Essays on Model Risk - The Role of Volatility for the Accuracy of Financial Risk Models

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Johannes Rohde
Hannover, November 2015

Abstract

By entry into force of the first two Basel Accords, financial institutions within major economies are urged to implement internal risk models in order to assess their exposures to credit risk and market risk. The requirement for an accurate modeling of financial risk resulted in the emergence of a new category of risk, which is induced by the usage of models and is termed *model risk*. Since volatility constitutes an integral component of each risk model, this thesis addresses the role of the volatility within different fields of financial risk management and examines the consequences that arise from an inaccurate representation of the volatility in financial risk models.

After Chapter 1 briefly introduces into the subject of the thesis, Chapter 2 deals with the computation of the credit default risk of an indebted firm and its classification in rating categories. On the basis of the Merton (1974) structural model, the volatility of the underlying equity is assumed to follow a process of the GARCH class of models. By computing probabilities of default for firms of the German DAX 30, it is shown that the disregard of specific characteristics of financial data may result in a different credit rating. Moreover, the impact of the type of the conditional distribution on the credit rating category is emphasized.

Chapter 3 provides an examination of the problems of the most common backtesting procedures for the evaluation of Value at Risk measures in view of regulatory aspects. By conducting a simulation study, standard approaches are compared with each other as well as with a procedure in which the volatility is corrected for estimation risk. The general results indicate that duration-based tests feature lower size distortions than frequency-based approaches. Even though the distortions can be reduced by accounting for the presence of estimation risk, the volatility-adjusted procedure still features significant oversized results.

In Chapter 4, a loss function-based framework for the comparison of the sensitivity of quantile risk measures with regard to a structural break in the volatility is developed. Using two types of loss functions, the theoretical results generally indicate that the lowest of the compared risk quantiles features the best responsiveness to the occurrence of a volatility break. Assuming various DGPs, different intensities of the break as well as realistic evaluation horizons, this result is confirmed within a broad comparative simulation study between Value at Risk and Expected Shortfall. An empirical application using data of several stock market indices additionally demonstrates the superiority of Expected Shortfall over Value at Risk.

Keywords: Credit risk, Market risk, Backtesting, Volatility break

Zusammenfassung

Seit Inkrafttreten der beiden ersten Baseler Eigenkapitalvereinbarungen sind Finanzinstitutionen großer Volkswirtschaften zur Verwendung interner Risikomodell angehalten um auf deren Basis ihre Kredit- und Marktrisiken zu bestimmen. Die Notwendigkeit einer möglichst präzisen Modellierung des Finanzrisikos hat zur Entstehung einer neuen Risikokategorie beigetragen, die aus der Verwendung von Modellen resultiert und gemeinhin als *Modellrisiko* bezeichnet wird. Angesichts der Tatsache, dass die Volatilität stets einen Hauptbestandteil eines jeden Risikomodells darstellt, setzt sich diese Dissertation mit der Rolle der Volatilität in verschiedenen Bereichen des Risikomanagements auseinander und untersucht die aus der Verwendung einer ungenauen Modellierung der Volatilität entstehenden Auswirkungen.

Während Kapitel 1 in die Grundthematik dieser Arbeit einführt, beschäftigt sich Kapitel 2 mit der Berechnung von Kreditausfallrisiken von Unternehmen und deren Klassifizierung in Ratingkategorien. Auf Basis des strukturellen Kreditrisikomodells von Merton (1974) wird dabei das Schwankungsverhalten des jeweiligen Eigenkapitals durch die Verwendung von Modellen bedingter Volatilität dargestellt. Indem Ausfallwahrscheinlichkeiten für Unternehmen des deutschen Aktienindex DAX 30 berechnet werden, wird aufgezeigt, dass die Vernachlässigung von für Finanzmarktdaten typischen Charakteristika zu einer Klassifizierung in eine abweichende Ratingkategorie führen kann. Außerdem wird auf den Einfluss der Art der bedingten Verteilung auf das Kreditrating eingegangen.

In Kapitel 3 werden die Probleme der gebräuchlichsten Backtestingverfahren zur Evaluation des Value at Risk hinsichtlich regulatorischer Vorschriften untersucht. In einer Simulationsstudie werden dabei Standardverfahren sowie eine alternative Herangehensweise, bei der die Volatilität zusätzlich vom auftretenden Schätzrisiko abhängig ist, miteinander vergleichen. Als generelles Resultat kann dabei festgehalten werden, dass auf der Zeitdauer zwischen zwei Unterschreitungen basierende Backtests geringere Verzerrungen der Size aufweisen als frequenzbasierte Backtests. Auch wenn die Verzerrung durch die Berücksichtigung des Schätzrisikos verringert werden kann, so weist auch die auf einer Varianzkorrektur basierende Testprozedur noch immer nach oben verzerrte Ergebnisse auf.

Ein auf Verlustfunktionen basierendes Modell zum Vergleich der Sensitivität von Quantilsrisikomaßen gegenüber Strukturbrüchen in der Volatilität wird in Kapitel 4 entwickelt. Dabei werden zwei verschiedene Arten von Verlustfunktionen unterstellt und theoretische Resultate hergeleitet, die eine zu bevorzugende Reaktionsfähigkeit des jeweils kleinsten Risikoquantils gegenüber einem Bruch in der Volatilität feststellen. Indem unterschiedliche datengenerierende Prozesse, verschiedene Bruchintensitäten sowie realistische Evaluationshorizonte unterstellt werden, können diese Ergebnisse auch in einer vergleichenden Simulationsstudie zwischen Value at Risk und Expected Shortfall zu Gunsten des letztgenannten Maßes bestätigt werden. In einer Anwendung auf Daten einiger großer Aktienindizes wird die Überlegenheit des Expected Shortfall nochmals aufgezeigt.

Schlüsselwörter: Kreditrisiko, Marktrisiko, Backtesting, Volatilitätsbruch

Contents

Chapter 1

Introduction

1 Introduction

Ever since the publication of the seminal works of mathematical finance in the 1960s and 1970s, such as by Mandelbrot (1963), Black and Scholes (1973) and Merton (1973), the quantitative measurement of financial risk has evolved as a field of significant importance to nearly all profit-seeking institutions. When mathematical models became available, the requirement for an accurate modeling of financial risk resulted in the emergence of a new category of risk. This comprises the risk which is induced by the usage of a model and is simply termed as model risk.

Several definitions of model risk have been offered in the course of an increasing literature on this field of research. Derman (1996) provides a rough specification by designating a model to be, at best, "a good scientific toy" which explains all the features that are most important to the user, but is incapable to depict every characteristic of the reality. A more detailed approach is presented by Kerkhof et al. (2010), who define estimation risk, misspecification risk, and identification risk to be potential sources of model risk and emphasize that capital reserves should depend on the reliability of the applied risk models. Sibbertsen et al. (2008) provide a statistical-based definition of model risk as being each type of risk that is caused by the application of a statistical model. In addition, they point out that the quantification of model risk demands a benchmark model by which the underlying model can be compared.

Since the amendments of the Basel Capital Accord became effective within the G-10 countries in 1998, financial institutions are allowed to use internal models for the assessment of capital requirements for both their exposures to credit risk and market risk, which arises from an institutions' trading activities. Following the outbreak of the subprime crisis of 2007 and 2008, for which the limited scope of the models used to value the credit status of mortgage borrowers has been blamed to be a key factor, an accurate risk management of financial institutions became a matter of public concern. As a result, statistical-based risk models are deemed to be indispensable for the institutions' decision making processes.

The volatility of the value of a financial instrument, which is closely linked to the perceived risk of an investor and the amount of uncertainty about future values, constitutes an integral component of each risk model. Since volatility is commonly considered to be the most sensitive parameter of a financial risk model, an appropriate measurement of the volatility is of crucial importance for the accuracy of the model employed. Moreover, Derman (2003) designates volatility to be the main driving factor of model risk, when it comes to the modeling of volatility smiles.

This thesis analyzes the role of the volatility within different fields of financial risk management. By considering several models for the assessment of risk or the evaluation of risk measures, it examines the consequences that arise from an inaccurate representation of the volatile components of the underlying model.

Chapter 2 deals with the computation of the credit default risk of an indebted firm and its classification in rating categories. The structural credit risk model proposed by Merton (1974) lays the groundwork for the quantitative assessment of a firm's credit risk in terms of its probability of default. Using this approach, the volatility of the underlying equity, which strongly affects the default probability, is assumed to follow a conditional volatility process. Since stock market data are well-known to feature specific characteristics, different types of GARCH models are considered in order to capture the respective properties, in particular the presence of leverage effects and long-range dependencies. In an empirical study using stock data of firms of the German DAX 30, default probabilities along with the corresponding credit ratings are calculated. In this process, the results are compared with credit ratings which are induced by the application of a conditional volatility model that disregards the specific properties of financial data. It becomes apparent that the ratings substantially differ in many cases. Hence, employing an incorrect model implies that the respective firms are classified in a different rating category. Moreover, the impact of the type of the conditional distribution on the credit rating is emphasized.

The further chapters examine aspects regarding the evaluation of risk measures. Financial institutions are required to compute minimum capital reserves subject to their credit risk, market risk, and operational risk since the Basel II regulations came into force. Since risk measures provide a tool to map profit and loss distributions to capital amounts (see Emmer et al. (2014)), the use of an adequate measure which produces robust risk estimates is of crucial importance for the institution as well as for the regulatory side. However, the accuracy of the methods used for evaluation depends on the appropriate specification of the volatility.

In particular, Chapter 3 focuses on the use of backtesting procedures for the evaluation of Value at Risk measures, which provide the preferred approach to assess market risk exposure by the second of the Basel Accords. However, the evaluation setting recommended therein entails significant statistical drawbacks when conducting backtests. For instance, a low number of violations of the estimated Value at Risk measure leads to heavy size distortions for most of the commonly used backtesting frameworks. In this chapter, different backtesting approaches are outlined and examined for these problems in view of regulatory aspects. By conducting a Monte Carlo study, the standard backtesting procedures are compared with the approach proposed by Escanciano and Olmo (2012). Within this backtesting framework, the volatility of the demeaned hit sequence is corrected for estimation risk, which describes the risk induced by the calculation of forecasts and provides a potential source of model risk. The results indicate that backtests which are based on the duration between two consecutive violations rather than on the plain hit sequence show the lowest size distortion, while even the tests accounting for estimation risk are not capable of significantly alleviating the distortions.

Due to several shortcomings of Value at Risk regarding mathematical and practical issues, the regulations of the Basel III accord mandate to replace Value at Risk as the preferred tool to compute market risk by Expected Shortfall by 2019. Chapter 4 provides a framework for both the theoretical comparison of quantile risk measures as well as a comparative evaluation of Value at Risk and Expected Shortfall in the presence of occasional structural breaks in the volatility of a profit and loss process, which represent a frequently documented characteristic of financial time series. Next to a break induced by the change of the variance of the innovation process, the possibility of a volatility break caused by a change in the innovation distribution is taken into consideration. By extending the approach introduced by Lopez (1998), a comparative evaluation technique is proposed which is based on the usage of loss functions of both a frequency type and a magnitude type. It can generally be derived that the risk measure on the basis of the lower of two quantiles features the higher responsiveness to a volatility break and is therefore superior by theoretical aspects in terms of the capability to identify the break. This result is confirmed within a broad comparative simulation study between Value at Risk and Expected Shortfall, for which different evaluation horizons, intensities of the volatility break as well as various DGPs for the modeling of the profit and loss series are assumed. An empirical application using data of several stock market indices additionally validates the findings and demonstrates the applicability of the proposed procedure.

Chapter 2

Credit Risk Modeling under Conditional Volatility

2 Credit Risk Modeling under Conditional Volatility

Co-authored with Philipp Sibbertsen

2.1 Introduction

Credit rating aims at the classification of credit applicants in rating categories. The accurate measurement of credit risk is of prime importance for the entire economic sector and equips rating agencies with significant power: Creditors are interested in an adequate credit rating that reflects the debtors' reliability, while borrowing firms strive for a preferably low interest on credits, which corresponds to a good rating, and a small amount of capital to keep in reserve, both of which are determined by their credit risk.

For a long time, the term *credit risk* featured only an abstract denotation. However, this changed since the enactments of the Basel II regulations issued by Basel Committee on Banking Supervision (2004a) mandatorily took effect in 2007 within the EU countries. One of the three pillars of Basel II addresses the maintenance of regulatory capital of credit institutes, between which in turn minimum capital requirements are imposed on a bank subject to its credit risk. Within the regulations, it is determined that corporate equity backing must depend on the probability of default of a firm. Thereby, credit risk becomes a quantifiable value which allows the evaluation of credit risk with quantitative methods.

The most popular approach to value credit risk in terms of probabilities of default involves the asset value model proposed by Merton (1974), which represents a generalization of the option pricing theory introduced by Black and Scholes (1973) and Merton (1973). In a commercial context, the Merton (1974) model was first applied in an adjusted form by Moody's KMV, which nowadays constitutes an industry standard tool for credit rating.

The probability of default commonly depends on a multiplicity of parameters. Among them, the most sensitive parameter, which severely reacts to extreme shocks and is therefore in the main focus of an investor's attention, is the volatility of the stock price, which directly affects the asset volatility and thereby also the probability of default. For this reason, it is of crucial interest to depict the stock volatility within the model framework in the most adequate way. The importance of the specification of volatility is referred to by Leland (2004), Jacobs and Li (2008), and Afik et al. (2012).

The well-known *stylized facts* refer to empirical findings in financial time series and comprise, among others, volatility clustering and leptokurtosis of returns, a negative correlation between past returns and future volatilities (the so-called leverage effect), and long-range dependencies (see Sewell (2011) for a comprehensive overview about characteristics of financial series). The presence of stylized facts within stock market time series constitutes an objective fact and is repeatedly proven, even for German stock market data (see, among others, Corhay and Rad (1994) and Sun et al. (2007)).

Several works exist which recognize the special role of volatility in credit risk valuation, but rather target to model the volatility as Itō stochastic process (see Heston (1993) for the most popular stochastic volatility approach and different extensions within the Merton framework such as Bu and Liao (2013)). Another strand of literature deals with implied volatilities, see, among others, the work by Hull, Nelken and White (2004), in which the parameters of the Merton model are estimated from options on the firm's stock.

However, while being considered when modeling stock market data, stylized facts are widely disregarded within the computation of credit risk. The main objective of this work is therefore to account for the existence of specific data characteristics by combining the Merton credit risk framework with conditional volatility models, which were primary introduced by Engle (1982). By employing conditional volatility models which use fractional integration, we allow shocks to die out at a hyperbolical rate and take account for the possibility of long-range dependencies within the conditional volatility equation as well. Furthermore, we show that the disrespect of leverage and long memory effects within the conditional volatility directly affects the credit rating of a firm. This in turn provides practical relevance regarding the resultant interest rate to be paid by the borrowing firm.

The remaining parts of this article are organized as follows: Section 2.2 presents Merton's structural approach to model corporate credit risk. Thereby, all relevant variables and determining factors of the underlying model are defined and a method to compute default probabilities is illustrated. In Section 2.3, several conditional volatility models (the GARCH class of models) are introduced which account for different stylized facts on financial market series. On the basis of German stock market data, the outlined approaches are combined in Section 2.4 in order to compute default probabilities and to quantify the risk of neglecting relevant properties of financial data. Section 2.5 concludes the article.

2.2 The Merton Credit Risk Model

Two approaches of credit risk modeling can be distinguished. On the one hand, the reduction approach derives the credit risk directly from the market price of corporate bonds, whereat the point of a firm's default can be considered as the first jump of a Poisson process, which (default) intensity is aligned to the given market values (see Duffie and Singleton (1994) for a more detailed overview of this model class). On the other hand, the most notable of the structural model approaches constitutes Robert Merton's (1974) credit risk model, which is based on the option pricing model proposed by Black and Scholes (1973) and Merton (1973). The main issue of this approach lies in the capital structure of a firm and in particular in the development of the firm's assets. Consequently, the possible default of the considered firm takes place endogenously and occurs if the firm's value falls short of a fixed boundary. Another advantage over the reduction approach, which assumes the default to be exogenous, is therefore the economic justification of default.

In order to introduce the Merton model, consider a firm whose capital structure contains an equity with a market value of E_t at time t. Moreover, the firm holds liabilities of constant face amount D, which only consist of a single debt taken up by a zero bond with debt maturity T. By assumption, the entire amount of liabilities has to be discharged at T without any priorities of order. [1] At maturity time $t = T$, the firm defaults if the firm's asset value A_t is too small to compensate its liabilities, i.e. $A_T < D$. Within this setting, it is assumed that the firm is conveyed to the creditors as soon as the credit is raised, while the firm is transferred back to the holders if the asset value is sufficiently large to repay the liabilities at T.

Thus, the holders possess a payoff function given by

$$\Lambda^H := \max\{0; A_T - D\}.$$

This is the same payoff structure as given by the long position of a European call option within the Black-Scholes model. Hence, the equity value can be considered to be a call option on the firm's asset value, $E(A_t, t)$. If the option is exercised by the firm holders, D is payed and debts are cleared, whereas D is considered to be the Black-Scholes strike price in the Merton setup. The firm's holders then earn $A_T - D$ for $A_T > D$ and zero otherwise, which is equivalent to the case in which the call is abandoned. Since all assumptions for a European call option are fulfilled, the Black-Scholes formula can be used to determine the value of the call, which represents the asset value in the specified setting.[2] Let $\tau = T - t$ denote the remaining time to maturity and $\Phi(\cdot)$ to be the $\mathcal{N}(0; 1)$ cumulative distribution function (cdf). Then, according to the Black-Scholes framework,

$$E(A_t, t) = A_t\, \Phi(v_1) - D\, \exp(-\mu_A\, \tau)\, \Phi(v_2) \tag{1}$$

depicts the equity value depending on t and the respective firm's asset A_t, whereby the inputs of the cdf's are defined by

$$v_1 = \frac{\ln\left(\frac{A_t}{D}\right) + \left(\mu_A + \frac{1}{2}\sigma_A^2\right)\tau}{\sigma_A\, \sqrt{\tau}} \tag{2}$$

and

$$v_2 = v_1 - \sigma_A\, \sqrt{\tau}. \tag{3}$$

The parameters $\mu_A \in \mathbb{R}$ and $\sigma_A > 0$ arise from the asset value process $\{A_t\}_{t \in \mathbb{R}_{\geq 0}}$, which follows (corresponding to Black-Scholes stock value) a Geometric Brownian Motion (GBM), solving the

[1] In addition, some of the usual assumptions in financial modeling are imposed, such as the absence of transaction costs or taxes as well as a constant risk-free interest rate.

[2] The situation from the creditors point of view can be considered by a payoff function of

$$\Lambda^C := \min\{D; A_T\} = D - \max\{0; D - A_T\},$$

i.e. D for $A_T > D$ or A_T if the firm defaults. If one takes a look at the latter term, it is quite interesting that $\max\{0; D - A_T\}$ is a measure for the credit risk of the creditors. It is zero in case that the firm does not default and takes the value $D - A_T$ in case of a default. As this depicts the payoff structure of a put option, the Black-Scholes formula for a European put option can likewise be used to calculate the credit risk.

stochastic differential equation (SDE)

$$dA_t = \mu_A \, A_t \, dt + \sigma_A \, A_t \, dW_t, \tag{4}$$

whereby $\{W_t\}_{t \in \mathbb{R}_{\geq 0}}$ is a standard Wiener process and μ_A depicts the expected return on assets. The diffusion parameter $\sigma_A > 0$ captures the level of the volatility of the asset value. By Itō's Lemma, the solution process for SDE (4) is given by

$$A_t = A_0 \exp\left((\mu_A - \frac{1}{2}\sigma_A^2)\, t + \sigma_A \, W_t \right).$$

The amount of credit risk can be derived from the Black-Scholes framework. A key figure for the valuation of the creditor's risk is the probability of the firm's default (PD), which occurs if the credit cannot fully be repaid at T. If one takes a look at the Gaussian cdf $\Phi(v_2)$, it is obvious that this specifies the probability for full repayment. Hence, the expression

$$PD := P(A_T < D) = \Phi(-v_2) = \Phi\left(\frac{\ln\left(\frac{D}{A_t}\right) - \left(\mu_A - \frac{1}{2}\sigma_A^2\right)\tau}{\sigma_A \sqrt{\tau}} \right) \tag{5}$$

denotes the probability of default by time T, whereat $\frac{D}{A_t}$ represents the debt financing ratio. Intuitively, increasing the debt financing ratio (thus meaning a higher amount of liabilities and a smaller asset value, resp.) leads to an increasing PD. Since the GBM A_t is log normal distributed, it follows that $\ln(A_t)$ follows a Gaussian distribution. Thus, $(\mu - \frac{1}{2}\sigma^2)\tau$ depicts the time-dependent expectation of the asset value, while $\sigma_A \sqrt{\tau}$ is the time-dependent asset volatility, increasing the probability of default for a high value of σ_A.

Within the Black-Scholes framework, $E(A_t, t)$ names the option value to be computed, depending on the observable stock price A_t. In contrast, the unobservable variable within the Merton approach is the asset value A_t (and thereby also its volatility σ_A), while the proportional equity value E_t is given by the stock price and thus represents a known value.

Since both variables are employed for the calculation of the PD (5), a system of equations depending on both variables needs to be solved prior to the computation of (5).

Using Itō's Lemma for the equity value $E(A_t, t)$, the equation

$$\sigma_E \, E_t = \frac{\partial E}{\partial A} \, A_t \, \sigma_A$$

holds (see Jones et al. (1984) for details), whereby σ_E is the instantaneous volatility of equity at time t. The derivative $\frac{\partial E}{\partial A}$ equals the European call option delta in the Black-Scholes framework. Thus,

$$\sigma_E = \Phi(v_1) \, \frac{A_t}{E_t} \, \sigma_A \tag{6}$$

forms the first part of the system of equations. Moreover, the Black-Scholes type formula for the equity value as given by (1), (2) and (3) is an equation in A_t and σ_A.

By solving (1) (in conjunction with (2) and (3)) and (6) for A_t and σ_A, the unobservable values can be obtained in order to compute the probability of default (5). The solution of this nonlinear system of equations of high grade demands the calculation of the parameters E_t, σ_E, μ_A, and the remaining time to maturity τ. Usually, the firm's stock price is used to model the equity value of the firm.

2.3 Modeling Conditional Volatility

The accurate modeling of the stock price volatility is of crucial relevance for the valuation of credit risk since high volatilities give rise to a high possibility of heavy amplitudes of the stock price process. Accounting for the stylized facts of financial time series (i.e. heteroskedastic volatilities along with volatility clustering, heavy tailed distributions of returns, the asymmetric response of conditional volatility to return shocks as well as the presence of long memory), the class of Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models needs to be employed in order to model the stock price volatility.

2.3.1 Symmetric and Asymmetric GARCH Models

The ARCH class of models proposed by Engle (1982) enables to describe the process volatility separately as a function of past squared innovations, $\varepsilon_{t-1}^2, \ldots, \varepsilon_{t-p}^2$. Employing Engle's ARCH model, Bollerslev (1986) remarked that a high lag order p cannot be avoided in order to obtain a good fit. Generalizing the work of Engle (1982), Bollerslev (1986) introduced the GARCH model, which allows the past variances to influence the instantaneous volatility as well.

Let $\{R_t\}_{t\in\mathbb{N}_0}$ be the mean process of a time series and assume $\{R_t\}$ to follow some ARMA(k,l) type process. Furthermore, let $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$ be the filtration generated by $\{R_t\}$, so that $\mathcal{F}_t = \sigma(R_s, s \leq t)$ applies. Then, the innovation process $\{\varepsilon_t\}_{t\in\mathbb{N}}$ follows a conditional distribution,

$$\varepsilon_t | \mathcal{F}_{t-1} \sim iid(0, \sigma_t^2), \tag{7}$$

depending on the information gathered by the past observations of the mean process. The conditional volatility of the residual process is then given by

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \, \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \, \sigma_{t-j}^2, \tag{8}$$

representing the GARCH(p,q) model, whereat $\omega > 0$, $\alpha_i \geq 0,\ i = 1, \ldots, p$ and $\beta_j \geq 0,\ j = 1, \ldots, q$ are imposed to ensure positivity of the conditional variance. However, Nelson and Cao (1992) show that positivity of (8) can be ensured without the non-negativity restrictions of the coefficients. The GARCH model features the stylized fact of volatility clustering as high values of elapsed conditional volatilities increase the probability to observe a high present conditional volatility. By transforming the GARCH(p, q) equation into its ARCH(∞) representation, it can easily be shown that an innovation observed infinitely long ago still influences the instantaneous variance by t. Bollerslev (1986) shows that (8) provides weak stationarity for $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

Since the past innovations influence the current volatility by its squared value, both negative and positive innovations have the same influence on (8). However, Black (1976) remarks that negative innovations cause a higher influence on the conditional volatility than positive ones. This is commonly known as the leverage effect, which is reasoned by a higher risk of default seized by the stock owners after a decreasing stock price as the liabilities D are constant and the ratio $\frac{D}{A_t}$ increases. This leads to a higher fluctuation of the stock price and a phase of high volatilities.

Ding, Engle and Granger (1993) generalize the GARCH model by accounting for the direction of the impact of the innovations. The assumption of the conditional variance, i.e. the squared volatility, to be the best method to model the conditional volatility is renounced and replaced by the volatility to the power of $\delta \in \mathbb{R}_{\geq 0}$. The conditional volatility of the Asymmetric Power ARCH (APARCH) of order (p, q, γ, δ) is then expressed by

$$\sigma_t^\delta = \omega + \sum_{i=1}^{p} \alpha_i \left(|\varepsilon_{t-i}| - \gamma_i \, \varepsilon_{t-i}\right)^\delta + \sum_{j=1}^{q} \beta_j \, \sigma_{t-j}^\delta. \tag{9}$$

The restrictions for the parameters α_i and β_j, $i = 1, \ldots, p, j = 1, \ldots, q$ are abided, while $\gamma_i \in (-1; 1), i = 1, \ldots, p$ is imposed on the leverage parameter to ensure positivity of (9). Besides, $\delta > 0$ is required. For $\gamma_i > 0$ negative innovations have a higher influence on the volatility than positive innovations (leverage effect). The power parameter δ describes a Box-Cox transformation of the volatility σ_t. Note that the GARCH model is nested by the APARCH model for $\delta = 2$ and $\gamma_i = 0 \, \forall i$.

By setting $\delta = 2$, it is assumed that the conditional volatility can be depicted best by the second centralized moment of $\{\varepsilon_t\}$, while the leverage effect is still taken into consideration. This case is covered by the GJR-GARCH introduced by Glosten et al. (1993), which imposes the restriction $\delta = 2$ within the APARCH conditional volatility (9). All further parameter restrictions stay the same as for the APARCH. Modeling a return series by GJR(p, q, γ), however, might rather be adequate if the innovations $\{\varepsilon_t\}$ follow a conditional Gaussian distribution. Within the work by Duan et al. (2006,) the GJR model is employed to represent the volatilities in option price models.

2.3.2 Long Memory GARCH Models

Another property which belongs to the well-known stylized facts on financial markets comprises the existence of a long term structure of dependence, i.e. innovations which occurred way back in the past still have a significant impact on present values of the process.

Within the mean equation the ARFIMA(k, d, l) model proposed by Granger and Joyeux (1980) accounts for the long term structure by introducing the memory parameter d, which represents the degree of persistence. Here, d is no longer restricted to be a natural number, but can embrace the set of real numbers. However, Harris and Nguyen (2011) refer to lots of empirical evidence for a more slowly declining autocorrelation function (ACF) of the past squared returns than a GARCH model, which is characterized by a geometrical decay of the ACF, could catch. Thus, modeling the long memory of the stock price only within the mean equation could not be sufficient since conditional volatilities may additionally be affected by past innovations.

When generalizing the GARCH model to allow for long term dependencies within the conditional volatility equation, it is practical to rewrite the GARCH conditional volatility equation (8) by its ARMA($p, \max(p, q)$)-in-squares form[3], which is given by

$$(1 - \alpha(L) - \beta(L))\,\varepsilon_t^2 = \omega + (1 - \beta(L))\,(\sigma_t^2 - \varepsilon_t^2) \tag{10}$$

when using the GARCH lag polynomial notation, whereby

$$\alpha(L) := \sum_{i=1}^{p} \alpha_i\, L^i \quad \text{and} \quad \beta(L) := \sum_{j=1}^{q} \beta_j\, L^j$$

as well as $L\sigma_t^2 = \sigma_{t-1}^2$ and $L\varepsilon_t^2 = \varepsilon_{t-1}$ applies. An alternative definition of the conditional variance of the GARCH equation (8) is then given by

$$\sigma_t^2 = \frac{\omega}{1 - \beta(L)} + \Theta(L)\varepsilon_t, \tag{11}$$

whereby $\Theta(L) := 1 - \frac{1-\alpha(L)-\beta(L)}{1-\beta(L)}$ holds. Note that each of the models introduced in the following are initially defined by the corresponding ARMA-in-squares representation for constructional reasons. We define the lag polynomial of the GARCH coefficients by

$$\varphi(L) = (1 - \alpha(L) - \beta(L))(1 - L)^{-d} \tag{12}$$

in order to obtain the Integrated GARCH (IGARCH) introduced by Engle and Bollerslev (1986) for $d = 1$ with

$$\varphi(L)\,(1 - L)\varepsilon_t^2 = \omega + (1 - \beta(L))(\sigma_t^2 - \varepsilon_t^2).$$

In contrast to the GARCH model, the IGARCH model comprises the possibility of a unit root for $1-\alpha(L)-\beta(L) = 0$. Nelson (1990) shows that the IGARCH unconditional volatility is infinite, while the first squared differences are stationary. Thus, the IGARCH model features infinite persistence, which, however, comprises commonly no property of financial series.

Baillie et al. (1996) provide the Fractionally Integrated GARCH (FIGARCH) model, which generalizes the degree of integration for the squared innovations to real numbers and is given by

$$\varphi(L)\,(1 - L)^d\varepsilon_t^2 = \omega + (1 - \beta(L))(\sigma_t^2 - \varepsilon_t^2) \qquad d \in \mathbb{R}, \tag{13}$$

[3]The order $\max(p, q)$ results from the dependence of the squared innovations from the GARCH coefficients.

whereat $\varphi(L)$ is defined by (12) for $d \in \mathbb{R}$. By transposition of (13) and definition of

$$\tilde{\omega} := \frac{\omega}{1-\beta(L)}$$

$$\text{and} \qquad \psi(L) := 1 - \frac{\varphi(L)}{1-\beta(L)}\,(1-L)^d,$$

the explicit form of the FIGARCH conditional volatility results in

$$\sigma_t^2 = \tilde{\omega} + \psi(L)\,\varepsilon_t^2, \tag{14}$$

whereby $d \in [0;1]$ and $\tilde{\omega} > 0$ ensure positive values of the conditional volatility. Further non-negativity restrictions are derived by Bollerslev and Mikkelsen (1996). Note that (14) depicts an ARCH(∞) representation with lag polynomial $\psi(L) = \sum_{i=1}^{\infty} \psi_i\, L^i$. For $d = 0$ and $d = 1$, FIGARCH results in GARCH and IGARCH, respectively.

Robinson (1991) uses the dissolved lag polynomial representation of $\psi(L)$ to show that the coefficients ψ_i for $d \in (0;1)$ decrease hyperbolically if $\forall i : \psi_i \geq 0$ holds. Baillie et al. (2007) remark that the series is sufficiently flexible to allow for slower hyperbolic rates of decay of the ACF, if d is an element of the relevant interval.

However, the unconditional variance of the FIGARCH model, given by

$$E[\varepsilon_t^2] = \frac{\tilde{\omega}}{1-\psi(1)}, \tag{15}$$

is infinite for values of $d \in (0;1)$. By developing the arguments of Nelson (1990), it is alleged by Baillie et al. (1996) that despite the lack of weakly stationarity the FIGARCH process is strongly stationary and ergodic. For a proof, see Caporin (2002). Kazakevicius and Leipus (1999) formulate a necessary condition for weak stationarity in the existence of summable ψ_i coefficients.

It has to be remarked that the properties of d varying in the range of $[0;1]$ is contrary to the modeling of the mean equation with an ARFIMA model since memory becomes shorter for the FIGARCH case when d is increasing. Consequently, it follows that for lower values of d, a longer memory is observed. Davidson (2004) refers this property to be counterintuitive as for the transition from $d \to 0$ to $d = 0$ memory jumps from infinite long memory to the short memory GARCH case and by transition from $d \to 1$ to $d = 1$ from short memory to infinite persistence (the IGARCH case). The reason for this finding is caused by the lag operator $(1-L)$ since it is connected to the squared residuals in the FIGARCH case (see (13)), while the lag operator is tied to the process values for the ARFIMA model.

Allowing again for asymmetric effects without neglecting long memory, the features of the APARCH and the FIGARCH model are combined within the Fractional Integrated Asymmetric Power ARCH (FIAPARCH) model developed by Tse (1998). The parameters $(\omega, p, d, q, \gamma, \delta)$ determine the model volatility, which is given pursuant to the ARMA-in-squares representation of the FIGARCH model

(13) by

$$\varphi(L)\,(1-L)^d\,(|\varepsilon_t|-\gamma\varepsilon_t)^\delta=\omega+(1-\beta(L))\left((|\varepsilon_t|-\gamma\varepsilon_t)^\delta-\varepsilon_t^\delta\right). \tag{16}$$

In analogy to the FIGARCH model, the explicit form of the conditional volatility can be written as

$$\sigma_t^\delta=\tilde{\omega}+\psi(L)\;(|\varepsilon_t|-\gamma\varepsilon_t)\,,$$

whereat $\delta>0$, $\forall i\in 1,\dots,q:\gamma_i=\gamma\in(-1;1)$, $d\in[0;1]$, $\tilde{\omega}:=\omega(1-\beta(L))^{-1}$, and $\varphi(L):=(1-\alpha(L)-\beta(L))(1-L)^d$ holds, while $\psi(L):=1-\left[\phi(L)(1-L)^d(1-\beta(L))^{-1}\right]$ represents the summarized back-shifted ARCH(∞) coefficients. Again, values of d, varying in $[0;1]$, ensure hyperbolic decreasing ACFs and strong stationarity (see Degiannakis (2004)). Correspondingly, weak stationarity is not achieved for $d\in(0;1)$. The parameter choice $\gamma=0$ an $\delta=2$ results in the FIGARCH alternative. Note that the FIAPARCH representation is exclusively able to picture the most frequently arising stylized facts within a single model: heavy tailed distribution of returns, volatility clustering, long memory, and asymmetric impacts of random shocks. A proof of weak stationarity of the FIAPARCH, however, is not available so far.

Combining the advantages of weak stationarity of the GARCH model and the ability of modeling long memory of the FIGARCH model, Davidson (2004) proposes the Hyperbolic GARCH (HYGARCH) model. By introducing the HYGARCH parameter η to the lagged squared residuals through the linear combination $((1-\eta)+\eta\,(1-L)^d)\varepsilon_t^2$, the ARMA-in-squares representation of the FIGARCH equation (13) results in

$$\varphi(L)(1+\eta[(1-L)^d-1])\varepsilon_t^2=\omega+(1-\beta(L))(\sigma_t^2-\varepsilon_t^2).$$

Thus, the explicit form of the conditional variance of the HYGARCH(p,d,q,η) model can be defined by

$$\sigma_t^2=\tilde{\omega}+\Xi(L)\varepsilon_t^2, \tag{17}$$

whereat $d\in[0;1]$, $\eta\in\mathbb{R}_{\geq 0}$, $\varphi(L):=(1-\alpha(L)-\beta(L))(1-L)^d$, $\Xi(L):=1-[\varphi(L)(1+\eta[(1-L)^d-1])(1-\beta(L))^{-1}]$, and $\tilde{\omega}:=\omega(1-\beta(L))^{-1}$ applies. By analogy with the FIGARCH case, (17) represents the ARCH(∞) form of the HYGARCH model, while $\Xi(L)\varepsilon_t^2$ represents the infinite sum of the lagged squared residuals (with coefficients Ξ_j). The HYGARCH model features weak stationarity under certain parameter restrictions and therefore existence of the variance.

Theorem. The HYGARCH model provides weak stationarity if both $1-\frac{\alpha(1)}{1-\beta(1)}>0$ and $\eta\in[0;1)$ hold.

Proof. Firstly, it is to show that the HYGARCH equation can be decomposed into a GARCH and a FIGARCH part. In continuation of the notation (see (11),(14) and (17)), we denote the

ARCH(∞) lag polynomials for GARCH, FIGARCH and HYGARCH, respectively, by

$$\begin{aligned}\Theta(L) &:= 1 - \frac{\varphi(L)}{1-\beta(L)}\\ \psi(L) &:= 1 - \frac{\varphi(L)(1-L)^d}{1-\beta(L)}\\ \Xi(L) &:= 1 - \frac{\phi(L)(1+\eta((1-L)^d-1))}{1-\beta(L)},\end{aligned}$$

whereby $d = 0$ holds for $\Theta(L)$. Then, it easily follows for $\Xi(L)$ by addition of an absolute zero

$$\begin{aligned}\Xi(L) &= \eta - \eta\frac{\phi(L)(1-L)^d}{1-\beta(L)} + (1-\eta) - (1-\eta)\frac{\phi(L)}{1-\beta(L)}\\ &= \eta\left(1 - \frac{\phi(L)(1-L)^d}{1-\beta(L)}\right) + (1-\eta)\left(\frac{\phi(L)}{1-\beta(L)}\right)\\ &= \eta\,\psi(L) + (1-\eta)\,\Theta(L).\end{aligned}$$

Apparently, for a higher value for η in this linear combination, we observe a higher influence of the long memory FIGARCH part at the expense of the short memory GARCH part.

Secondly, restrictions must be derived for which the process assures weak stationarity. Reminding of $E[\varepsilon_t] = 0\,\forall t$ and $Cov(\varepsilon_t, \varepsilon_{t-j}) = 0\,\forall t\,\forall j \in \mathbb{N}$ in the general case for the GARCH class of models, only $E[\varepsilon_t^2] = \dfrac{\tilde{\omega}}{1-\Xi(1)} < \infty$ is left to prove. For this purpose, consider

$$\Xi(1) = \sum_{i=1}^{\infty} \Xi_i = \eta\,\psi(1) + (1-\eta)\,\Theta(1)$$

and investigate the ARCH(∞) polynomials separately for covariance stationarity. Clearly, the GARCH polynomial provides weak stationarity if $\Theta(1) < 1$ is fulfilled (which is an alternative definition of the more common condition $\varphi(1) = 1 - \alpha(1) - \beta(1) > 0$ from the ARMA representation of the GARCH equation). However, since the FIGARCH model is not able to provide weak stationarity, $\psi(1) = 1$ for $d \in (0;1)$ must hold (see (15)). Thus,

$$\eta + (1-\eta)\,\Theta(1) < 1$$

is fulfilled, if

$$\Theta(1) = 1 - \frac{1-\alpha(1)-\beta(1)}{1-\beta(1)} = \frac{\alpha(1)}{1-\beta(1)} < 1 \tag{18}$$

holds and $\eta \in (0;1)$ generates a linear combination of the GARCH and the FIGARCH polynomial. Trivially, this is also true for $\eta = 0$ (GARCH case). Rewriting (18), the parameter restrictions for the HYGARCH model to be weak stationary result in

$$1 - \frac{\alpha(1)}{1-\beta(1)} > 0 \qquad \text{and} \qquad \eta \in [0;1). \tag{19}$$

□

Conrad (2010) points out that a weak stationary HYGARCH model under small modifications is possibly be obtained even for $\eta \geq 1$. Also note that an asymmetric version of the HYGARCH model is provided the HYAPARCH model proposed by Dark (2006), but is of less practical relevance.

2.4 Computing Default Probabilities

2.4.1 Data Description and Estimation Procedure

In this section, we want to bring together both the ideas of Merton's credit risk model and conditional volatility modeling with the GARCH class of models in order to compute probabilities of default (PD's) for a horizon of one year . We therefore consider daily stock data over an observation period from July 2002 to September 2007 of 24 firms which were part of German DAX 30 at that time, i.e. we observe 1370 trading days for each of the firms (with the exception of Lanxess, which stock market launch took place by February 2005, leaving only 695 observations here). Appendix A provides the plots of the log return series.

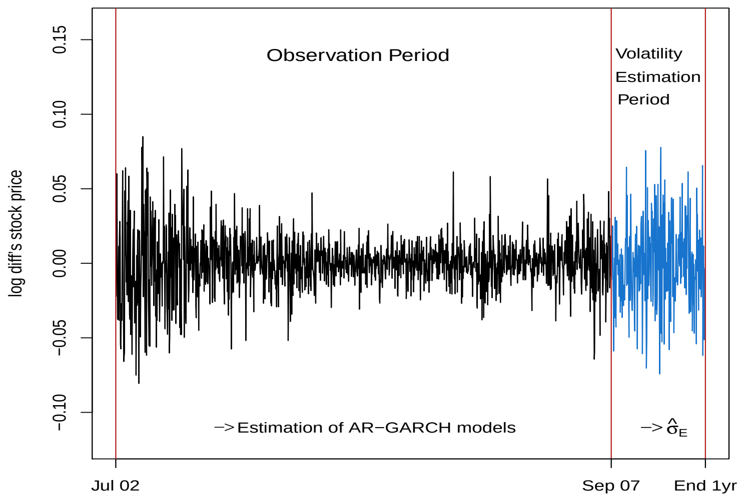


Figure 2.1: Visualization of the estimation procedure by means of the *Siemens* stock price log differences (Jul 02 - Sep 07) and one of 1,000 simulated trajectories over an one-year horizon generated by the best fitting DGP for the corresponding firm - AR(1)-GARCH(1,1) in this case (for parameter values see Appendix B).

In contribution of better understanding, the estimation procedure can be summarized as follows (see Figure 2.1): The first step comprises the estimation of different models of the GARCH class (GARCH, APARCH, GJR, FIGARCH, FIAPARCH, HYGARCH) from the observation period for the log differences of the stock price, which represents the proportional equity value. The DGP which describes the data best is then selected by the information criterion proposed by Hannan and

Quinn (1979). Subsequently, data for the selected model are simulated over the relevant horizon of one year, labeled as Volatility Estimation Period in Figure 2.1. By conducting 1,000 replications, the volatility parameter is then estimated from the simulated data. This parameter is needed to solve the non-linear system of equations represented by (1) and (6) in order to finally compute the PD's for firm i given by (5),

$$PD_i = \Phi\left(\frac{\ln\left(\frac{D_i}{A_{t,i}}\right) - \left(\mu_A - \frac{1}{2}\sigma_{A,i}^2\right)\tau}{\sigma_{A,i}\sqrt{\tau}}\right). \tag{20}$$

Note that μ_A may not be mixed up with the risk-free interest rate r, but denotes the expected return on assets, which has to be determined separately. Consistent with Campbell et al. (2008), we use a constant market risk premium $\mu_A = r + 0.06$, whereby $r = 0.04$ is the effective key interest rate set by ECB in June 2007. Several other approaches to determine μ_A exist. Some of them utilize the CAPM model (see Afik et al. (2012) for an overview), while Bharath and Shumway (2008) set the expected return assets equal to the stock return over the preceding year. The debt capital per share can be extracted from the annual business reports. However, it might fall short of considering only the short term debt as inauspicious developments could the firm require to preferentially serve long term credits. Therefore, as most of the more recent literature including the works by Bharath and Shumway (2008), Campbell et al. (2008), and Duffie et al. (2007), we use the KMV approach devised by Bohn and Crosbie (2003), for which the default barrier is composed of the short term debt plus half of the long term debt.

2.4.2 Results

For the estimation of the AR-GARCH models, let $R_t = \ln\left(\frac{E_t}{E_{t-1}}\right)$ be the log return of the stock prices E_t at time t. The mean equation of all models estimated in the following are represented by a simple AR(1) process, $R_t = \varrho R_{t-1} + \varepsilon_t$, whereby $\varepsilon_t = \sigma_t \nu_t$ with $\nu_t \sim iid(0; 1)\, \forall t$ holds and σ_t denotes the conditional volatility equation of the most suitable model. The usage of AR(1) for the mean can describe the observed log returns well and is in line with many other work on modeling finance data with AR-GARCH (see, among others, Ferenstein and Gasowski (2004)). Furthermore, in order to compare the effect on PD's resulting from the applied conditional distribution, we employ both a Gaussian and a Student-t distribution for all firms and models.

Different orders (p, q) for the GARCH part of all models are applied in the estimation process, but for the very most of cases the setting $p = q = 1$ outperforms all other combinations. Thus, only the models of GARCH order $(1, 1)$ with coefficients $\alpha := \alpha_1$ and $\beta := \beta_1$ are reported.

The full estimation results for the GARCH class of models for both assuming a Gaussian and a Student-t conditional distribution can be found in Appendix B. It is not surprising that a GARCH model is selected for only one firm (Siemens, which stock is commonly known for its stability and insensitivity for cycles) since typical properties of financial data are suppressed by the simple GARCH model. For the selected models, we mostly observe high significance for those parameters that indicate specific stylized facts (i.e. γ for the leverage effect (APARCH, GJR), d for long memory (FIGARCH) or both γ and d (FIAPARCH)) whenever the model features the effect in question. These results confirm that the well-known stylized facts need to be taken into account not only within the mean equation, but necessarily when modeling the conditional variance of stock market data as well. Notably, for the Gaussian conditional distribution, the HYGARCH parameter η is not significantly different from one in nearly each case, implying that the model falls back to the FIGARCH case, which is nested for $\eta = 1$. Assuming the Student-t conditional distribution, η clearly fails to be located within the interval that assures weak stationarity (see (19)). Thus, the HYGARCH model seems generally not to be appropriate to model stock market data.

Table 2.1 provides the selected models and the corresponding PD's for each firm when assuming a Gaussian and a Student-t conditional distribution within the volatility equation, respectively. In the majority of the cases, the selected models for both the conditional Gaussian and Student-t distribution are equal. For only nine firms, the best performing models are different, whereas only a marginal discrepancy exists for two of these firms as APARCH and GJR measure essentially the same effect. In contrast, for only one case, a rough deviance (APARCH vs. FIGARCH measuring different effects for Dt. Telekom) is observed. Note that the Student-t selected models always outnumber the Gaussian selected model by maximizing the Hannan-Quinn criterion (HQIC), which is in line with the findings by Corhay and Rad (1994).

Firm	Sel. Model & PD		Firm	Sel. Model & PD	
	Gaussian	*Student-t*		*Gaussian*	*Student-t*
Adidas	FIAPARCH 0.00004	FIGARCH 0.00006	E.ON	FIGARCH 0.00098	FIAPARCH 0.00095
Allianz	FIAPARCH 0.00000	FIAPARCH 0.00000	Fresenius MedCare	FIGARCH 0.00008	FIAPARCH 0.00014
BASF	APARCH 0.00015	APARCH 0.00016	Henkel	FIAPARCH 0.00013	FIAPARCH 0.00014
Bayer	GJR 0.00005	GJR 0.00003	Infineon	FIGARCH 0.00007	FIGARCH 0.00011
BMW	FIGARCH 0.00075	FIGARCH 0.00084	Lanxess	GJR 0.00009	GJR 0.00009
Continental	FIAPARCH 0.00029	FIAPARCH 0.00037	Linde	GJR 0.00018	APARCH 0.00006
Daimler	FIGARCH 0.00032	FIGARCH 0.00032	RWE	GJR 0.00025	GJR 0.00031
Dt. Bank	FIAPARCH 0.00104	GJR 0.00117	SAP	FIGARCH 0.00000	FIAPARCH 0.00000
Dt. Börse	APARCH 0.00037	GJR 0.00243	Siemens	GARCH 0.00012	GARCH 0.00017
Dt. Lufthansa	FIAPARCH 0.00043	FIGARCH 0.00050	ThyssenKrupp	FIGARCH 0.00045	FIGARCH 0.00048
Dt. Post	FIGARCH 0.01880	FIGARCH 0.01933	TUI	FIGARCH 0.00047	FIGARCH 0.00048
Dt. Telekom	APARCH 0.00070	FIGARCH 0.00072	Volkswagen	FIGARCH 0.00052	FIGARCH 0.00050

Table 2.1: Selected models and estimated PD's for DAX 30 firms for Gaussian and Student-t conditional distribution, respectively.

In most of the cases, the computed default probabilities are slightly higher for a Student-t conditional distribution than for a Gaussian, which can especially be compared when the selected models for one and the same firm are equal. Under identical conditions otherwise, this finding appears to be intuitive when comparing Gaussian and heavy tailed innovations. For three firms, we observe a higher PD for the Gaussian conditional distribution. It can also be derived from the results that those models which feature long memory tend to yield higher values of PD (of course, under validity of the assumption that equity quotas for two firms are nearly on an equal level, e.g. Henkel and Lanxess, Bayer and Infineon, Continental and RWE).

The next question arising is whether there is an effect on PD's when not the best model (selected by HQIC) is used to model the conditional volatility, but a "wrong" model. For this purpose, we employ the simple GARCH(1,1), insinuating to ignore special stylized facts such as leverage and long memory effects, one of which is found in nearly all data. The comparison between the selected and the GARCH model is exemplifically elaborated for the assumption of a Gaussian conditional distribution. Table 2.2 provides the PD's computed for both the actual selected model and under the assumption of GARCH innovations as well as the corresponding one year credit ratings as awarded by Standard & Poor's.

Firm	PD & Rating		Firm	PD & Rating	
	Selected Model	*GARCH*		*Selected Model*	*GARCH*
Adidas	0.00004 AAA	0.00001 AAA	E.ON	0.00098 A-	0.00085 A-
Allianz	0.00000 AAA	0.00000 AAA	Fresenius MedCare	0.00008 AAA	0.00000 AAA
BASF	0.00015 AA+	0.00017 AA+	Henkel	0.00013 AA+	0.00005 AAA
Bayer	0.00005 AAA	0.00005 AAA	Infineon	0.00007 AAA	0.00001 AAA
BMW	0.00075 A	0.00069 A	Lanxess	0.00009 AAA	0.00010 AA+
Continental	0.00029 AA	0.00020 AA	Linde	0.00018 AA+	0.00027 AA
Daimler	0.00032 AA-	0.00034 AA-	RWE	0.00025 AA	0.00031 AA-
Dt. Bank	0.00104 A-	0.00099 A-	SAP	0.00000 AAA	0.00000 AAA
Dt. Börse	0.00037 AA-	0.00258 BBB	Siemens	0.00012 AA+	-
Dt. Lufthansa	0.00043 A+	0.00050 A	ThyssenKrupp	0.00045 A+	0.00023 AA
Dt. Post	0.01880 BB-	0.01506 BB	TUI	0.00047 A+	0.00037 AA-
Dt. Telekom	0.00070 A	0.00070 A	Volkswagen	0.00052 A	0.00058 A

Table 2.2: Influence of "wrong" model on PD and S&P 1yr rating using Gaussian conditional distribution.

For those firms for which an APARCH/GJR was selected by HQIC, the PD's tend to be higher when the "wrong" GARCH model is used to model the conditional volatility (i.e. BASF, Bayer, Dt. Börse, Dt. Telekom, Lanxess, Linde, RWE). This effect is rather reverse for the models which account for long memory, even if not as distinct as for those which capture asymmetric reaction. This tendency might be explained by the fact that fractionally integrated conditional volatility models do not feature weak stationarity and therefore are prone to be explosive, although the very most of the estimated models are very mildly explosive if at all.

The impact resulting from the employment of the wrong model seems not to be decisive at first view. However, taking into consideration that the highest graded credit ratings are awarded only within an interval of $[0.0\%; 0.1\%]$ of PD and that a stock is already labeled to be speculative for a PD in excess of 0.94% (see Appendix C for an overview), the consequence from neglecting occurrent effects in stock data becomes more evident. At least for nearly 40% of the firms, the disregard of special characteristics of financial data entails a change of its credit rating. Four of these show a positive change in rating (Dt. Post, Henkel, ThyssenKrupp, TUI), while five firms are classified worse (Dt. Börse, Dt. Post, Lanxess, Linde, RWE). The degree of discrepancy yields one rating category each with the exception of ThyssenKrupp (improvement of two categories) and Dt. Börse, for which the degradation of five rating categories is striking. Certainly, all of these results come off by means of the S&P rating categorization - using a different classification of credit rating would possibly bring out different rating migrations as a result of which different firms could be affected.

For the sake of completeness, the empirical examination also involved constant stock price volatilities estimated from an AR(1) process. All of the results, however, yield significantly higher volatilities than under the assumption of conditional volatility which leads to higher PD's in consequence. This finding might be an explanation for the gap between the computed PD's and corresponding credit ratings and the actual rating of the firms in question, which tend to be worse than expectable under conditional volatility.

2.5 Conclusion

We combine the structural credit risk model proposed by Merton (1974) and the GARCH conditional volatility class of models in order to compute default probabilities in consideration of the presence of common characteristics of stock market data. This can be achieved by employing conditional volatility models which account for leverage effects and the existence of long memory, while credit risk is quantified by the probability of default of a firm subject to the Basel II regulations.

By applying this method to data of the German stock market, we thereby find strong evidence for the adequacy of conditional volatility models which are capable to capture specific properties as nearly all data sets contain leverage effects and/or long memory. One considered conditional volatility model using fractional integration (HYGARCH), whose weak stationarity is proved within the theoretical part of the article, turns out to be inappropriate to model stock market data.

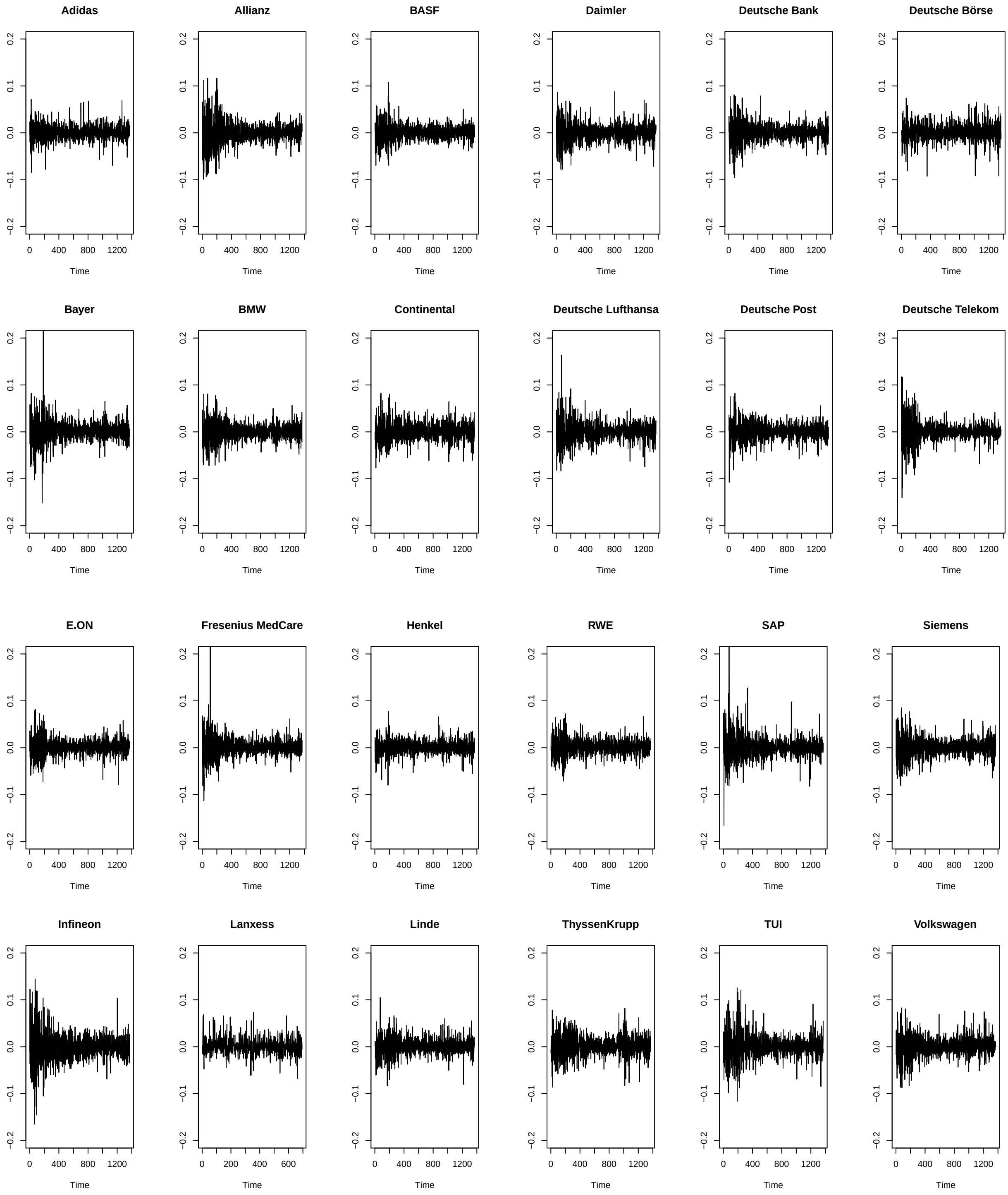
When computing one year default probabilities, slightly higher PD's result for the assumption of a conditional Student-t distribution than for imputing a Gaussian conditional distribution. In order to derive implications regarding the risk of neglecting specific stylized facts, we moreover examine the effects that arise from assuming a simple GARCH model instead of the selected model and obtain distinct credit ratings for one and the same firm in a considerable number of cases. The main finding therefore comprises the fact that the occurrence of specific characteristics of financial data needs to be considered not only within the mean equation of stock price series for the computation of PD's, but within the conditional volatility as well.

Practical relevance arises directly from the high share of discrepant ratings induced by the employment of an inferior model since credit ratings provide an indicating device for a firm's reliability and affect the interest rate which has to be paid out when raising a credit.

The computation of credit risk is a highly comprehensive topic as there are plenty of potential adjustable screws to rotate on. Along these lines, it would be reasonable to also implement conditional volatility within some of the large number of enhancements of the Merton approach. These include the *first passage* class, which assumes a time dependent exogenous default barrier, whereas default is possible to appear as stopping time before expiration (see Black and Cox (1976)), while Longstaff and Schwartz (1995) suggest the expected return to follow a stochastic process. Additionally, a more detailed empirical investigation which involves the influence of conditional volatility on mid and long term credit PD's would be important to determine the full credit risk that a firm has to bear. Even though the short term analysis already shows the importance of the consideration of the stylized facts, the examination of these issues remains an interesting topic for future research.

Appendix to Chapter 2

A Time Series Plots



B Estimation Results

Description: All estimates of the GARCH constant actually yield strictly positive values since with digits different from zero at least at sixth position after decimal point. - $(***), (**), (*)$ indicate significance of the coefficient to 1%, 5% and 10% level, respectively. - Note that $H_0 : \ln(\eta) = 0$ is tested for the HYGARCH parameters. - Highest HQIC values written in bold indicate the corresponding selected model. - ncr: No convergence reached for this model.

B.1 Results for the Assumption of a Gaussian Conditional Distribution

Adidas	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0444	0.0471	0.0470	0.0433	0.0478	0.0433
ω	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
d	-	-	-	0.1385***	0.2125***	0.0012**
α	0.0698*	0.0694***	0.0399	0.2409	0.1141	0.2186
β	0.8085***	0.8658***	0.8113***	0.3072	0.2592	0.2480
γ	-	0.5291**	0.0752*	-	0.7262**	-
δ	-	1.2496***	-	-	0.8862***	-
η	-	-	-	-	-	4.5396***
HQIC	5.647	5.650	5.649	5.651	**5.656**	5.649

Allianz	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0564**	0.05668*	0.0567**	0.0579**	0.0539*	0.0567**
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4345***	0.2262***	0.2126***
α	0.0794***	0.0754***	0.0427***	0.2612***	0.1552	0.2645*
β	0.9083***	0.9047***	0.9048***	0.5815***	0.2997*	0.4361**
γ	-	0.2452***	0.0741***	-	0.2092***	-
δ	-	2.0000***	-	-	2.5965***	-
η	-	-	-	-	-	1.3569
HQIC	5.311	5.321	5.319	5.314	**5.324**	5.313

BASF	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0372*	-0.03836	-0.0384	-0.0358	-0.0424	-0.0366
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3376***	0.1377	0.1608
α	0.0674***	0.0634***	0.0214*	0.2449**	0.1559	0.1807
β	0.9164***	0.9012***	0.9013***	0.04919***	0.2387	0.3264
γ	-	0.4179**	0.1060***	-	0.2670***	-
δ	-	2.0000***	-	-	2.8267***	-
η	-	-	-	-	-	1.4492
HQIC	5.784	**5.803**	5.802	5.783	5.780	5.781

Bayer	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0145	0.0140	0.0234	0.0144		0.0140
ω	0.0000***	0.0000***	0.0000***	0.0000***		0.0000***
d	-	-	-	0.8023***		0.8584***
α	0.0766***	0.8584***	0.0284***	0.1287		0.0971
β	0.9181***	0.0971	0.9399***	0.8482***		0.8694***
γ	-	0.8694***	0.9917***	-		-
δ	-	0.0056	-	-		-
η	-	-	-	-	-	0.9944
HQIC	5.263	5.309	**5.321**	5.261	ncr	5.258

BMW	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0149	0.0144	0.0152	0.0149	0.0155	0.0148
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4911***	0.04720***	0.4467***
α	0.0557***	0.0482***	0.0408***	0.2703***	0.2747***	0.2902***
β	0.9348***	0.9351***	0.9339***	0.7111***	0.7008***	0.6966***
γ	-	0.1502*	0.0276*	-	0.1436*	-
δ	-	2.1652***	-	-	1.9367***	-
η	-	-	-	-	-	1.0223
HQIC	5.566	5.562	5.565	**5.568**	5.564	5.564

Continental	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0138	-0.0101	-0.0144	-0.0127	-0.0100	-0.0128
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3435***	0.4705***	0.3713*
α	0.0727***	0.0731***	0.0313**	0.2195***	0.2692***	0.2157***
β	0.8976***	0.9086***	0.8957***	0.5021***	0.6980***	0.5164***
γ	-	0.5629***	0.0855***	-	0.7969*	-
δ	-	1.1405***	-	-	0.8736***	-
η	-	-	-	-	-	0.9740
HQIC	5.308	5.319	5.317	5.309	**5.324**	5.306

Daimler	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0079	0.0093	0.0091	0.0115	0.0119	0.0112
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3246***	0.3726***	0.5081*
α	0.0757***	0.0753***	0.0597***	0.1600*	0.1775**	0.1294
β	0.8932***	0.8953***	0.8951***	0.4633***	0.5237***	0.5736***
γ	-	0.0897	0.0260	-	0.0485	-
δ	-	1.8722***	-	-	1.7857***	-
η	-	-	-	-	-	0.9064*
HQIC	5.346	5.341	5.344	**5.348**	5.343	5.345

Deutsche Bank	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0488*	0.0464*	0.0464*	0.0539*	0.0494*	0.0543*
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4449***	0.4294***	0.5044**
α	0.0690***	0.0519***	0.0269**	0.2250***	0.2730***	0.2047**
β	0.9147***	0.9303***	0.9279***	0.6300***	0.6567***	0.6571***
γ	-	0.3240***	0.0608***	-	0.3168***	-
δ	-	1.9145***	-	-	1.7606***	-
η	-	-	-	-	-	0.9759
HQIC	5.536	5.540	5.542	5.539	**5.543**	5.536

Deutsche Börse	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0588*	0.0661***	0.0684**	0.0567*		0.0580*
ω	0.0000***	0.0000***	0.0000***	0.0000***		0.0000***
d	-	-	-	0.2928***		0.8308***
α	0.1251***	0.1139***	0.0688***	0.2103*		0.0562
β	0.7798***	0.8237***	0.7783***	0.3875***		0.6565***
γ	-	0.7052***	0.1250***	-		-
δ	-	0.5524***	-	-		-
η	-	-	-	-	-	0.8561*
HQIC	5.426	**5.449**	5.431	5.418	ncr	5.420

Deutsche Lufthansa	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0244	0.0277	0.0274	0.0380	0.0318	0.0359
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3945***	0.3393***	0.0017***
α	0.0408***	0.0327*	0.0168*	0.3674***	0.3818***	0.4217
β	0.9524***	0.9555***	0.9514***	0.6236***	0.5984***	0.4618
γ	-	0.4438	0.0475***	-	0.2864***	-
δ	-	1.8952***	-	-	2.0171***	-
η	-	-	-	-	-	107.7916***
HQIC	5.291	5.296	5.298	5.295	**5.301**	5.297

Deutsche Post	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0254	-0.0234	-0.0261	-0.0312	-0.0324	-0.0302
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4104***	0.3616***	0.2463
α	0.0405***	0.0471***	0.0449***	0.3249***	0.3434***	0.3618***
β	0.9495***	0.9730***	0.9503***	0.6701***	0.6499***	0.6321***
γ	-	-0.0334	-0.0076	-	-0.0393	-
δ	-	1.6654***	-	-	2.2019***	-
η	-	-	-	-	-	1.2246
HQIC	5.519	5.520	5.516	**5.523**	5.518	5.521

Deutsche Telekom	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0303	0.0304	0.0304	0.0381	0.0258	0.0366
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3770***	0.1476**	0.2645
α	0.0543***	0.0545***	0.0542***	0.2933***	0.3843***	0.3274***
β	0.9287***	0.9288***	0.9286***	0.6004***	0.4965***	0.5582***
γ	-	0.1733*	0.0006	-	-0.0127	-
δ	-	2.0304***	-	-	2.9276***	-
η	-	-	-	-	-	1.1417
HQIC	5.689	**5.697**	5.686	5.692	5.693	5.689

E.ON	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0386	-0.0398	-0.0398	-0.0422	-0.0411	-0.0401
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.8466***	0.9125***	0.9184***
α	0.0486***	0.0469***	0.0353***	0.2116***	0.1533*	0.1355*
β	0.9394***	0.9382***	0.9382***	0.9205***	0.9323***	0.9292***
γ	-	0.1428	0.0254	-	0.1481	-
δ	-	1.9979***	-	-	1.6992***	-
η	-	-	-	-	-	0.9882
HQIC	5.674	5.671	5.674	**5.675**	5.670	5.673

Fresenius MedCare	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0625**	-0.0476**	-0.0683**	-0.0570**	-0.0649**	-0.0694**
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.6206***	1.0000***	0.1875
α	0.0376***	0.0324***	0.0165*	0.4567***	0.1328**	0.7329***
β	0.9573***	0.9728***	0.9658***	0.8859***	0.9707***	0.8597***
γ	-	0.5021**	0.0301**	-	0.1689	-
δ	-	0.5307***	-	-	1.7987***	-
η	-	-	-	-	-	1.4977
HQIC	5.550	5.552	5.551	**5.5575**	5.557	5.556

Henkel	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0236	-0.0387	-0.0271	-0.0257	-0.0382	-0.0252
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.2815***	0.4018***	0.6001**
α	0.0722***	0.0620***	0.0109	0.4337***	0.3296***	0.3058**
β	0.8775***	0.9189***	0.8820***	0.6013***	0.6596***	0.7006***
γ	-	0.7606***	0.1060***	-	0.7775***	-
δ	-	0.8305***	-	-	0.8690***	-
η	-	-	-	-	-	0.8671*
HQIC	5.832	5.844	5.843	5.831	**5.848**	5.829

Infineon	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0175	0.0167	0.0173	0.0126	0.0122	0.0156
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4100***	0.3690***	0.1506
α	0.0654***	0.0612***	0.0557***	0.3743***	0.3928***	0.4395**
β	0.9188***	0.9182***	0.9186***	0.6666***	0.6495***	0.5637**
γ	-	0.0830	0.0199	-	0.0811	-
δ	-	2.1540***	-	-	2.1345***	-
η	-	-	-	-	-	1.5917
HQIC	4.793	4.788	4.790	**4.794**	4.789	4.791

Lanxess	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0178	0.0021	0.0215	0.0132	0.0241	0.0142
ω	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**
d	-	-	-	0.1393**	0.1387*	0.7904***
α	0.0638**	0.0648*	0.0138	0.0771	0.0433	0.0000
β	0.7682***	0.7622***	0.7634***	0.2021	0.1374	0.5181**
γ	-	0.1226	0.1364**	-	0.9261	-
δ	-	2.0011*	-	-	1.0149	-
η	-	-	-	-	-	0.6707
HQIC	5.168	5.172	**5.176**	5.155	5.160	5.155

Linde	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0421	-0.0362*	-0.0316	-0.0361		-0.0363
ω	0.0000***	0.0001***	0.0000***	0.0000***		0.0000***
d	-	-	-	0.2918***		0.0016***
α	0.0302***	0.0336***	0.0377**	0.5814***		0.8596***
β	0.9591***	0.9648***	0.9275***	0.7420***		0.9016***
γ	-	0.9787***	0.0941***	-		-
δ	-	0.5539***	-	-		-
η	-	-	-	-	-	96.6890***
HQIC	5.510	5.528	**5.529**	5.513	ncr	5.514

RWE	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0033	0.0033	0.0018	0.0032	0.0021	0.0031
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3428***	0.3551***	0.3135
α	0.0690***	0.0697***	0.0302*	0.5321***	0.4630***	0.5513***
β	0.9024***	0.8949***	0.8947***	0.7011***	0.6685***	0.7015***
γ	-	0.3392***	0.0830***	-	0.3106***	-
δ	-	1.7920***	-	-	1.6231***	-
η	-	-	-	-	-	1.0313
HQIC	5.604	5.610	**5.613**	5.607	5.612	5.604

SAP	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0321	-0.0227	-0.0240	-0.0191	-0.0183	-0.0191
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5721***	0.5817***	0.5744***
α	0.1304***	0.1375***	0.0974***	0.0491	0.0698	0.0489
β	0.8553***	0.8578***	0.8537***	0.5691***	0.5818***	0.5704***
γ	-	0.1597***	0.0751**	-	0.1150*	-
δ	-	1.7742***	-	-	1.9080***	-
η	-	-	-	-	-	0.9986
HQIC	5.198	5.198	5.200	**5.209**	5.206	5.206

Siemens	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0539*	0.0528*	0.0535*	0.0517*	0.0532*	0.0545*
ω	0.0000**	0.0000***	0.0000**	0.0000**	0.0000**	0.0000**
d	-	-	-	0.5085***	0.4089***	0.2450
α	0.0482***	0.0412***	0.0395***	0.2932***	0.3253***	0.3733***
β	0.9441***	0.9447***	0.9426***	0.7417***	0.6815***	0.6234***
γ	-	0.1135	0.0189	-	0.1453*	-
δ	-	2.2711***	-	-	2.1410***	-
η	-	-	-	-	-	1.2614
HQIC	**5.439**	5.435	5.438	5.435	5.432	5.433

ThyssenKrupp	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0306	0.0323	0.0305	0.0257	0.0274	0.0266
ω	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**
d	-	-	-	0.5620***	0.3783***	0.3779**
α	0.0543***	0.0339***	0.0548***	0.4197***	0.5222***	0.5242***
β	0.9383***	0.9437***	0.9386***	0.8248***	0.7650***	0.7882***
γ	-	-0.0492	-0.0014	-	-0.0051	-
δ	-	2.7987***	-	-	2.4798***	-
η	-	-	-	-	-	1.0964
HQIC	5.195	5.192	5.192	**5.202**	5.198	5.200

TUI	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0178	0.0174	0.0178	0.0161	0.0160	0.0160
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5141***	0.6959***	0.4964**
α	0.0520***	0.0668***	0.0482***	0.3375***	0.2468**	0.3458***
β	0.9397***	0.9358***	0.9404***	0.7648***	0.8401***	0.7589***
γ	-	0.0115	0.0060	-	0.0419***	-
δ	-	1.4548***	-	-	1.6153***	-
η	-	-	-	-	-	1.0059
HQIC	5.090	5.087	5.088	**5.092**	5.089	5.090

Volkswagen	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0793***	0.0785***	0.0787***	0.0802***	0.0803***	0.0798***
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4202***	0.4549***	0.3594
α	0.0834***	0.0819***	0.0657***	0.2594***	0.2697***	0.2762***
β	0.8912***	0.8879***	0.8883***	0.6135***	0.6562***	0.5850***
γ	-	0.1165*	0.0390*	-	0.1160	-
δ	-	2.0795***	-	-	1.9195***	-
η	-	-	-	-	-	1.0451
HQIC	5.258	5.255	5.258	**5.262**	5.259	5.259

B.2 Results for the Assumption of a Student-t Conditional Distribution

Adidas	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0165	0.0220	0.0168	0.0227	0.0235	0.0188
ω	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
d	-	-	-	0.2920***	0.4078***	0.0029***
α	0.0179***	0.0658***	0.0161**	0.6714***	0.3961***	0.8998***
β	0.9809***	0.9262***	0.9788***	0.8003***	0.6914***	0.9528***
γ	-	0.4548**	0.0065	-	0.5873*	-
δ	-	1.1189***	-	-	0.8930**	-
η	-	-	-	-	-	74.1804***
HQIC	5.771	5.767	5.769	**5.772**	5.769	5.771

Allianz	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0591**	0.0600**	0.0622**	0.0598**	0.0603**	0.0588**
ω	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**
d	-	-	-	0.4740***	0.2749***	0.2392
α	0.0930***	0.0706***	0.0444**	0.2338***	0.1524	0.2342
β	0.8976***	0.8900***	0.8989***	0.5940***	0.3488*	0.4316**
γ	-	0.2245***	0.0899***	-	0.2506***	-
δ	-	2.6010***	-	-	2.4183***	-
η	-	-	-	-	-	130744***
HQIC	5.321	5.328	5.3293	5.322	**5.3295**	5.320

BASF	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0460*	-0.0457*	-0.0456*	-0.0423	-0.0438	-0.0429
ω	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**
d	-	-	-	0.3714***	0.2588*	0.1441
α	0.0700***	0.0658***	0.0199	0.2022	0.2298	0.0998
β	0.9167***	0.9053***	0.9053***	0.4919***	0.4256*	0.2587
γ	-	0.4433*	0.1143***	-	0.3509**	-
δ	-	2.0102***	-	-	2.2460***	-
η	-	-	-	-	-	14200***
HQIC	5.802	**5.820**	5.818	5.801	5.813	5.800

Bayer	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0215	0.0241	0.0258	0.0248		0.0252
ω	0.0000***	0.0000***	0.0000***	0.0000***		0.0000***
d	-	-	-	0.3955***		0.2525
α	0.0706***	0.0433***	0.0253***	0.2553**		0.2451
β	0.9156***	0.9497***	0.9435***	0.5602***		0.4531*
γ	-	0.9878***	0.9877***	-		-
δ	-	1.2341***	-	-		-
η	-	-	-	-	-	199.099***
HQIC	5.333	5.355	**5.360**	5.332	ncr	5.330

BMW	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0178	0.0172	0.0175	0.0174	0.0158	0.0165
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5223***	0.4680***	0.3607*
α	0.0541***	0.0488**	0.0377**	0.2715***	0.3010***	0.3452***
β	0.9400***	0.9402***	0.9388***	0.7321***	0.7107***	0.6834***
γ	-	0.1683	0.0311	-	0.1601	-
δ	-	2.0706***	-	-	2.0309***	-
η	-	-	-	-	-	2248.7***
HQIC	5.589	5.585	5.588	**5.590**	5.586	5.587

Continental	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0184	-0.0164	-0.0205	-0.0160	-0.0154	-0.0158
ω	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**	0.0000**
d	-	-	-	0.3691***	0.5147***	0.3394
α	0.1007***	0.0976***	0.0501**	0.1349	0.2349***	0.1323
β	0.8635***	0.8840***	0.8667***	0.4259**	0.6724***	0.4044
γ	-	0.4398***	0.1032***	-	0.5496**	-
δ	-	1.2146***	-	-	1.0494***	-
η	-	-	-	-	-	416.089***
HQIC	5.347	5.352	5.352	5.348	**5.353**	5.344

Daimler	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0083	0.0111	0.0093	0.0135	0.0137	0.0123
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.0478***	0.5919***	0.2371
α	0.0678***	0.0817***	0.0503***	0.1567**	0.1516*	0.1997*
β	0.9239***	0.9165***	0.9209***	0.6218***	0.7065***	0.4873***
γ	-	0.1698*	0.0397***	-	0.1412	-
δ	-	1.6028***	-	-	1.7251***	-
η	-	-	-	-	-	622.478***
HQIC	5.409	5.407	5.409	**5.411**	5.407	5.408

Deutsche Bank	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0420*	0.0409	0.0407	0.0454*	0.0407	0.0447*
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5479***	0.3717***	0.3891**
α	0.0724***	0.0473**	0.0164	0.1913***	0.3053***	0.2450***
β	0.9226***	0.9386***	0.9336***	0.6931***	0.6251***	0.6279***
γ	-	0.5424**	0.0901***	-	0.4510***	-
δ	-	1.8653***	-	-	2.0079***	-
η	-	-	-	-	-	3116.6***
HQIC	5.567	5.578	**5.580**	5.569	5.579	5.567

Deutsche Börse	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0500*	0.0661*	0.0547*	0.0517*		0.0503*
ω	0.0000***	0.0000***	0.0000***	0.0000***		0.0000***
d	-	-	-	0.4106***		0.8750***
α	0.1744***	0.1331***	0.1019***	0.1503***		0.0453
β	0.7345***	0.8104***	0.7416***	0.3684***		0.06454***
γ	-	0.6114***	0.1556*	-		-
δ	-	0.5666**	-	-		-
η	-	-	-	-	-	102.044***
HQIC	5.510	5.513	**5.514**	5.505	ncr	5.505

Deutsche Lufthansa	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0002	0.0021	0.0023	0.0052	0.0049	0.0027
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4361***	0.4062***	0.0404
α	0.0788**	0.0981***	0.0560*	0.2653**	0.2703**	0.2892
β	0.9137***	0.9032***	0.9074***	0.5669***	0.5496***	0.3535
γ	-	0.2017**	0.0545*	-	0.1995**	-
δ	-	1.4724***	-	-	2.0163***	-
η	-	-	-	-	-	245.501***
HQIC	5.349	5.348	5.350	**5.352**	5.351	5.351

Deutsche Post	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0216	-0.0197	-0.0214	-0.0243	-0.0230	-0.0241
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4656***	0.4985***	0.2335
α	0.0864***	0.0934***	0.0840***	0.3426***	0.3305***	0.4538***
β	0.8994***	0.9000***	0.8981***	0.6918***	0.7050***	0.6457***
γ	-	0.0323	0.0068	-	0.0527	-
δ	-	1.7285***	-	-	1.8657***	-
η	-	-	-	-	-	401.336***
HQIC	5.569	5.564	5.567	**5.5710**	5.566	5.569

Deutsche Telekom	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0026	-0.0010	-0.0011	0.0028	0.0037	0.0013
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4593***	0.3564***	0.1431
α	0.0711***	0.0772***	0.0592***	0.2199**	0.1998*	0.2346
β	0.9241***	0.9189***	0.9200***	0.6028***	0.4863***	0.4128*
γ	-	0.1151	0.0328	-	0.1368	-
δ	-	1.9339***	-	-	2.2804***	-
η	-	-	-	-	-	79.0673***
HQIC	5.800	5.797	5.799	**5.801**	5.798	5.800

E.ON	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0371	-0.0322	-0.0379	-0.0341	-0.0292	-0.0342
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4680***	0.5283***	0.5127**
α	0.0658***	0.0698***	0.0341*	0.3118***	0.3029***	0.2942**
β	0.9196***	0.9229***	0.9149***	0.6740***	0.7227***	0.6919***
γ	-	0.5593**	0.0683**	-	0.7150***	-
δ	-	1.2616***	-	-	1.0080***	-
η	-	-	-	-	-	176.355***
HQIC	5.737	5.7381	5.7386	5.734	**5.7389**	5.731

Fresenius MedCare	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0761***	-0.0760***	-0.0769***	-0.0745**	-0.0736**	-0.0799***
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3781***	0.3487**	0.0227
α	0.0594***	0.0720***	0.0256***	0.5982***	0.6036***	0.9799***
β	0.9257***	0.9266***	0.9365***	0.7729***	0.7577***	0.9867***
γ	-	0.2835**	0.0581***	-	0.2367***	-
δ	-	1.5118***	-	-	2.0894***	-
η	-	-	-	-	-	372.747***
HQIC	5.635	5.638	5.640	5.6429	**5.643**	5.641

Henkel	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0474*	-0.0484*	-0.0502*	-0.0494*	-0.0484*	-0.0494*
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.2807***	0.3317***	0.2600
α	0.0685***	0.0629***	0.0475*	0.4159**	0.3356***	0.4219**
β	0.8878***	0.9005***	0.8806***	0.5762***	0.5914***	0.5706***
γ	-	0.7066**	0.1251***	-	0.6949**	-
δ	-	1.3280***	-	-	1.2819***	-
η	-	-	-	-	-	92.712***
HQIC	5.919	5.924	5.923	5.919	**5.927**	5.916

Infineon	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0283	0.0283	0.0283	0.0275	0.0282	0.0295
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.4922***	0.4306***	0.1979
α	0.0633***	0.0609***	0.0512***	0.3260***	0.3476***	0.4024***
β	0.9283***	0.9309***	0.9305***	0.7098***	0.6798***	0.5828***
γ	-	0.0875	0.0203	-	0.0803	-
δ	-	1.9860***	-	-	2.1650***	-
η	-	-	-	-	-	12510.2***
HQIC	4.822	4.817	4.820	**4.821**	4.816	4.819

Lanxess	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0183	-0.0179	-0.0179	-0.0261	-0.0210	-0.0236
ω	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
d	-	-	-	0.2222**	0.2904*	0.6730*
α	0.0577*	0.0645	0.0156	0.0258	0.3682	0.0000
β	0.8510***	0.8112***	0.8113***	0.2758	0.3682*	0.5025*
γ	-	0.5033***	0.1298*	-	0.5885	-
δ	-	0.8343	-	-	0.8722	-
η	-	-	-	-	-	146.028***
HQIC	5.222	5.226	**5.229**	5.218	5.218	5.216

Linde	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0479*	-0.0507*	-0.0390	-0.0460*	-0.0398*	-0.0436
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3785***	0.3623***	0.0015***
α	0.0350**	0.0316***	0.0709***	0.4714***	0.3533***	0.9345***
β	0.9627***	0.9739***	0.9049***	0.7038***	0.6073***	0.9652***
γ	-	0.9999***	0.3480***	-	0.4850**	-
δ	-	0.7783***	-	-	1.6697***	-
η	-	-	-	-	-	74.896***
HQIC	5.607	**5.615**	5.613	5.607	5.614	5.611

RWE	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0030	0.0021	0.0008	0.0045	0.0009	0.0045
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.3815***	0.3960***	0.2837
α	0.0619***	0.0700***	0.0343*	0.4646***	0.4178***	0.5166***
β	0.9200***	0.9064***	0.9059***	0.6870***	0.6676***	0.6761***
γ	-	0.2959**	0.0696**	-	0.2890**	-
δ	-	1.7333***	-	-	1.6355***	-
η	-	-	-	-	-	13436.1***
HQIC	5.621	5.623	**5.625**	5.622	5.624	5.619

SAP	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	-0.0130	-0.0099	-0.0093	-0.0148	0.0009	-0.0133
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5119***	0.3960***	0.2888
α	0.0713***	0.0753***	0.0358*	0.2344***	0.4178***	0.2915***
β	0.9252***	0.9364***	0.9390***	0.6742***	0.6676***	0.5756***
γ	-	0.2592**	0.0465**	-	0.2890**	-
δ	-	1.3090***	-	-	1.6355***	-
η	-	-	-	-	-	61.6701***
HQIC	5.335	5.338	5.336	5.335	**5.339**	5.334

Siemens	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0392	0.0382	0.0382	0.0409	0.0392	0.0409
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.7035***	0.4427**	0.3689
α	0.0509***	0.0388**	0.0378***	0.1883	0.3332***	0.3412***
β	0.9456***	0.9486***	0.9450***	0.8428***	0.7247***	0.7086***
γ	-	0.1470***	0.0269	-	0.1599*	-
δ	-	2.4089***	-	-	2.3055***	-
η	-	-	-	-	-	1327.03***
HQIC	**5.471**	5.468	5.470	5.469	5.466	5.466

ThyssenKrupp	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0227	0.0228	0.0227	0.0195	0.0189	0.0199
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.7145***	0.5618**	0.4187*
α	0.0624***	0.0671***	0.0583***	0.3192*	0.4143***	0.4968***
β	0.9348***	0.9320***	0.9328***	0.8697***	0.8166***	0.7978***
γ	-	0.0547	0.0115	-	0.0437	-
δ	-	1.8896***	-	-	2.2043***	-
η	-	-	-	-	-	883.331
HQIC	5.231	5.226	5.229	**5.234**	5.229	5.232

TUI	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0177	0.0148	0.0171	0.0139	0.0126	0.0159
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5858***	0.6132**	0.3419
α	0.0514***	0.0620***	0.0352**	0.3005***	0.2911*	0.4107***
β	0.9467***	0.9378***	0.9532***	0.7942***	0.8084***	0.7228***
γ	-	0.1105	0.0204	-	0.0813	-
δ	-	1.4791***	-	-	1.9194***	-
η	-	-	-	-	-	96.3127***
HQIC	5.1707	5.167	5.169	**5.1708**	5.166	5.168

Volkswagen	GARCH	APARCH	GJR	FIGARCH	FIAPARCH	HYGARCH
ϱ	0.0646**	0.0639**	0.0639**	0.0658**	0.0655**	0.0658**
ω	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***
d	-	-	-	0.5301***	0.5890***	0.5504*
α	0.0919***	0.0910***	0.0660***	0.2232***	0.2057**	0.2147
β	0.8902***	0.8914***	0.8913***	0.6809***	0.7264***	0.6894***
γ	-	0.1499*	0.0538*	-	0.1760**	-
δ	-	1.9965***	-	-	1.8784***	-
η	-	-	-	-	-	396.629***
HQIC	5.3182	5.315	5.3180	**5.3183**	5.316	5.315

C Standard & Poor's 1 Year Credit Ratings

The categories along with the ratings and PD's are adopted from Henking et al. (2006).

Rating	**PD (in %)**	**Rating category**	**Rating**	**PD (in %)**	**Rating category**
AAA	<0.01	Prime	BB+	<0.94	Speculative
AA+	<0.02	High grade	BB	<1.55	
AA	<0.03		BB-	<2.50	
AA-	<0.04		B+	<4.08	Highly speculative
A+	<0.05	Upper medium grade	B	<6.75	
A	<0.08		B-	<10.88	
A-	<0.13		CCC	<17.75	Extremely speculative
BBB+	<0.22	Lower medium grade	CC	<29.35	
BBB	<0.36		C	>29.35	
BBB-	<0.58		D		In default

Chapter 3

Model Risk in Backtesting Risk Measures

3 Model Risk in Backtesting Risk Measures

Co-authored with Corinna Evers

3.1 Introduction

Backtesting provides an instrument to analyze whether a model used for calculating risk measures is accurate. Since severe implications for the solvency capital arise from the calculation of risk measures, backtesting procedures are considered to be a core concern of supervisory activity, which strives to ensure the resilience of financial institutions in order to alleviate the impact of financial crisis.

The regulations issued by the Basel Committee on Banking Supervision (1996b) state that the calculation for the market capital requirement for the prevention of losses which result from adverse market conditions should be computed as follows: the maximum of either the 1% Value at Risk (VaR) or the average VaR reported over the previous 60 days is multiplied by a factor that depends on the sum of the VaR violations across the reporting period (traffic-light approach). Thus, the accuracy of the VaR model is closely linked to the regulatory framework. As defined by Kupiec (1995) and Christoffersen (1998), an accurate VaR model needs to satisfy two properties.

Firstly, the property of unconditional coverage claims that the probability of a violation equals the α level set for the VaR model. Unconditional coverage exists if

$$P(I(\alpha) = 1) = \alpha \tag{21}$$

holds, whereby $\{I_t\}$ denotes the hit sequence indicating whether or not a violation occurred at time t. The VaR model is deemed to be inaccurate (in the sense of failing to account for the incurred risk) if the number of violations exceeds the expected loss. The risk model is too conservative if the VaR model yields less violations than expected.

A second claim is the independence of the elements of the hit sequence. Contrary to a situation in which the violations are spread out evenly over the reporting horizon, the financial institution might not be able to tackle the losses if the violations occur in a cluster. Next to the property of unconditional coverage, an accurate VaR model is therefore characterized by satisfying the attribute of independence as well. This property is fulfilled if the hit sequence consists of independent Bernoulli random variables which are identically distributed with probability α, that is

$$I_t(\alpha) \overset{iid}{\sim} Ber(\alpha). \tag{22}$$

Backtests are statistical tests designed for determining the accuracy of VaR models. While several tests have been proposed for each of the two properties, joint tests determine whether the VaR model is entirely accurate in the sense of fulfilling both (21) and (22). However, joint tests are not considered to be universally preferable over single-property tests as they entail that the ability to detect the infringement of one of the two properties is decreasing (see Campbell (2005)). A type I error arises if an accurate model with a coverage of 99% is erroneously rejected. If the VaR model is inaccurate involving a lower coverage rate, e.g. 2%, a type II error represents the probability that the inaccurate model is not rejected. If the power of the backtest remains low, the probability of classifying an inaccurate model to be accurate (not rejecting the null) is comparatively high. Therefore, backtests should not be over- or undersized and feature high power. However, Escanciano and Olmo (2007) emphasize that standard backtesting approaches which neglect the presence of estimation risk are misleading.

This paper analyzes the problems of the most common backtesting procedures within a Monte Carlo study. The main result of this paper consists in the finding that even when accounting for the presence of estimation risk, the problems which arise from conducting common backtesting procedures cannot be alleviated, especially for the restrictions set by the regulation side. The remainder of the paper is organized as follows: subsequently, Section 3.2 describes the most relevant classes of backtesting procedures. In Section 3.3, we conduct a Monte Carlo study and examine the problems that arise when conducting univariate backtests in the view of regulatory aspects. The study includes very simple procedures as well as backtests which take the impact of estimation and misspecification risk into account. Finally, the Section 3.4 provides a conclusion.

3.2 Overview of Backtesting Procedures

Backtests can be distinguished by two categories: frequency-based and size-based tests. While frequency-based tests examine only the sequence which indicates whether a violation has occurred for the realized profit and loss series at the respective point in time, size-based tests are constructed from the size of the exceedance. As the regulatory framework is based upon the violations and not on their size, size-based tests are relatively rare to be found in the literature due to regulatory constraints (see Lopez (1998)).

3.2.1 Kupiec Tests for Unconditional Coverage

The most basic backtests for testing the unconditional coverage property are given by the time until first failure (TUFF) test and its generalization, the proportion of failures (POF) test, both suggested by Kupiec (1995). As shown by Kupiec (1995), the simplicity of the TUFF test entails that the total number of failures which occurred since the start of the monitoring is ignored. Thus, the POF test should always be run to validate potential loss estimates in place or in addition. In contrast to the TUFF framework, in which only the elapsed time until the first failure is considered, the POF uses the entire information. For this purpose (and all further analyses), consider a hit sequence $\{I_t\}_{t=1}^n$ of size n, whereby $\forall t: \ I_t \in \{0,1\}$ applies. The number of hits (i.e. the observations for which $I_t = 1$ is observed) is denoted by n_1, while $n_0 = n - n_1$ (i.e. $n_0 = \sharp(t: \ I_t = 0)$) stands for the number of observations without a violation. The probability of observing n_1 hits in a sample of size n is given by the probability function of the binomial distribution,

$$P(\sharp(t:\ I_t = 1) = n_1) = \binom{n}{n_1}(1-\alpha)^{n_0}\,\alpha^{n_1}.$$

For the null hypothesis of the POF test, $H_0 : \alpha = \hat{\Pi}$ with $\hat{\Pi} = \frac{n_1}{n}$, the associated test is a Likelihood Ratio (LR) test. The test statistic is given by

$$K = -2\,\log\left(L(\alpha)/L(\hat{\Pi})\right),$$

whereat α denotes the failure probability under the null, while $L(\cdot)$ represents the corresponding Likelihood function.

However, if the sample size is relatively small, both tests appear to have poor ability to distinguish between the underlying failure probability in the null hypothesis and failure probabilities which are slightly higher (see Kupiec (1995)). Thus, these frameworks might not be adequate for the analysis of the accuracy of VaR estimates which are evaluated over only a single trading year. Furthermore, a frequently arising problem consists in the absence of violations during the reporting period. This issue becomes most important if VaR models with a small failure probability are evaluated. In these cases, the Kupiec tests are not computable.

3.2.2 Christoffersen Tests for Independence and Conditional Coverage

When testing the *iid* hypothesis of the hit sequence, the autocorrelation of the sequence itself or the distance of the time span between consecutive violations is examined. Tests for independence of the observations require the complete specification of the alternative hypotheses in the sense that the structure in which violation clusters occur needs to be specified exactly. Autocorrelation-based tests can be constructed by testing the autocorrelation structure in the hit sequence $\{I_t\}$ itself or in the demeaned sequence $\{I_t - \alpha\}$, which forms a sequence of martingale difference summands (see Berkowitz et al. (2011)).

The LR-type test proposed by Christoffersen (1998) represents the first test of this kind. The basic idea behind this test consists in the following comparison: if there is no dependence between two consecutive observations, then the probability of monitoring no violation on the day after a violation occurred should be equal to the probability of monitoring no violation when no violation was observed on the day before.

Like the tests proposed by Kupiec (1995), a LR framework which is based on Markov chains is used for the test. The independence of the observations of the hit sequence is tested under the null against the alternative of a first-order Markov chain, in which the stochastic matrix

$$\Pi_1 = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix}$$

contains the transition probabilities $\pi_{i,j} = P(I_t = i | I_{t-1} = j)\,, i,j \in \{0,1\}$. Let n_{ij} be the number of observations, which yield the value $i \in \{0;1\}$ at some time t and the value $j \in \{0;1\}$ at time $t-1$. Then,

$$L(\Pi_1) := L(\Pi_1; \{I_t\}) = \pi_{00}^{n_{00}} \pi_{01}^{n_{01}} \pi_{10}^{n_{10}} \pi_{11}^{n_{11}}$$

constitutes the likelihood of the hit sequence $\{I_t\}$ under validity of the alternative model, while the likelihood for the null model can be computed by considering the stochastic matrix

$$\Pi_2 = \begin{pmatrix} 1-\pi_2 & \pi_2 \\ 1-\pi_2 & \pi_2 \end{pmatrix}.$$

The application of this model under the null makes it easy to see that the independence of the hit sequence is tested by this means since the rows all exhibit the same entries. Under the null, the previous observations do not influence the probability of monitoring a violation. The entries π_2 represent the probability of a violation. Accordingly, the number of observations are aggregated over index j as the past value j has no influence on the present value i. Thus, the probability of observing a violation is given by

$$\pi_2 = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}},$$

so that the likelihood function under the null model can be computed by

$$L(\Pi_2) := L(\Pi_2; \{I_t\}) = (1-\pi_2)^{(n_{00}+n_{10})}\, \pi_2^{(n_{01}+n_{11})}.$$

Using $L(\Pi_1)$ and $L(\Pi_2)$, the LR test statistic for the Christoffersen test of independence can be defined by

$$LR.IND = -2 \log \left(\frac{L(\Pi_1)}{L(\Pi_2)} \right).$$

Under validity of H_0, LR.IND is χ^2 distributed with one degree of freedom. Note that the Christoffersen (1998) test provides no opportunity of testing conditional coverage as LR.IND does not depend on the true coverage probability α. A joint test for both testing the independence and the conditional coverage property is provided below.

A problem which arises when using this procedure is that the Christoffersen (1998) test of independence only examines the dependence between two consecutive observations. Campbell (2005) refers to the possibility that the probability of monitoring a violation today may not be influenced by a yesterday's observation, but still could be influenced by prior observations.

Next to the test of independence of the hit sequence, Christoffersen (1998) introduces a test for unconditional coverage, testing $E[I_t] = \alpha$ against its alternative $E[I_t] \neq \alpha$. The joint test for the presence of both conditional coverage and independence, which is proposed by Christoffersen (1998) as well, combines the single-property tests in order to examine whether both properties are jointly fulfilled.

The basic idea is as simple as for the independence test: first, if the unconditional coverage property is fulfilled, then $\frac{n_{00}+n_{10}}{n_{00}+n_{01}+n_{10}+n_{11}} = \alpha$ must hold, implying that the number of violations matches with the hit probability α. Furthermore, as stated previously, the probability of a non-violation to follow a previous hit equals the probability of a non-violation to follow a previous non-violation. Thus, $\frac{n_{00}}{n_{00}+n_{01}} = \frac{n_{10}}{n_{10}+n_{11}}$ applies, if the independence property is fulfilled. Combining these considerations, both probabilities defined should match with the total proportion of non-violations, if the VaR measure meets the independence property. Thus, provided that the property of unconditional coverage is valid, this leads to

$$\frac{n_{00}}{n_{00}+n_{01}} = \frac{n_{10}}{n_{10}+n_{11}} = \frac{n_{00}+n_{01}}{n_{00}+n_{01}+n_{10}+n_{11}} = \alpha,$$

which denotes the hypothesis to be tested under the null. In terms of the LR framework, the likelihood of the null of the unconditional coverage test is tested against the alternative of the independence test. In effect, this forms a test for conditional coverage. Thus, the test statistics results in

$$LR.CC = -2\,\log\left(\frac{L(\alpha)}{L(\Pi_1)}\right).$$

Christoffersen (1998) shows that the limiting distribution of the joint test is $\chi^2(2)$. However, even if the utilization of a joint test might always seem preferable over the separate examination of the unconditional coverage and the independence property, it has to be remarked that joint tests ignore VaR measures which violate only a single property. As a result, the joint test may detect the violation of either unconditional coverage or independence in less cases than a test which covers only one of these properties. According to Campbell (2005), the usage of a test which comprises only a single property might be preferable if prior information about the VaR measure is available.

3.2.3 Escanciano/Olmo Tests for Unconditional Coverage

Escanciano and Olmo (2012) propose a test for unconditional coverage as well as a test for conditional coverage. Their analysis is based on a Monte Carlo study in which the unconditional and the conditional coverage tests are compared to a corrected version of these tests. The corrected versions account for the impact of estimation risk which is induced by the computation of forecasts. All tests are based on the demeaned hit sequence $\{I_t - \alpha\}$.

The test of unconditional coverage is derived from the validity of $E[I_t] = \alpha$ under the null model. The associated test statistic is given by

$$S_P = \frac{1}{\sqrt{P}} \sum_{t+R=1}^{P} (I_t - \alpha) \tag{23}$$

and is rested upon the unconditional coverage tests by Kupiec (1995) and Christoffersen (1998). It can easily be checked that $\frac{1}{\sigma}S_P$ converges against a standard normal distribution, whereby the term $\sigma = \sqrt{\alpha\,(1-\alpha)}$ denotes the standard deviation of the distribution of a single demeaned observation:

$$\frac{1}{\sigma\,\sqrt{P}} \sum_{t+R=1}^{P} (I_t - \alpha) \longrightarrow N(0;1).$$

When adjusting σ for estimation risk, it can be shown that the estimated standard deviation has the form

$$\sigma_{corr} = \Big(\alpha\,(1-\alpha) + \pi\,\hat{A}\,\hat{V}\,\hat{A}'\Big)^{-\frac{1}{2}}$$

by using a notation defined below. This expression holds for the assumption that the applied forecast scheme is set fixed and the underlying DGP is a GARCH process of order (1,1). Note that Escanciano and Olmo (2012) also provide adjusted tests for rolling and recursive forecast schemes.

Let $\pi = \lim\limits_{n\to\infty} \frac{P}{R}$ indicate the relation between the length P of the out-of-sample series and the length R of the in-sample period, which is used to estimate the process parameters. It is quite intuitive that for a large value of R in relation to P (and thus a relatively long in-sample series) the influence of estimation risk becomes negligibly small. Furthermore, let the matrix $V \sim (3 \times 3)$ contain the variances and covariances of the data generating process, while $A \sim (3 \times 1)$ denotes a vector containing the first derivatives of the DGP with respect to the GARCH parameters. Thereby, $\hat{A}$ and $\hat{V}$ denote consistent estimators for A and V. For a detailed derivation of A and V, see the Appendix. Note that the impact of estimation risk is asymptotically irrelevant for $\pi\hat{A}\hat{V}\hat{A}' = 0$. The resulting test statistic is given by

$$\tilde{S}_P = \frac{1}{\sqrt{n}\,\sigma_{corr}} \sum_{t=1}^{n} (I_t - \alpha), \tag{24}$$

whereby the limiting distribution is $N(0;1)$ for $n \to \infty$ under the null.

3.2.4 Duration-based Tests for Independence

The seminal duration-based backtesting approach is proposed by Christoffersen and Pelletier (2004). This class of backtests pursues the aim to overcome the pitfall of poor power in small samples of the backtests existing by then and strives to account not only for first order Markov dependencies, as is given by the independence test by Christoffersen (1998). The authors motivate their presented approach by the existence of no-hit periods which are either relatively short by reason of high market volatility or relatively long in the case that the markets calmed down. For this purpose, we define $d_i = t_i - t_{i-1}$, $i = 1, \ldots, I$ as the duration between hit number $i-1$ and hit number i, which occur at dates t_{i-1} and t_i ($t \in \{1, \ldots, n\}$), respectively.

To construct the test that assumes independence of the durations and thus a correctly specified

VaR model, a memoryless probability distribution is needed for modeling the durations. The only continuous distribution which includes a constant failure probability α is given by the exponential distribution, which is defined by the density

$$f^{Exp}(d) = \alpha \, \exp(-\alpha \, d),$$

whereby $\alpha \in \mathbb{R}_{>0}$ and $d \in \mathbb{R}_{\geq 0}$ holds. Note that the corresponding hazard function of the exponential distribution is $\lambda^{Exp}(d) = \alpha$. Thus, the null of independence checks whether the durations d_i come from an exponential distribution with likelihood function

$$\ln L(\alpha) = n \ln(\alpha) - \alpha \bar{d}.$$

For the alternative model, a duration distribution with a non-constant hazard rate is required. To this effect, the simplest case is represented by the Weibull distribution, which is defined by the density

$$f^W(d) = \alpha^b \, b \, d^{b-1} \, \exp(-(\alpha \, d)^b),$$

whereat $b \in \mathbb{R}_{>0}$ denotes a shape parameter. Note that the exponential distribution is nested by the Weibull distribution for $b = 1$. The Weibull hazard rate can easily be obtained by

$$\lambda^W(d) = \alpha^b \, b \, d^{b-1}.$$

For $b < 1$, the Weibull hazard rate is decreasing. Transferred to financial risk management, a decreasing λ^W indicates the tendency of the market to feature more extreme durations, i.e. periods of relatively short or relatively long duration. The log-likelihood function under the alternative is then given by

$$\ln L(\alpha; b) = \ln \lambda + \ln b + (b-1) \sum_i \ln d_i - \lambda \sum_i d_i^b.$$

Thereby, the pair of hypotheses can be reformulated in terms of the shape parameter b, that is $H_0 : b = 1$ vs. $H_1 : b \neq 1$.

The null of independence can be tested by a Likelihood ratio test, which test statistic is given by

$$LR_{Dur} = -2 \, \frac{\ln L(\alpha)}{\ln L(\alpha; b)}.$$

Under validity of H_0, LR_{Dur} follows a χ^2 distribution with two degrees of freedom.

In order to conduct the test, it is necessary to transform the hit sequence $\{I_t\}$ into a duration sequence $\{d_i\}_{i=1}^I$. When implementing the transformation, it has to be kept into account that the first and last duration is possibly censored, so that the duration of the first no-hit period could be longer than d_1 as there is no data available before. The only exception consists in the case that the first observation already features a hit. Likewise, the last duration could be longer than d_I if the last observation of $\{I_t\}$ involves no hit.

The above spanned framework provides the opportunity to model dependencies of higher order than it is possible with the Markov-type test. However, this test contains no information about the exact order of dependence.[4]

Another test for independence that does not exploit the hit sequence directly, but the properties of the durations between two consecutive hits, is proposed by Candelon et al. (2011). The major motivation behind the construction of this test is to overcome the drawback of low power in realistic sample sizes.

The test procedure exploits the following idea: an orthonormal polynomial can be associated to each distribution that belongs to the Pearson family of distributions. Orthonormal polynomials are composed by a sequence of polynomials in which each two polynomials are pairwise orthonormal under the L^2-inner product. Considering the duration sequence $\{d_i\}$ to be discrete, the orthonormal polynomial associated with the geometric distribution can be employed.

By defining the number of employed polynomials by $h \in \mathbb{N}$, the orthonormal polynomial associated with the geometric distribution with success probability β can be stated by the recursion

$$M_h = M_{j+1}(d;\beta) = \frac{(1-\beta)(2j+1)+\beta(j-d+1)}{(j+1)\sqrt{(1-\beta)}}\, M_j(d;\beta) - \frac{j}{j+1} M_{j-1}(d;\beta)$$

for any $j \in \mathbb{N}_0$, $\beta \in \mathbb{R}_{[0;1]}$, $d \in \mathbb{N}_0$, $d := d_i \, \forall i \in \{1,\ldots,I\}$ and initial values of $M_{-1}(d;\beta) = 0$ and $M_0(d;\beta) = 1$. Using the method of moments to estimate the parameters of the polynomial regression, efficient and consistent estimates can be obtained. Thus, under the null of conditional coverage, the moment condition

$$H_0 : E[M_j(d;\beta)] = 0.$$

is tested. The duration sequence follows a geometric distribution with hit probability β, which means that there is no correlation between two consecutive hits as the geometric distribution is memoryless.

In contrast to the duration-based test by Christoffersen and Pelletier (2004), this framework allows to separately test for unconditional coverage and the independence hypothesis. The reasoning is straightforward: since the expectation of a geometrically distributed random variable with parameter β is equal to $\frac{1}{\beta}$, it can be shown that this is equivalent to the condition for the orthonormal polynomial of order $h = 1$. This condition is tested under H_0 of unconditional coverage,

$$\begin{aligned} E[M_1(d;\beta)] &= E\left[\frac{1-\beta d}{\sqrt{1-\beta}}\right] \\ &= \frac{1-\beta\frac{1}{\beta}}{\sqrt{1-\beta}} = 0 \qquad \text{for } E\left[d\right] = \frac{1}{\beta}. \end{aligned}$$

[4]The order of dependence can be captured by the EACD framework introduced by Engle and Russell (1998).

The usage of orthonormal polynomials enables to run the test within the GMM framework with known asymptotic covariance matrices. The test statistic utilizing the polynomial order h can be stated by

$$C_{CC}^{G}(h) = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_j(d_i; \beta) \right)' \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_j(d_i; \beta) \right).$$

Under validity of H_0, $C_{CC}^{G}(h)$ has a χ^2 limit distribution with h degrees of freedom. Note that for the special case of unconditional coverage and $h = 1$, the test statistic obtains the form

$$C_{CC}^{G}(1) = C_{UC}^{G} = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_1(d_i; \beta) \right)^2.$$

When presuming continuity of $\{d_t\}$, the tests are run with the conditions adjusted for the exponential distribution and its corresponding orthonormal polynomials, which follow the recursion

$$L_h := L_{j+1}(d; \beta) = \frac{1}{n+1} \left[(2n + 1 - \beta d) \, L_j(d; \beta) - n L_{n-1}(d; \beta) \right],$$

whereby the initial values are given by $L_{-1} = 1$ and $L_1 = 1 - \beta d$, while L denotes a polynomial of the Laguerre family. The test statistic for the continuous case and the orthonormal polynomials associated with the exponential distribution is then given by

$$C_{CC}^{Exp}(h) = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} L_j(d_i; \beta) \right)' \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} L_j(d_i; \beta) \right).$$

Again, the test statistic follows a $\chi^2(h)$ distribution under the null.

3.3 Simulation Study

The following simulation study aims at the detection of the problems arising from conducting backtests with univariate time series. For this purpose, we simulate GARCH(1,1) processes, as given by

$$\begin{aligned} Y_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \theta_0 + \theta_1 Y_{t-1}^2 + \theta_2 \sigma_{t-1}^2. \end{aligned}$$

A parameter vector of $\theta' = (\theta_0, \theta_1, \theta_2) = (0.1, 0.1, 0.85)$ as well as different lengths of the in-sample period $R \in \{250, 500, 750, 1000, 1500\}$ and the out-of-sample horizon $P \in \{250, 500, 750, 1000, 1500\}$ are assumed. The in-sample period is used for the estimation of the respective parameters, while the out-of-sample period is used for the evaluation of the estimated risk measure. A VaR measure with exceedance level $\alpha = 0.01$ for the respective series is calculated in the next step, before the hit sequence $\{I_t\}$ is computed. In order to test the accuracy of the VaR estimates, the test statistic of the backtesting approaches described in the previous section are calculated. This procedure is replicated 5,000 times. Table 3.1 presents the results of the Monte Carlo study. For each combination of R and P, the respective empirical size is calculated from the computed test statistics. The first three columns summarize the results for the Kupiec (1995) test and the tests for independence and conditional coverage suggested by Christoffersen (1998), while the remaining columns show size results for the duration-based backtests by Candelon et al. (2011), for which the sequence $\{d_t\}$ of the time span between the respective hits of sequence $\{I_t\}$ has been taken into account. While the tests given by (4)-(6) are based on the null of a geometric distribution by assuming a number of orthonormal polynomial of $h = 1, 3, 5$, the columns (7)-(9) report the results for the tests, for which the distribution under the null is supposed to be continuous. For this purpose, we assume the same number of orthogonal polynomials as under the assumption of discreteness.

	P	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
R=250	250	0.0930	0.0322	0.0808	0.0486	0.0512	0.0334	0.0138	0.0134	0.0118
	500	0.2240	0.0428	0.1208	0.1758	0.1020	0.0730	0.0344	0.0390	0.0366
	750	0.2262	0.0578	0.1832	0.1840	0.1696	0.1392	0.0718	0.0746	0.0660
	1,000	0.2786	0.0684	0.2286	0.2396	0.2016	0.1660	0.0962	0.0952	0.0816
	1,500	0.3452	0.0756	0.3148	0.3454	0.2828	0.2426	0.1472	0.1458	0.1224
R=500	250	0.0664	0.0328	0.0622	0.0350	0.0388	0.0246	0.0066	0.0080	0.0072
	500	0.1682	0.0412	0.0802	0.1250	0.0682	0.0468	0.0224	0.0270	0.0250
	750	0.1612	0.0640	0.1300	0.1198	0.1128	0.0936	0.0470	0.0574	0.0524
	1,000	0.2138	0.0652	0.1712	0.1746	0.1454	0.1192	0.0666	0.0698	0.0600
	1,500	0.2472	0.0694	0.2296	0.2478	0.1834	0.1500	0.0872	0.0854	0.0744
R=750	250	0.0628	0.0368	0.0582	0.0314	0.0348	0.0236	0.0056	0.0064	0.0074
	500	0.1576	0.0414	0.0680	0.1102	0.0610	0.0456	0.0168	0.0234	0.0252
	750	0.1460	0.0605	0.1216	0.1065	0.0998	0.0849	0.0399	0.0514	0.0448
	1,000	0.1973	0.0621	0.1502	0.1581	0.1247	0.1000	0.0523	0.0589	0.0507
	1,500	0.2058	0.0748	0.2104	0.2064	0.1550	0.1260	0.0652	0.0764	0.0628
R=1,000	250	0.2058	0.0748	0.2104	0.2064	0.1550	0.1260	0.0652	0.0764	0.0628
	500	0.1430	0.0424	0.0634	0.1036	0.0556	0.0412	0.0166	0.0222	0.0230
	750	0.1300	0.0556	0.1076	0.0956	0.0918	0.0734	0.0378	0.0466	0.0394
	1,000	0.1678	0.0690	0.1440	0.1366	0.1096	0.0968	0.0568	0.0574	0.0508
	1,500	0.1877	0.0757	0.1941	0.1877	0.1522	0.1208	0.0673	0.0743	0.0625
R=1,500	250	0.1678	0.0690	0.1440	0.1366	0.1096	0.0968	0.0568	0.0574	0.0508
	500	0.1404	0.0378	0.0624	0.1000	0.0534	0.0384	0.0160	0.0224	0.0236
	750	0.1206	0.0620	0.1058	0.0890	0.0844	0.0674	0.0316	0.0402	0.0358
	1,000	0.1486	0.0604	0.1188	0.1152	0.0952	0.0822	0.0444	0.0494	0.0434
	1,500	0.1652	0.0752	0.1856	0.1656	0.1318	0.1062	0.0622	0.0678	0.0558

Table 3.1: Results for the simulation of the size of the following backtests ($\alpha = 0.01$): (1) Kupiec (1995), (2) Christoffersen (1998) test for independence and (3) conditional coverage, as well as duration-based tests for independence by Candelon et al. (2011) assuming a discrete distribution (4)-(6), and a continuous distribution (7)-(9), each based on orthonormal polynomials of orders 1, 3 and 5.

The first observation to be noted is that the majority of the backtests are oversized since the null is rejected too often. Thus, even if the null is true, the backtests classify the VaR to be inaccurate. However, some of the duration-based backtests tend to be undersized, especially if both P and R are indicated by small values. Secondly, if the choice of R and P induce a smaller value of the quotient which indicates the relation of out-of-sample length and in-sample length, a lower distortion can be observed, that is a smaller difference between the empirical and the nominal size. For example, while the Kupiec test is distorted by 33.52% for $R = 250$ and $P = 1,500$, the distortion becomes smaller if we assume a smaller length of the in-sample period. If the out-of-sample length is reduced to $P = 250$, the size is distorted by 8.3%. This is due to the reason that if a smaller number of observations in relation to P is available for the estimation of the parameters, the induced estimation risk increases. This leads to less accurate projections of VaR. Generally, duration-based backtests appear to have lower size distortions.

Respecting the presence of model risk, Escanciano and Olmo (2012) provide backtests which account for the presence of estimation risk. By the correction of the variance of the backtest provided by Kupiec (1995) and taking the demeaned hit sequence $\{I_t - \alpha\}$ into account, the test should not be rejected as often as for the uncorrected test. Therefore, it should be expected that the size distortions decrease by applying the corrected backtest by Escanciano and Olmo (2012). Again, we conduct a Monte Carlo experiment as outlined above with 5,000 replications for estimation and evaluations of $R, P \in \{250, 500, 750, 1000\}$ in order to compute the statistics S_P for testing unconditional coverage as well as the corrected statistics $\tilde{S}_P$, which are given by the equations (23) and (24). The size results are reported in Table 3.2.

	R = 250				R = 500			
P	250	500	750	1,000	250	500	750	1,000
S_P	0.138	0.182	0.250	0.268	0.108	0.154	0.228	0.194
$\tilde{S}_P$	0.088	0.096	0.082	0.118	0.074	0.078	0.092	0.074

	R = 750				R = 1,000			
P	250	500	750	1,000	250	500	750	1,000
S_P	0.128	0.142	0.228	0.184	0.100	0.090	0.180	0.156
$\tilde{S}_P$	0.090	0.098	0.084	0.064	0.084	0.062	0.078	0.084

Table 3.2: Results for the simulation of the size of the backtests proposed by Escanciano and Olmo (2012). The test for unconditional coverage is indicated by S_P, while the conditional coverage test is marked by $\tilde{S}_P$. A VaR exceedance level of $\alpha = 0.01$ is assumed.

For each combination of P and R, the variance correction results in a much lower empirical coverage for $\tilde{S}_P$, while the empirical and nominal coverage do hardly deviate from each other for a small value of the quotient π. However, for an evaluation sample of $P = 250$ observations and a VaR exceedance level of $\alpha = 0.01$, as it is recommended within the Basel II framework, the size distortions remain at a considerable level of about 7%. Therefore, the problem that the test rejects too often is not solved. Looking at the size distortions of the tests proposed by Escanciano and Olmo (2012), it can be noted that even when accounting for estimation risk the problem persists.

In Figure 3.1, the density of the true asymptotic distribution of S_P and $\tilde{S}_P$ (the Gaussian distribution) as well as the kernel density estimation of the test statistics S_P and $\tilde{S}_P$ (the corrected test) for $R = 250$, $P = 250$ and $\alpha = 0.01$ are plotted. While the density of S_P shows a considerable deviation from its asymptotic distribution, the kernel density estimation of the corrected backtest (given by $\tilde{S}_P$) provides a much better approximation.

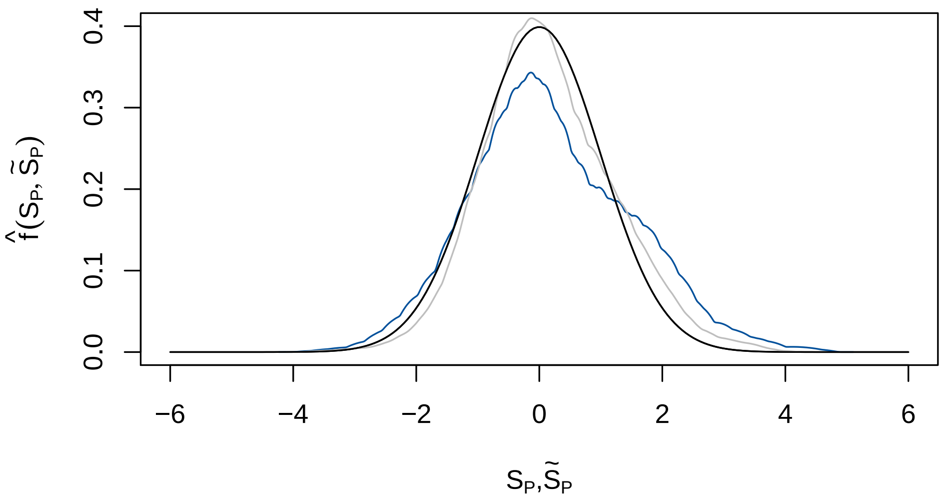


Figure 3.1: The density of the $N(0;1)$ distribution (black) compared with the kernel density estimate of S_P (blue) and the kernel density estimate of $\tilde{S}_P$ (gray) for $R = 250$, $P = 250$ and $\alpha = 0.01$

In another article (which accounts for misspecification risk in the proposed backtest framework), Escanciano and Olmo (2011) take into consideration that their modified test still suffers from problems of heavy size distortions even in the case of very small in-sample lengths. To put it in a nutshell, all classes of univariate backtests proposed so far feature the problem of size distortions within short in-sample horizons, even if this conclusion holds for duration-based backtests to a lesser extent.

Although the corrected backtests result in a reduction of the size distortion, the tests tend to reject too often. Despite of correcting for estimation risk, the problem especially persists in the setting recommended within the Basel II framework if a VaR exceedance level of $\alpha = 0.01$ is set. In this setting, duration-based backtests with orthonormal approximation of the distribution under the null seem to be the most promising alternative.

3.4 Conclusion

In this paper, we analyze the problems of backtests that have been suggested so far. Backtests which are based on hit and duration sequences in a univariate framework show heavy size distortions. The problems of univariate backtesting procedures consist in considerable size distortions for the Basel II setting. A possible solution of this problem is to account for model risk by correcting the asymptotic variance of the backtest in order to reduce the distortion. However, this issue cannot be alleviated by modifying backtests in a way that accounts for estimation risk. Financial institutions face restrictions for the conduction of backtesting from the regulatory side, which mandate to use an evaluation period of 250 observations. An alternative choice of the out-of-sample length does not suffice to reduce the empirical size. The application of inaccurate backtests entails severe implications and higher risk-based capital results because the factor for the calculation is directly linked to the number of hits.

A possible solution is the utilization of multivariate backtesting procedures in order to overcome these problems. Danciulescu (2010) and Berkowitz et al. (2011) argue that a multivariate framework induces a higher sample size and a more efficient usage of information. In our Monte Carlo study, backtests based on orthonormal polynomials performed best. The expansion of these procedures to a multivariate framework would therefore be an alternative to the approaches which are commonly used. Backtesting with multivariate orthonormal polynomials includes the assumption that the duration sequences follow an associated discrete or continuous multivariate distribution under the null and the approximation of these distributions by Laguerre polynomials in the continuous case. Therefore, the idea of multivariate backtesting with Laguerre polynomials is a topic to be pursued in further research.

Appendix to Chapter 3

Quasi-Maximum-Likelihood estimation of GARCH(1,1)

The following statements refer to Francq and Zakoïan as well as Escanciano and Olmo (2011).

The model is a pure GARCH(1,1) $Y_t = \mu + \sigma_t \varepsilon_t$ with $\sigma_t^2 = \theta_0 + \theta_1 Y_{t-1}^2 + \theta_2 \sigma_{t-1}^2$ with $\mu = 0$, innovation $\varepsilon_t = Y_t/\sigma_t \overset{iid}{\sim} t(\nu)$ and parameter vector $\theta = (\theta_0, \theta_1, \theta_2)$.

Asymptotic normality of QMLE:

$$\sqrt{T}(\hat{\theta} - \theta)' \xrightarrow{d} N(0, V)$$
$$V = J^{-1} I J^{-1}$$

Conditional Gaussian quasi-log-likelihood:

$$L = \prod \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{Y_t^2 - \mu}{2\sigma_t^2}\right)$$
$$\Rightarrow \qquad \tilde{l}_t = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma_t^2) - \frac{1}{2}\frac{Y_t^2}{\sigma_t^2} = -\frac{1}{2}\left\{\log(2\pi) + \log(\sigma_t^2) + \frac{Y_t^2}{\sigma_t^2}\right\}$$

Score:

$$\frac{\partial \tilde{l}_t}{\partial \theta} = -\frac{1}{2}\left\{\frac{\partial(\log(\sigma_t^2))}{\partial \theta} + \frac{\partial(\frac{Y_t^2}{\sigma_t^2})}{\partial \theta}\right\} = -\frac{1}{2}\left\{\frac{1}{\sigma_t^2}\frac{\partial \sigma_t^2}{\partial \theta} - \frac{Y_t^2}{\sigma_t^4}\frac{\partial \sigma_t^2}{\partial \theta}\right\}$$
$$= -\frac{1}{2}\left\{1 - \frac{Y_t^2}{\sigma_t^2}\right\}\left\{\frac{1}{\sigma_t^2}\frac{\partial \sigma_t^2}{\partial \theta}\right\} = -\frac{1}{2}\{1 - \varepsilon_t^2\}\left\{\frac{1}{\sigma_t^2}\frac{\partial \sigma_t^2}{\partial \theta}\right\}$$

Hessian:

$$\frac{\partial^2 \tilde{l}_t}{\partial \theta \partial \theta'} = -\frac{1}{2}\left\{-Y_t^2 \frac{\partial \sigma_t^{-2}}{\partial \theta}\frac{1}{\sigma_t^2}\frac{\partial \sigma_t^2}{\partial \theta} + \left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\left(\frac{\partial \sigma_t^{-2}}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta} + \frac{1}{\sigma_t^2}\frac{\partial^2 \sigma_t^2}{\partial \theta \partial \theta'}\right)\right\}$$
$$= -\frac{1}{2}\left\{-Y_t^2 \frac{\partial \sigma_t^{-2}}{\partial \theta}\frac{1}{\sigma_t^2}\frac{\partial \sigma_t^2}{\partial \theta} + \left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\left(\frac{\partial \sigma_t^{-2}}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta}\right) + \left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\left(\frac{1}{\sigma_t^2}\frac{\partial^2 \sigma_t^2}{\partial \theta \partial \theta'}\right)\right\}$$
$$= -\frac{1}{2}\left\{\frac{\partial \sigma_t^{-2}}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta}\left[-\frac{Y_t^2}{\sigma_t^2} + \left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\right] + \left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\left(\frac{1}{\sigma_t^2}\frac{\partial^2 \sigma_t^2}{\partial \theta \partial \theta'}\right)\right\}$$
$$= -\frac{1}{2}\left\{-\frac{1}{\sigma_t^4}\frac{\partial \sigma_t^2}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta}\left(1 - 2\frac{Y_t^2}{\sigma_t^2}\right) + \left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\left(\frac{1}{\sigma_t^2}\frac{\partial^2 \sigma_t^2}{\partial \theta \partial \theta'}\right)\right\}$$
$$= -\frac{1}{2}\left\{\left(1 - \frac{Y_t^2}{\sigma_t^2}\right)\left(\frac{1}{\sigma_t^2}\frac{\partial^2 \sigma_t^2}{\partial \theta \partial \theta'}\right) + \left(2\frac{Y_t^2}{\sigma_t^2} - 1\right)\frac{1}{\sigma_t^4}\frac{\partial \sigma_t^2}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta}\right\}$$
$$= -\frac{1}{2}\left\{(1 - \varepsilon_t^2)\left(\frac{1}{\sigma_t^2}\frac{\partial^2 \sigma_t^2}{\partial \theta \partial \theta'}\right) + (2\varepsilon_t^2 - 1)\frac{1}{\sigma_t^4}\frac{\partial \sigma_t^2}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta}\right\}$$

Expected value of Hessian, J:

$$J = E\left[-\frac{1}{2}\left\{(1-\varepsilon_t^2)\left(\frac{1}{\sigma_t^2}\frac{\partial^2\sigma_t^2}{\partial\theta\partial\theta'}\right) + (2\varepsilon_t^2-1)\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta}\right\}\right] = \frac{1}{2}\left\{E\left[\frac{1}{2}(2\varepsilon_t^2-1)\right]E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta}\right]\right\}$$
$$= \frac{1}{2}(2E(\varepsilon_t^2)-1)E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta}\right] = \frac{1}{2}E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right]$$

Expected value of squared score, I:

$$I = E\left[-\frac{1}{2}(1-\varepsilon_t^2)\frac{\partial\sigma_t^2}{\partial\theta}\frac{1}{\sigma_t^2}\left(-\frac{1}{2}(1-\varepsilon_t^2)\frac{\partial\sigma_t^2}{\partial\theta}\frac{1}{\sigma_t^2}\right)'\right] = E\left[\frac{1}{4}(1-\varepsilon_t^2)^2\right]E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right]$$
$$= \frac{1}{4}(E(\varepsilon_t^4)+1-E(\varepsilon_t^2))E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right] = \frac{1}{4}(E(\varepsilon_t^4)-1)E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right]$$
$$= \frac{1}{2}(E(\varepsilon_t^4)-1)\frac{1}{2}E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right] = \frac{1}{2}(E(\varepsilon_t^4)-1)J$$

Hence, asymptotic covariance matrix of QMLE, V:

$$V = J^{-1}\frac{1}{2}(E(\varepsilon_t^4)-1)JJ^{-1} = J^{-1}\frac{1}{2}(E(\varepsilon_t^4)-1)$$
$$= \frac{1}{2}(E(\varepsilon_t^4)-1)2\left[E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right]\right]^{-1} = (E(\varepsilon_t^4)-1)E\left[\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right]^{-1}$$

Consistent estimate of V:

$$\hat{V} = (\kappa-1)\left[P^{-1}\sum_{t=R+1}^{n}\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}\right]^{-1},$$

whereby

$$\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'} = \begin{pmatrix} \psi^2 & \psi\sum_{j=1}^{\infty}\theta_2^{j-1}Y_{t-j}^2 & \psi\sum_{j=1}^{\infty}\theta_2^{j-1}\sigma_{t-j}^2 \\ \psi\sum_{j=1}^{\infty}\theta_2^{j-1}Y_{t-j}^2 & \left(\sum_{j=1}^{\infty}\theta_2^{j-1}Y_{t-j}^2\right)^2 & \sum_{j=1}^{\infty}\theta_2^{j-1}Y_{t-j}^2\sum_{j=1}^{\infty}\theta_2^{j-1}\sigma_{t-j}^2 \\ \psi\sum_{j=1}^{\infty}\theta_2^{j-1}\sigma_{t-j}^2 & \sum_{j=1}^{\infty}\theta_2^{j-1}Y_{t-j}^2\sum_{j=1}^{\infty}\theta_2^{j-1}\sigma_{t-j}^2 & \left(\sum_{j=1}^{\infty}\theta_2^{j-1}\sigma_{t-j}^2\right)^2 \end{pmatrix}$$

applies, in which $\psi \equiv (1-\theta_2)^{-1}$ is set, while κ denotes the unstandardized kurtosis.

Consistent estimate of A:

$$\hat{A} = f(F_\varepsilon^{-1})F_\varepsilon^{-1}\frac{1}{P}\sum(\frac{1}{\sigma_t}\frac{\partial\sigma_t}{\partial\theta} = f(F_\varepsilon^{-1})F_\varepsilon^{-1}\frac{1}{P}\sum\begin{bmatrix}\frac{1}{2\sigma_t^2(1-\theta} \\ \frac{1}{\sigma_t^2}\sum_{j=1}^{\infty}\theta^{j-1}y_{t-j}^2 \\ \frac{1}{\sigma_t^2}\sum_{j=1}^{\infty}\theta j-1\sigma_{t-j}^2\end{bmatrix}$$

Chapter 4

Downside Risk Measure Performance in the Presence of Breaks in Volatility

4 Downside Risk Measure Performance in the Presence of Breaks in Volatility

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4.1 Introduction

During the past decades, a growing awareness for the importance of an accurate risk management of a financial institution has evolved. Since the 1996 amendment of the Basel Accord[5] on regulatory capital for market risk, banks are demanded to implement internal models for measuring market risk (BCBS (1996b, 1997)). A main objective of the Second Basel Accord (BCBS (2004a)) addresses the calculation of risk-sensitive minimum capital requirements and the definition of standards for the quantitative measurement of financial risk. In this context, Value at Risk (VaR) approaches are recommended as the appropriate instruments for assessing the market risk exposure of a financial institution and are widely used in financial risk management. However, the recent reviews of the Basel Accords redefine the capital rules for market risk and include the proposition to gradually replace VaR with Expected Shortfall (ES) by 2019 (see the consultative documents of the Third Basel Accord issued by BCBS (2012, 2013b)).

Hendricks and Hirtle (1997) point out that the benefit arising from a model-based capital requirement is undermined by the use of incorrect models, which indicates that the evaluation of the accuracy of the underlying risk models has to be of primary concern for banks and regulatory authorities. Backtesting frameworks represent the preferred tool to evaluate the performance of risk measures, even though numerous tests suffer from a lack of statistical power when following the recommendation of the Basel Committee to adopt an evaluation horizon of one year. This constitutes a widely examined issue which is described, among others, within the works of Lucas (2001), Campbell (2005), Nieppola (2009) and Røynstrand et al. (2012). In order to overcome this drawback, Lopez (1998) introduces loss function approaches as an alternative evaluation method which is not based on hypothesis testing, but draws upon forecast evaluation techniques. Campbell (2005) refers to the capability of targeting specific concerns of a financial institution by choosing a certain type of loss function and emphasizes the usefulness for the distinction between competing risk models.

Financial risk is often identified with the behavior of an asset's volatility. Consequently, the evaluation and the accuracy of the risk model strongly depends on the variance of the profit and loss series of the financial institution. A lot of evidence for occasional structural breaks in the volatility of financial time series is provided by, among others, Lamoureux and Lastrapes (1990) and Amihud and Mendelson (1991) and more recently within the works of Diamandis (2008) and Eichengreen et al. (2012). Hence, a financial institution should preferably employ a risk measure which is characterized by a reaction of sufficient sensitivity to the occurrence of a break in volatility in order to be

[5] All remarks about the Basel Accords refer to the frameworks issued by the Basel Committee on Banking Supervision (BCBS).

able to ensure an immediate adjustment of the underlying risk measure. While a variety of research addresses the development of testing procedures regarding the detection of a structural change (of unknown date) and the estimation of the break date (see Hansen (2001) and Perron (2006) for an overview of the testing and estimation methodology), the literature on the characteristics of risk measures thus far lacks an analysis of their performance in presence of a structural break in volatility and a substantiated recommendation on which measure to give priority in this matter.

This paper provides a theory-based comparison of downside risk measures regarding their responsiveness to structural breaks in volatility and distribution. A loss function-based framework for the theoretical design and the application performance of the comparative scenario study is proposed by extending the model comparison approach introduced by Lopez (1998). Even though the theoretical aspects generally address the comparison of any two quantile risk measures, the main focus of the application comprises the confrontation of VaR and ES.

The remainder of the paper is organized as follows: Section 4.2 provides a brief literature overview of the current status of research on risk measures and the previous utilization of loss functions in risk evaluation. In Section 4.3, the most common downside risk measures are reviewed and assessed whether they fulfill mathematical and practical requirements on measuring financial risk. Section 4.4 introduces the usage of loss functions for risk evaluation and develops a framework to compare the sensitivity of risk measures in response to a structural break in volatility as well as in reaction to a change in distribution. Moreover, theoretical results regarding the predominance of risk measures in presence of breaks are presented. The validness of these results for realistic evaluation horizons are examined within a broad simulation study presented in Section 4.5, which surveys the performance of VaR and ES for common DGPs and accounts for the direction and the intensity of the volatility break. In Section 4.6, the simulation results are reconfirmed by applying the proposed evaluation technique to several stock indices series. A conclusion of the work is provided in Section 4.7.

4.2 Literature Review

Even though VaR represents the most commonly used risk measure within financial risk management, the suitability of VaR has been questioned since it became the benchmark tool for assessing the exposure to market risk. Hendricks (1996) considers different VaR approaches for simulated portfolios. While he can attest an accurate performance to all examined methods at a 95% level, an understatement of the actual risk can be observed at a 99% level. This finding is endorsed by Bao et al. (2006), who investigate the predictive performance of VaR models in terms of several emerging Asian economies within the financial crisis of the late 90's. Berkowitz and O'Brien (2002) present evidence on the VaR model performance for large trading firms and conclude that the reported VaR estimates are not appropriate to indicate the firms' actual portfolio risk. Moreover, simple ARMA-GARCH models appear to outnumber VaR in terms of their forecasting performance, while VaR is not able to reflect volatility changes of the profit and loss series of the firms. Arising from all these shortcomings in performance and because of some further theoretical drawbacks (see Section 4.3.1 for a discussion), Acerbi and Tasche (2002) are surprised by the fact that VaR has been adopted by essentially all banks and regulators. Recent research about the improvement of the accuracy of VaR forecasts includes e.g. the article by Halbleib and Pohlmeier (2012).

ES has frequently been considered as an alternative to VaR for evaluating market risk exposure and numerous academic contributions of the more recent past deal with the comparison of VaR and ES by focussing on different aspects. Yamai and Yoshiba (2002) provide an overview of those studies by then and carve out that rational investors are often misled by employing VaR, which can be mitigated by adopting ES as main risk measure. However, an important requirement for its practicality constitutes the availability of efficient methods for backtesting ES. The findings of Basu (2006), who examines the impact of stress scenarios to the performance of VaR and ES, indicate that the responsiveness of VaR to shocks for historical simulations remains low, while ES is more suitable for capturing the impact of stress. Chen (2014) evaluates the effectiveness of the recent Basel reforms with regard to the regulatory reservations arising from the usage of VaR. While he criticizes ES for its lack of elicitability[6] and hence denies reliability of the results from backtesting ES, Acerbi and Székely (2014) propose three methodologies for backtesting ES and allege elicitability to be irrelevant for backtesting risk measures. Emmer et al. (2014) support this result, even though conceding that for these procedures more data is required than for backtesting VaR in order to reach an equivalent level of certainty. An analogue to the well-known conditional backtesting framework for VaR estimates is suggested by Escanciano and Du (2015) for the evaluation of ES forecasts.

Loss functions represent a widely used tool for assessing the prediction performance of competing models. After Lopez (1998) proposed three different types of loss functions and their utilization for measuring the accuracy of VaR estimates, this method became an established procedure for the evaluation of risk measures as well. Generalizations of this conception are provided by the works of Lopez (2001), in which economic loss functions are incorporated into a volatility forecasting framework, and Caporin (2008), who introduces a new set of loss functions for the purpose of comparing VaR measures in the presence of long memory effects. In further articles, loss function techniques are applied for the evaluation of the forecasting performance of several rival volatility models in VaR frameworks. These include González-Rivera et al. (2004), in which a goodness-of-fit loss function based on a VaR calculation is employed, and Amendola and Candila (2014), who suggest an asymmetric loss function for this purpose. Degiannakis et al. (2013) employ a quadratic loss function in order to examine whether conditional volatility models accounting for long memory outperform those implying short memory when forecasting VaR and ES. In a current paper, Abad et al. (2015) investigate whether the choice of a certain type of loss function affects the comparison of VaR models by additionally accounting either for the firm's or the regulator's point

[6]Elicitability represents a criterion for determining the optimal point forecast of a functional on a class of probability measures $\mathcal{P} \ni P$ (see Emmer et al. (2014)). In simple terms, a functional is elicitable relative to $\mathcal{P}$ if its optimal estimate $\hat{y}$ minimizes the expectation of a scoring function $\mathcal{S}(y, Z)$,

$$\hat{y} = \arg\min_{y} E_P[\mathcal{S}(y, Z)],$$

whereby Z denotes a random variable defined on $(\Omega, \mathcal{F}, P)$. Gneiting (2011) shows that ES fails to be elicitable and provides a discussion about the importance of elicitability for the comparison of different prediction methods. Ziegel (2014) generalizes this result to nearly all law-invariant spectral risk measures.

of view. Moreover, Campbell (2005) describes how loss function-based backtests can be conducted and remarks the enhanced flexibility of this approach.

4.3 Measuring Downside Risk

The intention pursued by applying a risk measure to some random variable X modeling the profit and loss (P&L) of a portfolio is to quantify its underlying risk and determine a minimum capital requirement to ensure that the risky position is acceptable to the regulatory authorities.

Following the axiomatic approach initiated by Artzner et al. (1999), a risk measure is supposed to feature certain desirable properties in order to be suitable for measuring financial risk. For this purpose, consider a linear space $\mathcal{H}$ of measurable functions $X : \Omega \to \mathbb{R}$, where Ω contains a fixed and finite set of possible future scenarios. Then, a mapping $\rho : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is called a coherent risk measure for $\mathcal{H}$ if the axioms (I)-(IV) are fulfilled:

$$\begin{aligned}
&\text{(I) Monotonicity:} && X_1 \overset{a.s.}{\leq} X_2;\ X_1, X_2 \in \mathcal{H} && \Rightarrow && \rho(X_2) \leq \rho(X_1) \\
&\text{(II) Subadditivity:} && X_1, X_2, X_1 + X_2 \in \mathcal{H} && \Rightarrow && \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2) \\
&\text{(III) Positive homogeneity:} && a \in \mathbb{R}_{\geq 0};\ X, aX \in \mathcal{H} && \Rightarrow && \rho(aX) = a\rho(X) \\
&\text{(IV) Translation invariance:} && a \in \mathbb{R};\ X \in \mathcal{H} && \Rightarrow && \rho(X + a) = \rho(X) - a
\end{aligned}$$

Note that for $a = 0$, axiom (III) implies normalization for ρ, i.e. $\rho(0) = 0$.

Föllmer and Schied (2002) propose a revision of the concept of coherent risk measures by replacing (II) and (III) by a weaker axiom: A risk measure which satisfies the axioms (I) and (IV) belongs to the class of convex risk measures if it additionally fulfills the axiom

$$\text{(V) Convexity:} \quad \rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$$

for $X_1, X_2 \in \mathcal{H}$ and $\lambda \in (0;1)$. Subject to validity of (III), the axioms (II) and (V) are equivalent (see Föllmer and Schied (2010)).

4.3.1 Value At Risk

Let $F_X(\cdot)$ be the cdf of the P&L random variable X. Then, the Value at Risk for an exogenously given confidence level $1 - \alpha$ is determined by

$$\begin{aligned}
\mathrm{VaR}_\alpha(X) &= \inf\{x \in \mathbb{R} : \mathrm{P}(X > x) \leq 1 - \alpha\} \\
&= \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\} = F_X^{-1}(\alpha),
\end{aligned} \tag{25}$$

whereby $\alpha := \mathrm{P}(X \leq \mathrm{VaR}_\alpha(X))$ holds and the second equality only applies for parametric VaR approaches. VaR can easily be interpreted as the return which is exceeded in $100 \cdot (1 - \alpha)\%$ of all periods. The simplicity of interpretation is one of the main reasons why VaR has evolved as an industry standard tool for financial institutions. Several techniques for the estimation of VaR exist, of which the historical simulation provides one of the simplest and most practical methods as it does not require any distributional assumptions. For a data set of m observations, the historical $\text{VaR}_\alpha(X)$ estimator is based on the sequence of past P&L realizations $\{x_t\}_{t=1}^m$ and can be defined by

$$\widehat{\text{VaR}}_\alpha(X) = q_\alpha\left(\{x_t\}_{t=1}^m\right), \tag{26}$$

whereby $q_\alpha(\cdot)$ denotes the quantile function for level α.

Despite its popularity in application, a couple of shortcomings of VaR include practical and intuitive issues as well as mathematical defects. Firstly, VaR considers only a single quantile of the underlying probability distribution, while all rare events of the downside tail are disregarded since the amount of the actual loss is not taken into account. Thus, a false sense of security could arise from the usage of VaR and lead to excessive risk taking (see Einhorn and Brown (2008)).

Furthermore, as can easily be shown by a simple counterexample (see Artzner et al. (1999) and Acerbi and Tasche (2001) for details), VaR fails to satisfy axiom (II) of subadditivity and thus does not represent a coherent risk measure. However, this contradicts the principle of diversification - one of the key concepts of modern portfolio theory, which consists in the postulation that the risk of an aggregate position should not be higher than the sum of the risks of the single positions. In terms of risk management, the possibility of reducing risk (and thus capital requirements) by splitting the risk up into its integral parts should be excluded by validity of (II). As a result of the aforementioned limitations of VaR, several alternative approaches of risk assessment have emerged.

4.3.2 Lower Partial Moments and Expected Shortfall

Risk measures which ensure the incorporation of the downside risk distribution provide an alternative to the frequently employed VaR approaches. Lower Partial Moments (LPM) were (mainly) introduced in financial economics by Fishburn (1977) and Bawa (1978) and define a family of downside risk measures specified by order $n \in \mathbb{N}_0$ and a target value $\tau \in \mathbb{R}$ from which the negative deviations are gauged.

Let X be a continuous and integrable random variable measuring a portfolio's P&L. Then, the general definition of LPM depending on n and τ is given by

$$\text{LPM}(X;\tau,n) = E\left[\max(\tau - x;0)^n\right] = \int_{-\infty}^{\tau} (\tau - X)^n \, dF_X,$$

whereby the latter equality holds if F_X represents a continuous distribution. LPM directly refer to the deviance from the reference level τ and are, unlike VaR, not related to a predetermined probability level. Depending on the problem under consideration, the reference level may be any suitable attractor, such as the expected return on portfolios, the rate of inflation or simply the point separating profits and losses. However, in terms of financial risk management the contemplated target frequently (and in the following) concerns VaR, i.e. $\tau \equiv \text{VaR}_\alpha(X)$.

Define $\text{LPM}(X; \text{VaR}_\alpha(X), n) =: \text{LPM}_{n,\alpha}(X)$ for simplification as well as $\mathbb{1}_{\{X \leq \text{VaR}_\alpha(X)\}}$ to be the indicator function for X falling short of $\text{VaR}_\alpha(X)$. In line with the work of Danielsson et al. (2006), the relation $\text{LPM}_{1,\alpha}(X) = \alpha \text{VaR}_\alpha(X) - E[X \, \mathbb{1}_{\{X \leq \text{VaR}_\alpha(X)\}}]$ holds when the LPM of order $n = 1$ is represented as a quantile of X. [7] However, as emphasized by Barbosa and Ferreira (2004), Lower Partial Moments do not belong to the class of coherent risk measures.

Expected Shortfall represents another downside risk measure, which constitutes a more established alternative in risk management than LPM measures. With respect to the target value $\text{VaR}_\alpha(X)$ its definition is given by

$$\text{ES}_\alpha(X) = E\left[X | X \leq \text{VaR}_\alpha(X)\right] = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\varphi(X) \, d\varphi = \text{VaR}_\alpha(X) - \frac{1}{\alpha} \text{LPM}_{1,\alpha}(X). \tag{27}$$

The first equality marks the character as conditional expectation of the $100 \cdot \alpha\%$ worst losses, while the second targets the property of ES to be the mean VaR over all levels lower than α. The last equality refers to the close relation to the class of LPM and VaR since ES results from the difference of the target value (VaR) and the scaled $\text{LPM}_{1,\alpha}(X)$.

Due to the fact that common values for α are 5% or 1%, ES usually assumes substantially larger values than LPM_1. The representation of ES in terms of the actual P&L distribution indicates that the ES-related quantile is given by the difference of $F_X^{-1}(\alpha)$ and $\text{ES}_\alpha(X)$. Hence, ES pays much more attention to the tail of the distribution than VaR and LPM_1.[8] Next to the attribute that it constitutes possibly the most intuitive perception of risk, ES overcomes the theoretical drawback of VaR and the class of LPM as it provides a coherent risk measure, which is shown by Artzner et al. (1999), and furthermore fulfills convexity as is proved by Rockafellar and Uryasev (2000).

ES can easily be estimated by taking advantage of its relationship with LPM of first order. For a sample of size m, the estimator of $\text{LPM}_{n,\alpha}(X)$ is given by

$$\widehat{\text{LPM}}_{n,\alpha}(X) = \frac{1}{m} \sum_{t=1}^{m} \max\left(\text{VaR}_\alpha(X) - X_t; 0\right)^n. \tag{28}$$

Consequently, it follows that $\widehat{\text{ES}}_\alpha(X) = \widehat{\text{VaR}}_\alpha(X) - \alpha^{-1} \, \widehat{\text{LPM}}_{1,\alpha}(X)$.

4.4 The Comparison of Risk Measures by Using Loss Functions

Next to the more familiar strand of literature concerning backtesting methods, loss function approaches constitute a second group of procedures to evaluate risk measure estimates (see Caporin (2008)), which provide the opportunity to compare risk measures across financial institutions. While the general idea of the most backtesting approaches is based on counting the pure number

[7] Commonly, only the values $n \in \{0, 1, 2\}$ are matter of main interest. For $n = 0$, the downside probability results, which shows the close relation to VaR. $\text{LPM}_{1,\alpha}(X)$ can be interpreted as downside expected value of X. The order $n = 2$ provides the expected squared deviation from VaR given by $\text{LPM}_{2,\alpha}(X) = \int_{-\infty}^{\text{VaR}_\alpha(X)} (\text{VaR}_\alpha(X) - x)^2 \, dF_X$. If $\tau \equiv E[X]$, $\text{LPM}_{2,\alpha}(X)$ equals the semivariance of X.

[8] Under the assumption of a continuous distribution ES is also known as Tail VaR.

of shortfalls below VaR, BCBS (1996b) suggests to attach importance to both the number and the magnitude of violations within an institution's risk evaluation.

4.4.1 Measuring Loss

The objective followed by using loss functions for risk evaluation consists in the minimization of costs utilizing a risk measure ρ. The value assigned by the loss function at a point in time is commonly termed as the loss function's score. From regulatory point of view lower scores are preferred over higher ones, i.e. each shortfall below ρ increases the cumulative loss.

For some reference value $\rho \in \Psi$ (fixed in the following) desired not to be underrun by an estimator $Y_{i,t} \in \mathcal{Y}_i$, define the mapping $\Gamma : \Psi \times \mathcal{Y}_i \mapsto \mathbb{R}_+$ as the loss function which assigns a non-negative valued score at time t for some P&L process $\{Y_{i,t}\}$ of type i. The actual type of the loss function is to be chosen subject to the matter of concern of the evaluating institution. Although many types can be constructed, two disparate approaches of assigning loss scores proposed by Lopez (1998) are focused on within this work (see Rosasco et al. (2003) for an overview and an examination about the impact of choosing different types of loss functions).

The most straightforward and elementary method to evaluate losses is presented by the binomial loss function $\Gamma^B(\rho, Y_{i,t})$, where at time t the score

$$\Gamma^B(\rho, Y_{i,t}) = \mathbb{1}_{\{Y_{i,t}<\rho\}} = \begin{cases} 1 & \text{if } Y_{i,t} < \rho \\ 0 & \text{if } Y_{i,t} \geq \rho, \end{cases} \tag{29}$$

is assigned. The binomial loss function attaches a score of one for an observation whenever it involves a violation of the threshold value set by the risk measure. As it only takes the frequency of extreme losses into account, this approach shares similarities with backtests which focus on the property of unconditional coverage.

The amount of shortfall below the risk measure in case of a hit, however, is not accommodated by Γ^B. Kiliç (2006) advocates the usage of magnitude-type loss functions since one immense single hit could already cause appreciable upheavals within the financial institution in question. Using the historical simulation approach for the evaluation of VaR models, Hendricks (1996) finds portfolio losses which exceed the corresponding VaR estimate by about 30% on average and extreme losses of much higher intensity. Incorporating these aspects the score of the quadratic loss function $\Gamma^Q(\rho; Y_{i,t})$ assigned at t is defined by

$$\Gamma^Q(\rho, Y_{i,t}) = \begin{cases} 1 + (Y_{i,t} - \rho)^2 & \text{if } Y_{i,t} < \rho \\ 0 & \text{if } Y_{i,t} \geq \rho. \end{cases} \tag{30}$$

As before in case of an exception, the score comprises a fixed value of one, though an additional score imposed by $\Gamma^Q(\rho, Y_{i,t})$ now increases quadratically with the magnitude of the occurred hit. Therefore, the quadratic loss approach might be more suitable for financial risk evaluation.

The cumulative loss of the entire evaluation period directly results from the sum of scores across all observations. Since the distribution of the observations at time t depends on $\sigma(Y_{i,s}; s < t)$, assumptions on their dependence are to be made in order to determine the actual score for the entire period. However, market risk amendments included in the Basel Accords (see BCBS (1997)), which mandate an evaluation period of only 250 observations, entail that the assumption of iid observations is usually inevitable (see Lopez (1998) and Dowd (2007)).

4.4.2 Risk Measure Performance in Presence of a Break in Volatility

In his seminal paper, Lopez (1998) introduces the utilization of loss functions for the evaluation of VaR models. On this basis, a procedure for the comparison of the behavior of two quantile risk measures in presence of a break in volatility is developed and presented in this subsection.

Consider the situation that a break in the volatility of the P&L process occurs at the very beginning of the evaluation period. Then, it would be desirable for the financial institution to identify the break and adjust the process as quickly as possible in order to ensure a suitable evaluation of the underlying risk measure. In this regard, the capability of identification connotes that the break is reflected by the score of the loss function. For this purpose, the risk measures are evaluated for two different settings: On the one hand, the process which was imputed prior to the break is incorrectly assumed to prevail within the evaluation period as well, and on the other hand, the underlying process for evaluation is correctly adjusted for the change in volatility. In order to construct a measure for the sensitivity of risk measures in response to a structural break, consider the expected score assigned for the incorrect process to be the numerator of a quotient and the expected score assigned for the properly adjusted process to be the corresponding denominator.

If the risk measure shows an appropriate response to the occurrence of the break, the quotient should be greater than 1 if the volatility declines after the occurrence of the break, while the quotient should display a value less than 1 if the volatility increases. Therefore, the quotient itself serves as a measure for the sensitivity to a structural break of the underlying measure. High responsiveness for both directions of the volatility change can be attested if the quotient value strongly deviates from one. When comparing two risk measures, a higher sensitivity is adjudged for the risk measure which (depending on the direction of the break) features the quotient of the more extreme value. Thus, the risk measure of the smaller of both quotient values shows a more preferable response to an increase in volatility, and vice versa. Since the quotient values only depend on the particular risk measure, this procedure is easily expandable to a comparison of more than two quantile risk measures. For a pairwise comparison of e.g. three risk measures, a transitive inequality relation on the real numbers applies.

The two risk measures to be compared are distinguished by the α-quantiles of the cdf of the mistakenly imputed process, which each mark the boundary between acceptable and undesirable risk, i.e. $\rho \equiv F^{-1}(\alpha)$.

Let X and Y be two random variables, whose unconditional variances are related by $\sigma_Y^2 = \kappa\sigma_X^2$ with $\kappa \in \mathbb{R}_+\backslash\{1\}$. Apart from this, the distributions of X and Y are identical. Assume that the observations of the P&L process prior to the break exhibit the distribution of X, while the observations after the break follow the distribution of Y. Moreover, let $F(\cdot)$ be the cdf of X and define $q_t := F^{-1}(\alpha)$ and $\tilde{q}_t := F^{-1}(\tilde{\alpha})$ to be the quantiles of X which represent the risk measures in question, whereby $\alpha > \tilde{\alpha}$, $q_t < 0$ and $\tilde{q}_t < 0$ holds. The sensitivity of the underlying risk measures is gauged by the quotient of the loss functions of X and Y with reference values q_t and $\tilde{q}_t$, respectively. These quotients are defined for loss functions each of type $m \in \{B, Q\}$ by

$$\Theta := \frac{\Gamma^m(X, q_t)}{\Gamma^m(Y, q_t)} \qquad \text{and} \qquad \tilde{\Theta} := \frac{\Gamma^m(X, \tilde{q}_t)}{\Gamma^m(Y, \tilde{q}_t)}. \tag{31}$$

Let the following assumptions (although not very restrictive within risk management) be imposed on both X and Y:

A1 The density functions are centered around 0.

A2 The probability measures are quasiconcave.

A3 The cdf's are strictly monotone on their entire supports.

Note that A1 implies $\frac{1}{2} > \alpha > \tilde{\alpha}$.

Proposition 1. *Let $m = B$. Under validity of A1-A3 the following conclusions apply:*

$$\text{For } \kappa < 1: \quad E[\Theta] < E[\tilde{\Theta}] \tag{32}$$

$$\text{For } \kappa > 1: \quad E[\Theta] > E[\tilde{\Theta}] \tag{33}$$

Proof. See Appendix A.1.

Proposition 2. *Let $m = Q$. Under validity of A1-A3 the following relations apply:*

$$\text{For } \kappa < 1: \quad E[\Theta] < E[\tilde{\Theta}] \qquad \text{if} \qquad \frac{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right) F(q_t)}{F\left(\frac{q_t}{\sqrt{\kappa}}\right) F(\tilde{q}_t)} < \frac{\kappa \tilde{q}_t^2 + q_t^2}{\kappa q_t^2 + \tilde{q}_t^2} \tag{34}$$

$$\text{For } \kappa > 1: \quad E[\Theta] > E[\tilde{\Theta}] \qquad \text{if} \qquad \frac{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right) F(q_t)}{F\left(\frac{q_t}{\sqrt{\kappa}}\right) F(\tilde{q}_t)} > \frac{\kappa \tilde{q}_t^2 + q_t^2}{\kappa q_t^2 + \tilde{q}_t^2} \tag{35}$$

Proof. See Appendix A.2.

For the evaluation by means of the binomial loss function (see (32) and (33)), the general conclusion of predominance of the risk measure which is represented by the lower of the compared quantiles can be drawn. Thus, the quantile risk measure $\tilde{q}_t$ features a higher sensitivity to the occurrence of a break than q_t - regardless of the direction of the volatility change.

Taking a closer look on (32), the quotient of expected scores should be preferably large if a break causes the volatility to decrease. As $\tilde{\Theta}$ involves the $\tilde{\alpha}$-quantile and $E[\Theta] < E[\tilde{\Theta}]$ applies for $\kappa < 1$, $\tilde{q}_t$ is more suitable to identify the break than q_t. The same result can be observed for $\kappa > 1$ (see (33)): Since the quotient values should be as small as possible when Y exhibits higher volatility than X and $E[\tilde{\Theta}] < E[\Theta]$ holds, the $\tilde{\alpha}$-based risk measure $\tilde{q}_t$ predominates the α-based risk measure

q_t for volatility increases as well.

If the binomial loss function is used for evaluation, these statements are true for any distribution which satisfies the assumptions A1-A3. In contrast, the implications drawn for assigning binomial loss scores only apply for the quadratic loss function (see (34) and (35)) if certain conditions are fulfilled. These depend on values of the cdf's of the α- and $\tilde{\alpha}$-quantiles of both X and Y, even though the quantiles of Y are converted into the respective cdf values of X for the propositions. Figure 4.1 illustrates the location of these quantiles with respect to the density of X. Note that the conditions for $m = Q$ do not depend on the actual level of the variances, but only on the volatility ratio κ. Obviously, the limit cases show Θ, $\tilde{\Theta} \to 0$ if $\kappa \to \infty$ and Θ, $\tilde{\Theta} \to \infty$ if $\kappa \to 0$.

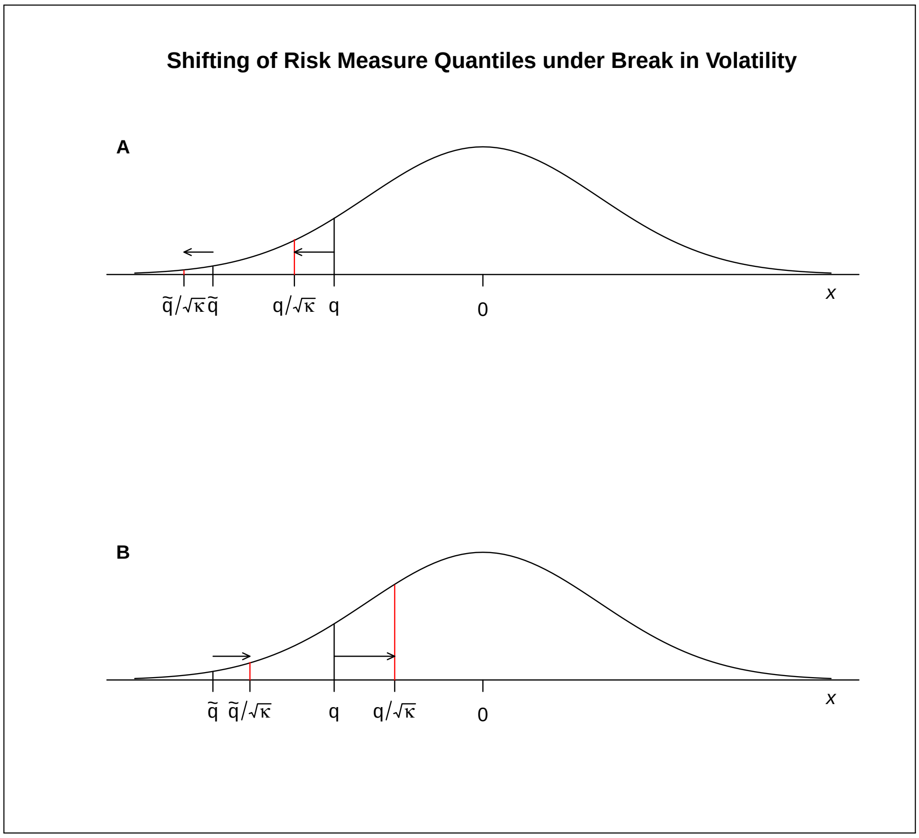

Figure 4.1: Risk measures represented by quantiles q and $\tilde{q}$ shift within the distribution of X when a break in volatility of intensity κ occurs. The quantiles of Y are notated in terms of X (red continuous lines). If $\kappa < 1$, quantiles decrease for X (as exemplified in Part A), while quantiles increase if $\kappa > 1$ (as exemplified in Part B).

However, the conditions set in (34) and (35) are met for any combination of quantiles and ratios of volatilities if the innovation generating distribution is either Gaussian or Student-t. Both distributions satisfy the imposed assumptions A1-A3 and represent the most of all applied distributions for modeling innovations in risk management.[9] In order to provide evidence for this argument, calculations regarding the fulfilment of these conditions are performed. The results are listed in Tables 4.1/a and 4.1/b.

[9]Further details about properties of probability distributions with focus on concavity are described in the work by Koenker and Mizera (2010).

The left hand side (the functional of cdf's, abbreviated by F) and the right hand side (the functional of quantiles, abbreviated by Q) of the conditions are stated for several combinations of $\tilde{\alpha}$ and κ for both the Gaussian (4.1/a) and the Student-t case (4.1/b). While the α-level for quantile q is fixed to 5%, the $\tilde{\alpha}$-quantile varies to regard different distances between the quantiles. As indicated by (34), Q>F holds within the upper parts of the tables (in which $\kappa < 1$ is valid), while the lower parts provide evidence for (35) as F>Q applies in each case. Both cases indicate that for any fixed α as $\tilde{q} \to q$ F$\nearrow$ Q holds if $\kappa < 1$ and F$\searrow$ Q if $\kappa > 1$, while for fixed quantiles F$\nearrow$ Q applies as $k \nearrow 1$, and vice versa.

a	κ							
$\alpha = 0.05$	0.1		0.5		0.9		0.95	
$\tilde{\alpha}$	F	Q	F	Q	F	Q	F	Q
0.0499	0.98559	0.99904	0.99835	0.99961	0.99981	0.99994	0.99991	0.99997
0.045	0.46431	0.95169	0.91627	0.98003	0.99018	0.99682	0.99533	0.99845
0.04	0.19548	0.90304	0.83038	0.95933	0.97924	0.99347	0.99010	0.99681
0.025	0.00579	0.75138	0.55720	0.89066	0.93624	0.98190	0.96923	0.99114
0.01	0.00001	0.57136	0.25039	0.79996	0.85590	0.96551	0.92886	0.98305
0.001	0.00000	0.37276	0.03101	0.68612	0.67778	0.94289	0.83162	0.97177
	1.05		1.1		1.5		2	
$\tilde{\alpha}$	F	Q	F	Q	F	Q	F	Q
0.0499	1.00008	1.00003	1.00015	1.00006	1.00057	1.00024	1.00087	1.00039
0.045	1.00426	1.00148	1.00816	1.00289	1.03056	1.01218	1.04666	1.02038
0.04	1.00907	1.00304	1.01741	1.00595	1.06602	1.02522	1.10167	1.04240
0.025	1.02876	1.00850	1.05569	1.01666	1.22199	1.07190	1.35441	1.12277
0.01	1.06921	1.01640	1.13642	1.03226	1.60389	1.14289	2.04202	1.25006
0.001	1.18186	1.02762	1.37606	1.05464	3.24389	1.25147	5.89878	1.45747

b	κ							
$\alpha = 0.05$	0.1		0.5		0.9		0.95	
$\tilde{\alpha}$	F	Q	F	Q	F	Q	F	Q
0.0499	0.99322	0.99821	0.99760	0.99944	0.99980	0.99992	0.99991	0.99996
0.045	0.68566	0.91064	0.87680	0.97154	0.98927	0.99597	0.99515	0.99804
0.04	0.42803	0.82196	0.74728	0.94173	0.97683	0.99167	0.98952	0.99593
0.025	0.03483	0.55879	0.33917	0.84122	0.92327	0.97644	0.96493	0.98846
0.01	0.00000	0.29467	0.02957	0.70848	0.80659	0.95405	0.90925	0.97735
0.001	0.00000	0.12390	0.00000	0.56300	0.51242	0.92351	0.74840	0.96200
	1.05		1.1		1.5		2	
$\tilde{\alpha}$	F	Q	F	Q	F	Q	F	Q
0.0499	1.00008	1.00004	1.00014	1.00007	1.00043	1.00030	1.00059	1.00050
0.045	1.00407	1.00187	1.00752	1.00366	1.02348	1.01546	1.03195	1.02590
0.04	1.00882	1.00389	1.01634	1.00760	1.05129	1.03231	1.06999	1.05444
0.025	1.02995	1.01112	1.05579	1.02182	1.17975	1.09490	1.24863	1.16333
0.01	1.08058	1.02205	1.15233	1.04350	1.52455	1.19628	1.75063	1.35003
0.001	1.25786	1.03757	1.51501	1.07468	3.26800	1.35603	4.67320	1.67328

Tables 4.1/a and 4.1/b: Calculations for the conditions set in (34) and (35) employing the Gaussian distribution (4.1/a) and the Student-t distribution (4.1/b). F and Q tag the left and the right hand side of the conditions, resp. The upper quantile level is fixed to $\alpha = 0.05$.

4.4.3 Risk Measure Performance in Presence of a Change in Distribution

Thus far, the switch in the volatility was assumed to be directly caused by a break in the second moment of the innovation process. In contrast, the focus of this subsection lies on volatility breaks induced by a change of the distribution of the innovation process and the comparative investigation of the ability of risk measures to discriminate between models of the same type, but with innovations drawn from distributions of different families. This aspect becomes practically relevant if e.g. a financial institution evaluates its utilized risk measure by assuming the wrong distribution. Under consideration of the most pertinent distributions in financial statistics, the Gaussian distribution is supposed to be erroneously postulated instead of the Student-t.

Maintaining the notation and the general setting of the previous subsection, consider two random variables $X \sim N(0;1)$ with cdf $F(\cdot)$ and $Y \sim t(\nu)$. While the Gaussian distribution prevails during the in-sample period and for the mistakenly perpetuated model for evaluation, the Student-t distribution holds true for the alternative model during the evaluation period. The sensitivity of the underlying quantiles q_t and $\tilde{q}_t$ to a change in distribution is measured by the quotients

$$\Theta^{\star} := \frac{\Gamma^m(X, q_t)}{\Gamma^m(Y, q_t)} \qquad \text{and} \qquad \tilde{\Theta}^{\star} := \frac{\Gamma^m(X, \tilde{q}_t)}{\Gamma^m(Y, \tilde{q}_t)}. \tag{36}$$

On this basis, the following proposition about the general comparative performance of risk measures in the presence of a change in distribution can be derived:

Proposition 3. *If $m \in \{B, Q\}$ and assumptions A1-A3 apply, $E[\Theta^{\star}] > E[\tilde{\Theta}^{\star}]$ holds.*

Proof. See Appendix A.3.

Since $Var(Y) \searrow Var(X) = 1$ as $\nu \to \infty$ and thus Y implicitly exhibits higher volatility than X for any choice of ν, the risk measure which shows the smaller quotient always outclasses the other. Hence, in accordance with the results for a break in volatility, this result indicates superiority for the risk measure involving the lower quantile $\tilde{q}$ in distinguishing between models which feature different distributions.

4.5 VaR vs. ES: A Comparative Simulation Study

The question arising is whether the theoretical results for the risk measure performance described in the previous section hold for simulations over realistic evaluation horizons. The simulation study is carried out as a comparison of two specific quantile risk measures, in fact VaR and ES (see (25) and (27)). By specifying ES in terms of a quantile of the P&L distribution (see Section 4.3.2), ES is considered to be the lower quantile $\tilde{q}_t$ and expected to outclass VaR (representing the higher quantile q_t) in distinguishing between different models.

4.5.1 Settings and DGP Configurations

The aim pursued in this section is to simulate the values of the quotients of the loss functions as defined by (31) and (36), respectively, under the assumption of certain settings. All examinations are performed in comparison to a reference time series model denoted by $\{X_{t,i}\}$ (in the following referred to as the "benchmark model") and premise on historical 1-day VaR and ES estimates. The benchmark model is valid during the in-sample period, from which the underlying risk measure is estimated. The estimated measures are appraised by a scenario analysis, which contrasts $\{X_{t,i}\}$

with the alternative model $\{Y_{t,i}\}$ over the course of the evaluation period. Certain properties of the alternative model, which depend on the problem to be evaluated, distinguishes the benchmark from the alternative model. The scores imposed on the benchmark model over the evaluation period are each compared with the scores generated by the alternative model. The index i tags the type of the data generating process (DGP).

When evaluating the performance for structural breaks in volatility, several scenarios for the alternative model are assumed. Different values of κ indicate the extent and the direction of the structural break. Thus, the alternative model features a volatility amounting the κ-fold of the benchmark model. In order to evaluate the response of the risk measures to volatility breaks of various intensity, volatility decreases of -50%, -35%, -20% and -10% as well as volatility increases of 10%, 20%, 35%, 50%, 75% and 100% with respect to the reference volatility of $\{X_{t,i}\}$ are considered for the alternative models $\{Y_{t,i}\}$. As defined in the previous section, the risk measure which provides a better ability to distinguish between the alternative model and the benchmark model should show the higher value of the quotients of loss functions (as defined by (31)) if the alternative model features the lower volatility, and vice versa.

Within the performance study regarding a change in distribution, the innovations of the benchmark model are $N(0;1)$, while Student-t distributions with different numbers of degrees of freedom (df) are assumed to generate the innovation process of the alternative model. Since all of them implicitly mark scenarios with higher volatilities than the benchmark model, the risk measure which shows the smaller quotients of the loss functions (as defined by (36)) is always expected to be the superior measure. The numbers of df ν are chosen in a way to generate increases of the unconditional variance of 10%, 20%, 35%, 50%, 75% and 100% as in the previous cases, namely $\nu \in \{22, 12, 7.71, 6, 4.67, 4\}$.

The occurrence of scenarios without any violation of the risk measure depends on the intensity of the volatility change and cannot be precluded, especially for small out-of-sample lengths. In order to enable a functioning evaluation and comparison for these special cases, some assumptions are to be made: In the event that no exceedance takes place for both the benchmark and the alternative model, a quotient value of one is assigned to the respective replication. A score of one plus a value reflecting the greatest finite percentile of the distribution of the quotients is assigned for scenarios in which no exceedance occurs for the alternative model. This avoids infinite values for a single replication and thus for the complete analysis. These substitution rules, however, do not affect the result of the comparison since scenarios in which the rules effectively apply are characterized by very high values of the respective quotient. This leads to the result that the value of such a quotient anyway surpasses the other quotient's value.

The simulation studies are conducted for measuring loss using both the binomial and the quadratic approach (see (29) and (30)) and 2,500 replications each. The in-sample length n_0 is chosen to comprise 2,000 data points, which approximately depicts eight trading years, while in line with the Basel Accords, the observation period n_1 is suggested to be 250 (representing 1 year of trading). However, this recommendation is frequently objected by both theorists (such as Best (2000), Pesaran and Zaffaroni (2004), Bams et al.(2005)) and economic authorities (such as National Bank of Austria (1999)). In order to accommodate for diverse point of views and to carve out the behavior of Θ and $\tilde{\Theta}$, different horizons for the observation period of $n_1 \in \{100, 175, 250, 500\}$ are imposed.

A number of standard stochastic processes are assumed as possible DGPs, whereby each benchmark model is defined for $t \in \{1, \ldots, n_0\}$ during the in-sample period and for $t \in \{n_0 + 1, \ldots, n_0 + n_1\}$ if the underlying process represents the benchmark or the alternative model within the evaluation period. The innovations of all model classes are assumed to be drawn from the Gaussian or the Student-t distribution. The following model classes are assumed to be the DGP i for both $\{X_{t,i}\}$ and $\{Y_{t,i}\}$ (for simplicity the DGPs are notated only in terms of the benchmark model):

DGP 1 White Noise: $X_t \overset{iid}{\sim} N(0, \sigma^2)$ (DGP 1a)
$X_t \overset{iid}{\sim} t(\nu)$ (DGP 1b)

DGP 2 ARMA(1,1): A simple linear model for the mean given by

$$X_t = \phi\, X_{t-1} + \varphi\, \varepsilon_{t-1} + \varepsilon_t,$$

whereby the iid innovations ε_t are drawn from a Gaussian distribution with parameters $(0; \sigma_\varepsilon^2)$ (DGP 2a) and from a Student-$t(\nu)$ distribution (DGP 2b), respectively.

DGP 3 GARCH(1,1), as proposed by Bollerslev (1986): For the mean $X_t = \varepsilon_t$ and $\varepsilon_t | \sigma(X_{t-1}, X_{t-2}, \ldots) \sim (0; \sigma_t^2)$, the model of conditional volatility is defined by

$$\begin{aligned} \varepsilon_t &= \sigma_t \xi_t \\ \sigma_t^2 &= \omega + \gamma\, \varepsilon_{t-1}^2 + \beta\, \sigma_{t-1}^2, \end{aligned}$$

whereby the iid innovations ξ_t are drawn from a Gaussian distribution (DGP 3a) and from a Student-$t(\nu)$ distribution (DGP 3b), respectively.

When assuming a break in the unconditional volatility, the first unconditional moment needs to stay unaffected. This is guaranteed for all DGPs by an unconditional expectation of 0. In order to avoid a change in the persistence of DGPs 2 and 3, the ARMA and GARCH coefficients stay unchanged by the volatility break. The volatility break in the ARMA process is implemented via a change of the error variance[10], while for the GARCH process, a volatility change can easily be obtained by varying the constant coefficient of the conditional variance equation.[11] A standard deviation of 0.02 for the benchmark process is always implied by an appropriate choice of the model parameters, which provides a realistic volatility level for financial log returns. The ARMA parameters of DGP 2 are assumed to be $\phi = 0.7$ and $\varphi = 0.1$. Parameters of $\gamma = 0.1$ and $\beta = 0.7$ for DGP 3 generate a heightened persistence of the GARCH models, while the constant of the conditional volatility of the benchmark models is chosen to be $\omega = 0.00008$.[12]

[10]The unconditional variance of an ARMA(1,1) is given by $Var[X_t] = \frac{1+\varphi^2-2\phi\varphi}{1-\phi^2}\sigma_\varepsilon^2$. The model provides weak stationarity for $|\phi| < 1$.

[11]The unconditional variance of a GARCH(1,1) is given by $Var[X_t] = \frac{\omega}{1-\gamma-\beta}$. The model provides weak stationary for $|\gamma + \beta| < 1$.

[12]For all models featuring the Student-t distribution, the volatility break needs to be implemented by a change of the degree of freedom ν as the unconditional variance of a r.v. $T \sim t(\nu)$ is given by $Var(T) = \frac{\nu}{\nu-2}$. Since $Var(T) \searrow 1$ as $\nu \to \infty$, the Student-t innovations need to be rescaled in order to ensure a standard deviation of 0.02.

4.5.2 Results: Break in Volatility

The ability of VaR and ES to distinguish models of different volatilities is initially compared for a VaR exceedance level of $\alpha = 0.05$, whereby VaR is computed by (26), while ES is estimated by utilizing the LPM approach given by (28). The results of the simulation study can be found in Appendices B.1 (measuring quadratic loss) and B.2 (measuring binomial loss), in which Θ and $\tilde{\Theta}$ measure the performance of VaR and ES, resp. The tables contain the results for all configurations of intensities of structural breaks (κ), lengths of evaluation periods (n_1), and DGPs, as described in the previous subsection. The values in parenthesis report the p-values for the t-statistic, testing $H_0 : \Theta \geq \tilde{\Theta}$ if $\kappa < 1$ and $H_0 : \Theta \leq \tilde{\Theta}$ if $\kappa > 1$.

A whole string of general conclusions can be drawn: First of all, it is to be noted that the loss quotient involving ES ($\tilde{\Theta}$) always holds the significantly larger value than the VaR quotient (Θ) for intensities $\kappa < 1$ across all evaluation sample sizes and DGPs. This finding validates the theoretical results given by (32) and (34) and attests the predominance of ES over VaR for volatility decreases. For the vast majority of cases, the superiority of ES can also be certified for scenarios involving volatility increases, where $\Theta > \tilde{\Theta}$ is expected to hold (see (33) and (35)). The sample length n_1 of the evaluation period, however, plays a more integral role here since the dominance of ES in terms of distinguishing between the benchmark and the alternative model becomes more severe, the longer the evaluation period lasts. While ES fails to outperform VaR for small volatility heightenings, especially in small samples, ES provides consistently and significantly better results than VaR for nearly all types of DGP and intensities of volatility breaks in mid-sized and large sample horizons, which includes a period of 250 observations, as recommended by the Basel Accords. Corresponding to intuition, the relative sensitivity in distinguishing processes with different volatilities is improving, the more extreme intensities of volatility breaks are assumed. This result is valid for small sample sizes as well.

Regarding the different kinds of loss functions, a very satisfying performance can be reasonably stated for both types considered. Only for a small number of cases the binomial and the quadratic loss approach indicate contrary results, thus neither of the types can systematically be preferred. However, this should not come as a surprise as the binomial aspect dominates the quadratic distance to VaR in case of a violation since the simulated series feature a very low (and hence an empirically realistic) volatility level within this part of the study.

The assumption of the data to be generated as simple White Noise comes closest to the character of the theoretical results as it models independent observations. This interlinks the introduced procedure with the familiar backtesting approach of testing whether any two elements of a hit sequence are independent from each other (see e.g. Campbell (2005)). In accordance with that, the Gaussian White Noise (see tables tagged with DGP 1a) presents a very good performance. While ES only fails to outclass VaR in small sample sizes for a volatility increase of 10%, ES provides results which perfectly fulfill the expectations derived from theory throughout for other intensities for both small and large numbers of observations. Sample sizes as of 250, however, are sufficient even for small volatility increases. If innovations are generated by a t-White Noise (DGP 1b), identification problems arise for small samples and volatility increases up to 20%. The relative performances of ES when employing mid-sized and large samples can fully keep pace with those of DGP 1a.

The ARMA class of models (DGP 2) shows a fairly sufficient performance for Gaussian innovations (DGP 2a), which equals that of White Noise, even though a slight lack of efficiency occurs for $\kappa = 1.1$ for $n_1 = 250$. In contrast, the analysis for Student-t innovations (DGP 2b) breaks down for volatility increases in small samples. Sample sizes of 500 observations are strongly recommended to obtain a sufficient responsiveness, especially if only small or mid-level breaks in volatility emerge. Altogether, ARMA-t yields the worst performance of all examined DGPs, although appropriate reactions to volatility decreases as well as to severe increases are still guaranteed if the evaluation horizon is long enough.

Even though the ARMA and the GARCH classes of models both feature serial dependence of observations, the GARCH results dwarf the outcomes for ARMA in terms of relative sensitivity by far. However, this comes as a less surprising result as GARCH models are able to capture volatility clustering. Assuming Gaussian innovations (DGP 3a), only minor difficulties in distinguishing between the different models in case of a 10% volatility increase are observed, for which even 250 observations are not sufficient to depict the outcome to be expected. Moreover, DGP 3b turns out to be the role model among all DGPs using Student-t innovations since even models capturing a volatility increase of only 20% can be discriminated well from the benchmark model.

Additionally, GARCH with Gaussian innovations as well as Gaussian White Noise yield excellent results even beyond the comparison of VaR and ES. Almost entirely in line with the theoretical findings, the values of both Θ and $\tilde{\Theta}$ cross the frontier of one from below after the intensities κ switch from volatility decrease to increase. Again, smaller deviations are observed in small samples and for low volatility heightenings only. This finding indicates a good ability to correctly distinguish between two processes of different volatilities, regardless whether VaR or ES is employed. However, the distance of the quotient of loss functions to the value of one is always higher for ES.[13]

All DGP classes show better performances if the innovations are drawn from a Gaussian distribution. While evaluation horizons of at most 500 observations suffice to demonstrate the superiority of ES for these models, marginal volatility breaks cannot be identified even in large samples if a Student-t distribution is employed instead. To sum up, it can be recorded that a small evaluation horizon is to be avoided for the distinction of processes of similar volatility in order to ensure superiority of the lower of two quantiles. The objective fact that the findings of the simulation study clearly give evidence for the theoretical results is not impaired by this.

4.5.3 Results: Change in Distribution

In order to compare the relative performance of VaR and ES in distinguishing between two processes employing different innovation distributions, the VaR level is set to be $\alpha = 0.05$ as in the previous section. According to (36), the study is carried out by simulating the quotients $\Theta^\star$ and $\tilde{\Theta}^\star$, whereat

[13] A value of one can be interpreted as utter inability to discriminate between the models.

the benchmark process utilizes $N(0;1)$ innovations, while the alternative model employs Student-t distributions of different numbers of df. The results for both assigning quadratic and binomial loss scores can be found in Appendix C. The tables contain the performances for all configurations of $t(\nu)$ distributions, lengths of evaluation periods (n_1), and DGPs presented in Section 4.5.1. The values in parenthesis report the p-values for the t-statistic, testing $H_0 : \Theta^\star \leq \tilde{\Theta}^\star$.

The principal finding lies in the fact that a diminishing number of df in the alternative model leads to a clearer predominance of ES over VaR in terms of their responsiveness to a change in distribution. This is again in line with the theory presented in Section 4.4.3, which gives rise to expect that $\Theta^\star > \tilde{\Theta}^\star$ holds. The superiority becomes more obvious for a growing number of observations in the evaluation sample, while small samples are already sufficient for a low number of df as these models yield the highest volatility and are easiest to distinguish from the benchmark model. Large sample sizes are able to allow to discriminate between models of similar volatility in most of the examined cases. The quotients show values of smaller than one with only a few exceptions. Longer sample horizons and smaller values of ν furthermore underpin these results.

White Noise and GARCH equally exhibit excellent ability to distinguish between the models and yield highly significant results for both employing quadratic and binomial losses. The only exception exists for models of nearly equal volatilities. Especially for $\tilde{\Theta}^\star$, values which are significantly lower than one are observed - in small samples even for the $t(12)$ alternative of relatively low volatility.

A weak performance in small samples can be attested for the ARMA class. When evaluating over horizons of only 100 data points, only major changes in the distribution (i.e. for models which feature a low number of df) can reliably be detected. For the $t(22)$ case, the ARMA model fails to capture the superiority of ES over VaR in the identification of the two processes even in samples of 250 observations. Unlike the results presented in Section 4.5.2, the binomial and quadratic loss approaches show smaller differences in favor of quadratic losses for small samples and for the benefit of binomial scores for a large evaluation horizon. This can be traced down to the fact that a standard deviation of 0.02 (as is valid in the previous part of the study) cannot be maintained since the benchmark model features $N(0;1)$ innovations. Hence, the quadratic loss function involves a quadratic component which is no longer dominated by the fixed part. Apart from this, even in the ARMA case, the procedure shows highly significant results in samples greater than 100 observations and for volatility increases of at least 20%.

At least for ARMA models, an evaluation sample of 250 observations is not sufficient in order to ensure a satisfying sensitivity of both risk measures. However, evidence for the validity of Proposition 3 can be found for all DGPs and numbers of df considered, implying that the risk measure which represents the lower of two quantiles features a better ability to discriminate between processes of similar volatility.

4.5.4 Robustness Checks

In order to check the generality of the conclusions drawn in the previous subsections, the simulations are rerun for some different parameter configurations than assumed so far. For the purpose of a manageable extent of results, the simulations are carried out by applying only the quadratic loss function.

Next to the 95% level, VaR estimates of a 99% confidence level are commonly utilized for measuring risk of financial institutions. Choosing $\alpha = 0.01$, nearly the same tendencies can be observed, even though the conclusions drawn for $\alpha = 0.05$ come only into force for larger sample sizes. That is, a downturn of the small sample performance across all DGPs can be noticed for structural breaks in volatility (see Appendix D.1). However, this can simply be explained by a small number of violations for both the benchmark and the alternative process, which inhibits a quick convergence to the result to be expected by lacking suitable observations (and is a well-known problem in many backtesting frameworks). While a sample size of at most $n_1 = 500$ is sufficient to demonstrate the superiority of ES for a 95% VaR confidence level, some DGPs demand larger evaluation horizons for $\alpha = 0.01$.[14] This point entails a minor difficulty of the introduced procedure. If the sensibility of the risk measures is evaluated for large volatility decreases ($\kappa = 0.5$), the issue of too few exceedances, especially of the alternative process, results in the inability to show the superiority of ES across all evaluation horizons. However, this problem seems more relevant for applying Gaussian innovations, while Student-t innovations are characterized by more extreme violations, which hence lead to more suitable ES quotients.

The same tendency is observed for the results of a change in distribution (see Appendix D.2), although to a less substantial extent. In very small samples, only changes to Student-t distributions with a low number of df are reliably identified, while samples of 250 and 500 observations provide results which are as good as for $\alpha = 0.05$. Some major differences only arise for the evaluation of ARMA models, for which a sample size of $n_1 = 500$ is recommended to ensure predominance of ES over VaR.

Another check is provided for alternative choices of the volatility level of the DGPs. While a standard deviation of 0.02 was assured within the preceding parts of the study, simulations for two alternative levels are supplementary run for assuming a break in volatility: On the one hand, a low-level standard deviation of 0.015 (see Appendix E.1), which corresponds to 56.25% percent of the initial variance, and on the other hand, a high-level standard deviation of 0.04 (see Appendix E.2), which amounts the fourfold of the initial variance. The simulation results for both the low- and high-level alternative volatility show no systematical different findings and comply with the outcomes as discussed in section 4.5.2 across all DGPs, intensities of breaks, and evaluation horizons. This however is perfectly in line with Propositions 1 and 2, which are indicated by independence from the actual variance level of the P&L process.

4.6 Empirical Application to Stock Indices

It remains to reconfirm the conclusions drawn from the simulation studies involving breaks in volatility by an application to empirical data sets. For this purpose, six time series of stock indices are analyzed, in fact the German DAX 30, the EURO STOXX 50, the FTSE 100 representing the

[14] Additional simulations are available for $n_1 = 1,000$ within this branch of the simulation study.

UK stock market, the Hang Seng Index of the Hong Kong stock market, the Japanese NIKKEI 225, and the US S&P 500. Each time series contains daily data from January 1990 up to and including March 2015. After performing a log transformation of the return series, 6,585 observations are each left for examination.

As a first step, the series are examined for structural breaks in the volatility by applying the CUSUM of squares test in the version of Deng and Perron (2008). The null of the absence of a structural break is rejected if the test statistic exceeds the 95% quantile of the limit distribution. By assuming a trimming parameter of 0.15, breaks are restricted to occur only within the central 70% of the observations. Thus, as is suggested by Bai and Perron (2006), a number of five breaks should not be exceeded within each entire series. In order to generate subsamples of lengths which guarantee robust estimations, a minimal distance of 10% of the entire sample between two breaks is respected (see Pesaran and Timmerman (2002)). Four breaks in volatility are each found for DAX 30, EURO STOXX 50, Hang Seng and S&P 500, while the series of NIKKEI 225 and FTSE 100 contain three and five breaks, resp. The plots of the log returns along with the estimated breaks are presented in Figure 4.3.

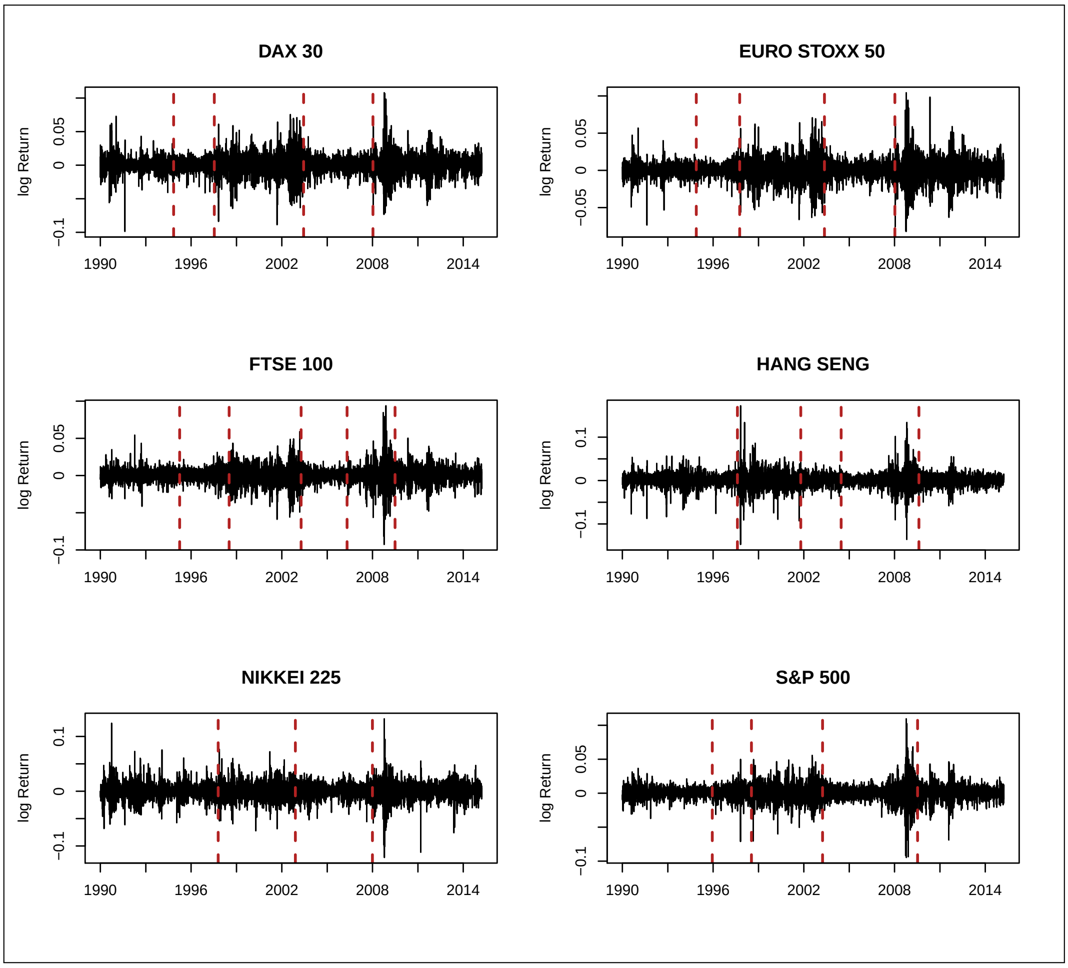

Figure 4.3: Plots of log returns of stock market indices from January 1990 to March 2015. The estimated structural breaks in volatility are indicated by vertical red dashed lines.

A number of k breaks splits the series into $k+1$ subperiods, so that VaR and ES are estimated from each of the first k subsamples. The risk measures estimated from subperiod $j \in \{1, \ldots, k\}$ are then evaluated within subperiod $j+1$ over horizons of $n_1 \in \{100, 175, 250, 375, 500, max\,EH\}$, whereat $max\,EH = 1,000$ is set, unless the length of the evaluation subsample is smaller than 1,000. In this case, $max\,EH$ denotes the length of the respective subperiod. The percentage changes of the variances of the evaluation samples, each in relation to the previous subperiod, are stated in Table 4.2 in chronological order of the breaks.

		$\mathbf{n_1}$					
Series	**Break No.**	**100**	**175**	**250**	**375**	**500**	$max\,EH$
DAX 30	I	-47.8	-51.9	-49.1	-50.9	-55.4	-44.5
	II	441.7	300.6	245.5	363.0	324.1	251.1
	III	-41.2	-57.6	-57.0	-63.7	-69.8	-73.5
	IV	192.4	144.8	498.8	453.3	368.4	246.2
		$Mean = 0.00029$ (0.1014)				$Variance = 0.00018$	
EURO STOXX 50	I	-48.1	-50.1	-48.9	-52.8	-54.3	-29.7
	II	299.4	242.9	326.0	396.3	324.4	269.2
	III	-48.6	-60.7	-62.7	-66.6	-71.6	-74.4
	IV	271.6	224.2	651.1	577.9	468.5	341.9
		$Mean = 0.00018$ (0.2654)				$Variance = 0.00020$	
FTSE 100	I	-51.1	-51.6	-50.9	-52.9	-50.7	-16.7
	II	423.9	324.3	246.1	201.6	207.1	169.0
	III	-57.5	-64.7	-68.5	-72.0	-75.6	-78.8
	IV	145.8	70.1	63.8	113.0	201.0	534.4
	V	-61.4	-61.8	-55.0	-56.3	-60.7	-58.0
		$Mean = 0.00016$ (0.2517)				$Variance = 0.00012$	
HANG SENG	I	670.7	570.0	457.8	407.5	323.2	198.0
	II	-66.3	-73.6	-71.1	-72.8	-74.5	-74.1
	III	-51.5	-55.1	-59.6	-59.2	33.5	154.8
	IV	-41.3	-47.0	-50.2	-57.9	-60.7	-54.4
		$Mean = 0.00033$ (0.1102)				$Variance = 0.00025$	
NIKKEI 225	I	133.9	73.7	64.6	60.2	36.7	20.0
	II	-40.2	-37.9	-30.7	-35.0	-36.8	-46.7
	III	213.0	140.1	529.6	423.0	327.4	184.6
		$Mean = -0.00011$ (0.5605)				$Variance = 0.00022$	
S&P 500	I	24.3	20.5	1.7	28.3	76.6	68.7
	II	248.6	169.3	135.2	101.6	121.4	102.5
	III	-53.3	-60.4	-65.0	-68.2	-71.1	-75.8
	IV	-41.4	-49.1	-30.0	-36.4	-44.3	-35.7
		$Mean = 0.00027$ (0.0544)				$Variance = 0.00013$	

Table 4.2: Percentage changes in volatility of stock market indices after the occurrence of a structural break by length n_1 of the evaluation horizon. The Roman numbers indicate the chronological occurrence of the break in the respective series, while $max\,EH$ denotes the length of the evaluation subsample, but not exceeding 1,000. Additionally, sample means along with their p-values and sample variances of the entire data sets are reported.

Various intensities of changes in volatility can be observed with each series containing at least one huge volatility increase. These increases are caused by the Russian financial crisis and the subsequent downfall of LTCM in 1998, the global financial crisis in late summer 2007 or early 2008 or its aftermaths[15]. After the period of high volatility beginning in 1998, a large volatility decrease is each observed in about spring of 2003 when the markets calmed down after the 2001 terrorist attacks, the Argentina economic crisis and the accusation of accounting fraud directed against Enron in 2002. Moreover, volatility changes of weaker intensity occur for many of the series. The percentage changes appear largely homogenous across the different out-of-sample sizes. A few exceptions can be observed, such as the breaks no. I of NIKKEI 225 and S&P 500, for which the intensities strongly depend on the evaluation horizon. For break no. III of the HANG SENG series, negative changes of volatility are present for short evaluation horizons, while the direction of the break switches for larger out-of-sample lengths. In addition, Table 4.2 provides the means and sample variances of the entire data, whereby the means largely show an insignificant difference from zero. Note that the levels of the variances are each in range of the low-level volatility assumed within the robustness check conducted in Section 4.5.4.

	Subperiod									
	1		2		3		4		5	
Series	**Model**	$\mathbf{n_0}$	**Model**	$\mathbf{n_0}$	**Model**	$\mathbf{n_0}$	**Model**	$\mathbf{n_0}$	**Model**	$\mathbf{n_0}$
DAX 30	3b	1266	3b	702	3a	1541	3b	1197		
EURO STOXX 50	3b	1277	3b	749	3b	1463	3b	1215		
FTSE 100	3b	1370	3b	854	3a	1242	3a	793	3b	828
HANG SENG	3b	1987	3b	1093	3b	697	3b	1342		
NIKKEI 225	3b	2035	3b	1332	3b	1330				
S&P 500	3b	1553	3b	676	3b	1227	3b	1640		

Table 4.3: Selected models and in-sample lengths by subsample of the stock indices series.

In a next step, the different models presented in Section 4.5.2 are estimated for each of the first k subsamples, from which the best performing model is selected by the criterion proposed by Schwarz (1978). The selected models along with the respective in-sample lengths n_0 are presented in Table 4.3. As a less surprising fact within financial data analysis, GARCH-t models perform best for most of the subsamples with few exceptions of GARCH models with Gaussian innovations. The last subperiod of each series only serves for the evaluation of the last estimated risk measure, so that no model needs to be estimated for subsample $k+1$. The described subsampling approach is used within many empirical applications in which time series are investigated for structural breaks, such as by Granger and Hyung (2004) within an application to S&P 500 absolute stock returns and by Rapach and Strauss (2008), who examine the empirical relevance of structural breaks in the unconditional variance of GARCH(1,1) models.

In accordance with the simulation studies presented before, the sensitivity of VaR and ES in response to a break in volatility is measured by the loss quotients given by (31). The selected

[15] The estimates for the exact dates of the structural breaks are itemized in Appendix F.1.

model which prevails during the in-sample period serves as the benchmark model, which does not account for the break.[16] The data of the respective subsample works as the alternative model of the application, which is confronted with simulations of the correct DGP of the preceding subperiod. The simulations of the benchmark model are carried out on the basis of 5,000 replications, while the quadratic loss function and a VaR level of 95% are applied for the evaluation. The results of the application can be found in Table 4.4, whereby the p-values of the respective one-sided t-tests are given in parenthesis.

	n_1	100		175		250		375		500		max EH	
	Subp.	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
DAX 30	1	1.6019	1.9986	2.6325	4.9829	1.9146	3.8399	2.4249	2.6357	2.5680	4.4237	1.8284	2.4703
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	2	0.5026	0.3499	0.5999	0.4242	0.6795	0.5786	0.6005	0.4162	0.6225	0.5548	0.7280	0.5399
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	3	2.3949	20.1246	2.7369	10.2511	3.0094	40.2556	4.4669	3.2339	6.0012	13.4401	6.1722	162.701
		(0.0000)		(0.0000)		(0.0000)		(1.0000)		(0.0000)		(0.0000)	
	4	0.7283	0.6411	0.6182	0.5489	0.5011	0.3653	0.5173	0.3813	0.5659	0.5778	0.6487	0.6287
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.6231)		(0.0159)	
EURO ST. 50	1	3.0726	3.2502	3.5192	5.1641	2.5272	1.2192	3.0922	3.5832	2.8583	3.0152	1.5038	1.7055
		(0.0000)		(0.0000)		(1.0000)		(0.0000)		(0.0000)		(0.0000)	
	2	0.6650	0.6200	0.6761	0.6493	0.5763	0.4165	0.5924	0.6305	0.6562	0.5905	0.7476	0.6977
		(0.0000)		(0.0017)		(0.0000)		(0.9998)		(0.0000)		(0.0001)	
	3	5.0131	10.0123	8.7114	17.7001	6.2924	40.2533	6.3536	79.1995	8.4197	91.9534	7.0599	161.258
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	4	0.6192	0.2539	0.5604	0.3333	0.4856	0.3976	0.5156	0.3642	0.5728	0.4333	0.6492	0.5216
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
FTSE 100	1	3.2943	2.2500	3.6602	7.7010	3.1864	3.2401	3.3812	3.6472	2.8829	3.9299	1.4155	1.8053
		(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	2	0.4819	0.3379	0.5587	0.4122	0.6264	0.5357	0.7166	0.5886	0.7087	0.6308	0.8666	0.7464
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	3	4.6742	1.3490	8.1626	17.0987	6.0471	30.2520	6.1308	40.2525	8.2786	57.4423	13.0176	138.784
		(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	4	0.6344	0.1674	0.8580	0.3611	0.9341	0.3321	0.8273	0.4881	0.6886	0.5328	0.6053	0.4317
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	5	5.6437	20.2520	9.3364	20.2482	2.1007	35.6527	2.0025	79.7716	3.2389	160.663	2.4324	6.4529
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
HANG SENG	1	0.4820	0.3027	0.5424	0.3251	0.5302	0.3526	0.5407	0.5079	0.5855	0.5463	0.7272	0.6773
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0001)		(0.0000)	
	2	5.5421	20.2760	9.2770	59.2046	6.6012	70.2277	9.6138	70.2937	12.5195	127.782	8.6407	93.3186
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	3	4.7157	13.7169	8.1208	20.2515	5.8063	76.8808	3.6676	81.6158	2.3240	7.7489	0.9748	0.6623
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)		(0.0000)	
	4	5.3499	1.2501	4.6449	5.5376	4.3006	6.8459	4.6061	9.3203	3.9414	16.1485	1.8002	3.1314
		(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
NIKKEI 225	1	0.6373	0.4500	0.7904	0.9306	0.5496	1.2000	0.6034	0.7038	0.7029	0.6820	0.8930	0.8434
		(0.0000)		(1.0000)		(1.0000)		(0.4990)		(0.0398)		(0.0000)	
	2	2.6685	1.2501	3.1822	4.9738	1.6307	0.9166	1.7791	2.5476	1.8616	2.1657	2.1684	4.0735
		(1.0000)		(0.0000)		(1.0000)		(0.0000)		(0.0000)		(0.0000)	
	3	0.5288	0.4316	0.5924	0.3409	0.4964	0.3902	0.5223	0.4284	0.5796	0.4251	0.8094	0.8083
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.4906)	
S&P 500	1	0.8278	0.5000	0.9971	0.9168	1.3222	0.5834	1.0388	0.7017	0.9650	0.6564	1.0317	0.6310
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	2	0.6076	0.6068	0.6684	0.4993	0.7195	0.8055	0.8120	0.6661	0.7732	0.3690	0.8770	0.8384
		(0.4702)		(0.0000)		(1.0000)		(0.0000)		(0.0000)		(0.0004)	
	3	5.1230	30.2504	9.0212	20.2473	12.6105	59.9850	18.6002	54.7629	24.7674	136.182	47.8045	215.660
		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
	4	1.6460	0.0144	1.7013	10.0016	1.4065	1.7501	1.7045	3.6651	1.7577	6.1638	1.5247	2.9361
		(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

Table 4.4: Results for the evaluation of VaR and ES by length of the evaluation horizon n_1 for the subsamples of different stock indices. The VaR confidence level is chosen to be 95%. The values in parentheses denote the p-values for the respective t-test (the directions of the breaks can be taken from Table 4.2).

The majority of the application results confirm the findings from the simulation studies conducted in Section 4.5 and are in line with the theoretical results. With some exceptions, which are mainly

[16] The estimated parameters of the selected models are listed in Appendix F.2.

present in evaluation samples of $n_1 = 100$, the comparison between both risk measures indicate highly significant differences for Θ and $\tilde{\Theta}$, each in favor of the direction to be expected. DAX break no. IV and HANG SENG break no. III mark the only exceptions in samples of $n_1 = 500$ for which ES is not preferred over VaR. Note that the latter break mentioned concerns the case in which the direction of the break switches shortly before the evaluation period ends. For the maximum evaluation sample size, ES is preferred over VaR with only a single exception (NIKKEI break no. III). The results for breaks of large intensity, for which ES appears to be superior in most of the cases, expand the findings of Basu (2006), who works out that ES is affected to extreme shocks, while VaR remains very sticky. However, the analysis works very satisfying even for breaks of weaker intensities, e.g. for small and mid-sized evaluation samples of S&P 500 break no. I. Apart from the comparative conclusions, it can be noted that for most of the cases both risk measures are able to identify the breaks, which is demonstrated by the respective values of Θ and $\tilde{\Theta}$. This result confirms Lopez (1998), who attests VaR a good ability to differentiate between the true and the false model when GARCH-$t(6)$ models are applied, and expands his findings to ES.

4.7 Conclusion

The accurate evaluation of a risk measure employed by a financial institution is of high importance in view of the institution's capital requirement. The most sensitive response to breaks in the volatility of the profit and loss process is a desirable property of the underlying measure. This paper proposes a loss function-based framework for the comparative measurement of the responsiveness of any two quantile downside risk measures to breaks in the volatility or in the distribution. For this purpose, the model comparison technique introduced by Lopez (1998) is exploited and extended. As a theoretical result, it can generally be noted that lower quantile risk measures are superior to higher risk quantiles concerning their ability to identify breaks in the volatility. VaR and ES are representatively contrasted within a broad simulation study and the theoretical results are validated for realistic evaluation horizons. Numerous settings involving volatility breaks of different intensities and several DGPs are checked by employing a frequency-type and a magnitude-type loss function. An empirical study additionally demonstrates the applicability of the procedure using data from six stock indices.

Both the simulation study and the empirical application strongly confirm the predominance of ES over VaR regarding their ability to respond to a volatility break. While for small evaluation samples the superiority of ES is not clearly identifiable for some DGPs, this result becomes more evident for increasing evaluation horizons. The conclusions drawn from the theoretical part of the comparison are met for the usage of all DGPs, even though the quality of performance for GARCH and White Noise models clearly surpasses that for ARMA models. While the choice of the loss function type carries secondary weight, models which involve Gaussian innovations provide better results in small samples than models whose innovations are drawn from a Student-t distribution. Only for breaks which lead to a slightly increasing volatility, the superior risk measure in theory is not reliably identifiable, while the procedure works well for volatility decreases of any intensity and increases of about 20%-35% over sufficiently large evaluation periods. In contrast to several other applications and practical considerations, this work suggests evaluation horizons of at least 250 observations for the evaluation of risk measures. However, even the recommendation by BCBS seems not to be sufficient for a limited set of scenarios in order to guarantee the better performance of the risk measure with the superior theoretical properties. This outcome is even stronger for lower VaR exceedance levels. The empirical application for breaks in volatility using a subsampling approach widely confirms these results for the selected and estimated models . In the absence of a suitable test for a structural break in distribution, the corresponding outcomes of the Monte Carlo study remain to be validated for empirical data.

The results of this work support the findings of prior research regarding the properties of VaR and ES within several stress scenarios. Considering the fact that literature in this particular field is still rare, this paper contributes to the expansion of the knowledge about the characteristics of risk measure in presence of structural breaks.

Appendix to Chapter 4

A Proofs

Under validity of assumptions A1-A3 and the notations given above and renaming $\alpha = F(q_t)$ and $\tilde{\alpha} = F(\tilde{q}_t)$, the proofs of propositions 1 and 2 are each carried out for $\kappa > 1$ (referring to equations (33) and (35)). For $\kappa < 1$ (referring to equations (32) and (34)), the same arguments apply.

A.1 Proof of Proposition 1

For any binomial loss involving observations which underrun the risk quantile with probability α, the expected value is given by $E\left[\Gamma^B\right] = \alpha$. The rest is straightforward:

$$\begin{aligned} & E[\Theta] > E[\tilde{\Theta}] \\ \Leftrightarrow \qquad & \frac{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right)}{F(\tilde{q}_t)} > \frac{F\left(\frac{q_t}{\sqrt{\kappa}}\right)}{F(q_t)} \end{aligned} \tag{37}$$

The last inequality holds as the assumptions of quasiconcavity and strict monotonicity are observed. Thus

$$\begin{aligned} & F\left(\kappa^{-\frac{1}{2}}\,\tilde{q}_t\right) - F(\tilde{q}_t) > F\left(\kappa^{-\frac{1}{2}} q_t\right) - F(q_t) \\ \Leftrightarrow \qquad & \frac{F\left(\kappa^{-\frac{1}{2}}\,\tilde{q}_t\right)}{F(\tilde{q}_t)} > 1 + \frac{F\left(\kappa^{-\frac{1}{2}} q_t\right) - F(q_t)}{F(\tilde{q}_t)} \\ \Rightarrow \qquad & \frac{F\left(\kappa^{-\frac{1}{2}}\,\tilde{q}_t\right)}{F(\tilde{q}_t)} > 1 + \frac{F\left(\kappa^{-\frac{1}{2}} q_t\right) - F(q_t)}{F(q_t)} = \frac{F\left(\kappa^{-\frac{1}{2}} q_t\right)}{F(q_t)} \end{aligned}$$

holds, whereby $F(q_t) > F(\tilde{q}_t)$ applies. This equals (37) and proves (33).

A.2 Proof of Proposition 2

The following statements apply for using the quadratic loss function:

$$\begin{aligned} & E[\Theta] > E[\tilde{\Theta}] \\ \Leftrightarrow \qquad & \frac{F(q_t)(1 + E\left[(X - q_t)^2\right])}{F\left(\frac{q_t}{\sqrt{\kappa}}\right)(1 + E\left[(Y - q_t)^2\right])} > \frac{F(\tilde{q}_t)(1 + E\left[(X - \tilde{q}_t)^2\right])}{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right)(1 + E\left[(Y - \tilde{q}_t)^2\right])} \\ \Leftrightarrow \qquad & \frac{F(q_t)(1 + E\left[X^2 - 2Xq_t + q_t^2\right])}{F\left(\frac{q_t}{\sqrt{\kappa}}\right)(1 + E\left[Y^2 - 2Yq_t + q_t^2\right])} > \frac{F(\tilde{q}_t)(1 + E\left[X^2 - 2X\tilde{q}_t + \tilde{q}_t^2\right])}{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right)(1 + E\left[Y^2 - 2Y\tilde{q}_t + \tilde{q}_t^2\right])} \\ \Leftrightarrow \qquad & \frac{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right)F(q_t)}{F\left(\frac{q_t}{\sqrt{\kappa}}\right)F(\tilde{q}_t)} > \frac{(1 + \sigma_X^2 + \tilde{q}_t^2)(1 + \sigma_Y^2 + q_t^2)}{(1 + \sigma_X^2 + q_t^2)(1 + \sigma_Y^2 + \tilde{q}_t^2)} \end{aligned}$$

$$\Leftrightarrow \qquad \frac{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right)F(q_t)}{F\left(\frac{q_t}{\sqrt{\kappa}}\right)F(\tilde{q}_t)} > \frac{\sigma_X^2\left(1+\kappa+\kappa\sigma_X^2\right)+1+q_t^2+\tilde{q}_t^2+q_t^2\tilde{q}_t^2+\sigma_X^2\overbrace{(\kappa\tilde{q}_t^2+q_t^2)}^{\mathcal{Q}_1}}{\underbrace{\sigma_X^2\left(1+\kappa+\kappa\sigma_X^2\right)+1+q_t^2+\tilde{q}_t^2+q_t^2\tilde{q}_t^2}_{a}+\underbrace{\sigma_X^2}_{b}\underbrace{\left(\kappa q_t^2+\tilde{q}_t^2\right)}_{\mathcal{Q}_2}}$$

$$\Rightarrow \qquad \frac{F\left(\frac{\tilde{q}_t}{\sqrt{\kappa}}\right)F(q_t)}{F\left(\frac{q_t}{\sqrt{\kappa}}\right)F(\tilde{q}_t)} > \frac{a+b(\kappa\tilde{q}_t^2+q_t^2)}{a+b(\kappa q_t^2+\tilde{q}_t^2)} > \frac{\kappa\tilde{q}_t^2+q_t^2}{\kappa q_t^2+\tilde{q}_t^2} > 1$$

The latter two inequalities hold as $a, b > 0$ and $\mathcal{Q}_1 > \mathcal{Q}_2$ for $\kappa > 1$, which proves equation (35) and confirms the computations given in Tables 4.1/a and 4.1/b.

A.3 Proof of Proposition 3

Let $X \sim N(0; \sigma_X^2 = 1)$ with cdf $F_N(\cdot)$ and $Y \sim t(\nu)$ with cdf $F_t(\cdot)$. The sensitivity functions $\Theta^\star$ and $\tilde{\Theta}^\star$ are given by (36). The proof is carried out for $m = Q$. Some steps are left out as being identical with those in Appendix A.2.

$$\begin{aligned} & E[\Theta^\star] > E[\tilde{\Theta}^\star] \\ \Leftrightarrow \qquad & \frac{P(Y \le \tilde{q}_t)\,F_N(q_t)}{P(Y \le q_t)\,F_N(\tilde{q}_t)} > \frac{(1+\sigma_X^2+\tilde{q}_t^2)(1+\sigma_Y^2+q_t^2)}{(1+\sigma_X^2+q_t^2)(1+\sigma_Y^2+\tilde{q}_t^2)} \\ \Leftrightarrow \qquad & \frac{F_t\left(\tilde{q}_t\right)\,F_N(q_t)}{F_t\left(q_t\right)F_N(\tilde{q}_t)} > \frac{2\left(1+\sigma_Y^2\right)+q_t^2\tilde{q}_t^2+\left(1+\sigma_Y^2\right)\tilde{q}_t^2+2q_t^2}{\underbrace{2\left(1+\sigma_Y^2\right)+q_t^2\tilde{q}_t^2}_{a>0}+\underbrace{\left(1+\sigma_Y^2\right)}_{b>2}q_t^2+2\tilde{q}_t^2} \end{aligned}$$

Since $a + b\,\tilde{q}_t^2 + 2q_t^2 > a + b\,q_t^2 + 2\tilde{q}_t^2$ is true for any $0 > q_t > \tilde{q}_t$, the right hand side of the latter inequality is larger than 1, so that

$$\frac{F_t(\tilde{q}_t)}{F_N(\tilde{q}_t)} > \frac{F_t(q_t)}{F_N(q_t)} \tag{38}$$

holds. By validity of assumptions A2 and A3, it follows that

$$F_t(\tilde{q}_t) - F_N(\tilde{q}_t) > F_t(q_t) - F_N(q_t),$$

whereby $F(\upsilon\tilde{q}_t) > F(\tilde{q}_t)$, $F(\upsilon q_t) > F(q_t)$ and $F(q_t) > F(\tilde{q}_t)$ applies, so that

$$\begin{aligned} \Leftrightarrow \qquad & \frac{F_t(\tilde{q}_t)}{F_N(\tilde{q}_t)} > 1 + \frac{F_t(q_t) - F_N(q_t)}{F_N(\tilde{q}_t)} \\ \Rightarrow \qquad & \frac{F_t(\tilde{q}_t)}{F_N(\tilde{q}_t)} > 1 + \frac{F_t(q_t) - F_N(q_t)}{F_N(q_t)} = \frac{F_t(q_t)}{F_N(q_t)} \end{aligned}$$

holds. This equals (38) and hence proves the proposition. The proof for $m = B$ equals those in Appendix A.1.

B Simulation Results: Break in Volatility

The following tables contain the results of the simulation study regarding the sensitiveness of Value at Risk (Θ) and Expected Shortfall ($\tilde{\Theta}$) to distinguish between the benchmark model and the alternative model as presented in Section 4.5.2. Each table contains results for all combinations of intensities of volatility breaks (κ) and lengths of evaluation periods (n_1). The values in parenthesis report the p-values for the t-statistic, testing $H_0 : \Theta \geq \tilde{\Theta}$ if $\kappa < 1$ and $H_0 : \Theta \leq \tilde{\Theta}$ if $\kappa > 1$. Each table is tagged with the respective DGP i, the type of loss function, and the VaR exceedance level (m-α%) in the upper left.

B.1 Results for Quadratic Loss

DGP 1a Q-5%	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.3106	7.7464	3.2395	5.4100	1.8925	4.0700	1.5264	2.8610	1.0848	1.1566
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9978)	
175	6.4073	8.7980	2.6681	5.8178	1.7562	2.5821	1.4290	1.7488	1.0678	1.0566
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.2898)	
250	5.5990	10.7815	2.6526	6.4125	1.7100	2.3067	1.4133	1.6378	1.0599	1.0346
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0836)	
500	4.5858	12.1893	2.4471	4.0020	1.6586	2.0118	1.3767	1.4877	1.0564	1.0122
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0025)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	0.9792	0.9492	0.8541	0.7448	0.7657	0.6337	0.6691	0.5308	0.6045	0.4743
	(0.0857)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	0.9615	0.8979	0.8456	0.7674	0.7653	0.6520	0.6749	0.5414	0.6749	0.5414
	(0.0003)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9552	0.8637	0.8513	0.7416	0.7686	0.6392	0.6794	0.5467	0.6172	0.4893
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9622	0.8838	0.8607	0.7526	0.7819	0.6644	0.6691	0.5770	0.6394	0.5126
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 1b Q-5%	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.1573	5.6718	1.6354	5.0585	1.4016	2.8868	1.3091	2.5355	1.2150	1.7207
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	2.0688	6.2213	1.5378	3.2636	1.3410	1.7324	1.2618	1.4820	1.1819	1.2343
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9859)	
250	1.9323	6.2341	1.5274	2.2890	1.3226	1.6020	1.2473	1.3741	1.1552	1.1669
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.7157)	
500	1.8374	4.0202	1.4602	2.0075	1.3211	1.4939	1.2389	1.3121	1.1518	1.1397
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.2397)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.1823	1.5175	1.1245	1.4080	1.1206	1.1190	1.0882	1.0949	1.0367	1.0528
	(1.0000)		(1.0000)		(0.4766)		(0.6023)		(0.7368)	
175	1.1309	1.1321	1.1192	1.0595	1.0954	1.0288	1.0492	0.9793	1.0272	0.9791
	(0.5206)		(0.0022)		(0.0007)		(0.0003)		(0.0093)	
250	1.1288	1.1051	1.1023	1.0603	1.0701	1.0201	1.0419	0.9469	1.0112	0.9157
	(0.1136)		(0.0164)		(0.0043)		(0.0000)		(0.0000)	
500	1.1288	1.1033	1.0993	1.0182	1.0756	1.0039	1.0463	0.9252	0.9961	0.9077
	(0.1079)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2a Q-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	12.4509	13.9057	7.0295	9.1720	2.8826	7.6616	1.9351	6.6179	1.2454	3.3440
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	9.3694	11.4105	3.8155	8.6495	1.9477	6.0081	1.5966	2.9443	1.1317	1.2370
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9999)	
250	7.5832	15.3081	3.1330	8.9290	1.9039	3.9196	1.4655	2.0343	1.1009	1.1046
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.5611)	
500	5.2424	9.2209	2.5314	5.0726	1.7174	2.2114	1.4157	1.5782	1.0836	1.0337
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0043)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.1143	1.8018	0.9373	1.0426	0.8232	0.7793	0.7019	0.6072	0.6279	0.4837
	(1.0000)		(0.9988)		(0.0922)		(0.0000)		(0.0000)	
175	1.0234	1.0031	0.8781	0.7678	0.7895	0.6725	0.6776	0.5538	0.6240	0.4978
	(0.2044)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9916	0.9286	0.8600	0.7481	0.7648	0.6742	0.6824	0.5634	0.6249	0.4995
	(0.0014)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9677	0.9027	0.8567	0.7770	0.7896	0.6672	0.6958	0.5723	0.6433	0.5133
	(0.0001)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2b Q-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.4240	8.2087	2.5194	9.2044	1.9356	8.4811	1.6049	9.1756	1.3769	5.3625
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	3.3091	9.6271	2.0510	8.0453	1.5742	5.0579	1.4260	4.7599	1.2475	1.9142
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
250	2.9489	12.7339	1.8873	7.4692	1.5039	2.5076	1.3473	1.8832	1.1993	1.3504
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
500	2.5268	14.4783	1.7233	2.9476	1.4327	1.7412	1.3197	1.4278	1.1748	1.1758
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.5194)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.3064	4.4500	1.2628	4.8108	1.1574	5.7033	1.1420	4.0091	1.0692	3.6839
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
175	1.1712	1.5724	1.1106	1.3632	1.0769	1.1573	1.0229	1.0319	0.9886	0.9451
	(1.0000)		(1.0000)		(0.9977)		(0.6339)		(0.0308)	
250	1.1495	1.2750	1.1026	1.0958	1.0487	1.0251	1.0065	0.9285	0.9724	0.9006
	(1.0000)		(0.3930)		(0.1580)		(0.0003)		(0.0001)	
500	1.1321	1.1151	1.0773	1.0187	1.0457	0.9456	0.9957	0.8877	0.9768	0.8427
	(0.2059)		(0.0016)		(0.0000)		(0.0000)		(0.0000)	

DGP 3a Q-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.0633	8.0556	3.0963	6.1853	1.8685	4.1122	1.5093	3.9228	1.1122	1.1561
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9508)	
175	5.7695	10.8826	2.5807	6.3667	1.7659	2.7959	1.4254	1.7218	1.0892	1.0797
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.3336)	
250	5.0701	10.5346	2.5498	6.5637	1.6858	2.2967	1.4049	1.6238	1.0705	1.0712
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.5144)	
500	4.3223	11.3582	2.3597	3.8389	1.6250	2.0191	1.3722	1.5048	1.0656	1.0269
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0094)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.0236	1.0700	0.8835	0.7419	0.7864	0.6955	0.6865	0.5732	0.6260	0.5064
	(0.9632)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	0.9798	0.9147	0.8681	0.7657	0.7858	0.6840	0.6765	0.5707	0.6258	0.4945
	(0.0005)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9792	0.8919	0.8588	0.7553	0.7746	0.6726	0.6898	0.5583	0.6302	0.5085
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9706	0.9024	0.8721	0.7615	0.7946	0.6742	0.7053	0.5763	0.6459	0.5253
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3b Q-5%	κ									
	0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.6511	4.2271	1.8981	3.4844	1.5359	2.8503	1.3916	2.2819	1.1671	1.2963
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	2.4691	4.8620	1.8585	2.4144	1.5239	1.8133	1.3342	1.4435	1.1407	1.1880
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9812)	
250	2.3997	3.3991	1.8221	2.1697	1.4802	1.5588	1.3497	1.4527	1.1450	1.1468
	(0.0000)		(0.0000)		(0.0005)		(0.0000)		(0.5356)	
500	2.3180	2.8070	1.7717	2.0295	1.4743	1.5355	1.3381	1.3820	1.1464	1.1366
	(0.0000)		(0.0000)		(0.0013)		(0.0112)		(0.2857)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.0843	1.2071	0.9981	0.9939	0.9225	0.8414	0.8188	0.7847	0.7621	0.6775
	(1.0000)		(0.4295)		(0.0000)		(0.0397)		(0.0000)	
175	1.0668	1.0491	0.9896	0.9040	0.9065	0.8625	0.8262	0.7505	0.7637	0.6689
	(0.1926)		(0.0000)		(0.0080)		(0.0000)		(0.0000)	
250	1.0723	1.0447	0.9747	0.9269	0.9198	0.8829	0.8370	0.7569	0.7748	0.6928
	(0.0751)		(0.0032)		(0.0200)		(0.0000)		(0.0000)	
500	1.0802	1.0546	0.9937	0.9580	0.9273	0.8637	0.8574	0.7841	0.7938	0.7090
	(0.0635)		(0.0141)		(0.0000)		(0.0000)		(0.0000)	

B.2 Results for Binomial Loss

DGP 1a B-5%	κ									
	0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.4538	7.8059	3.1183	5.4598	1.8402	3.5224	1.4570	3.0304	1.0842	1.1255
	(0.0013)		(0.0000)		(0.0000)		(0.0000)		(0.9555)	
175	5.9783	9.6006	2.8014	5.8475	1.7272	2.5040	1.4167	1.6701	1.0658	1.0476
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.1783)	
250	5.6378	9.5139	2.5520	5.8884	1.7171	2.2961	1.4148	1.5712	1.0678	1.0283
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0153)	
500	4.7482	11.7352	2.4329	4.0299	1.6595	2.0109	1.3753	1.5445	1.0660	1.0234
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0035)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	0.9786	0.9421	0.8685	0.7900	0.7693	0.6482	0.6751	0.5425	0.6083	0.4966
	(0.0437)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	0.9500	0.8716	0.8469	0.7378	0.7675	0.6432	0.6799	0.5469	0.6270	0.4747
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9837	0.9363	0.8452	0.7486	0.7690	0.6567	0.6859	0.5646	0.6273	0.4840
	(0.0031)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9659	0.8859	0.8600	0.7493	0.7865	0.6640	0.7030	0.6000	0.6396	0.5175
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 1b B-5%	κ									
	0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.2236	4.9218	1.5916	3.5749	1.3743	3.1461	1.3310	2.4361	1.1933	1.4154
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	1.9720	6.5540	1.5442	3.5989	1.3127	1.7136	1.2806	1.5487	1.1587	1.2490
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9990)	
250	1.9667	5.5552	1.5176	2.2134	1.3111	1.5893	1.2404	1.4158	1.1735	1.1883
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.7634)	
500	1.8500	3.9842	1.4786	1.9702	1.2972	1.4691	1.2460	1.3196	1.1519	1.1411
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.2607)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.1818	1.4040	1.1558	1.3911	1.1132	1.1892	1.0883	1.0912	1.0654	1.0336
	(1.0000)		(1.0000)		(0.9986)		(0.5474)		(0.0926)	
175	1.1484	1.1959	1.1169	1.0461	1.0839	1.0283	1.0649	0.9718	1.0402	0.9572
	(0.9807)		(0.0004)		(0.0036)		(0.0000)		(0.0000)	
250	1.1353	1.1228	1.1096	1.0651	1.0736	1.0004	1.0609	0.9816	1.0264	0.9190
	(0.2658)		(0.0109)		(0.0001)		(0.0000)		(0.0000)	
500	1.1336	1.0877	1.0973	1.0454	1.0765	1.0123	1.0513	0.9594	1.0364	0.9502
	(0.0030)		(0.0008)		(0.0000)		(0.0000)		(0.0000)	

DGP 2a B-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	10.8650	14.9116	7.9755	9.2406	2.6082	10.4300	1.8535	6.0509	1.2854	4.2785
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	12.1118	15.2674	3.6440	9.7292	2.0582	6.0617	1.5486	3.2486	1.1548	1.2510
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9994)	
250	7.6622	17.4049	3.0085	8.7170	1.8160	3.1848	1.4594	2.0166	1.0791	1.1294
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9808)	
500	5.2291	16.5410	2.5455	5.1707	1.6970	2.2320	1.3978	1.6332	1.0903	1.0346
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0019)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.1050	1.4959	0.8999	0.9270	0.8228	0.7220	0.7140	0.5963	0.6298	0.4824
	(1.0000)		(0.8413)		(0.0000)		(0.0000)		(0.0000)	
175	0.9995	1.0038	0.8782	0.8417	0.7915	0.7011	0.7004	0.5755	0.6303	0.5008
	(0.5693)		(0.0475)		(0.0000)		(0.0000)		(0.0000)	
250	0.9975	0.9279	0.8648	0.7903	0.7928	0.6670	0.6907	0.5599	0.6212	0.4874
	(0.0006)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9755	0.9008	0.8661	0.7711	0.7869	0.6710	0.6977	0.5794	0.6386	0.5188
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2b B-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	9.3847	10.5501	2.7776	10.2991	1.8807	7.7784	1.6456	7.0533	1.3537	4.7700
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	3.3483	11.0215	2.1084	7.5927	1.5947	6.6122	1.4380	5.3703	1.2048	2.0594
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
250	2.9767	14.8934	1.8539	8.0124	1.4808	2.3683	1.3655	1.8393	1.2143	1.3837
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
500	2.5048	13.3306	1.6926	2.6716	1.4103	1.7285	1.3196	1.4730	1.1800	1.1595
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.1658)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.2835	5.5721	1.2351	5.3479	1.1637	5.4208	1.1059	4.6756	1.0669	3.3103
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
175	1.1954	1.5434	1.1268	1.3215	1.0967	1.1348	1.0249	1.1033	0.9534	0.9826
	(1.0000)		(1.0000)		(0.9134)		(0.9981)		(0.0174)	
250	1.1634	1.2545	1.0898	1.0884	1.0572	1.0535	1.0026	0.9420	0.9653	0.8689
	(0.9996)		(0.4768)		(0.4584)		(0.0033)		(0.0000)	
500	1.1344	1.1111	1.0744	1.0260	2.5714	1.0571	2.4706	0.9998	0.9643	0.8430
	(0.1297)		(0.0072)		(0.0000)		(0.0000)		(0.0000)	

DGP 3a B-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.3894	9.3862	3.0866	6.1554	1.9702	4.1253	1.5137	3.1712	1.1087	1.3231
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	5.6606	10.2594	2.6661	8.4670	1.7245	2.6890	1.4261	1.6937	1.0804	1.0568
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.1299)	
250	5.0016	10.2310	2.5328	6.4984	1.6783	2.2706	1.4087	1.6119	1.0697	1.0668
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.4416)	
500	4.3062	11.1293	2.3487	3.7485	1.6265	2.0054	1.3683	1.5298	1.0747	1.0180
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0003)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	0.9983	1.0270	0.8804	0.7956	0.7961	0.6912	0.6842	0.5539	0.6358	0.4958
	(0.8793)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	0.9890	0.9330	0.8521	0.7947	0.7690	0.6517	0.6834	0.5540	0.6274	0.5290
	(0.0027)		(0.0006)		(0.0000)		(0.0000)		(0.0000)	
250	0.9736	0.9079	0.8621	0.7519	0.7818	0.6609	0.6943	0.5624	0.6318	0.4986
	(0.0001)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9762	0.9150	0.8745	0.7613	0.7885	0.6882	0.7091	0.5798	0.6513	0.5262
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3b B-5%	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.5841	4.8356	1.9332	4.0107	1.5635	2.9940	1.3650	2.0673	1.1907	1.3205
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	2.5189	4.6851	1.8668	2.2914	1.4844	1.7705	1.3565	1.4667	1.1341	1.1568
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.8533)	
250	2.4861	3.3245	1.8222	2.0951	1.5068	1.6214	1.3502	1.4725	1.1610	1.1430
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.1844)	
500	2.2940	2.7175	1.7776	1.9668	1.4864	1.5660	1.3378	1.3839	1.1481	1.1331
	(0.0000)		(0.0000)		(0.0001)		(0.0081)		(0.1932)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.1128	1.2070	1.0331	0.9896	0.9292	0.9049	0.8282	0.7987	0.7507	0.6903
	(0.9999)		(0.0238)		(0.1619)		(0.0624)		(0.0004)	
175	1.0780	1.0911	0.9976	0.9975	0.9143	0.8942	0.8316	0.7480	0.7639	0.6984
	(0.7343)		(0.4999)		(0.1372)		(0.0000)		(0.0000)	
250	1.0737	1.0762	1.0019	0.9848	0.9079	0.8794	0.8400	0.7472	0.7870	0.6998
	(0.5520)		(0.1717)		(0.0488)		(0.0000)		(0.0000)	
500	1.0841	1.0731	0.9945	0.9609	0.9327	0.8785	0.8584	0.7828	0.8001	0.7239
	(0.2575)		(0.0185)		(0.0002)		(0.0000)		(0.0000)	

C Simulation Results: Change in Distribution

The following tables contain the results for the simulation study presented in Section 4.5.3. The values in parenthesis report the p-values for a t-statistics, testing $H_0 : \Theta^\star \leq \tilde{\Theta}^\star$. The benchmark model always features $N(0;1)$ innovations. Each table is tagged with the respective DGP i, the type of loss function, and the VaR exceedance level (m-α%) in the upper left .

DGP 1 Q-5%	Alternative Distribution $t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.0418	1.0474	0.9088	0.7861	0.7564	0.6295	0.6648	0.5229	0.5451	0.4296	0.4910	0.3865
	(0.5886)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	1.0184	0.9586	0.8727	0.7482	0.7190	0.5988	0.6267	0.4968	0.5325	0.4194	0.4637	0.3646
	(0.0014)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9993	0.9276	0.8586	0.7412	0.7099	0.5791	0.6249	0.4893	0.5176	0.4048	0.4569	0.3607
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9934	0.9026	0.8538	0.7362	0.7097	0.5799	0.6174	0.4865	0.5102	0.3986	0.4440	0.3538
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 1 B-5%	Alternative Distribution $t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.0985	1.1015	1.0210	0.9048	0.9550	0.8051	0.9232	0.7097	0.8202	0.6150	0.7621	0.6127
	(0.5490)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	1.1091	1.0138	1.0108	0.8342	0.9382	0.7344	0.8799	0.6888	0.8141	0.6243	0.7647	0.5818
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	1.0866	1.0020	0.9985	0.9118	0.9176	0.7484	0.8686	0.6805	0.8149	0.6239	0.7821	0.5868
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9934	0.9026	0.8538	0.7362	0.7097	0.5799	0.6174	0.4865	0.5102	0.3986	0.4440	0.3538
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2 Q-5%	Alternative Distribution $t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.3588	5.6388	1.0834	2.1723	0.8581	1.0793	0.7516	0.7548	0.6131	0.5405	0.4946	0.4380
	(1.0000)		(1.0000)		(1.0000)		(0.5498)		(0.0003)		(0.0007)	
175	1.1487	1.3105	0.9281	0.9589	0.7770	0.6922	0.6432	0.5660	0.5416	0.4390	0.4735	0.3779
	(1.0000)		(0.8798)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	1.0790	1.0834	0.8960	0.8629	0.7378	0.6462	0.6325	0.5590	0.5296	0.4316	0.4496	0.3741
	(0.5667)		(0.0697)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	1.0201	0.9927	0.8800	0.7990	0.7134	0.6118	0.6068	0.5179	0.5022	0.4106	0.4329	0.3571
	(0.0896)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2	Alternative Distribution											
B-5%	$t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.2670	3.5443	1.0623	1.5159	0.9757	0.9584	0.8836	0.8462	0.8155	0.6555	0.7398	0.5728
	(1.0000)		(1.0000)		(0.2665)		(0.0763)		(0.0000)		(0.0000)	
175	1.1414	1.2702	1.0343	0.9559	0.9199	0.7910	0.8202	0.7358	0.7490	0.5863	0.7140	0.5442
	(1.0000)		(0.0007)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	1.0929	1.1248	1.0089	0.9353	0.8995	0.7786	0.8367	0.6857	0.7458	0.5879	0.6931	0.5498
	(0.9097)		(0.0003)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	1.0666	1.0528	0.9986	0.9012	0.8820	0.7696	0.8274	0.6910	0.7639	0.6175	0.7130	0.5634
	(0.2371)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3	Alternative Distribution											
Q-5%	$t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.0961	1.0827	0.9397	0.8184	0.8202	0.6677	0.7720	0.5628	0.6737	0.4856	0.6237	0.4410
	(0.3049)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	1.0383	0.9917	0.9206	0.7866	0.8090	0.6444	0.7443	0.5632	0.6551	0.4939	0.6243	0.4517
	(0.0130)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	1.0305	0.9393	0.9202	0.7851	0.8094	0.6315	0.7487	0.5696	0.6650	0.5002	0.6197	0.4620
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	1.0289	0.9291	0.9221	0.7782	0.8260	0.6594	0.7541	0.5914	0.6743	0.5271	0.6274	0.4850
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3	Alternative Distribution											
B-5%	$t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.0649	1.0807	0.9842	0.7888	0.8191	0.6702	0.7421	0.5866	0.6668	0.5222	0.6342	0.4531
	(0.7351)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	1.0439	0.9741	0.9215	0.8270	0.8222	0.6515	0.7360	0.5612	0.6771	0.5000	0.6176	0.4518
	(0.0003)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	1.0416	0.9702	0.9625	0.7860	0.8147	0.6453	0.7444	0.5713	0.6661	0.4970	0.6115	0.4591
	(0.0001)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	1.0394	0.9511	0.9226	0.7744	0.8216	0.6563	0.7541	0.5962	0.6772	0.5266	0.6318	0.4923
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

D Simulation Results: Alternative Choice of the VaR Level

The following tables contain the simulation results presented in Section 4.5.4 regarding the alternative choice of an 99% VaR level. Each table is tagged with the respective DGP i, the type of loss function, and the VaR exceedance level (m-α%) in the upper left.

D.1 Results for Breaks in Volatility

DGP 1a **Q-1%**	κ 0.5			0.65			0.8			0.9			1.1		
n_1	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)
100	5.7197	1.9829	(1.0000)	5.0420	3.8428	(1.0000)	4.0752	4.3059	(0.0001)	3.2773	3.1070	(0.9972)	2.0936	2.3040	(0.9997)
175	6.5127	3.9449	(1.0000)	6.0400	5.5169	(1.0000)	3.9314	4.1680	(0.0010)	3.6709	3.3850	(0.9992)	1.0692	2.1634	(1.0000)
250	9.7779	4.8655	(1.0000)	7.7140	5.3197	(1.0000)	3.8583	4.2809	(0.0000)	1.8766	3.9654	(0.0000)	1.0211	1.5107	(1.0000)
500	11.5000	6.7102	(1.0000)	7.0341	7.0321	(0.5065)	2.4056	4.3448	(0.0000)	1.6067	2.5198	(0.0000)	1.0256	0.9939	(0.0706)
1000	15.7711	11.8163	(1.0000)	4.6163	7.3578	(0.0000)	2.1250	3.0209	(0.0000)	1.5648	1.7735	(0.0000)	1.0291	0.9615	(0.0000)

	1.2			1.35			1.5			1.75			2		
n_1	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)
100	1.0954	2.4267	(1.0000)	0.7301	1.4121	(1.0000)	0.5775	0.6589	(0.9999)	0.4944	0.4422	(0.0013)	0.4261	0.3708	(0.0002)
175	0.8330	1.3710	(1.0000)	0.6782	0.7048	(0.9005)	0.5883	0.5243	(0.0002)	0.5067	0.4195	(0.0000)	0.4447	0.3899	(0.0002)
250	0.8609	0.7955	(0.0009)	0.7087	0.6919	(0.2023)	0.6047	0.5045	(0.0000)	0.5262	0.4431	(0.0000)	0.4596	0.3887	(0.0000)
500	0.8693	0.8209	(0.0052)	0.7196	0.6344	(0.0000)	0.6319	0.5470	(0.0000)	0.5457	0.4590	(0.0000)	0.4919	0.4244	(0.0000)
1000	0.8901	0.8215	(0.0000)	0.7595	0.6837	(0.0000)	0.6801	0.5977	(0.0000)	0.5907	0.5199	(0.0000)	0.5378	0.4719	(0.0000)

DGP 1b **Q-1%**	κ 0.5			0.65			0.8			0.9			1.1		
n_1	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)
100	5.0578	2.9374	(1.0000)	4.5289	3.5838	(1.0000)	3.6933	4.0656	(0.0000)	3.3001	3.1143	(0.9962)	2.3887	2.9855	(1.0000)
175	5.9962	5.7206	(1.0000)	4.1555	4.9457	(0.0000)	3.2325	3.5388	(0.0001)	1.7215	3.9897	(0.0000)	1.2980	2.9039	(1.0000)
250	7.1131	5.5849	(1.0000)	3.9538	4.7349	(0.0000)	1.8523	4.4377	(0.0000)	1.4285	3.8421	(0.0000)	1.2218	3.0573	(1.0000)
500	7.4850	6.9931	(0.9999)	2.1979	4.9014	(0.0000)	1.5698	3.5043	(0.0000)	1.3503	2.3296	(0.0000)	1.1860	1.2920	(1.0000)
1000	4.3653	9.0450	(0.0000)	2.0511	4.8562	(0.0000)	1.4990	1.9217	(0.0000)	1.3240	1.5074	(0.0000)	1.1513	1.1794	(0.9142)

	1.2			1.35			1.5			1.75			2		
n_1	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)	Θ	$\tilde{\Theta}$	(p)
100	2.6129	3.3442	(1.0000)	2.6617	2.5905	(0.1704)	1.5347	3.0577	(1.0000)	1.1860	2.3216	(1.0000)	1.1321	2.7487	(1.0000)
175	1.2668	2.6623	(1.0000)	1.1213	2.8429	(1.0000)	1.0434	2.6090	(1.0000)	0.8770	2.4154	(1.0000)	0.8592	1.9221	(1.0000)
250	1.1434	2.5509	(1.0000)	1.0372	2.3103	(1.0000)	0.9640	1.7998	(1.0000)	0.9000	1.1323	(1.0000)	0.8743	1.0945	(1.0000)
500	1.1041	1.2212	(1.0000)	1.0235	1.0248	(0.3511)	0.9814	0.9347	(0.0122)	0.9358	0.8353	(0.0000)	0.8921	0.8211	(0.0000)
1000	1.0909	1.0883	(0.4484)	1.0455	1.0045	(0.0140)	1.0072	0.9162	(0.0000)	0.9491	0.8803	(0.0000)	0.9217	0.8192	(0.0000)

DGP 2a Q-1%	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.6306	6.9009	8.6331	6.6665	6.5953	5.2897	6.9249	4.9379	4.4135	4.0581
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(0.0005)	
175	9.1840	5.8835	9.4738	5.5514	6.5173	4.8122	6.0858	8.4427	1.8025	3.7239
	(1.0000)		(1.0000)		(1.0000)		(0.0000)		(1.0000)	
250	11.5863	7.7755	11.0013	9.6656	6.2924	6.6292	4.5087	8.4830	1.2325	3.2720
	(1.0000)		(1.0000)		(0.0069)		(0.0000)		(1.0000)	
500	18.9935	10.4077	10.1836	8.8308	2.8504	6.9836	1.8519	4.3960	1.0575	1.2155
	(1.0000)		(1.0000)		(0.0000)		(0.0000)		(1.0000)	
1000	21.6042	15.4190	5.2743	10.7551	2.2796	4.5096	1.5999	1.9252	1.0267	1.0004
	(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0840)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	4.1554	5.9601	2.8425	3.0255	0.8533	2.8373	0.6229	2.3059	0.4494	0.5597
	(1.0000)		(0.9714)		(1.0000)		(1.0000)		(1.0000)	
175	1.1305	3.9492	0.7399	2.0041	0.6807	0.7158	0.4933	0.4470	0.4398	0.3736
	(1.0000)		(1.0000)		(0.9247)		(0.0037)		(0.0000)	
250	0.9889	2.6805	0.7380	0.8492	0.6241	0.6371	0.5192	0.4469	0.4599	0.3861
	(1.0000)		(1.0000)		(0.7339)		(0.0000)		(0.0000)	
500	0.8715	0.9007	0.7369	0.6384	0.6322	0.5509	0.5490	0.4726	0.4863	0.4219
	(0.9028)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
1000	0.8930	0.8453	0.7647	0.6838	0.6824	0.6001	0.5908	0.5137	0.5391	0.4652
	(0.0052)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2b Q-1%	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	10.1364	4.8860	9.5178	8.3722	6.9058	6.1119	6.3196	5.9088	6.2976	7.1459
	(1.0000)		(1.0000)		(1.0000)		(0.9999)		(1.0000)	
175	9.4891	6.8977	12.3915	6.2833	7.8434	5.5789	6.8347	5.9328	8.5364	6.0377
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(0.0000)	
250	12.0205	15.9042	9.0841	8.3552	6.3889	7.1395	5.2827	6.6031	1.9373	5.6785
	(0.0000)		(1.0000)		(0.0000)		(0.0000)		(1.0000)	
500	12.4043	12.8078	7.7031	8.7415	2.2239	6.7383	1.5852	5.8094	1.2157	3.4508
	(0.0189)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
1000	13.0849	13.7579	2.6505	7.2856	1.6849	4.4224	1.4235	2.1426	1.1440	1.2808
	(0.0005)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	5.4578	6.6728	5.5560	4.3752	5.6544	4.3422	4.3256	4.0205	4.3728	4.6133
	(1.0000)		(0.0000)		(0.0000)		(0.0027)		(0.9761)	
175	7.7265	4.3971	4.2548	4.1818	2.0091	3.9731	1.3863	4.2320	1.0706	7.0768
	(0.0000)		(0.2860)		(1.0000)		(1.0000)		(1.0000)	
250	1.5314	4.8377	1.1822	4.3458	1.0518	5.8713	1.0353	4.7275	0.8704	5.5424
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
500	1.0754	4.1908	1.0246	1.6846	0.9092	1.1303	0.8389	0.9646	0.8128	0.8189
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(0.6068)	
1000	1.0668	1.1323	0.9788	1.0342	0.9256	0.8709	0.8680	0.7910	0.8190	0.7862
	(0.9971)		(0.9931)		(0.0034)		(0.0000)		(0.0422)	

DGP 3a Q-1%	κ									
	0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	6.4172	3.8632	7.9423	3.6833	6.3192	3.4009	5.2829	3.1136	2.9237	3.1136
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(0.9894)	
175	8.6375	4.7948	6.9292	5.3375	4.9247	3.8086	3.4946	4.6270	1.2764	2.8862
	(1.0000)		(1.0000)		(1.0000)		(0.0000)		(1.0000)	
250	8.4146	6.6259	7.4802	6.7439	4.3584	5.4093	2.2590	4.5757	1.1508	2.7609
	(1.0000)		(1.0000)		(0.0000)		(0.0000)		(1.0000)	
500	12.4609	9.1181	6.6167	7.3741	2.4164	5.4834	1.6306	3.8222	1.0502	1.1087
	(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9891)	
1000	13.2097	10.9555	4.3104	7.4998	2.0247	3.0766	1.5032	1.7947	1.0065	1.0117
	(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.6099)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.1299	3.1971	0.9395	2.6120	0.6329	1.7596	0.5041	0.7662	0.4143	0.4087
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(0.3718)	
175	0.9101	2.2110	0.7132	1.9885	0.5905	0.6057	0.4927	0.4275	0.4189	0.3473
	(1.0000)		(1.0000)		(0.7787)		(0.0000)		(0.0000)	
250	0.8781	1.2858	0.8920	1.2679	0.7122	0.6511	0.4843	0.4241	0.4298	0.3525
	(1.0000)		(0.0012)		(0.0860)		(0.0000)		(0.0000)	
500	0.8527	0.7863	0.7083	0.6326	0.6119	0.5357	0.5145	0.4484	0.4586	0.3843
	(0.0001)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
1000	0.8699	0.8290	0.7386	0.6682	0.6486	0.5793	0.5571	0.4961	0.5046	0.4409
	(0.0070)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3b Q-1%	κ									
	0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	4.3100	3.5124	3.7911	3.3500	3.2246	4.8037	3.3665	3.8299	1.7073	2.7589
	(1.0000)		(1.0000)		(0.0000)		(0.0000)		(1.0000)	
175	4.7175	4.1045	4.0020	4.2915	1.9700	3.3747	1.8134	3.1966	1.2471	3.2860
	(1.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
250	4.8449	5.9004	2.4970	4.5913	1.7671	3.0293	1.4748	3.1908	1.2034	2.6290
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
500	2.8917	4.3427	2.0815	3.9896	1.6394	1.9185	1.4189	1.8452	1.1550	1.3510
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
1000	2.5533	3.2147	1.9213	2.1489	1.5407	1.6034	1.3765	1.4387	1.1634	1.1849
	(0.0000)		(0.0000)		(0.0062)		(0.0035)		(0.8547)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.6733	2.6570	1.2294	3.0653	0.8833	2.9020	0.8208	2.0899	0.7631	1.8946
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
175	1.1391	3.0788	0.9279	2.3680	0.8974	2.0210	0.7746	1.8313	0.6958	0.7963
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
250	1.0009	2.0405	0.9526	1.6053	0.9000	1.2076	0.7775	0.8116	0.6978	0.7683
	(1.0000)		(1.0000)		(1.0000)		(0.9461)		(0.9997)	
500	1.0518	1.1479	0.9622	0.9432	0.8893	0.8979	0.7988	0.8502	0.7360	0.6963
	(1.0000)		(0.1680)		(0.6675)		(0.9957)		(0.0111)	
1000	1.0972	1.0244	0.9924	0.9681	0.9307	0.9008	0.8527	0.8115	0.7875	0.7606
	(0.0000)		(0.0816)		(0.0433)		(0.0071)		(0.0496)	

D.2 Results for Change in Distribution

DGP 1 **Q-1%**	Alternative Distribution											
	$t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	2.1433	2.7911	0.8444	2.2164	0.5717	1.2507	0.4629	0.4613	0.3682	0.3167	0.3288	0.2715
	(1.0000)		(1.0000)		(1.0000)		(0.4671)		(0.0004)		(0.0000)	
175	0.9117	2.2150	0.6873	0.7236	0.4994	0.4163	0.4192	0.3407	0.3408	0.2773	0.3110	0.2654
	(1.0000)		(0.9400)		(0.0000)		(0.0000)		(0.0000)		(0.0003)	
250	0.8949	0.9887	0.6699	0.6080	0.4898	0.3973	0.4034	0.3366	0.3367	0.2815	0.3019	0.2594
	(0.9996)		(0.0009)		(0.0000)		(0.0000)		(0.0000)		(0.0002)	
500	0.8441	0.7337	0.6441	0.5492	0.4884	0.3997	0.4119	0.3372	0.3358	0.2839	0.2995	0.2607
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
1000	0.8515	0.7548	0.6505	0.5488	0.4940	0.4213	0.4188	0.3511	0.3362	0.2942	0.2995	0.2608
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2 **Q-1%**	Alternative Distribution											
	$t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	7.5680	8.2844	6.0021	6.0737	1.9341	4.5619	0.8813	2.7992	0.5355	2.3478	0.4228	0.5835
	(0.9997)		(0.6541)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
175	2.2152	6.2513	1.0558	7.4473	0.6696	1.6107	0.5267	0.6503	0.4112	0.3747	0.3654	0.3247
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(0.0124)		(0.0031)	
250	1.2204	5.6076	0.8041	1.3739	0.6122	0.6391	0.4994	0.4296	0.3957	0.3542	0.3540	0.3080
	(1.0000)		(1.0000)		(0.8918)		(0.0000)		(0.0029)		(0.0005)	
500	0.9942	1.1016	0.7743	0.6963	0.5688	0.5135	0.4821	0.4195	0.3975	0.3465	0.3486	0.3136
	(0.9999)		(0.0001)		(0.0006)		(0.0000)		(0.0002)		(0.0057)	
1000	0.9548	0.9086	0.7680	0.6902	0.5862	0.5229	0.4897	0.4305	0.4042	0.3570	0.3547	0.3236
	(0.0123)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0098)	

DGP 3 **Q-1%**	Alternative Distribution											
	$t(22)$		$t(12)$		$t(7.71)$		$t(6)$		t(4.67)		$t(4)$	
n_1	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$	$\Theta^\star$	$\tilde{\Theta}^\star$
100	1.4031	2.6554	0.7292	2.1693	0.5616	0.6163	0.4798	0.4241	0.4371	0.3540	0.3996	0.3273
	(1.0000)		(1.0000)		(0.9947)		(0.0007)		(0.0000)		(0.0000)	
175	0.9174	1.9998	0.7500	0.6728	0.5686	0.4603	0.4914	0.3882	0.4334	0.3508	0.4020	0.3279
	(1.0000)		(0.0003)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.8696	0.9087	0.6940	0.6228	0.5688	0.4745	0.5056	0.4029	0.4480	0.3652	0.4144	0.3483
	(0.9465)		(0.0001)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.8933	0.7986	0.7002	0.5916	0.5933	0.4850	0.5231	0.4421	0.4601	0.3976	0.4317	0.3777
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
1000	0.9091	0.8049	0.7480	0.6370	0.6316	0.5426	0.5642	0.4906	0.5043	0.4473	0.4715	0.4327
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

E Simulation Results: Alternative Choices of the Volatility Level

The following tables contain the simulation results presented in Section 4.5.4 regarding alternative choices of the variance of the innovation process. The VaR exceedance level is set to $\alpha = 0.05$. Each table is tagged with the respective DGP i, the type of loss function, the VaR exceedance level and the volatility level in relation to the initial standard deviation of $\sigma = 0.02$ (m-α%-volatility level) in the upper left.

E.1 Results for a Low-Level Volatility

The standard deviation of the DGPs is set to $\sigma = 0.015$.

DGP 1a Q-5%-Low	κ = 0.5			0.65			0.8			0.9			1.1		
n_1	Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$	
100	7.2971	7.7395	(0.0000)	3.2050	5.3392	(0.0000)	1.9360	4.2191	(0.0000)	1.4555	3.2898	(0.0000)	1.0808	1.1612	(0.9993)
175	6.2235	8.6825	(0.0000)	2.7833	5.8059	(0.0000)	1.7116	2.6718	(0.0000)	1.4214	1.7167	(0.0000)	1.0664	1.0414	(0.1091)
250	5.6391	10.1777	(0.0000)	2.6420	5.7858	(0.0000)	1.7088	2.3473	(0.0000)	1.4124	1.6302	(0.0000)	1.0679	1.0335	(0.0305)
500	4.6527	12.7864	(0.0000)	2.4268	4.1091	(0.0000)	1.6477	2.0275	(0.0000)	1.3736	1.5018	(0.0000)	1.0656	1.0075	(0.0001)

	1.2			1.35			1.5			1.75			2		
n_1	Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$	
100	0.9651	0.9347	(0.0804)	0.8434	0.7539	(0.0000)	0.7554	0.6440	(0.0000)	0.6776	0.5373	(0.0000)	0.6083	0.4774	(0.0000)
175	0.9526	0.8846	(0.0001)	0.8455	0.7621	(0.0000)	0.7724	0.6420	(0.0000)	0.6751	0.5383	(0.0000)	0.6195	0.4859	(0.0000)
250	0.9494	0.8690	(0.0000)	0.8493	0.7436	(0.0000)	0.7614	0.6386	(0.0000)	0.6791	0.5424	(0.0000)	0.6267	0.4897	(0.0000)
500	0.9664	0.8783	(0.0000)	0.8601	0.7543	(0.0000)	0.7847	0.6662	(0.0000)	0.6984	0.5730	(0.0000)	0.6394	0.5165	(0.0000)

DGP 1b Q-5%-Low	κ = 0.5			0.65			0.8			0.9			1.1		
n_1	Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$	
100	2.1568	4.9764	(0.0000)	1.6395	3.9619	(0.0000)	1.3896	3.1504	(0.0000)	1.3528	2.8194	(0.0000)	1.1931	1.7276	(1.0000)
175	2.0188	5.8923	(0.0000)	1.5383	3.4837	(0.0000)	1.3311	1.7414	(0.0000)	1.2783	1.6037	(0.0000)	1.1716	1.2384	(0.9974)
250	1.9368	6.1787	(0.0000)	1.4930	2.3393	(0.0000)	1.3232	1.5998	(0.0000)	1.2616	1.3791	(0.0000)	1.1558	1.1747	(0.8192)
500	1.8773	4.0469	(0.0000)	1.4684	2.0161	(0.0000)	1.3047	1.4737	(0.0000)	1.2405	1.3332	(0.0000)	1.1650	1.1524	(0.2336)

	1.2			1.35			1.5			1.75			2		
n_1	Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$	
100	1.1835	1.4433	(1.0000)	1.1382	1.3738	(1.0000)	1.1194	1.1710	(0.9754)	1.0722	1.0996	(0.8575)	1.0636	1.0708	(0.6105)
175	1.1364	1.1985	(0.9955)	1.1097	1.0945	(0.2412)	1.0844	1.0301	(0.0044)	1.0486	1.0301	(0.0005)	1.0417	0.9359	(0.0000)
250	1.1270	1.1096	(0.1870)	1.1017	1.0597	(0.0166)	1.0688	1.0037	(0.0002)	1.0418	0.9408	(0.0000)	1.0215	0.9142	(0.0000)
500	1.1258	1.1029	(0.0879)	1.0951	1.0145	(0.0000)	1.0791	0.9931	(0.0000)	1.0516	0.9519	(0.0000)	1.0238	0.9105	(0.0000)

DGP 2a Q-5%-Low	κ = 0.5			0.65			0.8			0.9			1.1		
n_1	Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$	
100	11.2476	11.5239	(0.0744)	8.5540	9.3109	(0.0001)	2.6547	6.0710	(0.0000)	1.9514	5.8547	(0.0000)	1.2401	3.5718	(1.0000)
175	12.8100	13.6281	(0.0017)	3.8325	9.1348	(0.0000)	2.0287	6.1726	(0.0000)	1.5671	2.9802	(0.0000)	1.1498	1.2801	(1.0000)
250	7.6772	17.7741	(0.0000)	2.9499	11.1736	(0.0000)	1.8185	3.3510	(0.0000)	1.5015	2.0317	(0.0000)	1.1058	1.1031	(0.4550)
500	5.1475	15.7450	(0.0000)	2.5900	5.1979	(0.0000)	1.6960	2.2577	(0.0000)	1.4018	1.6305	(0.0000)	1.0755	1.0502	(0.0921)

	1.2			1.35			1.5			1.75			2		
n_1	Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$		Θ	$\tilde{\Theta}$	
100	1.1248	2.0543	(1.0000)	0.9254	0.9653	(0.9223)	0.8231	0.7773	(0.0284)	0.7051	0.6129	(0.0000)	0.6294	0.5074	(0.0000)
175	0.9909	1.0619	(0.9968)	0.8795	0.7869	(0.0000)	0.7888	0.6804	(0.0000)	0.7016	0.5494	(0.0000)	0.6202	0.4946	(0.0000)
250	1.0047	0.9787	(0.1165)	0.8861	0.7552	(0.0000)	0.7848	0.6683	(0.0000)	0.6863	0.5538	(0.0000)	0.6176	0.4998	(0.0000)
500	0.9710	0.9086	(0.0002)	0.8662	0.7622	(0.0000)	0.7866	0.6681	(0.0000)	0.7020	0.5709	(0.0000)	0.6382	0.5162	(0.0000)

DGP 2b **Q-5%-Low**	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	9.4784	9.8515	2.6139	8.6163	1.9747	7.3725	1.6734	5.6298	1.4225	5.5154
	(0.0000)		(0.00000)		(0.0000)		(0.0000)		(1.0000)	
175	3.3640	13.5772	1.9553	8.3529	1.6006	5.6119	1.4300	5.3533	1.2143	2.0801
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
250	2.8600	12.9189	1.8830	7.1137	1.5190	2.4683	1.3805	1.9527	1.2134	1.3774
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
500	2.5331	11.5004	1.7327	2.7696	1.4379	1.7881	1.3197	1.4627	1.1812	1.1769
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.4190)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.2853	4.5390	1.2416	4.9514	1.1503	3.6707	1.0833	3.9376	1.0385	3.8141
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
175	1.1952	1.5484	1.1430	1.3950	1.1062	1.1628	1.0403	1.0133	0.9972	0.9496
	(1.0000)		(1.0000)		(0.9767)		(0.1491)		(0.0270)	
250	1.1665	1.2398	1.1030	1.1326	1.0608	1.0219	0.9980	0.9436	0.9947	0.8453
	(0.9964)		(0.8778)		(0.0512)		(0.0072)		(0.0000)	
500	1.1489	1.1000	1.0824	1.0218	1.0456	0.9718	1.0137	0.8889	0.9842	0.8486
	(0.0088)		(0.0011)		(0.0001)		(0.0000)		(0.0000)	

DGP 3a **Q-5%-Low**	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.6655	8.0081	3.0914	5.6735	1.8837	5.1955	1.5394	3.5573	1.1181	1.1583
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9377)	
175	5.6367	9.0326	2.7656	6.5675	1.7354	2.7662	1.4269	1.8699	1.0777	1.0816
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.5686)	
250	4.8603	10.3923	2.5379	5.7499	1.6841	2.3194	1.3900	1.7625	1.0716	1.0358
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0318)	
500	4.2889	11.4508	2.3448	3.6735	1.6247	1.9638	1.3851	1.4953	1.0692	1.0254
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0041)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	0.9945	1.0662	0.8748	0.7435	0.7907	0.7007	0.6860	0.5576	0.6331	0.4871
	(0.9974)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	0.9666	0.9007	0.8639	0.7626	0.7769	0.6571	0.6787	0.5507	0.6252	0.4931
	(0.0003)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9681	0.8830	0.8612	0.7547	0.7796	0.6651	0.6947	0.5561	0.6270	0.4991
	(0.0000)		(0.00000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9711	0.9003	0.8642	0.7676	0.7942	0.6839	0.7048	0.5812	0.6448	0.5211
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3b **Q-5%-Low**	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.6471	5.4180	1.9458	4.3118	1.5357	2.8559	1.3796	2.5645	1.1509	1.3162
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	2.4938	5.0713	1.8845	2.3902	1.5189	1.8103	1.3475	1.4481	1.1439	1.1348
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.3379)	
250	2.4048	3.3776	1.8330	2.0970	1.5016	1.6695	1.3396	1.4482	1.1432	1.1763
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9457)	
500	2.3098	2.8083	1.7706	1.9450	1.4676	1.5561	1.3409	1.3952	1.1531	1.1420
	(0.0000)		(0.0000)		(0.0000)		(0.0022)		(0.2622)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.0862	1.2234	0.9576	1.0734	0.9279	0.8452	0.8252	0.7774	0.7625	0.6717
	(1.0000)		(1.0000)		(0.0000)		(0.0070)		(0.0000)	
175	1.0741	1.1060	0.9861	0.9449	0.9211	0.8656	0.8281	0.7635	0.7574	0.6787
	(0.9322)		(0.0160)		(0.0012)		(0.0001)		(0.0000)	
250	1.0755	1.0454	0.9793	0.9201	0.9146	0.8356	0.8370	0.7480	0.7689	0.6905
	(0.0572)		(0.0004)		(0.0000)		(0.0000)		(0.0000)	
500	1.0880	1.0536	0.9995	0.9242	0.9284	0.8616	0.8483	0.7791	0.7937	0.7063
	(0.0208)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

E.2 Results for a High-Level Volatility

The standard deviation of the DGPs is set to $\sigma = 0.04$.

DGP 1a Q-5%-High	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.5664	6.9604	3.2385	6.5653	1.9023	3.7058	1.4525	2.7747	1.1105	1.1301
	(1.0000)		(0.0000)		(0.0000)		(0.0000)		(0.7714)	
175	6.3904	8.0061	7.5529	2.8648	1.7724	2.6017	1.4269	1.7683	1.0829	1.0489
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0471)	
250	6.0541	11.3196	2.6457	5.9930	1.7191	2.3256	1.3996	1.6426	1.0653	1.0357
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0523)	
500	4.7413	12.6762	2.4232	4.0468	1.6637	2.0436	1.3722	1.5337	1.0585	1.0119
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0014)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	0.9662	0.9320	0.8484	0.7470	0.7633	0.6368	0.6716	0.5294	0.6144	0.4707
	(0.0578)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
175	0.9534	0.8800	0.8431	0.7692	0.7625	0.6426	0.6704	0.5442	0.6156	0.4800
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9579	0.8664	0.8465	0.7472	0.7668	0.6417	0.6797	0.5449	0.6199	0.4917
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9588	0.8869	0.8623	0.7465	0.7858	0.6649	0.6975	0.5751	0.6426	0.5108
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 1b Q-5%-High	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.3075	5.7803	1.6652	5.4764	1.3996	2.8885	1.3427	2.7240	1.2207	1.5269
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	1.9887	7.2050	1.5206	4.3002	1.3414	1.7633	1.2486	1.6107	1.1718	1.2725
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
250	1.9571	10.9256	1.4920	2.2809	1.3297	1.7302	1.2454	1.3771	1.1609	1.1927
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9380)	
500	1.8655	4.0847	1.4707	2.0304	1.3053	1.5021	1.2477	1.3112	1.1509	1.1405
	(0.0000)		(0.0000)		(0.0000)		(0.0001)		(0.2712)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.1449	1.4473	1.1319	1.2645	1.1010	1.1499	1.0744	1.1064	1.0273	1.0677
	(1.0000)		(1.0000)		(0.9700)		(0.8931)		(0.9441)	
175	1.1308	1.1844	1.1145	1.0854	1.0892	1.0053	1.0393	0.9880	1.0270	0.9326
	(0.9885)		(0.0908)		(0.0000)		(0.0063)		(0.0000)	
250	1.1365	1.1051	1.1088	1.0593	1.0761	1.0056	1.0425	0.9459	1.0158	0.9186
	(0.0558)		(0.0057)		(0.0001)		(0.0000)		(0.0000)	
500	1.1257	1.0779	1.0870	1.0288	1.0727	0.9820	1.0410	0.9480	1.0202	0.8984
	(0.0023)		(0.0002)		(0.0000)		(0.0000)		(0.0000)	

DGP 2a Q-5%-High	κ 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	11.0008	11.4669	7.9591	8.6469	2.7828	6.1172	1.9818	5.9429	1.2593	3.9940
	(0.0370)		(0.0004)		(0.0000)		(0.0000)		(1.0000)	
175	11.5700	12.1330	3.7636	9.7698	2.0114	6.3601	1.5538	3.0183	1.1257	1.2163
	(0.0108)		(0.0000)		(0.0000)		(0.0000)		(0.9993)	
250	8.0676	16.0351	2.9188	8.6275	1.8850	3.4216	1.4914	1.9830	1.0881	1.1117
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.8376)	
500	5.2527	19.7073	2.4865	5.0921	1.7201	2.2416	1.3941	1.5552	1.0774	1.0597
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0727)	
	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.0503	1.8074	0.9213	0.9892	0.8171	0.7334	0.7040	0.6016	0.6338	0.4925
	(1.0000)		(0.9917)		(0.0002)		(0.0000)		(0.0000)	
175	1.0255	1.0592	0.8794	0.8245	0.7911	0.6712	0.6911	0.5684	0.6107	0.4978
	(0.8995)		(0.0056)		(0.0000)		(0.0000)		(0.0000)	
250	0.9581	0.9581	0.8653	0.7708	0.7750	0.6555	0.6987	0.5604	0.6243	0.4957
	(0.4367)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9662	0.9019	0.8550	0.7508	0.7943	0.6760	0.7055	0.5758	0.6422	0.5079
	(0.0001)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 2b Q-5%-High	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	9.3380	9.9788	2.5679	9.0470	1.8420	6.2951	1.5563	5.9220	1.3733	4.6345
	(0.0053)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	3.3801	13.7328	2.0667	7.8098	1.5732	6.6728	1.4039	5.5401	1.2266	1.8707
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
250	2.9659	12.4566	1.9221	8.3162	1.4802	2.5047	1.3541	1.8378	1.2064	1.2940
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9992)	
500	2.5349	11.1950	1.7510	2.9013	1.4456	1.6851	1.3300	1.4398	1.1767	1.1775
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.5142)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.2758	5.2476	1.2418	4.0127	1.1695	7.3567	1.1163	9.3445	1.0415	4.0359
	(1.0000)		(1.0000)		(1.0000)		(1.0000)		(1.0000)	
175	1.1516	1.7326	1.0929	1.3224	1.0695	1.1675	1.0397	1.0254	0.9858	0.9406
	(1.0000)		(1.0000)		(0.9997)		(0.2913)		(0.0336)	
250	1.1413	1.2579	1.0966	1.0906	1.0540	1.0260	1.0110	0.9496	0.9757	0.8773
	(1.0000)		(0.4036)		(0.1200)		(0.0032)		(0.0000)	
500	1.1322	1.0940	1.0745	0.9967	1.0434	0.9512	1.0100	0.8930	0.9683	0.8514
	(0.0320)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3a Q-5%-High	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	7.8296	8.0901	3.1164	5.6386	1.8648	4.3876	1.5626	4.3982	1.0948	1.2831
	(0.0411)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	5.6428	10.7867	2.7037	6.3107	1.7356	2.7717	1.4453	1.8479	1.0660	1.0966
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.9182)	
250	5.1276	11.0328	2.5537	6.5571	1.6910	2.3263	1.3994	1.6527	1.0817	1.0689
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.2608)	
500	4.2230	10.7951	2.3548	3.8214	1.6263	2.0310	1.3686	1.5213	1.0622	1.0309
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0287)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	0.9977	1.0663	0.8854	0.8244	0.7930	0.6912	0.6857	0.5563	0.6177	0.4889
	(0.9962)		(0.0021)		(0.0000)		(0.0000)		(0.0000)	
175	0.9696	0.9198	0.8665	0.7722	0.7750	0.6555	0.6910	0.5532	0.6249	0.4913
	(0.0057)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
250	0.9720	0.9169	0.8558	0.7520	0.7820	0.6615	0.6905	0.5573	0.6363	0.4974
	(0.0011)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	
500	0.9676	0.8996	0.8687	0.7540	0.7889	0.6757	0.7112	0.5847	0.6468	0.5232
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

DGP 3b Q-5%-High	κ = 0.5		0.65		0.8		0.9		1.1	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	2.6383	4.3702	1.9101	3.9463	1.5445	2.9216	1.3923	2.0611	1.1619	1.2968
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(1.0000)	
175	2.4544	5.0471	1.9238	2.3647	1.5045	1.8262	1.3671	1.4322	1.1608	1.1958
	(0.0000)		(0.0000)		(0.0000)		(0.0053)		(0.9375)	
250	2.4133	3.3469	1.8207	2.0979	1.4958	1.7236	1.3385	1.4553	1.1338	1.1596
	(0.0000)		(0.0000)		(0.0000)		(0.0000)		(0.8952)	
500	2.3082	2.7914	1.7610	1.9584	1.4685	1.5512	1.3382	1.3829	1.1422	1.1303
	(0.0000)		(0.0000)		(0.0000)		(0.0097)		(0.2459)	

	1.2		1.35		1.5		1.75		2	
n_1	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$	Θ	$\tilde{\Theta}$
100	1.0781	1.2342	0.9678	1.0938	0.9234	0.8334	0.8181	0.7710	0.7532	0.7254
	(1.0000)		(1.0000)		(0.0000)		(0.0076)		(0.0817)	
175	1.0697	1.0521	0.9861	0.9527	0.9138	0.8540	0.8283	0.7437	0.7579	0.6693
	(0.1941)		(0.0419)		(0.0006)		(0.0000)		(0.0000)	
250	1.0757	1.0434	0.9792	0.9264	0.9143	0.8781	0.8308	0.7730	0.7776	0.6977
	(0.0451)		(0.0012)		(0.0192)		(0.0002)		(0.0000)	
500	1.0837	1.0520	0.9979	0.9271	0.9404	0.8668	0.8545	0.7735	0.7967	0.7132
	(0.0296)		(0.0000)		(0.0000)		(0.0000)		(0.0000)	

F Supplementary Information on the Application to Stock Indices

F.1 Estimated Break Dates within the Stock Market Indices Series

Note that the dates are written in the order *month/day/year*.

	Estimated Break				
Series	I	II	III	IV	V
DAX 30	11/08/94	07/17/97	06/13/03	01/16/08	
EURO STOXX 50	11/23/94	10/08/97	10/23/01	01/14/08	
FTSE 100	04/03/95	07/10/98	04/15/03	04/28/06	07/01/09
HANG SENG	08/13/97	10/22/01	06/23/04	08/14/09	
NIKKEI	10/20/97	11/27/02	01/02/08		
S&P 500	12/14/95	07/17/98	04/01/03	07/14/09	

F.2 Estimated Parameters of Best Performing Models by Subperiods

Series	**Subp.**	**Estimated Parameters**
DAX 30	1	$\omega = 0.17235 \cdot 10^{-5}$, $\alpha = 0.07132$, $\beta = 0.91694$, $\nu = 5.30124$
	2	$\omega = 0.48050 \cdot 10^{-5}$, $\alpha = 0.06800$, $\beta = 0.86847$, $\nu = 8.22514$
	3	$\omega = 0.60932 \cdot 10^{-5}$, $\alpha = 0.09453$, $\beta = 0.88887$
	4	$\omega = 0.25548 \cdot 10^{-5}$, $\alpha = 0.07124$, $\beta = 0.90275$, $\nu = 10.56011$
EURO STOXX 50	1	$\omega = 0.35707 \cdot 10^{-5}$, $\alpha = 0.08515$, $\beta = 0.86858$, $\nu = 5.68607$
	2	$\omega = 0.07528 \cdot 10^{-5}$, $\alpha = 0.04945$, $\beta = 0.93916$, $\nu = 9.56355$
	3	$\omega = 0.50275 \cdot 10^{-5}$, $\alpha = 0.08479$, $\beta = 0.89855$, $\nu = 17.13093$
	4	$\omega = 0.28535 \cdot 10^{-5}$, $\alpha = 0.07633$, $\beta = 0.88882$, $\nu = 10.62949$
FTSE 100	1	$\omega = 0.30858 \cdot 10^{-5}$, $\alpha = 0.05245$, $\beta = 0.90178$, $\nu = 10.64346$
	2	$\omega = 0.02580 \cdot 10^{-5}$, $\alpha = 0.02904$, $\beta = 0.96754$, $\nu = 13.12938$
	3	$\omega = 0.56263 \cdot 10^{-5}$, $\alpha = 0.10960$, $\beta = 0.86287$
	4	$\omega = 0.10568 \cdot 10^{-5}$, $\alpha = 0.04774$, $\beta = 0.92647$
	5	$\omega = 0.16404 \cdot 10^{-5}$, $\alpha = 0.11824$, $\beta = 0.87772$, $\nu = 7.15715$
HANG SENG	1	$\omega = 0.62317 \cdot 10^{-5}$, $\alpha = 0.07788$, $\beta = 0.88970$, $\nu = 4.43131$
	2	$\omega = 2.05176 \cdot 10^{-5}$, $\alpha = 0.07458$, $\beta = 0.88195$, $\nu = 5.55199$
	3	$\omega = 1.03181 \cdot 10^{-4}$, $\alpha = 0.00100$, $\beta = 0.25236$, $\nu = 5.69186$
	4	$\omega = 0.07328 \cdot 10^{-5}$, $\alpha = 0.07652$, $\beta = 0.92682$, $\nu = 6.05149$
NIKKEI 225	1	$\omega = 0.37473 \cdot 10^{-5}$, $\alpha = 0.09151$, $\beta = 0.89576$, $\nu = 5.24517$
	2	$\omega = 0.89326 \cdot 10^{-5}$, $\alpha = 0.06486$, $\beta = 0.90087$, $\nu = 7.64209$
	3	$\omega = 0.13691 \cdot 10^{-5}$, $\alpha = 0.05615$, $\beta = 0.93551$, $\nu = 7.70700$
S&P 500	1	$\omega = 0.01487 \cdot 10^{-5}$, $\alpha = 0.02482$, $\beta = 0.97242$, $\nu = 5.24498$
	2	$\omega = 0.30272 \cdot 10^{-5}$, $\alpha = 0.04274$, $\beta = 0.92033$, $\nu = 5.91882$
	3	$\omega = 0.66309 \cdot 10^{-5}$, $\alpha = 0.07069$, $\beta = 0.89619$, $\nu = 9.05406$
	4	$\omega = 0.05614 \cdot 10^{-5}$, $\alpha = 0.06510$, $\beta = 0.93200$, $\nu = 7.27716$

Bibliography

Abad, P., Muela, S. B. and Martín, C. L. (2015). The role of the loss function in value-at-risk comparisons. *The Journal of Risk Model Validation*, 9(1), 1-19.

Acerbi, C. and Székely, B. (2014). Backtesting Expected Shortfall. *Risk*, 27(11).

Acerbi, C. and Tasche, D. (2002). Expected Shortfall: a natural coherent alternative to Value at Risk. *Economic Notes*, 31(2), 379-388.

Afik, Z., Arad, O., and Galil, K. (2012). Using Merton model: an empirical assessment of alternatives. *Available at SSRN 2032678.*

Amendola, A. and Candila, V. (2014). Evaluation of Volatility Forecasts in a VaR Framework. In *Mathematical and Statistical Methods for Actuarial Sciences and Finance*, 7-10, Springer.

Amihud, Y. and Mendelson, H. (1991). Volatility, efficiency, and trading: Evidence from the Japanese stock market. *The Journal of Finance*, 46(5), 1765-1789.

Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3), 203-228.

Bai, J. and Perron, P. (2006). Multiple structural change models: a simulation analysis. *Econometric Theory and Practice: Frontiers of Analysis and Applied Research*, 212-237, Cambridge University Press.

Baillie, R., Bollerslev T., and Mikkelsen, H. (1996). Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 74(1), 3-30.

Baillie, R., Han, Y.-W., Myers, R., and Song, J. (2007). Long memory models for daily and high frequency commodity futures returns. *Journal of Futures Markets*, 27(7), 643-668.

Bams, D., Lehnert, T., and Wolff, C. (2005). An evaluation framework for alternative VaR-models. *Journal of International Money and Finance*, 24(6), 944-958.

Bao, Y., Lee, T. H., and Saltoglu, B. (2006). Evaluating predictive performance of value-at-risk models in emerging markets: a reality check. *Journal of Forecasting*, 25(2), 101-128.

Barbosa, A. M. and Ferreira, M. A. (2004). Beyond Coherence and Extreme Losses: Root Lower Partial Moment as a Risk Measure. *SSRN Electronic Journal Paper 08(2004).*

Basel Committee on Banking Supervision (1996b). Supervisory Framework for the Use of "Backtesting" in Conjunction with the Internal Models Approach to Market Risk Capital Requirements. *Bank for International Settlements, Basel.*

Basel Committee on Banking Supervision (1997). Explanatory Note: Modification of the Basel Capital Accord of July 1988, as amended in January 1996. *Bank for International Settlements, Basel.*

Basel Committee on Banking Supervision (2004a). International Convergence of Capital Measurement and Capital Standards. *Bank for International Settlements, Basel.*

Basel Committee on Banking Supervision (2012). Consultative Document May 2012. Fundamental review of the trading book. *Bank for International Settlements, Basel.*

Basel Committee on Banking Supervision (2013b). Consultative Document October 2013. Fundamental review of the trading book: A revised market risk framework. *Bank for International Settlements, Basel.*

Basu, S. (2006). The impact of stress scenarios on VaR and expected shortfall. *Working Paper 1091462*, National Institute of Bank Management.

Bawa, V. S. (1978). Safety-first, stochastic dominance, and optimal portfolio choice. *Journal of Financial and Quantitative Analysis*, 13(02), 255-271.

Berkowitz, J., Christoffersen, P. F., and Pelletier, D. (2011). Evaluating value-at-risk models with desk-level data. *Management Science*, 57(12), 2213-2227.

Berkowitz, J. and O'Brien, J. (2002). How accurate are value-at-risk models at commercial banks?. *The Journal of Finance*, 57(3), 1093-1111.

Best, P. (2000). *Implementing Value at Risk.* Wiley.

Bharath, S. T. and Shumway, T. (2008). Forecasting default with the Merton distance to default model. *Review of Financial Studies*, 21(3), 1339-1369.

Black, F. (1976). The Pricing of Commodity Contracts. *Journal of Financial Econometrics*, 3(1), 167-179.

Black, F. and Cox, J. C. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2), 351-367.

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3).

Bohn, J. R. and Crosbie, P. (2003). Modeling default risk. *KMV Corporation.*

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31(3), 307-327.

Bollerslev, T. and Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, 73(1), 151-184.

Bu, D. and Liao, Y. (2013). Structural Credit Risk Model with Stochastic Volatility: A Particle-filter Approach. *Available at SSRN 2336452.*

Campbell, J., Hilscher, J., and Szilagyi, J. (2008). In search of distress risk. *The Journal of Finance*, 63(6), 2899-2939.

Campbell, S. D. (2005). A review of backtesting and backtesting procedures. *Divisions of Research & Statistics and Monetary Affairs*, Federal Reserve Board.

Candelon, B., Colletaz, G., Hurlin, C., and Tokpavi, S. (2011). Backtesting value-at-risk: a GMM duration-based test. *Journal of Financial Econometrics*, 9(2), 314-343.

Caporin, M. (2003). Stationarity, Memory and Parameter Estimation of FIGARCH Models. *GRETA Working Paper 0309*, University of Venice.

Caporin, M. (2008). Evaluating value-at-risk measures in the presence of long memory conditional volatility. *Journal of Risk*, 10(3).

Chen, J. M. (2014). Measuring Market Risk Under the Basel Accords: VaR, Stressed VaR, and Expected Shortfall. *IEB International Journal of Finance*, 8, 184-201.

Christoffersen, P. F. (1998). Evaluating interval forecasts. *International Economic Review*, 39(4), 841-862.

Christoffersen, P. F. and Pelletier, D. (2004). Backtesting value-at-risk: A duration-based approach. *Journal of Financial Econometrics*, 2(1), 84-108.

Conrad, C. (2010). Non-negativity conditions for the hyperbolic GARCH model. *Journal of Econometrics*, 157(2), 441-457.

Corhay, A. and Rad, A. T. (1994). Statistical properties of daily returns: Evidence from European stock markets. *Journal of Business Finance & Accounting*, 21(2), 271-282.

Danciulescu, C. (2010). Backtesting value-at-risk models: A multivariate approach. *Working Paper 004*, Center for Applied Economics & Policy Research.

Danielsson, J., Jorgensen, B. N., Sarma, M., and de Vries, C. G. (2006). Comparing downside risk measures for heavy tailed distributions. *Economics Letters*, 92(2), 202-208.

Dark, J. (2006). A New Long Memory Volatility Model with Time Varying Skewness and Kurtosis. *Manuscript*, Monash University, Australia.

Davidson, J. (2004). Moment and Memory Properties of Linear Conditional Heteroscedasticity Models, and a New Model. *Journal of Business & Economic Statistics*, 22(1), 16-29.

Degiannakis, S. (2004). Volatility Forecasting: Evidence from a Fractional Integrated Asymmetric Power ARCH Skewed-t Model. *Applied Financial Economics* (14), 1333-1342.

Degiannakis, S., Floros, C., and Dent, P. (2013). Forecasting value-at-risk and expected shortfall using fractionally integrated models of conditional volatility: International evidence. *International Review of Financial Analysis*, 27, 21-33.

Deng, A. and Perron, P. (2008). The limit distribution of the CUSUM of squares test under general mixing conditions. *Econometric Theory*, 24(03), 809-822.

Derman, E. (1996). Model Risk. *Quantitative Strategies Research Notes*, Goldman Sachs.

Derman, E. (2003). Laughter in the dark - The problem of the volatility smile. *University of Amsterdam*.

Diamandis, P. F. (2008). Financial liberalization and changes in the dynamic behaviour of emerging market volatility: Evidence from four Latin American equity markets. *Research in International Business and Finance*, 22(3), 362-377.

Ding, Z., Granger, C. W., and Engle, R. (1993). A Long Memory Property of Stock Market Returns and a new Model. *Journal of Empirical Finance* (1), 83-106.

Dowd, K. (2007). *Measuring Market Risk.* John Wiley & Sons.

Duan, J., Gauthier, G., Simonato, J., and Sasseville, C. (2006). Approximating the GJR-GARCH and EGARCH option pricing models analytically. *Journal of Computational Finance*, 9(3), 41.

Duffie, D., Saita, L., and Wang, K. (2007). Multi-period corporate default prediction with stochastic covariates. *Journal of Financial Economics*, 83(3), 635-665.

Duffie, D. and Singleton, K. (1999). Modeling term structures of defaultable bonds, *Review of Financial Studies*, 12 (4), 687-720.

Eichengreen, B., Mody, A., Nedeljkovic, M., and Sarno, L. (2012). How the subprime crisis went global: evidence from bank credit default swap spreads. *Journal of International Money and Finance*, 31(5), 1299-1318.

Einhorn, D. and Brown, A. (2008). Private profits and socialized risk. *Global Association of Risk Professionals*, 42, 10-26.

Emmer, S., Kratz, M., and Tasche, D. (2014). What is the best risk measure in practice? A comparison of standard measures. *Preprint*, ESSEC Business School, Paris.

Engle, R. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of the United Kingdom Inflation. *Econometrica*, 50, 987-1008.

Engle, R. and Bollerslev, T. (1986). Modeling the Persistence of Conditional Variances. *Econometric Reviews*, 5(1), 1-50.

Engle, R. F. and Russell, J. R. (1998). Autoregressive conditional duration: a new model for irregularly spaced transaction data. *Econometrica*, 1127-1162.

Escanciano, J. C. and Du, Z. (2015). Backtesting Expected Shortfall: Accounting for Tail Risk, *Working Paper 001*, Center for Applied Economics and Policy Research, Indiana University Bloomington.

Escanciano, J. C. and Olmo, J. (2007). Estimation risk effects on backtesting for parametric value-at-risk models. *Working Paper 005*, Center for Applied Economics and Policy Research, Indiana University Bloomington.

Escanciano, J. C. and Olmo, J. (2011). Robust backtesting tests for value-at-risk models. *Journal of Financial Econometrics*, 9(1), 132-161.

Escanciano, J. C. and Olmo, J. (2012). Backtesting parametric value-at-risk with estimation risk. *Journal of Business & Economic Statistics*, 28(1), 36-51.

Ferenstein, E. and Gasowski, M. (2004). Modelling stock returns with AR-GARCH processes. *SORT*, 28(1), 55-68.

Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *American Economic Review*, 67, 116–126

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, 6(4), 429-447.

Föllmer, H. and Schied, A. (2010). Convex risk measures. *Encyclopedia of Quantitative Finance*, Wiley.

Francq, C. and Zakoïan, J. M. (2004). Maximum likelihood estimation of pure GARCH and ARMA-GARCH processes. *Bernoulli*, 10(4), 605-637.

Glosten, L., Jagannathan, R., and Runkle, D. (1993). On the Relation between the Expected Value and the Volatility of the Normal Excess Return on Stocks. *Journal of Finance*, 48(5), 1779-1801.

Gneiting, T. (2011). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106(494), 746-762.

González-Rivera, G., Lee, T. H., and Mishra, S. (2004). Forecasting volatility: A reality check based on option pricing, utility function, value-at-risk, and predictive likelihood. *International Journal of Forecasting*, 20(4), 629-645.

Granger, C. W. and Hyung, N. (2004). Occasional structural breaks and long memory with an application to the S&P 500 absolute stock returns. *Journal of Empirical Finance*, 11(3), 399-421.

Granger, C. W. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis*, 1(1), 15-29.

Halbleib, R. and Pohlmeier, W. (2012). Improving the value at risk forecasts: Theory and evidence from the financial crisis. *Journal of Economic Dynamics and Control*, 36(8), 1212-1228.

Hannan, E. J. and Quinn, B. G. (1979). The determination of the order of an autoregression. *Journal of the Royal Statistical Society.* Series B (Methodological), 190-195.

Hansen, B. E. (2001). The new econometrics of structural change: Dating breaks in US labor productivity. *Journal of Economic perspectives*, 117-128.

Harris, R. D. and Nguyen, A. (2013). Long memory conditional volatility and asset allocation. *International Journal of Forecasting*, 29(2), 258-273.

Hendricks, D. (1996). Evaluation of Value-at-Risk Models Using Historical Data. *Economic Policy Review Federal Reserve Bank of New York*, 2(1), 39-67.

Hendricks, D. and Hirtle, B. (1997). Bank capital requirements for market risk: The internal models approach. *Economic Policy Review*, 3(4).

Henking, A., Bluhm, C., and Fahrmeir, L. (2006). *Kreditrisikomessung: Statistische Grundlagen, Methoden und Modellierung.* Springer.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2), 327-343.

Hull, J., Nelken, I., and White, A. (2004). Merton's model, credit risk, and volatility skews. *Journal of Credit Risk*, 1(1), 05.

Jacobs, K. and Li, X. (2008). Modeling the dynamics of credit spreads with stochastic volatility. *Management Science*, 54(6), 1176-1188.

Jones, E., Mason, S., and Rosenfeld, E. (1984). Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation. *Journal of Finance*, 39(3), 611-625.

Kazakevicius, V. and Leipus, R. (1999). On Stationarity in the ARCH(∞) Model. *Econometric Theory*, 18, 1-16.

Kerkhof, J., Melenberg, B., and Schumacher, H. (2010). Model risk and capital reserves. *Journal of Banking & Finance*, 34(1), 267-279.

Kiliç, E. (2006). Violation duration as a better way of VaR model evaluation: evidence from Turkish market portfolio. *MPRA Paper 5610, University Library of Munich.*

Koenker, R. and Mizera, I. (2010). Quasi-concave density estimation. *The Annals of Statistics*, 2998-3027.

Kupiec, P. H. (1995). Techniques for verifying the accuracy of risk measurement models. *The Journal of Derivatives*, 3(2).

Lamoureux, C. G. and Lastrapes, W. D. (1990). Persistence in variance, structural change, and the GARCH model. *Journal of Business & Economic Statistics*, 8(2), 225-234.

Leland, H. E. (2004). Predictions of default probabilities in structural models of debt. *Journal of Investment Management*, 2(2).

Longstaff, F. A. and Schwartz, E. S. (1995). A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3), 789-819.

Lopez, J. A. (1998). Methods for evaluating value-at-risk estimates. *Economic Policy Review*, 4(3).

Lopez, J. A. (2001). Evaluating the predictive accuracy of volatility models. *Journal of Forecasting*, 20(2), 87-109.

Lucas, A. (2001). Evaluating the Basle guidelines for backtesting banks' internal risk management models. *Journal of Money, Credit and Banking*, 826-846.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, 36, 394–419.

Merton, R. (1973). Theory of Rational Option Pricing. *The Bell Journal of Economics and Management Science*, 141-183.

Merton, R. (1974). On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance*, 29(2), 449-470.

National Bank of Austria (1999). Evaluation of Value-at-Risk Models. In *Guidelines on Market Risk*, Volume 3. National Bank of Austria.

Nelson, D. (1990). Stationarity and Persistence in the GARCH(1,1) Models. *Econometric Theory*, 6(3), 318-334.

Nelson, D. and Cao, C. Q. (1992). Inequality Constraints in the Univariate GARCH Model. *Journal of Business & Economic Statistics*, 10(2), 229-235.

Nieppola, O. (2009). Backtesting value-at-risk models. *Helsinki School of Economics.*

Perron, P. (2006). Dealing with structural breaks. *Palgrave Handbook of Econometrics*, 1, 278-352.

Pesaran, M. and Timmermann, A. (2002). Market timing and return prediction under model instability. *Journal of Empirical Finance*, 9(5), 495-510.

Pesaran, M. and Zaffaroni, P. (2004). Model averaging and value-at-risk based evaluation of large multi asset volatility models for risk management. *Research Paper*, Money Macro and Finance Research Group Conference.

Rapach, D. E. and Strauss, J. K. (2008). Structural breaks and GARCH models of exchange rate volatility. *Journal of Applied Econometrics*, 23(1), 65-90.

Robinson, P. (1991). Testing for strong serial Correlation and dynamic Conditional Heteroskedasticity in multiple Regression. *Journal of Econometrics*, 47(1), 67-84.

Rockafellar, R. T. and Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk*, 2, 21-42.

Rosasco, L., Vito, E., Caponnetto, A., Piana, M., and Verri, A. (2004). Are loss functions all the same?. *Neural Computation*, 16(5), 1063-1076.

Røynstrand, T., Nordbø, N., and Strat, V. (2012). Evaluating power of Value-at-Risk backtests.

Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of Statistics*, 6(2), 461-464.

Sewell, M. (2011). Characterization of financial time series. *RN 11(01)*, 1-35.

Sibbertsen, P., Stahl, G., and Luedtke, C. (2008). Measuring model risk. *Journal of Risk Model Validation*, 2(4), 65-81.

Sun, W., Rachev, S., and Fabozzi, F. J. (2007). Fractals or IID: evidence of long-range dependence and heavy tailedness from modeling German equity market returns. *Journal of Economics and Business*, 59(6), 575-595.

Tse, Y. K. (1998). The Conditional Heteroskedasticity of the Yen-Dollar Exchange Rate. *Journal of Applied Econometrics*, 13(1), 49-55.

Yamai, Y. and Yoshiba, T. (2002). On the validity of value-at-risk: comparative analyses with expected shortfall. *Monetary and Economic Studies*, 20(1), 57-85.

Ziegel, J. F. (2014). Coherence and elicitability. *Mathematical Finance.*