Torsors and generalized Cox rings for Manin's conjecture

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Kurzfassung. Eine Vermutung von Manin sagt die Verteilung rationaler Punkte auf Fano-Varietäten über Zahlkörpern vorher. Für manche Varietäten kann sie, nach einer Parametrisierung der Menge der rationalen Punkte durch ganze Punkte auf affinen Räumen, durch das Zählen von Gitterpunkt verifiziert werden. Wenn die Parametrisierung die Verbindung zwischen Cox-Ringen und universellen Torsoren verwendet, ist dieser Ansatz als Methode universeller Torsore bekannt.

In dieser Dissertation wird der Parametrisierungsschritt durch eine Konstruktion ganzer Modelle von Torsoren mit Hilfe von endlich erzeugten Cox-Ringen und durch eine explizite Beschreibung ihrer Mengen von ganzen Punkten präzisiert. Diese Ergebnisse werden dann auf die Parametrisierung der rationalen Punkte auf zerfallenden glatten del-Pezzo-Flächen von Grad 4 und auf einer zerfallenden singulären del-Pezzo-Fläche von Grad 4 angewendet. Sie werden auch verwendet, um Manins Vermutung für zerfallende glatte eigentliche torische Varietäten über imaginär-quadratischen Körpern herzuleiten.

Um die Methode universeller Torsore auf nicht-zerfallende Varietäten zu übertragen, führen wir verallgemeinerte Cox-Ringe und Cox-Garben ein und untersuchen ihre Eigenschaften und ihr Verhältnis zu Torsoren unter Quasitori. Um die Anwendung auf Manins Vermutung zu illustrieren, berechnen wir gewisse verallgemeinerte Cox-Ringe von Châtelet-Flächen.

Abstract. The distribution of rational points on Fano varieties over number fields is predicted by Manin's conjecture. For some varieties this can be verified via lattice point counting after a parameterization of the set of rational points via integral points on affine spaces. When the parameterization uses the connection between Cox rings and universal torsors, this approach is known as universal torsor method.

In this thesis the parameterization step is made precise by giving a construction of integral models of torsors via finitely generated Cox rings and by describing explicitely their sets of integral points. These results are then applied to parameterize the rational points on split smooth quartic del Pezzo surfaces and on a split singular quartic del Pezzo surface. They are also used to deduce Manin's conjecture for split smooth proper toric varieties over imaginary quadratic fields.

To adapt the universal torsor method to non-split varieties we introduce the notion of generalized Cox rings and Cox sheaves, and we study their properties and the relation to torsors under quasitori. To illustrate the application to Manin's conjecture, we compute certain generalized Cox rings of Châtelet surfaces.

Schlagworte: Cox ringe, rationale Punkte, Zahlkörpern

Keywords: Cox rings, rational points, number fields

Contents

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Introduction

The study of rational solutions of polynomial equations is a classical topic in number theory that dates back to the work of Diophantus of Alexandria in the third century. Since algebraic varieties are defined by systems of polynomial equations and the solutions of the equations defining a variety are its rational points, the task of solving diophantine equations translates into the problem of determining the set of rational points of certain varieties, or more generally schemes, in the language of algebraic geometry. The first natural question to ask about the set of rational points of a variety is whether it is nonempty. From the well known fact that the conic defined by the equation

$$x^2 + y^2 + z^2 = 0$$

has no rational points over $\mathbb{Q}$, but becomes isomorphic to $\mathbb{P}^1$ after a quadratic extension, it is clear that the behavior depends on the choice of the base field. If rational points exist, one may ask how many there are and how they are distributed. This last question is particularly interesting if the set of rational points is dense in the variety with respect to the Zariski topology, which is the case considered in this thesis.

In 1979 Schanuel [**Sch79**] proved that over number fields the number of rational points on $\mathbb{P}^n$ with Weil height at most B grows, asymptotically with B, as CB^{n+1}, where C is a constant depending only on n and on the base field. Inspired by Schanuel's result and by some numerical data about the distribution of $\mathbb{Q}$-rational points on a smooth cubic surface, Manin [**FMT89, BM90**] conjectured the following asymptotic formula for the number of rational points of anticanonical height at most B on suitable open subsets of Fano varieties with Zariski dense set of rational points over a number field:

$$CB(\log B)^{r-1}, \tag{0.1}$$

where r is the Picard rank of the variety and C is a positive constant depending on the geometry of the variety and on the base field. The reason for considering open subsets instead of the whole variety is that on some subvarieties, called accumulating, the number of rational points of bounded height grows faster than on their complement. For example, the 27 lines on a split smooth cubic surface are accumulating subvarieties [**FMT89**]. However, there are varieties for which the accumulating subvareties are Zariski dense (e.g. [**BT96**]). On the other hand, the asymptotic formula (0.1) holds also for a number of quasi-Fano varieties that are not Fano (e.g. [**BBD07**]). Hence, a comprehensive formulation of the conjecture remains to be found. See [**Rud14**] for a generalization of the conjecture that takes into account the case of Zariski dense accumulating subvarieties, and [**Pey95, Pey03**] for a generalization that considers quasi-Fano varieties and includes a precise interpretation of the constant C as the product of a geometric invariant and a Tamagawa number.

Manin's conjecture has been attacked with various techniques. The study of the zeta function attached to the height function via harmonic analysis on adelic points has been successful for certain smooth compactifications of algebraic groups over arbitrary number fields (e.g. [**BT98**] for tori and [**CLT02**] for additive groups). An application of the circle method introduced by Birch [**Bir62**], and extended by [**Ski97**] and others, verifies the conjecture for some complete intersections where the dimension of the variety is large with respect to the degree. A third approach, which was developed for some (weak) del Pezzo surfaces to which the aforementioned methods do not apply (e.g. [**Bre02, BBD07**]), consists of parameterizing the set of rational points via integral points on higher dimensional affine spaces, and then counting integral points, which are lattice points, on certain subsets of such affine spaces. Since the parameterizations are usually induced by a universal torsor, this approach is called the *universal torsor method*. Universal torsors have been introduced and studied by Colliot-Thélène and Sansuc [**CTS76**] as special torsors under quasitori (or groups of multiplicative type) to investigate the existence of rational points on geometrically rational varieties. They used torsors as descent varieties because a torsor Y over a variety X under an algebraic group G gives a partition of the set of rational points of X in terms of images of the points on its twists ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}Y \to X$:

$$X(k) = \bigsqcup_{\underline{\mathfrak{c}} \in H^1_{fppf}(k,G)} {}_{\underline{\mathfrak{c}}}\pi({}_{\underline{\mathfrak{c}}}Y(k)).$$

This holds in particular for universal torsors, and it remains true if we replace the number field k by its ring of integers (see Proposition 1.4 for a general version). For example, the universal torsor of the projective space $\mathbb{P}^n$ is $\mathbb{A}^{n+1} \smallsetminus \{0\}$ and the induced parameterization by integral points is

$$\mathbb{P}^n(k) = \bigsqcup_{\mathfrak{c}} \{(x_0 : \cdots : x_n) : x_1, \ldots, x_n \in \mathscr{O}_k, (x_1, \ldots, x_n) = \mathfrak{c}\},$$

where the disjoint union is over a set of integral representatives $\mathfrak{c}$ for the ideal class group of the ring of integers $\mathscr{O}_k$ of the number field k, which is isomorphic to the group $H^1_{fppf}(k, \mathbb{G}_m)$. The work of Schanuel for projective spaces uses this parameterization and is a fundamental example of proof via the universal torsor method.

Besides his work, until recently the proofs that can be considered under this approach concerned varieties defined over $\mathbb{Q}$. The universal torsor method in this case is easier because $\mathbb{Q}$ has class number 1 and $\mathbb{Z}$ has finite unit group: $\{\pm 1\}$. Hence, instead of counting rational points of bounded height on a variety one typically counts $\mathbb{Z}$-points on the universal torsor and then divides the result by 2^r, where r is the Picard rank. Moreover, so far universal torsor parameterizations are used mostly for *split* varieties, that is, with trivial Galois action on the geometric Picard group. In this situation two natural questions arise:

1. what is the universal torsor method over arbitrary number fields?

2. what is the right approach to *non-split* varieties?

In an attempt to answer them, the first task of this thesis is to generalize the universal torsor method to arbitrary number fields, at least the parameterization part.

The parameterization of the projective spaces presented above works because $\mathbb{P}^n(\mathscr{O}_k) = \mathbb{P}^n(k)$, because the natural morphism $\mathbb{A}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ is a torsor also over $\mathscr{O}_k$, and because $\mathbb{A}^{n+1} \smallsetminus \{0\}$ is embedded into an affine space. For a general variety X, analogous requirements are the following: X has a proper $\mathscr{O}_k$-model $\mathscr{X}$, so that $\mathscr{X}(\mathscr{O}_k) = X(k)$, and there is an $\mathscr{X}$-torsor $\mathscr{Y}$ under $\mathbb{G}^r_{m,\mathscr{O}_k}$ with an embedding as locally closed subset of an affine space $\mathbb{A}^N_{\mathscr{O}_k}$. In this setting, which is typical for split projective varieties with free finitely generated Picard group and finitely generated Cox ring (cf. Section 3.1), we compute the twisted torsors ${}_{\underline{\mathfrak{c}}}\mathscr{Y}$ as locally closed subsets of *twisted affine spaces* and we obtain the following explicit description of their sets of integral points as lattice points (cf. Theorem 1.10)

$$ {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) = \{\underline{y} \in \bigoplus_{i=1}^{N} \underline{\mathfrak{c}}^{m^{(i)}} : \sum_{i=1}^{t} f_i(\underline{y})\underline{\mathfrak{c}}^{-\deg f_i} = \mathscr{O}_k,\ g_j(\underline{y}) = 0, 1 \leq j \leq s\}, $$

where $\underline{\mathfrak{c}}^{m^{(i)}}$ are fractional ideals of $\mathscr{O}_k$ depending on $\underline{\mathfrak{c}}$ and on the action of the group $\mathbb{G}^r_{m,\mathscr{O}_k}$ on the torsor $\mathscr{Y}$, and g_j and f_i are the polynomials defining the closure of $\mathscr{Y}$ and of its complement, respectively, in $\mathbb{A}^N_{\mathscr{O}_k}$. In particular, to employ this result there is no need of computing the twists of $\mathscr{Y}$.

We face then the problem of finding $\mathscr{X}$ and $\mathscr{Y}$ that satisfy the requirements above. Salberger, who introduced the universal torsor method in [**Sal98**] to prove Manin's conjecture for smooth proper split toric varieties over $\mathbb{Q}$, used a parameterization via integral points on universal torsors of the natural $\mathbb{Z}$-models of toric varieties that are the $\mathbb{Z}$-toric schemes defined by the same fan. To compute the universal torsors he employed Cox's characterization of toric varieties as GIT quotients of certain open subsets of affine spaces that correspond to the homogeneous coordinate rings, today known as Cox rings, of the toric varieties [**Cox95**]. The realization of the universal torsor of a variety as an open subset of the spectrum of its Cox ring, if the latter is finitely generated, is a well known result over algebraically closed fields thanks to the work of Hassett [**Has09**] and others [**ADHL15**]. Moreover, Cox rings for many quasi-Fano varieties, in particular surfaces, have been computed by Batyrev, Popov [**BP04**], Hassett, Tschinkel [**HT04**], Hausen, Keicher and Laface [**HKL14**], among others. Except for some varieties for which the universal torsors were known or computed (e.g. [**Bre02, BBS14**]), the parameterizations used to verify special cases of Manin's conjecture were obtained by ad hoc manipulations of coordinates that, at least for split varieties, were usually inspired by the descriptions of the Cox rings in terms of generators and relations.

In Chapter 3 we show how to use such descriptions of finitely generated Cox rings over an algebraically closed field to produce integral models $\mathscr{Y} \to \mathscr{X}$ of universal torsors of certain split varieties as locally closed subsets of affine spaces by exhibiting the defining polynomial equations. We also give conditions under which the obtained model $\mathscr{Y}$ is a torsor and $\mathscr{X}$ has certain properties, like being smooth or projective. In Section 4.2 we apply this procedure to compute integral models of universal torsors of smooth split del Pezzo surfaces of degree four, that is, blowing-ups of $\mathbb{P}^2$ in five points

$$(1:0:0), \quad (0:1:0), \quad (0:0:1), \quad (1:1:1), \quad (a:b:c),$$

in general position. The second application that we present concerns the singular del Pezzo surface of degree 4 with singularity type $\mathbf{A}_3 + \mathbf{A}_1$ given in $\mathbb{P}^4$ by

$$x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0.$$

In this case, we compute an integral model of the universal torsor of the minimal desingularization X of the surface using the description of the Cox ring provided in [**Der14**], we prove that the model is a universal torsor in the sense of [**Sal98**] (cf. Proposition 5.2), and we produce an explicit parameterization of the set of rational points via lattice points (cf. Proposition 5.3), which agrees with the heuristic prediction in [**DF14a**]. In joint work with Frei [**FP14**] this parameterization is then used to prove Manin's conjecture for X over arbitrary number fields via refined lattice point counting techniques.

A third application of the universal torsor method is presented in Chapter 6. Here we consider the same family of toric varieties of [**Sal98**], and we extend Salberger's proof to imaginary quadratic fields. Toric varieties are a classical topic in the literature about Manin's conjecture. We recall that Schanuel's asymptotic formula for projective spaces [**Sch79**] inspired Manin's conjecture. Moreover, smooth projective toric varieties constituted an important example worked out via harmonic analysis on adelic points over arbitrary number fields [**BT98**] and over global fields of positive characteristic [**Bou11**], as well as via the universal torsor method over $\mathbb{Q}$ [**Sal98**]. Recently, a certain singular toric cubic surface was picked to investigate the universal torsor method over number fields beyond $\mathbb{Q}$ in [**DJ13, Fre13**]. Therefore, they represent a natural choice to illustrate how the results of Chapters 1 and 3 apply to a family of varieties. We restrict to imaginary quadratic fields, because a generalization of the the proofs in [**Sch79, Fre13**] does not seem straightforward.

Regarding the second of the two questions we asked above, we observe that some proofs of Manin's conjecture for non-split varieties make use of parameterizations induced by *intermediate* torsors. See, for example, [**BB07**] for a singular del Pezzo surface of degree 4, and [**BBP12**] for the family of Châtelet surfaces

$$x^2 + y^2 = \prod_{j=1}^{4} l_j(u,v), \tag{0.2}$$

where $l_1, \dots, l_4$ are pairwise non-proportional linear polynomials. In the first paper the parameterization is obtained by ad hoc computations unrelated to the equations defining the Cox ring, and it is vaguely attributed to a torsor that is not universal. In the second one, the universal torsors are explicitly computed as well as the intermediate torsors. In both cases, with the results of Section 2.4 it is possible to prove that the intermediate torsors are torsors under quasitori whose type is the natural injection of the Picard group of the variety into its geometric Picard group: *quasi-universal* torsors, in the terminology of [**CTS87**]. We do it explicitly for the Châtelet surfaces (0.2).

Since the parameterizations via lattice points on the universal torsors are computed using the natural embeddings into the spectra of the Cox rings, we are prompted to investigate whether a similar property holds for more general torsors under quasitori. We do it by introducing the notion of *generalized Cox rings* in Chapter 2. The Cox ring of a variety X over an algebraically closed field k is a $\mathrm{Pic}(X)$- or $\mathrm{Cl}(X)$-graded k-algebra which is isomorphic, as k-vector space, to the direct sum

$$\bigoplus_D H^0(X, \mathcal{O}_X(D)) \tag{0.3}$$

over a system of representatives D for $\mathrm{Pic}(X)$ (respectively, $\mathrm{Cl}(X)$). If $\mathrm{Pic}(X)$ (respectivley, $\mathrm{Cl}(X)$) is free, the Cox ring can be defined directly as a direct sum using a basis of the grading group [**HK00, EKW04**]. In general, the definition of a multiplication on (0.3) turning the direct sum into a graded ring is not obvious. This is done in [**BH03**] for $\mathrm{Pic}(X)$-graded Cox rings and in [**Hau08**] for $\mathrm{Cl}(X)$-graded Cox rings by introducing Cox sheaves, which are certain graded sheaves of $\mathcal{O}_X$-algebras isomorphic, as $\mathcal{O}_X$-modules, to the direct sum $\bigoplus_D \mathcal{O}_X(D)$ analogous to (0.3). Cox rings are then the rings of global sections of Cox sheaves, and Cox sheaves are essentially the structure sheaves of the universal torsors. We find a similar connection between generalized Cox rings and torsors under quasitori.

We first consider varieties over separably closed fields. We define the generalized Cox sheaves and rings from an axiomatic point of view, generalizing the axiomatic approach to Cox rings of [**DP14**]. The definition over arbitrary fields is then provided by Galois descent, once we specify what is a natural Galois action on a generalized Cox sheaf, or ring, of the base extension $X_{\overline{k}}$ of the variety X under the inclusion of the base field k in a separable closure $\overline{k}$. To each generalized Cox sheaf, or ring, is attached a homomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$ which is the type of the associated torsor under the following correspondence (cf. Corollary 2.40 and Corollary 2.46).

Theorem 0.1. *Let $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$ be a morphism of $\mathrm{Gal}(\overline{k}/k)$-modules. The contravariant functor*

$$\begin{array}{ccc} \{\textit{Cox sheaves of } X \textit{ of type } \lambda\} & \longrightarrow & \{X\textit{-torsors of type } \lambda\} \\ \mathcal{R} & \longmapsto & \mathrm{Spec}_X \mathcal{R} \end{array}$$

is an anti-equivalence of categories. If the image of λ is generated by effective divisor classes, then the covariant functor

$$\begin{array}{ccc} \{\textit{Cox sheaves of } X \textit{ of type } \lambda\} & \longrightarrow & \{\textit{Cox rings of } X \textit{ of type } \lambda\} \\ \mathcal{R} & \longmapsto & \mathcal{R}(X) \end{array}$$

induces a bijection between the set of isomorphism classes of Cox sheaves of X of type λ and the set of isomorphism classes of Cox rings of X of type λ.

After investigating the existence of generalized Cox sheaves and rings and their functoriality properties, we devote the last section of Chapter 2 to generalized Cox rings that are finitely generated as k-algebras. In particular, we prove that the torsor corresponding to a finitely generated generalized Cox ring R can be realized as an open subset of $\operatorname{Spec} R$, under conditions analogous to the ones in [**BH03**], and that such an embedding is computable in terms of polynomial equations. As an application we compute some Cox rings of identity and injective type over nonclosed fields for the family of Châtelet surfaces (0.2). In Chapter 3 we actually show how to produce integral models of torsors associated with certain finitely generated generalized Cox rings of projective varieties, not necessarily of identity type.

Standard notation

Throughout this thesis, we denote by k a field, by $\overline{k}$ a separable closure of k and by $\mathfrak{g}$ the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. If A is a ring, we write $H^1(A, \cdot)$ for the cohomology group $H^1(\operatorname{Spec} A, \cdot)$. Over a field we identify *étale* cohomology with Galois cohomology.

A k-variety is a separated scheme of finite type over k. The structure sheaf of a k-variety X is denoted by $\mathcal{O}_X$, and we write $k[X]$ for its ring of global sections $H^0(X, \mathcal{O}_X)$. The set of sections of an X-sheaf $\mathscr{L}$ over an open subset U of X are denoted by $\mathscr{L}(U)$ or by $H^0(U, \mathscr{L})$.

If $A \to A'$ is a homomorphism of rings and X is a scheme over A, we denote by $X_{A'}$ the base extension $X \times_{\operatorname{Spec} A} \operatorname{Spec} A'$.

If k is a number field, we denote by $\mathscr{O}_k$ its ring of integers, and by Ω_f, Ω_∞ and Ω_k the sets of finite places, infinite places and all places of k, respectively. For every place $\nu \in \Omega_k$, we denote by k_ν the completion of k at the place ν. For every $\nu \in \Omega_f$, if $\mathfrak{p}$ is the prime ideal of $\mathscr{O}_k$ corresponding to ν, we denote by $\mathbb{F}_\mathfrak{p} = \mathbb{F}_\nu = \mathscr{O}_k/\mathfrak{p}$ the residue field of k_ν.

If k'/k is an algebraic extension, we denote by $\mathfrak{N}_{k'/k}$ the associated norm, while $\mathfrak{N}$ denotes the absolute norm of a fractional ideal. If $a \in k$, we write $\mathfrak{N}(a)$ for the norm of the principal ideal $a\mathscr{O}_k$.

For every place $\nu \in \Omega_k$, we denote by $|\cdot|_\nu$ the absolute value of k_ν normalized as follows: if $\tilde{\nu}$ is the place of $\mathbb{Q}$ below ν and $\mathbb{Q}_{\tilde{\nu}}$ is the completion of $\mathbb{Q}$ at $\tilde{\nu}$, then $|\cdot|_\nu := |\,\mathfrak{N}_{k_\nu/\mathbb{Q}_{\tilde{\nu}}}(\cdot)|_{\tilde{\nu}}$, where $|\cdot|_{\tilde{\nu}}$ is the usual real or p-adic absolute value on $\mathbb{Q}_{\tilde{\nu}}$. We use the symbol $|\cdot|$ for the real absolute value, and we write $\#S$ for the cardinality of a set S.

Part 1

Torsors and generalized Cox rings

CHAPTER 1

Torsor parameterizations

This chapter gives an account of the parameterization properties of torsors, with special emphasis on torsors under $\mathbb{G}_m^r$ over schemes defined over Dedekind domains. The parameterization properties of torsors over varieties under algebraic groups have been studied in [**CTS87**] and [**Sko01**]. Here, we consider more general base schemes. Proposition 1.4 and the content of Section 1.2 are part of [**FP14**, §2].

1.1. Torsors

We start by recalling the definition and some basic properties of torsors. We refer to [**Mil80**, §III.4] and [**Sko01**, §2.2] for a more detailed discussion.

Definition 1.1. Let X be a scheme. A left (right) sheaf of torsors over X under a sheaf of groups $\mathcal{G}$ over X is an *fppf*-sheaf of sets $\mathcal{Y}$ over X with an action of $\mathcal{G}$ such that there exists an *fppf*-covering $(U_i \to X)$ such that $\mathcal{Y}|_{U_i}$ with the action of $\mathcal{G}|_{U_i}$ is isomorphic to $\mathcal{G}|_{U_i}$ with the left (respectively, right) action of $\mathcal{G}|_{U_i}$ on itself for all i.

If $\mathcal{G}$ and $\mathcal{Y}$ are representable by X-schemes G and Y, respectively, we say that Y is a torsor over X (or X-torsor) under G. In this case, Definition 1.1 is equivalent to the following characterization.

Definition 1.2. Let X be a scheme. A left (right) X-torsor under an X-group scheme G is an X-scheme Y endowed with a left (respectively, right) action of G compatible with the projections to X such that the structure morphism $Y \to X$ is *fppf* and the map $G \times_X Y \to Y \times_X Y$ given by $(g, y) \mapsto (g * y, y)$ is an isomorphism.

If X is a Z-scheme for a given scheme Z, G is any Z-group scheme and Y is an X-torsor under $G_X := G \times_Z X$, we say that Y is an X-torsor under G.

In this thesis we consider only torsors under commutative groups. From now on G denotes a commutative X- or Z-group scheme. For us, a morphism of torsors is always a morphism of schemes that preserves the torsor structures.

Isomorphism classes of torsors under commutative groups are classified by *fppf* cohomology as follows. See [**Mil80**, Proposition III.4.6] and [**Sko01**, §2.2].

Proposition 1.3. *Let X be a scheme. The set of isomorphism classes of sheaves of X-torsors under a sheaf of abelian groups $\mathcal{G}$ over X is in bijection with the group $H^1_{fppf}(X, \mathcal{G})$.*

If $\mathcal{G}$ is representable by an X-scheme G we write $H^1_{\mathit{fppf}}(X, G)$ instead of $H^1_{\mathit{fppf}}(X, \mathcal{G})$. We recall that if G is smooth, *fppf* cohomology can be replaced by *étale* cohomology (cf. [**BLR90**, p. 152]).

Given a torsor Y over X under G, its class $[Y]$ in $H^1_{\mathit{fppf}}(X, G)$ can be computed as follows. Let $(U_i \to X)$ be an *fppf*-covering that trivializes Y, and denote by $\beta_i : Y \times_X U_i \to G \times_X U_i$ the local isomorphisms. The cocycle $(\beta_{i,j})_{i,j}$ determined by the induced automorphisms $\beta_j \circ \beta_i^{-1}$ of $G \times_X U_i \times_X U_j$ represents the class $[Y]$ in $H^1_{\mathit{fppf}}(X, G)$. Moreover, Y is isomorphic to the torsor obtained by glueing $G \times_X U_i$ over any cocycle representing $[Y]$. We recall that glueing over an *fppf* covering does not always result in a scheme, as not every *fppf*-sheaf of algebras over X is representable by a scheme. See [**BLR90**, §6] for representability, and [**Mil80**, Theorem III.4.3] for representability of torsors.

We now introduce the notion of twisting X-torsors under G by elements in the cohomology group $H^1_{\mathit{fppf}}(X, G)$. Let Y be a torsor over X under G and $\alpha \in H^1_{\mathit{fppf}}(X, G)$. Let $(U_i \to X)$ be an *fppf*-covering that trivializes both Y and α, and let $\alpha_{i,j}$, $\beta_{i,j}$ be automorphisms of $G \times_X U_i \times_X U_j$ that define cocycles representing α and $[Y]$ respectively. Let $\tilde{\alpha} := (\alpha_{i,j})_{i,j}$. We denote by ${}_{\tilde{\alpha}}\mathcal{Y}$ the sheaf of X-torsors under G obtained by glueing $G \times_X U_i$ over the cocycle $(\alpha_{j,i} \circ \beta_{i,j})_{i,j}$. It is called the twist of Y by $\tilde{\alpha}$, and has class $[Y] - \alpha$ in $H^1_{\mathit{fppf}}(X, G)$. If ${}_{\tilde{\alpha}}\mathcal{Y}$ is representable, we denote by ${}_{\tilde{\alpha}}Y$ the corresponding torsor. For a definition of twisted torsors when G is not commutative, see [**Sko01**, p. 20]

We recall that an X-torsor Y under G is trivial (namely, isomorphic to the X-group G) exactly when $Y \to X$ has a section. Therefore, if X is a Z-scheme, G a Z-group scheme and $\pi : Y \to X$ a torsor under G, the fibers of π at Z-points of X are Z-torsors under G, and they contain a Z-point if and only if they are isomorphic to G. Although the morphism π is surjective by definition, it is not necessarily surjective at the level of Z-rational points, but the family of twists of Y by elements of $H^1(Z, G) \subseteq H^1(X, G)$ induce a partition on the set of Z-rational points of X as proven in the following proposition. This result is well known if Z is the spectrum of a field k, X is a k-variety and G an algebraic group. See [**CTS87**, (2.7.2)] and [**Sko01**, p. 22]. The proof given in [**Sko01**, p. 22] works also with $\operatorname{Spec} k$ replaced by an arbitrary base scheme Z. We include it here for completeness.

PROPOSITION 1.4. *Let Z be a scheme, G an abelian group scheme over Z, X a Z-scheme, and $\pi : Y \to X$ a torsor under $G_X := G \times_Z X$. Assume that the twisted torsors ${}_WY$ exist for all Z-torsors W under G (this is the case, for example, if G is affine over Z). Then*

$$X(Z) = \bigsqcup_{[W] \in H^1_{fppf}(Z,G)} {}_W\pi(({}_WY)(Z)),$$

where $\bigsqcup_{[W] \in H^1_{fppf}(Z,G)}$ is a disjoint union running through a system of representatives for the classes in $H^1_{fppf}(Z, G)$ and ${}_W\pi : {}_WY \to X$ is a twist of Y by $[W] \in H^1_{fppf}(Z, G)$.

PROOF. If $X(Z) = \emptyset$ then $Y(Z) = \emptyset$ for all X-torsors Y under G. Hence, we can assume that $X(Z) \neq \emptyset$. Let $W := Y \times_X Z$ be the fiber of π at a Z-point $\mathbf{x} : Z \to X$. Then W is a Z-torsor under G, because $fppf$-morphisms are stable under base extension and the following diagram is commutative

$$\begin{tikzcd} G \times_Z W \arrow[r, "(g,w) \mapsto (g*w,w)"'] \arrow[d, "\cong"'] & W \times_Z W \arrow[d, "\cong"] \\ (G_X \times_X Y) \times_X Z \arrow[r, "\cong", "(g,y,z) \mapsto (g*y,y,z)"'] & (Y \times_X Y) \times_X Z. \end{tikzcd}$$

The fiber $({}_W Y) \times_X Z$ of ${}_W\pi$ at the point $\mathbf{x}$ has a Z-rational point if and only if it admits a section, which is equivalent to the requirement that $({}_W Y) \times_X Z \cong G$. But twisting commutes with base change, thus $({}_W Y) \times_X Z \cong {}_W(Y \times_X Z) = {}_W W$ is a trivial torsor by [**Sko01**, p. 21].

If $\mathbf{x}' : Z \to X$ is another Z-point of X, then the fiber $({}_W Y) \times_X Z$ of ${}_W\pi$ at the point $\mathbf{x}'$ has a Z-point if and only if $W \cong Y \times_X Z$, where the fibered product on the right is induced by $\mathbf{x}'$. $\square$

1.2. Twisted torsors over Dedekind domains

We are going to describe the disjoint union of Proposition 1.4 in the special case where Z is the spectrum of a Dedekind domain A and Y is a locally closed subset of an affine space. In this case, the set of A-points on the twisted torsors can be expressed in terms of the polynomial equations defining Y and of ideals of A, without computing the twists explicitly.

In order to determine their sets of A-rational points, the twisted torsors will be given as open subschemes of closed subschemes of certain A-vector bundles that we call twisted affine spaces. Hence, we start with a definition of those.

DEFINITION 1.5. Let A be a Dedekind domain with fraction field k, and assume that we are given a $\mathbb{Z}^r$-grading on $k[y_1, \dots, y_N]$ defined by $\deg y_i = m^{(i)} \in \mathbb{Z}^r$. For any r-tuple $\underline{\mathfrak{c}} = (\mathfrak{c}_1, \dots, \mathfrak{c}_r)$ of nonzero fractional ideals of A, we define the *$\underline{\mathfrak{c}}$-twisted affine space over A* as the spectrum ${}_{\underline{\mathfrak{c}}}\mathbb{A}^N := \operatorname{Spec}({}_{\underline{\mathfrak{c}}}R)$ of the $\mathbb{Z}^r$-graded ring

$${}_{\underline{\mathfrak{c}}}R := \bigoplus_{m \in \mathbb{Z}^r} \underline{\mathfrak{c}}^{-m} R_m \subseteq k[y_1, \dots, y_N],$$

where $\underline{\mathfrak{c}}^{-m} := \mathfrak{c}_1^{-m_1} \cdots \mathfrak{c}_r^{-m_r}$ if $m = (m_1, \dots, m_r)$, and R_m is the degree-m-part of $A[y_1, \dots, y_N]$.

The twisted affine spaces defined above depend, of course, not only on N and $\underline{\mathfrak{c}}$, but also on the chosen $\mathbb{Z}^r$-grading. Here are some simple properties.

PROPOSITION 1.6. *The $\underline{\mathfrak{c}}$-twisted affine space over A defined above has the following properties.*

(i) There is a canonical isomorphism ${}_{\underline{\mathfrak{c}}}\mathbb{A}^N \times_{\operatorname{Spec} A} \operatorname{Spec} k \cong \mathbb{A}^N_k$.

(ii) Let $U = \operatorname{Spec}(A_U)$ *be an affine open subset of* $\operatorname{Spec}(A)$ *such that the fractional ideals* $\mathfrak{c}_1 A_U, \dots, \mathfrak{c}_r A_U$ *of* A_U *are principal. Then*

$${}_{\underline{\mathfrak{c}}}\mathbb{A}^N \times_{\operatorname{Spec}(A)} U \cong \mathbb{A}^N_A \times_{\operatorname{Spec}(A)} U.$$

(iii) *Via base extension and the canonical isomorphism from (i), we have*

$$_{\underline{\mathfrak{c}}}\mathbb{A}^N(A) = \{(y_1,\dots,y_N) \in k^N \mid y_i \in \underline{\mathfrak{c}}^{m^{(i)}} \text{ for all } 1 \le i \le N\}.$$

(iv) *The twisted affine space $_{\underline{\mathfrak{c}}}\mathbb{A}^N$ depends, up to isomorphism, only on the ideal classes of $\mathfrak{c}_1,\dots,\mathfrak{c}_r$.*

PROOF. The observation that $_{\underline{\mathfrak{c}}}R \otimes_A k \cong k[y_1,\dots,y_N]$ implies *(i)*. For $j \in \{1,\dots,r\}$, let c_j be a generator of $\mathfrak{c}_j A_U$ and, with $m \in \mathbb{Z}^r$, write $\underline{c}^m := c_1^{m_1}\cdots c_r^{m_r}$. Then $_{\underline{\mathfrak{c}}}R \otimes_A A_U \cong A_U[\underline{c}^{-m^{(1)}}y_1,\dots,\underline{c}^{-m^{(N)}}y_N] \cong A_U[y_1,\dots,y_N]$, which implies *(ii)*. For *(iii)*, we observe that every A-homomorphism $\varphi : {}_{\underline{\mathfrak{c}}}R \to A$ extends uniquely to a k-homomorphism $\varphi : k[y_1,\dots,y_N] \to k$. The k-homomorphisms coming from such A-homomorphisms are exactly those with $\varphi({}_{\underline{\mathfrak{c}}}R) \subseteq A$, that is, $\varphi(y_i) \in \underline{\mathfrak{c}}^{m^{(i)}}$ for all $i \in \{1,\dots,N\}$. To prove *(iv)*, let $\underline{b} = (b_1,\dots,b_r) \in (k^\times)^r$ and $\mathfrak{c}'_j := b_j\mathfrak{c}_j$ for $j \in \{1,\dots,r\}$. Then the k-automorphism of $k[y_1,\dots,y_N]$ mapping $y_i \mapsto \underline{b}^{-m^{(i)}}y_i$ induces an A-isomorphism between $_{\underline{\mathfrak{c}}}R$ and $_{\underline{\mathfrak{c}}'}R$. □

Next, we define twists of open subschemes of closed subschemes of $\mathbb{A}^N_A$ as certain subschemes of twisted affine spaces.

DEFINITION 1.7. With the hypotheses of Definition 1.5, let I_1, I_2 be $\mathbb{Z}^r$-homogeneous ideals of $A[y_1,\dots,y_N]$, and let Y be the subscheme of $\mathbb{A}^N_A$ defined by $Y := V(I_1) \smallsetminus V(I_2)$. With $I_{j,m}$ denoting the degree-m-part of I_j, we define the ideals

$$_{\underline{\mathfrak{c}}}I_j := \bigoplus_{m\in\mathbb{Z}^r} \underline{\mathfrak{c}}^{-m} I_{j,m} \subseteq {}_{\underline{\mathfrak{c}}}R.$$

The *twist of Y by $\underline{\mathfrak{c}}$* is the subscheme of $_{\underline{\mathfrak{c}}}\mathbb{A}^N$ defined by

$$_{\underline{\mathfrak{c}}}Y := V({}_{\underline{\mathfrak{c}}}I_1) \smallsetminus V({}_{\underline{\mathfrak{c}}}I_2).$$

PROPOSITION 1.8. *The twist of Y by $\underline{\mathfrak{c}}$ defined above has the following properties.*

(i) *The canonical isomorphism from Proposition 1.6, (i), induces an isomorphism $_{\underline{\mathfrak{c}}}Y \times_{\operatorname{Spec} A} \operatorname{Spec} k \cong Y_K$.*

(ii) *Let $U = \operatorname{Spec}(A_U)$ be an affine open subset of $\operatorname{Spec}(A)$ such that the fractional ideals $\mathfrak{c}_1 A_U,\dots,\mathfrak{c}_r A_U$ of A_U are principal. Then*

$$_{\underline{\mathfrak{c}}}Y \times_{\operatorname{Spec}(A)} U \cong Y \times_{\operatorname{Spec}(A)} U.$$

(iii) *Via base extension and the canonical isomorphism from (i), the set of A-points $_{\underline{\mathfrak{c}}}Y(A)$ is the subset of all $\underline{y} = (y_1,\dots,y_N) \in k^N$ with $a_i \in \underline{\mathfrak{c}}^{m^{(i)}}$ for all $i \in \{1,\dots,N\}$, such that*

$$\sum_{m\in\mathbb{Z}^r}\sum_{f\in I_{2,m}} f(\underline{y})\underline{\mathfrak{c}}^{-m} = A \tag{1.1}$$

and

$$g(\underline{y}) = 0 \text{ for all } g \in I_1. \tag{1.2}$$

(iv) *$_{\underline{\mathfrak{c}}}Y$ depends, up to isomorphism, only on the ideal classes of $\mathfrak{c}_1,\dots,\mathfrak{c}_r$.*

PROOF. Since the inclusion $A \to k$ is flat and ${}_{\underline{\mathfrak{c}}}I_1 \otimes_A k \cong I_1 \otimes_A k$ under the canonical isomorphism ${}_{\underline{\mathfrak{c}}}R \otimes_A k \cong k[y_1, \dots, y_N]$, we see that $V({}_{\underline{\mathfrak{c}}}I_1) \times_{\operatorname{Spec} A} \operatorname{Spec} k \cong V(I_1) \times_{\operatorname{Spec} A} \operatorname{Spec} k$. Let I_2 be generated by homogeneous polynomials $f_1, \dots, f_t \in A[y_1, \dots, y_N]$, and for every $i \in \{1, \dots, t\}$, let $b_{i,1}, \dots, b_{i,t_i}$ be generators of the fractional ideal $\underline{\mathfrak{c}}^{-\deg f_i}$. Then ${}_{\underline{\mathfrak{c}}}I_2$ is generated in ${}_{\underline{\mathfrak{c}}}R$ by the elements $b_{i,j}f_i$, and ${}_{\underline{\mathfrak{c}}}Y$ is covered by affine open subsets $\operatorname{Spec}(({}_{\underline{\mathfrak{c}}}R/{}_{\underline{\mathfrak{c}}}I_1)_{b_{i,j}f_i})$. Moreover,

$$({}_{\underline{\mathfrak{c}}}R/{}_{\underline{\mathfrak{c}}}I_1)_{b_{i,j}f_i} \otimes_A k \cong (A[y_1, \dots, y_N]/I_1)_{f_i} \otimes_A k$$

for every $i \in \{1, \dots, t\}$ and $j \in \{1, \dots, t_i\}$, which shows *(i)*.

For $j \in \{1, \dots, r\}$, let c_j be a generator of $\mathfrak{c}_j A_U$ and, with $m \in \mathbb{Z}^r$, write $\underline{c}^m := c_1^{m_1} \cdots c_r^{m_r}$. Let $\varphi_{\underline{c}} : A_U[y_1, \dots, y_N] \to A_U[\underline{c}^{-m^{(1)}} y_1, \dots, \underline{c}^{-m^{(N)}} y_N]$ be the isomorphism that sends $y_i \mapsto \underline{c}^{-m^{(i)}} y_i$. For every homogeneous $f \in I_2$ we obtain

$$\begin{aligned}(A[y_1, \dots, y_N]/I_1)_f \otimes_A A_U &\cong (A_U[y_1, \dots, y_N]/(I_1 \otimes_A A_U))_f \\ &\cong \big(\varphi_{\underline{c}}(A_U[y_1, \dots, y_N])/\varphi_{\underline{c}}(I_1 \otimes_A A_U)\big)_{\varphi_{\underline{c}}(f)} \\ &\cong \big(({}_{\underline{\mathfrak{c}}}R \otimes_A A_U)/({}_{\underline{\mathfrak{c}}}I_1 \otimes_A A_U)\big)_{\underline{c}^{-\deg f} f}.\end{aligned}$$

This proves *(ii)*, since $f \in I_2 \otimes_A A_U$ is equivalent to $\underline{c}^{-\deg f} f \in {}_{\underline{\mathfrak{c}}}I_2 \otimes_A A_U$. For *(iii)*, we first consider $V({}_{\underline{\mathfrak{c}}}I_1)(A)$. Via the identification in Proposition 1.6, *(iii)*, these points correspond to k-homomorphisms $\varphi : k[y_1, \dots, y_N] \to k$ with $\varphi(y_i) \in \underline{\mathfrak{c}}^{m^{(i)}}$ whose kernel contains the homogeneous ideal ${}_{\underline{\mathfrak{c}}}I_1 \otimes_A k = I_1 \otimes_A k$, that is, to points $\underline{y} \in k^N$ with $y_i \in \underline{\mathfrak{c}}^{m^{(i)}}$ and satisfying (1.2).

Next, let us consider $({}_{\underline{\mathfrak{c}}}\mathbb{A}^N \smallsetminus V({}_{\underline{\mathfrak{c}}}I_2))(A)$. These points correspond to A-homomorphisms $\varphi : {}_{\underline{\mathfrak{c}}}R \to A$ such that ${}_{\underline{\mathfrak{c}}}I_2 \not\subseteq \varphi^{-1}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of A. That is, $\varphi({}_{\underline{\mathfrak{c}}}I_2)A = A$. Keeping in mind that ${}_{\underline{\mathfrak{c}}}I_2$ is generated by its homogeneous elements and using the description of ${}_{\underline{\mathfrak{c}}}\mathbb{A}^N(A)$ from Proposition 1.6, *(iii)*, we see that $({}_{\underline{\mathfrak{c}}}\mathbb{A}^N \smallsetminus V({}_{\underline{\mathfrak{c}}}I_2))(A)$ corresponds to the set of all $\underline{y} \in k^N$ with $y_i \in \underline{\mathfrak{c}}^{m^{(i)}}$ and satisfying (1.1).

To prove *(iv)*, let $\underline{b} = (b_1, \dots, b_r) \in (k^\times)^r$ and $\mathfrak{c}'_j := b_j \mathfrak{c}_j$ for $j \in \{1, \dots, r\}$. Then the k-automorphism of $k[y_1, \dots, y_N]$ mapping $y_i \mapsto \underline{b}^{-m^{(i)}} y_i$ induces an A-isomorphism between ${}_{\underline{\mathfrak{c}}}R$ and ${}_{\underline{\mathfrak{c}}'}R$ which maps ${}_{\underline{\mathfrak{c}}}I_j$ isomorphically onto ${}_{\underline{\mathfrak{c}}'}I_j$, for $j \in \{1, 2\}$. $\square$

Now we can focus on the case where Y is a torsor over an A-scheme X. Throughout the rest of this section, we assume the following setup.

Let A be a Dedekind domain with fraction field k, and let there be a $\mathbb{Z}^r$-grading on $k[y_1, \dots, y_N]$ defined by $\deg y_i = m^{(i)} \in \mathbb{Z}^r$.

Let X be a separated scheme of finite type over A that admits an X-torsor $\pi : Y \to X$ under a split torus $\mathbb{G}_{m,X}^r$. We assume that there are $\mathbb{Z}^r$-homogeneous polynomials $f_1, \dots, f_t, g_1, \dots, g_s \in A[y_1, \dots, y_N]$ such that $Y = V(g_1, \dots, g_s) \smallsetminus V(f_1, \dots, f_t)$ as subscheme of $\mathbb{A}_A^N$. Moreover, we assume that the action of $\mathbb{G}_{m,X}^r$ on Y is induced by the following action on points:

$$(s_1, \dots, s_r) * (y_1, \dots, y_N) = (\underline{s}^{m^{(1)}} y_1, \dots, \underline{s}^{m^{(N)}} y_N)$$

for all $\underline{s} = (s_1, \dots, s_r) \in \mathbb{G}_{m,A}^r(A)$ and $(y_1, \dots, y_N) \in Y(A)$, where we write $\underline{s}^m := s_1^{m_1} \cdots s_r^{m_r}$ for $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$. Under these assumptions, we now define the twists of $\pi : Y \to X$.

Definition 1.9. Under the above hypotheses, let $\underline{\mathfrak{c}} = (\mathfrak{c}_1, \dots, \mathfrak{c}_r)$ be an r-tuple of nonzero fractional ideals of A, and let ${}_{\underline{\mathfrak{c}}}Y$ be the twist of Y from Definition 1.7. Then the *$\underline{\mathfrak{c}}$-twist of* $\pi : Y \to X$ is the morphism ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}Y \to X$ obtained by glueing the following morphisms:

$$ {}_{\underline{\mathfrak{c}}}\pi_U : {}_{\underline{\mathfrak{c}}}Y \times_{\mathrm{Spec}(A)} U \to X \times_{\mathrm{Spec}(A)} U, $$

where U runs through an open covering of $\mathrm{Spec}(A)$ by affine subschemes $U = \mathrm{Spec}(A_U)$ such that $\mathfrak{c}_1 A_U, \dots, \mathfrak{c}_r A_U$ are principal ideals of A_U, and ${}_{\underline{\mathfrak{c}}}\pi_U$ is defined as composition of π after the isomorphism $\phi_{\underline{c}} : {}_{\underline{\mathfrak{c}}}Y \times_{\mathrm{Spec}(A)} U \to Y \times_{\mathrm{Spec}(A)} U$ from Proposition 1.8, *(ii)*, induced by the isomorphism

$$ \varphi_{\underline{c}} : A_U[y_1, \dots, y_N] \to A_U[\underline{c}^{-m^{(1)}} y_1, \dots, \underline{c}^{-m^{(N)}} y_N], \quad y_i \mapsto \underline{c}^{-m^{(i)}} y_i, $$

where c_j is a generator of $\mathfrak{c}_j A_U$ for $j \in \{1, \dots, r\}$, and $\underline{c}^m := c_1^{m_1} \cdots c_r^{m_r}$ for all $m \in \mathbb{Z}^r$. The definition of ${}_{\underline{\mathfrak{c}}}\pi$ does not depend on the choice of the open subsets U nor on the choice of the generators c.

Now we are ready to state the main result of this section.

Theorem 1.10. *The $\underline{\mathfrak{c}}$ twists ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}Y \to X$ defined above have the following properties.*

(i) The morphism ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}Y \to X$ is a torsor over X under $\mathbb{G}_{m,X}^r$ of class $[Y] - [\underline{\mathfrak{c}}] \in H^1_{\acute{e}t}(X, \mathbb{G}_{m,X}^r)$.

(ii) Let $\mathcal{C}$ be a system of representatives for the class group $\mathrm{Pic}(A)$ *of A. If X is proper over A then, under base extension, the set of rational points on X_K decomposes as a disjoint union*

$$ X_K(k) = \bigsqcup_{\underline{\mathfrak{c}} \in \mathcal{C}^r} {}_{\underline{\mathfrak{c}}}\pi({}_{\underline{\mathfrak{c}}}Y(A)). $$

(iii) As a subset of k^N, the set ${}_{\underline{\mathfrak{c}}}Y(A)$ is equal to the set of all $\underline{y} \in k^N$ whose coordinates y_i lie in the fractional ideals $\underline{\mathfrak{c}}^{m^{(i)}}$, satisfying the coprimality conditions expressed by

$$ \sum_{i=1}^{t} f_i(\underline{y}) \underline{\mathfrak{c}}^{-\deg f_i} = A $$

and the torsor equations

$$ g_j(\underline{y}) = 0 \textit{ for all } j \in \{1, \dots, s\}. $$

Proof. For every choice of affine open subsets U, U' of $\mathrm{Spec}(A)$ as in Definition 1.9, and corresponding r-tuples $\underline{c}$, $\underline{c}'$ of generators for the principal fractional ideals over U, resp. U', let $\varphi_{\underline{c},\underline{c}'} : A_{U\cap U'}[y_1, \dots, y_N] \to A_{U\cap U'}[y_1, \dots, y_N]$ be the isomorphism induced by the automorphism of $k[y_1, \dots, y_N]$ mapping $y_i \mapsto \underline{c}^{-m^{(i)}} \underline{c}'^{m^{(i)}} y_i$, and $\phi_{\underline{c},\underline{c}'}$ the automorphism of $Y \times_{\mathrm{Spec}(A)} (U \cap U')$ induced by $\varphi_{\underline{c},\underline{c}'}$. Then $\phi_{\underline{c}} = \phi_{\underline{c},\underline{c}'} \circ \phi_{\underline{c}'}$ on ${}_{\underline{\mathfrak{c}}}Y \times_{\mathrm{Spec}(A)} (U \cap U')$. We observe that $\phi_{\underline{c},\underline{c}'}$ are the automorphisms induced by the $\mathbb{G}_{m,X}^r$-action of the cocycle $(c_1^{-1}c_1', \dots, c_r^{-1}c_r')_{U',U}$ that represents the class

$-[\underline{\mathfrak{c}}] \in H^1_{ét}(\mathrm{Spec}(A), \mathbb{G}^r_{m,A})$. Thus $\pi \circ \phi_{\underline{c},\underline{c}'} = \pi$ on $Y \times_{\mathrm{Spec}(A)} (U \cap U')$, as $\pi : Y \to X$ is a torsor under $\mathbb{G}^r_{m,X}$, and the morphism ${}_{\underline{\mathfrak{c}}}\pi$ is well defined. Since the automorphisms $\phi_{\underline{c},\underline{c}'}$ are $\mathbb{G}^r_{m,X}$-equivariant, the X-scheme ${}_{\underline{\mathfrak{c}}}Y$ is endowed with an action of $\mathbb{G}^r_{m,X}$, and the morphism ${}_{\underline{\mathfrak{c}}}\pi$ is an X-torsor under $\mathbb{G}^r_{m,X}$ of class $[Y] - [\underline{\mathfrak{c}}] \in H^1_{ét}(X, \mathbb{G}^r_{m,X})$, by construction.

We recall that two torsors with the same class in $H^1_{ét}(X, \mathbb{G}^r_{m,X})$ are X-isomorphic, so the images of their structure morphisms coincide as subsets of X. Moreover,

$$H^1_{fppf}(\mathrm{Spec}(A), \mathbb{G}^r_{m,A}) = H^1_{ét}(\mathrm{Spec}(A), \mathbb{G}^r_{m,A}) \cong \mathrm{Pic}(A)^r,$$

as étale cohomology commutes with direct sums (see [**Mil80**, Remark III.3.6 (d)]).

By the valuative criterion of properness, $X_k(k) = X(A)$ under base extension. Thus, property *(ii)* follows from Proposition 1.4.

Finally, *(iii)* was already proved in Proposition 1.8, *(iii)*. $\square$

A similar parameterization was found by Robbiani [**Rob98**] for smooth projective split toric varieties over number fields, and by Bourqui [**Bou09**] for varieties over global fields of positive characteristic.

CHAPTER 2

Generalized Cox rings

In this chapter we investigate the algebraic side of torsors under quasitori for varieties over arbitrary fields by introducing a generalization of the notions of Cox sheaf and Cox ring.

A Cox ring of a variety X over an algebraically closed field k is a $\mathrm{Pic}(X)$- or $\mathrm{Cl}(X)$-graded (depending on the definition) k-algebra which is isomorphic to the direct sum of the sets of global sections of the sheaves $\mathscr{O}_X(D)$ associated with the divisor classes in the grading group.

Homogeneous coordinate rings, or Cox rings, were introduced by Cox for toric varieties over $\mathbb{C}$ [**Cox95**] to study the analogies between toric varieties and the projective space; then by Hu and Keel for projective varieties with free finitely generated Picard group [**HK00**] to characterize Mori dream spaces; by Elizondo, Kurano and Watanabe for connected normal noetherian schemes with free finitely generated class group [**EKW04**] to study certain linear systems of projective varieties obtained by blowing-up projective spaces in finitely many points; by Berchtold and Hausen for varieties over algebraically closed fields both as Pic(X)-graded [**BH03**] and Cl(X)-graded rings [**Hau08**]. Recently, Cox rings for varieties over nonclosed fields were considered in joint work with Derenthal [**DP14**] (see also [**ADHL15**, §6.1]). The content of this chapter is a generalization of [**DP14**].

2.1. Torsors under quasitori

We start by recalling the classification of torsors under quasitori from [**CTS87**], and we show that the structure sheaves of such torsors are equipped with a natural grading by certain finitely generated Galois modules.

Let k be a field, fix a separable closure $\overline{k}$ of k, and let $\mathfrak{g}$ be the Galois group $\mathrm{Gal}(\overline{k}/k)$. For us, a quasitorus over k is a smooth k-group of multiplicative type of finite type, namely, a reduced affine algebraic group G over k such that $G_{\overline{k}} := G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$ is diagonalizable. See [**Gro64a, Gro64b**] for an exhaustive presentation of groups of multiplicative type and their properties. We recall that there is an antiequivalence of categories between quasitori over k and finitely generated $\mathfrak{g}$-modules with order of the torsion subgroup not divisible by the characteristic of k given by

$$G \mapsto \widehat{G_{\overline{k}}} := \mathrm{Hom}_{\overline{k}}(G_{\overline{k}}, \mathbb{G}_{m,\overline{k}}), \quad M \mapsto \widehat{M}^{\mathfrak{g}} := \mathrm{Spec}\, \overline{k}[M]^{\mathfrak{g}}, \qquad (2.1)$$

where $\overline{k}[M]^{\mathfrak{g}} := H^0(\mathfrak{g}, \overline{k}[M])$ is the subring of $\mathfrak{g}$-invariant elements of $\overline{k}[M]$. See [**Gro64a**, Proposition 2.1] and [**Gro64b**, Proposition 1.4].

Now we review some basic properties of torsors under quasitori. We refer to [**CTS87**] and [**Sko01**] for a detailed investigation. Let X be a k-variety and G a quasitorus over k. We recall from page 14 that $X_{\overline{k}} :=$

$X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, $G_X := G \times_{\operatorname{Spec} k} X$ and $G_{X_{\overline{k}}} := G_X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $\overline{k}[X_{\overline{k}}]$ denotes the ring of global sectios of the structure sheaf of $X_{\overline{k}}$.

According to [**Sko01**, Lemma 2.3.1] (cf. [**CTS87**, p. 408]), an X-torsor Y under G defines a homomorphism of $\mathfrak{g}$-modules

$$\mathtt{type}(Y) : \widehat{G_{\overline{k}}} \to \operatorname{Pic}(X_{\overline{k}}),$$

as follows. Let $(U_i \to X)$ be an *étale*-covering of X that trivializes Y, and $(\beta_{i,j})_{i,j}$ a cocycle representing $[Y]$ in $H^1(X, G_X)$. Let $\alpha \in \widehat{G_{\overline{k}}}$ considered as morphism $G_{X_{\overline{k}}} \to \mathbb{G}_{m,X_{\overline{k}}}$ after base change. Under the identification $\operatorname{Pic}(X_{\overline{k}}) = H^1(X_{\overline{k}}, \mathbb{G}_{m,X_{\overline{k}}})$, the image of α under $\mathtt{type}(Y)$ is the class of the cocycle $(\alpha(\beta_{i,j}))_{i,j}$. The homomorphism $\mathtt{type}(Y)$ is called the *type* of Y. If $\mathtt{type}(Y) = \mathrm{id}_{\operatorname{Pic}(X_{\overline{k}})}$, then Y is called a universal torsor of X. See [**CTS87**, §2.1]

Let $p : X \to \operatorname{Spec} k$ be the structure morphism of X, and assume that $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$. By [**CTS87**, Theorem 1.5.1] (cf. [**Sko01**, Corollary 2.3.9]), there is an exact sequence

$$0 \to H^1(k, G) \xrightarrow{p^*} H^1(X, G_X) \xrightarrow{\mathtt{type}} \operatorname{Hom}_{\mathfrak{g}}(\widehat{G_{\overline{k}}}, \operatorname{Pic}(X_{\overline{k}})) \to H^2(k, G), \quad (2.2)$$

where the cohomology groups are computed with respect to the *étale* topology, as G is smooth. In particular, if k is separably closed, the map $\mathtt{type}$ is an isomorphism (cf. [**Gro64b**, Proposition 1.4]).

REMARK 2.1. If $X(k) \neq \emptyset$, then the map $\mathtt{type}$ is surjective by [**CTS77**, Proposition 1]. In particular, X-torsors of every type exist.

The following proposition characterizes the structure sheaf of a torsor under a quasitorus, and offers an explanation for the study of the objects investigated in this chapter.

PROPOSITION 2.2. *Let X be a k-variety, G a quasitorus over k and $\pi : Y \to X$ a torsor under G. Then $\pi_* \mathcal{O}_{Y_{\overline{k}}}$ is a sheaf of $\widehat{G_{\overline{k}}}$-graded $\mathcal{O}_{X_{\overline{k}}}$-algebras.*

PROOF. Let $\lambda \in \operatorname{Hom}_{\mathfrak{g}}(\widehat{G_{\overline{k}}}, \operatorname{Pic}(X_{\overline{k}}))$ be the type of Y. For every $m \in \widehat{G_{\overline{k}}}$, let $\mathscr{L}(m)$ be an invertible sheaf on $X_{\overline{k}}$ of class $\lambda(m)$ in $\operatorname{Pic}(X_{\overline{k}})$.

Let $m_1, \dots, m_n$ be generators of $\widehat{G_{\overline{k}}}$, and let $\{U_i\}_i$ be an open covering of $X_{\overline{k}}$ that trivializes simultaneously $\mathscr{L}(m_1), \dots, \mathscr{L}(m_n)$. For every $m \in \widehat{G_{\overline{k}}}$, let $(\lambda_{i,j}(m))_{i,j}$ be a cocycle representing $\lambda(m) \in \operatorname{Pic}(X_{\overline{k}}) = H^1(X_{\overline{k}}, \mathbb{G}_{m,X})$. Then $[Y_{\overline{k}}] \in H^1(X_{\overline{k}}, G_{\overline{k}})$ is the class of the cocycle $(\beta_{i,j})_{i,j}$ with

$$\beta_{i,j} \in G_{\overline{k}}(U_i \cap U_j) \cong \operatorname{Hom}(\widehat{G_{\overline{k}}}, \mathcal{O}_{X_{\overline{k}}}(U_i \cap U_j))$$

defined by $\beta_{i,j}(m) := \lambda_{i,j}(m)$. (Recall that $G_{\overline{k}} \cong \operatorname{Spec} \overline{k}[\widehat{G_{\overline{k}}}]$ under the bijection (2.1)). Thus, $\{U_i\}_i$ trivializes π. Therefore,

$$\pi_* \mathcal{O}_{Y_{\overline{k}}}|_{U_i} \cong \mathcal{O}_{U_i}[\widehat{G_{\overline{k}}}]$$

for all i and these isomorphisms glue over the cocycle $(\beta_{i,j})_{i,j}$ to give an isomorphism of $\mathcal{O}_{X_{\overline{k}}}$-modules

$$\pi_* \mathcal{O}_{Y_{\overline{k}}} \cong \bigoplus_{m \in \widehat{G_{\overline{k}}}} \mathscr{L}(m).$$

This induces a $\widehat{G_{\overline{k}}}$-grading on $\pi_*\mathcal{O}_{Y_{\overline{k}}}$. □

2.2. Over separably closed fields

Motivated by Proposition 2.2, we introduce generalized Cox sheaves and Cox rings of varieties over separably closed fields. For the sake of brevity, we say simply Cox sheaves or Cox rings of a given type. Then the Cox sheaves and rings in the strict sense are called Cox sheaves and rings of identity type.

We fix some notation for the rest of this section. Let k be a separably closed field, and X an integral k-variety such that $k[X]^\times = k^\times$. Let $\mathrm{CaDiv}(X)$ be the group of Cartier divisors of X. For every divisor $D \in \mathrm{CaDiv}(X)$ we denote by $[D]$ its class in $\mathrm{Pic}(X)$. Given a Cartier divisor $D = \{(U_i, f_i)\}_i$, its associated sheaf $\mathcal{O}_X(D)$ is the invertible sheaf obtained by gluing $f_i^{-1}\mathcal{O}_{U_i}$. This is a subsheaf of the constant sheaf associated with the function field $k(X)$ of X. Therefore, given two Cartier divisors D_1, D_2 and sections $s_i \in H^0(U, \mathcal{O}_X(D_i))$, $i \in \{1,2\}$, the product $s_1s_2 \in H^0(U, \mathcal{O}_X(D_1 + D_2))$ is well-defined. Given a Cartier divisor $D = \{(U_i, f_i)\}_i$ and an element $f \in k(X)^\times$, we denote by $\mathrm{div}_D(f)$ the Cartier divisor $\{(U_i, ff_i)\}_i$. Note that $f \in H^0(X, \mathcal{O}(D))$ if and only if $\mathrm{div}_D(f)$ is an effective divisor. The support of a Cartier divisor $D = \{(U_i, f_i)\}_i$ is

$$\mathrm{Supp}(D) := \{x \in X : f_i \notin \mathcal{O}_{X,x}^\times \text{ if } x \in U_i\}.$$

Let M be a finitely generated abelian group, and assume that the characteristic of k does not divide the order of its torsion subgroup. Let $\lambda : M \to \mathrm{Pic}(X)$ be a group homomorphism, and

$$M_\lambda := \{(m, D) \in M \times \mathrm{CaDiv}(X) : [D] = \lambda(m)\}.$$

2.2.1. Generalized Cox sheaves. We start with an axiomatic definition of generalized Cox sheaves.

Definition 2.3. A *Cox sheaf of X of type λ* is a sheaf $\mathcal{R}$ of M-graded $\mathcal{O}_X$-algebras together with a family of isomorphisms of $\mathcal{O}_X$-modules

$$\{\phi_{m,D} : \mathcal{R}_m \to \mathcal{O}_X(D)\}_{(m,D)\in M_\lambda},$$

where $\mathcal{R}_m$ denotes the degree-m-part of $\mathcal{R}$, such that for every (m_1, D_1), $(m_2, D_2) \in M_\lambda$ there exists a nonzero constant $\alpha \in k$ that satisfies

$$\phi_{m_1,D_1}(s_1)\phi_{m_2,D_2}(s_2) = \alpha\phi_{m_1+m_2,D_1+D_2}(s_1s_2)$$

for all $s_i \in \mathcal{R}_{m_i}(U)$, $i \in \{1,2\}$, and all open subsets U of X.

A *morphism of Cox sheaves of X of type λ* is a morphism of M-graded $\mathcal{O}_X$-algebras.

A *structure of Cox sheaf of X of type λ* on a sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras consists of an M-grading on $\mathcal{R}$ and a family of isomorphisms $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ as above.

Via the following lemma, which characterizes the morphisms of $\mathcal{O}_X$-modules between invertible sheaves, and the remark that comes after we show that a structure of generalized Cox sheaf is actually determined by a subfamily of isomorphisms $\phi_{m,D}$. This lemma is also used later to determine the automorphism group of a generalized Cox sheaf.

LEMMA 2.4. *Let D, D' be two Cartier divisors and $\psi : \mathcal{O}_X(D) \to \mathcal{O}_X(D')$ a morphism of $\mathcal{O}_X$-modules. Then there exists $f \in k(X)$ such that $D = \mathrm{div}_{D'}(f)$ and $\psi(s) = fs$ for all $s \in H^0(U, \mathcal{O}_X(D))$ and all open subsets $U \subseteq X$. If $D = D'$, then $f \in k[X]$.*

PROOF. Without loss of generality, we can assume that D and D' are trivialized by the same open covering of X, say $D = \{(U_i, f_i)\}_i$ and $D' = \{(U_i, f_i')\}_i$. Let $s \in H^0(U_i, \mathcal{O}_X(D))$. Then $\psi(s) = \psi(f_i^{-1})f_i s$, and $\psi(f_i^{-1}) = u_i f_i'^{-1}$ for some $u_i \in H^0(U_i, \mathcal{O}_X)$. Moreover, $u_i f_i'^{-1} f_i = u_j f_j'^{-1} f_j$ in $k(X)$ for all i and j by restricting ψ to $U_i \cap U_j$. Take $f = u_i f_i'^{-1} f_i$. If $D = D'$, then $f \in H^0(U_i, \mathcal{O}_X)$ for all i. Hence, $f \in k[X]$. □

REMARK 2.5. The isomorphisms $\phi_{m,D}$ in Definition 2.3 are uniquely determined up to multiplication by nonzero elements of k. Therefore, given a Cox sheaf $\mathcal{R}$ of X of type λ, we say that two families $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ and $\{\phi'_{m,D}\}_{(m,D)\in M_\lambda}$ define the same Cox sheaf structure on $\mathcal{R}$ if for every $(m, D) \in M_\lambda$, there is $\alpha_{m,D} \in k^\times$ such that $\phi'_{m,D} = \alpha_{m,D}\phi_{m,D}$.

By Lemma 2.4, a Cox sheaf structure on $\mathcal{R}$ is determined once the isomorphism $\phi_{m,D}$ is known for a given $(m, D) \in M_\lambda$ for each $m \in M$.

REMARK 2.6. The sheaves defined in [**BH03**, Lemma 3.5], with the family of isomorphisms in [**BH03**, Lemma 3.5 (ii)], are generalized Cox sheaves of identity type in the sense of Definition 2.3. If X is locally factorial, the Cox sheaves constructed in [**ADHL15**, Construction 1.4.2.1], with the family of isomorphisms in [**ADHL15**, Lemma 1.4.3.4], are generalized Cox sheaves of identity type in the sense of Definition 2.3.

In [**BH03**], [**ADHL15**] and [**DP14**] Cox sheaves are constructed via a presentation of the grading group, $\mathrm{Pic}(X)$ or $\mathrm{Cl}(X)$, by a finitely generated free group of line bundles, Weil divisors and Cartier divisors, respectively. We show that generalized Cox sheaves of every type exist by means of a generalization of these constructions that makes use of a presentation of the grading group M by an abstract finitely generated free group. Later in this section we present another procedure, see Construction 2.18, which could seem a more natural generalization of the constructions mentioned above.

CONSTRUCTION 2.7. *Let*

$$0 \to \Lambda_0 \to \Lambda \xrightarrow{\varphi} M \to 0$$

be a presentation of M by a finitely generated free group Λ. Let $L_1, \ldots, L_n$ be a basis of Λ. For $1 \leq i \leq n$, let D_i be a Cartier divisor representing the class $\lambda(\varphi(L_i))$ in $\mathrm{Pic}(X)$. For every $L = \sum_{i=1}^n a_i L_i$ of Λ, let $\mathcal{S}_L := \mathcal{O}_X(\sum_{i=1}^n a_i D_i)$. Endow $\mathcal{S} := \bigoplus_{L\in\Lambda} \mathcal{S}_L$ with the multiplication of sections induced by the embeddings $\mathcal{S}_L \subseteq k(X)$. Let $\chi : \Lambda_0 \to k(X)^\times$ be a homomorphism of groups that satisfies $\mathrm{div}_0(\chi(E)) = \sum_{i=1}^n a_i D_i$ for all $E = \sum_{i=1}^n a_i L_i \in \Lambda_0$. We call χ a character associated with $\mathcal{S}$. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$ for all $E \in \Lambda_0$, where 1 has degree 0 and $\chi(E)$ has degree $-E$. We say that $\mathcal{I}$ is the sheaf of ideals of $\mathcal{S}$ defined by χ. Let $\mathcal{R} := \mathcal{S}/\mathcal{I}$ and $\pi : \mathcal{S} \to \mathcal{R}$ the projection. Then *(i)* $\mathcal{I}$ *is* M*-homogeneous, and every* Λ*-homogeneous section of* $\mathcal{I}$ *is zero;*

(ii) $\pi|_{\mathcal{S}_L} : \mathcal{S}_L \to \mathcal{R}_{\varphi(L)}$ *is an isomorphism of* $\mathcal{O}_X$*-modules for all* $L \in \Lambda$*;*

(iii) $\mathcal{R}$ *with any family of isomorphisms* $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ *induced by* $\{\pi|_{\mathcal{S}_L}^{-1}\}_{L\in\Lambda}$ *is a Cox sheaf of* X *of type* λ*.*

Proof. Since $\varphi(E) = 0$ for all $E \in \Lambda_0$, the sheaf $\mathcal{I}$ is M-homogeneous and $\mathcal{R}$ is M-graded. To prove *(i)*, let U be an open subset of X and $s \in H^0(U, \mathcal{I})$ a homogeneous section of degree $L \in \Lambda$. Let $\mathcal{V}$ be an open covering of U such that for all $V \in \mathcal{V}$ we can write $s|_V = \sum_{E\in\Lambda_0}(1 - \chi(E))s_E$ with $s_E \in H^0(V, \bigoplus_{E'\in\Lambda_0} \mathcal{S}_{L+E'})$ for all $E \in \Lambda_0$. Let $s_{E,E'} \in H^0(V, \mathcal{S}_{L+E'})$, such that $s_E = \sum_{E'\in\Lambda_0} s_{E,E'}$. Then

$$s|_V = \sum_{E,E'\in\Lambda_0} ((1 - \chi(E - E'))\chi(E')s_{E,E'} - (1 - \chi(-E'))\chi(E')s_{E,E'}),$$

where $\chi(E')s_{E,E'}$ is homogeneous of degree L. Hence, $s|_V = \sum_{E\in\Lambda_0}(1 - \chi(E))s'_E$ for suitable $s'_E \in H^0(V, \mathcal{S}_L)$. Since s is homogeneous of degree L, we get $s|_V = 0$ for all $V \in \mathcal{V}$. Hence, $s = 0$.

For *(ii)*, note that $\pi|_{\mathcal{S}_L}$ is injective for all $L \in \Lambda$ by *(i)*. Let U be an open subset of X and $L \in \Lambda$. We prove that $\pi|_{H^0(U,\mathcal{S}_L)} : H^0(U, \mathcal{S}_L) \to H^0(U, \mathcal{R}_{\varphi(L)})$ is surjective. Let $s \in H^0(U, \mathcal{R}_{\varphi(L)})$ and $\mathcal{V}$ an open covering of U such that for all $V \in \mathcal{V}$ there exists a section $s_V \in H^0(V, \bigoplus_{E\in\Lambda_0} \mathcal{S}_{L+E})$ with $\pi(s_V) = s|_V$. Write $s_V = \sum_{E\in\Lambda_0} s_{V,E}$ with $s_{V,E}$ homogeneous of degree $L + E$. Then $s'_V := s_V - \sum_{E\in\Lambda_0}(1 - \chi(E))s_{V,E}$ is homogeneous of degree L and $\pi(s'_V) = s|_V$. Since $s'_V|_{V\cap W} - s'_W|_{V\cap W}$ is a Λ-homogeneous section of $\mathcal{I}$, the sections s'_V glue by *(i)* to a section $s' \in H^0(U, \mathcal{S}_L)$ such that $\pi(s') = s$.

Finally, we prove *(iii)*. For every $(m, D) \in M_\lambda$, choose $L \in \varphi^{-1}(m)$ and an isomorphism of $\mathcal{O}_X$-modules $\psi_{L,D} : \mathcal{S}_L \to \mathcal{O}_X(D)$. Then

$$\left\{\phi_{m,D} := \psi_{L,D} \circ \pi|_{\mathcal{S}_L}^{-1} : \mathcal{R}_m \to \mathcal{O}_X(D)\right\}_{(m,D)\in M_\lambda}$$

is an arbitrary family of isomorphisms induced by $\{\pi|_{\mathcal{S}_L}^{-1}\}_{L\in\Lambda}$. By Lemma 2.4, for every $(m_1, D_1), (m_2, D_2) \in M_\lambda$ there exists a nonzero constant $\alpha \in k$ that satisfies $\phi_{m_1,D_1}(s_1)\phi_{m_2,D_2}(s_2) = \alpha\phi_{m_1+m_2,D_1+D_2}(s_1s_2)$ for all $s_i \in H^0(U, \mathcal{R}_{m_i})$, $i \in \{1, 2\}$, and all open subsets U of X. □

Remark 2.8. Since Λ is free and finitely generated, the same is true for Λ_0. Hence, characters χ as in Construction 2.7 always exist.

The next proposition shows that the construction above does not depend on the choice of the character χ.

Proposition 2.9. *The Cox sheaves defined in Construction 2.7 do not depend on the choice of the character* χ*, up to isomorphism of Cox sheaves.*

Proof. Let Λ, Λ_0 and $\mathcal{S}$ as in Construction 2.7. Let χ and χ' be two characters associated with $\mathcal{S}$, and $\mathcal{I}$ and $\mathcal{I}'$ the sheaves of ideals of $\mathcal{S}$ defined by χ and χ', respectively. Since Λ is free and finitely generated, k is separably closed and the oder of the torsion subgroup of M is not divisible by the characteristic of k, the character $\chi'\chi^{-1} : \Lambda_0 \to k^\times$ extends to a character $\alpha : \Lambda \to k^\times = k[X]^\times$, that defines an automorphism $\psi : \mathcal{S} \to \mathcal{S}$ by sending each homogeneous element s of degree L to $\alpha(L)s$. Since $\psi(1-\chi(E)) = 1-\alpha(E)\chi(E) = 1-\chi'(E)$ for all $E \in \Lambda_0$, the automorphism ψ maps $\mathcal{I}$ onto $\mathcal{I}'$, inducing an isomorphism $\mathcal{S}/\mathcal{I} \to \mathcal{S}/\mathcal{I}'$ of M-graded $\mathcal{O}_X$-algebras. $\square$

Finally we show that all Cox sheaves of X of type λ are isomorphic, by proving at the same time that every Cox sheaf of X of type λ is isomorphic to one defined in Construction 2.7 and that the Cox sheaves obtained from Construction 2.7 are all isomorphic.

PROPOSITION 2.10. *Let $\mathcal{R}$ be a Cox sheaf of X of type λ. For every $\mathcal{S}$ as in Construction 2.7, there exists a character χ associated with $\mathcal{S}$ and an isomorphism $\mathcal{R} \cong \mathcal{S}/\mathcal{I}$ of Cox sheaves of type λ, where $\mathcal{I}$ is the sheaf of ideals of $\mathcal{S}$ defined by χ.*

PROOF. Let Λ_0, Λ, φ and $\mathcal{S}$ be as in Construction 2.7. For every $m \in M$, we denote by $\mathcal{R}_m$ the degree-m-part of $\mathcal{R}$. Let $\mathcal{S}' := \bigoplus_{L\in\Lambda} \mathcal{R}_{\varphi(L)}$.

Let $\phi_{m,D} : \mathcal{R}_m \to \mathcal{O}_X(D)$, for $(m,D) \in M_\lambda$, be a family of isomorphisms associated with $\mathcal{R}$. Let $\mathcal{B}$ be the basis of Λ chosen in Construction 2.7, and let Λ_+ be the monoid generated by $\mathcal{B}$. For every $L \in \mathcal{B}$, denote by D_L the Cartier divisor representing the class $\lambda(\varphi(L))$ in $\mathrm{Pic}(X)$ such that $\mathcal{S}_L = \mathcal{O}_X(D_L)$. For every $L \in \Lambda_+$, write $L = \sum_{i=1}^n L_i$ with $L_i \in \mathcal{B}$, define $D_L := \sum_{i=1}^n D_{L_i}$, and let α_L be the unique element of $k^\times$ that satisfies

$$\prod_{i=1}^n \phi_{\varphi(L_i),D_{L_i}}(s_i) = \alpha_L \phi_{\varphi(L),D_L}(s_1\cdots s_n)$$

for all $s_i \in H^0(U,\mathcal{R}_{\varphi(L_i)})$ and all open $U \subseteq X$. For every $L \in \Lambda$, write $L = L^+ - L^-$ with $L^+, L^- \in \Lambda_+$, and define $\alpha_L := \alpha_{L^+}\alpha_{L^-}^{-1}\alpha^{-1}$, where $\alpha \in k^\times$ is the unique constant that satisfies $\phi_{\varphi(L),D_L}(s)\phi_{\varphi(L^-),D_{L^-}}(s') = \alpha\phi_{\varphi(L^+),D_{L^+}}(ss')$ for all $s \in H^0(U,\mathcal{R}_{\varphi(L)})$, $s' \in H^0(U,\mathcal{R}_{\varphi(L^-)})$ and all open $U \subseteq X$. The constant α_L does not depend on the choice of L^+ and L^-. The morphisms $\psi_L := \alpha_L\phi_{\varphi(L),D_L}$ induce an isomorphism of Λ-graded $\mathcal{O}_X$-algebras $\psi : \mathcal{S}' \to \mathcal{S}$.

The map $\chi : \Lambda_0 \to k(X)^\times$ defined by $\chi(E) = \psi_{-E}(1)$ for all $E \in \Lambda_0$, is a character associated with $\mathcal{S}$. Let $\mathcal{I}'$ be the kernel of the projection $\mathcal{S}' \to \mathcal{R}$. Then $\psi(\mathcal{I}')$ is the sheaf $\mathcal{I}$ of ideals of $\mathcal{S}$ defined by χ. Thus ψ induces an isomorphism $\mathcal{R} \cong \mathcal{S}/\mathcal{I}$. $\square$

COROLLARY 2.11. *There exists exactly one isomorphism class of Cox sheaves of X of type λ.*

REMARK 2.12. The corollary above agrees with the fact that X-torsors of type λ exist and are all isomorphic, as the map `type` in (2.2) is an isomorphism if k is separably closed.

Now we turn to the relation between generalized Cox sheaves and torsors under quasitori. This is made precise by the proposition below. We first point out some properties of the generalized Cox sheaves.

REMARK 2.13. Let $\mathcal{R}$ be a Cox sheaf of X of type λ. For every open subset U of X that trivializes all elements of $\lambda(M)$, there are isomorphisms

$$\mathcal{R}|_U \cong \mathcal{O}_U[M] \cong \bigoplus_{m\in M} \mathcal{O}_U.$$

Therefore, $\mathcal{R}$ is locally finitely generated as $\mathcal{O}_X$-algebra and locally free as $\mathcal{O}_X$-module.

PROPOSITION 2.14. *If $\pi : Y \to X$ is an X-torsor under $\widehat{M}$ of type λ, then $\pi_*\mathcal{O}_Y$ is a Cox sheaf of X of type λ. Conversely, if $\mathcal{R}$ is a Cox sheaf of X of type λ, then the relative spectrum* $\operatorname{Spec}_X \mathcal{R} \to X$ *is an X-torsor under $\widehat{M}$ of type λ.*

PROOF. Let $\pi : Y \to X$ be an X-torsor under $\widehat{M}$ of type λ. By Proposition 2.2 and the antiequivalence (2.1), $\pi_*\mathcal{O}_Y$ is a sheaf of M-graded $\mathcal{O}_X$-algebras. In particular, there is an open covering $\{U_i\}_i$ of X, Cartier divisors $D_m = \{(U_i, f_{i,m})\}_i$, for $m \in M$, such that $[D_m] = \lambda(m)$ for all $m \in M$, and an isomorphism of $\mathcal{O}_X$-modules

$$\phi : \pi_*\mathcal{O}_Y \to \bigoplus_{m\in M} \mathcal{O}_X(D_m)$$

such that $\phi|_{U_i} : \pi_*\mathcal{O}_Y|_{U_i} \to \bigoplus_{m\in M} f_{i,m}^{-1}\mathcal{O}_{U_i}$ is an isomorphism of $\mathcal{O}_{U_i}$-algebras for all i. For all $m \in M$, denote by $(\pi_*\mathcal{O}_Y)_m$ the degree-m-part of $\pi_*\mathcal{O}_Y$, and let ϕ_{m,D_m} be the isomorphism

$$\phi|_{(\pi_*\mathcal{O}_Y)_m} : (\pi_*\mathcal{O}_Y)_m \to \mathcal{O}_X(D_m).$$

Let U be an open subset of X, and $s_1, s_2 \in \pi_*\mathcal{O}_Y(U)$ homogeneous of degree m_1 and m_2, respectively. Then for all i,

$$\phi(s_1 s_2)|_{U\cap U_i} = f_{i,m_1+m_2}^{-1} f_{i,m_1}\phi(s_1)|_{U\cap U_i} f_{i,m_2}\phi(s_2)|_{U\cap U_i},$$

where the product on the right is computed in $k(X)$. Since the elements $f_{i,m_1+m_2}^{-1} f_{i,m_1} f_{i,m_2}$ belong to $H^0(U_i, \mathcal{O}_X)^\times$ for all i, and $f_{i,m_1+m_2}^{-1} f_{i,m_1} f_{i,m_2} = f_{j,m_1+m_2}^{-1} f_{j,m_1} f_{j,m_2}$ in $k(X)^\times$ for all i and j by restricting ϕ, s_1 and s_2 to $U \cap U_i \cap U_j$, the element $\alpha := f_{i,m_1+m_2} f_{i,m_1}^{-1} f_{i,m_2}^{-1}$ belongs to $k[X]^\times = k^\times$, and

$$\phi_{m,D_{m_1}}(s_1)\phi_{m,D_{m_2}}(s_2) = \alpha\phi_{m_1+m_2,D_{m_1+m_2}}(s_1 s_2),$$

for all $s_1, s_2 \in \pi_*\mathcal{O}_Y(U)$ homogeneous of degree m_1 and m_2, respectively.

Conversely, let $\mathcal{R}$ be a Cox sheaf of X of type λ. The morphism $\operatorname{Spec}_X \mathcal{R} \to X$ induced by $\mathcal{O}_X \subseteq \mathcal{R}$ is surjective. By Remark 2.13, $\mathcal{R}$ is locally free as $\mathcal{O}_X$-module and locally finitely generated as $\mathcal{O}_X$-algebra. Hence, the morphism π is flat and of finite type. Moreover, $\mathcal{R}$ is locally isomorphic to $\widehat{M} = \operatorname{Spec} \mathcal{O}_X[M]$ with the natural action of $\widehat{M}$ on itself. Hence, π is an X-torsor under $\widehat{M}$.

Let $\{U_i\}_i$ be an open covering that trivializes $\mathcal{R}$. The class of $\operatorname{Spec}_X \mathcal{R}$ in $H^1(X, \widehat{M}_X)$ is represented by the cocycle $(\beta_{i,j})_{i,j}$ with

$$\beta_{i,j} \in \operatorname{Hom}(M, \mathcal{O}_X(U_i \cap U_j)^\times)$$

defined by $\beta_{i,j}(m) := f_{i,m} f_{j,m}^{-1}$ for all $m \in M$, where $\{(U_i, f_{i,m})\}_i$ belongs to a fixed system of representatives for $\lambda(M)$. Therefore, $\texttt{type}([\operatorname{Spec}_X \mathcal{R}]) = \lambda$. □

REMARK 2.15. The proposition above shows that Cox sheaves and torsors of given type are equivalent notions. Therefore, Corollary 2.11 can be deduced from Proposition 2.14 and the known results about torsors under quasitori.

In the next proposition we define the pull-back of a Cox sheaf of X of type λ under a morphism $\varphi : M' \to M$ of finitely generated abelian groups. This pull-back of generalized Cox sheaves corresponds to the push-forward of torsors under quasitori under morphisms of quasitori (cf. [**Sko01**, p. 21] and Subsection 2.3.5).

Proposition 2.16. *Let $\varphi : M' \to M$ be a morphism of finitely generated abelian groups. Assume that the characteristic of k does not divide the order of the torsion subgroup of M'. If $\mathcal{R}$ is a Cox sheaf of X of type λ, then*

$$\varphi^*\mathcal{R} := \bigoplus_{m \in M'} \mathcal{R}_{\lambda(m)}$$

with the multiplication induced by $\mathcal{R}$ is a Cox sheaf of X of type $\lambda \circ \varphi$.

Proof. Let $M'_\lambda := \{(m', D) \in M' \times \mathrm{CaDiv}(X) : [D] = \lambda(\varphi(m))\}$. If $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ is a family of isomorphisms associated with the Cox sheaf $\mathcal{R}$, then $\{\phi_{\varphi(m'),D}\}_{(m',D)\in M'_\lambda}$, together with the M'-grading, define a Cox sheaf structure on $\varphi^*\mathcal{R}$. $\square$

Remark 2.17. As explained in [**Sko01**, p. 25], torsors of a given type can be obtained as push-forward of a universal torsor under the type homomorphism. Analogously, Cox sheaves of X of type λ can be obtained as pull-back of a Cox sheaf of X of identity type under λ by Proposition 2.16.

Inspired by Remark 2.17, we give a second construction of generalized Cox sheaves that coincides with Construction 2.7 if $\lambda = \mathrm{id}_{\mathrm{Pic}(X)}$.

Construction 2.18. *Let $D_1, \dots, D_n$ be Cartier divisors on X whose classes generate the subgroup $\lambda(M)$ of $\mathrm{Pic}(X)$. Let $\Lambda := \bigoplus_{i=1}^n \mathbb{Z}D_i$ and Λ_0 the kernel of the morphism $\Lambda \to \mathrm{Pic}(X)$ that sends an element $D \in \Lambda$ to the class $[D]$ of the corresponding Cartier divisor. Let*

$$\Lambda_\lambda := \{(m, D) \in M \times \Lambda : [D] = \lambda(m)\},$$

and for every $(m, D) \in \Lambda_\lambda$, let $\mathcal{S}_{(m,D)} := \mathcal{O}_X(D)$. Endow

$$\mathcal{S} := \bigoplus_{(m,D)\in\Lambda_\lambda} \mathcal{S}_{(m,D)}$$

with the multiplication induced by the inclusions $\mathcal{S}_{(m,D)} \subseteq k(X)$. Let $\chi : \Lambda_0 \to k(X)^\times$ be a homomorphism of groups that satisfies $\mathrm{div}_0(\chi(E)) = \sum_{i=1}^n a_i D_i$ for all $E = \sum_{i=1}^n a_i L_i \in \Lambda_0$. We call χ an identifying character for Λ. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$ for all $E \in \Lambda_0$, where 1 has degree $(0, 0)$ and $\chi(E)$ has degree $(0, E)$. We say that $\mathcal{I}$ is the sheaf of ideals of $\mathcal{S}$ defined by χ. Let $\mathcal{R} := \mathcal{S}/\mathcal{I}$ and $\pi : \mathcal{S} \to \mathcal{R}$ the projection. Then

- *(i) $\mathcal{I}$ is M-homogeneous, and every Λ_λ-homogeneous section of $\mathcal{I}$ is zero;*
- *(ii) $\pi|_{\mathcal{S}_{(m,D)}} : \mathcal{S}_{(m,D)} \to \mathcal{R}_m$ is an isomorphism of $\mathcal{O}_X$-modules for all $(m, D) \in \Lambda_\lambda$;*
- *(iii) $\mathcal{R}$ with any family of isomorphisms $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ induced by $\{\pi|^{-1}_{\mathcal{S}_{(m,D)}}\}_{(m,D)\in\Lambda_\lambda}$ is a Cox sheaf of X of type λ.*

Remark 2.19. The verification of the statements in Construction 2.18 is similar to the proof of Construction 2.7. The same argument that proves Proposition 2.9 shows that the Cox sheaves defined in Constructions 2.18 do not depend on the choice of the character χ, up to isomorphism of Cox sheaves. Moreover, one can prove a result analogous to Proposition 2.10 for the Cox sheaves obtained from Construction 2.18.

The following proposition shows that the groups Λ in Constructions 2.7 and 2.18 can be realized as groups of Cartier divisors.

Proposition 2.20. *Given a finite set $\{D_1, \dots, D_r\}$ of Cartier divisors, there exists a basis $\{D'_1, \dots, D'_r\}$ of a free subgroup of* $\mathrm{CaDiv}(X)$ *with $[D_i] = [D'_i]$ in* $\mathrm{Pic}(X)$ *for all $i \in \{1, \dots, r\}$.*

Proof. Without loss of generality, we can assume that $D_1, \dots, D_r$ are trivialized by the same affine open covering of X, say $D_i = \{(U_j, f_{i,j})\}_{j\geq 1}$. If there are $f_1, \dots, f_r \in k(X)^\times$ such that $f_1^{a_1} \cdots f_r^{a_r} \in H^0(U_1, \mathcal{O}_X)^\times$ with $a_1, \dots, a_r \in \mathbb{Z}$ if and only if $a_1 = \cdots = a_r = 0$, then the divisors $D'_i := \mathrm{div}_{D_i}(f_i f_{i,1}^{-1})$ with $i \in \{1, \dots, r\}$ generate a free subgroup of $\mathrm{CaDiv}(X)$.

To produce rational functions $f_1, \dots, f_r$ as above, one can proceed as follows. Let $A = H^0(U_1, \mathcal{O}_X)$. If X has dimension 0 this lemma is trivial. So, we assume that U has positive dimension, and hence infinitely many closed points. We construct $f_1, \dots, f_r$ by induction. Let $f_1 \in A$ be nonzero and noninvertible. For $1 < i \leq r$, let m_i be a maximal ideal of A not containing $g_{i-1} := \prod_{j=1}^{i-1} f_j$. Since $(g_{i-1}) + m_i = A$, there exists an element $f_i \in m_i$ such that $(g_{i-1}, f_i) = A$. Then $(f_j, f_i) = A$ for $1 \leq j \leq i-1$ and $f_i \neq 0$ since g_{i-1} is not invertible. Let $a_1, \dots, a_r \in \mathbb{Z}$ not all zero such that $\prod_{i=1}^r f_i^{a_i} \in A^\times$. Up to a permutation of the indices, we can assume that $a_1 \geq \cdots \geq a_r$. Since none of the f_i is invertible, $a_1 > 0$ and $a_r < 0$. We write $g := \prod_{a_i>0} f_i^{a_i}$ and $h := \prod_{a_i<0} f_i^{-a_i}$, so that $g, h \in A$ and $gh^{-1} \in A^\times$. Hence, $(h) = (g) \subseteq (f_1)$ in A. Then $(f_1) = (f_1, h) = A$, which contradicts the fact that f_1 is not a unit in A. □

The following two constructions are used in the next section to relate Cox sheaves to the existence of rational points. The first one is determined by a splitting of the natural exact sequence that relates the function field of X to its group of principal divisors.

Construction 2.21. *Let $\sigma : k(X)^\times \to k^\times$ be a splitting of the natural exact sequence of groups*

$$1 \to k^\times \to k(X)^\times \to k(X)^\times/k^\times \to 1. \tag{2.3}$$

For every $m \in M$, let D_m be a Cartier divisor representing the class $\lambda(m)$ in $\mathrm{Pic}(X)$ *and $\mathcal{R}_m := \mathcal{O}_X(D_m)$. Endow $\mathcal{R} := \bigoplus_{m\in M} \mathcal{R}_m$ with the following multiplication of sections induced by σ. For every open subset U of X and homogeneous sections $s_1, s_2 \in \mathcal{R}(U)$ of degree $m_1, m_2 \in M$, respectively, let $f \in k(X)^\times$ be the unique element such that $D_{m_1+m_2} = \mathrm{div}_{D_{m_1}+D_{m_2}}(f)$ and $\sigma(f) = 1$. Define $s_1 s_2 := f s_1 s_2 \in \mathcal{R}_{m_1+m_2}(U)$, where the product on the right is computed in $k(X)$. Then $\mathcal{R}$ is a Cox sheaf of X of type λ.*

Proof. By Remark 2.5, it is enough to determine the isomorphisms ϕ_{m,D_m} for $m \in M$. These are chosen to be the identity and are compatible with multiplication by construction. $\square$

The second construction is determined by a splitting of the natural exact sequence associated with the group of invertible sections on a small enough open subset of X.

Construction 2.22. *Let $U \subseteq X$ be a nonempty open subset such that $\lambda(M)$ is contained in the kernel of the natural morphism $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$. Let $\sigma : k[U]^\times \to k^\times$ be a splitting of the natural exact sequence of groups*

$$1 \to k^\times \to k[U]^\times \to k[U]^\times/k^\times \to 1. \tag{2.4}$$

For every $m \in M$, let D_m be a Cartier divisor supported on $X \smallsetminus U$ representing the class $\lambda(m)$ in $\mathrm{Pic}(X)$ and $\mathcal{R}_m := \mathcal{O}_X(D_m)$. Endow $\mathcal{R} := \bigoplus_{m\in M} \mathcal{R}_m$ with the following multiplication of sections induced by σ. For every open subset V of X and homogeneous sections $s_1, s_2 \in \mathcal{R}(V)$ of degree $m_1, m_2 \in M$, respectively, let $f \in k[U]^\times$ be the unique element such that $D_{m_1+m_2} = \mathrm{div}_{D_{m_1}+D_{m_2}}(f)$ and $\sigma(f) = 1$. Define $s_1s_2 := fs_1s_2 \in \mathcal{R}_{m_1+m_2}(V)$, where the product on the right is computed in $k(X)$. Then $\mathcal{R}$ is a Cox sheaf of X of type λ.

Remark 2.23. Every rational point $x \in U(k)$ defines a splitting $\sigma_x : k[U]^\times \to k^\times$ of (2.4) by $\sigma_x(f) := f(x)$ for all $f \in k[U]^\times$.

We conclude this first presentation of generalized Cox sheaves by determining their group of automorphisms.

Proposition 2.24. *Let $\mathcal{R}$ be a Cox sheaf of X of type λ. For every $h \in \widehat{M}(k) = \mathrm{Hom}(M, k^\times)$, let $\psi_h : \mathcal{R} \to \mathcal{R}$ be the map defined as scalar multiplication by $h(m)$ on $\mathcal{R}_m$ for all $m \in M$. Then $h \mapsto \psi_h$ defines an isomorphism between $\widehat{M}(k)$ and the group of Cox sheaf automorphisms of $\mathcal{R}$.*

Proof. By Lemma 2.4, a Cox sheaf automorphism ψ of $\mathcal{R}$ must be scalar multiplication by some $h_m \in k^\times$ on each homogeneous part $\mathcal{R}_m$. Moreover, $h_m h_{m'} = h_{m+m'}$ for all $m, m' \in M$ as ψ is compatible with the multiplication in $\mathcal{R}$. So, $m \mapsto h_m$ defines a group homomorphism $M \to k^\times$, and hence an element $h \in \widehat{M}(k)$ such that $\psi = \psi_h$. $\square$

2.2.2. Generalized Cox rings. In [**BH03, ADHL15**] a Cox ring is the ring of global sections of a Cox sheaf. We give an axiomatic definition of generalized Cox rings that is clearly satisfied by the rings of global sections of the generalized Cox sheaves defined previously in Section 2.2.

Definition 2.25. A *Cox ring of X of type λ* is an M-graded k-algebra R together with a family of isomorphisms of k-vector spaces $\phi_{m,D} : R_m \to H^0(X, \mathcal{O}_X(D))$ for every $(m, D) \in M_\lambda$, where R_m denotes the degree-m-part of R, such that for every $(m_1, D_1), (m_2, D_2) \in M_\lambda$ there exists a nonzero constant $\alpha \in k$ that satisfies $\phi_{m_1,D_1}(s_1)\phi_{m_2,D_2}(s_2) = \alpha\phi_{m_1+m_2,D_1+D_2}(s_1s_2)$ for all $s_i \in H^0(X, \mathcal{R}_{m_i})$, $i \in \{1, 2\}$.

A *structure of Cox ring of X of type* λ on a k-algebra R is an M-grading on R and a family of isomorphisms $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ as above.

We also define morphisms of generalized Cox rings of a given type.

DEFINITION 2.26. Let R and R' be two Cox rings of X of type λ. A morphism of M-graded k-algebras $\psi : R \to R'$ is a *morphism of Cox rings of X of type λ* if it is compatible with the families of isomorphisms $\phi_{m,D}$ and $\phi'_{m,D}$ associated with R and R', respectively, as follows: for every $(m, D) \in M_\lambda$ there is $\alpha_{m,D} \in k[X]$ such that $\phi'_{m,D} \circ \psi \circ \phi_{m,D}^{-1} = \alpha_{m,D} \operatorname{id}_{H^0(X,\mathcal{O}_X(D))}$.

REMARK 2.27. The isomorphisms $\phi_{m,D}$ in Definition 2.25 are uniquely determined up to multiplication by nonzero elements of k. Therefore, given a Cox ring R, we say that two families $\phi_{m,D}$ and $\phi'_{m,D}$ define the same Cox ring structure on R if for every $(m, D) \in M_\lambda$, there is $\alpha_{m,D} \in k^\times$ such that $\phi'_{m,D} = \alpha_{m,D}\phi_{m,D}$ (cf. Remark 2.5).

Moreover, a Cox ring structure on R is determined once the isomorphism $\phi_{m,D}$ is known for a chosen $(m, D) \in M_\lambda$ for each $m \in M$. Therefore, to check whether a morphism of k-algebras between Cox rings of given type is a morphism of Cox rings, it is enough to verify the condition in Definition 2.26 for a chosen $(m, D) \in M_\lambda$ for each $m \in M$.

REMARK 2.28. Morphisms of Cox rings of X of type λ respect the M-gradings. However, a k-algebra morphism respecting the M-grading is not necessarily a morphism of Cox rings of type λ. For example, for $X = \mathbb{P}^1_k$, the Picard group $\operatorname{Pic}(X)$ is free of rank 1, and $R \cong k[T_0, T_1]$ is a Cox ring of X of type $\operatorname{id}_{\operatorname{Pic}(X)}$, where the $\operatorname{Pic}(X)$-grading is the usual $\mathbb{Z}$-grading by the total degree, identifying effective divisor classes with nonnegative integers. Mapping T_0 and T_1 to any linearly independent linear polynomials in T_0, T_1 defines an automorphism of $k[T_0, T_1]$ respecting the grading, but any Cox ring automorphism of R in the sense of Definition 2.26 is multiplication by a scalar in $k^\times$.

An equivalent characterization of Cox rings of type λ, which is similar to [**DP14**, Definition 2.1], is provided by the following proposition.

PROPOSITION 2.29. *An M-graded k-algebra R has a structure of Cox ring of X of type λ if and only if there exists a map*

$$\operatorname{div} : \bigcup_{m\in M} (R_m \smallsetminus \{0\}) \to \operatorname{CaDiv}(X),$$

where R_m denotes the degree-m-part of R, such that for each $(m, D) \in M_\lambda$ there is a k-vector space isomorphism $\phi_{m,D} : R_m \to H^0(X, \mathcal{O}_X(D))$ satisfying $\operatorname{div}(s) = \operatorname{div}_D(\phi_{m,D}(s))$ *for all nonzero $s \in R_m$, and* $\operatorname{div}(s_1 s_2) = \operatorname{div}(s_1) + \operatorname{div}(s_2)$ *for all nonzero homogeneous elements s_1, s_2 of R.*

PROOF. If R is a Cox ring of X of type λ with family of isomorphisms $\phi_{m,D}$ as in Definition 2.25, for every $m \in M$ choose a divisor $D \in \operatorname{CaDiv}(X)$ such that $(m, D) \in M_\lambda$ and define $\operatorname{div}(s) := \operatorname{div}_D(\phi_{m,D}(s))$ for all nonzero $s \in R_m$.

For the converse implication, assume that R is an M-graded k-algebra endowed with a map div and isomorphisms $\phi_{m,D}$ as in the statement. We prove that the family of isomorphisms $\phi_{m,D}$ define a structure of Cox ring of X of type λ on R. For every $(m_1, D_1), (m_2, D_2) \in M_\lambda$ and $s_i \in H^0(X, \mathcal{R}_{m_i})$,

$i \in \{1,2\}$, the elements

$$\alpha_{m_1,m_2;D_1,D_2} := \phi_{m_1,D_1}(s_1)\phi_{m_2,D_2}(s_2)\phi_{m_1+m_2,D_1+D_2}(s_1s_2)^{-1}$$

do not depend on the chosen sections $s_i \in H^0(X,\mathcal{R}_{m_i})$ because the morphisms $\phi_{m,D}$ are linear. Moreover, $\alpha_{m_1,m_2;D_1,D_2}$ belong to $k^\times$ because

$$\operatorname{div}_{D_1+D_2}(\phi_{m_1+m_2,D_1+D_2}(s_1s_2)) = \operatorname{div}_{D_1}(\phi_{m_1,D_1}(s_1)) + \operatorname{div}_{D_2}(\phi_{m_2,D_2}(s_2)).$$

□

By definition, the degree-m-part of a Cox ring of X of type λ is nonzero if and only if $\lambda(m)$ is an effective class in $\operatorname{Pic}(X)$. We denote by M_{eff} the subgroup of M generated by the elements m such that $\lambda(m)$ is effective in $\operatorname{Pic}(X)$, and define $\lambda_{\text{eff}} := \lambda|_{M_{\text{eff}}} : M_{\text{eff}} \to \operatorname{Pic}(X)$.

Remark 2.30. A Cox ring of X of type λ is also a Cox ring of X of type λ', for all $\lambda' : M' \to \operatorname{Pic}(X)$ such that $M'_{\text{eff}} = M_{\text{eff}}$ and $\lambda'_{\text{eff}} = \lambda_{\text{eff}}$.

Since the degree-m-part of a Cox sheaf of X of type λ is always nonzero, we can expect that Cox sheaves of different types may have rings of global sections that are isomorphic generalized Cox rings. The relation between Cox rings and Cox sheaves of a given type is explained by the following proposition.

Proposition 2.31. *Let R be a Cox ring of X of type λ. Then R is isomorphic, as Cox ring of type λ, to the ring of global sections of a Cox sheaf of X of type λ.*

If $M = M_{\text{eff}}$, then every automorphism of R as Cox ring of X of type λ is induced by a unique automorphism of the underlying Cox sheaf of X of type λ.

Proof. For every $m \in M$, we denote by R_m the degree-m-part of R. Let $\phi_{m,D} : R_m \to H^0(X,\mathcal{O}_X(D))$, for $(m,D) \in M_\lambda$, be a family of isomorphisms associated with R.

Let Λ_0, Λ, φ and $\mathcal{S}$ be as in Construction 2.7. For every $L \in \Lambda$, let D_L be the Cartier divisor on X such that $\mathcal{S}_L = \mathcal{O}_X(D_L)$. Let Λ_{eff} be the subgroup of Λ generated by the elements L such that $\lambda(\varphi(L))$ is effective in $\operatorname{Pic}(X)$. We observe that $\Lambda_0 \subseteq \Lambda_{\text{eff}}$. Since Λ is free and finitely generated, the same is true for Λ_{eff}. Let $\mathcal{B}_{\text{eff}}$ be a basis of Λ_{eff}, and let $\Lambda_+ \subseteq \Lambda_{\text{eff}}$ be the monoid generated by $\mathcal{B}_{\text{eff}}$. For every $L \in \Lambda_+$, write $L = \sum_{i=1}^n L_i$ with $L_i \in \mathcal{B}_{\text{eff}}$, and let α_L be the unique element of $k^\times$ that satisfies

$$\prod_{i=1}^n \phi_{\varphi(L_i),D_{L_i}}(s_i) = \alpha_L \phi_{\varphi(L),D_L}(s_1\cdots s_n)$$

for all $s_i \in R_{\varphi(L_i)}$ with $i \in \{1,\dots,n\}$. For every $L \in \Lambda_{\text{eff}}$ such that $\lambda(\varphi(L))$ is effective in $\operatorname{Pic}(X)$, write $L = L^+ - L^-$ with $L^+, L^- \in \Lambda_+$, and define $\alpha_L := \alpha_{L^+}\alpha_{L^-}^{-1}\alpha^{-1}$, where $\alpha \in k^\times$ is the unique constant that satisfies $\phi_{\varphi(L),D_L}(s)\phi_{\varphi(L^-),D_{L^-}}(s') = \alpha\phi_{\varphi(L^+),D_{L^+}}(ss')$ for all $s \in R_{\varphi(L)}$ and $s' \in R_{\varphi(L^-)}$. The constant α_L does not depend on the choice of L^+ and L^-. The isomorphisms $\psi_L := \alpha_L^{-1}\phi_{\varphi(L),D_L}^{-1}$ induce a surjective morphism of graded k-algebras $\psi : H^0(X,\mathcal{S}) \to R$.

The map $\chi : \Lambda_0 \to k(X)^\times$ defined by $\chi(E) = \psi_{-E}^{-1}(1)$ for all $E \in \Lambda_0$, is a character associated with $\mathcal{S}$. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{S}$ associated with χ, and $\mathcal{R} := \mathcal{S}/\mathcal{I}$. Then $H^0(X,\mathcal{S})/H^0(X,\mathcal{I}) \cong H^0(X,\mathcal{R})$ by Construction 2.7 *(ii)*, and $H^0(X,\mathcal{I})$ is the kernel of ψ.

We show that the induced isomorphism $\psi : H^0(X,\mathcal{S})/H^0(X,\mathcal{I}) \to R$ is a morphism of Cox rings. The Cox sheaf $\mathcal{R}$ is endowed with the family of isomorphisms $\{\phi'_{m,D}\}_{(m,D)\in M_\lambda}$ induced by $\{\pi|_{\mathcal{S}_L}^{-1}\}_{L\in\Lambda}$, where $\pi : \mathcal{S} \to \mathcal{R}$ is the projection. For $m \in M$ such that $\lambda(m)$ is effective, let $L \in \Lambda_{\mathrm{eff}}$ such that $\varphi(L) = m$. Without loss of generality, we can assume that $\phi'_{m,D_L} = \pi|_{\mathcal{S}_L}^{-1}$. Hence,

$$\phi_{m,D_L} \circ \psi \circ (\phi'_{m,D_L})^{-1} = \alpha_L^{-1} \,\mathrm{id}_{H^0(X,\mathcal{O}_X(D_L))}\,.$$

Assume now that $M = M_{\mathrm{eff}}$, and let ψ be an automorphism of $\mathcal{R}(X)$. By definition of automorphism of a Cox ring of X of type λ, for every $m \in M$ such that $\lambda(m)$ is effective, there is a constant $h_m \in k^\times$ such that $\psi|_{\mathcal{R}(X)_m} = h_m \,\mathrm{id}_{\mathcal{R}(X)_m}$. Moreover, $h_m h_{m'} = h_{m+m'}$ for all $m, m' \in M$ as above as ψ is compatible with the multiplication in $\mathcal{R}$. Since $\lambda(M)$ is generated by effective divisor classes, $m \mapsto h_m$ defines a group homomorphism $M \to k^\times$, which is an element $h \in \widehat{M}(k)$, and hence an automorphism of $\mathcal{R}$ by Proposition 2.24. □

Remark 2.32. A morphism of Cox sheaves of X of type λ induces a morphism of Cox rings of X of type λ between the rings of global sections. Hence, for every λ, there exists exactly one isomorphism class of Cox rings of X of type λ.

Remark 2.33. As a consequence of Propositions 2.24 and 2.31, the group of Cox ring automorphisms of a Cox ring of X of type λ is isomorphic to $\widehat{M_{\mathrm{eff}}}(k)$.

Remark 2.34. If $\lambda(M)$ contains an ample divisor class, then $M = M_{\mathrm{eff}}$.

2.3. Over nonclosed fields

In this section we define generalized Cox sheaves and Cox rings of a variety over an arbitrary field by Galois descent, we classify them up to isomorphism, we explain the relation with torsors under quasitori, and we discuss some existence criteria. Moreover, we consider their functoriality properties.

We start by fixing some notation for the rest of this section. Let k be a field, fix a separable closure $\overline{k}$ of k, and let $\mathfrak{g}$ be the Galois group $\mathrm{Gal}(\overline{k}/k)$. Any algebraic extension of k mentioned in this section is contained in $\overline{k}$.

Let X be a geometrically integral k-variety such that $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, where $X_{\overline{k}} := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$. The action of $\mathfrak{g}$ on $\overline{k}$ induces an action on $A \otimes_k \overline{k}$ (with $\mathrm{g} \in \mathfrak{g}$ acting via $\mathrm{id}_A \otimes \mathrm{g}$) for any k-algebra A, and similarly on $\mathcal{O}_{X_{\overline{k}}} = \mathcal{O}_X \otimes_k \overline{k}$. The action of an element $\mathrm{g} \in \mathfrak{g}$ on $X_{\overline{k}}$ is the one induced by the action of g^{-1} on $\mathcal{O}_{X_{\overline{k}}}$. For every $\mathrm{g} \in \mathfrak{g}$, we denote by $\mathrm{g}D$ the natural Galois action on a divisor $D \in \mathrm{CaDiv}(X_{\overline{k}})$, by $\mathrm{g}(f)$ the natural Galois action on an element $f \in \overline{k}(X)$. All these actions are *continuous* (with respect to the Krull topology on $\mathfrak{g}$ and the discrete topology on the other objects). We will denote by $g * s$ an action of $g \in \mathfrak{g}$ on a section s of a Cox sheaf (or an element s of a Cox ring) of $X_{\overline{k}}$.

Let M be a $\mathfrak{g}$-module which is finitely generated as abelian group, and assume that the characterstic of k does not divide the order of its torsion subgroup. Let $\lambda : M \to \operatorname{Pic}(X_{\overline{k}})$ be a homomorphism of $\mathfrak{g}$-modules.

2.3.1. Natural Galois actions. To define generalized Cox sheaves and rings of X by Galois descent we need to specify what is a Galois action compatible with the structure of Cox sheaf or ring on a generalized Cox sheaf or ring, respectively, of $X_{\overline{k}}$.

Definition 2.35. A continuous $\mathfrak{g}$-action on a Cox sheaf $\mathcal{R}$ of $X_{\overline{k}}$ of type λ is called *natural* if, given an associated family of isomorphisms $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$, for every $g \in \mathfrak{g}$ and $(m, D) \in M_\lambda$, the automorphism of $\mathcal{R}$ defined by the action of g restricts to an isomorphism $\mathcal{R}_m \to \mathcal{R}_{gm}$ such that $g^{-1} \circ \phi_{gm,gD} \circ g \circ \phi_{m,D}^{-1}$ is an automorphism of $\mathcal{O}_{X_{\overline{k}}}(D)$. A $\mathfrak{g}$-*equivariant Cox sheaf of $X_{\overline{k}}$ of type* λ is a Cox sheaf of $X_{\overline{k}}$ of type λ with a natural $\mathfrak{g}$-action.

Analogously, a continuous $\mathfrak{g}$-action on a Cox ring R of $X_{\overline{k}}$ of type λ is called *natural* if, given an associated family of isomorphisms $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$, for every $g \in \mathfrak{g}$ and $(m, D) \in M_\lambda$, the automorphism of R defined by the action of g restricts to an isomorphism $R_m \to R_{gm}$ such that $g^{-1} \circ \phi_{gm,gD} \circ g \circ \phi_{m,D}^{-1} = \alpha \operatorname{id}_{H^0(X_{\overline{k}},\mathcal{O}_{X_{\overline{k}}}(D))}$ with $\alpha \in \overline{k}^\times$. A $\mathfrak{g}$-*equivariant Cox ring of $X_{\overline{k}}$ of type* λ is a Cox ring of $X_{\overline{k}}$ of type λ with a natural $\mathfrak{g}$-action.

A *morphism of $\mathfrak{g}$-equivariant Cox sheaves (rings) of $X_{\overline{k}}$ of type* λ is a $\mathfrak{g}$-equivariant morphism of Cox sheaves (respectively, rings) of $X_{\overline{k}}$ of type λ.

Remark 2.36. Using the equivalent definition of generalized Cox ring provided by Proposition 2.29, a continuous $\mathfrak{g}$-action on a Cox ring R of $X_{\overline{k}}$ of type λ is *natural* if and only if $\operatorname{div}(g * s) = g \operatorname{div}(s)$ for all nonzero homogeneous $s \in R$ and all $g \in \mathfrak{g}$.

A natural $\mathfrak{g}$-action on a generalized Cox sheaf $\mathcal{R}$ of $X_{\overline{k}}$ of type λ induces a natural $\mathfrak{g}$-action on the Cox ring $\mathcal{R}(X_{\overline{k}})$ of $X_{\overline{k}}$ of type λ. The next proposition shows that the converse holds if $M = M_{\text{eff}}$, where M_{eff} is the effective subgroup of M defined before Remark 2.30.

Proposition 2.37. *Let $\mathcal{R}$ be a Cox sheaf of $X_{\overline{k}}$ of type $M = M_{\text{eff}}$. Then every natural $\mathfrak{g}$-action on $\mathcal{R}(X_{\overline{k}})$ is induced by a natural $\mathfrak{g}$-action on $\mathcal{R}$.*

Proof. Assume that $\mathcal{R}(X_{\overline{k}})$ is endowed with a natural $\mathfrak{g}$-action. Let $\phi_{m,D} : \mathcal{R}_m \to \mathcal{O}_{X_{\overline{k}}}(D)$, for $(m, D) \in M_\lambda$, be a family of isomorphisms associated with $\mathcal{R}$. For every $g \in \mathfrak{g}$ and $(m, D) \in M_\lambda$ such that $\lambda(m)$ is effective in $\operatorname{Pic}(X_{\overline{k}})$, let $\alpha_{g,m,D} \in \overline{k}^\times$ be the unique constant such that

$$\phi_{gm,gD} \circ g = \alpha_{g,m,D} g \circ \phi_{m,D}.$$

For an arbitrary $(m, D) \in M_\lambda$, let $(m^+, D^+), (m^-, D^-) \in M_\lambda$ such that $m = m^+ - m^-$, $D = D^+ - D^-$, and $\lambda(m^+)$ and $\lambda(m^-)$ are both effective in $\operatorname{Pic}(X_{\overline{k}})$. We define $\alpha_{g,m,D} := \alpha_{g,m^+,D^+}\alpha_{g,m^-,D^-}^{-1}\beta g(\gamma^{-1})$, where $\beta, \gamma \in \overline{k}^\times$ are the unique constants such that

$$\phi_{m,D}(s)\phi_{m^-,D^-}(s') = \gamma\phi_{m^+,D^+}(ss')$$

and
$$\phi_{\mathrm{g}m,\mathrm{g}D}(\mathrm{g}*s)\phi_{\mathrm{g}m^-,\mathrm{g}D^-}(\mathrm{g}*s') = \beta\phi_{\mathrm{g}m^+,\mathrm{g}D^+}(\mathrm{g}*(ss'))$$
for all $s \in \mathcal{R}(X_{\overline{k}})_m$ and $s' \in \mathcal{R}(X_{\overline{k}})_{m^-}$. The constant $\alpha_{\mathrm{g},m,D}$ do not depend on the choice of (m^+, D^+) and (m^-, D^-).

For every homogeneous section s of $\mathcal{R}$ and every $\mathrm{g} \in \mathfrak{g}$, let $\mathrm{g} * s := \alpha_{\mathrm{g},m,D}(\phi^{-1}_{\mathrm{g}m,\mathrm{g}D} \circ \mathrm{g} \circ \phi_{m,D})(s)$ for any $(m, D) \in M_\lambda$ such that s has degree m. This definition does not depend on the choice of the representative D for $\lambda(m)$ and induces a natural action of $\mathfrak{g}$ on $\mathcal{R}$ in the sense of Definition 2.35. □

2.3.2. Generalized Cox sheaves and rings. We are ready to define generalized Cox sheaves and Cox rings of X by Galois descent.

Definition 2.38. A *Cox sheaf of X of type λ* is a sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras together with a structure of Cox sheaf of $X_{\overline{k}}$ of type λ on $\mathcal{R}_{\overline{k}} := \mathcal{R} \otimes_k \overline{k}$ such that $\mathcal{R}_{\overline{k}}$ with the induced action of $\mathfrak{g}$ is a $\mathfrak{g}$-equivariant Cox sheaf of $X_{\overline{k}}$. Analogously, a *Cox ring of X of type λ* is a k-algebra R together with a structure of Cox ring of $X_{\overline{k}}$ of type λ on $R_{\overline{k}} := R \otimes_k \overline{k}$ such that $R_{\overline{k}}$ with the induced action of $\mathfrak{g}$ is a $\mathfrak{g}$-equivariant Cox ring of $X_{\overline{k}}$.

A *morphism of Cox sheaves of X of type λ* is a morphism of $\mathcal{O}_X$-algebras $\psi : \mathcal{R} \to \mathcal{R}'$ such that $\psi \otimes \mathrm{id}_{\overline{k}} : \mathcal{R}_{\overline{k}} \to \mathcal{R}'_{\overline{k}}$ is a morphism of Cox sheaves of $X_{\overline{k}}$ of type λ. Analogously, a *morphism of Cox rings of X of type λ* is a morphism of k-algebras $\psi : R \to R'$ such that $\psi \otimes \mathrm{id}_{\overline{k}} : R_{\overline{k}} \to R'_{\overline{k}}$ is a morphism of Cox rings of $X_{\overline{k}}$ of type λ.

The following proposition show that the notions of generalized Cox sheaves and rings of X are equivalent to the notions of $\mathfrak{g}$-equivariant generalized Cox sheaves and rings of $X_{\overline{k}}$, respectively.

Proposition 2.39. *The covariant functors*
$$\begin{array}{ccc} \{\textit{Cox sheaves of } X \textit{ of type } \lambda\} & \longrightarrow & \left\{\begin{array}{c}\mathfrak{g}\textit{-equivariant Cox sheaves}\\ \textit{of } X_{\overline{k}} \textit{ of type } \lambda\end{array}\right\} \\ \mathcal{R} & \longmapsto & \mathcal{R}_{\overline{k}} \end{array}$$
and
$$\begin{array}{ccc} \{\textit{Cox rings of } X \textit{ of type } \lambda\} & \longrightarrow & \left\{\begin{array}{c}\mathfrak{g}\textit{-equivariant Cox rings}\\ \textit{of } X_{\overline{k}} \textit{ of type } \lambda\end{array}\right\} \\ R & \longmapsto & R_{\overline{k}} \end{array}$$
are equivalences of categories, with inverse functor $H^0(\mathfrak{g}, .)$.

Proof. Let R be a $\mathfrak{g}$-equivariant Cox ring of $X_{\overline{k}}$ of type λ, and $R^{\mathfrak{g}} := H^0(\mathfrak{g}, R)$ its subring of $\mathfrak{g}$-invariant elements. Since the action of $\mathfrak{g}$ on R is continuous, there is an isomorphism $R^{\mathfrak{g}} \otimes_k \overline{k} \cong R$ by [**Mil12**, Proposition 16.15]. Similarly, if $\mathcal{R}$ is a $\mathfrak{g}$-equivariant Cox sheaf of $X_{\overline{k}}$ of type λ, then the sheaf $\mathcal{R}^{\mathfrak{g}}$ defined by $\mathcal{R}^{\mathfrak{g}}(U) := H^0(\mathfrak{g}, \mathcal{R}(U_{\overline{k}}))$ for all open subsets U of X is a Cox sheaf of X of type λ. Moreover, if $\psi : \mathcal{R} \to \mathcal{R}'$ is a $\mathfrak{g}$-equivariant morphism of $\mathfrak{g}$-equivariant Cox sheaves of $X_{\overline{k}}$ of type λ, then $\psi(\mathcal{R}(U_{\overline{k}})^{\mathfrak{g}}) \subseteq \mathcal{R}'(U_{\overline{k}})^{\mathfrak{g}}$ for every open subset U of X. Hence, ψ restricts to a unique morphism $\psi^{\mathfrak{g}} : \mathcal{R}^{\mathfrak{g}} \to \mathcal{R}'^{\mathfrak{g}}$ of Cox sheaves of X of type λ such that $\psi = \psi^{\mathfrak{g}} \otimes \mathrm{id}_{\overline{k}}$ under the identifications $\mathcal{R} = (\mathcal{R}^{\mathfrak{g}})_{\overline{k}}$ and $\mathcal{R}' = (\mathcal{R}'^{\mathfrak{g}})_{\overline{k}}$. □

The following corollary establishes the connection between generalized Cox sheaves and Cox rings of varieties over arbitrary fields.

COROLLARY 2.40. *If $\mathcal{R}$ is a Cox sheaf of X of type λ, then $\mathcal{R}(X)$ is a Cox ring of X of type λ.*

If $M = M_{\mathrm{eff}}$, then this induces a bijection between the set of isomorphism classes of Cox sheaves of X of type λ and the set of isomorphism classes of Cox rings of X of type λ.

PROOF. Let $\mathcal{R}$ be a Cox sheaf of X of type λ. By Definition 2.38, $\mathcal{R}_{\overline{k}}$ is a $\mathfrak{g}$-equivariant Cox sheaf of $X_{\overline{k}}$. Hence, $\mathcal{R}_{\overline{k}}(X_{\overline{k}})$ is a Cox ring of $X_{\overline{k}}$ of type λ according to Definition 2.25. Since the induced $\mathfrak{g}$-action turns it into a $\mathfrak{g}$-equivariant Cox ring of type λ, the ring of $\mathfrak{g}$-invariants $\mathcal{R}(X) = \mathcal{R}_{\overline{k}}(X_{\overline{k}})^{\mathfrak{g}}$ is a Cox ring of X of type λ (cf. Proposition 2.39).

Assume now that $M = M_{\mathrm{eff}}$. Clearly isomorphic Cox sheaves of X of type λ give isomorphic Cox rings of X of type λ if we take rings of global sections. Hence, the map $\mathcal{R} \mapsto \mathcal{R}(X)$ is well defined on isomorphism classes. Surjectivity is a consequence of Propositions 2.31, 2.37 and 2.39. For injectivity, assume that $\mathcal{R}$ and $\mathcal{R}'$ are Cox sheaves of X satisfying $\mathcal{R}(X) \cong \mathcal{R}'(X)$. This isomorphism of Cox rings of X induces an isomorphism of $\mathfrak{g}$-equivariant Cox rings $\mathcal{R}(X)_{\overline{k}} \cong \mathcal{R}'(X)_{\overline{k}}$ that extends to a unique isomorphism of Cox sheaves $\psi : \mathcal{R}_{\overline{k}} \to \mathcal{R}'_{\overline{k}}$ by Proposition 2.31. Since $\mathrm{g}^{-1} \circ \psi^{-1} \circ \mathrm{g} \circ \psi$ is an automorphism on $\mathcal{R}_{\overline{k}}$ that induces the identity on $\mathcal{R}_{\overline{k}}(X_{\overline{k}})$ for every $\mathrm{g} \in \mathfrak{g}$, by the uniqueness statement of Proposition 2.31, $\mathrm{g}^{-1} \circ \psi^{-1} \circ \mathrm{g} \circ \psi$ is the identity as well, which means that ψ is $\mathfrak{g}$-equivariant, and hence, restricts to an isomorphism $\mathcal{R} \cong \mathcal{R}'$. □

We apply descent theory (see [**Ser02**, §III.1] or [**Mil12**, §16]) to classify Cox sheaves and Cox rings of X up to isomorphism. Recall that by Corollary 2.11, every two Cox sheaves of X of type λ become isomorphic after the field extension $k \subseteq \overline{k}$. Moreover, if ψ is an isomorphism of Cox sheaves of type λ between two $\mathfrak{g}$-equivariant Cox sheaves of $X_{\overline{k}}$ of type λ, then ${}^{\mathrm{g}}\psi := \mathrm{g} \circ \psi \circ \mathrm{g}^{-1}$ is another isomorphism for all $\mathrm{g} \in \mathfrak{g}$, and sending g to the automorphism $\psi^{-1} \circ {}^{\mathrm{g}}\psi$ defines a map $\mathfrak{g} \to \widehat{M}(\overline{k})$ by Proposition 2.24, which turns out to be a 1-cocycle. The generalized Cox sheaves of X of type λ are then classified as follows.

PROPOSITION 2.41. *Assume that X has a Cox sheaf $\mathcal{R}$ of type λ. Sending a Cox sheaf $\mathcal{R}'$ of X of type λ with a Cox sheaf isomorphism $\psi : \mathcal{R}_{\overline{k}} \to \mathcal{R}'_{\overline{k}}$ to the class of the cocycle*

$$\mathfrak{g} \to \mathrm{Aut}(\mathcal{R}_{\overline{k}}) = \widehat{M}(\overline{k}), \quad \mathrm{g} \mapsto \psi^{-1} \circ {}^{\mathrm{g}}\psi$$

defines an bijective map from the set of isomorphism classes of Cox sheaves of X of type λ to $H^1(k, \widehat{M})$.

PROOF. The class of the cocycle $\mathrm{g} \mapsto \psi^{-1} \circ {}^{\mathrm{g}}\psi$ does not depend on the choice of ψ. Moreover, if $\mathcal{R}'$, $\mathcal{R}''$ are two Cox sheaves of X of type λ, and $\psi' : \mathcal{R}_{\overline{k}} \to \mathcal{R}'_{\overline{k}}$, $\psi'' : \mathcal{R}_{\overline{k}} \to \mathcal{R}''_{\overline{k}}$ are two Cox sheaf isomorphisms, the associated cocycles have the same class in $H^1(k, \widehat{M})$ if and only if $\psi'' \circ \psi'^{-1} : \mathcal{R}' \to \mathcal{R}''$ is $\mathfrak{g}$-equivariant. Hence, the map in the statement is injective.

For surjectivity, note that a cocycle $\sigma : \mathfrak{g} \to \widehat{M}(\overline{k})$ defines a twisted action

$$\mathfrak{g} \times \mathcal{R}_{\overline{k}} \to \mathcal{R}_{\overline{k}}, \qquad (\mathrm{g}, s) \mapsto (\psi_{\sigma_{\mathrm{g}}} \circ (\mathrm{id}_{\mathcal{R}} \otimes \mathrm{g}))(s),$$

where $\psi_{\sigma_{\mathrm{g}}}$ is the automorphism defined in Proposition 2.24. Since

$$H^1(\mathfrak{g}, \widehat{M}(\overline{k})) = \varinjlim_{k'/k} H^1(\mathrm{Gal}(k'/k), \widehat{M}(k')),$$

where the direct limit is taken over the finite Galois extensions k' of k inside $\overline{k}$, the twisted action defined by σ is continuous, and hence it is a natural $\mathfrak{g}$-action on $\mathcal{R}_{\overline{k}}$ according to Definition 2.35. By Proposition 2.39, the sheaf of invariants $\mathcal{R}^\sigma$ is a Cox sheaf of X of type λ such that $\mathcal{R}_{\overline{k}} \cong \mathcal{R}^\sigma \otimes_k \overline{k}$. The cocycle associated with this isomorphism is $(\mathrm{id}_{\mathcal{R}_{\overline{k}}} \circ \psi_{\sigma_{\mathrm{g}}} \circ \mathrm{g} \circ \mathrm{id}_{\mathcal{R}_{\overline{k}}} \circ \mathrm{g}^{-1})_{\mathrm{g} \in \mathfrak{g}} = \sigma$. □

Remark 2.42. As a consequence of Corollary 2.40 and Proposition 2.41, the map in Proposition 2.41 defines a bijection between the set of isomorphism classes of Cox rings of X of type λ and $H^1(k, \widehat{M_{\mathrm{eff}}})$.

The inverse map to the bijection in Proposition 2.41 is obtained by twisting a Cox sheaf of X of type λ by cocycles. Hence, we introduce the notion of twisted generalized Cox sheaf.

Definition 2.43. For every Cox sheaf $\mathcal{R}$ of X of type λ and every cocycle $\sigma : \mathfrak{g} \to \widehat{M}(\overline{k})$, we denote by $\mathcal{R}^\sigma$ the *twisted Cox sheaf* constructed in the proof of Proposition 2.41.

2.3.3. Relation to torsors under quasitori. Now we explore the connection between Cox sheaves of X of type λ and X-torsors of the same type, and prove that they are equivalent notions. We start by showing that the relative spectrum functor Spec_X sends generalized Cox sheaves to torsors of the same type and is compatible with twisting.

Proposition 2.44. *If $\mathcal{R}$ is a Cox sheaf of X of type λ, then* $\mathrm{Spec}_X \mathcal{R}$ *is an X-torsor of type λ. Moreover, for each cocycle $\sigma : \mathfrak{g} \to \widehat{M}(\overline{k})$, the X-torsor* $\mathrm{Spec}_X \mathcal{R}^\sigma$ *has class* $[\mathrm{Spec}_X \mathcal{R}] - [\sigma]$ *in* $H^1(X, \widehat{M}_X)$.

Proof. Let $\mathcal{R}$ be a Cox sheaf of X of type λ. Since $\mathcal{R}_{\overline{k}}$ is a $\mathfrak{g}$-equivariant Cox sheaf of $X_{\overline{k}}$, the action of $\widehat{M}$ on $\mathcal{R}_{\overline{k}}$ induced by the M-grading descends to an action of $\widehat{M}^{\mathfrak{g}} = \mathrm{Spec}\,\overline{k}[M]^{\mathfrak{g}}$ on $\mathcal{R}$, where $\overline{k}[M]^{\mathfrak{g}}$ is the subring of $\mathfrak{g}$-invariant elements of $\overline{k}[M]$, and the canonical morphism $\mathrm{Spec}_X \mathcal{R} \to X$ is an X-torsor of type λ by Proposition 2.14 and *fpqc*-descent.

Let $\sigma : \mathfrak{g} \to \widehat{M}(\overline{k})$ be a cocycle. By Proposition 2.41, the twisted Cox sheaf $\mathcal{R}^\sigma$ corresponds to the $\mathfrak{g}$-equivariant Cox sheaf $\mathcal{R}_{\overline{k}}$ of $X_{\overline{k}}$ with the twisted action of $\mathfrak{g}$ given by $(\mathrm{g}, s) \mapsto \sigma_{\mathrm{g}}(\mathrm{g}(s))$ under the bijection of Proposition 2.39. Thus, $\mathrm{Spec}_X \mathcal{R}^\sigma$ is obtained by Galois descent from $\mathrm{Spec}_{X_{\overline{k}}} \mathcal{R}_{\overline{k}}$ with the twisted $\mathfrak{g}$-action $(\mathrm{g}, x) \mapsto \sigma_{\mathrm{g}}^{-1}(\mathrm{g}(x))$. Therefore, $[\mathrm{Spec}_X \mathcal{R}^\sigma] = [\mathrm{Spec}_X \mathcal{R}] - [\sigma]$ in $H^1(X, \widehat{M}_X)$ (cf. [**Sko01**, Example 2, p. 21]). □

The next proposition shows that the direct image of the structure sheaf of an X-torsor of type λ under the torsor morphism is a generalized Cox sheaf of the same type, and that the functor Spec_X in the previous proposition is essentially surjective.

PROPOSITION 2.45. *If $\pi : Y \to X$ is an X-torsor of type λ, then $\pi_*\mathcal{O}_Y$ is a Cox sheaf of X of type λ, $\mathcal{O}_Y(Y)$ is a Cox ring of X of type λ, and* $\operatorname{Spec}_X \pi_*\mathcal{O}_Y \cong Y$.

PROOF. Let $\pi : Y \to X$ be an X-torsor of type λ. Let $\overline{\pi} : Y_{\overline{k}} \to X_{\overline{k}}$ be the induced $X_{\overline{k}}$-torsor of type λ. Then $\overline{\pi}_*\mathcal{O}_{Y_{\overline{k}}}$ is a Cox sheaf of $X_{\overline{k}}$ of type λ by Proposition 2.14. The induced $\mathfrak{g}$-action on $\overline{\pi}_*\mathcal{O}_{Y_{\overline{k}}}$ is a natural $\mathfrak{g}$-action in the sense of Definition 2.35, as it is given by automorphisms that are compatible with the $\mathfrak{g}$-action on M. Hence, $\pi_*\mathcal{O}_Y$ is a Cox sheaf of X of type λ according to Definition 2.38, and $\mathcal{O}_Y(Y) = (\pi_*\mathcal{O}_Y)(X)$ is a Cox ring of X of type λ by Corollary 2.40. Moreover, π is affine by [**MFK94**, Proposition 0.7]. Hence, $\operatorname{Spec}_X \pi_*\mathcal{O}_Y \cong Y$. $\square$

The relation between generalized Cox sheaves and torsors under quasitori is summarized by the following corollary.

COROLLARY 2.46. *The contravariant functor*

$$\begin{aligned}\{\textit{Cox sheaves of } X \textit{ of type } \lambda\} &\longrightarrow \{X\textit{-torsors of type } \lambda\}\\ \mathcal{R} &\longmapsto \operatorname{Spec}_X \mathcal{R}\end{aligned}$$

is an anti-equivalence of categories with inverse functor

$$\begin{aligned}\{X\textit{-torsors of type } \lambda\} &\longrightarrow \{\textit{Cox sheaves of } X \textit{ of type } \lambda\}\\ \pi : Y \to X &\longmapsto \pi_*\mathcal{O}_Y.\end{aligned}$$

PROOF. By Propositions 2.44 and 2.45, both functors are well-defined and essentially surjective. Moreover, the functor Spec_X is fully faithful. $\square$

2.3.4. Existence. Now we turn to existence criteria for generalized Cox sheaves and rings of varieties over nonclosed fields. We start with an example that shows that Cox rings of identity type do not always exist.

EXAMPLE 2.47. Let $X \subset \mathbb{P}^2_{\mathbb{R}}$ be the conic defined by $x^2 + y^2 + z^2 = 0$, with $X(\mathbb{R}) = \emptyset$. It is the image of the closed immersion

$$\psi : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}}, \qquad (u : v) \mapsto (2uv : u^2 - v^2 : \mathrm{i}(u^2 + v^2)).$$

A Cox ring of $\mathbb{P}^1_{\mathbb{C}}$ of type $\mathrm{id}_{\operatorname{Pic}(\mathbb{P}^1_{\mathbb{C}})}$ is $\mathbb{C}[u, v]$, where $u, v \in H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(1))$ vanish in $(0 : 1), (1 : 0)$, respectively. If X has a Cox ring R of type $\mathrm{id}_{\operatorname{Pic}(X_{\mathbb{C}})}$ over $\mathbb{R}$, then $R_{\mathbb{C}}$ is endowed with a natural action of $\mathfrak{g} = \operatorname{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \mathrm{g}\}$. Via the isomorphism ψ, this action pulls back to an action of $\mathfrak{g}$ on the Cox ring $\mathbb{C}[u, v]$ of $\mathbb{P}^1_{\mathbb{C}}$ with the following properties. Since g exchanges $(0 : 1 : \mathrm{i}) = \psi((1 : 0))$ and $(0 : 1 : -\mathrm{i}) = \psi((0 : 1))$, there exists $\alpha \in \mathbb{C}^\times$ such that $\mathrm{g}(u) = \alpha v$. Then $\mathrm{g}^2(u) = u$ gives $\mathrm{g}(v) = \mathrm{g}(\alpha)^{-1} u$. Furthermore, g exchanges $(1 : 0 : \mathrm{i}) = \psi((1 : 1))$ and $(1 : 0 : -\mathrm{i}) = \psi((1 : -1))$, hence $\mathrm{g}(u - v) = \alpha v - \mathrm{g}(\alpha)^{-1} u$ should be a scalar multiple of $u + v$. This implies $\alpha = -\mathrm{g}(\alpha)^{-1}$, which is impossible for $\alpha \in \mathbb{C}^\times$. Therefore, the conic X without $\mathbb{R}$-rational points does not have a Cox ring of type $\mathrm{id}_{\operatorname{Pic}(X_{\mathbb{C}})}$.

REMARK 2.48. By Proposition 2.41, the existence of a Cox sheaf, or Cox ring, of a given type λ implies that all Cox sheaves (or rings) of $X_{\overline{k}}$ of type λ are $\mathfrak{g}$-equivariant. Therefore, we say that Cox sheaves (or rings) of X of type λ exist (cf. (2.2)).

The next proposition relates the existence of Cox sheaves to the existence of $\mathfrak{g}$-equivariant characters in Construction 2.7.

PROPOSITION 2.49. *Let $\mathcal{M} \subseteq M$ be a finite $\mathfrak{g}$-invariant set of generators for M, and $\mathcal{D}$ a finite $\mathfrak{g}$-invariant set of Cartier divisors such that $\lambda(\mathcal{M}) = \{[D] : D \in \mathcal{D}\}$. Let*

$$\mathcal{M}_\lambda := \{(m, D) \in \mathcal{M} \times \mathcal{D} : [D] = \lambda(m)\}$$

with the componentwise action of $\mathfrak{g}$, and let $\Lambda := \bigoplus_{L \in \mathcal{M}_\lambda} \mathbb{Z}L$ with the induced $\mathfrak{g}$-action. Let $\varphi : \Lambda \to M$ be defined by $\varphi((m, D)) = m$ for all $(m, D) \in \Lambda$, and $\mathcal{S} := \bigoplus_{(m,D)\in\Lambda} \mathcal{O}_{X_{\overline{k}}}(D)$ the associated $\mathcal{O}_{X_{\overline{k}}}$-algebra as in Construction 2.7.

Then a Cox sheaf of X of type λ exists if and only if there is a $\mathfrak{g}$-equivariant character associated with $\mathcal{S}$.

PROOF. Let Λ_0 be the kernel of φ, and let $\mathcal{S}_{(m,D)} := \mathcal{O}_{X_{\overline{k}}}(D)$ for all $(m, D) \in \Lambda$. Given a $\mathfrak{g}$-equivariant character $\chi : \Lambda_0 \to \overline{k}(X_{\overline{k}})^\times$ associated with $\mathcal{S}$. The sheaf $\mathcal{I}$ of ideals of $\mathcal{S}$ defined by χ is invariant under the action of $\mathfrak{g}$ on $\mathcal{S}$ induced by the natural $\mathfrak{g}$-action on $\overline{k}(X_{\overline{k}})$. Therefore, $\mathcal{R} := \mathcal{S}/\mathcal{I}$ is a $\mathfrak{g}$-equivariant Cox sheaf of $X_{\overline{k}}$ of type λ, and the sheaf of invariants $\mathcal{R}^{\mathfrak{g}}$ is a Cox sheaf of X of type λ.

Assume now that $\mathcal{R}$ is a Cox sheaf of X of type λ. Then $\mathcal{R}_{\overline{k}}$ is a $\mathfrak{g}$-equivariant Cox sheaf of $X_{\overline{k}}$ of type λ. By Proposition 2.10, we can assume that $\mathcal{R}_{\overline{k}} = \mathcal{S}/\mathcal{I}$, where $\mathcal{I}$ is the sheaf of ideals of $\mathcal{S}$ defined by a character $\chi : \Lambda_0 \to \overline{k}(X_{\overline{k}})^\times$ associated with $\mathcal{S}$. Let $\pi : \mathcal{S} \to \mathcal{R}_{\overline{k}}$ be the projection.

Let $\phi_{m,D}$, for $(m, D) \in M_\lambda$, be a family of isomorphisms associated with $\mathcal{R}_{\overline{k}}$. Without loss of generality, we can assume that $\phi_{m,D} = \pi|_{\mathcal{S}_{(m,D)}}^{-1}$ for all $(m, D) \in \Lambda$. By Definition 2.35 and Lemma 2.4, for every $\mathrm{g} \in \mathfrak{g}$ and every $(m, D) \in \Lambda$, there exists a constant $\alpha_{\mathrm{g},m,D} \in \overline{k}^\times$ such that

$$(\pi|_{\mathcal{S}_{(\mathrm{g}m,\mathrm{g}D)}}^{-1} \circ \mathrm{g} \circ \pi|_{\mathcal{S}_{(m,D)}})(s) = \alpha_{\mathrm{g},m,D}\mathrm{g}(s)$$

for all sections s of $\mathcal{S}_{(m,D)} = \mathcal{O}_{X_{\overline{k}}}(D)$. These constants satisfy

$$\alpha_{\mathrm{g},m+m',D+D'} = \alpha_{\mathrm{g},m,D}\alpha_{\mathrm{g},m',D'} \qquad \alpha_{\mathrm{g}\mathrm{g}',m,D} = \alpha_{\mathrm{g},\mathrm{g}'m,\mathrm{g}'D}\mathrm{g}(\alpha_{\mathrm{g}',m,D})$$

for all $(m, D), (m', D') \in \Lambda$ and all $\mathrm{g}, \mathrm{g}' \in \mathfrak{g}$. Moreover, $\chi((0, \mathrm{g}D)) = \alpha_{\mathrm{g},0,-D}\mathrm{g}(\chi((0, D)))$ for all $(0, D) \in \Lambda_0$. Fix a system $\mathcal{M}'_\lambda \subseteq \mathcal{M}_\lambda$ of representatives for the orbits under the action of $\mathfrak{g}$. For every $(m, D) \in \mathcal{M}'_\lambda$ and every $\mathrm{g} \in \mathfrak{g}$, let $\beta_{(\mathrm{g}m,\mathrm{g}D)} := \alpha_{\mathrm{g},m,D}$. Given $L' \in \Lambda$, write $L' = \sum_{L\in\mathcal{M}_\lambda} a_L L$ with $a_L \in \mathbb{Z}$, and define

$$\beta_{L'} := \prod_{L\in\mathcal{M}_\lambda} \beta_L^{a_L}.$$

The homomorphism $\chi' : \Lambda_0 \to \overline{k}(X_{\overline{k}})^\times$ that sends $E \in \Lambda_0$ to $\beta_E\chi(E)$ is $\mathfrak{g}$-equivariant and satisfies $\mathrm{div}_0(\chi'((0, D))) = D$ for all $(0, D) \in \Lambda_0$. $\square$

From [**CTS87**, Proposition 2.2.8] we know that the existence of universal torsors (that is, torsors of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$) for a *smooth* variety X over a nonclosed field k is equivalent to the existence of a $\mathfrak{g}$-equivariant splitting of the exact sequence

$$1 \to \overline{k}^\times \to \overline{k}(X_{\overline{k}})^\times \to \overline{k}(X_{\overline{k}})^\times/\overline{k}^\times \to 1. \tag{2.5}$$

Moreover, if U is a nonempty open subset of X, and M is the kernel of the natural homomorphism $\mathrm{Pic}(X_{\overline{k}}) \to \mathrm{Pic}(U_{\overline{k}})$, the existence of torsors of injective type $M \subseteq \mathrm{Pic}(X_{\overline{k}})$ is equivalent to the existence of a $\mathfrak{g}$-equivariant splitting of the exact sequence

$$1 \to \overline{k}^\times \to \overline{k}[U_{\overline{k}}]^\times \to \overline{k}[U_{\overline{k}}]^\times/\overline{k}^\times \to 1. \tag{2.6}$$

As a consequence of Corollaries 2.40 and 2.46 and of Remark 2.48, this is also equivalent to the existence of Cox sheaves (and Cox rings if $M = M_{\mathrm{eff}}$) of X of the same type.

In our more general setting, where X can be singular, we show how to construct a Cox sheaf of X of type λ starting from $\mathfrak{g}$-equivariant splittings of such exact sequences. Then, Proposition 2.44 gives a torsor of X of type λ in this setting, generalizing the above results of [**CTS87**].

Proposition 2.50. *(i) If* (2.5) *admits a $\mathfrak{g}$-equivariant splitting, then Cox sheaves and Cox rings of X of type λ exist.*

(ii) If there exists a $\mathfrak{g}$-equivariant splitting of the exact sequence (2.6) *associated with a nonempty open subset U of X such that $\lambda(M)$ is contained in the kernel of the natural morphism* $\mathrm{Pic}(X_{\overline{k}}) \to \mathrm{Pic}(U_{\overline{k}})$*, then Cox sheaves and Cox rings of X of type λ exist.*

Proof. We prove *(i)*. Let $\sigma : k(X)^\times \to k^\times$ be a $\mathfrak{g}$-equivariant splitting of (2.5). For $m \in M$, define D_m and $\mathcal{R}_m$ as in Construction 2.21. Then the Cox sheaf $\mathcal{R} := \bigoplus_{m\in M} \mathcal{R}_m$ of $X_{\overline{k}}$ of type λ defined in Construction 2.21 is equivariant with respect to the following action of $\mathfrak{g}$. For every $\mathrm{g} \in \mathfrak{g}$ and $m \in M$, let $f \in \overline{k}(X_{\overline{k}})^\times$ be the unique element such that $D_{\mathrm{g}m} = \mathrm{div}_{\mathrm{g}D_m}(f)$ and $\sigma(f) = 1$. Define $\mathrm{g} * s := f\mathrm{g}(s) \in \mathcal{R}_{\mathrm{g}m}$ for all sections $s \in \mathcal{R}_m$, where the product on the right is computed in $\overline{k}(X_{\overline{k}})$. The sheaf of invariants $\mathcal{R}^{\mathfrak{g}}$ is then a Cox sheaf of X of type λ by Proposition 2.39, and $\mathcal{R}^{\mathfrak{g}}(X)$ is a Cox ring of X of type λ by Corollary 2.40. The proof of *(ii)* is similar using Construction 2.22. ☐

Another way to produce generalized Cox sheaves and rings of varieties over nonclosed fields is by pulling-back a given one.

Remark 2.51. If X has a Cox sheaf of type λ and $\varphi : M' \to M$ is a morphism of finitely generated $\mathfrak{g}$-modules, then the pull-back of a $\mathfrak{g}$-equivariant Cox sheaf $\mathcal{R}$ of $X_{\overline{k}}$ of type λ under φ,

$$\varphi^*\mathcal{R} := \bigoplus_{m\in M'} \mathcal{R}_{\varphi(m)},$$

defined in Proposition 2.16 inherits from $\mathcal{R}$ and action of $\mathfrak{g}$ that turns it into a $\mathfrak{g}$-equivariant Cox sheaf. Hence, Cox sheaves of X of type $\lambda \circ \varphi$ exist by Proposition 2.39. In particular, if X has a Cox sheaf of injective type $M'' \subseteq \mathrm{Pic}(X_{\overline{k}})$ for any $\mathfrak{g}$-invariant subgroup M'' of $\mathrm{Pic}(X_{\overline{k}})$ such that

$\lambda(M) \subseteq M''$, then Cox sheaves of X of type λ exist. Therefore, if X has a Cox sheaf of identity type, then all Cox sheaves of X of all types exist.

We show now that a splitting of (2.6) is equivalent to the existence of certain Cox sheaves of X of injective type, at least if $X_{\overline{k}}$ is locally factorial (cf. [**CTS87**, Proposition 2.2.8 (v)]).

PROPOSITION 2.52. *If $X_{\overline{k}}$ is locally factorial, and X has a Cox sheaf of injective type λ, then the natural exact sequence* (2.6) *has a $\mathfrak{g}$-equivariant splitting for every nonempty open subset $U \subseteq X$ such that $\lambda(M)$ is the kernel of the natural morphism* $\mathrm{Pic}(X_{\overline{k}}) \to \mathrm{Pic}(U_{\overline{k}})$. *If in addition k is perfect and $\lambda = \mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$, also the exact sequence* (2.5) *admits a $\mathfrak{g}$-equivariant splitting.*

PROOF. Let $U \subseteq X$ be a nonempty open subset such that $\lambda(M)$ is contained in the kernel of the natural morphism $\mathrm{Pic}(X_{\overline{k}}) \to \mathrm{Pic}(U_{\overline{k}})$. Then the group Λ of Cartier divisors on $X_{\overline{k}}$ supported outside $U_{\overline{k}}$ is free, finitely generated and has a $\mathfrak{g}$-invariant basis (consisting of the prime divisors supported outside $U_{\overline{k}}$). The group $\overline{k}[U_{\overline{k}}]^\times/\overline{k}^\times$ is naturally identified with the subgroup $\Lambda_0 \subseteq \Lambda$ of principal divisors. By Proposition 2.49, there exists a $\mathfrak{g}$-equivariant character $\chi : \Lambda_0 \to \overline{k}(X_{\overline{k}})^\times$ associated with $\mathcal{S} := \bigoplus_{D\in\Lambda} \mathcal{O}_X(D)$. Since every element of Λ_0 is supported outside $U_{\overline{k}}$, the image of χ is contained in $\overline{k}[U_{\overline{k}}]^\times$. Hence, χ is a $\mathfrak{g}$-equivariant splitting of the exact sequence (2.6) associated with U.

Assume now that k is perfect and that $\lambda = \mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$. Let X' be the smooth locus of $X_{\overline{k}}$ and $U' = U_{\overline{k}} \cap X'$. Since $X_{\overline{k}}$ is normal, its singular locus has codimension ≥ 2. Hence, $\overline{k}(X_{\overline{k}}) = \overline{k}(X')$ and $\overline{k}[U_{\overline{k}}] = \overline{k}[U']$. Then, according to [**CTS87**, Proposition 2.2.8], the exact sequence (2.5) admits a $\mathfrak{g}$-equivariant splitting whenever (2.6) does.

□

REMARK 2.53. As in Remark 2.23, a k-rational point on U defines a $\mathfrak{g}$-equivariant splitting of (2.6). Moreover, every smooth k-rational point on X defines a $\mathfrak{g}$-equivariant splitting of (2.5) by [**CTS87**, Remarque 2.2.3]. Therefore, if X has a smooth k-rational point, Cox sheaves and Cox rings of X of any type exist by Proposition 2.50. If $X(k) \neq \emptyset$, the same result can be obtained combining Remark 2.1 with Proposition 2.45.

The existence of a k-rational point on X allows us to construct explicitly a Cox sheaf of X of type λ as follows (cf. [**ADHL15**, Construction 1.4.2.3]).

CONSTRUCTION 2.54. *Let $x \in X(k)$. Let $\mathcal{M} \subseteq M$ be a finite $\mathfrak{g}$-invariant set of generators for M, and $\mathcal{D}$ a finite $\mathfrak{g}$ invariant set of Cartier divisors of $X_{\overline{k}}$ supported outside x such that $\lambda(\mathcal{M}) = \{[D] : D \in \mathcal{D}\}$. Let*

$$\mathcal{M}_\lambda := \{(m, D) \in \mathcal{M} \times \mathcal{D} : [D] = \lambda(m)\}$$

with the componentwise action of $\mathfrak{g}$, and let $\Lambda := \bigoplus_{L\in\mathcal{M}_\lambda} \mathbb{Z}L$ with the induced $\mathfrak{g}$-action. Let $\varphi : \Lambda \to M$ be defined by $\varphi((m, D)) = m$ for all $(m, D) \in \Lambda$, and $\mathcal{S} := \bigoplus_{(m,D)\in\Lambda} \mathcal{O}_{X_{\overline{k}}}(D)$ the associated $\mathcal{O}_{X_{\overline{k}}}$-algebra as in Construction 2.7. Let Λ_0 be the kernel of φ. The morphism $\chi : \Lambda_0 \to \overline{k}(X_{\overline{k}})^\times$ that sends $(0, D) \in \Lambda_0$ to the unique element $f \in \overline{k}(X_{\overline{k}})^\times$ such that $\mathrm{div}_0(f) = D$ *and $f(x) = 1$ is a $\mathfrak{g}$-equivariant character associated with $\mathcal{S}$. Let $\mathcal{I}$ be the* *sheaf of ideals of $\mathcal{S}$ defined by χ as in Construction 2.7. Then the Cox sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ is $\mathfrak{g}$-equivariant by Proposition 2.49. Let $\pi : \operatorname{Spec}_X \mathcal{R}^{\mathfrak{g}} \to X$ be the induced torsor of type λ, then $x \in \pi((\operatorname{Spec}_X \mathcal{R}^{\mathfrak{g}})(k))$.*

Proof. The set of Cartier divisors of $X_{\overline{k}}$ that do not contain x in their support form a $\mathfrak{g}$-invariant group that generates $\operatorname{Pic}(X_{\overline{k}})$. Indeed: if $D = \{(U_i, f_i)\}_i$ is a Cartier divisor on $X_{\overline{k}}$ that contains x in its support, take j such that $x \in U_j$. Then the divisor $\operatorname{div}_D(f_j^{-1})$ is linearly equivalent to D and is supported outside x. Therefore, it is always possible to choose a set $\mathcal{D}$ as in the statement. The character χ defined by x as above is $\mathfrak{g}$-equivariant because x is $\mathfrak{g}$-invariant.

Let U be an affine open neighborhood of x in X. Then the homomorphism $\psi : \mathcal{S}(U_{\overline{k}}) \to \overline{k}$ defined by $\psi(s) := s(x)$ for all homogeneous sections $s \in \mathcal{S}(U_{\overline{k}})$ is well defined because D is supported outside x for all $(m, D) \in \Lambda$, and $\mathfrak{g}$-equivariant because x is $\mathfrak{g}$-invariant. Since $\chi(E)(x) = 1$ for all $E \in \Lambda_0$, the homomorphism ψ factors through $\mathcal{R}(U_{\overline{k}})$ and defines a k-rational point on $\pi^{-1}(x)$ by Galois descent. □

The following example shows that the existence of a k-rational point on X is not necessary for the existence of Cox rings of X of arbitrary type. We recall that if the $\mathfrak{g}$-action on M is trivial, a Cox ring of X of type $\lambda : M \to \operatorname{Pic}(X_{\overline{k}})$ always exists.

Example 2.55. Let k be an arbitrary number field. Let X be the smooth projective fourfold over k in [**Sme14**, Theorem 3.6]. Then $X(k) = \emptyset$ and there is no étale (and hence, no algebraic) Brauer–Manin obstruction to the Hasse principle. Moreover, $\operatorname{Pic}(X_{\overline{k}})$ is a finitely generated abelian group, as the Albanese variety of X is trivial. Therefore, X has a universal torsor by [**Sko01**, Corollary 6.1.3], and Cox rings and Cox sheaves of X of all types exist by Proposition 2.45 and Remark 2.51.

2.3.5. Functoriality. We discuss now the functoriality properties of generalized Cox sheaves and rings of varieties over an arbitrary field. We refer to [**CTS87**] and [**BH03**] for the analogous properties of torsors under quasitori and of Cox rings of identity type, respectively.

By [**CTS87**, Proposition 1.5.2], the exact sequence (2.2) is functorial in X and G, covariant in G and contravariant in X. If we denote by M the $\mathfrak{g}$-module $\widehat{G_{\overline{k}}}$ dual to G under (2.1), the exact sequence (2.2) is functorial in X and M, contravariant with respect to both.

We consider first the functoriality with respect to morphisms of quasitori $G \to G'$, or equivalently, with respect to the induced morphism of dual $\mathfrak{g}$-modules $\varphi : M' \to M$. If $\mathcal{R}$ is a Cox sheaf of X of type $\lambda : M \to \operatorname{Pic}(X_{\overline{k}})$, the pull-back $\varphi^*\mathcal{R} := (\varphi^*\mathcal{R}_{\overline{k}})^{\mathfrak{g}}$ defined in Remark 2.51 is endowed with a natural morphism of graded $\mathcal{O}_X$-algebras compatible with φ

$$\varphi^*\mathcal{R} \to \mathcal{R}, \tag{2.7}$$

obtained by Galois descent from the $\mathfrak{g}$-equivariant morphism $\varphi^*\mathcal{R}_{\overline{k}} \to \mathcal{R}_{\overline{k}}$ that restricts to the identity $(\varphi^*\mathcal{R}_{\overline{k}})_m = (\mathcal{R}_{\overline{k}})_{\varphi(m)}$ for all $m \in M'$. Moreover, each morphism $\psi : \mathcal{R} \to \mathcal{R}'$ of Cox sheaves of X of type λ pulls back under φ to a morphism $\varphi^*(\psi) : \varphi^*\mathcal{R} \to \varphi^*\mathcal{R}'$ of Cox sheaves of X of type $\lambda \circ \varphi$

such that the following diagram is commutative

$$\begin{array}{ccc} \varphi^*\mathcal{R} & \xrightarrow{\varphi^*(\psi)} & \varphi^*\mathcal{R}' \\ \downarrow & & \downarrow \\ \mathcal{R} & \xrightarrow{\psi} & \mathcal{R}'. \end{array}$$

The pull-back of Cox sheaves under φ corresponds to the push-forward of torsors defined in [**Sko01**, Example 3, p. 21] under the dual morphism of quasitori $\widehat{\varphi} : G \to G'$. In particular, $\widehat{\varphi}_* \operatorname{Spec}_X \mathcal{R} \cong \operatorname{Spec}_X \varphi^*\mathcal{R}$.

We consider now the functoriality with respect to morphisms of k-varieties. Let $p : X' \to X$ be a morphism of k-varieties and $\mathcal{R}$ a Cox sheaf of X of type $\lambda : M \to \operatorname{Pic}(X_{\overline{k}})$. We denote by $p^* : \operatorname{Pic}(X_{\overline{k}}) \to \operatorname{Pic}(X'_{\overline{k}})$ the pull-back of divisor classes under p. We recall that the property of being a torsor over X under G is stable under base extension. Therefore, $\operatorname{Spec}_{X'} p^*\mathcal{R} \cong X' \times_X \operatorname{Spec}_X \mathcal{R}$ is an X'-torsor under $\widehat{M^{\mathfrak{g}}}$.

Proposition 2.56. *Let $p : X' \to X$ be a morphism of k-varieties, $p^* : \operatorname{Pic}(X_{\overline{k}}) \to \operatorname{Pic}(X'_{\overline{k}})$ the induced pull-back of divisor classes, and $\mathcal{R}$ a Cox sheaf of X of type $\lambda : M \to \operatorname{Pic}(X_{\overline{k}})$. Then, $p^*\mathcal{R}$ is a Cox sheaf of X' of type $p^* \circ \lambda : M \to \operatorname{Pic}(X'_{\overline{k}})$.*

Proof. If $\{\phi_{m,D}\}_{(m,D)\in M_\lambda}$ is a family of isomorphisms associated with the Cox sheaf $\mathcal{R}$, then the family of isomorphisms

$$p^*(\phi_{m,D}) : p^*\mathcal{R}_m \to p^*\mathcal{O}_{X'}(D),$$

for $(m, D) \in M_\lambda$, defines a structure of Cox sheaf of X' of type $p^* \circ \lambda$ on $p^*\mathcal{R}$, as p^* commutes with direct limits. □

We summarize the results of this subsection in the following corollary.

Corollary 2.57. *Let $p : X' \to X$ be a morphism of k-varieties, and $\varphi : M' \to M$ a morphism of $\mathfrak{g}$-modules.*

(1) If $\mathcal{R}$ is a Cox sheaf of X of type $\lambda : M \to \operatorname{Pic}(X_{\overline{k}})$, then $p^*\varphi^*\mathcal{R}$ is a Cox sheaf of X' of type

$$p^* \circ \lambda \circ \varphi : M' \to \operatorname{Pic}(X'_{\overline{k}}).$$

(2) If $\psi : \mathcal{R} \to \mathcal{R}'$ is a morphism of Cox sheaves of X of type λ, then

$$p^*(\varphi^*(\psi)) : p^*\varphi^*\mathcal{R} \to p^*\varphi^*\mathcal{R}'$$

is a morphism of Cox sheaves of X' of type $p^* \circ \lambda \circ \varphi$.

(3) The the double pull-back $p^* \circ \varphi^*$ is a covariant functor from the category of Cox sheaves of X of type λ to the category of Cox sheaves of X' of type $p^* \circ \lambda \circ \varphi$.

(4) There is an isomorphism of functors between $p^* \circ \varphi^*$ and $\varphi^* \circ p^*$.

Proof. Parts *(1)* and *(2)* are a consequence of Proposition 2.56 and the discussion above.

For *(3)* we observe that φ^* and p^* are both covariant functors. For *(4)*, we observe that $p^*\varphi^*\mathcal{R} = \varphi^*p^*\mathcal{R}$ since p^* commutes with direct limits, and hence, also $p^*(\varphi^*(\psi)) = \varphi^*(p^*(\psi))$, as $\varphi^*(\psi)$ and $\varphi^*(p^*(\psi))$ restrict to ψ and $p^*(\psi)$, respectively, on the homogeneous parts of the Cox sheaves. $\square$

REMARK 2.58. The analogous functoriality properties of generalized Cox rings can be deduced from Corollary 2.57 via Corollary 2.40.

REMARK 2.59. As in Corollary 2.57, let $p : X' \to X$ be a morphism of k-varieties, assume that X' has a Cox sheaf $\mathcal{R}'$ of identity type, and let $\mathcal{R}$ be a Cox sheaf of X of type λ. Up to twisting $\mathcal{R}'$ by an element of $H^1(k, \widehat{M})$ we can assume that $p^*\mathcal{R} \cong (p^* \circ \lambda)^*\mathcal{R}'$. Then there exists a natural morphism of graded $\mathcal{O}_{X'}$-algebras $p^*\mathcal{R} \to \mathcal{R}'$ as in (2.7), and also a morphism of graded $\mathcal{O}_X$-algebras $\mathcal{R} \to p_*\mathcal{R}'$, as p^* and p_* are adjoint functors. This gives a new proof of the existence statement in [**BH03**, Proposition 5.3] by considering $k = \overline{k}$ and $\lambda = \mathrm{id}_{\mathrm{Pic}(X)}$. We observe that in general $p_*\mathcal{R}'$ is not a generalized Cox sheaf of X.

2.4. Finitely generated generalized Cox rings

We dedicate this section to investigate some properties of generalized Cox rings that are finitely generated as k-algebras, with the purpose of realizing the corresponding torsors under quasitori as locally closed subsets of finite dimensional affine spaces. We first observe that the pull-back of a generalized Cox ring under a morphism of grading groups, which we define in the remark below by analogy with the pull-back of generalized Cox sheaves, preserves finite generation.

REMARK 2.60. Assume that $k = \overline{k}$. Let M' be a finitely generated abelian group and assume that the characteristic of k does not divide the order of the torsion subgroup of M'. Using Proposition 2.16 and Proposition 2.31 we define the pull-back of a Cox ring R of X of type λ under a group homomorphism $\varphi : M' \to M$ to be $\varphi^*R := \bigoplus_{m \in M'} R_{\varphi(m)}$. If R is a finitely generated k-algebra, then also φ^*R is a finitely generated k-algebra by [**ADHL15**, Proposition 1.1.2.4].

2.4.1. Embedded torsors. From [**ADHL15**, Construction 1.6.3.1] (cf. [**BH03**, Proposition 3.10], we know that given a variety X over an algebraically closed field k with a Cox sheaf $\mathcal{R}$ of identity type such that $\mathcal{R}(X)$ is a finitely generated k-algebra, then the universal torsor $\mathrm{Spec}_X \mathcal{R}$ embeds into $\mathrm{Spec}\, \mathcal{R}(X)$ as an open subset, under some assumptions on X, which are satisfied, for example, if X is locally factorial or projective. The next proposition gives an analogous result for finitely generated generalized Cox rings and arbitrary torsors under quasitori. We first describe the action on a torsor of type λ induced by the grading on the associated generalized Cox sheaf.

REMARK 2.61. We recall from Proposition 2.44 that, given a Cox sheaf $\mathcal{R}$ of X of type λ, the relative spectrum $\mathrm{Spec}_X \mathcal{R}$ is an X-torsor of type λ under the action of $\widehat{M}^{\mathfrak{g}}$ on $\mathrm{Spec}_X \mathcal{R}$ induced by the M-grading on $\mathcal{R}_{\overline{k}}$. In particular, $\widehat{M}^{\mathfrak{g}} = \mathrm{Spec}\, \overline{k}[M]^{\mathfrak{g}}$ and the action induced by the M-grading is determined by the $\mathfrak{g}$-equivariant $\widehat{M}$-action on $\mathrm{Spec}_{X_{\overline{k}}} \mathcal{R}_{\overline{k}}$ defined by

$$\mathcal{R}_{\overline{k}}(U) \to \overline{k}[M] \otimes_{\overline{k}} \mathcal{R}_{\overline{k}}(U), \quad \sum_{i=1}^{n} s_i \mapsto \sum_{i=1}^{n} m_i \otimes s_i$$

for all open subsets U of $X_{\overline{k}}$ and all homogeneous sections $s_i \in \mathcal{R}_{\overline{k}}(U)_{m_i}$. We observe that this defines also an action of $\widehat{M}^{\mathfrak{g}}$ on $\mathrm{Spec}\,\mathcal{R}(X)$, and that the natural morphism $\mathrm{Spec}_X \mathcal{R} \to \mathrm{Spec}\,\mathcal{R}(X)$ is $\widehat{M}^{\mathfrak{g}}$-equivariant.

Proposition 2.62. *Assume that $k = \overline{k}$, that $\mathcal{R}$ is a Cox sheaf of X of type λ such that $\mathcal{R}(X)$ is finitely generated as a k-algebra, and that there are nonzero homogeneous sections $f_1, \dots, f_t \in \mathcal{R}(X)$ of degrees $m_1, \dots, m_t \in M$, respectively, such that the open subsets $X \smallsetminus \mathrm{Supp}(\mathrm{div}_{D_i}(\phi_{m_i,D_i}(f_i))$ are affine and cover X. Then the natural morphism $\mathrm{Spec}_X \mathcal{R} \to \mathrm{Spec}\,\mathcal{R}(X)$ is a $\widehat{M}$-equivariant open immersion and the complement of the image is defined by the ideal $\sqrt{(f_1, \dots, f_t)}$ of $\mathcal{R}(X)$.*

Proof. Let $\pi : \mathrm{Spec}_X \mathcal{R} \to X$ be the morphism induced by $\mathcal{O}_X \subseteq \mathcal{R}$. The open subsets $\pi^{-1}(X \smallsetminus \mathrm{Supp}(\mathrm{div}_{D_i}(\phi_{m_i,D_i}(f_i)))$ are affine and cover $\mathrm{Spec}_X \mathcal{R}$. Moreover, $\mathcal{R}(X \smallsetminus \mathrm{Supp}(\mathrm{div}_{D_i}(\phi_{m_i,D_i}(f_i))) = \mathcal{R}(X)[f_i^{-1}]$ for all $i \in \{1, \dots, t\}$. Hence, $\mathrm{Spec}_X \mathcal{R} \to \mathrm{Spec}\,\mathcal{R}(X)$ is an open immersion whose image is the union of the principal open subset of $\mathrm{Spec}\,\mathcal{R}(X)$ defined by f_i for $i \in \{1, \dots, t\}$. $\square$

Remark 2.63. The assumption of Proposition 2.62 on the affine open covering is equivalent to the requirement that there are effective Cartier divisors $D_1, \dots, D_t$ on X such that $[D_i] \in \lambda(M)$ for all $i \in \{1, \dots, t\}$ and such that the open subsets $X \smallsetminus \mathrm{Supp}(D_i)$ are affine and cover X. If $\lambda(M) = \mathrm{Pic}(X)$ this is the definition of divisorial variety [**Bor63**, §3]. Among those there are quasi-projective varieties and locally factorial varieties [**Bor63**, §4].

If $\lambda(M)$ contains the class of a very ample invertible sheaf on X, then the hypothesis of Proposition 2.62 on the affine open covering is satisfied, and the open immersion $\mathrm{Spec}_X \mathcal{R} \to \mathrm{Spec}\,\mathcal{R}(X)$ can be characterized as follows (cf. [**ADHL15**, Corollary 1.6.3.6]).

Corollary 2.64. *Assume that $k = \overline{k}$, that X is projective and has a Cox sheaf $\mathcal{R}$ of type λ such that $\mathcal{R}(X)$ is a finitely generated k-algebra, and there is $m \in M$ such that $\lambda(m)$ is very ample. Then $\mathrm{Spec}_X \mathcal{R} \to \mathrm{Spec}\,\mathcal{R}(X)$ is a $\widehat{M}$-equivariant open immersion and the complement of the image is defined by the ideal $\sqrt{\langle \mathcal{R}(X)_m \rangle}$ of $\mathcal{R}(X)$, where $\langle \mathcal{R}(X)_m \rangle$ is the ideal generated by the degree-m-part of $\mathcal{R}(X)$.*

2.4.2. Generators and relations. The following proposition and the remark below explain how to realize a finitely generated generalized Cox ring as a quotient of a polynomial ring. Without loss of generality, we assume that $M = M_{\mathrm{eff}}$ (cf. Remark 2.30).

Proposition 2.65. *Assume that $k = \overline{k}$. Let $m_1, \dots, m_N \in M$ be a set of generators for M_{eff}. Let $\Lambda := \bigoplus_{i=1}^{N} \mathbb{Z} m_i$, and let Λ_0 be the kernel of the natural homomorphism $\varphi : \Lambda \to M$. For $1 \leq i \leq N$, let D_i be a Cartier* *divisor representing the class $\lambda(m_i)$ in $\operatorname{Pic}(X)$, and for every $L = \sum_{i=1}^N a_i m_i$ of Λ, let $D_L := \sum_{i=1}^N a_i D_i$. Let $\chi : \Lambda_0 \to k(X)^\times$ be a character associated with the $\mathcal{O}_X$-algebra $\mathcal{S} := \bigoplus_{L\in\Lambda} \mathcal{O}_X(D_L)$.*

Endow $S := k[\eta_1, \dots, \eta_N]$ with the M-grading induced by assigning degree m_i to η_i for each $i \in \{1, \dots, N\}$. For every $L \in \Lambda$, let

$$\phi_L : S_{\varphi(L)} \to H^0(X, \mathcal{O}_X(D_L))$$

be the linear map that sends $\eta_1^{a_1} \cdots \eta_N^{a_N}$ to $\chi(\sum_{i=1}^N a_i m_i - L)$ for all $(a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N$ such that $\sum_{i=1}^N a_i m_i = \varphi(L)$ in M. Let $g_1, \dots, g_s \in S$ be homogeneous elements such that for each $i \in \{1, \dots, s\}$, if g_i has degree $\varphi(L_i)$ with $L_i \in \Lambda$, then $\phi_{L_i}(g_i) = 0$.

Then $R := S/(g_1, \dots, g_s)$ is a Cox ring of X of type λ if and only if the linear map

$$\phi_{\varphi(L),D_L} : R_{\varphi(L)} \to H^0(X, \mathcal{O}_X(D_L))$$

induced by ϕ_L is an isomorphism for all $L \in \Lambda$.

Conversely, if R is a finitely generated Cox ring of X of type λ, then there are generators $m_1, \dots, m_N$ of M_{eff}, a character χ and polynomials $g_1, \dots, g_s \in k[\eta_1, \dots, \eta_N]$ as above such that $R \cong k[\eta_1, \dots, \eta_N]/(g_1, \dots, g_s)$.

PROOF. For the first statement, we notice that for all $L_1, L_2 \in \Lambda$,

$$\phi_{L_1}(s_1)\phi_{L_2}(s_2) = \phi_{L_1+L_2}(s_1 s_2)$$

for all $s_1, s_2 \in S$ homogeneous of degrees $\varphi(L_1)$ and $\varphi(L_2)$, respectively, because χ is a group homomorphism.

Conversely, assume that R is a finitely generated Cox ring of X of type λ, and let $s_1, \dots, s_N$ be a finite set of homogeneous elements that generate R. Let for every $i \in \{1, \dots, N\}$, let m_i be the degree of s_i. Sending $\eta_i \mapsto s_i$ defines a surjective homomorphism $\pi : k[\eta_1, \dots, \eta_N] \to R$ of M-graded rings, where the grading on $S := k[\eta_1, \dots, \eta_N]$ is defined by assigning degree m_i to η_i for $i \in \{1, \dots, N\}$. Since S is noetherian, the kernel of π is generated by finitely many homogeneous elements $g_1, \dots, g_s$.

Since $s_1, \dots, s_N$ generate R, the elements $m_1, \dots, m_N$ generate M_{eff}. Let $\Lambda := \bigoplus_{i=1}^N \mathbb{Z} m_i$, and let Λ_0 be the kernel of the natural homomorphism $\varphi : \Lambda \to M$. For $1 \leq i \leq N$, let D_i be a Cartier divisor representing the class $\lambda(m_i)$ in $\operatorname{Pic}(X)$, and for every $L = \sum_{i=1}^N a_i m_i$ of Λ, let $D_L := \sum_{i=1}^N a_i D_i$.

Let $\phi_{m,D}$, for $(m, D) \in M_\lambda$, be a family of isomorphisms associated with R. Let $\Lambda_+ := \bigoplus_{i=1}^N \mathbb{Z}_{\geq 0} m_i$. For every $L = \sum_{i=1}^N a_i m_i \in \Lambda_+$, let $\alpha_L := \phi_{\varphi(L),D_L}(s_1^{a_1} \cdots s_N^{a_N})$. For every $L \in \Lambda$ such that $\lambda(\varphi(L))$ is an effective class, write $L = L^+ - L^-$ with $L^+, L^- \in \Lambda_+$, and define $\alpha_L := \alpha_{L^+}\alpha_{L^-}^{-1}\alpha$, where $\alpha \in k^\times$ is the unique constant such that $\phi_{\varphi(L),D_L}(s)\phi_{\varphi(L^-),D_{L^-}}(s') = \alpha\phi_{\varphi(L^+),D_{L^+}}(ss')$ for all $s \in R_{\varphi(L)}$ and $s' \in R_{\varphi(L^-)}$. The constant α_L does not depend on the choice of L^+ and L^-. If $\lambda(\varphi(L))$ is not effective, take $\alpha_L := 1$.

The isomorphisms $\phi_L := \alpha_L^{-1}\phi_{\varphi(L),D_L}$ satisfy

$$\phi_{L_1}(s_1)\phi_{L_2}(s_2) = \phi_{L_1+L_2}(s_1 s_2)$$

for all $s_1, s_2 \in S$ homogeneous of degrees $\varphi(L_1)$ and $\varphi(L_2)$, respectively, for all $L_1, L_2 \in \Lambda$. Then, the map $\chi : \Lambda_0 \to k(X)^\times$ defined by $\chi(E) = \phi_{-E}(1)$

for all $E \in \Lambda_0$, is a character associated with $\bigoplus_{L\in\Lambda} \mathcal{O}_X(D_L)$ that defines on $k[\eta_1,\dots,\eta_N]/(g_1,\dots,g_s)$ a structure of Cox ring of X of type λ which coincides with the one induced by R via π. $\square$

Over nonclosed fields we consider the following $\mathfrak{g}$-equivariant version of Proposition 2.65.

Remark 2.66. Let k be an arbitrary field, and consider the construction in Proposition 2.65 for $X_{\overline{k}}$. If the set $\{(m_1, D_1), \dots, (m_N, D_N)\}$ is $\mathfrak{g}$-invariant with respect to the componentwise action of $\mathfrak{g}$, and the character χ is $\mathfrak{g}$-equivariant, then the morphisms ϕ_L defined by χ are $\mathfrak{g}$-equivariant for all $L \in \Lambda$. Therefore, if $R = S/(g_1,\dots,g_s)$ is a Cox ring of $X_{\overline{k}}$ of type λ, it is endowed with a natural $\mathfrak{g}$-action, and descends to a Cox ring $R^{\mathfrak{g}}$ of X of type λ by Proposition 2.39. The ring $R^{\mathfrak{g}}$ is a finitely generated k-algebra by faithfully flat descent.

2.4.3. Example: Cox rings for a family of Châtelet surfaces. As an example, we describe Cox rings of Châtelet surfaces of identity type and of injective type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ as quotients of polynomial rings.

A *Châtelet surface* X over a field k is a smooth compactification of an affine surface defined in $\mathbb{A}^3_k$ by an equation of the form

$$x^2 - ay^2 = P(z),$$

where $a \in k^\times$ and P is a separable polynomial of degree 4. See [**CTSSD87a, CTSSD87b**].

We consider the family of Châtelet surfaces from [**BBP12**]. Assume that $a = -1$ is not a square in k and that P is a product of four linear polynomials $l_j(u,v) = a_j u + b_j v$ with pairwise non-proportional $(a_j, b_j) \in k^2$. Let $X_1 \subseteq \mathbb{P}^2_k \times \mathbb{A}^1_k$ be defined by

$$x^2 + y^2 = t^2 \prod_{j=1}^{4} l_j(u,1),$$

and let $X_2 \subseteq \mathbb{P}^2_k \times \mathbb{A}^1_k$ be defined by

$$x^2 + y^2 = t^2 \prod_{j=1}^{4} l_j(1,v).$$

Let X be the Châtelet surface obtained by glueing X_1 and X_2 along the isomorphism

$$X_1 \cap \{u \neq 0\} \to X_2 \cap \{v \neq 0\}, \quad ((x:y:t),u) \mapsto ((x:y:u^2 t), 1/u).$$

An anticanonical morphism $\psi : X \to \mathbb{P}^4_k$, maps X onto the singular quartic del Pezzo surface $X' \subset \mathbb{P}^4_k$ defined by

$$x_0x_2 - x_1^2 = x_3^2 + x_4^2 - (ax_0 + bx_1 + cx_2)(a'x_0 + b'x_1 + c'x_2) = 0,$$

where

$$(a,b,c) = (a_1a_2, a_1b_2 + a_2b_1, b_1b_2),\ (a',b',c') = (a_3a_4, a_3b_4 + a_4b_3, b_3b_4).$$

The singularities of $X'_{\overline{k}}$ are $P^{\pm} = (0:0:0:1:-(\pm\mathrm{i}))$, of type $\mathbf{A}_1$, where $\mathrm{i}^2 = -1$, and $X'_{\overline{k}}$ contains precisely eight lines of $\mathbb{P}^4_{\overline{k}}$:

$$D'^{\pm}_j = \{x_3 - (\pm\mathrm{i})x_4 = a_j x_1 + b_j x_2 = a_j x_0 + b_j x_1 = 0\}, \quad 1 \leq j \leq 4.$$

The anticanonical morphism ψ is a minimal desingularization of X'. Hence, the Châtelet surface X is a weak del Pezzo surface. The (-1)-curves on $X_{\overline{k}}$ are precisely the strict transforms of the lines on $X'_{\overline{k}}$, namely $D^{\pm}_j$ given by

$$D^{\pm}_j \cap X_{1,\overline{k}} = \{x - (\pm\mathrm{i})y = l_j(u,1) = 0\}$$

for $1 \leq j \leq 4$. The (-2)-curves on $X_{\overline{k}}$ are the inverse images $D^{\pm}_0$ of the singularities $P^{\pm} \in X'_{\overline{k}}$. They are determined by

$$D^{\pm}_0 \cap X_{1,\overline{k}} = \{x - (\pm\mathrm{i})y = t = 0\}.$$

The Dynkin diagram in Figure 1 encodes the configuration of negative curves on $X_{\overline{k}}$. For any two curves D, D' in the diagram, the number of edges between D and D' is the intersection number $[D].[D']$.

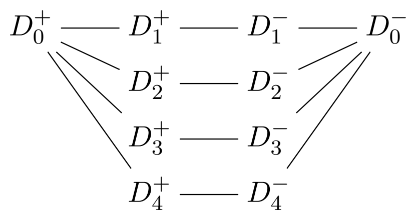

FIGURE 1. Configuration of curves on $X_{\overline{k}}$.

We observe that the lines $D^{\pm}_j$ are all defined over $k(\mathrm{i})$. In particular, they are fixed under the action of $\mathrm{Gal}(\overline{k}/k(\mathrm{i}))$, while $\mathrm{Gal}(k(\mathrm{i})/k)$ exchanges D^+_j with D^-_j for all $j \in \{0, \dots, 4\}$. Let $\Lambda \cong \mathbb{Z}^{10}$ be the free abstract group with basis $\{D^{\pm}_j\}_{0 \leq j \leq 4}$, and Λ_0 the kernel of the homomorphism $\Lambda \to \mathrm{Pic}(X_{\overline{k}})$ that sends a divisor to its class. According to [**BBP12**, (2.2), (2.3)], $\mathrm{Pic}(X_{\overline{k}}) \cong \mathbb{Z}^6$ is generated by the divisors in Λ, and $\Lambda_0 \cong \mathbb{Z}^4$ is generated by

$$E_{i,j} := (D^+_i + D^-_i) - (D^+_j + D^-_j),$$
$$E_{\{i,j\},\{l,m\}} := (D^+_0 + D^+_i + D^+_j) - (D^-_0 + D^-_l + D^-_m),$$

for $\{i,j,l,m\} = \{1,2,3,4\}$. Note that $E_{1,2}, E_{1,3}, E_{1,4}, E_{\{1,2\},\{3,4\}}$ form a basis for Λ_0, because

$$E_{i,j} = -E_{j,i}, \quad E_{i,j} + E_{j,l} = E_{i,l}, \quad E_{\{i,j\},\{l,m\}} - E_{i,l} = E_{\{j,l\},\{i,m\}} \tag{2.8}$$

for all choices of $\{i,j,l,m\} = \{1,2,3,4\}$.

We show now that the rings in [**BBP12**, §4] are Cox rings of X of identity type according to Proposition 2.65 and Remark 2.66.

PROPOSITION 2.67. *The $\overline{k}$-algebra*

$$\overline{R} := \overline{k}[\eta^{\pm}_0, \dots, \eta^{\pm}_4]/(\Delta_{i,j}\eta^+_l\eta^-_l + \Delta_{j,l}\eta^+_i\eta^-_i + \Delta_{l,i}\eta^+_j\eta^-_j)_{1 \leq i < j < l \leq 4}, \tag{2.9}$$

where $\Delta_{i,j} = a_ib_j - a_jb_i$ *for all* $1 \leq i,j \leq 4$ *and* $\eta_j^{\pm}$ *has degree* $D_j^{\pm}$ *for all* $0 \leq j \leq 4$*, is a Cox ring of* $X_{\overline{k}}$ *of type* $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$*, together with the character* $\chi : \Lambda_0 \to \overline{k}(X_{\overline{k}})^{\times}$ *defined by*

$$\chi(E_{i,j}) := \psi^* \left(\frac{a_ix_1 + b_ix_2}{a_jx_1 + b_jx_2}\right),$$
$$\chi(E_{\{i,j\},\{l,m\}}) := \psi^* \left(\frac{x_3 - \mathrm{i}x_4}{a_la_mx_0 + (a_lb_m + a_mb_l)x_1 + b_mb_lx_2}\right)$$

for all choices of $\{i,j,l,m\} = \{1,2,3,4\}$.

Proof. The map χ is a group homomorphism because it respects the equalities (2.8). This is verified by direct computations recalling that $x_0x_2 = x_1^2$. Moreover, by the definition of χ and since it is a group homomorphism, $\mathrm{div}_0(\chi(E)) = E$ for all $E \in \Lambda_0$.

Let $\{\phi_L\}_{L\in\Lambda}$ be the linear maps for $\overline{k}[\eta_0^{\pm}, \dots, \eta_4^{\pm}]$ defined by χ in Proposition 2.65.

We observe that the polynomials

$$g_{i,j,l} := \Delta_{i,j}\eta_l^+\eta_l^- + \Delta_{j,l}\eta_i^+\eta_i^- + \Delta_{l,i}\eta_j^+\eta_j^- \tag{2.10}$$

have all the same degree $[D_1^+ + D_1^-] = \cdots = [D_4^+ + D_4^-]$ in $\mathrm{Pic}(X_{\overline{k}})$, and that

$$\phi_{D_m^+ + D_m^-}(g_{i,j,l}) = \Delta_{i,j}\chi(E_{l,m}) + \Delta_{j,l}\chi(E_{i,m}) + \Delta_{l,i}\chi(E_{j,m}) = 0$$

for all $1 \leq i < j < l \leq 4$ and all $m \in \{1, \dots, 4\}$. By Proposition 2.65, the $\mathrm{Pic}(X_{\overline{k}})$-graded $\overline{k}$-algebra $\overline{R}$ is a Cox ring of $X_{\overline{k}}$ as long as the induced morphisms $\phi_{[D],D}$ are isomorphisms for all $D \in \Lambda$. The results of [**Der06**, 6.4(iii)] and [**ADHL15**, Theorem 5.4.4.5(3)], ensure that the ten generators corresponding to the negative curves and the relations $g_{i,j,l}$ in degree $[D_1^+ + D_1^-]$ are sufficient, namely, that each $\phi_{[D],D}$ is an isomorphism. □

We recall that the set $\{D_0^{\pm}, \dots, D_4^{\pm}\}$ is $\mathfrak{g}$-equivariant, and that each divisor is invariant under the action of $\mathrm{Gal}(\overline{k}/k(\mathrm{i}))$. Let $\mathrm{g} \in \mathrm{Gal}(k(\mathrm{i})/k) \cong \mathbb{Z}/2\mathbb{Z}$ be the nontrivial element. Then $\mathrm{g}(D_j^+) = D_j^-$ for all $j \in \{0, \dots, 4\}$.

Proposition 2.68. *The* k*-algebra*

$$R := k[\xi_j, \mu_j, j \in \{0, \dots, 4\}]/(\tilde{g}_{i,j,l})_{1\leq i<j<l\leq 4},$$

with

$$\tilde{g}_{i,j,l} := \Delta_{i,j}(\xi_l^2 + \mu_l^2) + \Delta_{j,l}(\xi_i^2 + \mu_i^2) + \Delta_{l,i}(\xi_j^2 + \mu_j^2) \tag{2.11}$$

for all $1 \leq i < j < l \leq 4$*, is a Cox ring of* X *of type* $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$.

Proof. The character χ defined in Proposition 2.67 is $\mathfrak{g}$-equivariant because for all choices of $\{i,j,l,m\} = \{1,2,3,4\}$,

$$\mathrm{g}(E_{i,j}) = E_{i,j}, \qquad \mathrm{g}(E_{\{i,j\},\{l,m\}}) = E_{\{l,m\},\{i,j\}}$$

and

$$x_3^2+x_4^2 = (a_la_mx_0+(a_lb_m+a_mb_l)x_1+b_lb_mx_2)(a_ia_jx_0+(a_ib_j+a_jb_i)x_1+b_ib_jx_2)$$

is the second equation of X'. Then the Cox ring $\overline{R}$ defined in Proposition 2.67 with the action of g that exchanges η_j^+ and η_j^- for all $0 \leq j \leq 4$ is $\mathfrak{g}$-equivariant by Remark 2.66. Hence, the subring $\overline{R}^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant elements of $\overline{R}$ is a Cox ring of X of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$ by Proposition 2.39. We observe that $R = \overline{R}^{\mathfrak{g}}$ via

$$\xi_j := (\eta_j^+ + \eta_j^-)/2, \quad \mu_j := (\eta_j^+ - \eta_j^-)/(2\mathrm{i}),$$

so that $\eta_j^\pm = \xi_j \pm \mathrm{i}\mu_j$ for all $j \in \{0,\dots,4\}$. □

Now we describe representatives for all isomorphism classes of Cox rings of X of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$. We twist the Cox ring R in Proposition 2.68 by an element of $H^1(k, G)$, where $G = \operatorname{Spec} \overline{k}[\mathrm{Pic}(X_{\overline{k}})]^{\mathfrak{g}}$ is the k-quasitorus dual to $\mathrm{Pic}(X_{\overline{k}})$. By [**GS06**, Proposition 3.3.14], there is an exact sequence

$$0 \to H^1(\mathrm{Gal}(k(\mathrm{i})/k), G(k(\mathrm{i}))) \to H^1(\mathfrak{g}, G(\overline{k})) \to H^1(\mathrm{Gal}(\overline{k}, k(\mathrm{i})), G_{k(\mathrm{i})}(\overline{k})).$$

We recall that $\mathrm{Gal}(\overline{k}, k(\mathrm{i}))$ acts trivially on $\mathrm{Pic}(X_{\overline{k}}) \cong \mathbb{Z}^6$, so that $\mathrm{Pic}(X_{k(\mathrm{i})})$ equals $\mathrm{Pic}(X_{\overline{k}})$. Then $G_{k(\mathrm{i})} \cong \mathbb{G}_{m,k(\mathrm{i})}^6$ and the last group in the exact sequence above is trivial. Therefore, we always represent an element of $H^1(k, G)$ by a cocycle $\sigma : \mathrm{Gal}(k(\mathrm{i})/k) \to G(k(\mathrm{i}))$, which is determined by $\sigma_{\mathrm{g}} \in G(k(\mathrm{i})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X_{k(\mathrm{i})}), k(\mathrm{i})^\times)$, where $\mathrm{g} \in \mathrm{Gal}(k(\mathrm{i})/k) \cong \mathbb{Z}/2\mathbb{Z}$ is the nontrivial element, as above.

Proposition 2.69. *The k-algebra*

$$R^\sigma := k[\xi_j, \mu_j, j \in \{0,\dots,4\}]/(\tilde{g}^\sigma_{i,j,l})_{1\le i<j<l\le 4},$$

with

$$\tilde{g}^\sigma_{i,j,l} := \Delta_{i,j}(\xi_l^2+\mu_l^2) + \Delta_{j,l} n^\sigma_{i,l}(\xi_i^2+\mu_i^2) + \Delta_{l,i} n^\sigma_{j,l}(\xi_j^2+\mu_j^2) \tag{2.12}$$

for all $1 \le i < j < l \le 4$, where

$$n^\sigma_{i,j} := \sigma_{\mathrm{g}}([D_j^+ - D_i^+]) \in k^\times$$

for $i,j \in \{1,\dots,4\}$, is a Cox ring of X of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$.

Proof. The natural action of g on $\overline{R}$ twisted by σ exchanges η_j^- with $\sigma_{\mathrm{g}}([D_j^+])\eta_j^+$ for $j \in \{0,\dots,4\}$, as in the proof of Proposition 2.41. Then

$$\xi_j := (\sigma_{\mathrm{g}}([D_j^+])\eta_j^+ + \eta_j^-)/2, \quad \mu_j := (\sigma_{\mathrm{g}}([D_j^+])\eta_j^+ - \eta_j^-)/(2\mathrm{i}),$$

are invariant under the twisted action and

$$k[\xi_j, \mu_j, j \in \{0,\dots,4\}] \otimes_k \overline{k} = \overline{k}[\eta_j^\pm, j \in \{0,\dots,4\}]$$

as

$$\eta_j^+ = \sigma_{\mathrm{g}}([D_j^+])^{-1}(\xi_j + \mathrm{i}\mu_j) \quad \text{and} \quad \eta_j^- = \xi_j - \mathrm{i}\mu_j. \tag{2.13}$$

Substituting (2.13) into (2.10) and multiplying by $\sigma_{\mathrm{g}}([D_l^+])$, we obtain (2.12). Moreover, we use the cocycle condition and the fact that $\sigma_{\mathrm{g}}(E_{i,j}) = 1$ for all $i,j \in \{1,\dots,4\}$ to check that $\mathrm{g}(n^\sigma_{i,j}) = n^\sigma_{i,j}$ for all $i,j \in \{1,\dots,4\}$. □

For $k = \mathbb{Q}$, the polynomials $\tilde{g}^\sigma_{i,j,l}$ defining the twisted Cox rings R^σ in Proposition 2.69 are the same as the equations [**BBP12**, (4.2)], since we can choose $(n_1,n_2,n_3,n_4) \in (\mathbb{Z} \smallsetminus \{0\})^4$ such that $n^\sigma_{i,j} = n_i/n_j$ for all $i,j \in \{1,\dots,4\}$. Note that [**BBP12**, Proposition 4.9] shows that every k-rational point on X can be lifted to a universal torsor corresponding to a twisted Cox ring as above with (n_1,n_2,n_3,n_4) such that $n_1n_2n_3n_4$ is a square in $\mathbb{Z}$. We show that twisted Cox rings as above are characterized by the fact that $n_1n_2n_3n_4$ is a sum of two squares.

PROPOSITION 2.70. *Let* $(n_1, n_2, n_3, n_4) \in (k^\times)^4$. *There is a cocycle* σ : $\mathrm{Gal}(k(\mathrm{i})/k) \to G(k(\mathrm{i}))$ *such that* $n_{i,j}^\sigma = n_i/n_j$ *for all* $i, j \in \{1, \dots, 4\}$ *if and only if* $n_1n_2n_3n_4 = \alpha^2 + \beta^2$ *with* $\alpha, \beta \in k$.

PROOF. Given a cocycle σ as in the statement,

$$n_1n_2n_3n_4 = n_1^4 n_{1,1}^\sigma n_{2,1}^\sigma n_{3,1}^\sigma n_{4,1}^\sigma = n_1^4 \sigma_{\mathrm{g}}([D_0^+ + 2D_1^+])\mathrm{g}(\sigma_{\mathrm{g}}([D_0^+ + 2D_1^+])),$$

using $[E_{\{3,4\},\{1,2\}} - E_{1,2}] = [0] \in \mathrm{Pic}(X_{k(\mathrm{i})})$ and the cocycle condition. Hence, $n_1n_2n_3n_4 = \alpha^2 + \beta^2$ with $\alpha, \beta \in k$ such that $\sigma_{\mathrm{g}}([D_0^+ + 2D_1^+]) = (\alpha + \mathrm{i}\beta)/n_1^2$. Conversely, if $n_1n_2n_3n_4 = \alpha^2 + \beta^2$ with $\alpha, \beta \in k$,

$$\sigma_{\mathrm{g}}([D_0^+]) := \alpha + \mathrm{i}\beta, \quad \sigma_{\mathrm{g}}([D_0^-]) := (\alpha - \mathrm{i}\beta)^{-1},$$

and

$$\sigma_{\mathrm{g}}([D_j^+]) := n_j^{-1}, \quad \sigma_{\mathrm{g}}([D_j^-]) := n_j, \quad j \in \{1, \dots, 4\},$$

defines a cocycle $\sigma : \mathrm{Gal}(k(\mathrm{i})/k) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X_{k(\mathrm{i})}), k(\mathrm{i})^\times)$. □

Since X has a Cox ring of identity type, Cox rings of X of arbitrary type exist by Remark 2.51. The next proposition computes a Cox ring of X of injective type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$. We recall that $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\overline{k}})^{\mathfrak{g}} \cong \mathbb{Z}^2$ is generated by $[D_0^+ + D_0^-]$ and $[D_1^+ + D_1^-]$. Let R be the Cox ring of X defined in Proposition 2.68.

PROPOSITION 2.71. *Every Cox ring of* X *of injective type* $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ *is isomorphic to the* $\mathrm{Pic}(X)$*-graded* k*-algebra*

$$k[x, y, t, u, v]/(x^2 + y^2 - t^2 \prod_{j=1}^{4}(a_ju + b_jv)),$$

where x *and* y *have degree* $[D_0^+ + D_0^- + 2(D_1^+ + D_1^-)]$, t *has degree* $[D_0^+ + D_0^-]$, *and* u *and* v *have degree* $[D_1^+ + D_1^-]$.

PROOF. By Remark 2.51 and the fact that $\mathfrak{g}$ acts trivially on $\mathrm{Pic}(X)$, every Cox ring R' of injective type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ is isomorphic to the $\mathfrak{g}$-invariant subring of

$$\bigoplus_{m \in \mathrm{Pic}(X)} \overline{R}_m,$$

where $\overline{R}$ is the Cox ring defined in Proposition 2.67. For $m \in \mathrm{Pic}(X_{\overline{k}})$, the $\overline{k}$-vector space $\overline{R}_m$ is generated by the monomials $\prod_{j=0}^4 (\eta_j^+)^{e_j^+}(\eta_j^-)^{e_j^-}$ with $e_j^\pm \in \mathbb{Z}_{\geq 0}$, for $0 \leq j \leq 4$, and such that $[\sum_{j=0}^4 (e_j^+ D_j^+ + e_j^- D_j^-)] = m$. Solving these linear equations with $m \in \mathrm{Pic}(X)$ for nonnegative $e_0^\pm, \dots, e_4^\pm$, one finds that $R'_{\overline{k}}$ is isomorphic to the subring of $\overline{R}$ generated by the elements

$$\theta^+ := \eta_0^+ \prod_{j=0}^{4} \eta_j^+, \quad \theta^- := \eta_0^- \prod_{j=0}^{4} \eta_j^-, \quad \theta_j := \eta_j^+ \eta_j^-, \ j \in \{0, \dots, 4\},$$

that satisfy the relations

$$\theta^+\theta^- = \theta_0 \prod_{j=0}^{4} \theta_j, \qquad \Delta_{i,j}\theta_l + \Delta_{j,l}\theta_i + \Delta_{l,i}\theta_j = 0,\ 1 \le i < j < l \le 4.$$

The elements x, y, t, u, v of $R'_{\overline{k}}$ determined by

$$x + \mathrm{i}y = \eta_0^+ \prod_{j=0}^{4} \eta_j^+, \quad x - \mathrm{i}y = \eta_0^- \prod_{j=0}^{4} \eta_j^-,$$
$$t = \eta_0^+\eta_0^-, \quad a_j u + b_j v = \eta_j^+\eta_j^-,\ j \in \{1, \dots, 4\},$$

are $\mathfrak{g}$-invariant and generate $R'_{\overline{k}}$. The kernel of the induced homomorphism $\overline{k}[x, y, t, u, v] \to \overline{R}$ is generated by

$$x^2 + y^2 - t^2 \prod_{j=1}^{4} (a_j u + b_j v).$$

Hence, $R' \cong k[x, y, t, u, v]/(x^2 + y^2 - t^2 \prod_{j=1}^{4} (a_j u + b_j v))$. □

Remark 2.72. The Cox ring in Proposition 2.71 corresponds to the torsor in [**BBP12**, Definition 4.1].

CHAPTER 3

Integral models of torsors

This chapter is devoted to descent properties of torsors under tori over certain projective varieties. Given a noetherian integral domain A with fraction field k, and a projective k-variety with a torsor of a given type embedded in the spectrum of a finitely generated Cox ring of the same type, we explain how to construct A-models of the torsor, of the variety and of the torsor morphism so that the last one is a torsor. We also give some criteria to determine certain properties of the models by looking at analogous properties of the Cox ring. The content of this chapter is a generalization of [**FP14**, §3].

3.1. A construction

We fix a separable closure $\overline{k}$ of the fraction field k of A. Let $\overline{X}$ be an integral projective $\overline{k}$-variety. Assume that $\overline{X}$ has a finitely generated Cox ring $\overline{R}$ of type $\lambda : \mathbb{Z}^r \to \operatorname{Pic}(\overline{X})$ such that $\lambda(\mathbb{Z}^r)$ contains an ample divisor class. By Proposition 2.65, we can write

$$\overline{R} = \overline{k}[\eta_1, \dots, \eta_N]/I,$$

where $\eta_1, \dots, \eta_N$ are $\mathbb{Z}^r$-homogeneous, and I is a homogeneous ideal.

Let $\overline{Y}$ be an $\overline{X}$ torsor of type λ. By Corollary 2.64, we can assume, without loss of generality, that $\overline{Y}$ is an open subset of $\operatorname{Spec} \overline{R}$ whose complement is defined by monic monomials

$$f_1, \dots, f_t \in \overline{k}[\eta_1, \dots, \eta_N] \smallsetminus \sqrt{I}.$$

For $i \in \{1, \dots, N\}$, let $m^{(i)} \in \mathbb{Z}^r$ be the degree of η_i. We recall that the action of $\widehat{\mathbb{Z}^r} = \mathbb{G}^r_{m,\overline{k}}$ on $\overline{Y}$ is induced by the action of $\mathbb{G}^r_{m,\overline{k}}$ on $\operatorname{Spec} \overline{R}$ defined by the homomorphism

$$\overline{R} \to \overline{k}[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_{\overline{k}} \overline{R}, \quad \eta_j \mapsto \underline{z}^{m^{(j)}} \otimes \eta_j,$$

where $z^m := z_1^{m_1} \cdots z_r^{m_r}$ for all $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$ (cf. Remark 2.61).

Construction 3.1. *Assume that the ideal I is generated by polynomials $g_1, \dots, g_s \in A[\eta_1, \dots, \eta_N]$. Let*

$$R := A[\eta_1, \dots, \eta_N]/(g_1, \dots, g_s),$$

and let Y be the complement of the closed subset of $\operatorname{Spec} R$ *defined by $f_1, \dots, f_t$. For $i \in \{1, \dots, t\}$, let $U_i := \operatorname{Spec} R[f_i^{-1}]$ and*

$$\overline{U}_i := U_i \times_{\operatorname{Spec} A} \operatorname{Spec} \overline{k} \cong \operatorname{Spec} \overline{R}[f_i^{-1}].$$

Then $\{U_i\}_{1 \le i \le t}$ is an affine open covering of Y, the family $\{\overline{U}_i\}_{1 \le i \le t}$ is an affine open covering of $\overline{Y}$, and $Y_{\overline{k}} \cong \overline{Y}$.

The $\mathbb{Z}^r$-grading of $\overline{R}$ induces a $\mathbb{Z}^r$-grading on R by assigning the degrees of $\eta_1, \dots, \eta_N$. We assume that $(R; f_1, \dots, f_t)$ satisfies the following condition:

$$\text{for every } i, j \in \{1, \dots, t\}, \text{ there is a homogeneous invertible element of } R[f_i^{-1}] \text{ of degree a multiple of } \deg f_j. \tag{3.1}$$

For $i \in \{1, \dots, t\}$, let R_i be the degree-0-part of the ring $R[f_i^{-1}]$ and $V_i := \operatorname{Spec}(R_i)$. Then $R_i \otimes_A \overline{k}$ is the degree-0-part of $\overline{R}[f_i^{-1}]$ for all $i \in \{1, \dots, t\}$. Since $\overline{Y} \to \overline{X}$ is a geometric quotient under the action of $\mathbb{G}^r_{m,\overline{k}}$, gluing the family of schemes $\{\operatorname{Spec}(R_i \otimes_A \overline{k})\}_{1 \le i \le t}$ yields a variety isomorphic to $\overline{X}$. Let X be the A-scheme obtained by gluing $\{V_i\}_{1 \le i \le t}$. Then X is a model of $\overline{X}$ over A and comes endowed with a natural morphism $\pi : Y \to X$ induced by the inclusions $R_i \to R[f_i^{-1}]$ for $i \in \{1, \dots, t\}$. Since the inclusions $R_i \to R[f_i^{-1}]$ induce surjective morphisms $U_i \to V_i$ for all $i \in \{1, \dots, t\}$, the morphism π is surjective. Moreover, π is of finite presentation because X is noetherian and $R[f_i^{-1}]$ is a finitely generated R_i-algebra for every $i \in \{1, \dots, t\}$. Since $f_1, \dots, f_t$ are $\mathbb{Z}^r$-homogeneous, the homomorphism

$$R \to A[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_A R, \quad \eta_j \mapsto \underline{z}^{m(j)} \otimes \eta_j$$

induces an action of $\mathbb{G}^r_{m,A}$ on Y which is given by

$$\underline{s} * (y_1, \dots, y_N) = (\underline{s}^{m(1)} y_1, \dots, \underline{s}^{m(N)} y_N) \tag{3.2}$$

on A-points $\underline{s} = (s_1, \dots, s_r) \in \mathbb{G}^r_{m,A}(A)$ and $(y_1, \dots, y_N) \in Y(A)$, where $\underline{s}^m := s_1^{m_1} \cdots s_r^{m_r}$ for all $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$.

Moreover, π is an X-torsor under $\mathbb{G}^r_{m,A}$ (compatible with the structure of $\overline{X}$-torsor of type λ on $\overline{Y}$) if and only if π is flat and the morphism of schemes $\psi : \mathbb{G}^r_{m,A} \times_{\operatorname{Spec} A} Y \to Y \times_X Y$ that sends $(\underline{s}, \underline{y}) \mapsto (\underline{s} * \underline{y}, \underline{y})$, obtained by gluing the morphisms

$$\psi_i : R[f_i^{-1}] \otimes_{R_i} R[f_i^{-1}] \to A[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_A R[f_i^{-1}],$$
$$\eta_j \otimes \eta_l \mapsto \underline{z}^{m(j)} \otimes \eta_j \eta_l$$

for $1 \le i \le t$, is an isomorphism (cf. Definition 1.2).

REMARK 3.2. If $A = k$, then $\pi : Y \to X$ is a torsor of type $\lambda : \mathbb{Z}^r \to \operatorname{Pic}(\overline{X})$, where the action of $\operatorname{Gal}(\overline{k}/k)$ on $\mathbb{Z}^r$ is trivial, by *fpqc* descent.

For X flat over A, we define the type of an X-torsor under $\mathbb{G}^r_{m,A}$ by means of the following exact sequence analogous to (2.2)

$$0 \to H^1(A, \mathbb{G}^r_{m,A}) \longrightarrow H^1(X, \mathbb{G}^r_{m,X}) \xrightarrow{\texttt{type}} \operatorname{Hom}_{\mathfrak{g}}(\widehat{\mathbb{G}^r_{m,A}}, \operatorname{Pic}_{X/A}), \tag{3.3}$$

which is obtained from the second exact sequence in [**CTS87**, p. 387] with base scheme $\operatorname{Spec} A$ and group $\mathbb{G}^r_{m,A}$ via [**CTS87**, Proposition 1.4.1]. We recall that Salberger introduced universal torsors for certain schemes defined over noetherian base schemes [**Sal98**, Definition 5.14] by considering the same exact sequence [**Sal98**, 5.13 (b)] for torsors under A-tori.

Under reasonable hypotheses, the following theorem shows that $\pi : Y \to X$ is a torsor under $\mathbb{G}^r_{m,A}$ whose type $\widehat{\mathbb{G}^r_{m,A}} \to \operatorname{Pic}_{X/A}$ induces λ at the level of $\overline{k}$-rational points. In particular, π is a universal torsor in the sense of Salberger's definition if $\lambda = \mathrm{id}_{\mathrm{Pic}(\overline{X})}$.

THEOREM 3.3. *Let π be as in Construction 3.1. If $(R; f_1, \ldots, f_t)$ satisfies the condition that*

$$\text{the degrees of the homogeneous invertible elements of } R[f_i^{-1}] \text{ generate } \mathbb{Z}^r \text{ for all } i \in \{1, \ldots, t\}, \tag{3.4}$$

then π is an X-torsor under $\mathbb{G}_{m,A}^r$. If we additionally assume that $X(A) \neq \emptyset$, that X is smooth, projective, of constant relative dimension, and with geometrically integral fibers over A, and that for every prime ideal $\mathfrak{p}$ of A the cohomology groups $H^i(X_{k(\mathfrak{p})}, \mathcal{O}_{X_{k(\mathfrak{p})}})$ vanish for $i \in \{1,2\}$, then π is an X-torsor under $\mathbb{G}_{m,A}^r$ whose type $\widehat{\mathbb{G}_{m,A}^r} \to \mathrm{Pic}_{X/A}$ induces λ at the level of $\overline{k}$-rational points. In particular, if $\lambda = \mathrm{id}_{\mathrm{Pic}(\overline{X})}$, then π is a universal torsor of X.

PROOF. Flatness of π is equivalent to flatness of all the inclusions $R_i \to R[f_i^{-1}]$, namely, to injectivity of the induced morphisms $J \otimes_{R_i} R[f_i^{-1}] \to R[f_i^{-1}]$ for all ideals J of R_i. Fix $i \in \{1, \ldots, t\}$. Let J be an ideal of R_i. A general element in the kernel of the induced morphism $J \otimes_{R_i} R[f_i^{-1}] \to R[f_i^{-1}]$ is $h = \sum_{j=1}^n h_j \otimes h'_j$, where $h_j \in J$ has degree 0 and $h'_j \in R[f_i^{-1}]$, and such that $\sum_{j=1}^n h_j h'_j = 0$ in $R[f_i^{-1}]$. Since $R[f_i^{-1}]$ is a graded ring, it is enough to consider homogeneous elements h, i.e., with all h'_j homogeneous of fixed degree $\deg h \in \mathbb{Z}^r$. Since the degrees of the homogeneous invertible elements of $R[f_i^{-1}]$ generate $\mathbb{Z}^r$, there exists $f \in R[f_i^{-1}]^\times$ of degree $\deg h$. Then $h = (\sum_{j=1}^n h_j h'_j f^{-1}) \otimes f = 0$ in $J \otimes_{R_i} R[f_i^{-1}]$.

In order to prove that ψ is an isomorphism, it suffices to prove that all ψ_i are isomorphisms. Let $\ell_1, \ldots, \ell_r$ be the canonical basis of $\mathbb{Z}^r$. For every $i \in \{1, \ldots, t\}$ and $l \in \{1, \ldots, r\}$, let $h_{i,l} \in R[f_i^{-1}]^\times$ be a homogeneous element of degree ℓ_l. Then the morphism

$$\tilde{\psi}_i : A[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}] \otimes_A R[f_i^{-1}] \to R[f_i^{-1}] \otimes_{R_i} R[f_i^{-1}]$$

that sends

$$1 \otimes \eta_j \mapsto 1 \otimes \eta_j \quad \text{and} \quad z_l \otimes 1 \mapsto h_{i,l} \otimes h_{i,l}^{-1}$$

for all $j \in \{1, \ldots, N\}$ and $l \in \{1, \ldots, r\}$, is well defined and inverse to ψ_i, for all $i \in \{1, \ldots, t\}$.

Under the additional assumptions, the relative étale Picard functor of X over A is representable by a twisted constant A-group scheme $\mathrm{Pic}_{X/A}$ by [**Pir12**, Proposition 2.1]. By [**Har77**, Corollary III.12.9], $R^2 p_* \mathcal{O}_X = 0$, where $p : X \to \mathrm{Spec}(A)$ is the structure morphism. Since Y_k is an X_k-torsor of type λ by Remark 3.2 and the exact sequence (3.3) is functorial, the type of the torsor $Y \to X$ induces λ at the level of k-rational points. If $\lambda = \mathrm{id}_{\mathrm{Pic}(\overline{X})}$, then $\mathrm{Pic}(\overline{X}) \cong \mathbb{Z}^r$ has trivial Galois action by Remark 3.2. Hence, the group scheme $\mathrm{Pic}_{X/A}$ is constant and represented by $\mathbb{Z}^r$ by étale descent, and the morphism

$$\mathrm{Hom}_A(\widehat{\mathbb{G}_{m,A}^r}, \mathrm{Pic}_{X/A}) \to \mathrm{Hom}_k(\widehat{\mathbb{G}_{m,k}^r}, \mathrm{Pic}_{X_k/k})$$

is injective. □

3.2. Some properties

This section provides sufficient conditions to check the various hypotheses of Theorem 3.3. We start by showing that the model X of Construction 3.1 is independent of the choice of $f_1, \dots, f_m$ under some conditions.

LEMMA 3.4. *Let $f_1', \dots, f_{t'}' \in \overline{k}[\eta_1, \dots, \eta_N] \smallsetminus \sqrt{I}$ be monic monomials such that $(R; f_1, \dots, f_t, f_1', \dots, f_{t'}')$ satisfies the condition* (3.1). *Let C_A and C_A' be the ideals of $A[\eta_1, \dots, \eta_N]$ generated by $f_1, \dots, f_t, g_1, \dots, g_s$, and $f_1', \dots, f_{t'}', g_1, \dots, g_s$, respectively, and assume that $\sqrt{C_A'} = \sqrt{C_A}$. Then X is isomorphic to the A-model X' of $\overline{X}$ constructed using $f_1', \dots, f_{t'}'$ in Construction 3.1.*

PROOF. For every $i \in \{t+1, \dots, t+t'\}$, let $f_i := f_{i-t}'$, and $V_i := \mathrm{Spec}(R_i)$, where R_i is the degree-0-part of $R[{f_i}^{-1}]$. For $1 \le i, j \le t+t'$, let $h_{i,j} \in R[f_i^{-1}]^\times$ be a homogeneous element of degree $-n_{i,j} \deg f_j$ for some positive integer $n_{i,j}$, and let $V_{i,j} := \mathrm{Spec}(R_i[(f_j^{n_{i,j}} h_{i,j})^{-1}]) \subseteq V_i$. Since $\sqrt{C_A'} = \sqrt{C_A}$, the ideal of R_i generated by

$$f_{t+1}{}^{n_{i,t+1}} h_{i,t+1}, \dots, f_{t+t'}{}^{n_{i,t+t'}} h_{i,t+t'}$$

contains $f_i^n f_i^{-n} = 1$ for some positive integer n. Hence, $V_i = \bigcup_{j=t+1}^{t+t'} V_{i,j}$ for every $i \in \{1, \dots, t\}$. Likewise, $V_i = \bigcup_{j=1}^{m} V_{i,j}$ for every $i \in \{t+1, \dots, t+t'\}$.

The identifications $R_i[(f_j^{n_{i,j}} h_{i,j})^{-1}] = R_j[(f_i^{n_{j,i}} h_{j,i})^{-1}]$ inside $R[(f_i f_j)^{-1}]$ induce isomorphisms $V_{i,j} \cong V_{j,i}$, for all $i, j \in \{1, \dots, t+t'\}$, that are compatible on the intersections. The schemes X and X' are the gluing of $\{V_i\}_{1 \le i \le t}$, and $\{V_i\}_{t+1 \le i \le t+t'}$, respectively, along the isomorphisms mentioned above. Since $\{V_{i,j}\}_{1 \le i \le t, t+1 \le j \le t+t'}$ is an open covering of X, $\{V_{j,i}\}_{1 \le i \le t, t+1 \le j \le t+t'}$ is an open covering of X', and all the isomorphisms $V_{i,j} \cong V_{j,i}$ are compatible on the intersections, they glue to a global isomorphism $X \cong X'$. □

The next three propositions provide sufficient conditions for X having geometrically integral fibers, and being smooth and projective over A.

PROPOSITION 3.5. *If $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ has geometrically integral fibers, then $X \to \mathrm{Spec}(A)$ has geometrically integral fibers.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of A, and let k' be an algebraic extension of the residue field $k(\mathfrak{p})$. Since $R \otimes_A k'$ is an integral domain by hypothesis, the ring $R_i \otimes_A k'$ is an integral domain for all $i \in \{1, \dots, t\}$. Thus, $X_{k'}$ is covered by a family of integral open subsets $\{W_i := \mathrm{Spec}(R_i \otimes_A k')\}_{1 \le i \le t}$ such that $W_i \cap W_j$ is nonempty for all nonempty W_i and W_j. Indeed, for $i, j \in \{1, \dots, t\}$, the intersection $W_i \cap W_j$ is the spectrum of the degree-0-part of the ring $(R \otimes_A k')[(f_i f_j)^{-1}]$, which is nonzero whenever f_i and f_j are nonzero elements of $R \otimes_A k'$.

Given any nonempty open subset U of $X_{k'}$ and nonzero sections $s_1, s_2 \in \mathcal{O}_{X_{k'}}(U)$, there exist $i_1, i_2 \in \{1, \dots, t\}$ such that $s_j|_{U \cap W_{i_j}} \neq 0$ for $j \in \{1, 2\}$. Therefore, $s_j|_{U \cap W_{i_1} \cap W_{i_2}} \neq 0$ for $j \in \{1, 2\}$ as W_{i_1}, W_{i_2} are integral, and $U \cap W_{i_1}$, $W_{i_1} \cap W_{i_2}$ are dense in W_{i_1}. Thus, $(s_1 s_2)|_{U \cap W_{i_1} \cap W_{i_2}} \neq 0$ and $s_1 s_2 \neq 0$ in $\mathcal{O}_{X_{k'}}(U)$. □

PROPOSITION 3.6. *Assume that A is a Dedekind domain, $\overline{X}$ is smooth, R is an integral domain,* $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ *has geometrically integral fibers, and π is flat (the last holds, for example, if* (3.4) *is satisfied). If the Jacobian matrix*

$$\left(\frac{\partial g_i}{\partial \eta_j}(\underline{y})\right)_{\substack{1\le i\le s\\ 1\le j\le N}}$$

has rank $N - \dim \overline{X} - r$ for all $\underline{y} \in Y(\overline{k(\mathfrak{p})})$ and $\mathfrak{p} \in \mathrm{Spec}(A)$, where $\overline{k(\mathfrak{p})}$ is an algebraic closure of the residue field $k(\mathfrak{p})$, then X is smooth over A.

PROOF. We prove first that Y is smooth over A. By [**Gro67**, Proposition 17.8.2], the scheme Y is smooth over A if and only if $Y \to \mathrm{Spec}(A)$ is flat and $Y_{k(\mathfrak{p})}$ is smooth over $k(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Since R is an integral domain and $A \to R$ is injective, R is a flat A-algebra by [**Liu02**, Corollary 1.2.14] as A is a Dedekind domain, and in particular Y is flat over A.

Since $\overline{\pi} : \overline{Y} \to \overline{X}$ is a torsor under $\mathbb{G}^r_{m,\overline{X}}$, the fiber $\overline{Y}_x$ of $\overline{\pi}$ at a point $x \in \overline{X}$ is a trivial $k(x)$-torsor under $\mathbb{G}^r_{m,k(x)}$, where $k(x)$ is the residue field of $\overline{X}$ at x, (see [**Mil80**, Corollary III.4.7 and Lemma III.4.10]). Hence, $\overline{Y}_x \cong \mathbb{G}^r_{m,k(x)}$ has dimension r for all $x \in \overline{X}$, and $\overline{Y}$ has dimension $\dim \overline{X} + r$ by [**Har77**, Exercise II.3.22]. Then $\dim Y_{k(\mathfrak{p})} \ge \dim \overline{X} + r$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$ by [**Gro66**, Lemme 13.1.1]. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. By the assumptions on the Jacobian matrix and [**Har77**, Theorem I.3.2 (c), Theorem I.5.1 and Proposition I.5.2A], we conclude that $\dim Y_{k(\mathfrak{p})} = \dim \overline{X} + r$ and $Y_{\overline{k(\mathfrak{p})}}$ is regular at all its closed points. Then $Y_{k(\mathfrak{p})}$ is smooth over $k(\mathfrak{p})$.

Therefore, X is smooth over A by [**Gro67**, Proposition 17.7.7], as Y is smooth over A and π is flat and surjective. □

REMARK 3.7. Assume that $\overline{X}$ is smooth and $\mathrm{Pic}(\overline{X})$ is finitely generated. By [**ADHL15**, Theorem 1.5.1.1], a Cox ring of $\overline{X}$ of identity type is an integral domain, hence also $\overline{R}$ is an integral domain, as the grading group of $\overline{R}$ is free. Then R is an integral domain if and only if $I \cap A[\eta_1, \dots, \eta_N] = (g_1, \dots, g_s)$.

PROPOSITION 3.8. *Assume that $f_1, \dots, f_t$ have all the same degree m such that $\lambda(m)$ is very ample. Let $C_{\overline{k}}$ and C_A be the ideals of $\overline{k}[\eta_1, \dots, \eta_N]$ and $A[\eta_1, \dots, \eta_N]$, respectively, generated by $f_1, \dots, f_t, g_1, \dots, g_s$. If*

$$\sqrt{C_{\overline{k}}} \cap A[\eta_1, \dots, \eta_N] = \sqrt{C_A},$$

then X is projective over A.

PROOF. Since R is a finitely generated A-algebra, the Veronese subring $\bigoplus_{n\in\mathbb{N}} R_{nm}$, where R_{nm} denotes the degree-nm-part of R, is a finitely generated A-algebra by [**ADHL15**, Proposition 1.1.2.4]. By [**Gro61**, Lemme 2.1.6], there exists a positive integer d such that $R' := \bigoplus_{n\in\mathbb{N}} R_{ndm}$ is generated by R_{dm} as A-algebra. Let $f'_1, \dots, f'_{t'}$ be generators of the A-module R_{dm}.

For all $i \in \{1, \dots, t'\}$, denote by R'_i the degree-0-part of $R'[f_i'^{-1}]$, which is generated by $f'_1 f_i'^{-1}, \dots, f'_{t'} f_i'^{-1}$ and coincides with the degree-0-part of $R[f_i'^{-1}]$. We recall that $\mathrm{Proj}(R')$ is defined as gluing of the affine schemes $\{V'_i := \mathrm{Spec}(R'_i)\}_{1\le i\le t'}$ along the isomorphisms on principal open subsets induced by the identifications $R'_i[f'_i f'^{-1}_j] = R'_j[f'_j f'^{-1}_i]$ inside $R[(f'_i f'_j)^{-1}]$ for $1 \leq i, j \leq t'$.

Let C'_A be the ideal of $A[\eta_1, \dots, \eta_N]$ generated by $f'_1, \dots, f'_{t'}, g_1, \dots, g_s$. Since $\sqrt{C_A} = \sqrt{C_A^d}$ and $C_A^d \subseteq C'_A$ by construction, there is an inclusion of radical ideals $\sqrt{C_A} \subseteq \sqrt{C'_A}$. By Corollary 2.64, the polynomials $f'_1, \dots, f'_{t'}$ and $g_1, \dots, g_s$ generate an ideal of $\overline{k}[\eta_1, \dots, \eta_N]$ whose radical is $\sqrt{C_{\overline{k}}}$. Hence, $\sqrt{C'_A} \subseteq \sqrt{C_A}$. Since $(R; f_1, \dots, f_t, f'_1, \dots, f'_{t'})$ satisfies the condition (3.1), there is an isomorphism $X \cong \operatorname{Proj}(R')$ by Lemma 3.4. $\square$

In the applications that we have in mind, $\overline{X}$ is obtained from $\mathbb{P}^2_{\overline{k}}$ by a chain of blowing-ups at closed points. The next proposition provides some conditions that make Construction 3.1 compatible with such blowing-ups for Cox rings of identity type. This can be used to verify the cohomology conditions of Theorem 3.3.

In the situation of Construction 3.1, we assume that $\overline{X}$ is a smooth surface and that $\lambda = \mathrm{id}_{\operatorname{Pic}(\overline{X})}$. We assume that the effective divisor on $\overline{X}$ corresponding to the section η_i is an integral curve D_i for all $i \in \{1, \dots, N\}$, and that D_N is a (-1)-curve on $\overline{X}$. Let $b : \overline{X} \to \overline{X}'$ be a birational morphism that contracts exactly D_N according to Castelnuovo's criterion. For every $i \in \{1, \dots, N-1\}$, let $D'_i = b(D_i)$. Assume that $x = b(D_N)$ belongs to D'_i exactly for $i \in \{1, 2\}$, and $D_1 \cap D_2 = \emptyset$. Then a Cox ring of $\overline{X}'$ is $\overline{R}' \cong \overline{R}/(\eta_N - 1)$ by [**HKL14**, Proposition 2.2], and the canonical pull-back of sections is defined by

$$b^* : \overline{R}' \to \overline{R}, \quad \eta_i \mapsto \begin{cases} \eta_i \eta_N & \text{if } i \in \{1, 2\}; \\ \eta_i & \text{otherwise.} \end{cases}$$

Let $\overline{Y}' \subseteq \operatorname{Spec} \overline{R}'$ be the characteristic space of $\overline{X}'$, and let $f'_1, \dots, f'_{t'} \in \overline{k}[\eta_1, \dots, \eta_{N-1}]$ be monic monomials that define the closed subset of $\operatorname{Spec} \overline{R}'$ complement to $\overline{Y}'$. Let I' be the ideal of $\overline{k}[\eta_1, \dots, \eta_N]$ generated by $g_1, \dots, g_s$ and $\eta_N - 1$. Assume that $I \cap A[\eta_1, \dots, \eta_N] = (g_1, \dots, g_s)$ and also $I' \cap A[\eta_1, \dots, \eta_N] = (g_1, \dots, g_s, \eta_N - 1)$. Let $R' = R/(\eta_N - 1)$, and let $Y' \to X'$ be the A-model of the universal torsor $\overline{Y}' \to \overline{X}'$ defined in Construction 3.1. Let C_A and C'_A be the ideals of $A[\eta_1, \dots, \eta_N]$ generated by $f_1, \dots, f_t, g_1, \dots, g_s$ and $b^*(f'_1)\eta_1, \dots, b^*(f'_{t'})\eta_1, b^*(f'_1)\eta_2, \dots, b^*(f'_{t'})\eta_2, g_1, \dots, g_s$, respectively. We assume that $\sqrt{C_A} = \sqrt{C'_A}$, that (3.4) holds for both $(R; f_1, \dots, f_t)$ and $(R'; f'_1, \dots, f'_{t'})$, and that (η_1, η_2) is a prime ideal in $\overline{R}'$.

Proposition 3.9. *Under the hypotheses listed above, X is a blowing-up of X' with center the closed subscheme defined by η_1, η_2.*

Proof. Let $f' \in \{f'_1, \dots, f'_{t'}\}$ and $f := b^*(f')$. Since $\operatorname{Pic} \overline{X} \cong \operatorname{Pic}(\overline{X}') \oplus \mathbb{Z}[D_N]$ and $\deg \eta_j = \deg b^* \eta_j - [D_N]$ for $j \in \{1, 2\}$, the degrees of the homogeneous invertible elements of $R[(f\eta_j)^{-1}]$ generate $\operatorname{Pic}(\overline{X})$ for $j \in \{1, 2\}$. Hence, (3.4) holds for $(R; b^*(f'_1)\eta_1, \dots, b^*(f'_{t'})\eta_1, b^*(f'_1)\eta_2, \dots, b^*(f'_{t'})\eta_2)$. Let $X'_{f'}$ be the spectrum of the degree-0-part R'_0 of the ring $R'[f'^{-1}]$, and let $X_{f\eta_j}$ be the spectrum of the degree-0-part $R[(f\eta_j)^{-1}]_0$ of the ring $R[(f\eta_j)^{-1}]$

for $j \in \{1,2\}$. Let $\overline{X}'_{f'}$ be the complement in $\overline{X}'$ of the support of the effective divisor corresponding to the section f', analogously we define $\overline{X}_f$ and $\overline{X}_{f\eta_j}$ for $j \in \{1,2\}$. By [**ADHL15**, Corollary 1.6.3.5], $\overline{X}'_{f'} = \operatorname{Spec}(R'_0 \otimes_A \overline{k})$. Since $D_1 \cap D_2 = \emptyset$ in $\overline{X}$, $\overline{X}_f = \overline{X}_{f\eta_1} \cup \overline{X}_{f\eta_2}$.

Let $h_1, h_2 \in R'[f'^{-1}]^\times$ be homogeneous elements of degrees $-\deg \eta_1$, $-\deg \eta_2$, respectively. Then $(\eta_1 h_1, \eta_2 h_2)$ is the ideal of $R'_0 \otimes_A \overline{k}$ defining $\{x\} \cap \overline{X}'_{f'}$.

If $f' \in (\eta_1, \eta_2)$ in R', then $x \notin \overline{X}'_{f'}$, and b induces an isomorphism between $\overline{X}_f = b^{-1}(\overline{X}'_{f'})$ and $\overline{X}'_{f'}$. That is, b^* induces an isomorphism between the degree-0-part of $\overline{R}'[f'^{-1}]$ and the degree-0-part of $\overline{R}[f^{-1}]$ that descends to an isomorphism between R'_0 and the degree-0-part $R[f^{-1}]_0$ of $R[f^{-1}]$ with the quotient morphism as inverse. Moreover, $X_{f\eta_j}$ is the spectrum of the degree-0-part of $R[(f\eta_j\eta_N)^{-1}]$ for $j \in \{1,2\}$, as f is a multiple of η_N in R. Then $X_{f\eta_1} \cup X_{f\eta_2} = \operatorname{Spec}(R[f^{-1}]_0)$, as $1 \in (\eta_1\eta_N b^*(h_1), \eta_2\eta_N b^*(h_2))$ in $R[f^{-1}]_0$.

If $f' \notin (\eta_1, \eta_2)$ in R', then $x \in \overline{X}'_{f'}$, and $\overline{X}_f = b^{-1}(\overline{X}'_{f'})$ is the blowing-up of $\overline{X}'_{f'}$ with center x. The blowing-up of $X'_{f'}$ with center $V(\eta_1 h_1, \eta_2 h_2)$ is covered by two open subsets that are the spectra of the degree-0-parts of the localizations of $\bigoplus_{d \geq 0} (\eta_1 h_1, \eta_2 h_2)^d$ at its degree-1-elements $\eta_j h_j$ for $j \in \{1,2\}$, respectively. Such an open covering is isomorphic to the gluing of the spectra of $R'_0[\eta_i h_i (\eta_j h_j)^{-1}]$, for $\{i,j\} = \{1,2\}$. Since, for $\{i,j\} = \{1,2\}$, b^* induces an isomorphism $R'_0[\eta_i h_i(\eta_j h_j)^{-1}] \to R[(f\eta_j)^{-1}]_0$ with the quotient morphism as inverse, the gluing of $X_{f\eta_1}$ and $X_{f\eta_2}$ is the blowing-up of $X'_{f'}$ with center $V(\eta_1 h_1, \eta_2 h_2)$.

By Lemma 3.4, the scheme X is isomorphic to the gluing of $X_{b^*(f')\eta_j}$ for $f' \in \{f'_1, \ldots, f'_{t'}\}$ and $j \in \{1,2\}$. Hence, it is a blowing-up of X' with center the closed subscheme defined by η_1, η_2. $\square$

Part 2

Applications to Manin's conjecture

CHAPTER 4

The universal torsor method

This chapter is devoted to a presentation of the universal torsor method for Manin's conjecture. After a historical introduction, we gather the results of Chapters 1 and 3, and we explain their application. The actual application of the method is then supported and further illustrated by the three examples that constitute the rest of this thesis.

In the first section we introduce the method and fix some notation that will be used in this and in the remaining chapters. The second section presents some computations for the torsor parameterization in the case of split smooth del Pezzo surfaces of degree 4.

4.1. The method

The universal torsor method is one of the techniques developed in response to the following conjecture about the distribution of rational points on Fano varieties formulated by Manin around 1989.

Conjecture 4.1 (Manin). *Let X be a Fano variety over a number field k, and let $H : X(k) \to \mathbb{R}_{\geq 0}$ be the height function induced by an adelic metrization on the anticanonical line bundle of X. If $X(k)$ is Zariski dense in X, then there exists an open subset U of X such that*

$$\#\{x \in U(k) : H(x) \leq B\} \sim CB(\log B)^{r-1}$$

as $B \to \infty$, where C is a constant depending on X and k, and r is the Picard rank of X.

The conjecture was inspired by a result of Schanuel for projective spaces [**Sch79**], which determines an asymptotic formula for the number of rational points of bounded Weil height

$$\mathbb{P}^n_k(k) \to \mathbb{R}_{\geq 0}, \quad (x_0 : \cdots : x_n) \mapsto \prod_{\nu \in \Omega_k} \max_{0 \leq i \leq n} |x_i|_\nu, \tag{4.1}$$

over arbitrary number fields k. We recall from page 14 that Ω_k and Ω_∞ denote the sets of all places and of the infinite places of k, respectively, and that for every $\nu \in \Omega_k$, the absolute value $|\cdot|_\nu$ of k_ν is normalized as follows: if $\tilde{\nu}$ is the place of $\mathbb{Q}$ below ν and $\mathbb{Q}_{\tilde{\nu}}$ is the completion of $\mathbb{Q}$ at $\tilde{\nu}$, then $|\cdot|_\nu := |\mathfrak{N}_{k_\nu/\mathbb{Q}_{\tilde{\nu}}}(\cdot)|_{\tilde{\nu}}$, where $|\cdot|_{\tilde{\nu}}$ is the usual real or p-adic absolute value on $\mathbb{Q}_{\tilde{\nu}}$. In general, if $X \to \mathbb{P}^n_k$ is a morphism induced by the anticanonical sheaf, the function $X(k) \to \mathbb{R}_{\geq 0}$ obtained by composition with the Weil height (4.1) is an anticanonical height on X.

The conjecture appeared first in [**FMT89**], where it is proven to hold for generalized flag varieties, to be invariant under fiber product and compatible with the predictions of the circle method for complete intersections in projective spaces. In the same paper, it is observed that a variety may contain proper closed subvarieties where the number of rational points of bounded height grows faster than on their complement, and hence, dominate the asymptotic behavior. For example, this happens for the 27 lines on smooth cubic surfaces. These subvarieties are called accumulating subvarieties, and the subset U in the conjecture is usually their complement. But a variety may contain infinitely many Zariski dense accumulating subvarieties [**BT96**], actually at most countably many [**BM90**]. In all the known examples the rational points on the accumulating subvarieties form a *thin* set in the sense of Serre [**Ser89**], and one can decide to count points on its complement (e.g. [**Rud14**]). The constant C in the asymptotic formula has been interpreted in terms of local densities on the set of adelic points by Peyre [**Pey95, Pey03**], refining thus the conjecture and extending it to a wider class of varieties that are called *quasi-Fano*.

Manin's conjecture has been attacked via harmonic analysis on adelic points for compactification of certain algebraic groups (e.g. tori [**BT96**] and additive groups [**CLT02**]), via the circle method for certain complete intersections in projective spaces (e.g. [**Pey95, Lou15**]), and via other lattice point counting techniques for varieties where the previous methods do not apply [**Bre02, BBD07**]. One of the common features of the proofs via lattice points counting, is the preparatory parameterization of the set of rational points by integral points on certain higher dimensional affine spaces. In some cases considering the affine cone of a projective variety is enough, especially for varieties of Picard number 1. In general it seems that the parameterizations are induced by torsors of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$, which are universal torsors if the variety X is *split*, that is, the Galois action on the geometric Picard group $\mathrm{Pic}(X_{\overline{k}})$ is trivial. In some papers the role of torsors is explicit (e.g. [**Sal98, BBP12, BBS14**]), but in most cases it is just mentioned or guessed (e.g. [**BBD07, BB07**]).

Until recently, except for [**Sch79**], all proofs making use of parameterizations explicitly or implicitly induced by torsors considered varieties over $\mathbb{Q}$. The first attempt to apply the universal torsor method over number fields beyond $\mathbb{Q}$ (for a toric cubic surface over imaginary quadratic fields of class number 1 [**DJ13**], for the same surface over arbitrary number fields [**Fre13**], for some other singular del Pezzo surfaces over imaginary quadratic fields [**DF14a, DF14b, DF15**]) led to a better understanding of what is a parameterization via integral points on universal torsors for varieties over arbitrary number fields. This was presented for the first time in a general and systematic way in joint work with Frei [**FP14**], which contains the first proof of Manin's conjecture over arbitrary number fields for a non-toric variety via the universal torsor method (cf. Chapter 5).

The universal torsor method for Manin's conjecture consists of a parameterization step and a counting step. We fix a Dedekind domain A with field of fractions k, and a separable closure $\overline{k}$ of k. In the applications k will be a number field and A its ring of integers. The parameterization step is stated at this level of generality by the following theorem that gathers some results from Theorems 1.10 and 3.3.

THEOREM 4.2. *Let X be a smooth projective geometrically integral k-variety with $X(k) \neq \emptyset$. Assume that $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, that $\mathrm{Pic}(X) \cong \mathbb{Z}^r$, and that $X_{\overline{k}}$ has a Cox ring $\overline{R}$ of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ that is finitely generated as $\overline{k}$-algebra. Let $\pi : \overline{Y} \to X_{\overline{k}}$ be a torsor of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$. Write $\overline{R} = \overline{k}[\eta_1, \dots, \eta_N]/(g_1, \dots, g_s)$ with $g_1, \dots, g_s \in A[\eta_1, \dots, \eta_N]$, and let $R := A[\eta_1, \dots, \eta_N]/(g_1, \dots, g_s)$. Assume that $\mathrm{Spec}\,\overline{R} \smallsetminus \overline{Y}$ is defined by monic monomials $f_1, \dots, f_t$ in the variables $\eta_1, \dots, \eta_N$, and that the degrees of the homogeneous invertible elements in $R[f_i^{-1}]$ generate $\mathrm{Pic}(X)$ for all $i \in \{1, \dots, t\}$.*

Then

(1) *there exists A-schemes $\mathscr{X}$ and $\mathscr{Y}$ and a morphism $\tilde{\pi} : \mathscr{X} \to \mathscr{Y}$, such that $\mathscr{X}_k \cong X$, $\mathscr{Y}_{\overline{k}} \cong \overline{Y}$, $\tilde{\pi}$ is an $\mathscr{X}$-torsor under $\mathbb{G}_{m,A}^r$ and induces π under base extension.*

(2) *If $\mathscr{X}$ is proper,*

$$X(k) = \bigsqcup_{\mathfrak{c} \in \mathcal{C}^r} {}_{\mathfrak{c}}\tilde{\pi}({}_{\mathfrak{c}}\mathscr{Y}(A))$$

under the inclusion $\mathscr{X}(A) \subseteq X(k)$, where $\mathcal{C}$ is a system of representatives for the class group of A and ${}_{\mathfrak{c}}\tilde{\pi} : {}_{\mathfrak{c}}\mathscr{Y} \to \mathscr{X}$ is a twist of $\tilde{\pi}$ by $\mathfrak{c}$.

(3) *For every $\underline{\mathfrak{c}} \in \mathcal{C}$, the set ${}_{\mathfrak{c}}\mathscr{Y}(A)$ is equal to the set of all $\underline{y} \in k^N$ whose coordinates y_i lie in the fractional ideals $\underline{\mathfrak{c}}^{\deg \eta_i}$, satisfying the coprimality conditions expressed by*

$$\sum_{i=1}^{t} f_i(\underline{y})\underline{\mathfrak{c}}^{-\deg f_i} = A$$

and the torsor equations

$$g_j(\underline{y}) = 0 \text{ for all } j \in \{1, \dots, s\}.$$

We restrict now to the case of a number field, and we denote by $\mathscr{O}_k$ its ring of integers. Assume that X is quasi-Fano, let H be an anticanonical height, and assume that the complement of the union of the accumulating subvarieties in X is an open subset U. We keep using the notation introduced in the theorem. After the parameterization step, the cardinality $N_{U,H,k}(B)$ of the set $\{x \in U(k) : H(x) \le B\}$ is

$$\sum_{\mathfrak{c} \in \mathcal{C}^r} \# \left(\{\underline{y} \in {}_{\mathfrak{c}}\mathscr{Y}(\mathscr{O}_k) \cap \tilde{\pi}^{-1}(U(k)) : H(\pi(\underline{y})) \le B\}/(\mathscr{O}_k^\times)^r \right).$$

If we decompose $\mathscr{O}_k^\times$ as a direct product of its torsion subgroup u_k with a free subgroup $\mathscr{U}_k$,

$$N_{U,H,k}(B) = \frac{1}{\# u_k^r} \sum_{\mathfrak{c} \in \mathcal{C}^r} \# \left(\{\underline{y} \in {}_{\mathfrak{c}}\mathscr{Y}(\mathscr{O}_k) \cap \tilde{\pi}^{-1}(U(k)) : H(\pi(\underline{y})) \le B\}/\mathscr{U}_k^r \right).$$

Let $\mathcal{F}$ be a fundamental domain for the action of $\mathscr{U}_k^r$ on $\mathscr{Y}_k(k) \subseteq k^N$, then the counting problem consists of estimating the cardinality of the sets $A_{\mathfrak{c}}(B)$ of points $\underline{y}$ in the lattice $\bigoplus_{i=1}^N \underline{\mathfrak{c}}^{\deg \eta_i} \subseteq k^N$ such that

(a) $g_i(\underline{y}) = 0$ for all $i \in \{1, \dots, s\}$;

(b) $\sum_{i=1}^{t} f_i(\underline{y})\underline{\mathfrak{c}}^{-\deg f_i} = \mathscr{O}_k$;

(c) $\pi(\underline{y}) \in U$;

(d) $\underline{y} \in \mathcal{F}$;

(e) $H(\pi(\underline{y})) \le B$.

Usually, condition (d) is the hardest to handle because different choices of the fundamental domain $\mathcal{F}$ may lead to very different subsets of k^N defined by (a),. . . ,(e), usually unbounded. This explains why most of the applications of the universal torsor method over number fields beyond $\mathbb{Q}$ is restricted to imaginary quadratic fields. The condition (b) is typically handled by Möbius inversion, see Section 6.2 for an example. In some cases (cf. Proposition 4.7 and Section 5.2) condition (c) enables to express some coordinates in terms of the others via (a). Moreover, if (c) forces some coordinates y_i to be nonzero, the condition $y_i \in \underline{\mathfrak{c}}^{\deg \eta_i}$ gives $\mathfrak{N}(y_i) \ge \mathfrak{N}(\underline{\mathfrak{c}}^{\deg \eta_i}) > 0$. This new condition combined with the height condition (e) could lead to count lattice points in bounded regions inside the $\mathbb{R}$-vector space $\prod_{\nu \in \Omega_\infty} k_\nu$ (cf. Section 6.3).

4.2. Smooth quartic del Pezzo surfaces

We now compute an integral model of a universal torsor for smooth split del Pezzo surfaces of degree 4 over number fields. We also describe the lifting of the height function to the twisted torsors and the preimage of the open subset complement to the accumulating subvarieties.

Let k be a number field. We denote by $\mathscr{O}_k$ its ring of integers, and we fix an algebraic closure $\overline{k}$ of k. Let X be a smooth del Pezzo surface of degree 4 over k such that the action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathrm{Pic}(X_{\overline{k}})$ is trivial. Wihtout loss of generality we can assume that X is a blowing-up $\psi : X \to \mathbb{P}^2_k$ in the following 5 points

$$P_1 := (1:0:0), \quad P_2 := (0:1:0), \quad P_3 := (0:0:1),$$
$$P_4 := (1:1:1), \quad P_5 := (a:b:c),$$

where $a \ne b \ne c \ne a$ and $a, b, c \in \mathscr{O}_k \smallsetminus \{0\}$. For $1 \le i \le 5$, let $D_i := \psi^{-1}(P_i)$ be the exceptional divisor associated to P_i, and $\ell_i := [D_i]$ its class in $\mathrm{Pic}(X)$. Let ℓ_0 be the class of the inverse image under ψ of a line in $\mathbb{P}^2_k$. Then $\ell_0, \dots, \ell_5$ form a basis of $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\overline{k}}) \cong \mathbb{Z}^6$.

For every $1 \le i < j \le 5$, let $D_{i,j}$ be the strict transform under ψ of the line in $\mathbb{P}^2_k$ passing through P_i and P_j. Let D_0 be the strict transform under ψ of the conic in $\mathbb{P}^2_k$ passing through $P_1, \dots, P_5$. Then $[D_{i,j}] = \ell_0 - \ell_i - \ell_j$ for all $1 \le i < j \le 5$, and $[D_0] = 2\ell_0 - \ell_1 - \ell_2 - \ell_3 - \ell_4 - \ell_5$.

By [**Tsc09**, Example 5.3.2], a Cox ring $\overline{R}$ of $X_{\overline{k}}$ of identity type is a $\mathrm{Pic}(X_{\overline{k}})$-graded $\overline{k}$-algebra with 16 homogeneous generators and 20 homogeneous relations. We denote the generators by η_i for $0 \le i \le 5$ and $\eta_{i,j}$ for $1 \le i < j \le 5$, where η_i has degree $[D_i]$ and $\eta_{i,j}$ has degree $[D_{i,j}]$ for all i

and j. According to [**Tsc09**, Example 5.3.2], the 20 relations are

$$\eta_{2,3}\eta_{4,5} + \eta_{2,4}\eta_{3,5} - \eta_{2,5}\eta_{3,4}, \quad a\eta_0\eta_1 + b\eta_{2,4}\eta_{3,5} - c\eta_{2,5}\eta_{3,4} \tag{4.2}$$

of degree $[D_0] + \ell_1$,

$$\eta_{1,3}\eta_{4,5} + \eta_{1,4}\eta_{3,5} - \eta_{1,5}\eta_{3,4}, \quad a\eta_0\eta_2 + b\eta_{1,4}\eta_{3,5} - a\eta_{1,5}\eta_{3,4} \tag{4.3}$$

of degree $[D_0] + \ell_2$,

$$\eta_{1,2}\eta_{4,5} + \eta_{1,4}\eta_{2,5} - \eta_{1,5}\eta_{2,4}, \quad a\eta_0\eta_3 + c\eta_{1,4}\eta_{2,5} - a\eta_{1,5}\eta_{2,4} \tag{4.4}$$

of degree $[D_0] + \ell_3$,

$$\eta_{1,5}\eta_{2,3} + \eta_{1,2}\eta_{3,5} - \eta_{1,3}\eta_{2,5}, \quad a\eta_0\eta_4 + (b-a)\eta_{1,2}\eta_{3,5} - (c-a)\eta_{1,3}\eta_{2,5} \tag{4.5}$$

of degree $[D_0] + \ell_4$,

$$\eta_{1,4}\eta_{2,3} + \eta_{1,2}\eta_{3,4} - \eta_{1,3}\eta_{2,4}, \quad a^2\eta_0\eta_5 + c(b-a)\eta_{1,2}\eta_{3,4} - b(c-a)\eta_{1,3}\eta_{2,4} \tag{4.6}$$

of degree $[D_0] + \ell_5$,

$$\eta_{1,4}\eta_4 + \eta_{1,3}\eta_3 - \eta_{1,2}\eta_2, \quad a\eta_{1,5}\eta_5 + b\eta_{1,3}\eta_3 - c\eta_{1,2}\eta_2 \tag{4.7}$$

of degree $\ell_0 - \ell_1$,

$$\eta_{2,4}\eta_4 + \eta_{2,3}\eta_3 - \eta_{1,2}\eta_1, \quad a\eta_{2,5}\eta_5 + b\eta_{2,3}\eta_3 - a\eta_{1,2}\eta_1 \tag{4.8}$$

of degree $\ell_0 - \ell_2$,

$$\eta_{3,4}\eta_4 + \eta_{2,3}\eta_2 - \eta_{1,3}\eta_1, \quad a\eta_{3,5}\eta_5 + c\eta_{2,3}\eta_2 - a\eta_{1,3}\eta_1 \tag{4.9}$$

of degree $\ell_0 - \ell_3$,

$$\eta_{1,4}\eta_1 + \eta_{3,4}\eta_3 - \eta_{2,4}\eta_2, \quad a\eta_{4,5}\eta_5 + (b-a)\eta_{3,4}\eta_3 - (c-a)\eta_{2,4}\eta_2 \tag{4.10}$$

of degree $\ell_0 - \ell_4$, and

$$a\eta_{1,5}\eta_1 + b\eta_{3,5}\eta_3 - c\eta_{2,5}\eta_2, \quad a\eta_{4,5}\eta_4 + (b-a)\eta_{3,5}\eta_3 - (c-a)\eta_{2,5}\eta_2 \tag{4.11}$$

of degree $\ell_0 - \ell_5$.

We denote by K a canonical divisor on X. The anticanonical class $[-K] = 3\ell_0 - \ell_1 - \ell_2 - \ell_3 - \ell_4 - \ell_5$ is very ample and induces a closed immersion $X \to \mathbb{P}^4_k$, whose image is defined by two homogeneous polynomials of degree 2. In the next proposition we compute a basis of $H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}(-K))$.

PROPOSITION 4.3. *The monomials*

$$\eta_1\eta_{2,5}\eta_{1,3}\eta_{1,4},\ \eta_2\eta_{1,3}\eta_{2,4}\eta_{2,5},\ \eta_3\eta_{2,4}\eta_{1,3}\eta_{3,5},\ \eta_4\eta_{3,5}\eta_{1,4}\eta_{2,4},\ \eta_5\eta_{1,4}\eta_{2,5}\eta_{3,5} \tag{4.12}$$

form a basis of the $\overline{k}$-vector space $\overline{R}_{[-K]}$.

PROOF. Since $X_{\overline{k}}$ is a smooth del Pezzo surface and $\overline{R}_{[-K]}$ is isomorphic to $H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}(-K))$, the dimension of $\overline{R}_{[-K]}$ is computed as $9-d$, where d is the degree of $X_{\overline{k}}$. The $\overline{k}$-vector space $\overline{R}_{[-K]}$ is generated by the monomials of degree $[-K]$ in the generators $\eta_i, \eta_{i,j}$. These are the 10 monomials $\eta_0\eta_i\eta_j\eta_{i,j}$ for $1 \le i < j \le 5$ and the 30 monomials $h_{i,j,l} := \eta_i\eta_{j,l}\eta_{i,m}\eta_{i,n}$ for $\{i,j,l,m,n\} = \{1,\dots,5\}$. Here $\eta_{i,j} := \eta_{j,i}$ if $i > j$. Using the relations

(4.2),...,(4.6) among the generators of $\overline{R}$, we see that $\overline{R}_{[-K]}$ is generated by the 30 monomials $h_{i,j,l}$, as

$$
\begin{aligned}
a\eta_0\eta_1\eta_2\eta_{1,2} &= ch_{2,3,4} - bh_{2,3,5}, & a\eta_0\eta_2\eta_3\eta_{2,3} &= ah_{3,1,5} - bh_{3,1,4},\\
a\eta_0\eta_1\eta_3\eta_{1,3} &= ch_{3,2,5} - bh_{3,2,4}, & a\eta_0\eta_2\eta_4\eta_{2,4} &= ah_{4,1,5} - bh_{4,3,5},\\
a\eta_0\eta_1\eta_4\eta_{1,4} &= ch_{4,2,5} - bh_{4,3,5}, & a\eta_0\eta_2\eta_5\eta_{2,5} &= ah_{5,3,4} - bh_{5,1,4},\\
a\eta_0\eta_1\eta_5\eta_{1,5} &= ch_{5,3,4} - bh_{5,2,4}, & a\eta_0\eta_3\eta_4\eta_{3,4} &= ah_{4,1,5} - ch_{4,2,5},\\
a\eta_0\eta_4\eta_5\eta_{4,5} &= (c-a)h_{5,1,3} - (b-a)h_{5,1,2}, & a\eta_0\eta_3\eta_5\eta_{3,5} &= ah_{5,2,4} - ch_{5,1,4}.
\end{aligned}
$$

Using the other relations one proves that the monomials $h_{i,j,l}$ all belong to the vector space spanned by $h_{1,2,5}, h_{2,1,3}, h_{3,2,4}, h_{4,3,5}, h_{5,1,4}$ in the following order: $h_{2,3,5}$ and $h_{5,2,4}$ via (4.7), $h_{1,3,5}$ and $h_{3,1,4}$ via (4.8), $h_{2,1,4}$ and $h_{4,2,5}$ via (4.9), $h_{3,2,5}$ and $h_{5,1,3}$ via (4.10), $h_{1,2,4}$ and $h_{4,1,3}$ via (4.11), $h_{2,3,4}$ and $h_{5,3,4}$ via (4.7), $h_{1,4,5}$ and $h_{3,4,5}$ via (4.8), $h_{2,1,5}$ and $h_{4,1,5}$ via (4.9), $h_{3,1,2}$ and $h_{5,1,2}$ via (4.10), $h_{1,2,3}$ and $h_{4,2,3}$ via (4.11), $h_{2,4,5}$ and $h_{5,2,3}$ via (4.7), $h_{1,3,4}$ and $h_{3,1,5}$ via (4.8), and $h_{4,1,2}$ via (4.9). $\square$

Using the basis in Proposition 4.3, we can find the equations defining the image of an anticanonical embedding of $X_{\overline{k}}$ in $\mathbb{P}^4_{\overline{k}}$.

Proposition 4.4. *An anticanonical image of $X_{\overline{k}}$ in $\mathbb{P}^4_{\overline{k}}$ is the closed subvariety defined by the equations*

$$
\begin{cases}
cx_2(x_4 - x_5) + (a-b)x_3(x_1 - x_5) = 0\\
x_1(cx_4 + (c-b)x_3) + x_5(bx_3 - cx_2) = 0
\end{cases}
$$

in the homogeneous coordinates $(x_1 : \cdots : x_5)$ on $\mathbb{P}^4_{\overline{k}}$.

Proof. The morphism $\Psi : \overline{Y} \to \mathbb{P}^4_{\overline{k}}$ that sends $(\eta_i, \eta_{j,l})_{\substack{0\le i\le 5\\ 1\le j<l\le 5}}$ to

$$(\eta_1\eta_{2,5}\eta_{1,3}\eta_{1,4} : \eta_2\eta_{1,3}\eta_{2,4}\eta_{2,5} : \eta_3\eta_{2,4}\eta_{1,3}\eta_{3,5} : \eta_4\eta_{3,5}\eta_{1,4}\eta_{2,4} : \eta_5\eta_{1,4}\eta_{2,5}\eta_{3,5})$$

factors as the composition of a closed immersion $X_{\overline{k}} \to \mathbb{P}^4_{\overline{k}}$ induced by a basis of $H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}(-K))$ corresponding to the elements (4.12) after the universal torsor morphism $\overline{Y} \to X_{\overline{k}}$. The image of Ψ is the closed subvariety of $\mathbb{P}^4_{\overline{k}}$ defined by the two quadratic equations in the statement. $\square$

Now we choose suitable monomials that define the complement of a universal torsor in $\operatorname{Spec}\overline{R}$ and satisfy the condition (3.4) with respect to the integral model of $\overline{R}$ that we consider below. Let I be the ideal of $\overline{R}$ generated by the monomials (4.12). Let J be the ideal of $\overline{R}$ generated by the monomials

$$\eta_{1,2}\eta_{1,5},\ \eta_{1,2}\eta_{2,3},\ \eta_{1,5}\eta_{4,5},\ \eta_{2,3}\eta_{3,4},\ \eta_{3,4}\eta_{4,5}. \tag{4.13}$$

Proposition 4.5. *The open subscheme $\overline{Y}$ complement to $V(IJ)$ in* $\operatorname{Spec}\overline{R}$ *is a universal torsor of $X_{\overline{k}}$.*

Proof. Since $-K$ is very ample, the open subscheme $\overline{Y}$ complement to $V(I)$ in $\operatorname{Spec}\overline{R}$ is a universal torsor of $X_{\overline{k}}$ by Corollary 2.64. As in the proof of Proposition 4.3, we denote by $h_{i,j,l}$ the monomial $\eta_i\eta_{j,l}\eta_{i,m}\eta_{i,n}$ for $\{i,j,l,m,n\} = \{1,\dots,5\}$. The monomials

$$h_{1,3,4},\ h_{2,4,5},\ h_{3,1,5},\ h_{4,1,2},\ h_{5,2,3} \in J$$

form a basis of $\overline{R}_{[-K]}$, because the relations (4.7),...,(4.11) give

$$\begin{aligned}
h_{1,2,5} &= h_{1,3,4} - h_{2,4,5} - h_{4,1,2} + (c-a)^{-1}((b-c)h_{3,1,5} + ah_{5,2,3}),\\
h_{2,1,3} &= \frac{(b-a)h_{1,3,4} - bh_{4,1,2}}{b-c} - h_{2,4,5} + \frac{(b-a)h_{3,1,5} + ah_{5,2,3}}{c-a},\\
h_{3,2,4} &= \frac{(c-a)h_{1,3,4} - ch_{4,1,2}}{b-c} + h_{3,1,5} + \frac{ah_{5,2,3} - ch_{2,4,5}}{b},\\
h_{4,3,5} &= h_{1,3,4} - h_{3,1,5} - h_{4,1,2} - b^{-1}((b-c)h_{2,4,5} + ah_{5,2,3}),\\
h_{5,1,4} &= h_{1,3,4} - h_{4,1,2} - h_{5,2,3} + a^{-1}((c-a)h_{2,4,5} - bh_{3,1,5}).
\end{aligned} \tag{4.14}$$

Therefore, $I \subseteq J$, and the ideals I and IJ define the same closed subset of $\operatorname{Spec}\overline{R}$. □

Let $\mathscr{O}_k$ be the ring of integers of k. We construct now an $\mathscr{O}_k$-model of the universal torsor $\overline{Y} \to X_{\overline{k}}$ that is a torsor under $\mathbb{G}^6_{m,\mathscr{O}_k}$.

Let $f_1, \dots, f_{25}$ be the monomials that we obtain by multiplying the monomials (4.12) with the monomials (4.13). Then $f_1, \dots, f_{25}$ generate the ideal IJ, and hence, define the complement of the universal torsor $\overline{Y}$ in $\operatorname{Spec}\overline{R}$ by Proposition 4.5. Let R be the quotient of the polynomial ring over $\mathscr{O}_k$ in the 16 variables $\eta_i, \eta_{j,l}$, with $0 \le i \le 5$ and $1 \le j < l \le 5$, by the ideal generated by the polynomials (4.2),...,(4.11), and let $\mathscr{Y} \to \mathscr{X}$ be the $\mathscr{O}_k$-model of the universal torsor $\overline{Y} \to X_{\overline{k}}$ defined by $f_1, \dots, f_{25}$ in Construction 3.1. We observe that the degrees of the variables η_j and $\eta_{j,l}$ appearing in f_i generate $\operatorname{Pic}(X_{\overline{k}})$ for all $i \in \{1, \dots, 25\}$. Since these variables are invertible in $R[f_i^{-1}]$, the morphism $\mathscr{Y} \to \mathscr{X}$ is a torsor under $\mathbb{G}^6_{m,\mathscr{O}_k}$ by Theorem 3.3.

Let I' and I'' be the ideals of R generated by the monomials (4.12) and by $f_1, \dots, f_{25}$, respectively. From now on we assume that $\sqrt{I'} = \sqrt{I''}$, namely, that the complement of $\mathscr{Y}$ in $\operatorname{Spec} R$ is defined by the monomials (4.12). This holds, for example, if a, b, $b-c$ and $c-a$ belong to $\mathscr{O}_k^\times$, because under such assumption the equations (4.14) have coefficients in $\mathscr{O}_k$. For a concrete example, let k be a number field that contains a primitive 6-th root of unity ζ, and take $a = \zeta$, $b = \zeta^2$ and $c = \zeta + \zeta^2$.

To describe the lifting of the anticanonical height function induced by the embedding in Proposition 4.4 to the twisted torsors, we consider a system $\mathcal{C}$ of representatives for the class group of $\mathscr{O}_k$. For any given $\underline{\mathfrak{c}} = (\mathfrak{c}_0, \dots, \mathfrak{c}_5) \in \mathcal{C}^6$, we denote by ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}\mathscr{Y} \to \mathscr{X}$ the twist of $\mathscr{Y}$ constructed in Definition 1.9. The height $H : X(k) \to \mathbb{R}_{\ge 0}$, defined as the Weil height (4.1) after the anticanonical embedding in $\mathbb{P}^4_k$ given in Proposition 4.4, lifts to the twisted torsors as follows.

Proposition 4.6. *Let $\underline{\mathfrak{c}} \in \mathcal{C}$. If $\sqrt{I'} = \sqrt{I''}$,*

$$H({}_{\underline{\mathfrak{c}}}\pi(\underline{y})) = \mathfrak{N}(\mathfrak{c}_0^{-3}\mathfrak{c}_1 \cdots \mathfrak{c}_5) \prod_{\nu \in \Omega_\infty} \max \left\{ \begin{matrix} |y_1 y_{2,5} y_{1,3} y_{1,4}|_\nu, |y_2 y_{1,3} y_{2,4} y_{2,5}|_\nu, \\ |y_3 y_{2,4} y_{1,3} y_{3,5}|_\nu, |y_4 y_{3,5} y_{1,4} y_{2,4}|_\nu, \\ |y_5 y_{1,4} y_{2,5} y_{3,5}|_\nu \end{matrix} \right\}.$$

for all $\underline{y} = (y_i, y_{j,l})_{\substack{0 \le i \le 5 \\ 1 \le j < l \le 5}} \in {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$.

PROOF. For $\underline{y} \in {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$, let

$$\begin{aligned} h_1(\underline{y}) &:= y_1 y_{2,5} y_{1,3} y_{1,4}, \quad h_2(\underline{y}) := y_2 y_{1,3} y_{2,4} y_{2,5}, \quad h_3(\underline{y}) := y_3 y_{2,4} y_{1,3} y_{3,5}, \\ h_4(\underline{y}) &:= y_4 y_{3,5} y_{1,4} y_{2,4}, \quad h_5(\underline{y}) := y_5 y_{1,4} y_{2,5} y_{3,5}. \end{aligned} \tag{4.15}$$

By Theorem 1.10, every $\underline{y} \in {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$ satisfy

$$\sum_{i=1}^{5} h_i(\underline{y}) \mathfrak{c}_0^{-3} \mathfrak{c}_1 \cdots \mathfrak{c}_5 = \mathscr{O}_k,$$

and $h_i(\underline{y})\mathfrak{c}_0^{-3}\mathfrak{c}_1 \cdots \mathfrak{c}_5$ is an integral ideal of $\mathscr{O}_k$ for all $i \in \{1, \dots, 5\}$. For every prime ideal $\mathfrak{p}$ of $\mathscr{O}_k$, let $n_\mathfrak{p} \in \mathbb{Z}$ such that $\mathfrak{c}_0^{-3}\mathfrak{c}_1 \cdots \mathfrak{c}_5 = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}}$, where $\prod_\mathfrak{p}$ denotes a product over the prime ideals of $\mathscr{O}_k$. For every prime ideal $\mathfrak{p}$ of $\mathscr{O}_k$, there exists $i \in \{1, \dots 5\}$ such that $\mathfrak{p}$ does not divide the ideal $h_i(\underline{y})\mathfrak{c}_0^{-3}\mathfrak{c}_1 \cdots \mathfrak{c}_5$, that is, $|h_i(\underline{y})|_\nu = \mathfrak{N}(\mathfrak{p})^{n_\mathfrak{p}} \geq |h_j(\underline{y})|_\nu$ for all $j \in \{1, \dots, 5\}$, where ν is the finite place of k associated with $\mathfrak{p}$. □

The preimage of the open subset U of X complement of the union of the 16 lines D_i, $D_{j,l}$, with $0 \leq i \leq 5$ and $1 \leq j < l \leq 5$, under the torsor morphism $\pi : \mathscr{Y}_k \to X$ is defined by

$$\prod_{i=0}^{5} \eta_i \prod_{1 \leq j < l \leq 5} \eta_{j,l} \neq 0.$$

Therefore, it is contained in an closed subset of a 10-dimensional affine space defined by the two cubic equations in the proposition below.

PROPOSITION 4.7. *Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$. If $\sqrt{I'} = \sqrt{I''}$ and $a \in \mathscr{O}_k^\times$, the set ${}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) \cap (\pi^{-1}(U))(k)$ is equal to the set of all*

$$\underline{y} = (y_1, y_2, y_3, y_4, y_5, y_{1,3}, y_{1,4}, y_{2,4}, y_{2,5}, y_{3,5}) \in (k^\times)^{10}$$

whose coordinates $y_i, y_{i,j}$ lie in the fractional ideals $\underline{\mathfrak{c}}^{\deg \eta_i}$, $\underline{\mathfrak{c}}^{\deg \eta_{i,j}}$, respectively, satisfying the coprimality conditions

$$\sum_{i=1}^{5} h_i(\underline{y}) \mathfrak{c}_0^{-3} \mathfrak{c}_1 \cdots \mathfrak{c}_5 = \mathscr{O}_k,$$

with h_i defined in (4.15)*, and the two equations*

$$\begin{aligned} (b-a) y_1 y_3 y_{1,3} + (b-a) y_1 y_4 y_{1,4} - b y_2 y_4 y_{2,4} + a y_2 y_5 y_{2,5} &= 0, \\ (c-a) y_1 y_3 y_{1,3} + c y_1 y_4 y_{1,4} - c y_2 y_4 y_{2,4} + a y_3 y_5 y_{3,5} &= 0. \end{aligned}$$

PROOF. By Theorem 1.10, the set ${}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$ is equal to the set of $\underline{y} = (y_0, \dots, y_5, y_{1,2}, \dots, y_{4,5}) \in k^{16}$, with $y_i \in \underline{\mathfrak{c}}^{\deg \eta_i}$ for all $0 \leq i \leq 5$ and $y_{i,j} \in \underline{\mathfrak{c}}^{\deg \eta_{i,j}}$ for all $1 \leq i < j \leq 5$, that are common zeros of the polynomials (4.2),…,(4.11) and satisfy

$$\sum_{i=1}^{5} h_i(\underline{y}) \mathfrak{c}_0^{-3} \mathfrak{c}_1 \cdots \mathfrak{c}_5 = \mathscr{O}_k.$$

If $\prod_{i=0}^{5} y_i \prod_{1\le j<l\le 5} y_{j,l} \neq 0$, then

$$y_{1,2} = y_2^{-1}(y_3y_{1,3} + y_4y_{1,4}) \text{ via (4.7)},$$
$$y_{1,5} = (ay_1)^{-1}(cy_2y_{2,5} - by_3y_{3,5}) \text{ via (4.11)},$$
$$y_{2,3} = (y_2y_3)^{-1}(y_1y_3y_{1,3} + y_1y_4y_{1,4} - y_2y_4y_{2,4}) \text{ via (4.9) and (4.10)},$$
$$y_{3,4} = y_3^{-1}(y_2y_{2,4} - y_1y_{1,4}) \text{ via (4.10)},$$
$$y_{4,5} = (ay_4)^{-1}((c-a)y_2y_{2,5} - (b-a)y_3y_{3,5}) \text{ via (4.11)},$$
$$y_0 = (ay_1y_3)^{-1}(cy_2y_{2,4}y_{2,5} - cy_1y_{1,4}y_{2,5} - by_3y_{2,4}y_{3,5}),$$

where the last equality is obtained from (4.3) and from the previous computations. The localization of the ring R at the element $\prod_{i=0}^{5} \eta_i \prod_{1\le j<l\le 5} \eta_{j,l}$ is hence isomorphic to the quotient of the $\mathscr{O}_k$-algebra

$$\mathscr{O}_k[\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_{1,3}, \eta_{1,4}, \eta_{2,4}, \eta_{2,5}, \eta_{3,5}][(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_{1,3}\eta_{1,4}\eta_{2,4}\eta_{2,5}\eta_{3,5})^{-1}]$$

by the ideal generated by

$$(b-a)\eta_1\eta_3\eta_{1,3} + (b-a)\eta_1\eta_4\eta_{1,4} - b\eta_2\eta_4\eta_{2,4} + a\eta_2\eta_5\eta_{2,5} = 0,$$
$$(c-a)\eta_1\eta_3\eta_{1,3} + c\eta_1\eta_4\eta_{1,4} - c\eta_2\eta_4\eta_{2,4} + a\eta_3\eta_5\eta_{3,5} = 0. \qquad \square$$

CHAPTER 5

A singular quartic del Pezzo surface

Let k be a number field, and let S be the the anticanonically embedded singular del Pezzo surface of degree 4 and type $\mathbf{A}_3+\mathbf{A}_1$ given in $\mathbb{P}^4_k$ by the equations

$$x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0. \tag{5.1}$$

Let U be the complement of the lines in S, and let H be the anticanonical height on $S(k)$ induced by the Weil height on $\mathbb{P}^4_k(k)$,

$$H(x_0 : \cdots : x_4) := \prod_{\nu\in\Omega_k} \max\{|x_0|_\nu, \dots, |x_4|_\nu\}.$$

We are interested in the asymptotic behavior of

$$N_{U,H,k}(B) := \#\{x \in U(k) : H(x) \leq B\} \tag{5.2}$$

as $B \to \infty$.

Manin's conjecture for S over arbitrary number fields is proven in joint work with Frei [**FP14**]. There, we parameterize the set of k-rational points on U via integral points on an $\mathscr{O}_k$-model of the universal torsor of a minimal desingularization of S, and then we prove Manin's conjecture for S via refined lattice points counting techniques. We include here the parameterization step of the universal torsor method from the same paper.

5.1. An integral model

Let $\overline{k}$ be an algebraic closure of k, and $X_{\overline{k}}$ the minimal desingularization of $S_{\overline{k}}$ as in [**Der14**]. The aim of this section is to apply Theorem 1.10 to an $\mathscr{O}_k$-model of a universal torsor of $X_{\overline{k}}$ obtained by Construction 3.1 in order to get a parameterization of $U(k)$ via integral points on twisted torsors. An elementary application of the results in [**DF14a**] would lead to the same parameterization.

We start by describing the universal torsor of $X_{\overline{k}}$ inside the spectrum of the associated Cox ring. By the data provided in [**Der14**, §3.4], $X_{\overline{k}}$ is a blowing-up of $\mathbb{P}^2_{\overline{k}}$ in five points in almost general position with Picard group $\operatorname{Pic}(X_{\overline{k}}) \cong \mathbb{Z}^6$, and a Cox ring of $X_{\overline{k}}$ of identity type is a $\operatorname{Pic}(X_{\overline{k}})$-graded $\overline{k}$-algebra with nine generators and one homogeneous relation:

$$\overline{R} = \overline{k}[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7).$$

For $i \in \{1, \dots, 9\}$, the degree of η_i is $[D_i] \in \operatorname{Pic}(X_{\overline{k}})$, where $[D_i]$ are the divisor classes listed below. Let $\ell_0, \dots, \ell_5$ be the basis of $\operatorname{Pic}(X_{\overline{k}})$ given in [**Der14**]. Then the intersection form is defined by $\ell_0^2 = 1$, $\ell_i^2 = -1$ for $1 \leq i \leq 5$, and $\ell_i.\ell_j = 0$ for all $0 \leq i < j \leq 5$. The classes

$$[D_1] = \ell_5, \quad [D_2] = \ell_4, \quad [D_5] = \ell_3$$

are the (-1)-curves on $X_{\overline{k}}$,

$$[D_3] = \ell_1 - \ell_2, \quad [D_4] = \ell_2 - \ell_3,$$
$$[D_6] = \ell_0 - \ell_1 - \ell_4 - \ell_5, \quad [D_7] = \ell_0 - \ell_1 - \ell_2 - \ell_3$$

are the (-2)-curves on $X_{\overline{k}}$, and

$$[D_8] = \ell_0 - \ell_4, \quad [D_9] = \ell_0 - \ell_5.$$

The Dynkin diagram in Figure 1 encodes the configuration of curves on $X_{\overline{k}}$. For any $i \neq j$ the number of edges between D_i and D_j is the intersection

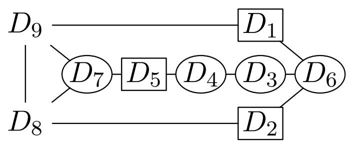

FIGURE 1. Configuration of curves on $X_{\overline{k}}$.

number $[D_i].[D_j]$. For all $1 \leq i < j \leq 9$ such that $[D_i].[D_j] = 0$, we call $J_{i,j}$ the ideal of $\overline{R}$ generated by η_i and η_j, and we define

$$J := \prod_{\substack{1 \leq i < j \leq 9 \\ \overline{D}_i.\overline{D}_j = 0}} J_{i,j}. \tag{5.3}$$

The following computations prove that the open subscheme $\overline{Y}$ of $\operatorname{Spec} \overline{R}$ complement to $V(J)$ is defined by the global sections of a very ample divisor on $X_{\overline{k}}$. Therefore, it is a universal torsor of $X_{\overline{k}}$ by Corollary 2.64.

For all $1 \leq i < j \leq 9$, let $A_{i,j} := \prod_{l \in \{1,\dots,9\} \setminus \{i,j\}} \eta_l$, and $A_{7,8,9} := \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6$. Let J' be the ideal of $\overline{R}$ generated by the following monomials:

$$A_{7,8,9}, \quad A_{1,6}, \quad A_{1,9}, \quad A_{2,6}, \quad A_{2,8}, \quad A_{3,4}, \quad A_{3,6}, \quad A_{4,5}, \quad A_{5,7}. \tag{5.4}$$

Let $[D] := 9\ell_0 - 3\ell_1 - 2\ell_2 - \ell_3 - \ell_4 - \ell_5$, and let I be the ideal of $\overline{R}$ generated by $\bigoplus_{n \in \mathbb{Z}_{>0}} \overline{R}_{n[D]}$. Let I' be the ideal of $\overline{R}$ generated by the following monomials, coming from the relations in the right column:

$$
\begin{aligned}
A_{1,2,6} &:= \eta_3\eta_4\eta_5\eta_7\eta_8\eta_9 & [D] &= [4D_3 + 9D_4 + 15D_5 + 7D_7 + D_8 + D_9],\\
A_{1,3,6} &:= \eta_2\eta_4\eta_5\eta_7\eta_8\eta_9 & [D] &= [4D_2 + D_4 + 3D_5 + 3D_7 + 5D_8 + D_9],\\
A_{2,3,6} &:= \eta_1\eta_4\eta_5\eta_7\eta_8\eta_9 & [D] &= [4D_1 + D_4 + 3D_5 + 3D_7 + D_8 + 5D_9],\\
A_{4,5,8} &:= \eta_1\eta_2\eta_3\eta_6\eta_7\eta_9 & [D] &= [7D_1 + 2D_2 + D_3 + 3D_6 + D_7 + 5D_9],\\
A_{4,5,9} &:= \eta_1\eta_2\eta_3\eta_6\eta_7\eta_8 & [D] &= [2D_1 + 7D_2 + D_3 + 3D_6 + D_7 + 5D_8],\\
A_{5,7,8} &:= \eta_1\eta_2\eta_3\eta_4\eta_6\eta_9 & [D] &= [8D_1 + 5D_2 + 3D_3 + D_4 + 6D_6 + 3D_9],\\
A_{5,7,9} &:= \eta_1\eta_2\eta_3\eta_4\eta_6\eta_8 & [D] &= [5D_1 + 8D_2 + 3D_3 + D_4 + 6D_6 + 3D_8],\\
A_{7,8,9} &:= \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6 & [D] &= [8D_1 + 8D_2 + 6D_3 + 4D_4 + 3D_5 + 9D_6],\\
A_{1,2,8,9} &:= \eta_3\eta_4\eta_5\eta_6\eta_7 & [D] &= [6D_3 + 12D_4 + 19D_5 + D_6 + 8D_7],\\
A_{1,3,4,9} &:= \eta_2\eta_5\eta_6\eta_7\eta_8 & [D] &= [6D_2 + D_5 + D_6 + 2D_7 + 6D_8],\\
A_{2,3,4,8} &:= \eta_1\eta_5\eta_6\eta_7\eta_9 & [D] &= [6D_1 + D_5 + D_6 + 2D_7 + 6D_9].
\end{aligned}
$$

PROPOSITION 5.1. *The open subscheme $\overline{Y}$ complement to $V(J)$ in $\operatorname{Spec}\overline{R}$ is a universal torsor of $X_{\overline{k}}$.*

PROOF. According to [**Der14**, §3.4], the surface $X_{\overline{k}}$ is a blowing-up of $\mathbb{P}^2_{\overline{k}}$ in five points. Such a description of $X_{\overline{k}}$ allows us to determine the irreducible curves and the intersection pairing on $X_{\overline{k}}$ (see [**Har77**, §V.3]), and to show that the divisor class $[D]$ is ample by the Nakai-Moishezon criterion. Then, by Corollary 2.64, the complement of the closed subset of $\operatorname{Spec}\overline{R}$ defined by I is a universal torsor of $X_{\overline{k}}$.

To finish the proof, it suffices to show that $\sqrt{I} = \sqrt{J}$. Computing generators of the product (5.3) shows that $J \subseteq J' \subseteq \sqrt{J}$. The ideal I is generated by the monomials $\prod_{i=1}^{9} \eta_i^{e_i}$ of degree $n[D]$ for $n > 0$. Solving the linear equations $\sum_{i=1}^{9} e_i[D_i] = n[D]$ for nonnegative integers $e_1, \dots, e_9$ and arbitrary $n > 0$, one finds that $I \subseteq I' \subseteq \sqrt{I}$. Moreover, $J' \subseteq I'$. Using the relation among the generators of $\overline{R}$, we prove that $I' \subseteq \sqrt{J}$ by the following computations:

$$\begin{aligned}
&\eta_8 A_{1,6} + \eta_9 A_{2,6} = -\eta_3\eta_4^2\eta_5^3\eta_7 A_{1,2,6}, && \eta_4^2\eta_5^3\eta_7 A_{1,6} + \eta_9 A_{3,6} = -\eta_2\eta_8 A_{1,3,6},\\
&\eta_8 A_{3,6} + \eta_4^2\eta_5^3\eta_7 A_{2,6} = -\eta_1\eta_9 A_{2,3,6}, && \eta_2\eta_3\eta_4\eta_5^2\eta_7 A_{2,8} + \eta_2 A_{4,5} = -\eta_1\eta_9 A_{4,5,8},\\
&A_{1,9} + A_{2,8} = -\eta_3\eta_4^2\eta_5^3\eta_7 A_{1,2,8,9}, && \eta_2\eta_3\eta_4^2\eta_5^2 A_{2,8} + \eta_2 A_{5,7} = -\eta_1\eta_9 A_{5,7,8},\\
&\eta_4\eta_5^3\eta_7 A_{2,8} + A_{3,4} = -\eta_1\eta_9 A_{2,3,4,8}, && \eta_1\eta_3\eta_4\eta_5^2\eta_7 A_{1,9} + \eta_1 A_{4,5} = -\eta_2\eta_8 A_{4,5,9},\\
&\eta_4\eta_5^3\eta_7 A_{1,9} + A_{3,4} = -\eta_2\eta_8 A_{1,3,4,9}, && \eta_1\eta_3\eta_4^2\eta_5^2 A_{1,9} + \eta_1 A_{5,7} = -\eta_2\eta_8 A_{5,7,9}.
\end{aligned}$$

□

We construct now an $\mathscr{O}_k$-model of the universal torsor $\overline{Y} \to X_{\overline{k}}$ which is a universal torsor over a projective $\mathscr{O}_k$-model of $X_{\overline{k}}$.

Let $f_1, \dots, f_9$ be the following monomials of degree $[D]$:

$$\begin{array}{lll}
\eta_1^8\eta_2^8\eta_3^6\eta_4^4\eta_5^3\eta_6^9, & \eta_2^3\eta_3\eta_4^3\eta_5^6\eta_7^4\eta_8^4\eta_9, & \eta_2^5\eta_3\eta_4^2\eta_5^4\eta_6\eta_7^3\eta_8^5,\\
\eta_1^3\eta_3\eta_4^3\eta_5^6\eta_7^4\eta_8\eta_9^4, & \eta_1^5\eta_3\eta_4^2\eta_5^4\eta_6\eta_7^3\eta_9^5, & \eta_1^5\eta_2\eta_5\eta_6\eta_7^2\eta_8\eta_9^5,\\
\eta_1^2\eta_2^2\eta_4\eta_5^3\eta_7^3\eta_8^3\eta_9^3, & \eta_1^6\eta_2^3\eta_3\eta_6^3\eta_7\eta_8\eta_9^4, & \eta_1^7\eta_2^6\eta_3^3\eta_4\eta_6^6\eta_8\eta_9^2.
\end{array}$$

Comparing $f_1, \dots, f_9$ with the generators of J', we see that the radical of the ideal of $\overline{R}$ generated by $f_1, \dots, f_9$ is $\sqrt{J'} = \sqrt{J}$. Hence, $f_1, \dots, f_9$ define the complement of the universal torsor $\overline{Y}$ of $X_{\overline{k}}$ in $\operatorname{Spec}\overline{R}$. Let $R := \mathscr{O}_k[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7)$, and let $\mathscr{Y} \to \mathscr{X}$ be the $\mathscr{O}_k$-model of the universal torsor $\overline{Y} \to X_{\overline{k}}$ defined by $f_1, \dots, f_9$ in Construction 3.1. Some properties of this model are described in the following proposition, which is an application of the results of Chapter 3.

PROPOSITION 5.2. *(i) The scheme $\mathscr{X}$ is smooth, projective, and with geometrically integral fibers over $\mathscr{O}_k$.*

(ii) For every prime ideal $\mathfrak{p}$ of $\mathscr{O}_k$, the fibre $\mathscr{X}_{k(\mathfrak{p})}$ is obtained from $\mathbb{P}^2_{k(\mathfrak{p})}$ by a chain of 5 blowing-ups at $k(\mathfrak{p})$-points.

(iii) The morphism $\mathscr{Y} \to \mathscr{X}$ is a universal torsor under $\mathbb{G}^6_{m,\mathscr{X}}$.

PROOF. Simple computations show that the degrees of the variables η_j appearing in f_i generate $\operatorname{Pic}(X_{\overline{k}})$ for all $i \in \{1, \dots, 9\}$. Since these η_j are invertible in $R[f_i{}^{-1}]$, the condition (3.4) holds for $(R; f_1, \dots, f_9)$.

Since $g := \eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7$ is irreducible in $\overline{k(\mathfrak{p})}[\eta_1, \ldots, \eta_9]$ for all prime ideals $\mathfrak{p}$ of $\mathscr{O}_k$, the $\mathscr{O}_k$-scheme $\mathscr{X}$ has geometrically integral fibers by Proposition 3.5.

The Jacobian matrix $(\partial g/\partial\eta_i)_{1\le i\le 9}$ is

$$(\eta_9,\ \eta_8,\ \eta_4^2\eta_5^3\eta_7,\ 2\eta_3\eta_4\eta_5^3\eta_7,\ 3\eta_3\eta_4^2\eta_5^2\eta_7,\ 0,\ \eta_3\eta_4^2\eta_5^3,\ \eta_2,\ \eta_1),$$

and has rank 1 on $\mathscr{Y}(\overline{k(\mathfrak{p})})$ because the monomials $f_1, \ldots, f_9$ belong to the ideal generated by η_1, η_2. Then $\mathscr{X}$ is smooth by Proposition 3.6.

To verify the hypotheses of Proposition 3.8, we define $C'_{\overline{k}}$ and $C'_{\mathscr{O}_k}$ as the ideals of $\overline{k}[\eta_1, \ldots, \eta_9]$ and $\mathscr{O}_k[\eta_1, \ldots, \eta_9]$ generated by the monic monomials in (5.4) and g. One can check that $C'_{\overline{k}}$ has a Gröbner basis $\{h_1, \ldots, h_l\} \subseteq C'_{\mathscr{O}_k}$ consisting of polynomials whose coefficients are all equal to 1. This implies that $C'_{\overline{k}} \cap \mathscr{O}_k[\eta_1, \ldots, \eta_9] = (h_1, \ldots, h_l) = C'_{\mathscr{O}_k}$.

Since the radical of the ideal $C_{\overline{k}}$ (resp. $C_{\mathscr{O}_k}$) generated by $f_1, \ldots, f_9$ and g in $\overline{k}[\eta_1, \ldots, \eta_9]$ (resp. in $\mathscr{O}_k[\eta_1, \ldots, \eta_9]$) coincides with the radical of $C'_{\overline{k}}$ (resp. of $C'_{\mathscr{O}_k}$), the hypotheses of Proposition 3.8 are satisfied. Hence, $\mathscr{X}$ is projective over $\mathscr{O}_k$.

To prove *(ii)*, let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_k$, and $\overline{k(\mathfrak{p})}$ an algebraic closure of the residue field $k(\mathfrak{p})$. By the data provided in [**Der14**], $X_{\overline{k}}$ is a blowing-up of a split toric $\overline{k}$-variety $X'_{\overline{k}}$ at a closed point and with exceptional divisor corresponding to the section $\eta_1 \in \overline{R}$. The center of the blowing-up $b : X_{\overline{k}} \to X'_{\overline{k}}$ is the intersection of the prime divisors of $X'_{\overline{k}}$ corresponding to the sections η_6, η_9 in the Cox ring $\overline{R}'$ of $X'_{\overline{k}}$ under the identification

$$\overline{R}' \cong \overline{R}/(\eta_1 - 1) \cong \overline{k}[\eta_2, \ldots, \eta_9]/(\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7)$$

provided by [**HKL14**, Proposition 2.2]. The rays of the fan Δ defining $X'_{\overline{k}}$ correspond to the generators $\eta_2, \ldots, \eta_8$ of $\overline{R}'$. We denote them by $\rho_2, \ldots, \rho_8$ (see Figure 2). Let $\mathscr{X}'$ be the $\mathscr{O}_k$-toric scheme defined by Δ as in [**Sal98**,

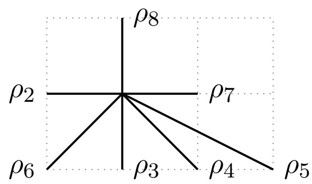

FIGURE 2. The fan Δ.

Remarks 8.6 (b)], and

$$R' := \mathscr{O}_k[\eta_2, \ldots, \eta_9]/(\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7) \cong \mathscr{O}_k[\eta_2, \ldots, \eta_8].$$

The fan Δ has 7 maximal cones. For $1 \le i \le 7$, let f'_i be the product $\prod \eta_j$ running over the indices $j \in \{2, \ldots, 8\}$ such that the ray ρ_j does not belong to the i-th maximal cone. By [**Sal98**, §8], the monomials $f'_1, \ldots, f'_7$ define the complement of the universal torsor of $X'_{\overline{k}}$ contained in $\operatorname{Spec} \overline{R}'$.

For every $i \in \{1, \ldots, 7\}$, the open affine toric subvariety of $X'_{\overline{k}}$ corresponding to the i-th maximal cone has trivial Picard group, and its complement consists of the effective divisor defined by the section f'_i. Hence,

$(R'; f'_1, \dots, f'_7)$ satisfies (3.4), and $\mathscr{X}'$ is the $\mathscr{O}_k$-model of X' defined by Construction 3.1.

Recall the notation before Proposition 3.9. Since the radical of the ideal of $\mathscr{O}_k[\eta_1, \dots, \eta_9]$ generated by $f_1, \dots, f_9$ is the radical of the ideal generated by $b^*(f'_1)\eta_6, \dots, b^*(f'_7)\eta_6, b^*(f'_1)\eta_9, \dots, b^*(f'_7)\eta_9$, the model $\mathscr{X}$ is a blowing-up of $\mathscr{X}'$ with center the closed subscheme defined by η_6, η_9 by Proposition 3.9.

Since this closed subscheme is flat over $\mathscr{O}_k$, the variety $\mathscr{X}_{k(\mathfrak{p})}$ is the blowing-up of $\mathscr{X}'_{k(\mathfrak{p})}$ in the $k(\mathfrak{p})$-point defined by η_6, η_9. Moreover, $\mathscr{X}'_{k(\mathfrak{p})}$ is the split toric $k(\mathfrak{p})$-variety defined by Δ, which is obtained from $\mathbb{P}^2_{k(\mathfrak{p})}$ by four toric blowing-ups at $k(\mathfrak{p})$-points. Therefore, $H^i(\mathscr{X}_{k(\mathfrak{p})}, \mathcal{O}_{\mathscr{X}_{k(\mathfrak{p})}}) = 0$ for $i \in \{1, 2\}$ by [**Har77**, Proposition V.3.4] and [**Ful93**, p. 74]. Hence, all the hypotheses of Theorem 3.3 are satisfied. $\square$

5.2. Parameterization

The action of $\mathbb{G}^6_{m,\mathscr{O}_k}(\mathscr{O}_k) \cong (\mathscr{O}_k^\times)^6$ on $\mathscr{Y}(\mathscr{O}_k)$ is given by (3.2), where $m^{(1)}, \dots, m^{(9)} \in \mathbb{Z}^6$ denote the degrees of $\eta_1, \dots, \eta_9$, respectively, under the identification $\operatorname{Pic}(X_{\overline{k}}) \cong \mathbb{Z}^6$ provided by the basis $\ell_0, \dots, \ell_5$. Namely,

$$m^{(1)} = (0,0,0,0,0,1), m^{(2)} = (0,0,0,0,1,0), m^{(3)} = (0,1,-1,0,0,0),$$
$$m^{(4)} = (0,0,1,-1,0,0), m^{(5)} = (0,0,0,1,0,0), m^{(6)} = (1,-1,0,0,-1,-1),$$
$$m^{(7)} = (1,-1,-1,-1,0,0), m^{(8)} = (1,0,0,0,-1,0), m^{(9)} = (1,0,0,0,0,-1).$$

Before we apply Theorem 1.10 to obtain a parameterization of $U(k)$ by integral points on twists of $\mathscr{Y}$, we describe the preimage of U inside the universal torsor, and we fix some more notation.

Let $X := \mathscr{X}_k$, $Y := \mathscr{Y}_k$, and $\pi : Y \to X$ the base change of the universal torsor morphism $\mathscr{Y} \to \mathscr{X}$ under the inclusion $\mathscr{O}_k \subseteq k$. We observe that π is a universal torsor of X by Remark 3.2.

Let $\overline{\Psi} : \overline{Y} \to S_{\overline{k}}$ be the composition of the universal torsor morphism $\overline{Y} \to X_{\overline{k}}$ with the minimal desingularization morphism $X_{\overline{k}} \to S_{\overline{k}}$. According to [**Der14**, §3.4], the map $\overline{\Psi} : \overline{Y}(\overline{k}) \to S_{\overline{k}}(\overline{k})$ sends a point $(y_1, \dots, y_9) \in \overline{Y}(\overline{k})$ to the point

$$(y_2y_3y_4y_5y_6y_7y_8 : y_1^2y_2^2y_3^2y_4y_6^3 : y_1y_2y_3^2y_4^2y_5^2y_6^2y_7 : y_1y_3y_4y_5y_6y_7y_9 : y_7y_8y_9) \tag{5.5}$$

in $S_{\overline{k}}(\overline{k}) \subseteq \mathbb{P}^4(\overline{k})$. Since $\overline{\Psi}$ is defined over k, it induces a morphism $\Psi : Y \to S \subseteq \mathbb{P}^4_k$ which is given by (5.5) on k-rational points.

Since $\pi : Y \to X$ is a geometric quotient, the invariant morphism Ψ factors through a minimal desingularization $\gamma : X \to S$, which is a model of the minimal desingularization $X_{\overline{k}} \to S_{\overline{k}}$.

We recall that U is defined as the complement of the lines in S. By [**Der14**, Table 6], the surface $S_{\overline{k}}$ contains exactly three lines of $\mathbb{P}^4_{\overline{k}}$. These are defined over k and an easy computation shows that

$$S \smallsetminus U = \{x_0x_1 = x_0x_3 = x_1x_3 = x_2 = 0\}.$$

Then $\Psi^{-1}(S \smallsetminus U) = \{\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7 = 0\}$, and $\Psi^{-1}(U(k)) = Y(k) \cap (k^\times)^7 \times k^2$.

Let $\mathcal{C}$ be a fixed system of integral representatives for the class group of $\mathscr{O}_k$, that is, it contains exactly one integral ideal from each class. For any given $\underline{\mathfrak{c}} = (\mathfrak{c}_0, \dots, \mathfrak{c}_5) \in \mathcal{C}^6$, we denote by ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}\mathscr{Y} \to \mathscr{X}$ the twist of $\mathscr{Y}$ constructed as in Definition 1.9. We write

$$\underline{\mathfrak{c}}_*^{m^{(j)}} := \begin{cases} \underline{\mathfrak{c}}^{m^{(j)}} \smallsetminus \{0\} & \text{if } j \in \{1, \dots, 7\}, \\ \underline{\mathfrak{c}}^{m^{(j)}} & \text{if } j \in \{8, 9\}. \end{cases}$$

For $\nu \in \Omega_k$ and $(x_1, \dots, x_8) \in k_v^8$ with $x_1 \neq 0$, we write

$$h_\nu(x_1, \dots, x_8) := \max \left\{ \begin{array}{l} |x_2x_3x_4x_5x_6x_7x_8|_\nu, |x_1^2x_2^2x_3^2x_4x_6^3|_\nu, \\ |x_1x_2x_3^2x_4^2x_5^2x_6^2x_7|_\nu, \\ |x_3x_4x_5x_6x_7(x_3x_4^2x_5^3x_7 + x_2x_8)|_\nu, \\ \left|\dfrac{x_2x_7x_8^2 + x_3x_4^2x_5^3x_7^2x_8}{x_1}\right|_\nu \end{array} \right\}.$$

Let $\mathcal{F}$ be a fundamental domain for the action

$$\text{of} \quad \mathscr{U}_k \times (\mathscr{O}_k^\times)^5 \quad \text{on} \quad (k^\times)^7 \times k^2,$$

induced by (3.2), where $\underline{u} = (u_0, \dots, u_5)$ maps $(y_1, \dots, y_9)$ to

$$(\underline{u}^{m^{(1)}} \cdot y_1, \dots, \underline{u}^{m^{(9)}} \cdot y_9). \tag{5.6}$$

After all these preparations, we define $M_{\underline{\mathfrak{c}}}(B)$ as the set of all

$$(y_1, \dots, y_9) \in (\underline{\mathfrak{c}}_*^{m^{(1)}} \times \dots \times \underline{\mathfrak{c}}_*^{m^{(9)}}) \cap \mathcal{F}$$

that satisfy the *height conditions*

$$\prod_{\nu \in \Omega_\infty} h_\nu(\sigma_v(y_1, \dots, y_8)) \leq \mathfrak{N}(\mathfrak{c}_0^3\mathfrak{c}_1^{-1} \cdots \mathfrak{c}_5^{-1})B, \tag{5.7}$$

the *torsor equation*

$$y_1y_9 + y_2y_8 + y_3y_4^2y_5^3y_7 = 0, \tag{5.8}$$

and the *coprimality conditions*

$$y_i\underline{\mathfrak{c}}^{-m^{(i)}} + y_j\underline{\mathfrak{c}}^{-m^{(j)}} = \mathscr{O}_k \tag{5.9}$$

for all distinct nonadjacent vertices D_i, D_j in Figure 1. We can now reduce the estimation of $N_{U,H,k}(B)$ to counting the $M_{\underline{\mathfrak{c}}}(B)$.

PROPOSITION 5.3. *With the sets $M_{\underline{\mathfrak{c}}}(B)$ defined as above and $N_{U,H,k}(B)$ as in 5.2, we have*

$$N_{U,H,k}(B) = \frac{1}{\# u_k} \sum_{\underline{\mathfrak{c}} \in \mathcal{C}^6} \# M_{\underline{\mathfrak{c}}}(B).$$

PROOF. Since U is contained in the smooth locus of S, the minimal desingularization $\gamma : X \to S$ induces an isomorphism $\gamma^{-1}(U) \to U$, so

$$N_{U,H,k}(B) = \#\{x \in \gamma^{-1}(U)(k) \mid H(\gamma(x)) \leq B\}.$$

By Theorem 1.10, *(ii)*, there is a disjoint union

$$\gamma^{-1}(U)(k) = \bigsqcup_{\underline{\mathfrak{c}} \in \mathcal{C}^6} {}_{\underline{\mathfrak{c}}}\pi({}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) \cap \Psi^{-1}(U(k))).$$

Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$. Since, for any $y_j \in \underline{\mathfrak{c}}^{m^{(j)}}$, the associated ideal $y_j \underline{\mathfrak{c}}^{-m^{(j)}}$ is an integral ideal, the coprimality condition

$$\prod_{\substack{1 \le i < j \le 9 \\ \overline{D}_i . \overline{D}_j = 0}} \left(y_i \underline{\mathfrak{c}}^{-m^{(i)}} + y_j \underline{\mathfrak{c}}^{-m^{(j)}} \right) = \mathscr{O}_k$$

is equivalent to the coprimality conditions (5.9). Then by Theorem 1.10, *(iii)*, ${}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) \cap \Psi^{-1}(U(k))$ is the set of all

$$(y_1, \dots, y_9) \in (\underline{\mathfrak{c}}_*^{m^{(1)}} \times \cdots \times \underline{\mathfrak{c}}_*^{m^{(9)}})$$

that satisfy (5.8) and (5.9).

Let $(y_1, \dots, y_9) \in {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$. Using the torsor equation (5.8) to eliminate y_9, we see that

$$H(\Psi(y_1, \dots, y_9)) = \prod_{v \in \Omega_k} h_v(\sigma_v(y_1, \dots, y_8)).$$

Moreover, the coprimality conditions (5.9) imply that

$$\prod_{v \in \Omega_f} h_v(\sigma_v(y_1, \dots, y_8)) = \mathfrak{N}(y_2 y_3 y_4 y_5 y_6 y_7 y_8 \mathscr{O}_k + \cdots + y_7 y_8 y_9 \mathscr{O}_k)^{-1} =$$

$$= \mathfrak{N}(\mathfrak{c}_0^{-3} \mathfrak{c}_1 \cdots \mathfrak{c}_5).$$

Thus the condition $H(\Psi(y_1, \dots, y_9)) \le B$ is equivalent to our height conditions (5.7).

Let $\mathcal{F}' \subseteq \mathcal{F}$ be a fundamental domain for the action of $\mathbb{G}_{m,\mathscr{O}_k}^6(\mathscr{O}_k) = (\mathscr{O}_k^\times)^6$ on $(k^\times)^7 \times k^2$ and consider the set $M'_{\underline{\mathfrak{c}}}(B)$ of all

$$(y_1, \dots, y_9) \in (\underline{\mathfrak{c}}_*^{m^{(1)}} \times \cdots \times \underline{\mathfrak{c}}_*^{m^{(9)}}) \cap \mathcal{F}'$$

that satisfy (5.7), (5.8) and (5.9).

Since the action of $(\mathscr{O}_k^\times)^6$ on ${}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$ is free, each orbit is the union of $\# u_k$ orbits of the induced action of $\mathscr{U}_k \times (\mathscr{O}_k^\times)^5$. Each of these orbits has exactly one representative in $\mathcal{F}$, so $\# M_{\underline{\mathfrak{c}}}(B) = \# u_k \# M'_{\underline{\mathfrak{c}}}(B)$.

Finally, we observe that the fibers of ${}_{\underline{\mathfrak{c}}}\pi$ are the orbits of the action of $\mathbb{G}_{m,\mathscr{X}}^6$ on ${}_{\underline{\mathfrak{c}}}\mathscr{Y}$. Hence, there is a bijection between the sets $U(k)$ and $\bigsqcup_{\underline{\mathfrak{c}} \in \mathcal{C}^6} ({}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) \cap \Psi^{-1}(U(k)) \cap \mathcal{F}')$, so

$$N_{U,H,k}(B) = \sum_{\underline{\mathfrak{c}} \in \mathcal{C}^6} \# M'_{\underline{\mathfrak{c}}}(B) = \frac{1}{\# u_k} \sum_{\underline{\mathfrak{c}} \in \mathcal{C}^6} \# M_{\underline{\mathfrak{c}}}(B). \qquad \square$$

5.3. Manin's conjecture

In joint work with Frei [**FP14**], we use the parameterization provided by Proposition 5.3, together with a suitable choice of the fundamental domain $\mathcal{F}$, to prove the following theorem.

THEOREM 5.4. *Let k be a number field of degree d, let S be given in $\mathbb{P}_k^4$ by* (5.1)*, let U be the complement of the lines in S, and let $\epsilon > 0$. As $B \to \infty$,*

$$N_{U,H,k}(B) = c_{S,H} B (\log B)^5 + O(B (\log B)^{5 - 1/d + \epsilon}),$$

with an explicit $c_{S,H} > 0$*. This formula agrees with Peyre's refined version of Manin's conjecture* [**Pey03**, Formule empirique 5.1]*. The implicit constant in the error term depends on* k *and* ϵ*.*

The proof (cf. [**FP14**, §5-§13]) involves refined techinques, including o-minimality, to count lattice points in certain unbounded subsets of $\mathbb{R}^{8d}$.

CHAPTER 6

Toric varieties

In this chapter we generalize Salberger's proof of Manin's conjecture for toric varieties over $\mathbb{Q}$ [**Sal98**], by proving the following theorem.

THEOREM 6.1. *Let X be a smooth equivariant compactification of a split torus T over an imaginary quadratic number field k. Assume that the anticanonical sheaf of X is generated by its global sections. Let H be the toric anticanonical height function on $X(k)$ defined in* [**Sal98**, §10]*, and let $N_{T,H,k}(B)$ be the number of k-rational points on T of toric anticanonical height at most B. Then, for all $\varepsilon > 0$,*

$$N_{T,H,k}(B) = C_{X,H,k} B(\log B)^{r-1} + O(B(1 + \log B)^{r-2+1/f+\varepsilon}),$$

where r is the rank of the Picard group of X, f is the smallest positive integer such that there exist f rays of the fan Σ defining X not contained in a cone of Σ, and $C_{X,H,k}$ is the constant predicted by Peyre in [**Pey95**].

Manin's conjecture for toric varieties has been considered also by Batyrev and Tschinkel [**BT98**] and Bourqui [**Bou11**], who proved it for smooth projective toric varieties over arbitrary number fields and over global fields of positive characteristic, respectively, via harmonic analysis techniques. Moreover, the error term in [**Sal98**] was improved by de la Bretèche in [**Bre01**] by means of the precise estimations of multiple sums of arithmetic functions. We also recall that Schanuel's precursory work [**Sch79**] constitutes a proof of Manin's conjecture for projective spaces via universal torsor method over arbitrary number fields, and that recently the same method has been applied over arbitrary number fields for a certain singular toric cubic surface [**Fre13**].

Our proof is an explicative application of the universal torsor method for a family of varieties over number fields beyond $\mathbb{Q}$. We restrict to the case of imaginary quadratic fields because, even though the choice of a suitable fundamental domain for the action of $(\mathscr{O}_k^\times)^r$ is clear for some specific varieties [**Sch79, Fre13**], it strongly depends on the features of the variety, and hence, a generalization of the strategies of those papers to the family considered above does not seem straightforward.

The chapter is organized as follows. In Section 6.1 we apply the results of Chapter 1 to recover the parameterization via lattice points on the integral universal torsors constructed in [**Sal98**] that is described in [**Rob98**], and we show that the anticanonical height function defined in [**Sal98**] lifts to the integral points on the twisted torsors involving just the absolute values at the infinite places of the number field (cf. Proposition 6.2). In Section 6.2 we study a multiplicative function, introduced in [**Pey95**] and used also in

[**Sal98**], attached to the characteristic function of the set of integral coordinates that belong to the universal torsor, and we perform Möbius inversion to get rid of the coprimality conditions. Thus, we reduce to count lattice points in bounded subsets $D(B)$ of an $\mathbb{R}$-vector space $\mathbb{C}^N \cong \mathbb{R}^{2N}$. Section 6.3 contains the proof of Theorem 6.1. Since the height function depends only on the absolute value of the coordinates, we partition $\mathbb{C}^N$ in strongly convex rational polyhedral cones, each of them spanned by a fundamental domain of the lattice which is a parallelotope F with the property that, for every lattice point x in the cone, $x + F$ intersects the boundary of the region $D(B)$ defined by the height function if and only if the lifted height of x is at most B and the height of $x + \gamma$ is strictly greater than B, where γ is the long diagonal of the parallelotope (cf. Subsection 6.3.2). For each cone C, we compare the number of lattice points in $C \cap D(B)$ with the volume of the region $C \cap D(B)$ (cf. Proposition 6.14), and we estimate the difference by counting the number of lattice points $x \in C \cap D(B)$ such that $x + F \nsubseteq C \cap D(B)$ (cf. Proposition 6.13). In the last section we show that Peyre's conjecture [**Pey95**] is satisfied by the constant in the main term of the asymptotic formula obtained in Section 6.3:

$$C_{X,H,k} = \alpha(X)\kappa|\Delta_k|^{-N/2} h_k^r w_k^{-r} (2\pi)^N \#\Sigma_{max}, \tag{6.1}$$

where $\alpha(X)$ is the constant defined in [**Pey95**, §2] (cf. [**Sal98**, §7]), κ is a constant depending on X and k (see (6.7)), Δ_k is the discriminant of k, h_k is the class number of k, w_k is the number of roots of 1 contained in k, $\#\Sigma_{max}$ is the number of maximal cones of the fan defining X. The results of Sections 6.1 and 6.2 up to §6.2.2 hold for arbitrary number fields. The content of this chapter first appeared in [**Pie15**].

6.1. Universal torsors and heights on toric varieties

6.1.1. Parameterization via universal torsors. Let X be a smooth complete toric variety over a number field k that is an equivariant compactification of a split torus T. We denote by T_* the lattice of cocharactes of T and by Σ the fan in $T_{*\mathbb{R}} := T_* \otimes_{\mathbb{Z}} \mathbb{R}$ that defines X.

By [**CTS76**, §4] and [**Sal98**, §8] (cf. [**Mad05**, Appendix]), we know that X has a universal torsor $\pi : Y \to X$, which is unique up to isomorphism and can be realized as the the toric variety defined by the pullback of Σ under the morphism of lattices

$$\Lambda_* \to T_*, \quad \rho \mapsto n_\rho,$$

where n_ρ is the unique generator of $\rho \cap T_*$, $\Lambda_* := \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\rho$ and $\Sigma(1)$ denotes the set of rays of Σ. We denote by $\Lambda = \pi^{-1}(T)$ the torus contained in Y whose group of cocharacters is Λ_*. By [**Cox95**] a Cox ring of X of identity type is a Pic(X)-graded polynomial ring in $N := \#\Sigma(1)$ variables

$$R = k[x_\rho : \rho \in \Sigma(1)],$$

where $\Sigma(1)$ is the set of rays of Σ, and the degree of the variable x_ρ is the class of the prime toric invariant divisor D_ρ corresponding to the ray ρ. Moreover, there is an open embedding $Y \to \mathbb{A}_k^N := \operatorname{Spec} R$ whose complement is the closed subset defined by the monomials

$$\underline{x}^{D_\sigma} := \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} x_\rho, \quad \sigma\in\Sigma_{max},$$

where Σ_{max} is the set of maximal cones of Σ, and $\sigma(1)$ is the set of rays of Σ contained in the cone σ (see [**Sal98**, §8]).

Let $\mathscr{O}_k$ be the ring of integers of k. By [**Sal98**, Remark 8.6] (cf. Theorem 3.3), the universal torsor $Y\to X$ admits an $\mathscr{O}_k$-model $\mathscr{Y}\to\mathscr{X}$, which is a universal torsor of the $\mathscr{O}_k$-toric scheme $\mathscr{X}$ defined by the fan Σ, and

$$\mathscr{Y}\cong \mathbb{A}^N_{\mathscr{O}_k}\smallsetminus V(\underline{x}^{D_\sigma} : \sigma\in\Sigma_{max}). \tag{6.2}$$

Fix a basis $\ell_1,\dots,\ell_r$ of $\mathrm{Pic}(X)$ (namely, an isomorphism $\mathrm{Pic}(X)\cong\mathbb{Z}^r$). Let $\mathcal{C}$ be a system of representatives for the class group of k. By Theorem 1.10, the universal torsor $\mathscr{Y}\to\mathscr{X}$ induces the following parameterization of the set of rational points of X:

$$X(k) = \bigsqcup_{\underline{\mathfrak{c}}\in\mathcal{C}^r} {}_{\underline{\mathfrak{c}}}\tilde\pi({}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)), \tag{6.3}$$

where ${}_{\underline{\mathfrak{c}}}\tilde\pi : {}_{\underline{\mathfrak{c}}}\mathscr{Y}\to\mathscr{X}$ is the twist of $\mathscr{Y}\to\mathscr{X}$ according to Definition 1.9. The parameterization can be made explicit as follows. For every $\underline{\mathfrak{c}}=(\mathfrak{c}_1,\dots,\mathfrak{c}_r)\in\mathcal{C}^r$ and every divisor D of X, let $\underline{\mathfrak{c}}^D := \prod_{i=1}^r \mathfrak{c}_i^{a_i}$, where $a_1,\dots,a_r\in\mathbb{Z}$ are determined by the equality $[D]=\sum_{i=1}^r a_i\ell_i$ in $\mathrm{Pic}(X)$. For every $\underline{x} = (x_\rho)_{\rho\in\Sigma(1)}\in k^N$ and $\underline{\mathfrak{c}}\in\mathcal{C}^r$, let $\underline{\mathfrak{c}}_{\underline{x}} = (\underline{\mathfrak{c}}_{x_\rho})_{\rho\in\Sigma(1)}$ be defined by $\underline{\mathfrak{c}}_{x_\rho} := x_\rho\underline{\mathfrak{c}}^{-D_\rho}$ for all $\rho\in\Sigma(1)$. For every $\sigma\in\Sigma_{max}$, let $D_\sigma := \sum_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} D_\rho$ and $\underline{\mathfrak{c}}_{\underline{x}}^{D_\sigma} := \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)}\underline{\mathfrak{c}}_{x_\rho}$. With this notation,

$${}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) = \{\underline{x}\in\bigoplus_{\rho\in\Sigma(1)}\underline{\mathfrak{c}}^{D_\rho} : \sum_{\sigma\in\Sigma_{max}}\underline{\mathfrak{c}}_{\underline{x}}^{D_\sigma} = \mathscr{O}_k\}\subseteq k^N \tag{6.4}$$

for all $\underline{\mathfrak{c}}\in\mathcal{C}^r$, by Theorem 1.10 (cf. [**Rob98**, p. 15]).

6.1.2. Anticanonical height function. We recall the anticanonical height function defined by Salberger in [**Sal98**, §10] and list some of its properties. We need first some notation.

For every torus invariant divisor $D=\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ of X and for every $\sigma\in\Sigma_{max}$, let $u_{\sigma,D}$ be the character of T determined by $u_{\sigma,D}(n_\rho)=a_\rho$ for all $\rho\in\sigma(1)$, and define $D(\sigma) := D-\sum_{\rho\in\Sigma(1)} u_{\sigma,D}(n_\rho)D_\rho$.

We denote by $-K$ the anticanonical divisor $\sum_{\rho\in\Sigma(1)} D_\rho$ of X. For every $\sigma\in\Sigma_{max}$ and $\rho\in\Sigma(1)$, let $\alpha_{\sigma,\rho} := 1-u_{\sigma,-K}(n_\rho)$, so that $-K(\sigma) = \sum_{\rho\in\Sigma(1)}\alpha_{\sigma,\rho}D_\rho$. By [**Sal98**, Proposition 8.7 (a)], we know that if $\mathcal{O}_X(-K)$ is generated by its global sections, then $-K(\sigma)$ is an effective divisor for all $\sigma\in\Sigma_{max}$, and $\alpha_{\sigma,\rho}=0$ for all $\rho\in\sigma(1)$. From now on, we assume that $\mathcal{O}_X(-K)$ is generated by its global sections.

Let $\nu\in\Omega_k$. For every $\underline{x}=(x_\rho)_{\rho\in\Sigma(1)}\in k_\nu^N$ and every effective divisor $D=\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ of X, we denote by $\underline{x}^D$ the product $\prod_{\rho\in\Sigma(1)} x_\rho^{a_\rho}$.

Let $H : X(k)\to\mathbb{R}_{\geq 0}$ be the height function corresponding to the anticanonical divisor $-K$ defined in [**Sal98**, (10.4)]. For every $\underline{x}\in Y(k)$,

$$H(\pi(\underline{x})) = \prod_{\nu\in\Omega_k}\sup_{\sigma\in\Sigma_{max}} |\underline{x}^{-K(\sigma)}|_\nu \tag{6.5}$$

by [**Sal98**, Proposition 10.14]. For integral points on the twisted torsors that appear in (6.3), the height (6.5) can be expressed as product involving just the archimedean places as the following proposition shows.

PROPOSITION 6.2. *Let* $\underline{\mathfrak{c}} \in \mathcal{C}^r$ *and* $\underline{x} \in {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k)$, *then*

$$H(\pi(\underline{x})) = \frac{1}{\mathfrak{N}(\underline{\mathfrak{c}}^{-K})} \prod_{\nu \in \Omega_\infty} \sup_{\sigma \in \Sigma_{max}} |\underline{x}^{-K(\sigma)}|_\nu.$$

PROOF. For every $\rho \in \Sigma(1)$, let $m_{\rho,\mathfrak{p}}, n_{\rho,\mathfrak{p}} \in \mathbb{Z}$ be defined by the equalities $\underline{\mathfrak{c}}^{D_\rho} = \prod_{\mathfrak{p}} \mathfrak{p}^{m_{\rho,\mathfrak{p}}}$ and $x_\rho \mathscr{O}_k = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\rho,\mathfrak{p}}}$, where $\prod_{\mathfrak{p}}$ denotes a product over the prime ideals of $\mathscr{O}_k$. Let $\nu \in \Omega_f$ be a finite place associated to a prime ideal $\mathfrak{p}$ of $\mathscr{O}_k$. Then

$$\sup_{\sigma \in \Sigma_{max}} |\underline{x}^{-K(\sigma)}|_\nu = \left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{\min_{\sigma \in \Sigma_{max}} \sum_{\rho \in \Sigma(1)} \alpha_{\sigma,\rho} n_{\rho,\mathfrak{p}}}.$$

We write

$$\begin{aligned} \min_{\sigma \in \Sigma_{max}} \sum_{\rho \in \Sigma(1)} \alpha_{\sigma,\rho} n_{\rho,\mathfrak{p}} &= \min_{\sigma \in \Sigma_{max}} \left(\sum_{\rho \in \Sigma(1)} \alpha_{\sigma,\rho}(n_{\rho,\mathfrak{p}} - m_{\rho,\mathfrak{p}}) + \sum_{\rho \in \Sigma(1)} \alpha_{\sigma,\rho} m_{\rho,\mathfrak{p}} \right) \\ &= \min_{\sigma \in \Sigma_{max}} \sum_{\rho \in \Sigma(1) \smallsetminus \sigma(1)} \alpha_{\sigma,\rho}(n_{\rho,\mathfrak{p}} - m_{\rho,\mathfrak{p}}) + \sum_{\rho \in \Sigma(1)} m_{\rho,\mathfrak{p}}, \end{aligned}$$

as $\alpha_{\sigma,\rho} = 0$ for all $\rho \in \sigma(1)$ and $[-K(\sigma)] = [-K]$ in $\mathrm{Pic}(X)$ for all $\sigma \in \Sigma_{max}$.

Since $x_\rho \in \underline{\mathfrak{c}}^{D_\rho}$ for all $\rho \in \Sigma(1)$, the inequality $n_{\rho,\mathfrak{p}} \geq m_{\rho,\mathfrak{p}}$ holds for all $\rho \in \Sigma(1)$. The coprimality condition $\sum_{\sigma \in \Sigma_{max}} \underline{\mathfrak{c}}_{\underline{x}}^{D_\sigma} = \mathscr{O}_k$ gives

$$\min_{\sigma \in \Sigma_{max}} \sum_{\rho \in \Sigma(1) \smallsetminus \sigma(1)} (n_{\rho,\mathfrak{p}} - m_{\rho,\mathfrak{p}}) = 0.$$

Hence, $\min_{\sigma \in \Sigma_{max}} \sum_{\rho \in \Sigma(1)} \alpha_{\sigma,\rho} n_{\rho,\mathfrak{p}} = \sum_{\rho \in \Sigma(1)} m_{\rho,\mathfrak{p}}$, as $-K(\sigma)$ is effective for all $\sigma \in \Sigma_{max}$. □

LEMMA 6.3. *Let* $\nu \in \Omega_k$ *and* $\underline{x} \in Y(k_\nu)$ *then*

$$|\underline{x}^{-K}|_\nu \leq \sup_{\sigma \in \Sigma_{max}} |\underline{x}^{-K(\sigma)}|_\nu.$$

PROOF. By [**Sal98**, Proposition 9.2] there exists $\sigma \in \Sigma_{max}$ such that $\left|\frac{\underline{x}^{-K}}{\underline{x}^{-K(\sigma)}}\right|_\nu \leq 1$. Then $|\underline{x}^{-K}|_\nu \leq |\underline{x}^{-K(\sigma)}|_\nu \leq \sup_{\sigma \in \Sigma_{max}} |\underline{x}^{-K(\sigma)}|_\nu$. □

6.2. Möbius inversion

6.2.1. Generalized Möbius function. We introduce a generalized Möbius function, like in [**Sal98**, §11] and [**Pey95**, §8.5], that we use to get rid of the coprimality condition in (6.4) via Möbius inversion. In order to do so, we fix some notation.

Let $\mathcal{I}$ be the set of nonzero ideals of $\mathscr{O}_k$. For every $\underline{\mathfrak{d}} = (\mathfrak{d}_\rho)_{\rho \in \Sigma(1)} \in \mathcal{I}^N$, let $\mathfrak{N}(\underline{\mathfrak{d}}) := \prod_{\rho \in \Sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho) \in \mathbb{Z}_{>0}$. For every prime ideal $\mathfrak{p}$ of $\mathscr{O}_k$, we denote by $\sum_{\underline{\mathfrak{d}},\mathfrak{p}}$ a sum over the set of $\underline{\mathfrak{d}} = (\mathfrak{d}_\rho)_{\rho \in \Sigma(1)} \in \mathcal{I}^N$ such that $\mathfrak{d}_\rho$ is a nonnegative power of $\mathfrak{p}$ for all $\rho \in \Sigma(1)$. We denote by $\prod_{\mathfrak{p}}$ a product over all prime ideals of $\mathscr{O}_k$, and by $\sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b}$, $\sum_{\mathfrak{N}(\underline{\mathfrak{d}})\geq b}$ a sum over the set of $\underline{\mathfrak{d}} \in \mathcal{I}^N$ that satisfy $\mathfrak{N}(\underline{\mathfrak{d}}) \leq b$, $\mathfrak{N}(\underline{\mathfrak{d}}) \geq b$ respectively.

For every $\underline{\mathfrak{b}} = (\mathfrak{b}_\rho)_{\rho\in\Sigma(1)} \in \mathcal{I}^N$ and for every effective divisor D of the form $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ on X, let $\underline{\mathfrak{b}}^D := \prod_{\rho\in\Sigma(1)} \mathfrak{b}_\rho^{a_\rho}$. We denote by

$$\chi : \mathcal{I}^N \to \{0,1\}$$

the characteristic function of the subset

$$\{\underline{\mathfrak{b}} \in \mathcal{I}^N : \sum_{\sigma\in\Sigma_{max}} \underline{\mathfrak{b}}^{D_\sigma} = \mathscr{O}_k\} \ \subseteq\ \mathcal{I}^N.$$

For every $\underline{\mathfrak{d}} \in \mathcal{I}^N$, let

$$\chi_{\underline{\mathfrak{d}}} : \mathcal{I}^N \to \{0,1\}$$

be the characteristic function of the subset

$$\{\underline{\mathfrak{b}} \in \mathcal{I}^N : \underline{\mathfrak{d}}|\underline{\mathfrak{b}}\} \ \subseteq\ \mathcal{I}^N,$$

where $\underline{\mathfrak{d}}|\underline{\mathfrak{b}}$ means $\mathfrak{b}_\rho \subseteq \mathfrak{d}_\rho$ for all $\rho \in \Sigma(1)$.

By [**Pey95**, Lemme 8.5.1], there exists a unique function $\mu : \mathcal{I}^N \to \mathbb{Z}$ such that

$$\chi = \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \mu(\underline{\mathfrak{d}})\chi_{\underline{\mathfrak{d}}}.$$

Moreover, for every $\underline{\mathfrak{d}} \in \mathcal{I}^N$,

$$\mu(\underline{\mathfrak{d}}) = \prod_{\mathfrak{p}} \mu(\underline{\mathfrak{d}}_{\mathfrak{p}}), \tag{6.6}$$

where $\underline{\mathfrak{d}}_{\mathfrak{p}} := (\mathfrak{d}_{\rho,\mathfrak{p}})_{\rho\in\Sigma(1)}$ with $\mathfrak{d}_{\rho,\mathfrak{p}}$ a nonnegative power of $\mathfrak{p}$ such that $\mathfrak{d}_\rho = \prod_{\mathfrak{p}} \mathfrak{d}_{\rho,\mathfrak{p}}$ holds in $\mathscr{O}_k$ for all $\rho \in \Sigma(1)$.

The next proposition shows some properties of μ that we use later.

Proposition 6.4. *Let $f \in \mathbb{Z}_{>0}$ be the smallest number of rays of Σ not contained in a cone of Σ. Then*

(i) the product $\prod_{\mathfrak{p}} \sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{\mu(\underline{\mathfrak{d}})}{\mathfrak{N}(\underline{\mathfrak{d}})^s}$ and the sum $\sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \frac{\mu(\underline{\mathfrak{d}})}{\mathfrak{N}(\underline{\mathfrak{d}})^s}$ converge absolutely and coincide for $s > 1/f$;
(ii) $\sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b} |\mu(\underline{\mathfrak{d}})| = O(b^{1/f+\varepsilon})$, for all $\varepsilon > 0$;
(iii) $\sum_{\mathfrak{N}(\underline{\mathfrak{d}})\geq b} (|\mu(\underline{\mathfrak{d}})|/\,\mathfrak{N}(\underline{\mathfrak{d}})) = O(b^{1/f-1+\varepsilon})$, for all $\varepsilon > 0$.

Proof. To prove (i), we note that (6.6) gives

$$\sum_{\substack{\underline{\mathfrak{d}}\in\mathcal{I}^N\\ \mathfrak{N}(\underline{\mathfrak{d}})\leq b}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^s} = \sum_{\substack{\underline{\mathfrak{d}}\in\mathcal{I}^N\\ \mathfrak{N}(\underline{\mathfrak{d}})\leq b}} \prod_{\mathfrak{p}} \frac{|\mu(\underline{\mathfrak{d}}_{\mathfrak{p}})|}{\mathfrak{N}(\underline{\mathfrak{d}}_{\mathfrak{p}})^s} \leq \prod_{\mathfrak{p}} \sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^s}$$

for all $s \in \mathbb{R}$ and $b > 0$, and

$$\lim_{b\to\infty} \sum_{\substack{\underline{\mathfrak{d}}\in\mathcal{I}^N\\ \mathfrak{N}(\underline{\mathfrak{d}})\leq b}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^s} = \prod_{\mathfrak{p}} \sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^s}.$$

Hence, it suffices to show that $\prod_{\mathfrak{p}} \sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^s}$ converges for $s > 1/f$.

Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_k$. The sum $\sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})}$ is finite, because $\mu((\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)}) = 0$ for all N-tuples of non-negative integers $(e_\rho)_{\rho\in\Sigma(1)}$ such that $e_\rho \geq 2$ for some $\rho \in \Sigma(1)$. If $(e_\rho)_{\rho\in\Sigma(1)}$ is an N-tuple of non-negative integers, not all 0, and $e_\rho = 0$ for all $\rho \in \Sigma(1) \smallsetminus \sigma(1)$ for a maximal cone $\sigma \in \Sigma_{max}$, then $\mu((\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)}) = 0$. Therefore, if $(e_\rho)_{\rho\in\Sigma(1)}$ is an N-tuple of non-negative integers, not all 0, such that $\mu((\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)}) \neq 0$, then $\mathfrak{N}(\mathfrak{p})^f \,|\, \mathfrak{N}((\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)})$. Then we can write

$$\sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^s} = 1 + \frac{1}{\mathfrak{N}(\mathfrak{p})^{fs}} Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right),$$

where Q is a polynomial with non-negative integer coefficients. For every N-tuple $(e_\rho)_{\rho\in\Sigma(1)}$ of non-negative integers, $\mu((\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)})$ does not depend on the choice of the prime ideal $\mathfrak{p}$. Hence, Q is independent of the choice of $\mathfrak{p}$. Thus

$$\sum_{\mathfrak{p}} \frac{1}{\mathfrak{N}(\mathfrak{p})^{fs}} Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right) \leq [k:\mathbb{Q}]Q(1) \sum_{n\in\mathbb{Z}_{>0}} \frac{1}{n^{fs}}$$

is convergent for $s > 1/f$. Here $\sum_{\mathfrak{p}}$ denotes a sum over the prime ideals of $\mathscr{O}_k$.

Property (ii) follows from (i) because for every $\varepsilon > 0$,

$$\sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b} |\mu(\underline{\mathfrak{d}})| \leq b^{1/f+\varepsilon} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^{1/f+\varepsilon}} \leq b^{1/f+\varepsilon} \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})^{1/f+\varepsilon}}.$$

Property (iii) can be proven using (ii) as follows:

$$\begin{aligned}
\sum_{\mathfrak{N}(\underline{\mathfrak{d}})\geq b} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})} &= \sum_{b'\geq b} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b'} \frac{|\mu(\underline{\mathfrak{d}})|}{b'} - \sum_{b'\geq b-1} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b'} \frac{|\mu(\underline{\mathfrak{d}})|}{b'+1} = \\
&= \sum_{b'\geq b} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b'} \frac{|\mu(\underline{\mathfrak{d}})|}{b'(b'+1)} - \frac{1}{b} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b-1} |\mu(\underline{\mathfrak{d}})| \leq \\
&\leq \sum_{b'\geq b} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b'} \frac{|\mu(\underline{\mathfrak{d}})|}{b'^2} - \frac{1}{b} \sum_{\mathfrak{N}(\underline{\mathfrak{d}})\leq b-1} |\mu(\underline{\mathfrak{d}})| = \\
&= O\left(\sum_{b'\geq b} b'^{1/f-2+\varepsilon}\right) + O\left(\frac{(b-1)^{1/f+\varepsilon}}{b}\right). \qquad \square
\end{aligned}$$

After Proposition 6.4 (i) we define

$$\kappa := \prod_{\mathfrak{p}} \sum_{\underline{\mathfrak{d}},\mathfrak{p}} \frac{\mu(\underline{\mathfrak{d}})}{\mathfrak{N}(\underline{\mathfrak{d}})} = \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \frac{\mu(\underline{\mathfrak{d}})}{\mathfrak{N}(\underline{\mathfrak{d}})}. \tag{6.7}$$

6.2.2. Möbius inversion for imaginary quadratic fields. From now on we assume that k is an imaginary quadratic extension of $\mathbb{Q}$. Let w_k be the cardinality of the group of units $\mathscr{O}_k^\times$ of $\mathscr{O}_k$.

We identify with $\mathbb{C}$ the completion of k at its only infinite place and we denote by $|\cdot|_\infty$ the corresponding absolute value normalized as at page 14.

For every $\underline{x} \in \mathbb{C}^N$, let

$$h(\underline{x}) := \sup_{\sigma \in \Sigma_{max}} |\underline{x}^{-K(\sigma)}|_\infty.$$

For every $\underline{\mathfrak{c}} \in \mathcal{C}^r$, every $\underline{\mathfrak{d}} \in \mathcal{I}^N$ and every $B > 0$, we define

$$A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) := \{\underline{x} \in \Lambda(k) \cap \bigoplus_{\rho \in \Sigma(1)} \mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho} : h(\underline{x}) \leq \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B\},$$
$$C_{\underline{\mathfrak{c}}}(B) := \{\underline{x} \in {}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) \cap \Lambda(k) : H(\pi(\underline{x})) \leq B\}.$$

Let $\underline{\mathfrak{o}} \in \mathcal{I}^N$ be defined by $\mathfrak{o}_\rho := \mathscr{O}_k$ for all $\rho \in \Sigma(1)$. Then

$$A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) = \{\underline{x} \in A_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}(B) : \underline{\mathfrak{d}} | \underline{\mathfrak{c}}_{\underline{x}}\}.$$

Proposition 6.5. *The sets $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ and $C_{\underline{\mathfrak{c}}}(B)$ are finite for all $\underline{\mathfrak{c}} \in \mathcal{C}^r$, all $\underline{\mathfrak{d}} \in \mathcal{I}^N$ and all $B > 0$. In particular $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) = \emptyset$ if $\mathfrak{N}(\underline{\mathfrak{d}}) > B$.*

Proof. Let $\underline{\mathfrak{c}} \in \mathcal{C}^r$, $\underline{\mathfrak{d}} \in \mathcal{I}^N$, $B > 0$ and $\underline{x} \in A_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}(B)$. If $\underline{\mathfrak{d}} | \underline{\mathfrak{c}}_{\underline{x}}$, Lemma 6.3 and the definition of $-K$ give

$$\mathfrak{N}(\underline{\mathfrak{d}}) = \mathfrak{N}(\underline{\mathfrak{d}}^{-K}) \leq \mathfrak{N}(\underline{\mathfrak{c}}_{\underline{x}}^{-K}) = \frac{\mathfrak{N}(\underline{x}^{-K})}{\mathfrak{N}(\underline{\mathfrak{c}}^{-K})} \leq \frac{h(\underline{x})}{\mathfrak{N}(\underline{\mathfrak{c}}^{-K})} \leq B,$$

and hence, $|x_\rho|_\infty \leq \mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1} B$ for all $\rho \in \Sigma(1)$, as $x_\rho \neq 0$ for all $\rho \in \Sigma(1)$. Thus $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ is finite, and $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) = \emptyset$ if $\mathfrak{N}(\underline{\mathfrak{d}}) > B$. Moreover, $C_{\underline{\mathfrak{c}}}(B)$ is finite as it is a subset of $A_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}(B)$ by (6.4) and Proposition 6.2. □

Proposition 6.6. *With $N_{T,H,k}$ defined in Theorem 6.1 and the notation above,*

$$N_{T,H,k}(B) = \frac{1}{w_k^r} \sum_{\underline{\mathfrak{c}} \in \mathcal{C}^r} \# C_{\underline{\mathfrak{c}}}(B)$$

for all $B > 0$.

Proof. Recall that $N_{T,H,k}(B)$ is the cardinality of the set

$$\{\underline{x} \in T(k) : H(\pi(\underline{x})) \leq B\}.$$

Since $\pi^{-1}(T) = \Lambda$, the parameterization (6.3) gives

$$T(k) = \bigsqcup_{\underline{\mathfrak{c}} \in \mathcal{C}^r} {}_{\underline{\mathfrak{c}}}\tilde{\pi}({}_{\underline{\mathfrak{c}}}\mathscr{Y}(\mathscr{O}_k) \cap \Lambda(k)).$$

Let $\underline{\mathfrak{c}} \in \mathcal{C}^r$. Since ${}_{\underline{\mathfrak{c}}}\tilde{\pi}$ is a torsor over $\mathscr{X}$ under $\mathbb{G}_{m,\mathscr{X}}^r$ by Theorem 1.10, its fibers over $\mathscr{O}_k$-rational points of $\mathscr{X}$ are either empty or isomorphic to $(\mathscr{O}_k^\times)^r$. Hence, each nonempty fiber of ${}_{\underline{\mathfrak{c}}}\tilde{\pi}$ over an $\mathscr{O}_k$-rational point of $\mathscr{X}$ contains exactly w_k^r points. □

Proposition 6.7. *For every $\underline{\mathfrak{c}} \in \mathcal{C}^r$ and every $B > 0$,*

$$\# C_{\underline{\mathfrak{c}}}(B) = \sum_{\underline{\mathfrak{d}} \in \mathcal{I}^N} \mu(\underline{\mathfrak{d}}) \# A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B).$$

PROOF. The definition of χ, μ and $\chi_{\underline{\mathfrak{d}}}$, together with (6.4) and Proposition 6.5, gives

$$\begin{aligned}\#C_{\underline{\mathfrak{c}}}(B) &= \sum_{\underline{x}\in A_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}(B)} \chi(\underline{\mathfrak{c}}_{\underline{x}}) = \sum_{\underline{x}\in A_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}(B)} \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \mu(\underline{\mathfrak{d}})\chi_{\underline{\mathfrak{d}}}(\underline{\mathfrak{c}}_{\underline{x}}) = \\ &= \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \mu(\underline{\mathfrak{d}}) \sum_{\underline{x}\in A_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}(B)} \chi_{\underline{\mathfrak{d}}}(\underline{\mathfrak{c}}_{\underline{x}}) = \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} \mu(\underline{\mathfrak{d}})\#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B). \qquad \square\end{aligned}$$

6.3. Verification of Manin's conjecture

We prove Theorem 6.1 by estimating the cardinality of the sets $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$, namely by counting lattice points in the following sets.

For every $\underline{\mathfrak{c}} \in \mathcal{C}^r$ and every $B > 0$, let

$$D_{\underline{\mathfrak{c}}}(B) := \{\underline{x} \in \mathbb{C}^N : |x_\rho|_\infty \geq \mathfrak{N}(\underline{\mathfrak{c}}^{D_\rho}) \ \forall \rho \in \Sigma(1),\ h(\underline{x}) \leq \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B\}.$$

We note that $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) = D_{\underline{\mathfrak{c}}}(B) \cap \bigoplus_{\rho\in\Sigma(1)} \mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}$ for all $\underline{\mathfrak{c}} \in \mathcal{C}^r$ and $\underline{\mathfrak{d}} \in \mathcal{I}^N$. In Proposition 6.14 we approximate the cardinality of the sets $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ by the volume of $D_{\underline{\mathfrak{c}}}(B)$ with respect to the Haar measure $\mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x}$ on $\mathbb{C}^N \cong \mathbb{R}^{2N}$ normalized such that the volume of a fundamental domain for the lattice $\bigoplus_{\rho\in\Sigma(1)} \mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}$ is 1. Hence, we first compute the volume of $D_{\underline{\mathfrak{c}}}(B)$.

6.3.1. Volume computation. Following [**Sal98**, Notation 9.1], we consider

$$\mathcal{L} : T(\mathbb{C}) = (\mathbb{C}^\times)^{\dim X} \to T_{*\mathbb{R}}, \quad \underline{x} \mapsto (\log |x_i|_\infty)_{1\leq i\leq \dim X}.$$

For every maximal cone $\sigma \in \Sigma_{max}$, let C_σ be the closure of $\mathcal{L}^{-1}(-\sigma)$ in $X(\mathbb{C})$.

REMARK 6.8. We observe that $\underline{x} \in \pi^{-1}(C_\sigma)$ if and only if $|\underline{x}^{D_\rho - D_{\rho(\sigma)}}|_\infty \leq 1$ for all $\rho \in \sigma(1)$, as the proof of [**Sal98**, Proposition 11.22] works with $\mathbb{R}$ replaced by $\mathbb{C}$, and in this case $h(\underline{x}) = |\underline{x}^{-K(\sigma)}|_\infty$ (cf. the proof of [**Sal98**, Proposition 9.8]).

PROPOSITION 6.9. *Assume that $r > 1$. For every $\underline{\mathfrak{c}} \in \mathcal{C}^r$ and $\underline{\mathfrak{d}} \in \mathcal{I}^N$,*

$$\mathfrak{N}(\underline{\mathfrak{d}}) \int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} = \frac{(2\pi)^N \#\Sigma_{max}\alpha(X)}{(\sqrt{|\Delta_k|})^N} B(\log B)^{r-1} + O(B(1+\log B)^{r-2})$$

where $\alpha(X)$ is the constant $\alpha_{Peyre}(X)$ defined in [**Sal98**, §7]*, and the implicit constant in the error term does not depend on $\underline{\mathfrak{d}}$.*

PROOF. Fix $\underline{\mathfrak{c}} \in \mathcal{C}^r$. For every $B > 0$, the subset $D_{\underline{\mathfrak{c}}}(B) \subset \mathbb{C}^N$ is bounded by Lemma 6.3. For every $\sigma \in \Sigma_{max}$, let $D_{\underline{\mathfrak{c}},\sigma}(B) := D_{\underline{\mathfrak{c}}}(B) \cap \pi^{-1}(C_\sigma)$. Let $\sigma' \neq \sigma$ in Σ_{max} and $\rho \in \sigma(1) \smallsetminus \sigma'(1)$. Since $|\underline{x}^{D_\rho - D_{\rho(\sigma)}}|_\infty = 1$ for all $\underline{x} \in D_{\underline{\mathfrak{c}},\sigma}(B) \cap D_{\underline{\mathfrak{c}},\sigma'}(B)$ (cf. proof of [**Sal98**, Proposition 11.22]), and $u_{\sigma,D_\rho} \neq 0$, the set $D_{\underline{\mathfrak{c}},\sigma}(B) \cap D_{\underline{\mathfrak{c}},\sigma'}(B)$ is contained in a codimension 1 subvariety of $\mathbb{C}^N \cong \mathbb{R}^{2N}$. Hence, $\int_{D_{\underline{\mathfrak{c}},\sigma}(B)\cap D_{\underline{\mathfrak{c}},\sigma'}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} = 0$, and

$$\int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} = \sum_{\sigma\in\Sigma_{max}} \int_{D_{\underline{\mathfrak{c}},\sigma}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x}.$$

Fix $\sigma \in \Sigma_{max}$. By [**Lan70**, Lemma V.2.2], the lattice $\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}$ in $\mathbb{C} \cong \mathbb{R}^2$ has determinant $\mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})\sqrt{|\Delta_k|}/2$ for all $\rho \in \Sigma(1)$. Therefore,

$$\int_{D_{\underline{\mathfrak{c}},\sigma}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} = \frac{2^N}{\mathfrak{N}(\underline{\mathfrak{d}})\,\mathfrak{N}(\underline{\mathfrak{c}}^{-K})(\sqrt{|\Delta_k|})^N} \int_{D_{\underline{\mathfrak{c}},\sigma}(B)} \mathrm{d}\underline{x},$$

where $\mathrm{d}\underline{x}$ is the usual Lebesgue measure on $\mathbb{C}^N \cong \mathbb{R}^{2N}$.

Passing to polar coordinates with $y_\rho := |x_\rho|_\infty / \mathfrak{N}(\underline{\mathfrak{c}}^{D_\rho})$ for all $\rho \in \Sigma(1)$ gives

$$\int_{D_{\underline{\mathfrak{c}},\sigma}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} = \frac{(2\pi)^N}{\mathfrak{N}(\underline{\mathfrak{d}})(\sqrt{|\Delta_k|})^N} \int_{D'_{\underline{\mathfrak{c}},\sigma}(B)} \mathrm{d}\underline{y},$$

where $\mathrm{d}\underline{y}$ is the usual Lebesgue measure on $\mathbb{R}^N$ and $D'_{\underline{\mathfrak{c}},\sigma}(B)$ is the set of $\underline{y} \in \mathbb{R}^N$ that satisfy

$$\min_{\rho\in\Sigma(1)} y_\rho \geq 1, \quad \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} y^{\alpha_{\sigma,\rho}} \leq B, \quad y_\rho \leq \prod_{\rho'\in\Sigma(1)\smallsetminus\sigma(1)} y_{\rho'}^{-u_{\sigma,D_\rho}(n_{\rho'})} \ \forall \rho \in \sigma(1)$$

(cf. Remark 6.8). Moreover, [**Sal98**, (11.41)] gives

$$\int_{D'_{\underline{\mathfrak{c}},\sigma}(B)} \mathrm{d}\underline{y} = \alpha(X)B(\log B)^{r-1} + O(B(1+\log B)^{r-2}). \qquad \square$$

As explained in the introduction, we compare the cardinality of the sets $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ with the volume of $D_{\underline{\mathfrak{c}}}(B)$ after intersecting both with a suitable partition of $\mathbb{C}^N$ in strongly convex rational polyhedral cones. The next two lemmas are the tool to produce such a partition.

6.3.2. Ideal lattices in $\mathbb{C} \cong \mathbb{R}^2$. We show how to produce a partition of $\mathbb{C} \cong \mathbb{R}^2$ in six cones generated by bases of an ideal lattice consisting of vectors of small length with respect to the determinant of the lattice, with the property that in each cone the sum of the generators is longer than their difference.

LEMMA 6.10. *Let $\mathfrak{a}$ be a nonzero fractional ideal of k. Then there exist $v_1, \dots, v_6 \in \mathfrak{a}$ such that, if we set $v_7 := v_1$, then*

(1) $[0,1)v_i + [0,1)v_{i+1}$ is a fundamental domain for the lattice $\mathfrak{a} \subseteq \mathbb{C} \cong \mathbb{R}^2$ for all $i \in \{1,\dots,6\}$;
(2) we can write $v_i = (\eta_i, \theta_i)$ in polar coordinates for $i \in \{1,\dots,6\}$ so that if we set $\theta_7 := 2\pi + \theta_1$, then $0 \leq \theta_{i+1} - \theta_i \leq \frac{\pi}{2}$ for all $i \in \{1,\dots,6\}$;
(3) $|v_i|_\infty \leq 16\pi^{-2}|\Delta_k|\,\mathfrak{N}(\mathfrak{a})$ for all $i \in \{1,\dots,6\}$.

PROOF. Let $w_1, w_2 \in \mathfrak{a} \subseteq \mathbb{R}^2$ be R-linearly independent elements such that $|w_1|_\infty^{1/2}$ and $|w_2|_\infty^{1/2}$ are respectively the first and the second successive minimum of the lattice $\mathfrak{a} \subseteq \mathbb{R}^2$ with respect to the unit ball. Let $w_3 := w_1 + w_2$. For $i \in \{1,2,3\}$, write $w_i = (\eta'_i, \theta'_i)$ in polar coordinates with $\theta'_i \in [0, 2\pi)$. Without loss of generality we can assume that $\frac{\pi}{2} \leq |\theta'_1 - \theta'_2| < \pi$. Then the inequalities $|w_1|_\infty^{1/2} \leq |w_2|_\infty^{1/2} \leq |w_1 + w_2|_\infty^{1/2}$ force $|\theta'_3 - \theta'_1| \leq \frac{\pi}{2}$ and $|\theta'_3 - \theta'_2| \leq \frac{\pi}{3}$. Let $u_1 := w_1$, $u_2 := w_3$, $u_3 := w_2$, $u_4 := -w_1$, $u_5 := -w_3$ and $u_6 := -w_2$. For $i \in \{1,\dots,6\}$, let $v_i := u_i$ if $\theta'_2 - \theta'_1 > 0$, and $v_i := u_{7-i}$ if $\theta'_2 - \theta'_1 < 0$.

We recall that the lattice $\mathfrak{a} \subseteq \mathbb{R}^2$ has determinant $\mathfrak{N}(\mathfrak{a})\sqrt{|\Delta_k|}/2$ (see [**Lan70**, Lemma V.2.2] for example). Then [**Cas59**, §VIII.4.3] gives

$$|w_1|_\infty |w_2|_\infty \leq 4\pi^{-2}|\Delta_k|\,\mathfrak{N}(\mathfrak{a})^2.$$

Hence, $|w_1|_\infty, |w_2|_\infty \leq 4\pi^{-2}|\Delta_k|\,\mathfrak{N}(\mathfrak{a})$ as $|w_i|_\infty \geq \mathfrak{N}(\mathfrak{a})$ for $i \in \{1,2\}$. Therefore, $|v_i|_\infty \leq 2(|w_1|_\infty + |w_2|_\infty) \leq 16\pi^{-2}|\Delta_k|\,\mathfrak{N}(\mathfrak{a})$ for all $i \in \{1,\dots,6\}$. $\square$

LEMMA 6.11. *Let L be a lattice in $\mathbb{C} \cong \mathbb{R}^2$. Let $v_1, v_2 \in L$ such that $[0,1)v_1 + [0,1)v_2$ is a fundamental domain for L and there are $\theta_1, \theta_2 \in \mathbb{R}$ with $0 < \theta_2 - \theta_1 \leq \frac{\pi}{2}$ and $v_i = (\eta_i, \theta_i)$ for $i \in \{1,2\}$ in polar coordinates. Then*

$$|w|_\infty \leq |w + x|_\infty < |w + v_1 + v_2|_\infty$$

for all $w \in \mathbb{Z}_{\geq 0}v_1 + \mathbb{Z}_{\geq 0}v_2$ and all $x \in [0,1]v_1 + [0,1]v_2$, $x \neq v_1 + v_2$.

PROOF. Let $w \in \mathbb{Z}_{\geq 0}v_1 + \mathbb{Z}_{\geq 0}v_2$ and $x \in [0,1]v_1 + [0,1]v_2$, $x \neq v_1 + v_2$. We can write $w = (\eta, \theta)$ and $x = (\eta', \theta')$ in polar coordinates, with $\theta_1 \leq \theta, \theta' \leq \theta_2$. Then

$$|w + x|_\infty = |w|_\infty + |x|_\infty - 2\eta\eta' \cos(\pi - \theta + \theta'),$$

and $\cos(\pi - \theta + \theta') \leq 0$, as $|\theta' - \theta| \leq \pi/2$. Similarly,

$$|w + v_1 + v_2|_\infty \geq |w + x|_\infty + |v_1 + v_2 - x|_\infty,$$

and $|v_1 + v_2 - x|_\infty > 0$ as $x \neq v_1 + v_2$. $\square$

In the rest of this section, if $L_1, \dots, L_n$ are lattices (or fundamental domain of lattices) in $\mathbb{C} \cong \mathbb{R}^2$, we denote by $\bigoplus_{i=1}^n L_i$ the lattice (respectively, the fundamental domain) in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ obtained as direct sum of the L_i.

Before proceeding with the comparison between the cardinality of the sets $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ and the volume of $D_{\underline{\mathfrak{c}}}(B)$, we estimate the number of lattice points close to the border of the domains $D_{\underline{\mathfrak{c}}}(B)$ where we count them.

6.3.3. Boundary estimation. First comes a technical result analogous to [**Sal98**, Sublemma 11.24].

LEMMA 6.12. *Let $e, e_1, \dots, e_r \in \mathbb{Z}_{\geq 0}$, $e > 0$, $B \geq 1$, $D \in (0,1]$ such that $DB \geq 1$. Let $\epsilon_r := \max\{e_1, \dots, e_r\}/e$. Then*

$$\sum \prod_{i=1}^r y_i^{\frac{e_i}{e} - 1} \leq \max\{1, \epsilon_r 2^{\epsilon_r}\}(DB)^{1/e}(1 + \log B)^r$$

where the sum runs through the set of positive integers $y_1, \dots, y_r$ that satisfy

$$\max\{y_1, \dots, y_r\} \leq B \text{ and } \prod_{i=1}^r y_i^{e_i} \leq DB.$$

PROOF. The proof goes by induction on r. Let $S(D, B, e; e_1, \dots, e_r)$ be the sum in the statement. Assume first that $r = 1$. If $e_1 = 0$, then

$$S(D, B, e; 0) \leq 1 + \log B \leq (DB)^{1/e}(1 + \log B) \tag{6.8}$$

as $DB \geq 1$. If $e_1 > 0$, then

$$S(D, B, e; e_1) \leq \int_0^{(DB)^{1/e_1}+1} y^{\frac{e_1}{e} - 1}\,\mathrm{d}y \leq \frac{e_1}{e} 2^{\frac{e_1}{e}}(DB)^{1/e} \leq$$
$$\leq \epsilon_1 2^{\epsilon_1}(DB)^{1/e}(1 + \log B)$$

as $(DB)^{1/e_1} \geq 1$ and $B \geq 1$. Assume now that $r > 1$. If $e_r = 0$, then $\epsilon_{r-1} = \epsilon_r$ and

$$S(D,B,e;e_1,\dots,e_r) = S(D,B,e;e_1,\dots,e_{r-1})S(D,B,e;0) \leq \\ \leq \max\{1,\epsilon_r 2^{\epsilon_r}\}(DB)^{1/e}(1+\log B)^{r-1}(1+\log B)$$

by the induction assumption and (6.8). If $e_1,\dots,e_r > 0$, then

$$S(D,B,e;e_1,\dots,e_r) = \sum_{y_r \leq (DB)^{1/e_r}} y_r^{\frac{e_r}{e}-1} S(y_r^{-e_r}D,B,e;e_1,\dots,e_{r-1}) \leq \\ \leq \max\{1,\epsilon_r 2^{\epsilon_r}\}(DB)^{1/e}(1+\log B)^{r-1}S(D,B,e;0)$$

as $(DB)^{1/e_r} \leq B$ and $\epsilon_{r-1} \leq \epsilon_r$. The expected result follows by (6.8). $\square$

The next proposition, inspired by [**Sal98**, Lemma 11.25(b)], estimates the number of lattice points near the border of certain subdomains of $D_{\underline{\mathfrak{c}}}(B)$.

PROPOSITION 6.13. *Assume that $r > 1$. Let $\underline{\mathfrak{c}} \in \mathcal{C}^r$, $\underline{\mathfrak{d}} \in \mathcal{I}^N$, $B \geq 1$ and $\tilde{\rho} \in \Sigma(1)$. For every $\rho \in \Sigma(1)$, let $v_{\rho,1}, v_{\rho,2} \in \mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}$ that satisfy Lemma 6.11 for the lattice $\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}$ in $\mathbb{C} \cong \mathbb{R}^2$. Let*

$$F := \bigoplus_{\rho \in \Sigma(1)} ([0,1)v_{\rho,1} + [0,1)v_{\rho,2}),$$

and let $\delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde{\rho}}(F;B)$ be the set

$$\{\underline{x} \in A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) \cap \bigoplus_{\rho \in \Sigma(1)} (\mathbb{Z}_{\geq 0} v_{\rho,1} + \mathbb{Z}_{\geq 0} v_{\rho,2}) : h(\underline{x}+\gamma) > \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B\},$$

where $\gamma = (\gamma_\rho)_{\rho \in \Sigma(1)} \in \mathbb{C}^N$ is defined by $\gamma_\rho := 0$ if $\rho \neq \tilde{\rho}$, $\gamma_{\tilde{\rho}} := v_{\tilde{\rho},1} + v_{\tilde{\rho},2}$. Then there exist positive constants B_1 and C_1, both independent of B and $\underline{\mathfrak{d}}$, such that

$$\#\delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde{\rho}}(F;B) \leq C_1 \, \mathfrak{N}(\underline{\mathfrak{d}})^{-1} B (1+\log B)^{r-2} \min\{\mathfrak{N}(\underline{\mathfrak{d}}), 1+\log B\}.$$

for all $B \geq B_1$.

PROOF. For every $\sigma \in \Sigma_{max}$, let δ_σ be the set of $\underline{x} \in \delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde{\rho}}(F;B)$ such that $\pi(\underline{x}+\gamma) \in C_\sigma$. Then $\#\delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde{\rho}}(F;B) \leq \sum_{\sigma \in \Sigma_{max}} \#\delta_\sigma$ and it suffices to show that for every $\sigma \in \Sigma_{max}$ there are positive constants B' and C', both independent of B and $\underline{\mathfrak{d}}$, such that for all $B \geq B'$,

$$\#\delta_\sigma \leq C' \, \mathfrak{N}(\underline{\mathfrak{d}})^{-1} B (1+\log B)^{r-2} \min\{\mathfrak{N}(\underline{\mathfrak{d}}), 1+\log B\}.$$

Fix $\sigma \in \Sigma_{max}$. If $\underline{x} \in \delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde{\rho}}(F;B)$, then $\pi(\underline{x}+\gamma) \in C_\sigma$ if and only if

$$\prod_{\rho' \in \Sigma(1)} |x_{\rho'} + \gamma_{\rho'}|_\infty^{u_{\sigma,D_\rho}(n_{\rho'})} \leq 1$$

for all $\rho \in \sigma(1)$, and in this case

$$h(\underline{x}+\gamma) = \prod_{\rho \in \Sigma(1) \smallsetminus \sigma(1)} |x_\rho + \gamma_\rho|_\infty^{\alpha_{\sigma,\rho}},$$

as recalled in Remark 6.8. Therefore, if $\alpha_{\sigma,\tilde{\rho}} = 0$ then $\delta_\sigma = \emptyset$ for all $B > 0$. Hence, we assume that $\alpha_{\sigma,\tilde{\rho}} > 0$. Thus, δ_σ is the set of $\underline{x} \in \bigoplus_{\rho\in\Sigma(1)}(\mathbb{Z}_{\geq 0}v_{\rho,1} + \mathbb{Z}_{\geq 0}v_{\rho,2})$ that satisfy

$$|x_\rho|_\infty \geq \mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}) \quad \forall \rho \in \Sigma(1), \tag{6.9}$$

$$|x_\rho|_\infty \leq \prod_{\rho'\in\Sigma(1)\smallsetminus\sigma(1)} |x_{\rho'} + \gamma_{\rho'}|_\infty^{-u_{\sigma,D_\rho}(n_{\rho'})} \quad \forall \rho \in \sigma(1), \tag{6.10}$$

$$h(\underline{x}) \leq \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B, \tag{6.11}$$

$$\prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} |x_\rho + \gamma_\rho|_\infty^{\alpha_{\sigma,\rho}} > \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B. \tag{6.12}$$

Let $S := \Sigma(1) \smallsetminus (\sigma(1) \cup \{\tilde{\rho}\})$. Then Lemma 6.3 , (6.9) and (6.11) give

$$\prod_{\rho\in S} |x_\rho|_\infty \leq \frac{\mathfrak{N}(\underline{\mathfrak{c}}^{\sum_{\rho\in S} D_\rho})}{\prod_{\rho\in\Sigma(1)\smallsetminus S} \mathfrak{N}(\mathfrak{d}_\rho)} B \quad \text{and} \quad \prod_{\rho\in S} |x_\rho|_\infty^{\alpha_{\sigma,\rho}} \leq \frac{\mathfrak{N}(\underline{\mathfrak{c}}^{-K-\alpha_{\sigma,\tilde{\rho}}D_{\tilde{\rho}}})}{\mathfrak{N}(\mathfrak{d}_{\tilde{\rho}})^{\alpha_{\sigma,\tilde{\rho}}}} B \tag{6.13}$$

for every $\underline{x} \in \delta_\sigma$. Let $S(B)$ be the set of $(x_\rho)_{\rho\in S} \in \bigoplus_{\rho\in S}(\mathbb{Z}_{\geq 0}v_{\rho,1} + \mathbb{Z}_{\geq 0}v_{\rho,2})$ that satisfy (6.13) and such that $x_\rho \neq 0$ for all $\rho \in S$.

For every $\underline{x} \in \delta_\sigma$, condition (6.11) gives

$$|x_{\tilde{\rho}}|_\infty \leq \left(\frac{\mathfrak{N}(\underline{\mathfrak{c}}^{-K})B}{\prod_{\rho\in S} |x_\rho|_\infty^{\alpha_{\sigma,\rho}}} \right)^{\frac{1}{\alpha_{\sigma,\tilde{\rho}}}}. \tag{6.14}$$

Moreover, (6.12) can be written as

$$|x_{\tilde{\rho}} + \gamma_{\tilde{\rho}}|_\infty^{\alpha_{\sigma,\tilde{\rho}}} \prod_{\rho\in S} |x_\rho|_\infty^{\alpha_{\sigma,\rho}} > \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B,$$

and together with the triangular inequality for $|\cdot|_\infty^{1/2}$ gives

$$|x_{\tilde{\rho}}|_\infty > \left(\left(\frac{\mathfrak{N}(\underline{\mathfrak{c}}^{-K})B}{\prod_{\rho\in S} |x_\rho|_\infty^{\alpha_{\sigma,\rho}}} \right)^{\frac{1}{2\alpha_{\sigma,\tilde{\rho}}}} - |\gamma_{\tilde{\rho}}|_\infty^{\frac{1}{2}} \right)^2. \tag{6.15}$$

Therefore, for every $(x_\rho)_{\rho\in S} \in S(B)$ there are at most

$$\frac{\pi|\gamma_{\tilde{\rho}}|_\infty^{\frac{1}{2}}}{\mathfrak{N}(\mathfrak{d}_{\tilde{\rho}}\underline{\mathfrak{c}}^{D_{\tilde{\rho}}})} \left(\frac{\mathfrak{N}(\underline{\mathfrak{c}}^{-K})B}{\prod_{\rho\in S} |x_\rho|_\infty^{\alpha_{\sigma,\rho}}} \right)^{\frac{1}{2\alpha_{\sigma,\tilde{\rho}}}}$$

elements of $\mathbb{Z}_{\geq 0}v_{\tilde{\rho},1} + \mathbb{Z}_{\geq 0}v_{\tilde{\rho},2}$ that satisfy the conditions (6.14) and (6.15).

By (6.10), for every $(x_\rho)_{\rho\in S} \in S(B)$ and $x_{\tilde{\rho}}$ as above, there are at most

$$A := \prod_{\rho\in\sigma(1)} \left(\frac{C_0}{\mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})} \prod_{\rho'\in\Sigma(1)\smallsetminus\sigma(1)} |x_{\rho'} + \gamma_{\rho'}|_\infty^{-u_{\sigma,D_\rho}(n_{\rho'})} \right)$$

elements $(x_\rho)_{\rho\in\sigma(1)} \in \bigoplus_{\rho\in\sigma(1)}(\mathbb{Z}_{\geq 0}v_{\rho,1} + \mathbb{Z}_{\geq 0}v_{\rho,2})$ such that $(x_\rho)_{\rho\in\Sigma(1)} \in \delta_\sigma$, where $C_0 = (\pi + 64|\Delta_k|\pi^{-1})/2$. The equality $\sum_{\rho\in\sigma(1)} D_\rho(\sigma) = -K(\sigma) - \sum_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} D_\rho$ together with the triangular inequality for $|\cdot|_\infty^{1/2}$, Lemma

6.10 and (6.9) gives

$$
\begin{aligned}
A &= \frac{C_0^{N-r}}{\prod_{\rho\in\sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})} \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} |x_\rho+\gamma_\rho|_\infty^{\alpha_{\sigma,\rho}-1} = \\
&= \frac{C_0^{N-r}}{\prod_{\rho\in\sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})} \left(\frac{|x_{\tilde\rho}+\gamma_{\tilde\rho}|_\infty}{|x_{\tilde\rho}|_\infty}\right)^{\alpha_{\sigma,\tilde\rho}-1} \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} |x_\rho|_\infty^{\alpha_{\sigma,\rho}-1} \leq \\
&\leq \frac{C_0^{N-r}}{\prod_{\rho\in\sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})} \left(1+\frac{4|\Delta_k|^{\frac12}}{\pi}\right)^{2(\alpha_{\sigma,\tilde\rho}-1)} \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} |x_\rho|_\infty^{\alpha_{\sigma,\rho}-1}.
\end{aligned}
$$

Then (6.14) above gives

$$
A \leq \frac{C_0^{N-r}(\mathfrak{N}(\underline{\mathfrak{c}}^{-K})B)^{\frac{\alpha_{\sigma,\tilde\rho}-1}{\alpha_{\sigma,\tilde\rho}}}}{\prod_{\rho\in\sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho})} \left(1+\frac{4|\Delta_k|^{\frac12}}{\pi}\right)^{2(\alpha_{\sigma,\tilde\rho}-1)} \prod_{\rho\in S} |x_\rho|_\infty^{\frac{\alpha_{\sigma,\rho}}{\alpha_{\sigma,\tilde\rho}}-1}.
$$

Hence, there is a positive constant C independent of B and $\underline{\mathfrak{d}}$ such that for every $(x_\rho)_{\rho\in S}\in S(B)$ there are at most

$$
\frac{C|\gamma_{\tilde\rho}|_\infty^{\frac12} B^{1-\frac{1}{2\alpha_{\sigma,\tilde\rho}}}}{\prod_{\rho\in\Sigma(1)\smallsetminus S}\mathfrak{N}(\mathfrak{d}_\rho)} \prod_{\rho\in S} |x_\rho|_\infty^{\frac{\alpha_{\sigma,\rho}}{2\alpha_{\sigma,\tilde\rho}}-1}
$$

elements $(x_\rho)_{\rho\in\Sigma(1)\smallsetminus S}\in\bigoplus_{\rho\in\Sigma(1)\smallsetminus S}(\mathbb{Z}_{\geq0}v_{\rho,1}+\mathbb{Z}_{\geq0}v_{\rho,2})$ such that $(x_\rho)_{\rho\in\Sigma(1)}\in\delta_\sigma$.

For every $\rho\in S$, let $y_\rho := |x_\rho|_\infty/\mathfrak{N}(\mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho})$. Then

$$
\#\delta_\sigma \leq \frac{(2C_0)^{r-1}C\,\mathfrak{N}(\underline{\mathfrak{c}}^{\sum_{\rho\in S}(\frac{\alpha_{\sigma,\rho}}{2\alpha_{\sigma,\tilde\rho}}-1)D_\rho})|\gamma_{\tilde\rho}|_\infty^{\frac12}}{\mathfrak{N}(\underline{\mathfrak{d}})\prod_{\rho\in S}\mathfrak{N}(\mathfrak{d}_\rho)^{-\frac{\alpha_{\sigma,\rho}}{2\alpha_{\sigma,\tilde\rho}}}} B^{1-\frac{1}{2\alpha_{\sigma,\tilde\rho}}} \sum\prod_{\rho\in S} y_\rho^{\frac{\alpha_{\sigma,\rho}}{2\alpha_{\sigma,\tilde\rho}}-1}, \tag{6.16}
$$

where the sum runs through the set of $(y_\rho)_{\rho\in S}\in(\mathbb{Z}_{>0})^{r-1}$ that satisfy

$$
\prod_{\rho\in S} y_\rho \leq B\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1}, \qquad \prod_{\rho\in S} y_\rho^{\alpha_{\sigma,\rho}} \leq B \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho)^{-\alpha_{\sigma,\rho}}.
$$

If $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)=\emptyset$, then $\delta_\sigma=\emptyset$. Hence, we assume that $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)\neq\emptyset$. Then $\mathfrak{N}(\underline{\mathfrak{d}})\leq B$ by Proposition 6.5, and $\prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)}\mathfrak{N}(\mathfrak{d}_\rho)^{\alpha_{\sigma,\rho}}\leq B$ by (6.9) and (6.11) and the fact that $\alpha_{\sigma,\rho}=0$ for all $\rho\in\sigma(1)$.

If $\mathfrak{N}(\underline{\mathfrak{d}})\prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)}\mathfrak{N}(\mathfrak{d}_\rho)^{-\alpha_{\sigma,\rho}}\leq 1$, by Lemma 6.12 and Lemma 6.10 there is a positive constant C'' independent of B and $\underline{\mathfrak{d}}$ such that

$$
\#\delta_\sigma \leq C''\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1}B(1+\log B)^{r-1}.
$$

If $\mathfrak{N}(\underline{\mathfrak{d}})\prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)}\mathfrak{N}(\mathfrak{d}_\rho)^{-\alpha_{\sigma,\rho}}\geq 1$, then the inequality (6.16) holds with the sum running through the set of $(y_\rho)_{\rho\in S}\in(\mathbb{Z}_{>0})^{r-1}$ that satisfy

$$
\prod_{\rho\in S} y_\rho,\quad \prod_{\rho\in S} y_\rho^{\alpha_{\sigma,\rho}} \quad\leq\quad B \prod_{\rho\in\Sigma(1)\smallsetminus\sigma(1)} \mathfrak{N}(\mathfrak{d}_\rho)^{-\alpha_{\sigma,\rho}}.
$$

By Lemma 6.12 and Lemma 6.10 there exists a positive constant C''' independent of B and $\underline{\mathfrak{d}}$ such that

$$
\#\delta_\sigma \leq C'''\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1}B(1+\log B)^{r-1}.
$$

For every $\rho \in S$, let $z_\rho := |x_\rho|_\infty / \mathfrak{N}(\underline{\mathfrak{c}}^{D_\rho})$. We use Lemma 6.10 to estimate $|\gamma_{\tilde\rho}|_\infty$. Then

$$\#\delta_\sigma \leq \frac{(2C_0)^{r-1} C 4\pi^{-1} |\Delta_k|^{\frac{1}{2}} \mathfrak{N}(\mathfrak{d}_{\tilde\rho}\underline{\mathfrak{c}}^{D_{\tilde\rho}})^{\frac{1}{2}}}{\mathfrak{N}(\underline{\mathfrak{c}}^{\sum_{\rho\in S}(1-\frac{\alpha_{\sigma,\rho}}{2\alpha_{\sigma,\tilde\rho}})D_\rho}) \prod_{\rho\in\Sigma(1)\smallsetminus S} \mathfrak{N}(\mathfrak{d}_\rho)} B^{1-\frac{1}{2\alpha_{\sigma,\tilde\rho}}} \sum \prod_{\rho\in S} z_\rho^{\frac{\alpha_{\sigma,\rho}}{2\alpha_{\sigma,\tilde\rho}}-1},$$

where the sum runs through the set of $(z_\rho)_{\rho\in S} \in (\mathbb{Z}_{>0})^{r-1}$ that satisfy

$$\prod_{\rho\in S} z_\rho, \quad \prod_{\rho\in S} z_\rho^{\alpha_{\sigma,\rho}} \quad \leq \quad B.$$

We recall that $\mathfrak{N}(\mathfrak{d}_{\tilde\rho})^{1/2} \prod_{\rho\in\Sigma(1)\smallsetminus S} \mathfrak{N}(\mathfrak{d}_\rho)^{-1} \leq 1$. By [**Sal98**, Sublemma 11.24] there exist positive constants $B' \geq 1$ and C'''', both independent of B and $\underline{\mathfrak{d}}$, such that

$$\#\delta_\sigma \leq C'''' B(1+\log B)^{r-2} \quad \text{for all } B \geq B'.$$

Take $C' := \max\{C'', C''', C''''\}$. □

6.3.4. Lattice point counting. The next proposition compares the cardinality of the sets $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ with the volume of $D_{\underline{\mathfrak{c}}}(B)$ computed above.

Proposition 6.14. *Assume that $r > 1$. Let $\underline{\mathfrak{c}} \in \mathcal{C}^r$, $\underline{\mathfrak{d}} \in \mathcal{I}^N$. Then there are positive constants B_2 and C_2, both independent of $\underline{\mathfrak{d}}$ and of B, such that*

$$\left| \#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) - \int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} \right| \leq C_2\, \mathfrak{N}(\underline{\mathfrak{d}})^{-1} B(1+\log B)^{r-2} \min\{\mathfrak{N}(\underline{\mathfrak{d}}), 1+\log B\}.$$

for all $B \geq B_2$.

Proof. If $\mathfrak{N}(\underline{\mathfrak{d}}) > B$, then $\#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) = 0$ by Proposition 6.5. By Proposition 6.9 there are positive constants B' and C', both independent of $\underline{\mathfrak{d}}$ and of B, such that $\int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} \leq C'\, \mathfrak{N}(\underline{\mathfrak{d}})^{-1} B(\log B)^{r-1}$ for all $B \geq B'$.

Assume now that $\mathfrak{N}(\underline{\mathfrak{d}}) \leq B$. For every $\rho \in \Sigma(1)$, let $v_{\rho,1}, \ldots, v_{\rho,6} \in \mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho}$ that satisfy Lemma 6.10, and let $v_{\rho,7} := v_{\rho,1}$. Let $I := \{1, \ldots, 6\}^{\Sigma(1)}$.

For every $\underline{i} \in I$, let $D_{\underline{i}} := D_{\underline{\mathfrak{c}}}(B) \cap \bigoplus_{\rho\in\Sigma(1)} (\mathbb{R}_{\geq 0} v_{\rho,i_\rho} + \mathbb{R}_{\geq 0} v_{\rho,i_\rho+1})$. Then,

$$\int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} = \sum_{\underline{i}\in I} \int_{D_{\underline{i}}} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x}. \tag{6.17}$$

Recall that $A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) = D_{\underline{\mathfrak{c}}}(B) \cap \bigoplus_{\rho\in\Sigma(1)} \mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho}$. Fix $\underline{i} \in I$. Let $A_{\underline{i}} := A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) \cap D_{\underline{i}}$. Since $\bigcup_{\underline{i}\in I} D_{\underline{i}} = D_{\underline{\mathfrak{c}}}(B)$, we can compute $\#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B)$ using the inclusion-exclusion principle. For every $\underline{i}, \underline{i}' \in I$, $\underline{i} \neq \underline{i}'$, there exists $\rho \in \Sigma(1)$ and $\underline{j} \in \{\underline{i}, \underline{i}'\}$ such that $A_{\underline{i}} \cap A_{\underline{i}'} \subseteq L_{\underline{j},\rho}(B)$, where

$$L_{\underline{j},\rho}(B) := (\mathbb{Z}_{\geq 0} v_{\rho,j_\rho+1} \oplus \bigoplus_{\rho'\in\Sigma(1)\smallsetminus\{\rho\}} (\mathbb{Z}_{\geq 0} v_{\rho',j_{\rho'}} + \mathbb{Z}_{\geq 0} v_{\rho',j_{\rho'}+1})) \cap D_{\underline{\mathfrak{c}}}(B).$$

Then

$$\left| \#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) - \sum_{\underline{i}\in I} \#A_{\underline{i}} \right| \leq 2^{\#I} \max_{\underline{i}\in I, \rho\in\Sigma(1)} \#L_{\underline{i},\rho}(B), \tag{6.18}$$

and

$$\left|\#A_{\mathfrak{c},\mathfrak{d}}(B) - \int_{D_{\mathfrak{c}}(B)} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x}\right| \le 2^{\# I} \max_{\underline{i}\in I, \rho\in\Sigma(1)} \# L_{\underline{i},\rho}(B) + \sum_{\underline{i}\in I}\left|\#A_{\underline{i}} - \int_{D_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x}\right|.$$

Fix $\underline{i} \in I$. We compare $\#A_{\underline{i}}$ with the volume of $D_{\underline{i}}$ by counting the number of translated fundamental domains

$$F_{\underline{i}} := \bigoplus_{\rho\in\Sigma(1)} ([0,1)v_{\rho,i_\rho} + [0,1)v_{\rho,i_\rho+1})$$

of $\bigoplus_{\rho\in\Sigma(1)} \mathfrak{d}_\rho \underline{\mathfrak{c}}^{D_\rho}$ contained in $D_{\underline{i}}$ and those that intersect the boundary

$$\tilde{D}_{\underline{i}} := \{\underline{x} \in D_{\underline{i}} : h(\underline{x}) = \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B\}.$$

Let $\underline{x}$ be an element of

$$L_{\underline{i}} := \bigoplus_{\rho\in\Sigma(1)} (\mathbb{Z}_{\ge 0}v_{\rho,i_\rho} + \mathbb{Z}_{\ge 0}v_{\rho,i_\rho+1}).$$

By Lemma 6.11,

$$|x_\rho|_\infty \le |x_\rho + x'_\rho|_\infty < |x_\rho + v_{\rho,i_\rho} + v_{\rho,i_\rho+1}|_\infty$$

for all $\rho \in \Sigma(1)$ and all $\underline{x}' \in F_{\underline{i}}$. Let $\gamma_{\underline{i}} := (v_{\rho,i_\rho} + v_{\rho,i_\rho+1})_{\rho\in\Sigma(1)}$. Then

$$h(\underline{x}) \le h(\underline{x} + \underline{x}') < h(\underline{x} + \gamma_{\underline{i}}) \tag{6.19}$$

for all $\underline{x}' \in F_{\underline{i}}$. Hence, $\underline{x} + F_{\underline{i}}$ intersects $\tilde{D}_{\underline{i}}$ if and only if

$$h(\underline{x}) \le \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B < h(\underline{x} + \gamma_{\underline{i}}).$$

Let

$$I_{\underline{i}} := \{\underline{x} \in A_{\underline{i}} : \underline{x} + F_{\underline{i}} \subseteq D_{\underline{i}}\},$$
$$S_{\underline{i}}(B) := \{\underline{x} \in A_{\underline{i}} : h(\underline{x} + \gamma_{\underline{i}}) > \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B\}.$$

Then $\#A_{\underline{i}} = \#I_{\underline{i}} + \#S_{\underline{i}}(B)$ and $\int_{D_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x} \ge \#I_{\underline{i}}$.

Write $\int_{D_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x} = \int_{D'_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x} + \int_{D_{\underline{i}}\smallsetminus D'_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x}$ with

$$D'_{\underline{i}} := D_{\underline{i}} \cap \{\underline{x} \in \mathbb{C}^N : |x_\rho|_\infty \ge \mathfrak{N}(\mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho}),\ \forall \rho \in \Sigma(1)\}.$$

Then

$$\int_{D_{\underline{i}}\smallsetminus D'_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x} \le \int_{\{\underline{x}\in\mathbb{C}^N : |x_\rho|_\infty \le \mathfrak{N}(\mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho}),\ \forall\rho\in\Sigma(1)\}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x} \le (2\pi|\Delta_k|^{-1/2})^N$$

$$\int_{D'_{\underline{i}}} \mathrm{d}_{\mathfrak{c},\mathfrak{d}}\underline{x} \le \#\{\underline{x} \in L_{\underline{i}} : (\underline{x} + F_{\underline{i}}) \cap D'_{\underline{i}} \ne \emptyset\}.$$

Let $\underline{x} \in L_{\underline{i}}$ such that $(\underline{x} + F_{\underline{i}}) \cap D'_{\underline{i}} \ne \emptyset$. Then $h(\underline{x}) \le \mathfrak{N}(\underline{\mathfrak{c}}^{-K})B$ by (6.19). If $\underline{x} \notin A_{\underline{i}}$, then there exists $\tilde{\rho} \in \Sigma(1)$ such that $x_{\tilde{\rho}} = 0$. Let $\underline{z} \in L_{\underline{i}}$ be defined by $z_\rho := v_{\rho,i_\rho+1}$ for all $\rho \in \Sigma(1)$. Then $|z_\rho|_\infty \le 16\pi^{-2}|\Delta_k|\,\mathfrak{N}(\mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho})$ for all $\rho \in \Sigma(1)$ by Lemma 6.10. Let $\underline{y} \in (\underline{x} + F_{\underline{i}}) \cap D'_{\underline{i}}$. For all $\rho \in \Sigma(1)$, then $|x_\rho|_\infty \le |y_\rho|_\infty$ by Lemma 6.11, and $|y_\rho|_\infty \ge \mathfrak{N}(\mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho})$ since $\underline{y} \in D'_{\underline{i}}$. Hence

$$|x_\rho + z_\rho|_\infty \le 2(|x_\rho|_\infty + |z_\rho|_\infty) \le 2(|y_\rho|_\infty + 16\pi^{-2}|\Delta_k|\,\mathfrak{N}(\mathfrak{d}_\rho\underline{\mathfrak{c}}^{D_\rho})) \le$$
$$\le 2(1 + 16\pi^{-2}|\Delta_k|)|y_\rho|_\infty$$

Then $\underline{x}+\underline{z} \in L_{\underline{i},\tilde\rho}(CB)$, where $C := (2(1+16\pi^{-2}|\Delta_k|))^{\max_{\sigma\in\Sigma_{max}}\sum_{\rho\in\Sigma(1)}\alpha_{\sigma,\rho}}$, and

$$\int_{D'_{\underline{i}}} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} \leq \#A_{\underline{i}} + \sum_{\rho\in\Sigma(1)} \#L_{\underline{i},\rho}(CB).$$

For every $\rho \in \Sigma(1)$ and every $\underline{x} \in L_{\underline{i},\rho}(B)$ there exists an integer $m \geq 0$ such that $\underline{x} + mv_{\rho,i_\rho} \in S_{\underline{i}}(B)$. Hence, $\#L_{\underline{i},\rho}(B) \leq \#S_{\underline{i}}(B)$.

Thus,

$$\int_{D_{\underline{i}}} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x} \leq \#I_{\underline{i}} + (2\pi|\Delta_k|^{-1/2})^N + \#S_{\underline{i}}(B) + N\#S_{\underline{i}}(CB),$$

and

$$\left|\#A_{\underline{i}} - \int_{D_{\underline{i}}} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x}\right| \leq (2\pi|\Delta_k|^{-1/2})^N + (N+2)\max\{\#S_{\underline{i}}(B), \#S_{\underline{i}}(CB)\},$$

and

$$\left|\#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) - \int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x}\right| \leq$$
$$\leq \#I(2\pi|\Delta_k|^{-1/2})^N + (2^{\#I} + \#I(N+2))\max_{\underline{i}\in I}\{\#S_{\underline{i}}(B), \#S_{\underline{i}}(CB)\}.$$

It remains to estimate $\max_{\underline{i}\in I}\{\#S_{\underline{i}}(B), \#S_{\underline{i}}(CB)\}$. Let $\underline{i} \in I$. For every $\underline{x} \in S_{\underline{i}}(B)$, there exists $\gamma \in \bigoplus_{\rho\in\Sigma(1)}\{0, v_{\rho,i_\rho} + v_{\rho,i_\rho+1}\}$ and $\tilde\rho \in \Sigma(1)$ such that $\underline{x} + \gamma \in \delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde\rho}(F_{\underline{i}}; B)$. Hence, $\#S_{\underline{i}}(B) \leq 2^N \sum_{\tilde\rho\in\Sigma(1)} \#\delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde\rho}(F_{\underline{i}}; B)$.

By Proposition 6.13 there exist constants $B'' \geq 1$ and $C'' \geq \pi^N$, both independent of $\underline{\mathfrak{d}}$ and B, such that

$$\#\delta_{\underline{\mathfrak{c}},\underline{\mathfrak{d}},\tilde\rho}(F_{\underline{i}}; B) \leq C''\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1}B(1+\log B)^{r-2}\min\{\mathfrak{N}(\underline{\mathfrak{d}}), 1+\log B\}$$

for all $B \geq B''$, all $\tilde\rho \in \Sigma(1)$ and all $\underline{i} \in I$. Since $C \geq 1$,

$$\max_{\underline{i}\in I}\{\#S_{\underline{i}}(B), \#S_{\underline{i}}(CB)\} \leq$$
$$\leq N2^N C''\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1}CB(1+\log(CB))^{r-2}\min\{\mathfrak{N}(\underline{\mathfrak{d}}), 1+\log(CB)\}$$

for all $B \geq B''$. Take $B_2 := \max\{B', B''\}$ and

$$C_2 := \max\{C', (2^{\#I} + \#I(N+2))N2^{N+1}C''C(1+\log C)^{r-1}\}. \qquad \square$$

Proof of Theorem 6.1. For $r = 1$ see [**Sch79**]. For $r > 1$, Proposition 6.6 gives

$$N_{T,H,k}(B) = \frac{1}{w_k^r}\sum_{\underline{\mathfrak{c}}\in\mathcal{C}^r} \#C_{\underline{\mathfrak{c}}}(B).$$

Let C_2 and B_2 be the constants in Proposition 6.14. For all $B \geq B_2$, Proposition 6.7, (6.7), Proposition 6.14 and Proposition 6.4 (ii) and (iii) give

$$\left|\#C_{\underline{\mathfrak{c}}}(B) - \kappa \int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{o}}}\underline{x}\right| \le \sum_{\underline{\mathfrak{d}}\in\mathcal{I}^N} |\mu(\underline{\mathfrak{d}})| \left|\#A_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}(B) - \int_{D_{\underline{\mathfrak{c}}}(B)} \mathrm{d}_{\underline{\mathfrak{c}},\underline{\mathfrak{d}}}\underline{x}\right| \le$$

$$\le C_2 B(1+\log B)^{r-2}\left(\sum_{\substack{\underline{\mathfrak{d}}\in\mathcal{I}^N\\ \mathfrak{N}(\underline{\mathfrak{d}})\le 1+\log B}} |\mu(\underline{\mathfrak{d}})| + (1+\log B)\sum_{\substack{\underline{\mathfrak{d}}\in\mathcal{I}^N\\ \mathfrak{N}(\underline{\mathfrak{d}})> 1+\log B}} \frac{|\mu(\underline{\mathfrak{d}})|}{\mathfrak{N}(\underline{\mathfrak{d}})}\right)$$

$$= O(B(1+\log B)^{r-2+1/f+\varepsilon})$$

for all $\varepsilon > 0$. Apply Proposition 6.9 with $\underline{\mathfrak{d}} = \underline{\mathfrak{o}}$. □

6.4. Compatibility with Peyre's conjecture

We conclude this chapter by showing that the leading constant $C_{X,H,k}$ in Theorem 6.1 satisfies Peyre's conjecture [**Pey95**, Conjecture 2.3.1].

We denote by $\mathbf{A}_k$ the ring of adeles of k. Let $X(\mathbf{A}_k)^0$ be the inverse image of 0 under the map

$$X(\mathbf{A}_k) \to \mathrm{Hom}_{\mathbb{Z}}(H^2_{\acute{e}t}(X,\mathbb{G}_m)/H^2_{\acute{e}t}(k,\mathbb{G}_m),\mathbb{Q}/\mathbb{Z})$$

in [**Sal98**, Proposition 6.7], and let $\overline{X(k)}$ be the closure of the diagonal embedding of $X(k)$ in $X(\mathbf{A}_k)$. Since X satisfies weak approximation by [**Sko01**, Theorem 5.1.2], the two inclusions

$$\overline{X(k)} \subseteq X(\mathbf{A}_k)^0 \subseteq X(\mathbf{A}_k)$$

are equalities. We recall that since X is split there is just one class of universal torsors over X. By [**Sal98**, Remarks 6.13, 7.8], Peyre's conjecture for X coincides then with [**Sal98**, Conjectures 7.12]:

$$C_{X,H,k} = \alpha(X)\tau(X, \|\|_X), \tag{6.20}$$

where $\alpha(X)$ is the constant defined in [**Pey95**, §2] (cf. [**Sal98**, §7]), and by [**Sal98**, Theorem 6.19], $\tau(X,\|\|_X)$ is the Tamagawa number defined in [**Sal98**, Definition 6.18] associated to the class of the universal torsor $\pi : Y \to X$ and to the adelic norm $\|\|_X$ for X that defines H (cf. [**Sal98**, (10.4)]).

We recall that Ω_f denotes the set of finite places of k. For all $\nu\in\Omega_f$, let $\mathscr{O}_\nu$ be the ring of integers of k_ν. Since the model $\mathscr{Y}\to\mathscr{X}$ is defined over $\mathscr{O}_k$, [**Sal98**, Proposition 9.14(a)] and [**Sal98**, Proposition 5.20(c)][1] give the following expression for $\tau(X,\|\|_X)$

$$\overline{\Theta}^1(T^1_{NS}(\mathbf{A}_k)/T_{NS}(k))m_\infty(X_\infty(\mathbf{A}_k)^0\cap\pi(Y(\mathbb{C})))\prod_{\nu\in\Omega_f} n_\nu(\mathscr{Y}(\mathscr{O}_\nu)),$$

[1] cf. [**Sal98**, (5.21)].

where $T_{NS} \cong \mathbb{G}_{m,k}^r$ is the Neron Severi torus of X, that is, the torus dual to the geometric Picard group of X; $T_{NS}^1(\mathbf{A}_k)$ is the kernel of the homomorphism

$$T_{NS}(\mathbf{A}_k) \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{R}) \cong \mathbb{R}^r, \quad (x_{i,\nu})_{\substack{1\leq i\leq r \\ \nu\in\Omega_\infty}} \mapsto \left(\log \prod_{\nu\in\Omega_k} |x_{i,\nu}|_\nu\right)_{1\leq i\leq r} \tag{6.21}$$

(cf. [**Sal98**, (5.18)]); $\overline{\Theta}^1$ is the Haar measure on $T_{NS}^1(\mathbf{A}_k)/T_{NS}(k)$ induced by the Haar measure Θ_∞ on $T_{NS}(\mathbf{A}_k)$ under the bijection established in [**Sal98**, p. 167], where Θ_∞ is determined by the adelic order norm in [**Sal98**, Remarks 5.9(b)] and by the convergence factors $\beta_\infty = 1$, $\beta_\nu = L_\nu(1, T_{NS})$ for $\nu \in \Omega_f$ defined in the proof of [**Sal98**, Lemma 5.16]; m_∞ is the Borel measure on $X(\mathbb{C})$ defined by the adelic norm $\|\|_X$; $X_\infty(\mathbf{A}_k)^0$ is the compact open subset of $X(\mathbb{C})$ such that $X(\mathbf{A}_k)^0 = X_\infty(\mathbf{A}_k)^0 \times \prod_{\nu\in\Omega_f} \mathscr{X}(\mathscr{O}_\nu)$, where $X(\mathbf{A}_k)^0$ is defined in [**Sal98**, Notation 6.8]; n_ν is the Borel measure on $Y(k_\nu)$ defined by the ν-adic norm $\|\|_{X\to Y}$ of [**Sal98**, Theorem 5.17].

We are ready to show that the constant (6.1) in Theorem 6.1 satisfies (6.20).

Proposition 6.15. *With the notation above*

(i) $\overline{\Theta}^1(T_{NS}^1(\mathbf{A}_k)/T_{NS}(k)) = (2\pi h_k w_k^{-1})^r$;

(ii) $m_\infty(X_\infty(\mathbf{A}_k)^0 \cap \pi(Y(\mathbb{C}))) = (2\pi)^{N-r}\#\Sigma_{max}$;

(iii) $\prod_{\nu\in\Omega_f} n_\nu(\mathscr{Y}(\mathscr{O}_\nu)) = \kappa|\Delta_k|^{-N/2}$.

Proof. We first prove *(i)*. According to [**Sal98**, p. 167], $T(k)$ is endowed with the counting measure, and $T_{NS}(\mathbf{A}_k)/T_{NS}^1(\mathbf{A}_k)$ is endowed with the Haar measure pullback of the usual Lebesgue measure on $\mathbb{R}^r$ under the isomorphism induced by (6.21). By [**Sal98**, 3.28–3.30] and [**Sal98**, Theorem 4.14],

$$\Theta_\infty = \prod_{\nu\in\Omega_k} \beta_\nu\omega_\nu$$

where ω_ν are the local Haar measures on $T_{NS}(k_\nu)$ canonically induced by an invariant differential form of degree r on T_{NS} as in [**Wei82**, §2.2] (cf. the lines before Theorem 4.14 in [**Sal98**]). By [**Sal98**, Lemma 5.16] the convergence factors β_ν coincide with the canonical correcting factors for T_{NS} defined in [**Ono61**, §3.3]. Then,

$$\overline{\Theta}^1(T_{NS}^1(\mathbf{A}_k)/T_{NS}(k)) = c(\Theta_\infty; \widehat{T_{NS}}) = |\Delta_k|^{r/2}\gamma(T_{NS}; k/k)$$

by [**Ono61**, (3.2.1)], where $c(\Theta_\infty; \widehat{T_{NS}})$ and $\gamma(T_{NS}; k/k)$ are the numbers defined in [**Ono61**, §3.2] and [**Ono61**, §3.5] respectively. We recall that $T_{NS} \cong \mathbb{G}_{m,k}^r$ is a split torus. Therefore, denoting by ζ_k the Dedekind zeta function of k,

$$\gamma(T_{NS}; k/k) = \gamma(\mathbb{G}_{m,k}; k/k)^r = \left(\lim_{s\to 1}(s-1)\zeta_k(s)\right)^r = (|\Delta_k|^{-1/2}2\pi h_k w_k^{-1})^r$$

by [**Ono61**, Proposition 3.5.1, Theorem 3.5.1] and [**Wei67**, Corollary to Theorem VII.6.3].

To prove *(ii)*, we recall that $X(\mathbf{A}_k)^0 = X(\mathbf{A}_k)$, and that $X(\mathbf{A}_k) = X(\mathbb{C})\times\prod_{\nu\in\Omega_f}\mathscr{X}(\mathscr{O}_\nu)$ as X is proper. Hence, $X_\infty(\mathbf{A}_k)^0 = X(\mathbb{C})$. Moreover, $\pi(Y(\mathbb{C})) = X(\mathbb{C})$ by [**CTS87**, (2.7.2)], and $m_\infty(X(\mathbb{C})) = (2\pi)^{N-r}\#\Sigma_{max}$ by [**Sal98**, Proposition 9.16].

We now prove *(iii)*. Let $\nu\in\Omega_f$ and denote by $\mathfrak{p}$ the corresponding prime ideal of $\mathscr{O}_k$. By [**Sal98**, Corollary 2.15, Proposition 9.14],

$$n_\nu(\mathscr{Y}(\mathscr{O}_\nu)) = \#\mathscr{Y}(\mathbb{F}_\mathfrak{p})\left(\frac{\mu_\nu(\mathscr{O}_\nu)}{\#\mathbb{F}_\mathfrak{p}}\right)^N$$

where μ_ν is the Haar measure on the additive locally compact group k_ν normalized such that $\mu_\nu(\mathscr{O}_\nu)\mu_\nu(\mathscr{O}_\nu^D) = 1$. Here $\mathscr{O}_\nu^D$ is the inverse different of $\mathscr{O}_\nu$.

For $\underline{x}\in\mathbb{F}_\mathfrak{p}^N$, let $\underline{x}\mathscr{O}_k := (x_1'\mathscr{O}_k,\dots,x_N'\mathscr{O}_k)$ where $(x_1',\dots,x_N')\in\mathscr{O}_k^N$ is a representative of the class $\underline{x}\in(\mathscr{O}_k/\mathfrak{p})^N$. Then, with the notation of Subsection 6.2.1,

$$\#\mathscr{Y}(\mathbb{F}_\mathfrak{p}) = \sum_{\underline{x}\in\mathbb{F}_\mathfrak{p}^N}\chi((\underline{x}\mathscr{O}_k)_\mathfrak{p}) = \sum_{\underline{x}\in\mathbb{F}_\mathfrak{p}^N}\sum_{\underline{\mathfrak{d}},\mathfrak{p}}\mu(\underline{\mathfrak{d}})\chi_{\underline{\mathfrak{d}}}(\underline{x}\mathscr{O}_k) = \sum_{\underline{\mathfrak{d}},\mathfrak{p}}\mu(\underline{\mathfrak{d}})\sum_{\underline{x}\in\mathbb{F}_\mathfrak{p}^N}\chi_{\underline{\mathfrak{d}}}(\underline{x}\mathscr{O}_k)$$

Since $\mu((\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)}) = 0$ for all N-tuples of non-negative integers $(e_\rho)_{\rho\in\Sigma(1)}$ such that $e_\rho\geq 2$ for some $\rho\in\Sigma(1)$, we can replace $\sum_{\underline{\mathfrak{d}},\mathfrak{p}}$ by the sum $\sum'_{\underline{\mathfrak{d}}}$ running through the set of $\underline{\mathfrak{d}} = (\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)}$ with $e_\rho\in\{0,1\}$ for all $\rho\in\Sigma(1)$.

Let $i\in\{1,\dots,N\}$. If $\mathfrak{d}_i = \mathscr{O}_k$, then $\mathfrak{d}_i \mid x_i'\mathscr{O}_k$ for all $\underline{x}\in\mathbb{F}_\mathfrak{p}^N$. If $\mathfrak{d}_i = \mathfrak{p}$, then $\mathfrak{d}_i \mid x_i'\mathscr{O}_k$ if and only if $x_i = 0$ in $\mathbb{F}_\mathfrak{p}$. Hence, given $\underline{\mathfrak{d}} = (\mathfrak{p}^{e_\rho})_{\rho\in\Sigma(1)}$ with $e_\rho\in\{0,1\}$,

$$\sum_{\underline{x}\in\mathbb{F}_\mathfrak{p}^N}\chi_{\underline{\mathfrak{d}}}(\underline{x}\mathscr{O}_k) = \#\mathbb{F}_\mathfrak{p}^{\{\rho\in\Sigma(1):e_\rho=0\}} = \#\mathbb{F}_\mathfrak{p}^N\,\mathfrak{N}(\underline{\mathfrak{d}})^{-1}.$$

Then,

$$n_\nu(\mathscr{Y}(\mathscr{O}_\nu)) = \mu_\nu(\mathscr{O}_\nu)^N\sum_{\underline{\mathfrak{d}},\mathfrak{p}}\frac{\mu(\underline{\mathfrak{d}})}{\mathfrak{N}(\underline{\mathfrak{d}})}.$$

Moreover, $\prod_{\nu\in\Omega_f}\mu_\nu(\mathscr{O}_\nu) = |\Delta_k|^{-1/2}$ because of the normalization of the measures μ_ν, and $\kappa = \prod_\mathfrak{p}\sum_{\underline{\mathfrak{d}},\mathfrak{p}}\frac{\mu(\underline{\mathfrak{d}})}{\mathfrak{N}(\underline{\mathfrak{d}})}$ by (6.7). □

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