{J}(t)(\rho) := \mathrm{tr}_{\mathcal{K}}(L(t)\rho L(t)^*)$ as in theorem 4.4.

The next lemma is a simple technical trick. As we later see it is behind the mentioned equivalence.

Lemma 7.12. *Let $t \to f(t)$ for $t \in [0,T]$ be a Banach space valued Bochner integrable function, then:*

$$\frac{1}{T} \left\| \int_0^T f \mathrm{d}s \right\|^2 \leq \int_0^T \|f\|^2 \mathrm{d}s. \tag{7.31}$$

Proof. For a complex valued function $c(t) \in L^1([0,T])$ we have:

$$\int_0^T |f|^2 \mathrm{d}s - \frac{1}{T} \left(\int_0^T |f| \mathrm{d}s \right)^2 = \int_0^T \left(|f| - \frac{1}{T} \int |f| \mathrm{d}s \right)^2 \mathrm{d}s \geq 0. \tag{7.32}$$

The result follows from the inequality $\left\| \int f \mathrm{d}s \right\| \leq \int \|f\| \mathrm{d}s$ for the Bochner integral valued integration [34, theorem II.2.4]. □

7.4.2. Dilations well defined

Almost all important properties of the single-step dilation V_i are a direct consequence of the following technical estimate. It is in turn a direct consequence of lemma 7.12.

Lemma 7.13. *For any $k \in \mathbb{N}$, $\psi \in \mathcal{D} \otimes \mathbb{C}^k$ and $X \in \mathfrak{B}(\mathcal{H} \otimes \mathbb{C}^k)$ we have:*

$$\|(\mathbb{1}_{\mathcal{K}} \otimes X)V_{1,i} \otimes \mathbb{1}_k \psi\|^2$$

$$\leq \int_{t_{i-1}}^{t_i} \|(\mathbb{1}_{\mathcal{K}} \otimes X)\left((\mathbb{1}_{\mathcal{K}} \otimes U(t_i,s))L(s)U(s,t_{i-1}) \otimes \mathbb{1}_k\right)\psi\|^2 \,\mathrm{d}s. \tag{7.33}$$

$$\tag{7.34}$$

Proof. We insert definition 7.1 and apply lemma 7.12:

$$\|(\mathbb{1}_{\mathcal{K}} \otimes X)V_{1,i} \otimes \mathbb{1}_k \psi\|^2 \tag{7.35}$$

$$= \frac{1}{\tau_i} \left\| \int_{t_{i-1}}^{t_i} (\mathbb{1}_{\mathcal{K}} \otimes X)\left((\mathbb{1}_{\mathcal{K}} \otimes U(t_i,s))L(s)U(s,t_{i-1}) \otimes \mathbb{1}_k\right)\psi \mathrm{d}s \right\|^2 \tag{7.36}$$

$$\leq \int_{t_{i-1}}^{t_i} \|(\mathbb{1}_{\mathcal{K}} \otimes X)\left((\mathbb{1}_{\mathcal{K}} \otimes U(t_i,s))L(s)U(s,t_{i-1}) \otimes \mathbb{1}_k\right)\psi\|^2 \,\mathrm{d}s. \tag{7.37}$$

□

In the preceding lemma in the case $k = 1$ we can use that $(\mathcal{K} \otimes \mathcal{H}, L(s))$ is an exit space for $U(t,s)$, as in definition 4.13, to show that definition 7.1 is well defined, i.e. a bounded operator. In other words the following corollary asserts that the "event map" $V_{1,i}$ does not "create" more probability than the no event evolution $V_{0,i}$ "destroys". In fact it turns out that our approach always systematically underestimates the probability of events, i.e. V_i is not an isometry but a contraction. This is a simple consequence of the fact that we include at most one event per time-step.

Corollary 7.14. *In the notation of definition 7.1 and for $\Theta \in \mathfrak{Z}([0,T])$ the operators $V_{i,0}$ and $V_{i,1}$ are bounded for every $i \in I(\Theta)$. Precisely for $\phi \in \mathcal{H}$:*

$$\|V_{i,1}\varphi\|^2 \leq \|\varphi\|^2 - \|V_{i,0}\varphi\|^2, \tag{7.38}$$

and hence V_i is a contraction $\|V_i\| \leq 1$.

Proof. Apply lemma 7.13 for $k = 1$ and $\mathbb{1} = B \in \mathfrak{B}(\mathcal{H})$. If we view the integrator of (7.33) as a function of $s \in [t_{i-1}, t_i]$, we get:

$$\|(\mathbb{1}_{\mathcal{K}} \otimes U(t_i,s))L(s)U(s,t_{i-1})\psi\|^2 \leq \|L(s)U(s,t_{i-1})\psi\|^2 \tag{7.39}$$

$$\leq 2\,\mathfrak{Re}\,\langle U(s,t_{i-1})\psi, KU(s,t_{i-1})\psi\rangle = -\frac{\mathrm{d}}{\mathrm{d}s}\|U(s,t_{i-1})\psi\|^2. \tag{7.40}$$

Where we used the exit space property from the assumptions 1.

Reinserting this in (7.33) we get $\|V_{1,i}\psi\|^2 \leq \|\psi\|^2 - \|U(t_i,t_{i-1})\psi\|^2$ for $\psi \in \mathcal{D}$. So by denseness of $\mathcal{D}$ the operator $V_{1,i}$ can be uniquely extended to a bounded operator $V_{1,i} : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ on the whole of $\mathcal{H}$, such that: $\|V_{1,i}\varphi\|^2 \leq \|\varphi\|^2 - \|U(t_i,t_{i-1})\varphi\|^2$ for all $\varphi \in \mathcal{H}$.

Furthermore since $\|V_i\varphi\|^2 = \|V_{i,0}\varphi\|^2 + \|V_{i,1}\varphi\|^2$ we get that V_i is a contraction for each $i \in I(\Theta)$. □

Note that the "one-event" part in each step $V_{i,1}$ is bounded by the norm loss of the no event evolution $U(t,s)$ over that same step. In particular V_i is never an isometry.

7.4.3. Proof of convergence

The convergence proof for our construction is astonishingly simple. The central point is that our approximation leads to a increasing net of completely-positive operators, which is furthermore bounded. Just as increasing bounded sequences such nets converge.

The following lemma does ensure the convergence of our dilation construction. Loosely speaking it says that the limit has the structure of a projective limit. Precisely it gives the spaces $V_\Theta(\mathcal{H}) \subset \mathcal{K}_\Theta$ a projective structure, similar to definition 2.10.

Lemma 7.15. *$J^*_{\Xi,\Theta} \otimes \mathbb{1} V_\Xi \varphi = V_\Theta$ or equivalently $J_{\Xi,\Theta} \otimes \mathbb{1}_\mathcal{H} V_\Theta = P_\Theta \otimes \mathbb{1}_\mathcal{H} V_\Xi$.*

Proof. To see this we just evaluate the explicit expressions on the dense set $\psi \in \mathcal{D}$

$$\begin{aligned} J^*_{\Xi,\Theta} \otimes \mathbb{1}_\mathcal{H} V_\Xi \psi = & \prod_{i \in I(\Theta)} \left(J_i^* \otimes \mathbb{1}_\mathcal{H} V_{\Xi|i} \right) \psi \\ = & \bigotimes_{i \in I(\Theta)} |0\rangle \otimes \prod_{j \in I(\Xi|i)} V_{j,0} \psi \\ & + \sum_{j \in I(\Theta|i)} \sqrt{\frac{\tau_j}{\tau_i}} \prod_{j<k \in I(\Xi|i)} (\mathbb{1} \otimes V_{k,0}) \circ V_{j,1} \circ \prod_{j>k \in I(\Xi|i)} V_{k,0} \psi \\ = & V_\Theta \psi. \end{aligned}$$

□

The proof of the convergence theorem, is now a simple consequence. It rests on the observation that net of completely-positive operators $\widehat{\mathbb{E}}_\Theta$, which is induced by the net V_Θ is monotonous and bounded.

Proof of Theorem 7.3. Define $\mathbb{E}_\Theta(X) = V_\Theta^* \mathbb{1}_{K_\Theta} \otimes X V_\Theta$. Lemma 7.13 shows that this net is bounded, i.e. $\|E_\Theta\| \leq 1$ for all $\Theta \in \mathfrak{Z}([s,t])$. Furthermore the net is monotonous, i.e.,for $\Theta \subset \Xi$ we have:

$$(\mathbb{E}_\Xi - \mathbb{E}_\Theta)(X) = ((\mathbb{1} - P_\Theta) \otimes \mathbb{1} V_\Xi)^* \mathbb{1} \otimes X (\mathbb{1} - P_\Theta) \otimes \mathbb{1} V_\Xi. \tag{7.41}$$

Hence the differences $\mathbb{E}_\Xi - \mathbb{E}_\Theta$ are completely-positive and thus the net $\mathbb{E}_\Theta$ converges point-wise in weak topology, since for $B \in \mathfrak{B}(\mathcal{H})$ and $\rho \in \mathfrak{B}(\mathcal{H})^*$ the net $\rho \circ \mathbb{E}_\Theta(X)$ is a bounded monotonous and real valued net, i.e. convergent. Consequently for $\varphi \in \mathcal{H} \otimes \mathbb{C}^k$ the following net is Cauchy, which is the statement of the theorem, i.e.

$$\| \left(V_\Xi - J_{\Xi\Theta} \otimes \mathbb{1} V_\Theta \right) \otimes \mathbb{1}_k \varphi \|^2 = \langle \varphi, (\mathbb{E}_\Xi - \mathbb{E}_\Theta) \otimes \mathrm{id}(\mathbb{1}) \varphi \rangle . \tag{7.42}$$

□

7.4.4. Infinitesimal properties

The idea of our dilation construction was to approximate a short time step to first order in the number of events. Here we show that this is indeed the case. The central technical ingredient is lemma 7.17.

From the construction it is clear that the "intensity" of events is bounded by the norm loss of the no-event evolution system $U(t,s)$. This idea leads to the following lemma.

Lemma 7.16. *Let V_Θ be the dilation from definition 7.1 and $U(t,s)$ an evolution system for the no-event part. We then have for $\phi \in \mathcal{H}$*

$$\|V_\Theta\varphi - |0\rangle \otimes U(T,0)\varphi\| \leq \|U(T,0)\varphi\|^2 - \|\varphi\|^2. \tag{7.43}$$

Proof. Let $\Theta \in \mathfrak{Z}([0,T])$ be an interval decomposition. Denote by $\widehat{\mathbb{E}}_\Theta : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ the map $B \to V_\Theta^* \mathbb{1}_{\mathcal{K}_\Theta} \otimes BV_\Theta$. Observe that we can write this map as an iteration of the single steps:

$$\widehat{\mathbb{E}}_\Theta(X) = \prod_{i \in I(\Theta)} (\mathbb{F}_{0,i} + \mathbb{F}_{1,i}). \tag{7.44}$$

Furthermore the difference between this approximation and $\mathbb{F}_0$ is completely-positive. Telescoping the difference over the interval decomposition Θ we get the desired bound. For $\psi \in \mathcal{D}$:

$$\left\langle \varphi, \left(\widehat{\mathbb{E}}_\Theta - \mathbb{F}_0(0,T)\right)(\mathbb{1}_\mathcal{H})\varphi \right\rangle \tag{7.45}$$

$$\leq \sum_{i \in I(\Theta)} \langle \varphi, (\mathbb{F}_0(0,t_{i-1})\mathbb{F}_{i,1})(\mathbb{1})\varphi\rangle = \sum_{i \in I(\Theta)} \|V_{i,1}U(t_{i-1},0)\varphi\|^2 \tag{7.46}$$

$$\leq \|U(T,0)\varphi\|^2 - \|\varphi\|^2. \tag{7.47}$$

In the first inequality we used the contractivity of the map $(\mathbb{F}_{0,i} + \mathbb{F}_{1,i})(B) = V_i^* \mathbb{1} \otimes BV_i$. In the second we used the norm estimate from corollary 7.14 □

The following technical lemma ensures that the dilation has the desired behavior in first order.

Lemma 7.17. *For $t_{i-1} \leq t_i$ let V_i be the single step dilation from definition 7.1 corresponding to that time-step and let $\mathcal{M}(\sigma) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K} \otimes \mathcal{H})$ be the first-arrival time measure for the semigroup $\mathbb{F}_{0*}(s,t)$, generated by the exit space $\mathcal{K} \otimes H, L$ for $U(t,s)$. Then we have for $\psi \in \mathcal{D}$:*

$$\lim_{t_i \searrow t_{i-1}} \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|^2 \mathrm{d}s - \frac{1}{\tau_i}\|V_{1,i}\psi\|^2 = 0. \tag{7.48}$$

Proof. Inserting definitions we get:

$$\frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|^2 \mathrm{d}s - \left\| \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \mathbb{1} \otimes U(t_i,s)L(s)U(s,t_{i-1})\psi \mathrm{d}s \right\|^2. \tag{7.49}$$

A null addition of $\left(\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|\mathrm{d}s\right)^2$ splits this difference into two positive terms. The first is:

$$\frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|^2 \mathrm{d}s - \left(\frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|\mathrm{d}s\right)^2. \tag{7.50}$$

By lemma 7.12 this term is positive. It goes to zero in the limit $\tau_i \to 0$ because of the Fundamental theorem of calculus. The second term too is evidently positive, it looks like:

$$\left(\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|\mathrm{d}s\right)^2 - \left\|\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \mathbb{1}_{\mathcal{K}}\otimes U(t_i,s)L(s)U(s,t_{i-1})\psi\mathrm{d}s\right\|^2. \quad (7.51)$$

We bound it in the fashion $a^2 - b^2 \leq (2a)(a-b)$ for $a \geq b$. To get the convergence we exploit contractivity of $U(t,s)$ and replace the integrand in the second term by $U(T,s)L(s)U(s,t_{i-1})\psi$ for an arbitrary $T \geq t_i$. The fundamental theorem of calculus for the Bochner integral [34][II.2.Theorem 9] then shows, that the term is bounded by $\|L(t_i)\psi\| - \|\mathbb{1}_{\mathcal{K}} \otimes U(T)L(t_i)\psi)\|$. The free choice of T ensures the convergence. The boundedness of a is ensured by the following calculation:

$$\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|\mathrm{d}s \leq \frac{1}{\sqrt{\tau_i}}\sqrt{\int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|^2\mathrm{d}s} \quad (7.52)$$

$$\leq\sqrt{2\|\psi\|\frac{1}{\tau_i}\|U(t_i,t_{i-1})\psi-\psi\|}. \quad (7.53)$$

□

The importance of this lemma is clearer, when we transfer the equation to Schrödinger picture. There we have:

$$\lim_{t_i \searrow t_{i-1}} \frac{1}{\tau_i} \operatorname{tr}\left(\left(\mathcal{M}([t_{i-1},t_i]) - \mathbb{F}_{1,i*}\right)(|\psi\rangle\langle\psi|)\right) = 0. \quad (7.54)$$

That is, up to first order the one event evolution $\mathbb{F}_{1,i}$ captures the behavior of perturbation construction from section 4.3.

It is now a direct consequence that the behavior of the limit dilation $V_{[s,t]}$ is up to first order captured by the approximate single-step dilation. We do not strictly need the following result to proof theorem 7.5, since we can use results from section 4 to take a shortcut. Nonetheless the result is an important technical tool in the application of cMPS, since it often allows to replace the abstract limit dilation with the concretely given one-step dilation.

Corollary 7.18. *For $t_{i-1} \leq t_i$ let $V_{[t_{i-1},t_i]}$ be the limit dilation from definition 7.4 and V_i the single step dilation from definition 7.1 corresponding to the same time step. Then for $\psi \in \mathcal{D}$:*

$$\lim_{t_i \to t_{i-1}} \frac{1}{\tau_i}\|\left(V_{[t_{i-1},t_i]} - J_{\{t_{i-1},t_i\}}V_i\right)\psi\|^2 \quad (7.55)$$

Here $J_{\{t_{i-1},t_i\}} : \mathcal{K}\oplus\mathbb{C} \to \mathcal{K}_{[t_{i-1},t_i]}$ denotes the embedding of the one-step dilation space into the limit dilation space.

Proof. Let $\psi \in \mathcal{D}$. Since the difference $B \to V^*_{[t_{i-1},t_i]}\mathbb{1}_{\mathcal{K}_{[t_{i-1},t_i]}} \otimes BV_{[t_{i-1},t_i]} - V_i\mathbb{1}_{\mathcal{K}} \otimes BV_i$ is a completely-positive map by lemma 7.15, we only have to show:

$$\frac{1}{\tau_i}\left\langle\psi, \left(\left(V^*_{[t_{i-1},t_i]}V_{[t_{i-1},t_i]} - \mathbb{F}_0(t_{i-1},t_i)(\mathbb{1}_{\mathcal{H}}) - \mathbb{F}_{i,1}(\mathbb{1}_{\mathcal{H}})\right)\right)\psi\right\rangle. \quad (7.56)$$

Note that lemma 7.13 shows that:

$$\left\langle \psi, \left(V^*_{[t_{i-1},t_i]} V_{[t_{i-1},t_i]} - \mathbb{F}_0(t_{i-1},t_i)(\mathbb{1}_{\mathcal{H}}) \right) \psi \right\rangle \leq \int_0^T \langle \psi, \mathbb{F}_0(0,t)\Lambda_t(\mathbb{1})\psi, \rangle \, \mathrm{d}t \tag{7.57}$$

as this is true for every discretization V_Θ. Using positivity the corollary is then a direct consequence of lemma 7.17. □

Note that due to the order structure of the associated completely-positive maps, which we exploited in the proof of the convergence theorem 7.3, the dilations at any discretization level coincide to first order. Precisely we have that for any two interval decompositions $\Theta, \Xi \in \mathfrak{Z}([t_{i-1}, t_i])$ with $\Theta \leq \Xi$ we have :

$$\| \left(V_{[t_{i-1},t_i]} - J_{\{t_{i-1},t_i\}} V_i \right) \psi \|^2 \geq \| \left(V_\Xi - J_{\Xi\Theta} V_\Theta \right) \psi \|^2. \tag{7.58}$$

As already announced the limit dilation $V_{[s,t]}$ is a dilation of the minimal solution, i.e., bounded by any other solution so the Lindblad equation. This property is shared by the single-step solution.

Lemma 7.19. *For any completely-positive evolution system $\widetilde{\mathbb{E}}$, solving the Cauchy equation* (7.16)*, the map*

$$\widetilde{\mathbb{E}}(t_{i-1}, t_i) - \mathbb{F}_{0,i} - \mathbb{F}_{1,i} \tag{7.59}$$

is completely-positive.

Proof. The following proof is actually best stated in Schrödinger picture, using the theory developed in section 4. To keep these two sections as independent as possible we stick to the technically slightly less elegant Heisenberg picture.

Note that we can rephrase lemma 7.13 in the form:

$$\int_{t_{i-1}}^{t_i} \mathbb{F}_0(t_i, s) \circ \mathcal{J} \circ \mathbb{F}_0(s, t_i) \mathrm{d}s - \mathbb{F}_{1,i} \text{ is completely-positive.} \tag{7.60}$$

The integral is defined as a weak integral on functionals of the form $B \to \langle \psi, B\psi \rangle$ for $\psi \in \mathcal{D}$ and $B \in \mathfrak{B}(\mathcal{H})$. Since $\mathcal{D}$ is by assumption dense in $\mathcal{H}$ and the integral is bounded this definition extends to a bounded operator on $\mathfrak{B}(\mathcal{H})$. The boundedness of the integral is easily seen in the original form of the expression in equation (7.33).

Applying the integral form of the Cauchy equation twice, with an intermediate application of (7.60) we see that every such $\widetilde{\mathbb{E}}(s,t)$ bounds our solution as a completely-positive map.

$$\widetilde{\mathbb{E}}(t_{i-1}, t_i) - \mathbb{F}_{i,0} - \mathbb{F}_{i,1} \tag{7.61}$$

$$= \int_{t_{i-1}}^{t_i} \mathbb{F}_0(t_{i-1}, s) \Lambda \widetilde{\mathbb{E}}(s, t_i) \mathrm{d}s - \mathbb{F}_{i,1} \tag{7.62}$$

$$\overset{(7.60)}{\geq} \int_{t_{i-1}}^{t_i} \mathbb{F}_0(t_{i-1}, s) \Lambda (\widetilde{\mathbb{E}}(s, t_i) - \mathbb{F}_0(s, t_i)) \mathrm{d}s \tag{7.63}$$

$$= \int_{t_{i-1}}^{t_i} \mathbb{F}_0(t_{i-1}, s) \Lambda \left(\int_s^{t_i} \mathbb{F}_0(s,t) \Lambda \mathbb{F}(t, t_i) \mathrm{d}t \right) \mathrm{d}s. \tag{7.64}$$

The last expression is evidently completely-positive. To ensure that the expression is well defined the inner integral has to be defined as described above.

□

Notice that the preceding lemma allows to compare the current approach with the discussion in chapter 4. We note that in the Schrödinger picture equation (7.60) shows the complete positivity of:

$$\int_{t_{i-1}}^{t_i} \mathbb{F}_{0,*}(t_i, s) \circ \mathcal{M}_{t_{i-1}}(\mathrm{d}s) - \mathbb{F}_{1,i*} \tag{7.65}$$

On the other hand the lemma shows that the two regularizations of the reinsertion event coincide to "first order". That is in the current approach we approach the regularization from chapter 4 from below.

The missing step in out proof of theorem 7.5 is to show that the limit $V_{[s,t]}$ is a dilation of the minimal solution. In other words the infinitesimal behavior of the induced evolution coincides with that of a solution of the Cauchy equation. The following lemma asserts this property. It is again a direct consequence of the Cauchy equation and lemma 7.17.

Corollary 7.20. *For $t_{i-1} \leq t_i$ let V_i be the single step dilation from definition 7.1 corresponding to that time-step. Let $\widetilde{\mathbb{E}}(s,t)$ be any contractive solution of the Cauchy equation* (7.11)*, i.e. a possibly non minimal one. Then for $\psi \in \mathcal{D}$:*

$$\lim_{t_i \to t_{i-1}} \frac{1}{\tau_i} \left\langle \psi, \left(\widehat{\mathbb{E}}(t_{i-1}, t_i)(B) - (\mathbb{F}_{0,i} + \mathbb{F}_{1,i})(\mathcal{B})\right) \psi \right\rangle = 0. \tag{7.66}$$

Proof. We already know from lemma 7.19 that difference between $\widetilde{\mathbb{E}}(s,t)$ and the single step dilation is completely-positive. Hence it is sufficient if we evaluate it for $B = \mathbb{1}_{\mathcal{H}}$:

$$\frac{1}{\tau_i} \left\langle \psi, \left(\left(\widetilde{\mathbb{E}}(t_{i-1}, t_i) - \mathbb{F}_0(t_{i-1}, t_i) - \mathbb{F}_{i,1}\right)(\mathbb{1})\right) \psi \right\rangle \tag{7.67}$$

Applying the Cauchy equation (7.11) in integral form, see corollary 3.8, and using that $\widetilde{\mathbb{E}}(s,t)$ is contractive, we get:

$$\frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \mathrm{tr}\left(|\psi\rangle\langle\psi| \mathbb{F}(t_{i-1}, s) \circ \Lambda \circ \widetilde{\mathbb{E}}(s, t_i)(\mathbb{1})\right) - \frac{1}{\tau_i} \mathrm{tr}\left(|\psi\rangle\langle\psi| \mathbb{F}_1(\mathbb{1})\right) \tag{7.68}$$

$$\leq \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \|L(s)U(s,t_{i-1})\psi\|^2 \mathrm{d}s$$

$$- \left\| \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \mathbb{1} \otimes U(t_i, s) L(s) U(s, t_{i-1}) \psi \mathrm{d}s \right\|^2. \tag{7.69}$$

□

To proof theorem 7.5, we now just have to collect the preceding lemmas and corollaries.

Proof of Theorem 7.5. The weak-$*$ continuity of $\mathbb{E}(s,t)$ is follows immediately from lemma 7.16.

To see that $\widehat{\mathbb{E}}(s,t)$, as defined in definition 7.4, is indeed an evolution system in the sense that $\widehat{\mathbb{E}}(r,s)\widehat{\mathbb{E}}(s,t) = \widehat{\mathbb{E}}(r,t)$, we show that the limits $\widehat{\mathbb{E}}_\Lambda(B) := V_\Lambda^* \mathbb{1} \otimes BV_\Lambda$ and $\widehat{\mathbb{E}}_\Xi \widehat{\mathbb{E}}_\Theta B$, defined analogously, coincide. To see that keep in mind that $\widehat{\mathbb{E}}(s,t)$ is a normal map and hence has a well defined pre-adjoint on the space of trace class operators. For $\Lambda \in \mathfrak{Z}(r,t)$, $\Theta \in \mathfrak{Z}([s,t])$ and $\Xi \in \mathfrak{Z}(r,s)$ we have:

$$
\begin{aligned}
&\rho \circ (\widehat{\mathbb{E}}(r,t) - \widehat{\mathbb{E}}(r,s)\widehat{\mathbb{E}}(s,t))(X) && (7.70)\\
\leq& \rho \circ (\widehat{\mathbb{E}}(r,t) - \widehat{\mathbb{E}}_\Lambda)(X)\\
&\quad + \rho \circ \widehat{\mathbb{E}}(r,s) \circ (\widehat{\mathbb{E}}(s,t) - \widehat{\mathbb{E}}_\Theta)(X)\\
&\quad\quad + \rho \circ (\widehat{\mathbb{E}}(r,s) - \widehat{\mathbb{E}}_\Xi) \circ \widehat{\mathbb{E}}_\Theta(s,t)(X)) && (7.71)
\end{aligned}
$$

The result now follows via a standart 3ϵ argument.

We already know that a minimal solution of the Cauchy equation exists by theorem 4.16. By corollary 7.20 we know that $\widehat{\mathbb{E}}(s,t)$ has the same infinitesimal behavior as any solution, i.e. it is a solution itself. And from lemma 7.19 it follows that the net $\mathbb{E}_\Theta$ approaches every solution from below, i.e. $\widehat{\mathbb{E}}(s,t)$ is a minimal solution.

□

7.5. Proof: bounded case

In the case of a bounded Lindbladian we can obviously get far better norm estimates than in the unbounded case. In particular the contribution of event term is small in norm, this later allows us to ignore terms with more than one arrival.

Corollary 7.21. *Under the assumptions 2 and with the notation from definitions 7.1 and 7.11 we have:*

$$\|\mathbb{F}_{1,i}\| = \|V_{1,i}\|^2 \leq 2C\tau_i. \tag{7.72}$$

A straight forward calculation using lemma 7.12 shows the desired result.

Proof.

$$
\begin{aligned}
\|V_{1,i}\|^2 &\leq \int_{t_{i-1}}^{t_i} \|U(t_i,s)L(s)U(s,t_{i-1})\|^2 \,\mathrm{d}s && (7.73)\\
&\leq \int_{t_{i-1}}^{t_i} \|L(s)\|^2 \,\mathrm{d}s \leq \int_{t_{i-1}}^{t_i} \|K(s) + K^*(s)\| \mathrm{d}s. && (7.74)
\end{aligned}
$$

□

We need the following definition to separate contributions to the limit, containing maximally one event per subinterval from those with more events.

Definition 7.22. Let $\Theta \subset \Xi \in \mathfrak{Z}([0,T])$. For every $i \in I(\Theta)$ let $P_{0,i}$ and $P_{1,i}$ be the projectors on the zero and one eigenspace of the number operator restricted to the interval $i \in I(\Theta)$, i.e. $P_{0,i} = |0\rangle\langle 0|$ and if $\alpha \in \mathbb{A}$ labels a basis of $\mathcal{K}$ $P_{1,i} = \sum_{\alpha\in\mathbb{A}} \sum_{j\in I(\Xi|i)} |\alpha@j\rangle\langle\alpha@j|$

$$P_{1,i} = \sum_{j\in I(\Xi|i)} \left(\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{\mathcal{K}} \end{pmatrix} @j \right). \tag{7.75}$$

We then denote the projector onto the space of at most one event per subinterval $i \in I(\Theta)$ as $P_{\mathcal{S}\Theta} \in \mathfrak{B}(\mathcal{K}_\Xi)$. It is defined as:

$$P_{\mathcal{S}\Theta} = \bigotimes_{i\in I(\Theta)} (P_{0,i} + P_{1,i}) \tag{7.76}$$

Remember that $P_\Theta = J_{\Xi\Theta} J^*_{\Xi\Theta}$ denotes the projector onto $\mathcal{K}_\Theta$ considered as a subspace of $\mathcal{K}_\Xi$. It is easy to see that $P_{\mathcal{S}\Theta} \geq P_\Theta$. That is because the $\mathcal{K}_\Theta$ as a subspace of $\mathcal{K}_\Xi$ contains only events which are averaged over the subintervals $i \in I(\Theta)$.

With this definition we now get a clean separation of the different contributions.

Corollary 7.23. *With the definitions given above we have that:*

$$\|\mathbb{E}_\Xi - \mathbb{E}_\Theta\|_{cb} = \|\mathbb{E}_\Xi((\mathbb{1} - P_{\mathcal{S}\Theta}) \otimes \mathbb{1})\| + \|\mathbb{E}_\Xi((P_{\mathcal{S}\Theta} - P_\Theta) \otimes \mathbb{1})\|. \tag{7.77}$$

Proof. This is a simple consequence of the order structure of the maps $\mathbb{E}_\Xi$ we saw in lemma 7.15.

$$\|\mathbb{E}_\Xi - \mathbb{E}_\Theta\|_{\mathrm{cb}} = \|(\mathbb{E}_\Xi((\mathbb{1} - P_\Theta) \otimes \mathbb{1})\| \tag{7.78}$$
$$= \|\mathbb{E}_\Xi(((\mathbb{1} - P_{\mathcal{S}}) + (P_{\mathcal{S}} - P_\Theta)) \otimes \mathbb{1})\| \tag{7.79}$$
$$= \|((\mathbb{1} - P_{\mathcal{S}}) + (P_{\mathcal{S}} - P_\Theta)) \otimes \mathbb{1})V_\Xi\|^2 \tag{7.80}$$
$$= \|\mathbb{E}_\Xi((\mathbb{1} - P_{\mathcal{S}}) \otimes \mathbb{1})\| + \|\mathbb{E}_\Xi((P_{\mathcal{S}} - P_\Theta) \otimes \mathbb{1})\|. \tag{7.81}$$

□

We now separately evaluate the two expressions on the right side of the equation.

The first one vanishes in the limit because, the occurrence of more than one event in a short subinterval becomes negligible.

Corollary 7.24. *With the definitions given above we have that:*

$$\|\mathbb{E}_\Xi((\mathbb{1} - P_{\mathcal{S}\Theta}) \otimes \mathbb{1})\| \leq \sum_{i\in I(\Xi)} 2C^2\tau_i^2. \tag{7.82}$$

Proof. Denote $\mathbb{E}_{\Xi|i\mathcal{S}}(X) := \mathbb{E}_{\Xi|i}((P_0 + P_1) \otimes X)$. We evidently have:

$$\|\mathbb{E}_\Xi((\mathbb{1} - P_{\mathcal{S}\Theta}) \otimes \mathbb{1})\| = \left\| \prod_{i\in I(\Xi)} \mathbb{E}_{\Xi|i} - \prod_{i\in I(\Xi)} \mathbb{E}_{\Xi|i\mathcal{S}} \right\|. \tag{7.83}$$

Telescoping the product and using the contractivity of the maps $\mathbb{E}_{\Xi|i}$ for $i \in I(\Theta)$ we get the following bound.

$$\sum_{i\in I(\Theta)} \left\|\mathbb{E}_{\Xi|i} - \mathbb{E}_{\Xi|i\mathcal{S}}\right\|. \tag{7.84}$$

Observe that:

$$\mathbb{E}_{\Xi|i\mathcal{S}} = \prod_{j\in I(\Xi|i)} \mathbb{F}_{0,j} + \sum_{j\in I(\Xi|i)} \prod_{j>k\in I(\Xi|i)} \mathbb{F}_{0,k}\mathbb{F}_{1,j} \prod_{j<k\in I(\Xi|i)} \mathbb{F}_{0,k} \tag{7.85}$$

$$\mathbb{E}_{\Xi|i} = \prod_{j\in I(\Xi|i)} \mathbb{F}_{0,j} + \mathbb{F}_{1,j}. \tag{7.86}$$

Telescoping the difference twice we get:

$$\sum_{\underline{j}<\underline{k}\in I(\Xi|i)} \prod_{\underline{l}<j\in I(\Xi|i)} \mathbb{F}_{0,j}\mathbb{F}_{1,j} \prod_{j<\underline{l}<k\in I(\Xi|i)} \mathbb{F}_{0,l}\mathbb{F}_{1,k} \prod_{k<\underline{l}\in I(\Xi|i)} (\mathbb{F}_{0,l}+\mathbb{F}_{1,l})\,. \tag{7.87}$$

Using that all terms are contractive and $\mathbb{F}_{1,j} \leq 2C\tau_j$ an application of the submultiplicativity of the norm yields the desired result. □

The other term in corollary 7.23 compares the slightly different behavior of the two dilations for the case of one event per coarse subinterval. The two terms mainly differ in the way the occurrence of the event is averaged.

Corollary 7.25. *We have the following simple estimate.*

$$\|\mathbb{E}_\Xi((P_{\mathcal{S}\Theta} - P_\Theta)\otimes \mathbb{1})\| \tag{7.88}$$

$$\leq 2\sum_{i\in I(\Theta)} \tau_i \left(2\tau_i C(C+1) + \sup_{t_{i-1}\leq s\leq t_i} \|L(s)-L(t_i)\|\right). \tag{7.89}$$

Proof. As a first step we again telescope over the subintervals $i \in I(\Theta)$ and use the contractivity to arrive at the following bound. In the last step we inserted definitions and used that vectors $|\varphi@j\rangle$ and $|\psi@k\rangle$ are orthogonal for $j \neq k$

$$\|\mathbb{E}_\Xi((P_{\mathcal{S}\Theta} - P_\Theta)\otimes \mathbb{1})\| \tag{7.90}$$

$$\leq \sum_{i\in I(\Theta)} \|\mathbb{E}_{\Xi|i}((P_{0,i}+P_{1,i}-P_\Theta)\otimes \mathbb{1})\| \tag{7.91}$$

$$= \sum_{i\in I(\Theta)}\sum_{j\in I(\Xi|i)} \tau_j \left\| \left(\frac{1}{\sqrt{\tau_j}}\mathbb{1}_{\mathcal{K}}\otimes U(t_i,t_j)V_{j,1}U(t_{j-1},t_{i-1}) - \frac{1}{\sqrt{\tau_i}}V_{i,1}\right)\varphi\right\|^2.$$

To get an estimate on these terms we compare separately for each $i \in I(\Theta)$ the subtrahend and minuend to $L(t_i)$. Inserting definitions we get:

$$\left\|\frac{1}{\sqrt{\tau_i}}V_{i,1} - L(t_i)\right\| \tag{7.92}$$

$$\leq \left\|\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} U(t_i,s)L(s)U(s,t_{i-1}) - L(t_i)\mathrm{d}s\right\| \tag{7.93}$$

$$\leq 2\tau_i C(C+1) + \sup_{t_{i-1}\leq s\leq t_i}\|L(s)-L(t_i)\|. \tag{7.94}$$

In the last step we inserted some straight forward bounds, i.e. $\|U(t_i,t_{i-1}) - \mathbb{1}\| \leq \tau_i C$ and $\|L(s)\| \leq C+1$. The second term can be estimated in exactly the same manner. We arrive at the desired bound. □

Piecing together the previous corollaries we get a proof of the main theorem on cMPS with bounded generator.

Proof of theorem 7.6. From the above estimates it is clear that the construction converges. If we additionally assume that the function $t \to L(t)$ is Lipschitz continuous with constant D we have that:

$$\|\mathbb{E}_\Xi - \mathbb{E}_\Theta\|_{\mathrm{cb}} \leq 2\left(C^2 + 2C(C+1) + D\right) \sum_{i \in I(\Xi)} \tau_i^2 \tag{7.95}$$

The rest of the theorem is a direct consequence of the unbounded case and the boundedness of the generator $\mathcal{L}$. □

8. Continuous Measurements

In this chapter we take a look at operators on the limit space $\mathcal{K}_{[s,t]}$, which, as we established in section 6.4 can be used to describe the information an open quantum system emits to the environment in continuous time. In other words such operators describe continuous time measurements.

We are particularly interested to us operators on $\mathcal{K}_{[s,t]}$ to describe counting statistics of measurements which can be described by cMPS, see section 7.2. That is we want adapt out treatment of observables to continuous measurement description we worked out in section 7. In this sense this chapter completes our description of continuous time measurement in the sense of figure 8.1. The following treatment builds on the discussion of operators between inductive limit spaces in section 2.1.5.

Since we already identified $\mathcal{K}_{[s,t]}$ as being unitarily equivalent to Bose-Fock space, it is clear that Weyl operators will play a major role in the coming discussion. It turns out that they are intimately related to the Lindblad form, because they also arise naturally as a representation of the gauge freedom of the Lindblad equation. Also their generators the Bose field operators serve as a connection to the theory of arrival time measures discussed in section 4.

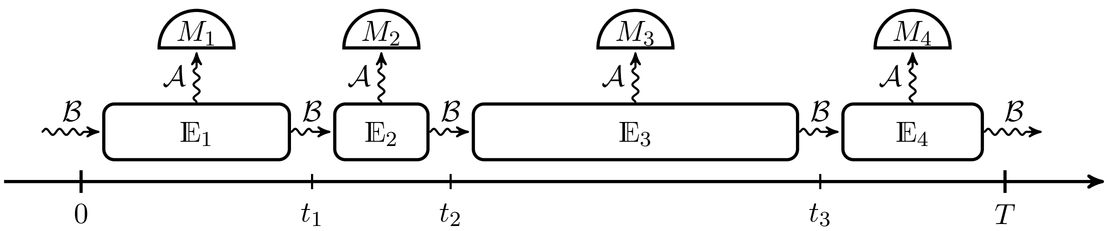


Figure 8.1.: A discrete time evolution with a compatible measurement

As $\mathcal{K}_{[s,t]}$ is also unitarily equivalent to Fermi-Fock space we of course also comment briefly on this case. It should however be clear that the main focus is on the Bose-Fock space. Nonetheless our main results are naturally independent of the "interpretation" of $\mathcal{K}_{[s,t]}$ either as a Fermi- or Bose- Fock space.

Throughout this section we shall stick to the principle of working in the discrete picture as much as possible.

8.1. Main results

Main results of this section are the construction and discussion of Weyl operators on the one hand. On the other hand we reduce the calculation of expectation values of arbitrary operators on $\mathcal{K}_{[s,t]}$ under cMPS to the calculation evolution systems on the system Hilbert space $\mathcal{H}$ alone. This greatly simplifies calculations and generalizes of one of the defining properties of FCS. This second result can also be interpreted as a Levy-Khinchin formula.

8.1.1. Notation

One can view any Bose-Fock space as an irreducible representation of the canonical commutation relations in Weyl form. Furthermore the exponential property of Fock space allows us to write $\mathfrak{B}\left(\Gamma_+(\mathrm{L}^2([0,T]))\right)$ as a tensor product as $\mathrm{L}^2([0,T])$ can naturally be written as a direct sum over subintervals, i.e. for $i \in \mathfrak{Z}([0,T])$ we have:

$$\mathrm{L}^2([0,T]) = \bigoplus_{i \in I(\Theta)} \mathrm{L}^2([t_{i-1}, t_i]) \tag{8.1}$$

$$\Gamma_+\left(\mathrm{L}^2([0,T])\right) = \bigotimes_{i \in I(\Theta)} \Gamma_+\left(\mathrm{L}^2([t_{i-1}, t_i])\right). \tag{8.2}$$

The Weyl operators are compatible with this "exponential structure" in the sense that the following operators are mapped to each other by the canonical unitary equivalence underlying the last equivalence. In other words let $\lambda_{[0,T]} \in \mathrm{L}^2([0,T], \mathbb{C})$ and $\lambda_{[s,t]}$ be its restriction to the interval $[s,t] \subset [0,T]$, then:

$$W(\lambda_{[0,T]}) \in \mathfrak{B}\left(\Gamma_+(\mathrm{L}([0,T]))\right) \tag{8.3}$$

$$\simeq \bigotimes_{i \in I(\Theta)} W(\lambda_{[t_{i-1},t_i]}) \in \bigotimes_{i \in I(\Theta)} \mathfrak{B}\left(\Gamma_+(\mathrm{L}([t_{i-1}, t_i]))\right). \tag{8.4}$$

Note that this is basically the quasi-local structure of the CCR-algebra over $\mathrm{L}^2(\mathbb{R}, \mathbb{C})$ [15], which we needed to define "infinitely extended cMPS in section 7.2.2. In the following discussion we shall concentrate on operators of the form $\bigotimes_{i \in I(\Theta)} O_i$ for $O_i \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K})$. As we shall see these operators lead to a natural analogue of Weyl operators for the discrete case.

8.1.2. Weyl operators

We start by collecting our results on the construction of Weyl operators $W(\lambda)$ on $\mathcal{K}_{[0,T]}$ as the limit of a net of discrete "Weyl operators" $W_\Theta(\lambda)$ on $\mathcal{K}_\Theta$. Only later, in section 8.2, we motivate our construction of $W_\Theta(\lambda)$ by showing its intimate connections with the gauge invariance of bounded Lindblad generators. The proof of the results are collected in section 8.4.

The importance of Weyl operators is clear anyway as they are "characteristic" functions of field operators, i.e. they can be used to calculate important expectation values of the field. A discrete approximation for Weyl operator then allows to reliably approximate such quantities. Solid understanding of the limit then allows to understand how well certain expectation values can be approximated in the discrete picture.

Our construction of W_Θ is independent of the explicit structure of the space $\mathcal{K}$. In particular $\mathcal{K}$ could be finite dimensional. Hence it is already clear that the operators $W_\Theta(\lambda)$ can not satisfy the canonical commutation relations, since this is impossible on finite dimensional spaces. Depending on the dimension of $\mathcal{K}$ the space $\mathcal{K}_\Theta$ could be finite dimensional.

We can not expect to find discrete approximations in operator norm. Instead we focus on strong convergence, which we are able to proof in most cases.

Our discrete Weyl operators are basically a series of "infinitesimal" rotations of each of the spaces $\mathbb{C} \oplus \mathcal{K}$ constituting a discrete dilation space K_Θ. These rotations are "perpendicular" to the partitioning of the one-step dilation space into event and no-event part, i.e. they mix event and no-event part. Hence we have to be very careful with their scaling.

Definition 8.1. Let $\lambda \in \mathcal{K}$. We define the following operator:

$$\Phi(\sqrt{\tau}\lambda) \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K}) \qquad \Phi(\sqrt{\tau}\lambda) := \begin{pmatrix} 0 & \sqrt{\tau}\langle\lambda| \\ -\sqrt{\tau}|\lambda\rangle & 0 \end{pmatrix}. \tag{8.5}$$

Let $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ and $\Theta \in \mathfrak{Z}([0,T])$. For $i \in I(\Theta)$ set $\lambda_i := \lambda(t_i)$, as usual $\tau_i = t_i - t_{i-1}$ for $t_i \in \mathfrak{Z}([0,T])$. We define the discrete *Weyl operator* for λ to be:

$$W_\Theta(\lambda) := \bigotimes_{i \in I(\Theta)} \exp(\Phi(\sqrt{\tau_i}\lambda_i)) \tag{8.6}$$

$$= \bigotimes_{i \in I(\Theta)} \begin{pmatrix} \cos(\theta) & \sin(\theta)\langle e_\lambda| \\ -\sin(\theta)|e_\lambda\rangle & \mathbb{1} + (\cos(\theta) - 1)P_\lambda \end{pmatrix}, \tag{8.7}$$

where e_Θ is the unit vector in direction λ, P_λ is the projection onto λ and the scaling is given by $\theta_i = \sqrt{\tau_i}\|\lambda\|$

The last equation is obvious when we go to a basis including $|0\rangle$ and $|\lambda\rangle$. It is clear that the operators $W_\Theta(\lambda)$ do not have the right commutation relations/group structure. However, observe that $t \mapsto W_\Theta(t\lambda)$ is clearly a unitary one-parameter group, since $i\Phi(\sqrt{\tau}\lambda)$ is a self-adjoint operator. These properties are conserved throughout the limit process.

Sometimes the Weyl operators on $\Gamma_+(\mathrm{L}([0,T],\mathcal{K})$ are extended to also give a representation of the unitary group on $\mathcal{K}$. Such a representation is quite easy to obtain. Deviating slightly from the standard approach we include time dependent unitaries. This allows to describe characteristic functions of "time-dependent" measurements.

Definition 8.2. Let $U : [0,T] \to \mathfrak{B}(\mathcal{K})$ be an operator valued function : Given $\Theta \in \mathfrak{Z}([0,T])$ we set:

$$U_i \in \mathfrak{B}(\mathcal{K}) \qquad U_i := U(t_i) \tag{8.8}$$

$$U_\Theta \in \mathfrak{B}(\mathcal{K}_\Theta) \qquad U_\Theta := \bigotimes_{i \in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & U_i \end{pmatrix}. \tag{8.9}$$

We now have to show, that the two definitions 8.1 and 8.2 actually do what they are supposed to do. This is they converge to a representation of the CCR algebra on $\mathcal{K}_{[0,T]}$. That is the content of the following series of theorems and definitions.

Theorem 8.3. *Let $U : [0,T] \to \mathfrak{B}(\mathcal{K})$ be an (unitary) operator valued function continuous in the strong topology and let $U_\Theta \in \mathfrak{B}(\mathcal{K}_\Theta)$ be as in definition 8.2 then the net $\Theta \mapsto J_\Theta U_\Theta J_\Theta^* \in \mathfrak{B}(\mathcal{K}_{[0,T]})$ for $\Theta \in \mathfrak{Z}([0,T])$ converges in the strong topology to a unitary operator, i.e. for any convergent net $\Theta \mapsto \varphi_\Theta \in \mathcal{K}_\Theta$ and every $\epsilon > 0$ there is a $\Theta_\epsilon \in \mathfrak{Z}([0,T])$ such that: $\|U_\Lambda J_{\Lambda\Xi}\varphi_\Xi - J_{\Lambda\Xi}U_\Xi\varphi_\Xi\| \leq \epsilon$ for all $\Theta_\epsilon \leq \Xi \leq \Lambda \in \mathfrak{Z}([0,T])$.*

Let $\lambda : [0,T] \to \mathcal{K}$ be a continuous function and $W_\Theta(\lambda)$ as in definition 8.1 then the net $\Theta \mapsto J_\Theta W_\Theta(\lambda) J_\Theta^*$ converges in the strong topology to a unitary operator.

This allows us to define operators on the limit space $\mathcal{K}_{[0,T]}$, which turn out to be the Weyl operators.

Definition 8.4. Let $U : [0,T] \to \mathfrak{B}(\mathcal{K})$ be an unitary operator valued function continuous in the strong topology and let $\lambda : [0,T] \to \mathcal{K}$ be a continuous function. We define:

$$W(\lambda) \in \mathfrak{Z}([0,T]) \qquad W(\lambda) = \operatorname*{s-lim}_{\Theta\in\mathfrak{Z}([0,T])} W_\Theta \tag{8.10}$$

$$W(\lambda, U) \in \mathfrak{Z}([0,T]) \qquad W(\lambda, U) = W(\lambda) \operatorname*{s-lim}_{\Theta\in\mathfrak{Z}([0,T])} U_\Theta \tag{8.11}$$

$$\tag{8.12}$$

For an arbitrary $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ we define $W(\lambda) := \text{s-lim}_{n\to\infty} W(\lambda_n)$ where λ_n is a sequence of continuous functions converging to λ.

The last part of the definition is justified only later in corollary 8.24. It allows to define Weyl operators for arbitrary functions in $\mathrm{L}^2([0,T])$. These operators still behave as expected, i.e. fulfill the Weyl form of the canonical commutation relations.

Theorem 8.5. *In the situation of definition 8.4 and with the function $\lambda \in \mathrm{L}^2([0,T])$ not necessarily continuous the operator $W(\lambda,U)$ defined above is unitary and fulfills the Weyl commutation relations. That is for $\lambda, \mu \in \mathrm{L}^2([0,T],\mathcal{K})$ and U_1, U_2 strongly-continuous and unitary operator valued as in the previous definition we have:*

$$W(\lambda, U_1)W(\mu, U_2) = \exp\left(-\mathfrak{Im}\,\langle\lambda, U_1\mu\rangle\right) W((\lambda + U_2\mu), U_1U_2) \tag{8.13}$$

Furthermore for every λ the function $t \mapsto W(t\lambda)$ for $t \in \mathbb{R}$ is a strongly-continuous one parameter group of unitary operators.

8.1.3. Expectation values

Expectation values of cMPS have the important property that they satisfy a basic version of the Ito calculus of quantum stochastic calculus [72]. In our case the statement is, that expectations of operators which lead to a semigroup, need only be calculated to “first” order. A comparison to the quantum stochastic calculus case shows that we exactly reproduce their Ito table.

We shall mainly use the following result to derive expressions for the calculation of expectation values of Weyl operators on the Hilbert space $\mathcal{H}$ alone. The operators which can be calculated by the following lemma are best thought of as characteristic functions of stochastic processes. We further elucidate these connections in section 8.3.1.

Lemma 8.6. *Let $W \in \mathfrak{B}(\mathcal{K})$ be an operator, s.t. for any $\Theta \in \mathfrak{Z}([0,T])$ and up to terms of higher order in $\sqrt{\tau_i}$ we can write:*

$$J_\Theta^* W J_\Theta = \begin{pmatrix} 1 + \tau_i c(t_i) & \sqrt{\tau_i}\langle\lambda_1(t_i)| \\ \sqrt{\tau_i}|\lambda_2(t_i)\rangle & O(t_i) + \mathbb{1} \end{pmatrix}. \tag{8.14}$$

with functions $c \in \mathrm{L}^1([0,T],\mathcal{K})$, $\lambda_1, \lambda_2 \in \mathrm{L}^2([0,T],\mathcal{K})$ and $O \in \mathrm{L}^1([0,T],\mathfrak{B}(\mathcal{K}))$.

Under the assumptions 1 on page 115 let $\mathbb{E}_{[0,T]} : \mathfrak{B}(\mathcal{K}_{[0,T]} \otimes \mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a cMPS solving the Cauchy equation with generator:

$$\langle\psi, \mathcal{L}(t)(B)\varphi\rangle = \langle K(s)\psi, B\varphi\rangle + \langle\psi, BK(s)\varphi\rangle + \langle L(s)\psi, \mathbb{1}_\mathcal{K} \otimes BL(s)\varphi\rangle \tag{8.15}$$

Then $\mathbb{E}_W(s,t)(X) := \mathbb{E}_{[s,t]}(W \otimes X)$ *defines an weak-∗-continuous evolution system solving the Cauchy equation with the following perturbed Lindbladian:*

$$\left.\frac{\partial}{\partial t}\mathbb{E}_{s,t}\right|_{t=s} = \langle \psi, (\mathcal{L}(t) + c(t)\mathrm{id})(X)\psi\rangle + \langle L_t\psi, O(t) \otimes X L_t \psi\rangle \\ + \langle \lambda_1 \otimes \psi, \mathbb{1} \otimes X L\psi\rangle + \langle L\psi, \mathbb{1} \otimes X\lambda_2 \otimes \psi\rangle \,, \qquad (8.16)$$

for $\psi \in \mathcal{D}$

The proof of this lemma can be found in section 8.4. Observe that the operator valued function O is not assumed to map into the unitary operators. This allows us to describe expectation values of general POVM in addition to expectation values of projection valued measures, which would correspond to to projection valued measures.

In the case that the Lindbladian, as well as the perturbed generator are time-independent, the preceding lemma is basically some kind of Levy-Khintchine formula in disguise.

Often one is interested only in low order moments of the characteristic function of a stochastic process rather than in the full counting statistics. Moments of the probability distribution are differentials of the characteristic function and hence correspond to expectation values of the "generators".

An example is the case where $W = W(\lambda)$ is a Weyl operator. One would then be interested in the behavior of the family of evolution systems $\mathbb{E}_{W(\lambda)}(s,t)$ as a function of $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$. Directional derivatives then correspond to expectation values of the smoothed field operators.

Since the generators of the Weyl operators are are unbounded operators, the technical treatment of products of generators becomes quite complicated when treated via their "canonical" discretization as in definition 8.9.

Alternatively one can calculate low order moments of the characteristic function by using "perturbation theory" of differential equations as demonstrated for the example of quasi free semigroups in section 9.2.4.

The formulas we obtain by this method seem to be consistent with those one would obtain when taking limits of products of discretized field operators as in definition 8.8. However in both cases it seems to be difficult to state conditions under which the necessary objects exist.

8.2. Weyl operators and gauge symmetry

To better understand the occurrence of Weyl operators in the continuous limit and their relation to it, we examine again the gauge invariance of the Lindblad equation, which was already discussed in section 3.3. Now our focus lies on understanding how this gauge freedom "interacts" with the continuous Stinespring dilation.

From an abstract viewpoint it is quite clear that the invariance of the Lindblad generator under some group should lead to a representation of this group on the dilation space. This would be an analogue of point two of the Stinespring theorem 2.18. That is we can connect "equivalent" dilations through unitary operators. Holevo already studied the gauge freedom of the Lindblad equation in [53]. The following discussion, should also extend to further symmetries of the Lindbladian.

Let us shortly remind ourselves of the most important properties of the gauge group of the Lindblad generator, which we already discussed in section 3.3.

Reminder 6. Let $\mathcal{L}(t)$ be a Lindblad generator, i.e. there exists separable Hilbert spaces $\mathcal{H}, \mathcal{K}$ and two families of unbounded operators

$$K(t) : \mathrm{dom}(K(t)) \to \mathcal{H} \tag{8.17}$$
$$L(t) : \mathrm{dom}(K(t)) \to \mathcal{K} \otimes \mathcal{H} \tag{8.18}$$

fulfilling the assumptions 1. Furthermore let for each t be $(U(t), \lambda(t), x(t))$ a gauge triple for $\mathcal{L}(t)$. As usual we denote:

$$\widetilde{L}(t) = U(t) \otimes \mathbb{1}_{\mathcal{H}} L(t) + |\lambda(t)\rangle \otimes \mathbb{1}_{\mathcal{H}} \tag{8.19}$$
$$\widetilde{K}(t) = K(t) - \frac{1}{2}\|\lambda(t)\|^2 - (\langle U(t)^* \lambda| \otimes \mathbb{1})L(t) + ix(t)\mathbb{1}_{\mathcal{H}} \tag{8.20}$$

If we denote by (K, L) the tuple of operators constituting a Lindblad equation and by $g(\lambda(t), U(t), x(t)$ the transformation $(K(t), L(t)) \mapsto (\widetilde{K}(t), \widetilde{L}(t))$. Then the family of such transformations has the following group law:

$$g(\lambda_2, U_2, x_2) g(\lambda_1, U_1, x_1) = g(\lambda_2 + U_2\lambda_1, U_2 U_1, x_1 + x_2 - \mathfrak{Im}\,\langle \lambda_2 | U_2 \lambda_1 \rangle) \tag{8.21}$$

We are now going to try to unitarily represent this gauge group on the dilation space $\mathcal{K}_\Theta$. A "time-slice-wise" representation of the subgroup $g(0, U, 0)$ on $\mathcal{K}_\Theta$ is easy to find. It is clear that if we equip $L^2([0,T], \mathfrak{B}(\mathcal{K}))$ with the point-wise multiplication then $U \mapsto W_\Theta(0, U_\Theta)$ is a representation of $L^2([0,T], \mathfrak{B}(\mathcal{K}))$ on $\mathcal{K}_\Theta$. The limit properties of this representation were described in theorem 8.3.

The subgroup generated by elements of the form $g(\lambda, \mathbb{1}, 0)$ is harder to represent, as this requires an infinite dimensional dilation space. This is the case, because the subgroup $g(t\lambda, \mathbb{1}, 0)$ for $t \in \mathbb{R}$ should only act on a two dimensional space. Which is a consequence of treating all possible dilation spaces $\mathcal{K}$ on the same basis, i.e. we want to be able to restrict to a single Lindblad operator without changing the nature of the representation. However a unitary representation of this subgroup would be a set of Weyl operators. Hence it is clear that we can get a representation only in the limit, where the dilation space becomes infinite dimensional, and that the restriction to a time slice diverges.

A short and easy " calculation" shows that up to "relevant order" the operator $\exp(\Phi(\lambda))$ implements the gauge shift on a short time slice.

Theorem 8.7. *Let $\mathcal{L}(t)$ be a bounded Lindblad generator, i.e. there exists Hilbert spaces $\mathcal{H}, \mathcal{K}$ and two families of operators $K(t) \in \mathfrak{B}(\mathcal{H})$ and $K(t)\mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ fulfilling the assumptions 2. Furthermore let (U, λ, x) be a continuously differentiable gauge for $\mathcal{L}(t)$. As usual we denote:*

$$\widetilde{L}(t) = U(t) \otimes \mathbb{1}_{\mathcal{H}} L(t) + |\lambda(t)\rangle \otimes \mathbb{1}_{\mathcal{H}} \tag{8.22}$$
$$\widetilde{K}(t) = K(t) - \frac{1}{2}\|\lambda(t)\|^2 - (\langle U(t)^* \lambda| \otimes \mathbb{1})L(t) + ix(t)\mathbb{1}_{\mathcal{H}}. \tag{8.23}$$

Iff $V_{[s,t]}$ is the continuous Stinespring dilation generated by the pair (K, L) and $\widetilde{V}_{[s,t]}$ the dilation generated by the pair $(\widetilde{K}, \widetilde{L})$ then:

$$\widetilde{V}_{[s,t]} = \exp\left(i \int_s^t x(t) \mathrm{d}t \right) W(\lambda, U) \otimes \mathbb{1}_{\mathcal{H}} V_{[s,t]}. \tag{8.24}$$

The easiest way to show this result is to use the alternative definition of the one step dilations in equation (7.19) and (7.20). The theorem is then a direct consequence of lemma 7.7.

8.3. Applications

We spend the rest of this section applying lemma 8.6. While doing that we try to profit as much as possible from the limit structure of all our constructions. That is we try to relate "easy" to understand discrete constructions with their continuous counterparts.

A nice example of a discrete construction which is "compatible" with our limit for $\mathcal{K}_{[0,T]}$ is the second quantization of POVMs. We shall see how essentially the same discrete "second quantiation" leads to quite different interpretations for different classes of POVMs. Constructions for the second quantization of POVMs can be found in the literature, e.g. a short introduction can be found in the beginning of [61]. One can of course use these constructions and the equivalence of $\mathcal{K}_{[0,T]}$ with Bose- and Fermi- Fock space to describe field observables.

We are however interested in describing those observables which can be efficiently described using cMPS and which are in their structure compatible with our discretization. Loosely speaking, these are "beam-type" measurements, i.e. all the measurements where we shine a beam onto detector array. The arising problem of a description of possibly infinitely extended beams, i.e. measurements over arbitrarily long times and stationary beams, is moreover elegantly solved by the underlying evolution system/semigroup structure, as described in section 7.2.

The beam type measurements fall in two different classes. Firstly "point-process" type measurements, i.e. those where the outcome of a measurement is a (finite) set of arrival times and associated measurement outcomes. In other words, measurements where the outcome is a number of distinct points in a set. An example is a measurement which counts all arriving photons with a specific polarization and wave length, such that the detector only fires "once in a while". These measurements should be thought of generalizations of a Poisson process. We describe this case in section 8.3.3.

The second class of measurements are those with a "continuous path" as outcome, e.g. the continuous measurements of phase fluctuations with respect to a fixed reference field, i.e. homodyne detection. The outcome of such measurements should be thought of as a generalization of Brownian motion. These measurements require a bit more care in the discrete to continuous limit, since one has to be careful with the scaling. We describe this case in section 8.3.2.

The distinction between those two possible contributions to a measurement is essentially an analogue of the Levy-Khintchine representation for classical Levy-processes.

8.3.1. Discrete picture

The basic idea of discrete "beam-type" observables is that we measure the same, or rather a predetermined, observable in every time step, i.e. on every "chunk" of field arriving.

In terms of our continuous limit construction for the quantum field we can approximate the field in every time step as consisting of zero or one particle. That is of the form $\mathcal{K}_\Theta$ for $\Theta \in \mathfrak{Z}([0,T])$ where the measurement goes over the time interval $[0,T]$ and Θ fixes the field discretization.

A POVM describing a measurement in exactly one "time-slice", i.e. sub-interval $i \in I(\Theta)$, and ignoring the outcomes in the other subintervals clearly has the form $(A_i(x)@i)$ where $x \mapsto A_i(x)$ is a POVM in $\mathfrak{B}(\mathbb{C} \otimes \mathcal{K})$. Here we used the notation for operators on $\mathcal{K}_{[0,T]}$ introduced in definition 6.15. Sums of products of such operators clearly capture the whole observable algebra $\mathfrak{B}(\mathcal{K}_\Theta)$.

So the following definition captures "beam-type" measurements. The operators A_i should be interpreted as POVM elements, i.e. effects $0 \leq A_i \leq \mathbb{1}$.

Definition 8.8. Given $\Theta \in \mathfrak{Z}([0,T],\mathcal{K})$ let $A_i \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K})$ for $i \in I(\Theta)$ be a family of operators. We define:

$$\Gamma_\Theta\left((A_i)\right) = \sum_{i\in I(\Theta)} (A(t_i)@i) \tag{8.25}$$

$$\Gamma_{\Theta-}\left((A_i)\right) = \sum_{i\in I(\Theta)} (A(t_i)@i)_- \,. \tag{8.26}$$

The first operator corresponds to second quantization for Bose field, while the second one describes Fermi fields. This becomes clearer when we take a look at the commutators of discretized field operators. For the Bose case we have:

$$[\Phi_\Theta(\lambda), \Phi_\Theta(\mu)] = \Phi_\Theta(\lambda)\Phi_\Theta(\mu) - \Phi_\Theta(\mu)\Phi_\Theta(\lambda) \tag{8.27}$$

$$=\Gamma\left(\tau_i \begin{pmatrix} -2i\,\mathfrak{Im}\,\langle (t_i)\lambda, \mu(t_i)\rangle & 0 \\ 0 & -|\lambda(t_i)\rangle\langle\mu(t_i)| + |\mu(t_i)\rangle\langle\lambda(t_i)| \end{pmatrix}\right) \tag{8.28}$$

And in the Fermi case we have:

$$\{\Phi_{\Theta-}(\lambda), \Phi_{\Theta-}(\mu)\} = \Phi_{\Theta-}(\lambda)\Phi_{\Theta-}(\mu) + \Phi_{\Theta-}(\mu)\Phi_{\Theta-}(\lambda) \tag{8.29}$$

$$=\Gamma\left(\tau_i \begin{pmatrix} -2i\,\mathfrak{Re}\,\langle (t_i)\lambda, \mu(t_i)\rangle & 0 \\ 0 & -|\lambda(t_i)\rangle\langle\mu(t_i)| - |\mu(t_i)\rangle\langle\lambda(t_i)| \end{pmatrix}\right) \tag{8.30}$$

Note that in the Fermi case one should concentrate on the operators $\Phi_{\Theta-}(\lambda)+i\Phi_{\Theta-}(-i\lambda)$, and their adjoints. That is the usual creation and annihilation operators, instead of the "Majorana" operators.

The weak-∗-limit of these commutators clearly approaches $-2i\,\mathfrak{Im}\,\langle (t_i)\lambda, \mu(t_i)\rangle$ for the commutation relations and $-2i\,\mathfrak{Re}\,\langle (t_i)\lambda, \mu(t_i)\rangle$ for the anti-commutation relations, which is a direct consequence of lemma 8.17.

As already mentioned there are essentially two different classes of POVMs and related stochastic processes one can look at. The two classes correspond to "random-walk-like" measurements and measurements on arrival. The requirements for the existence of a limit as well as the structure of the limit are quite different. They are the content of the next two sections.

8.3.2. Field operators and diffusive measurements

One of the things one should notice, when looking at definition 8.8 is that the generators of the discrete Weyl operators $W_\Theta(\lambda) = \exp\left(i\Phi_\Theta(\lambda)\right)$ have exactly this form. More formally we define

Definition 8.9. Given a function $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ and $\Theta \in \mathfrak{Z}([0,T])$.

$$\Phi_\Theta(\lambda) := \Gamma_\Theta(\Phi(\sqrt{\tau_i}\lambda(t_{i-1}))) \tag{8.31}$$

$$\Phi_{\Theta-}(\lambda) := \Gamma_{\Theta-}(\Phi(\sqrt{\tau_i}\lambda(t_{i-1}))). \tag{8.32}$$

These operators are discrete analogues of the position/momentum operators, i.e. the field operators for the quantum field, as they are generators of Weyl operators. And Weyl operators implement phase space translations. Observe that we suppress the explicit imaginary unit i in the definition of the generator, i.e. $\Phi_\Theta(\lambda)$ is anti-hermitian, instead of hermitian.

Corollary 8.10. *In the situation of the previous definition we have:*

$$W_\Theta(t\lambda) = \exp(t\Phi_\Theta(\lambda)). \tag{8.33}$$

From the strong convergence of the discrete Weyl operators, it is clear that also the generators, do converge in some sense. More concretely, we get convergence in the strong topology for the resolvents of the generators. For a precise statement and proof see corollary 8.25.

A straightforward calculation shows that that the discretized field operators almost fulfill the canonical (anti-) commutation relations, as we saw at the end of the previous section.

If we want to understand the stochastic process behind the discrete Weyl operators, we have to look at the POVM, or rather projection valued measure, which belongs to the operator $\Phi_\Theta(\lambda)$, i.e. its spectral projections. Since $i\Phi_\Theta(\lambda)$ is a self-adjoint operator we know that, by the spectral theorem, it has well defined spectral projections. Actually already each of the operators $i\left(\Phi(\sqrt{\tau_i}\lambda(t_i)\right)$ is self adjoint, so that we can try to understand the measurements on the individual sub-intervals.

Corollary 8.11. *Given $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ we set:*

$$M_-(\lambda(t_i)) \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K}) \qquad M_-(\lambda(t_i)) := \frac{1}{2}\begin{pmatrix} 1 & -i\langle e_{\lambda(t_i)}| \\ i|e_{\lambda(t_i)}\rangle & P_{\lambda(t_i)} \end{pmatrix} \tag{8.34}$$

$$M_+(\lambda(t_i)) \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K}) \qquad M_+(\lambda(t_i)) := \frac{1}{2}\begin{pmatrix} 1 & i\langle e_{\lambda(t_i)}| \\ -i|e_{\lambda(t_i)}\rangle & P_{\lambda(t_i)} \end{pmatrix} \tag{8.35}$$

$$M_0(\lambda(t_i)) \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K}) \qquad M_0(\lambda(t_i)) := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} - P_{\lambda(t_i)} \end{pmatrix}, \tag{8.36}$$

where $e_{\lambda(t_i)}$ is the unit vector in direction $\lambda(t_i)$ and $P_{\lambda(t_i)}$ the projection onto $\lambda(t_i)$.

Clearly all three operators are projections and we have:

$$\Phi\left(\sqrt{\tau_i}\lambda(t_i)\right) = \sqrt{\tau_i}\|\lambda(t_i)\|M_+(\lambda(t_i)) - \sqrt{\tau_i}\|\lambda(t_i)\|M_-(\lambda(t_i)). \tag{8.37}$$

The measurement described by the two projections M_+ and M_- can be understood as a measurement of the phase fluctuations of the light field in the time-step $i \in I(\Theta)$.

The operator $\Phi\left(\sqrt{\tau_i}\lambda(t_i)\right)$ describes an infinitesimal time-step of a homodyne detection, i.e. the superposition of the field with a fixed reference field in a coherent state with a 50:50 beam splitter. Depending on the phase shift of the field with respect to the reference field photons end up in a different arm of the interferometer as depicted in figure 8.2.

The operator $\Phi_\Theta(\lambda)$ is then best understood as a stochastic process monitoring the integrated difference between the intensities in the two arms of the interferometer. In other word the operator $P_\Theta(\lambda)$ for $\Theta \in \mathfrak{Z}([0,T])$ describes a family of random walks after a fixed time interval $[0,T]$. The family converges to a Brownian motion/Wiener process over the time interval $[0,T]$. This explains the square root needed for the correct scaling. Looking at the characteristic functions of the random walks, i.e. expectations of $W_\Theta(\lambda)$ this correspondence can be made precise.

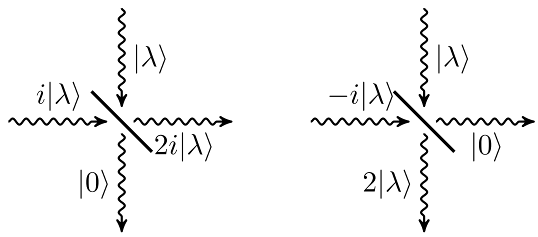


Figure 8.2.: Homodyne detection

8.3.3. Measurements on arrival

The other class of of measurements we are interested in are those measurements where we measure a fixed observable on every arrival. That is the "point-process" type observables described above. We have already taken a look at these processes in section 4.

Luckily we have already studied the prime example of such an observable: the number operator in definition 6.16. It is clear that the number operator has the desired form, it simply measures if there is an event in a subinterval and then sums over all the outcomes. However, now we are not only interested in the total number of events but also in their distribution in time.

More general, we want to second quantize an observable/POVM, or a family of observables, on $\mathcal{K}$ to an observable on $\mathcal{K}_{0,T}$. To make this possible we pair the measurement on $\mathcal{K}$ with a measurement of the arrival time, giving us a legitimate observable on $\mathrm{L}^2([0,T],\mathcal{K})$. The resulting class of observables is essentially the one described in section 4. This idea is captured in the following definition:

Definition 8.12. Let $\Theta \in \mathfrak{Z}([0,T])$ and $O_i \in \mathfrak{B}(\mathcal{K})$ for $i \in I(\Theta)$ be a family of operators. In abuse of notation we define:

$$A_i \in \mathfrak{B}(\mathbb{C} \oplus \mathcal{K}) \qquad A_i := \begin{pmatrix} 0 & 0 \\ 0 & O_i \end{pmatrix} \tag{8.38}$$

$$\Gamma_\Theta(O_i) \in \mathfrak{B}(\mathcal{K}_\Theta) \qquad \Gamma_\Theta(O_i) := \Gamma_\Theta(A_i). \tag{8.39}$$

It is already clear from section 4 that the stochastic processes described by such operators should be understood as point processes, i.e. random sums of Dirac measures. To describe these point processes we have to include in our setup an explicit description of the possibility that there is no arrival during a specific time interval. The operator describing this situation th from the previous definition, which described a measurement on arrival under the condition that an arrival has happened.

Definition 8.13. Let (X,Ω) be a measurable set with σ-algebra Ω. Let $\Theta \in \mathfrak{Z}([0,T])$ and for every $i \in I(\Theta)$ let $F_i : \Omega \to \mathfrak{B}(\mathcal{K})$ be a POVM.

A set $\xi \in \{0\} \cup X$ is measurable iff $\xi \cap X$ is measurable in X. We denote the σ-algebra of the set $\{0\} \cup X$ by $\mathfrak{X}$. Define $M_i : \mathfrak{X} \to \mathfrak{B}(\mathbb{C} \oplus \mathcal{K})$ by setting for $\mathcal{X} \in \Omega$:

$$M_i(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M_i(\mathcal{X}) = \begin{pmatrix} 0 & 0 \\ 0 & F(\mathcal{X}) \end{pmatrix}. \tag{8.40}$$

The measure of a set in $\{0\} \cup X$ is defined to be the sum of its measures in $\{0\}$ and $\{X\}$.

In the preceding definition we equipped the set $\{0\}$ with the only possible σ-algebra, containing the empty set and the whole set. And the disjoint union $\{0\} \cup X$ with its canonical σ-algebra, which is defined to be the finest σ algebra such that the two canonical embeddings are measurable. Each M_i for $i \in I(\Theta)$ is clearly a POVM.

With the preceding definition we can now easily describe discrete measurement-on-arrival processes as point processes with state space $\Theta \times X$. Every possible combination of arrivals is just described by the corresponding tensor product of operators.

Definition 8.14. In the situation of the previous definition we define a probability measure on the space $\prod_{i\in I(\Theta)}(\{0\}\cup X)$ equipped with the product sigma algebra.

With $\xi_i \in \mathfrak{X}$ for all $i \in I(\Theta)$ and a given state $\rho \in \mathfrak{T}(\mathcal{K}_\Theta)$ we define a measure ν_ρ by:

$$\nu_\rho\left(\prod_{i\in I(\Theta)} \xi_i\right) \mapsto \operatorname{tr}\left(\rho \bigotimes_{i\in I(\Theta)} M_i(\xi_i)\right) \tag{8.41}$$

Let $\mathcal{E}$ be the product σ-algebra on $[0,T]\times X$. Then we define a point process with state space $[0,T]\times X$ by the following kernel, which just counts the events falling in a given region:

$$\begin{aligned} k:\left(\prod_{i\in I(\Theta)} (\{0\}\cup X)\right) \times \mathcal{E} : \mathbb{R} \\ (x_i)_{i\in I(\Theta)} \times \xi \mapsto \#\{(t_{i-1},x_i) | (t_{i-1},x_i)\in\xi\}. \end{aligned} \tag{8.42}$$

If $x_i = 0$, i.e no-event, it is of course understood that $(i, x_i) \neq \xi$ for arbitrary ξ.

To be able to better compare the point processes for different interval decompositions Ξ we defined them all as point processes with state space $[0,T]\times X$, i.e. as discrete approximations of the same continuous process. For our discrete approximation it does not matter that we choose the events in the subinterval $i \in I(\Theta)$ to happen at the left boundary t_{i-1}. One could have chosen any point in $[t_{i-1}, t_i]$. If we want to treat all the discretizations as point processes with a common probability space we could use the following embedding, and push forward the measure ν_ρ:

$$\begin{aligned} \prod_{i\in I(\Theta)} (\{0\}\cup X) \to \bigcup_{n=0}^{\infty} (\Delta_n([0,T])\times X^n) \\ (0,\ldots,\underset{\substack{\uparrow\\ \text{pos. } i_1}}{x_1},0,\ldots,0,\underset{\substack{\uparrow\\ \text{pos. } i_n}}{x_n},\ldots,0) \mapsto (t_{i_1-1},\ldots,t_{i_n-1})\times(x_1,\ldots x_n) \end{aligned} \tag{8.43}$$

The limit of these point processes should be seen as an analogy of a sum of scaled Bernoulli processes converging to a Poisson process.

The connection of these kernels to the second quantization of measurements on arrival is best understood in terms of the characteristic functionals of the point processes. It is clear that in definition 8.13, for every $i \in I(\Theta)$ M_i is a POVM on $\mathfrak{B}(\mathbb{C}\oplus\mathcal{K})$. The POVM M_i describes the situation where in the time interval i either no particle arrives, $M_i(0)$, or where we measure according to the POVM F on arrival. Using this we can naturally describe a discrete time measurement on arrival situation by summing up the measurements on the individual subintervals.

Corollary 8.15. *Let $\rho \in \mathfrak{T}(\mathcal{K}_{[0,T]})$ be a state, i.e. of norm 1 and positive.*
Let $F : X \to \mathfrak{B}(\mathcal{K})$ for $t \in [0,T]$ be a projection valued measure. We set:

$$U : [0,T] \to \mathfrak{B}(\mathcal{K}) \qquad t \mapsto \int_X \exp(if(t,x))F(\mathrm{d}x). \tag{8.44}$$

For every continuous function $f : [0,T] \times X \to \mathbb{R}$ and every $\Theta \in \mathfrak{Z}([0,T])$ the following functional functional is the characteristic functional for the point process given in definition 8.14 iff we set $F_i = F$ for $i \in I(\Theta)$ in the notation of definition 8.12:

$$f \mapsto \mathrm{tr}\left(J_\Theta^* \rho J_\Theta W_\Theta(0,U)\right), \tag{8.45}$$

The net of point processes converges, in the sense that their characteristic functions converge.

Proof. The convergence of the characteristic functionals is a direct consequence of theorem 8.3. We denote $\rho_\Theta := J_\Theta^* \rho J_\Theta$. To see that the expectation value of the Weyl operators is the characteristic function of the point process we just insert the definitions:

$$\begin{aligned}
&\mathrm{tr}\left(\rho_\Theta \exp\left(i\Gamma_\Theta\left(\int_X f(t_{i-1},x)F(t_{i-1},x)\mathrm{d}x\right)\right)\right)\\
&= \mathrm{tr}\left(\rho_\Theta \bigotimes_{i\in I(\Theta)} \exp\left(i\left(\int_X f(t_{i-1},x)F(t_{i-1},x)\mathrm{d}x\right)\right)\right)\\
&= \mathrm{tr}\left(\rho_\Theta \bigotimes_{i\in I(\Theta)} \left(M(0) + \int_X \exp\left(if(t_{i-1},x)\right)F(t_{i-1},x)\mathrm{d}x\right)\right).
\end{aligned} \tag{8.46}$$

□

Though it seems clear, it is beyond the scope of this work, that for a cMPS ρ the limit point process is exactly of the form described in the end of section 4. Lemma 8.6 could be used to calculate the expectation values.

Of course convergence of the underlying stochastic processes can also be shown for general POVMs using the setup from corollary 8.15. One looses however the connection with Weyl operators, but can still define the characteristic functions through equation (8.46).

Since we only need to show weak-$*$ convergence for the operators, convergence can easily be shown using the technical lemma 8.17. In this case one can also drop the restriction to functionals on real valued function.

With the right continuity conditions it is clear that we can also take the family of POVMs to be time dependent. We arrive at the following corollary describing second quantization of general POVMs and the associated continuous limits.

Corollary 8.16. *Let $\rho \in \mathfrak{T}(\mathcal{K}_{[0,T]})$ be a state, i.e. of norm 1 and positive.*
Let $F : [0,T] \times X \to \mathfrak{B}(\mathcal{K})$ for $t \in [0,T]$ be a POVM.

Given a measurable function $f : [0,T] \times X \to \mathbb{C}$ and $\Theta \in \mathfrak{Z}([0,T])$ we set for every $i \in I(\Theta)$

$$O_i(f) : [0,T] \to \mathfrak{B}(\mathcal{K}) \qquad t \mapsto \frac{1}{\tau_i} \int_{[t_{i-1},t_i]\times X} \exp(if(t,x))F(t,x)\mathrm{d}t\mathrm{d}x. \tag{8.47}$$

The following net of characteristic functionals of point processes converges:

$$\mathcal{C}_\Theta : \mathrm{L}([0,T]\otimes X) \to C \qquad C_\Theta(f) := \mathrm{tr}\left(J_\Theta^* \rho J_\Theta \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & O_i(f) \end{pmatrix} \right). \tag{8.48}$$

Here $\mathrm{L}([0,T]\otimes X)$ *in abuse of notation denotes the set of measurable functions.*

Note that the characteristic functional might evaluate to ∞ as we do not require integrability of f.

8.4. Proofs of the main results

We start with a simple lemma, which is of invaluable use when it comes to calculating weak-$*$ limits of nets of operators in K_Θ. It is clear that the map $J^*_{\Xi,\Theta} : \mathcal{K}_\Xi \to \mathcal{K}_\Theta$ can be seen as an averaging operation. Similarly $J_{\Xi,\Theta}$ also introduces and averaging on the level of operators. The exact form of this averaging procedure is the content of the following lemma:

Lemma 8.17. *Let $\Theta \subset \Xi \in \mathfrak{Z}([0,T])$, let $\mathcal{K}$ be a separable Hilbert space and let for each $j \in I(\Xi)$: $\lambda_j, \nu_j \in \mathcal{K}$, $c_j \in \mathbb{C}$ and $O_j \in \mathfrak{B}(\mathcal{K})$, such that we can define the following family of operators:*

$$W_j \in \mathfrak{B}(\mathbb{C}\oplus\mathcal{K}) \qquad W_j = c_j \begin{pmatrix} 1 & \sqrt{\tau_j}\langle\nu_j| \\ \sqrt{\tau_j}|\lambda_j\rangle & O_j + \tau_j|\lambda_j\rangle\langle\nu_j| \end{pmatrix}. \tag{8.49}$$

Then we have:

$$J_{\Xi,\Theta} \bigotimes_{j\in I(\Xi)} W_j J^*_{\Xi,\Theta} = \bigotimes_{i\in I(\Theta)} \widetilde{W}_i, \tag{8.50}$$

where:

$$\widetilde{W}_i = \left(\prod_{j\in I(\Xi|i)} c_j \right) \begin{pmatrix} 1 & \sqrt{\tau_i}\langle\overline{\nu}_i| \\ \sqrt{\tau_i}|\overline{\lambda}_j\rangle & \overline{O}_i + \tau_i|\overline{\lambda}_i\rangle\langle\overline{\nu}_i| \end{pmatrix}. \tag{8.51}$$

And with the notation $\overline{\lambda}_i := \mathfrak{M}_i(\lambda_j)$, $\overline{\nu}_i := \mathfrak{M}_i(\nu_j)$ $\overline{O}_i := \mathfrak{M}_i(O_j)$ with $\mathfrak{M}_i$ being the average over the sub interval i for $i \in I(\Theta)$, i.e.:

$$\mathfrak{M}_i(x_j) = \sum_{j\in I(\Xi|i)} \frac{\tau_j}{\tau_i} x_j. \tag{8.52}$$

Proof. It is clear that to calculate the matrix elements of W_i for $i \in I(\Theta)$, it is sufficient to restrict to $\mathcal{K}_{\Xi|i}$. The calculation of the matrix elements is then a straightforward calculation, e.g.:

$$\left\langle J_i|0\rangle, \bigotimes_{j\in I(\Xi|i)} W_j J_i|0\rangle \right\rangle = \prod_{j\in I(\Xi|i)} c_j \tag{8.53}$$

Only the lower right matrix element requires a bit more care, since every vector $|\alpha@j\rangle$ leads to two "contributions", i.e. for $j,k \in I(\Xi|_i)$ we have:

$$\left\langle \alpha@j \middle| \bigotimes_{j\in I(\Xi|i)} W_j \middle| \beta@k \right\rangle = \langle\alpha|O_j|\beta\rangle + \sqrt{\tau_j\tau_k}\,\langle\alpha,\lambda_j\rangle\,\langle\nu_k,\beta\rangle \tag{8.54}$$

The second term reflects the situation where $\bigotimes_{j\in I(\Xi|i)} W_j$ "destroys" an event at time k and "creates" an event at time j during a single time-step i. The first term is just the "conservation" of events. □

The above theorem is a key ingredient in defining rigorous limits of operators on K_Θ. It can only be used to obtain limits in weak-$*$ topology. However, as we shall see for the strong convergence of nets of unitary operators it is enough to show that the nets are "unitary" in a weak-$*$ sense. For a precise statement see the proof of lemma 8.19.

Convergence to the identity in the weak-$*$ topology is thus particularly interesting for us. There the "relevant order", in terms of the square root of the interval length $\sqrt{\tau}$, clearly depends on the "component" of the operator. The following corollary is a direct consequence of lemma 8.17.

Corollary 8.18. *Let $\lambda : [0,T] \to \mathcal{K}$, $c : [0,T] \to \mathbb{C}$ and $O : [0,T] \to \mathfrak{B}(\mathcal{K})$ be functions, s.t. $\sup_{t\in[0,T]} \|\lambda(t)\| \le C$ and analogously for O and c. Furthermore for a given $\Theta \in \mathfrak{Z}([0,T])$ we set $\lambda_i := \lambda(t_i)$ and analogously for O and c.*

Under these conditions the following net of operators converges to the identity in weak-$*$ topology.

$$W_\Theta = \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1+\tau_i^2 c_i & \tau_i^{3/2}\langle\psi_i| \\ \tau_i^{3/2}|\phi_i\rangle & \mathbb{1}+\tau_i O_i \end{pmatrix}. \tag{8.55}$$

8.4.1. Weyl operators

We start by proving the various part of the theorems 8.3 and 8.5. We treat the two "types" of Weyl operators separately and start with the implementations of the unitary rotations.

Lemma 8.19. *Let $U : [0,T] \to \mathfrak{B}(\mathcal{H})$ be a unitary operator valued function continuous in the strong topology. Then $W(0,U) := \text{s-lim}_\Theta U_\Theta$ exists.*

Proof. Using the unitarity of U and hence of U_Θ, we only have to show:

$$\|U_\Lambda J_{\Lambda\Theta}\varphi - U_\Xi J_{\Xi\Theta}\varphi\|^2 = 2 - 2\,\mathfrak{Re}\,\langle J_{\Xi\Theta}\varphi, J^*_{\Lambda\Xi} U_\Lambda J_{\Lambda\Xi} U_\Xi J_{\Xi\Theta}\varphi\rangle \tag{8.56}$$

Evaluating the explicit form of $J^*_{\Lambda\Xi} U_\Lambda J_{\Lambda\Xi}$ this is easy to see. By lemma 8.17 we have:

$$J^*_{\Xi,\Theta} U_\Xi J_{\Xi,\Theta} = \bigotimes_{i \in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & \sum_{j \in I(\Xi|i)} \frac{\tau_j}{\tau_i} U_j \end{pmatrix}. \tag{8.57}$$

□

Next we show the convergence of the "usual" Weyl operators for continuous functions.

Lemma 8.20. *Let $\lambda : [0, T] \to \mathcal{K}$ be a continuous function then $W(\lambda) := \text{s-lim}_\Theta W_\Theta$ exists.*

Proof. First note that since λ is continuous it is also uniformly bounded in norm. To show strong convergence we have to show that for every $\Theta \in \mathfrak{Z}([0, T])$ and for every $\varphi_\Theta \in \mathcal{K}_\Theta$ the following norm difference vanished in the limit.

$$\begin{aligned} &\|W_\Lambda(\lambda) J_{\Lambda\Theta}\varphi - J_{\Lambda\Xi} W_\Xi(\lambda) J_{\Xi\Theta}\varphi\|^2 \\ =&2 - 2\,\mathfrak{Re}\,\langle J_{\Xi\Theta}\varphi, (J^*_{\Lambda\Xi} W_\Lambda(\lambda) J_{\Lambda\Xi} W_\Xi) J_{\Xi\Theta}\varphi\rangle \end{aligned} \tag{8.58}$$

That is, we have to show that $W_\Xi J^*_{\Lambda\Xi} W_\Lambda J_{\Lambda\Xi}$ and its adjoint converge to the identity in the weak-$*$ topology, i.e. we are going to use corollary 8.18. To the relevant order the Weyl operators have the form:

$$W_\Theta(\lambda) \approx \bigotimes_{i \in I(\Theta)} \begin{pmatrix} 1 + \tau_i \|\lambda_i\|^2 & \sqrt{\tau_i}\,\langle\lambda_i| \\ -\sqrt{\tau_i}\,|\lambda_i\rangle & \mathbb{1} \end{pmatrix}. \tag{8.59}$$

A straightforward matrix multiplication then shows that the Weyl operators converge for continuous functions λ. □

It remains to check the basic properties of the Weyl operators. We start with the commutation relations of Weyl operators with continuous arguments. A central property which we shall leverage to get the promised more general results.

Lemma 8.21. *Let $\lambda, \mu \in \mathrm{L}^2([0, T], \mathcal{K})$ and $U_1, U_2 : [0, T] \to \mathfrak{B}(\mathcal{K})$ a unitary operator valued function continuous in the strong topology. Then:*

$$W(\lambda, U_1) W(\mu, U_2) = \exp\left(-\,\mathfrak{Im}\,\langle\lambda, U_1\mu\rangle\right) W((\lambda + U_2\mu), U_1U_2). \tag{8.60}$$

Proof. Since the operators $W(\lambda)$ are bounded their strong limits commute with the operator product. First assume that λ, μ are continuous. Explicit evaluation of the following expression together with corollary 8.18 shows the identity for $U_1 = U_2 = \mathbb{1}$

$$\text{s-}\lim_\Theta W_\Theta(\lambda) W_\Theta(\mu) W_\Theta(-(\lambda + \mu)))\exp\left(\mathfrak{Im}\,\langle\lambda, \mu\rangle\right) = \mathbb{1} \tag{8.61}$$

This extends to the case of general λ and μ by denseness.

Observe that we clearly have: $W(0, U^*) W(\lambda) W(0, U) = W(U^*\lambda)$, as this is exactly true in every discrete approximation for continuous λ. Here $U\lambda$ is defined by pointwise application. Furthermore it is obvious that $W(0,U_1)W(0,U_2) = W(0,U_1U_2)$, where again the composition is pointwise. Hence:

$$
\begin{aligned}
&W(\lambda,U_1)W(\mu,U_2) = W(\lambda)W(0,U_1)W(\mu)W(0,U_2) &(8.62)\\
=&W(\lambda)W(0,U_1)W(\mu)W(0,U_1^*)W(0,U_1)W(0,U_2)\\
=&W(\lambda)W(U_1\mu)W(0,U_1U_2), &(8.63)
\end{aligned}
$$

which yields the desired result together with the previous equation.

□

The following discrete counterpart of lemma 7.12 is needed as a technical tool for the main technical lemma in this section.

Lemma 8.22. *Let $\Theta \in \mathfrak{Z}([s,t])$ and $f_i \in \mathbb{C}$ for $i \in I(\Theta)$. Then we have:*

$$
\left(\sum_{i\in I(\Theta)} \frac{\tau_i}{t-s}|f_i|\right)^2 \leq \sum_{i\in I(\Theta)} \frac{\tau_i}{t-s}|f_i|^2 \tag{8.64}
$$

Proof. Analogous to the continuous case we have:

$$
\begin{aligned}
&\sum_{i\in I(\Theta)} \frac{\tau_i}{t-s}|f_i|^2 - \left(\sum_{i\in I(\Theta)} \frac{\tau_i}{T}|f_i|\right)^2 &(8.65)\\
=&\sum_{i\in I(\Theta)} \frac{\tau_i}{t-s}\left(|f_i| - \sum_{k\in I(\Theta)} \frac{\tau_k}{t-s}|f_k|\right)^2 &(8.66)
\end{aligned}
$$

□

The following lemma is a discrete analogue of the fact that the field operators, i.e. the generators of the Weyl operators, are relatively bounded by the number operator. Assume that Θ partitions $[0,T]$ in n equal subintervals, we see that the generator generator of $W(\lambda)$ is bounded by the "event-density" n/T, which is consistent with the interpretation of $\mathcal{K}_{[0,T]}$ as a Bose-Fock space. This is the central technical lemma on Weyl operators. All other interesting properties of Weyl operators are essentially consequences of this lemma and the commutation relations when restricted to continuous functions. The proof is essentially a straightforward calculation.

Lemma 8.23. *For $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ we have*

$$
\lim_{\Xi} \|\Phi_\Xi(\lambda)J_\Theta\| \leq \sqrt{\left(\frac{1}{\min\{\tau_i | i \in I(\Theta)\}} + 2\right)}\|\lambda\| \tag{8.67}
$$

Proof. Choose a continuous function μ such that $\|\mu(t)\| \leq \|\lambda(t)\|$ We shall bound the square of the operator in terms of μ, by explicit calculation, i.e:

$$
\lim_{\Xi} \sum_{j,l} J^*_{\Xi\Theta}\left(\Phi(\sqrt{\tau_j}\lambda_j)@j\right)\left(\Phi(\sqrt{\tau_l}\lambda_l)@k\right)J_{\Xi\Theta} \tag{8.68}
$$

When evaluating the operator product we have to distinguish two different cases. The case where j, l fall in the same subinterval of Θ, and where they do not.

If j, l fall in the same subinterval of Θ we have again to distinguish between two cases. The cases $j = l$ and $j \neq k$. In order to apply lemma 8.17 we have to write $\Phi(\sqrt{\tau_j}\lambda_j) + \mathbb{1} - \mathbb{1}$ and expand the product. A simple calculation yields:

$$\sum_{i \in I(\Theta)} \sum_{j \in I(\Xi|i)} \tau_j \|\lambda_j\|^2 \left(\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{1} + \frac{\tau_j}{\tau_i}\left(P_{\lambda_j} - \mathbb{1}\right) \end{pmatrix} @i \right)$$
$$+ \sum_{i \in I(\Theta)} \left(\begin{pmatrix} 0 & 0 \\ 0 & 2\sum_{j \neq l \in I(\Xi|i)} \frac{\tau_j \tau_l}{\tau_i} |\lambda_j\rangle\langle\lambda_l| \end{pmatrix} @i \right) \tag{8.69}$$

We can point-wise bound $\|\lambda(t)\|$ by $\|\mu(t)\|$. For a continuous function on a compact domain $\sum_j \tau_j \|\mu(t_k)\|^2$ clearly converges to the integral. Applying the triangle inequality to the first sum it is hence clear that in the limit along Ξ it is bounded by the L^2-norm $\|\mu\|$.

The last sum in the preceding equation hast the following form:

$$2 \sum_{i \in I(\Theta)} \left(\begin{pmatrix} 0 & 0 \\ 0 & \tau_i |\mathfrak{M}_i(\lambda_j)\rangle\langle\mathfrak{M}_i(\lambda_l)| - \tau_i \mathfrak{M}_i\left(|\lambda_j\rangle\langle\lambda_j|\right) \end{pmatrix} @i \right) \tag{8.70}$$

It is easy to see that each term in this sum is a scaled projector and hence positive. Thus each term is dominated as a positive operator by $(|\mathfrak{M}_i(\lambda_j)\rangle\langle\mathfrak{M}_i(\lambda_l)|@j)$. Applying the triangle inequality to the sum and using lemma 8.22 we see that this term is bounded by $2\|\mu\|$.

However we only count $1\|\mu\|$ against our final estimate because we use half of the term in equation (8.70) in the next estimate.

In the case where j and k do not fall in the same subinterval of Θ the form of the operators almost does not change. We only replace the vectors by their averaged counterparts:

$$\sum_{i \neq k \in I(\Theta)} \left(\Phi\left(\sqrt{\tau_i}\mathfrak{M}_i(\lambda_j)\right) @i\right) \left(\Phi\left(\sqrt{\tau_k}\mathfrak{M}_i(\lambda_j)\right) @i\right). \tag{8.71}$$

The preceding sum goes over i and k leaving out all terms where $i = k$. Each of the operators $\left(\Phi\left(\sqrt{\tau_i}\mathfrak{M}_i(\lambda_j)\right) @i\right)$ is norm clearly bounded in operator by $\sqrt{\tau_i}\|\mathfrak{M}_i(\lambda_j)\|$. Again we apply the triangle inequality and get the bound:

$$\left(\sum_i \frac{1}{\sqrt{\tau_i}} \left\| \sum_{j \in I(\Xi|i)} \tau_j \lambda(t_i) \right\| \right)^2. \tag{8.72}$$

Here we have used half of equation (8.70) to add in the "diagonal". Using the triangle inequality again and replacing $\|\lambda(t)\|$ by $\|\mu(t)\|$, we see that this term is certainly smaller than the following term, where again the norm is the L^2-norm:

$$\frac{1}{\min\{\tau_i | i \in I(\Theta)\}} \|\mu\|^2 \tag{8.73}$$

Hence in total we get the desired bound in terms of any possible μ. Denseness of continuous functions in L^2 ensures that we can replace $\|\mu\|$ by $\|\lambda\|$. □

This lemma has three important consequences. Firstly it allows us to extend the notion of Weyl operators to arbitrary square integrable functions, instead of continuous functions.

Corollary 8.24. *Let λ_n be a Cauchy sequence of continuous functions in $\mathrm{L}^2([0,T],\mathcal{K})$ such that $\lambda := \lim_{n\to\infty}\lambda_n$. Then $W(\lambda) := \text{s-lim}\, W(\lambda_n)$ exists and fulfills the CCR in Weyl form.*

Proof. Due to lemma 8.21 it is sufficient to show that $\text{s-lim}\, W(\lambda_n) = \mathbb{1}$ for sequences converging to 0. This is a direct consequence of lemma 8.23 since:

$$\begin{aligned} \|W(\lambda_n)J_\Theta\varphi_\Theta - J_\Theta\varphi_\Theta\|^2 &= \lim_\Xi \|\left(W_\Xi(\lambda_n) - \mathbb{1}\right) J_{\Xi\Theta}\varphi_\Theta\|^2 \\ = \lim_\Xi &\left\|\left(\int_0^1 W_\Xi(s\lambda_n)\mathrm{d}s\Gamma_\Xi(\Phi(\lambda_n))\right) J_{\Xi\Theta}\varphi_\Theta\right\|^2 \\ \le \lim_\Xi &\|\Gamma_\Xi(\Phi(\lambda_n))J_{\Xi\Theta}\varphi_\Theta\|^2 . \end{aligned} \tag{8.74}$$

□

The second consequence is the convergence of the generators of the discrete Weyl operators in the strong resolvent sense.

Corollary 8.25. *For every $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ the function $t \mapsto W(t\lambda)$ for $t \in \mathbb{R}$ defines a unitary strongly-continuous one-parameter group.*

If $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ is a continuous function the net of operators $\Gamma_\Theta(\Phi(\lambda))$ for $\Theta \in \mathfrak{Z}([0,T])$ converges to the generator of $W(t\lambda)$ in the strong resolvent sense, i.e.

$$\lim_\Xi J_\Xi(\Gamma_\Xi(\lambda) - \mu\mathbb{1})^{-1}J_{\Xi\Theta} = \int_0^\infty e^{-t\mu}W(t\lambda)J_{\Xi\Theta}\mathrm{d}t. \tag{8.75}$$

Proof. By the previous corollary we have:

$$\|\left(W(h\lambda) - \mathbb{1}\right) J_\Theta\varphi_\Theta\| \le \lim_\Xi |h|\, \|\Gamma_\Xi(\Phi(\lambda))J_{\Xi\Theta}\varphi_\Theta\| . \tag{8.76}$$

This immediately shows strong continuity in h. Furthermore it is clear that the sign of h does not matter. The other semigroup properties are a direct consequence of the commutation relations in lemma 8.21.

The convergence of the generators is a direct consequence of equality between resolvents of the generator and Laplace transforms of the semigroup, i.e. the formula:

$$(\Gamma_\Theta(\lambda) - \mu\mathbb{1})^{-1} = \int_0^\infty e^{-t\mu}W_\Theta(t\lambda)\mathrm{d}t. \tag{8.77}$$

A closer inspection of the proof of lemma 8.20 shows that the rate of convergence of $W_\Theta(t\lambda)$ decreases with growing t as $\|t\lambda\|$ increases. Hence to see convergence we restrict to a finite t and bound the tail of the integral by the exponential decrease. □

The other important consequence of lemma 8.23 is that it allows us to partly describe the domain of the Weyl operators.

Corollary 8.26. *$J_\Theta\mathcal{K}_\Theta$ is is the domain of the generator of the semigroup $W(t\lambda)$ for all $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$.*

Proof. Firstly we observe that, by the proof of the previous corollary, the relevant limit does not diverge, since

$$\frac{1}{h}\| \left(W(h\lambda) - \mathbb{1}\right) J_\Theta \varphi_\Theta\| \leq \lim_{\Xi} \|\Gamma_\Xi(\Phi(\lambda)) J_{\Xi\Theta}\varphi_\Theta\| \tag{8.78}$$

Since Hilbert spaces are reflexive, this is enough to ensure convergence of the limit by [76, theorem 10.7.2]. □

Collecting the results in this section we get a proof of the theorems 8.3 and 8.5.

Proof of theorem 8.3. The theorem is a direct consequence of the lemmas 8.19 and 8.20 □

Proof of theorem 8.5. The theorem is a direct consequence of the lemma 8.21 and corollary 8.24. □

8.4.2. Expectation values

The proof of the central lemma 8.6 of this section is surprisingly straightforward, since most of the work was already done in section 7. In the end the proof is mainly an application of lemma 7.17.

Proof of lemma 8.6. By construction it is clear that for any $\Theta \in I(\Xi)$ there exists a family of operators $W_i \in \mathfrak{B}(\mathcal{K}_{[t_{i-1},t_i]})$, such that for any $\omega \in \mathfrak{T}(\mathcal{K}_{[0,T]})$ and with the canonical isomorphism between $\mathfrak{B}(\mathcal{K}_{[0,T]}) = \bigotimes_{i\in I(\Theta)} \mathfrak{B}(\mathcal{K}_{[t_{i-1},t_i]})$, it holds that

$$\operatorname{tr}(\omega W) = \left(\omega \bigotimes_{i\in I(\Theta)} W_i\right). \tag{8.79}$$

Hence $\mathbb{E}_{[s,t]}(W \otimes X)$ is clearly an evolution system. We have to show continuity and the form of the generator. It is again sufficient to restrict our attention to a single short subinterval We have to show that the following expression converges:

$$\frac{1}{\tau_i} \operatorname{tr}\left(\rho \mathbb{E}_{[t_{i-1},t_i]}(W \otimes \mathbb{1})\right) \tag{8.80}$$

$$= \frac{1}{\tau_i} \operatorname{tr}\left(\rho \mathbb{E}_{[t_{i-1},t_i]})(W \otimes \mathbb{1} - P_{\{t_{i-1},t_i\}} W P_{\{t_{i-1},t_i\}} \otimes \mathbb{1})\right)$$
$$+ \frac{1}{\tau_i} \operatorname{tr}\left(\rho \mathbb{E}_{[s,t+h]}(P_{\{t_{i-1},t_i\}} W P_{\{t_{i-1},t_i\}} \otimes \mathbb{1})\right) \tag{8.81}$$

We use lemma 7.18 to see that the first term on the right side converges for $\tau_i \to 0$ and every $t_i \in [0,T]$. The convergence of the second term is a consequence of lemma 7.17, the continuity properties of the no event semigroup and an expansion to the relevant order.

$$\frac{1}{\tau_i}\operatorname{tr}\left(|\psi\rangle\langle\psi|V_i^* J_i^* W J_i \otimes X V_i\right) \tag{8.82}$$

$$\approx \frac{1}{\tau_i}\left\langle V_i\psi, \begin{pmatrix} 1+\tau_i c(t_i) & \sqrt{\tau_i}\langle\lambda_1(t_i)| \\ \sqrt{\tau_i}|\lambda_2(t_i)\rangle & (O(t_i)+\mathbb{1}) \end{pmatrix} \otimes X V_i \psi\right\rangle \tag{8.83}$$

$$= \frac{1}{\tau_i}\big(\mathbb{F}_{0,[t_i,t_{i-1}]}(X) + \mathbb{F}_{1,[t_i,t_{i-1}]}((O_i+\mathbb{1})\otimes X)\big) + c(t_i)\mathbb{F}_{0,[t_i,t_{i-1}]}(X)$$
$$+ \frac{1}{\sqrt{\tau_i}}\left(\langle V_0\psi, \langle\lambda_1| \otimes X V_1 \psi\rangle + \langle V_1\psi, |\lambda_2\rangle \otimes (X V_0)\psi\rangle\right). \tag{8.84}$$

□

9. Examples of cMPS

The arguably simplest example of a Lindblad equation and associated cMPS is the is the following memory-less memory system, i.e. we take a trivial Lindblad generator, $K = L = 0$ and apply a gauge transformation. The main interest of this example lies in the fact that it is a generic example we can always add terms of the following form. This leads to a class of “Gaussian” cMPS.

A far more general class of cMPS with infinite dimensional space $\mathcal{H}$, for which one can still calculate many properties is the class cMPS belonging to quasi free evolutions on $\mathcal{H}$.

9.1. Gaussian states

Let $\mathcal{H} = \mathbb{C}$ and $\mathcal{K} = C^n$ for some $n \in \mathbb{N}$. For $1 \leq \alpha \leq n$ let $t \to \phi_\alpha(t)$ be continuously differentiable functions. We are interested in the following gauge of trivial Lindblad generator:

$$\mathcal{L}(x) = 0 = -\sum_\alpha \|\phi_\alpha(t)\|^2 x + \sum_\alpha \overline{\phi}_\alpha(t) x \phi_\alpha(t)\rangle \tag{9.1}$$

That is, there is no time evolution and we are looking at pure gauge processes. By lemma 8.6 we need to solve the following differential equation for the characteristic function of the associated measurement on arrival for a given $\mathfrak{B}(\mathcal{K})$ valued POVM F:

$$\dot{C}(f) = \mathrm{tr}\left(\sigma(t)\int_X \left(e^{if(t)} - 1\right) F(\mathrm{d}x)\right) C(f) \tag{9.2}$$

with $\sigma(t) = \sum_\alpha |\phi_\alpha(t)|^2 |\alpha\rangle\langle\alpha|$ and hence

$$C(f) = \exp\left(\int_0^T \int_X \left(e^{if(t)} - 1\right) \mathrm{tr}\left(\sigma(t)F(\mathrm{d}x)\right) \mathrm{d}t\right) \tag{9.3}$$

which is the characteristics function of a process producing states in state $\sigma(t)/tr(\sigma(t))$ with intensity $\mathrm{tr}(\sigma(t))$. Hence if we work in the “wrong” gauge for a given Lindbladian, we add random Poisson noise to our description.

Alternatively we can analyze the statistics of the field operators via their characteristic function, i.e. calculate the expectation of the Weyl operators. for $\lambda \in \mathrm{L}^2([0,T],\mathcal{K})$ and $\phi(t) := \sum_\alpha \phi_\alpha(t)|\alpha\rangle$ we get, using lemma 8.6:

$$\mathrm{tr}\left(\rho W(\lambda)\right) := \exp\left(\int_0^T -\frac{1}{2}\|\lambda\|^2 + 2i\,\mathfrak{Im}\,\langle\lambda(t),\phi(t)\rangle\,\mathrm{d}t\right) \tag{9.4}$$

Where ρ is the cMPS generated by the given Lindbladian. In other words the cMPS is a Gaussian state

9.2. Quasi-free cMPS

A more interesting class of examples is given by the following class of quasi-free semigroups for Boson systems. For a nice overview see [1], but note that we are following different conventions on the generators of the Weyl systems, i.e. those in [72].

The underlying theory of quasi-free maps goes back to [32, 33, 81]. We however are not interested in quasi-free states on a general CCR-algebra, but only in the case where the CCR-algebra is represented on Fock space. Analogous results on quasi free systems of fermions [40, 41] could be obtained by corresponding techniques. There are also strong connections to [17].

A similar class of cMPS was previously studied by Eisert [38].

9.2.1. Lindblad generators and setup

As the system Hilbert space we fix for the rest of this section the Fock space $\mathcal{H} = \Gamma_+(\mathcal{H}_1)$, for a separable one-particle Hilbert space $\mathcal{H}_1$. For the class of quasi-free evolutions we are interested in it is convenient to choose $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_1$ as the single step dilation space. The set $\mathbb{A}$ is taken to enumerate a basis of $\mathcal{H}_1$. And not as usual a basis of $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_1$ for reasons which shall become apparent shortly.

To start our analysis of the cMPS generated by quasi-free evolutions we first of all have to describe the Lindblad generators leading to quasi-free evolutions. These are those evolutions where particles are created in some mixture of pure states and absorbed in some other states. In between the state undergoes a free unitary evolution, i.e. the evolution can be performed on one-particle level. The presence of absorption and creation of particles explains the choice of $\mathcal{K}$ as we want to be able to describe absorption of and creation in an arbitrary state.

Such systems are described by an Lindblad equation of the following form.

$$\mathcal{L}(X) = \sum_\alpha a^*\left(\sqrt{\Lambda}\phi_\alpha\right) X a\left(\sqrt{\Lambda}\phi_\alpha\right) + a\left(\sqrt{\Delta}\phi_\alpha\right) X a^*\left(\sqrt{\Delta}\phi_\alpha\right) \tag{9.5}$$

$$-\frac{1}{2}\Gamma_+\left(\Lambda + \Delta - 2iH\right) X - \frac{1}{2} X \Gamma_+\left(\Lambda + \Delta + 2iH\right) - \mathrm{tr}(\Delta) X \tag{9.6}$$

Where $\Lambda, \Delta \in \mathfrak{B}(\mathcal{H}_1)$ are positive operators and H is self-adjoined. The whole expression can be rigorously defined on the domain of exponential vectors. Γ_+ denotes the Fock functor, i.e. the second quantization of the observables.

Δ describes the creation of new particles, and Λ describes the destruction of particles. The operator $\mathcal{H}$ is the self-adjoined Hamiltonian of the free evolution.

For simplicity we took $\mathcal{L}$ to be time independent. Furthermore we assume the operators Λ and Δ to be supported on a finite dimensional subspace. We choose to identify the label space with the one particle space of the underlying Fock space, i.e. $\mathcal{K} := \mathcal{H}_1 \oplus \mathcal{H}_1$, where $\mathcal{H} = \Gamma_+(\mathcal{H}_1)$. Note that the following solution works as well with time-dependent generators. We restrict to the case of a time-independent generator mostly for notational convenience.

Since $\mathcal{L}$ is time-independent it is fairly easy to check that it is a valid Lindblad generator. We only have to assert that $\Gamma_+\left(\Lambda + \Delta + 2iH\right)$ is the generator of a semigroup, which is immediate from our assumptions since $\Lambda + \Delta + 2iH$ is essentially a bounded perturbation of iH which is the generator of a semigroup by Stone's theorem. As $\mathcal{D}$ we can formally choose exp(dom(H)) the set of exponential vectors over the domain of H.

It is easy to see that the generator $\mathcal{L}$ formally conserve probability, i.e. $\mathcal{L}(\mathbb{1}) = 0$. We shall see that the resulting evolution also conserves probability.

9.2.2. Quasi free evolutions

From here on we denote Weyl operators acting on $\mathcal{K}_\infty$ by capital letter $W(\lambda)$ and Weyl operators acting on $\mathcal{H}$ by lowercase letter $w(\varphi)$. The generator of the Weyl operators is defined through: $w(t\varphi) = \exp(-it\Psi(\varphi))$. The annihilation operators are hence given as: $a(\varphi) := \frac{1}{2}\left(-\Psi(i\varphi) + i\Psi(\varphi)\right)$ and the creation operators as their adjoint.

The following commutation relations for Weyl operators, which can also be rigorously defined on the domain of exponential vectors, greatly help in solving the Lindblad equation.

$$w(\varphi)a(\psi)w(-\varphi) = a(\psi) - \langle\psi, \varphi\rangle\mathbb{1} \tag{9.7}$$

$$w(\varphi)a^*(\psi)w(-\varphi) = a^*(\psi) - \langle\varphi, \psi\rangle\mathbb{1} \tag{9.8}$$

A short calculation shows that:

$$\mathcal{L}(w(\varphi)) = \left(i\Psi\left(\left(+\frac{1}{2}(\Lambda - \Delta) - iH\right)\varphi\right) - \left\langle\lambda, \frac{1}{2}(\Lambda + \Delta) + iH\varphi\right\rangle\right)w(\varphi) \tag{9.9}$$

This property allows us to rigorously solve the Lindblad equations. The main trick lies in comparing the above expression with directional derivatives of the Weyl operators. Note that directional derivatives of Weyl operators are given as:

$$\begin{aligned}\frac{\partial}{\partial\psi}w(\varphi) &:= \lim_{h\to 0}\left(w(\varphi + h\psi) - w(\varphi)\right) &(9.10)\\ &= \lim_{h\to 0}\left(w(h\psi)e^{ih\,\mathfrak{Im}\langle\psi,\varphi\rangle} - \mathbb{1}\right)w(\varphi) &(9.11)\\ &= \left(-i\Psi(\psi) + i\,\mathfrak{Im}\langle\psi, \varphi\rangle\right)w(\varphi) &(9.12)\end{aligned}$$

The consequence being that $\mathcal{L}$ maps Weyl operators to a sum of Weyl operator and their first order derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}t}w(\varphi(t)) = \left(-i\Psi\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\right) + i\,\mathfrak{Im}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t), \varphi(t)\right\rangle\right)w(\varphi(t)) \tag{9.13}$$

We denote $\widehat{\mathbb{E}}(t) = \exp(t\mathcal{L})$. The Ansatz $\widehat{\mathbb{E}}(t)(w(\varphi)) = f(t)w(\varphi(t))$ with differentiable functions $\varphi(t)$ and $f(t)$ leads to the following equation:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}f(t) - i\Psi\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\right)f(t) + i\,\mathfrak{Im}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t), \varphi(t)\right\rangle f(t) &(9.14)\\ =&i\Psi\left(\left(\frac{1}{2}(\Lambda - \Delta) - iH\right)\varphi(t)\right)f_t - \left\langle\varphi(t), \frac{1}{2}(\Lambda + \Delta) + iH\varphi(t)\right\rangle f_t &(9.15)\end{aligned}$$

That is, we have to solve the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left(-\frac{1}{2}(\Lambda - \Delta) + iH\right)\varphi(t) \tag{9.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = -\left\langle\varphi(t), \frac{1}{2}(\Lambda + \Delta)\varphi(t)\right\rangle f(t) \tag{9.17}$$

With the initial conditions $\varphi(0) = \varphi$ and $f(0) = 1$. This clearly gives the solutions:

$$\varphi(t) = \exp\left(t\left(-\frac{1}{2}(\Lambda - \Delta) + iH\right)\right)\phi \tag{9.18}$$

$$f(t) = \exp\left(-\int_0^t \left\langle \varphi(t), \frac{1}{2}(\Lambda + \Delta)\varphi(t)\right\rangle \mathrm{d}s\right) \tag{9.19}$$

That is $\phi(t)$ is given through the evolution of a semigroup. Hence to understand the evolution in $\Gamma_+(\mathcal{H}_1)$ we only have to solve a semigroup on $\mathcal{H}_1$. This is a huge simplification. Another consequence is that the evolution generated by $\mathcal{L}$ maps Weyl operators to Weyl operators. This property is the what is used to define quasi free states on general CCR-algebras.

9.2.3. Characteristic functions

Using exactly the same tricks, we can calculate the characteristic functions for the field operators on the output field. Let $\lambda, \nu \in \mathrm{L}^2([0,T], \mathcal{H}_1)$. We are interested in calculating expectation values of the family of operators $\mathbb{E}_{[s,t]}(W((\lambda, \bar{\nu}))\otimes X)$ where $\bar{\nu}$ is the point-wise complex conjugate of the function ν.

According to 8.6 to get the characteristic function we have to solve the Cauchy equation, for which the formal generator is a perturbation of the original Lindbladian, i.e. $\mathcal{L} + \widetilde{\mathcal{L}}_t^{(\lambda,\bar{\nu})}$. The explicit form of the disturbance can readily be read of from lemma 8.6.

$$\begin{aligned}
&\widetilde{\mathcal{L}}^{(\lambda,\bar{\nu})}(w(\varphi)) \\
&= -\frac{1}{2}\left(\|\lambda(t)\|^2 + \|\nu(t)\|^2\right) w(\varphi) && (9.20)\\
&\quad + \sum_\alpha \left(\bar{\lambda}_\alpha(t) w(\varphi) a(\sqrt{\Lambda}|\alpha\rangle) - a^*(\sqrt{\Lambda}|\alpha\rangle\phi_\alpha) w(\varphi)\lambda_\alpha(t)\right) && (9.21)\\
&\quad + \sum_\alpha \left(\nu\alpha(t) w(\varphi) a^*(\sqrt{\Delta}|\alpha\rangle) - a(\sqrt{\Delta}|\alpha\rangle) w(\varphi)\bar{\nu}_\alpha(t)\right) && (9.22)
\end{aligned}$$

Again commuting the Weyl operator to the right, we see that also this generator maps Weyl operators to Weyl operators plus a directional derivative.

$$\begin{aligned}
\widetilde{\mathcal{L}}^{(\lambda,\bar{\nu})}(w(\varphi)) = \Bigg(& i\Psi\left(\sqrt{\Lambda}\lambda(t) - \sqrt{\Delta}\mu(t)\right) - \frac{1}{2}\left(\|\lambda(t)\|^2 + \|g(t)\|^2\right) \\
& - \left(\left\langle \sqrt{\Lambda}\lambda(t), \varphi\right\rangle + \left\langle \varphi, \sqrt{\Delta}\nu(t)\right\rangle\right)\Bigg) w(\varphi)
\end{aligned} \tag{9.23}$$

We can again simplify this differential equation by trying the Ansatz $\widehat{\mathbb{E}}(t)(w(\varphi)) = f(t)w(\varphi(t))$, which already allowed us to solve the "free" case.

We end up with the following set of first order differential equations. Contrary to the undisturbed evolution, we now have to solve an inhomogeneous equation. This is easily done because the semigroup $\exp\left(t\left(-\frac{1}{2}(\Lambda - \Delta) + iH\right)\right)$ solving the "free" case can be used to construct the inhomogeneous solution.

$$\dot{\varphi}(t) = \left(-\frac{1}{2}(\Lambda - \Delta) + iH\right)\varphi(t) - \sqrt{\Lambda}\lambda(t) + \sqrt{\Delta}\nu(t) \tag{9.24}$$

$$\dot{f}_t = -\Bigg(\left\langle \varphi(t), \frac{1}{2}(\Lambda + \Delta)\varphi(t)\right\rangle + \frac{1}{2}(\|\lambda(t)\|^2 + \|\nu(t)\|^2) \tag{9.25}$$
$$+ \mathfrak{Re}\left\langle \left(\sqrt{\Lambda}\lambda(t) + \sqrt{\Delta}\nu(t)\right), \lambda_t\right\rangle\Bigg) f(t)$$

Again we have the initial conditions $\varphi(0) = \varphi$ and $f(0) = 1$. If we require that $\lambda(t), \mu(t) \in \operatorname{dom}(H)$ for all t and furthermore that the functions are continuous, then the standard semigroup theory allows us to solve the first differential equation. The existence of a solution to the second equation does not need any further assumptions. Formal solutions of these equations can be written down in exactly the same manner as in the homogeneous case.

$$\varphi(t) = \exp\left(-\frac{1}{2}t(\Lambda - \Delta) + iH\right)\varphi$$
$$- \int_0^t \exp\left(-\frac{1}{2}(t-s)(\Lambda - \Delta) + iH\right)\left(\sqrt{\Lambda}\lambda(s) + \sqrt{\Delta}\nu(s)\right) \mathrm{d}s \tag{9.26}$$

And the corresponding exponential for $f(t)$.

Sadly this solution for the characteristic functional is not very practical when we want to calculate functional derivatives of it. Furthermore the same solution technique does not work as well for the characteristic functions associated to the measurement of second quantized POVMs, i.e., the point processes described in 8.3.3. Here the same ansatz leads to a coupled system of second order differential equations, which seems to be a lot harder to solve.

9.2.4. Perturbative calculations

Luckily there is an alternative method to calculate low order differentials of the characteristic functions, based on "perturbation theory" of differential equation. The existence of the $n-th$ moment measures of the characteristic function is equivalent to the existence of the functional derivatives to the same order [25, chapter 9.5].

Let $\mathbb{F}(\lambda)(s,t)$ be the evolution system describing the characteristic function, as in lemma 8.6, we are interested in. Here λ describes the parameters the characteristic functional depends upon, e.g. a function in $\mathrm{L}^2([0,T], \mathcal{H}_1 \oplus \mathcal{H}_1)$ in the case of the field operators. Similarly we denote the generator of this evolution by $\mathcal{L}_t^\lambda$. When we assume that the time derivative and the functional derivatives commute, we can write, for $s \in [0,T]$ and ξ in the same space as λ:

$$\frac{\partial}{\partial s}\frac{\partial}{\partial \xi}\mathbb{F}(\lambda)(s,T) = \frac{\partial}{\partial \xi}\frac{\partial}{\partial s}\mathbb{F}(\lambda)(s,T) = -\frac{\partial}{\partial \xi}\mathcal{L}(\lambda,s)\mathbb{F}(\lambda)(s,T) \tag{9.27}$$

$$= -\left(\frac{\partial}{\partial \xi}\mathcal{L}(\lambda,s)\right)\mathbb{F}(\lambda)(s,T) - \mathcal{L}(\lambda,s)\left(\frac{\partial}{\partial \xi}\mathbb{F}(\lambda)(s,T)\right). \tag{9.28}$$

If we iterate this construction, we can express the higher order functional derivatives of the characteristic functional as a system of ordinary differential equations. Moreover under the right conditions all the equations are inhomogeneous versions of the same homogeneous Cauchy equation.

Explicitly we introduce the notation: $\mathcal{C}_\lambda^{\xi_1,\xi_2,\dots} = \cdots \frac{\partial}{\partial\xi_2}\frac{\partial}{\partial\xi_1}C(\lambda)$ for $C(\lambda) \in \mathfrak{B}(\mathcal{H})$ we have to solve the following system of differential equations:

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}C_{\lambda,t} \\ \frac{\mathrm{d}}{\mathrm{d}t}C_{\lambda,t}^{\xi_1} \\ \frac{\mathrm{d}}{\mathrm{d}t}C_{\lambda,t}^{\xi_2} \\ \frac{\mathrm{d}}{\mathrm{d}t}C_{\lambda,t}^{\xi_1,\xi_2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathcal{L}_t^\lambda & & & & \\ \frac{\partial}{\partial\xi_1}\mathcal{L}_t^\lambda & \mathcal{L}_t^\lambda & & & \\ \frac{\partial}{\partial\xi_2}\mathcal{L}_t^\lambda & & \mathcal{L}_t^\lambda & & \\ \frac{\partial^2}{\partial\xi_2\partial\xi_1}\mathcal{L}_t^\lambda & \frac{\partial}{\partial\xi_2}\mathcal{L}_t^\lambda & \frac{\partial}{\partial\xi_1}\mathcal{L}_t^\lambda & \mathcal{L}_t^\lambda & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} C_{\lambda,t} \\ C_{\lambda,t}^{\xi_1} \\ C_{\lambda,t}^{\xi_2} \\ C_{\lambda,t}^{\xi_1,\xi_2} \\ \vdots \end{pmatrix} \tag{9.29}$$

The solution to the first differential equation is given by:

$$\mathcal{C}_{\lambda,s} = \mathbb{F}(\lambda)(s,T)(C), \tag{9.30}$$

for some final value $C \in \mathfrak{B}(\mathcal{H})$. Solutions to the other equations, can then be constructed exploiting the fact that the evolution system $\mathbb{F}(\lambda)(s,T)$ is a Greens function for the problem. For the first order derivatives we get:

$$C_\lambda^\xi(T) = \int_0^T \mathbb{F}(\lambda)(0,s) \circ \frac{\partial}{\partial\xi}\mathcal{L}_s^\lambda \circ \mathbb{F}(\lambda)(s,T)(C)\mathrm{d}s \tag{9.31}$$

For the second order moments we get expressions of the form:

$$\begin{aligned} C_\lambda^{\xi,\mu}(T) = \int_0^t \mathbb{F}(\lambda)(0,s) \circ \frac{\partial^2}{\partial\xi\partial\mu}\mathcal{L}_s^\lambda \circ \mathbb{F}(\lambda)(s,t)\mathrm{d}s + \\ \int_0^T\int_0^T \mathbb{F}(\lambda)(0,r) \circ \frac{\partial}{\partial\xi}\mathcal{L}_r^\lambda \circ \\ \mathbb{F}(\lambda)(r,s) \circ \frac{\partial}{\partial\mu}\mathcal{L}_s^\lambda \circ \mathbb{F}(\lambda)(s,T)(C)\mathrm{d}r\mathrm{d}s. \end{aligned} \tag{9.32}$$

Higher order can be calculated accordingly. Our potential to calculate moments of the stochastic process, thus rests on us being able to construct the family of evolution systems $\mathbb{F}(\lambda)(s,t)$. When we evaluate the derivatives of the characteristic function at 0 then $\mathbb{F}(0)(s,t)$ is just the quasi free semigroup, for which we constructed a solution earlier. If we furthermore used the identity operator $\mathbb{1}_\mathcal{H}$ and an invariant state ρ to define the cMPS, the calculations get even simpler, and calculation of the first order moments reduces to integration of a scalar function.

Application to field operators Applicability of this method to higher order terms is facilitated by the fact, that by lemma 8.6 generator of the characteristic function for the field operators is a second order polynomial in λ. This means differentials of $\mathcal{L}_t^\lambda$ of order higher than 2 vanish, which greatly simplifies the systems of differential equations we have to solve.

So when we apply the technique described above to the calculation of moments of the stochastic process describing the measurement of field operators we have to calculate functional derivatives of the generator $\widetilde{\mathcal{L}}^{(\lambda,\bar{\nu})}$ given in equation (9.23). A short calculation, where we again express the generator as a derivative of the Weyl operator, yields:

$$\begin{aligned}
&\frac{\partial}{\partial(h,\bar{g})}\widetilde{\mathcal{L}}^{(\lambda,\bar{\nu})}(w(\varphi)) \\
&= -\,\mathfrak{Re}\left(\langle\lambda(t),h(t)\rangle\langle\bar{\nu}(t),\bar{g}(t)\rangle + \left\langle\sqrt{\Lambda}h(t),\varphi\right\rangle\right. \\
&\qquad\left.+\left\langle\varphi,\sqrt{\Delta}\bar{g}(t)\right\rangle\right)w(\varphi) \\
&\qquad\qquad+\frac{\partial}{\partial(-\sqrt{\Lambda}h_t+\sqrt{\Delta}\bar{g}_t)}w(\varphi)
\end{aligned}$$

and

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2}\mathcal{L}_t^{\lambda}(w(\mu)) = -\,\mathfrak{Re}\langle\xi_2,\xi_1\rangle w(\mu) \tag{9.33}$$

9.2.5. Characteristic functions of point processes

We can of course use the same technique to calculate the characteristic function corresponding for the point processes. For simplicity we restrict to describing POVMs where we independently monitor particle creation and particle reductions, i.e. POVMs $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_1$ which are "block diagonal" in the sense that, given a measure space $(X, \mathcal{X})$ and two POVMs $M_1, M_2 : \mathcal{X} \to \mathfrak{B}(H_1)$, we are interested in POVMs of the form:

$$M : \mathcal{X}^2 \to \mathfrak{B}(\mathcal{K}) \qquad (\xi_1,\xi_2) \mapsto M((\xi_1,\xi_2))\begin{pmatrix} M_1(\xi_1) & 0 \\ 0 & M_2(\xi_1)\end{pmatrix} \tag{9.34}$$

Where $\mathcal{X}^2$ is the product σ-algebra of $X \times X$. The perturbation we have to add to the Lindbladian for the characteristic functions follows readily from corollary 8.16 together with lemma 8.6.

$$\widetilde{\mathcal{L}}_t^{g,h}(w(\mu)) \tag{9.35}$$

$$=\sum_{\alpha,\beta}\left\langle\alpha\middle|M_1\left[e^{ig(t)}-1\right]\middle|\beta\right\rangle a^*\left(\sqrt{\Lambda}|\alpha\rangle\right)w(\varphi)a\left(\sqrt{\Lambda}|\beta\rangle\right)$$

$$+\left\langle\alpha\right|, M_2\left[e^{ig(t)}-1\right]\left|\beta\right\rangle a\left(\sqrt{\Delta}|\alpha\rangle\right)w(\varphi)a^*\left(\sqrt{\Delta}|\beta\rangle\right) \tag{9.36}$$

As before we commute the Weyl operators to the right, which leads to the following form:

$$\widetilde{\mathcal{L}}_t^{g,h}(w(\mu))$$

$$=\sum_{\alpha,\beta}\left(\langle\alpha|M_1\left[e^{ig(t)}-1\right]+M_2\left[e^{ig(t)}-1\right]^*|\beta\rangle\right. \tag{9.37}$$

$$\left.a^*\left(\sqrt{\Lambda}|\alpha\rangle\right)a\left(\sqrt{\Lambda}|\beta\rangle\right)+a\left(\sqrt{\Delta}|\alpha\rangle\right)a^*\left(\sqrt{\Delta}|\beta\rangle\right)\right)w(\varphi) \tag{9.38}$$

$$-\left(a^*\left(\sqrt{\Lambda}M_1\left[e^{ig(t)}-1\right]\sqrt{\Lambda}\varphi\right)\right. \tag{9.39}$$

$$\left.+\,a\left(\sqrt{\Lambda}M_2\left[e^{ig(t)}-1\right]\sqrt{\Lambda}\varphi\right)\right)w(\varphi) \tag{9.40}$$

This can be re-expressed as a second order polynomial in directional derivatives applied to $w(\mu)$ using:

$$a(\varphi)w(\mu) = \frac{1}{2}\left(\langle\varphi,\mu\rangle - \left(\frac{\partial}{\partial\varphi} + i\frac{\partial}{\partial i\varphi}\right)\right)w(\psi) \tag{9.41}$$

$$a^*(\varphi)w(\mu) = \frac{1}{2}\left(\langle\mu,\varphi_i\rangle + \left(\frac{\partial}{\partial\varphi} - i\frac{\partial}{\partial i\varphi}\right)\right)w(\psi) \tag{9.42}$$

Many interesting properties of the point process can be calculated from their factorial moments, as discussed in section 2.2.5. Remember that the generating functional for them is related to the characteristic function of the point process $\mathcal{C}(f) = \widehat{\mathcal{C}}(\exp(if) - 1)$, where $\mathcal{C}(f)$ is the characteristic functional and $\widehat{\mathcal{C}}$ generates the factorial moments. The factorial moments have the advantage that they are far easier to calculate in a perturbative manner, because the perturbation to the Cauchy equation we have to solve for them looks like:

$$\widetilde{\mathcal{L}}_t^{g,h}(w(\mu))$$

$$= \sum_{\alpha,\beta}\Bigg(\langle\alpha|M_1\left[g(t)\right] + M_2\left[g(t)\right]^*|\beta\rangle \tag{9.43}$$

$$\left(a^*\left(\sqrt{\Lambda}|\alpha\rangle\right)a\left(\sqrt{\Lambda}|\beta\rangle\right) + a\left(\sqrt{\Delta}|\alpha\rangle\right)a^*\left(\sqrt{\Delta}|\beta\rangle\right)\right)\Bigg)w(\varphi) \tag{9.44}$$

$$-\Big(a^*\left(\sqrt{\Lambda}M_1\left[g(t)\right]\sqrt{\Lambda}\varphi\right) \tag{9.45}$$

$$+\, a\left(\sqrt{\Lambda}M_2\left[g(t)\right]\sqrt{\Lambda}\varphi\right)\Big)w(\varphi) \tag{9.46}$$

It follows that functional derivatives of second order already vanish.

10. Conclusion and open problems

10.1. Conclusion

In this thesis we studied the problem of describing all continuous-time measurements which are compatible with the given time evolution of an open quantum system. Here compatible means that the measurement does not change the systems evolution if one chooses to ignore all measurement results. And a continuous-time measurement is a description which allows to restrict observations to arbitrary time-intervals.

Since the goal was to describe all possible measurement we actually constructed a description of the quantum-information an open systems emits to its environment adapted to a continuous-time viewpoint. That is, we described observations of the quantum system as delayed-choice measurements, i.e. without fixing the observable. This viewpoint on measurements is heavily tied to the Stinespring dilation of completely-positive maps, which is a central pillar of all our constructions.

The first half of this thesis was then devoted to the study of the natural class of systems which allow for such an description. These are those quantum systems which are the minimal solution of a Lindblad equation. To enable us to describe the largest possible class of systems, we included time-dependence wherever possible.

Apart from getting a better understanding of the Lindblad equation and the related notion of quantum jumps this part also served a second purpose. We tried to explore how the class of solutions of Lindblad equations relates to the class of Markovian open quantum systems. Here Markovian means that we can stop and restart the evolution at arbitrary time-points, that is the description has the structure of a semigroup, or evolution system. To this extend we studied an old (counter-)example due to Holevo of a semigroup which is not quite of Lindblad form, that is a semigroup which is not a minimal solution of a Lindblad equation. We also got a partial description of continuous-time measurements compatible with this slightly larger class of quantum evolutions.

In the second half of this thesis, we solved the problem of describing all continuous-time measurements compatible with a given time evolution, whenever the evolution is the minimal solution of a Lindblad equation. Our solution has the form of a discrete-to continuous-time limit. This has the advantage that we can build on the discrete case, which is solved and does not pose any interpretational problems, thereby avoiding interpretational problems in the continuous-time case.

Our limit construction is based ob the following three observations.

1. Short time-steps are easy to describe approximately.

2. Piecing together solutions for time-steps is easy.

3. Comparing approximate short time-steps is easy.

The limit procedure has the form of a limit of finitely correlated states, which are also known as matrix product states. This structure allows easily to extend the description to measurements over arbitrarily long times and of beams in the stationary limit.

The limit of states was complemented by a limit of observables. That is we constructed continuous-time observables from their discrete time counterparts, thereby also avoiding interpretational problems. The results can be read as a Levy-Khintchine formula. We also discussed the relations between the gauge freedom of the Lindblad equation and the existence of Weyl operators on the limit dilation space.

Our investigation ended with the discussion of an example of a class of continuous-time evolutions: the quasi free evolutions. These evolutions are an interesting model for sources of particle beams.

10.2. Outlook and open problems

There are still lots of interesting open questions related to both parts of this thesis. There is for example the big question for applications of the formalism we build up. A possible concrete application could be a rigorous study of the ground state of the fractional quantum hall effect, as it was proposed in [87, 39].

Another whole line of research would be to transfer all those interesting and useful properties finitely correlated states (FCS) have to the continuous case. For example it is known that every state on a spin chain can be expressed as a FCS. And many people are interested in finding a good generalization of the parent Hamiltonian construction to the continuous case. We will not go into details here, but concentrate on four possible research directions.

Firstly Holevo's counterexample and related constructions, secondly the existence of a dominant Kraus operator for generic quantum evolutions, thirdly the question of minimality of our construction and last but not least the addition of feedback to our description of measurements.

10.2.1. Holevo's counterexample

Apart from the proof of its existence, which we reproduced in section 5 Holevo's counterexample is still far from being understood. For example it is entirely unclear, at least to the author, if the construction is in any way generic, or just a very particular case.

To get a better understanding of this example, it would be interesting to get a better description of the manner in which the example escapes the description by cMPS. That is one would like to understand in which way the event which reinserts the system in a fixed state is different from the first set of Lindblad operators, which amount to differentiating the function. In some way the second type of event in equation (5.56) eludes having a diffusive description, at least if it is added to the first Lindblad equation (5.29). That is the second type of event can only be measured in a measurement-on-arrival-type setup while the statistics of the first one can either be measured in such a way or in a homodyne measurement setup.

The other big question is the existence of other similar examples. For example it still remains to be studied if Holevo's example can be understood as merely having a degenerate domain algebra in the sense of Arveson [3]. And furthermore if semigroups with degenerate domain algebras, such as the one at the end of Arveson's paper or [44], are other examples of semigroups which are not minimal solutions to a Lindblad equation.

And of course it is still to be answered how one should best modify the Lindblad form to encompass all Markov evolutions into the same framework. There is also the question if there exist completely-positive and strongly-continuous semigroups, which can not even be described by the framework introduced in section 4.

10.2.2. Dominant Kraus operator

It would be interesting to find out if a generalization of the notion of a dominant Kraus operator leads to a better understanding of the Lindblad form or possibly to an extension.

A dominant Kraus operator would be something like the following definition:

Definition 10.1. Let $\mathcal{H}$ be a Hilbert space and $\mathbb{E}_*(t) : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ a strongly-continuous semigroup of completely-positive maps, we say that a family of operators $U(t) : \mathcal{H} \to \mathcal{H}$ for $t \in [0, T]$ is a *dominant Kraus operator* for $\mathbb{E}_*(t)$ iff

1. $\lim_{t\to 0} \|\mathbb{E}_*(t)(\rho) - U(t)\rho U^*(t)\|_{\mathrm{tr}} = 0$.
2. $\mathbb{E}_*(t) - \mathrm{ad}_{U(t)}$ is completely-positive for all t, where $\mathrm{ad}_{U(t)}(\rho) := U(t)\rho U^*(t)$.

From the Stinespring dilation theorem it is clear that dominant Kraus operators always exists. The interesting question is: Can the family $U(t)$ be chosen continuous? Actually one would like to find a semigroup $U(t)$ which is a dominant Kraus operator. For minimal solutions to Lindblad equations the no-event semigroup always defines a dominant Kraus operator. It is an intriguing question, if it is always possible to find a semigroup, which is a dominant Kraus operator. Furthermore classification of all dominant Kraus operators with semigroup structure for a fixed completely-positive semigroup would be interesting, as this question is intimately connected with the gauge freedom of the Lindblad equation.

10.2.3. Minimality

From a more abstract viewpoint our limit construction of FCS as well as the construction of Parthasarathy in [71], which we described in section 6.5.2 both are continuous Stinespring dilations. That is common dilations for Kernels of the following form:

$$\left\langle \varphi, \mathbb{E}_{t_0,t_1}\Big(X_1^* \dots E_{t_{n-2},t_{n-1}}\big(X_{n-1}^* \mathbb{E}_{t_{n-1},t_n}(X_n^* Y_n) Y_{n-1}\big) \dots Y_1\Big)\psi \right\rangle \tag{10.1}$$

Where $\mathbb{E}(s,t) \in \mathfrak{CP}(\mathfrak{B}(\mathcal{H}))$ is a given evolution system, and $X_i, Y_i \in \mathfrak{B}(\mathcal{H})$ as well as $t_i \in [0, T]$ can be chosen arbitrary.

This observation leads to the following definition:

Definition 10.2. Given a two-parameter family of Hilbert spaces $K(s,t)$ with an associated family of unitary operators: $U(r,s,t) : \mathcal{K}(s,r) \otimes \mathcal{K}(s,t) \to \mathcal{K}(r,t)$.

A continuous Stinespring dilation for a family of completely-positive maps $\mathbb{E}(s,t) \in \mathfrak{CP}(\mathfrak{B}(\mathcal{H}))$ is a family of maps $V_{s,t} : \mathcal{H} \to \mathcal{K}(s,t) \otimes \mathcal{H}$ such that

$$\mathbb{E}_{s,t} = V_{t,s}^* \mathbb{1}_{\mathcal{K}(s,t)} \otimes X V_{t,s} \tag{10.2}$$

h

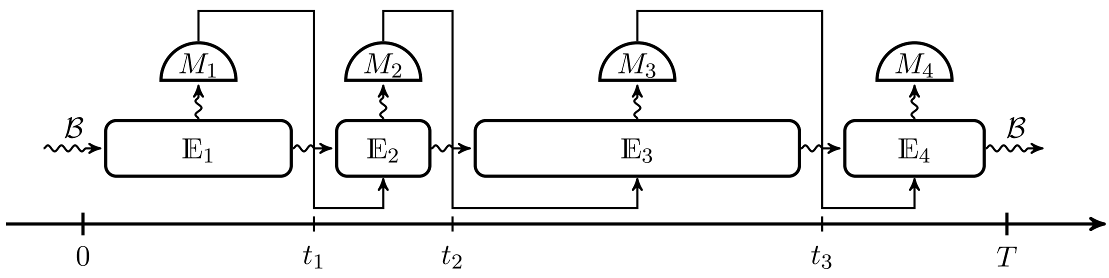


Figure 10.1.: A possible discrete time version of instantaneous feedback

and

$$V_{t,s} \circ V_{s,r} = U(r,s,t) \otimes \mathbb{1} V(t,r) \tag{10.3}$$

A continuous dilation is minimal if the span of the following sets is dense in $\mathcal{K}(s,r)$ for Θ

$$\left\{ \prod_{i \in I(\Theta)} (\mathbb{1} \otimes X_i) V_{t_i, t_{i-1}} \varphi \middle| \Theta \in \mathfrak{Z}([0,T]), \varphi \in \mathcal{H}, X_i \in \mathfrak{B}(\mathcal{H}) \right\} \tag{10.4}$$

If $V_{t,s}$ is a continuous Stinespring dilation for $\mathbb{E}(s,t)$ we have:

$$\left\langle \varphi, \mathbb{E}_{t_0,t_1} \Big(X_1^* \dots E_{t_{n-2},t_{n-1}} \big(X_{n-1}^* \mathbb{E}_{t_{n-1},t_n} (X_n^* Y_n) Y_{n-1} \big) \dots Y_1 \Big) \psi \right\rangle \tag{10.5}$$

$$= \left\langle \prod_{i=1}^{n} (\mathbb{1} \otimes X_i) V_{t_i,t_{i-1}} \varphi, \prod_{i=1}^{n} (\mathbb{1} \otimes Y_i) V_{t_i,t_{i-1}} \psi \right\rangle \tag{10.6}$$

It is now an interesting question if the continuous Stinespring dilation $V_{t,s} := V_{[s,t]}$ defined in chapter 7 is minimal. We conjecture that this is the case for dilations associated to strongly-continuous semigroups which are minimal solutions of Lindblad equations.

10.2.4. Feedback

As we already know the formalism we build up can be used to describe measurements on open quantum system in continuous time. Feedback seems to be a natural addition to this formalism. That means we would like to make the evolution at some time-point dependent on the measurements we performed on the system up to that time-point. This version of feedback is known as classical feedback.

Classical feedback fits naturally into the theory we already build up, because our approach already incorporates two important ingredients for its description. Firstly, to describe feedback one needs a good description of the measurements up to a certain time. Secondly, the time-dependence of the generator in our approach could probably be interpreted as a classical control. A nice introduction into feedback can be found in [14].

The last important point about feedback is, that its description is completely clear in a discrete-time setup. It is thus natural hope, that our discrete to continuous time limit extends to the case with feedback. A discrete time picture of a possible feedback scheme can be found in figure 10.1.

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