On Microelectromechanical Systems with General Permittivity

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Zusammenfassung

Gegenstand der vorliegenden Arbeit ist die mathematische Untersuchung von Systemen gekoppelter partieller Differentialgleichungen, die von der Modellierung elektrostatisch betriebener mikroelektromechanischer Systeme mit allgemeiner Permittivität herrühren. Eine Herleitung verschiedener Modelle wird vorgestellt, die es dem Leser ermöglicht, eine Einsicht in die diversen physikalischen Aspekte zu erlangen, die entsprechend der jeweiligen Anwendung in Betracht gezogen werden können. In jedem Fall koppeln alle adäquaten Systeme ein entweder semi- oder quasilineares hyperbolisches oder parabolisches Evolutionsproblem für die Auslenkung einer elastischen Membran mit einem elliptischen freien Randwertproblem, das das elektrostatische Potential in dem Gebiet zwischen der elastischen Membran und einer starren Bodenplatte determiniert.

In der Folge wird das qualitative Verhalten der Lösungen zweier verschiedener gekoppelter Probleme studiert. Genauer beinhalten beide betrachteten Systeme das elliptische freie Randwertproblem zur Bestimmung des elektrostatischen Potentials, das lediglich entsprechend der Wahl des Permittivitätsprofils variiert. Eher kleine oder große Auslenkungen der Membran beschreibend, kommt ein entweder semi- oder quasilineares parabolisches Evolutionsproblem hinzu. Für beide Systeme wird gezeigt, dass sie für alle beliebigen positiven Werte λ der angelegten Spannung lokal bezüglich Zeit wohlgestellt sind. Kleine Werte λ der angelegten Spannung, die einen gewissen kritischen Wert λ_* nicht überschreiten, lassen sogar global in der Zeit existierende Lösungen zu. Im semilinearen Fall wird für ein gegen Null konvergierendes Längenverhältnis des Geräts Konvergenz der Lösungen des vollen gekoppelten Problems gegen diejenigen des entkoppelten sogenannten *Small-Aspect Ratio Models* nachgewiesen.

Des Weiteren wird ein Thema behandelt, das erst mit der Berücksichtigung nicht-konstanter Permittivitätsprofile Bedeutung erlangt – die Richtung der Membranauslenkung oder, in mathematischer Ausdrucksweise, das Vorzeichen der Lösung des Evolutionsproblems. Mit Hilfe des parabolischen Vergleichsprinzips werden strukturelle Bedingungen an das Potential sowie das Permittivitätsprofil spezifiziert, die Nicht-Positivität der Membranauslenkung garantieren. Für gewisse Permittivitätsprofile wird schließlich bewiesen, dass Singularitäten nach endlicher Zeit auftreten können, sobald die angelegte Spannung einen bestimmten kritischen Wert λ^* überschreitet. Die Arbeit schließt mit einer numerischen Analyse des semilinearen Problems, die insbesondere die Betrachtung des vollen gekoppelten Problems rechtfertigt, indem sie wesentliche qualitative Unterschiede zwischen den Lösungen des weitverbreiteten *Small-Aspect Ratio Models* und denen des vollen gekoppelten Modells aufzeigt.

Schlüsselwörter: Mikroelektromechanische Systeme (MEMS), Permittivität, partielle Differentialgleichungen, freie Randwertprobleme, nichtlineare Evolutionsgleichungen, Singularitäten nach endlicher Zeit

Abstract

Of concern is the mathematical investigation of systems of coupled partial differential equations arising from the modelling of electrostatically actuated microelectromechanical systems with general permittivity profile. A derivation of different models is presented that enables the reader to establish an understanding of the various physical modelling aspects that might be taken into account according to the particular application. Howsoever, all suitable systems couple an either semi- or quasilinear hyperbolic or parabolic evolution problem for the displacement of an elastic membrane with an elliptic moving boundary problem that determines the electrostatic potential in the region between the elastic membrane and a rigid ground plate.

Subsequently the qualitative behaviour of the solutions of two different coupled problems is studied. More precisely, both systems under consideration consist of the elliptic free boundary problem for the determination of the electrostatic potential, which varies solely according to the choice of the permittivity profile. Describing rather small or large deflections of the membrane, an either semi- or quasilinear parabolic evolution problem is added. Both systems are shown to be well-posed locally in time for all arbitrary positive values λ of the applied voltage. Small values λ of the applied voltage, that do not exceed a certain critical value λ_*, do even allow globally in time existing solutions. For the semilinear case we establish the convergence of solutions to the full coupled problem towards those of the decoupled so-called small-aspect ratio model, as the aspect ratio of the device tends to zero.

Furthermore, a topic is addressed that is of note not till non-constant permittivity profiles are taken into account – the direction of the membrane's deflection or, in mathematical parlance, the sign of the solution to the evolution problem. By means of the parabolic comparison principle structural conditions on the potential and on the permittivity profile are specified which guarantee non-positivity of the membrane's displacement. For certain permittivity profiles we finally prove that finite-time singularities may occur as soon as the applied voltage exceeds a certain critical value λ^*. We complete the work by a numerical analysis of the semilinear problem that in particular justifies the consideration of the full coupled problem by revealing substantial qualitative differences of the solutions to the widely-used small-aspect ratio model and the full coupled problem.

Keywords: Microelectromechanical systems (MEMS), permittivity, partial differential equations, free boundary value problem, nonlinear evolution equations, finite-time singularities

Résumé

La thèse concerne l'investigation mathématique des systèmes d'équations aux dérivées partielles couplées, qui découlent de la modélisation des microsystèmes électromécaniques avec une permittivité générale. Une dérivation des différents modèles est présentée, ce qui permet au lecteur d'acquérir une compréhension des nombreux aspects physiques qui peuvent être pris en considération conformément à l'application visée. Quoi qu'il en soit, tous les systèmes appropriés couplent une équation d'évolution semi- ou quasilinéaire qui est soit hyperbolique soit parabolique pour modéliser la déformation d'une membrane élastique et un problème elliptique à frontière libre. Ce dernier détermine le potentiel électrique dans la région située entre la membrane élastique et une plaque à la masse. Ci-après le comportement qualitatif des solutions de deux différents problèmes couplés est étudié. Plus précisément, les deux systèmes considérés se composent d'un problème elliptique à frontière libre pour la détermination du potentiel électrique, qui varie exclusivement en fonction du choix du profil de permittivité. Un problème d'évolution parabolique semilinéaire ou quasilinéaire est ajouté, décrivant respectivement des petites ou des grandes déformations de la membrane.

Il est montré que les deux systèmes sont localement bien posés dans le temps pour n'importe quelle valeur $\lambda > 0$ de la tension électrique appliquée. Pour de petites valeurs λ de la tension électrique appliquée, n'excédant pas une certaine valeur critique λ^*, permettent même une solution unique qui existe globalement et pas que localement. Pour le cas semilinéaire la convergence des solutions du problème couplé vers celles du modèle élancé (*small-aspect ratio model*) est établie, lorsque le rapport hauteur/largeur tend vers zéro.

De plus, l'utilisation de profils de permittivité non-constants rend non-triviale l'étude du signe de la solution du problème d'évolution ou en termes mécaniques l'étude de la direction de la déformation de la membrane. En employant le principe du maximum parabolique des conditions structurelles au potentiel et au profil de permittivité sont spécifiées pour garantir la non-positivité de la déformation de la membrane. Enfin, la formation de singularités en temps fini pour certains profils de permittivité du moment que la tension électrique excède une certaine valeur critique λ^* est prouvée. Le travail est terminé par une analyse numérique du problème semilinéaire, qui en particulier justifie la considération du problème entier couplé en démontrant des différences qualitatives entre les solutions du *small-aspect ratio model* communément utilisé et celles du problème couplé.

Mots clés: Microsystèmes électromécaniques (MEMS), permittivité, équations aux dérivées partielles, problème à frontière libre, équation d'évolution nonlinéaire, singularités en temps fini

Contents

1 Introduction **1**

2 The Modelling **5**
2.1 A Nonlinear Elasticity Model . . . 6
2.2 A Simplified Linear Elasticity Model . . . 18
2.3 The Mathematical Models Under Study . . . 20

3 Local Well-Posedness and Global Existence **25**
3.1 On the Semilinear Case . . . 25
3.2 On the Quasilinear Case . . . 42

4 The Small-Aspect Ratio Limit **57**

5 On Some Qualitative Properties of Solutions **77**
5.1 Non-Positivity of the Membrane's Displacement . . . 77
5.2 Non-Existence of Global Solutions . . . 84
5.2.1 Finite-Time Singularities in the Semilinear Setting . . . 85
5.2.2 Finite-Time Singularities in the Quasilinear Setting . . . 95

6 Numerical Investigations **109**
6.1 Approximate Solution of the Elliptic Moving Boundary Problem . . . 110
6.2 Approximate Solution of the Parabolic Evolution Problem . . . 116
6.3 Numerical Results . . . 124
Bibliography . . . 133

1 | Introduction

Moving boundary problems or synonymously *free boundary value problems* frequently arise in a natural way when describing complex physical or chemical phenomena in nature and technique. They denote systems of partial differential equations which are in particular characterised by the fact that they are to be solved for a domain whose boundary is not known a priori and thus itself constitutes a part of the solution. Due to this coupling between the components of the full solution moving boundary problems are inherently nonlinear, making their analytical and numerical investigation evidently rather involved. On the other hand their intricate and nonlinear nature provides a more accurate description of complex processes than linear or nonlinear models on fixed domains. As a consequence in the last decades the investigation of moving boundary problems has received remarkable attention in applied mathematics. In this spirit, the present thesis is devoted to an analysis of free boundary value problems describing the dynamic behaviour of *microelectromechanical systems*.

Microelectromechanical systems (MEMS) constitute a technology of miniaturised devices whose dimensions range between some micrometres and one millimetre. Being typically made up of a sensor, a transistor as well as a mechanical actuator, MEMS sense the environment and act on it by combining microelectronics with non-electronic activities from micromechanics, fluidics or optics. Whereas the component of microsensors is already well developed, the understanding and construction of microactuators still pose a challenge and thus also deserve study in different fields of science [9]. The underlying technology is based on the approach of generating mechanical motion by for instance electrostatic, thermal, hydraulic, magnetic or other forces which act by reason of a perception of the environment by a sensor.

Due to their low manufacturing costs, their low demand for energy, their high reliability and in particular their tremendous versatility, MEMS have found their way into numerous branches of industry and science. The automotive industry, telecommunications or the biomedical industry shall be instanced here in order to provide an insight into the enormous range of applications. As inertial sensors MEMS are used for the activation of airbags [6] and for the protection of hard disks or for mechanical image stabilisation in optic devices, to mention only few examples. Furthermore, MEMS are applied as micro pumps [11] and micro valves [20] in micro fluidics.

As mentioned above there are various different microactuation principles, each having advantages for particular requirements. For instance micromagentic actuation exhibits remarkable advantages such as *high forces, large deflections, low input impedances and thus, the involvement of only low voltages* [9], once it is integrated in MEMS devices. However, since key components for micromagnetic actuation are three-dimensional, other microactuation principles are still favoured in general, but nonetheless, micromagnetic actuators are for instance beneficial in the context of MEMS devices with high aspect ratio. In the framework of this thesis MEMS devices are studied which perform mechanical motion by electrostatic actuation. Being initially in a configuration in which the mechanical components are separate, a voltage is applied across the device such that the components are at different electric potentials/electric charges. This imbalance of potentials/charges acting on each other induces attractive or repulsive forces which are described by *Coulomb's law*.

More precisely, a certain type of an idealised electrostatically actuated MEMS device is investigated which consists of a rigid ground plate and an elastic membrane that is suspended above the former and held fixed at its boundary. Moreover, the deformable elastic membrane is assumed to be of infinitely small thickness and features a certain dielectric permittivity profile. In order to cause a mechanical deflection of the latter, a voltage is applied across the device such that the ground plate and the membrane are at different electric potentials which induces a *Coulomb force* and thus gives rise to a deformation of the membrane. A sketch of such a MEMS device is offered in Figure 1.1. A necessity in order to understand the mode of operation of the device is to gain knowledge about the membrane's deformation on the one hand and about the electrostatic potential in the region occupied by the ground plate and the membrane on the other hand.

In fact, in the mathematical modelling of the dynamics of electrostatically actuated MEMS devices a strong coupling between those two quantities becomes apparent. More precisely, an elliptic problem is to be solved for the electrostatic potential in a domain whose boundary evolves with time as the membrane deflects with time. To describe the dynamics of the free boundary a further partial differential equation is to be specified.

In order to avoid the handling of the resultant difficulties, researchers have heretofore exploited the fact that in a multitude of applications the aspect ratio of the device, i.e. the ratio of height and length of the device, is rather small. More precisely, the assumption of a negligibly small aspect ratio allows an explicit expression for the electrostatic potential, whereby the coupled problem is reduced to a single evolution equation whose right-hand side features a singularity in the moment the membrane touches down on the ground plate. However, it is worthwhile to mention that the assumption of a vanishing aspect ratio is not reasonable in all applications [1]. As examples for MEMS devices high aspect ratio turbines and micromotors may be mentioned.

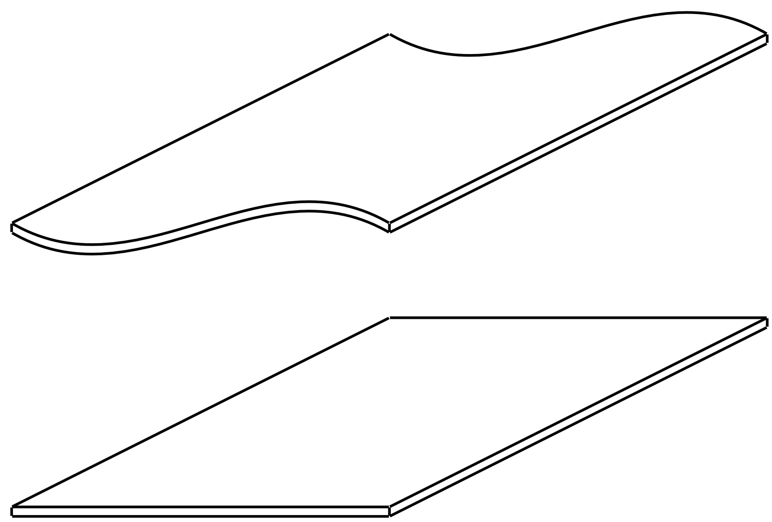


Figure 1.1: Sketch of the investigated idealised MEMS device.

Hitherto, various theoretical contributions in engineering science, physics and mathematics have been dedicated to the investigation of MEMS devices in order to better understand their behaviour and thus to advance the corresponding technology. Whereas a multitude of them treats the case of a vanishing aspect ratio (see for instance [18, 20, 21, 25, 26, 27, 29, 33, 39]), pioneering results on the coupled problem go back to Escher, Laurençot and Walker. In their recent contributions the authors take different physical modelling aspects into account but always assume the permittivity profile f to be constant. The work [32] deals with stationary solutions in the semilinear regime, whereas in [14] the semilinear evolution problem is investigated. Moreover, the reader shall be referred to the works [13, 15, 34, 35], each of them again assuming the permittivity to be constant but taking other different physical aspects, such as large deflections or bending effects, into account. Further investigations of qualitative properties of MEMS systems may be found in [36, 37, 38]. However, none of the above mentioned works is concerned with a coupled system, including the additional feature of a general permittivity profile $f = f(x, u(t, x))$. To the best of the author's knowledge, this thesis together with the related papers [41, 40, 17, 16, 12] constitute the first contribution in that direction.

It is the intention of this thesis to analyse different coupled systems of partial differential equations characterising the dynamic behaviour of MEMS devices constructed as described above. In order to specify the different components of the analysis, the introduction is closed by outlining the organisation of this thesis.

The purpose of Chapter 2 is to provide an overview of the various mathematical models describing the dynamic behaviour of electrostatically actuated MEMS devices according to their appearance in applications. We end up with two coupled systems consisting of an either semi- or quasilinear parabolic evolution problem for the membrane's displacement and an elliptic moving boundary problem determining the electrostatic potential in the region between the deformable membrane and the rigid ground plate.

Chapters 3–5 are then devoted to a qualitative analysis of these two coupled problems. More precisely, in Chapter 3 both problems are shown to be well-posed locally in time for all arbitrarily large values λ of the applied voltage. Moreover, it is proved that the solutions exists even globally in time, provided that the applied voltage does not exceed a critical value λ_*; see also [40] for the results on the semilinear case.

Chapter 4 is restricted to the case of a semilinear evolution problem describing the membrane's displacement. The convergence of solutions to the coupled problem towards those of the widely-used reduced *small-aspect ratio model* is established, as the aspect ratio tends to zero [40].

The direction of the membrane's deflection as well as finite-time singularities are the subjects teated in Chapter 5, c.f. also the works [41, 17, 16]. Structural conditions are specified for different classes of permittivity profiles which ensure that the membrane deflects towards the ground plate. In addition, these non-positive solutions are shown to cease to exist after a finite time of existence under certain additional assumptions.

The thesis is completed by a numerical investigation of the semilinear coupled problem. *Finite elements* and the *Crank–Nicolson method* are introduced as they are used to serve the purpose of numerically computing approximate solutions to the full coupled problem. The results reveal in particular considerable differences in the qualitative behaviour of solutions to the semilinear coupled problem and its decoupled counterpart [12].

2 | The Modelling

In this chapter the equations governing the dynamic behaviour of an idealised electrostatically actuated MEMS device with general permittivity profile are derived.

The investigated type of MEMS devices consists of two quadrilateral components – a flat rigid ground plate and an elastic membrane that is suspended above the former. The elastic membrane is coated with a thin conducting film on its upper surface and it features in addition a certain dielectric permittivity profile.

In our investigations all ingredients of the system are assumed to be homogeneous in one lateral direction so that we may in fact restrict the analysis to a cross section of the device. Denoting by $\tilde{x}$ and $\tilde{z}$ the horizontal and vertical direction, respectively, we consider the ground plate to be located at height $\tilde{z} = -h$ and the undeflected membrane at $\tilde{z} = 0$, both having the length $2l$. Moreover, the length $2l$ of the device is assumed to be large compared to the gap size h of the undeformed configuration, which means that we are in the regime of a small *aspect ratio* $\varepsilon = \frac{h}{l} \ll 1$.

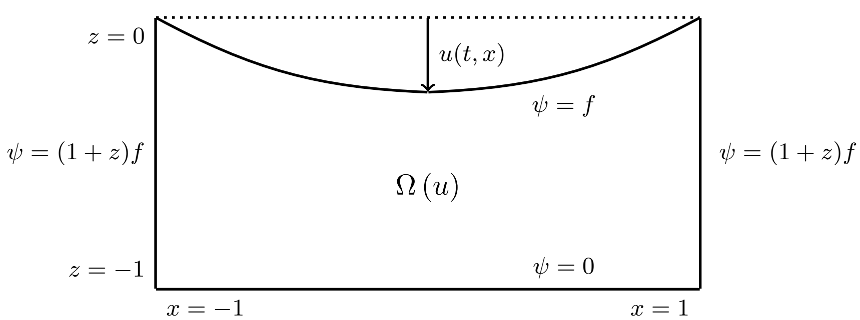


Figure 2.1: Cross section of the investigated idealised MEMS device.

An application of a voltage V to the conducting film on the membrane, such that the grounded plate and the membrane are at different electric potentials, induces a deformation of the elastic membrane assumed to be only in $\tilde{z}$-direction. We denote the deformation at time $\tilde{t} \geq 0$ and position $\tilde{x} \in L := (-l, l)$ by $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x})$. The second quantity of general interest, the electrostatic potential at time $\tilde{t} \geq 0$ and a certain position $(\tilde{x}, \tilde{z})$ in the region between the ground plate and the elastic membrane is denoted by $\tilde{\psi} = \tilde{\psi}(\tilde{t}, \tilde{x}, \tilde{z})$. It is worthwhile to mention again that the shape of this region changes with time as the membrane deflects with time. Finally we denote the permittivity profile of the membrane by $f = f\big(\tilde{x}, \tilde{u}(\tilde{t}, \tilde{x})\big)$.

2.1 | A Nonlinear Elasticity Model

For the nonce the time variable $\tilde{t}$ appears as a parameter, whence it is temporarily suppressed in the notation.

Governing Equations for the Electrostatic Potential. Pursuant to *Gauß' law of electrodynamics* the electrostatic potential is harmonic in the region

$$\tilde{\Omega}(\tilde{u}) := \{(\tilde{x}, \tilde{z}); -l < \tilde{x} < l, -h < \tilde{z} < \tilde{u}(\tilde{x})\}$$

between the rigid ground plate and the membrane, that is

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{\psi}_{\tilde{z}\tilde{z}} = 0, \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}).$$

Furthermore, the fixed plate at $\tilde{z} = -h$ is grounded, i.e. at zero potential, whereas the membrane is at potential $V f\big(\tilde{x}, \tilde{u}(\tilde{x})\big)$. These boundary conditions are expressed by the equations

$$\tilde{\psi}(\tilde{x}, -h) = 0, \quad \tilde{\psi}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) = V f\big(\tilde{x}, \tilde{u}(\tilde{x})\big), \quad \tilde{x} \in L.$$

Governing Equations for the Membrane's Deformation. By means of nonlinear elasticity theory we first derive the governing equations for the case of static plate deformations under the *hypotheses of Love–Kirchhoff*. In particular this includes the assumption that vectors normal to the middle surface remain normal to the middle surface after deformation. We refer the reader for instance to [8] for a detailed view on these hypotheses. Finally we assume the elastic plate to be of infinitely small thickness which reduces the more general model for plate deformations to one describing deformations of elastic membranes. If no ambiguity is to be feared we use both expressions suitably.

The total *potential energy* E_p of the configuration, which is generated due to the deformation of the elastic plate, is constituted by the pointwise sum of *stretching energy* E_s, *bending energy* E_b and *electrostatic energy* E_e, i.e. it holds

$$E_p(\tilde{u}) = E_s(\tilde{u}) + E_b(\tilde{u}) + E_e(\tilde{u}).$$

Denoting by $\tau > 0$ the tension constant of the plate, the *stretching energy* is given by

$$E_s(\tilde{u}) = \tau \int_{-l}^{l} \left(\sqrt{1 + \left(\tilde{u}_{\tilde{x}}(\tilde{x})\right)^2} - 1 \right) d\tilde{x}. \tag{2.1}$$

The integral describes the variation of the plate's length from $2l$, i.e. from the situation in which deformation is absent.

The likewise involved *bending energy* is proportional to the L_2-norm of the plate's curvature. More precisely, it is given by

$$E_b(\tilde{u}) = \frac{b}{2} \int_{-l}^{l} \left(\partial_x \left(\frac{\tilde{u}_{\tilde{x}}(\tilde{x})}{\sqrt{1 + (\tilde{u}_{\tilde{x}}(\tilde{x}))^2}} \right) \right)^2 \sqrt{1 + (\tilde{u}_{\tilde{x}}(\tilde{x}))^2}\, d\tilde{x},$$

where the coefficient b, describing the flexural rigidity of the plate, is defined as

$$b = \frac{2\alpha^3 Y}{3(1-\nu)^2}.$$

The parameters in this ratio denote the thickness α of the plate, the *Young modulus* Y and the *Poisson ratio* ν.

Finally, the *electrostatic energy* is given by

$$E_e(\tilde{u}) = -\frac{\varepsilon_0}{2} \int_{-l}^{l} \int_{-h}^{\tilde{u}(\tilde{x})} \left(\nabla \tilde{\psi}(\tilde{x}, \tilde{z}) \right)^2 d\tilde{z}\, d\tilde{x} = -\frac{\varepsilon_0}{2} \int_{\tilde{\Omega}(\tilde{u})} \left(\nabla \tilde{\psi}(\tilde{x}, \tilde{z}) \right)^2 d(\tilde{x}, \tilde{z}),$$

with ε_0 being the permittivity of free space. The variation of E_e corresponds to the work of the force on the elastic plate that is induced by the electric field with potential $\tilde{\psi}(\tilde{x}, \tilde{z})$.

Consequently, the total potential energy of the system is given by

$$E_p(\tilde{u}) = \tau \int_{-l}^{l} \left(\sqrt{1 + (\tilde{u}_{\tilde{x}}(\tilde{x}))^2} - 1 \right) d\tilde{x} \; + \; \frac{b}{2} \int_{-l}^{l} \left(\partial_{\tilde{x}} \left(\frac{\tilde{u}_{\tilde{x}}(\tilde{x})}{\sqrt{1 + (\tilde{u}_{\tilde{x}}(\tilde{x}))^2}} \right) \right)^2 \sqrt{1 + (\tilde{u}_{\tilde{x}}(\tilde{x}))^2} d\tilde{x}$$
$$- \frac{\varepsilon_0}{2} \int_{\tilde{\Omega}(\tilde{u})} \left(\nabla \tilde{\psi}(\tilde{x}, \tilde{z}) \right)^2 d(\tilde{x}, \tilde{z}).$$

Derivation of the Euler-Lagrange Equation. Due to *Hamilton's principle of least action* the partial differential equation describing the dynamics of the plate's deformation is the *Euler–Lagrange equation* which is obtained by minimising a suitable energy functional.

The time dependent part of the problem is considered in a second step, whereas in a first step we derive the Euler–Lagrange equation in terms of static deflections. To this end we define the *Lagrangian*

$$\mathcal{L} : L \times W_2^4(L) \longrightarrow \mathbb{R}$$

by

$$\begin{aligned}
\mathcal{L}(\tilde{x}, \tilde{u}) = & -\tau \left(\sqrt{1 + (\tilde{u}_{\tilde{x}})^2} - 1 \right) - \frac{b}{2} \partial_{\tilde{x}} \left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1 + (\tilde{u}_{\tilde{x}})^2}} \right)^2 \sqrt{1 + (\tilde{u}_{\tilde{x}})^2} \\
& + \frac{\varepsilon_0}{2} \int_{-h}^{\tilde{u}} \left(\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{z}) \right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{z}) \right)^2 d\tilde{z} \\
= & -\tau \left(\sqrt{1 + (\tilde{u}_{\tilde{x}})^2} - 1 \right) - \frac{b}{2} \frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1 + (\tilde{u}_{\tilde{x}})^2)^{5/2}} + \frac{\varepsilon_0}{2} \int_{-h}^{\tilde{u}} \left(\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{z}) \right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{z}) \right)^2 d\tilde{z}
\end{aligned}$$

and minimise the according energy functional

$$\int_{-l}^{l} \mathcal{L}(\tilde{x}, \tilde{u}) \, d\tilde{x} \tag{2.2}$$

by means of *calculus of variations*. The problem of minimising an integral over an infinite dimensional function space is then treated as the problem of minimising a function of a single real-valued variable.[1]

In order to accomplish the latter problem assume that $\tilde{u} = \tilde{u}(\tilde{x})$ is the current minimiser of (2.2), satisfying

$$\tilde{u} \in W_{2,D}^4(L), \quad \tilde{u}(\tilde{x}) > -h, \quad \tilde{x} \in [-l, l]. \tag{2.3}$$

Then, given $\sigma \in \mathbb{R}$ and a function $v \in C_c^\infty(L)$, we introduce the notation

$$w(\sigma)(\tilde{x}) := \tilde{u}(\tilde{x}) + \sigma v(\tilde{x}), \quad \tilde{x} \in [-l, l],$$

and derive the necessary condition for $\tilde{u}$ being a minimiser of (2.2) by computing the *first variation*

$$\delta E_p(\tilde{u}; v) = \frac{d}{d\sigma} E_p(\tilde{u} + \sigma v)|_{\sigma=0} = \delta \Big(E_s(\tilde{u}; v) + E_b(\tilde{u}; v) + E_e(\tilde{u}; v) \Big)|_{\sigma=0} \tag{2.4}$$

[1] In physics and engineering it is common to consider regularity assumptions as physically given and thus to presume the validity of the Euler–Lagrange equations. Mathematically speaking we therefore just verify the necessary condition for the existence of an extremum of the functional.

and finally checking the condition $\delta E_p(\tilde{u};v)=0$. For the stretching term one obtains

$$\frac{d}{d\sigma}E_s(\tilde{u}+\sigma v)=\frac{d}{d\sigma}\left(\tau\int_{-l}^{l}\sqrt{1+(w_{\tilde{x}})^2}-1\,d\tilde{x}\right)=\tau\int_{-l}^{l}\frac{\tilde{u}_{\tilde{x}}\,v_{\tilde{x}}+\sigma(v_{\tilde{x}})^2}{\sqrt{1+(\tilde{u}_{\tilde{x}})^2+2\sigma\tilde{u}_{\tilde{x}}\,v_{\tilde{x}}+\sigma^2(v_{\tilde{x}})^2}}\,d\tilde{x}$$

and therefore, using the fact that v is compactly supported in the interval L,

$$\delta E_s(\tilde{u};v)=\tau\int_{-l}^{l}\frac{\tilde{u}_{\tilde{x}}v_{\tilde{x}}}{\sqrt{1+(\tilde{u}_{\tilde{x}})^2}}=-\tau\int_{-l}^{l}v\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1+(\tilde{u}_{\tilde{x}})^2}}\right)d\tilde{x}. \tag{2.5}$$

For the bending term, note that

$$\partial_{\tilde{x}}\left(\frac{w_{\tilde{x}}}{\sqrt{1+(w_{\tilde{x}})^2}}\right)=\frac{w_{\tilde{x}\tilde{x}}}{(1+(\tilde{w}_{\tilde{x}})^2)^{3/2}},$$

whence we may write

$$\begin{aligned}\frac{d}{d\sigma}E_b(\tilde{u}+\sigma v)&=\frac{d}{d\sigma}\left(\frac{b}{2}\int_{-l}^{l}\left(\partial_{\tilde{x}}\left(\frac{w_{\tilde{x}}}{\sqrt{1+(w_{\tilde{x}})^2}}\right)\right)^2\sqrt{1+(w_{\tilde{x}})^2}\,d\tilde{x}\right)\\&=\frac{d}{d\sigma}\left(\frac{b}{2}\int_{-l}^{l}\frac{(w_{\tilde{x}\tilde{x}})^2}{(1+(w_{\tilde{x}})^2)^{5/2}}\,d\tilde{x}\right)\\&=b\int_{-l}^{l}\frac{(\tilde{u}_{\tilde{x}\tilde{x}}+\sigma v_{\tilde{x}\tilde{x}})v_{\tilde{x}\tilde{x}}}{(1+(\tilde{u}_{\tilde{x}})^2+2\sigma\tilde{u}_{\tilde{x}}v_{\tilde{x}}+\sigma^2(v_{\tilde{x}})^2)^{5/2}}\,d\tilde{x}-\frac{5b}{2}\int_{-l}^{l}\frac{\big(\tilde{u}_{\tilde{x}}v_{\tilde{x}}+\sigma(v_{\tilde{x}})^2\big)\big(\tilde{u}_{\tilde{x}\tilde{x}}+\sigma(v_{\tilde{x}\tilde{x}})\big)^2}{\big(1+\tilde{u}_{\tilde{x}}^2+2\sigma\tilde{u}_{\tilde{x}}v_{\tilde{x}}+\sigma^2(v_{\tilde{x}})^2\big)^{7/2}}\,d\tilde{x}\end{aligned}$$

and again using that $v(\pm l)=0$ we obtain

$$\begin{aligned}\delta E_b(\tilde{u};v)&=b\int_{-l}^{l}\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1+(\tilde{u}_{\tilde{x}})^2)^{5/2}}v_{\tilde{x}\tilde{x}}\,d\tilde{x}-\frac{5b}{2}\int_{-l}^{l}\frac{\tilde{u}_{\tilde{x}}(\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1+(\tilde{u}_{\tilde{x}})^2)^{7/2}}v_{\tilde{x}}\,d\tilde{x}\\&=b\int_{-l}^{l}\partial_{\tilde{x}}^2\left(\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1+(\tilde{u}_{\tilde{x}})^2)^{5/2}}\right)v\,d\tilde{x}+\frac{5b}{2}\int_{-l}^{l}\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}(\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1+(\tilde{u}_{\tilde{x}})^2)^{7/2}}\right)v\,d\tilde{x}.\end{aligned} \tag{2.6}$$

It finally remains to take the electrostatic energy into account and to calculate $\delta E_e(\tilde{u}+\sigma v)$. In the sequel this is done by an application of the *transport theorem*, c.f. [4, XII, Theorem 2.11] or [28, Theorem 5.2.2] for instance. To this end, given $\sigma\in\mathbb{R}, v\in C_c^\infty(L)$ and $w(\sigma)(\tilde{x})=\tilde{u}(\tilde{x})+\sigma v(\tilde{x})$ as above, we pick $\sigma_0>0$ such that the choice of $\tilde{u}$ as in (2.3) implies that $w(\sigma)(\tilde{x})>-h$ for all $\tilde{x}\in[-l,l]$ and all $\sigma\in[-\sigma_0,\sigma_0]$ and such that we may introduce the well-defined and connected open set

$$\tilde{\Omega}_\sigma:=\{(\tilde{x},\tilde{z})\in L\times(-h,\infty);-h<\tilde{z}<w(\sigma)(\tilde{x})\},\quad \sigma\in[-\sigma_0,\sigma_0].$$

In addition, there exists a representation

$$\tilde{\Omega}_\sigma=\phi\big(\sigma;\tilde{\Omega}(\tilde{u})\big),$$

of $\tilde{\Omega}_\sigma$ via the (global) diffeomorphism

$$\phi(\sigma;\tilde{x},\tilde{z}) := \left(\tilde{x}, \tilde{z} + \sigma v(\tilde{x}) \frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right), \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u}) = \tilde{\Omega}_0.$$

In order to be able to handle the electrostatic energy with a variational approach it is necessary to investigate the problem for $\tilde{\psi}$ corresponding to the variation w of the minimiser $\tilde{u}$ in direction v. For this purpose denote by $\tilde{\psi}(\sigma;\tilde{u},v) \in W_2^2(\tilde{\Omega}_\sigma)$ the solution to

$$\tilde{\psi}_{\tilde{x}\tilde{x}}(\sigma;\tilde{u},v) + \tilde{\psi}_{\tilde{z}\tilde{z}}(\sigma;\tilde{u},v) = 0, \qquad (\tilde{x},\tilde{z}) \in \tilde{\Omega}_\sigma, \tag{2.7}$$

$$\tilde{\psi}(\sigma;\tilde{u},v) = \frac{h+\tilde{z}}{h+w(\sigma)(\tilde{x})} V f\big(\tilde{x}, w(\sigma)(\tilde{x})\big), \quad (\tilde{x},\tilde{z}) \in \partial\tilde{\Omega}_\sigma. \tag{2.8}$$

Moreover, we introduce the *velocity* $\mathcal{V}$ of the path $\{\tilde{\psi}(\sigma;\tilde{u},v); \sigma \in (-\sigma_0,\sigma_0)\}$, defined as[2]

$$\mathcal{V} := \frac{d}{d\sigma}\psi(\sigma;\tilde{u},v)|_{\sigma=0}, \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u}). \tag{2.9}$$

and show that also $\mathcal{V}$ satisfies (2.7)–(2.8) in the limit $\sigma = 0$. To this end, observe that (2.7) is equivalent to[3]

$$\tilde{\psi}_{\tilde{x}\tilde{x}}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right) + \tilde{\psi}_{\tilde{z}\tilde{z}}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right) = 0, \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u}),$$

whence a differentiation of this equation with respect to σ yields

$$\begin{aligned} &\tilde{\psi}_{\tilde{x}\tilde{x}\sigma}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right) + \tilde{\psi}_{\tilde{x}\tilde{x}\tilde{z}}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right) v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})} \\ &+ \tilde{\psi}_{\tilde{z}\tilde{z}\sigma}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right) + \tilde{\psi}_{\tilde{z}\tilde{z}\tilde{z}}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right) v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})} = 0, \end{aligned} \tag{2.10}$$

for all $(\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u})$. Then, letting $\sigma \to 0$ in (2.10), we first find that

$$\mathcal{V}_{\tilde{x}\tilde{x}}(\tilde{x},\tilde{z}) + \mathcal{V}_{\tilde{z}\tilde{z}}(\tilde{x},\tilde{z}) + v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\Big(\tilde{\psi}_{\tilde{x}\tilde{x}\tilde{z}}(\tilde{x},\tilde{z}) + \tilde{\psi}_{\tilde{z}\tilde{z}\tilde{z}}(\tilde{x},\tilde{z})\Big) = 0, \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u}),$$

whence by (2.7)

$$\mathcal{V}_{\tilde{x}\tilde{x}}(\tilde{x},\tilde{z}) + \mathcal{V}_{\tilde{z}\tilde{z}}(\tilde{x},\tilde{z}) = 0, \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u}).$$

[2]Note that in fact $\mathcal{V}$ is a function of the variables $\tilde{x}$ and $\tilde{z}$ in the sense that

$$\mathcal{V}(\tilde{x},\tilde{z}) = \frac{d}{d\sigma}\psi(\sigma;\tilde{u},v)|_{\sigma=0}(\tilde{x},\tilde{z}).$$

[3]In fact $\tilde{\psi}(\sigma;\tilde{u},v)$ is a function of the variables $\tilde{x}$ and $\tilde{z}$. For the sake of simplicity we suppress the dependence of $\tilde{\psi}$ of $\tilde{u}$ and v and use the notation $\tilde{\psi}(\sigma;\tilde{x},\tilde{z})$.

In addition, one can infer from the boundary condition (2.8) that

$$\tilde{\psi}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right)=\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}Vf\big(\tilde{x},\tilde{u}(\tilde{x})+\sigma v(\tilde{x})\big),\quad (\tilde{x},\tilde{z})\in\partial\tilde{\Omega}(\tilde{u}),$$

and differentiating this identity with respect to σ yields

$$\begin{aligned}\tilde{\psi}_\sigma\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right)+\tilde{\psi}_{\tilde{z}}\left(\sigma;\tilde{x},\tilde{z}+\sigma v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right)v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\\ =\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}Vf_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})+\sigma v(\tilde{x})\big)v(\tilde{x})\end{aligned}$$

for $(\tilde{x},\tilde{z})\in\partial\tilde{\Omega}(\tilde{u})$. Finally, as $\sigma\to 0$ we find that

$$\mathcal{V}(\tilde{x},\tilde{z})=v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\left(Vf_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)-\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\right),\quad (\tilde{x},\tilde{z})\in\partial\tilde{\Omega}(\tilde{u}). \tag{2.11}$$

Since $v\in C_c^\infty(L)$ and $h+\tilde{z}=0$ for $\tilde{z}=-h$, one may in particular extract from equation (2.11) the identities

$$\begin{aligned}\mathcal{V}(\pm l,\tilde{z})&=0,\quad \tilde{z}\in(-h,0),\\ \mathcal{V}(\tilde{x},-h)&=0,\quad \tilde{x}\in L,\\ \mathcal{V}(\tilde{x},\tilde{u}(\tilde{x}))&=v(\tilde{x})\left(Vf_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)-\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right),\quad \tilde{x}\in L.\end{aligned} \tag{2.12}$$

Having this preliminary knowledge at hand, we are finally prepared to consider the energy

$$E_e(\tilde{u}+\sigma v)=-\frac{\varepsilon_0}{2}\int_{\tilde{\Omega}_\sigma}\big(\tilde{\psi}_{\tilde{x}}(\sigma;\tilde{x},\tilde{z})\big)^2+\big(\tilde{\psi}_{\tilde{z}}(\sigma;\tilde{x},\tilde{z})\big)^2\,d(\tilde{x},\tilde{z})$$

or, more precisely, its derivative with respect to σ at $\sigma=0$. Firstly, invoking [28, Thm. 5.2.2] yields the identity

$$\begin{aligned}\delta E_e(\tilde{u};v)=&-\varepsilon_0\int_{\tilde{\Omega}(\tilde{u})}\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{z})\mathcal{V}_{\tilde{x}}(\tilde{x},\tilde{z})+\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\mathcal{V}_{\tilde{z}}(\tilde{x},\tilde{z})\,d(\tilde{x},\tilde{z})\\ &-\varepsilon_0\int_{\tilde{\Omega}(\tilde{u})}\mathrm{div}\left(\frac{\big(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{z})\big)^2+\big(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\big)^2}{2}\phi_\sigma(0,\tilde{x},\tilde{z})\right)d(\tilde{x},\tilde{z})\end{aligned} \tag{2.13}$$

and using (2.7) one can readily see that

$$\begin{aligned}\mathrm{div}\left(\mathcal{V}\left(\tilde{\psi}_{\tilde{x}},\tilde{\psi}_{\tilde{z}}\right)\right)&=\mathcal{V}_{\tilde{x}}\tilde{\psi}_{\tilde{x}}+\mathcal{V}\tilde{\psi}_{\tilde{x}\tilde{x}}+\mathcal{V}_{\tilde{z}}\tilde{\psi}_{\tilde{z}}+\mathcal{V}\tilde{\psi}_{\tilde{z}\tilde{z}}\\ &=\mathcal{V}_{\tilde{x}}\tilde{\psi}_{\tilde{x}}+\mathcal{V}_{\tilde{z}}\tilde{\psi}_{\tilde{z}}+\mathcal{V}\left(\tilde{\psi}_{\tilde{x}\tilde{x}}+\tilde{\psi}_{\tilde{z}\tilde{z}}\right)\\ &=\mathcal{V}_{\tilde{x}}\tilde{\psi}_{\tilde{x}}+\mathcal{V}_{\tilde{z}}\tilde{\psi}_{\tilde{z}}\end{aligned} \tag{2.14}$$

holds true for all $(\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u})$. Then, fusing the findings (2.13) and (2.14) leads to the equation

$$\begin{aligned}\delta E_e(\tilde{u};v) = & -\varepsilon_0 \int_{\tilde{\Omega}(\tilde{u})} \operatorname{div}\Big(\mathcal{V}(\tilde{x},\tilde{z})\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{z}),\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\right)\Big)\, d(\tilde{x},\tilde{z}) \\ & -\varepsilon_0 \int_{\tilde{\Omega}(\tilde{u})} \operatorname{div}\left(\frac{\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{z})\right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\right)^2}{2}\phi_\sigma(0,\tilde{x},\tilde{z})\right) d(\tilde{x},\tilde{z}).\end{aligned}$$

Allowing for the identity

$$\phi_\sigma(0;\tilde{x},\tilde{z}) = \left(0, v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\right), \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}(\tilde{u}),$$

an application of the *Green–Riemann integration theorem* reveals

$$\begin{aligned}\delta E_e(\tilde{u};v) = & -\varepsilon_0 \int_{\partial\tilde{\Omega}(\tilde{u})} \mathcal{V}(\tilde{x},\tilde{z})\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{z})\, d\tilde{z} + \varepsilon_0 \int_{\partial\tilde{\Omega}(\tilde{u})} \mathcal{V}(\tilde{x},\tilde{z})\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\, d\tilde{z} \\ & + \frac{\varepsilon_0}{2}\int_{\partial\tilde{\Omega}(\tilde{u})} v(\tilde{x})\frac{h+\tilde{z}}{h+\tilde{u}(\tilde{x})}\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{z})\right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{z})\right)^2\right) d\tilde{x}.\end{aligned}$$

Then, exploiting the relations $v(\pm l) = 0$, $h + \tilde{z} = 0$ for $\tilde{z} = -h$, and the boundary conditions (2.12) for $\mathcal{V}$, the above integrals vanish at the lateral boundaries and on the ground plate at $\tilde{z} = -h$, whereby we obtain

$$\begin{aligned}\delta E_e(\tilde{u};v) = & \, \varepsilon_0 V \int_{-l}^{l} v(\tilde{x}) f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{u}_{\tilde{x}}(\tilde{x}) - \tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right) d\tilde{x} \\ & -\varepsilon_0 \int_{-l}^{l} v(\tilde{x})\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{u}_{\tilde{x}}(\tilde{x}) - \left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2\right) d\tilde{x} \\ & -\frac{\varepsilon_0}{2}\int_{-l}^{l} v(\tilde{x})\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2\right) d\tilde{x}.\end{aligned}$$

From the boundary condition $\tilde{\psi}\big(\tilde{x},\tilde{u}(\tilde{x})\big) = V f\big(\tilde{x},\tilde{u}(\tilde{x})\big)$, $\tilde{x} \in L$, we may deduce the equality

$$\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{u}_{\tilde{x}}(\tilde{x}) = V\Big(f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big) + f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{u}_{\tilde{x}}(\tilde{x})\Big) - \tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big),$$

and it follows that

$$\begin{aligned}\delta E_e(\tilde{u};v) = & \, \varepsilon_0 V \int_{-l}^{l} v(\tilde{x}) f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\Big(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{u}_{\tilde{x}}(\tilde{x}) - \tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\Big) d\tilde{x} \\ & -\varepsilon_0 V \int_{-l}^{l} v(\tilde{x})\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\Big(f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big) + f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\tilde{u}_{\tilde{x}}(\tilde{x})\Big) d\tilde{x} \\ & +\frac{\varepsilon_0}{2}\int_{-l}^{l} v(\tilde{x})\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x},u(\tilde{x}))\right)^2\right) d\tilde{x}.\end{aligned}$$

This equation may finally be rewritten as

$$\begin{aligned}\delta E_e(\tilde{u};v) = \frac{\varepsilon_0}{2}\int_{-l}^{l} v(\tilde{x})\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2+\left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2\right)d\tilde{x}\\ -\varepsilon_0 V\int_{-l}^{l} v(\tilde{x})\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)+\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right)d\tilde{x}.\end{aligned} \tag{2.15}$$

Recalling (2.4) as well as the equality

$$\delta E_p(\tilde{u};v) = \delta\Big(E_s(\tilde{u};v)+E_b(\tilde{u};v)+E_e(\tilde{u};v)\Big) = 0$$

as a necessary condition for $\tilde{u}$ being a minimiser of the energy functional (2.2), we may see by (2.5), (2.6) and (2.15) that this is satisfied for all suitable functions v, if and only if $\tilde{u}$ complies with the Euler–Lagrange equation

$$\begin{aligned}0 = \tau\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1+(\tilde{u}_{\tilde{x}})^2}}\right)-b\partial_{\tilde{x}}^2\left(\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1+(\tilde{u}_{\tilde{x}})^2)^{5/2}}\right)-\frac{5b}{2}\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}(\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1+(\tilde{u}_{\tilde{x}})^2)^{7/2}}\right)\\ -\frac{\varepsilon_0}{2}\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2+\left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2\right)+\varepsilon_0 V\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)+\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right).\end{aligned}$$

Heretofore, static deflections of the elastic plate are discussed and it remains to take the dynamics into account. This means that from now on the time variable $\tilde{t}$ explicitly returns to the notation. More precisely, denoting by ρ the mass density per unit volume of the plate and recalling that α denotes its thickness, due to *Newton's Second Law* the sum of all forces is equal to $\rho\alpha\tilde{u}_{\tilde{t}\tilde{t}}(\tilde{t},\tilde{x})$ and we get

$$\begin{aligned}\rho\alpha\tilde{u}_{\tilde{t}\tilde{t}}(\tilde{t},\tilde{x})-\tau\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1+(\tilde{u}_{\tilde{x}})^2}}\right)+b\partial_{\tilde{x}}^2\left(\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1+(\tilde{u}_{\tilde{x}})^2)^{5/2}}\right)+\frac{5b}{2}\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}(\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1+(\tilde{u}_{\tilde{x}})^2)^{7/2}}\right)\\ =-\frac{\varepsilon_0}{2}\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2+\left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2\right)+\varepsilon_0 V\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)+\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right).\end{aligned}$$

Lastly, the superposition of the elastic and electrostatic forces is combined with a *damping force* $-a\tilde{u}_{\tilde{t}}$ which is linearly proportional to the velocity $\tilde{u}_{\tilde{t}}$ with damping constant a. That is, we obtain

$$\begin{aligned}\rho\alpha\tilde{u}_{\tilde{t}\tilde{t}}(\tilde{t},\tilde{x})+a\tilde{u}_{\tilde{t}}-\tau\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1+(\tilde{u}_{\tilde{x}})^2}}\right)+b\partial_{\tilde{x}}^2\left(\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1+(\tilde{u}_{\tilde{x}})^2)^{5/2}}\right)+\frac{5b}{2}\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}(\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1+(\tilde{u}_{\tilde{x}})^2)^{7/2}}\right)\\ =-\frac{\varepsilon_0}{2}\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2+\left(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\right)^2\right)+\varepsilon_0 V\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)+\tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right).\end{aligned}$$

Fusing the above considerations we end up with the following coupled system of partial differential equations. The elliptic free boundary value problem for the electrostatic potential in the region determined by the grounded plate at $\tilde{z}=-h$ and the membrane at $\tilde{z}=\tilde{u}(\tilde{t},\tilde{x})$, both of length $2l$, reads

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{\psi}_{\tilde{z}\tilde{z}} = 0, \qquad \tilde{t} > 0,\ (\tilde{x}, \tilde{z}) \in \tilde{\Omega}(\tilde{u}), \tag{2.16}$$

$$\tilde{\psi}(\tilde{t}, \tilde{x}, \tilde{z}) = \frac{h + \tilde{z}}{h + \tilde{u}(\tilde{t}, \tilde{x})} f\big(\tilde{x}, \tilde{u}(\tilde{t}, \tilde{x})\big), \quad \tilde{t} > 0,\ (\tilde{x}, \tilde{z}) \in \partial\tilde{\Omega}(\tilde{u}), \tag{2.17}$$

where the conditions $\tilde{\psi} = 0$ and $\tilde{\psi} = V f\big(\tilde{x}, \tilde{u}(\tilde{t}, \tilde{x})\big)$ on the ground plate and the membrane, respectively, are continuously extended to the lateral boundaries $(\pm l, \tilde{z})$, $\tilde{z} \in (-h, 0)$. The dynamics of the deflection $\tilde{u}$ is thus described by the fourth-order equation

$$\begin{aligned} \rho\alpha \tilde{u}_{\tilde{t}\tilde{t}} + a\tilde{u}_{\tilde{t}} + \tilde{A}_1(\tilde{u}) = &- \frac{\varepsilon_0}{2} \Big(\big(\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x}))\big)^2 + \big(\tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{u}(\tilde{x}))\big)^2 \Big) \\ &+ \varepsilon_0 V \Big(\tilde{\psi}_{\tilde{x}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) f_{\tilde{x}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) + \tilde{\psi}_{\tilde{z}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) f_{\tilde{u}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) \Big), \end{aligned} \tag{2.18}$$

where $\tilde{A}_1(\tilde{u})$ is the quasilinear fourth-order differential operator defined by

$$\tilde{A}_1(\tilde{u}) := -\tau \partial_{\tilde{x}} \left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1 + (\tilde{u}_{\tilde{x}})^2}} \right) + b\partial_{\tilde{x}}^2 \left(\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1 + (\tilde{u}_{\tilde{x}})^2)^{5/2}} \right) + \frac{5b}{2} \partial_{\tilde{x}} \left(\frac{\tilde{u}_{\tilde{x}} (\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1 + (\tilde{u}_{\tilde{x}})^2)^{7/2}} \right).$$

Furthermore, we assume the membrane to be clamped at its boundary $(\pm l, 0)$ and to have a certain initial deflection $\tilde{u}_*(\tilde{x})$ at time $\tilde{t} = 0$. This is expressed by the clamped boundary conditions

$$\tilde{u}(\tilde{t}, \pm l) = \tilde{u}_{\tilde{x}}(\tilde{t}, \pm l) = 0, \quad \tilde{t} > 0,$$

and the initial conditions

$$\tilde{u}(0, \tilde{x}) = \tilde{u}_*(\tilde{x}), \quad \tilde{u}_{\tilde{t}}(0, \tilde{x}) = \tilde{u}_{**}(\tilde{x}), \quad \tilde{x} \in L,$$

respectively.

2.1.1 Remark
We briefly discuss two variants of the above modelling by distinguishing energy conserving and energy dissipating systems. Whereas the first occurs when damping effects are neglected, the latter corresponds to the case of no inertial effects.

(1) *We assume to be in an energy conserving Hamiltonian regime in which damping is not taken into account. The* total energy *of the system is defined as the pointwise difference of* kinetic energy E_k *and potential energy* E_p*. The kinetic energy at any instant in time is described by the functional*

$$E_k(\tilde{u}) = \frac{\rho\alpha}{2} \int_{-l}^{l} \big(\tilde{u}_{\tilde{t}}\big)^2 d\tilde{x}.$$

Immediately taking dynamics into account, given $0 < t_1 < t_2 < \infty$*, Hamilton's principle means* *to minimise the* action *of the system, i.e. the double integral*

$$\int_{t_1}^{t_2} \int_{-l}^{l} \mathcal{L}(\tilde{t}, \tilde{x}, \tilde{u})\, d\tilde{x}\, d\tilde{t},$$

where the Lagrangian $\mathcal{L}$ is now given by[4] $\mathcal{L} : (t_1, t_2) \times L \times W_2^{2,4}\big((t_1, t_2) \times L\big) \to \mathbb{R}$,

$$\begin{aligned}\mathcal{L}(\tilde{t}, \tilde{x}, \tilde{u}) = {} & \frac{\rho\alpha}{2}\tilde{u}_{\tilde{t}}^2 - \tau\left(\sqrt{1 + (\tilde{u}_{\tilde{x}})^2} - 1\right) - \frac{b}{2}\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(1 + (\tilde{u}_{\tilde{x}})^2)^{5/2}} \\ & + \frac{\varepsilon_0}{2}\int_{-h}^{\tilde{u}} \left(\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{z})\right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{z})\right)^2 d\tilde{z},\end{aligned}$$

and the corresponding Euler–Lagrange equation, obtained by a straightforward adaption of the above calculations, reads

$$\begin{aligned}\rho\alpha\tilde{u}_{\tilde{t}\tilde{t}} + \tilde{A}_1(\tilde{u}) = {} & -\frac{\varepsilon_0}{2}\left(\left(\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{u}(\tilde{x}))\right)^2 + \left(\tilde{\psi}_{\tilde{z}}(\tilde{x}, \tilde{u}(\tilde{x}))\right)^2\right) \\ & + \varepsilon_0 V\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) f_{\tilde{x}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) + \tilde{\psi}_{\tilde{z}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big) f_{\tilde{u}}\big(\tilde{x}, \tilde{u}(\tilde{x})\big)\right).\end{aligned}$$

(2) *Being in the energy dissipating regime where inertial effects are neglected, we shall see in the following that the corresponding evolution equation may formally be perceived as a* gradient flow system.

(i) *Let H be a Hilbert space over $\mathbb{R}$ with inner product $(\cdot, \cdot)_H$ and let $E \in C(H, \mathbb{R})$ denote a continuous functional on H. Given $v \in H$, assume that*

$$\delta E(v; w) := \frac{d}{d\sigma}E(v + \sigma w)|_{\sigma=0}$$

exists in H for all $w \in H$. Under this hypothesis assume in addition that there is a $z(v) \in H$ such that

$$\big(z(v), w\big)_H = \delta E(v; w), \quad w \in H.$$

Note that $z(v)$ is uniquely determined if it exists. We call $z(v)$ the generalised gradient *of E at v and use the notation*

$$\nabla E(v) := z(v).$$

If $E \in C^1(H, \mathbb{R})$ then $\nabla E(v)$ exists for all $v \in H$ with

$$DE(v)w = \big(\nabla E(v), w\big)_H, \quad w \in H.$$

(ii) *Given $T > 0$, consider $v \in C^1\big((0, T), H\big)$ and assume that $\nabla E\big(v(t)\big)$ exists in H for all*

[4] Note that $W_2^{2,4}\big((t_1, t_2) \times L\big)$ denotes the usual anisotropic Sobolev space with respect to $\tilde{t}$ and $\tilde{x}$.

$t \in (0,T)$. If v complies with the equation

$$v'(t) = -\nabla E\big(v(t)\big), \quad t \in (0,T), \tag{2.19}$$

then we say that v is a solution to the gradient flow system associated to E on $(0,T)$.

(iii) Suppose that E is contained in $C^1(H,\mathbb{R})$ and $v \in C^1\big((0,T),H\big)$ is a solution to (2.19) on $(0,T)$. Then $E\big(v(t)\big)$ is decreasing on $(0,T)$. Indeed $E\big(v(\cdot)\big)$ is differentiable on $(0,T)$ and the chain rule yields

$$\frac{d}{dt}E\big(v(t)\big) = \big(\nabla E(v(t)), v'(t)\big)_H = -\|\nabla E\big(v(t)\big)\|_H^2, \quad t \in (0,T). \tag{2.20}$$

Interpreting E as an energy the last equation reveals the energy dissipation of the system. Moreover, if the path $v(t)$ avoids any critical point of E the dissipation is strict.

(iv) Taking $H = L_2(L)$ and $E(\tilde{u}) = E_p(\tilde{u})$ with $\tilde{u} \in W_{2,D}^4(L)$ we deduce from (2.5), (2.6) and (2.15) that formally

$$\begin{aligned}\nabla E_p(\tilde{u}) = &-A_1(\tilde{u}) - \frac{\varepsilon_0}{2}\left(\big(\tilde{\psi}_{\tilde{x}}(\tilde{x},\tilde{u}(\tilde{x}))\big)^2 + \big(\tilde{\psi}_{\tilde{z}}(\tilde{x},\tilde{u}(\tilde{x}))\big)^2\right)\\ &+\varepsilon_0 V\left(\tilde{\psi}_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{x}}\big(\tilde{x},\tilde{u}(\tilde{x})\big) + \tilde{\psi}_{\tilde{z}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)f_{\tilde{u}}\big(\tilde{x},\tilde{u}(\tilde{x})\big)\right).\end{aligned}$$

This means that if $\rho = 0$ and $a = 1$ equation (2.18) may be perceived as the gradient flow system associated to E_p in $L_2(L)$.

SCALING – INTRODUCTION OF DIMENSIONLESS VARIABLES. Now dimensionless variables are introduced and the above terms and equations are rewritten in dimensionless form. To that effect, the electrostatic potential is scaled with the applied voltage,

$$\psi = \frac{\tilde{\psi}}{V},$$

the time is scaled with a damping timescale of the system,

$$t = \frac{\tau}{al^2}\tilde{t},$$

and the variables $\tilde{x}$ and $\tilde{z}$ as well as $\tilde{u}$ are scaled with the length l and the gap size h of the undeflected configuration, respectively,

$$x = \frac{\tilde{x}}{l}, \quad z = \frac{\tilde{z}}{h}, \quad u = \frac{\tilde{u}}{h}. \tag{2.21}$$

Furthermore, the aspect ratio of the device is denoted by $\varepsilon = h/l$. The rescaled dimensionless problem for the electrostatic potential thus reads

$$\begin{aligned}
\varepsilon^2\psi_{xx}+\psi_{zz}&=0, && t>0,\ (x,z)\in\Omega(u(t)),\\
\psi(t,x,z)&=\frac{1+z}{1+u(t,x)}f\big(x,u(t,x)\big), && t>0,\ (x,z)\in\partial\Omega(u(t)),
\end{aligned}$$

where the region $\Omega(u(t))$ is now given by

$$\Omega(u(t))=\{(x,z)\in(-1,1)\times(-1,\infty);-1<z<u(t,x)\}\,.$$

In dimensionless form the evolution of the membrane's deflection is specified by the equation

$$\begin{aligned}
\rho\alpha\frac{h\tau^2}{a^2l^4}u_{tt}+\frac{h\tau}{l^2}u_t+A_1(u)=&-\frac{\varepsilon_0V^2}{2}\left(\frac{1}{l^2}\big(\psi_x(x,u(x))\big)^2+\frac{1}{h^2}\big(\psi_z(x,u(x))\big)^2\right)\\
&+\varepsilon_0V^2\left(\frac{1}{l^2}\psi_x\big(x,u(x)\big)f_x\big(x,u(x)\big)+\frac{1}{h^2}\psi_z\big(x,u(x)\big)f_u\big(x,u(x)\big)\right),
\end{aligned}\tag{2.22}$$

with $A_1(u)$ given by

$$A_1(u)=-\frac{\tau\varepsilon}{l}\partial_x\left(\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right)+\frac{b\varepsilon}{l^3}\partial_x^2\left(\frac{u_{xx}}{(1+\varepsilon^2(u_x)^2)^{5/2}}\right)+\frac{5b\varepsilon^3}{2l^3}\partial_x\left(\frac{u_x(u_{xx})^2}{(1+\varepsilon^2(u_x)^2)^{7/2}}\right).$$

Multiplying (2.22) by $l^2/h\tau$ and using the definition of ε then leads to the equation

$$\begin{aligned}
\frac{\rho\alpha\tau}{a^2l^2}u_{tt}+u_t+A(u)=&-\frac{\varepsilon_0V^2}{2\varepsilon^2h\tau}\Big(\varepsilon^2\big(\psi_x(x,u(x))\big)^2+\big(\psi_z(x,u(x))\big)^2\Big)\\
&+\frac{\varepsilon_0V}{\varepsilon^2h\tau}\Big(\varepsilon^2\psi_x\big(x,u(x)\big)f_x\big(x,u(x)\big)+\psi_z\big(x,u(x)\big)f_u\big(x,u(x)\big)\Big),
\end{aligned}$$

with the rescaled quasilinear fourth-order differential operator

$$A(u):=\frac{l^2}{h\tau}A_1(u)=-\partial_x\left(\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right)+\frac{b}{l^2\tau}\partial_x^2\left(\frac{u_{xx}}{(1+\varepsilon^2(u_x)^2)^{5/2}}\right)+\frac{5b\varepsilon^2}{2l^2\tau}\partial_x\left(\frac{u_x(u_{xx})^2}{(1+\varepsilon^2(u_x)^2)^{7/2}}\right).$$

Lastly, by introduction of the parameters

$$\gamma:=\frac{\sqrt{\rho\alpha\tau}}{al},\quad \beta:=\frac{b}{\tau l^2},\quad \lambda=\lambda(\varepsilon):=\frac{\varepsilon_0V^2}{2\varepsilon^2h\tau},$$

the deflection of the thin elastic plate in terms of nonlinear elasticity may be determined by the evolution equation

$$\gamma^2 u_{tt} + u_t + A(u) = -\lambda\left(\varepsilon^2\big(\psi_x(x,u(x))\big)^2 + \big(\psi_z(x,u(x))\big)^2\right) \\ + 2\lambda\Big(\varepsilon^2\psi_x\big(x,u(x)\big)f_x\big(x,u(x)\big) + \psi_z\big(x,u(x)\big)f_u\big(x,u(x)\big)\Big),$$

with

$$A(u) = -\partial_x\left(\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right) + \beta\partial_x^2\left(\frac{u_{xx}}{(1+\varepsilon^2(u_x)^2)^{5/2}}\right) + \frac{5}{2}\beta\varepsilon^2\partial_x\left(\frac{u_x(u_{xx})^2}{(1+\varepsilon^2(u_x)^2)^{7/2}}\right)$$

and the according boundary and initial conditions

$$u(t,\pm 1) = u_x(t,\pm 1) = 0, \quad t>0,$$
$$u(0,x) = u_*(x), \quad u_t(0,x) = u_{**}(x), \quad x\in(-1,1).$$

Here, γ is the systems *quality factor*[5], β measures the relative importance of tension and rigidity and λ is a ratio of a reference electrostatic force to a reference elastic force. It is proportional to the square of the applied voltage and serves as a *tuning parameter* for the system.

2.2 | A Simplified Linear Elasticity Model

In the previous section, a general model for the dynamic behaviour of an electrostatically actuated MEMS device has been derived by means of nonlinear elasticity theory. Allowing also for large deflections of the membrane, it is the characteristic of the governing elasticity terms to be nonlinear. However, in many engineering applications it is reasonable to only require the device to feature *small membrane deflections* and thus to restrict the mathematical investigations to a linear elasticity model. It is the purpose of this section to derive the analgon of the above model by means of linear elasticity theory.

Starting from the unscaled regime, in a first step, we assume $(\tilde{u}_{\tilde{x}})^2$ to be small, i.e. $(\tilde{u}_{\tilde{x}})^2 \ll 1$, and consider the Taylor series expansion

$$\sqrt{1+(\tilde{u}_{\tilde{x}})^2} \simeq 1 + \frac{1}{2}(\tilde{u}_{\tilde{x}})^2 + \dots$$

of the term $\sqrt{1+(\tilde{u}_{\tilde{x}})^2}$ around $(\tilde{u}_{\tilde{x}})^2 = 0$, ignoring all but the first two terms.[6] The *linearised*

[5] Recall that $\gamma = \sqrt{\rho\alpha\tau}/al$ is a measure for the damping of an oscillating system. Small values γ refer to strongly damped systems and thus indicate a large rate of decay of oscillations.

[6] Note that the constant first term in the Taylor series expansion just voids the constant length in the stretching energy, whence we include the second term $(\tilde{u}_{\tilde{x}})^2/2$ as well.

stretching energy may then be written as

$$E_s(\tilde{u}) = \frac{\tau}{2}\int_{-l}^{l} \big(\tilde{u}_{\tilde{x}}(\tilde{x})\big)^2 \, d\tilde{x}.$$

As before, given $\sigma \in \mathbb{R}$ and a function $v \in C_c^\infty(L)$, we introduce for $\tilde{x} \in [-l, l]$ the variation $w(\sigma)(\tilde{x}) = \tilde{u}(\tilde{x}) + \sigma v(\tilde{x})$ of $\tilde{u}(\tilde{x})$ in the direction of v. We then find that

$$\frac{d}{d\sigma} E_s(\tilde{u} + \sigma v) = \frac{d}{d\sigma}\left(\frac{\tau}{2}\int_{-l}^{l} (w_{\tilde{x}})^2 \, d\tilde{x}\right) = \tau \int_{-l}^{l} \tilde{u}_{\tilde{x}} v_{\tilde{x}} + \sigma (v_{\tilde{x}})^2 \, d\tilde{x}$$

whence, using that v is compactly supported in L,

$$\delta E_s(\tilde{u}; v) = \tau \int_{-l}^{l} \tilde{u}_{\tilde{x}} v_{\tilde{x}} \, d\tilde{x} = -\tau \int_{-l}^{l} \tilde{u}_{\tilde{x}\tilde{x}} v \, d\tilde{x}.$$

We proceed similarly in order to obtain the a linearised version of the bending term. Again requiring $(\tilde{u}_{\tilde{x}})^2 \ll 1$ to be small, we consider the Taylor series expansion

$$\left(\partial_{\tilde{x}}\left(\frac{\tilde{u}_{\tilde{x}}}{\sqrt{1 + (\tilde{u}_{\tilde{x}})^2}}\right)\right)^2 \sqrt{1 + (\tilde{u}_{\tilde{x}})^2} = \frac{(\tilde{u}_{\tilde{x}\tilde{x}})^2}{(1 + (\tilde{u}_{\tilde{x}})^2)^{5/2}} \simeq (\tilde{u}_{\tilde{x}\tilde{x}})^2 + \ldots$$

around $(\tilde{u}_{\tilde{x}})^2 = 0$, whence the *linearised bending energy* reads

$$E_b(\tilde{u}) = \frac{b}{2}\int_{-l}^{l} \big(\tilde{u}_{\tilde{x}\tilde{x}}(\tilde{x})\big)^2 \, d\tilde{x}.$$

Therefore, we find that

$$\frac{d}{d\sigma} E_b(\tilde{u} + \sigma v) = \frac{d}{d\sigma}\left(\frac{b}{2}\int_{-l}^{l} (w_{\tilde{x}\tilde{x}})^2 \, d\tilde{x}\right) = b \int_{-l}^{l} \tilde{u}_{\tilde{x}\tilde{x}} v_{\tilde{x}\tilde{x}} + \sigma (v_{\tilde{x}\tilde{x}})^2 \, d\tilde{x}$$

and thus finally

$$\delta E_b(\tilde{u}; v) = b \int_{-l}^{l} \tilde{u}_{\tilde{x}\tilde{x}} v_{\tilde{x}\tilde{x}} \, d\tilde{x} = b \int_{-l}^{l} \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} v \, d\tilde{x}.$$

With the same scaling as above, the Euler–Lagrange equation in the regime of linear elasticity reads

$$\begin{aligned} \gamma^2 u_{tt} + u_t - u_{xx} + \beta u_{xxxx} = {} & -\lambda\left(\varepsilon^2\big(\psi_x(x, u(x))\big)^2 + \big(\psi_z(x, u(x))\big)^2\right) \\ & + 2\lambda\Big(\varepsilon^2 \psi_x\big(x, u(x)\big) f_x\big(x, u(x)\big) + \psi_z\big(x, u(x)\big) f_u\big(x, u(x)\big)\Big) \end{aligned} \tag{2.23}$$

for $t > 0$ and $x \in (-1, 1)$.

2.3 | The Mathematical Models Under Study

Based on the previous two sections, it is the purpose of the present one to give a brief overview of the different variants of the very general nonlinear and linear elasticity models, respectively, which reflect different physical assumptions, as they are adequate for different applications. Even if there are more variants conceivable, the presented elaboration is restricted to those models which are investigated more detailed in the subsequent chapters.

To this end, denoting by $u = u(t,x),\ t > 0,\ x \in I := (-1,1)$, the membrane's deformation, the elliptic problem governing the electrostatic potential of the system at any instant $t \geq 0$ of time always reads

$$\psi_{xx} + \psi_{zz} = 0, \qquad t > 0,\ (x,z) \in \Omega(u), \tag{2.24}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f\big(x, u(t,x)\big), \quad t > 0,\ (x,z) \in \partial\Omega(u), \tag{2.25}$$

where the region $\Omega(u(t))$ between the rigid ground plate at $z = -1$ and the elastic membrane at $z = u(t,x)$ at any instant of time is given by

$$\Omega(u) = \{(x,z) \in (-1,1) \times (-1,\infty);\ -1 < z < u(t,x)\}.$$

Depending on the choice of the evolution equation for the membrane's displacement, only the permittivity profile might vary, being either a function $f = f(x),\ f = f\big(u(t,x)\big)$ or $f = f\big(x, u(t,x)\big)$. In the above elliptic moving boundary value problem this does only influence the boundary condition accordingly.

The situation is more involved for the choice of an appropriate model describing the dynamics of the thin elastic plate's displacement. Within the scope of both approaches – the linear and the nonlinear elasticity theory – in the following analysis we make two physical assumptions which have significant effects on the mathematical classification of the resulting equations.

First of all, we restrict the further investigations to *viscosity-dominated systems*, i.e. to a setting in which damping effects dominate over inertial effects. More precisely, this means that the parameter γ appearing in front of the inertial term is assumed to be very small, i.e.

$$\gamma = \frac{\sqrt{\rho\alpha\tau}}{al} \ll 1,$$

and thus that we ignore the inertial term $\gamma^2 u_{tt}$ in the equations. Note that the highest-order time derivative thus appears in the shape of the damping term u_t which is of first order. This restriction is of course not relevant for all possible MEMS devices but for instance to model the dynamic behaviour of micro pumps [48] or micro grippers [49].

One may furthermore act on the assumption that membranes or infinitely thin plates do not resist bending, i.e. they have no flexural rigidity. This is the case if

$$b = \frac{2\alpha^3 Y}{3(1-\nu)^2} = 0 \quad \Longrightarrow \quad \beta = \frac{b}{\tau l^2} = 0,$$

and thus the spatial higher-oder terms

$$\beta \partial_x^2 \left(\frac{u_{xx}}{(1+\varepsilon^2 (u_x)^2)^{5/2}} \right) + \frac{5}{2} \beta \varepsilon^2 \partial_x \left(\frac{u_x (u_{xx})^2}{(1+\varepsilon^2 (u_x)^2)^{7/2}} \right) \quad \text{or} \quad \beta u_{xxxx},$$

in the nonlinear or linear elasticity regime, respectively, are eliminated. Deformations due to bending are thus neglected, which means that the governing equations are reduced from fourth-order to second-order (spatial) equations. Representatives for MEMS devices for which the suppression of bending effects is reasonable are certain micro pumps or the Grating Light Valve, respectively [45, p. 239].

Combining the above two physical assumptions we end up with a model influenced by stretching, damping and electrostatic forces. Note that those models are not admissible for all kinds of applications, but that on the other hand there exist applications for which a negligence of inertial and bending effects is reasonable.

It finally remains to take different varying permittivity profiles into account. The simplest case of a constant permittivity $f \equiv 1$ has extensively been studied in the recent time and is thus not a subject in the present study. It is rather the main objective of this thesis to consider the case in which the membrane exhibits a certain varying dielectric permittivity profile, itself depending either on the spatial variable $x \in I$, the membrane's displacement $u = u(t, x)$, or even both. More precisely, the permittivity profile is given by a function of one of the three following types:

- $[x \mapsto f(x)] : I \to \mathbb{R}$;
- $[u \mapsto f(u)] : (-1, \infty) \to \mathbb{R}$;
- $[(x, u) \mapsto f(x, u)] : I \times (-1, \infty) \to \mathbb{R}$.

Depending on the choice of the dielectric permittivity profile, also the right-hand side of the evolution equation differs. Denoting the right-hand side in any case by $g_{\varepsilon,\lambda}(u)$, if $f = f(x)$, it is given by

$$g_{\varepsilon,\lambda}(u) := -\lambda \left(\varepsilon^2 \big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2 \right) + 2\lambda \varepsilon^2 \psi_x(x,u) f'(x), \tag{2.26}$$

where f' denotes the derivative of f with respect to x. In the case $f = f(u)$, the right-hand side reads

$$g_{\varepsilon,\lambda}(u) := -\lambda\left(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\right) + 2\lambda\psi_z(x,u)f'(u), \tag{2.27}$$

f' denoting the derivative of f with respect to u, and finally in the case $f = f(x,u)$ we have

$$g_{\varepsilon,\lambda}(u) := -\lambda\left(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\right) + 2\lambda\Big(\varepsilon^2\psi_x(x,u)f_x(x,u) + \psi_z(x,u)f_u(x,u)\Big), \tag{2.28}$$

f_x and f_u denoting the partial derivatives of f with respect to its first and second variable, respectively.

Reviewing the above considerations as a whole, we end up with the quasilinear parabolic initial-boundary value problem

$$u_t - \partial_x\left(\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right) = g_{\varepsilon,\lambda}(u) \quad t>0,\, x\in I, \tag{2.29}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{2.30}$$

$$u(0,x) = u_*(x), \quad x\in I, \tag{2.31}$$

in the regime of nonlinear elasticity, whence in the linear elasticity setting the analogue problem is a semilinear parabolic initial-boundary value problem which reads

$$u_t - u_{xx} = g_{\varepsilon,\lambda}(u) \quad t>0,\, x\in I, \tag{2.32}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{2.33}$$

$$u(0,x) = u_*(x), \quad x\in I, \tag{2.34}$$

with a right-hand side according to the choice of the dielectric permittivity profile f.

2.3.1 Remark *(1) Note that in both cases,* (2.29)–(2.31) *and* (2.32)–(2.34), *the evolution problem for the membrane's displacement is strongly coupled to the elliptic moving boundary problem* (2.24)–(2.25) *in the following way. On the one hand the solution to the elliptic free boundary value problem is to be determined in the domain* $\Omega(u(t))$ *which changes its shape with time as the membrane deflects with time. The coupling is thus observably in the boundary conditions for* ψ*. On the other hand, the right-hand side of the evolution equation for the membrane's deformation* u *exhibits a nonlinear and nonlocal dependence of the gradient of the potential* ψ*.*

(2) Observe that the above reasoning is formal *in the sense that several regularity properties are used which are not verified rigorously, e.g.*

- *the Gâteaux-differentiability of* E_p *at* $\tilde{u}$ *in* (2.4);

- *the differentiability of the path* $\{\psi(\sigma;u);\sigma\in(-\sigma_0,\sigma_0)\}$ *with respect to* σ, *c.f.* (2.9),
- *the additional spatial regularity of* ψ *used to derive* (2.10)

(3) In the mathematical and numerical analysis in this thesis the main attention is devoted to the linear elasticity model (2.32)–(2.34) *with the most general permittivity profile* $f = f(x,u)$, *as far as possible. Nonetheless, the nonlinear elasticity model* (2.29)–(2.31) *is analysed precisely. It should be mentioned that for the latter model the presented results are partly based on joint works with Joachim Escher.*

3 | Local Well-Posedness and Global Existence

As a first aspect in the mathematical analysis of the coupled systems derived in Chapter 2 we address the questions of existence and uniqueness of solutions. Both when the membrane's displacement is determined in the semilinear regime (2.32)–(2.34) as well as when it is described by the quasilinear problem (2.29)–(2.31), it turns out that the answers to those questions strongly depend on the applied voltage. More precisely, we show that the systems possess locally in time existing unique solutions for all arbitrarily large values λ of the applied voltage, and that solutions exist even globally in time, provided that the applied voltage does not exceed a certain critical value λ_*.

Section 3.1 deals with the semilinear problem (2.32)–(2.34) arising from linear elasticity theory, whereas Section 3.2 is addressed to its quasilinear counterpart (2.29)–(2.31) arising from nonlinear elasticity theory.

3.1 | On the Semilinear Case

Based on the work [40] this section is devoted to results on local well-posedness and global existence of solutions to the coupled system of partial differential equations consisting of the semilinear parabolic initial boundary value problem

$$\begin{aligned} u_t - u_{xx} &= -\lambda \left(\varepsilon^2 \big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\right) \\ &\qquad + 2\lambda\Big(\varepsilon^2 \psi_x(x,u) f_x(x,u) + \psi_z(x,u) f_u(x,u)\Big), \quad t>0,\ x \in I, \qquad (3.1) \\ u(t,\pm 1) &= 0, \quad t>0, \qquad (3.2) \\ u(0,x) &= u_*(x), \quad x \in I, \qquad (3.3) \end{aligned}$$

describing the time evolution of the displacement $u = u(t,x)$ of the membrane, and the elliptic free boundary value problem

$$\varepsilon^2\psi_{xx} + \psi_{zz} = 0, \qquad t > 0,\ (x,z) \in \Omega(u(t)), \tag{3.4}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f\big(x, u(t,x)\big), \quad t > 0,\ (x,z) \in \partial\Omega(u(t)), \tag{3.5}$$

characterising the electrostatic potential $\psi = \psi(t,x,z)$ in the region

$$\Omega(u(t)) = \{(x,z) \in (-1,1) \times (-1,\infty); -1 < z < u(t,x)\},$$

determined by the rigid ground plate at $z = -1$ and the elastic membrane at $z = u(t,x)$. It is worthwhile to mention again the meaning of the two parameters ε and λ occurring in the above equations. The first one, $\varepsilon = h/l > 0$, denotes the *aspect ratio* of the device, i.e. the ratio of the gap size h between the two plates in the undeformed configuration, to the half l of the device length before scaling. The second parameter $\lambda > 0$ is proportional to the square of the applied voltage and is shown to have a considerable influence on the behaviour of the solution to (3.1)–(3.5). In particular, the system (3.1)–(3.5) is shown to be locally well-posed for all arbitrarily large values $\lambda > 0$ of the applied voltage. Moreover, we prove that the solution might even exist forever, provided that the applied voltage is small enough, i.e. smaller than a certain critical value λ_*.

Before going into detail, it is valuable to briefly outline the general ideas of the proof. As already performed in [40] we follow the lines of [14], where the authors study the above system with constant permittivity $f \equiv 1$. According to that the problems (3.1)–(3.3) and (3.4)–(3.5) are considered separately. In a first step the moving boundary problem (3.4)–(3.5) for ψ is transformed into an elliptic boundary problem on the fixed rectangle $\Omega := I \times (-1,0)$. Indeed, to stand to benefit from the fixed geometry is not totally free of cost as we now have to deal with an elliptic differential operator of second order with non-constant coefficients, depending on u, u_x and u_{xx}. However, the latter problem is shown to be well-posed for a given function u (see Theorem 3.1.3 below). Having the solution of the transformed elliptic problem at hand, the second step consists in investigating the evolution problem (3.1)–(3.3) for the membrane's displacement u. This problem can be characterised as a nonlocal semilinear heat equation, whereby one may reformulate it as an abstract parameter dependent Cauchy problem and finally apply a fixed-point argument in order to infer that also the evolution problem is well-posed.

Although one may basically apply the methods used in [14, Theorem1 & 2], the reasoning requires additional endeavour for handling a general varying permittivity profile $f = f\big(x, u(t,x)\big)$. Revealing of that are the following two lemmas on the regularity of the Nemitskii operator induced by the function $f : [-1,1] \times [-1,\infty) \to \mathbb{R}$.

3.1.1 Lemma (Global Lipschitz Continuity of the Nemitskii Operator, [40, Lemma 3.1])
Given $f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$ and $S \subset W_2^2(I)$, consider the Nemitskii operator

$$N_f : S \longrightarrow W_2^2(I), \qquad v \longmapsto f(\cdot, v(\cdot))$$

induced by f. If S is bounded in $W_2^2(I)$ then N_f is globally Lipschitz continuous. That is, there exists a constant $c_{f,L} = c_{f,L}(S) > 0$ such that

$$\|N_f(v_1) - N_f(v_2)\|_{W_2^2(I)} \,\leq\, c_{f,L}\,\|v_1 - v_2\|_{W_2^2(I)}$$

for all $v_1, v_2 \in S$.

The proof is an immediate consequence of the mean value theorem in integral form applied to $N_f(v_1) - N_f(v_2)$ and its derivatives of first and second order in the $L_2(I)$-norm.

3.1.2 Corollary (Boundedness of N_f, [40, Corollary 3.2])
Under the assumptions of Lemma 3.1.1 the operator N_f is uniformly bounded, i.e. there exists a constant $c_{f,B} = c_{f,B}(S) > 0$ such that

$$\|N_f(v)\|_{W_2^2(I)} \,\leq\, c_{f,B}$$

for all $v \in S$.

If no ambiguity is to be feared both, the function $f : [-1,1]\times[-1,\infty) \to \mathbb{R}$ and the Nemitskii operator N_f are subsequently denoted by f, i.e. we write $N_f(v) = f(v)$ for $v \in W_2^2(I)$.

Following the lines of [14] we now realise the above introduced concept for the proof of local well-posedness of the coupled system (3.1)–(3.5) and transform the moving boundary problem (3.4)–(3.5) to the fixed rectangle $\Omega := I \times (0,1)$.

Given $q \in (2,\infty)$ and an arbitrary function $v \in W_q^2(I)$ taking values in $(-1,\infty)$, we define the diffeomorphism

$$T_v : \overline{\Omega}(v) \longrightarrow \overline{\Omega}, \quad T_v(x,z) := \left(x, \frac{1+z}{1+v(x)}\right). \tag{3.6}$$

The according inverse is given by

$$T_v^{-1}(x,\eta) = \left(x, (1+v(x))\,\eta - 1\right), \quad (x,\eta) \in \overline{\Omega}. \tag{3.7}$$

Introducing the function $\varphi : \overline{\Omega} \to \mathbb{R}$, defined as the composition $\varphi := \psi \circ T_{u(t)}^{-1}$, the membrane's deformation $u = u(t,x)$ may be determined according to the transformed evolution problem

$$u_t - u_{xx} = -\lambda\Bigg(\varepsilon^2\left(-\big(f_x(x,u)\big)^2 + \big((f_u(x,u)u_x\big)^2\right) - 2\frac{1+\varepsilon^2(u_x)^2}{1+u} f_u(x,u)\varphi_\eta(t,x,1) + \frac{1+\varepsilon^2(u_x)^2}{(1+u)^2}\big(\varphi_\eta(t,x,1)\big)^2\Bigg),\ t>0,\ x\in I, \tag{3.8}$$

$$u(t,\pm 1) = 0, \quad t > 0, \tag{3.9}$$

$$u(0,x) = u_*(x), \quad x \in I. \tag{3.10}$$

The equivalent formulation of (3.4)–(3.5) on the fixed rectangle Ω reads

$$\left(\mathcal{L}_{u(t)}\varphi\right)(t,x,\eta) = 0, \qquad t>0, (x,\eta)\in\Omega, \tag{3.11}$$

$$\varphi(t,x,\eta) = \eta f(x,u), \quad t>0, (x,\eta)\in\partial\Omega, \tag{3.12}$$

with the transformed v-dependent elliptic differential operator

$$\mathcal{L}_v w := \varepsilon^2 w_{xx} - 2\varepsilon^2\eta\frac{v_x}{1+v}w_{x\eta} + \frac{1+\varepsilon^2\eta^2(v_x)^2}{(1+v)^2}w_{\eta\eta} + \varepsilon^2\eta\left(2\left(\frac{v_x}{1+v}\right)^2 - \frac{v_{xx}}{1+v}\right)w_\eta \tag{3.13}$$

of second order. Finally, (c.f. [14]) for $q\in[2,\infty)$ and $\kappa\in(0,1)$ the set

$$S_q(\kappa) := \left\{u\in W^2_{q,D}(I);\ \|u\|_{W^2_{q,D}(I)} < 1/\kappa \ \text{ and } \ -1+\kappa < u(x) \text{ for } x\in I\right\},$$

with

$$W^{2\alpha}_{q,D}(I) := \begin{cases} W^{2\alpha}_q(I), & 2\alpha\in[0,1/q) \\ \left\{u\in W^{2\alpha}_q(I);\ u(\pm1)=0\right\}, & 2\alpha\in(1/q,2] \end{cases}$$

is introduced.

We are now in a position to prove that for a given membrane's displacement u the transformed elliptic boundary value problem (3.11)–(3.12) on Ω possesses a unique solution.

3.1.3 Theorem (Solution to the Elliptic Problem on Ω, [40, Theorem 3.3])
Let $\kappa\in(0,1), \varepsilon>0$, and $q>2$. Given $v\in S_q(\kappa)$ and $f\in C^3\big([-1,1]\times[-1,\infty),\mathbb{R}\big)$ there is a unique solution $\varphi_v\in W^2_2(\Omega)$ to the problem

$$\left(\mathcal{L}_v\varphi_v\right)(x,\eta) = 0, \qquad (x,\eta)\in\Omega, \tag{3.14}$$

$$\varphi_v(x,\eta) = \eta f(x,v), \quad (x,\eta)\in\partial\Omega. \tag{3.15}$$

In addition, defining the function $\tilde{v}$ *by* $\tilde{v}(x) := v(-x)$ *for* $x \in I$ *and assuming that* $f\big(x, v(x)\big) = f\big(-x, v(x)\big)$ *for* $x \in I$*, one obtains*

$$\varphi_{\tilde{v}}(x,\eta) = \varphi_v(-x,\eta), \quad (x,\eta) \in \Omega.$$

Proof. (i) Given $v \in S_q(\kappa)$, for $(x,\eta) \in \Omega$ define the function

$$\begin{aligned} F_v(x,\eta) &:= \mathcal{L}_v\big(\eta f(x,v)\big) \\ &= \varepsilon^2\eta\Big(f_{xx}(x,v) + 2f_{xv}(x,v)v_x + f_{vv}(x,v)(v_x)^2 + f_v(x,v)v_{xx}\Big) \\ &\quad - 2\varepsilon^2\eta\frac{v_x}{1+v}\Big(f_x(x,v) + f_v(x,v)v_x\Big) + \varepsilon^2\eta\left(2\left(\frac{v_x}{1+v}\right)^2 - \frac{v_{xx}}{1+v}\right)f(x,v). \end{aligned} \tag{3.16}$$

Since $v \in S_q(\kappa)$ and $f(v) \in W_2^2(I)$ thanks to Corollary 3.1.2, one may verify that F_v belongs to $L_2(\Omega)$ with

$$\|F_v\|_{L_2(\Omega)} \leq c(\kappa,\varepsilon). \tag{3.17}$$

Therefore the assumptions of [14, Lemma 6] are fulfilled,[1] whence there exists a unique solution $\phi_v \in W_{2,D}^2(\Omega)$ to the problem

$$-\mathcal{L}_v\phi_v = F_v, \quad (x,\eta) \in \Omega, \tag{3.18}$$

$$\phi_v = 0, \quad (x,\eta) \in \partial\Omega, \tag{3.19}$$

with homogenised boundary conditions, satisfying

$$\|\phi_v\|_{W_2^2(\Omega)} \leq c(\kappa,\varepsilon)\,\|F_v\|_{L_2(\Omega)}\,. \tag{3.20}$$

The function φ_v, defined by

$$\varphi_v(x,\eta) := \phi_v(x,\eta) + \eta f(x,v), \quad (x,\eta) \in \overline{\Omega},$$

then obviously solves (3.11)–(3.12). Furthermore, combining (3.17) and (3.20) with the fact that $\|f(v)\|_{W_2^2(I)} \leq c_{f,B}$, one obtains

$$\|\varphi_v\|_{W_2^2(\Omega)} \leq \|\phi_v\|_{W_2^2(\Omega)} + \|\eta f(x,v)\|_{W_2^2(\Omega)} \leq c(\kappa,\varepsilon). \tag{3.21}$$

Eventually, the uniqueness of $\phi_v \in W_{2,D}^2(\Omega)$ implies that $\varphi_v \in W_2^2(\Omega)$ is the unique solution to (3.11)–(3.12).

(ii) It remains to prove that φ_v is even with respect to $x \in I$. Given $v \in S_q(\kappa)$, the function $\tilde{v}$ defined

[1] The proof of [14, Lemma 6] is based on [22, Theorem. 8.3] and [31, Theorem. 9.1].

by $\tilde{v}(x) := v(-x)$ for $x \in I$, obviously belongs to $S_q(\kappa)$. The properties of $\mathcal{L}_{\tilde{v}}$ and $F_{\tilde{v}}$ together with the assumption $f\big(x, v(x)\big) = f\big(-x, v(x)\big)$, $x \in I$, ensure that the function $(x,\eta) \mapsto \phi_v(-x,\eta) =: \tilde{\phi}(x,\eta)$ solves (3.18)–(3.19) with $\tilde{v}$ instead of v:

$$\begin{aligned}
-\mathcal{L}_{\tilde{v}}\tilde{\phi}(x,\eta) = & -\varepsilon^2\tilde{\phi}_{xx}(x,\eta) + 2\varepsilon^2\eta\frac{\tilde{v}_x(x)}{1+\tilde{v}(x)}\tilde{\phi}_{x\eta}(x,\eta) - \frac{1+\varepsilon^2\eta^2\big(\tilde{v}_x(x)\big)^2}{\big(1+\tilde{v}(x)\big)^2}\tilde{\phi}_{\eta\eta}(x,\eta) \\
& -\varepsilon^2\eta\left(2\left(\frac{\tilde{v}_x(x)}{1+\tilde{v}(x)}\right)^2 - \frac{\tilde{v}_{xx}(x)}{1+\tilde{v}(x)}\right)\tilde{\phi}_\eta(x,\eta) \\
= & -\mathcal{L}_v\phi_v(-x,\eta) \\
= & F_v(-x,\eta) \\
= & \varepsilon^2\eta\Big(f_{xx}\big(-x,v(-x)\big) + 2f_{xv}\big(-x,v(-x)\big)v_x(-x) \\
& + f_{vv}\big(-x,v(-x)\big)\big(v_x(-x)\big)^2 + f_v\big(-x,v(-x)\big)v_{xx}(-x)\Big) \\
& -2\varepsilon^2\eta\frac{v_x(-x)}{1+v(-x)}\Big(f_x\big(-x,v(-x)\big) + f_v\big(-x,v(-x)\big)v_x(-x)\Big) \\
& +\varepsilon^2\eta\left(2\left(\frac{v_x(-x)}{1+v(-x)}\right)^2 - \frac{v_{xx}(-x)}{1+v(-x)}\right)f\big(-x,v(-x)\big) \\
= & F_{\tilde{v}}(x,\eta).
\end{aligned}$$

Consequently, $\tilde{\phi}(x,\eta) = \phi_v(-x,\eta)$ solves (3.18)–(3.19) with $\tilde{v}$ instead of v and the uniqueness of the solution to (3.18)–(3.19) implies that

$$\phi_{\tilde{v}}(x,\eta) = \phi_v(-x,\eta), \quad (x,\eta) \in \Omega.$$

The definition of $\varphi_v(x,\eta) = \phi_v(x,\eta) + \eta f(x,v)$ together with the fact that f is even with respect to $x \in I$ then readily yields

$$\varphi_{\tilde{v}}(x,\eta) = \varphi_v(-x,\eta), \quad (x,\eta) \in \Omega.$$

This completes the proof. □

Having solved the transformed elliptic boundary problem (3.11)–(3.12) on the fixed rectangle Ω for a given displacement u, in pursuance of the introductory words on the concept of this section we are now left with handling the evolution problem (3.8)–(3.10). For this purpose we prove in the subsequent lemma that the right-hand side of (3.8) is globally Lipschitz continuous and bounded as a function $g_\varepsilon : S_q(\kappa) \to W^{2\sigma}_{2,D}(I)$, where $\sigma \in [0, 1/2)$. Those two properties do then give rise to the fact that the evolution problem for the membrane's displacement may be solved by methods of the semigroup theory.

3.1.4 Lemma (Properties of g_ε, [40, Lemma 3.4])
Let $\kappa \in (0,1)$, $\varepsilon > 0$ and $q > 2$. Moreover, let $f \in C^3\big([-1,1]\times[-1,\infty),\mathbb{R}\big)$. Then, with $\varphi_v \in W_2^2(\Omega)$ denoting the unique solution to (3.14)–(3.15), for $2\sigma \in [0,1/2)$ the mapping

$$\begin{aligned} g_\varepsilon \colon S_q(\kappa) &\longrightarrow W_{2,D}^{2\sigma}(I),\\ v &\longmapsto \varepsilon^2\Big(-\big(f_x(x,v)\big)^2 + \big(f_v(x,v)\big)^2 (v_x)^2\Big)\\ &\qquad - 2\frac{1+\varepsilon^2(v_x)^2}{1+v} f_v(x,v)\partial_\eta\varphi_v(\cdot,1) + \frac{1+\varepsilon^2(v_x)^2}{(1+v)^2}\big(\partial_\eta\varphi_v(\cdot,1)\big)^2 \end{aligned}$$

has the following properties:

(i) g_ε is globally Lipschitz continuous. That is, there is a constant $c_L = c_L(\kappa,\varepsilon) > 0$ such that

$$\|g_\varepsilon(v_1) - g_\varepsilon(v_2)\|_{W_{2,D}^{2\sigma}(I)} \le c_L(\kappa,\varepsilon)\, \|v_1 - v_2\|_{W_q^2(I)}$$

for all $v_1, v_2 \in S_q(\kappa)$.

(ii) g_ε is uniformly bounded. That is, there exists a constant $c_B = c_B(\kappa,\varepsilon) > 0$ such that

$$\|g_\varepsilon(v)\|_{W_{2,D}^{2\sigma}(I)} \le c_B(\kappa,\varepsilon)$$

for every $v \in S_q(\kappa)$.

Proof. (i) Given $v \in S_q(\kappa)$, define the bounded linear operator

$$\mathcal{A}(v) \in \mathcal{L}\big(W_{2,D}^2(\Omega), L_2(\Omega)\big), \qquad \mathcal{A}(v)\phi := -\mathcal{L}_v\phi.$$

[14, Lemma 6] guarantees that $\mathcal{A}(v)$ is invertible with inverse operator $\mathcal{A}(v)^{-1} \in \mathcal{L}\big(L_2(\Omega), W_{2,D}^2(\Omega)\big)$, satisfying

$$\big\|\mathcal{A}(v)^{-1}\big\|_{\mathcal{L}(L_2(\Omega),W_{2,D}^2(\Omega))} \le c(\kappa,\varepsilon). \tag{3.22}$$

As mentioned in the proof of [14, Proposition 5], by arguments concerning the continuity of pointwise multiplication in Sobolev spaces one obtains

$$\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\|_{\mathcal{L}(W_{2,D}^2(\Omega),L_2(\Omega))} \le c(\kappa,\varepsilon)\, \|v_1 - v_2\|_{W_q^2(I)} \tag{3.23}$$

for $v_1, v_2 \in S_q(\kappa)$. Moreover, again as in the proof of [14, Proposition 5] the identity

$$\mathcal{A}(v_1)^{-1} - \mathcal{A}(v_2)^{-1} = \mathcal{A}(v_1)^{-1}\left[\mathcal{A}(v_2) - \mathcal{A}(v_1)\right]\mathcal{A}(v_2)^{-1}$$

in $\mathcal{L}\big(L_2(\Omega), W^2_{2,D}(\Omega)\big)$ allows to infer from (3.22) and (3.23) that

$$\begin{aligned}
\big\|\mathcal{A}(v_1)^{-1} - \mathcal{A}(v_2)^{-1}\big\|&_{\mathcal{L}(L_2(\Omega),W^2_{2,D}(\Omega))} \\
&\le \big\|\mathcal{A}(v_1)^{-1}\big\|_{\mathcal{L}(L_2(\Omega),W^2_{2,D}(\Omega))} \|\mathcal{A}(v_2)-\mathcal{A}(v_1)\|_{\mathcal{L}(W^2_{2,D}(\Omega),L_2(\Omega))} \\
&\quad \big\|\mathcal{A}(v_2)^{-1}\big\|_{\mathcal{L}(L_2(\Omega),W^2_{2,D}(\Omega))} \\
&\le c(\kappa,\varepsilon)\,\|\mathcal{A}(v_2)-\mathcal{A}(v_1)\|_{\mathcal{L}(W^2_{2,D}(\Omega),L_2(\Omega))} \\
&\le c(\kappa,\varepsilon)\,\|v_1-v_2\|_{W^2_q(I)}
\end{aligned} \tag{3.24}$$

where $v_1, v_2 \in S_q(\kappa)$. Furthermore, owing to the boundedness and the Lipschitz continuity of $f(v)$ in $W^2_2(I)$ (cf. Lemma 3.1.1 and Corollary 3.1.2), for $v_1, v_2 \in S_q(\kappa)$ there holds

$$\|F_{v_1} - F_{v_2}\|_{L_2(\Omega)} \le c(\kappa,\varepsilon)\,\|v_1-v_2\|_{W^2_q(I)}\,. \tag{3.25}$$

A combination of (3.24), (3.25) and (3.22) with the Lipschitz continuity of $f(v)$ in $W^2_2(I)$ yields[2] the existence of a constant $c(\kappa,\varepsilon)>0$ such that

$$\begin{aligned}
\|\varphi_{v_1} - \varphi_{v_2}&\|_{W^2_2(\Omega)} \\
&\le \|\phi_{v_1}-\phi_{v_2}\|_{W^2_2(\Omega)} + 2\,\|f(v_1)-f(v_2)\|_{W^2_2(I)} \\
&\le \big\|\big(\mathcal{A}(v_1)^{-1}-\mathcal{A}(v_2)^{-1}\big)\,F_{v_1}\big\|_{W^2_{2,D}(\Omega)} + \big\|\mathcal{A}(v_2)^{-1}(F_{v_1}-F_{v_2})\big\|_{W^2_{2,D}(\Omega)} \\
&\qquad + 2\,\|f(v_1)-f(v_2)\|_{W^2_2(I)} \\
&\le c(\kappa,\varepsilon)\,\|v_1-v_2\|_{W^2_q(I)} + c(\kappa,\varepsilon)\,\|F_{v_1}-F_{v_2}\|_{L_2(\Omega)} \\
&\qquad + 2\,\|f(v_1)-f(v_2)\|_{W^2_2(I)} \\
&\le c(\kappa,\varepsilon)\,\|v_1-v_2\|_{W^2_q(I)}
\end{aligned} \tag{3.26}$$

for $v_1, v_2 \in S_q(\kappa)$. One may then invoke [43, Chapter 2, Theorem 5.4] to obtain

$$\|\partial_\eta \varphi_v(\cdot,1)\|_{W^{1/2}_2(I)} \le c\,\|\varphi_v\|_{W^2_2(\Omega)}\,. \tag{3.27}$$

Fusing (3.26) and (3.27) leads to

$$\|\partial_\eta\varphi_{v_1}(\cdot,1) - \partial_\eta\varphi_{v_2}(\cdot,1)\|_{W^{1/2}_2(I)} \le c(\kappa,\varepsilon)\,\|v_1-v_2\|_{W^2_q(I)} \tag{3.28}$$

for $v_1, v_2 \in S_q(\kappa)$, whence the mapping

$$S_q(\kappa) \longrightarrow W^{1/2}_2(I), \quad v \longmapsto \partial_\eta \varphi_v(\cdot,1) \tag{3.29}$$

[2] Observe that $\|\eta f(v)\|_{W^2_2(\Omega)} \le \sqrt{2}\|f(v)\|_{W^2_2(I)} \le 2\|f(v)\|_{W^2_2(I)}$.

is globally Lipschitz continuous. The continuity of pointwise multiplication[3]

$$W_2^{1/2}(I) \cdot W_2^{1/2}(I) \hookrightarrow W_2^{2\sigma_1}(I), \quad 2\sigma_1 < 1/2,$$

according to [2, Theorem 4.1] implies that the mapping

$$S_q(\kappa) \longrightarrow W_2^{2\sigma_1}(I), \quad v \longmapsto \big(\partial_\eta \varphi_v(\cdot, 1)\big)^2 \tag{3.30}$$

is globally Lipschitz continuous[4]. That $W_q^2(I)$ is continuously embedded in $W_\infty^1(I)$, together with the continuity of pointwise multiplication $W_\infty^1(I) \cdot W_q^1(I) \hookrightarrow W_q^1(I)$ yields that the mapping

$$S_q(\kappa) \longrightarrow W_q^1(I), \quad v \longmapsto \frac{1 + \varepsilon^2 (v_x)^2}{(1+v)^2} \tag{3.31}$$

is globally Lipschitz continuous. Applying the continuity of pointwise multiplication

$$W_q^1(I) \cdot W_2^{2\sigma_1}(I) \hookrightarrow W_2^{2\sigma}(I) = W_{2,D}^{2\sigma}(I), \quad 2\sigma < 2\sigma_1 < 1/2,$$

to the Lipschitz continuous mappings (3.30) and (3.31) one may deduce the Lipschitz continuity of

$$S_q(\kappa) \longrightarrow W_{2,D}^{2\sigma}(I), \quad v \longmapsto \frac{1 + \varepsilon^2 (v_x)^2}{(1+v)^2} \big(\partial_\eta \varphi_v(\cdot, 1)\big)^2.$$

Thanks to Lemma 3.1.1 the mapping

$$S_q(\kappa) \longrightarrow W_2^1(I), \quad v \longmapsto f_v(x, v) \tag{3.32}$$

is globally Lipschitz continuous and using the continuity of the embeddings

$$W_q^1(I) \cdot W_q^1 \hookrightarrow W_q^1(I), \qquad W_\infty^1(I) \cdot W_q^1 \hookrightarrow W_q^1(I)$$

one obtains the global Lipschitz continuity of the mapping

$$S_q(\kappa) \longrightarrow W_q^1(I), \quad v \longmapsto \frac{1 + \varepsilon^2 (v_x)^2}{1+v}. \tag{3.33}$$

As a consequence of the continuity of pointwise multiplication $W_q^1(I) \cdot W_2^1(I) \hookrightarrow W_2^1(I)$ and

$$W_2^1(I) \cdot W_2^{1/2}(I) \hookrightarrow W_2^{\rho}(I) = W_{2,D}^{\rho}(I), \quad 0 < 2\sigma < 2\sigma_1 < \rho < 1/2,$$

[3] In what follows all arguments concerning the continuity of pointwise multiplication in Sobolev spaces are due to [2].

[4] In the whole contribution *global Lipschitz continuity* means that the Lipschitz constant does not depend on $v_1, v_2 \in S_q(\kappa)$ but only on the parameters ε and κ.

applied to the mappings (3.29), (3.32) and (3.33), their pointwise product

$$S_q(\kappa) \longrightarrow W^{\rho}_{2,D}(I), \quad v \longmapsto 2\frac{1+\varepsilon^2(v_x)^2}{1+v} f_v(x,v)\partial_\eta \varphi_v(\cdot,1)$$

is globally Lipschitz continuous. Moreover, one may invoke Lemma 3.1.1 as well as Corollary 3.1.2 and the continuity of pointwise multiplication $W^1_2(I)\cdot W^1_2(I) \hookrightarrow W^1_2(I)$ to conclude that the mapping

$$S_q(\kappa) \longrightarrow W^1_2(I), \quad v \longmapsto \big(f_x(x,v)\big)^2$$

is globally Lipschitz continuous. Finally, by combining some of the already mentioned arguments one obtains that the mapping

$$S_q(\kappa) \longrightarrow W^1_2(I), \quad v \longmapsto \big(f_v(x,v)\big)^2 (v_x)^2$$

is globally Lipschitz continuous. As a sum of globally Lipschitz continuous functions, eventually the mapping $g_\varepsilon : S_q(\kappa) \to W^{2\sigma}_{2,D}(I)$ is globally Lipschitz continuous. This yields the first assertion of the lemma.

(ii) First of all, thanks to part (i) there exists a constant $c_L = c_L(\kappa,\varepsilon) > 0$ such that

$$\|g_\varepsilon(v_1) - g_\varepsilon(v_2)\|_{W^{2\sigma}_{2,D}(I)} \le c_L(\kappa,\varepsilon)\,\|v_1 - v_2\|_{W^2_q(I)}$$

for all $v_1, v_2 \in S_q(\kappa)$. Furthermore, by definition of g_ε there holds

$$\begin{aligned} \|g_\varepsilon(0)\|_{W^{2\sigma}_{2,D}(I)} \le \varepsilon^2 \|\big(f_x(x,0)\big)^2\|_{W^{2\sigma}_{2,D}(I)} + 2\|f_v(x,0)\partial_\eta\varphi_0(x,1)\|_{W^{2\sigma}_{2,D}(I)} \\ + \|\big(\partial_\eta\varphi_0(x,1)\big)^2\|_{W^{2\sigma}_{2,D}(I)}. \end{aligned} \tag{3.34}$$

The first term on the right-hand side of (3.34) may be estimated by means of the continuity of the embedding $W^2_2(I) \hookrightarrow W^{2\sigma}_{2,D}(I)$, $2\sigma < 1/2$, as in (i), and the boundedness of $f(v)$ in $W^2_2(I)$, so that

$$\varepsilon^2 \|\big(f_x(x,0)\big)^2\|_{W^{2\sigma}_{2,D}(I)} \le c(\varepsilon) c^2_{f,B}. \tag{3.35}$$

To control the second term of (3.34) one can invoke the continuity of pointwise multiplication

$$W^1_2(I) \cdot W^{1/2}_2(I) \hookrightarrow W^{2\sigma}_{2,D}(I),$$

together with [43, Chapter 2, Theorem 5.4] as well as (3.21) and again the boundedness of $f(v)$ in $W^2_2(I)$. Altogether this leads to

$$\|f_v(x,0)\partial_\eta\varphi_0(x,1)\|_{W^{2\sigma}_{2,D}(I)} \le c_{f,B}\, c(\kappa,\varepsilon). \tag{3.36}$$

As for (3.36), combining the continuity of the embedding of $W_2^{1/2}(I)$ in $W_{2,D}^{2\sigma}(I)$ with [43, Chapter 2, Theorem 5.4] and (3.21) yields

$$\|\big(\partial_\eta \varphi_0(x,1)\big)^2\|_{W_{2,D}^{2\sigma}(I)} \leq c(\kappa,\varepsilon). \tag{3.37}$$

Fusing (3.35), (3.36) and (3.37) one finally obtains

$$\|g_\varepsilon(0)\|_{W_{2,D}^{2\sigma}(I)} \leq c(\kappa,\varepsilon).$$

Therefore, observing that $0 \in S_q(\kappa)$, one may deduce that $g_\varepsilon : S_q(\kappa) \to W_{2,D}^{2\sigma}(I)$ is bounded by a constant depending only on κ and ε:

$$\begin{aligned}
\|g_\varepsilon(v)\|_{W_{2,D}^{2\sigma}(I)} &\leq \|g_\varepsilon(v) - g_\varepsilon(0)\|_{W_{2,D}^{2\sigma}(I)} + \|g_\varepsilon(0)\|_{W_{2,D}^{2\sigma}(I)} \\
&\leq c_L(\kappa,\varepsilon)\, \|v\|_{W_q^2(I)} + \|g_\varepsilon(0)\|_{W_{2,D}^{2\sigma}(I)} \\
&\leq \frac{c_L(\kappa,\varepsilon)}{\kappa} + c(\kappa,\varepsilon) \\
&=: c_B(\kappa,\varepsilon).
\end{aligned}$$

This completes the proof. □

Thanks to the above lemma we are now in a position to employ arguments from the semigroup theory in order to verify the local existence of a unique solution (u,ψ) to the coupled problem (3.1)–(3.5).

3.1.5 Theorem (Local Well-Posedness, [40, Theorem 3.5])
Let $q \in (2,\infty)$ and $\varepsilon > 0$. Given an initial value $u_* \in W_{q,D}^2(I)$ with $u_*(x) > -1$ for $x \in I$, and $f \in C^3\big([-1,1] \times [-1,\infty), \mathbb{R}\big)$, the following holds true:

(i) *For each voltage value $\lambda > 0$ there exists a unique maximal solution (u,ψ) to (3.1)–(3.5) on the maximal interval $[0,T)$ of existence in the sense that*

$$u \in C^1\big([0,T), L_q(I)\big) \cap C\big([0,T), W_{q,D}^2(I)\big)$$

satisfies (3.1)–(3.3) with

$$u(t,x) > -1, \quad t \in [0,T),\, x \in I,$$

and $\psi \in W_2^2\big(\Omega(u(t))\big)$ solves (3.4)–(3.5) on $\Omega(u(t))$ for each $t \in [0,T)$.

(ii) *If for each $\tau > 0$ there is $\kappa(\tau) \in (0,1)$ such that $u(t) \in S_q(\kappa(\tau))$ for $t \in [0,T) \cap [0,\tau]$, then the solution (u,ψ) to (3.1)–(3.5) exists globally in time, which means that $T = \infty$.*

(iii) *If $u_*(x) = u_*(-x)$ and $f\big(x,u(t,x)\big) = f\big(-x,u(t,x)\big)$ for all $t \in [0,T)$ and $x \in I$, then for every $t \in [0,T)$, $u = u(t,x)$ and $\psi = \psi(t,x,z)$ are even with respect to $x \in I$ as well.*

Again the proof of this result relies on the proof of [14, Theorem 1].

Proof. (i) In order to stand to benefit from arguments of the semigroup theory, (3.8) subject to the boundary condition (3.9) and the initial condition (3.10) must be reformulated as a Cauchy problem. For that purpose let $p \in (1,\infty)$ and define the differential operator[5]

$$A \in \mathcal{L}\big(W^2_{p,D}(I), L_p(I)\big), \qquad Av := -\partial_x^2 v. \tag{3.38}$$

Then, (3.8) subject to (3.9) and (3.10) may be perceived as the abstract parameter-dependent semilinear Cauchy problem

$$\dot{u} + Au = -\lambda g_\varepsilon(u), \quad t > 0, \tag{3.39}$$

$$u(0) = u_*, \tag{3.40}$$

with the function g_ε introduced in Lemma 3.1.4. The proof is now performed by employing a fixed point argument to (3.39)–(3.40). To this end let

$$\big\{e^{-tA}; t \geq 0\big\}$$

denote the (analytic) heat semigroup on $L_p(I)$ corresponding to $-A$. By assumption there is a $\kappa \in (0, 1/2)$ such that

$$u_* \in \overline{S}_q(2\kappa). \tag{3.41}$$

Fixing $1/2 - 1/q < 2\sigma < 1/2$ with $2\sigma \neq 1/q$, [14, Lemma 7] guarantees the existence of constants $M \geq 1$ and $\omega > 0$ such that[6]

$$\big\|e^{-tA}\big\|_{\mathcal{L}(W^2_{q,D}(I))} + t^{-\sigma+1+\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\,\big\|e^{-tA}\big\|_{\mathcal{L}(W^{2\sigma}_{2,D}(I),W^2_{q,D}(I))} \leq Me^{-\omega t}, \quad t \geq 0. \tag{3.42}$$

Given $\kappa_* := \kappa/M$, thanks to Lemma 3.1.4 there exist positive constants $c_L(\kappa,\varepsilon)$ and $c_B(\kappa,\varepsilon)$ such that

$$\|g_\varepsilon(v_1) - g_\varepsilon(v_2)\|_{W^{2\sigma}_{2,D}(I)} \leq c_L(\kappa,\varepsilon)\,\|v_1 - v_2\|_{W^2_{q,D}(I)}\,, \quad v_1, v_2 \in \overline{S}_q(\kappa_*), \tag{3.43}$$

and

$$\|g_\varepsilon(v)\|_{W^{2\sigma}_{2,D}(I)} \leq c_B(\kappa,\varepsilon), \quad v \in \overline{S}_q(\kappa_*), \tag{3.44}$$

respectively. For $\tau > 0$ we now define the space

$$\mathcal{V}_\tau := C\big([0,\tau], \overline{S}_q(\kappa_*)\big)$$

[5] Since for $v \in W^2_{p,D}(I)$ it holds that $Av = -\partial_x^2 v \in L_p(I) \hookrightarrow L_r(I)$ for $r \leq p$, we write $A = A_p$ for $A_p \in \mathcal{L}\big(W^2_{p,D}(I), L_p(I)\big)$.

[6] The proof of [14, Lemma 7] is based on results of [3], [23] and [50].

and subsequently for $t \in [0,\tau]$ and $v \in \mathcal{V}_\tau$ the operator

$$G(v)(t) := e^{-tA}u_* - \lambda \int_0^t e^{-(t-s)A} g_\varepsilon\big(v(s)\big)\, ds. \tag{3.45}$$

With the objective of establishing the existence of a fixed point of (3.45) we now verify that G defines a contraction on $\mathcal{V}_\tau$ for a certain τ. To this end we introduce the functional

$$\mathcal{I}(\tau) := \int_0^\tau e^{-\omega s} s^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\, ds \le \mathcal{I}(\infty) := \int_0^\infty e^{-\omega s} s^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\, ds, \tag{3.46}$$

which is finite thanks to the positivity of ω and the choice of σ. Given $v \in \mathcal{V}_\tau$ and $t \in [0,\tau]$ it follows from (3.41), (3.42) and (3.44) that

$$\begin{aligned}
\|G(v)(t)\|_{W^2_{q,D}(I)} &\le \big\|e^{-tA}u_*\big\|_{W^2_{q,D}(I)} + \lambda \int_0^t \Big\|e^{-(t-s)A} g_\varepsilon\big(v(s)\big)\Big\|_{W^2_{q,D}(I)}\, ds \\
&\le \big\|e^{-tA}\big\|_{\mathcal{L}(W^2_{q,D}(I))} \|u_*\|_{W^2_{q,D}(I)} \\
&\qquad + \lambda \int_0^t \Big\|e^{-(t-s)A}\Big\|_{\mathcal{L}(W^{2\sigma}_{2,D}(I), W^2_{q,D}(I))} \big\|g_\varepsilon\big(v(s)\big)\big\|_{W^{2\sigma}_{2,D}(I)}\, ds \\
&\le \frac{1}{2\kappa} \big\|e^{-tA}\big\|_{\mathcal{L}(W^2_{q,D}(I))} + \lambda c_B(\kappa,\varepsilon) \int_0^t \Big\|e^{-(t-s)A}\Big\|_{\mathcal{L}(W^{2\sigma}_{2,D}(I), W^2_{q,D}(I))}\, ds \\
&\le \frac{1}{2\kappa} M e^{-\omega t} + \lambda M c_B(\kappa,\varepsilon) \int_0^t e^{-\omega(t-s)} (t-s)^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\, ds \\
&\le \frac{M}{2\kappa} + \lambda M c_B(\kappa,\varepsilon) \mathcal{I}(\tau).
\end{aligned} \tag{3.47}$$

Note that the heat semigroup is a positive contraction semigroup on $L_\infty(I)$, whence it is additionally submarkovian.[7] Due to this fact and since $W^2_{q,D}(I) \hookrightarrow L_\infty(I)$ with embedding constant 2 and $u_* > 2\kappa - 1$, one can deduce from (3.42) and (3.44) that

$$\begin{aligned}
G(v)(t) &\ge e^{-tA}(2\kappa - 1) - \lambda \int_0^t e^{-(t-s)A} g_\varepsilon\big(v(s)\big)\, ds \\
&\ge 2\kappa - 1 - \lambda \int_0^t \Big\|e^{-(t-s)A} g_\varepsilon\big(v(s)\big)\Big\|_{L_\infty(I)}\, ds \\
&\ge 2\kappa - 1 - 2\lambda \int_0^t \Big\|e^{-(t-s)A}\Big\|_{\mathcal{L}\left(W^{2\sigma}_{2,D}(I), W^2_{q,D}(I)\right)} \big\|g_\varepsilon\big(v(s)\big)\big\|_{W^{2\sigma}_{2,D}(I)}\, ds \\
&\ge 2\kappa - 1 - 2\lambda M c_B(\kappa,\varepsilon) \int_0^t e^{-\omega(t-s)} (t-s)^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\, ds \\
&\ge -1 + 2\kappa - 2\lambda M c_B(\kappa,\varepsilon) \mathcal{I}(\tau).
\end{aligned} \tag{3.48}$$

[7] This follows from the parabolic maximum principle.

Finally, (c.f. [14]) one may infer from (3.42) and (3.43) that

$$\begin{aligned}
&\|G(v_1) - G(v_2)\|_{W^2_{q,D}(I)} \\
&\qquad = \lambda \left\| \int_0^t e^{-(t-s)A} \left[g_\varepsilon\big(v_1(s)\big) - g_\varepsilon\big(v_2(s)\big) \right] \, ds \right\|_{W^2_{q,D}(I)} \\
&\qquad \leq \lambda \int_0^t \left\| e^{-(t-s)A} \right\|_{\mathcal{L}(W^{2\sigma}_{2,D}(I), W^2_{q,D}(I))} \left\| g_\varepsilon\big(v_1(s)\big) - g_\varepsilon\big(v_2(s)\big) \right\|_{W^{2\sigma}_{2,D}(I)} \, ds \\
&\qquad \leq \lambda M c_L(\kappa, \varepsilon) \int_0^t e^{-\omega(t-s)} (t-s)^{\sigma - 1 - \frac{1}{2}(\frac{1}{2} - \frac{1}{q})} \|v_1(s) - v_2(s)\|_{W^2_{q,D}(I)} \, ds \\
&\qquad \leq \lambda M c_L(\kappa, \varepsilon) \mathcal{I}(\tau) \|v_1 - v_2\|_{C([0,\tau], W^2_{q,D}(I))} .
\end{aligned} \tag{3.49}$$

Availing ourselves of the fact that $\mathcal{I}(\tau) \to 0$ as $\tau \to 0$, the estimates (3.47), (3.48) and (3.49) imply that there exists $\tau_* := \tau_*(\lambda, \kappa, \varepsilon, q, \sigma) > 0$ sufficiently small such that

$$G \colon \mathcal{V}_{\tau_*} \longrightarrow \mathcal{V}_{\tau_*}$$

defines a contraction. Since $\mathcal{V}_{\tau_*}$ is a complete metric space one may eventually invoke *Banach's fixed-point theorem* to conclude that there exists a unique fixed point

$$u = G(u) \in \mathcal{V}_{\tau_*}.$$

This shows that (3.39) possesses a unique mild solution on $[0, \tau_*]$ with $u_* \in W^2_{q,D}(I)$. It follows from general parabolic theory (c.f. [3]) that this mild solution is also a strong solution

$$u \in C^1\big([0,T), L_q(I)\big) \cap C\big([0,T), W^2_{q,D}(I)\big) \cap \ C\big((0,T)), W^{2+2\sigma}_{2,D}(I)\big)$$

for some maximal time $T \in [\tau_*, \infty)$. It satisfies

$$u(t,x) > -1, \quad t \in [0,T),\, x \in I.$$

Lastly, observe that

$$\psi(t) = \varphi_{u(t)} \circ T_{u(t)} \tag{3.50}$$

belongs to $W^2_2\big(\Omega(u(t))\big)$ and solves (3.4)–(3.5) for each $t \in [0,T)$, with $T_{u(t)}$ as introduced in (3.6).

(ii) We prove the contraposition of the assertion. Assume that $T < \infty$. Then there exist $\tau_* > 0$ and $t_* \in [0,T) \cap [0, \tau_*]$ such that $u(t_*) \notin S_q(\kappa(\tau_*))$ for all $\kappa(\tau_*) \in (0,1)$. This means that either

$$\lim_{t \to T} \min_{x \in [-1,1]} u(t,x) = -1 \qquad \text{or} \qquad \lim_{t \to T} \|u(t)\|_{W^2_q(I)} = \infty.$$

(iii) It remains to show that u and ψ are even with respect to $x \in I$, provided that u_* and f are. For that purpose suppose that $u_*(x) = u_*(-x)$ for all $x \in I$, and denote by u be the corresponding maximal solution to (3.39)–(3.40) with maximal time $T \in (0, \infty]$ of existence. Then, introducing the function $\tilde{u}$ defined by $\tilde{u}(t,x) := u(t,-x),\ t \in [0,T)$, $x \in I$, and using the additional assumption that $f\big(x, u(t,x)\big) = f\big(-x, u(t,x)\big)$ for $t \in [0,T)$, $x \in I$, one may infer from (3.39) that

$$\tilde{u}_t(t,x) + A\tilde{u}(t,x) = -\lambda g_\varepsilon\big(-x, u(t,-x)\big).$$

The definitions of g_ε and $\tilde{u}$ yield

$$\begin{aligned}
-\lambda g_\varepsilon\big(-x,&u(t,-x)\big) \\
&= -\lambda\Bigg(\varepsilon^2\left(-\big(f_x(-x,u(t,-x))\big)^2 + \big(f_u(-x,u(t,-x))u_x(t,-x)\big)^2\right) \\
&\qquad - 2\frac{1+\varepsilon^2\big(u_x(t,-x)\big)^2}{1+u(t,-x)} f_u\big(-x,u(t,-x)\big)\partial_\eta\varphi_u(-x,1) \\
&\qquad + \frac{1+\varepsilon^2\big(u_x(t,-x)\big)^2}{\big(1+u(t,-x)\big)^2}\big(\partial_\eta\varphi_u(-x,1)\big)^2\Bigg), \\
&= -\lambda\Bigg(\varepsilon^2\left(-\big(f_x(-x,\tilde{u}(t,x))\big)^2 + \big(f_u(-x,\tilde{u}(t,x))\tilde{u}_x(t,x)\big)^2\right) \\
&\qquad - 2\frac{1+\varepsilon^2\big(\tilde{u}_x(t,x)\big)^2}{1+\tilde{u}(t,x)} f_u\big(-x,\tilde{u}(t,x)\big)\partial_\eta\varphi_{\tilde{u}}(x,1) \\
&\qquad + \frac{1+\varepsilon^2\big(\tilde{u}_x(t,x)\big)^2}{\big(1+\tilde{u}(t,x)\big)^2}\big(\partial_\eta\varphi_{\tilde{u}}(x,1)\big)^2\Bigg).
\end{aligned}$$

Finally, using the assumption that $f\big(x, u(t,x)\big) = f\big(-x, u(t,x)\big)$ for all $x \in I$ one may observe that

$$-\lambda g_\varepsilon\big(-x, u(t,-x)\big) = -\lambda g_\varepsilon\big(x, \tilde{u}(t,x)\big),$$

which eventually leads to

$$\tilde{u}_t(t,x) + A\tilde{u}(t,x) = -\lambda g_\varepsilon\big(x, \tilde{u}(t,x)\big).$$

The uniqueness of the solution to (3.39)–(3.40) therefore implies that u coincides with $\tilde{u}$ so that $u(t,\cdot)$ is even on I for all $t \in [0,T)$. That $\psi(t,\cdot,z)$ is even follows from (3.50), using the fact that $\varphi_{u(t)}(t,\cdot,\eta)$ is even thanks to Theorem 3.1.3. Thus, the proof is complete. □

Before proving the at the outset mentioned result on global existence, it is worthwhile to make the following observation, which is of particular relevance in further parts of this work.

3.1.6 Corollary

Let $u_* \in S_q(\kappa)$ satisfy $u_*(x) \leq 0$ for all $x \in I$, and assume that the implication

$$v \in S_q(\kappa),\ v(x) \leq 0\ \forall x \in I \quad \Longrightarrow \quad g_\varepsilon(v) \in S_q(\kappa),\ g_\varepsilon\big(v(x)\big) \geq 0\ \forall x \in I \tag{3.51}$$

holds true. Then the solution u to (3.1)–(3.3) *satisfies*

$$u(t,x) \leq 0, \quad t \in [0,T),\ x \in I.$$

Proof. Pick $T_* \in (0,T)$ and introduce the set

$$S_q^-(\kappa) := \{v \in \overline{S}_q(\kappa);\ v(x) \leq 0 \text{ for } x \in I\}.$$

Given $u_* \in S_q^-(\kappa)$, it then suffices to show that $u(t,x) \leq 0$ for all $t \in [0,T_*]$ and $x \in I$. To this end, let $\tau \in (0,T_*]$ and introduce the space

$$\mathcal{V}_\tau^- := C\big([0,\tau], S_q^-(\kappa)\big).$$

The proof of Theorem 3.1.5, c.f. (3.47), (3.48) and (3.49), already provides evidence that there exists $\tau_* \leq \tau$ such that

$$G : \mathcal{V}_{\tau_*} \longrightarrow \mathcal{V}_{\tau_*}$$

is a contractive self mapping. Since the semigroup $\{e^{-tA}; t \geq 0\}$ corresponding to $-A$ is positive, the assumption (3.51) yields that, given $v \in \mathcal{V}_{\tau_*}^-$, it follows that

$$G(v)(t) = e^{-tA}u_* - \lambda \int_0^t e^{-(t-s)A} g_\varepsilon\big(v(s)\big)\, ds \leq 0, \quad t \in [0,\tau_*].$$

Therefore also

$$G : \mathcal{V}_{\tau_*}^- \longrightarrow \mathcal{V}_{\tau_*}^-$$

defines a contractive self mapping. As in the proof of Theorem 3.1.5 we can now conclude that G possesses a unique fixed point $G(u)(t) = u(t) \in S_q^-(\kappa)$ on $[0,\tau_*]$, being the unique solution $u \in C^1\big([0,\tau_*), L_q(I)\big) \cap C\big([0,\tau_*), W_q^2(I)\big)$ which satisfies

$$u(t,x) \leq 0, \quad t \in [0,\tau_*],\ x \in [-1,1].$$

By a finite number of the above fixed-point iteration this reasoning may be extended to the interval $[0,T_*]$. □

In Theorem 3.1.5 it is proved that the solution to (3.1)–(3.5) exists locally in time for all voltage values λ. However, the next theorem is addressed to global existence. It turns out that the solution

(u, ψ) to (3.1)–(3.5) exists even globally in time, if λ is smaller than a critical value λ_*.

3.1.7 Theorem (Global Existence, [40, Theorem 3.7])
Let $q \in (2, \infty)$, $\varepsilon > 0$ and $\lambda > 0$. Furthermore, given $f \in C^3\big([-1,1]\times[-1,\infty),\mathbb{R}\big)$ and $u_* \in W^2_{q,D}(I)$, satisfying $-1 < u_*(x)$ for $x \in I$, let (u, ψ) denote the corresponding solution to (3.1)–(3.5) on the maximal interval $[0, T)$ of existence. Then, given $\kappa \in (0, 1)$, there exist $\lambda_* := \lambda_*(\kappa, \varepsilon) > 0$ and $\kappa_* := \kappa_*(\kappa, \varepsilon) > 0$ such that $T = \infty$ and $u(t) \in S_q(\kappa_*)$ for $t \geq 0$, provided that $\lambda \in (0, \lambda_*)$.

The proof performs exactly as the one in [14] for the case of constant permittivity. Hence, merely the main steps of the proof are mentioned here for the sake of completeness.

Proof. Given $u_* \in S_q(\kappa)$, let (u, ψ) be the corresponding solution to (3.1)–(3.5) on the maximal interval $[0, T)$ of existence. We pick $\kappa_* := \kappa/M$ with M as in (3.42) and put $\lambda_* := \lambda_*(\kappa, \varepsilon, q, \sigma) > 0$ such that

$$\lambda_* M \max\left\{c_L(\kappa,\varepsilon), c_B(\kappa,\varepsilon)\right\} \mathcal{I}(\infty) \leq \frac{1}{2} < \frac{1}{2\kappa_*} \tag{3.52}$$

and

$$2\lambda_* M c_B(\kappa,\varepsilon)\mathcal{I}(\infty) \leq \kappa_*. \tag{3.53}$$

Using $\lambda \leq \lambda_*$ and recalling the relation (3.46), i.e. $\mathcal{I}(\tau) \leq \mathcal{I}(\infty) < \infty$, one may infer from the estimates (3.47)–(3.49) that for each $\tau > 0$ the mapping G defines a contractive self mapping on $\mathcal{V}_\tau$. More precisely, G complies with the estimates

$$\|G(u)(t)\|_{W^2_{q,D}(I)} \leq \frac{M}{2\kappa} + \lambda_* M \max\left\{c_L(\kappa,\varepsilon), c_B(\kappa,\varepsilon)\right\} \mathcal{I}(\infty) < \frac{1}{\kappa_*}$$

and

$$G(u)(t) \geq -1 + 2\kappa - 2\lambda_* M c_B(\kappa,\varepsilon)\mathcal{I}(\infty) \geq -1 + \kappa_*,$$

implying that $G\colon \mathcal{V}_\tau \to \mathcal{V}_\tau$ is a self mapping. In addition, it is shown in (3.49) that

$$\|G(v_1)(t) - G(v_2)(t)\|_{W^2_{q,D}(I)} \leq \lambda M c_L(\kappa,\varepsilon)\mathcal{I}(\tau) \left\|v_1 - v_2\right\|_{C\left([0,\tau], W^2_{q,D}(I)\right)},$$

where

$$\lambda M c_L(\kappa,\varepsilon)\mathcal{I}(\tau) \leq \lambda_* M c_L(\kappa,\varepsilon)\mathcal{I}(\infty) < 1.$$

Thus, $G\colon \mathcal{V}_\tau \to \mathcal{V}_\tau$ is a contraction which allows of invoking Banach's fixed-point theorem as in the proof of Theorem 3.1.5 to deduce that

$$G(u)(t) = e^{-tA}u_* - \lambda \int_0^t e^{-(t-s)A} g_\varepsilon\big(u(s)\big)\, ds, \quad t \in [0,\tau],$$

possesses a unique fixed point $u \in \mathcal{V}_\tau$. Since the above reasoning is true for all arbitrarily chosen $\tau > 0$, we may eventually conclude that $T = \infty$ and $u(t) \in \overline{S}_q(\kappa_*)$ for all $t \geq 0$. □

3.2 | On the Quasilinear Case

It is the intention of this section to present a local well-posedness as well as a global existence result for the coupled system consisting of the quasilinear evolution problem

$$
\begin{aligned}
u_t - \partial_x \left(\frac{u_x}{\sqrt{1+\varepsilon^2 (u_x)^2}} \right) &= -\lambda \big(\varepsilon^2 \psi_x^2(x,u) + \psi_z^2(x,u) \big) \\
&\quad + 2\lambda \big(\varepsilon^2 \psi_x(x,u) f_x(x,u) + \psi_z(x,u) f_u(x,u) \big), \quad t > 0,\ x \in I, \qquad (3.54) \\
u(t, \pm 1) &= 0, \quad t > 0, \qquad (3.55) \\
u(0,x) &= u_*(x), \quad x \in I, \qquad (3.56)
\end{aligned}
$$

characterising the time evolution $u = u(t,x)$ of the membrane's displacement, and the elliptic moving boundary problem

$$
\begin{aligned}
\varepsilon^2 \psi_{xx} + \psi_{zz} &= 0, & t > 0,\ (x,z) \in \Omega(u(t)), \qquad (3.57) \\
\psi(t,x,z) &= \frac{1+z}{1+u(t,x)} f\big(x, u(t,x)\big), & t > 0,\ (x,z) \in \partial\Omega(u(t)), \qquad (3.58)
\end{aligned}
$$

describing the behaviour of the electrostatic potential $\psi = \psi(t,x,z)$ in the region

$$
\Omega(u(t)) = \{(x,z) \in I \times (-1,\infty); -1 < z < u(t,x)\}
$$

between the ground plate and the elastic membrane. Similar to what is shown in the previous section we verify local existence and uniqueness of the solution (u,ψ) to (3.54)–(3.58) for all arbitrary values λ of the applied voltage, as well as global existence, provided that the applied voltage does not exceed a critical value λ_*.

As in [17] for the case of a spatially varying permittivity profile $f = f(x)$ and in [16] for $f = f\big(u(t,x)\big)$, the performance of the proofs is based on the methods used in [15].

Following the lines of Section 3.1, the first step in the investigation of (3.54)–(3.58) is to observe that for a given $v \in S_q(\kappa)$ the transformed elliptic moving boundary problem on the fixed rectangular domain $\Omega = I \times (-1,0)$ possesses a unique solution $\varphi \in W_2^2(\Omega)$. Since the statement is exactly the same as in Theorem 3.1.3 in Section 3.1 we just replicate the result without giving the proof.

Let $\kappa \in (0,1)$, $\varepsilon > 0$, and $q \in (2,\infty)$. Given $v \in S_q(\kappa)$ and $f \in C^3\big([-1,1] \times \mathbb{R}, \mathbb{R}\big)$, there is a unique *solution $\varphi_v \in W_2^2(\Omega)$ to the problem*

$$(\mathcal{L}_v\varphi_v)(x,\eta) = 0, \qquad (x,\eta) \in \Omega, \tag{3.59}$$

$$\varphi_v(x,\eta) = \eta f(x,v), \quad (x,\eta) \in \partial\Omega, \tag{3.60}$$

satisfying the a priori estimate

$$\|\varphi_v\|_{W_2^2(\Omega)} \leq c(\kappa,\varepsilon) \tag{3.61}$$

with a positive constant $c(\kappa,\varepsilon)$. In addition, defining the function $\tilde{v}$ by $\tilde{v}(x) := v(-x)$ for all $x \in I$ and assuming that $f\big(x,v(x)\big) = f\big(-x,v(x)\big)$, $x \in I$, one obtains that

$$\varphi_{\tilde{v}}(x,\eta) = \varphi_v(-x,\eta), \quad (x,\eta) \in \Omega.$$

The basic idea for handling the evolution problem (3.54)–(3.56) may be described as follows. Denoting for the sake of better readability by $\varphi \in W_2^2(\Omega)$ the unique solution to (3.59)–(3.60), the transformed quasilinear initial boundary value problem

$$u_t - \partial_x\left(\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right) = -\lambda g_\varepsilon(u), \quad t>0,\ x \in I, \tag{3.62}$$

$$u(t,\pm 1) = 0, \quad t > 0, \tag{3.63}$$

$$u(0,x) = u_*(x), \quad x \in I, \tag{3.64}$$

with g_ε given by

$$\begin{aligned} g_\varepsilon(u) = {} & \varepsilon^2\Big(-\big(f_x(x,u)\big)^2 + \big((f_u(x,u)u_x\big)^2\Big) \\ & - 2\frac{1+\varepsilon^2(u_x)^2}{1+u} f_u(x,u)\varphi_\eta(t,x,1) + \frac{1+\varepsilon^2(u_x)^2}{(1+u)^2}\big(\varphi_\eta(t,x,1)\big)^2 \end{aligned}$$

is perceived as an abstract parameter-dependent quasilinear Cauchy problem. At this point the main two differences between the semilinear problem and its quasilinear counterpart become apparent. In order to employ the instrument of semigroup theory to the latter, firstly a suitable evolution operator $U_{A_\varepsilon(w)}$, induced by the quasilinear second-order differential operator $A_\varepsilon(w)v = -v_{xx}/\big(1+\varepsilon^2(w_x)^2\big)^{3/2}$, has to be introduced. One can see later on that in this context the results derived in [15] may be literally adopted. As also pointed out in [15], the second difference consists in proving regularity properties of the right-hand side of (3.62). More precisely, on the one hand, the transformed right-hand side $g_\varepsilon(u)$ has to be globally bounded as a mapping $g_\varepsilon\colon S_q(\kappa) \to W_{2,D}^{2\sigma}(I)$, $2\sigma \in [0,1/2)$, as shown in Lemma 3.1.4. On the other hand, the quasilinear nature of the evolution problem requires that the Lipschitz continuity of its right-hand side is to be verified in weaker norms than for the semilinear case. These explanatory notes are formalised in Proposition 3.2.4 stated below.

The general concept for proving well-posedness of the quasilinear problem is similar to the one in Section 3.1 for the semilinear case. We first prove a Lipschitz estimate for the Nemitskii operator N_f induced by the permittivity profile f. It becomes clear in the course of this section that also the Lipschitz continuity of N_f is to be proved in the above mentioned weaker norms. Having this Lipschitz estimate at hand we may then derive some technical auxiliary results which in the end enable us to apply the arguments of the proof of [15, Theorem 1.1] in order to proof local well-posedness and global existence.

3.2.1 Lemma (Global Lipschitz Continuity of the Nemitskii Operator II)
Given $q \in (2,\infty)$ and $\kappa \in (0,1)$, let $S_q(\kappa)$ be defined as above. Moreover, let $\xi \in [0,(q-1)/q)$. Then the Nemitskii operator

$$N_f \colon W_q^{2-\xi}(I) \longrightarrow W_q^{2-\xi}(I), \quad v \longmapsto f(\cdot, v(\cdot))$$

induced by $f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$ is globally Lipschitz continuous in the following sense. There exists a constant $c_{f,L} > 0$ such that

$$\|N_f(v_1) - N_f(v_2)\|_{W_q^{2-\xi}(I)} \leq c_{f,L}\|v_1 - v_2\|_{W_q^{2-\xi}(I)}$$

for all $v_1, v_2 \in S_q(\kappa)$.

As usual we denote both the function $f \colon [-1,1]\times\mathbb{R} \to \mathbb{R}$ as well as the Nemitskii operator N_f, by f if no ambiguity is to be feared, i.e. $N_f(v) = f(x,v)$ for $v \in S_q(\kappa)$. Moreover, given $v \in S_q(\kappa)$, in the following the notation $N_f(v)' = f_x(x,v) + f_v(x,v)v_x$ is used for the total derivative of $N_f(v)$ with respect to x, and we write $v' = v_x$ if this is appropriate for the sake of better readability.

Proof. Given $v_1, v_2 \in S_q(\kappa)$, observe that v_1, v_2 and hence for $s \in [0,1]$ also $v_1 + s[v_2 - v_1]$ is bounded in $C^1\big([-1,1],\mathbb{R}\big)$ by a uniform constant depending only on κ. As a simple consequence of the *mean value theorem in integral form* and the continuity of the embedding $W_q^{2-\xi}(I) \hookrightarrow L_q(I)$ we therefore obtain

$$\begin{aligned}\|f(x,v_1)-f(x,v_2)\|^q_{L_q(I)} &= \int_I |f(x,v_1(x))-f(x,v_2(x))|^q dx\\ &= \int_I \left|\int_0^1 f_v\big(x,v_1(x)+s[v_2(x)-v_1(x)]\big)\,(v_2(x)-v_1(x))\,ds\right|^q dx\\ &\le \max_{\substack{x\in[-1,1]\\ s\in[0,1]}} \Big(f_v\big(x,v_1(x)+s[v_2(x)-v_1(x)]\big)\Big)^q \int_I |v_2(x)-v_1(x)|^q\,dx\\ &\le c\|v_1-v_2\|^q_{L_q(I)} \le c\|v_1-v_2\|^q_{W^{2-\xi}_q(I)}.\end{aligned}$$

It remains to verify the estimates

$$\|f_x(x,v_1)-f_x(x,v_2)\|_{W^{1-\xi}_q(I)} \le c\,\|v_1-v_2\|_{W^{2-\xi}_q(I)} \tag{3.65}$$

and

$$\|f_v(x,v_1)v_1'-f_v(x,v_2)v_2'\|_{W^{1-\xi}_q(I)} \le c\,\|v_1-v_2\|_{W^{2-\xi}_q(I)} \tag{3.66}$$

for suitable constants $c>0$. To this end, firstly observe that by means of the mean value theorem in integral form one may prove $N_{f_x}, N_{f_v} \in \mathrm{Lip}\big(W^1_q(I), W^1_q(I)\big)$. Since $W^{2-\xi}_q(I)$ is continuously embedded in $W^1_q(I)$ this implies in addition that

$$N_{f_x}, N_{f_v} \in \mathrm{Lip}\big(W^{2-\xi}_q(I), W^1_q(I)\big).$$

The inequality (3.65) now follows immediately and we are left with proving (3.66). To this end note that pointwise multiplication $W^1_q(I)\cdot W^{1-\xi}_q(I) \hookrightarrow W^{1-\xi}_q(I)$ is continuous (c.f. [2, Theorem 4.1]), whereby we obtain

$$\begin{aligned}&\|f_v(x,v_1)v_1'-f_v(x,v_2)v_2'\|_{W^{1-\xi}_q(I)}\\ &\qquad\le \|f_v(x,v_1)\left(v_1'-v_2'\right)\|_{W^{1-\xi}_q(I)} + \|\big(f_v(x,v_1)-f_v(x,v_2)\big)v_2'\|_{W^{1-\xi}_q(I)}\\ &\qquad\le c\|f_v(x,v_1)\|_{W^1_q(I)}\|v_1'-v_2'\|_{W^{1-\xi}_q(I)} + c\|f_v(x,v_1)-f_v(x,v_2)\|_{W^{1-\xi}_q(I)}\|v_2'\|_{W^1_q(I)}.\end{aligned} \tag{3.67}$$

Using in (3.67) that f_v is contained in $\mathrm{Lip}\big(W^{2-\xi}_q(I), W^1_q(I)\big)$ and that $v_2\in S_q(\kappa)$ finally yields the existence of a constant $c_{f,L}=c_{f,L}(\kappa,f)>0$ such that

$$\|f_v(x,v_1)v_1'-f_v(x,v_2)v_2'\|_{W^{1-\xi}_q(I)} \le c_{f,L}\|v_1-v_2\|_{W^{2-\xi}_q(I)}. \tag{3.68}$$

This completes the proof. □

Prior to proving Lipschitz continuity of g_ε in suitable norms, Lipschitz continuity of φ_v with respect to $v\in S_q(\kappa)$ in suitable norms is to be verified (see Lemma 3.2.3 below). The associated proof is an adaption of [15, Lemma 2.6] to a non-constant permittivity profile so that the following auxiliary result on the function F_v introduced in (3.16) is indispensable.

3.2.2 Lemma

Let $f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$*. Then, given* $\xi \in [0,(q-1)/q)$ *and* $\alpha \in (\xi,1)$*, there exists a constant* $c = c(\kappa,\varepsilon) > 0$ *such that*

$$\|F_{v_1} - F_{v_2}\|_{W^{-\alpha}_{2,D}(\Omega)} \le c(\kappa,\varepsilon)\|v_1 - v_2\|_{W^{2-\xi}_q(I)}, \quad v_1, v_2 \in S_q(\kappa).$$

Proof. Recall that, given $v \in S_q(\kappa)$, the function F_v, defined on Ω as

$$\begin{aligned} F_v(x,\eta) &= \mathcal{L}_v\big(\eta f(x,v)\big) \\ &= \varepsilon^2\eta\big(f_{xx}(x,v) + 2f_{xv}(x,v)v_x + f_{vv}(x,v)(v_x)^2 + f_v(x,v)v_{xx}\big) \\ &\quad - 2\varepsilon^2\eta\frac{v'}{1+v}\big(f_x(x,v) + f_v(x,v)v_x\big) + \varepsilon^2\eta\left(2\left(\frac{v_x}{1+v}\right)^2 - \frac{v_{xx}}{1+v}\right)f(x,v) \\ &= \varepsilon^2\eta N_f(v)'' - 2\varepsilon^2\eta\frac{v'}{1+v}N_f(v)' + \varepsilon^2\eta\left(2\left(\frac{v_x}{1+v}\right)^2 - \frac{v_{xx}}{1+v}\right)N_f(v) \end{aligned}$$

satisfies $F_v \in L_2(\Omega) \hookrightarrow W^{-\alpha}_{2,D}(\Omega)$ as stated in (3.17). Now, given $v_1, v_2 \in S_q(\kappa)$, we show that there exists a constant $c > 0$ such that the dual pairing $\langle (F_{v_1} - F_{v_2}), \mu\rangle$ of $F_{v_1} - F_{v_2} \in W^{-\alpha}_{2,D}(\Omega)$ and $\mu \in W^{\alpha}_{2,D}(\Omega) \setminus \{0\}$ complies with the estimate

$$\langle (F_{v_1} - F_{v_2}), \mu\rangle \le \left|\int_\Omega (F_{v_1} - F_{v_2})\mu \, d(x,\eta)\right| \le c\|v_1 - v_2\|_{W^{2-\xi}_q(I)}\|\mu\|_{W^{\alpha}_{2,D}(\Omega)},$$

whereby one may then conclude that

$$\|F_{v_1} - F_{v_2}\|_{W^{-\alpha}_{2,D}(\Omega)} := \sup_{\mu \in W^{\alpha}_{2,D}(\Omega)\setminus\{0\}} \frac{\langle F_{v_1} - F_{v_2}, \mu\rangle}{\|\mu\|_{W^{\alpha}_{2,D}(\Omega)}} \le c\|v_1 - v_2\|_{W^{2-\xi}_q(I)}.$$

To this end we observe firstly that

$$\begin{aligned} &\left|\int_\Omega (F_{v_1} - F_{v_2})\mu \, d(x,\eta)\right| \\ &\quad \le \varepsilon^2\left|\int_\Omega \big(N_f(v_1)'' - N_f(v_2)''\big)\mu \, d(x,\eta)\right| \\ &\quad + 2\varepsilon^2\left|\int_\Omega \left(\frac{v_1'}{1+v_1}N_f(v_1)' - \frac{v_2'}{1+v_2}N_f(v_2)'\right)\mu \, d(x,\eta)\right| \\ &\quad + \varepsilon^2\left|\int_\Omega \left(2\left(\frac{v_1'}{1+v_1}\right)^2 - \frac{v_1''}{1+v_1}\right)N_f(v_1) - \left(2\left(\frac{v_2'}{1+v_2}\right)^2 - \frac{v_2''}{1+v_2}\right)N_f(v_2)\mu \, d(x,\eta)\right| \end{aligned} \tag{3.69}$$

and estimate the terms on the right-hand side separately. For this purpose we firstly observe that the differential operator[8] $D_x^2\colon W_2^2(\Omega)\to L_2(\Omega)$, $h(x,\eta)\mapsto \partial_x^2 h(x,\eta)$, is linear and bounded and may be extended such that $D_x^2\in\mathcal{L}\big(W_2^{2-\alpha}(\Omega),W_2^{-\alpha}(\Omega)\big)$. Observing that

$$\begin{aligned}\left|\int_\Omega \big(N_f(v_1)''-N_f(v_2)''\big)\mu\,d(x,\eta)\right| &\le \left\|N_f(v_1)''-N_f(v_2)''\right\|_{W_2^{-\alpha}(\Omega)}\|\mu\|_{W_{2,D}^{\alpha}(\Omega)}\\ &\le \|N_f(v_1)-N_f(v_2)\|_{W_2^{2-\alpha}(\Omega)}\|\mu\|_{W_{2,D}^{\alpha}(\Omega)},\end{aligned}$$

the continuity of the embedding $W_q^{2-\xi}(I)\hookrightarrow W_2^{2-\alpha}(I)$ together with the Lipschitz continuity of the Nemitskii operator $N_f(v)=f(x,v)$, proved in Lemma 3.2.1, thus lead to the estimate

$$\begin{aligned}\left|\int_\Omega \big(N_f(v_1)''-N_f(v_2)''\big)\mu\,d(x,\eta)\right| &\le c\,\|N_f(v_1)-N_f(v_2)\|_{W_q^{2-\xi}(I)}\|\mu\|_{W_{2,D}^{\alpha}(\Omega)}\\ &\le c\|v_1-v_2\|_{W_q^{2-\xi}(I)}\|\mu\|_{W_{2,D}^{\alpha}(\Omega)}\end{aligned}\tag{3.70}$$

for the first term. Regarding the second term, one readily obtains

$$\begin{aligned}&\left|\int_\Omega\left(\frac{v_1'}{1+v_1}N_f(v_1)'-\frac{v_2'}{1+v_2}N_f(v_2)'\right)\mu\,d(x,\eta)\right|\\ &\quad\le\left(\left\|\frac{v_1'}{1+v_1}\big(N_f(v_1)'-N_f(v_2)'\big)\right\|_{W_2^{-\alpha}(\Omega)}+\left\|\left(\frac{v_1'}{1+v_1}-\frac{v_2'}{1+v_2}\right)N_f(v_2)'\right\|_{W_2^{-\alpha}(\Omega)}\right)\|\mu\|_{W_{2,D}^{\alpha}(\Omega)}\end{aligned}$$

so that the continuity of the pointwise multiplications

$$W_q^1(\Omega)\cdot W_2^{-\alpha}(\Omega)\hookrightarrow W_2^{-\alpha}(\Omega)\qquad\text{and}\qquad W_q^{1-\xi}(I)\cdot W_q^1(I)\hookrightarrow W_q^{1-\xi}(I)\hookrightarrow W_2^{-\alpha}(I)$$

leads to the inequality

$$\begin{aligned}&\left|\int_\Omega\left(\frac{v_1'}{1+v_1}N_f(v_1)'-\frac{v_2'}{1+v_2}N_f(v_2)'\right)\mu\,d(x,\eta)\right|\\ &\quad\le\left\|\frac{v_1'}{1+v_1}\right\|_{W_q^1(I)}\left\|N_f(v_1)'-N_f(v_2)'\right\|_{W_2^{-\alpha}(\Omega)}\|\mu\|_{W_{2,D}^{\alpha}(\Omega)}\\ &\qquad+\left\|\frac{v_1'}{1+v_1}-\frac{v_2'}{1+v_2}\right\|_{W_q^{1-\xi}(I)}\left\|N_f(v_2)'\right\|_{W_q^1(I)}\|\mu\|_{W_{2,D}^{\alpha}(\Omega)}.\end{aligned}$$

Moreover, again thanks to Lemma 3.2.1 by additionally using the facts that $W_q^{1-\xi}(I)$ is a multiplication algebra and that each $v\in S_q(\kappa)$ is bounded in $C^1\big([-1,1],\mathbb{R}\big)$ by a uniform constant $c(\kappa)>0$,

[8] If $h\in W_2^2(\Omega)$ depends only on $x\in I$ the simplified notation $D_x^2h(x)=h''(x)$ is used.

we find that

$$\begin{aligned}\Big|\int_\Omega\Big(\frac{v_1'}{1+v_1}N_f(v_1)' - \frac{v_2'}{1+v_2}N_f(v_2)'\Big)\,\mu\,d(x,\eta)\Big|\\ &\le c(\kappa)\,\|N_f(v_1)-N_f(v_2)\|_{W_q^{2-\xi}(I)}\,\|\mu\|_{W^\alpha_{2,D}(\Omega)}\\ &\quad + c(\kappa,f)\|v_1'-v_2'\|_{W_q^{1-\xi}(I)}\|v_1-v_2\|_{W_q^{1-\xi}(I)}\|\mu\|_{W^\alpha_{2,D}(\Omega)}\\ &\le c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}\|\mu\|_{W^\alpha_{2,D}(\Omega)}.\end{aligned} \tag{3.71}$$

Finally, fusing the arguments used in [15, Lemma 2.5] with Lemma 3.2.1 in order to estimate the third term, one may verify that also the inequality

$$\begin{aligned}\Big|\int_\Omega\Big(2\Big(\frac{v_1'}{1+v_1}\Big)^2 - \frac{v_1''}{1+v_1}\Big)N_f(v_1) - \Big(2\Big(\frac{v_2'}{1+v_2}\Big)^2 - \frac{v_2''}{1+v_2}\Big)N_f(v_2)\mu\,d(x,\eta)\Big|\\ \le c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}\|\mu\|_{W^\alpha_{2,D}(\Omega)}\end{aligned} \tag{3.72}$$

holds true. A combination of the estimates (3.70)–(3.72) eventually yields

$$\Big|\int_\Omega (F_{v_1}-F_{v_2})\mu\,d(x,\eta)\Big| \le c(\kappa,\varepsilon,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}\|\mu\|_{W^\alpha_{2,D}(\Omega)}$$

and with the introductory words the proof is complete. □

As a combination of the previous lemma with results from [15] we obtain Lipschitz continuity of φ_v as formulated in the following lemma.

3.2.3 Lemma
Let $f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$. Then, given $\xi \in [0,(q-1)/q)$ and $\alpha \in (\xi,1)$, there exists a constant $c = c(\kappa,\varepsilon) > 0$ such that

$$\|\varphi_{v_1}-\varphi_{v_2}\|_{W^{2-\alpha}_{2,D}(\Omega)} \le c(\kappa,\varepsilon)\|v_1-v_2\|_{W_q^{2-\xi}(I)}, \quad v_1,v_2 \in S_q(\kappa).$$

Proof. As in [15, Section 2] we pick $v \in S_q(\kappa)$ and introduce the bounded linear operator

$$\mathcal{A}(v) \in \mathcal{L}\big(W^1_{2,D}(\Omega), W^{-1}_{2,D}(\Omega)\big) \cap \mathcal{L}\big(W^2_{2,D}(\Omega), L_2(\Omega)\big),$$

defined as

$$\mathcal{A}(v)\phi := -\mathcal{L}_v\phi, \quad \phi \in W^1_{2,D}(\Omega).$$

[15, Lemma 2.2] yields the existence of a unique solution $\phi_v \in W^2_{2,D}(\Omega)$ to the problem

$$\begin{aligned} -\mathcal{L}_v\phi_v &= F_v, \quad (x,\eta)\in\Omega,\\ \phi_v &= 0, \quad\ \ (x,\eta)\in\partial\Omega. \end{aligned}$$

This implies in particular that $\mathcal{A}(v)$ is invertible and that the relation $\phi_v = \mathcal{A}(v)^{-1}F_v$ holds true on Ω. Moreover, one may check that $\varphi_v = \phi_v + \eta F_v$ (c.f. also the proof of Theorem 3.1.3 in Section 3.1). In addition, it is proved in [15, Lemma 2.3] that, given $\theta \in [0,1]\setminus\{1/2\}$, there exists a uniform constant $c = c(\kappa,\varepsilon) > 0$ such that

$$\|\mathcal{A}(v)^{-1}\|_{\mathcal{L}\left(W^{\theta-1}_{2,D}(\Omega),W^{\theta+1}_{2,D}(\Omega)\right)} \le c(\kappa,\varepsilon), \quad v\in S_q(\kappa).$$

Finally, [15, Lemma 2.4] states that, given $\xi \in [0,(q-1)/q)$ and $\alpha\in(\xi,1)$, there exists a further constant $c = c(\kappa,\varepsilon) > 0$ such that $\mathcal{A}(v)$ is Lipschitz continuous in the following norms:

$$\|\mathcal{A}(v_1)-\mathcal{A}(v_2)\|_{\mathcal{L}\left(W^2_{2,D}(\Omega),W^{-\alpha}_{2,D}(\Omega)\right)} \le c(\kappa,\varepsilon)\|v_1-v_2\|_{W^{2-\xi}_q(I)}, \quad v_1,v_2\in S_q(\kappa).$$

Having these preliminaries at hand pick $v_1, v_2 \in S_q(\kappa)$ and note that $\varphi_{v_1}-\varphi_{v_2} = \phi_{v_1}-\phi_{v_2} \in W^{2-\alpha}_{2,D}(\Omega)$. This allows the following calculation.

$$\begin{aligned} \|\varphi_{v_1}-\varphi_{v_2}\|_{W^{2-\alpha}_{2,D}(\Omega)} &\le \|\mathcal{A}(v_1)^{-1}\big(F_{v_1}-F_{v_2}\big)\|_{W^{2-\alpha}_{2,D}(\Omega)} + \|\big(\mathcal{A}(v_1)^{-1}-\mathcal{A}(v_2)^{-1}\big)F_{v_2}\|_{W^{2-\alpha}_{2,D}(\Omega)}\\ &\le \|\mathcal{A}(v_1)^{-1}\|_{\mathcal{L}\left(W^{-\alpha}_{2,D}(\Omega),W^{2-\alpha}_{2,D}(\Omega)\right)}\|F_{v_1}-F_{v_2}\|_{W^{-\alpha}_{2,D}(\Omega)}\\ &\quad + \|\mathcal{A}(v_1)^{-1}[\mathcal{A}(v_2)-\mathcal{A}(v_1)]\mathcal{A}(v_2)^{-1}\|_{\mathcal{L}\left(W^{-\alpha}_{2,D}(\Omega),W^{2-\alpha}_{2,D}(\Omega)\right)}\|F_{v_2}\|_{W^{2-\alpha}_{2,D}(\Omega)}\\ &\le \|\mathcal{A}(v_1)^{-1}\|_{\mathcal{L}\left(W^{-\alpha}_{2,D}(\Omega),W^{2-\alpha}_{2,D}(\Omega)\right)}\|F_{v_1}-F_{v_2}\|_{W^{-\alpha}_{2,D}(\Omega)}\\ &\quad + \|\mathcal{A}(v_1)^{-1}\|_{\mathcal{L}\left(W^{-\alpha}_{2,D}(\Omega),W^{2-\alpha}_{2,D}(\Omega)\right)}\|\mathcal{A}(v_1)-\mathcal{A}(v_2)\|_{\mathcal{L}\left(W^2_{2,D}(\Omega),W^{-\alpha}_{2,D}(\Omega)\right)}\\ &\quad\quad \cdot \|\mathcal{A}(v_2)^{-1}\|_{\mathcal{L}\left(L_2(\Omega),W^2_{2,D}(\Omega)\right)}\|F_{v_2}\|_{L_2(\Omega)} \end{aligned}$$

Now using that

$$\|\mathcal{A}(v)^{-1}\|_{\mathcal{L}\left(W^{\theta-1}_{2,D}(\Omega),W^{\theta+1}_{2,D}(\Omega)\right)} \le c(\kappa,\varepsilon), \quad v\in S_q(\kappa),$$

for all $\theta\in[0,1]\setminus\{1/2\}$ and that

$$\|\mathcal{A}(v_1)-\mathcal{A}(v_2)\|_{\mathcal{L}\left(W^2_{2,D}(\Omega),W^{-\alpha}_{2,D}(\Omega)\right)} \le c(\kappa,\varepsilon)\|v_1-v_2\|_{W^{2-\xi}_q(I)}, \quad v_1,v_2\in S_q(\kappa),$$

thanks to [15, Lemma 2.3 & Lemma 2.4], one may conclude that

$$\|\varphi_{v_1}-\varphi_{v_2}\|_{W^{2-\alpha}_{2,D}(\Omega)} \le c(\kappa,\varepsilon)\|F_{v_1}-F_{v_2}\|_{W^{-\alpha}_{2,D}(\Omega)} + c(\kappa,\varepsilon)\|F_{v_2}\|_{L_2(\Omega)}\|v_1-v_2\|_{W^{2-\xi}_q(I)}.$$

Finally, recalling that F_v is uniformly bounded in $L_2(\Omega)$ for all $v \in S_q(\kappa)$, we invoke Lemma 3.2.2 and end up with

$$\|\varphi_{v_1} - \varphi_{v_2}\|_{W^{2-\alpha}_{2,D}(\Omega)} \leq c(\kappa,\varepsilon)\|v_1 - v_2\|_{W^{2-\xi}_q(I)}, \quad v_1, v_2 \in S_q(\kappa).$$

This completes the proof. □

Fusing the above presented results we are now able to verify that the evolution equations's right-hand side g_ε is globally Lipschitz continuous and uniformly bounded in suitable norms.

3.2.4 Proposition (Properties of g_ε)
Let $\kappa \in (0,1), \varepsilon > 0$ and $q \in (2,\infty)$. Moreover, let $f \in C^3\big([-1,1] \times \mathbb{R}, \mathbb{R}\big)$. Then, with $\varphi_v \in W^2_2(\Omega)$ denoting the unique solution to (3.59)–(3.60), for $2\sigma \in [0,1/2)$, the mapping

$$\begin{aligned} g_\varepsilon \colon S_q(\kappa) &\longrightarrow W^{2\sigma}_{2,D}(I),\\ v &\longmapsto \varepsilon^2\Big(-\big(f_x(x,u)\big)^2 + \big((f_u(x,u)u_x\big)^2\Big)\\ &\qquad - 2\frac{1+\varepsilon^2(u_x)^2}{1+u} f_u(x,u)\partial_\eta\varphi_v(\cdot,1) + \frac{1+\varepsilon^2(u_x)^2}{(1+u)^2}(\partial_\eta\varphi_v(\cdot,1))^2 \end{aligned}$$

has the following properties:

(i) g_ε is uniformly bounded in $W^{2\sigma}_{2,D}(I)$. That is, there is a constant $c_B = c_B(\kappa,\varepsilon) > 0$ such that

$$\|g_\varepsilon(v)\|_{W^{2\sigma}_{2,D}(I)} \leq c_B(\kappa,\varepsilon)$$

for every $v \in S_q(\kappa)$.

(ii) Given $\xi \in [0,1/2)$ and $\nu \in [0,(1-2\xi)/2)$, g_ε complies with the following global Lipschitz inequality. There exists a constant $c_L = c_L(\kappa,\varepsilon) > 0$ such that

$$\|g_\varepsilon(v_1) - g_\varepsilon(v_2)\|_{W^{\nu}_{2,D}(I)} \leq c_L(\kappa,\varepsilon)\|v_1 - v_2\|_{W^{2-\xi}_{q,D}(I)}$$

for all $v_1, v_2 \in S_q(\kappa)$.

Proof. Part (i) of the proposition contains exactly the same statement as in Lemma 3.1.4 of Section 3.1. We are thus left with proving part (ii). To this end, given $v \in S_q(\kappa)$, we introduce the notation

$$h_1(v) := \varepsilon^2\Big(-\big(f_x(x,v)\big)^2 + \big((f_v(x,v)v'\big)^2\Big), \quad h_2(v) := \frac{1+\varepsilon^2(v')^2}{1+v} f_v(x,v)\partial_\eta\varphi_v(t,x,1)$$

as well as

$$h_3(v) := \frac{1+\varepsilon^2(v')^2}{(1+v)^2}\big(\partial_\eta\varphi_v(t,x,1)\big)^2$$

and write $g_\varepsilon(v)$ as

$$g_\varepsilon(v) := h_1(v) - 2h_2(v) + h_3(v).$$

Having the previous Lemma 3.2.3 in mind, the Lipschitz continuity of h_3 with respect to $v \in S_q(\kappa)$ in the corresponding topologies may be literally adopted from the proof of [15, Proposition 2.1]. Thus, there exists a constant $c = c(\kappa, \varepsilon) > 0$ such that

$$\|h_3(v_1) - h_3(v_2)\|_{W^\nu_{2,D}(I)} \le c(\kappa, \varepsilon)\|v_1 - v_2\|_{W^{2-\xi}_{q,D}(I)}, \quad v_1, v_2 \in S_q(\kappa), \tag{3.73}$$

for $\xi \in [0, 1/2)$ and $\nu \in [0, (1-2\xi)/2)$. In order to verify the Lipschitz continuity of h_2 we see that

$$\begin{aligned} \|h_2(v_1) - h_2(v_2)\|_{W^\nu_2(I)} \le & \left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1} f_v(x, v_1)\Big(\partial_\eta \varphi_{v_1}(x,1) - \partial_\eta \varphi_{v_2}(x,1)\Big)\right\|_{W^\nu_2(I)} \\ & + \left\|\left(\frac{1+\varepsilon^2(v_1')^2}{1+v_1} f_v(x,v_1) - \frac{1+\varepsilon^2(v_2')^2}{1+v_2} f_v(x,v_2)\right)\partial_\eta\varphi_{v_2}(x,1)\right\|_{W^\nu_2(I)} \end{aligned} \tag{3.74}$$

and introduce the notation

$$\begin{aligned} N_1 &:= \left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1} f_v(x, v_1)\Big(\partial_\eta \varphi_{v_1}(x,1) - \partial_\eta \varphi_{v_2}(x,1)\Big)\right\|_{W^\nu_2(I)}, \\ N_2 &:= \left\|\left(\frac{1+\varepsilon^2(v_1')^2}{1+v_1} f_v(x,v_1) - \frac{1+\varepsilon^2(v_2')^2}{1+v_2} f_v(x,v_2)\right)\partial_\eta\varphi_{v_2}(x,1)\right\|_{W^\nu_2(I)}. \end{aligned}$$

Since pointwise multiplication

$$W^1_q(I) \cdot W^1_2(I) \cdot W^{1/2-\alpha}_2(I) \hookrightarrow W^\nu_2(I)$$

is continuous (c.f. [2, Theorem 4.1]) N_1 complies with the estimate

$$N_1 \le c \left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1}\right\|_{W^1_q(I)} \|f_v(x,v_1)\|_{W^1_2(I)} \|\partial_\eta \varphi_{v_1}(x,1) - \partial_\eta \varphi_{v_2}(x,1)\|_{W^{1/2-\alpha}_2(I)}.$$

Moreover, using that $v_1 \in S_q(\kappa)$ and that $f_v(x, v_1)$ is bounded in the sense of Corollary 3.1.2 it follows from the properties of the trace operator as stated in [24, Theorem 1.5.1.1] that

$$N_1 \le c(\kappa, f)\, \|\varphi_{v_1} - \varphi_{v_2}\|_{W^{2-\alpha}_2(\Omega)}.$$

Finally using the Lipschitz continuity of φ_v in the sense of Lemma 3.2.3 we end up with

$$N_1 = \left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1} f_v(x, v_1)\Big(\partial_\eta \varphi_{v_1}(x,1) - \partial_\eta \varphi_{v_2}(x,1)\Big)\right\|_{W^\nu_2(I)} \le c(\kappa, f)\|v_1 - v_2\|_{W^{2-\xi}_q(I)}. \tag{3.75}$$

As for N_2 observe that

$$\begin{aligned}
N_2 \le & \left(\left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1}\big(f_v(x,v_1)-f_v(x,v_2)\big) \right\|_{W_q^{1-\xi}(I)} \right. \\
& \left. + \left\| \left(\frac{1+\varepsilon^2(v_1')^2}{1+v_1} - \frac{1+\varepsilon^2(v_2')^2}{1+v_2} \right) f_v(x,v_2) \right\|_{W_q^{1-\xi}(I)} \right) \|\partial_\eta \varphi_{v_2}(x,1)\|_{W_2^{1/2}(I)} \\
\le & \left(\left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1} \right\|_{W_q^1(I)} \|f_v(x,v_1)-f_v(x,v_2)\|_{W_q^{1-\xi}(I)} \right. \\
& \left. + \left\| \frac{1+\varepsilon^2(v_1')^2}{1+v_1} - \frac{1+\varepsilon^2(v_2')^2}{1+v_2} \right\|_{W_q^{1-\xi}(I)} \|f_v(x,v_2)\|_{W_q^{1-\xi}(I)} \right) \|\partial_\eta \varphi_{v_2}(x,1)\|_{W_2^{1/2}(I)}
\end{aligned}$$

since pointwise multiplications

$$W_q^{1-\xi}(I)\cdot W_2^{1/2}(I) \hookrightarrow W_2^{\nu}(I) \quad \text{and} \quad W_q^1(I)\cdot W_q^{1-\xi}(I) \hookrightarrow W_q^{1-\xi}(I)$$

are continuous. Again using that $v_2 \in S_q(\kappa)$ and once more invoking [24, Theorem 1.5.1.1] to see that $\|\partial_\eta \varphi_{v_2}(x,1)\|_{W_2^{1/2}(I)} \le \|\varphi_{v_2}\|_{W_2^2(\Omega)} \le c$ leads to the estimate

$$N_2 \le c(\kappa,f)\left(\|f_v(x,v_1)-f_v(x,v_2)\|_{W_q^{1-\xi}(I)} + \|v_1'-v_2'\|_{W_q^{1-\xi}(I)} \right).$$

In the proof of Lemma 3.2.1 it is shown that

$$\|f_v(x,v_1)-f_v(x,v_2)\|_{W_q^{1-\xi}(I)} \le c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)},$$

whereby N_2 eventually satisfies

$$\begin{aligned}
N_2 &= \left\| \left(\frac{1+\varepsilon^2(v_1')^2}{1+v_1} f_v(x,v_1) - \frac{1+\varepsilon^2(v_2')^2}{1+v_2} f_v(x,v_2) \right) \partial_\eta \varphi_{v_2}(x,1) \right\|_{W_2^{\nu}(I)} \\
&\le c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}.
\end{aligned} \tag{3.76}$$

Combining (3.75) and (3.76) proves the Lipschitz continuity of h_2, i.e. the inequality

$$\|h_2(v_1)-h_2(v_2)\|_{W_2^{\nu}(I)} \le c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}. \tag{3.77}$$

We are hence left with verifying an analogue estimate for h_1. For this purpose observe that continuity of pointwise multiplication leads to

$$\begin{aligned}\|h_1(v_1)-h_1(v_2)\|_{W_2^\nu(I)} \\ \leq \left\|\big(f_x(x,v_1)\big)^2-\big(f_x(x,v_2)\big)^2\right\|_{W_2^\nu(I)}+\left\|\big(f_v(x,v_1)\big)^2(v_1')^2-\big(f_v(x,v_2)\big)^2(v_2')^2\right\|_{W_2^\nu(I)}.\end{aligned} \tag{3.78}$$

Regarding the Lipschitz continuity of the first term on the right-hand side of (3.78) observe that pointwise multiplication $W_q^{1-\xi}(I)\cdot W_2^1(I)\hookrightarrow W_2^\nu(I)$ is continuous and recall that (c.f. the proof of Lemma 3.2.1) $N_{f_x}\in \mathrm{Lip}\big(W_q^{2-\xi}(I),W_q^1(I)\big)$. This readily yields

$$\begin{aligned}\left\|\big(f_x(x,v_1)\big)^2-\big(f_x(x,v_2)\big)^2\right\|_{W_2^\nu(I)} \\ \leq \|f_x(x,v_1)-f_x(x,v_2)\|_{W_q^{1-\xi}(I)}\,\|f_x(x,v_1)+f_x(x,v_2)\|_{W_2^1(I)} \\ \leq c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}.\end{aligned} \tag{3.79}$$

Furthermore, the second term on the right-hand side of (3.78) may be treated as follows. Using again that pointwise multiplication $W_q^{1-\xi}(I)\cdot W_2^1(I)\hookrightarrow W_2^\nu(I)$ is continuous and moreover that $W_2^1(I)$ is a multiplication algebra, we find that

$$\begin{aligned}&\left\|\big(f_v(x,v_1)\big)^2(v_1')^2-\big(f_v(x,v_2)\big)^2(v_2')^2\right\|_{W_2^\nu(I)} \\ &\leq \left\|f_v(x,v_1)v_1'-f_v(x,v_2)v_2'\right\|_{W_q^{1-\xi}(I)}+\left\|f_v(x,v_1)v_1'+f_v(x,v_2)v_2'\right\|_{W_2^1(I)} \\ &\leq \left\|f_v(x,v_1)v_1'-f_v(x,v_2)v_2'\right\|_{W_q^{1-\xi}(I)} \\ &\quad+\Big(\|f_v(x,v_1)\|_{W_2^1(I)}\|v_1'\|_{W_2^1(I)}+\|f_v(x,v_2)\|_{W_2^1(I)}\|v_2'\|_{W_2^1(I)}\Big).\end{aligned}$$

We finally recall that $v_1,v_2\in S_q(\kappa)$ and that $f_v(x,v_1)$ and $f_v(x,v_2)$ are uniformly bounded in $W_2^2(I)$ to deduce from (3.68) in Lemma 3.2.1 that

$$\left\|(f_v(x,v_1))^2(v_1')^2-(f_v(x,v_2))^2(v_2')^2\right\|_{W_2^\nu(I)}\leq c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)}. \tag{3.80}$$

Fusing (3.79) and (3.80) yields the Lipschitz continuity of h_1 with respect to $v\in S_q(\kappa)$ in the sense:

$$\|h_1(v_1)-h_1(v_2)\|_{W_2^\nu(I)}\leq c(\kappa,f)\|v_1-v_2\|_{W_q^{2-\xi}(I)},\quad v_1,v_2\in S_q(\kappa), \tag{3.81}$$

whereby the assertion is proved after combining (3.73), (3.77) and (3.81). □

We are now prepared to verify the following result on local existence and uniqueness of solutions to (3.54)–(3.58). As already mentioned, the proof is performed as the one of [15, Theorem 1.1].

3.2.5 Theorem (Local Well-Posedness)
Let $q\in(2,\infty)$ and $\varepsilon>0$. Given an initial value $u_*\in W_{q,D}^2(I)$ with $u_*(x)>-1$ for all $x\in I$, and

$f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$, *the following holds true:*

(i) For each voltage value $\lambda > 0$ *there exists a unique maximal solution* (u,ψ) *to* (3.54)–(3.58) *on the maximal interval* $[0,T)$ *of existence in the sense that*

$$u \in C^1\big([0,T),L_q(I)\big)\cap C\big([0,T),W^2_{q,D}(I)\big)$$

satisfies (3.54)–(3.56) *with*

$$u(t,x) > -1, \quad t\in[0,T), x\in I,$$

and $\psi \in W^2_2\big(\Omega(u(t))\big)$ *solves* (3.57)–(3.58) *for each* $t\in[0,T)$.

(ii) If for each $\tau > 0$ *there is a* $\kappa(\tau)\in(0,1)$ *such that* $u(t)\in S_q\big(\kappa(\tau)\big)$ *for* $t\in[0,T)\cap[0,\tau]$, *then the solution* (u,ψ) *to* (3.54)–(3.58) *exists globally in time, meaning that* $T=\infty$.

(iii) If $u_*(x) = u_*(-x)$ *and* $f\big(x,u(t,x)\big) = f\big(-x,u(t,x)\big)$ *for all* $t\in[0,T)$ *and* $x \in I$, *then* $u=u(t,x)$ *and* $\psi=\psi(t,x,z)$ *are even with respect to* $x\in I$ *for all times* $t\in[0,T)$ *as well.*

Proof. (i) Following [15] we fix $q\in(2,\infty), \kappa\in(0,1)$ and $\xi\in(0,(q-1)/q)$ and introduce the set

$$Z_q(\kappa) := \left\{ w \in W^{2-\xi}_{q,D}(I); \|w\|_{W^{2-\xi}_q(I)} \le 1/\kappa \right\}.$$

The identity

$$\partial_x\left(\frac{v_x}{\sqrt{1+\varepsilon^2(v_x)^2}}\right) = -\frac{v_{xx}}{\big(1+\varepsilon^2(v_x)^2\big)^{3/2}}$$

then serves as a motivation to define for $w\in Z_q(\kappa)$ the linear differential operator

$$A_\varepsilon(w)\in\mathcal{L}\big(W^2_{q,D}(I),L_q(I)\big), \quad A_\varepsilon(w)v := -\frac{v_{xx}}{\big(1+\varepsilon^2(w_x)^2\big)^{3/2}}$$

of second order such that (3.54)–(3.56) may be perceived as the abstract Cauchy problem

$$\dot{u} + A_\varepsilon(u)u = -\lambda g_\varepsilon(u), \quad t>0, \tag{3.82}$$

$$u(0) = u_*. \tag{3.83}$$

Moreover, thanks to [15, Lemma 3.1] we know that the operator $A_\varepsilon(w)$ generates a strongly continuous analytic semigroup on $L_q(I)$.[9] The Cauchy problem (3.82)–(3.83) may be solved by introducing so-called *evolution operators* in the sense of [3, Section II]. To this end, fix $\rho\in(0,1)$ and $\tau>0$ and

[9] Note that the proof of [15, Lemma 3.1] uses the general statement in [3, Section I, Theorem 1.2.2].

define the set

$$\mathcal{W}_\tau(\kappa) := \left\{ w\colon [0,\tau] \to Z_q(\kappa);\ \max_{0\le t,s\le\tau} \frac{\|w(t)-w(s)\|_{W_q^{2-\xi}(I)}}{|t-s|^\rho} < \infty \right\}.$$

It is proved in [15, Proposition 3.1], that if $w = w(t)$ is Hölder continuous in t in the sense that w is contained in $\mathcal{W}_\tau(\kappa)$, then $A_\varepsilon(w)$ generates a unique parabolic evolution operator $U_{A_\varepsilon(w)}(t,s)$, $0 \le s \le t \le \tau$, possessing $W_{q,D}^2(I)$ as a regularity subspace. Furthermore, each linear operator $U_{A_\varepsilon(w)}(t,s)$ is positive from $L_q(I)$ to $L_q(I)$ for all $0 \le s \le t \le \tau$. Having the boundedness and Lipschitz continuity of g_ε in terms of Proposition 3.2.4 at hand one may now literally follow the lines of the proof of [15, Theorem 1.1] to conclude that there exists a constant $\kappa_* \le \kappa$ such that[10]

$$G(v)(t) := U_{A_\varepsilon(v)}(t,0)u_* - \lambda \int_0^t U_{A_\varepsilon(v)}(t,s) g_\varepsilon\big(v(s)\big)\, ds, \quad t \in [0,\tau],$$

defines for each $v \in \mathcal{W}_\tau(\kappa)$ with $v(t) \in \overline{S}_q(\kappa_*)$, $t \in [0,\tau]$, a contractive self mapping on the complete metric space

$$\mathcal{V}_\tau := \Big(\big\{ v \in \mathcal{W}_\tau(\kappa);\ v(t) \in \overline{S}_q(\kappa_*), t \in [0,\tau] \big\}, d \Big),$$

where d denotes the metric induced by $C\big([0,\tau], W_q^{2-\xi}(I)\big)$. Therefore one may deduce from Banach's fixed-point theorem that there exists a unique fixed point

$$u = G(u) \in \mathcal{V}_\tau$$

of G. As in the semilinear case this finally proves part (i) of the theorem, as it implies that for each $\lambda > 0$ (3.82)–(3.83) possesses a unique non-extendable solution

$$u \in C^1\big([0,T), L_q(I)\big) \cap C\big([0,T), W_{q,D}^2(I)\big)$$

for some $T \in (\tau, \infty]$, satisfying

$$u(t,x) > -1, \quad t \in [0,T),\ x \in I.$$

Parts (ii) and (iii) may be proved as in the semilinear case as well. □

3.2.6 Theorem (Global Existence)
Let $q \in (2,\infty), \varepsilon > 0$ and $\lambda > 0$. Furthermore, given $f \in C^3\big([-1,1] \times \mathbb{R}, \mathbb{R}\big)$ and $u_* \in W_{q,D}^2(I)$ satisfying $u_*(x) > -1$ for $x \in I$, let (u,ψ) denote the corresponding solution to (3.54)–(3.58) on the maximal interval $[0,T)$ of existence. Then, given $\kappa \in (0,1)$, there exist $\lambda_* = \lambda_*(\kappa) > 0$ and

[10] In fact κ_* is given as $\kappa_* = \kappa / c_*(\kappa)$, where $c_*(\kappa) \ge 1$ is a suitable constant related to the evolution operator $U_{A(w)}$; see (3.11) in [15].

$c = c(\kappa) > 0$ such that $T = \infty$ with $u(t,x) \geq -1+\kappa$ for all $t \in [0,\infty)$ and $x \in I$, provided that $\lambda \in (0,\lambda_*)$ and $\|u_*\|_{W_q^2(I)} \leq c(\kappa)$. In that case u enjoys the following additional regularity properties:

$$u \in BUC^{\rho}\big([0,\infty), W_{q,D}^{2-\rho}(I)\big) \cap L_\infty\big([0,\infty), W_{q,D}^2(I)\big) \tag{3.84}$$

for some small $\rho > 0$.

3.2.7 Remark *(1) The proof of Theorem 3.2.6 may be literally adopted from the one of [15, Theorem 1.1 (iv)], where the authors demonstrate that G is a contractive self mapping on $\mathcal{V}_\tau$ for all $\tau \geq 0$, provided that the applied voltage $\lambda > 0$ is smaller than a certain critical value λ_* and the initial value u_* is bounded in the $W_q^2(I)$-norm by a certain constant $c(\kappa) > 0$. Note that the latter condition is not required in the semilinear case.*

(2) It is worthwhile to mention that temporally global solutions u do never touch down on the ground plate, not even in infinite time. Moreover, note that they are bounded in $W_q^2(I)$ by a uniform constant.

(3) It follows from general parabolic theory (c.f. [3]), that the globally existing mild solution is also a strong solution enjoying the additional regularity stated in (3.84).

The section is completed by making the observation that Corollary 3.1.6 in Section 3.1 on the sign property of the solution u does likewise hold true in the quasilinear case. Thus, we have the following corollary.

3.2.8 Corollary
Let $u_* \in S_q(\kappa)$ satisfy $u_*(x) \leq 0$ for all $x \in I$ and assume that the implication

$$v \in S_q(\kappa),\ v(x) \leq 0\ \forall x \in I \quad \Longrightarrow \quad g_\varepsilon(v) \in S_q(\kappa),\ g_\varepsilon(v(x)) \geq 0\ \forall x \in I \tag{3.85}$$

holds true. Then the solution u to (3.54)–(3.56) *satisfies*

$$u(t,x) \leq 0, \quad t \in [0,T),\ x \in I.$$

4 | The Small-Aspect Ratio Limit

As mentioned in the introduction the analysis of *coupled* systems of partial differential equations has only recently become part of the mathematical investigation of microelectromechanical systems. Irrespective of the precise physical regime for an adequate choice of the governing equations describing the dynamics of the membrane's displacement u – suppose for instance the linear elasticity setting (2.32)–(2.34) or its nonlinear elasticity counterpart (2.29)–(2.31) – the displacement of the elastic membrane causes a change of the shape of the domain $\Omega\big(u(t)\big)$ occupied by the ground plate and the overlying membrane. This gives rise to a coupling between the problem for the electrostatic potential ψ and the membrane's displacement u. On the one hand ψ is to be determined by a free boundary value problem in the domain $\Omega(u(t))$. On the other hand the right-hand side of the evolution equation for u involves the partial derivatives of the potential ψ signifying a further coupling in the system. Roughly speaking, this strong coupling between the two problems makes their mathematical analysis rather complex, whereby it has heretofore been and it still is a quite common approach in MEMS research to make an assumption which reduces the initial nonlocal coupled problem to an uncoupled semilinear evolution equation for u.

For pioneering contributions to the understanding of the full coupled problem the reader is again referred to the works [32] and [14]. It is the intention of this chapter to generalise a convergence result on the small-aspect ratio limit obtained in [14] for $f \equiv 1$ to the case of a general permittivity profile $f = f\big(x, u(t,x)\big)$. To this end, consider the system

$$u_t - u_{xx} = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\Big) + 2\lambda\Big(\varepsilon^2\psi_x(x,u)f_x(x,u) + \psi_z(x,u)f_u(x,u)\Big), \quad t>0,\ x\in I, \tag{4.1}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{4.2}$$

$$u(0,x) = u_*(x), \quad x \in I, \tag{4.3}$$

arising from the linear elasticity approach for the membrane's displacement, together with the elliptic moving boundary problem

$$\varepsilon^2\psi_{xx} + \psi_{zz} = 0, \qquad (x,z) \in \Omega(u(t)), \tag{4.4}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f\big(x,u(t,x)\big), \quad (x,z) \in \partial\Omega(u(t)), \tag{4.5}$$

for the electrostatic potential in the region

$$\Omega\big(u(t)\big) := \{(x,z) \in I \times (-1,\infty); -1 < z < u(t,x)\}.$$

A common approach in order to decouple the problems (4.1)–(4.3) and (4.4)–(4.5) is to consider the aspect ratio ε of the respective MEMS device to be fairly small, i.e. $\varepsilon \ll 1$, or in fact even $\varepsilon = 0$. In this case the potential is computed as if the two plates were locally parallel and the resulting explicit expression for ψ avoids the coupling via the right-hand side of (4.1). The full coupled problem is then reduced to a semilinear parabolic initial boundary value problem possessing a singularity in the instant the elastic membrane touches down on the ground plate. In detail, setting $\varepsilon = 0$ in (4.4) yields the reduced problem

$$\psi_{zz}(t,x,z) = 0, \qquad t>0,\ (x,z) \in \Omega(u(t)), \tag{4.6}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f\big(x,u(t,x)\big), \quad t>0,\ (x,z) \in \partial\Omega(u(t)), \tag{4.7}$$

for the electrostatic potential whose solution $\psi := \psi_0$ may be explicitly stated as

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f\big(x,u(t,x)\big), \quad t>0,\ (x,z) \in I \times (-1,0). \tag{4.8}$$

Inserting the likewise computable partial derivative

$$\psi_z(t,x,z) = \frac{f\big(x,u(t,x)\big)}{1+u(t,x)}$$

into the evolution equation for the membrane displacement in the case $\varepsilon = 0$, i.e. into the equation

$$u_t - u_{xx} = -\lambda\big(\psi_z(t,x,u)\big)^2 + 2\lambda\psi_z(t,x,u) f_u(x,u), \quad t>0,\ x \in I, \tag{4.9}$$

implies that the displacement $u := u_0$ finally satisfies the so-called *small-aspect ratio model*

$$u_t - u_{xx} = -\lambda \left(\frac{f\big(x, u(t,x)\big)}{1+u(t,x)} \right)^2 + 2\lambda \frac{f\big(x,u(t,x)\big)}{1+u(t,x)} f_u\big(x, u(t,x)\big), \quad t>0,\ x \in I, \tag{4.10}$$

$$u(t,\pm 1) = 0, \qquad t>0, \tag{4.11}$$

$$u(0,x) = u_*(x), \qquad x \in I. \tag{4.12}$$

Problem (4.10)–(4.12) is a reduced model for the elastic behaviour of the system. It is uncoupled from the potential equation and may be solved independently. However, note that the evolution equation (4.10) is still nonlinear.

Denoting for $\varepsilon > 0$ the solution to (4.1)–(4.5) by $(u_\varepsilon, \psi_\varepsilon)$, we shall see in this section that as ε tends to zero, the corresponding sequence $(u_\varepsilon, \psi_\varepsilon)_\varepsilon$ converges in a certain sense to the solution (u_0, ψ_0) of the small-aspect ratio model (4.10)–(4.12) with ψ_0 given in (4.8). More precisely, we prove the following result.

4.0.9 Theorem (Small-Aspect Ratio Limit, [40, Theorem 4.1])
Let $\lambda > 0, q \in (2,\infty), \kappa \in (0,1), f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$, and let $u_* \in S_q(\kappa)$ with $u_* < 1 + K_0$ for $x \in I$. For $\varepsilon > 0$ let $(u_\varepsilon, \psi_\varepsilon)$ be the unique solution to (4.1)–(4.5) on the maximal interval $[0,T)$ of existence. Then there are $\tau > 0$, $\varepsilon_* \in (0,1)$, and $\kappa_* \in (0,1)$, depending only on q and κ, such that $T \geq \tau$ and $u_\varepsilon(t) \in S_q(\kappa_*)$ for all $t \in [0,\tau]$ and for all $\varepsilon \in (0,\varepsilon_*)$. Moreover, the small-aspect ratio problem (4.10)–(4.12) has a unique solution

$$u_0 \in C^1\big([0,\tau], L_q(I)\big) \cap C\big([0,\tau], W^2_{q,D}(I)\big)$$

satisfying $u_0(t) \in S_q(\kappa_*)$ for all $t \in [0,\tau]$ and such that the convergences

$$u_\varepsilon \longrightarrow u_0 \quad \text{in} \quad C^{1-\theta}\big([0,\tau], W^{2\theta}_q(I)\big), \quad \theta \in (0,1),$$

and

$$\psi_\varepsilon(t)\chi_{\Omega(u_\varepsilon(t))} \longrightarrow \psi_0(t)\chi_{\Omega(u_0(t))} \quad \text{in } L_2\big(I \times (-1,0), \mathbb{R}\big), \quad t \in [0,\tau], \tag{4.13}$$

hold true as $\varepsilon \to 0$. Here, ψ_0 is the potential given in (4.8). Furthermore, there exists a $\Lambda(\kappa) > 0$ such that the above results hold true for each $\tau > 0$ provided that $\lambda \in (0, \Lambda(\kappa))$.

In order to prove Theorem 4.0.9 first of all some preparations are done. For that purpose fix $\lambda > 0$, $q \in (2,\infty)$, $\kappa \in (0,1)$, and let $u_* \in S_q(\kappa)$ with $u_*(x) < 1 + K_0$ for $x \in I$. For $\varepsilon > 0$ let $(u_\varepsilon, \psi_\varepsilon)$ denote the unique solution to (4.1)–(4.5) which is defined on the maximal interval $[0,T)$ of existence. In the following, $(K_i)_{i\geq 1}$ and K denote positive constants depending only on q and κ, but not on $\varepsilon > 0$ sufficiently small.

Set

$$\kappa_* := \frac{\kappa}{2M} < \kappa, \tag{4.14}$$

where $M \geq 1$ is the constant defined in (3.42). Moreover, define

$$\tau_\varepsilon := \sup \left\{ t \in [0,T); u_\varepsilon(s) \in \overline{S}_q(\kappa_*) \ \forall \ s \in [0,t] \right\}.$$

The choice of $\kappa_* < \kappa$ implies that if u_* belongs to $S_q(\kappa)$, then we also have $u_* \in S_q(\kappa_*)$. Since by Theorem 3.1.5 the solution u_ε is continuous in $t \in [0,T)$ for all sufficiently small $\varepsilon > 0$, there must exist $t > 0$ such that $u_\varepsilon(s) \in \overline{S}_q(\kappa_*)$ for all $s \in [0,t]$. Consequently we have that $\tau_\varepsilon > 0$. Furthermore, the definition of $S_q(\kappa_*)$ together with the continuity of the embedding of $W_q^2(I)$ in $C^1\big([-1,1],\mathbb{R}\big)$ yields the existence of a constant $K_1 > 0$ such that for all $\varepsilon > 0$

$$-1 + \kappa_* \leq u_\varepsilon(t,x) \leq 1 + K_0, \quad t \in [0,\tau_\varepsilon],\, x \in [-1,1], \tag{4.15}$$

$$\|u_\varepsilon(t)\|_{W_q^2(I)} + \|u_\varepsilon(t)\|_{W_\infty^1(I)} \leq K_1, \qquad t \in [0,\tau_\varepsilon]. \tag{4.16}$$

As a consequence of (4.16) and since $f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$, cf. Corollary 3.1.2, there is an $\varepsilon_* \in (0,1)$, depending only on q and κ, such that

$$\varepsilon_*^2 \,\|\partial_x u_\varepsilon(t)\|_{L_\infty(I)}^2 \leq K_2 < 1 \tag{4.17}$$

and

$$\varepsilon_*^2 \left(\big\|f_x\big(u_\varepsilon(t)\big)\big\|_{W_2^1(I)}^2 + \big\|f_u\big(u_\varepsilon(t)\big)\big\|_{W_2^1(I)}^2 + \big\|f_x\big(u_\varepsilon(t)\big)\big\|_{L_\infty(I)}^2 \right) \leq K_3 \tag{4.18}$$

for $t \in [0,\tau_\varepsilon],\ \varepsilon \in (0,\varepsilon_*]$. For $\varepsilon \in (0,\varepsilon_*)$ set

$$\varphi_\varepsilon(t) := \varphi_{u_\varepsilon(t)} = \psi_\varepsilon(t) \circ T_{u_\varepsilon(t)}^{-1}, \quad t \in [0,\tau_\varepsilon],$$

with $T_{u_\varepsilon(t)}^{-1}$ given by (3.7) and

$$\phi_\varepsilon(t,x,\eta) := \varphi_\varepsilon(t,x,\eta) - \eta f\big(x,u_\varepsilon(t,x)\big), \quad t \in [0,\tau_\varepsilon],\, (x,\eta) \in \overline{\Omega}. \tag{4.19}$$

The groundwork for the proof of Theorem 4.0.9 is the derivation of appropriate a priori estimates on the family $(\phi_\varepsilon)_\varepsilon$, implying that it converges to zero in $L_2(\Omega)$ as $\varepsilon \to 0$, c.f. (4.55). This yields in particular the convergence stated in (4.13). It is additionally crucial for the convergence of the according displacements $(u_\varepsilon)_\varepsilon$, c.f. the proof of Theorem 4.0.9. For this purpose in Lemma 4.0.11 the analysis of [32, Section 3] and [14, Lemma 12] is extended to the case of a nonconstant permittivity profile. We start by giving an $L_\infty(\Omega)$-bound for $\phi_u(t)$.

4.0.10 Lemma ([40, Lemma 4.2])
Let $\kappa \in (0,1)$, $\varepsilon > 0$ *and* $q \in (2,\infty)$. *Given* $f \in C^3\big([-1,1]\times\mathbb{R},\mathbb{R}\big)$ *and* $u \in S_q(\kappa)$, *there is a constant* $K_4 > 0$ *such that*

$$\|\phi_u(t)\|_{L_\infty(\Omega)} \leq K_4$$

for $t \in [0,T)$.

Proof. From the Sobolev embedding theorem combined with Corollary 3.1.2 we get

$$\|f\big(u(t)\big)\|_{L_\infty(I)} \leq c\|f\big(u(t)\big)\|_{W_2^2(I)} \leq cc_{f,B} := \tilde{K}_4.$$

Defining on $\overline{\Omega}$ the function $\overline{w}$ by $\overline{w}(x,\eta) := \tilde{K}_4$ we observe that

$$\begin{aligned} (\mathcal{L}\overline{w})(x,\eta) &= 0, & t > 0, (x,\eta) \in \Omega, \\ \overline{w}(x,\eta) &= \tilde{K}_4 \geq \eta\|f\big(u(t)\big)\|_{L_\infty(I)} \geq \eta f(x,u), & t > 0, (x,\eta) \in \partial\Omega, \end{aligned}$$

whence the maximum principle yields that $\overline{w}$ is a supersolution to (3.11)–(3.12) on $\overline{\Omega}$, i.e.

$$\overline{w}(x,\eta) \geq \varphi_u(x,\eta), \quad (x,\eta) \in \overline{\Omega}. \tag{4.20}$$

Similarly, we define for $(x,\eta) \in \overline{\Omega}$ the function $\underline{w}(x,\eta) := -\tilde{K}_4$, to see that

$$\begin{aligned} (\mathcal{L}\underline{w})(x,\eta) &= 0, & t > 0, (x,\eta) \in \Omega, \\ \underline{w}(x,\eta) &= -\tilde{K}_4 \leq -\eta\|f\big(u(t)\big)\|_{L_\infty(I)} \leq \eta f(x,u), & t > 0, (x,\eta) \in \partial\Omega. \end{aligned}$$

Again by the maximum principle we obtain that $\underline{w}$ is a subsolution to (3.11)–(3.12) on $\overline{\Omega}$, i.e.

$$\underline{w}(x,\eta) \leq \varphi_u(x,\eta), \quad (x,\eta) \in \overline{\Omega}. \tag{4.21}$$

Finally, (4.20) and (4.21) may be used to conclude that

$$-\tilde{K}_4 - \eta f(x,u) \leq \varphi_u(x,\eta) - \eta f(x,u) \leq \tilde{K}_4 - \eta f(x,u)$$

and thus

$$-2\tilde{K}_4 \leq \phi_u(x,\eta) \leq 2\tilde{K}_4,$$

whence eventually

$$\|\phi_u(t)\|_{L_\infty(\Omega)} \leq 2\tilde{K}_4 =: K_4$$

for all $t \in [0,T)$. This proves the assertion. □

4.0.11 Lemma ([40, Lemma 4.3])
There is a constant $K_5 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_*)$ and $t \in [0, \tau_\varepsilon]$, there holds

$$\|\partial_x \phi_\varepsilon(t)\|_{L_2(\Omega)} + \frac{1}{\varepsilon}\Big(\|\phi_\varepsilon(t)\|_{L_2(\Omega)} + \|\partial_\eta \phi_\varepsilon(t)\|_{L_2(\Omega)}\Big) \le K_5, \tag{4.22}$$

$$\frac{1}{\varepsilon}\|\partial_{x\eta}\phi_\varepsilon(t)\|_{L_2(\Omega)} + \frac{1}{\varepsilon^2}\|\partial_{\eta\eta}\phi_\varepsilon(t)\|_{L_2(\Omega)} \le K_5, \tag{4.23}$$

$$\frac{1}{\varepsilon}\|\partial_\eta \phi_\varepsilon(t, \cdot, 1)\|_{W_2^{1/2}(I)} \le K_5. \tag{4.24}$$

Proof. Fix $\varepsilon \in (0, \varepsilon_*)$ and $t \in [0, \tau_\varepsilon]$. Then note that by Lemma 4.0.10 there exists a constant $K_4 > 0$ such that

$$\|\phi_u(t)\|_{L_\infty(\Omega)} \le K_4, \quad t \in [0, \tau_\varepsilon],$$

and thanks to (4.15), (4.16) and the boundedness of $f\big(u_\varepsilon(t)\big)$ in $W_2^2(I)$ the function

$$\begin{aligned} F_\varepsilon(t,x,\eta) &:= F_{u_\varepsilon(t)}(x,\eta) \\ &= \varepsilon^2\eta\Big(f_{xx}(x,u_\varepsilon) + 2f_{xu}(x,u_\varepsilon)\partial_x u_\varepsilon + f_{uu}(x,u_\varepsilon)\big(\partial_x u_\varepsilon\big)^2 + f_u(x,u_\varepsilon)\partial_{xx}u_\varepsilon\Big) \\ &\quad - 2\varepsilon^2\eta\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\Big(f_x(x,u_\varepsilon) + f_u(x,u_\varepsilon)\partial_x u_\varepsilon\Big) + \varepsilon^2\eta\left(2\left(\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\right)^2 - \frac{\partial_{xx}u_\varepsilon}{u_\varepsilon}\right) f(x,u_\varepsilon) \end{aligned}$$

complies for $t \in [0, \tau_\varepsilon]$ and $(x,\eta) \in \Omega$ with the estimate

$$\begin{aligned} &\|F_\varepsilon(t)\|_{L_q(\Omega)} \\ &\le \varepsilon^2\Bigg(\|f_{xx}(x,u_\varepsilon)\|_{L_q(I)} + 2\|f_{uu}(x,u_\varepsilon)\|_{L_q(I)}\|\partial_x u_\varepsilon\|^2_{L_\infty(I)} \\ &\qquad + \|f_u(x,u_\varepsilon)\|_{L_\infty(I)}\|\partial_{xx}u_\varepsilon\|_{L_q(I)} + 2\left\|\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\right\|_{L_\infty(I)}\|f_x(x,u_\varepsilon)\|_{L_q(I)} \\ &\qquad + 2\left\|\frac{\big(\partial_x u_\varepsilon\big)^2}{1+u_\varepsilon}\right\|_{L_\infty(I)}\|f_u(x,u_\varepsilon)\|_{L_q(I)} + 2\left\|\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\right\|^2_{L_\infty(I)}\|f(x,u_\varepsilon)\|_{L_q(I)} \\ &\qquad + \left\|\frac{\partial_{xx}u_\varepsilon}{1+u_\varepsilon}\right\|_{L_q(I)}\|f(x,u_\varepsilon)\|_{L_\infty(I)}\Bigg) \\ &\le K_6\varepsilon^2. \end{aligned}$$

Together with *Hölder's inequality* this leads to

$$\|F_\varepsilon(t)\|_{L_p(\Omega)} \le 2^{(q-p)/pq}\|F_\varepsilon(t)\|_{L_q(\Omega)} \le K_7\varepsilon^2, \quad p \in [1,q). \tag{4.25}$$

Multiplying (3.18) by ϕ_ε, integrating over Ω and using the Green–Riemann formula together with the boundary condition (3.19) as in [32, Lemma 11] or [14], respectively, leads to the equation

$$\begin{aligned} &\int_\Omega F_\varepsilon \phi_\varepsilon \, d(x,\eta) \\ &\quad = \varepsilon^2 \int_\Omega \partial_x\phi_\varepsilon - \eta \frac{\partial_x u_\varepsilon}{1+u_\varepsilon} \big(\partial_\eta \phi_\varepsilon\big)^2 \, d(x,\eta) + \int_\Omega \left(\frac{\partial_\eta \phi_\varepsilon}{1+u_\varepsilon}\right)^2 d(x,\eta) \\ &\qquad + \varepsilon^2 \int_\Omega \eta \left(2\left(\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\right)^2 - \frac{\partial_{xx} u_\varepsilon}{1+u_\varepsilon}\right) \phi_\varepsilon \partial_\eta \phi_\varepsilon \, d(x,\eta). \end{aligned} \tag{4.26}$$

Introducing for $t \in [0,\tau_\varepsilon]$ and $(x,\eta) \in \Omega$ the function

$$\mu(t,x,\eta) := \varepsilon^2 \eta \left(2\left(\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\right)^2 - \frac{\partial_{xx} u_\varepsilon}{1+u_\varepsilon}\right),$$

(4.26) is equivalent to the identity

$$\begin{aligned} &\int_\Omega \phi_\varepsilon \big(F_\varepsilon - \mu \partial_\eta \phi_\varepsilon\big) d(x,\eta) \\ &\quad = \varepsilon^2 \int_\Omega \left(\partial_x \phi_\varepsilon - \eta \frac{\partial_x u_\varepsilon}{1+u_\varepsilon} \partial_\eta \phi_\varepsilon\right)^2 d(x,\eta) + \int_\Omega \left(\frac{\partial_\eta \phi_\varepsilon}{1+u_\varepsilon}\right)^2 d(x,\eta). \end{aligned} \tag{4.27}$$

By means of the inequality $(a-b)^2 \geq a^2/2 - b^2$, as well as (4.15) and (4.17), the right-hand side of (4.27) may be estimated from below as

$$\begin{aligned} &\int_\Omega \phi_\varepsilon \big(F_\varepsilon - \mu \partial_\eta \phi_\varepsilon\big) \, d(x,\eta) \\ &\quad \geq \frac{\varepsilon^2}{2} \|\partial_x \phi_\varepsilon\|_{L_2(\Omega)}^2 - \varepsilon^2 \|\partial_x u_\varepsilon\|_{L_\infty(I)}^2 \left\| \frac{\partial_\eta \phi_\varepsilon}{1+u_\varepsilon} \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial_\eta \phi_\varepsilon}{1+u_\varepsilon} \right\|_{L_2(\Omega)}^2 \\ &\quad \geq \frac{\varepsilon^2}{2} \|\partial_x \phi_\varepsilon\|_{L_2(\Omega)}^2 + K_8 \|\partial_\eta \phi_\varepsilon\|_{L_2(\Omega)}^2, \end{aligned} \tag{4.28}$$

with $K_8 = (1-K_2)/(1+K_0)^2 < 1$. Next, thanks to (4.15) we obtain the relation

$$\|\mu\|_{L_q(\Omega)} \leq \frac{2\varepsilon^2}{\kappa_*^2} \|\partial_x u_\varepsilon\|_{L_{2q}(I)}^2 + \frac{\varepsilon^2}{\kappa_*} \|\partial_{xx} u_\varepsilon\|_{L_q(I)} \leq K_9 \varepsilon^2. \tag{4.29}$$

In addition, we clearly have

$$\int_\Omega \phi_\varepsilon \big(F_\varepsilon - \mu \partial_\eta \phi_\varepsilon\big) \, d(x,\eta) \leq \|\phi_\varepsilon (F_\varepsilon - \mu \partial_\eta \phi_\varepsilon)\|_{L_1(\Omega)}.$$

Together with Lemma 4.0.10, (4.25) and Hölder's inequality this yields

$$\begin{aligned}\int_\Omega \phi_\varepsilon\big(F_\varepsilon - \mu\partial_\eta\phi_\varepsilon\big)\, d(x,\eta) \\ &\le c\|\phi_\varepsilon\|_{L_\infty(\Omega)}\left(\|F_\varepsilon\|_{L_1(\Omega)} + \|\mu\|_{L_2(\Omega)}\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}\right) \\ &\le cK_4\left(K_7\varepsilon^2 + K_9\varepsilon^2\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}\right) \\ &\le K_{10}\varepsilon^2 + K_{10}\varepsilon^2\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}.\end{aligned} \tag{4.30}$$

Fusing (4.28) and (4.30) we obtain

$$\frac{\varepsilon^2}{2}\|\partial_x\phi_\varepsilon\|^2_{L_2(\Omega)} + K_8\|\partial_\eta\phi_\varepsilon\|^2_{L_2(\Omega)} \le K_{10}\varepsilon^2 + K_{10}\varepsilon^2\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)},$$

whence finally

$$\varepsilon^2\|\partial_x\phi_\varepsilon\|^2_{L_2(\Omega)} + \|\partial_\eta\phi_\varepsilon\|^2_{L_2(\Omega)} \le K_{11}\varepsilon^2. \tag{4.31}$$

For $x \in I$ there holds $\phi_\varepsilon(x,1) = 0$ and therefore $\|\phi_\varepsilon\|_{L_2(\Omega)} \le \sqrt{2}\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}$. A combination of this fact with (4.31) then readily gives

$$\|(\phi_\varepsilon)_x\|_{L_2(\Omega)} + \frac{1}{\varepsilon}\left(\|\phi_\varepsilon\|_{L_2(\Omega)} + \|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}\right) \le \tilde{K}_5,$$

which is (4.22), i.e. the first statement of the lemma.

In a next step (4.23) is verified. To this end we define the functions

$$\xi_\varepsilon := \partial_{\eta\eta}\phi_\varepsilon \qquad \text{and} \qquad \omega_\varepsilon := \partial_{x\eta}\phi_\varepsilon,$$

multiply (3.18) by ξ_ε, integrate over Ω and use [24, Lemma 4.3.1.2 & Lemma 4.3.1.3] to find that

$$\begin{aligned}-\int_\Omega \big(F_\varepsilon + \mu\partial_\eta\phi_\varepsilon\big)\xi_\varepsilon\, d(x,\eta) \\ &= \int_\Omega \varepsilon^2\partial_{xx}\phi_\varepsilon\xi_\varepsilon - 2\varepsilon^2\eta\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\omega_\varepsilon\xi_\varepsilon + \frac{1+\varepsilon^2\eta^2(\partial_x u_\varepsilon)^2}{(1+u_\varepsilon)^2}\xi_\varepsilon^2\, d(x,\eta) \\ &= \int_\Omega \left(\frac{\xi_\varepsilon^2}{(1+u_\varepsilon)^2} + \varepsilon^2\left(\omega_\varepsilon - \eta\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\xi_\varepsilon\right)^2\right) d(x,\eta).\end{aligned} \tag{4.32}$$

In order to estimate the right-hand side of (4.32) from below we use again the inequality $(a-b)^2 \ge a^2/2 - b^2$ to obtain

$$-\int_\Omega (F_\varepsilon + \mu\partial_\eta\phi_\varepsilon)\xi_\varepsilon\, d(x,\eta) \ge \int_\Omega \left(\frac{\xi_\varepsilon^2}{(1+u_\varepsilon)^2} + \frac{\varepsilon^2}{2}\omega_\varepsilon - \varepsilon^2\eta^2\left(\frac{\partial_x u_\varepsilon}{1+u_\varepsilon}\right)^2\xi_\varepsilon^2\right) d(x,\eta).$$

(4.17) and (4.15) then lead to leads to

$$
\begin{aligned}
-\int_\Omega (F_\varepsilon + \mu\partial_\eta\phi_\varepsilon)\xi_\varepsilon \, d(x,\eta) &\geq \int_\Omega (1-K_2)\frac{\xi_\varepsilon^2}{(1+u_\varepsilon)^2} + \frac{\varepsilon^2}{2}\omega_\varepsilon^2 \, d(x,\eta) \\
&\geq K_{12}\left(\|\xi_\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon^2\|\omega_\varepsilon\|_{L_2(\Omega)}^2\right).
\end{aligned}
\tag{4.33}
$$

For the right-hand side of (4.33) we introduce the notation

$$
Q_\varepsilon := \sqrt{\|\xi_\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon^2\|\omega_\varepsilon\|_{L_2(\Omega)}^2}.
$$

By means of Hölder's inequality, (4.25) and (4.29) this term may be estimated as follows.

$$
\begin{aligned}
Q_\varepsilon^2 &\leq \frac{1}{K_{12}}\|(F_\varepsilon + \mu\partial_\eta\phi_\varepsilon)\xi_\varepsilon\|_{L_1(\Omega)} \\
&\leq \frac{1}{K_{12}}\left(\|F_\varepsilon\|_{L_2(\Omega)} + \|\mu\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}\right)\|\xi_\varepsilon\|_{L_2(\Omega)} \\
&\leq \frac{1}{K_{12}}\left(K_7\varepsilon^2 + \|\mu\|_{L_2(\Omega)}\|\partial_\eta\phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)}\right)\|\xi_\varepsilon\|_{L_2(\Omega)} \\
&\leq K_{13}\varepsilon^2\left(1 + \|\partial_\eta\phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)}\right)Q_\varepsilon.
\end{aligned}
$$

Hence, we have

$$
Q_\varepsilon \leq K_{13}\varepsilon^2\left(1 + \|\partial_\eta\phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)}\right). \tag{4.34}
$$

We now want to further estimate Q_ε by considering the term $\|\partial_\eta\phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)}$. For this purpose we use the *Gagliardo–Nirenberg inequality* [44]

$$
\|\partial_\eta\phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)} \leq K_{14}\|\partial_\eta\phi_\varepsilon\|_{W_2^1(\Omega)}^{2/q}\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}^{(q-2)/q}
$$

and observe that by (4.31)

$$
\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}^{(q-2)/q} \leq K_{15}\varepsilon^{(q-2)/q}.
$$

Fusing the last two relations leads to the estimate

$$
\begin{aligned}
\|\partial_\eta\phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)} &\leq K_{14}\|\partial_\eta\phi_\varepsilon\|_{W_2^1(\Omega)}^{2/q}\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}^{(q-2)/q} \\
&\leq K_{14}K_{15}\varepsilon^{(q-2)/q}\|\partial_\eta\phi_\varepsilon\|_{W_2^1(\Omega)}^{2/q} \\
&= K_{14}K_{15}\varepsilon^{(q-4)/q}\left(\varepsilon^2\|\partial_\eta\phi_\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon^2\|\xi_\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon^2\|\omega_\varepsilon\|_{L_2(\Omega)}^2\right)^{1/q} \\
&\leq K_{14}K_{15}\varepsilon^{(q-4)/q}\left(K_{11}\varepsilon^4 + Q_\varepsilon^2\right)^{1/q} \\
&\leq K_{16}\left(\varepsilon + \varepsilon^{(q-4)/q}Q_\varepsilon^{2/q}\right).
\end{aligned}
\tag{4.35}
$$

Combining (4.34) and (4.35) we find that

$$\begin{aligned} Q_\varepsilon &\le K_{13}\varepsilon^2 \left(1 + \|\partial_\eta \phi_\varepsilon\|_{L_{2q/(q-2)}(\Omega)}\right) \\ &\le K_{13}\varepsilon^2 \left(1 + K_{16}\left(\varepsilon + \varepsilon^{(q-4)/q} Q_\varepsilon^{2/q}\right)\right) \\ &\le K_{17}\left(\varepsilon^2 + \varepsilon^{(3q-4)/q} Q_\varepsilon^{2/q}\right). \end{aligned}$$

An application of *Young's inequality* yields

$$K_{17}\varepsilon^{(3q-4)/q} Q_\varepsilon^{2/q} \le K_{18}\varepsilon^{(3q-4)/(q-2)} + \frac{2}{q} Q_\varepsilon,$$

whence

$$\begin{aligned} Q_\varepsilon &\le K_{17}\varepsilon^2 + K_{17}\varepsilon^{(3q-4)/q} Q_\varepsilon^{2/q} \\ &\le K_{17}\varepsilon^2 + \frac{2}{q} Q_\varepsilon + K_{18}\varepsilon^{(3q-4)/(q-2)} \\ &\le K_{19}\varepsilon^2 \left(1 + \varepsilon^{q/(q-2)}\right) \\ &\le K_{20}\varepsilon^2. \end{aligned} \tag{4.36}$$

Having (4.36) at hand we may conclude that

$$\|\xi_\varepsilon\|_{L_2(\Omega)} + \varepsilon\|\omega_\varepsilon\|_{L_2(\Omega)} \le \sqrt{2} Q_\varepsilon \le \sqrt{2} K_{20}\varepsilon^2$$

and dividing both sides of this inequality by ε^2 we end up with

$$\frac{1}{\varepsilon^2}\|\xi_\varepsilon\|_{L_2(\Omega)} + \frac{1}{\varepsilon}\|\omega_\varepsilon\|_{L_2(\Omega)} \le \sqrt{2} K_{20},$$

which is (4.23), i.e. the second statement of the lemma.

Lastly, it remains to prove (4.24). For this purpose observe that as a consequence of (4.22) and (4.23) we have

$$\begin{aligned} \frac{1}{\varepsilon}\|\partial_\eta \phi_\varepsilon\|_{W_2^1(\Omega)} &\le \frac{1}{\varepsilon}\|\partial_\eta \phi_\varepsilon\|_{L_2(\Omega)} + \frac{1}{\varepsilon^2}\|\partial_{\eta\eta} \phi_\varepsilon\|_{L_2(\Omega)} + \frac{1}{\varepsilon}\|\partial_{x\eta} \phi_\varepsilon\|_{L_2(\Omega)} \\ &\le \frac{1}{\varepsilon}\|\partial_\eta \phi_\varepsilon\|_{L_2(\Omega)} + \sqrt{2} K_{20} \\ &\le \tilde{K}_5 + \sqrt{2} K_{20}. \end{aligned} \tag{4.37}$$

Together with [43, Chapter 2, Theorem 5.4] this implies

$$\|\partial_\eta \phi_\varepsilon(\cdot, 1)\|_{W_2^{1/2}(I)} \le c\|\partial_\eta \phi_\varepsilon\|_{W_2^1(\Omega)} \le \overline{K}_5 \varepsilon.$$

This is (4.24), whence the last assertion of the lemma is verified and with $K_5 \geq \max\{\tilde{K}_5, \sqrt{2}K_{20}, \overline{K}_5\}$ the proof is complete. □

As a corollary of Lemma 4.0.11 we obtain the subsequent lemma.

4.0.12 Lemma ([40, Lemma 4.4]) *(i) There is a $\tau > 0$, depending only on q, λ and κ, such that $\tau_\varepsilon \geq \tau$ for all $\varepsilon \in (0, \varepsilon_*)$.*

(ii) There is $\Lambda := \Lambda(\kappa) > 0$ such that $\tau_\varepsilon = T = \infty$ for all $\varepsilon \in (0, \varepsilon_*)$ provided that $\lambda \in (0, \Lambda)$.

In other words Lemma 4.0.12 says that for all arbitrarily small $\varepsilon \in (0, \varepsilon_*)$ the maximal time T of existence is strictly positive such that for $\varepsilon \in (0, \varepsilon_*)$ the solutions $(u_\varepsilon, \psi_\varepsilon)$ to (4.1)–(4.5) have a common interval of existence. Again the corresponding proof works as the one of [14, Lemma 13], except that one has to handle some additional terms which come into play due to the fact that f is not assumed to be constant.

Proof. (i) We show that $u_\varepsilon(t) \in S_q(\kappa_*)$ for all $t \in [0, \tau] \cap [0, \tau_\varepsilon]$, whence the assertion follows from the definition of τ_ε. Using the results in [2] on pointwise multiplication in Sobolev spaces as in the proof of Lemma 3.1.4, a combination of the relations (4.15), (4.16), (4.18), (4.24) and Corollary 3.1.2 implies that, given $2\sigma \in (1/2 - 1/q, 1/2)$, there exists a constant $K_{21} > 0$ such that

$$\|g_\varepsilon\big(u_\varepsilon(t)\big)\|_{W_2^{2\sigma}(I)} \leq K_{21}. \tag{4.38}$$

Having (4.38) at hand, by means of (3.42) and the fact that $u_* \in S_q(\kappa)$ we may deduce from the *variation-of-constant formula* that (cf. (3.47)) for $t \in [0, \tau_\varepsilon]$,

$$\begin{aligned} \|u_\varepsilon(t)\|_{W_{q,D}^2(I)} &\leq M e^{-\omega t}\|u_*\|_{W_{q,D}^2(I)} \\ &\qquad + \lambda \int_0^t e^{-\omega(t-s)}(t-s)^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|g_\varepsilon\big(u_\varepsilon(t)\big)\|_{W_{2,D}^{2\sigma}(I)}\, ds \\ &\leq \frac{Me^{-\omega t}}{\kappa} + \lambda K_{21} M \int_0^t e^{-\omega(t-s)}(t-s)^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\, ds \\ &\leq \frac{M}{\kappa} + \lambda M K_{21}\mathcal{I}(t). \end{aligned} \tag{4.39}$$

Furthermore, as in (3.48) by additionally using (4.38) we obtain that

$$\begin{aligned} u_\varepsilon(t) &\geq (\kappa - 1) - \lambda \int_0^t \|e^{-(t-s)A} g_\varepsilon\big(u_\varepsilon(s)\big)\|_{L_\infty(I)}\, ds \\ &\geq (\kappa - 1) - 2\lambda \int_0^t e^{-\omega(t-s)}(t-s)^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|g_\varepsilon\big(u_\varepsilon(s)\big)\|_{W_{2,D}^{2\sigma}(I)}\, ds \\ &\geq -1 + \kappa - 2\lambda K_{21} M\mathcal{I}(t). \end{aligned} \tag{4.40}$$

Since $\mathcal{I}(t) \to 0$ as $t \to 0$ we can conclude that there exists $\tau > 0$, depending only on q and κ, such that

$$\mathcal{I}(t) < \frac{1}{\lambda \kappa K_{21}} \qquad \text{and} \qquad \mathcal{I}(t) < \frac{(2M-1)\kappa}{4\lambda M^2 K_{21}}$$

holds true for all $t \in [0,\tau]$. Fusing the first inequality for $\mathcal{I}$ with (4.39) and the second one with (4.40) leads to

$$\|u_\varepsilon(t)\|_{W^2_{q,D}(I)} \leq \frac{M}{\kappa} + \lambda M K_{21} \mathcal{I}(t) < \frac{2M}{\kappa} = \frac{1}{\kappa_*}$$

and

$$u_\varepsilon(t) \geq -1 + \kappa - 2\lambda K_{21} M \mathcal{I}(t) > -1 + \frac{\kappa}{2M} = -1 + \kappa_*,$$

both inequalities holding for all $t \in [0,\tau]$. Hence the first assertion of the lemma is verified.

(ii) In order to prove the second statement of the lemma we set

$$\Lambda_*(\kappa) := \frac{1}{\kappa K_{21} \mathcal{I}(\infty)}, \quad \Lambda_{**}(\kappa) := \frac{(2M-1)\kappa}{4M^2 K_{21} \mathcal{I}(\infty)}$$

and

$$\Lambda(\kappa) := \min\left\{\Lambda_*(\kappa), \Lambda_{**}(\kappa)\right\},$$

and take $\lambda \in (0, \Lambda(\kappa))$. This implies that for all $t > 0$ we obtain the relations

$$\frac{1}{\lambda \kappa K_{21}} > \frac{1}{\Lambda(\kappa)\kappa K_{21}} \geq \frac{1}{\Lambda_*(\kappa)\kappa K_{21}} = \mathcal{I}(\infty) \geq \mathcal{I}(t)$$

and

$$\frac{(2M-1)\kappa}{4\lambda M^2 K_{21}} > \frac{(2M-1)\kappa}{4\Lambda(\kappa) M^2 K_{21}} \geq \frac{(2M-1)\kappa}{4\Lambda_{**}(\kappa) M^2 K_{21}} = \mathcal{I}(\infty) \geq \mathcal{I}(t),$$

whence we conclude that, given $\lambda \in (0, \Lambda(\kappa))$, the inequalities

$$\mathcal{I}(t) < \frac{1}{\lambda \kappa K_{21}} \qquad \text{and} \qquad \mathcal{I}(t) < \frac{(2M-1)\kappa}{4\lambda M^2 K_{21}}$$

hold true for every $\tau > 0$. Due to (i) this implies that $\tau_\varepsilon \geq \tau$ for every $\tau > 0$ and therefore $\tau_\varepsilon = T = \infty$ for all $\varepsilon \in (0, \varepsilon_*)$. This completes the proof. □

With the above preparations we are now able to present the proof of Theorem 4.0.9.

Proof of Theorem 4.0.9:

In a first step we prove that the family $(u_\varepsilon)_\varepsilon$ converges to the solution u_0 of the small-aspect ratio model in $C^{1-\theta}\big([0,\tau], W^{2\theta}_q(I)\big)$, as ε to zero. Since

$$\partial_t u_\varepsilon - \partial_{xx} u_\varepsilon = -\lambda g_\varepsilon\big(u_\varepsilon(t)\big), \quad t \in [0,\tau],\ x \in I, \tag{4.41}$$

with g_ε defined in Lemma 3.1.4 and τ as in Lemma 4.0.12, it follows from the definition of g_ε, (4.15), (4.16), (4.17), and the continuity of the embeddings $W_2^{1/2}(I) \hookrightarrow L_{2q}(I) \hookrightarrow L_q(I)$ that, for $t \in [0,\tau]$,

$$\begin{aligned}
\|\partial_t u_\varepsilon(t)\|_{L_q(I)} &\leq \|\partial_{xx} u_\varepsilon(t)\|_{L_q(I)} + \lambda \left\|g_\varepsilon\big(u_\varepsilon(t)\big)\right\|_{L_q(I)} \\
&\leq K_{22} + \lambda\Bigg(\varepsilon^2 \left\|f_x\big(x,u_\varepsilon(t)\big)\right\|^2_{L_{2q}(I)} + c\varepsilon^2 \left\|f_u\big(x,u_\varepsilon(t)\big)\right\|^2_{L_\infty(I)} \|\partial_x u_\varepsilon(t)\|^2_{L_\infty(I)} \\
&\quad + 2\left\|\frac{1+\varepsilon^2\big(\partial_x u_\varepsilon(t)\big)^2}{1+u_\varepsilon(t)}\right\|_{L_\infty(I)} \left\|f_u\big(x,u_\varepsilon(t)\big)\right\|_{L_\infty(I)} \|\partial_\eta \varphi_\varepsilon(t,\cdot,1)\|_{L_q(I)} \\
&\quad + \left\|\frac{1+\varepsilon^2\big(\partial_x u_\varepsilon(t)\big)^2}{\big(1+u_\varepsilon(t)\big)^2}\right\|_{L_\infty(I)} \|\partial_\eta \varphi_\varepsilon(t,\cdot,1)\|^2_{L_{2q}(I)}\Bigg) \\
&\leq K_{22} + \lambda\Bigg(c\varepsilon^2 \left\|f_x\big(x,u_\varepsilon(t)\big)\right\|^2_{L_\infty(I)} + cK_2 \left\|f\big(x,u_\varepsilon(t)\big)\right\|^2_{W^1_\infty(I)} \\
&\quad + \frac{2c}{\kappa_*}(1+K_2) \left\|f\big(x,u_\varepsilon(t)\big)\right\|^2_{W^1_\infty(I)} \left(\|\partial_\eta \phi_\varepsilon(t,\cdot,1)\|_{W_2^{1/2}(I)} + c\|f\big(x,u_\varepsilon(t)\big)\|^2_{W_2^2(I)}\right) \\
&\quad + \frac{c}{\kappa_*^2}(1+K_2)\Big(\|\partial_\eta \phi_\varepsilon(t,\cdot,1)\|^2_{W_2^{1/2}(I)} \\
&\quad + 2c\|\partial_\eta \phi_\varepsilon(t,\cdot,1)\|_{W_2^{1/2}(I)} \|f\big(x,u_\varepsilon(t)\big)\|_{W_2^2(I)} + c\|f\big(x,u_\varepsilon(t)\big)\|^2_{W_2^2(I)}\Big)\Bigg).
\end{aligned}$$

Finally, again using [43, Chapter 2, Theorem 5.4], the boundedness of $f\big(u_\varepsilon(t)\big)$ in $W_2^2(I)$ due to Corollary 3.1.2 and (4.18) we end up with

$$\begin{aligned}
\|\partial_t u_\varepsilon(t)\|_{L_q(I)} &\leq K_{22} + \lambda\Bigg(c\varepsilon^2 \left\|f_x\big(x,u_\varepsilon(t)\big)\right\|^2_{L_\infty(I)} + cK_2 \left\|f\big(x,u_\varepsilon(t)\big)\right\|^2_{W^1_\infty(I)} \\
&\quad + \frac{2c}{\kappa_*}(1+K_2) \left\|f\big(x,u_\varepsilon(t)\big)\right\|^2_{W^1_\infty(I)} \left(\|\phi_\varepsilon\|_{W_2^2(I)} + c\|f\big(x,u_\varepsilon(t)\big)\|^2_{W_2^2(I)}\right) \\
&\quad + \frac{c}{\kappa_*^2}(1+K_2)\Big(c\|\phi_\varepsilon\|^2_{W_2^2(I)} + 2c\|\phi_\varepsilon\|_{W_2^2(I)} \|f\big(x,u_\varepsilon(t)\big)\|_{W_2^2(I)} \\
&\quad + c\|f\big(x,u_\varepsilon(t)\big)\|^2_{W_2^2(I)}\Big)\Bigg) \\
&\leq K(\lambda).
\end{aligned}$$

Having in mind that in addition $\|u_\varepsilon(t)\|_{W_q^2(I)} \leq K_1$ for $t \in [0,\tau]$ by (4.16), one may observe that the family $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}$ is bounded in

$$C^1\big([0,\tau], L_q(I)\big) \cap C\big([0,\tau], W_q^2(I)\big)$$

and thus also in $C^{1-\theta}\big([0,\tau], W_q^{2\theta}(I)\big)$, $\theta \in (0,1)$. This enables us to deduce from [51, Corollary

4] that the sequence $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}$ is relatively compact in $C\big([0,\tau],W_q^{2\theta}(I)\big)$, whence there exists a subsequence $(\varepsilon_k)_{k\geq 1}$ of positive real numbers, $\varepsilon_k \searrow 0$, and $u_0 \in C\big([0,\tau],W_q^{2\theta}(I)\big)$ such that

$$u_{\varepsilon_k} \longrightarrow u_0 \quad \text{in } C\big([0,\tau],W_q^{2\theta}(I)\big) \tag{4.42}$$

as $k\to\infty$. Moreover, for $\theta \in ((q+1)/2q,1)$ the embedding $W_q^{2\theta}(I) \hookrightarrow W_\infty^1(I)$ is continuous, whence one may conclude that

$$u_{\varepsilon_k} \longrightarrow u_0 \quad \text{in } C\big([0,\tau],W_\infty^{1}(I)\big). \tag{4.43}$$

Since with $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}$ also the subsequence $(u_{\varepsilon_k})_{k\geq 1}$ is contained in $C^{1-\theta}\big([0,\tau],W_q^{2\theta}(I)\big)$, the convergence in (4.42) implies that

$$\|u_0(t)-u_0(s)\|_{W_q^{2\theta}(I)} = \lim_{k\to\infty}\|u_{\varepsilon_k}(t)-u_{\varepsilon_k}(s)\|_{W_q^{2\theta}(I)} \leq c|t-s|^{1-\theta}$$

for all $s,t\in[0,\tau]$, $s\neq t$, and $\theta\in(0,1)$. Eventually, we have $u_0 \in C^{1-\theta}\big([0,\tau],W_q^{2\theta}(I)\big)$, and as a consequence of (4.15) and (4.43)

$$-1+\kappa_* \leq u_0(t,x) \leq 1+K_0, \quad t\in[0,\tau],\ x\in[-1,1]. \tag{4.44}$$

Next, we prove that the right-hand side of the full evolution equation (4.1) converges to the right-hand side of the small-aspect ratio model (4.10) in an appropriate sense. Using the relations (4.19) and (4.24) leads to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\big\|\partial_\eta\varphi_\varepsilon(t,\cdot,1) - f\big(u_\varepsilon(t)\big)\big\|_{W_2^{1/2}(I)}\\
&\qquad = \lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\|\partial_\eta\phi_\varepsilon(t,\cdot,1)\|_{W_2^{1/2}(I)} \leq \lim_{\varepsilon\to 0}\varepsilon K_5 = 0.
\end{aligned} \tag{4.45}$$

Similarly, using in addition the uniform boundedness of $f\big(u_\varepsilon(t)\big)$ in $W_2^2(I)$, as well as the continuity of the embeddings $W_2^{1/2}(I)\hookrightarrow L_{2q}(I)\hookrightarrow L_q(I)$, one obtains

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\Big\|\big(\partial_\eta\varphi_\varepsilon(t,\cdot,1)\big)^2-\big(f(u_\varepsilon(t))\big)^2\Big\|_{L_q(I)}\\
&\quad\leq \lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\left(\Big\|\big(\partial_\eta\phi_\varepsilon(t,\cdot,1)\big)^2\Big\|_{L_q(I)} + 2\left\|\partial_\eta\phi_\varepsilon(t,\cdot,1)f\big(u_\varepsilon(t)\big)\right\|_{L_q(I)}\right)\\
&\quad\leq \lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\Big(\|\partial_\eta\phi_\varepsilon(t,\cdot,1)\|^2_{L_{2q}(I)} + 2cc_{f,B}\,\|\partial_\eta\phi_\varepsilon(t,\cdot,1)\|_{L_q(I)}\Big)\\
&\quad\leq \lim_{\varepsilon\to 0}\sup_{t\in[0,\tau]}\Big(c\,\|\partial_\eta\phi_\varepsilon(t,\cdot,1)\|^2_{W_2^{1/2}(I)} + c\,\|\partial_\eta\phi_\varepsilon(t,\cdot,1)\|_{W_2^{1/2}(I)}\Big)\\
&\quad\leq \lim_{\varepsilon\to 0}\big(c\varepsilon^2K_5^2 + c\varepsilon K_5\big)\\
&\quad= 0.
\end{aligned} \tag{4.46}$$

Furthermore, the convergence $u_{\varepsilon_k} \to u_0$ in $C\big([0,\tau], W^1_\infty(I)\big)$ implies that

$$\lim_{k\to\infty} \sup_{t\in[0,\tau]} \left\| \frac{1}{\big(1+u_{\varepsilon_k}(t)\big)^2} - \frac{1}{\big(1+u_0(t)\big)^2} \right\|_{L_\infty(I)} = 0. \tag{4.47}$$

Finally, using (4.44), Corollary 3.1.2 and once more (4.43) one may invoke the mean value theorem for integrals to obtain

$$\begin{aligned}
&\lim_{k\to\infty} \sup_{t\in[0,\tau]} \left\| \frac{f_u\big(x,u_{\varepsilon_k}(t)\big)}{1+u_{\varepsilon_k}(t)} - \frac{f_u\big(x,u_0(t)\big)}{1+u_0(t)} \right\|_{L_\infty(I)} \\
&\quad \le \lim_{k\to\infty} \sup_{t\in[0,\tau]} \Bigg(\left\| f_u\big(x,u_{\varepsilon_k}(t)\big) \left(\frac{1}{1+u_{\varepsilon_k}(t)} - \frac{1}{1+u_0(t)} \right) \right\|_{L_\infty(I)} \\
&\qquad\qquad + \left\| \frac{1}{1+u_0(t)} \left[f_u\big(x,u_{\varepsilon_k}(t)\big) - f_u\big(x,u_0(t)\big) \right] \right\|_{L_\infty(I)} \Bigg) \\
&\quad \le \lim_{k\to\infty} \sup_{t\in[0,\tau]} \Bigg(c_{f,B} \left\| \frac{1}{1+u_{\varepsilon_k}(t)} - \frac{1}{1+u_0(t)} \right\|_{L_\infty(I)} \\
&\qquad\qquad + \frac{1}{\kappa_*} \| f_u\big(x,u_{\varepsilon_k}(t)\big) - f_u\big(x,u_0(t)\big) \|_{L_\infty(I)} \Bigg) \\
&\quad \le \lim_{k\to\infty} \sup_{t\in[0,\tau]} \frac{1}{\kappa_*} \sup_{s\in[0,1]} \| f_{uu}\big(x, u_{\varepsilon_k}(t) + s[u_0(t)-u_{\varepsilon_k}(t)]\big) \|_{L_\infty(I)} \| u_{\varepsilon_k}(t) - u_0(t) \|_{L_\infty(I)} \\
&\quad = 0.
\end{aligned} \tag{4.48}$$

We now introduce the function

$$h(v) := \left(\frac{f(x,v)}{1+v} \right)^2 - 2\frac{f(x,v)}{1+v} f_v(x,v), \quad v \in W^{2\theta}_q(I),$$

and show that $g_{\varepsilon_k}(u_{\varepsilon_k})$ converges to $h(u_0)$ in $C\big([0,\tau], L_q(I)\big)$ as $k\to\infty$. To this end, observe that

$$\begin{aligned}
&\lim_{k\to\infty} \sup_{t\in[0,\tau]} \big\| g_{\varepsilon_k}\big(u_{\varepsilon_k}(t)\big) - h\big(u_0(t)\big) \big\|_{L_q(I)} \\
&\le \lim_{k\to\infty} \sup_{t\in[0,\tau]} \Bigg(\varepsilon_k^2 \left\| \big(f_x(x,u_{\varepsilon_k}(t))\big)^2 \right\|_{L_q(I)} + \varepsilon_k^2 \left\| \big(f_u(x,u_{\varepsilon_k}(t))\big)^2 \big(\partial_x u_{\varepsilon_k}(t)\big)^2 \right\|_{L_q(I)} \\
&\qquad + 2 \left\| \frac{1+\varepsilon_k^2\big(\partial_x u_{\varepsilon_k}(t)\big)^2}{1+u_{\varepsilon_k}(t)} f_u\big(x,u_{\varepsilon_k}(t)\big) \partial_\eta \varphi_{\varepsilon_k}(t,\cdot,1) - \frac{f\big(x,u_0(t)\big)}{1+u_0(t)} f_u\big(x,u_0(t)\big) \right\|_{L_q(I)} \\
&\qquad + \left\| \frac{1+\varepsilon_k^2\big(\partial_x u_{\varepsilon_k}(t)\big)^2}{\big(1+u_{\varepsilon_k}(t)\big)^2} \big(\partial_\eta \varphi_{\varepsilon_k}(t,\cdot,1)\big)^2 - \left(\frac{f\big(x,u_0(t)\big)}{1+u_0(t)} \right)^2 \right\|_{L_q(I)} \Bigg)
\end{aligned}$$

and thus

$$
\begin{aligned}
&\lim_{k\to\infty}\sup_{t\in[0,\tau]}\left\|g_{\varepsilon_k}\big(u_{\varepsilon_k}(t)\big)-h\big(u_0(t)\big)\right\|_{L_q(I)}\\
&\le \lim_{k\to\infty}\sup_{t\in[0,\tau]}\Bigg(\varepsilon_k^2\left\|f_x\big(x,u_{\varepsilon_k}(t)\big)\right\|^2_{L_\infty(I)}+\varepsilon_k^2 c\left\|f_u\big(x,u_{\varepsilon_k}(t)\big)\right\|^2_{L_\infty(I)}\|\partial_x u_{\varepsilon_k}(t)\|^2_{L_\infty(I)}\\
&\qquad + 2c\,\|\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)\|_{L_q(I)}\left\|\frac{f_u\big(x,u_{\varepsilon_k}(t)\big)}{1+u_{\varepsilon_k}(t)}-\frac{f_u\big(x,u_0(t)\big)}{1+u_0(t)}\right\|_{L_\infty(I)}\\
&\qquad + 2c\left\|\frac{f_u\big(x,u_0(t)\big)}{1+u_0(t)}\right\|_{L_\infty(I)}\left\|\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)-f\big(x,u_0(t)\big)\right\|_{L_q(I)}\\
&\qquad + 2\varepsilon_k^2 c\left\|\frac{\big(\partial_x u_{\varepsilon_k}(t)\big)^2}{1+u_{\varepsilon_k}(t)}\right\|_{L_\infty(I)}\left\|f_u\big(x,u_{\varepsilon_k}(t)\big)\right\|_{L_\infty(I)}\|\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)\|_{L_q(I)}\\
&\qquad + c\,\|\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)\|^2_{L_{2q}(I)}\left\|\frac{1}{\big(1+u_{\varepsilon_k}(t)\big)^2}-\frac{1}{\big(1+u_0(t)\big)^2}\right\|_{L_\infty(I)}\\
&\qquad + c\left\|\frac{1}{\big(1+u_0(t)\big)^2}\right\|_{L_\infty(I)}\left\|\big(\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)\big)^2-\big(f(x,u_0(t))\big)^2\right\|_{L_q(I)}\\
&\qquad + c\varepsilon_k^2\left\|\frac{\partial_x u_{\varepsilon_k}(t)}{1+u_{\varepsilon_k}(t)}\right\|^2_{L_\infty(I)}\|\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)\|^2_{L_{2q}(I)}\Bigg).
\end{aligned}
$$

Then combining (4.16) and (4.24) with the boundedness of $f\big(u_{\varepsilon_k}(t)\big)$ and $f\big(u_0(t)\big)$ in $W_2^2(I)$ and with the relations (4.45)–(4.48) one ends up with

$$
\begin{aligned}
&\lim_{k\to\infty}\sup_{t\in[0,\tau]}\left\|g_{\varepsilon_k}\big(u_{\varepsilon_k}(t)\big)-h\big(u_0(t)\big)\right\|_{L_q(I)}\\
&\le \lim_{k\to\infty}\sup_{t\in[0,\tau]}\Bigg(c\varepsilon_k^2+c\varepsilon_k^2+\Big(c\,\|\partial_\eta\phi_{\varepsilon_k}(t,\cdot,1)\|_{W_2^{1/2}(I)}+c\Big)\left\|\frac{f_u\big(x,u_{\varepsilon_k}(t)\big)}{1+u_{\varepsilon_k}(t)}-\frac{f_u\big(x,u_0(t)\big)}{1+u_0(t)}\right\|_{L_\infty(I)}\\
&+ c\left\|\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)-f\big(x,u_0(t)\big)\right\|_{L_q(I)}\\
&+ \varepsilon_k^2\Big(c\left\|\partial_\eta\phi_{\varepsilon_k}(t,\cdot,1)+f\big(x,u_{\varepsilon_k}(t)\big)\right\|_{W_2^{1/2}(I)}+c\Big)\\
&+ \Big(c\,\|\partial_\eta\phi_{\varepsilon_k}(t,\cdot,1)\|^2_{W_2^{1/2}(I)}+c\,\|\partial_\eta\phi_{\varepsilon_k}(t,\cdot,1)\|_{W_2^{1/2}(I)}+c\Big)\left\|\frac{1}{\big(1+u_{\varepsilon_k}(t)\big)^2}-\frac{1}{\big(1+u_0(t)\big)^2}\right\|_{L_\infty(I)}\\
&+ c\left\|\big(\partial_\eta\varphi_{\varepsilon_k}(t,\cdot,1)\big)^2-\big(f(x,u_0(t))\big)^2\right\|_{L_q(I)}\\
&+ \varepsilon_k^2\Big(c\,\|\partial_\eta\phi_{\varepsilon_k}(t,\cdot,1)\|^2_{W_2^{1/2}(I)}+c\,\|\partial_\eta\phi_{\varepsilon_k}(t,\cdot,1)\|_{W_2^{1/2}(I)}+c\Big)\Bigg)\\
&=0,
\end{aligned}
$$

that is,

$$g_{\varepsilon_k}\big(u_{\varepsilon_k}(t)\big) \longrightarrow h\big(u_0(t)\big) = \left(\frac{f\big(x,u_0(t)\big)}{1+u_0(t)}\right)^2 - 2\frac{f\big(x,u_0(t)\big)}{1+u_0(t)} f_u\big(x,u_0(t)\big) \tag{4.49}$$

in $C\big([0,\tau],L_q(I)\big)$.

We are now left with showing that u_0 is the unique solution to the small-aspect ratio model (4.10). Rewriting (4.10) as the abstract Cauchy problem

$$\begin{aligned} \dot{v}_0 + Av_0 &= -\lambda h\big(v_0(t)\big), \quad t \in [0,\tau], \\ v_0(0) &= u_*, \end{aligned} \tag{4.50}$$

with the operator A as in (3.39), the unique solution to (4.50) is given by the variation-of-constant formula

$$v_0(t) = e^{-tA}u_* - \lambda \int_0^t e^{-(t-s)A} h\big(v_0(s)\big)\, ds, \quad t \in [0,\tau].$$

Furthermore, recalling the identity (4.41) the fact that $(u_{\varepsilon_k})_{k\geq 1}$ is a subsequence of $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}$ implies that for every $k \geq 1$ it holds that

$$\begin{aligned} \dot{u}_{\varepsilon_k} + Au_{\varepsilon_k} &= -\lambda g_{\varepsilon_k}\big(u_{\varepsilon_k}(t)\big), \quad t \in [0,\tau], \\ u_{\varepsilon_k}(0) &= u_*, \end{aligned} \tag{4.51}$$

with unique solution

$$u_{\varepsilon_k}(t) = e^{-tA}u_* - \lambda \int_0^t e^{-(t-s)A} g_{\varepsilon_k}\big(u_{\varepsilon_k}(s)\big)\, ds, \quad t \in [0,\tau].$$

Let in addition w be the unique solution to the Cauchy problem

$$\begin{aligned} \dot{w} + Aw &= -\lambda h\big(u_0(t)\big), \quad t \in [0,\tau], \\ w(0) &= u_*. \end{aligned} \tag{4.52}$$

Now defining for each $k \geq 1$ the function

$$\vartheta_k(t) := u_{\varepsilon_k}(t) - w(t), \quad t \in [0,\tau],$$

a combination of (4.51) and (4.52) yields

$$\begin{aligned} \dot{\vartheta}_k + A\vartheta_k &= -\lambda\Big(g_{\varepsilon_k}\big(u_{\varepsilon_k}(t)\big) - h\big(u_0(t)\big)\Big), \quad t \in [0,\tau], \\ \vartheta_k(0) &= 0. \end{aligned} \tag{4.53}$$

Since $g_{\varepsilon_k}(u_{\varepsilon_k}) - h(u_0) \in C\big([0,\tau],L_q(I)\big)$ by (4.49) one may apply [42, Lemma 7.1.1] to conclude that $\vartheta_k \in C^{1-\theta}\big([0,\tau], W^{2\theta}_{q,D}(I)\big)$ for $\theta \in (0,1)$, $k \geq 1$, and that in addition there exists a constant $C > 0$, not depending on τ, g_{ε_k}, and h, such that

$$\lim_{k\to\infty} \|u_{\varepsilon_k}(t) - w(t)\|_{C^{1-\theta}([0,\tau],W^{2\theta}_q(I))} \leq \lim_{k\to\infty} C\lambda \left\|g_{\varepsilon_k}(u_{\varepsilon_k}) - h(u_0)\right\|_{L_\infty([0,\tau],L_q(I))} = 0.$$

In other words,

$$u_{\varepsilon_k} \longrightarrow w \quad \text{in } C^{1-\theta}\big([0,\tau], W^{2\theta}_q(I)\big)$$

as $k \to \infty$ for $\theta \in (0,1)$. In view of (4.43) the uniqueness of the limit function implies that $w = u_0 \in C^{1-\theta}\big([0,\tau], W^{2\theta}_q(I)\big)$ so that (4.52) may be rewritten as

$$\begin{aligned} \dot{u}_0 + Au_0 &= -\lambda h\big(u_0(t)\big), \quad t \in [0,\tau], \\ u_0(0) &= u_*. \end{aligned} \tag{4.54}$$

The uniqueness of the solution to the small-aspect ratio model (4.10) implies in addition that the solutions v_0 to (4.50) and u_0 to (4.54), respectively, coincide. Thus, one may conclude that $v_0 = u_0 \in C^{1-\theta}\big([0,\tau], W^{2\theta}_q(I)\big)$ is the unique solution to the small-aspect ratio model (4.10). Lastly, u_0 belongs to $\overline{S}_q(\kappa_*)$ for all $t \in [0,\tau]$ thanks to (4.44) and the continuity properties of u_ε. This implies in particular that not only a subsequence but the whole family $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}$ converges to u_0 in $C^{1-\theta}\big([0,\tau], W^{2\theta}_q(I)\big)$, $\theta \in (0,1)$, as ε tends to 0.

Finally, we are left with verifying the convergence in (4.13). To this end, recall that for $\varepsilon > 0$

$$T_\varepsilon : \overline{\Omega}(u_\varepsilon) \longrightarrow \overline{\Omega}, \quad T_\varepsilon(x,z) := \left(x, \frac{1+z}{1+u_\varepsilon(x)}\right),$$

such that the corresponding Jacobian is given by

$$DT_\varepsilon(x,z) = \begin{pmatrix} 1 & 0 \\ -\frac{(1+z)\partial_x u_\varepsilon}{(1+u_\varepsilon)^2} & \frac{1}{1+u_\varepsilon} \end{pmatrix}$$

with determinant

$$\det\big(DT_\varepsilon(x,z)\big) = \frac{1}{1+u_\varepsilon(x)} \geq \frac{1}{1+K_0}.$$

Since

$$\varphi_\varepsilon(t) = \varphi_{u_\varepsilon}(t) = \psi_\varepsilon(t) \circ T_\varepsilon^{-1} \quad \text{and} \quad \phi_\varepsilon(t) = \varphi_\varepsilon(t) - \eta f\big(u_\varepsilon(t)\big),$$

the transformation formula for integrals yields

$$\begin{aligned}\|\varphi_\varepsilon(t)-\eta f\big(u_\varepsilon(t)\big)\|^2_{L_2(\Omega)} &= \int_{-1}^{1}\int_{-1}^{u_\varepsilon(t)} \left(\psi_\varepsilon(t)-\frac{1+z}{1+u_\varepsilon(t)}f\big(u_\varepsilon(t)\big)\right)^2 \frac{d(x,z)}{1+u_\varepsilon(t)} \\ &\geq \frac{1}{1+K_0}\left\|\psi_\varepsilon(t)-\frac{1+z}{1+u_\varepsilon(t)}f\big(u_\varepsilon(t)\big)\right\|^2_{L_2(\Omega(u_\varepsilon(t)))}.\end{aligned} \tag{4.55}$$

Additionally observing that thanks to (4.22) in Lemma 4.0.11 there holds

$$\lim_{\varepsilon\to 0}\|\varphi_\varepsilon(t)-\eta f\big(u_\varepsilon(t)\big)\|^2_{L_2(\Omega)} = \lim_{\varepsilon\to 0}\|\phi_\varepsilon(t)\|^2_{L_2(\Omega)} \leq \lim_{\varepsilon\to 0}\varepsilon^2 K_2^2 = 0,$$

together with (4.54) this implies

$$\lim_{\varepsilon\to 0}\left\|\psi_\varepsilon(t)-\frac{1+z}{1+u_\varepsilon(t)}f\big(u_\varepsilon(t)\big)\right\|_{L_2(\Omega(u_\varepsilon(t)))} = 0.$$

In other words,

$$\psi_\varepsilon(t)\chi_{\Omega(u_\varepsilon(t))} \longrightarrow \psi_0(t)\chi_{\Omega(u_0(t))} \quad \text{in} \quad L_2\big(I\times(-1,0),\mathbb{R}\big),$$

as $\varepsilon\to 0$, where ψ_0 is given in (4.8) with $u=u_0$. This completes the proof. □

5 | On Some Qualitative Properties of Solutions

In the previous parts of this work we have seen different mathematical models for the characterisation of the dynamic behaviour of MEMS devices. In addition to choosing an either linear or nonlinear elasticity approach, we have in particular distinguished between the small-aspect ratio model and the full problem, coupling the moving boundary problem for the potential ψ with an either semi- or quasilinear evolution problem for the membrane's displacement u. Moreover, different permittivity profiles f give rise to different equations and might thus have a certain influence on the qualitative behaviour of solutions. It turns out, that there exist indeed qualitative differences of the solutions to the different systems and that these differences become apparent not till non-constant permittivity profiles are taken into account.

This chapter is divided into two sections. The first one, Section 5.1, is devoted to sign-properties of the solution u to the evolution problem for the displacement of the elastic membrane. It deals with the question, if the membrane always deflects towards the ground plate or if other scenarios, such as a sign-changing or a positive deflection, are possible. Section 5.2 is concerned with the phenomenon of the so-called *pull-in instability*, i.e. with the situation in which the pull-in voltage exceeds a certain critical value and thus causes a singularity of the solution after finite time.

5.1 | Non-Positivity of the Membrane's Displacement

Since parabolic maximum principles[1] are available for both settings the semilinear as well as the quasilinear evolution problem (see i.e. [19, 47, 30]) we do not explicitly distinguish between these two cases. More precisely, in the regime of a positive aspect ratio $\varepsilon > 0$ we consider the moving

[1] The established literature frequently refers to *the maximum principle*, including in this notion in fact several different maximum principles as well as comparison principles.

boundary problem

$$\varepsilon^2\psi_{xx} + \psi_{zz} = 0, \qquad t > 0,\ (x,z) \in \Omega(u(t)), \tag{5.1}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)}f, \quad t > 0,\ (x,z) \in \partial\Omega(u(t)), \tag{5.2}$$

coupled with an either semi- or quasilinear initial boundary value problem for the displacement u of the membrane. In the sequel this evolution problem for u is rewritten as the general abstract parameter-dependent Cauchy problem

$$u_t + A(u)u = -\lambda g_\varepsilon(u), \quad t > 0, \tag{5.3}$$

$$u(0) = u_*, \tag{5.4}$$

where for a given $v \in S_q(\kappa)$ the differential operator $A(v) \in \mathcal{L}(W^2_{q,D}(I), L_q(I)),\ q > 2$, is defined as[2]

$$A(v)u := -\frac{u_{xx}}{\big(1+\varepsilon^2(v_x)^2\big)^{3/2}}, \quad u \in W^2_{q,D}(I),$$

in the quasilinear case arising from the nonlinear elasticity theory, whereas for the semilinear case, arising from a linear elasticity approach, we set $A(v) \equiv A(0)$ for all $v \in S_q(\kappa)$ and obtain

$$A(0)u = -u_{xx}, \quad u \in W^2_{q,D}(I).$$

As before, given $v \in S_q(\kappa)$, we denote by $\{e^{-tA(v)}; t \geq 0\}$ the semigroup on $L_q(I)$ corresponding to $-A(v)$. The exact structure of the right-hand side $-\lambda g_\varepsilon(u)$ is then determined by the choice of the permittivity profile f. Almost the same notation is used for the small-aspect ratio model, i.e. in the situation of a formally vanishing aspect ratio $\varepsilon = 0$. As we have seen in the previous chapter, given an explicit expression for the potential ψ, the small-aspect ratio model may be rewritten as

$$u_t + A(u)u = -\lambda g_0(u), \quad t > 0, \tag{5.5}$$

$$u(0) = u_*. \tag{5.6}$$

In the same way as for $\varepsilon > 0$ the structure of the right-hand side $-\lambda g_0(u)$ might vary, depending on the choice of the function f. In order to be able to reveal sign-properties of the according solutions by means of the parabolic maximum principle, the challenge is thus to investigate the respective right-hand side $-\lambda g_\varepsilon(u)$ or $-\lambda g_0(u)$ of the evolution equation regarding its sign. It is worthwhile to explicitly mention again, that this challenge strongly depends on the choice of the permittivity profile f.

[2]Note that as in (3.38) in Section 3.1 the subscript q for the operator $A_q(v) := A(v) \in \mathcal{L}\big(W^2_{q,D}, L_q(I)\big)$ is suppressed since $A_q(v)w \in L_q \hookrightarrow L_p$ for $1 < q \leq p < \infty$.

More precisely, in the case of a constant permittivity profile $f \equiv 1$ the full evolution equation (5.3) reads

$$u_t + A(u)u = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\Big), \quad t > 0. \tag{5.7}$$

With $\psi(t,x,z) = (1+z)/(1+u(t,x))$ for $t > 0,\ (x,z) \in I \times (-1,0)$, the corresponding small-aspect ratio equation (5.5) is given by

$$u_t + A(u)u = -\frac{\lambda}{(1+u)^2}, \quad t > 0. \tag{5.8}$$

It may be readily deduced from the parabolic maximum principle that, given a non-positive initial value $u_* \leq 0$, both equations (5.7) and (5.8) always provide non-positive solutions u. In other words, a constant permittivity profile $f \equiv 1$ immediately implies that the membrane always deflects towards the ground plate.

The situation is rather different in the case of a spatially varying permittivity profile $f = f(x)$. Denoting by $f'(x)$ the derivative of f with respect to x, the full evolution equation (5.3) is given by

$$u_t + A(u)u = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2 - 2\varepsilon^2\psi_x(x,u)f'(x)\Big), \quad t > 0, \tag{5.9}$$

whereas the according small-aspect ratio equation (5.5) for a computed $\psi(t,x,z) = f(x)(1+z)/(1+u),\ t > 0,\ (x,z) \in I \times (-1,0)$, reads

$$u_t + A(u)u = -\lambda\left(\frac{f(x)}{1+u}\right)^2, \quad t > 0. \tag{5.10}$$

Invoking again the parabolic maximum principle, one may observe that for $f = f(x)$ the small-aspect ratio model always possesses non-positive solutions, provided that the initial value u_* is non-positive. On the other hand, due to the additional term $2\varepsilon^2\psi_x(x,u)f'(x)$ in (5.9), this is not at all clear for the full problem. Although the initial deflection u_* is non-positive, after a certain time the deflection might become positive or change its sign.

In the setting where f depends only on the deformation u of the membrane, i.e. when $f = f(u)$ the full evolution equations reads

$$u_t + A(u)u = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2 - 2\psi_z(x,u)f'(u)\Big), \quad t > 0, \tag{5.11}$$

with $f'(u)$ denoting the derivative of f with respect to u. With $\psi(t,x,z) = f(u)(1+z)/(1+u),\ t > 0,\ (x,z) \in I \times (-1,0)$, the associated small-aspect ratio equation is given by

$$u_t + A(u)u = -\lambda\left(\left(\frac{f(u)}{1+u}\right)^2 - 2\frac{f(u)}{1+u}f'(u)\right), \quad t > 0. \tag{5.12}$$

One may thus observe that in the case $f = f(u)$ neither in the coupled setting nor in the small-aspect ratio regime an immediate statement about the sign of the solution u is possible. Additional information on the potential ψ and on the permittivity profile f is necessary in order to deduce a statement from the maximum principle.

The situation is similar when the permittivity profile f depends on both x and u. The full equation is then given by

$$\begin{aligned} u_t + A(u)u = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2 \\ + 2\big(\varepsilon^2\psi_x(x,u)f_x(x,u) + \psi_z(x,u)f_u(x,u)\big)\Big), \quad t > 0, \end{aligned} \tag{5.13}$$

whereas the small-aspect ratio equation reads

$$u_t + A(u)u = -\lambda\left(\left(\frac{f(x,u)}{1+u}\right)^2 - 2\frac{f(x,u)}{1+u}f_u(x,u)\right), \quad t > 0. \tag{5.14}$$

The remaining part of this section is eventually devoted to the proof of non-positivity of the membrane's displacement u, provided that the initial displacement u_* is non-positive and the potential ψ satisfies certain boundary conditions.

The corresponding results have already been published in [41] for the case of a spatially varying permittivity $f = f(x)$ and in [16] for the case in which f depends on the membrane's displacement. In the scope of this work the proof is in addition extended to the most general setting $f = f(x,u)$. To this end, pick $\tau \in (0,T)$. It suffices to show that $u(t) \leq 0$ on $[0,\tau]$. Since u is obtained by a fixed-point iteration based on the variation-of-constant formula induced by (5.3)–(5.4), we may assume without loss of generality[3] that u is represented by the identity

$$u(t) = U_{A(u)}(t,0)u_* - \lambda\int_0^t U_{A(u)}(t,s)g_\varepsilon\big(u(s)\big)\,ds$$

in $C\big([0,\tau], W_q^2(I)\big)$, where $U_{A(u)}$ denotes the evolution operator introduced in Section 3.2. Thanks to the positivity of the heat semigroup it is thus enough to prove that $g_\varepsilon(v(t,x)) \geq 0$ for a given $v \in W_q^2(I)$ with $v(t,x) \leq 0$ for $(t,x) \in [0,\tau] \times I$. It turns out that in what follows the time variable $t \in [0,\tau]$ appears as a parameter. In order to lighten the notation, we therefore omit the time and introduce the following general notation.

- $v \in W_q^2(I)$, $q \in (2,\infty)$, such that $v(x) \leq 0$ for all $x \in I$;
- $\Omega(v)$ is the domain corresponding to v;
- $\psi \in W_2^2(\Omega(v))$ is the solution to (5.1)–(5.2).

[3] In general, $\tau > 0$ is obtained by a continued but finite application of Banach's fixed point theorem.

As a combination of the *Cauchy–Schwarz* inequality and the *mean value theorem for integrals* we obtain the following lemma.

5.1.1 Lemma
Let $f \in C\big([-1,1]\times[-1,0],\mathbb{R}\big)$. Then, given $x \in I$, there exists $z_0 = z_0(x) \in [-1, v]$ such that

$$\left(\frac{f(x,v(x))}{1+v(x)}\right)^2 \le (\psi_z(x,z_0))^2.$$

Proof. As in [13, Lemma 7] and [41, Lemma 4.1] we deduce from the boundary condition for the solution ψ to (5.1)–(5.2) and the Cauchy–Schwarz inequality that

$$\begin{aligned}\frac{\big(f(x,v(x))\big)^2}{1+v(x)} &= \frac{\big(\psi(x,v(x))-\psi(x,-1)\big)^2}{1+v(x)}\\ &= \frac{1}{1+v(x)}\left(\int_{-1}^{v(x)} \psi_z(x,z)\,dz\right)^2 \\ &\le \int_{-1}^{v(x)} \big(\psi_z(x,z)\big)^2\,dz, \quad x\in I.\end{aligned} \tag{5.15}$$

By the mean value theorem for integrals we find that given $x \in I$, there exists a $z_0 = z_0(x) \in [-1, v(x)]$ such that

$$\int_{-1}^{v(x)} \big(\psi_z(x,z)\big)^2\,dz = \big(v(x)+1\big)\big(\psi_z(x,z_0)\big)^2. \tag{5.16}$$

Combining (5.15) and (5.16), one finally obtains

$$\left(\frac{f\big(x,v(x)\big)}{1+v(x)}\right)^2 \le \big(\psi_z(x,z_0)\big)^2, \quad x\in I,$$

and the proof is complete. □

The following theorem is a generalisation of [41] and [16], where the cases $f = f(x)$ and $f = f(u)$ are treated, respectively.

5.1.2 Theorem (Non-Positivity of u)
Let $f \in C^1\big([-1,1]\times[-1,0],\mathbb{R}\big)$ be positive and assume that the boundary conditions

$$\psi_{zz}(x,-1)\ge 0 \quad \textit{and} \quad \psi_{zz}(x,v(x))\ge 0, \quad x\in I, \tag{5.17}$$

hold true for the solution ψ to (5.1)–(5.2)*. Then, if*

$$0 < \varepsilon^2 \leq \min_{\substack{x\in[-1,1],\\ r\in[-1,0]}} \frac{\big(f(x,r)\big)^2 - 4\big(f_r(x,r)\big)^2}{2\big(f_x(x,r)\big)^2}. \tag{5.18}$$

and $u_*(x) \leq 0$, $x \in I$, the unique solution u to (5.3)–(5.4) *satisfies*

$$u(t,x) \leq 0, \quad (t,x) \in [0,T) \times I.$$

Proof. Let $v \in W_q^2(I)$ with $v(x) \leq 0$ for all $x \in I$. We prove that $g_\varepsilon(v) \geq 0$. To this end, firstly note that the elementary inequalities

$$2\psi_x(x,v(x))f_x(x,v(x)) \leq \big(\psi_x(x,v(x))\big)^2 + \big(f_x(x,v(x))\big)^2, \quad x \in I,$$

and

$$\psi_z(x,v(x))f_v(x,v(x)) \leq \frac{1}{4}\big(\psi_z(x,v(x))\big)^2 + \big(f_v(x,v(x))\big)^2, \quad x \in I,$$

hold true. They readily yield the estimate

$$\begin{aligned} g_\varepsilon(v) &= \varepsilon^2\big(\psi_x(x,v(x))\big)^2 + \big(\psi_z(x,v(x))\big)^2 - 2\varepsilon^2\psi_x(x,v(x))f_x(x,v(x)) \\ &\quad - 2\psi_z(x,v(x))f_v(x,v(x)) \\ &\geq \big(\psi_z(x,v(x))\big)^2 - \varepsilon^2\big(f_x(x,v(x))\big)^2 - 2\psi_z(x,v(x))f_v(x,v(x)) \\ &\geq \frac{1}{2}\big(\psi_z(x,v(x))\big)^2 - \varepsilon^2\big(f_x(x,v(x))\big)^2 - 2\big(f_v(x,v(x))\big)^2. \end{aligned}$$

Moreover, as a consequence of the non-positivity of v one obtains the inequality

$$\left(\frac{f(x,v(x))}{1+v(x)}\right)^2 \geq \big(f(x,v(x))\big)^2, \quad x \in I.$$

Together with the assumption (5.18) on ε this leads to the estimate

$$g_\varepsilon(v) \geq \frac{1}{2}\Big(\big(\psi_z(x,v(x))\big)^2 - \big(f(x,v(x))\big)^2\Big) \geq \frac{1}{2}\left(\big(\psi_z(x,v(x))\big)^2 - \left(\frac{f(x,v(x))}{1+v(x)}\right)^2\right). \tag{5.19}$$

Fusing (5.19) with Lemma 5.1.1 eventually yields

$$g_\varepsilon(v) \geq \frac{1}{2}\Big(\big(\psi_z(x,v(x))\big)^2 - \big(\psi_z(x,z_0)\big)^2\Big),$$

where $z_0 = z_0(x) \in [-1, v(x)]$. Thanks to *Hopf's maximum principle* we have that

$$\psi_z(x,-1) \geq 0, \quad x \in I. \tag{5.20}$$

Moreover, since ψ satisfies (5.1), the function η, defined by $\eta(x,z) := \psi_{zz}(x,z)$ for $(x,z) \in \Omega(v)$, so does as well. Additionally observing that on the lateral boundary it holds that

$$\eta(\pm 1, z) = 0, \quad z \in (-1, 0),$$

and by assumption

$$\eta(x,-1) \geq 0 \quad \text{and} \quad \eta(x, u(x)) \geq 0, \quad x \in I,$$

an application of the elliptic maximum principle yields

$$\psi_{zz}(x,z) \geq 0, \quad (x,z) \in \Omega(v). \tag{5.21}$$

From (5.20) and (5.21) one may now infer that $\psi_z(x,z)$ is non-negative on $\Omega(v)$ and increasing in $z \in [-1, v(x))$, implying that

$$0 \leq \psi_z(x, z_0) \leq \psi_z(x, v(x)), \quad x \in I. \tag{5.22}$$

This finally proves

$$g_\varepsilon(v) \geq 0.$$

To summarise, we have shown that $v \leq 0$ implies $g_\varepsilon(v) \geq 0$ and with the introductory words of this section the proof is complete. □

It remains to discuss the above Theorem 5.1.2 for permittivity profile f depending only on the spatial variable $x \in I$ or on the membrane's displacement u, respectively. Recall that in the general setting $f = f(x,u)$ the fundamental condition (c.f. (5.18)) on ε or f, respectively, reads

$$0 < \varepsilon^2 \leq \min_{\substack{x \in [-1,1], \\ r \in [-1,0]}} \frac{\big(f(x,r)\big)^2 - 4\big(f_r(x,r)\big)^2}{2\big(f_x(x,r)\big)^2}. \tag{5.23}$$

Thus, if f depends only on the membrane's displacement u then condition (5.18) is modified to

$$\min_{r \in [-1,0]} f(r) \geq 2 \min_{r \in [-1,0]} |f'(r)|. \tag{5.24}$$

Note that this condition does not depend on $\varepsilon > 0$.

In the case $f = f(x)$ the term $-4\big(f_r(x,r)\big)^2$ in (5.18) vanishes and the proof of Theorem 5.1.2 applies to the case $f = f(x)$ if we require that (c.f. [41])

$$0 < \varepsilon \leq \min_{x \in [-1,1]} \frac{f(x)}{\sqrt{2}|f'(x)|}. \tag{5.25}$$

However, one may observe that (5.25) is not sharp. Following the lines of the proof of Theorem 5.1.2, a direct calculation shows that (5.25) might be improved such that the theorem holds true under the condition

$$0 < \varepsilon \leq \min_{x\in[-1,1]} \frac{f(x)}{|f'(x)|}. \tag{5.26}$$

The above observations are summarised in the following corollary.

5.1.3 Corollary
(i) Given a positive $f \in C^1([-1,1],\mathbb{R})$, assume that the solution ψ to (5.1)–(5.2) complies with the inequalities

$$\psi_{zz}(x,-1) \geq 0 \quad \text{and} \quad \psi_{zz}(x,v(x)) \geq 0, \quad x \in I, \tag{5.27}$$

Then, if the condition

$$0 < \varepsilon \leq \min_{x\in[-1,1]} \frac{f(x)}{|f'(x)|}. \tag{5.28}$$

is satisfied and $u_*(x) \leq 0$ for all $x \in I$ the unique solution u to (5.3)–(5.4) satisfies

$$u(t,x) \leq 0, \quad (t,x) \in [0,T) \times I.$$

(ii) Let $f \in C^1\big([-1,0],\mathbb{R}\big)$ be positive and assume that the inequalities

$$\psi_{zz}(x,-1) \geq 0 \quad \text{and} \quad \psi_{zz}(x,v(x)) \geq 0, \quad x \in I, \tag{5.29}$$

hold true for the solution ψ to (5.1)–(5.2). Then, if f complies with the condition

$$\min_{r\in[-1,0]} f(r) \geq 2 \min_{r\in[-1,0]} |f'(r)|. \tag{5.30}$$

and $u_*(x) \leq 0$ for all $x \in I$, the unique solution u to (5.3)–(5.4) satisfies

$$u(t,x) \leq 0, \quad (t,x) \in [0,T) \times I.$$

5.2 | Non-Existence of Global Solutions

Depending on the individual application of the MEMS-based device it might be either an explicitly desired effect to apply a voltage value that leads to a touchdown of the membrane on the ground plate, or, in contrast, the contact of the two plates could damage the device. The understanding of this touchdown behaviour is one of the major objectives in the mathematical investigation of MEMS-based devices. In the present work this topic is addressed as follows. In Chapter 3 it is shown for the semilinear as well as the quasilinear setting that there exists a critical value $\lambda_* > 0$

such that the unique solution (u, ψ) to the coupled problem exists forever, provided that the applied voltage $\lambda > 0$ is smaller than λ_*. In this case we have uniform bounds on u in the $W_q^2(I)$-norm and the membrane does never touch down on the ground plate, not even in infinite time. Contrariwise, we shall see in this section that there is another critical value $\lambda^* \geq \lambda_*$ such that the solution u ceases to exists after a finite time T of existence, provided that $\lambda > \lambda^*$ and ε is small enough.[4] In this case the membrane's displacement develops a singularity in the sense that one of the following two phenomena may be observed. Either the membrane touches down on the ground plate, i.e.

$$\liminf_{t \to T} \min_{x \in [-1,1]} u(t,x) = -1,$$

or u becomes unbounded in the $W_q^2(I)$-norm, i.e.

$$\limsup_{t \to T} \|u(t)\|_{W_q^2(I)} = \infty.$$

For a constant permittivity profile $f \equiv 1$ this is shown in [14, Theorem 2(ii)]. The present work covers spatially varying permittivity profiles $f = f(x)$, $x \in I$, for both the semilinear and the quasilinear case (c.f. [41] and [17], respectively) and permittivity profiles $f = f(u)$ for the quasilinear case (c.f. [16]). Hitherto it is still an open problem to verify the existence of finite-time singularities when $f = f(x, u)$ depends on both $x \in I$ and the displacement u of the membrane.

The general concept of the according proofs is to derive a differential inequality for a certain energy functional and to integrate this inequality with respect to the time t in order to get an upper bound for the maximal time T of existence. The main difference between the semilinear and the quasilinear case consists in the choice of the functional, as we will see in the subsequent paragraphs.

5.2.1 | Finite-Time Singularities in the Semilinear Setting

Restricting the analysis to the case of spatially varying permittivity profiles, we now address the appearance of finite-time singularities for the system consisting of the elliptic moving boundary problem

$$\varepsilon^2 \psi_{xx} + \psi_{zz} = 0, \qquad t > 0,\ (x,z) \in \Omega(u), \tag{5.31}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f(x), \quad t > 0,\ (x,z) \in \partial\Omega(u), \tag{5.32}$$

[4] If $f = f(u)$ the result may even be improved as in this case there is no condition on $\varepsilon > 0$.

for the electrostatic potential ψ, coupled with the semilinear initial boundary value problem

$$u_t - u_{xx} = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\Big) + 2\lambda\varepsilon^2\psi_x(x,u)f'(x), \quad t>0,\ x\in I, \tag{5.33}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{5.34}$$

$$u(0,x) = u_*(x), \quad x\in I. \tag{5.35}$$

We shall see that there exists a voltage value $\lambda^* > 0$ such that the solution u to (5.33)–(5.35) cannot exist globally in time, provided that $\varepsilon > 0$ is small enough. More precisely, we prove the following result.

5.2.1 Theorem (Finite-Time Singularities, [41, Theorem 5.1])
Let $f \in C^2([-1,1],\mathbb{R})$ be positive with $f(-1) = f(1)$ and denote by $u \in C\big([0,T), W_q^2(I)\big)$ the solution to (5.33)–(5.35)*. Assume in addition that*

$$u(t,x) \leq 0, \quad (t,x)\in [0,T)\times I.$$

Then there exists $\lambda^* > 0$ such that $T<\infty$, provided that $\lambda > \lambda^*$ and $\varepsilon \in \big(0, 1/\sqrt{\lambda}\big]$. That is, we have either

$$\limsup_{t\to T} \|u(t)\|_{W_q^2(I)} = \infty \qquad \text{or} \qquad \liminf_{t\to T} \min_{x\in[-1,1]} u(t,x) = -1.$$

Before proving the above theorem, it is convenient to provide some preliminary definitions and results. In the subsequent argumentation we follow the lines of [14, Theorem 1.2 (ii)]. For $x \in I$ define

$$\varphi(x) := \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right) \quad \text{and} \quad \mu := \frac{\pi^2}{4}. \tag{5.36}$$

Then μ is the principal eigenvalue of the $L_2(I)$-realisation of $-\partial_x^2$ subject to homogeneous Dirichlet boundary conditions, i.e.

$$-\varphi_{xx} = \mu\varphi \quad \text{in } I, \qquad \varphi(\pm 1) = 0. \tag{5.37}$$

Observe in addition that $\|\varphi\|_{L_1(I)} = 1$. Denoting by u be the solution to (5.33)–(5.35) on its maximal interval $[0,T)$ of existence we introduce the functional

$$E_\alpha(t) := \int_I \varphi(x)\left(u + \frac{\alpha}{2}u^2\right)(t,x)\,dx, \quad t\in[0,T), \tag{5.38}$$

where $\alpha \in (0,1)$ is a free parameter to be determined later. Recall that by Theorem 3.1.5 we know that $u(t,x) > -1$ and by assumption one has $u(t,x) \leq 0$. In summary,

$$-1 < u(t,x) \leq 0, \quad (t,x)\in[0,T)\times I. \tag{5.39}$$

As a first consequence of (5.39) notice that

$$-1 \leq \frac{\alpha - 2}{2} \leq E_\alpha(t) \leq 0, \quad t \in [0, T).$$

In addition to the above facts, by differentiating the boundary condition (5.32) one obtains the identity

$$\psi_x\big(t, x, u(t,x)\big) = f'(x) - \psi_z\big(t, x, u(t,x)\big) u_x(t,x), \tag{5.40}$$

for all $(t,x) \in (0,T) \times I$. Equation (5.40) plays a crucial role in the proof of the above theorem as it is used several times in the subsequent reasoning. To summarise, we state the following general assumptions for the lemmas in this section:

- $f \in C^2\big([-1,1], \mathbb{R}\big)$ with $f(x) > 0$ for all $x \in I$ and $f(-1) = f(1)$;
- $u \in C\big([0,T), W_q^2(I)\big)$, $q \in (2, \infty)$, is the solution to (5.33)–(5.35), satisfying (5.39);
- $\psi \in W_2^2\big(\Omega(u(t))\big)$ is the solution to (5.31)–(5.32);
- $\varepsilon, \lambda, \alpha, \beta > 0$ are parameters, being determined later;
- μ, φ and E_α are given as defined in (5.36) and (5.38), respectively.

The rough concept of the proof of Theorem 5.2.1 is to derive a differential inequality for the functional E_α and to integrate this inequality with respect to t in order to get an upper bound for the maximal time T of existence.

According to this concept, the first lemma yields a differential equation for the functional E_α, which is the basis for the following estimates.

5.2.2 Lemma ([41, Lemma 5.2])
Given $t \in (0,T)$, there holds[5]

$$\begin{aligned} &\frac{dE_\alpha}{dt} + \mu E_\alpha + \alpha \int_I \varphi (u_x)^2 \, dx \\ &\qquad = -\lambda \int_I \varphi (1 + \alpha u)\big(1 + \varepsilon^2 (u_x)^2\big)\big(\psi_z(x,u)\big)^2 \, dx + \lambda \varepsilon^2 \int_I \varphi (1 + \alpha u)(f')^2 \, dx. \end{aligned}$$

Proof. Having (5.40) at hand, multiplication of the evolution equation (5.33) by $\varphi(1 + \alpha u)$ and

[5]For the sake of better readability we suppress the variables in the calculations if no ambiguity is possible.

integration over I with respect to x leads to

$$\begin{aligned}\int_I \varphi(1+\alpha u)(u_t - u_{xx})\,dx = &-\lambda \int_I \varphi(1+\alpha u)\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\Big)\,dx \\ &+ 2\lambda\varepsilon^2 \int_I \varphi(1+\alpha u)\psi_x(x,u) f'\,dx \\ = &-\lambda \int_I \varphi(1+\alpha u)\big(1+\varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\,dx \\ &+ \lambda\varepsilon^2 \int_I \varphi(1+\alpha u)(f')^2\,dx.\end{aligned} \tag{5.41}$$

Moreover, using the definition of E_α, one may verify that

$$\frac{dE_\alpha}{dt} = \frac{d}{dt}\int_I \varphi\left(u + \frac{\alpha}{2}u^2\right)\,dx = \int_I \varphi(1+\alpha u)u_t\,dx, \quad t \in (0,T). \tag{5.42}$$

Next, thanks to the eigenvalue problem (5.37) and the boundary condition (5.34), twice integrating by parts yields

$$\begin{aligned}\mu E_\alpha &= \mu \int_I \varphi\left(u + \frac{\alpha}{2}u^2\right)\,dx \\ &= -\int_I \varphi_{xx}\left(u + \frac{\alpha}{2}u^2\right)\,dx \\ &= \int_I \varphi_x(1+\alpha u)u_x\,dx \\ &= -\int_I \varphi(1+\alpha u)u_{xx}\,dx - \alpha\int_I \varphi(u_x)^2\,dx, \quad t \in (0,T).\end{aligned} \tag{5.43}$$

Combining (5.42) and (5.43), one obtains

$$\frac{dE_\alpha}{dt} + \mu E_\alpha + \alpha\int_I \varphi u_x^2\,dx = \int_I \varphi(1+\alpha u)(u_t - u_{xx})\,dx, \quad t \in (0,T),$$

and finally, fusing this equation with (5.41), we end up with

$$\begin{aligned}&\frac{dE_\alpha}{dt} + \mu E_\alpha + \alpha\int_I \varphi(u_x)^2\,dx \\ &\quad = -\lambda\int_I \varphi(1+\alpha u)\big(1+\varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\,dx + \lambda\varepsilon^2\int_I \varphi(1+\alpha u)(f')^2\,dx\end{aligned}$$

for $t \in (0,T)$. This is the assertion of the lemma. □

As one may see later on, the following lemma serves as a useful manipulation in order to estimate the right-hand side of the above obtained differential equation for E_α.

5.2.3 Lemma ([41, Lemma 5.3])
Given $t \in [0,T)$, it holds

$$\begin{aligned}
&\int_I \varphi f\big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)\,dx \\
&\quad = \int_{\Omega(u(t))} \varphi\left(\varepsilon^2(\psi_x)^2+(\psi_z)^2+\frac{\mu\varepsilon^2}{2}\psi^2\right)\,d(x,z) - \frac{\mu\varepsilon^2}{6}(f(1))^2 \\
&\qquad - \frac{\varepsilon^2}{2}\int_I \varphi_x f^2 u_x\,dx + \varepsilon^2\int_I \varphi f f' u_x\,dx.
\end{aligned}$$

Proof. The idea of the proof is to multiply equation (5.31) by $\varphi\psi$ and to integrate over $\Omega(u(t))$ with respect to x and z. Thanks to the Green–Riemann integration formula as well as to the boundary conditions for ψ and φ, respectively, for $t \in [0,T)$ we calculate the following:

$$\begin{aligned}
0 &= \varepsilon^2\int_\Omega \psi_{xx}\varphi\psi\,d(x,z) + \int_\Omega \psi_{zz}\varphi\psi\,d(x,z) \\
&= -\int_\Omega \varphi\big(\varepsilon^2(\psi_x)^2+(\psi_z)^2\big)\,d(x,z) - \varepsilon^2\int_\Omega \varphi_x\psi\psi_x\,d(x,z) \\
&\qquad + \int_I \varphi\psi(x,u)\left(-\varepsilon^2\psi_x(x,u)u_x+\psi_z(x,u)\right)\,dx \\
&= -\int_\Omega \varphi\big(\varepsilon^2(\psi_x)^2+(\psi_z)^2\big)\,d(x,z) - \frac{\varepsilon^2}{2}\int_\Omega \varphi_x(\psi^2)_x\,d(x,z) \\
&\qquad + \int_I \varphi f\left(-\varepsilon^2\psi_x(x,u)u_x+\psi_z(x,u)\right)\,dx \\
&= -\int_\Omega \varphi\big(\varepsilon^2(\psi_x)^2+(\psi_z)^2\big)\,d(x,z) - \frac{\varepsilon^2}{2}\int_\Omega \varphi_x(\psi^2)_x\,d(x,z) \\
&\qquad + \int_I \varphi f\big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)\,dx - \varepsilon^2\int_I \varphi f f' u_x\,dx,
\end{aligned}$$

where in the last step again the identity (5.40) is used. Finally, due to (5.37) and (5.32) there holds

$$\begin{aligned}
\int_\Omega \varphi_x(\psi^2)_x\,d(x,z) &= -\int_\Omega \varphi_{xx}\psi^2\,d(x,z) - \int_I \varphi_x\big(\psi(x,u)\big)^2 u_x\,dx \\
&\qquad - \int_{-1}^0 \varphi_x(-1)(1+z)^2\big(f(-1)\big)^2\,dz + \int_{-1}^0 \varphi_x(1)(1+z)^2\big(f(1)\big)^2\,dz \\
&= -\int_\Omega \varphi_{xx}\psi^2\,d(x,z) - \int_I \varphi_x f^2 u_x\,dx - \big[\varphi_x(-1)-\varphi_x(1)\big]\frac{\big(f(1)\big)^2}{3} \\
&= -\int_\Omega \varphi_{xx}\psi^2\,d(x,z) - \int_I \varphi_x f^2 u_x\,dx + \frac{\big(f(1)\big)^2}{3}\int_I \varphi_{xx}\,dx \\
&= \mu\int_\Omega \varphi\psi^2\,d(x,z) - \int_I \varphi_x f^2 u_x\,dx - \frac{\mu\big(f(1)\big)^2}{3}
\end{aligned}$$

for $t \in [0,T)$, whereby one obtains

$$\begin{aligned}\int_I \varphi f\big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)\,dx \\ &= \int_\Omega \varphi\big(\varepsilon^2(\psi_x)^2+(\psi_z)^2+\frac{\mu\varepsilon^2}{2}\psi^2\big)\,d(x,z) - \frac{\mu\varepsilon^2}{6}\big(f(1)\big)^2 \\ &\quad - \frac{\varepsilon^2}{2}\int_I \varphi_x f^2 u_x\,dx + \varepsilon^2\int_I \varphi f f' u_x\,dx\end{aligned}$$

for $t \in [0,T)$, as claimed. □

The next result is an easy consequence of the Cauchy–Schwarz inequality and the boundary condition (5.32).

5.2.4 Lemma ([41, Lemma 5.4])
Given $t \in [0,T)$, it holds

$$\int_I \varphi \frac{f^2}{1+u}\,dx \le \int_{\Omega(u(t))} \varphi(\psi_z)^2\,d(x,z).$$

Proof. As performed in (5.15) in Lemma 5.1.1 one may deduce from the boundary condition (5.32) for ψ and from the Cauchy–Schwarz inequality that

$$\frac{f^2}{1+u} \le \int_{-1}^{u} (\psi_z)^2\,dz, \quad (t,x) \in [0,T)\times I.$$

Owing to the non-negativity of φ, we may multiply both sides of this inequality by φ and integrate over I with respect to x to obtain

$$\int_I \varphi \frac{f^2}{1+u}\,dx \le \int_\Omega \varphi\psi_z^2\,d(x,z), \quad t \in [0,T).$$

This completes the proof. □

As a last auxiliary step for the proof of Theorem 5.2.1 we define for $t \in [0,T)$ the functional

$$\Phi_\lambda(t) := \int_I \varphi\big(1+\varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\,dx \tag{5.44}$$

and derive a lower bound for it in the subsequent lemma.

5.2.5 Lemma ([41, Lemma 5.5])
Given $t \in [0,T)$ and $\beta > 0$, there holds

$$\Phi_\lambda(t) \ge \frac{1}{\beta}\left(-\frac{\mu\varepsilon^2 M^2}{6} - \frac{M^2}{4\beta} - \varepsilon^2 c_f + \frac{m^2}{1+E_\alpha(t)}\right) - \frac{M^2\varepsilon^2}{4\beta^2}\int_I \varphi(u_x)^2\,dx,$$

where the constants m, M and c_f are defined by

$$m := \min_{x\in[-1,1]} f(x), \quad M := \max_{x\in[-1,1]} f(x), \quad \text{and} \quad c_f := \int_I \varphi f|f''|\, dx. \tag{5.45}$$

Proof. Fix $t \in [0,T)$. Since we have

$$\int_I \varphi f\big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)\, dx \le \beta\Phi_\lambda + \frac{1}{4\beta}\int_I \varphi\big(1+\varepsilon^2(u_x)^2\big)f^2\, dx, \quad t\in[0,T),$$

for $\beta > 0$, by the *weighted Young inequality*, we may apply Lemma 5.2.3 to obtain

$$\begin{aligned}
\Phi_\lambda(t) = &\int_I \varphi\big(1+\varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\, dx \\
\ge &\frac{1}{\beta}\int_I \varphi f\big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)\, dx - \frac{1}{4\beta^2}\int_I \varphi f^2\, dx - \frac{\varepsilon^2}{4\beta^2}\int_I \varphi f^2 (u_x)^2\, dx \\
= &\frac{1}{\beta}\Bigg(\int_\Omega \varphi(\psi_z)^2\, d(x,z) - \frac{\mu\varepsilon^2\big(f(1)\big)^2}{6} - \frac{\varepsilon^2}{2}\int_I \varphi_x f^2 u_x\, dx + \varepsilon^2\int_I \varphi f f' u_x\, dx \\
&\quad + \varepsilon^2\int_\Omega \varphi(\psi_x)^2\, d(x,z) + \frac{\mu\varepsilon^2}{2}\int_\Omega \varphi\psi^2\, d(x,z)\Bigg) \\
&\quad - \frac{1}{4\beta^2}\int_I \varphi f^2\, dx - \frac{\varepsilon^2}{4\beta^2}\int_I \varphi f^2(u_x)^2\, dx
\end{aligned}$$

for $t \in [0,T)$. With the constants m, M and c_f, introduced in (5.45), we may infer from Lemma 5.2.4, the non-negativity of φ, and the integration by parts formula that

$$\begin{aligned}
\Phi_\lambda(t) \ge &\frac{1}{\beta}\left(m^2\int_I \frac{\varphi}{1+u}\, dx - \frac{\mu\varepsilon^2 M^2}{6} - \frac{\varepsilon^2}{2}\int_I \varphi_x f^2 u_x\, dx + \varepsilon^2\int_I \varphi f f' u_x\, dx - \frac{M^2}{4\beta}\right) \\
&\quad - \frac{\varepsilon^2 M^2}{4\beta^2}\int_I \varphi(u_x)^2\, dx \\
= &\frac{1}{\beta}\Bigg(m^2\int_I \frac{\varphi}{1+u}\, dx - \frac{\mu\varepsilon^2 M^2}{6} + \frac{\varepsilon^2}{2}\int_I u\varphi_{xx} f^2\, dx + \varepsilon^2\int_I u\varphi_x f f'\, dx \\
&\quad + \varepsilon^2\int_I \varphi f f' u_x\, dx - \frac{M^2}{4\beta}\Bigg) - \frac{\varepsilon^2 M^2}{4\beta^2}\int_I \varphi(u_x)^2\, dx \\
= &\frac{1}{\beta}\Bigg(m^2\int_I \frac{\varphi}{1+u}\, dx - \frac{\mu\varepsilon^2 M^2}{6} + \frac{\varepsilon^2}{2}\int_I u\varphi_{xx} f^2\, dx - \varepsilon^2\int_I \varphi u (f')^2\, dx \\
&\quad - \varepsilon^2\int_I \varphi u f f''\, dx - \frac{M^2}{4\beta}\Bigg) - \frac{\varepsilon^2 M^2}{4\beta^2}\int_I \varphi(u_x)^2\, dx
\end{aligned}$$

for $t \in [0,T)$. At this point, we may deduce from (5.37), again the non-negativity of φ, and the non-positivity of u that, given $t \in [0,T)$, it holds that

$$\begin{aligned}\Phi_\lambda(t) &\geq \frac{1}{\beta}\left(m^2 \int_I \frac{\varphi}{1+u}\,dx - \frac{\mu\varepsilon^2 M^2}{6} - \frac{\mu\varepsilon^2}{2}\int_I \varphi u f^2\,dx - \varepsilon^2 \int_I \varphi u f f''\,dx - \frac{M^2}{4\beta}\right)\\ &\quad - \frac{\varepsilon^2 M^2}{4\beta^2}\int_I \varphi (u_x)^2\,dx\\ &\geq \frac{1}{\beta}\left(m^2 \int_I \frac{\varphi}{1+u+\frac{\alpha}{2}u^2}\,dx - \frac{\mu\varepsilon^2 M^2}{6} - \varepsilon^2 \int_I \varphi u f f''\,dx - \frac{M^2}{4\beta}\right)\\ &\quad - \frac{\varepsilon^2 M^2}{4\beta^2}\int_I \varphi (u_x)^2\,dx.\end{aligned}$$

Observing that we may apply *Jensen's inequality* with the convex function $[r \mapsto 1/(1+r)]$ and the probability measure $\varphi(x)\,dx$, using the definition of the constant c_f and the fact that $-1 < u(t,x)$ for all $(t,x) \in [0,T) \times I$, we finally end up with

$$\Phi_\lambda(t) \geq \frac{1}{\beta}\left(-\frac{\mu\varepsilon^2 M^2}{6} - \frac{M^2}{4\beta} - \varepsilon^2 c_f + \frac{m^2}{1+E_\alpha}\right) - \frac{\varepsilon^2 M^2}{4\beta^2}\int_I \varphi (u_x)^2\,dx, \quad t \in [0,T).$$

Eventually the proof is complete. □

With the preliminary material from the above lemmas we are now able to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let $\alpha \in (0,1)$ to be determined later. We first derive a differential inequality for the energy functional E_α. Invoking Lemma 5.2.2, for $t \in (0,T)$ we have

$$\begin{aligned}&\frac{dE_\alpha}{dt} + \mu E_\alpha + \alpha \int_I \varphi (u_x)^2\,dx\\ &\quad = -\lambda \int_I \varphi (1+\alpha u)\big(1+\varepsilon^2 (u_x)^2\big)\big(\psi_z(x,u)\big)^2\,dx + \lambda\varepsilon^2 \int_I \varphi(1+\alpha u)(f')^2\,dx.\end{aligned}$$

Using the fact that φ is non-negative and that $1+\alpha u \geq 1-\alpha$ for $(t,x) \in (0,T) \times I$, c.f. (5.39), one further obtains

$$\begin{aligned}&\frac{dE_\alpha}{dt} + \mu E_\alpha + \alpha \int_I \varphi (u_x)^2\,dx\\ &\quad \leq -\lambda(1-\alpha)\Phi_\lambda(t) + \lambda\varepsilon^2 \int_I \varphi(1+\alpha u)(f')^2\,dx, \quad t \in (0,T),\end{aligned} \tag{5.46}$$

with Φ_λ introduced in (5.44). Lemma 5.2.5 yields the following estimate for $\Phi_\lambda(t)$ on $(0,T)$:

$$\Phi_\lambda(t) \geq \frac{1}{\beta}\left(-\frac{\mu\varepsilon^2 M^2}{6} - \frac{M^2}{4\beta} - \varepsilon^2 c_f + \frac{m^2}{1+E_\alpha}\right) - \frac{M^2\varepsilon^2}{4\beta^2}\int_I \varphi (u_x)^2\,dx. \tag{5.47}$$

Here, $\beta > 0$ is a further free parameter to be determined later and the constants m, M and c_f are defined in (5.45). Fusing (5.46) and (5.47) leads to

$$\frac{dE_\alpha}{dt} + \mu E_\alpha + \alpha \int_I \varphi(u_x)^2\,dx \leq \frac{\lambda(1-\alpha)}{\beta}\left(\frac{\mu\varepsilon^2 M^2}{6} + \frac{M^2}{4\beta} + \varepsilon^2 c_f - \frac{m^2}{1+E_\alpha}\right) + \frac{\lambda(1-\alpha)M^2\varepsilon^2}{4\beta^2}\int_I \varphi(u_x)^2\,dx + \lambda\varepsilon^2 \int_I \varphi(f')^2\,dx, \quad t \in (0,T).$$

Since $-1 \leq E_\alpha(t)$ for all $t \in [0,T)$ by introducing the further constant

$$d_f := \int_I \varphi(f')^2\,dx$$

one obtains

$$\frac{dE_\alpha}{dt} + \left(\alpha - \frac{\lambda(1-\alpha)M^2\varepsilon^2}{4\beta^2}\right)\int_I \varphi(u_x)^2\,dx \leq \mu + \frac{\lambda(1-\alpha)}{\beta}\left(\frac{\mu\varepsilon^2 M^2}{6} + \frac{M^2}{4\beta} + \varepsilon^2 c_f - \frac{m^2}{1+E_\alpha}\right) + \lambda\varepsilon^2 d_f, \quad t \in (0,T).$$

Now we use the freedom of choosing α in such a way that we are able to control the λ-dependent term u_x. More precisely, we choose

$$\alpha = \frac{\lambda(1-\alpha)M^2\varepsilon^2}{4\beta^2},$$

which is equivalent to

$$\alpha = \frac{\lambda M^2\varepsilon^2}{4\beta^2 + \lambda M^2\varepsilon^2} \in (0,1).$$

Consequently, we obtain the following differential inequality for E_α on $(0,T)$:

$$\frac{dE_\alpha}{dt} \leq \mu + \lambda\varepsilon^2 d_f + \frac{\lambda}{\beta}\left(\frac{\mu\varepsilon^2 M^2}{6} + \frac{M^2}{4\beta} + \varepsilon^2 c_f - \frac{m^2}{1+E_\alpha}\right).$$

Choosing

$$\varepsilon \leq \frac{1}{\sqrt{\lambda}} \qquad \text{and} \qquad \beta = \sqrt{\lambda}$$

then implies

$$\frac{dE_\alpha}{dt} \leq \mu + d_f + \sqrt{\lambda}\left(\frac{\mu M^2}{6\lambda} + \frac{M^2}{4\sqrt{\lambda}} + \frac{c_f}{\lambda} - \frac{m^2}{1+E_\alpha}\right), \quad t \in (0,T).$$

Denoting the right-hand side of the last inequality by $F_\lambda(E_\alpha)$, i.e. defining

$$F_\lambda(E_\alpha) := \mu + d_f + \sqrt{\lambda}\left(\frac{\mu M^2}{6\lambda} + \frac{M^2}{4\sqrt{\lambda}} + \frac{c_f}{\lambda} - \frac{m^2}{1+E_\alpha}\right), \quad t \in (0,T),$$

one may observe that F_λ is increasing on $(-1,\infty)$ and that E_α is non-positive. This yields

$$\frac{dE_\alpha(t)}{dt} \leq F_\lambda(E_\alpha(t)) \leq F_\lambda(0), \quad t \in (0,T). \tag{5.48}$$

Since

$$F_\lambda(0) = \mu + d_f + \sqrt{\lambda}\left(\frac{\mu M^2}{6\lambda} + \frac{M^2}{4\sqrt{\lambda}} + \frac{c_f}{\lambda} - m^2\right)$$

is strictly decreasing in λ and positive for small values of λ, it follows that one may find $\lambda^* > 0$ large enough, such that $F_{\lambda^*}(0) = 0$. Integrating inequality (5.48) with respect to t then implies

$$T \leq -\frac{1}{F_\lambda(0)} < \infty,$$

provided that $\lambda > \lambda^*$, and the proof is complete.

□

5.2.6 Remark *(1) Observe that λ^* depends only on the constants*

$$m = \min_{x\in[-1,1]} f(x), \qquad M = \max_{x\in[-1,1]} f(x),$$
$$c_f = \int_I \varphi f|f''|\,dx \qquad d_f = \int_I \varphi (f')^2\,dx,$$

but not on any further properties of f.

(2) Theorem 5.2.1 provides an upper bound for the maximal time T of existence for all $\varepsilon \in (0, 1/\sqrt{\lambda}]$.

(3) Observe that Lemma 5.2.2 and Lemma 5.2.3 require only that $f \in C^1\big([-1,1],\mathbb{R}\big)$ and Lemma 5.2.4 does even hold true for $f \in C\big([-1,1],\mathbb{R}\big)$.

Finally, we complete this section by fusing some of the previous observations. In Theorem 5.1.2 conditions are specified which ensure that solutions emerging from non-positive initial values stay non-positive for all times t of existence. For spatially varying permittivity profiles the above mentioned conditions are even improved in Corollary 5.1.3 (i). In addition, the non-positivity of u appears as a crucial condition for the verification of the appearance of finite-time singularities, c.f. Theorem 5.2.1. Thus, combining Theorem 5.1.2 with Corollary 5.1.3 and Theorem 5.2.1 advices us to introduce the constant

$$m_{f,\lambda} := \min\left\{\min_{x\in[-1,1]} \frac{f(x)}{|f'(x)|}, \frac{1}{\sqrt{\lambda}}\right\}$$

and to formulate the following corollary. Note that we are still restricted to the regime of spatially varying permittivity profiles.

5.2.7 Corollary

Let $f \in C^2\big([-1,1,\mathbb{R}]\big)$ be positive with $f(-1) = f(1)$ and denote by $u \in C\big([0,T), W_q^2(I)\big)$ the solution to (5.33)–(5.35) *with corresponding initial value $u_*(x) \le 0$, $x \in I$. Assume in addition that*

$$\psi_{zz}(x,-1) \ge 0 \quad \text{and} \quad \psi_{zz}(x, v(x)) \ge 0 \tag{5.49}$$

for all $x \in I$ and all $v \in W_q^2(I)$, satisfying $v(x) \le 0$, $x \in I$. Then there exists a critical voltage value $\lambda^* > 0$ such that $T < \infty$, provided that $\lambda > \lambda^*$ and $\varepsilon \in (0, m_{f,\lambda})$. That is, we have either

$$\limsup_{t\to T} \|u(t)\|_{W_q^2(I)} = \infty \qquad \text{or} \qquad \liminf_{t\to T} \min_{x\in[-1,1]} u(t,x) = -1.$$

5.2.2 | Finite-Time Singularities in the Quasilinear Setting

Similar to what is done in the semilinear case, we now study conditions which ensure that the solution u to the quasilinear evolution problem develops a singularity after a finite time of existence.

For constant permittivity profiles $f \equiv 1$ the according result is published in [13], where the authors study the time evolution of a certain energy functional in order to derive an upper bound for maximal time T of existence. The same approach with even the same energy functional may be used for non-constant permittivity profiles $f = f(x)$ and $f(u)$, respectively, as we will see in the following (c.f. also [17, 16]). In order to not go beyond the scope of this work we focus here on the case in which the permittivity profile f depends on the membrane's displacement u. The case of spatially varying permittivity profiles is just briefly discussed without going into details.

Spatially Varying Permittivity Profiles; $f = f(x)$. Given a spatially varying permittivity profile $f = f(x)$, $x \in I$, we consider the moving boundary problem

$$\varepsilon^2\psi_{xx} + \psi_{zz} = 0, \qquad t > 0, (x,z) \in \Omega(u(t)), \tag{5.50}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f(x), \quad t > 0, (x,z) \in \partial\Omega(u(t)), \tag{5.51}$$

coupled with the quasilinear parabolic evolution problem

$$u_t - \partial_x\left(\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right) = -\lambda\left(\varepsilon^2\big(\psi_x(x,u(t,x))\big)^2 + \big(\psi_z(x,u(t,x))\big)^2\right)$$
$$+ 2\lambda\varepsilon^2\psi_x(x,u(t,x))f'(x), \quad t>0, x\in I, \tag{5.52}$$
$$u(t,\pm 1) = 0, \quad t>0, \tag{5.53}$$
$$u(0,x) = u_*(x), \quad x\in I. \tag{5.54}$$

The following theorem on finite-time singularities of the solution u to (5.52)–(5.54) may be verified.

5.2.8 Theorem (Finite-Time Singularity; $f = f(x)$; [17])
Let $q\in(2,\infty), \varepsilon>0$ and $\lambda>0$. Moreover, given a positive $f\in C^1\big([-1,1],\mathbb{R}\big)$ and an initial datum $u_*\in W^2_{q,D}(I)$, satisfying $-1<u_*(x)\le 0$ for all $x\in I$, denote by (u,ψ) the unique solution to (5.50)–(5.54) *on the maximal interval* $[0,T)$ *of existence and assume that*

(A_1) $\max_{x\in[-1,1]} f(x) < \sqrt{2}\min_{x\in[-1,1]} f(x)$;

(A_2) $\min_{x\in[-1,1]} f(x) = f(-1) = f(1)$;

(A_3) $u(t,x)\le 0, \quad (t,x)\in[0,T)\times I$.

Then there exist $\varepsilon_*>0$ and $\lambda^* = \lambda^*(\varepsilon_*)>0$ such that the maximal existence time T of the unique solution u to (5.52)–(5.54) *is finite,*[6] *provided that $\varepsilon\in(0,\varepsilon_*)$ and $\lambda>\lambda^*$. In this case either*

$$\liminf_{t\to T}\min_{[-1,1]} u(t,x) = -1 \qquad \text{or} \qquad \limsup_{t\to T}\|u(t)\|_{W^2_q(I)} = \infty.$$

As mentioned above we omit the according proof in order to not going beyond the scope of this thesis and refer the reader to [17] for more detailed information. Nevertheless it is worthwhile to mention that the non-positivity of u as stated in (A_3) is crucial for the proof.

PERMITTIVITY PROFILES DEPENDING ON THE MEMBRANE'S DISPLACEMENT; $f = f(u)$. In this paragraph we study the coupled system consisting of the elliptic free boundary value problem

$$\varepsilon^2\psi_{xx} + \psi_{zz} = 0, \qquad t>0, (x,z)\in\Omega(u(t)), \tag{5.55}$$
$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)} f\big(u(t,x)\big), \quad t>0, (x,z)\in\partial\Omega(u(t)), \tag{5.56}$$

[6] Letting $\delta := 2\min(f^2) - \max(f^2)$, we have in fact $\lambda^* = 1/(\delta\varepsilon)$.

and the quasilinear parabolic evolution problem

$$u_t - \partial_x \left(\frac{u_x}{\sqrt{1+\varepsilon^2 (u_x)^2}} \right) = -\lambda \left(\varepsilon^2 \big(\psi_x(x,u(t,x))\big)^2 + \big(\psi_z(x,u(t,x))\big)^2 \right)$$
$$+ 2\lambda \psi_z(x,u(t,x)) f'\big(u(t,x)\big), \quad t>0,\ x\in I, \tag{5.57}$$
$$u(t,\pm 1) = 0, \quad t>0, \tag{5.58}$$
$$u(0,x) = u_*(x), \quad x \in I. \tag{5.59}$$

The permittivity profile is considered to be a function depending on the displacement u of the membrane and $f'(u)$ denotes the derivative of f with respect to u. We prove that under certain conditions the solution u to the quasilinear parabolic initial boundary value problem (5.57)–(5.59) develops a singularity after a finite time T of existence. More precisely, as in the semilinear and the quasilinear setting with $f = f(x)$ either a blow-up of the $W_q^2(I)$-norm of u or a touchdown takes place.

5.2.9 Theorem (Finite-Time Singularity; $f = f\big(u(t,x)\big)$, [16, Theorem 3.4])
Let $q \in (2,\infty), \varepsilon > 0$ and $\lambda > 0$. Moreover, given a positive $f \in C^2\big([-1,0],\mathbb{R}\big)$ and an initial datum $u_*(x) \le 0$, $x \in I$, denote by (u,ψ) the unique solution to (5.55)–(5.59) on the maximal interval $[0,T)$ of existence and assume that the following conditions hold true:

(A_1) $\max_{x\in[-1,0]} f(r) < \sqrt{2} \min_{r\in[-1,0]} f(r)$;

(A_2) $f'(r) \le 0, \quad r \in [-1,0]$,

(A_3) $u(t,x) \le 0, \quad (t,x) \in [0,T) \times I$;

(A_4) $\psi_{zz}(t,x,-1) \ge 0$ *and* $\psi_{zz}\big(t,x,u(t,x)\big) \ge 0, \quad t\in[0,T),\ x \in I$.

Then the maximal existence time T of the unique solution u to (5.57)–(5.59) is finite. More precisely, there exists [7] $\lambda^* > 0$ such that either

$$\liminf_{t\to T} \min_{x\in[-1,1]} u(t,x) = -1 \qquad \textit{or} \qquad \limsup_{t\to T} \|u(t)\|_{W_q^2(I)} = \infty$$

for all $\lambda > \lambda^*$.

As already mentioned the proof of this result relies on an appropriate estimate of an energy functional which then leads to an upper bound for the maximal time T of existence of the solution (u,ψ) to (5.55)–(5.59). In the sequel we provide the essential steps for the estimate of the energy functional in the form of several technical lemmas. Those auxiliary results are finally fused in a separate proof of Theorem 5.2.9.

[7] Letting $\delta := 2\min(f^2) - \max(f^2)$, we have in fact $\lambda^* = 1/(\delta\varepsilon)$.

As a starting point we state a representation of ψ_x on the membrane u

$$\psi_x(x,u(t,x)) = \big(f'(u(t,x)) - \psi_z(x,u(t,x))\big)\, u_x(t,x), \quad (t,x) \in (0,T)\times I, \tag{5.60}$$

which is of special importance as it is frequently used in the following. Furthermore, ψ_x vanishes on the ground plate, i.e.

$$\psi_x(x,-1) = 0, \quad x \in I. \tag{5.61}$$

Note that both identities, (5.60) and (5.61) may be derived from the boundary condition (5.56).

The following lemma provides an integral identity based on the equation (5.55) for ψ.

5.2.10 Lemma ([16, Lemma 3.5])
Given $f \in C^1\big([-1,0],\mathbb{R}\big)$ there holds

$$\begin{aligned}
&\frac{1}{2}\int_I \big(1{+}\varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\,dx \\
&\qquad = \int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\,dx + \frac{1}{2}\int_I \big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1)f(u)\,dx \\
&\qquad\quad - \varepsilon^2\int_I f(u)f'(u)(u_x)^2\,dx + \frac{\varepsilon^2}{2}\int_I \big(f'(u)u_x\big)^2\,dx - \varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z).
\end{aligned}$$

Proof. Thanks to *Fubini's theorem* and the Green–Riemann integration formula and the boundary condition (5.61) we find that

$$\begin{aligned}
&-\varepsilon^2\int_\Omega \psi_x\big(\psi_{zx} - f'(u)u_x\big)\,d(x,z) \\
&\qquad = -\frac{\varepsilon^2}{2}\int_\Omega \big((\psi_x)^2\big)_z\,d(x,z) + \varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z) \\
&\qquad = -\frac{\varepsilon^2}{2}\int_I \big(\psi_x(x,u)\big)^2\,dx + \varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z).
\end{aligned}$$

Using the identity (5.60) then leads to

$$\begin{aligned}
&-\varepsilon^2\int_\Omega \psi_x\big(\psi_{zx} - f'(u)u_x\big)\,d(x,z) \\
&\qquad = -\frac{\varepsilon^2}{2}\int_I \big(f'(u)u_x - \psi_z(x,u)u_x\big)^2\,dx + \varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z) \\
&\qquad = -\frac{\varepsilon^2}{2}\int_I \big(f'(u)u_x\big)^2\,dx + \varepsilon^2\int_I \psi_z(x,u)f'(u)(u_x)^2\,dx \\
&\qquad\quad - \frac{\varepsilon^2}{2}\int_I \big(\psi_z(x,u)u_x\big)^2\,dx + \varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z).
\end{aligned} \tag{5.62}$$

As above, invoking the Green–Riemann integration formula as well as the boundary conditions (5.56) and (5.60), we obtain

$$\begin{aligned}
\varepsilon^2 \int_\Omega & \psi_{xx}\big(\psi_z - f(u)\big)\, d(x,z) \\
&= \varepsilon^2 \int_\Omega \big(\psi_x(\psi_z - f(u))\big)_x\, d(x,z) - \varepsilon^2 \int_\Omega \psi_x\big(\psi_{zx} - f'(u)u_x\big)\, d(x,z) \\
&= \varepsilon^2 \int_{\partial\Omega} \psi_x\big(\psi_z - f(u)\big)\, dz - \varepsilon^2 \int_\Omega \psi_x\big(\psi_{zx} - f'(u)u_x\big)\, d(x,z) \\
&= \varepsilon^2 \int_I \big(\psi_z(x,u)u_x\big)^2\, dx - \varepsilon^2 \int_I \psi_z(x,u)f(u)(u_x)^2\, dx \\
&\quad - \varepsilon^2 \int_I \psi_z(x,u)f'(u)(u_x)^2\, dx + \varepsilon^2 \int_I f(u)f'(u)(u_x)^2\, dx \\
&\quad - \varepsilon^2 \int_\Omega \psi_x\big(\psi_{zx} - f'(u)u_x\big)\, d(x,z).
\end{aligned} \tag{5.63}$$

Fusing (5.62) and (5.63) then yields

$$\begin{aligned}
\varepsilon^2 \int_\Omega \psi_{xx}\big(\psi_z - f(u)\big)\, d(x,z) &= \varepsilon^2 \int_\Omega \psi_x f'(u)u_x\, d(x,z) - \frac{\varepsilon^2}{2} \int_I \big(f'(u)u_x\big)^2\, dx \\
&+ \frac{\varepsilon^2}{2} \int_I \big(\psi_z(x,u)u_x\big)^2\, dx + \varepsilon^2 \int_I \big(f'(u) - \psi_z(x,u)\big) f(u)(u_x)^2\, dx.
\end{aligned} \tag{5.64}$$

Similarly, again due to Fubini's theorem we may derive the identity

$$\begin{aligned}
\int_\Omega & \psi_{zz}\big(\psi_z - f(u)\big)\, d(x,z) \\
&= -\int_\Omega \psi_{zz} f(u)\, d(x,z) + \frac{1}{2} \int_I \big(\psi_z(x,u)\big)^2 - \big(\psi_z(x,-1)\big)^2\, dx \\
&= \int_I \big(\psi_z(x,-1) - \psi_z(x,u)\big) f(u)\, dx + \frac{1}{2} \int_I \big(\psi_z(x,u)\big)^2 - \big(\psi_z(x,-1)\big)^2\, dx.
\end{aligned} \tag{5.65}$$

Multiplying now equation (5.50) by $\psi_z - f(u)$, integrating over Ω and using the above equations (5.64) and (5.65) we find that

$$\begin{aligned}
0 &= \int_\Omega \big(\varepsilon^2 \psi_{xx} + \psi_{zz}\big)\big(\psi_z - f(u)\big)\, d(x,z) \\
&= \varepsilon^2 \int_\Omega \psi_x f'(u)u_x\, d(x,z) - \frac{1}{2} \int_I \big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1)f(u)\, dx \\
&\quad + \frac{1}{2} \int_I \big(1 + \varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\, dx - \int_I \big(1 + \varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\, dx \\
&\quad + \varepsilon^2 \int_I f(u)f'(u)(u_x)^2\, dx - \frac{\varepsilon^2}{2} \int_I \big(f'(u)u_x\big)^2\, dx.
\end{aligned}$$

Finally, the last equation is equivalent to

$$\begin{aligned}\frac{1}{2}\int_I &\big(1+\varepsilon^2(u_x)^2\big)\big(\psi_z(x,u)\big)^2\,dx \\ &= \int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\,dx + \frac{1}{2}\int_I \big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1)f(u)\,dx \\ &\quad - \varepsilon^2\int_I f(u)f'(u)(u_x)^2\,dx + \frac{\varepsilon^2}{2}\int_I \big(f'(u)u_x\big)^2\,dx - \varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z),\end{aligned}$$

whence the proof is complete. □

Subsequently, a further manipulation of the first term of the above obtained integral equality is verified.

5.2.11 Lemma ([16, Lemma 3.6])
Given $f \in C^1\big([-1,0],\mathbb{R}\big)$, the following equation holds true:

$$\begin{aligned}\int_I &\big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\,dx \\ &= \int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\,d(x,z) + \varepsilon^2\int_I f(u)f'(u)(u_x)^2\,dx \\ &\quad - f(0)\,\varepsilon^2\int_{-1}^0 (1+z)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\,dz.\end{aligned}$$

Proof. Using the boundary condition (5.56) for ψ it follows that

$$\int_{\partial\Omega} \psi_z\psi\,dx = -\int_I \psi_z(x,u)f(u)\,dx. \tag{5.66}$$

Similarly, by recalling in addition the identity (5.60), we find that

$$\begin{aligned}\int_{\partial\Omega}\psi_x\psi\,dz &= -\int_I \psi_x(x,u)f(u)u_x\,dx + f(0)\int_{-1}^0 (1+z)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\,dz \\ &= -\int_I \big(f'(u)-\psi_z(x,u)\big)\,f(u)(u_x)^2\,dx + f(0)\int_{-1}^0 (1+z)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\,dz.\end{aligned} \tag{5.67}$$

Now multiplying the equation (5.55) by ψ and integrating over Ω one obtains

$$\begin{aligned}0 &= \int_\Omega \big(\varepsilon^2\psi_{xx}+\psi_{zz}\big)\psi\,d(x,z) \\ &= -\int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\,d(x,z) + \int_\Omega \varepsilon^2(\psi_x\psi)_x + (\psi_z\psi)_z\,d(x,z).\end{aligned}$$

A reapplication of the Green–Riemann integration formula together with (5.66) and (5.67) finally yields

$$\begin{aligned}0 &= -\int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\, d(x,z) + \varepsilon^2 \int_{\partial\Omega} \psi_x\psi\, dz - \int_{\partial\Omega} \psi_z\psi\, dx \\ &= -\int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\, d(x,z) + \int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\, dx \\ &\quad - \varepsilon^2 \int_I f(u)f'(u)(u_x)^2\, dx + \varepsilon^2 f(0)\int_{-1}^0 (1+z)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\, dz.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\, dx \\ &\qquad = \int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\, d(x,z) + \varepsilon^2\int_I f(u)f'(u)(u_x)^2\, dx \\ &\qquad\quad - \varepsilon^2 f(0)\int_{-1}^0 (1+z)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\, dz,\end{aligned}$$

whereby the proof is complete. □

In the following lemma we provide a subsolution to the elliptic problem (5.55)–(5.56) for ψ.

5.2.12 Lemma ([17, Lemma 3.7])
Given a positive $f \in C\big([-1,0],\mathbb{R}\big)$ we introduce the notation

$$m := \min_{r\in[-1,0]} f(r)$$

and define $\eta(x,z) := (1+z)m$ for $(x,z)\in\overline{\Omega}$. Then η is a subsolution to (5.55)–(5.56), i.e. we have

$$\eta(x,z) \le \psi(x,z), \quad (x,z)\in\overline{\Omega}.$$

Proof. It is clear that η satisfies the equation (5.55), i.e.

$$\varepsilon^2\eta_{xx} + \eta_{zz} = 0 = \varepsilon^2\psi_{xx} + \psi_{zz}, \quad (x,z)\in\Omega.$$

Moreover, on the lateral components of the boundary it holds that

$$\eta(\pm 1,z) = (1+z)m \le (1+z)f(0) = \psi(\pm 1,z), \quad z\in(-1,0).$$

Finally, we have

$$\eta(x,-1) = 0 = \psi(-1,z), \quad x\in I,$$

on the ground plate, as well as

$$\eta\big(x, u(x)\big) = (1+u)m \le (1+u)f\big(u(x)\big) \le f\big(u(x)\big) = \psi\big(x, u(x)\big), \quad x \in I,$$

on the membrane.[8] Thus, the elliptic maximum principle yields the assertion. □

Using the fact that η is a subsolution to (5.55)–(5.56) now leads to the following result on the sign of ψ_x on the lateral boundaries. In some sense the results is reminiscent of *Hopf's maximum principle.*

5.2.13 Lemma ([16, Lemma 3.8])
Given a positive $f \in C([-1,0],\mathbb{R})$ with $m = f(0)$, the potential ψ satisfies

$$\pm\psi_x(\pm 1, z) \le 0, \quad z \in (-1,0).$$

Proof. The statement readily follows from an application of Lemma 5.2.12:

$$\begin{aligned}\psi_x(1,z) &= \lim_{h\searrow 0} \frac{\psi(1-h,z)-\psi(1,z)}{-h} \\ &= \lim_{h\searrow 0} \frac{\psi(1-h,z)-(1+z)f(0)}{-h} \\ &\le \lim_{h\searrow 0} \frac{\eta(1-h,z)-(1+z)f(0)}{-h} \\ &= 0.\end{aligned}$$

Similarly one deduces $\psi_x(-1,z) \ge 0$ for all $z \in (-1,0)$. □

5.2.14 Corollary ([16, Corollary 3.9])
Given a positive $f \in C^1\big([-1,0],\mathbb{R}\big)$, the inequality

$$\int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\,dz \ge \int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\,d(x,z) + \varepsilon^2 \int_I f(u)f'(u)(u_x)^2\,dx$$

holds true.

Proof. From Lemma 5.2.11 we know that the identity

$$\begin{aligned}\int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f(u)\,dx &= \int_\Omega \varepsilon^2(\psi_x)^2 + (\psi_z)^2\,d(x,z) \\ &\quad + \varepsilon^2 \int_I f(u)f'(u)(u_x)^2\,dx - f(0)\,\varepsilon^2 \int_{-1}^0 (1+z)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\,dz\end{aligned}$$

[8] Recall that $u(t,x) \le 0$ on $[0,T)\times I$ by Theorem 5.1.2.

holds true. Having in mind that f is positive by assumption we may invoke Lemma 5.2.13 to deduce that

$$-f(0)\big(\psi_x(1,z)-\psi_x(-1,z)\big)\geq 0, \quad z\in(-1,0),$$

whereby the assertion immediately follows. □

By simple calculations one may derive the following two auxiliary results.

5.2.15 Lemma ([16, Lemma 3.10])
The estimate

$$\int_I \big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1)f(u)\,dx \geq -\int_I \big(f(u)\big)^2\,dx$$

holds true for every $f\in C\big([-1,0],\mathbb{R}\big)$.

Proof. We readily see that

$$\big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1)f(u) + \big(f(u)\big)^2 = \big(\psi_z(x,-1)-f(u)\big)^2 \geq 0,$$

whereby

$$\big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1)f(u) \geq -\big(f(u)\big)^2.$$

An integration over I with respect to x completes the proof. □

5.2.16 Lemma ([16, Lemma 3.11])
Given an $f\in C^1\big([-1,0],\mathbb{R}\big)$, *we obtain the inequality*

$$-\varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z) \geq -\frac{\varepsilon^2}{4}\int_\Omega (\psi_x)^2\,d(x,z) - \varepsilon^2\int_I \big(f'(u)u_x\big)^2 dx.$$

Proof. Again we use the elementary observation

$$0\leq \left(\frac{1}{2}\psi_x - f'(u)u_x\right)^2 = \frac{1}{4}(\psi_x)^2 - \psi_x f'(u)u_x + \big(f'(u)u_x\big)^2$$

to get the inequality

$$-\psi_x f'(u)u_x \geq -\frac{1}{4}(\psi_x)^2 - \big(f'(u)u_x\big)^2.$$

Integration over Ω yields

$$-\varepsilon^2\int_\Omega \psi_x f'(u)u_x\,d(x,z) \geq -\frac{\varepsilon^2}{4}\int_\Omega (\psi_x)^2\,d(x,z) - \varepsilon^2\int_\Omega \big(f'(u)u_x\big)^2 d(x,z).$$

Finally, using $u(x) \leq 0$ for all $x \in I$, guaranteed by Theorem 5.1.2, Fubini's theorem leads to

$$\begin{aligned}-\varepsilon^2 \int_\Omega \psi_x f'(u) u_x \, d(x,z) &\geq -\frac{\varepsilon^2}{4} \int_\Omega (\psi_x)^2 \, d(x,z) - \varepsilon^2 \int_I (u+1)\big(f'(u)u_x\big)^2 \, dx \\ &\geq -\frac{\varepsilon^2}{4} \int_\Omega (\psi_x)^2 \, d(x,z) - \varepsilon^2 \int_I \big(f'(u)u_x\big)^2 dx\end{aligned}$$

and the proof is complete. □

Given $t \in [0,T)$, we introduce the *Dirichlet form* associated to (5.55), i.e.

$$\Phi_\lambda(t) := \frac{\lambda}{2} \int_I \varepsilon^2 \big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2 \, dx \tag{5.68}$$

and fuse the above lemmas to obtain a lower bound for the above introduced Dirichlet form of (5.55)–(5.56) in terms of a weighted $L_2(I)$-norm of the permittivity profile f. Similar as above we use the notation

$$m := \min_{r \in [-1,0]} f(r) \quad \text{and} \quad M := \max_{r \in [-1,0]} f(r).$$

5.2.17 Lemma
Given $f \in C\big([-1,0],\mathbb{R}\big)$, there holds

$$\int_\Omega \frac{3}{4}\varepsilon^2(\psi_x)^2 + (\psi_z)^2 d(x,z) \geq \int_I \frac{\big(f(u)\big)^2}{1+u} \, dx.$$

Proof. Again as in (5.15) in Lemma 5.1.1 we deduce from the boundary condition (5.56) for ψ and a trivial application of Cauchy–Schwarz's inequality that

$$\frac{\big(f(u)\big)^2}{1+u} \leq \int_{-1}^u (\psi_z)^2 \, dz.$$

Integrating this inequality with respect to $x \in I$ and using Fubini's theorem yields

$$\int_I \frac{\big(f(u)\big)^2}{1+u} \, dx \leq \int_\Omega (\psi_z)^2 \, d(x,z) \leq \int_\Omega \frac{3}{4}\varepsilon^2 (\psi_x)^2 + (\psi_z)^2 \, d(x,z),$$

which is the statement of the lemma. □

5.2.18 Lemma ([16, Lemma 3.12])
Let $f \in C^1\big([-1,0],\mathbb{R}\big)$ be positive with $m = f(0)$. Then the functional $\Phi_\lambda(t)$, introduced in (5.68)*,*

complies with the inequality

$$\Phi_\lambda(t) \geq \lambda \left(\frac{2m^2}{1-E(t)} - M^2 - \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx \right).$$

Proof. First we use the identity (5.60) to find that

$$\begin{aligned}
\Phi_\lambda(t) &= \frac{\lambda}{2} \int_I \varepsilon^2 \big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2 \, dx \\
&= \frac{\lambda}{2} \int_I \big(1 + \varepsilon^2 (u_x)^2\big) \big(\psi_z(x,u)\big)^2 \, dx + \frac{\lambda \varepsilon^2}{2} \int_I \big(f'(u) u_x\big)^2 \, dx \\
&\quad - \lambda \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx.
\end{aligned}$$

Invoking Lemma 5.2.10 and Corollary 5.2.14 we obtain

$$\begin{aligned}
\Phi_\lambda(t) &= \lambda \int_I \big(1 + \varepsilon^2 (u_x)^2\big) \psi_z(x,u) f(u) \, dx + \frac{\lambda}{2} \int_I \big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1) f(u) \, dx \\
&\quad - \lambda \varepsilon^2 \int_I f(u) f'(u)(u_x)^2 \, dx + \lambda \varepsilon^2 \int_I \big(f'(u) u_x\big)^2 \, dx - \lambda \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx \\
&\quad - \lambda \varepsilon^2 \int_\Omega \psi_x f'(u) u_x \, d(x,z) \\
&\geq \lambda \int_\Omega \varepsilon^2 (\psi_x)^2 + (\psi_z)^2 \, d(x,z) - \lambda \varepsilon^2 \int_\Omega \psi_x f'(u) u_x \, d(x,z) + \lambda \varepsilon^2 \int_I \big(f'(u) u_x\big)^2 \, dx \\
&\quad + \frac{\lambda}{2} \int_I \big(\psi_z(x,-1)\big)^2 - 2\psi_z(x,-1) f(u) \, dx - \lambda \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx.
\end{aligned}$$

Hence, thanks to Lemma 5.2.15 and Lemma 5.2.16 we obtain the estimate

$$\Phi_\lambda(t) \geq \lambda \int_\Omega \frac{3}{4} \varepsilon^2 (\psi_x)^2 + (\psi_z)^2 \, d(x,z) - \lambda \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx - \frac{\lambda}{2} \int_I (f(u))^2 \, dx.$$

Recalling Corollary 5.2.17 and applying Jensen's inequality to the convex function $[r \mapsto 1/(1+r)]$ and the probability measure $dx/2$ we finally end up with

$$\begin{aligned}
\Phi_\lambda(t) &\geq \lambda \int_I \frac{\big(f(u)\big)^2}{1+u} \, dx - \lambda \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx - \frac{\lambda}{2} \int_I \big(f(u)\big)^2 \, dx \\
&\geq \frac{2\lambda m^2}{1-E(t)} - \lambda \varepsilon^2 \int_I \psi_z(x,u) f'(u)(u_x)^2 \, dx - \lambda M^2,
\end{aligned}$$

which completes the proof. □

We are finally prepared to prove Theorem 5.2.8. From now on we explicitly mention the time variable t when it is requested from the context.

Proof of Theorem 5.2.8:
Given $t \in [0,T)$, we introduce the functional

$$E(t) := -\frac{1}{2}\int_I u(t,x)\,dx.$$

Since we know that $-1 < u(t,x) \leq 0$ for all $(t,x) \in [0,T) \times I$, cf. Theorem 5.1.2, it follows that

$$0 \leq E(t) < 1, \quad t \in [0,T).$$

Using the evolution equation (5.57) and the definition of $\Phi_\lambda(t)$ gives

$$\begin{aligned}\frac{dE}{dt}(t) &= -\frac{1}{2}\left[\frac{u_x}{\sqrt{1+\varepsilon^2(u_x)^2}}\right]_{x=-1}^{x=1} + \frac{\lambda}{2}\int_I \varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\,dx \\ &\quad - \lambda\int_I \psi_z(x,u)f'(u)\,dx \\ &\geq -\frac{1}{\varepsilon} + \Phi_\lambda(t) - \lambda\int_I \psi_z(x,u)f'(u)\,dx.\end{aligned} \tag{5.69}$$

Fusing this inequality with the estimate

$$\Phi_\lambda(t) \geq \lambda\left(\frac{2m^2}{1-E(t)} - M^2 - \varepsilon^2\int_I \psi_z(x,u)f'(u)(u_x)^2\,dx\right)$$

from Lemma 5.2.18 leads to

$$\frac{dE}{dt}(t) \geq -\frac{1}{\varepsilon} + \lambda\left(\frac{2m^2}{1-E(t)} - M^2\right) - \lambda\int_I \big(1+\varepsilon^2(u_x)^2\big)\psi_z(x,u)f'(u)\,dx.$$

By a combination of the assumption (A_2) with the fact that $\psi_z(x,z) \geq 0$, cf. (5.22), we end up with the differential inequality

$$\frac{dE}{dt}(t) \geq -\frac{1}{\varepsilon} + \lambda\left(\frac{2m^2}{1-E(t)} - M^2\right) := F_\lambda(E(t)).$$

Observe that F_λ is (strictly) increasing on $[0,1)$ which implies that

$$\frac{dE}{dt}(t) \geq F_\lambda(E(t)) \geq F_\lambda(0). \tag{5.70}$$

Furthermore, evaluating F_λ in $E \equiv 0$ yields

$$F_\lambda(0) = -\frac{1}{\varepsilon} + \lambda\left(2m^2 - M^2\right).$$

By assumption we know that $\delta := 2m^2 - M^2$ is positive. Thus, if $\lambda > \lambda^* := 1/(\delta\varepsilon)$ it follows that

$F_\lambda(0) > 0$. Integrating the inequality (5.70) with respect to t then implies that $1 \geq E(0) + F_\lambda(0)T$ and eventually

$$T < \frac{1}{F_\lambda(0)} < \infty.$$

This completes the proof.

□

5.2.19 Remark

It is worthwhile to compare the assumptions of Theorem 5.2.9 with those of Theorem 5.2.8, where the case $f = f(x)$ is treated.

(1) *Theorem 5.2.9 holds true for any $\varepsilon > 0$ (provided that λ is accordingly large enough). This is in contrast to Theorem 5.2.8, where we have to assume that $\varepsilon > 0$ is small (and λ accordingly large enough).*

(2) *As one may see in Section 5.1, the condition (A_4) on ψ_{zz} to be non-negative on the membrane and on the ground plate is crucial in order to prove non-positivity of the membrane's displacement $u(t,x)$. But, moreover, in the present study, where the permittivity profile f depends on u, this assumption on ψ_{zz} is also necessary in order to verify the occurrence of finite-time singularities, even if we already know that $u(t,x) \leq 0$.*

6 | Numerical Investigations

This chapter is devoted to the numerical investigation of the system coupling the semilinear evolution problem, arising from a linear elasticity approach, with the associated elliptic moving boundary problem.

As performed in Chapter 3 for the analytical investigation, we first transform the elliptic moving boundary problem for the electric potential ψ to the fixed rectangle $\Omega := I \times (-1, 0)$. In the numerical computations we may thus benefit from a relatively simple geometry. The price to pay for this advantage is that the transformed elliptic problem has non-constant coefficients depending on the displacement u and its spatial derivatives up to order two. However, in the subsequent sections we consider the system coupling the evolution problem

$$\begin{aligned} u_t - u_{xx} = -\lambda \Bigg(& \varepsilon^2 \Big(-\big(f_x(x,u)\big)^2 + \big(f_u(x,u)u_x\big)^2 \Big) \\ & - 2\frac{1+\varepsilon^2(u_x)^2}{1+u} f_u(x,u)\varphi_\eta(t,x,1) + \frac{1+\varepsilon^2(u_x)^2}{(1+u)^2}\big(\varphi_\eta(t,x,1)\big)^2 \Bigg), \quad t>0,\ x\in I, \end{aligned} \tag{6.1}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{6.2}$$

$$u(0,x) = u_*(x), \quad x \in I, \tag{6.3}$$

for the membrane's displacement with the elliptic problem

$$\big(\mathcal{L}_{u(t)}\varphi\big) = 0, \qquad t>0,\ (x,\eta)\in\Omega, \tag{6.4}$$

$$\varphi(t,x,\eta) = \eta f(x,u), \quad t>0,\ (x,\eta)\in\partial\Omega, \tag{6.5}$$

determining the transformed electric potential in the region $\Omega = I \times (-1, 0)$. Recall that for $u(t) \in S_q(\kappa)$, the differential operator $\mathcal{L}_{u(t)}$ is given by

$$\mathcal{L}_{u(t)}\varphi = \varepsilon^2\varphi_{xx} - 2\varepsilon^2\eta\frac{u_x}{1+u}\varphi_{x\eta} + \frac{1+\varepsilon^2\eta^2(u_x)^2}{(1+u)^2}\varphi_{\eta\eta} + \varepsilon^2\eta\left(2\left(\frac{u_x}{1+u}\right)^2 - \frac{u_{xx}}{1+u}\right)\varphi_\eta.$$

Separating the treatments of the semilinear parabolic problem for u and the the elliptic problem for φ, we address in Section 6.1 the solution of the elliptic problem by means of the *Finite-element method*. Section 6.2 is then concerned with the problem of numerically determining the membrane's displacement u using the *Crank–Nicolson method*. There is a wide range of literature on the mentioned numerical methods. Howsoever the reader is referred to the textbooks [8, 5, 7] as pertinent references for more details on the following elaboration.

6.1 | Approximate Solution of the Elliptic Moving Boundary Problem

Let $u = u(t) \in S_q(\kappa)$ be a given membrane's displacement at a fixed time $t \geq 0$ and $f \in C^3\big([-1,1] \times \mathbb{R}\big)$.[1] We consider the elliptic boundary value problem

$$\left(\mathcal{L}_u\varphi\right)(t,x,\eta) = 0, \qquad t > 0,\ (x,\eta) \in \Omega, \tag{6.6}$$

$$\varphi(t,x,\eta) = \eta f(x,u), \quad t > 0,\ (x,\eta) \in \partial\Omega, \tag{6.7}$$

of second order in the region $\Omega = I \times (-1,0)$. From Theorem 3.1.3 we know that this problem possesses a unique solution $\varphi = \varphi(t) \in W_2^2(\Omega)$. In this section the basic concepts for a numerical treatment of (6.6)–(6.7) are presented.

Homogenisation of the Boundary Conditions. As it is common practice we start by reducing the above problem to one with homogeneous boundary conditions. To this end, recall from the proof of Theorem 3.1.3 that the function $\phi = \phi(u) \in H_0^1(\Omega)$, defined by

$$\phi(x,\eta) := \varphi(x,\eta) - \eta f(x,u), \quad (x,\eta) \in \overline{\Omega},$$

is the unique solution to the homogeneous boundary value problem

$$-\mathcal{L}_u\phi = F_u, \quad (x,\eta) \in \Omega, \tag{6.8}$$

$$\phi = 0, \quad (x,\eta) \in \partial\Omega, \tag{6.9}$$

where $F_u(x,\eta) := \mathcal{L}_u\big(\eta f(x,u)\big)$ for $(x,\eta) \in \Omega$.

[1] Again the time t appears as a parameter, whence it is omitted in the notation.

A Variational Formulation. In the following a suitable variational formulation of the problem (6.8)–(6.9) is derived which then serves as a basis for the numerical computation of $\varphi = \phi + \eta f(x,u)$ by means of the finite-element method. For this purpose, we consider the operator $\mathcal{L}_u$ in divergence form

$$-\mathcal{L}_u\phi = -\partial_x\big(a_{11}(u)\phi_x + a_{12}(u)\phi_\eta\big) - \partial_\eta\big(a_{21}(u)\phi_x + a_{22}(u)\phi_\eta\big) - b_1(u)\phi_x - b_2(u)\phi_\eta,$$

where

$$\begin{aligned} a_{11}(u) &:= \varepsilon^2, & a_{12}(u) &:= -\varepsilon^2\eta\frac{u_x}{1+u}, \\ a_{21}(u) &:= a_{12}(u), & a_{22}(u) &:= \frac{1+\varepsilon^2\eta^2(u_x)^2}{(1+u)^2}, \\ b_1(u) &:= \varepsilon^2\frac{u_x}{1+u}, & b_2(u) &:= -\varepsilon^2\eta\left(\frac{u_x}{1+u}\right)^2, \end{aligned}$$

multiply the equation (6.8) by a testfunction $\mu \in H_0^1(\Omega)$ and integrate over Ω. This leads to

$$\begin{aligned} &\int_\Omega F_u\,\mu\,d(x,\eta) = -\int_\Omega \big(\mathcal{L}_u\phi\big)\mu\,d(x,\eta) \\ &= -\int_\Omega \partial_x\left(\varepsilon^2\phi_x - \varepsilon^2\eta\frac{u_x}{1+u}\phi_\eta\right)\mu\,d(x,\eta) - \int_\Omega \partial_\eta\left(-\varepsilon^2\eta\frac{u_x}{1+u}\phi_x + \frac{1+\varepsilon^2\eta^2(u_x)^2}{(1+u)^2}\phi_\eta\right)\mu\,d(x,\eta) \\ &\quad - \varepsilon^2\int_\Omega \frac{u_x}{1+u}\phi_x\mu\,d(x,\eta) + \varepsilon^2\int_\Omega \eta\left(\frac{u_x}{1+u}\right)^2\phi_\eta\mu\,d(x,\eta). \end{aligned}$$

Thanks to the Green–Riemann integration theorem and the fact that μ vanishes at the boundary $\partial\Omega$ we then find that

$$\begin{aligned} &\int_\Omega F_u\,\mu\,d(x,\eta) \\ &= \varepsilon^2\int_\Omega \phi_x\mu_x\,d(x,\eta) - \varepsilon^2\int_\Omega \eta\frac{u_x}{1+u}\phi_\eta\mu_x\,d(x,\eta) - \varepsilon^2\int_\Omega \eta\frac{u_x}{1+u}\phi_x\mu_\eta\,d(x,\eta) \\ &\quad + \int_\Omega \frac{1+\varepsilon^2\eta^2(u_x)^2}{(1+u)^2}\phi_\eta\mu_\eta\,d(x,\eta) - \varepsilon^2\int_\Omega \frac{u_x}{1+u}\phi_x\mu\,d(x,\eta) + \varepsilon^2\int_\Omega \eta\left(\frac{u_x}{1+u}\right)^2\phi_\eta\mu\,d(x,\eta). \end{aligned} \tag{6.10}$$

Given $\phi, \mu \in H_0^1(\Omega)$ we now define the bilinear form

$$a(\phi,\mu) := \int_\Omega (\nabla\phi)^T A\nabla\mu + b^T\nabla\phi\,\mu\,d(x,\eta), \tag{6.11}$$

with the matrix A given by

$$A(x,\eta,u) := \begin{pmatrix} \varepsilon^2 & -\varepsilon^2\eta\frac{u_x}{1+u} \\ -\varepsilon^2\eta\frac{u_x}{1+u} & \frac{1+\varepsilon^2\eta^2(u_x)^2}{(1+u)^2} \end{pmatrix} \tag{6.12}$$

and the vector b defined by

$$b(x,\eta,u) := \begin{pmatrix} -\varepsilon^2 \frac{u_x}{1+u} \\ \varepsilon^2 \eta \left(\frac{u_x}{1+u}\right)^2 \end{pmatrix}. \tag{6.13}$$

6.1.1 Lemma (Continuity of a)
The bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined in (6.11) *is continuous, i.e. there exists a constant $C > 0$ such that*

$$|a(\phi,\mu)| \leq C \|\phi\|_{H^1(\Omega)} \|\mu\|_{H^1(\Omega)}$$

for all $\phi, \mu \in H_0^1(\Omega)$.

Proof. Since $u = u(t) \in S_q(\kappa)$ we know that $\|u\|_{C^1([-1,1])} \leq c$ and $\|1/(1+u)\|_{L_\infty([-1,1])} \leq c$ for a positive constant c that depends on κ. Using in addition the fact that $0 \leq \eta \leq 1$ the Cauchy–Schwarz inequality readily yields the existence of a constant $C = C(\varepsilon,\kappa)$ such that

$$|a(\phi,\mu)| \leq C \|\phi\|_{H^1(\Omega)} \|\mu\|_{H^1(\Omega)}$$

for all $\phi, \mu \in H_0^1(\Omega)$, which is the assertion. □

Though the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is continuous, it is in general not coercive. Nonetheless, introducing the *principal part* $a_\pi : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ of a, defined by

$$a_\pi(\phi,\mu) := \int_\Omega (\nabla\phi)^T A \nabla\mu \, d(x,\eta), \quad \phi,\, \mu \in H_0^1(\Omega)$$

with A as in (6.12), we may state the following results in this direction.

6.1.2 Lemma
The principal part $a_\pi : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ of the bilinear form a is elliptic (uniformly in $u \in S_q(\kappa)$). That is, there exists a constant $C = C(\varepsilon,\kappa) > 0$ such that

$$a_\pi(\phi,\phi) \geq C \|\phi\|_{H^1(\Omega)}^2$$

for all $\phi \in H_0^1(\Omega)$.

Proof. As in [32, Lemma 5] we show that the matrix

$$A(x,\eta,u) := \begin{pmatrix} \varepsilon^2 & -\varepsilon^2 \eta \frac{u_x}{1+u} \\ -\varepsilon^2 \eta \frac{u_x}{1+u} & \frac{1+\varepsilon^2\eta^2 (u_x)^2}{(1+u)^2} \end{pmatrix}$$

defined in (6.12) is positive definite uniformly in $u \in S_q(\kappa)$. To this end, denote by

$$t := \varepsilon^2 + \frac{1+\varepsilon^2\eta^2(u_x)^2}{(1+u)^2} \quad \text{and} \quad d := \frac{\varepsilon^2}{(1+u)^2}$$

the trace and the determinant of A, respectively. As A is obviously symmetric both eigenvalues

$$\mu_\pm = \frac{1}{2}\left(t \pm \sqrt{t^2-4d}\right)$$

are real-valued, which in particular implies that $t^2 \geq 4d$. In order to prove that A is positive definite (uniformly in $u \in S_q(\kappa)$) we are thus left with showing uniform positivity of μ_-. For this purpose observe that the fact that u is contained in $S_q(\kappa)$ implies that $-1+\kappa \leq u$ for all $x \in I$ and that there exists a constant $c = c(\kappa) > 0$ such that $\|u\|_{C^1([-1,1])} \leq c(\kappa)$. Due to this we may deduce the inequalities

$$\frac{1}{c(\kappa)} + \varepsilon^2 \leq t \leq \varepsilon^2 + \frac{1+\varepsilon^2 c(\kappa)^2}{\kappa^2} \quad \text{and} \quad d \geq \frac{\varepsilon^2}{c(\kappa)^2}.$$

Together with the relation

$$\mu_- \geq \frac{1}{2}\left(t - \sqrt{t^2-4d}\right) \geq \frac{d}{t} \Longleftrightarrow t - 2\frac{d}{t} \geq \sqrt{t^2-4d} \Longleftrightarrow 4\frac{d^2}{t^2} \geq 0$$

this yields

$$\mu_+ \geq \mu_- \geq \frac{d}{t} \geq \frac{\varepsilon^2}{c(\kappa)^2 t} \geq \frac{\varepsilon^2\kappa^2}{c(\kappa)^2\left(\varepsilon^2\kappa^2+1+\varepsilon^2 c(\kappa)^2\right)} > 0.$$

Having the positive definiteness of A at hand, we may now readily infer that the principal part a_π of a is coercive. More precisely, we have

$$a_\pi(\phi,\phi) = \int_\Omega (\nabla\phi)^T A \nabla\phi \, d(x,\eta) \geq \mu_- \int_\Omega |\nabla\phi|^2 d(x,\eta) \geq c\|\phi\|_{H^1(\Omega)}, \quad \phi \in H_0^1(\Omega), \tag{6.14}$$

as a consequence of the *Poincaré–Friedrichs inequality.* □

Given the coercivity of the principal part a_π of a, it is worthwhile to discuss a condition under which the entire bilinear form a is coercive. This is realised by the following corollary and the subsequent remark.

6.1.3 Corollary

Given $u = u(t) \in S_q(\kappa)$, the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is coercive if the inequality

$$\inf_{(x,\eta)\in\Omega} \left(\frac{\mu_-}{c_p} - \frac{\operatorname{div} b(x,\eta)}{2}\right) =: \alpha_0 > 0 \tag{6.15}$$

is satisfied, where c_p denotes Poincaré constant *of Ω.*

Proof. We know from (6.14) in the proof of Lemma 6.1.2 that

$$a(\phi,\phi)=\int_\Omega(\nabla\phi)^T A\nabla\phi\,d(x,\eta)+\int_\Omega b^T\nabla\phi\,\phi\,d(x,\eta)\geq\mu_-\int_\Omega|\nabla\phi|^2d(x,\eta)+\int_\Omega b^T\nabla\phi\,\phi\,d(x,\eta).$$

Thanks to the identity

$$\int_\Omega b^T\nabla\phi\,\phi\,d(x,\eta)=\frac{1}{2}\int_\Omega b^T\nabla(\phi^2)d(x,\eta)=-\frac{1}{2}\int_\Omega\operatorname{div}b\,\phi^2d(x,\eta),$$

together with the assumption and the definition of c_p this may be further estimated as

$$\begin{aligned}a(\phi,\phi)&\geq\mu_-\int_\Omega|\nabla\phi|^2d(x,\eta)-\frac{1}{2}\int_\Omega\operatorname{div}b\,\phi^2d(x,\eta)\\&=\left(\left(\mu_--\frac{\alpha_0}{c_p}\right)+\frac{\alpha_0}{c_p}\right)\int_\Omega|\nabla\phi|^2d(x,\eta)-\frac{1}{2}\int_\Omega\operatorname{div}b\,\phi^2d(x,\eta)\\&\geq\frac{\alpha_0}{c_p}\int_\Omega|\nabla\phi|^2d(x,\eta)+\int_\Omega\left(\left(\frac{\mu_-}{c_p}-\alpha_0\right)-\frac{\operatorname{div}b}{2}\right)\phi^2d(x,\eta)\\&\geq\frac{\alpha_0}{c_p}\int_\Omega|\nabla\phi|^2d(x,\eta).\end{aligned}$$

The equivalence of the norms $||\nabla\phi||_{L_2(\Omega)}$ and $||\phi||_{H^1(\Omega)}$ finally completes the proof. □

6.1.4 Remark

Condition (6.15) *to ensure coercivity of* a *is obviously true if* div b *is non-positive which is equivalent to the relation*

$$2\frac{(u_x)^2}{1+u}\leq u_{xx}$$

pointwise on I*. However, this inequality is only meaningful if* u_{xx} *is non-negative. In the stationary case with* $f\equiv 1$ *the evolution equation becomes*

$$-u_{xx}=-\lambda\left(\varepsilon^2\big(\psi_x(x,u)\big)^2+\big(\psi_z(x,u)\big)^2\right),\quad x\in I,$$

whence u_{xx} *is obviously non-negative. In the more general case of a varying permittivity profile Section 5.1 provides conditions which ensure the non-positivity of the according right-hand side.*

However, even if the bilinear form a *is in general not coercive it is proved in Section 3.1 that the elliptic moving boundary problems* (6.6)–(6.7) *and* (6.8)–(6.9) *possess locally in time existing unique solutions.*

Having the definition of the bilinear form a at hand and using the notation $(\cdot,\cdot)_{L_2(\Omega)}$ for the $L_2(\Omega)$-inner product, a suitable variational formulation[2] of (6.8)–(6.9) reads as follows:

[2]Note that the above proceeding does not uniquely lead to the bilinear form a defined in (6.11). In (6.10) the

$$\text{Find } \phi \in H_0^1(\Omega) \text{ such that } \quad a(\phi,\mu) = \big(F_u,\mu\big)_{L_2(\Omega)} \quad \text{for all } \mu \in H_0^1(\Omega).$$

Since $a(\eta f,\mu) = \big(F_u,\mu\big)$ we may equivalently use the variational formulation

$$\text{Find } \varphi \in H^1(\Omega) \text{ such that } \quad a(\varphi,\mu) = 0 \quad \text{for all } \mu \in H_0^1(\Omega) \quad \text{and} \quad \varphi - \eta f \in H_0^1(\Omega).$$

Discretisation. In order to apply the finite-element method we start by partitioning the given domain Ω into finitely many subdomains, also called *elements*. More precisely, we consider the uniform and regular partition

$$\mathcal{R}_N := \{R_1, R_2, \ldots, R_N\}$$

of $\Omega = I \times (-1,0)$ into $N = N_x N_\eta$ rectangles of horizontal length $h_x = 2/N_x$ and vertical length $h_\eta = 1/N_\eta$.

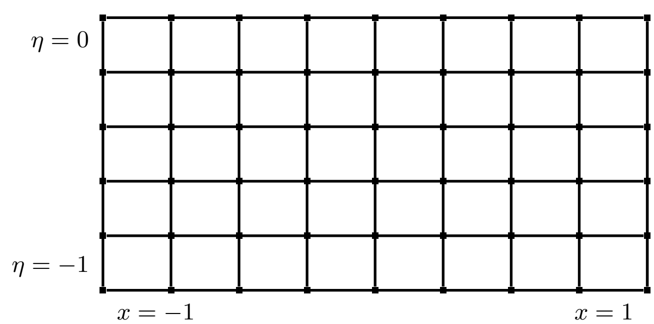


Figure 6.1: Partition of Ω with $N_x = 8$ and $N_\eta = 5$.

Referring to

$$\mathcal{P}_m := \Big\{\mu(x,\eta) = \sum_{\substack{k,l\geq 0,\\ k+l\leq m}} a_{kl} x^k \eta^l\Big\}$$

as the set of *polynomials of degree* $\leq m$ in two variables, we chose the D-dimensional subspace

$$V_N := \big\{\mu \in C(\overline{\Omega}); \mu|_R \in \mathcal{P}_1(x,\eta)\, \forall R \in \mathcal{R}_N \text{ and } \mu|_{\partial\Omega} = 0\big\}$$

of $H_0^1(\Omega)$ as *ansatz space*, with $D < \infty$. We are now in a position to define the *discrete variational formulation* of (6.8)–(6.9). It reads

$$\text{Find } \phi_N \in V_N \text{ such that } \quad a\big(\phi_N,\mu_N\big) = \big(F_u,\mu_N\big)_{L_2(\Omega)} \quad \text{for all } \mu_N \in V_N.$$

integration by parts of the mixed term, which includes $\phi_{x\eta}$, is performed with respect to the variable η. One might as well perform the integration with respect to x.

Given a basis $\{\gamma_1, \dots, \gamma_D\}$ of the ansatz space V_N, the Galerkin approximation ϕ_N may be written as

$$\phi_N(x,\eta) = \sum_{i=1}^{D} \phi_N(x_i,\eta_i)\gamma_i(x,\eta)$$

and is characterised by the identities

$$a\big(\phi_N, \gamma_j\big) = \big(F_u, \gamma_j\big)_{L_2(\Omega)}, \quad j = 1, \dots, D,$$

or equivalently by the equations

$$\sum_{i=1}^{D} \phi_N(x_i,\eta_i) a\big(\gamma_i, \gamma_j\big) = \big(F_u, \gamma_j\big)_{L_2(\Omega)}, \quad j = 1, \dots, D.$$

Thus, introducing the matrix

$$\mathbf{A}_N := [a(\gamma_i, \gamma_j)]_{i,j=1}^{D}$$

as well as the vectors

$$\mathbf{\Phi}_N := [\phi_N(x_j, \eta_j)]_{j=1}^{D} \quad \text{and} \quad \mathbf{F}_N := \Big[\big(F_u, \gamma_j\big)_{L_2(\Omega)}\Big]_{j=1}^{D}$$

we may finally state the linear system of equations

$$\mathbf{A}_N \mathbf{\Phi}_N = \mathbf{F}_N$$

which is to be solved in order to determine the Galerkin approximation.

6.2 | Approximate Solution of the Parabolic Evolution Problem

For the time being we assume $\varphi \in H^2(\Omega)$ to be given and devote this section to the computation of an approximative solution to the semilinear parabolic initial boundary value problem

$$\begin{aligned} u_t - u_{xx} = -\lambda\Bigg(&\varepsilon^2\Big(-\big(f_x(x,u)\big)^2 + \big(f_u(x,u)u_x\big)^2\Big) \\ &- 2\frac{1+\varepsilon^2(u_x)^2}{1+u} f_u(x,u)\varphi_\eta(t,x,1) + \frac{1+\varepsilon^2(u_x)^2}{(1+u)^2}\big(\varphi_\eta(t,x,1)\big)^2\Bigg), \quad t > 0,\ x \in I, \quad (6.16) \end{aligned}$$

$$u(t, \pm 1) = 0, \quad t > 0, \quad (6.17)$$

$$u(0,x) = u_*(x), \quad x \in I, \quad (6.18)$$

for the membrane's displacement u. The general approach is to firstly discretise with respect to time which yields to an elliptic boundary problem for each discrete instant of time. This elliptic problem is then solved by means of the Galerkin method as in Section 6.1.

For the sake of better readability we denote the right-hand side of the evolution equation by $-\lambda g_\varepsilon(u)$ so that (6.16) may be rewritten as

$$u_t - u_{xx} = -\lambda g_\varepsilon(u), \quad t > 0,\, x \in I.$$

The Variational Formulation. In order to obtain the variational formulation of the above problem we fix $t \in [0,T)$ and multiply the evolution equation (6.16) by a testfunction $v \in H_0^1(I)$. Integration over I with respect to x yields

$$\int_I (u_t - u_{xx}) v \, dx = -\lambda \int_I g_\varepsilon(u) v \, dx,$$

and via integration by parts

$$\int_I u_t v + u_x v_x \, dx = -\lambda \int_I g_\varepsilon(u) v \, dx.$$

Introducing the bilinear form

$$a(u,v) := \int_I u_x(x) v_x(x) \, dx$$

and the variational formulation corresponding to (6.16)–(6.18) reads

Find $u : (0,T) \to H_0^1(I)$ such that for all $t \in (0,T)$

$$(u_t(t), v)_{L_2(I)} + a(u(t), v) = -\lambda (g_\varepsilon(u(t)), v)_{L_2(I)} \quad \text{for all } v \in H_0^1(I),$$
$$u(0) = u_*.$$

One-Step Time Discretisation. In order to approximately solve the above stated variational formulation of (6.16)–(6.18) we subdivide the interval $[0,T)$ of existence[3] into N_t equidistant subintervals of length $\tau := T/N_t$, i.e. we have

$$0 =: t_0 < t_1 < \ldots < t_{N_t - 1} < t_{N_t} := T.$$

[3]If $T = \infty$ we consider τ to be given and introduce the time levels $t_n := n\tau$ for $n = 0, 1, 2, \ldots$. That is, we formally set $N_t = \infty$.

The solution is then computed by a so-called *one-step procedure*, i.e. the solution $u(t_i) \in H_0^1(I)$ at a time level t_i is computed by exploiting the knowledge about $u(t_{i-1}) \in H_0^1(I)$ at the previous time level t_{i-1}. More precisely, the term u_t is approximated by the difference quotient

$$u_t(t) \approx \frac{u(t_i) - u(t_{i-1})}{\tau}.$$

Furthermore, by introducing the parameter $\theta \in [0,1]$ and using the approximations

$$a\big(u(t_{i-1} + \theta\tau), v\big) \approx \theta a\big(u(t_i), v\big) + (1-\theta) a\big(u(t_{i-1}), v\big)$$

and

$$\big(-\lambda g_\varepsilon(u(t_{i-1} + \theta\tau)), v\big)_{L_2(I)} \approx \theta\big(-\lambda g_\varepsilon(u(t_i)), v\big)_{L_2(I)} + (1-\theta)\big(-\lambda g_\varepsilon(u(t_{i-1})), v\big)_{L_2(I)}$$

at a time $t_{i-1}+\theta\big(t_i - t_{i-1}\big) = t_{i-1}+\theta\tau$ between two time levels t_{i-1} and t_i we obtain the approximation

$$\begin{aligned}\big(u(t_i), v\big)_{L_2(I)} - \big(u(t_{i-1}), v\big)_{L_2(I)} &+ \tau \left(\theta a\big(u(t_i), v\big) + (1-\theta) a\big(u(t_{i-1}), v\big)\right) \\ &= \tau \left(\theta\big(-\lambda g_\varepsilon(u(t_i)), v\big)_{L_2(I)} + (1-\theta)\big(-\lambda g_\varepsilon(u(t_{i-1})), v\big)_{L_2(I)}\right)\end{aligned}$$

for all $v \in H_0^1(I)$. Eventually, we conclude that the approximate solution to (6.16)–(6.18) is computed by successively solving for each time level t_i, $i = 0, 1, \ldots, N_t$, a sequence of variational problems which read as follows:

Find $u_\tau(t_i) \in H_0^1(I)$ such that

$$\begin{aligned}\big(u_\tau(t_i), v\big)_{L_2(I)} &+ \tau\theta \left(a\big(u_\tau(t_i), v\big) - \big(-\lambda g_\varepsilon(u_\tau(t_i)), v\big)_{L_2(I)}\right) \\ &= \big(u_\tau(t_{i-1}), v\big)_{L_2(I)} - \tau(1-\theta) \left(a\big(u_\tau(t_{i-1}), v\big) - \big(-\lambda g_\varepsilon(u_\tau(t_{i-1})), v\big)_{L_2(I)}\right)\end{aligned}$$

for all $v \in H_0^1(I)$.

It is worthwhile to briefly contemplate different values of the parameter $\theta \in [0,1]$. For $\theta = 0$ the above approach yields the *explicit Euler scheme*

$$\big(u_\tau(t_i), v\big)_{L_2(I)} + \tau a\big(u_\tau(t_{i-1}), v\big) = \big(u_\tau(t_{i-1}), v\big)_{L_2(I)} + \tau\big(-\lambda g_\varepsilon(u_\tau(t_{i-1})), v\big)_{L_2(I)},$$

whereas the choice $\theta = 1$ leads to the *implicit Euler scheme*

$$\big(u_\tau(t_i), v\big)_{L_2(I)} + \tau a\big(u_\tau(t_i), v\big) = \big(u_\tau(t_{i-1}), v\big)_{L_2(I)} + \tau\big(-\lambda g_\varepsilon(u_\tau(t_i)), v\big)_{L_2(I)}.$$

The case $\theta = 1/2$ is here of special interest as it used to obtain the numerical results presented in Section 6.3. It leads to the Crank–Nicolson method in which the solution is determined according to the scheme

$$\big(u_\tau(t_i),v\big)_{L_2(I)} + \frac{\tau}{2}a\big(u_\tau(t_i),v\big) = \big(u_\tau(t_{i-1}),v\big)_{L_2(I)} - \frac{\tau}{2}a\big(u_\tau(t_{i-1}),v\big)$$
$$+\frac{\tau}{2}\Big(\big(-\lambda g_\varepsilon(u_\tau(t_i)),v\big)_{L_2(I)} + \big(-\lambda g_\varepsilon(u_\tau(t_{i-1})),v\big)_{L_2(I)}\Big).$$

In order to integrate this one-step time discretisation procedure into an applicable numerical scheme a space discretisation remains to be implemented. The corresponding relevant aspects are introduced in the subsequent paragraph.

Space Discretisation. For the spatial discretisation again the finite-element method is used. To this end we subdivide the interval $I = (-1, 1)$ into N_x subintervals of equidistant length $h := h_x$, i.e. we have

$$\mathcal{I} := \{I_1, I_2, \ldots, I_{N_x}\},$$

where each subinterval $I_k = [x_{k-1}, x_k]$, $k = 1, \ldots, N_x$, is of length $h = 2/N_x$. Note that for the sake of simplification we use the same values N_x and $h = h_x$ as for the discretisation of the elliptic problem in the previous section.

As finite-element space we chose the $(N_x - 1)$-dimensional subspace $V_h \subset H_0^1(I)$ as

$$V_h := \big\{v \in C\big([-1,1]\big); v|_I \in \mathcal{P}_1(x)\, \forall I \in \mathcal{I} \text{ and } v(\pm 1) = 0\big\}.$$

We are now in a position to formulate the *fully discrete variational formulation* of (6.16)–(6.18) at each time level t_i, $i = 0, 1, \ldots, N_t$. It reads

Find $u_{\tau,h}(t_i) \in V_h$ such that

$$\big(u_{\tau,h}(t_i),v_h\big)_{L_2(I)} + \frac{\tau}{2}\Big(a\big(u_{\tau,h}(t_i),v_h\big) - \big(-\lambda g_\varepsilon(u_{\tau,h}(t_i)),v_h\big)_{L_2(I)}\Big)$$
$$= \big(u_{\tau,h}(t_{i-1}),v_h\big)_{L_2(I)} - \Big(\frac{\tau}{2}a\big(u_{\tau,h}(t_{i-1}),v_h\big) - \big(-\lambda g_\varepsilon(u_{\tau,h}(t_{i-1})),v_h\big)_{L_2(I)}\Big).$$

for all $v_h \in V_h$.

In order to represent the Galerkin approximation and the ansatz functions we introduce the nodal basis given by $\gamma_k \in V_h$, $k = 1, \ldots, d$, with $d := N_x - 1$, such that

$$\gamma_k(x) = \begin{cases} (x - x_{k-1})/(x_k - x_{k-1}), & x_{k-1} \le x \le x_k, \\ (x_{k+1} - x)/(x_{k+1} - x_k), & x_k \le x \le x_{k+1}, \\ 0, & \text{elsewise.} \end{cases}$$

The Galerkin approximation $u_{\tau,h}$ may then be written as

$$u_{\tau,h}(x) = \sum_{k=1}^{d} u_{\tau,h}(x_k)\gamma_k(x)$$

and is at each time level t_i characterised by the identities

$$\begin{aligned}\big(u_{\tau,h}(t_i),\gamma_l\big)_{L_2(I)} + \frac{\tau}{2}a\big(u_{\tau,h}(t_i),\gamma_l\big) &= \big(u_{\tau,h}(t_{i-1}),\gamma_l\big)_{L_2(I)} - \frac{\tau}{2}a\big(u_{\tau,h}(t_{i-1}),\gamma_l\big)\\ &\quad + \frac{\tau}{2}\Big(\big(-\lambda g_\varepsilon(u_{\tau,h}(t_i)),\gamma_l\big)_{L_2(I)} + \big(-\lambda g_\varepsilon(u_{\tau,h}(t_{i-1})),\gamma_l\big)_{L_2(I)}\Big),\end{aligned}$$

for $l = 1,\dots,d$, or equivalently by the equations

$$\begin{aligned}&\sum_{k=1}^{d} u_{\tau,h}(t_i,x_k)\big(\gamma_k,\gamma_l\big)_{L_2(I)} + \frac{\tau}{2}\sum_{k=1}^{d} u_{\tau,h}(t_i,x_k)a\big(\gamma_k,\gamma_l\big)\\ &\qquad = \sum_{k=1}^{d} u_{\tau,h}(t_{i-1},x_k)\big(\gamma_k,\gamma_l\big)_{L_2(I)} - \frac{\tau}{2}\sum_{k=1}^{d} u_{\tau,h}(t_{i-1},x_k)a\big(\gamma_k,\gamma_l\big)\\ &\qquad + \frac{\tau}{2}\left[\left(-\lambda g_\varepsilon\left(\sum_{k=1}^{d} u_{\tau,h}(t_i,x_k)\gamma_k\right),\gamma_l\right)_{L_2(I)} + \left(-\lambda g_\varepsilon\left(\sum_{k=1}^{d} u_{\tau,h}(t_{i-1},x_k)\gamma_k\right),\gamma_l\right)_{L_2(I)}\right],\end{aligned}$$

for $l = 1,\dots,d$.

Similar as in the previous section we now define the *stiffness matrix* $\mathbf{K}$ and the *mass matrix* $\mathbf{M}$ as

$$\mathbf{K} := [a(\gamma_k,\gamma_l)]_{k,l=1}^{d} \quad \text{and} \quad \mathbf{M} := \left[(\gamma_k,\gamma_l)_{L_2(I)}\right]_{k,l=1}^{d},$$

respectively, as well as the vectors

$$\mathbf{u}_{t_i,h} := [u_{\tau,h}(t_i,x_k)]_{k=1}^{d} \quad \text{and} \quad \mathbf{G}\big(\mathbf{u}_{t_i,h}\big) := \left[\left(-\lambda g_\varepsilon\left(\sum_{k=1}^{d} u_{\tau,h}(t_i,x_k)\gamma_k\right),\gamma_l\right)_{L_2(I)}\right]_{l=1}^{d}. \tag{6.19}$$

We are then in a position to state the nonlinear system of equations

$$\left(\mathbf{M} + \frac{\tau}{2}\mathbf{K}\right)\mathbf{u}_{t_i,h}^T - \frac{\tau}{2}\mathbf{G}\big(\mathbf{u}_{t_i,h}\big) = \left(\mathbf{M} - \frac{\tau}{2}\mathbf{K}\right)\mathbf{u}_{t_{i-1},h}^T + \frac{\tau}{2}\mathbf{G}\big(\mathbf{u}_{t_{i-1},h}\big), \tag{6.20}$$

which is to be solved at each time level t_i in order to determine the Galerkin approximation of (6.16)–(6.18).

Finally, a justification for using the above described Crank-Nicolson method is left open. In this regard the following remark is formulated, c.f. for instance [5, 46].

6.2.1 Remark

Using piecewise linear elements the fully discrete one-step θ-method complies for $\theta \in [1/2, 1]$ with the estimate

$$\|u(t_i) - u_{\tau,h}(t_i)\|_{H^1(I)} \leq c(\tau + h)$$

for the global truncation error *if $u \in C^2\big([0,T], H_0^1(I)\big) \cap C\big([0,T], H^2(I)\big)$. Under stronger regularity assumptions on u the according estimate may even be improved in the case $\theta = 1/2$, i.e. for the Crank-Nicolson method. More precisely, it holds that*

$$\|u(t_i) - u_{\tau,h}(t_i)\|_{H^1(I)} \leq c(\tau^2 + h)$$

if $u \in C^3\big([0,T], H_0^1(I)\big) \cap C\big([0,T], H^2(I)\big)$. Given $\theta \in [0, 1/2]$ one obtains

$$\|u(t_i) - u_{\tau,h}(t_i)\|_{H^1(I)} \leq ch$$

under the additional assumption $\tau \leq c(\theta)h^2$ if $u \in C^2\big([0,T], H_0^1(I)\big) \cap C\big([0,T], H^2(I)\big)$. Note that the respective regularity assumptions on u are satisfied if the initial datum is sufficiently smooth.

Solving the Nonlinear System of Equations Via Newton's Method. In the context of this work the new iterate $u_{\tau,h}(t_i)$ at each time level t_i, $i = 1, \ldots, N_t$, is computed by solving the above derived nonlinear system of equations (6.20) via *Newton's method*. For this purpose we rewrite (6.20) in the form

$$\mathbf{F}\big(\mathbf{u}_{t_i,h}\big) := \Big(\mathbf{M} + \frac{\tau}{2}\mathbf{K}\Big)\,\mathbf{u}_{t_i,h}^T - \frac{\tau}{2}\mathbf{G}\big(\mathbf{u}_{t_i,h}\big) - \Big[\Big(\mathbf{M} - \frac{\tau}{2}\mathbf{K}\Big)\,\mathbf{u}_{t_{i-1},h}^T + \frac{\tau}{2}\mathbf{G}\big(\mathbf{u}_{t_{i-1},h}\big)\Big] = 0. \tag{6.21}$$

Note that the term in the squared brackets is a known quantity from the last preceding time level t_{i-1}. Due to the structure of the remaining two terms in (6.21) we obtain the Jacobian matrix

$$\mathbf{F}'\big(\mathbf{u}_{t_i,h}\big) = \Big(\mathbf{M} + \frac{\tau}{2}\mathbf{K}\Big) - \frac{\tau}{2}\mathbf{G}'\big(\mathbf{u}_{t_i,h}\big). \tag{6.22}$$

of $\mathbf{F}\big(\mathbf{u}_{t_i,h}\big)$ with respect to $\mathbf{u}_{t_i,h}^T$.

Newton's method applied to (6.21) at the time level t_i thus results in the following scheme, whose description is inspired by the notation used in MATLAB.

$\mathbf{u}_0 = \mathbf{u}_{t_{i-1},h}$;
$\mathbf{F}_0 = \mathbf{F}(\mathbf{u}_0)$;
$\mathbf{F}_k = \mathbf{F}_0$;

$k = 0$;

while $\|\mathbf{F}_k\|/\|\mathbf{F}_0\| > \kappa$

Compute Jacobian matrix $\mathbf{F}'\big(\mathbf{u}_k\big) = \big(\mathbf{M} + \frac{\tau}{2}\mathbf{K}\big) - \frac{\tau}{2}\mathbf{G}'\big(\mathbf{u}_k\big)$;

$\mathbf{s}_k = \mathbf{F}'\big(\mathbf{u}_k\big)\backslash\mathbf{F}_k$;

$\mathbf{u}_{k+1} = \mathbf{u}_k - \mathbf{s}_k$;

Compute potential $\mathbf{\Phi} = \mathbf{\Phi}\big(\mathbf{u}_{k+1}\big)$;

Evaluate right-hand side $\mathbf{G}\big(\mathbf{u}_{k+1}\big)$ at current $\mathbf{\Phi}\big(\mathbf{u}_{k+1}\big)$;

Evaluate nonlinear function $\mathbf{F}_{k+1} = \mathbf{F}\big(\mathbf{u}_{k+1}\big)$ at current iterate $\mathbf{u}_{k+1}$;

Replace $k = k + 1$;

end

$\mathbf{u}_{t_i,h} = \mathbf{u}_k$;

It is worthwhile to finally examine the computation of the Jacobian matrix in (6.22) carefully. More precisely, since the first summand $\big(\mathbf{M} + \frac{\tau}{2}\mathbf{K}\big)$ is known, we may focus our attention on the computation of $\mathbf{G}'\big(\mathbf{u}_{t_i,h}\big)$. To this end, we slightly modify the notation and write $g_\varepsilon\big(x, u, u_x, \varphi_\eta(x,1)\big)$ instead of $g_\varepsilon(u)$ in order express the actual dependence of g_ε on x, u_x and φ_η, where by reason of the coupling φ_η itself depends on u. Moreover, we abstain from mentioning the time level t_i explicitly in the notation. According to (6.19) the l-th component $\mathbf{G}_l\big(\mathbf{u}_{t_i,h}\big)$ of the vector $\mathbf{G}\big(\mathbf{u}_{t_i,h}\big)$ is hence given as

$$\mathbf{G}_l\big(\mathbf{u}_{t_i,h}\big) = -\lambda \int_I g_\varepsilon\big(x, u_{\tau,h}, u'_{\tau,h}, \varphi_\eta(x,1)\big)\gamma_l(x)\,dx.$$

Using the notation $u_k := u_{\tau,h}(t_i, x_k)$ we may then compute the entry $\big[\mathbf{G}'\big(\mathbf{u}_{t_i,h}\big)\big]_{lk} = \partial\mathbf{G}_l\big(\mathbf{u}_{t_i,h}\big)/\partial u_k$ of the tangent matrix $\mathbf{G}'\big(\mathbf{u}_{t_i,h}\big)$ as follows.[4]

$$\frac{\partial\mathbf{G}_l\big(\mathbf{u}_{t_i,h}\big)}{\partial u_k} = -\lambda \int_I \left(\partial_2 g_\varepsilon \gamma_k \gamma_l + \partial_3 g_\varepsilon \gamma'_k \gamma_l + \partial_4 g_\varepsilon \frac{\partial\varphi_\eta(x,1)}{\partial u_k}\gamma_l\right)dx.$$

It remains to consider the derivative $\frac{\partial\varphi_\eta(x,1)}{\partial u_k}$. For this purpose define $\beta(x,\eta) := \eta f\big(x, u(t,x)\big)$, $(x,\eta) \in \Omega$, and recall that $\varphi(x,\eta) = \phi(x,\eta) + \beta(x,\eta)$ depends on u due to the coupling, whence we may write

$$\frac{\partial\varphi}{\partial u_k} = \frac{\partial\phi}{\partial u_k} + \frac{\partial\beta}{\partial u_k}.$$

Following the assumption $\partial_u\partial_\eta\varphi(t_i,x,1) = \partial_\eta\partial_u\varphi(t_i,x,1)$ the idea is to firstly compute the derivative $\partial\phi/\partial u_k$. To this end recall that the approximate solution to the elliptic problem (6.6)–(6.7) is

[4] Note that $\partial_2 g_\varepsilon, \partial_3 g_\varepsilon$ and $\partial_4 g_\varepsilon$ denote the partial derivatives of g_ε with respect to its second, third and fourth component, respectively.

determined by the solution $\mathbf{\Phi}_N$ of the linear system

$$\mathbf{A}_N \mathbf{\Phi}_N = \mathbf{F}_N \tag{6.23}$$

of equations. Thus, we have $\mathbf{\Phi}_N = \mathbf{A}_N^{-1}\mathbf{F}_N$ and therefore

$$\frac{\partial \mathbf{\Phi}_N}{\partial u_k} = \frac{\partial \mathbf{A}_N^{-1}}{\partial u_k}\mathbf{F}_N + \mathbf{A}_N^{-1}\frac{\partial \mathbf{F}_N}{\partial u_k} = -\mathbf{A}_N^{-1}\frac{\partial \mathbf{A}_N}{\partial u_k}\mathbf{A}_N^{-1}\mathbf{F}_N + \mathbf{A}_N^{-1}\frac{\partial \mathbf{F}_N}{\partial u_k}.$$

Since the same discretisation with respect to x is used for both φ and u, at each η-discretisation level η_n, $n \in \{1, \dots, N_\eta + 1\}$ we have the representation

$$\phi(x, \eta_n) = \sum_{j=1}^{D} \phi(x_j, \eta_n)\gamma_j(x).$$

Therefore, the i-th component $[\mathbf{F}_N]_i$ of the vector $\mathbf{F}_N$ in (6.23) evaluated at $\eta = \eta_n$ is given by

$$[\mathbf{F}_N]_i = \int_I F_u(\eta_n, x, u)\gamma_i(x)\, dx,$$

whence

$$\left[\frac{\partial [\mathbf{F}_N]_i}{\partial u_k}\right] = \int_I \frac{\partial F_u(\eta_n, x, u)}{\partial u}\gamma_k(x)\gamma_i(x)\, dx. \tag{6.24}$$

Moreover, with $A = A(\eta_n)$ and $b = b(\eta_n)$ both evaluated at η_n,

$$[\mathbf{A}_N]_{lm} = \int_I \gamma_l'(x) A\big(x, u_{t_i,h}, u_{t_i,h}'\big)\gamma_m'(x) b\big(u_{t_i,h}'\big)^T \gamma_l'(x)\gamma_m(x)\, dx$$

is the lm-th element of the matrix $\mathbf{A}_N$ at $\eta = \eta_n$, whence we may compute the lmk-th element

$$\left[\frac{\partial [\mathbf{A}_N]}{\partial u_k}\right]_{lmk} = \int_I \gamma_l'(x)\left(\partial_2 A(x, u_{t_i,h}, u_{t_i,h}')\gamma_k(x) + \partial_3 A(x, u_{t_i,h}, u_{t_i,h}')\gamma_k'(x)\right)\gamma_m'(x) + \frac{\partial b^T}{\partial u_k}\gamma_k'(x)\gamma_l'(x)\gamma_m(x)\, dx$$

of the tensor of order three as the derivative of $\mathbf{A}_N$ with respect to u_k. Thus, we have

$$\frac{\partial \mathbf{\Phi}_N}{\partial u_k}$$

and it remains to calculate $\partial\vec{\beta}_N/\partial u_k$, with $\vec{\beta}_N$ denoting the vector of function values of β at the grid points, if

$$\vec{\varphi}_N := \mathbf{\Phi}_N + \vec{\beta}_N$$

is considered to determine the approximate solution φ_N corresponding to φ. It turns out that in fact $\partial\vec{\beta}_N/\partial u_k$ is already known from the computation in (6.24) as a consequence of the definition of β.

Finally, in the implementation the derivative of $\partial\varphi/\partial u$ with respect to η, evaluated at $\eta = 1$, is realised by the difference quotient

$$\frac{\partial\varphi_\eta(t_i, x, 1)}{\partial u} = \frac{1}{-h_\eta}\left(\frac{\partial\varphi}{\partial u}(t_i, x_k, 1 - h_\eta) - \frac{\partial\varphi}{\partial u}(t_i, x_k, 1)\right).$$

6.3 | Numerical Results

This section is devoted to the presentation of a selection of numerical results obtained by the above described procedure. According to the analytical investigation in the previous chapters the focus is not on the performance of the developed numerical methods but on illustrating the qualitative behaviour of solutions. In particular differences between the full coupled problem and the small-aspect ratio limit are revealed.

As in Chapter 2 we consider the general coupled system consisting of the semilinear evolution problem

$$u_t - u_{xx} = g_{\varepsilon,\lambda}(u), \quad t > 0,\ x \in I, \tag{6.25}$$

$$u(t, \pm 1) = 0, \quad t > 0, \tag{6.26}$$

$$u(0, x) = u_*(x), \quad x \in I, \tag{6.27}$$

together with the moving boundary problem

$$\varepsilon^2\psi_{xx}(x, u) + \psi_{zz}(x, u) = 0, \quad t > 0,\ (x, z) \in \Omega\big(u(t)\big), \tag{6.28}$$

$$\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)} f, \quad t > 0,\ (x, z) \in \partial\Omega\big(u(t)\big), \tag{6.29}$$

2 where the right-hand side $g_{\varepsilon,\lambda}(u)$ of the evolution equation (6.25) as well as the boundary condition (6.29) are specified corresponding to the choice of the permittivity profile f, being as usual either a function $f = f(x)$, $f = f(u(t, x))$ or $f = f(x, u(t, x))$.

The implementation is performed in MATLAB.[5] For all subsequently presented results the discretisation is implemented with $N_x = 80$ and $N_\eta = 40$. Furthermore, the numerical integration is realised by a Gauß–Legendre quadrature (see e.g. [46, 10]) with two Gauß points in each direction, i.e. we use two Gauß points per interval $\mathcal{I}_i$ and four Gauß points per rectangle $\mathcal{R}_i$.

[5] Version 8.4.0.150421 (R2014b).

Constant Permittivity. In this paragraph the coupled problem (6.25)–(6.29) is considered for a constant permittivity $f \equiv 1$. In this case the right-hand side of (6.25) is given by

$$g_{\varepsilon,\lambda}(u) = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2 + \big(\psi_z(x,u)\big)^2\Big), \quad t > 0,\, x \in I.$$

The solution of the coupled problem is compared to the solution of its reduced counterpart, where the potential is given by

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)}, \quad t \geq 0,\, (x,z) \in \overline{\Omega}(u(t)),$$

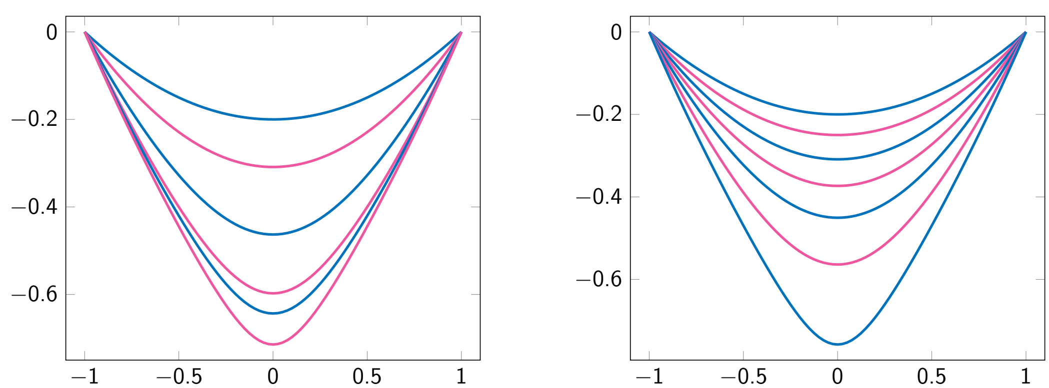
and the membrane's displacement satisfies the small-aspect ratio model

$$u_t - u_{xx} = -\lambda \frac{1}{\big(1+u(t,x)\big)^2}, \quad t > 0,\, x \in I, \tag{6.30}$$

$$u(t,\pm 1) = 0, \quad t > 0, \tag{6.31}$$

$$u(0,x) = u_*(x), \quad x \in I. \tag{6.32}$$

Figure 6.2, to be read top down, illustrates the time evolution of the membrane's displacement for the coupled as well as for the reduced problem. In both settings the initial deflection is chosen to be $u_*(x) = \big(x^2-1\big)/5$ and the value of the applied voltage amounts $\lambda = 0.6$. Although a singularity in form of a touchdown at $x = 0$ seems likely, this phenomenon may not yet be observed by the methods implemented in the framework of this thesis.



(a) Coupled problem (b) Small-aspect ratio model

Figure 6.2: Comparison of the coupled problem and the small-aspect ratio model for $f \equiv 1$ and $\lambda = 0.6$.

Howsoever, an improvement of the implementation is worthwhile so that the phenomenon of the *pull-in instability* becomes numerically visible.

A PERMITTIVITY PROFILE $f = f(x, u)$. We consider in this paragraph the coupled problem (6.25)–(6.29) for permittivity profiles $f = f\big(x, u(t, x)\big)$, whence the right-hand side of (6.25) is given by

$$g_{\varepsilon,\lambda}(u) = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2+\big(\psi_z(x,u)\big)^2\Big)+2\lambda\Big(\varepsilon^2\psi_x(x,u)f_x(x,u)+\psi_z(x,u)f_u(x,u)\Big), \quad t > 0,\ x \in I.$$

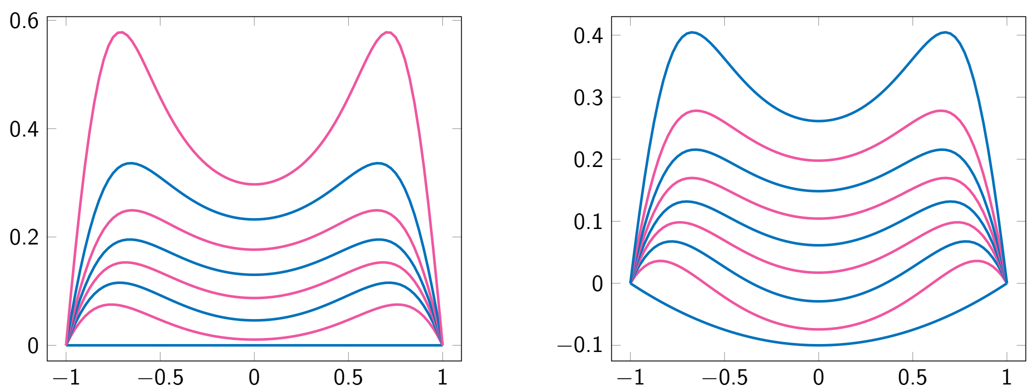
Figure 6.3 illustrates the evolution of the membrane's displacement for a permittivity profile

$$f\big(x, u(t,x)\big) := x^2\big(1 + u(t,x)\big)^4 + 0.1$$

and different initial data.

Reading from the bottom up we see in Figure 6.3(a) that, starting from the undeflected configuration $u_* \equiv 0$, the solution instantaneously becomes positive in all points except the boundary points $x = \pm 1$.

In Figure 6.3(b), to be read from the bottom up, we see that even if the initial deflection is negative (except in $x = \pm 1$), for instance if $u_* = (x^2 - 1)/10$ the solution u becomes positive.



(a) $u_* \equiv 0$ (b) $u_*(x) = (1 + x)^2/10$

Figure 6.3: Approximate solution $u_{\tau,h}$ for $\varepsilon = 0.25, \lambda = 0.5, f(x, u) = x^2(1 + u)^4 + 0.1$ and different initial data.

QUALITATIVE DIFFERENCES BETWEEN THE COUPLED PROBLEM AND THE CORRESPONDING SMALL-ASPECT RATIO MODEL FOR $f = f(x)$. The reduced small-aspect ratio model (6.33)–(6.35) is able to capture various qualitative properties of the coupled system (6.25)–(6.29), such as evenness with respect to $x \in I$, the existence of a pull-in voltage λ^*, as well as global existence for small values $\lambda < \lambda_*$ of the applied voltage. In the case of a constant permittivity profile even the sign property (of u) is conserved when reducing the model to the small-aspect ratio regime. Moreover, the sequence $(\psi_\varepsilon, u_\varepsilon)_\varepsilon$ of unique solutions to (6.25)–(6.29) converges to the unique solution (ψ_0, u_0) of (6.33)–(6.35) as ε tends to zero. However, this paragraph serves the purpose of providing numerical evidence which strengthens the conjecture that there are phenomena in the coupled system which cannot be observed in the small-aspect ratio model. More precisely, we specify permittivity profiles $f = f(x)$ leading to positive deformations u of the membrane in the coupled setting, whereas in stark contrast to that positivity of u_0 is impossible as a consequence of the maximum principle (c.f. the discussion in Section 5.1). The results presented in this paragraph are based on the work [12], jointly with Joachim Escher and Pierre Gosselet.

Given spatially varying permittivity profiles $f = f(x)$, we compare the coupled system (6.25)–(6.29), with $g_{\varepsilon,\lambda}(u)$ given by

$$g_{\varepsilon,\lambda}(u) = -\lambda\Big(\varepsilon^2\big(\psi_x(x,u)\big)^2+\big(\psi_z(x,u)\big)^2\Big) + 2\lambda\varepsilon^2\psi_x(x,u)f'(x), \quad t>0,\, x\in I,$$

to its reduced counterpart, where the potential may be explicitly stated as

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)}f(x), \quad t \geq 0,\, (x,z)\in \overline{\Omega}(u(t)),$$

and the membrane's displacement evolves according to the small-aspect ratio model

$$u_t - u_{xx} = -\lambda\left(\frac{f(x)}{1+u(t,x)}\right)^2, \quad t>0,\, x\in I, \tag{6.33}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{6.34}$$

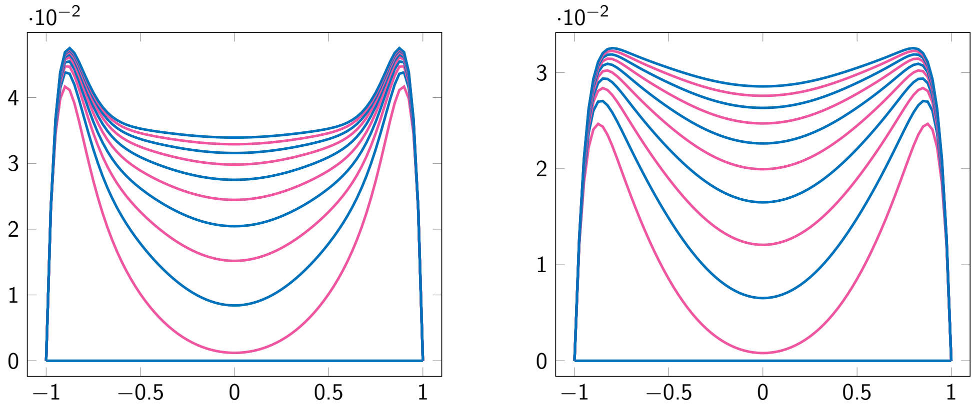
$$u(0,x) = u_*(x), \quad x\in I. \tag{6.35}$$

Figures 6.4 and 6.5 illustrate for a permittivity profile

$$f(x) := x^8 + 0{,}1$$

the approximate solution to (6.25)–(6.29) at different time levels for decreasing values of the aspect ratio ε and $\lambda = 1$. More precisely, the initial displacement is chosen to be $u_* \equiv 0$ and the remaining curves – to be read from bottom up – represent the approximate membrane's displacement at every tenth time level, where $T = 1$ and $N_t = 100$.

In the numerical experiments depicted in Figure 6.4 one may observe that for $\varepsilon \in \{0.4, 0.6\}$, starting from $u_* \equiv 0$, the solution immediately becomes positive at all interior points $x \in I$ and increases with time.

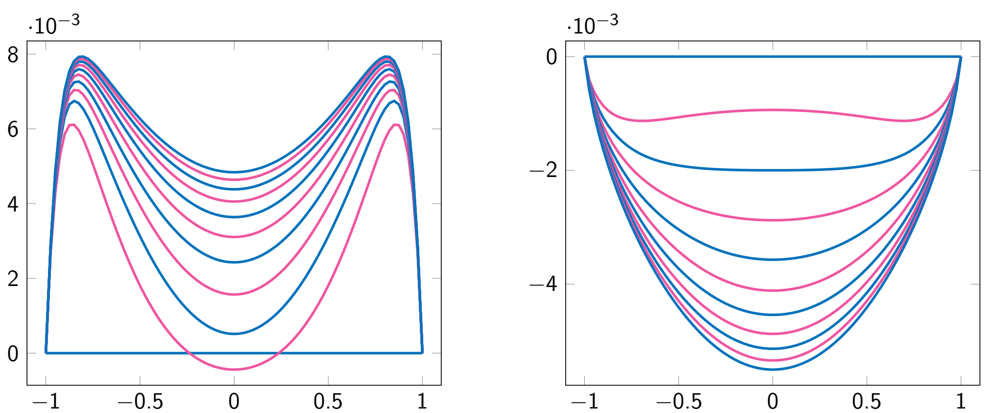


(a) $\varepsilon = 0.6$

(b) $\varepsilon = 0.4$

Figure 6.4: Approximate solution $u_{\tau,h}$ for $f(x) = x^8 + 0.1$ with $\lambda = 1$ and different $\varepsilon > 0$.

For $\varepsilon = 0.2, \varepsilon = 0.15$ and $\varepsilon = 0.1$ (c.f. Figure 6.5) the solution shows a different behaviour. Given $\varepsilon = 0.2$, the temporarily increasing solution develops also negative values at small time levels before it becomes strictly positive everywhere except in the boundary points (c.f. Figure 6.5(a)). In contrast to that is the evolution of the approximate solution for $\varepsilon = 0.1$ illustrated in Figure 6.5(b). Also starting from the zero displacement, the membrane abruptly deflects towards the ground plate in all points $x \in I$.



(a) $\varepsilon = 0.2$

(b) $\varepsilon = 0.1$

Figure 6.5: Approximate solution $u_{\tau,h}$ for $f(x) = x^8 + 0.1$ with $\lambda = 1$ and $\varepsilon \in \{0.1, 0.2\}$.

Moreover, the value $\varepsilon = 0.15$ seems to be a threshold value with regard to the evolution of the solution's sign. Starting from the initial deflection $u_* \equiv 0$, the magenta curve in Figure 6.6 illustrates the approximate solution at time level $t = 1/100$, whence the blue curve represents all further iterates at time levels $t = 10/100, 20/100, 30/100, \ldots$ until the maximal computing time $T = 2$ is reached. That is, the blue curve in Figure 6.6 might be a steady state for the aspect ratio $\varepsilon = 0.15$ and a value $\lambda = 1$ of the applied voltage. Since for $\varepsilon = 0.2$ the deflection becomes positive after a certain number of time steps and for $\varepsilon = 0.1$ the solution immediately becomes negative, it seems as if for $\varepsilon = 0.15$ the solution *is hesitant to decide whether it behaves according to the full coupled problem or according to the small-aspect ratio model.*

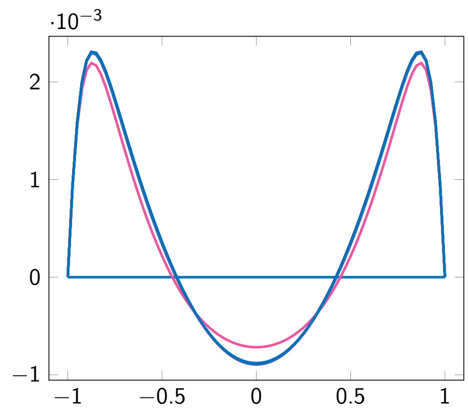
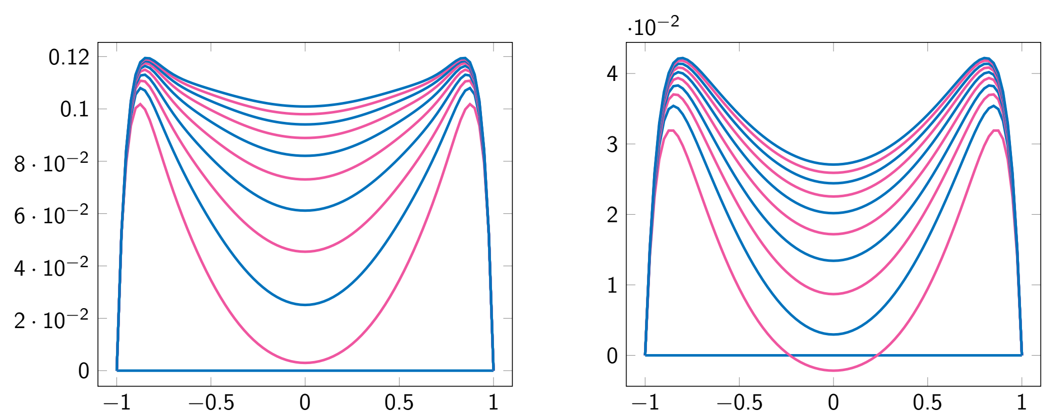


Figure 6.6: Approximate solution $u_{\tau,h}$ for $f(x) = x^8 + 0.1$ with $\lambda = 1$ and $\varepsilon = 0.15$.

For larger values λ of the applied voltage the situation is similar as one may see in Figures 6.7 and 6.8 that the displacement of the membrane evolves as for $\lambda = 1$ but with larger absolute function values.



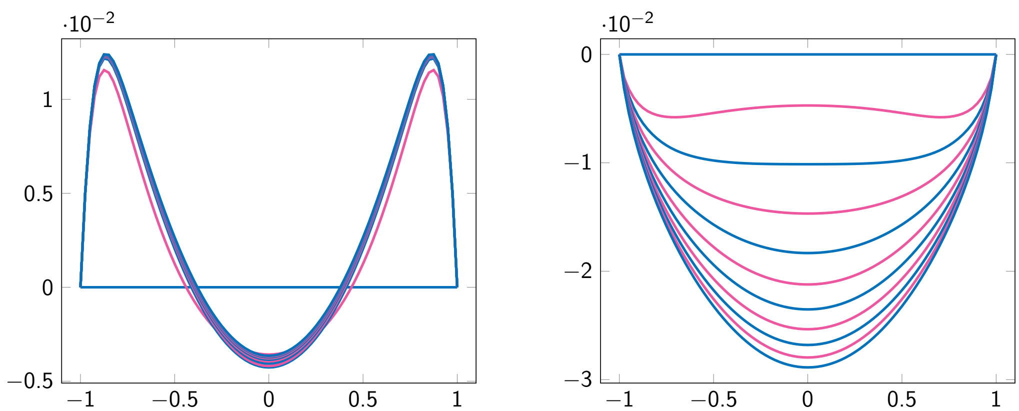
(a) $\varepsilon = 0.4$

(b) $\varepsilon = 0.2$

Figure 6.7: Approximate solution $u_{\tau,h}$ for $f(x) = x^8 + 0.1$ with $\lambda = 5$ and different $\varepsilon > 0$.

Figure 6.7, to be read from the bottom up, illustrates the time evolution of the membrane's displacement for $\lambda = 5$ and $\varepsilon \in \{0.2, 0.4\}$. As in the case $\lambda = 1$ Figure 6.7(a) shows that in the regime $\varepsilon = 0.4$ the solution instantaneously becomes positive at all interior points $x \in I$, whereas u initially takes also negative values when choosing $\varepsilon = 0.2$.

That also for $\lambda = 5$ in the setting $\varepsilon = 0.1$ the membrane instantaneously evolves towards to ground plate and that there seems to be a steady state for $\varepsilon = 0.15$ may be observed in Figure 6.8.

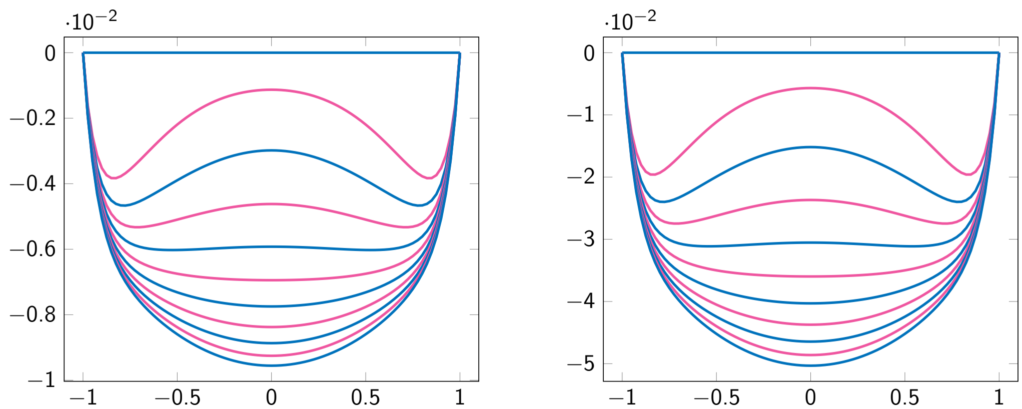


(a) $\varepsilon = 0.15$ (b) $\varepsilon = 0.1$

Figure 6.8: Approximate solution $u_{\tau,h}$ for $f(x) = x^8 + 0.1$ with $\lambda = 5$ and different $\varepsilon > 0$.

A question arising from the presented results might be if though u becomes *less positive* as ε decreases, the results hypothesise that, for every fixed $\varepsilon > 0$, one may find a voltage value $\lambda > 0$ and a permittivity profile $f = f(x)$ such that the solution u always becomes positive in the course of time. It is worthwhile to mention again that this is not possible in the small-aspect ratio regime.

In contrast to what is observed for the coupled system, Figures 6.9(a) and 6.9(b) show that the situation is rather different in the small-aspect ratio regime. In both figures the time evolution of the membrane's displacement is to be read top down. Choosing the same permittivity profile and applying voltages $\lambda \in \{1, 5\}$ as in Figure 6.4, 6.5 and 6.8, respectively, one may thus observe that in the small-aspect ratio regime the solution immediately becomes strictly negative in all interior points $x \in I$ and is decreasing in time.



(a) $\lambda = 1$

(b) $\lambda = 5$

Figure 6.9: Approximate solution $u_{\tau,h}$ to the small-aspect ratio model with $u_* \equiv 0$ for $f(x) = x^8 + 0.1$ and different λ.

The above illustrated results suggest that there is a change of the qualitative behaviour of the solution to the full coupled problem for a threshold value $\varepsilon > 0$ as ε tends to zero in the following sense. Starting from the zero initial displacement, in particular for *large* values of ε the membrane's displacement immediately becomes positive and is increasing in time (c.f. Figure 6.4). However, another interesting observation is illustrated in Figure 6.10.

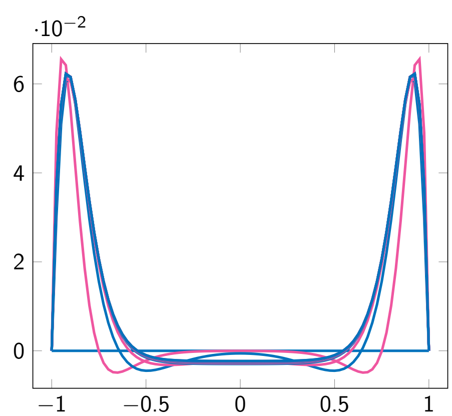


Figure 6.10: Approximate solution $u_{\tau,h}$ with $u_* \equiv 0$ for $\varepsilon = 0.85, \lambda = 1$ and $f(x) = x^8 + 0.1$.

It seems that not only for an aspect ratio ε tending to zero the qualitative behaviour of the solution differs in a way that increasing positive solutions become sign-changing and finally negative decreasing solutions. Figure 6.10 shows the approximate displacement of the membrane at different time levels for $\varepsilon = 0.85$. One may observe that the solution is sign-changing with positive peaks near the boundary points $x = \pm 1$ and non-positive/negative function values in a subinterval of I around $x = 0$.

6.3.1 Remark *(1) Theorem 4.0.9 apparently implies that given a positive permittivity profile $f = f(x)$, there exists an $\varepsilon_0 > 0$ such that $u(t, x) \leq 0$ on $[0, T) \times [-1, 1]$ for all $\varepsilon \in (0, \varepsilon_0)$.*

(2) The above presented numerical results suggest the supposition that there are $\varepsilon > 0, \lambda > 0$, and smooth permittivity profiles $f = f(x) > 0$ such that $u(t, x) > 0$ for all $x \in I$ and all $t \in (0, T)$. If this is true in view of Theorem 5.2.1 there must be $\varepsilon > 0$, $\lambda > 0$, and $f = f(x)$ such that a touchdown of the membrane in finite time is impossible. This means in particular that either the corresponding solution exists globally in time or a blow-up of the $W_q^2(I)$-norm takes place in finite time.

(3) The phenomena observed in the above illustrations are not restricted to the permittivity profile $f(x) = x^8 + 0.1$. Similar results may be obtained for instance if $f(x) = x^{2k} + 0.1$ with $k \in \{1, 2, 3, \ldots\}$ or if $f(x) = \exp(ax^2)$ with $a \in [1, 3]$.

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