From finite to infinite: cluster algebras as colimits, and mutating torsion pairs in discrete cluster categories

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Abstract

This thesis explores some questions regarding the combinatorial structure of cluster algebras and cluster categories, with a strong focus on cluster algebras and cluster categories of infinite rank.

Recently, cluster algebras of infinite rank have received more and more attention. We formalize the way in which one can think about cluster algebras of infinite rank by showing that every rooted cluster algebra of infinite rank can be written as a colimit of rooted cluster algebras of finite rank. Relying on the proof of the positivity conjecture for skew-symmetric cluster algebras of finite rank by Lee and Schiffler, it follows as a direct consequence that the positivity conjecture holds for skew-symmetric cluster algebras of infinite rank.

The framework for our colimit construction is the category of rooted cluster algebras introduced by Assem, Dupont and Schiffler. We further investigate this category and give a sufficient and necessary condition for a ring homomorphism between cluster algebras to give rise to a rooted cluster morphism without specializations. Assem, Dupont and Schiffler proposed the problem of a classification of ideal rooted cluster morphisms. We provide a partial solution by showing that every rooted cluster morphism without specializations is ideal, but in general rooted cluster morphisms are not ideal.

We further investigate the combinatorial structure of cluster categories of infinite rank by studying mutation of torsion pairs in the important example of discrete cluster categories of Dynkin type A, which are cluster categories of infinite rank studied by Igusa and Todorov. Work in progress by Holm and Jørgensen combinatorially classifies torsion pairs in discrete cluster categories of Dynkin type A. Relying on this classification, we provide a complete combinatorial model for mutation of torsion pairs in these categories.

Torsion pairs in cluster categories of finite Dynkin type D have been classified by Holm, Jørgensen and Rubey. We combinatorially describe mutations of torsion pairs in these cluster categories. The situation displays less symmetry than in Dynkin type A, providing additional challenges, but nevertheless allows for a nice combinatorial description.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit Fragen zur kombinatorischen Struktur von Cluster-Algebren und Cluster-Kategorien, mit einem Fokus auf Cluster-Algebren und Cluster-Kategorien von unendlichem Rang.

In den letzten Jahren wurde Cluster-Algebren von unendlichem Rang mehr und mehr Beachtung geschenkt. Wir zeigen, dass jede Cluster-Algebra von unendlichem Rang als Colimes von Cluster-Algebren endlichen Ranges geschrieben werden kann. Eine wichtige Konsequenz dieses Resultats erlaubt uns den Beweis der Positivitätsvermutung für schiefsymmetrische Cluster-Algebren von unendlichem Rang. Die Positivitätsvermutung war eine lange offenstehende Vermutung in der Cluster-Theorie, die erst kürzlich von Lee und Schiffler für schiefsymmetrische Cluster-Algebren von endlichem Rang bewiesen wurde. Der Rahmen für unsere Colimeskonstruktion ist die von Assem, Dupont und Schiffler eingeführte Kategorie von verwurzelten Cluster-Algebren. Wir befassen uns weiter mit dieser Kategorie und geben eine hinreichende und notwendige Bedingung, wann ein Ring-Homomorphismus zwischen Cluster-Algebren einen verwurzelten Cluster-Morphismus ohne Spezialisierungen induziert. Ausserdem geben wir eine Teilantwort auf die Frage von Assem, Dupont und Schiffler, welche verwurzelten Cluster-Morphismen ideal sind. Anders als von ihnen vermutet, ist nicht jeder verwurzelte Cluster-Morphismus ideal und wir präsentieren ein Gegenbeispiel. Wir zeigen ausserdem, dass jeder verwurzelte Cluster-Morphismus ohne Spezialisierungen ideal ist.

Des Weiteren untersuchen wir die kombinatorische Struktur von Cluster-Kategorien von unendlichem Rang, indem wir Mutationen von Torsionspaaren im wichtigen Beispiel von diskreten Cluster-Kategorien von Dynkin-Typ A untersuchen. Dies sind von Igusa und Todorov untersuchte Cluster-Kategorien von unendlichem Rang. Laufende Studien von Holm und Jørgensen klassifizieren Torsionspaare in diskreten Cluster-Kategorien von Dynkin-Typ A durch einen kombinatorischen Zugang. Wir nutzen diese Klassifizierung, um ein vollständiges kombinatorisches Modell für Mutationen von Torsionspaaren in diskreten Cluster-Kategorien von Dynkin-Typ A zu geben.

Torsionspaare in den Cluster-Kategorien von endlichem Dynkin-Typ D wurden von Holm, Jørgensen und Rubey klassifiziert. Wir beschreiben Mutationen von Torsionspaaren in diesen Cluster-Kategorien. Die Situation weist weniger Symmetrie auf als in Dynkin-Typ A, was uns vor neue Herausforderungen stellt. Dennoch ist eine kombinatorisch klare Interpretation der Mutationen möglich.

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Keywords: Cluster algebras of infinite rank, cluster categories, mutation of torsion pairs in triangulated categories
Schlagworte: Cluster-Algebren von unendlichem Rang, Cluster-Kategorien, Mutation von Torsionspaaren in triangulierten Kategorien

Contents

Chapter 1

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [FZ1] at the beginning of this millennium, motivated by the study of total positivity and dual canonical bases in Lie theory. They have grown increasingly popular over the past decade and enjoy the attention of an active research community; not least because they are of interest in a vast variety of mathematical fields, reaching far beyond their original ties with combinatorial algebra. Despite their young age, deep connections between cluster algebras and diverse areas of mathematics such as Teichmüller theory, Poisson geometry, mathematical physics, integrable systems and the representation theory of finite dimensional algebras have emerged. A collection of interesting problems and open conjectures concerning cluster algebras – made even more beautiful by virtue of being easy to state yet hard to prove – more than justify the standing of cluster theory as a mathematical field in its own right. A nice example is the positivity conjecture, which – having been conjectured by Fomin and Zelevinsky in [FZ1] – persisted as an open problem for more than ten years until being solved very recently for skew-symmetric cluster algebras of finite rank by Lee and Schiffler [LS].

Put briefly, cluster algebras are commutative rings with a combinatorial structure. Classically, to give a presentation of an algebra, one specifies a set of generators and defining relations. The fundamental idea of a cluster algebra is different: We start with a so-called initial seed Σ, consisting of a distinguished subset of generators X called a cluster, which is simply a set of indeterminates over $\mathbb{Q}$, and a combinatorial rule, encoded in a skew-symmetrizable locally finite integer matrix B. Inductively, by a process called mutation, we obtain a family of seeds from this initial seed, each new seed consisting again of a cluster and a skew-symmetrizable matrix. All clusters are of the same cardinality, and the union of their elements, which are called cluster variables, generate the cluster algebra associated to the seed Σ. The initial seed Σ is by no means unique – every other seed which we obtain from Σ by mutation gives rise to the same cluster algebra.

This thesis is concerned with several topics in the intersection of combinatorics, algebra, and category theory that fall into the framework of cluster algebras and categories. The main theme of the work is the study of infinite clusters in various guises. What follows serves as both a gentle introduction to and rough outline of the results we present, beginning with the category of rooted cluster algebras, passing via cluster categories, and ending with mutation of torsion pairs in discrete cluster categories of Dynkin type A and in cluster categories of finite Dynkin type D.

Classically, clusters are finite. However, the theory can be extended naturally to allow infinite clusters, giving rise to cluster algebras of infinite rank. While most of the research on cluster algebras in the past decade has focused on cluster algebras of finite rank, recently an interest in cluster algebras of infinite rank has arisen, appearing for example in work by Hernandez and Leclerc [HL] as well as in joint work with Grabowski [GG]. Cluster algebras of infinite rank are the main focus of our studies in Chapter 2. We formalize the way in which one can think about cluster algebras of infinite rank by showing that we can consider them as colimits of cluster algebras of finite rank. The context for these considerations is the category *Clus* of rooted cluster algebras, introduced by Assem, Dupont and Schiffler [ADS]. Rooted cluster algebras are pointed versions of cluster algebras, that is we fix an initial seed. This allows for a rigorous definition of what it means for a ring homomorphism between rooted cluster algebras to commute with mutation – which is exactly what we want for a natural map between cluster algebras. This idea gives rise to the concept of rooted cluster morphisms, which provide the morphisms in the category *Clus*, while rooted cluster algebras are the objects. The main result Theorem 2.4.7 of Chapter 2 can be stated as follows.

Theorem. *Every rooted cluster algebra of infinite rank is isomorphic to a colimit of rooted cluster algebras of finite rank.*

This theorem provides useful insights into the nature of cluster algebras of infinite rank and we expect it to facilitate the generalization of a range of properties for cluster algebras of finite rank to cluster algebras of infinite rank. Notably, the positivity conjecture, which was proved by Lee and Schiffler [LS] for skew-symmetric cluster algebras of finite rank, holds for skew-symmetric cluster algebras of infinite rank as a consequence of Theorem 2.4.7. This is shown in Theorem 2.4.10.

On the way to our main result, we encounter and solve a few problems concerning rooted cluster morphisms, including the question from [ADS] asking whether or not every rooted cluster morphism is ideal – the answer is no in general as we show in Theorem 2.3.16, but yes for the important class of rooted cluster morphisms without specializations (see Proposition 2.3.33).

The second part of this thesis concerns cluster categories. Various classical problems for cluster algebras profited from the categorical approach to cluster theory provided by Buan, Marsh, Reineke, Reiten and Todorov [BMRRT] with the introduction of cluster categories and an alternative approach to categorification via preprojective algebras by Geiss, Leclerc and Schröer (see [GLS] for a comprehensive overview). Chapter 3 provides a short overview of the theory of cluster categories. The basic idea is that all combinatorial aspects of cluster algebras find a translation into cluster categories: There will be a categorical analogue of clusters, of cluster variables and of mutation. Like many aspects of cluster algebras, the research on the infinite case is strongly promoted by the study of their categorical counterparts and in recent years, the work on cluster categories of infinite rank by Holm and Jørgensen [HJ] as well as Igusa and Todorov (for example [IT1], [IT2] and [IT3]) has provided meaningful insights into the combinatorial structure of infinite versions of cluster categories of Dynkin type A. These cluster categories form the content of Section 3.3, where we first consider the cluster category of infinite Dynkin type A_∞ in Section 3.3.1 studied by Holm and Jørgensen and later its generalization due to Igusa and Todorov via discrete cluster categories of Dynkin type A in Section 3.3.2. Very roughly speaking, discrete cluster categories of Dynkin type A have a combinatorial interpretation via arcs in the closed disc with discrete sets of endpoints on its boundary. Indecomposable objects of the discrete cluster category $\mathcal{C}(\mathcal{Z})$ associated to the subset $\mathcal{Z} \subseteq S^1$ correspond to arcs with endpoints in $\mathcal{Z}$ and its subcategories correspond to sets of such arcs.

The combinatorial structure on discrete cluster categories of Dynkin type A not only allows for a canonical generalization of the cluster structures from finite Dynkin type A, it also enables further combinatorial generalizations. In particular, work in progress by Holm and Jørgensen combinatorially classifies torsion pairs in discrete cluster categories of Dynkin type A, generalizing the classification of torsion pairs in the cluster category of infinite Dynkin type A_∞ by Ng [Ng] and in cluster categories of finite Dynkin type A by Holm, Jørgensen and Rubey [HJR1]. Holm and Jørgensen show that torsion pairs in the discrete cluster category $\mathcal{C}(\mathcal{Z})$ associated to $\mathcal{Z} \subseteq S^1$ are in one-to-one correspondence with certain sets of arcs with endpoints in $\mathcal{Z}$, which we call Ptolemy diagrams of $\mathcal{Z}$. Torsion pairs in triangulated categories were introduced by Iyama and Yoshino [IY], providing a triangulated version of torsion pairs in abelian categories due to Dickson [D]. In the same paper [IY], Iyama and Yoshino introduced mutation in triangulated categories, providing a generalization of the categorical version of mutation of clusters. Zhou and Zhu [ZZ2] have shown that in nice enough circumstances, mutation of a torsion pair in a triangulated category T gives rise to another torsion pair in T . Chapter 4 is devoted to the study of torsion pairs in triangulated categories and their mutations. In Section 4.3, we present a combinatorial model for mutation of torsion pairs in discrete cluster categories of Dynkin type A via mutation of the corresponding Ptolemy diagrams. The main result of Section 4.3 can be loosely stated as follows – the precise statement is given in Theorem 4.3.10.

Theorem. *Mutation of a torsion pair in the discrete cluster category $\mathcal{C}(\mathcal{Z})$ corresponds to mutation of the associated Ptolemy diagram of $\mathcal{Z}$.*

This generalizes results by Zhou and Zhu [ZZ2], who combinatorially described mutation of torsion pairs in the cluster categories of finite Dynkin type A and of infinite Dynkin type A_∞.

The combinatorial model for discrete Dynkin type A is very nice, as it is quite symmetric. It gets more complicated once we move on to Dynkin type D: Using the combinatorial model for finite Dynkin type D introduced by Fomin and Zelevinsky [FZ3], Holm, Jørgensen and Rubey [HJR2] classified torsion pairs in cluster categories of finite Dynkin type D. Some of the symmetry of the Dynkin type A case gets lost, since we have to deal with the exceptional vertices of Dynkin diagrams of type D. However, inspired by the approach for Dynkin type A, a clean combinatorial classification of torsion pairs in the cluster category of Dynkin type D_n for $n \geq 4$ via so-called Ptolemy diagrams of Dynkin type D_n is possible. In Section 4.4 we use this classification to describe mutation of torsion pairs in cluster categories of finite Dynkin type D combinatorially. The main result of this section, Theorem 4.4.21, can be loosely stated as follows.

Theorem. *Mutation of a torsion pair in the cluster category of Dynkin type D_n corresponds to mutation of the associated Ptolemy diagram of Dynkin type D_n.*

This thesis is organized as follows. Chapter 2 is concerned with the study of the category of rooted cluster algebras. In Section 2.2 we review the most important definitions along with explanations and examples. In Section 2.3 we present the category of rooted cluster algebras as introduced by Assem, Dupont and Schiffler [ADS]. We prove helpful new facts about the morphisms in this category, which are called rooted cluster morphisms. In particular, we answer an open question from [ADS] by showing that not every rooted cluster morphism is ideal in Section 2.3.2 and we give a complete characterization of rooted cluster morphisms without specializations in Section 2.3.6. Section 2.4 deals with colimits in the category of rooted cluster algebras and contains our main result Theorem 2.4.7 stating that every rooted cluster algebra can be written as a colimit of rooted cluster algebras of finite rank. Section 2.4.3 presents, as an important application of our main result, the generalization of Lee and Schiffler's [LS] solution to the positivity conjecture for skew-symmetric cluster algebras of finite rank to skew-symmetric cluster algebras of infinite rank.

In Chapter 3 we review cluster categories. Section 3.2 presents an overview of the most relevant features of cluster categories to the results in this thesis. We explain how the combinatorial structures of cluster categories and cluster algebras are linked. In line with our interest in cluster algebras of infinite rank we dedicate Section 3.3 to the work on cluster categories of infinite rank as studied by Holm and Jørgensen [HJ] and Igusa and Todorov ([IT1] and [IT3]).

Mutation of torsion pairs in triangulated categories forms the content of Chapter 4. Section 4.2 presents an overview of the concepts of torsion pairs and mutation in triangulated categories and we see how they relate to cluster structures on cluster categories. In Section 4.3 we present a combinatorial model for the mutation of torsion pairs in discrete cluster categories of Dynkin type A and in Section 4.4 we present a combinatorial model for the mutation of torsion pairs in cluster categories of finite Dynkin type D.

Chapter 2

Cluster algebras of infinite rank as colimits

2.1 Introduction

This chapter is concerned with cluster algebras, with the aim of presenting a better understanding of cluster algebras of infinite rank. In general, when passing from finite to infinite cardinality, it is natural to consider limits or colimits in an appropriate category. The optimal framework for our purposes is given by the category *Clus* of rooted cluster algebras, which was introduced by Assem, Dupont and Schiffler [ADS]. The objects of *Clus* are what can be thought of as pointed versions of cluster algebras; they are pairs consisting of a cluster algebra and a fixed initial seed. Fixing a distinguished initial seed allows for the definition of natural maps between cluster algebras, so-called rooted cluster morphisms, which are ring homomorphisms commuting with mutation and which provide the morphisms for the category *Clus*. We review the most important definitions in Section 2.2.

The category *Clus* of rooted cluster algebras and its morphisms are very new concepts, only having been introduced in Assem, Dupont and Schiffler's paper [ADS] from 2014. In Section 2.3 we devote some space to the study of rooted cluster morphisms and show a few useful properties. In particular we answer an open question on ideal rooted cluster morphisms. A rooted cluster morphism is ideal, if its image coincides with the rooted cluster algebra generated by the image of the initial seed. Assem, Dupont and Schiffler ask in [ADS, Problem 2.12] for a characterization of ideal rooted cluster morphisms. We answer part of this question by showing that not every rooted cluster morphism is necessarily ideal in Theorem 2.3.16. The counterexample we provide is a rooted cluster morphism with specializations, that is, some cluster variables get sent to integers. Rooted cluster morphisms without specializations are more nicely behaved and we characterize them by a necessary and sufficient combinatorial condition. As a result we show that every rooted cluster morphism without specializations is ideal (see Proposition 2.3.33).

In Section 2.4 we proceed to study colimits in the category *Clus*. We show that the category *Clus* is neither complete nor cocomplete, that is, limits and colimits do not in general exist. However, it has sufficient colimits to express any cluster algebra of infinite rank as a colimit of cluster algebras of finite rank, as we show in our main result (see Theorem 2.4.7).

Theorem. *Every rooted cluster algebra of infinite rank can be written as a colimit of rooted cluster algebras of finite rank in the category Clus.*

We expect this statement to be a useful tool in extending results that are known for (certain) cluster algebras of finite rank to cluster algebras of infinite rank. As an important application, we show in Theorem 2.4.10 that the positivity conjecture, as shown by Lee and Schiffler in [LS] for skew-symmetric cluster algebras of finite rank, holds for skew-symmetric cluster algebras of infinite rank.

Important sources of (rooted) cluster algebras are triangulations of marked surfaces, as studied for triangulations of surfaces with finitely many marked points by Fomin, Shapiro and Thurston [FST]. An important inspiration for our work on cluster algebras of infinite rank stems from recent work on cluster categories of infinite rank as carried out by Holm and Jørgensen in [HJ] and by Igusa and Todorov in [IT1] and [IT3], which uses countable triangulations of the closed disc with infinitely many marked points as a combinatorial model. (Serving as a purely motivational concept here, we will talk about cluster categories in more detail in Chapter 3.) Section 2.4.4 is concerned with cluster algebras associated to countable triangulations of the closed disc, providing an algebraic interpretation of these cluster categories. It is a direct consequence of our main result, Theorem 2.4.7, that every rooted cluster algebra arising from a countable triangulation of the closed disc can be written as a colimit of finite rooted cluster algebras. We show that all the finite rooted cluster algebras occurring in this colimit can be taken to be of finite Dynkin type A.

2.2 Rooted cluster algebras

Cluster algebras have been introduced by Fomin and Zelevinsky [FZ1]. Throughout this thesis we work with cluster algebras of geometric type and in this chapter we consider their rooted versions, which we obtain by fixing an initial seed. Rooted cluster algebras are the objects in the category *Clus* we want to work in, and which was introduced by Assem, Dupont and Schiffler [ADS].

2.2.1 Seeds

All the information we need to construct a (rooted) cluster algebra is contained in a so-called seed. Along with a distinguished subset of generators for our cluster algebra, it contains a rule that describes how a prescribed set of generators and the relations between them can be obtained. This rule can be encoded in a skew-symmetrizable integer matrix. A *skew-symmetrizable integer matrix* is a square integer matrix B such that there exists a diagonal matrix D with positive integer entries and a skew-symmetric integer matrix S with $S = DB$.

Definition 2.2.1 ([FZ2, Section 1.2])**.** A *seed* is a triple $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$, where

- X is a countable set of indeterminates over $\mathbb{Z}$, i.e. the field $\mathcal{F}_\Sigma = \mathbb{Q}(x \mid x \in \mathrm{X})$ of rational functions in X is a purely transcendental field extension of $\mathbb{Q}$. The set X is called the *cluster of* Σ.

- $\mathrm{ex} \subseteq \mathrm{X}$ is a subset of the cluster. The elements of ex are called the *exchangeable variables of* Σ. The elements $\mathrm{X} \setminus \mathrm{ex}$ are called the *coefficients of* Σ.

- $B = (b_{vw})_{v,w \in \mathrm{X}}$ is a skew-symmetrizable integer matrix with rows and columns labelled by X, which is locally finite, i.e. for every $v \in \mathrm{X}$ there are only finitely many non-zero entries b_{vw} and b_{uv}. The matrix B is called the *exchange matrix of* Σ.

The field $\mathcal{F}_\Sigma = \mathbb{Q}(x \mid x \in \mathrm{X})$ is called the *ambient field* of the seed Σ. Two seeds $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w \in \mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{vw})_{v,w \in \mathrm{X}'})$ are called *isomorphic*, and we write $\Sigma \cong \Sigma'$, if there exists a bijection $f\colon\ \mathrm{X} \to \mathrm{X}'$ inducing a bijection $f\colon\ \mathrm{ex} \to \mathrm{ex}'$ such that for all $v, w \in \mathrm{X}$ we have $b_{vw} = b'_{f(v)f(w)}$.

Remark 2.2.2. The assumption of countability of the cluster X in a seed is not necessary for any of our results to hold (up to a minor change in Theorem 2.4.7, cf. Remark 2.4.9). However, as we will see in Remark 2.3.22, from a combinatorial viewpoint one does not observe any new phenomena by considering uncountable seeds. Where appropriate, we will include a short remark clarifying the situation for uncountable clusters.

Often when giving examples it is more intuitive to think of the combinatorics of a seed as encoded in a quiver instead of in a matrix. This is possible if the exchange matrix is skew-symmetric.

Remark 2.2.3. If the exchange matrix B of the seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ is skew-symmetric, we can express it via a quiver Q_B. The vertices of Q_B are labelled by elements in the cluster X and there are b_{vw} arrows from v to w whenever $b_{vw} \geq 0$. The quiver Q_B is locally finite, i.e. there are only finitely many arrows incident with every vertex. For a seed whose exchange matrix is skew-symmetric by abuse of notation we will often write $\Sigma = (\mathrm{X}, \mathrm{ex}, Q_B)$ for the seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$.

Conversely, to any locally finite quiver without loops or 2-cycles we can associate a locally finite skew-symmetric matrix, with rows and columns labelled by the vertices and with entries

$$b_{ij} = \begin{cases} \#\{\text{arrows from } i \text{ to } j\}, & \text{if there are arrows from } i \text{ to } j \\ -\#\{\text{arrows from } j \text{ to } i\}, & \text{if there are arrows from } j \text{ to } i \\ 0, & \text{otherwise.} \end{cases}$$

Thus we can use locally finite skew-symmetric matrices and locally finite quivers interchangeably.

To any seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ we can naturally associate its *opposite seed* $\Sigma^{op} = (\mathrm{X}, \mathrm{ex}, -B)$, by reversing all signs in the exchange matrix B. If B is skew-symmetric, this corresponds to reversing all arrows in the associated quiver Q_B which gives rise to the opposite quiver Q_B^{op}.

Notation 2.2.4. From now on, when we consider a seed with a skew-symmetric exchange matrix pictured as a quiver, we will mark vertices associated to coefficients with squares.

Example 2.2.5. Consider the seed

$$\Sigma = (\{x_1, x_2, x_3, x_4\}, \{x_1, x_2\}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}).$$

Its exchange matrix is skew-symmetric, and we can express it via a quiver:

$$\Sigma = (\{x_1, x_2, x_3, x_4\}, \{x_1, x_2\}, \begin{tikzcd} & \boxed{x_3} \arrow[dl] \\ x_1 \arrow[r] & x_2 & \boxed{x_4} \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}).$$

Its opposite seed is given by

$$\Sigma^{op} = (\{x_1, x_2, x_3, x_4\}, \{x_1, x_2\}, \begin{tikzcd} & \boxed{x_3} \\ x_1 & x_2 \arrow[l] \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & \boxed{x_4} \end{tikzcd}).$$

An important source of seeds is provided by triangulations of surfaces with (possibly infinitely many) marked points. Throughout this chapter we will follow the example of countable triangulations of the closed disc with marked points on the boundary. This provides a connection to the work of Holm and Jørgensen [HJ] and Igusa and Todorov ([IT3, Section 2.4] and [IT1]), covering cluster categories of countable rank which have combinatorial models via triangulations of the closed disc. In this chapter, cluster categories do not appear, except for the aforementioned motivational purpose and we will provide an overview of cluster categories later in Chapter 3 with a short introduction to the cluster categories of infinite rank studied by Holm and Jørgensen and Igusa and Todorov in Sections 3.3.1 and 3.3.2.

Let us start by defining what we mean by a triangulation of the closed disc $\overline{D}_2$. We cover the boundary $\partial\overline{D}_2 = S^1$ of the closed disc by $\mathbb{R}$ in the usual way: $e\colon \mathbb{R} \to S^1, x \mapsto e(x) := e^{ix}$.

Notation 2.2.6. For any two elements $a \neq b \in S^1$ choose a lifting $\tilde{a} \in \mathbb{R}$ of a and $\tilde{b} \in \mathbb{R}$ of b under the map e such that $\tilde{a} \leq \tilde{b} < \tilde{a} + 2\pi$. Then we denote by $[a, b]$ the image

$$[a, b] = e([\tilde{a}, \tilde{b}]).$$

We define the open interval (a, b) and the half-open intervals $[a, b)$ and $(a, b]$ analogously.

We view $\overline{D}_2 \subseteq \mathbb{R}^2$ as a topological space with the standard topology. Let $\mathcal{Z} \subseteq S^1$ be a subset of the boundary of $\overline{D}_2$. To rule out trivial cases, throughout we assume that any such subset contains at least two elements, i.e. $|\mathcal{Z}| \geq 2$.

Definition 2.2.7. An *arc of* $\mathcal{Z}$ is a two-element subset of $\mathcal{Z}$, i.e. a set $\{x_0, x_1\} \subseteq \mathcal{Z}$ with $x_0 \neq x_1$. An arc $\{x_0, x_1\}$ of $\mathcal{Z}$ is called an *edge of* $\mathcal{Z}$ if $(x_0, x_1) \cap \mathcal{Z} = \emptyset$ or $(x_1, x_0) \cap \mathcal{Z} = \emptyset$. An arc of $\mathcal{Z}$ that is not an edge of $\mathcal{Z}$ is called an *internal arc of* $\mathcal{Z}$.

Two arcs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are said to *cross* if either $y_0 \in (x_0, x_1)$ and $y_1 \in (x_1, x_0)$ or $y_1 \in (x_0, x_1)$ and $y_0 \in (x_1, x_0)$, i.e. if the straight line connecting x_0 and x_1 crosses the straight line connecting y_0 and y_1 in the closed disc.

A *triangulation of the closed disc with marked points* $\mathcal{Z}$ is a maximal collection $\mathcal{T}$ of pairwise non-crossing arcs of $\mathcal{Z}$, i.e. a collection $\mathcal{T}$ of non-crossing arcs of $\mathcal{Z}$ such that every arc of $\mathcal{Z}$ that is not contained in $\mathcal{T}$ crosses at least one arc in $\mathcal{T}$. We call a triangulation $\mathcal{T}$ a *countable triangulation of the closed disc*, if the set $\mathcal{T}$ is countable.

Remark 2.2.8. Note that in order for a triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ to be countable, the set $\mathcal{Z} \subseteq S^1$ does not need to be countable. Consider for example $\mathcal{Z} = S^1$ and the triangulation

$$\mathcal{T} = \{\left\{e\left(\frac{m\pi}{2^n}\right), e\left(\frac{(m+1)\pi}{2^n}\right)\right\} \mid n \geq 0, 0 \leq m < 2^{n+1}\}$$

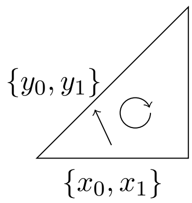
of the closed disc with marked points S^1, where the endpoints of the arcs in $\mathcal{T}$ are a countable dense subset of S^1 (see Figure 3.4 for a picture). Thus $\mathcal{T}$ is a countable triangulation of the closed disc with uncountably many marked points $\mathcal{Z} = S^1$. Similarly, any subset $\mathcal{Z} \subseteq S^1$ allows a countable triangulation of the closed disc with marked points $\mathcal{Z}$.

Remark 2.2.9. An edge of a subset $\mathcal{Z} \subseteq S^1$ crosses no other arcs of $\mathcal{Z}$. Thus by definition every triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ must contain all edges of $\mathcal{Z}$. Note that the set of edges can be empty, for example if we have $\mathcal{Z} = S^1$.

To any countable triangulation of the closed disc we can associate a seed, via the same method that has been introduced by Fomin, Shapiro and Thurston [FST, Definition 4.1 and Section 5] for finite triangulations of surfaces.

Definition 2.2.10. Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. The *seed* $\Sigma_\mathcal{T}$ *associated to* $\mathcal{T}$ is the skew-symmetric seed $\Sigma_\mathcal{T} = (\mathcal{T}, \mathrm{ex}_\mathcal{T}, Q_\mathcal{T})$ defined as follows.

- The elements in the cluster are labelled by the arcs in $\mathcal{T}$.
- An arc $\{x_0, x_1\} \in \mathcal{T}$ is called *exchangeable in* $\mathcal{T}$, if it is the diagonal of a quadrilateral in $\mathcal{T}$, i.e. if there exist vertices $y_0, y_1 \in \mathcal{Z}$ with $y_0 \in (x_0, x_1)$ and $y_1 \in (x_1, x_0)$ such that $\{x_0, y_0\}, \{y_0, x_1\}, \{x_1, y_1\}$ and $\{y_1, x_0\}$ lie in $\mathcal{T}$. The exchangeable variables $\mathrm{ex}_\mathcal{T}$ are labelled by exchangeable arcs in $\mathcal{T}$.
- The exchange matrix of $\Sigma_\mathcal{T}$ is skew-symmetric and we express it via the quiver $Q_\mathcal{T}$: The vertices of $Q_\mathcal{T}$ are labelled by the arcs in $\mathcal{T}$, and for $\{x_0, x_1\}, \{y_0, y_1\} \in \mathcal{T}$ there is an arrow $\{x_0, x_1\} \to \{y_0, y_1\}$ in $Q_\mathcal{T}$ if and only if the arcs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are sides of a common triangle in $\mathcal{T}$ and $\{y_0, y_1\}$ lies in a clockwise direction from $\{x_0, x_1\}$:



Remark 2.2.11. If we omit the countability assumption on the cluster of a seed (cf. Remark 2.2.2), we do not need countable triangulations, but rather any triangulation of the closed disc will give rise to a seed with possibly uncountable cluster.

Because every arc in $\mathcal{T}$ is the side of at most two triangles in $\mathcal{T}$, the quiver $Q_\mathcal{T}$ is locally finite and the seed $\Sigma_\mathcal{T}$ associated to a triangulation $\mathcal{T}$ of the closed disc is indeed a seed in the sense of Definition 2.2.1 in light of Remark 2.2.3.

Remark 2.2.12. An exchangeable arc in a triangulation $\mathcal{T}$ of the closed disc is always internal, as every edge is adjacent to at most one triangle in $\mathcal{T}$ and hence cannot be the diagonal of a quadrilateral in $\mathcal{T}$. However, not every internal arc is necessarily exchangeable. Consider for example the subset

$$\mathcal{Z} = \{e\left(\frac{\pi}{k}\right) | k \in \mathbb{Z} \setminus \{0\}\} \subseteq S^1$$

which has exactly one limit point at 1, and the triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ whose internal arcs are given by

$$\mathcal{T}_{int} = \{\{e\left(\frac{\pi}{2}\right), e\left(\frac{\pi}{k}\right)\} | k \in \mathbb{Z}_{>3}\} \cup \{\{e\left(-\frac{\pi}{2}\right), e\left(-\frac{\pi}{k}\right)\} | k \in \mathbb{Z}_{>3}\} \cup \{\{e\left(\frac{\pi}{2}\right), e\left(-\frac{\pi}{2}\right)\}\}$$

(see Figure 2.1), i.e. $\mathcal{T}$ consists of the union of $\mathcal{T}_{int}$ and all edges of $\mathcal{Z}$. The arc $\{e(\frac{\pi}{2}), e(-\frac{\pi}{2})\} \in \mathcal{T}$ is internal. However, it is not exchangeable: If it was, then it would

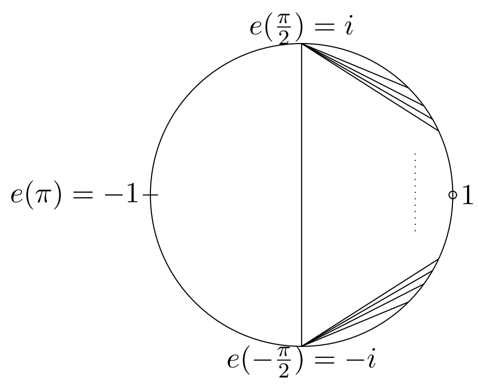


Figure 2.1: In this example of a triangulation the internal arc $\{-i, i\}$ is not exchangeable.

have to be contained in a quadrilateral in $\mathcal{T}$, so there would exist a $z \in (e(-\frac{\pi}{2}), e(\frac{\pi}{2})) \cap \mathcal{Z}$ with $\{e(\frac{\pi}{2}), z\}, \{z, e(-\frac{\pi}{2})\} \in \mathcal{T}$. However, if $z \in (1, e(\frac{\pi}{2}))$ then the arc $\{z, e(-\frac{\pi}{2})\}$ intersects infinitely many of the arcs in $\{\{e(\frac{\pi}{2}), e(\frac{\pi}{k})\} | k \in \mathbb{Z}_{>2}\} \subseteq \mathcal{T}$ and otherwise, if $z \in (e(-\frac{\pi}{2}), 1)$, the arc $\{e(\frac{\pi}{2}), z\}$ intersects infinitely many of the arcs in $\{\{e(-\frac{\pi}{2}), e(\frac{\pi}{k})\} | k \in \mathbb{Z}_{<2}\} \subseteq \mathcal{T}$. This leads to a contradiction, since arcs in $\mathcal{T}$ have to be pairwise non-crossing.

2.2.2 Mutation

A seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ contains all of the data that is needed to construct the associated (rooted) cluster algebra. In order to actually obtain generators for the cluster algebra, a combinatorial process, which is called mutation, is applied. The information needed to perform mutation is encoded in the exchange matrix B.

Definition 2.2.13 ([FZ2, Definition 1.1])**.** Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a seed and let $x \in \mathrm{ex}$ be an exchangeable variable of Σ. We denote the *mutation of* Σ *at* x by $\mu_x(\Sigma) = (\mu_x(\mathrm{X}), \mu_x(\mathrm{ex}), \mu_x(B))$. It is defined by the following data.

- For any $y \in X$ the mutation of y at x is defined by

$$\mu_x(y) = y, \text{ if } y \neq x$$

and

$$\mu_x(x) = \frac{\prod\limits_{v \in \mathrm{X}:\, b_{xv} > 0} v^{b_{xv}} + \prod\limits_{v \in \mathrm{X}:\, b_{xv} < 0} v^{-b_{xv}}}{x} \in \mathcal{F}_\Sigma. \tag{2.1}$$

The equations of the form (2.1) are called *exchange relations*. The cluster, respectively the exchangeable variables, of the seed $\mu_x(\Sigma)$ thus are

$$\mu_x(\mathrm{X}) = \{\mu_x(y) | y \in \mathrm{X}\} = (\mathrm{X} \setminus x) \cup \mu_x(x) \text{ and}$$
$$\mu_x(\mathrm{ex}) = \{\mu_x(y) | y \in \mathrm{ex}\} = (\mathrm{ex} \setminus x) \cup \mu_x(x).$$

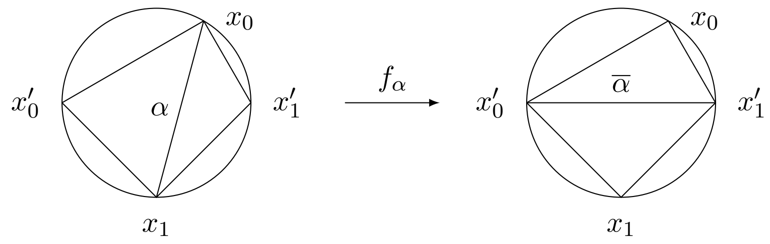


Figure 2.2: Diagonal flip at the arc $\alpha = \{x_0, x_1\}$

- The matrix $\mu_x(B) = (\tilde{b}_{\tilde{v}\tilde{w}})_{\tilde{v},\tilde{w}\in\mu_x(\mathrm{X})}$ is given by *matrix mutation of B at x*: For $\tilde{v} = \mu_x(v)$ and $\tilde{w} = \mu_x(w)$ set

$$\tilde{b}_{\tilde{v}\tilde{w}} = \mu_x(b_{vw}) = \begin{cases} -b_{vw} \text{ if } v = x \text{ or } w = x, \\ b_{vw} + \frac{1}{2}(|b_{vx}|b_{xw} + b_{vx}|b_{xw}|), \text{ otherwise.} \end{cases}$$

Remark 2.2.14. The following facts are well-known and straightforward to check.

(1) Mutation is involutive, i.e. for a seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and any $x \in \mathrm{ex}$ we have $\mu_{\mu_x(x)} \circ \mu_x(\Sigma) = \Sigma$.

(2) Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a seed and let $x \in \mathrm{ex}$. The cluster $\mu_x(\mathrm{X})$ of the seed $\mu_x(\Sigma)$ is a transcendence basis of the ambient field $\mathcal{F}_\Sigma = \mathbb{Q}(X)$ of Σ.

In the case where the exchange matrix B is skew-symmetric, mutation of B corresponds to quiver mutation of the associated quiver Q_B, where mutation of the quiver Q_B at a vertex v of Q_B is denoted by $\mu_v(Q_B) := Q_{\mu_v(B)}$.

Consider our standard example of a seed $\Sigma_\mathcal{T} = (\mathrm{X}_\mathcal{T}, \mathrm{ex}_\mathcal{T}, Q_\mathcal{T})$ associated to a countable triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. Geometrically, mutation of $\Sigma_\mathcal{T}$ at an exchangeable variable in $\mathrm{ex}_\mathcal{T}$ can be represented by a so-called diagonal flip of $\mathcal{T}$. Every exchangeable arc $\{x_0, x_1\} \in \mathrm{ex}_\mathcal{T}$ is the diagonal of a unique quadrilateral with vertices x_0, x_1, x_0' and x_1' in $\mathcal{Z}$, whose sides $\{x_0, x_0'\}$, $\{x_0', x_1\}$, $\{x_1, x_1'\}$ and $\{x_1', x_0\}$ are all contained in $\mathcal{T}$. The *diagonal flip of $\mathcal{T}$ at $\alpha = \{x_0, x_1\}$* is the map $f_\alpha \colon \mathcal{T} \to (\mathcal{T} \setminus \alpha) \cup \overline{\alpha}$ which replaces the arc α in $\mathcal{T}$ by the arc $\overline{\alpha} = \{x_0', x_1'\}$ of $\mathcal{Z}$ and leaves all other arcs invariant, see Figure 2.2.

It is well-known for finite triangulations of the closed disc that for any exchangeable arc $\alpha \in \mathcal{T}$ we have $\mu_\alpha(Q_\mathcal{T}) = Q_{f_\alpha(\mathcal{T})}$. Since mutations of quivers and diagonal flips are defined locally, only a finite subquiver of $Q_\mathcal{T}$ is affected by the mutation at α: This is the full subquiver consisting of α and those vertices of $Q_\mathcal{T}$ that are labelled by the arcs of the unique quadrilateral in $\mathcal{T}$ that has α as a diagonal. Therefore the equality $\mu_\alpha(Q_\mathcal{T}) = Q_{f_\alpha(\mathcal{T})}$ remains true for infinite triangulations.

2.2.3 Rooted cluster algebras

Mutation of a seed at any exchangeable variable in its cluster yields another seed, which again can be mutated at any exchangeable variable in its respective cluster. Thus we can successively mutate a seed Σ along what are called Σ-admissible sequences. Mutation along all possible Σ-admissible sequences will provide a prescribed set of generators of the cluster algebra associated to the seed Σ, the definition of which we will recall in this section.

Definition 2.2.15 ([ADS, Definition 1.3])**.** Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a seed. For $l \geq 1$ a sequence $(x_1, \ldots, x_l)$ is called Σ-*admissible* if $x_1 \in \mathrm{ex}$ and for every $2 \leq k \leq l$, we have $x_k \in \mu_{x_{k-1}} \circ \ldots \circ \mu_{x_1}(\mathrm{ex})$. The empty sequence of length $l = 0$ is Σ-admissible for every seed Σ and mutation of Σ along the empty sequence leaves Σ invariant. We denote by

$$\mathrm{Mut}(\Sigma) = \{\mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma) \mid l \geq 0, (x_1, \ldots, x_l)\ \Sigma\text{-admissible}\}$$

the set of all seeds which can be reached from Σ by iterated mutation along Σ-admissible sequences and call it the *mutation class of* Σ.

Since mutation is involutive (see Remark 2.2.14 (1)), it is clear that mutation along Σ-admissible sequences induces an equivalence relation on seeds, where two seeds Σ and Σ' are *mutation equivalent* if and only if there exists a Σ-admissible sequence $(x_1, \ldots, x_l)$ with $\mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma) = \Sigma'$. The mutation class of a seed Σ is thus really an equivalence class. Analogously, mutation equivalence of locally finite, skew-symmetrizable exchange matrices is defined.

Remark 2.2.16. Note that it is a direct consequence of Definition 2.2.13 that if two seeds Σ and Σ' are mutation equivalent, then the coefficients of Σ are precisely the coefficients of Σ' and that by Remark 2.2.14 (2), any two mutation equivalent seeds give rise to the same ambient field $\mathcal{F}_\Sigma = \mathcal{F}_{\Sigma'}$.

By mutating a seed Σ along all possible Σ-admissible sequences we obtain the mutation class $\mathrm{Mut}(\Sigma)$ of Σ and with it a collection of overlapping clusters. Let $P(\mathcal{F}_\Sigma)$ denote the powerset (i.e. the set of all subsets) of the ambient field $\mathcal{F}_\Sigma$, and let

$$cl_\Sigma \colon \mathrm{Mut}(\Sigma) \to P(\mathcal{F}_\Sigma), (\tilde{\mathrm{X}}, \tilde{\mathrm{ex}}, \tilde{B}) \mapsto \tilde{\mathrm{X}}$$

be the map assigning to each seed in the mutation class of Σ its cluster. We now define the cluster algebra associated to a given seed Σ. The original definition for cluster algebras of finite rank is given by Fomin and Zelevinsky in [FZ1, Definition 2.3].

Definition 2.2.17. Let Σ be a seed. The *cluster algebra associated to* Σ is the $\mathbb{Z}$-subalgebra of its ambient field $\mathcal{F}_\Sigma$ given by

$$\mathcal{A}(\Sigma) = \mathbb{Z}[x \mid x \in cl_\Sigma(\mathrm{Mut}(\Sigma))] \subseteq \mathcal{F}_\Sigma.$$

The elements of $cl_\Sigma(\mathrm{Mut}(\Sigma))$ are called the *cluster variables* and the coefficients of Σ are called the *coefficients* of the cluster algebra $\mathcal{A}(\Sigma)$. We call the cluster algebra $\mathcal{A}(\Sigma)$ *coefficient-free*, if $\mathrm{X} = \mathrm{ex}$ and we call it *skew-symmetric*, if the matrix B is skew-symmetric. The *rank* of the cluster algebra $\mathcal{A}(\Sigma)$ is defined as the cardinality of the cluster of Σ.

Remark 2.2.18. Traditionally, the rank of a cluster algebra $\mathcal{A}(\Sigma)$ is defined as the cardinality of the set of exchangeable variables of Σ, while we define it as the cardinality of the cluster of Σ. A major point of interest in this thesis are cluster algebras of infinite rank, and when we talk about those we explicitely want to include cluster algebras associated to seeds with infinitely many coefficients but only finitely many exchangeable variables.

Example 2.2.19. For a seed $\Sigma = (X, \emptyset, B)$ with no exchangeable cluster variables, we have $\mathrm{Mut}(\Sigma) = \{\Sigma\}$ and the cluster algebra $\mathcal{A}(\Sigma)$ is isomorphic to the polynomial algebra $\mathbb{Z}[x \mid x \in X]$. The empty seed $\Sigma_\emptyset = (\emptyset, \emptyset, \emptyset)$ gives rise to the cluster algebra $\mathcal{A}(\Sigma_\emptyset) \cong \mathbb{Z}$.

Two seeds in the same mutation class give rise to the same cluster algebra. To rigorously define morphisms between cluster algebras in the sense of [ADS] (as we will do in Section 2.3) it is necessary to fix an initial seed.

Definition 2.2.20 ([ADS, Definition 1.4])**.** For any given seed Σ the *rooted cluster algebra with initial seed* Σ is the pair $(\mathcal{A}(\Sigma), \Sigma)$, where $\mathcal{A}(\Sigma)$ is the cluster algebra associated to Σ. We call the rooted cluster algebra $(\mathcal{A}(\Sigma), \Sigma)$ *coefficient-free*, respectively *skew-symmetric*, if $\mathcal{A}(\Sigma)$ is coefficient-free, respectively skew-symmetric. The *cluster variables*, respectively the *coefficients*, of the rooted cluster algebra $(\mathcal{A}(\Sigma), \Sigma)$ are defined as the cluster variables, respectively the coefficients of $\mathcal{A}(\Sigma)$ and the *rank* of $(\mathcal{A}(\Sigma), \Sigma)$ is defined as the rank of $\mathcal{A}(\Sigma)$.

Two distinct seeds in the same mutation class do not give rise to the same rooted cluster algebra. We can think of rooted cluster algebras as pointed versions of cluster algebras.

Example 2.2.21. Consider the seed

$$\Sigma = (\{x_1, x_2\}, \{x_2\}, \boxed{x_1} \longrightarrow x_2).$$

There are only two seeds in the mutation class of Σ, namely Σ itself and

$$\mu_{x_2}(\Sigma) = (\{x_1, \frac{x_1+1}{x_2}\}, \{\frac{x_1+1}{x_2}\}, \boxed{x_1} \longleftarrow \tfrac{x_1+1}{x_2}).$$

The rooted cluster algebra $(\mathcal{A}(\Sigma), \Sigma)$ is of rank two, its cluster variables are x_1, x_2 and $\frac{x_1+1}{x_2}$ and it has one coefficient x_1. As a ring, the cluster algebra $\mathcal{A}(\Sigma)$ is of the form $\mathcal{A}(\Sigma) \cong \mathbb{Z}[x_1, x_2, x_2'] \big/ (x_2 x_2' = x_1 + 1)$.

Remark 2.2.22. Combinatorially, the coefficients of a cluster algebra are not very interesting: They do not get changed under mutation and we just "carry them along". However, cluster algebras occurring in nature frequently have coefficients. For instance, the $\mathbb{Z}$-form of the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(2,n)]$ of the Grassmannian of planes in $\mathbb{C}^n$, considered as a projective variety via the Plücker embedding, carries the structure of a cluster algebra with coefficients. Thus it is important to develop the theory in this generality. Fomin and Zelevinsky [FZ2] assumed coefficients to be invertible. Contrary to this we, as in [ADS], do not assume invertibility of coefficients. We could pass to the case of invertible coefficients simply by localizing at coefficients.

Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. Recall that the cluster variables in the associated seed $\Sigma_{\mathcal{T}}$ are labelled by the arcs of $\mathcal{T}$ and mutation is represented by diagonal flips. We denote by $R_{\mathcal{T}}$ the set of arcs

$$R_{\mathcal{T}} = \{\mu_{\alpha_l} \circ \ldots \circ \mu_{\alpha_1}(\alpha) \mid l \geq 0, \alpha \in \mathcal{T}, (\alpha_1, \ldots, \alpha_l) \text{ is } \Sigma_{\mathcal{T}}\text{-admissible}\}$$

and call its elements the *arcs that can be reached from* $\mathcal{T}$. These are all of the arcs of $\mathcal{Z}$ we obtain from $\mathcal{T}$ by finite sequences of successive diagonal flips, and they correspond to the cluster variables of $\mathcal{A}(\Sigma_{\mathcal{T}})$, which effectively are all variables we obtain from the initial cluster by successive mutation at exchangeable variables.

Remark 2.2.23. If $\mathcal{T}$ is finite, then all arcs of $\mathcal{Z}$ can be reached from $\mathcal{T}$. However, this is not necessarily the case if $\mathcal{T}$ is infinite. For example, as in Remark 2.2.12 consider the subset

$$\mathcal{Z} = \{e\left(\frac{\pi}{k}\right) | k \in \mathbb{Z} \setminus \{0\}\} \subseteq S^1$$

and the triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ whose internal arcs are given by

$$\mathcal{T}_{int} = \{\{e\left(\frac{\pi}{2}\right), e\left(\frac{\pi}{k}\right)\} | k \in \mathbb{Z}_{>3}\} \cup \{\{e\left(-\frac{\pi}{2}\right), e\left(-\frac{\pi}{k}\right)\} | k \in \mathbb{Z}_{>3}\} \cup \{\{e\left(\frac{\pi}{2}\right), e\left(-\frac{\pi}{2}\right)\}\}$$

(see Figure 2.1). The arc $\{e(-\frac{\pi}{4}), e(\frac{\pi}{4})\}$ of $\mathcal{Z}$ cannot be reached from $\mathcal{T}$. If it could, then there would be an $l \geq 0$, a $\Sigma_{\mathcal{T}}$-admissible sequence $(\alpha_1, \ldots, \alpha_l)$ of arcs of $\mathcal{Z}$ and an arc $\alpha \in \mathcal{T}$, such that

$$\{e(-\frac{\pi}{4}), e(\frac{\pi}{4})\} = \mu_{\alpha_l} \circ \ldots \circ \mu_{\alpha_1}(\alpha) \in \mu_{\alpha_l} \circ \ldots \circ \mu_{\alpha_1}(\mathcal{T}).$$

However, the two infinite triangulations $\mu_{\alpha_l} \circ \ldots \circ \mu_{\alpha_1}(\mathcal{T})$ and $\mathcal{T}$ differ only by finitely many elements. Since $\{e(-\frac{\pi}{4}), e(\frac{\pi}{4})\}$ crosses infinitely many arcs in $\mathcal{T}$ it also crosses infinitely many arcs in $\mu_{\alpha_l} \circ \ldots \circ \mu_{\alpha_1}(\mathcal{T})$. This contradicts the fact that $\mu_{\alpha_l} \circ \ldots \circ \mu_{\alpha_1}(\mathcal{T})$ is a triangulation.

The exchange relations (cf. Equation (2.1) in Definition 2.2.13) for mutation of seeds in $\mathrm{Mut}(\Sigma_{\mathcal{T}})$ are the *Plücker relations*: For any two arcs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ in $R_{\mathcal{T}}$, such that $\{x_0, x_1\}$ and $\{y_0, y_1\}$ cross, we have

$$\{x_0, x_1\}\{y_0, y_1\} = \{x_0, y_0\}\{x_1, y_1\} + \{x_0, y_1\}\{x_1, y_0\}.$$

We denote the ideal generated by the Plücker relations in $\mathcal{T}$ by $J_{\mathcal{T}}$. Then the cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is the ring generated by all arcs of $R_{\mathcal{T}}$ being subject to the Plücker relations:

$$\mathcal{A}(\Sigma_{\mathcal{T}}) = \mathbb{Z}[\alpha \mid \alpha \in R_{\mathcal{T}}]/J_{\mathcal{T}}.$$

Remark 2.2.24. If $\mathcal{T}$ is a finite triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ of cardinality $|\mathcal{Z}| = n+3$ for an $n \geq 1$, then the ring $\mathcal{A}(\Sigma_{\mathcal{T}})$ is a cluster algebra of Dynkin type A_n, i.e. it is skew-symmetric and the full subquiver of the exchange quiver of $\Sigma_{\mathcal{T}}$ consisting of the vertices associated to the exchangeable variables of $\Sigma_{\mathcal{T}}$ is mutation equivalent to an orientation of the Dynkin diagram A_n. There is no ambiguity here, as all orientations of the Dynkin diagram A_n are mutation equivalent as quivers. (This can be checked for example by successive quiver mutations at sources and sinks.) In particular, up to ring isomorphism, there is exactly one coefficient-free cluster algebra of Dynkin type A_n. We say that the cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$, respectively the rooted cluster algebra $(\mathcal{A}(\Sigma_{\mathcal{T}}), \Sigma_{\mathcal{T}})$ is *of finite Dynkin type A*. A closer look reveals that base change to $\mathbb{C}$ yields the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(2, n+3)]$ of the Grassmannian of planes in $\mathbb{C}^{n+3}$ via the Plücker embedding, as shown by Fomin and Zelevinsky [FZ2, Proposition 12.7].

In the case where $\mathcal{Z} \subseteq S^1$ is discrete with exactly one limit point the cluster algebras associated to triangulations of $\mathcal{Z}$ have been studied in [GG]. After base change to $\mathbb{C}$ they are subrings of the homogeneous coordinate ring of the doubly infinite Grassmannian of planes via the Plücker embedding. Note that this is in analogy with our observations above on cluster algebras of finite Dynkin type A.

2.3 Rooted cluster morphisms and the category of rooted cluster algebras

When working with cluster algebras, it is natural to wonder what a "morphism of cluster algebras" should be. Intuitively we want such maps to be ring homomorphisms commuting with mutation. In [FZ2, Section 1.2], Fomin and Zelevinsky considered what they called strong isomorphisms of cluster algebras. These are isomorphisms of rings between cluster algebras that map each seed to an isomorphic seed. This idea was generalized by Assem, Schiffler and Shramchenko in [ASS] via the notion of cluster automorphisms. A cluster automorphism is a ring automorphism of a cluster algebra which sends a distinguished seed Σ to another seed $f(\Sigma)$ in the mutation class of Σ, such that f commutes with mutation at every variable in the two clusters. Again, only ring homomorphisms between isomorphic rings are considered. Furthermore, only coefficient-free cluster algebras are considered and cluster automorphisms always bijectively map clusters to clusters: There is no way to "delete" cluster variables.

2.3.1 Rooted cluster morphisms

In [ADS] Assem, Dupont and Schiffler introduced the notion of rooted cluster morphisms. Passing from cluster algebras to rooted cluster algebras by fixing an initial seed allows for a rigorous definition of what one means for a ring homomorphism between not necessarily ring isomorphic cluster algebras to commute with mutation.

Definition 2.3.1 ([ADS, Definition 2.1])**.** Let Σ and Σ' be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a map between their associated cluster algebras, see Definition 2.2.17. A Σ-admissible sequence $(x_1, \ldots, x_l)$ whose image $(f(x_1), \ldots, f(x_l))$ is Σ'-admissible is called (f, Σ, Σ')-*biadmissible*.

Definition 2.3.2 ([ADS, Definition 2.2])**.** Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ be seeds and let $(\mathcal{A}(\Sigma), \Sigma)$ and $(\mathcal{A}(\Sigma'), \Sigma')$ be the corresponding rooted cluster algebras, see Definition 2.2.20. A *rooted cluster morphism* from $(\mathcal{A}(\Sigma), \Sigma)$ to $(\mathcal{A}(\Sigma'), \Sigma')$ is a ring homomorphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ of unital rings, i.e. a ring homomorphism with $f(1) = 1$, satisfying the following conditions:

CM1 $f(\mathrm{X}) \subseteq \mathrm{X}' \cup \mathbb{Z}$.

CM2 $f(\mathrm{ex}) \subseteq \mathrm{ex}' \cup \mathbb{Z}$.

CM3 The homomorphism f commutes with mutation along (f, Σ, Σ')-biadmissible sequences, i.e. for every (f, Σ, Σ')-biadmissible sequence $(x_1, \ldots, x_l)$ we have

$$f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y))$$

for all $y \in \mathrm{X}$ with $f(y) \in \mathrm{X}'$.

Notation 2.3.3. From now on by abuse of notation we write $\mathcal{A}(\Sigma)$ for the rooted cluster algebra $(\mathcal{A}(\Sigma), \Sigma)$.

Remark 2.3.4. Every cluster automorphism in the sense of Assem, Schiffler und Shramchenko [ASS] can be viewed as a rooted cluster morphism from a skew-symmetric, coefficient-free rooted cluster algebra $\mathcal{A}(\Sigma)$ of finite rank to itself (where skew-symmetry, finite rank and coefficient-freeness are the assumptions in [ASS] for the definition of a cluster automorphism). Thus rooted cluster morphisms really provide a generalization of the concept of cluster automorphisms.

The following example includes some of the more interesting things that can happen with rooted cluster morphisms: Firstly, they may exist between non-isomorphic rings, further we may "delete" cluster variables by sending them to integers and we may "defreeze" coefficients by sending them to exchangeable cluster variables.

Example 2.3.5. Consider the seeds

$$\Sigma = (\{x_1, x_2, x_3\}, \{x_2, x_3\}, \begin{tikzcd} \boxed{x_1} \arrow[r] & x_2 & x_3 \arrow[l] \end{tikzcd})$$

and
$$\Sigma' = (\{y_1, y_2\}, \{y_1, y_2\}, \; y_1 \longrightarrow y_2 \;).$$
The associated cluster algebras are as rings isomorphic to
$$\mathcal{A}(\Sigma) \cong \mathbb{Z}[x_1, x_2, x_3, \frac{x_1x_3+1}{x_2}, \frac{x_2+1}{x_3}, \frac{x_1x_3+x_2+1}{x_2x_3}]$$
$$\mathcal{A}(\Sigma') \cong \mathbb{Z}[y_1, y_2, \frac{y_1+1}{y_2}, \frac{y_2+1}{y_1}, \frac{y_1+y_2+1}{y_1y_2}].$$
Consider the ring homomorphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ we obtain from the projection of x_3 to 1, which acts on the cluster variables of Σ as $x_i \mapsto y_i$ for $i = 1, 2$ and $x_3 \mapsto 1$. This ring homomorphism satisfies axioms CM1 and CM2 by definition. The only exchangeable cluster variable in Σ whose image is exchangeable in Σ' is x_2 with $f(x_2) = y_2$, so the first entry of every (f, Σ, Σ')-biadmissible sequence has to be x_2. We have
$$f(\mu_{x_2}(x_2)) = f\left(\frac{x_1x_3+1}{x_2}\right) = \frac{y_1+1}{y_2} = \mu_{y_2}(y_2) = \mu_{f(x_2)}(f(x_2))$$
and, since $f(x_i) \neq f(x_2)$ for $i = 1, 3$, we have $f(\mu_{x_2}(x_i)) = f(x_i) = \mu_{f(x_2)}(f(x_i))$. Furthermore, the only exchangeable cluster variable in $\mu_{x_2}(\Sigma)$ whose image is exchangeable in $\mu_{y_2}(\Sigma')$ is $\mu_{x_2}(x_2)$ with $f(\mu_{x_2}(x_2)) = \mu_{y_2}(y_2)$, so all (f, Σ, Σ')-biadmissible sequences have alternating entries x_2 and $\mu_{x_2}(x_2)$. Since mutation is involutive (see Remark 2.2.14), the ring homomorphism f commutes with mutation along any of these sequences. Thus axiom CM3 is satisfied and f is a rooted cluster morphism.

The following proposition shows that the conditions for a map $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ to be a rooted cluster morphism are preserved under mutation along biadmissible sequences.

Proposition 2.3.6. *Let Σ and Σ' be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. Then for every (f, Σ, Σ')-biadmissible sequence $(x_1, \ldots, x_l)$, the map f induces a rooted cluster morphism $f\colon \mathcal{A}(\tilde{\Sigma}) \to \mathcal{A}(\tilde{\Sigma}')$ between the rooted cluster algebras with initial seeds $\tilde{\Sigma} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma)$ and $\tilde{\Sigma}' = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(\Sigma')$.*

Proof. Because Σ and $\tilde{\Sigma}$, respectively Σ' and $\tilde{\Sigma}'$, are mutation equivalent, we have $\mathcal{A}(\Sigma) = \mathcal{A}(\tilde{\Sigma})$ and $\mathcal{A}(\Sigma') = \mathcal{A}(\tilde{\Sigma}')$ as algebras, so $f\colon \mathcal{A}(\tilde{\Sigma}) \to \mathcal{A}(\tilde{\Sigma}')$ is well-defined as a ring homomorphism. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ and let $\tilde{\Sigma} = (\tilde{\mathrm{X}}, \tilde{\mathrm{ex}}, \tilde{B})$ and $\tilde{\Sigma}' = (\tilde{\mathrm{X}}', \tilde{\mathrm{ex}}', \tilde{B}')$. Then every element $\tilde{x}$ of $\tilde{\mathrm{X}}$ (respectively of $\tilde{\mathrm{ex}}$) is of the form $\tilde{x} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(x)$ for an $x \in \mathrm{X}$ (respectively $x \in \mathrm{ex}$). If $f(x) \in \mathbb{Z}$, then because $(x_1, \ldots, x_l)$ is (f, Σ, Σ')-biadmissible we have $x \neq x_i$ for all $1 \leq i \leq l$. Thus $\tilde{x} = x$ and $f(\tilde{x}) = f(x) \in \mathbb{Z}$. On the other hand, if $f(x) \notin \mathbb{Z}$, then by axiom CM1 (respectively CM2) we have $f(x) \in \mathrm{X}'$ (respectively $f(x) \in \mathrm{ex}'$). Thus by axiom CM3 for $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ we have
$$f(\tilde{x}) = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(x))$$
which lies in $\tilde{\mathrm{X}}'$ (respectively in $\tilde{\mathrm{ex}}'$). Thus $f\colon \mathcal{A}(\tilde{\Sigma}) \to \mathcal{A}(\tilde{\Sigma}')$ satisfies axioms CM1 and CM2. Because $\tilde{\mathrm{ex}} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(\mathrm{ex})$ and $\tilde{\mathrm{ex}}' = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(\mathrm{ex}')$, every $(f, \tilde{\Sigma}, \tilde{\Sigma}')$-biadmissible sequence $(y_1, \ldots, y_m)$ gives rise to a (f, Σ, Σ')-biadmissible

sequence $(x_1, \ldots, x_l, y_1, \ldots, y_m)$. Let now $\tilde{y} \in \tilde{\mathrm{X}}$ be such that $f(\tilde{y}) \in \tilde{\mathrm{X}}'$. We have $\tilde{y} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ for a $y \in \mathrm{X}$. If $f(y) \in \mathbb{Z}$, then with the same argument as above we have $\tilde{y} = y$ and thus $f(\tilde{y}) \in \mathbb{Z}$. Therefore whenever we have $f(\tilde{y}) \in \tilde{\mathrm{X}}'$ we have $f(y) \in \mathrm{X}'$ and by axiom CM3 for $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ we have

$$\begin{aligned} f(\mu_{y_m} \circ \ldots \circ \mu_{y_1}(\tilde{y})) &= f(\mu_{y_m} \circ \ldots \circ \mu_{y_1} \circ \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) \\ &= \mu_{f(y_m)} \circ \ldots \circ \mu_{f(y_1)} \circ \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y)) \\ &= \mu_{f(y_m)} \circ \ldots \circ \mu_{f(y_1)}(f(\tilde{y})) \end{aligned}$$

Thus axiom CM3 is satisfied for $f\colon \mathcal{A}(\tilde{\Sigma}) \to \mathcal{A}(\tilde{\Sigma}')$. □

We will show in Proposition 2.3.9 that a rooted cluster morphism is quite limited in its action on exchangeable variables of the initial seed: It has to be injective on the exchangeable variables that are not being sent to integers. Furthermore, it cannot map any coefficients to the same cluster variable to which it maps an exchangeable variable. It may however send two coefficients to the same cluster variable, as long as it is careful about their exchangeable neighbours.

Definition 2.3.7. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in\mathrm{X}})$ be a seed and let $x \in \mathrm{X}$ be a cluster variable in Σ. We call a cluster variable $y \in \mathrm{X}$ *a neighbour of x in Σ*, if $b_{xy} \neq 0$.

Remark 2.3.8. Note that being neighbours is a symmetric relation: For a given seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ a cluster variable $x \in \mathrm{X}$ is a neighbour of $y \in \mathrm{X}$ in Σ if and only if y is a neighbour of x in Σ. We then say that *x and y are neighbours in* Σ.

Proposition 2.3.9. *Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in\mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. If $x \neq y$ are cluster variables of Σ with $f(x) = f(y) \in \mathrm{X}'$, then both x and y are coefficients of Σ. In that case for any $z \in \mathrm{ex}$ that is a neighbour of both x and y in Σ and such that $f(z) \in \mathrm{ex}'$, the entries b_{zx} and b_{zy} have the same sign.*

Proof. Let $x \in \mathrm{ex}$ with $f(x) \in \mathrm{X}'$. We want to show that $f(y) \neq f(x)$ for every cluster variable $y \in \mathrm{X} \setminus x$. By axiom CM2 we have $f(x) \in \mathrm{ex}'$ and the sequence (x) is (f, Σ, Σ')-biadmissible. Let $y \in \mathrm{X}$ with $y \neq x$. If $f(y) \in \mathbb{Z}$ then we have $f(x) \neq f(y)$, thus assume $f(y) \in \mathrm{X}'$. By axiom CM3 we obtain

$$f(y) = f(\mu_x(y)) = \mu_{f(x)}(f(y)).$$

Assume for a contradiction that $f(y) = f(x) =: z' \in \mathrm{ex}'$. This would imply $z' = \mu_{z'}(z')$. Writing $B' = (b'_{xy})_{x,y\in\mathrm{X}'}$ we thus would have

$$(z')^2 = z'\mu_{z'}(z') = \prod_{v\in\mathrm{X}'\colon b'_{z'v}>0} v^{b'_{z'v}} + \prod_{v\in\mathrm{X}'\colon b'_{z'v}<0} v^{-b'_{z'v}},$$

which contradicts algebraic independence of the cluster variables in X'.

We now prove the second part of the statement. Let $x \neq y \in \mathrm{X} \setminus \mathrm{ex}$ be coefficients of Σ. Assume for a contradiction that $f(x) = f(y) = x' \in \mathrm{X}'$ and there exists a $z \in \mathrm{ex}$ which is a neighbour of both x and y in Σ with $f(z) \in \mathrm{ex}'$ such that b_{zx} and b_{zy} have opposite signs. Without loss of generality assume $b_{zx} > 0$ and $b_{zy} < 0$. Then we have

$$\begin{aligned} f(z\mu_z(z)) &= \prod_{v \in \mathrm{X}\colon b_{zv} > 0} f(v)^{b_{zv}} + \prod_{v \in \mathrm{X}\colon b_{zv} < 0} f(v)^{-b_{zv}} \\ &= x' \left(\prod_{x \neq v \in \mathrm{X}\colon b_{zv} > 0} f(v)^{b_{zv}} + \prod_{y \neq v \in \mathrm{X}\colon b_{zv} < 0} f(v)^{-b_{zv}} \right). \end{aligned}$$

By axiom CM3 this has to be equal to

$$f(z)\mu_{f(z)}(f(z)) = \prod_{v' \in \mathrm{X}'\colon b'_{f(z)v'} > 0} (v')^{b'_{f(z)v'}} + \prod_{v' \in \mathrm{X}'\colon b'_{f(z)v'} < 0} (v')^{-b'_{f(z)v'}},$$

and since we either have $b'_{f(z)x'} \geq 0$ or $b'_{f(z)x'} < 0$ the cluster variable x' cannot divide both summands on the right hand side. This contradicts algebraic independence of the variables in X'.

□

Corollary 2.3.10. *Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w \in \mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. Consider any (f, Σ, Σ')-biadmissible sequence $(x_1, \dots, x_l)$. If $x \neq y$ are cluster variables in $\mu_{x_l} \circ \dots \circ \mu_{x_1}(\mathrm{X})$ with $f(x) = f(y) \in \mu_{f(x_l)} \circ \dots \mu_{f(x_1)}(\mathrm{X}')$, then both x and y are coefficients of Σ. In that case, for any exchangeable neighbour $z \in \mu_{x_l} \circ \dots \circ \mu_{x_1}(\mathrm{ex})$ of both x and y in $\mu_{x_l} \circ \dots \circ \mu_{x_1}(\Sigma)$ with $f(z) \in \mu_{f(x_l)} \circ \dots \circ \mu_{f(x_1)}(\mathrm{ex}')$ the entries $\mu_{x_l} \circ \dots \circ \mu_{x_1}(b_{zx})$ and $\mu_{x_l} \circ \dots \circ \mu_{x_1}(b_{zy})$ of the matrix $\mu_{x_l} \circ \dots \circ \mu_{x_1}(B)$ have the same sign.*

Proof. By Remark 2.2.16 the coefficients of $\mu_{x_l} \circ \dots \circ \mu_{x_1}(\Sigma) =: \tilde{\Sigma}$ are precisely the coefficients of Σ. The statement follows from Proposition 2.3.9 by using Proposition 2.3.6 to view f as a rooted cluster morphism with source $\mathcal{A}(\tilde{\Sigma})$. □

2.3.2 Ideal rooted cluster morphisms

An ideal rooted cluster morphism is a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ whose image $f(\mathcal{A}(\Sigma))$ is the rooted cluster algebra with initial seed $f(\Sigma)$ the image of Σ. In the discussion before [ADS, Problem 2.12] (which asks for a characterization of all ideal rooted cluster morphisms), the authors asked whether every rooted cluster morphism was ideal. In this section we answer the question by showing that not every rooted cluster morphism is ideal.

Definition 2.3.11 ([ADS, Definition 2.8])**.** Let

$$\Sigma = (\mathrm{X}, \mathrm{ex}, B) \quad \text{and} \quad \Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{vw})_{v,w \in \mathrm{X}'})$$

be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. Then the image $f(\Sigma)$ of the seed Σ under the morphism f is the seed

$$f(\Sigma) = (f(\mathrm{X}) \cap \mathrm{X}', f(\mathrm{ex}) \cap \mathrm{ex}', f(B) = (b'_{vw})_{v,w \in f(\mathrm{X}) \cap \mathrm{X}'}).$$

Note that the exchangeable variables of the image seed $f(\Sigma)$ all are images of exchangeable variables of Σ. We might well have exchangeable variables of Σ' that lie in the image $f(\mathrm{X} \setminus \mathrm{ex})$ of the coefficients of Σ – these are not exchangeable variables of $f(\Sigma)$.

Example 2.3.12. Consider the seeds

$$\Sigma = (\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \{x_1, x_2, x_3\}, \begin{tikzcd} & & \boxed{x_5} \arrow[r] \arrow[dl] & x_3 \arrow[r] & \boxed{x_6} \\ x_1 \arrow[r] & x_2 & \boxed{x_4} \arrow[l, shift left] \arrow[l, shift right] \arrow[ur] & & \boxed{x_7} \end{tikzcd})$$

and

$$\Sigma' = (\{y_1, y_2, y_3, z_1, z_2, a\}, \{y_1, y_2, y_3, a\}, \begin{tikzcd} y_1 \arrow[r] & y_2 & z_1 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[rr, bend left] & y_3 \arrow[l, shift left] \arrow[l, shift right] & z_2 \arrow[l] & a \arrow[l] \end{tikzcd})$$

and the map $f\colon \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \to \{y_1, y_2, y_3, z_1, z_2, a\}$ which maps

$$\begin{aligned} x_i &\mapsto y_i \text{ for } i = 1, 2, 3 \\ x_i &\mapsto z_1 \text{ for } i = 4,5,7 \\ x_6 &\mapsto z_2. \end{aligned}$$

As we will see in Example 2.3.40 this map induces a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$. The image $f(\Sigma)$ is the seed

$$f(\Sigma) = (\{y_1, y_2, y_3, z_1, z_2\}, \{y_1, y_2, y_3\}, \begin{tikzcd} y_1 \arrow[r] & y_2 & \boxed{z_1} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[rr, bend left] & y_3 \arrow[l, shift left] \arrow[l, shift right] & \boxed{z_2} \arrow[l] \end{tikzcd}).$$

If $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is a rooted cluster morphism, then the seed $f(\Sigma)$ is an example of what is called a full subseed of the seed Σ'.

Definition 2.3.13 ([ADS, Definition 4.9])**.** Let $\Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{vw})_{v,w \in \mathrm{X}'})$ be a seed. A *full subseed* of Σ' is a seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w \in \mathrm{X}})$ such that

- $\mathrm{X} \subseteq \mathrm{X}'$,
- $\mathrm{ex} \subseteq \mathrm{ex}'$,
- B is the submatrix of B' formed by the entries labelled by $\mathrm{X} \times \mathrm{X}$, i.e. for all $v, w \in \mathrm{X}$ we have $b_{vw} = b'_{vw}$.

Remark 2.3.14. Note that while all exchangeable variables of a full subseed of Σ' have to be exchangeable variables of Σ', cluster variables which are coefficients in the full subseed are not necessarily coefficients in Σ'.

Definition 2.3.15 ([ADS, Definition 2.11])**.** A rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is called *ideal* if its image is the rooted cluster algebra with initial seed $f(\Sigma)$, i.e. if $f(\mathcal{A}(\Sigma)) = \mathcal{A}(f(\Sigma))$.

In [ADS, Lemma 2.10] the authors showed that the inclusion $\mathcal{A}(f(\Sigma)) \subseteq f(\mathcal{A}(\Sigma))$ holds for any rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$. We can show that the converse is not true in general.

Theorem 2.3.16. *Not every rooted cluster morphism is ideal.*

Proof. We give an example of a rooted cluster morphism that is not ideal. Consider the seeds

$$\Sigma = (\{a_1, a_2, x\}, \{x\}, \boxed{a_1} \longrightarrow x \longrightarrow \boxed{a_2})$$

and

$$\Sigma' = (\{y_1, y_2\}, \{y_1, y_2\},\ y_1 \longrightarrow y_2\ ).$$

As rings, the cluster algebras are isomorphic to

$$\mathcal{A}(\Sigma) \cong \mathbb{Z}[a_1, a_2, x, x'] \big/ \langle xx' = a_1 + a_2 \rangle$$

and

$$\mathcal{A}(\Sigma') \cong \mathbb{Z}\left[y_1, y_2, \frac{1+y_1}{y_2}, \frac{1+y_2}{y_1}, \frac{1+y_1+y_2}{y_1 y_2}\right].$$

Consider the ring homomorphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ defined by the algebraic extension of the map which sends

$$\begin{aligned} a_1 &\mapsto 1 & \qquad a_2 &\mapsto -1 \\ x &\mapsto 0 & \qquad x' &\mapsto y_1. \end{aligned}$$

Because $f(xx') = 0 = f(a_1 + a_2)$ this is well-defined. Furthermore, it satisfies the axioms CM1 and CM2 for a rooted cluster morphism and because there are no (f, Σ, Σ')-biadmissible sequences it trivially satisfies axiom CM3 and thus is a rooted cluster morphism. The image of the seed Σ is $f(\Sigma) = (\emptyset, \emptyset, \emptyset)$ and thus as a ring we have $\mathcal{A}(f(\Sigma)) \cong \mathbb{Z}$. However, the image of the cluster algebra $\mathcal{A}(\Sigma)$ is $f(\mathcal{A}(\Sigma)) \cong \mathbb{Z}[y_1]$. $\square$

2.3.3 The category of rooted cluster algebras

Considering rooted cluster algebras and rooted cluster morphisms gives rise to a category.

Definition 2.3.17 ([ADS, Definition 2.6])**.** The *category of rooted cluster algebras Clus* is the category which has as objects rooted cluster algebras and as morphisms rooted cluster morphisms.

In [ADS, Section 2] it was shown that *Clus* satisfies the axioms of a category. In particular, axiom CM2 for rooted cluster morphisms is necessary to ensure that compositions of rooted cluster morphisms are again rooted cluster morphisms, as the following example illustrates.

Example 2.3.18. Consider the seeds

$$\Sigma_1 = \Big(\{x_1, x_2, x_3\}, \{x_2\}, \boxed{x_1} \longrightarrow x_2 \longrightarrow \boxed{x_3}\Big),$$
$$\Sigma_2 = \Big(\{z\}, \emptyset, \boxed{z}\Big)$$
$$\Sigma_3 = (\{y_1, y_2\}, \{y_1, y_2\},\ y_1 \longrightarrow y_2\ )$$

with associated cluster algebras

$$\mathcal{A}(\Sigma_1) = \mathbb{Z}[x_1, x_2, x_3, \frac{x_1 + x_3}{x_2}], \quad A(\Sigma_2) = \mathbb{Z}[z]$$

and

$$\mathcal{A}(\Sigma_3) = \mathbb{Z}[y_1, y_2, \frac{1 + y_2}{y_1}, \frac{1 + y_1}{y_2}, \frac{1 + y_1 + y_2}{y_1 y_2}].$$

Consider the ring homomorphisms $f\colon \mathcal{A}(\Sigma_1) \to \mathcal{A}(\Sigma_2)$, which is defined by sending $x_i \mapsto z$ for all $i = 1, 2, 3$, and $g\colon \mathcal{A}(\Sigma_2) \to \mathcal{A}(\Sigma_3)$ defined by sending $z \mapsto y_1$. Both $f\colon \mathcal{A}(\Sigma_1) \to \mathcal{A}(\Sigma_2)$ and $g\colon \mathcal{A}(\Sigma_2) \to \mathcal{A}(\Sigma_3)$ satisfy axiom CM1, but f does not satisfy axiom CM2. Axiom CM3 is satisfied trivially by both f and g, since there are neither (f, Σ_1, Σ_2) nor (g, Σ_2, Σ_3)-biadmissible sequences. However, the composition $g \circ f$ does not satisfy axiom CM3: Consider the $(g \circ f, \Sigma_1, \Sigma_3)$-biadmissible sequence (x_2). We have

$$g \circ f(\mu_{x_2}(x_2)) = g \circ f\left(\frac{x_1 + x_3}{x_2}\right) = g(2) = 2$$

but

$$\mu_{g \circ f(x_2)}(g \circ f(x_2)) = \mu_{y_1}(y_1) = \frac{1 + y_2}{y_1}.$$

2.3.4 Coproducts and connectedness of seeds

Assem, Dupont and Schiffler showed in [ADS, Lemma 5.1] that countable coproducts exist in the category *Clus* of rooted cluster algebras. Taking coproducts of a countable family $\{\mathcal{A}(\Sigma_i)\}_{i \in I}$ of rooted cluster algebras amounts to taking what can be intuitively described as the disjoint union Σ of their seeds. The seeds Σ_i will be full subseeds of the seed Σ which are mutually disconnected.

Definition 2.3.19. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a seed. A sequence $x_0, x_1, \ldots, x_l$ of cluster variables in X with $l \geq 0$ such that for any $0 \leq i < l$ the cluster variables x_i and x_{i+1} are neighbours in Σ is called a *path of length l in Σ*. We call two cluster variables $x, y \in \mathrm{X}$ *connected in Σ*, if there exists a path $x_0, \ldots, x_l$ of finite length $l \geq 0$ in Σ such that $x = x_0$ and $y = x_l$. We call the seed Σ *connected* if any two cluster variables $x, y \in \mathrm{X}$ are connected in Σ. We call a rooted cluster algebra $\mathcal{A}(\Sigma)$ *connected* if its initial seed Σ is connected.

Remark 2.3.20. If $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ is a seed with skew-symmetric exchange matrix B, it is connected if and only if the underlying graph of the associated quiver Q_B is connected.

We can decompose any seed into its connected components. For a seed

$$\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in\mathrm{X}})$$

and an element $x \in \mathrm{X}$ of its cluster, the *connected component of x in* Σ is the full connected subseed Σ_x of Σ consisting of those cluster variables in X that are connected to x and such that all coefficients in Σ_x are also coefficients in Σ, i.e.

$$\Sigma_x = (\mathrm{X}_x, \mathrm{ex}\cap\mathrm{X}_x, B_x = (b_{vw})_{v,w\in\mathrm{X}_x}),$$

where $\mathrm{X}_x = \{y \in \mathrm{X} \mid x \text{ and } y \text{ are connected in } \Sigma\}$. The decomposition is given as follows: Let $\{\Sigma_j = (\mathrm{X}_j, \mathrm{ex}_j, B_j = (b^j_{vw})_{v,w\in\mathrm{X}_j})\}_{j\in I}$ for a countable index set I be the set of mutually distinct connected components in Σ. Since no vertex in X_i is connected to any vertex in X_j for $i \neq j \in I$, we have $b_{xy} = 0$ for $x \in \mathrm{X}_i$ and $y \in \mathrm{X}_j$ with $i \neq j \in I$. Thus we have $\mathrm{X} = \bigcup_{j\in I}\mathrm{X}_j$ and $\mathrm{ex} = \bigcup_{j\in I}\mathrm{ex}_j$, and the matrix B has the matrices B_j for $j \in I$ as block-diagonal entries, i.e. $b_{vw} = b^j_{vw}$ if $v, w \in \mathrm{X}_j$ for some $j \in I$ and $b_{vw} = 0$ otherwise.

Remark 2.3.21. It is a direct consequence of the definition of matrix mutation (cf. Definition 2.2.13) that mutation of seeds respects connected components.

Conversely we can build a new seed from a countable collection of seeds by taking the disjoint union of the clusters and the exchangeable variables and constructing a big matrix which contains all of their exchange matrices as block-diagonal entries: Let $\{\Sigma_j = (\mathrm{X}_j, \mathrm{ex}_j, B_j)\}_{j\in I}$ be a countable collection of seeds. Denote by $\bigsqcup$ the disjoint union and set

$$\bigsqcup_{j\in I}\Sigma_j := (\bigsqcup_{j\in I}\mathrm{X}_j, \bigsqcup_{j\in I}\mathrm{ex}_j, B),$$

where B is the block-diagonal matrix with blocks B_j for $j \in I$. The analogous construction for rooted cluster algebras is taking coproducts; by [ADS, Lemma 5.1], the category *Clus* of rooted cluster algebras admits countable coproducts $\coprod$ and for a countable index set I we have

$$\coprod_{j\in I}\mathcal{A}(\Sigma_j) \cong \mathcal{A}(\bigsqcup_{j\in I}\Sigma_j).$$

The seeds Σ_j for $j \in I$ are mutually disconnected full subseeds of Σ. On the other hand, since we can decompose any given seed into its connected components and there are only countably many cluster variables, given a rooted cluster algebra $\mathcal{A}(\Sigma)$ we can write it as a countable coproduct of connected rooted cluster algebras.

Remark 2.3.22. If we omit the countability assumption for clusters of seeds (cf. Remark 2.2.2), then uncountable coproducts exist: This follows directly from the proof of [ADS, Lemma 5.1], where the countability assumption is solely needed for the cluster of the coproduct to be countable. All of the other arguments go through directly.

In fact, having uncountably many connected components is the only way for a seed to have an uncountable cluster; if a seed is connected, then the fact that it has a countable cluster is automatic: Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a connected seed and let $x \in \mathrm{X}$. For $l \geq 0$ we set $\mathrm{X}_l = \{y \in \mathrm{X} \mid\ x$ and y are connected by a path of length l in $\Sigma\}$. Because B is locally finite, for each $l \geq 0$, the set X_l is finite and because Σ is connected, we have $\mathrm{X} = \bigcup_{l\geq 0} \mathrm{X}_l$. Therefore, X is countable.

Thus every connected component of a seed has a countable cluster. As a consequence, one does not currently gain much from considering uncountable clusters. The usual operations on seeds, namely mutations along (finite) admissible sequences, affect only finitely many connected components and hence only operate on a countable full subseed which is not connected to its invariant complement. So for all practical purposes one can restrict to working with countable seeds without any substantial loss of generality.

Let us consider again the example of a countable triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. Note that by definition of the seed $\Sigma_{\mathcal{T}}$ associated to $\mathcal{T}$ two cluster variables $\alpha, \beta \in \mathcal{T}$ are neighbours in $\Sigma_{\mathcal{T}}$ if and only if the arcs α and β are sides of a common triangle in $\mathcal{T}$ and they are connected in $\Sigma_{\mathcal{T}}$ if and only if there exists a $k \geq 0$ and a sequence of arcs $\gamma_0, \ldots, \gamma_k$, such that $\alpha = \gamma_0$ and $\beta = \gamma_k$ and for all $0 \leq i \leq k$ the arcs γ_i and γ_{i+1} are sides of a common triangle in $\mathcal{T}$. It turns out that the connected components of $\Sigma_{\mathcal{T}}$ depend on the behaviour of arcs in $\mathcal{T}$ in the neighbourhood of limit points of $\mathcal{Z}$.

Definition 2.3.23. Let $\mathcal{Z} \subseteq S^1$. We say that a sequence $\{z_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of points in $\mathcal{Z}$ converging to z converges to $z \in S^1$ *from the right*, if for any $x \in S^1$ the set $[x, z) \cap \mathcal{Z}$ is infinite and the set $(z, x] \cap \mathcal{Z}$ is finite. We say that it converges to $z \in S^1$ *from the left*, if for any $x \in S^1$ the set $[x, z) \cap \mathcal{Z}$ is finite and the set $(z, x] \cap \mathcal{Z}$ is infinite. We say that it converges to $z \in S^1$ *from both sides*, if for any $x \in S^1$ both the set $[x, z) \cap \mathcal{Z}$ and the set $(z, x] \cap \mathcal{Z}$ are infinite.

Let $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence of arcs of $\mathcal{Z}$ and let the sequence of endpoints converge to $a = \varinjlim a_i \in S^1$ and $b = \varinjlim b_i \in S^1$. If both sequences of endpoints $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ and $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ are non-constant, we say that the sequence $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs is a *nest* if $a = b$ and we say that it is a *half-nest* if $a \neq b$.

If the sequence $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is constant and the sequence $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is non-constant, we say that the sequence $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs is a *right-fountain at a converging to b*, if $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to b from the right, we say that it is a *left-fountain at a converging to b*, if $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to b from the left and we say that it is a *fountain at a converging to b*, if $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to b from both sides. We call a sequence $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs in $\mathcal{Z}$ a *split fountain converging to b*, if it can be partitioned into a left fountain $\{a_l, b_i\}_{i\in\mathbb{Z}_{<0}}$ at $a_l \in \mathcal{Z}$ converging to $b \in S^1$ and a right fountain $\{a_r, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ at $a_r \in \mathcal{Z}$ converging to b with $a_l \neq a_r$.

To determine the connected components of the seed $\Sigma_{\mathcal{T}}$ associated to a given countable triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ it is helpful to view any half-nest, fountain and right-or left-fountain $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ as converging to an arc $\{a, b\}$ of the

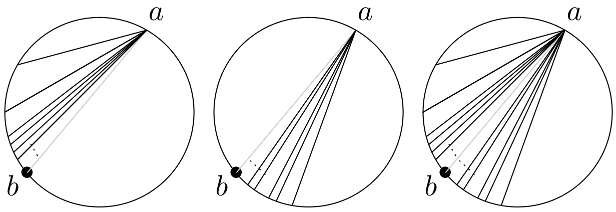


Figure 2.3: Triangulations of the closed disc consisting of (from left to right) a right fountain at $a \in \mathcal{Z}$, a left fountain at a and a fountain at a, all converging to the limit arc $\{a, b\}$ of the respective triangulation

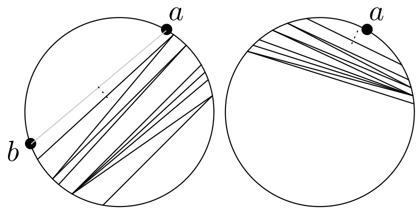


Figure 2.4: A triangulation of the closed disc consisting of a half nest converging to the limit arc $\{a, b\}$ of the triangulation, and a nest where the sequence of endpoints converges to a

topological closure $\overline{\mathcal{Z}}$ of $\mathcal{Z} \subseteq S^1$, where $a = \varinjlim a_i$ and $b = \varinjlim b_i$. Let $\mathcal{T}$ be a triangulation of the closed disc with marked points $\mathcal{Z}$ and let $\{a, b\}$ be an arc of $\overline{\mathcal{Z}}$ such that there is a half-nest, fountain and right-or left-fountain in $\mathcal{T}$ converging $\{a, b\}$. Then we call $\{a, b\}$ a *limit arc of* $\mathcal{T}$. Figure 2.3 provides an illustration of a left-fountain, a right-fountain and a fountain, while Figure 2.4 illustrates a half-nest and a nest.

Lemma 2.3.24. *Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. Two arcs $\{x_0, x_1\} \neq \{y_0, y_1\}$ are connected in $\Sigma_{\mathcal{T}}$ if and only if there is no limit arc $\{a, b\}$ of $\mathcal{T}$ such that $x_0, x_1 \in [a, b]$ and $y_0, y_1 \in [b, a]$ or vice-versa.*

Proof. First assume that there exists a limit arc $\{a, b\}$ of $\mathcal{T}$. It is straightforward to check that there cannot be a finite sequence of arcs connecting any arc with endpoints in $[a, b]$ with a distinct arc with endpoints in $[b, a]$, since there are infinitely many arcs from the right-or left-fountain, fountain or half-nest in $\mathcal{T}$ converging to the limit arc $\{a, b\}$ of $\mathcal{T}$ in between.

On the other hand, assume that $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are not connected. Without loss of generality let $(x_0, x_1) \subseteq (x_0, y_0) \subseteq (x_0, y_1)$. We can construct a sequence of arcs $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ by setting $a_0 = x_0$ and $b_0 = x_1$ and for $i \geq 1$ choosing $\{a_i, b_i\}$ such that $\{a_{i-1}, b_{i-1}\}$ and $\{a_i, b_i\}$ are sides of a common triangle in $\mathcal{T}$ and such that $a_i \in [y_1, a_{i-1}]$ and $b_i \in [b_{i-1}, y_0]$. Because $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are not connected, a_i and b_i are well-defined for all $i \geq 0$ and at least one of the sequences $\{a_i\}_{i \in \mathbb{Z}}$ and $\{b_i\}_{i \in \mathbb{Z}}$ is not constant. Both sequences of endpoints $\{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ are monotone and bounded above and below and thus $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a half-nest, fountain or right-or left-fountain converging

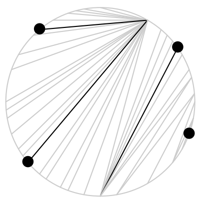


Figure 2.5: This picture illustrates a partition of a triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ into connected components. The arcs in $\mathcal{T}$ are drawn in grey, the limit arcs of $\mathcal{T}$ are drawn in black. Limit points of $\mathcal{Z}$ are marked by bullets.

to a limit arc $\{a, b\}$ of $\mathcal{T}$ such that $x_0, x_1 \in [a, b]$ and $y_0, y_1 \in [b, a]$. □

Remark 2.3.25. For a given countable triangulation $\mathcal{T}$ with marked points $\mathcal{Z} \subseteq S^1$ the limit arcs $\{a, b\}$ of $\mathcal{T}$ partition the seed $\Sigma_{\mathcal{T}}$ associated to $\mathcal{T}$ into connected components. If a limit arc $\{a, b\}$ of $\mathcal{T}$ is not an arc of $\mathcal{Z}$ (in particular this is always the case if $\mathcal{Z}$ is discrete) then it divides $\Sigma_{\mathcal{T}}$ into two mutually disconnected components. If the limit arc $\{a, b\}$ of $\mathcal{T}$ is an arc of $\mathcal{Z}$, then it is an arc in $\mathcal{T}$ (because it cannot cross any arc of $\mathcal{T}$) and it provides an additional connected component, consisting only of the arc $\{a, b\}$ itself. Figure 2.5 provides an illustration of the partition of a triangulation into connected components.

Lemma 2.3.26. *Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ and let $\Sigma_{\mathcal{T}}$ be the associated seed. Then the rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is isomorphic to a coproduct of connected rooted cluster algebras associated to countable triangulations of the closed disc.*

Proof. The limit arcs of $\mathcal{T}$ partition the seed $\Sigma_{\mathcal{T}}$ associated to $\mathcal{T}$ into countably many (since $\mathcal{T}$ is countable) connected components $\{\Sigma_{\mathcal{T}_i}\}_{i \in I}$ for some countable index set I. Thus by the discussion in Section 2.3.4 the rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is isomorphic to the countable coproduct $\coprod_{i \in I} \mathcal{A}(\Sigma_{\mathcal{T}_i})$. □

2.3.5 Isomorphisms of rooted cluster algebras

An isomorphism of rooted cluster algebras implies a close combinatorial relation between their initial seeds. First, we introduce some useful terminology.

Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a seed. We say that two cluster variables $x, y \in \mathrm{X}$ are *connected by exchangeable variables in* Σ if there exists a path $x_0, x_1, \ldots, x_l$ of finite length $l \geq 0$ in Σ (see Definition 2.3.19), such that $x = x_0$, $y = x_l$, and $x_1, \ldots, x_{l-1}$ lie in ex. Further, if $l \in \{0, 1\}$ then at least one of x_0 and x_l has to lie in ex. Thus, two coefficients that are neighbours are not necessarily connected by exchangeable variables and a coefficient is not necessarily connected to itself by exchangeable variables. For an exchangeable variable $x \in \mathrm{ex}$ we define the *exchangeably connected component of x in* Σ to be the full subseed $\Sigma_x^{ex} = (\mathrm{X}_x^{ex}, \mathrm{ex} \cap \mathrm{X}_x^{ex}, B_x = (b_{vw})_{v,w \in \mathrm{X}_x^{ex}})$ of Σ where

$$\mathrm{X}_x^{ex} = \{y \in \mathrm{X} \mid x \text{ and } y \text{ are connected by exchangeable variables in } \Sigma\}.$$

Partitioning a seed into its exchangeably connected components can be useful when studying mutations of a seed; mutations within an exchangeably connected component leave all other exchangeably connected components unchanged.

Remark 2.3.27. We can decompose any seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ into its exchangeably connected components $\{\Sigma_j = (\mathrm{X}_j, \mathrm{ex}_j, B_j = (b^j_{vw})_{v,w \in \mathrm{X}_j})\}_{j \in I}$, where I is a countable index set and $\mathrm{ex} = \sqcup_{j \in I} \mathrm{ex}_j$. Mutation along a Σ_j-admissible sequence leaves all other exchangeably connected components unchanged (up to entries of the exchange matrix labelled by coefficients), i.e. if $(x_1, \ldots, x_l)$ is a Σ_i-admissible sequence and $y \in \mathrm{ex}_j$ with $i \neq j \in I$ we have

$$(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma))_y = (\mathrm{X}_j, \mathrm{ex}_j, \tilde{B}_j = (\tilde{b}^j_{vw})_{v,w \in \mathrm{X}_j}),$$

where $\tilde{b}^j_{vw} = b^j_{vw}$ for all $v, w \in \mathrm{ex}^j$. This follows directly from the fact that mutation at an exchangeable variable x of Σ only affects entries in the exchange matrix that are labelled by neighbours of x in Σ. No entries that are labelled by exchangeable variables of any other exchangeably connected component are affected.

In particular, the same holds true for connected components rather than just exchangeably connected components: Mutation in one connected component does not affect any other connected component, since two cluster variables in different connected components are necessarily in different exchangeably connected components.

Definition 2.3.28. We call two seeds $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w \in \mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{vw})_{v,w \in \mathrm{X}'})$ similar, if there exists a bijection $\varphi\colon \mathrm{X} \to \mathrm{X}'$ restricting to a bijection $\varphi\colon \mathrm{ex} \to \mathrm{ex}'$ such that for every exchangeable variable $x \in \mathrm{X}$ the exchangeably connected component Σ_x^{ex} of x in Σ is isomorphic (cf. Definition 2.2.1) to the exchangeably connected component $\Sigma^{ex}_{\varphi(x)}$ of $\varphi(x)$ in Σ' or to its opposite seed $(\Sigma^{ex}_{\varphi(x)})^{op}$.

Example 2.3.29. Consider the seeds

$$\Sigma_1 = \left(\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_2, x_4, x_5\}, \begin{tikzcd} & & & & \boxed{x_6} \arrow[dl] \\ x_1 \arrow[r] & x_2 \arrow[r] & \boxed{x_3} \arrow[r] & x_4 & x_5 \arrow[l] \end{tikzcd}\right),$$

$$\Sigma_2 = \left(\{y_1, y_2, y_3, y_4, y_5, y_6\}, \{y_1, y_2, y_4, y_5\}, \begin{tikzcd} & & & & \boxed{y_6} \\ y_1 \arrow[r] & y_2 \arrow[r] & \boxed{y_3} & y_4 \arrow[l] \arrow[ur] \arrow[r] & y_5 \end{tikzcd}\right)$$

and

$$\Sigma_3 = \left(\{z_1, z_2, z_3, z_4, z_5, z_6\}, \{z_1, z_2, z_4, z_5\}, \begin{tikzcd} & & & & \boxed{z_6} \arrow[dl] \\ z_1 \arrow[r] & z_2 & \boxed{z_3} \arrow[l] \arrow[r] & z_4 & z_5 \arrow[l] \end{tikzcd}\right).$$

Each of the three seeds consists of two distinct exchangeably connected components, which for Σ_1 are the full subseeds

$$(\Sigma_1)^{ex}_{x_1} = (\{x_1, x_2, x_3\}, \{x_1, x_2\}, \; x_1 \to x_2 \to \boxed{x_3})$$

and

$$(\Sigma_1)^{ex}_{x_4} = (\{x_3, x_4, x_5, x_6\}, \{x_4, x_5\}, \begin{tikzcd} & & \boxed{x_6} \arrow[dl] \\ \boxed{x_3} \arrow[r] & x_4 & x_5 \arrow[l] \end{tikzcd}).$$

Similarly, we can determine the two distinct exchangeably connected components of Σ_2 and Σ_3 and we see that Σ_1 and Σ_2 are similar $((\Sigma_1)^{ex}_{x_1} \cong (\Sigma_2)^{ex}_{y_1}$ and $(\Sigma_1)^{ex}_{x_4} \cong ((\Sigma_2)^{ex}_{y_4})^{op})$, but neither is similar to Σ_3: There is no exchangeably connected component of Σ_2 nor of Σ_3 that is isomorphic to $(\Sigma_3)^{ex}_{z_1}$ or $((\Sigma_3)^{ex}_{z_1})^{op}$.

Theorem 2.3.30. *The rooted cluster algebras $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ are isomorphic if and only if their initial seeds Σ and Σ' are similar.*

This statement can be derived from [ADS, Section 3]. However, we consider the case where a seed might consist of several exchangeably connected components, so for the convenience of the reader we give a short proof.

Proof. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in \mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{vw})_{v,w\in \mathrm{X}'})$ be similar via a bijection $\varphi\colon \mathrm{X} \to \mathrm{X}'$. It involves some calculations to check that φ induces a rooted cluster morphism; in the interest of not giving a rather technical argument twice, we refer to a result that will be proved in Section 2.3.6: In Theorem 2.3.37 we give three necessary and sufficient conditions for a map between clusters of seeds to give rise to a rooted cluster morphism. It is straightforward to check that φ satisfies all of these: Since it is a bijection restricting to a bijection from ex to ex$'$ it satisfies conditions (1) and (2). It satisfies condition (3) because for every two exchangeable cluster variables x and y in the same exchangeably connected component of Σ we have $b'_{\varphi(x)w'} = b_{x\varphi^{-1}(w')}$ and $b'_{\varphi(y)w'} = b_{y\varphi^{-1}(w')}$ or $b'_{\varphi(x)w'} = -b_{x\varphi^{-1}(w')}$ and $b'_{\varphi(y)w'} = -b_{y\varphi^{-1}(w')}$ for all $w' \in \mathrm{X}'$. Thus it induces a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$. For the same reasons, the inverse $\varphi^{-1}\colon \mathrm{X}' \to \mathrm{X}$ induces a rooted cluster morphism $g\colon \mathcal{A}(\Sigma') \to \mathcal{A}(\Sigma)$. It remains to check that f and g are mutual inverses as rooted cluster morphisms.

Let x be a cluster variable in $\mathcal{A}(\Sigma)$. It is of the form $x = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ for some $y \in \mathrm{X}$ and a Σ-admissible sequence $(x_1, \ldots, x_l)$. By induction on the length l of the admissible sequence, we show that $g \circ f(x) = x$ and thus $g \circ f$ is the identity on $\mathcal{A}(\Sigma)$: If $l = 0$ we have $g \circ f(x) = \varphi^{-1} \circ \varphi(x) = x$. If $g \circ f$ is the identity on all cluster variables which can be written as a mutation along a Σ-admissible sequence of length $l - 1$, then in particular $g \circ f(\mu_{x_{l-1}\circ\ldots\circ\mu_{x_1}}(y)) = \mu_{x_{l-1}} \circ \ldots \circ \mu_{x_1}(y)$ and $g \circ f(x_l) = x_l$ and $(x_1, \ldots, x_l)$

is $(g \circ f, \Sigma, \Sigma)$-biadmissible. Thus by axiom CM3 for $g \circ f$ we have

$$\begin{aligned} g \circ f(x) &= \mu_{g \circ f(x_l)} \circ \ldots \circ \mu_{g \circ f(x_1)}(g \circ f(y)) \\ &= \mu_{x_l} \circ \mu_{g \circ f(x_{l-1})} \circ \ldots \circ \mu_{g \circ f(x_1)}(g \circ f(y)) \\ &= \mu_{x_l}(g \circ f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y))) \\ &= \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y) = x. \end{aligned}$$

The argument for $f \circ g$ being the identity on $\mathcal{A}(\Sigma')$ is symmetric and thus f is a rooted cluster isomorphism.

On the other hand, if $f \colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is a rooted cluster isomorphism then by [ADS, Corollary 3.2] it induces a bijection $\varphi \colon \mathrm{X} \to \mathrm{X}'$. By condition (3) of Theorem 2.3.37 it follows that Σ and Σ' are similar under this bijection. $\square$

Two rooted cluster algebras with mutation equivalent initial seeds are in general not isomorphic in *Clus*.

Example 2.3.31. Consider the seed

$$\Sigma = (\{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}, \ x_1 \to x_2 \leftarrow x_3 \ )$$

and its mutation at x_1

$$\Sigma' := \mu_{x_1}(\Sigma) = (\{x_1' = \frac{x_2 + 1}{x_1}, x_2, x_3\}, \{x_1', x_2, x_3\}, \ x_1' \leftarrow x_2 \leftarrow x_3 \ ).$$

The seeds Σ and Σ' are not similar, thus the associated cluster algebras $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ are not isomorphic. Indeed, if $f \colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ were an isomorphism of rooted cluster algebras, then it would be an isomorphism of rings with $f(\{x_1, x_2, x_3\}) = \{x_1', x_2, x_3\}$. Thus we would have $f(x_i) = x_2$ for some $i \in \{1, 2, 3\}$. To avoid confusion, denote mutation of the seed Σ by μ^Σ and mutation of the seed Σ' by $\mu^{\Sigma'}$. We have

$$f(\mu^{\Sigma}_{x_i}(x_i)) = \begin{cases} \frac{f(x_2)+1}{f(x_1)} = \frac{f(x_2)+1}{x_2}, \text{ if } i = 1 \\ \frac{f(x_1)f(x_3)+1}{f(x_2)} = \frac{f(x_1)f(x_3)+1}{x_2}, \text{ if } i = 2 \\ \frac{f(x_2)+1}{f(x_3)} = \frac{f(x_2)+1}{x_2}, \text{ if } i = 3 \end{cases}$$

none of which can be equal to

$$\mu^{\Sigma'}_{x_2}(x_2) = \frac{x_1' + x_3}{x_2}.$$

So f does not satisfy axiom CM3.

2.3.6 Rooted cluster morphisms without specializations

The definition of rooted cluster morphisms (see Definition 2.3.2) allows cluster variables to be sent to integers. Sending a cluster variable to an integer is called a *specialization*. Given the seeds $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ we call a rooted cluster morphism $f \colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ a *rooted cluster morphism without specializations*, if $f(\mathrm{X}) \subseteq \mathrm{X}'$, i.e. if all cluster variables get sent to cluster variables.

Lemma 2.3.32. *Let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism without specializations. Then every Σ-admissible sequence is (f, Σ, Σ')-biadmissible.*

Proof. We prove the claim by induction on the length l of a Σ-admissible sequence. It is satisfied trivially for sequences of length $l = 0$. Assume now that it is satisfied for all Σ-admissible sequences of lengths at most $l \geq 0$ and let $(x_1, \ldots, x_{l+1})$ be a Σ-admissible sequence of length $l + 1$. By Definition 2.2.15 we have $x_{l+1} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ for some $y \in \mathrm{ex}$ and thus – because, by induction hypothesis, $(x_1, \ldots, x_l)$ is (f, Σ, Σ')-biadmissible – $f(x_{l+1}) = f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y))$ by axiom CM3. By axiom CM2 $f(y) \in \mathrm{ex}'$ so we have $f(x_{l+1}) \in \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(ex')$ and thus $(x_1, \ldots, x_{l+1})$ is (f, Σ, Σ')-biadmissible. $\square$

Lemma 2.3.32 helps us to further understand ideal rooted cluster morphisms (see Definition 2.3.15): In [ADS, Problem 2.12], Assem, Dupont and Schiffler asked for a classification of ideal rooted cluster morphisms. We provide a partial answer via the following consequence.

Proposition 2.3.33. *Every rooted cluster morphism without specializations is ideal.*

Proof. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism without specializations. Every element of $f(\mathcal{A}(\Sigma))$ can be written as an integer polynomial in the images of cluster variables of $\mathcal{A}(\Sigma)$. A cluster variable $x \in \mathcal{A}(\Sigma)$ is of the form $x = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ for $y \in \mathrm{X}$ and a Σ-admissible sequence $(x_1, \ldots, x_l)$. Because f is without specializations we have $f(y) \in \mathrm{X}'$ and by Lemma 2.3.32 $(f(x_1), \ldots, f(x_l))$ is Σ'-admissible. By axiom CM3 we obtain $f(x) = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y))$. This is an element of $\mathcal{A}(f(\Sigma))$ and thus $f(\mathcal{A}(\Sigma)) \subseteq \mathcal{A}(f(\Sigma))$. The other inclusion always holds and was proved in [ADS, Lemma 2.1]. $\square$

Generally, if we have a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ the combinatorial structures of the two seeds Σ and Σ' are linked via those exchangeable cluster variables in the cluster of Σ that do not get sent to integers. This provides a particularly strong combinatorial link between two rooted cluster algebras $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ for which there exists a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ without specializations.

Lemma 2.3.34. *Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w \in \mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{v'w'})_{v',w' \in \mathrm{X}'})$ be seeds and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. Let $x \in \mathrm{ex}$ be an exchangeable variable of Σ with $f(x) \in \mathrm{ex}'$. Consider the exchangeably connected component*

$$f(\Sigma)^{ex}_{f(x)} = (f(\mathrm{X})^{ex}_{f(x)} \cap \mathrm{X}', f(\mathrm{ex})^{ex}_{f(x)} \cap \mathrm{ex}', (b'_{vw})_{v,w \in f(\mathrm{X})^{ex}_{f(x)} \cap \mathrm{X}'})$$

of $f(x)$ in the full subseed $f(\Sigma) \subseteq \Sigma'$. Then we have

$$b'_{f(y)v'} = \sum_{v \in \mathrm{X}\colon f(v) = v'} b_{yv} \text{ for all } f(y) \in f(\mathrm{ex})^{ex}_{f(x)} \cap \mathrm{ex}' \text{ and all } v' \in \mathrm{X}' \text{ or}$$

$$b'_{f(y)v'} = -\sum_{v \in \mathrm{X}\colon f(v) = v'} b_{yv} \text{ for all } f(y) \in f(\mathrm{ex})^{ex}_{f(x)} \cap \mathrm{ex}' \text{ and all } v' \in \mathrm{X}',$$

where the empty sum is taken to be 0*. In particular, if* $x, y \in \mathrm{ex}$ *with* $f(x), f(y) \in \mathrm{ex}'$*, then we have* $b'_{f(x)f(y)} = \pm b_{xy}$*.*

Before we give the proof for Lemma 2.3.34, we want to explore its meaning in more detail by visualizing the statement for skew-symmetric rooted cluster algebras. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, Q)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', Q')$ be skew-symmetric seeds with their combinatorial data encoded in the quivers Q, respectively Q', and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. Then Lemma 2.3.34 implies that for every $x \in \mathrm{ex}$ with $f(x) \in \mathrm{ex}'$ one of the following two holds:

- The arrows incident with x "are invariant under f", i.e. the number of arrows starting (respectively ending) in x as a vertex of Q is equal to the number of arrows starting (respectively ending) in $f(x)$ as a vertex of Q'.
- The arrows incident with x "change direction under f", i.e. the number of arrows starting (respectively ending) in x as a vertex of Q is equal to the number of arrows ending (respectively starting) in $f(x)$ as a vertex of Q'.

Furthermore, for every $y \in \mathrm{ex}$ whose image $f(y)$ lies in the same exchangeably connected component of the image seed $f(\Sigma)$ as $f(x)$, the arrows incident with y are invariant under f if and only if the arrows incident with x are, and equivalently, the arrows incident with y change direction under f if and only if the arrows incident with x do. We now prove Lemma 2.3.34.

Proof. Let $\tilde{\mathrm{X}} = \{x \in \mathrm{X} \mid f(x) \in \mathrm{X}'\}$ be the set of cluster variables in X that get mapped to cluster variables in X′ and let $x \in \tilde{\mathrm{X}} \cap \mathrm{ex}$ with $f(x) = x'$. Because f is a ring homomorphism we have

$$\begin{aligned} f(\mu_x(x)x) =& f\left(\prod_{v \in \mathrm{X}\colon b_{xv}>0} v^{b_{xv}} + \prod_{v \in \mathrm{X}\colon b_{xv}<0} v^{-b_{xv}}\right) \\ =& k_1 \prod_{v \in \tilde{\mathrm{X}}\colon b_{xv}>0} f(v)^{b_{xv}} + k_2 \prod_{v \in \tilde{\mathrm{X}}\colon b_{xv}<0} f(v)^{-b_{xv}} \end{aligned}$$

for some $k_1, k_2 \in \mathbb{Z}$. By axiom CM3 this has to be equal to

$$\mu_{f(x)}(f(x))f(x) = \mu_{x'}(x')x' = \prod_{v' \in \mathrm{X}'\colon b'_{x'v'}>0} (v')^{b'_{x'v'}} + \prod_{v' \in \mathrm{X}'\colon b'_{x'v'}<0} (v')^{-b'_{x'v'}}.$$

We set

$$\begin{aligned} M_1 &= \prod_{v \in \tilde{\mathrm{X}}\colon b_{xv}>0} f(v)^{b_{xv}} & M_2 &= \prod_{v \in \tilde{\mathrm{X}}\colon b_{xv}<0} f(v)^{-b_{xv}} \\ M_1' &= \prod_{v' \in \mathrm{X}'\colon b'_{x'v'}>0} (v')^{b'_{x'v'}} & M_2' &= \prod_{v' \in \mathrm{X}'\colon b'_{x'v'}<0} (v')^{-b'_{x'v'}}, \end{aligned}$$

and thus have $k_1M_1 + k_2M_2 = M_1' + M_2'$, where M_1, M_2, M_1' and M_2' are non-zero monic monomials in X′ over $\mathbb{Z}$. Assume first that $b_{x'v'} = 0$ for all $v' \in \mathrm{X}'$. Then we have

$M_1' = M_2' = 1$, which implies $k_1 M_1 + k_2 M_2 = 2$ and by algebraic independence of variables in X' we have $M_1 = M_2 = 1$. Therefore we have $b_{xv} = 0$ for all $v \in \tilde{\mathrm{X}}$ and in particular for all $v' \in \mathrm{X}'$ we have $0 = b'_{x'v'} = \sum_{f(v)=v'} b_{xv} = 0$. Assume now that there exists a $z' \in \mathrm{X}'$ with $b'_{x'z'} \neq 0$. Without loss of generality we may assume $b'_{x'z'} > 0$. Then z' divides M_1'. Since z' does not divide M_2', and $M_1', M_2' \neq 0$, by algebraic independence of variables in X' either z divides M_1 or z divides M_2. If z divides M_1, by comparing coefficients of z' we obtain $k_1 M_1 = M_1'$ and $k_2 M_2 = M_2'$ and if z divides M_2 we obtain $k_1 M_1 = M_2'$ and $k_2 M_2 = M_1'$. Either way, since M_1' and M_2' are monic, we get

$$k_1 = k_2 = 1 \tag{2.2}$$

and the first case implies

$$b'_{f(x)v'} = b'_{x'v'} = \sum_{v \in \mathrm{X}\colon f(v)=v'} b_{xv},$$

for all $v' \in \mathrm{X}'$ and the second case implies

$$b'_{f(x)v'} = b'_{x'v'} = -\sum_{v \in \mathrm{X}\colon f(v)=v'} b_{xv}$$

for all $v' \in \mathrm{X}'$. In particular, if $x, y \in \mathrm{ex} \cap \tilde{\mathrm{X}}$ then by Proposition 2.3.9 we have $b'_{f(x)f(y)} = \pm b_{xy}$.

Let now $x, y \in \tilde{\mathrm{X}} \cap \mathrm{ex}$ be cluster variables such that their images $f(x)$ and $f(y)$ are cluster variables in the same exchangeably connected component of $f(\Sigma)$ and let them be connected by the path $f(x) = f(x_0), f(x_1), \ldots, f(x_l) = f(y)$ with $x_1, \ldots, x_{l-1} \in \mathrm{ex}$. Assume that we have

$$b'_{f(x)v'} = \sum_{v \in \mathrm{X}\colon f(v)=v'} b_{xv}$$

for all $v' \in \mathrm{X}'$, i.e. no signs occur. By the above argument, this is the case if and only if $b'_{f(x)f(x_1)} = b_{xx_1}$. Iteratively applying the same argument to x_i for $i = 0, \ldots, l-1$, yields that this holds if and only if $b'_{f(x_i)f(x_{i+1})} = b_{x_i x_{i+1}}$ for all $0 \leq i < l$; in particular if and only if $b'_{f(x_{l-1})f(y)} = b_{x_{l-1}y}$, which holds if and only if

$$b'_{f(y)v'} = \sum_{v \in \mathrm{X}\colon f(v)=v'} b_{yv}$$

for all $v' \in \mathrm{X}'$. This proves the claim.

$\square$

Remark 2.3.35. The proof of Lemma 2.3.34 also tells us something about specializations of cluster variables. If $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B)$ are seeds and $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is a rooted cluster morphism, then for an $x \in \mathrm{ex}$ with $f(x) \in \mathrm{ex}'$ the following holds: If $f(x)$ has at least one neighbour in $f(\Sigma)$, i.e. if it is not its own connected component in $f(\Sigma)$, then all neighbours of x in Σ that do not get mapped to cluster variables get mapped into $\{\pm 1\}$. This follows directly from Equation (2.2) in the proof.

In the following, we want to characterize rooted cluster morphisms without specializations. Before we do that, we observe the following useful fact.

Remark 2.3.36. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ be seeds. Any map $f\colon \mathrm{X} \to \mathrm{X}'$ gives rise to a unique ring homomorphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{F}_{\Sigma'}$, because all elements of the ring $\mathcal{A}(\Sigma)$ are Laurent polynomials in X. Thus in particular every rooted cluster morphism without specializations is uniquely determined by its values on the cluster of the initial seed.

Theorem 2.3.37. *Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in\mathrm{X}})$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B' = (b'_{vw})_{v,w\in\mathrm{X}'})$ be seeds and let $f\colon \mathrm{X} \to \mathrm{X}'$ be a map. Then the algebraic extension of f to $\mathcal{A}(\Sigma)$ gives rise to a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ if and only if the following hold:*

(1) *The map f restricts to an injection $f\,|_{\mathrm{ex}}\colon \mathrm{ex} \to \mathrm{ex}'$.*

(2) *If $f(x) = f(y)$ for some $x \neq y \in \mathrm{X}$ then both x and y are coefficients of Σ. In that case for any Σ-admissible sequence $(x_1, \ldots, x_l)$, setting $\mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma) =: (\tilde{\mathrm{X}}, \tilde{\mathrm{ex}}, \tilde{B} = (\tilde{b}_{vw})_{v,w\in\tilde{\mathrm{X}}})$, for any neighbour $z \in \tilde{\mathrm{ex}}$ of both x and y in $\tilde{\Sigma}$ the entries $\tilde{b}_{zx}$ and $\tilde{b}_{zy}$ have the same sign.*

(3) *Let $x \in \mathrm{ex}$ and consider the exchangeably connected component*

$$f(\Sigma)^{ex}_{f(x)} = (f(\mathrm{X})^{ex}_{f(x)}, f(\mathrm{ex})^{ex}_{f(x)}, (b'_{vw})_{v,w\in f(\mathrm{X})^{ex}_{f(x)}})$$

of $f(x)$ in the full subseed $f(\Sigma) \subseteq \Sigma'$. Then we have

$$b'_{f(y)v'} = \sum_{v\in\mathrm{X}\colon f(v)=v'} b_{yv} \text{ for all } f(y) \in f(\mathrm{ex})^{ex}_{f(x)} \text{ and all } v' \in \mathrm{X}' \text{ or}$$
$$b'_{f(y)v'} = -\sum_{v\in\mathrm{X}\colon f(v)=v'} b_{yv} \text{ for all } f(y) \in f(\mathrm{ex})^{ex}_{f(x)} \text{ and all } v' \in \mathrm{X}',$$

where the empty sum is taken to be 0*.*

Remark 2.3.38. Condition (2) of Theorem 2.3.37 is not always easy to check for two given seeds $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ and a map $f\colon \mathrm{X} \to \mathrm{X}'$. However, it is useful for checking when such a map does not induce a rooted cluster morphism. On the other hand, if for all $x, y \in \mathrm{X} \setminus \mathrm{ex}$ with $f(x) = f(y)$ we have $b_{xv} = b_{yv}$ for all $v \in \mathrm{ex}$ then it is straightforward to check using the definition of matrix mutation in Definition 2.2.13 that condition (2) is satisfied.

Proof. Assume first that the map f extends to a rooted cluster morphism. By axiom CM2 and Proposition 2.3.9 point (1) holds. By Lemma 2.3.32 every Σ-admissible sequence is (f, Σ, Σ')-biadmissible and thus by Corollary 2.3.10 point (2) holds. By Lemma 2.3.34 point (3) is satisfied.

Assume, on the other hand, that $f\colon \mathrm{X} \to \mathrm{X}'$ is a map satisfying conditions (1) to (3). By Remark 2.3.36, it gives rise to a unique ring homomorphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{F}_{\Sigma'}$. This ring homomorphism satisfies axioms CM1 and CM2 by definition and condition (1). It remains to check axiom CM3 and that the image $f(\mathcal{A}(\Sigma))$ is contained in $\mathcal{A}(\Sigma')$.

We show the following points for every Σ-admissible sequence $(x_1, \ldots, x_l)$ by induction on the length l.

(a) The sequence $(x_1, \ldots, x_l)$ is (f, Σ, Σ')-biadmissible.

(b) For any $y \in \mathrm{X}$ we have

$$f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y)).$$

(c) Set

$$\mu_{x_l} \circ \ldots \mu_{x_1}(\Sigma) =: \tilde{\Sigma} = (\tilde{\mathrm{X}}, \tilde{\mathrm{ex}}, \tilde{B} = (\tilde{b}_{vw})_{v,w \in \tilde{\mathrm{X}}})$$

to be the mutation of the seed Σ along $(x_1, \ldots, x_l)$ and

$$\mu_{f(x_l)} \circ \ldots \mu_{f(x_1)}(\Sigma') =: \tilde{\Sigma}' = (\tilde{\mathrm{X}}', \tilde{\mathrm{ex}}', \tilde{B}' = (\tilde{b}'_{vw})_{v,w \in \tilde{\mathrm{X}}'})$$

to be the mutation of Σ' along $(f(x_1), \ldots, f(x_l))$. If $f(x) = f(y)$ for some $x \neq y \in \tilde{X}$, then both x and y are coefficients of $\tilde{\Sigma}$. (This is equivalent to saying that for any $x \in \tilde{\mathrm{ex}}$ and any $y \in \tilde{\mathrm{X}}$ with $x \neq y$ we have $f(x) \neq f(y)$.)

(d) For every $v \in \tilde{\mathrm{ex}}$ we have

$$\tilde{b}'_{f(y)v'} = \sum_{v \in \tilde{\mathrm{X}}\colon f(v) = v'} \tilde{b}_{yv} \qquad \text{for all } v' \in \tilde{\mathrm{X}}' \text{ or}$$
$$\tilde{b}'_{f(y)v'} = -\sum_{v \in \tilde{\mathrm{X}}\colon f(v) = v'} \tilde{b}_{yv} \qquad \text{for all } v' \in \tilde{\mathrm{X}}',$$

and for all $y \in \tilde{\mathrm{ex}}$ such that $f(x)$ and $f(y)$ lie in the same exchangeably connected component of $f(\tilde{\Sigma})$ we have

$$\tilde{b}'_{f(x)v'} = \sum_{v \in \tilde{\mathrm{X}}\colon f(v) = v'} \tilde{b}_{xv}$$

for all $v' \in \tilde{\mathrm{X}}'$ if and only if

$$\tilde{b}'_{f(y)v'} = \sum_{v \in \tilde{\mathrm{X}}\colon f(v) = v'} \tilde{b}_{yv}$$

for all $v' \in \tilde{\mathrm{X}}'$.

If these conditions are satisfied for every Σ-admissible sequence $(x_1, \ldots, x_l)$, then by condition (b) axiom CM3 is satisfied and by conditions (a) and (b) the image of $\mathcal{A}(\Sigma)$ under the algebraic extension of f lies in $\mathcal{A}(\Sigma')$. Conditions (c) and (d) are used to help prove conditions (a) and (b).

We check conditions (a) to (d) for arbitrary Σ-admissible sequences by induction on their length l. For a Σ-admissible sequence of length $l = 0$ conditions (a) and (b) are satisfied trivially, condition (c) is satisfied by condition (2) and condition (d) is satisfied by condition (3). Assume now that they are satisfied for all Σ-admissible sequences of length $\leq l$ and let $(x_1, \ldots, x_{l+1})$ be a Σ-admissible sequence of length $l+1$. We set

$$\mu_{x_l} \circ \ldots \mu_{x_1}(\Sigma) =: \tilde{\Sigma} = (\tilde{\mathrm{X}}, \tilde{\mathrm{ex}}, \tilde{B} = (\tilde{b}_{vw})_{v,w \in \tilde{\mathrm{X}}})$$

and

$$\mu_{f(x_l)} \circ \ldots \mu_{f(x_1)}(\Sigma') =: \tilde{\Sigma}' = (\tilde{\mathrm{X}}', \tilde{\mathrm{ex}}', \tilde{B}' = (\tilde{b}'_{vw})_{v,w \in \tilde{\mathrm{X}}'})$$

as above.

(a) We have $x_{l+1} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ for some $y \in \mathrm{ex}$ and thus $f(x_{l+1}) = f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y))$ by induction assumption (b) on the sequence $(x_1, \ldots, x_l)$. By condition (1) we have $f(y) \in \mathrm{ex}'$ and thus $f(x_{l+1}) \in \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(\mathrm{ex}')$ and $(x_1, \ldots, x_{l+1})$ is (f, Σ, Σ')-biadmissible.

(b) Let $y \in \mathrm{X}$. We have $\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y) \in \tilde{X}$ and $x_{l+1} \in \tilde{\mathrm{ex}}$. If we have $x_{l+1} \neq \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ this implies

$$f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) \neq f(x_{l+1})$$

by induction assumption (c). In this case mutation at x_{l+1}, respectively at $f(x_{l+1})$ acts trivially on $\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$, respectively on $\mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y))$ and we obtain

$$\begin{aligned} f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) &= f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) \\ &= \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y)) \\ &= \mu_{f(x_{l+1})} \circ \ldots \circ \mu_{f(x_1)}(f(y)), \end{aligned}$$

where the second equality follows from induction assumption (b) on the sequence $(x_1, \ldots, x_l)$. If, on the other hand, $x_{l+1} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)$ then we have

$$f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) = \frac{\prod\limits_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} > 0} f(v)^{\tilde{b}_{x_{l+1}v}} + \prod\limits_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} < 0} f(v)^{-\tilde{b}_{x_{l+1}v}}}{f(x_{l+1})}. \tag{2.3}$$

By induction assumption (d) we have

$$\begin{aligned} \tilde{b}'_{f(x_{l+1})v'} &= \sum_{v \in \tilde{\mathrm{X}}:\ f(v) = v'} \tilde{b}_{x_{l+1}v} \text{ for all } v' \in \tilde{\mathrm{X}}' \text{ or} \\ \tilde{b}'_{f(x_{l+1})v'} &= -\sum_{v \in \tilde{\mathrm{X}}:\ f(v) = v'} \tilde{b}_{x_{l+1}v} \text{ for all } v' \in \tilde{\mathrm{X}}'. \end{aligned}$$

Without loss of generality, assume that the first equation holds (otherwise we can simply change the signs below accordingly). By condition (2), for any $v' \in \tilde{\mathrm{X}}'$ all non-trivial summands in

$$\sum_{v \in \tilde{\mathrm{X}}:\ f(v)=v'} \tilde{b}_{x_{l+1}v}$$

have the same sign. Therefore for any $v' \in \tilde{\mathrm{X}}'$, we have $\tilde{b}_{f(x_{l+1})v'} \geq 0$ if and only if $\tilde{b}_{x_{l+1}v} \geq 0$ for all $v \in \mathrm{X}$ with $f(v) = v'$, and $\tilde{b}_{f(x_{l+1})v'} = 0$ if and only if $\tilde{b}_{x_{l+1}v} = 0$ for all $v \in \tilde{\mathrm{X}}$ with $f(v) = v'$. We get

$$\begin{aligned}
\prod_{v' \in \tilde{\mathrm{X}}': \tilde{b}'_{f(x_{l+1})v'} > 0} (v')^{\tilde{b}'_{f(x_{l+1})v'}} &= \prod_{v' \in \tilde{\mathrm{X}}': \tilde{b}'_{f(x_{l+1})v'} > 0} (v')^{\left(\sum_{v \in \tilde{\mathrm{X}}:\ f(v)=v'} \tilde{b}_{x_{l+1}v}\right)} \\
&= \prod_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} > 0} f(v)^{\tilde{b}_{x_{l+1}v}}
\end{aligned}$$

and the analogous statement for the product over $v' \in \tilde{\mathrm{X}}'$ with $\tilde{b}'_{f(x_{l+1})v'} < 0$. Substituting into Equation (2.3) we obtain

$$f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) = \frac{\prod_{v' \in \tilde{\mathrm{X}}': \tilde{b}'_{f(x_{l+1})v'} > 0} (v')^{\tilde{b}'_{f(x_{l+1})v'}} + \prod_{v' \in \tilde{\mathrm{X}}': \tilde{b}'_{f(x_{l+1})v'} < 0} (v')^{-\tilde{b}'_{f(x_{l+1})v'}}}{f(x_{l+1})},$$

which by definition of mutation is equal to

$$\mu_{f(x_{l+1})}(f(x_{l+1})) = \mu_{f(x_{l+1})}(f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y))).$$

By induction assumption (b) we obtain

$$f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_{l+1})} \circ \ldots \circ \mu_{f(x_1)}(f(y)).$$

(c) Let now $x \in \mu_{x_{l+1}}(\tilde{\mathrm{ex}})$ and $y \in \mu_{x_{l+1}}(\tilde{\mathrm{X}})$ with $x \neq y$. We want to show that $f(x) \neq f(y)$. We have $x = \mu_{x_{l+1}}(\tilde{x})$ and $y = \mu_{x_{l+1}}(\tilde{y})$ for some $\tilde{x} \in \tilde{\mathrm{ex}}$ and $\tilde{y} \in \tilde{\mathrm{X}}$ with $\tilde{x} \neq \tilde{y}$. If both $\tilde{x} \neq x_{l+1}$ and $\tilde{y} \neq x_{l+1}$, then $x = \tilde{x} \in \tilde{\mathrm{ex}}$ and $y = \tilde{y} \in \tilde{\mathrm{X}}$ and by induction assumption (c) we have $f(x) \neq f(y)$. Thus assume without loss of generality that $\tilde{x} = x_{l+1}$ and $\tilde{y} \neq x_{l+1}$. Then we have

$$\begin{aligned}
f(x)f(x_{l+1}) &= f(\mu_{x_{l+1}}(x_{l+1}))f(x_{l+1}) \\
&= \prod_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} > 0} f(v)^{\tilde{b}_{x_{l+1}v}} + \prod_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} < 0} f(v)^{-\tilde{b}_{x_{l+1}v}}
\end{aligned}$$

and thus $f(x)$ divides the right hand side of the equation. On the other hand, we have $f(y) = f(\mu_{x_{l+1}}(\tilde{y})) = f(\tilde{y}) \in \tilde{\mathrm{X}}'$. Assume for a contradiction that $f(x) = f(y)$.

In particular, this implies $f(x) \in \tilde{\mathrm{X}}'$. By algebraic independence of the elements of $\tilde{\mathrm{X}}'$, $f(x)$ must divide both

$$\prod_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} > 0} f(v)^{\tilde{b}_{x_{l+1}v}}$$

and

$$\prod_{v \in \tilde{\mathrm{X}}: \tilde{b}_{x_{l+1}v} < 0} f(v)^{-\tilde{b}_{x_{l+1}v}}.$$

This would mean that there exist $v \neq w \in \tilde{X}$ with $f(v) = f(w)$ and such that $\tilde{b}_{x_{l+1}v} > 0$ and $\tilde{b}_{x_{l+1}w} < 0$, which contradicts condition (2). Thus we have $f(x) \neq f(y)$.

(d) Set now

$$\mu_{x_{l+1}}(\tilde{B}) =: \mathcal{B} = (\beta_{vw})_{v,w \in \mu_{x_{l+1}}(\tilde{\mathrm{X}})}$$

and

$$\mu_{f(x_{l+1})}(\tilde{B}') =: \mathcal{B}' = (\beta'_{vw})_{v,w \in \mu_{f(x_{l+1})}(\tilde{\mathrm{X}}')}.$$

Fix $v = \mu_{x_{l+1}}(\tilde{v}) \in \mu_{x_{l+1}}(\tilde{\mathrm{ex}})$. By definition of matrix mutation (Definition 2.2.13), for all $w = \mu_{x_{l+1}}(\tilde{w}) \in \mu_{x_{l+1}}(\tilde{\mathrm{X}})$ we have

$$\beta_{vw} = \mu_{x_{l+1}}(\tilde{b}_{\tilde{v}\tilde{w}}) = \begin{cases} -\tilde{b}_{\tilde{v}\tilde{w}}, & \text{if } \tilde{v} = x_{l+1} \text{ or } \tilde{w} = x_{l+1} \\ \tilde{b}_{\tilde{v}\tilde{w}} + \frac{1}{2}(|\tilde{b}_{\tilde{v}x_{l+1}}|\tilde{b}_{x_{l+1}\tilde{w}} + \tilde{b}_{\tilde{v}x_{l+1}}|\tilde{b}_{x_{l+1}\tilde{w}}|), & \text{else.} \end{cases}$$

We have shown that condition (b) holds for the sequence $(x_1, \ldots, x_{l+1})$ and thus we have $f(v) = \mu_{f(x_{l+1})}(f(\tilde{v}))$. Thus for every $w' = \mu_{f(x_{l+1})}(\tilde{w}') \in \mu_{f(x_{l+1})}(\tilde{\mathrm{X}}')$ we have

$$\beta'_{f(v)w'} = \mu_{f(x_{l+1})}(\tilde{b}'_{f(\tilde{v})\tilde{w}'}) = \begin{cases} -\tilde{b}'_{f(\tilde{v})\tilde{w}'}, & \text{if } f(\tilde{v}) = f(x_{l+1}) \text{ or } \tilde{w}' = f(x_{l+1}) \\ \tilde{b}'_{f(\tilde{v})\tilde{w}'} + \frac{1}{2}(|\tilde{b}'_{f(\tilde{v})f(x_{l+1})}|\tilde{b}'_{f(x_{l+1})\tilde{w}'} + \tilde{b}'_{f(\tilde{v})f(x_{l+1})}|\tilde{b}'_{f(x_{l+1})\tilde{w}'}|), \\ \text{else.} \end{cases}$$

By induction assumption (d) we have

$$\tilde{b}'_{f(\tilde{v})\tilde{u}'} = \pm \sum_{\tilde{u} \in \tilde{\mathrm{X}}:\ f(\tilde{u}) = \tilde{u}'} \tilde{b}_{\tilde{v}\tilde{u}} \text{ and } \tilde{b}'_{f(x_{l+1})\tilde{u}'} = \pm \sum_{\tilde{u} \in \tilde{\mathrm{X}}:\ f(\tilde{u}) = \tilde{u}'} \tilde{b}_{x_{l+1}\tilde{u}}$$

for all $\tilde{u}' \in \tilde{\mathrm{X}}'$ and the signs of the two sums are the same if $f(\tilde{v})$ and $f(x_{l+1})$ are connected by a path of variables in $f(\tilde{\mathrm{ex}})$, hence in particular if $\tilde{b}'_{f(v)f(x_{l+1})} \neq 0$. Note further that since $\tilde{v} \in \tilde{\mathrm{ex}}$ we have $\tilde{b}'_{f(\tilde{v})f(x_{l+1})} = \pm\tilde{b}_{\tilde{v}x_{l+1}}$ by induction assumption (d) and $f(\tilde{v}) = f(x_{l+1})$ if and only if $\tilde{v} = x_{l+1}$ by assumption (c). Setting

$$S_{f(x_{l+1})} := \sum_{\tilde{w} \in \tilde{\mathrm{X}}:\ f(\tilde{w}) = \tilde{w}'} \tilde{b}_{x_{l+1}\tilde{w}}$$

we obtain

$$\beta'_{f(v)w'} = \begin{cases} -(\pm \sum\limits_{\tilde{w}\in\tilde{\mathrm{X}}\colon f(\tilde{w})=\tilde{w}'} \tilde{b}_{\tilde{v}\tilde{w}}), \text{ if } \tilde{v} = x_{l+1} \text{ or } \tilde{w}' = f(x_{l+1}) \\ \pm \sum\limits_{\tilde{w}\in\tilde{\mathrm{X}}\colon f(\tilde{w})=\tilde{w}'} \tilde{b}_{\tilde{v}\tilde{w}} + \frac{1}{2}\Big(|\tilde{b}_{\tilde{v}x_{l+1}}|(\pm S_{f(x_{l+1})}) \pm \tilde{b}_{\tilde{v}x_{l+1}}|S_{f(x_{l+1})}|\Big), \text{ else.} \end{cases}$$

Pulling out the sum yields

$$\begin{aligned} \beta'_{f(v)w'} &= \begin{cases} \pm \sum\limits_{\tilde{w}\in\tilde{\mathrm{X}}\colon f(\tilde{w})=\tilde{w}'} (-\tilde{b}_{\tilde{v}\tilde{w}}), \text{ if } \tilde{v} = x_{l+1} \\ \pm(-\tilde{b}_{\tilde{v}x_{l+1}}), \text{ if } \tilde{w}' = f(x_{l+1}) \\ \pm \sum\limits_{\tilde{w}\in\tilde{\mathrm{X}}\colon f(\tilde{w})=\tilde{w}'} \Big(\tilde{b}_{\tilde{v}\tilde{w}} + \frac{1}{2}\Big(|\tilde{b}_{\tilde{v}x_{l+1}}|\tilde{b}_{x_{l+1}\tilde{w}} + \tilde{b}_{\tilde{v}x_{l+1}}|\tilde{b}_{x_{l+1}\tilde{w}}|\Big)\Big), \text{ else.} \end{cases} \\ &= \pm \sum_{\tilde{w}\in\tilde{\mathrm{X}}\colon f(\tilde{w})=\tilde{w}'} \mu_{x_{l+1}}(\tilde{b}_{\tilde{v}\tilde{w}}) \\ &= \pm \sum_{w\in\mu_{x_{l+1}}(\tilde{\mathrm{X}})\colon f(w)=w'} \beta_{vw}, \end{aligned}$$

where the last equality holds because for $w = \mu_{x_{l+1}}(\tilde{w})$ by condition (b) we have

$$f(w) = f(\mu_{x_{l+1}}(\tilde{w})) = \mu_{f(x_{l+1})}(f(\tilde{w})).$$

Thus for every $w' = \mu_{f(x_{l+1})}(\tilde{w}')$ we have $f(w) = w' = \mu_{f(x_{l+1})}(\tilde{w}')$ if and only if $f(\tilde{w}) = \tilde{w}'$.

Observe that by definition of matrix mutation, if for $x, y \in \mu_{x_{l+1}}(\mathrm{X})$ with $x = \mu_{x_{l+1}}(\tilde{x})$ and $y = \mu_{x_{l+1}}(\tilde{y})$ we have $\beta'_{f(x)f(y)} \neq 0$, then we have $\tilde{b}'_{f(\tilde{x})f(\tilde{y})} \neq 0$ or both $\tilde{b}'_{f(\tilde{x})f(x_{l+1})} \neq 0$ and $\tilde{b}'_{f(\tilde{y})f(x_{l+1})} \neq 0$. Therefore, if two variables $f(x) = \mu_{f(x_{l+1})}(f(\tilde{x})) \in f(\mu_{x_{l+1}}(\mathrm{X}))$ and $f(y) = \mu_{f(x_{l+1})}(f(\tilde{y})) \in f(\mu_{x_{l+1}}(\mathrm{X}))$ are exchangeably connected in $f(\mu_{x_{l+1}}(\tilde{\Sigma}))$, then $f(\tilde{x})$ and $f(\tilde{y})$ are exchangeably connected in $f(\tilde{\Sigma})$. Thus the signs of the sums in a given exchangeably connected component of $f(\Sigma)$ carry over from $\tilde{B}'$ to $\mathcal{B}'$ and we obtain by induction assumption (d) that

$$\beta'_{f(x)u'} = \sum_{u\in\mu_{x_{l+1}}(\tilde{\mathrm{X}})\colon f(u)=u'} \beta_{xw}$$

for all $u' \in \mu_{f(x_{l+1})}(\tilde{\mathrm{X}}')$ if and only if

$$\beta'_{f(y)u'} = \sum_{u\in\mu_{x_{l+1}}\colon f(u)=u'(\tilde{\mathrm{X}})} \beta_{yw}$$

for all $u' \in \mu_{f(x_{l+1})}(\tilde{\mathrm{X}}')$.

□

Remark 2.3.39. Theorem 2.3.37 implies that, for a rooted cluster morphism $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ without specializations, the full subseed $(\mathrm{ex}, \mathrm{ex}, (b_{vw})_{v,w\in\mathrm{ex}})$ of exchangeable variables of $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in\mathrm{X}})$ is similar to a full subseed of Σ'.

Theorem 2.3.37 shows that rooted cluster morphisms without specializations are quite restrictive. It is helpful to visualize this via skew-symmetric rooted cluster algebras where the exchange matrices can be encoded in quivers. Let $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ be skew-symmetric rooted cluster algebras with initial seeds $\Sigma = (\mathrm{X}, \mathrm{ex}, Q)$ and $\Sigma' = (\mathrm{X}', \mathrm{ex}', Q')$, where the exchange matrices are represented via the quivers Q and Q' respectively, and let $f\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism without specializations. Two coefficients $x \neq y \in \mathrm{X} \setminus \mathrm{ex}$ may get sent to the same cluster variable if and only if there is no path in Q of length two between x and y that passes through an exchangeable variable. In any case, the number of arrows between $f(x) \in f(\mathrm{ex})$ and $f(y) \in f(\mathrm{X})$ in Q' is equal to the sum of the number of arrows between x and the preimages of y (where we do not worry about the directions of the arrows). The following example highlights most interesting features of rooted cluster morphisms without specializations: We may glue vertices to images of coefficients and we may glue together coefficients of Σ while keeping track of all the arrows that directly connect them to exchangeable variables. Furthermore, we can always add or remove arrows between images of coefficients of Σ and we can turn any coefficient into an exchangeable variable.

Example 2.3.40. Consider the seeds

$$\Sigma = (\mathrm{X} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \{x_1, x_2, x_3\}, \begin{tikzcd} & & \boxed{x_5} \arrow[r] \arrow[dl] & x_3 \arrow[r] & \boxed{x_6} \\ x_1 \arrow[r] & x_2 & \boxed{x_4} \arrow[l, shift left] \arrow[l, shift right] \arrow[ur] & & \boxed{x_7} \end{tikzcd})$$

and

$$\Sigma' = (\mathrm{X}' = \{y_1, y_2, y_3, z_1, z_2, a\}, \{y_1, y_2, y_3, a\}, \begin{tikzcd} y_1 \arrow[r] & y_2 & z_1 \arrow[l, shift left=1.5] \arrow[l] \arrow[l, shift right=1.5] \arrow[rr, bend left] & y_3 \arrow[l, shift left] \arrow[l, shift right] & z_2 \arrow[l] & a \arrow[l] \end{tikzcd})$$

and the map $f\colon \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \to \{y_1, y_2, y_3, z_1, z_2, a\}$ which maps

$$\begin{aligned} x_i &\mapsto y_i \text{ for } i = 1, 2, 3 \\ x_i &\mapsto z_1 \text{ for } i = 4{,}5{,}7 \\ x_6 &\mapsto z_2. \end{aligned}$$

We check that f satisfies conditions (1), (2) and (3) from Theorem 2.3.37. By definition of f, the restriction of f to the exchangeable variables of Σ is an injection that maps into the exchangeable variables of Σ', thus condition (1) is satisfied.

The variables x_4, x_5 and x_7 all get mapped to the same variable, so we have to check condition (2) for those. Firstly, they are all coefficients. Let now $(x_1, \ldots, x_l)$ be a Σ-admissible sequence and set $\tilde{\Sigma} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma)$ with $\tilde{\Sigma} = (\tilde{\mathrm{X}}, \tilde{\mathrm{ex}}, \tilde{Q})$. We have to check that in $\tilde{Q}$ there are no paths of length 2 passing through an exchangeable vertex $v \in \mathrm{ex}$ from any of x_4, x_5 and x_7 to any of x_4, x_5 or x_7 (i.e. no paths of the form $x_i \to v \to x_j$ for $i, j \in \{4, 5, 7\}$). Since x_7 is its own connected component in Σ, and therefore also in $\tilde{\Sigma}$, we have no arrow between x_7 and any $v \in \tilde{\mathrm{ex}}$ in $\tilde{Q}$. Furthermore, we can check the condition for the two exchangeably connected components $\Sigma_{x_1}^{ex} = (\mathrm{X}_{x_1}^{ex}, \mathrm{ex}_{x_1}^{ex}, Q_{x_1}^{ex})$ and $\Sigma_{x_3}^{ex} = (\mathrm{X}_{x_3}^{ex}, \mathrm{ex}_{x_3}^{ex}, Q_{x_3}^{ex})$ individually, by Remark 2.3.27. It is straightforward but tedious to check that both $Q_{x_1}^{ex}$ and $Q_{x_3}^{ex}$ are mutation finite along Σ_{x_1}-, respectively Σ_{x_3}-admissible sequences, with ten quivers in the mutation class of $Q_{x_1}^{ex}$ and two quivers in the mutation class of $Q_{x_3}^{ex}$ and that the condition (2) holds for all of them.

Finally, we can see that in the exchangeably connected component $f(\mathrm{X})_{f(x_1)}^{ex}$ of the image seed $f(\Sigma)$ (see Example 2.3.12) the number of arrows from $f(x) \in f(\mathrm{ex})$ to $f(y) \in f(\mathrm{X})$ is equal to the sum of arrows from x to the preimages of y and vice versa. In the other exchangeably connected component $f(\mathrm{X})_{f(x_3)}^{ex}$ of $f(\Sigma)$, the number of arrows from $f(x) \in f(\mathrm{ex})$ to $f(y) \in f(\mathrm{X})$ is equal to the sum of arrows from the preimages of y to x and vice versa. Thus condition (3) is satisfied.

2.4 Rooted cluster algebras of infinite rank as colimits of rooted cluster algebras of finite rank

In this section, we show that every rooted cluster algebra of infinite rank can be written as a linear colimit of rooted cluster algebras of finite rank. This yields a formal way to manipulate cluster algebras of infinite rank by viewing them locally as cluster algebras of finite rank.

2.4.1 Colimits and limits in *Clus*

We start by recalling the notion of limit and colimit. Let $\mathcal{C}$ and $\mathcal{J}$ be categories and let $F\colon \mathcal{J} \to \mathcal{C}$ be a diagram of type $\mathcal{J}$ in the category $\mathcal{C}$, i.e. a functor from $\mathcal{J}$ to $\mathcal{C}$.

The *limit* $\lim(F)$ *of* F (if it exists) is an object $\lim(F) \in \mathcal{C}$ together with a family of morphisms $f_i\colon \lim(F) \to F(i)$ in $\mathcal{C}$ indexed by the objects $i \in \mathcal{J}$ such that for any morphism $f_{ij}\colon i \to j$ in $\mathcal{J}$ we have $F(f_{ij}) \circ f_i = f_j$ and $\lim(F)$ is universal with this property. That is, for any object $C \in \mathcal{C}$ with a family of morphisms $g_i\colon C \to F(i)$ in $\mathcal{C}$ for objects $i \in \mathcal{J}$ such that $F(f_{ij}) \circ g_i = g_j$ for all morphisms $f_{ij}\colon i \to j$ in $\mathcal{J}$ there exists a unique morphism $h\colon C \to \lim(F)$ such that the following diagram commutes.

$$\begin{tikzcd} & C \arrow[d, "h"] \arrow[ddl, "g_i"'] \arrow[ddr, "g_j"] & \\ & \lim(F) \arrow[dl, "f_i"] \arrow[dr, "f_j"'] & \\ F(i) \arrow[rr, "F(f_{ij})"] & & F(j) \end{tikzcd}$$

The dual notion of the limit of F is the *colimit* $\mathrm{colim}(F)$ *of* F. If it exists, it is an object $\mathrm{colim}(F) \in \mathcal{C}$ together with a family of morphisms $f_i\colon F(i) \to \mathrm{colim}(F)$ in $\mathcal{C}$ indexed by the objects $i \in \mathcal{J}$ such that for any morphism $f_{ij}\colon i \to j$ in $\mathcal{J}$ we have

$f_j \circ F(f_{ij}) = f_i$ and for any object $C \in \mathcal{C}$ with a family of morphisms $g_i \colon F(i) \to C$ in $\mathcal{C}$ for objects $i \in \mathcal{J}$ such that $g_j \circ F(f_{ij}) = g_i$ for all morphisms $f_{ij} \colon i \to j$ in $\mathcal{J}$ there exists a unique morphism $h \colon \operatorname{colim}(F) \to C$ such that the following diagram commutes.

$$\begin{tikzcd}
 & C & \\
 & \operatorname{colim}(F) \arrow[u, "h"] & \\
F(i) \arrow[uur, "g_i"] \arrow[ur, "f_i"'] \arrow[rr, "F(f_{ij})"] & & F(j) \arrow[uul, "g_j"'] \arrow[ul, "f_j"]
\end{tikzcd}$$

A limit $\lim(F)$ or colimit $\operatorname{colim}(F)$ is called *finite*, respectively *infinite* if the index category $\mathcal{J}$ in the diagram $F \colon \mathcal{J} \to \mathcal{C}$ is finite, respectively infinite. It is called *small* if the index category $\mathcal{J}$ in the diagram $F \colon \mathcal{J} \to \mathcal{C}$ is small. A category is called *complete*, respectively *cocomplete*, if it has all small limits, respectively colimits.

Remark 2.4.1. Products are examples of limits. They are limits of diagrams $F \colon \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is a discrete category, i.e. a category with no morphisms except the identity morphisms. Dually, coproducts are examples of colimits.

Coequalizers are examples for finite colimits. They are colimits of diagrams $G : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is the category with two objects i_1 and i_2 and two parallel morphisms $i_1 \rightrightarrows i_2$ in addition to the identity morphisms. Dually, equalizers are examples of finite limits.

In fact, these are rather important examples as having equalizers and small products is necessary and sufficient for a category to be complete, and dually a category is cocomplete if and only if it has coequalizers and small coproducts, see for example Mac Lane's book [ML, Chapter V].

Theorem 2.4.2. *The category $Clus$ is neither complete nor cocomplete.*

Proof. If the category $Clus$ were complete, then finite products would exist, cf. Remark 2.4.1. However, by [ADS, Proposition 5.4], the category $Clus$ does not admit finite products, hence it cannot be complete.

Furthermore, if $Clus$ were cocomplete then coequalizers would exist. However, consider the seeds

$$\Sigma_0 = (\{x_0, x_1\}, \{x_0, x_1\}, x_0 \to x_1) \text{ and } \Sigma_1 = (\{y_0, y_1\}, \{y_0, y_1\}, y_0 \to y_1)$$

and the parallel rooted cluster isomorphisms defined by the algebraic extensions of

$$f \colon \begin{cases} \mathcal{A}(\Sigma_0) \to \mathcal{A}(\Sigma_1) \\ x_i \mapsto y_i \text{ for } i = 0, 1 \end{cases} \quad \text{and } g \colon \begin{cases} \mathcal{A}(\Sigma_0) \to \mathcal{A}(\Sigma_1) \\ x_i \mapsto y_{1-i} \text{ for } i = 0, 1. \end{cases}$$

Assume for a contradiction that there exists a coequalizer for f and g, i.e. a rooted cluster algebra $\mathcal{A}(\Sigma)$ with initial seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ with a rooted cluster morphism

$\varphi\colon \mathcal{A}(\Sigma_1) \to \mathcal{A}(\Sigma)$ such that $\varphi \circ f = \varphi \circ g$ and it is universal with this property. Because φ is a rooted cluster morphism and $\varphi \circ f = \varphi \circ g$ we have $\varphi(y_0) = \varphi(y_1) \in \mathrm{ex} \cup \mathbb{Z}$. By Proposition 2.3.9 two distinct exchangeable variables of Σ_1 cannot be sent to the same exchangeable variable via a rooted cluster morphism. Thus we must have $\varphi(y_0) = \varphi(y_1) \in \mathbb{Z}$. Consider the empty seed $\Sigma_\emptyset = (\emptyset, \emptyset, \emptyset)$. As a ring, we have $\mathcal{A}(\Sigma_\emptyset) \cong \mathbb{Z}$. Consider the rooted cluster morphisms $\psi_1\colon \mathcal{A}(\Sigma_1) \to \mathcal{A}(\Sigma_\emptyset)$, defined by sending all cluster variables in $\mathcal{A}(\Sigma_1)$ to 0, and $\psi_2\colon \mathcal{A}(\Sigma_1) \to \mathcal{A}(\Sigma_\emptyset)$ defined by evaluating both y_0 and y_1 at 1. Because a rooted cluster morphism is a ring homomorphism between unital rings, any rooted cluster morphism from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(\Sigma_\emptyset)$ acts as the identity on the subring $\mathbb{Z}$. Thus, if $\varphi(y_0) = \varphi(y_1) \neq 0$, then ψ_1 does not factor through φ and if $\varphi(y_0) = \varphi(y_1) = 0$, then ψ_2 does not factor through φ. Therefore there exists no coequalizer for f and g and *Clus* is not cocomplete. □

2.4.2 Rooted cluster algebras of infinite rank as colimits

Even though colimits do not in general exist in *Clus*, we can show that there are sufficient colimits such that every rooted cluster algebra of infinite rank is isomorphic to a colimit of rooted cluster algebras of finite rank. More precisely, we can write them as linear colimits. A colimit $\mathrm{colim}(F)$ in a category $\mathcal{C}$ is called *linear*, if the index category $\mathcal{J}$ of the diagram $F\colon \mathcal{J} \to \mathcal{C}$ is a set endowed with a linear order viewed as a category. A diagram $F\colon \mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is endowed with a linear order $\leq$ is just a *linear system* of objects in $\mathcal{C}$, that is a family of objects $\{C_i\}_{i \in \mathcal{J}}$ and a family of morphisms $\{f_{ij}\}_{i \leq j \in \mathcal{J}}$ such that $f_{jk} \circ f_{ij} = f_{ik}$ and $f_{ii} = \mathrm{id}_{C_i}$ for all $i \leq j \leq k$ in $\mathcal{J}$. In order to explicitly construct a suitable linear system of rooted cluster algebras of finite rank, we use the fact that in certain nice cases inclusions of subseeds give rise to rooted cluster morphisms.

In general, if Σ is a full subseed of Σ' (see Definition 2.3.13), the natural inclusion of rings $\mathcal{A}(\Sigma) \to \mathcal{F}_{\Sigma'}$ does not give rise to a rooted cluster morphism $\mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$, see [ADS, Remark 4.10]. However, we can fix this with an additional condition which has to do with how the subseed is connected to the bigger seed.

Definition 2.4.3. Let $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ be a seed with a full subseed $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w \in \mathrm{X}})$ such that for every $x \in \mathrm{X}$ with a neighbour $y \in \mathrm{X}' \setminus \mathrm{X}$ in Σ' we have $x \in \mathrm{X} \setminus \mathrm{ex}$, i.e. x is a coefficient of Σ. Then we say that Σ *and* Σ' *are connected only by coefficients of* Σ.

Example 2.4.4. Consider the seed

$$\Sigma' = (\mathrm{X}' = \{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_5\}, \begin{tikzcd} & & x_5 \arrow[dl] \arrow[r] & x_3 \arrow[r] & \boxed{x_6} \\ x_1 \arrow[r] & x_2 & \boxed{x_4} \arrow[l, shift left] \arrow[l, shift right] \arrow[ur] & & \end{tikzcd} \quad)$$

and its full subseed

$$\Sigma_1 = (\mathrm{X}_1 = \{x_1, x_2, x_4, x_5\}, \{x_1, x_2\}, \begin{tikzcd} & & \boxed{x_5} \arrow[dl] \\ x_1 \arrow[r] & x_2 & \boxed{x_4} \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}).$$

The seeds Σ_1 and Σ' are only connected by coefficients of Σ_1: The only elements of X_1 which have neighbours belonging to $\mathrm{X}' \setminus \mathrm{X}_1$ in Σ' are x_4 and x_5 (they have both the neighbour x_3), and both x_4 and x_5 are coefficients of Σ_1. Consider now the full subseed

$$\Sigma_2 = (\mathrm{X}_2 = \{x_1, x_2\}, \{x_1, x_2\}, \; x_1 \to x_2 \quad)$$

of Σ'. The seeds Σ_2 and Σ' are not connected only by coefficients of Σ_2: The element $x_2 \in \mathrm{X}_2$ is an exchangeable variable of Σ_2 and it has neighbours x_4 and x_5 in Σ'.

The condition of being connected only by coefficients is transitive.

Lemma 2.4.5. *If $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ is a full subseed of $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ and Σ' is a full subseed of $\Sigma'' = (\mathrm{X}'', \mathrm{ex}'', B'')$, such that Σ and Σ' are only connected by coefficients in Σ and such that Σ' and Σ'' are only connected by coefficients in Σ', then Σ and Σ'' are only connected by coefficients in Σ.*

Proof. If $x \in \mathrm{ex}$ is an exchangeable variable of Σ then, by the definition of full subseed, it is an exchangeable variable of Σ'. Because Σ' and Σ'' are only connected by coefficients of Σ', x cannot have a neighbour in Σ'' that lies in $\mathrm{X}'' \setminus \mathrm{X}'$. All neighbours of x in Σ'' thus lie in X', and, because Σ' is a full subseed of Σ'', these are exactly those variables that are neighbours of x in Σ'. Because Σ and Σ' are only connected by coefficients in Σ, these neighbours must be elements of X. $\square$

If Σ is a full subseed of Σ', such that the seeds are connected only by coefficients of Σ, then the inclusion of Σ in Σ' induces a rooted cluster morphism.

Lemma 2.4.6. *Let $\Sigma = (\mathrm{X}, \mathrm{ex}, B)$ be a full subseed of $\Sigma' = (\mathrm{X}', \mathrm{ex}', B')$ such that Σ and Σ' are connected only by coefficients of Σ. Then the inclusion $f \colon \mathrm{X} \to \mathrm{X}'$ gives rise to a rooted cluster morphism $f \colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$.*

Proof. This follows directly from Theorem 2.3.37. $\square$

For any given rooted cluster algebra $\mathcal{A}(\Sigma)$ we can build a linear system $\{\mathcal{A}(\Sigma_i)\}_{i \in \mathbb{Z}}$ of rooted cluster algebras whose initial seeds are finite full subseeds Σ_i of Σ such that for all $i \in \mathbb{Z}$, the seeds Σ_i and Σ are only connected by coefficients of Σ_i. Further, we can construct it in a way, such that for all $i \leq j$ the seed Σ_i is a full subseed of Σ_j and the two are connected only by coefficients of Σ_i. This construction yields a linear system of rooted cluster algebras of finite rank which has the desired rooted cluster algebra $\mathcal{A}(\Sigma)$ as its colimit.

Theorem 2.4.7. *Every rooted cluster algebra is isomorphic to a linear colimit of rooted cluster algebras of finite rank in the category Clus of rooted cluster algebras.*

Proof. Let $\mathcal{A}(\Sigma)$ be a rooted cluster algebra with initial seed $\Sigma = (\mathrm{X}, \mathrm{ex}, B = (b_{vw})_{v,w\in \mathrm{X}})$. Let $\Sigma = \bigsqcup_{j\in J} \Sigma^j$ be its decomposition into connected seeds with $\Sigma^j = (\mathrm{X}^j, \mathrm{ex}^j, B^j)$ for $j \in J$, where J is some countable index set (since the cluster X is countable by Definition 2.2.1, there are only countably many connected components). We can thus write the rooted cluster algebra $\mathcal{A}(\Sigma)$ as the countable coproduct of the connected rooted cluster algebras $\mathcal{A}(\Sigma^j)$:

$$\mathcal{A}(\Sigma) \cong \coprod_{j\in J} \mathcal{A}(\Sigma^j).$$

For notational simplicity we assume $J = \{0, 1, \dots, n\}$ for some $n \in \mathbb{Z}_{\geq 0}$ if J is finite, and $J = \mathbb{Z}_{\geq 0}$ if J is infinite. We construct a linear system of rooted cluster algebras as follows. For $j \in J$ choose $x_0^j \in \mathrm{X}^j$ and inductively define full subseeds Σ_i^j of Σ by

$$\begin{aligned}
\Sigma_0^j &= (\mathrm{X}_0^j, \mathrm{ex}_0^j, B_0^j) = (\{x_0^j\}, \emptyset, [0]) \\
\Sigma_{i+1}^j &= (\mathrm{X}_{i+1}^j, \mathrm{ex}_{i+1}^j, B_{i+1}^j) \\
&= \left(\mathrm{X}_i^j \cup \{w \in \mathrm{X} \mid b_{vw} \neq 0 \text{ for some } v \in \mathrm{X}_i^j\}, \mathrm{X}_i^j \cap \mathrm{ex}, B_{i+1}^j\right), \text{ for } i \geq 0
\end{aligned}$$

where B_{i+1}^j is the full submatrix of B formed by the entries labelled by $\mathrm{X}_{i+1}^j \times \mathrm{X}_{i+1}^j$. Note that because B is skew-symmetrizable $b_{vw} \neq 0$ is equivalent to $b_{wv} \neq 0$. Because B^j is locally finite, for all $i \geq 0$ the cluster X_i^j in the seed Σ_i^j is finite. We set

$$\tilde{\Sigma}_i := \coprod_{j\in J: j\leq i} \Sigma_{i-j}^j$$

and write $\tilde{\Sigma}_i = (\tilde{\mathrm{X}}_i, \tilde{\mathrm{ex}}_i, \tilde{B}_i = ((\tilde{b}_i)_{vw})_{v,w\in\tilde{X}_i})$. Because the cluster in each of the seeds Σ_{i-j}^j for $j \in J$ with $0 \leq j \leq i$ is finite, so is the cluster $\tilde{\mathrm{X}}_i$ of $\tilde{\Sigma}_i$. By definition, the seed $\tilde{\Sigma}_i$ is a full subseed of the seed $\tilde{\Sigma}_{i+1}$ for all $i \geq 0$ and all the seeds $\tilde{\Sigma}_i$ are full subseeds of Σ.

We now want to show that for all $i \geq 0$ the seeds $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_{i+1}$ are connected only by coefficients of $\tilde{\Sigma}_i$. From that it follows by Lemma 2.4.5, that $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_j$ for all $i \leq j$ are connected only by coefficients of $\tilde{\Sigma}_i$. Because the subseeds Σ_i^j and $\Sigma_{i'}^{j'}$ are by definition mutually disconnected for $j \neq j'$ in J and any $i, i' \in \mathbb{Z}_{\geq 0}$, it is enough to check that Σ_i^j and Σ_{i+1}^j are connected only by coefficients of Σ_i^j for any $i \in \mathbb{Z}$ and $j \in J$. Let $x \in \mathrm{ex}_i^j$ and $y \in \mathrm{X}_{i+1}^j$ with $b_{xy} \neq 0$. We want to show that this implies $y \in \mathrm{X}_i^j$. We have $i > 0$, since $\mathrm{ex}_0^j = \emptyset$ for all $j \in J$. It follows that $x \in \mathrm{ex}_i^j = \mathrm{X}_{i-1}^j \cap \mathrm{ex} \subseteq \mathrm{X}_{i-1}^j$ and thus $y \in \{w \in \mathrm{X} \mid b_{vw} \neq 0 \text{ for some } v \in \mathrm{X}_{i-1}^j\} \subseteq \mathrm{X}_i^j$. Therefore $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_{i+1}$ are connected only by coefficients of $\tilde{\Sigma}_i$. The same argument shows that for any $i \geq 0$ the seeds $\tilde{\Sigma}_i$ and Σ are connected only by coefficients of $\tilde{\Sigma}_i$.

By Lemma 2.4.6 for $0 \leq i \leq j$, the natural inclusion $f_{ij}\colon \tilde{\mathrm{X}}_i \to \tilde{\mathrm{X}}_j$ gives rise to a rooted cluster morphism $f_{ij}\colon \mathcal{A}(\tilde{\Sigma}_i) \to \mathcal{A}(\tilde{\Sigma}_j)$. For all $0 \leq i \leq j \leq k$ we have $f_{jk} \circ f_{ij} = f_{ik}$ and $f_{ii} = \mathrm{id}_{\mathcal{A}(\tilde{\Sigma}_i)}$, so the morphisms form a linear system of rooted cluster algebras of finite rank. Further, again by Lemma 2.4.6, for $i \geq 0$ the natural inclusion $f_i \colon \tilde{\mathrm{X}}_i \to \mathrm{X}$ gives rise to a rooted cluster morphism $f_i \colon \mathcal{A}(\tilde{\Sigma}_i) \to \mathcal{A}(\Sigma)$. We show that $\mathcal{A}(\Sigma)$ together with the maps $f_i \colon \mathcal{A}(\tilde{\Sigma}_i) \to \mathcal{A}(\Sigma)$ for $i \geq 0$ is in fact the colimit of this linear system in the category of rooted cluster algebras.

Because for any $j \in J$, the seed Σ^j is connected, we have $\mathrm{X}^j = \bigcup_{i \geq 0} \mathrm{X}_i^j$ and thus

$$\mathrm{X} = \bigsqcup_{j \in J} \mathrm{X}^j = \bigsqcup_{j \in J} \bigcup_{i \geq 0} \mathrm{X}_i^j = \bigcup_{i \geq 0} \bigsqcup_{j \in J} \mathrm{X}_i^j = \bigcup_{i \geq 0} \tilde{\mathrm{X}}_i.$$

Because every exchange relation in $\mathcal{A}(\Sigma)$ lifts to an exchange relation in $\mathcal{A}(\tilde{\Sigma}_i)$ for all i big enough (by virtue of the exchange matrices $\tilde{B}_i$ being arbitrarily large restrictions of the exchange matrix B), any fixed element of $\mathcal{A}(\Sigma)$ is contained in $\mathcal{A}(\tilde{\Sigma}_i)$ for all i sufficiently large.

Let $\Sigma' = (\mathrm{X}', \mathrm{ex}', Q')$ be a seed such that for all $i \geq 0$ there are rooted cluster morphisms $g_i \colon \mathcal{A}(\tilde{\Sigma}_i) \to \mathcal{A}(\Sigma')$ compatible with the linear system $f_{ij} \colon \mathcal{A}(\tilde{\Sigma}_i) \to \mathcal{A}(\tilde{\Sigma}_j)$. We define a ring homomorphism $f \colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ by $f(x) = g_i(x)$, whenever $x \in \mathcal{A}(\tilde{\Sigma}_i)$, i.e. it is the unique ring homomorphism making the following diagram commute.

$$\begin{tikzcd}
 & \mathcal{A}(\Sigma') & \\
 & \mathcal{A}(\Sigma) \arrow[u, dashed, "f"] & \\
\mathcal{A}(\tilde{\Sigma}_i) \arrow[uur, "g_i"] \arrow[ur, "f_i"'] \arrow[rr, "f_{ij}"] & & \mathcal{A}(\tilde{\Sigma}_j) \arrow[uul, "g_j"'] \arrow[ul, "f_j"]
\end{tikzcd}$$

For every $x \in \mathrm{X}$ (respectively $x \in \mathrm{ex}$), there exists a $k \geq 0$ such that $x \in \tilde{\mathrm{X}}_i$ (respectively $x \in \tilde{\mathrm{ex}}_i$) for all $i \geq k$. Thus $f(x) = g_i(x)$ for all $i \geq k$ lies in X' (respectively in ex'), because g_i is a rooted cluster morphism for all $i \geq 0$. Thus the ring homomorphism f satisfies axioms CM1 and CM2. Let now $(x_1, \ldots, x_l)$ be a (f, Σ, Σ')-biadmissible sequence and let $y \in \mathrm{X}$ such that $f(y) \in \mathrm{X}'$. Then there exists an $i \geq 0$ such that $y \in \tilde{\mathrm{X}}_i$ and the sequence $(x_1, \ldots, x_l)$ is $(g_i, \tilde{\Sigma}_i, \Sigma')$-biadmissible. Thus we get

$$\begin{aligned}
f(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) &= f \circ f_i(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) \\
&= g_i(\mu_{x_l} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{g_i(x_l)} \circ \ldots \circ \mu_{g_i(x_1)}(g_i(y))) \\
&= \mu_{f(x_l)} \circ \ldots \circ \mu_{f(x_1)}(f(y)).
\end{aligned}$$

Therefore the ring homomorphism f satisfies CM3 and is a rooted cluster morphism. Thus $\mathcal{A}(\Sigma)$ satisfies the required universal property. □

Remark 2.4.8. Work in progress by Stovicek and van Roosmalen shows the analogue of Theorem 2.4.7 for cluster categories of infinite rank. However, their approach is different and it is not clear that either result can be easily obtained from the other.

Remark 2.4.9. The proof of Theorem 2.4.7 assumes that the seed of our cluster algebra has a countable cluster. We can omit this assumption, but the price we pay is that the colimit is no longer linear. If we allow seeds with uncountable clusters, by Remark 2.3.22 every connected component is still countable. For any given rooted cluster algebra of possibly uncountable rank, we can take the decomposition of its initial seed into (possibly uncountably many) connected components. This allows us to write our rooted cluster algebra as a (possibly uncountable) coproduct of linear colimits of rooted cluster algebras of finite rank, which – since taking coproducts is a special example of a colimit – is a colimit of rooted cluster algebras of finite rank.

2.4.3 Positivity for cluster algebras of infinite rank

Fomin and Zelevinsky showed in [FZ1, Theorem 3.1] that every cluster variable of a cluster algebra of finite rank is a Laurent polynomial in the elements of its initial cluster over $\mathbb{Z}$ and they conjectured that the coefficients in this Laurent polynomial are nonnegative. The so-called positivity conjecture has been a central problem in the theory of cluster algebras and has recently been solved by Lee and Schiffler [LS] for all skew-symmetric cluster algebras of finite rank. Previously, the problem had been solved via different approaches for important special cases, such as for acyclic cluster algebras by Kimura and Qin [KQ] and for cluster algebras from surfaces by Musiker, Schiffler and Williams [MSW].

Theorem 2.4.10. *The positivity conjecture holds for every skew-symmetric cluster algebra of infinite rank, i.e. for every skew-symmetric cluster algebra $\mathcal{A}(\Sigma)$ of infinite rank associated to a seed $\Sigma = (\mathrm{X}, \mathrm{ex}, Q)$, every cluster variable in $\mathcal{A}(\Sigma)$ is a Laurent polynomial in X over $\mathbb{Z}$ with nonnegative coefficients.*

Proof. Let $\Sigma = (\mathrm{X}, \mathrm{ex}, Q)$ be a skew-symmetric cluster algebra of infinite rank. Using the construction in the proof of Theorem 2.4.7, the associated rooted cluster algebra $\mathcal{A}(\Sigma)$ can be written as a linear colimit $\mathcal{A}(\Sigma) = \mathrm{colim}(\mathcal{A}(\Sigma_i))$ of a linear system $\{\mathcal{A}(\Sigma_i)\}_{i\in\mathbb{Z}}$ of skew-symmetric rooted cluster algebras of finite rank with seeds $\Sigma_i = (\mathrm{X}_i, \mathrm{ex}_i, B_i)$ and with canonical inclusions $f_i \colon \mathcal{A}(\Sigma_i) \to \mathcal{A}(\Sigma)$ for $i \in \mathbb{Z}$. Let $\tilde{x} \in \mathcal{A}(\Sigma)$ be a cluster variable, thus $\tilde{x} = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(x)$ for some $x \in \mathrm{X}$ and some Σ-admissible sequence $(x_1, \ldots, x_l)$. Then there exists an $i \in \mathbb{Z}$ such that $x \in \mathrm{X}_i$ and $(x_1, \ldots, x_l)$ is Σ_i-admissible. Set $y = \mu_{x_l} \circ \ldots \circ \mu_{x_1}(x)$ in $\mathcal{A}(\Sigma_i)$. By axiom CM3 for f_i we have $f_i(y) = \tilde{x}$. By [LS, Theorem 4.2], the cluster variable $y \in \mathcal{A}(\Sigma_i)$ is a Laurent polynomial in X_i over $\mathbb{Z}$ with nonnegative coefficients. Since f_i is a ring homomorphism (with $f_i(1) = 1$) the image $\tilde{x} = f_i(y)$ is a Laurent polynomial in $f_i(\mathrm{X}_i) \subseteq \mathrm{X}$ over $\mathbb{Z}$ with nonnegative coefficients. □

Remark 2.4.11. The positivity conjecture still holds if we allow uncountable clusters: Let $\mathcal{A}(\Sigma)$ be a rooted cluster algebra of uncountable rank. We can decompose it into its connected components $\mathcal{A}(\Sigma_i)$ with seeds $\Sigma_i = (\mathrm{X}_i, \mathrm{ex}_i, B_i)$ for $i \in I$ for some uncountable index set I. By Remark 2.3.22, for all $i \in I$ the rooted cluster algebra $\mathcal{A}(\Sigma_i)$ is of countable rank and, by the defintion of coproduct, every cluster variable x in $\mathcal{A}(\Sigma)$ lives in the cluster algebra $\mathcal{A}(\Sigma_i)$ of countable rank for a unique $i \in I$. Since the positivity conjecture holds for $\mathcal{A}(\Sigma_i)$, the cluster variable x is a Laurent Polynomial in X_i with nonnegative integer coefficients, and thus in particular a Laurent polynomial in X with nonnegative integer coefficients.

2.4.4 Rooted cluster algebras from infinite triangulations of the closed disc

It follows from Theorem 2.4.7 that every rooted cluster algebra arising from a countable triangulation of the closed disc can be written as a colimit of rooted cluster algebras of finite rank. Moreover, as we will see in this section, it can be written as a linear colimit of rooted cluster algebras that arise from finite triangulations of the closed disc. Thus we obtain a formal way of treating cluster algebras associated to infinite triangulations of the closed disc as infinite versions of cluster algebras of Dynkin type A. This provides the algebraic analogue of the work of Holm and Jørgensen [HJ] and Igusa and Todorov ([IT1], [IT3, Section 2.4]), who introduced infinite versions of cluster categories of Dynkin type A. A short introduction to these cluster categories will be given in Sections 3.3.1 and 3.3.2 of Chapter 3.

Theorem 2.4.12. *Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z}$. Then the associated rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is isomorphic to a countable coproduct $\mathcal{A}(\Sigma_{\mathcal{T}}) \cong \coprod_{j\in I} \mathcal{A}(\Sigma_{\mathcal{T}_j})$ of linear colimits $\mathcal{A}(\Sigma_{\mathcal{T}_j}) \cong \operatorname{colim}(\mathcal{A}(\Sigma_{\mathcal{T}_i^j})$ of rooted cluster algebras $\mathcal{A}(\Sigma_{\mathcal{T}_i^j})$ of finite Dynkin type A.*

Proof. We can directly translate the proof of Theorem 2.4.7 to this situation. Let first $\mathcal{T}$ be a connected triangulation. We can build a linear system of rooted cluster algebras associated to finite triangulations of the closed disc as follows. Let $\{x_0, x_1\} \in \mathcal{T}$ and set $\mathcal{T}_0 = \{\{x_0, x_1\}\}$ and for all $i \geq 0$ set

$$\mathcal{T}_{i+1} = \mathcal{T}_i \cup \left\{ \alpha \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{there exists a } \beta \in \mathcal{T}_i \text{ such that } \alpha \text{ and } \beta \\ \text{are sides of a common triangle in } \mathcal{T} \end{array} \right\},$$

where, for all $i \geq 0$, $\mathcal{T}_i$ as a triangulation of the closed disc with marked points $\mathcal{Z}_i$ being the endpoints of arcs in $\mathcal{T}_i$. We pass from $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$ by glueing triangles to all of those edges of $\mathcal{Z}_i$ that are not edges of $\mathcal{Z}$. We can write $\mathcal{T}$ as the countable union of these finite triangulations of the closed disc which are ordered by inclusion:

$$\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}_i, \text{ with } \mathcal{T}_i \subseteq \mathcal{T}_j \text{ for all } j \geq i \geq 0.$$

By Lemma 2.4.6, the natural inclusions $f_{ij}\colon \mathcal{A}(\Sigma_{\mathcal{T}_i}) \to \mathcal{A}(\Sigma_{\mathcal{T}_j})$ for $0 \leq i \leq j$ provide a linear system of rooted cluster algebras and following the lines of the proof of Theorem

2.4.7 there is an isomorphism of rooted cluster algebras

$$\mathcal{A}(\Sigma_{\mathcal{T}}) \cong \operatorname{colim}(\mathcal{A}(\Sigma_{\mathcal{T}_i})).$$

The rooted cluster algebras $\mathcal{A}(\Sigma_{\mathcal{T}_i})$ are associated to finite triangulations of the closed disc and thus are of finite Dynkin type A. By Lemma 2.3.26, every rooted cluster algebra associated to a triangulation of the closed disc is isomorphic to a coproduct of rooted cluster algebras associated to connected triangulations of the closed disc. This proves the claim. □

Remark 2.4.13. The idea of the proof of Theorem 2.4.12 follows the construction in the proof of Theorem 2.4.7. Glueing on arcs to the edges of the triangulations corresponds to glueing on new cluster variables to coefficients.

In the case where the set of marked points $\mathcal{Z} \subseteq S^1$ has precisely one limit point, the cluster algebras associated to triangulations of $\mathcal{Z}$ have been classified by their connected components in [GG].

In the language of the category *Clus* we can reformulate the main result from [GG] as follows.

Theorem 2.4.14 ([GG, Theorems 3.11 and 3.16])**.** *Let $\mathcal{Z}$ be a discrete subset of S^1 with exactly one limit point and let $\mathcal{T}$ be a triangulation of $\mathcal{Z}$. Then one of the following holds:*

(1) *The triangulation $\mathcal{T}$ has a nest and the rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is isomorphic to an infinite linear colimit of rooted cluster algebras of finite Dynkin type A.*

(2) *The triangulation $\mathcal{T}$ has a fountain and the rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is isomorphic to the coproduct of two infinite linear colimits of rooted cluster algebras of finite Dynkin type A.*

(3) *The triangulation $\mathcal{T}$ has a split fountain and the rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}})$ is isomorphic to the coproduct of a rooted cluster algebra of finite Dynkin type A and two infinite linear colimits of rooted cluster algebras of finite Dynkin type A.*

Remark 2.4.15. In a similar fashion it is possible to classify rooted cluster algebras associated to arbitrary triangulations of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. The decomposition of a fixed triangulation $\mathcal{T}$ into connected components can be worked out directly with the help of Lemma 2.3.24 and Remark 2.3.25.

The work in [GG] was inspired by Holm and Jørgensen's study of the cluster category of infinite Dynkin type A_∞. Igusa and Todorov introduced generalizations of this cluster category in [IT1] and [IT3]. More details on these cluster categories will be given in Section 3.3 of Chapter 3. All of these cluster categories have combinatorial interpretations via countable triangulations of the closed disc and thus find their algebraic counterparts in our cluster algebras associated to triangulations of the closed disc. In Section 3.3.2

we explicitly describe how to obtain the rooted cluster algebra that allows the same combinatorics as the continuous cluster category of Dynkin type A (as studied in [IT1]) as a colimit of rooted cluster algebras of finite Dynkin type A.

Chapter 3

Cluster categories

3.1 Introduction

An important step towards a better understanding of cluster algebras has been made by their categorification, which started with the introduction of cluster categories by Buan, Marsh, Reineke, Reiten and Todorov in [BMRRT]. In Section 3.2.1 we focus on this construction, which works for skew-symmetric, coefficient-free cluster algebras of finite rank, whose exchange quivers are mutation-equivalent to an acyclic quiver.

This categorification allowed for elegant proofs of structural properties of cluster algebras, that had not been known before, such as the proof of the positivity conjecture for cluster algebras of simply laced Dynkin type by Caldero and Keller in [CK1] and the denominator conjecture for acyclic cluster algebras, i.e. cluster algebras whose exchange quivers are mutation-equivalent to an acyclic quiver, by the same authors in [CK2]. The basic idea of a categorical version of cluster algebras is that all the combinatorial concepts we know from cluster algebras will reappear: The cluster variables will find their analogue in indecomposable objects, the clusters in certain subcategories, which are sometimes also called clusters, and we will have a concept of mutation, that allows us to uniquely replace an indecomposable object in a cluster by a new one in order to obtain another cluster. Cluster categories in the sense of [BMRRT] are triangulated categories by Keller [K], and the explicit description of mutation on the categorical level relies on this triangulated structure.

There are more triangulated categories than just the classical cluster categories which mirror the combinatorics of cluster algebras. Buan, Iyama, Reiten and Scott [BIRS] introduced the notion of cluster structures on triangulated categories. The classical cluster categories from [BMRRT] carry a natural cluster structure in the expected way. However, these are not the only categories with cluster structures. In particular, categories with a cluster structure might have infinite clusters and thus provide analogues of cluster algebras of infinite rank. Important examples of cluster categories of infinite rank have been studied by Holm and Jørgensen (see [HJ] for a cluster algebra of infinite Dynkin type A) and by Igusa and Todorov (see [IT1] and [IT3] for the continuous cluster category of Dynkin type A and, amongst more general examples, discrete cluster categories of Dynkin type A).

Conventions Throughout the rest of this thesis, we work over an algebraically closed field k. All triangulated categories are assumed to be k-linear, Hom-finite and Krull-Schmidt and functors are assumed to be k-linear. All subcategories of a triangulated category T are assumed to be full and closed under isomorphisms, direct summands and finite direct sums and we write $X \subseteq T$, if X is a subcategory of T. In the same vein we write $x \in X$ for an object x in X. If A is a collection of objects in T, we denote by $\operatorname{add} A$ its additive hull.

We express any skew-symmetric cluster algebra without coefficients via a quiver Q, cf. Remark 2.2.3. We associate to a locally finite quiver without loops or 2-cycles a seed $\Sigma_Q = (X_Q, X_Q, Q)$ (see Definition 2.2.1) whose cluster variables are labelled by vertices of Q and all of which are exchangeable. We write $\mathcal{A}_Q$ for the coefficient-free cluster algebra $\mathcal{A}(\Sigma_Q)$ associated to Σ_Q (see Definition 2.2.17) and call it the *cluster algebra associated to Q*.

3.2 Cluster categories as a categorification of cluster algebras

In [BMRRT], Buan, Marsh, Reineke, Reiten and Todorov introduced the cluster category $\mathcal{C}_{\mathcal{H}}$ associated to a finite dimensional hereditary algebra $\mathcal{H}$, i.e. a finite dimensional algebra of global dimension at most one. Every such algebra $\mathcal{H}$ is derived equivalent to the path algebra kQ of a finite quiver Q without oriented cycles (in fact, if in addition $\mathcal{H}$ is basic it is even Morita equivalent to such a path algebra, see or example Assem, Simson and Skowronski's book [ASiSk, Chapter VII, Theorem 1.7]). The cluster category $\mathcal{C}_{\mathcal{H}}$ yields a categorical interpretation of the combinatorics of the cluster algebra $\mathcal{A}_Q$ associated to the quiver Q.

3.2.1 Cluster categories

In this section, we recall the definition of cluster categories as introduced in [BMRRT]. Let $\mathcal{H}$ be a finite dimensional hereditary algebra and consider the bounded derived category $D^b(\operatorname{mod} \mathcal{H})$ of finitely generated right $\mathcal{H}$-modules. Since $\mathcal{H}$ is hereditary, the objects of $D^b(\operatorname{mod} \mathcal{H})$ are finite sums of shifts of indecomposable objects of $\operatorname{mod} \mathcal{H}$. The morphisms are given by (shifts of) morphisms and extensions in $\operatorname{mod} \mathcal{H}$ (see for example Happel's book [H2]). This category is triangulated and has Auslander-Reiten triangles (see Reiten and Van den Bergh's article [RVdB, Section I.2] for the definition), as shown by Happel [H1, Section 3.6]. Denote by Σ the shift functor and by τ the Auslander-Reiten translation of $D^b(\operatorname{mod}\mathcal{H})$. The *cluster category* associated to $\mathcal{H}$ is defined as the orbit category

$$\mathcal{C}_{\mathcal{H}} := D^b(\operatorname{mod}\mathcal{H})\big/\tau^{-1}\Sigma.$$

That is, the objects of $\mathcal{C}_{\mathcal{H}}$ are just the objects of $D^b(\operatorname{mod}\mathcal{H})$ and the morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}_{\mathcal{H}}}(x,y) = \coprod_{i\in\mathbb{Z}} \operatorname{Hom}_{D^b(\operatorname{mod}\mathcal{H})}((\tau^{-1}\Sigma)^i(x),y),$$

for $x, y \in \mathcal{C}_{\mathcal{H}}$. By [BMRRT] the cluster category $\mathcal{C}_{\mathcal{H}}$ is a k-linear, Hom-finite, Krull-Schmidt category and by Keller [K], it is canonically triangulated with shift functor Σ – that is, it inherits the shift functor from $D^b(\operatorname{mod}\mathcal{H})$. By Reiten and Van den Bergh [RVdB, Theorem I.2.4] the category $D^b(\operatorname{mod}\mathcal{H})$ having Auslander-Reiten triangles is equivalent to it having a Serre functor S, which is then given by $S = \tau\Sigma$ and the cluster category $\mathcal{C}_{\mathcal{H}}$ inherits the Serre functor S. In general, a *Serre functor* on a triangulated category T is an exact functor $S\colon T \to T$, such that Serre duality is satisfied: For any two objects $x, y \in T$

$$\operatorname{Hom}_T(x,y) \cong \operatorname{Hom}_T(y,Sx)^*,$$

where * denotes the dual space. If a Serre functor exists it is unique up to unique natural isomorphism. A triangulated category with shift functor Σ and a Serre functor S is called *n-Calabi-Yau*, if there exists an isomorphism of functors $S \cong \Sigma^n$.

Remark 3.2.1. On any triangulated category T with shift functor Σ, Auslander-Reiten translation τ and Serre functor S we have $S = \tau\Sigma$, and thus $\tau \cong \Sigma$ as functors on T if and only if T is 2-Calabi-Yau. Since $\tau \cong \Sigma$ on $\mathcal{C}_{\mathcal{H}}$, the cluster category $\mathcal{C}_{\mathcal{H}}$ associated to $\mathcal{H}$ is 2-Calabi-Yau.

Notation 3.2.2. For a triangulated category T with shift functor Σ we set

$$\operatorname{Ext}^1_T(x,y) := \operatorname{Hom}_T(x,\Sigma y).$$

This notation is standard and extends the usual notion of extensions: If $\mathcal{A}$ is an abelian category and $x, y \in \mathcal{A}$, then

$$\operatorname{Hom}_{D^b(\mathcal{A})}(x,\Sigma y) \cong \operatorname{Ext}^1_{\mathcal{A}}(x,y).$$

So, using the isomorphism $\operatorname{Hom}_{\mathcal{C}_{\mathcal{H}}}(x,\Sigma y) \cong \operatorname{Hom}_{\mathcal{C}_{\mathcal{H}}}(\Sigma y,\Sigma^2 x)^*$ in $\mathcal{C}_{\mathcal{H}}$ coming from Serre duality, we get

$$\operatorname{Ext}^1_{\mathcal{C}_{\mathcal{H}}}(x,y) \cong \operatorname{Ext}^1_{\mathcal{C}_{\mathcal{H}}}(y,x)^*.$$

Hence the dimension of Ext^1 is symmetric in its two arguments. This symmetry will be implicitly used throughout the rest of this thesis.

For combinatorial computations, as we will carry out in Chapter 4, it is useful to consider Auslander-Reiten quivers of categories. The Auslander-Reiten quiver $\Gamma(C)$ of an abelian category C with Auslander-Reiten sequences, or of a triangulated category

C with Auslander-Reiten triangles (see for example [ASiSk, Chapter IV] for a thorough introduction to Auslander-Reiten theory for finite dimensional algebras, and [H2, Chapter I.4] for the triangulated setting) is the quiver with vertices given by isomorphism classes of indecomposable objects in C and number of arrows given by the dimension of the space of irreducible maps between them. Consider the case where $\mathcal{H}$ is of finite representation type, i.e. by Gabriel's theorem it is isomorphic to the path algebra kQ of a simply laced Dynkin quiver Q. By [H1, Corollary 4.5(i)] the Auslander-Reiten quiver $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$ is the repetitive quiver $\mathbb{Z}Q$. Its vertices are pairs (i, v) with $i \in \mathbb{Z}$ and $v \in Q_0$, where Q_0 is the set of vertices of Q. For every arrow $\alpha\colon v \to w$ in Q there are arrows $\alpha_i\colon (i, v) \to (i, w)$ and $\sigma(\alpha_i)\colon (i-1, w) \to (i, v)$ in $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$ for all $i \in \mathbb{Z}$.

We consider the example where our quiver has underlying diagram D_n for $n \geq 4$. This is the case which will be considered in Section 4.4 of Chapter 4, when we study mutation of torsion pairs in cluster categories of finite Dynkin type D. The module categories of the path algebras of any two orientations of a Dynkin diagram are derived equivalent (this is a well-known fact and can be shown using BGP-reflection functors as introduced by Bernstein, Gelfand and Ponomarev [BGP]). Let Q have underlying diagram D_n. Since we are only interested in $D^b(\operatorname{mod} kQ)$ and its orbit category $\mathcal{C}_{kQ}$, we can, without loss of generality, assume that Q is a linear orientation of D_n and label its vertices in the following way:

$$\begin{tikzcd}
 & & & & (n-1)_+ \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-2) \arrow[u] \arrow[d] \\
 & & & & (n-1)_-
\end{tikzcd}$$

We write $D^b(\operatorname{mod} kQ) = D^b(\operatorname{mod} kD_n)$, and $\mathcal{C}_{kQ} = \mathcal{C}_{kD_n}$, since as noted above

$$D^b(\operatorname{mod} kQ) \cong D^b(\operatorname{mod} kQ') \text{ and } \mathcal{C}_{kQ} \cong \mathcal{C}_{kQ'}$$

for any other orientation Q' of the Dynkin diagram D_n. For any $n \geq 4$, the cluster category $\mathcal{C}_{kD_n}$ is called a *cluster category of finite Dynkin type D*. Figure 3.1 provides an illustration of the Auslander-Reiten quiver $\Gamma(D^b(\operatorname{mod} kD_n))$. For a general finite dimensional hereditary algebra $\mathcal{H}$, the functor $\tau^{-1}\Sigma$ is an auto-equivalence of the category $D^b(\operatorname{mod}\mathcal{H})$ and thus induces an action on the vertices of $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$. The Auslander-Reiten translation τ acts on the vertices of the Auslander-Reiten quiver by sending each vertex to its left-most neighbour, i.e. $\tau\colon (i, v) \mapsto (i-1, v)$ for all $i \in \mathbb{Z}$ and all vertices v of Q. Note how this is reflected in the way we draw the Auslander-Reiten quiver; for any indecomposable object m of $D^b(\operatorname{mod}\mathcal{H})$and its Auslander-Reiten translation τm, the corresponding vertices in $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$ are drawn on the same horizontal level. In Figure 3.1 we depict the action of the Auslander-Reiten translation by dashed arrows. The natural embedding of the category $\operatorname{mod}\mathcal{H}$ into the bounded derived category $D^b(\operatorname{mod}\mathcal{H})$ induces an embedding of the Auslander-Reiten quiver $\Gamma(\operatorname{mod}\mathcal{H})$ into $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$ and

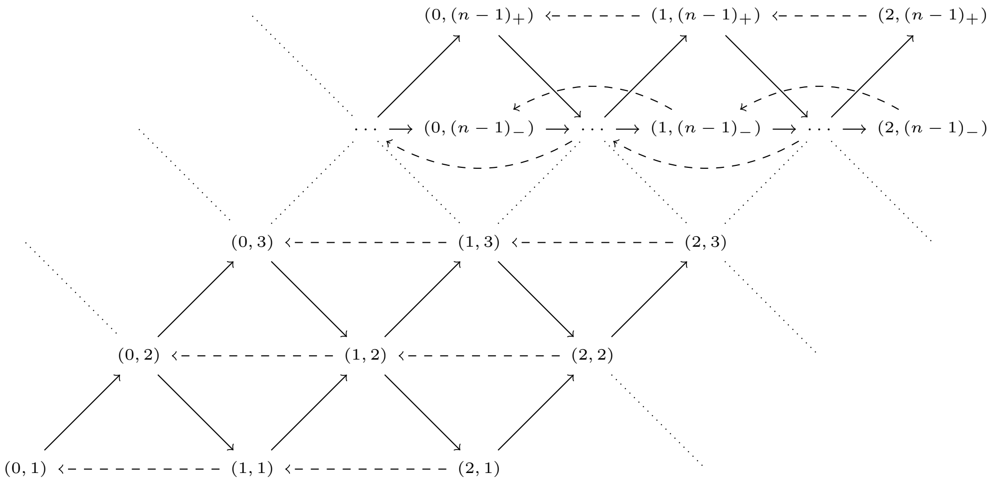


Figure 3.1: The Auslander-Reiten quiver $\Gamma(D^b(\operatorname{mod} kD_n))$ with the action of the Auslander-Reiten translation indicated by dashed arrows

we can view $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$ as being naturally covered by the shifted copies of $\Gamma(\operatorname{mod}\mathcal{H})$. This explains the action of Σ on vertices of $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$. Identifying the vertices in the orbits of $\tau^{-1}\Sigma$ on $\Gamma(D^b(\operatorname{mod}\mathcal{H}))$ gives rise to the Auslander-Reiten quiver $\Gamma(\mathcal{C}_{\mathcal{H}})$ of the cluster category. For an explicit combinatorial description of the action of $\tau^{-1}\Sigma$ on the vertices of $\Gamma(D^b(\operatorname{mod} kQ))$ for a simply laced Dynkin quiver Q, we refer the reader to Table 1 in Miyachi and Yekutieli's paper [MY] and restrict ourselves to the example of Dynkin type D_n. In $D^b(\operatorname{mod} kD_n)$ the auto-equivalence $\tau^{-1}\Sigma$ acts on the vertices of $\Gamma(D^b(\operatorname{mod} kD_n))$ as

$$\tau^{-1}\Sigma : \begin{cases} (i,j) \mapsto (i+n,j) \text{ for } 1 \leq j < i+n, \\ (i,(n-1)_{\pm}) \mapsto \begin{cases} (i+n,(n-1)_{\pm}) \text{ if } n \text{ is even,} \\ (i+n,(n-1)_{\mp}) \text{ if } n \text{ is odd.} \end{cases} \end{cases}$$

Note that this action depends on the parity of n.

We use the coordinate system induced from the one on $\Gamma(D^b(\operatorname{mod} kD_n))$ to label the vertices of $\Gamma(\mathcal{C}_{D_n})$, using 0 up to $(n-1)$ as first coordinates, i.e. choosing a fundamental domain as indicated in Figure 3.2.

3.2.2 Cluster categories and cluster algebras

Let Q be a finite connected acyclic quiver; more generally, we could take a finite connected quiver mutation equivalent to an acyclic quiver Q', as then the associated cluster algebras are equal, i.e. $\mathcal{A}_Q = \mathcal{A}_{Q'}$, and we can consider the cluster category $\mathcal{C}_{kQ'}$. A particularly nice feature of the cluster category $\mathcal{C}_{kQ}$ is that it models the combinatorial structure of the cluster algebra $\mathcal{A}_Q$ associated to the quiver Q. This means that we have analogues in the cluster category $\mathcal{C}_{kQ}$ for each of the basic combinatorial elements of the cluster algebra $\mathcal{A}_Q$: We find a natural concept of clusters, cluster variables and mutation in $\mathcal{C}_{kQ}$.

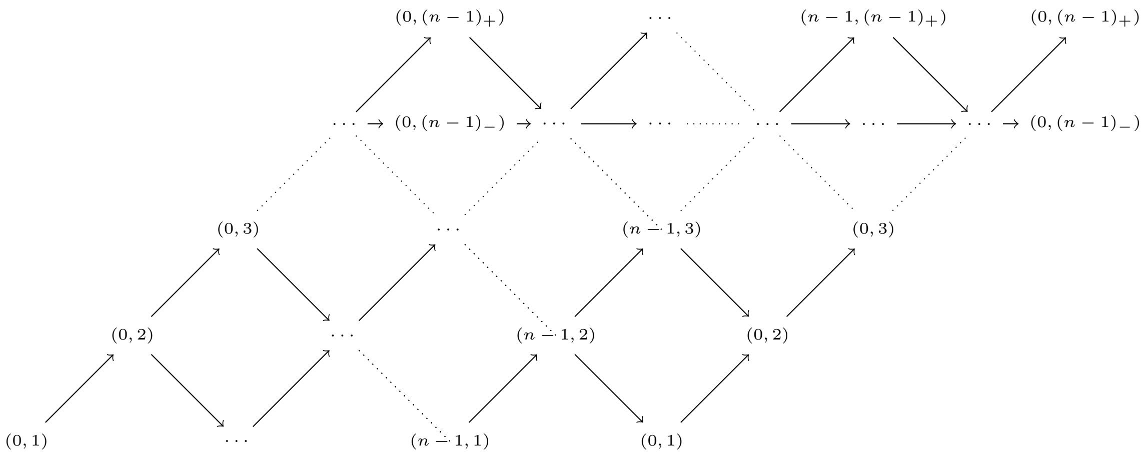


Figure 3.2: In this picture of the Auslander-Reiten quiver $\Gamma(\mathcal{C}_{kD_n})$, any two vertices with the same labelling are identified. Note that the morphisms "wrap around" and we can picture $\Gamma(\mathcal{C}_{kD_n})$ as lying on a cylinder.

First, there is a bijection between indecomposable objects in $\mathcal{C}_{kQ}$ and cluster variables of $\mathcal{A}_Q$. In general, an explicit description of this bijection is given by the Caldero-Chapoton map as introduced by Caldero and Chaperon [CC] for Q a simply laced Dynkin quiver and by Caldero and Keller [CK2] for Q acyclic. Those indecomposable objects that come from projective kQ-modules (via the composition of the natural embedding $\operatorname{mod} kQ \to D^b(\operatorname{mod} kQ)$ with the canonical projection $D^b(\operatorname{mod} kQ) \to \mathcal{C}_{kQ}$) correspond to cluster variables in the seed Σ_Q associated to Q. In the simply laced Dynkin case, the indecomposable objects of $\mathcal{C}_{kQ}$ and the cluster variables of $\mathcal{A}_Q$ are both in bijection with the almost positive roots (i.e. the union of the positive roots and the simple negative roots) of the simple Lie-algebra associated to the underlying diagram of Q, as shown by Fomin and Zelevinsky in [FZ2, Theorem 1.9].

The analogues of clusters in $\mathcal{C}_{kQ}$ are given by cluster tilting subcategories, as they are called by Buan, Iyama, Reiten and Scott in [BIRS], see also Iyama's paper [I], where they are called maximal 1-orthogonal subcategories, and [BMRRT], where, without the assumption of functorial finiteness, they are called Ext-configurations. A subcategory $X \subseteq T$ of a triangulated category T is called *functorially finite*, if for all $t \in T$ there exists a *right X-approximation*, i.e. a morphism $f\colon x \to t$ with $x \in X$ such that all morphisms from an object of X to t factor through f, and a *left X-approximation*, i.e. a morphism $g\colon t \to x'$ with $x' \in X$ such that all morphisms from t to an object of X factor through g. Let us introduce some notation before we give the definition of a cluster tilting subcategory.

Notation 3.2.3. For a subcategory X of a triangulated category T, we denote by $X^\perp$ the subcategory $X^\perp = \{t \in T \mid \operatorname{Hom}_T(x,t) = 0 \ \forall x \in X\}$ and dually by ${}^\perp X$ the subcategory ${}^\perp X = \{t \in T \mid \operatorname{Hom}_T(t,x) = 0 \ \forall x \in X\}$.

Definition 3.2.4. Let T be a triangulated category with shift functor Σ (recall that we assume it to be k-linear, Hom-finite and Krull-Schmidt.) A subcategory $X \subseteq T$ is called *rigid*, if $\mathrm{Ext}^1(x, y) = 0$ for all $x, y \in X$. It is called *maximal rigid*, if it is maximal with this property, i.e. if it is rigid and if $X \subseteq Y$ with Y a rigid subcategory of T, then $X = Y$. A subcategory $X \subseteq T$ is called a *cluster tilting subcategory of* T if it is functorially finite and $X = (\Sigma^{-1}X)^{\perp} = {}^{\perp}(\Sigma X)$.

Remark 3.2.5. While cluster tilting subcategories are always maximal rigid by definition, maximal rigid subcategories need not be cluster tilting. Counterexamples were given for example by Burban, Iyama, Keller and Reiten [BIKR] in stable categories of maximal Cohen-Macaulay modules over odd-dimensional isolated hypersurface singularities or by Buan, Marsh and Vatne [BMV] in cluster tubes.

If T is 2-Calabi-Yau – in particular if $T = \mathcal{C}_{kQ}$ – then by Serre duality we have $\mathrm{Ext}^1(x, y) \cong \mathrm{Ext}^1(y, x)^*$ for any two objects $x, y \in T$, so $(\Sigma^{-1}X)^{\perp} = {}^{\perp}(\Sigma X)$ holds true for any subcategory $X \subseteq T$.

The "tilting" in the name cluster tilting subcategory stems from tilting theory: A tilting module in the module category $\operatorname{mod} \mathcal{H}$ of a finite dimensional basic hereditary algebra $\mathcal{H}$ is a $\mathcal{H}$-module x with $\mathrm{Ext}^1_{\operatorname{mod} \mathcal{H}}(x, x) = 0$, and such that x has n non-isomorphic indecomposable summands, where n is the number of simple $\mathcal{H}$-modules, i.e. the number of vertices of Q if $\mathcal{H}$ is Morita equivalent to kQ. It has been shown in [BMRRT, Theorem 3.3] that every tilting module in $\operatorname{mod} kQ$ induces a cluster tilting subcategory of $\mathcal{C}_{kQ}$ (through the functor from $\operatorname{mod} kQ$ to $\mathcal{C}_{kQ}$ given by the composition of the natural inclusion $\operatorname{mod} kQ \to D^b(\operatorname{mod} kQ)$ and the natural projection $D^b(\operatorname{mod} kQ) \to \mathcal{C}_{kQ}$). On the other hand every cluster tilting subcategory of $\mathcal{C}_{kQ}$ is induced by a tilting module in some module category $\operatorname{mod} \mathcal{H}$, where $\mathcal{H}$ is a finite dimensional hereditary algebra derived equivalent to kQ. In particular, up to isomorphism there are n distinct indecomposable objects in a cluster tilting subcategory of $\mathcal{C}_{kQ}$. This number coincides with the cardinality of the clusters of $\mathcal{A}_Q$. Moreover, the Caldero-Chapoton map induces a bijection between cluster tilting subcategories of $\mathcal{C}_{kQ}$ and clusters of $\mathcal{A}_Q$. For example, the additive hull of the indecomposable objects that come from projective kQ-modules forms a cluster tilting subcategory of $\mathcal{C}_{kQ}$, since the projective modules give rise to a tilting object in $\operatorname{mod} kQ$. This subcategory corresponds to the cluster in the seed Σ_Q associated to Q.

If $X = \operatorname{add}\{x_1, \dots, x_n\}$ is a cluster tilting subcategory with mutually non-isomorphic indecomposable objects $x_1, \dots, x_n$, then by [BMRRT, Theorem 5.1] for every $i = 1, \dots, n$ there exists a up to isomorphism unique $x_i^* \neq x_i$, such that

$$\mu_{x_i}(X) := \operatorname{add}\{x_1, \dots, x_{i-1}, x_i^*, x_{i+1}, \dots, x_n\}$$

is again a cluster tilting subcategory. Thus we have a concept of mutation of cluster tilting subcategories in $\mathcal{C}_{\mathcal{H}}$.

Consider the case where Q is an orientation of a simply laced Dynkin quiver. As with any acyclic quiver Q, the indecomposable objects of $\mathcal{C}_{kQ}$ are in bijection with the cluster variables of $\mathcal{A}_Q$. In this case there are only finitely many cluster variables by [FZ2, Theorem 1.5]. Thus every subcategory of $\mathcal{C}_{kQ}$ is automatically functorially finite and in this situation the cluster tilting subcategories are just the maximal rigid subcategories of $\mathcal{C}_{kQ}$ (for example by [ZZ1, Theorem 2.6]). We can view them as maximal collections of pairwise non-isomorphic and mutually compatible indecomposable objects, where we say that two indecomposable objects $x, y \in \mathcal{C}_{kQ}$ are compatible if $\mathrm{Ext}^1_{\mathcal{C}_{kQ}}(x, y) = 0$. Then the cluster tilting subcategories of $\mathcal{C}_{kQ}$ are of the form $\mathrm{add}\, X$, where X is a maximal set of pairwise mutually compatible indecomposable objects of $\mathcal{C}_{kQ}$.

Remark 3.2.6. From Remark 2.2.24 in Chapter 2 we know that finite triangulations of the closed disc provide a combinatorial model for cluster algebras of finite Dynkin type A: Exchangeable cluster variables correspond to internal arcs of a fixed finite subset $\mathcal{Z} \subseteq S^1$ and clusters correspond to triangulations of the closed disc with marked points $\mathcal{Z}$. Via the bijection between cluster variables in $\mathcal{A}_Q$ and indecomposable objects in $\mathcal{C}_{kQ}$ this provides a combinatorial model for cluster categories of finite Dynkin type A, i.e. of cluster categories associated to a hereditary algebra kQ, where Q is an orientation of a Dynkin diagram A_n for some $n \geq 1$. In this case, compatibility of two objects is encoded by non-crossing of the corresponding arcs.

3.2.3 Cluster structures

The combinatorial structure on a category mirroring the structure of a cluster algebra can also be found in other categories besides the cluster categories associated to finite dimensional hereditary algebras. In particular, the cluster categories of infinite rank which we will consider in Section 3.3 require a more general approach to cluster categories. The basic combinatorial structure we want to have on triangulated categories to provide a categorification of cluster algebras is formalized by the concept of cluster structures on triangulated categories.

Definition 3.2.7 ([BIRS, Section I.1])**.** Let T be a triangulated category. (Recall that we assume it to be Hom-finite, k-linear and Krull-Schmidt.) A *cluster structure on* T is a collection of sets X of indecomposable objects, called *clusters*, such that the following axioms CS1, CS2, CS3 and CS4 are satisfied.

- CS1 For every cluster X and each indecomposable object $x \in X$ there exists a up to isomorphism unique indecomposable object $x^* \in T$ that is not isomorphic to x, such that $\mu_x(X) := (X \setminus \{x\}) \bigcup \{x^*\}$ is also a cluster. The pair (x, x^*) is called an *exchange pair* and we call the cluster $\mu_x(X)$ the *mutation of* X *at* x.

- CS2 Consider the subcategory $D := \mathrm{add}(X \setminus \{x\})$. For an exchange pair (x, x^*) as above there exist distinguished triangles

$$x^* \xrightarrow{f} d \xrightarrow{g} x \longrightarrow \Sigma x^*$$

and

$$x \xrightarrow{s} d' \xrightarrow{t} x^* \longrightarrow \Sigma x$$

with $d, d' \in D$ and where f and s are left D-approximations and g and t are right D-approximations. These distinguished triangles are called *exchange triangles.*

For any subcategory $C \subseteq T$ the quiver of C is defined as the quiver of its opposite endomorphism algebra $\mathrm{End}(C)^{op}$. Here, the endomorphism algebra of C is

$$\mathrm{End}(C) = \bigoplus_{x,y \in \mathrm{Ind}\, C} \mathrm{Hom}_C(x, y),$$

where Ind_C is a set of representatives of the isomorphism classes of indecomposable objects in C. The multiplication is induced by composition.

CS3 For every cluster X the quiver of the subcategory $\mathrm{add}\, X$ has no loops or 2-cycles.

CS4 The quiver of the subcategory $\mathrm{add}\, \mu_x(X)$ is the mutation of the quiver of $\mathrm{add}\, X$ at the vertex x (see Definition 2.2.13).

Axiom CS1 tells us that mutation of clusters is defined, and, analogously to mutation of clusters in cluster algebras, is given by uniquely replacing one indecomposable object (which represents a cluster variable). Axiom CS2 further provides an analogue of the exchange relations in the cluster algebra via exchange triangles. For a formal way of connecting the concept of exchange triangles in cluster categories associated to finite dimensional hereditary algebras to exchange relations of their algebraic counterparts, we refer the interested reader to the work of Caldero and Keller [CK1] and [CK2].

Axiom CS3 is needed for axiom CS4 to make sense – quiver mutation is only defined for quivers without loops or 2-cycles (i.e. those quivers that are associated to skew-symmetric matrices, cf. Remark 2.2.3). Axiom CS4 finally tells us that mutation of clusters in a cluster structure follows the same combinatorial rules as mutation of clusters in cluster algebras.

3.3 Cluster categories of infinite rank

In general, if a triangulated category T has cluster tilting subcategories whose quivers have no loops or 2-cycles, they form a cluster structure on T by [BIRS, Proposition I.1.6]. These cluster tilting subcategories may have infinitely many indecomposable objects up to isomorphism. If T has a cluster structure with clusters of infinite cardinality, we call T a *cluster category of infinite rank.* Cluster categories of infinite rank provide a categorical interpretation of cluster algebras of infinite rank. The examples in this section provide a categorical analogue for the cluster algebras of infinite rank coming from triangulations of the closed disc studied in Section 2.4.4.

3.3.1 A cluster category of infinite Dynkin type A

A nice concrete example of a cluster category of infinite rank has been studied by Holm and Jørgensen in [HJ]. Consider the quiver Q given by a sink-source orientation of the infinite Dynkin diagram A_∞:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \cdots$$

and the derived category $D^f(\mathrm{mod}\, kQ)$ of complexes of kQ-modules with finite dimensional total homology. Let Σ denote its shift functor and τ its Auslander-Reiten translation. Analogously to the case for finite acyclic quivers, we consider the category

$$\mathcal{C}_{A_\infty} := D^f(\mathrm{mod}\, kQ)\Big/\tau^{-1}\Sigma$$

which we call the *cluster category of infinite Dynkin type* A_∞. We recall from [HJ] that this is a 2-Calabi-Yau triangulated category. It was shown in [HJ, Section 5] that the cluster tilting subcategories of the category $\mathcal{C}_{A_\infty}$ form a cluster structure on $\mathcal{C}_{A_\infty}$. In [HJ, Section 3] a combinatorial model was introduced for the cluster structure on $\mathcal{C}_{A_\infty}$ via triangulations of the ∞-gon, by which we mean the integers equipped with their natural linear order. In keeping with the convention of Chapter 2 we describe the model via triangulations of the closed disc with marked points $\mathcal{Z} \subseteq S^1$, such that $\mathcal{Z}$ has exactly one limit point, e.g.

$$\mathcal{Z} = \{e\left(\frac{\pi}{m}\right) | m \in \mathbb{Z} \setminus \{0\}\} \subseteq S^1$$

which has its unique limit point at 1 (cf. Section 2.2.1 for notation). Note that the two combinatorial models are completely analogous – we obtain the ∞-gon by cutting S^1 with marked points $\mathcal{Z}$ at the limit point of $\mathcal{Z}$.

Let $\mathcal{Z} = \{e\left(\frac{\pi}{m}\right) | m \in \mathbb{Z} \setminus \{0\}\} \subseteq S^1$. The indecomposable objects in $\mathcal{C}_{A_\infty}$ are in bijection with the internal arcs of $\mathcal{Z}$, such that for objects $x, y \in \mathcal{C}_{A_\infty}$ we have $\mathrm{Ext}^1(x, y) = 0$ if and only if the arcs corresponding to x and y do not cross. As in finite Dynkin type A we have a notion of compatibility between two indecomposable objects (cf. Remark 3.2.6): Two indecomposable objects are compatible if the Ext^1-space between them vanishes, which is the case if and only if the corresponding arcs do not cross. The maximal rigid subcategories of $\mathcal{C}_{A_\infty}$ thus correspond to triangulations of the closed disc with marked points $\mathcal{Z}$. Unlike in finite Dynkin type A, not all maximal rigid subcategories of $\mathcal{C}_{A_\infty}$ are cluster tilting subcategories, since we do not get functorial finiteness for free. By [HJ, Theorem 4.4] the cluster tilting subcategories correspond to those triangulations that have either a nest or a fountain (cf. Definition 2.3.23). Triangulations with a split fountain correspond to maximal rigid subcategories that are not functorially finite. Mutation of cluster tilting subcategories corresponds to diagonal flips (cf. Figure 2.2).

Here we see very nicely from a combinatorial viewpoint why maximal rigid subcategories are not necessarily enough for a cluster structure: Consider for example the triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ from Remark 2.2.12. It has internal arcs given by

$$\mathcal{T}_{int} = \{\{e\left(\frac{\pi}{2}\right), e\left(\frac{\pi}{k}\right)\} | k \in \mathbb{Z}_{>3}\} \cup \{\{e\left(-\frac{\pi}{2}\right), e\left(-\frac{\pi}{k}\right)\} | k \in \mathbb{Z}_{>3}\} \cup \{\{e\left(\frac{\pi}{2}\right), e\left(-\frac{\pi}{2}\right)\}\}$$

(see Figure 2.1), i.e. $\mathcal{T}$ is the union of $\mathcal{T}_{int}$ and all edges of $\mathcal{Z} = \{e\left(\frac{\pi}{m}\right) | m \in \mathbb{Z}\setminus\{0\}\} \subseteq S^1$.

This is a triangulation with a split fountain and thus corresponds to a maximal rigid subcategory of $\mathcal{C}_{A_\infty}$ that is not functorially finite. Recall from Remark 2.2.12 that the arc $\alpha = \{e(\frac{\pi}{2}), e(-\frac{\pi}{2})\} \in \mathcal{T}$ is internal but not exchangeable. Thus the subcategory $\operatorname{add} X_{\mathcal{T}}$ of $\mathcal{C}_{A_\infty}$ with indecomposable objects $X_{\mathcal{T}}$ corresponding to $\mathcal{T}$ is not mutable at the indecomposable object $x_\alpha \in X_{\mathcal{T}}$ corresponding to α: The indecomposable x_α is (up to isomorphism) the only indecomposable object in ${}^{\perp}(\Sigma \operatorname{add}(X_{\mathcal{T}} \setminus \{x_\alpha\}))$ that is not already contained in $\operatorname{add}(X_{\mathcal{T}} \setminus \{x_\alpha\})$.

An algebraic interpretation of the cluster structure on the category $\mathcal{C}_{A_\infty}$ has been given in joint work with Grabowski [GG]: We classified cluster algebras associated to triangulations of the ∞-gon. Not all triangulations are mutation equivalent via finite admissible sequences of diagonal flips; in fact there are uncountably many mutation classes. They give rise to different cluster algebras. Briefly put, triangulations without a fountain or split fountain give rise to cluster algebra structures on the homogeneous coordinate ring of an infinite version of the Grassmannian (see [GG] for more detail) and other triangulations yield cluster structures on proper subalgebras of the coordinate ring. In light of our results from Chapter 2, they are all colimits of finite coproducts of cluster algebras of finite Dynkin type A (cf. Theorem 2.4.14).

3.3.2 Discrete and continuous cluster categories of Dynkin type A

Instead of just considering triangulations of the closed disc with marked points $\mathcal{Z}$, where $\mathcal{Z}$ has only one limit point, we could allow multiple limit points. We consider first the discrete case.

Definition 3.3.1. A subset $\mathcal{Z} \subseteq S^1$ is called *admissible*, if it is discrete, contains at least four points and satisfies the *two-sided limit condition*: For every sequence in $\mathcal{Z}$ converging to a limit point $a \in S^1$ from the left, there is a sequence in $\mathcal{Z}$ converging to a from the right and vice versa (see Definition 2.3.23 for terminology).

We discussed rooted cluster algebras associated to countably infinite triangulations of the closed disc – including those whose marked points form an admissible set – in Section 2.4.4. For admissible sets of marked points, Igusa and Todorov [IT3] provided the categorical analogue. Let $\mathcal{Z} \subseteq S^1$ be an admissible subset. The *discrete cluster category* $\mathcal{C}(\mathcal{Z})$ *of Dynkin type* A *associated to* $\mathcal{Z}$ is the k-linear, additive category with

- indecomposable objects given by internal arcs of $\mathcal{Z}$ and

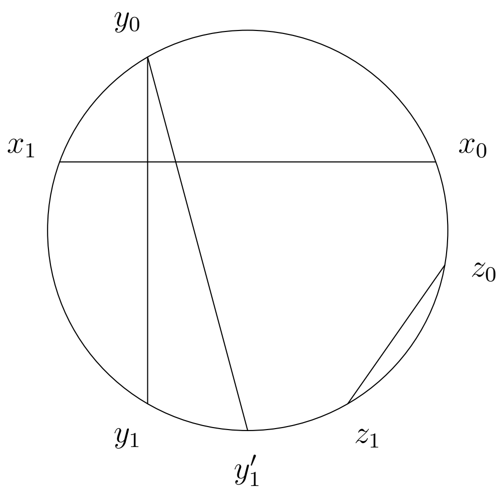


Figure 3.3: There are morphisms from $\{x_0, x_1\}$ to $\{y_0, y_1\}$ and vice versa and morphisms from $\{x_0, x_1\}$ to $\{y_0, y_1'\}$ and vice versa. There are morphisms from $\{y_0, y_1\}$ to $\{y_0, y_1'\}$, but none from $\{y_0, y_1'\}$ to $\{y_0, y_1\}$ and every morphism from $\{x_0, x_1\}$ to $\{y_0, y_1'\}$ factors through $\{y_0, y_1\}$. There are no morphisms in either direction between $\{x_0, x_1\}$ and $\{z_0, z_1\}$.

- morphisms between indecomposable objects given by

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{Z})}(\{x_0, x_1\}, \{y_0, y_1\}) \cong \begin{cases} k, & \text{if } y_0 \in [x_0, x_1) \text{ and } y_1 \in [x_1, x_0) \text{ or vice versa} \\ 0, & \text{otherwise,} \end{cases}$$

  such that

- a non-zero morphism in $\mathrm{Hom}_{\mathcal{C}(\mathcal{Z})}(\{x_0, x_1\}, \{y_0, y_1\})$ factors through $\{z_0, z_1\}$ if and only if $z_0 \in [x_0, y_0]$ and $z_1 \in [x_1, y_1]$.

Figure 3.3 provides an example of indecomposable objects in $\mathcal{C}(\mathcal{Z})$ and in the caption we describe if morphisms exist between them. Note that if two arcs cross, there are always non-zero morphisms in both directions between the two corresponding indecomposable objects.

Remark 3.3.2. Igusa and Todorov define discrete cluster categories in the slightly more general setting where $\mathcal{Z}$ is a cyclically ordered set with an admissible automorphism, see [IT3, Theorem 2.4.1] and [IT3, Lemma 2.4.12]. We restrict ourselves to those discrete cluster categories associated to admissible subsets of S^1, since they correspond to special cases of the ongoing example from Chapter 2 of rooted cluster algebras coming from triangulations of the closed disc. Further, for admissible subsets $\mathcal{Z} \subseteq S^1$, torsion pairs in $\mathcal{C}(\mathcal{Z})$ are classified in work in progress by Holm and Jørgensen. This will be of interest in Chapter 4, when we study mutation of torsion pairs in discrete cluster categories of Dynkin type A.

It can be seen directly from the combinatorial models that in the case where $\mathcal{Z}$ has exactly one limit point, the category $\mathcal{C}(\mathcal{Z})$ is isomorphic to the category $\mathcal{C}_{A_\infty}$ of infinite Dynkin type A_∞ described in Section 3.3.1, and when $|\mathcal{Z}| = n$ is finite, it is isomorphic to the classical cluster category of Dynkin type A_n (see Remark 3.2.6). Igusa and Todorov constructed the category $\mathcal{C}(\mathcal{Z})$ as the stable category of a Frobenius category and thus it is triangulated. This is where (implicitly) the admissibility condition on the subset $\mathcal{Z} \subseteq S^1$, and in particular the two-sided limit condition, is used. Recall that we did not have these restrictions in Chapter 2 when studying rooted cluster algebras associated to triangulations of the closed disc.

The discreteness and two-sided limit condition for $\mathcal{Z}$ imply that every point $x \in \mathcal{Z}$ has a successor $s(x)$, i.e. a unique point $s(x) \in \mathcal{Z}$, such that $\mathcal{Z} \cap (x, s(x)) = \emptyset$, and a predecessor $p(x)$, i.e. a unique point $p(x) \in \mathcal{Z}$, such that $\mathcal{Z} \cap (p(x), x) = \emptyset$. On objects, the shift functor is given by $\Sigma(\{x_0, x_1\}) = \{p(x_0), p(x_1)\}$ and we obtain (cf. Notation 3.2.2)

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{C}(\mathcal{Z})}(\{x_0, x_1\}, \{y_0, y_1\}) &= \operatorname{Hom}_{\mathcal{C}(\mathcal{Z})}(\{x_0, x_1\}, \Sigma\{y_0, y_1\}) \\ &\cong \begin{cases} k, & \text{if } \{x_0, x_1\} \text{ and } \{y_0, y_1\} \text{ cross} \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

It follows directly that $\operatorname{Ext}^1_{\mathcal{C}(\mathcal{Z})}(X, Y) \cong \operatorname{Ext}^1_{\mathcal{C}(\mathcal{Z})}(Y, X)$ for any two objects $X, Y \in \mathcal{C}(\mathcal{Z})$ and it was further shown in [IT3, Theorem 2.4.5] that the discrete cluster category $\mathcal{C}(\mathcal{Z})$ is 2-Calabi-Yau.

Adapting the methods from [HJ], it was shown in [IT3, Section 2.4.2] that $\mathcal{C}(\mathcal{Z})$ has a cluster structure with clusters given by those triangulations of the closed disc with marked points $\mathcal{Z}$, in which for every right-fountain converging to a limit point a, there is a left-fountain converging to a and vice versa (cf. Definition 2.3.23). Mutation is, as for cluster algebras associated to triangulations of the closed disc, given by diagonal flips (cf. Figure 2.2).

One can even take the construction one step further and consider a continuous set of marked points, e.g. $\mathcal{Z} = S^1$. In [IT1], Igusa and Todorov described a category which has precisely this underlying combinatorial model. We will not go into detail here, as this category is not relevant for the rest of this thesis. Let us just mention that it provides a nice example of a cluster category of infinite rank categorifying a cluster algebra of infinite rank which occurs as a colimit of cluster algebras of finite Dynkin type A. We refer the interested reader to [IT1] as well as [IT2] for a thorough explanation. The central idea is to consider the limit of the cluster categories of Dynkin type A_n as n goes to infinity. The *continuous cluster category $\mathcal{C}_\pi$ of Dynkin type A* is the k-linear, additive category whose indecomposable objects are given by arcs of S^1 and morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}_\pi}(\{x_0, x_1\}, \{y_0, y_1\}) \cong \begin{cases} k, & \text{if } y_0 \in [x_0, x_1) \text{ and } y_1 \in [x_1, x_0) \text{ or vice versa} \\ 0, & \text{otherwise,} \end{cases}$$

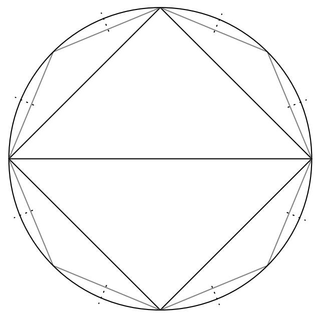


Figure 3.4: The standard cluster $\mathcal{T}_{st}$

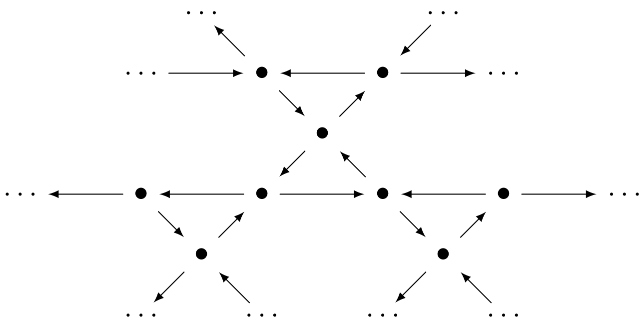


Figure 3.5: Quiver associated to the standard cluster $\mathcal{T}_{st}$

where a non-zero morphism in $\mathrm{Hom}_{\mathcal{C}_\pi}(\{x_0, x_1\}, \{y_0, y_1\})$ factors through $\{z_0, z_1\}$ if and only if $z_0 \in [x_0, y_0]$ and $z_1 \in [x_1, y_1]$. Again, Igusa and Todorov showed that this category is equivalent to the stable category of a Frobenius category and the category $\mathcal{C}_\pi$ is therefore triangulated. Igusa and Todorov showed that $\mathcal{C}_\pi$ has a cluster structure. The clusters are given by triangulations of the closed disc that are all equivalent to the so-called standard cluster, where we say that two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of the closed disc are equivalent, if there exists an orientation-preserving homeomorphism $\varphi\colon S^1 \to S^1$, such that $\varphi(\mathcal{T}_1) = \mathcal{T}_2$. The *standard cluster* is defined as the triangulation

$$\mathcal{T}_{st} = \{\left\{e\left(\frac{m\pi}{2^n}\right), e\left(\frac{(m+1)\pi}{2^n}\right)\right\} \mid n \geq 0, 0 \leq m < 2^{n+1}\},$$

see Figure 3.4. Mutation is, as usual, given by diagonal flips. Informally speaking, up to rearranging the endpoints of arcs in the triangulation while preserving their cyclic order, all clusters look the same. In particular, they all give rise to the same exchange quiver $Q_{\mathcal{T}_{st}}$, in the sense of Definition 2.2.10, see Figure 3.5. The quiver $Q_{\mathcal{T}_{st}}$ has a countably infinite set of vertices and every vertex of $Q_{\mathcal{T}_{st}}$ is a vertex in exactly two oriented three-cycles. A very interesting feature, that we can also observe when looking at diagonal flips in $\mathcal{T}_{st}$, is that mutation at any vertex leaves the quiver $Q_{\mathcal{T}_{st}}$ invariant.

The continuous cluster category $\mathcal{C}_\pi$ of Dynkin type A is the categorical version of the cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}_{st}})$, in the sense that they share the same combinatorics. Our work from Section 2.4.4 in Chapter 2 provides further support for Igusa and Todorov's idea that the continuous cluster category is a limit of cluster categories of Dynkin type A_n as

n goes to infinity: By Theorem 2.4.12 the rooted cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}_{st}})$ is isomorphic to a countable coproduct of linear colimits of cluster algebras of finite Dynkin type A. In fact, it is isomorphic to a linear colimit of rooted cluster algebras of finite Dynkin type A: We can construct the triangulation $\mathcal{T}_{st}$ by starting with a triangulation of the closed disc with four marked points and successively glueing on triangles to all the edges, thus really taking the limit in every direction. More precisely, we can construct a linear system of rooted cluster algebras $\mathcal{A}(\Sigma_{\mathcal{T}_n})$ for $n \geq 1$, where

$$\mathcal{T}_n = \{\{e\left(\frac{m\pi}{2^n}\right), e\left(\frac{(m+1)\pi}{2^n}\right)\} \mid 0 \leq m < 2^{n+1}\}$$

and the rooted cluster morphism $f_{mn}\colon \mathcal{A}(\Sigma_{\mathcal{T}_m}) \to \mathcal{A}(\Sigma_{\mathcal{T}_n})$ for $n \geq m$ is defined by the natural embedding (see Lemma 2.4.6). Then the rooted cluster algebra $\mathcal{A}(\Sigma_{st})$ is the colimit of this system and for all $n \geq 1$ the cluster algebra $\mathcal{A}(\Sigma_{\mathcal{T}_n})$ is of finite Dynkin type A (cf. the proof of Theorem 2.4.12).

Chapter 4

Mutation of torsion pairs

4.1 Introduction

In this chapter, we discuss mutation of torsion pairs in triangulated categories from a combinatorial perspective. An important motivation for mutation in triangulated categories stems from cluster theory. As we have seen in Chapter 3, cluster categories mimic the combinatorics of cluster algebras and we obtain a categorical interpretation of mutation using the categories' triangulated structures.

Iyama and Yoshino [IY] introduced a more general concept of mutation in triangulated categories. Every subcategory X of T can be mutated in two directions with respect to a rigid subcategory $D \subseteq T$, yielding two subcategories $\mu_D(X)$ and $\mu_D^-(X)$. The mutation of cluster tilting subcategories in the sense of Buan, Iyama, Reiten and Scott [BIRS] (see Definition 3.2.7) is a special case of this notion of mutation in a triangulated category.

Not all triangulated categories have cluster tilting subcategories, for example cluster tubes as shown by Buan, Marsh and Vatne [BMV, Corollary 2.7]. However, triangulated categories always contain torsion pairs, which were introduced by Iyama and Yoshino in [IY] and which we will discuss in more detail in Section 4.2.1: A torsion pair in a triangulated category T is a pair of subcategories (X, Y), such that there are no non-zero morphisms from X to Y and every object in T can be written as an extension of an object in Y by an object in X. This provides a triangulated version of the classical notion of torsion pairs in abelian categories due to Dickson [D].

Any torsion pair (X, Y) is defined uniquely by the subcategory X, which is called its torsion part or equivalently by the subcategory Y, which is called its torsion-free part. Cluster tilting subcategories can be viewed as a special case of torsion pairs, as every cluster tilting subcategory of T is the torsion part of a torsion pair in T (cf. Example 4.2.3). It is natural to ask how to define mutation of torsion pairs to provide a generalization of mutation of cluster tilting subcategories. It was shown by Zhou and Zhu [ZZ2] that if a triangulated category T (as usual we assume k-linearity, Hom-finiteness and the Krull-Schmidt property) has Auslander-Reiten triangles, then mutation of a torsion pair in T

in the sense of Iyama and Yoshino [IY] with respect to a suitably nice rigid subcategory D yields another torsion pair in T.

In Section 4.3 we provide a combinatorial model for mutation of torsion pairs in discrete cluster categories of Dynkin type A, relying on work in progress by Holm and Jørgensen, who classify torsion pairs in these categories. In the discrete cluster category $\mathcal{C}(\mathcal{Z})$ associated to an admissible subset $\mathcal{Z} \subseteq S^1$, torsion pairs are in one-to-one correspondence with nice Ptolemy diagrams of $\mathcal{Z}$. In general, we call a set of internal arcs of $\mathcal{Z}$ a diagram of $\mathcal{Z}$ and Ptolemy diagrams of $\mathcal{Z}$ are diagrams of $\mathcal{Z}$ satisfying a certain combinatorial property: Roughly speaking, whenever two arcs in a Ptolemy diagram cross, their convex hull also needs to be contained in the Ptolemy diagram (see Section 4.3.1 for a precise definition). Holm and Jørgensen use the fact that the dimension of Ext^1-spaces between two indecomposable objects can be read off the combinatorial model by counting how many times (in this case either once or not at all) the arcs corresponding to the objects cross. This allows one to determine which diagrams of $\mathcal{Z}$ are associated to torsion parts of torsion pairs. They turn out to be Ptolemy diagrams and vice versa, every Ptolemy diagram represents the torsion part of a torsion pair. Just like mutation of a torsion pair in $\mathcal{C}(\mathcal{Z})$ depends on a nice rigid subcategory $D \subseteq \mathcal{C}(\mathcal{Z})$, mutation of the corresponding Ptolemy diagram of $\mathcal{Z}$ is defined with respect to the diagram $\mathscr{D}$ of $\mathcal{Z}$ corresponding to the subcategory D (cf. Definition 4.3.5). Since D is rigid, any such diagram $\mathscr{D}$ consists of pairwise non-crossing arcs and we geometrically define the mutations $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^{-}$ as mutually inverse bijections on internal arcs of $\mathcal{Z}$ that do not cross any arcs of $\mathscr{D}$. We show in Theorem 4.3.10 that mutation of Ptolemy diagrams of $\mathcal{Z}$ provides a combinatorial model for mutation of torsion pairs in the discrete cluster category $\mathcal{C}(\mathcal{Z})$. As we have seen in Section 3.3.2, the cluster categories of finite Dynkin type A and of infinite Dynkin type A_∞ (see Section 3.3.1) are special examples of discrete cluster categories of Dynkin type A. Our work in Section 4.3 generalizes results by Zhou and Zhu [ZZ2] who provide a combinatorial model for those two special cases, where they rely on the classification of torsion pairs in the cluster category of type A_∞ due to Ng [Ng] and of finite Dynkin type A due to Holm, Jørgensen and Rubey [HJR1].

In Section 4.4 we provide a combinatorial description of mutation of torsion pairs in cluster categories of finite Dynkin type D. The situation is more complicated than in type A, because we have to deal with the indecomposable objects in the cluster category which arise from the exceptional vertices of Dynkin diagrams of type D. Holm, Jørgensen and Rubey [HJR2] classified torsion pairs in the cluster category of Dynkin type D_n for $n \geq 4$ using Ptolemy diagrams of Dynkin type D_n. They used a combinatorial model for Dynkin type D_n which was first introduced by Fomin and Zelevinsky in [FZ3] and which is closely related to the model for the cluster category of Dynkin type D_n of Schiffler [Sch] using triangulations of the punctured disc. We will discuss this model in detail in Section 4.4.1: For the cluster category of Dynkin type D_n with $n \geq 4$ consider the regular $2n$-gon $\mathcal{P}_{2n}$. An arc of $\mathcal{P}_{2n}$ is a pair of vertices of $\mathcal{P}_{2n}$ and an arc that is invariant under rotation by π is called a diameter. We consider rotationally symmetric pairs of arcs and introduce two copies of each diameter; a red one and a green one. Then indecomposable objects in the cluster category $\mathcal{C}_{kD_n}$ are identified with rotationally symmetric pairs of arcs and red and green diameters in the regular $2n$-gon. Subcategories (which as usual are assumed to be full and closed under isomorphisms, direct summands and finite direct sums) thus correspond to collections of arcs, that are invariant under rotation by π and which we call diagrams of Dynkin type D_n.

It was shown by Holm, Jørgensen and Rubey in [HJR2] that torsion parts of torsion pairs in $\mathcal{C}_{kD_n}$ correspond to diagrams of Dynkin type D_n with a distinctive combinatorial property, called Ptolemy diagrams of Dynkin type D_n (see Section 4.4.2). They resemble Ptolemy diagrams of $\mathcal{Z}$ for admissible subsets $\mathcal{Z} \subseteq S^1$ and an integral part of the idea for the classification is again that the dimension of Ext^1-spaces between two indecomposable objects can be read off from the combinatorial model by counting the number of times the corresponding pairs of arcs, respectively diameters cross. In this case, we can have Ext^1-spaces of dimension zero, one or two. Again, we define mutation of a Ptolemy diagram of Dynkin type D_n with respect to a subdiagram $\mathscr{D}$ corresponding to a rigid subcategory D. Any such subdiagram $\mathscr{D}$ consists of pairwise non-crossing arcs and divides the $2n$-gon into convex polygons which we call $\mathscr{D}$-cells of Dynkin type D_n. As in discrete Dynkin type A, we define the mutations $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^-$ as mutually inverse bijections on arcs that do not cross any arcs of $\mathscr{D}$ (cf. Definition 4.4.15). Essentially, the mutations $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^-$ can be thought of as rotating the arcs within each of the $\mathscr{D}$-cells of Dynkin type D_n in a clockwise respectively anticlockwise direction. We show in Theorem 4.3.10 that mutation of Ptolemy diagrams of Dynkin type D_n provides a combinatorial model for mutation of torsion pairs in the cluster category $\mathcal{C}_{kD_n}$.

Conventions We still use the same conventions as in Chapter 3. For instance, recall that all triangulated categories are assumed to be k-linear, Hom-finite and Krull-Schmidt and all subcategories are assumed to be full and closed under isomorphisms, direct summands and finite direct sums. In particular this means that to describe a subcategory X of a triangulated category T, it is sufficient to identify its indecomposable objects up to isomorphism.

4.2 Torsion pairs and mutation in triangulated categories

We have seen in Chapter 3 how certain triangulated categories can be used to model the combinatorial structure of cluster algebras. Classically, the cluster structure is given by cluster tilting subcategories. However, not all triangulated categories have cluster tilting subcategories (for example cluster tubes, as shown by Buan, Marsh and Vatne [BMV]), but they always allow torsion pairs (for example the trivial one $(T, 0)$). Every cluster tilting subcategory gives rise to a torsion pair (as we will see in Example 4.2.3), so we can view torsion pairs as a generalization of cluster tilting subcategories. Before we study mutation of torsion pairs in discrete cluster categories of Dynkin type A and in cluster categories of finite Dynkin type D in Sections 4.3 and 4.4, we review the general concepts of torsion pairs and mutation in triangulated categories.

4.2.1 Torsion pairs in triangulated categories

Torsion pairs in triangulated categories were introduced by Iyama and Yoshino in [IY]. From now on let T be a triangulated category with shift functor Σ and recall that we assume it to be k-linear, Hom-finite and Krull-Schmidt.

Definition 4.2.1 ([IY, Definition 2.2]). A *torsion pair* in T is a pair (X, Y) of subcategories of T such that

TP1 $\mathrm{Hom}_T(x, y) = 0$ for all $x \in X$ and $y \in Y$.

TP2 For each $t \in T$ there exists a distinguished triangle

$$x \to t \to y \to \Sigma x$$

with $x \in X$ and $y \in Y$.

In a torsion pair (X, Y), the subcategory X is called the *torsion part* and the subcategory Y is called the *torsion-free part* of (X, Y).

Remark 4.2.2. Definition 4.2.1 provides a triangulated version of torsion pairs in abelian categories as introduced by Dickson [D]. The terminology originates in the notion of a torsion pair in the special case of the abelian category $\mathrm{mod}\,\mathbb{Z}$ of finitely generated abelian groups. Every finitely generated abelian group is a (split) extension of a torsion-free group by a torsion one. Furthermore, there are no maps from torsion abelian groups to torsion-free abelian groups. So the finitely generated torsion abelian groups and the finitely generated torsion-free abelian groups form a torsion pair in the abelian category $\mathrm{mod}\,\mathbb{Z}$.

Before we provide an example and a short proof of some well-known properties, we introduce some terminology: A subcategory $X \subseteq T$ is called *contravariantly finite*, if every $t \in T$ has a *right X-approximation*, i.e. a morphism $f : x \to t$ with $x \in X$, such that every morphism from an object of X into t factors through f. It is called *covariantly finite*, if every $t \in T$ has a *left X-approximation*, i .e. a morphism $g : t \to x'$ with $x' \in X$, such that every morphism from t into an object of X factors through g. Recall (from the paragraph before Notation 3.2.3) that the subcategory $X \subseteq T$ is called *functorially finite*, if it is both contravariantly and covariantly finite.

Example 4.2.3. Every cluster tilting subcategory $X \subseteq T$ (see Definition 3.2.4) gives rise to a torsion pair $(X, X^{\perp})$ (cf. Notation 3.2.3): Axiom TP1 is satisfied by definition. Let now $t \in T$. Because X is a cluster tilting subcategory, it is functorially finite and thus there exists a right X-approximation $f\colon x \to t$, which we complete to a triangle:

$$x \to t \to y \to \Sigma x.$$

For any $\tilde{x} \in X$, applying $\mathrm{Hom}_T(\tilde{x}, -)$ to the triangle yields the long exact sequence

$$\ldots \to \mathrm{Hom}_T(\tilde{x}, x) \to \mathrm{Hom}_T(\tilde{x}, t) \to \mathrm{Hom}_T(\tilde{x}, y) \to \mathrm{Hom}_T(\tilde{x}, \Sigma x) \to \ldots$$

Because X is cluster tilting, we have $\mathrm{Hom}_T(\tilde{x}, \Sigma x) = 0$ and because $f\colon x \to t$ is a right X-approximation, the morphism $\mathrm{Hom}_T(\tilde{x}, x) \to \mathrm{Hom}_T(\tilde{x}, t)$ is surjective. It follows that $\mathrm{Hom}_T(\tilde{x}, y) = 0$ for all $\tilde{x} \in X$ and thus $y \in X^{\perp}$. Therefore, axiom TP2 is satisfied.

The following lemma summarizes some well-known and useful properties for torsion pairs. As we could find no convenient reference for all of the statements given, we include a proof for the convenience of the reader.

Lemma 4.2.4. *Let (X, Y) be a torsion pair in T. Then the following hold.*

(i) $Y = X^{\perp}$*, so the torsion-free part is uniquely determined by the torsion part.*

(ii) $X = {}^{\perp}Y$*, so the torsion part is uniquely determined by the torsion-free part.*

(iii) X *is contravariantly finite and extension closed.*

(iv) Y *is covariantly finite and extension closed.*

(v) $X = {}^{\perp}(X^{\perp})$ *and* $Y = ({}^{\perp}Y)^{\perp}$*.*

Proof. (i) It follows directly from axiom TP1 that $Y \subseteq X^{\perp}$. Let now $t \in X^{\perp}$. By axiom TP2 there exists a distinguished triangle

$$x \xrightarrow{f} t \longrightarrow y \longrightarrow \Sigma x$$

with $x \in X$ and $y \in Y$. Because $t \in X^{\perp}$, we have $f = 0$ and thus $y \cong t \oplus \Sigma x$ and because Y is closed under direct summands, we have $t \in Y$.

(ii) This is proved dually to (i).

(iii) Let $t \in T$. By axiom TP2, there exists a distinguished triangle

$$x \xrightarrow{f} t \longrightarrow y \longrightarrow \Sigma x$$

with $x \in X$ and $y \in Y$. For any $\tilde{x} \in X$, applying $\mathrm{Hom}_T(\tilde{x}, -)$ to the triangle yields the long exact sequence

$$\ldots \to \mathrm{Hom}_T(\tilde{x}, x) \to \mathrm{Hom}_T(\tilde{x}, t) \to \mathrm{Hom}_T(\tilde{x}, y) \to \mathrm{Hom}_T(\tilde{x}, \Sigma x) \to \ldots$$

Because by axiom TP1 $\mathrm{Hom}_T(\tilde{x}, y) = 0$, the morphism $\mathrm{Hom}_T(\tilde{x}, x) \to \mathrm{Hom}_T(\tilde{x}, t)$ is surjective, i.e. every morphism from $\tilde{x}$ to t factors through f. Thus f is a right X-approximation and X is contravariantly finite.

Consider now a distinguished triangle

$$\tilde{x} \longrightarrow t \longrightarrow \tilde{x}' \longrightarrow \Sigma\tilde{x}$$

with $\tilde{x}, \tilde{x}' \in X$. For a $y \in Y$, applying $\mathrm{Hom}_T(-, y)$ to the triangle yields the long exact sequence

$$\ldots \to \mathrm{Hom}_T(\Sigma\tilde{x}, y) \to \mathrm{Hom}_T(\tilde{x}', y) \to \mathrm{Hom}_T(t, y) \to \mathrm{Hom}_T(\tilde{x}, y) \to \ldots$$

By axiom TP1, we have $\mathrm{Hom}_T(\tilde{x}', y) = 0 = \mathrm{Hom}_T(\tilde{x}, y)$ and therefore $\mathrm{Hom}_T(t, y) = 0$. Therefore $t \in {}^{\perp}Y$, which by part (ii) is equal to X, and thus X is extension closed.

(iv) This is proved dually to (iii).

(v) This follows directly from (i) and (ii).

□

Remark 4.2.5. By Lemma 4.2.4, every torsion pair in T is of the form $(X, X^{\perp})$, where $X \subseteq T$ is a contravariantly finite and extension closed subcategory. Instead of thinking of a torsion pair as a pair of subcategories, it is common to just think about it in terms of its torsion part (or equivalently its torsion-free part). It is for purely historical reasons (cf. Remark 4.2.2) that the term torsion pair prevails.

In fact, conditions (iii) and (v) in Lemma 4.2.4 are not only necessary, but sufficient. Iyama and Yoshino prove the following:

Proposition 4.2.6 ([IY, Proposition 2.3])**.** *A subcategory $X \subseteq T$ is the torsion part of a torsion pair if and only if it is a contravariantly finite subcategory with ${}^{\perp}(X^{\perp}) = X$.*

Studying torsion pairs in T thus boils down to studying contravariantly finite subcategories X with ${}^{\perp}(X^{\perp}) = X$. This point of view turns out to be particularly useful when classifying torsion pairs in those cluster categories of Dynkin type which have combinatorial models via triangulations of surfaces with marked points. These are generally modelled such that dimensions of Ext^1-spaces can be read off by counting crossings of arcs. The condition ${}^{\perp}(X^{\perp}) = X$ can, in the examples we provide in the following chapters, be nicely translated into certain configurations of arcs in the respective combinatorial models.

Example 4.2.7. Recall from Remark 2.2.24 that, for $n \geq 1$, triangulations of the closed disc with marked points $\mathcal{Z}$, where $|\mathcal{Z}| = n+3$ provide a combinatorial model for a cluster algebra of Dynkin type A_n. The exchangeable cluster variables correspond to internal arcs of $\mathcal{Z}$, clusters correspond to triangulations of the closed disc with marked points $\mathcal{Z}$ and mutations to diagonal flips. Analogously, this model works for the cluster category $\mathcal{C}_{kA_n}$ (which encodes the coefficient-free cluster algebra of Dynkin type A_n): Indecomposable objects correspond to internal arcs of $\mathcal{Z}$ and cluster tilting subcategories to triangulations of the closed disc with marked points $\mathcal{Z}$. Further, the dimension of the Ext^1-space between two indecomposable objects can be read off directly from the geometric picture: It is one-dimensional if the corresponding arcs cross and zero otherwise.

If $X \subseteq \mathcal{C}_{kA_n}$ is a subcategory whose indecomposable objects correspond to a set of arcs $\mathscr{X}$, then ${}^{\perp}(\Sigma X) = (\Sigma^{-1}X)^{\perp}$ (the two are equal, since $\mathcal{C}_{kA_n}$ is 2-Calabi-Yau, see for example Remark 3.2.5) has as indecomposable objects all those corresponding to internal arcs that do not cross any of the arcs in $\mathscr{X}$ and these will be denoted by

$$\mathrm{nc}\,\mathscr{X} = \{\alpha \text{ an internal arc of } \mathcal{Z} \mid \alpha \text{ does not cross any arc in } \mathscr{X}\}.$$

The condition ${}^{\perp}(X^{\perp}) = X$ translates to the combinatorial condition $\mathrm{nc}(\mathrm{nc}\,\mathscr{X}) = \mathscr{X}$ by [HJR1, Proposition 2.3]. Since there are only finitely many indecomposable objects (up to isomorphism) in $\mathcal{C}_{kA_n}$ and $\mathcal{C}_{kA_n}$ is Hom-finite, every subcategory of $\mathcal{C}_{kA_n}$ is functorially finite. The subcategories of $\mathcal{C}_{kA_n}$ that are the torsion part of a torsion pair are thus precisely those that correspond to a set of arcs $\mathscr{X}$ with $\mathrm{nc}(\mathrm{nc}\,\mathscr{X}) = \mathscr{X}$.

Holm, Jørgensen and Rubey [HJR1, Theorem A] classified torsion pairs in $\mathcal{C}_{kA_n}$ using this idea. They showed that torsion pairs are in a one-to-one correspondence with so-called Ptolemy diagrams of $\mathcal{Z}$, with $|\mathcal{Z}| = n+3$: A subcategory $X \subseteq \mathcal{C}_{kA_n}$ is the torsion part of a torsion pair if and only if the corresponding set of arcs $\mathscr{X}$ is a Ptolemy diagram of $\mathcal{Z}$. A set of arcs $\mathcal{P}$ of an admissible subset $\mathcal{Z} \subseteq S^1$ (see Definition 3.3.1) is called a *Ptolemy diagram of* $\mathcal{Z}$ if it satisfies the *Ptolemy condition*: For every two arcs $\{x_0, x_1\}, \{y_0, y_1\} \in \mathcal{P}$ such that $\{x_0, x_1\}$ and $\{y_0, y_1\}$ cross, every arc in their *convex hull*, i.e. in the collection of arcs $\{x_0, y_0\}, \{y_0, x_1\}, \{x_1, y_1\}$ and $\{y_1, x_0\}$ of $\mathcal{Z}$, is an edge of $\mathcal{Z}$ or contained in $\mathcal{P}$ (see Section 4.3.1 for more details). This concept was inspired by work of Ng [Ng], who classified torsion pairs in the cluster category $\mathcal{C}_{A_\infty}$ of infinite Dynkin type A_∞. We will look at torsion pairs in discrete cluster categories of Dynkin type A more generally in Section 4.3, which will include both finite Dynkin type A and infinite Dynkin type A_∞ as special cases.

In this example we see combinatorially how cluster tilting subcategories are torsion parts of torsion pairs in $\mathcal{C}_{kA_n}$: They correspond to triangulations of the closed disc with marked points $\mathcal{Z}$ with $|\mathcal{Z}| = n+3$. Triangulations are Ptolemy diagrams of $\mathcal{Z}$, since the Ptolemy condition is satisfied trivially as no two arcs in a triangulation cross.

4.2.2 Mutation in triangulated categories

Mutation in triangulated categories with respect to rigid subcategories has been studied by Iyama and Yoshino [IY] and in cluster categories provides a generalization of mutation of cluster tilting subcategories. Zhou and Zhu [ZZ2] have shown that applying this general mutation with respect to a suitably nice rigid subcategory to a torsion pair produces another torsion pair.

Definition 4.2.8 ([IY, Definition 2.5])**.** Fix a rigid subcategory D of T (see Definition 3.2.4). For a subcategory $M \subseteq T$, the *mutations of M with respect to D* are the subcategories

1. $\mu_D^-(M)$ of objects $t \in {}^\perp(\Sigma D)$ such that there exists a distinguished triangle

$$m \xrightarrow{f} d \longrightarrow t \longrightarrow \Sigma m$$

   with $m \in M$ and $d \in D$.

2. $\mu_D(M)$ of objects $t \in (\Sigma^{-1} D)^\perp$ such that there exists a distinguished triangle

$$t \longrightarrow d \xrightarrow{g} m \longrightarrow \Sigma t$$

   with $m \in M$ and $d \in D$.

A pair (M, N) of subcategories $M, N \subseteq T$ is called a *D-mutation pair* if

$$D \subseteq N \subseteq \mu_D^-(M) \quad \text{and} \quad D \subseteq M \subseteq \mu_D(N).$$

Remark 4.2.9. The assumption in 1. of Definition 4.2.8 that $t \in {}^\perp(\Sigma D)$ is in fact equivalent to asking that the morphism f in the distinguished triangle is a left D-approximation: Consider any distinguished triangle

$$m \xrightarrow{f} d \longrightarrow t \longrightarrow \Sigma m$$

with $m \in M$ and $d \in D$. For any $\tilde{d} \in D$, applying $\mathrm{Hom}_T(-, \tilde{d})$ to the triangle yields the long exact sequence

$$\ldots \to \mathrm{Hom}_T(t, \tilde{d}) \to \mathrm{Hom}_T(d, \tilde{d}) \to \mathrm{Hom}_T(m, \tilde{d}) \to \mathrm{Hom}_T(t, \Sigma\tilde{d}) \to \mathrm{Hom}_T(d, \Sigma\tilde{d}) \to \ldots$$

Because D is rigid, $\mathrm{Hom}_T(d, \Sigma\tilde{d}) = 0$ and so the morphism $\mathrm{Hom}_T(m, \tilde{d}) \to \mathrm{Hom}_T(t, \Sigma\tilde{d})$ is surjective. Thus $\mathrm{Hom}_T(t, \Sigma\tilde{d}) = 0$ if and only if the morphism $\mathrm{Hom}_T(d, \tilde{d}) \to \mathrm{Hom}_T(m, \tilde{d})$ is surjective. Thus, since $\tilde{d} \in D$ was chosen arbitrarily, $t \in {}^\perp(\Sigma D)$ if and only if f is a left D-approximation.

Dually, the assumption in 2. of Definition 4.2.8 that $t \in (\Sigma^{-1} D)^\perp$ is equivalent to asking that the morphism g in the distinguished triangle is a right D-approximation.

Example 4.2.10. Assume that T has cluster tilting subcategories which form a cluster structure and let X be a cluster. The subcategory $\operatorname{add} X$ is cluster tilting, and in particular rigid. Let $x \in X$ be any indecomposable object in X and consider the subcategory $D := \operatorname{add}(X \setminus x) \subseteq \operatorname{add} X$, which, as a subcategory of a rigid subcategory, is also rigid.

Then comparing Definition 4.2.8 with Definition 3.2.7, CS2, the mutation of add X with respect to D corresponds to the mutation of X at x, i.e.

$$\mathrm{add}(\mu_x(X)) = \mu_D(X) = \mu_D^-(X).$$

In this sense, we can view mutation in triangulated categories as a generalization of mutation of cluster tilting subcategories. Note that in this special case, the mutations $\mu_D(X)$ and $\mu_D^-(X)$ are equal. This is not generally the case (see for example Figure 4.14).

The following statement is well-known, but we give a short proof for the convenience of the reader.

Lemma 4.2.11. *Mutation with respect to a rigid subcategory $D \subseteq T$ leaves D invariant, i.e. we have $\mu_D(D) = \mu_D^-(D) = D$.*

Proof. Let $d \in D$ and let

$$d \longrightarrow d' \longrightarrow t \xrightarrow{0} \Sigma d$$

be a distinguished triangle with $d' \in D$ and $t \in {}^{\perp}(\Sigma D)$. Since the map $t \to \Sigma d$ is zero, the triangle splits and we get $d' \cong d \oplus t$ and because every subcategory is assumed to be closed under direct summands, we have $t \in D$. Thus we have $\mu_D^-(D) \subseteq D$. On the other hand for any object $\tilde{d}$ in D, the distinguished triangle $0 \longrightarrow \tilde{d} = \tilde{d} \longrightarrow 0$ exists and thus we have equality; $\mu_D^-(D) = D$. Dually one shows that $\mu_D(D) = D$. □

Remark 4.2.12. By [IY, Proposition 2.6], for any D-mutation pair (M, N) in T we have $M = \mu_D(N)$ and $N = \mu_D^-(M)$. This implies that the mutations μ_D and μ_D^- are mutually inverse, i.e. $\mu_D(\mu_D^-(M)) = M$ and $\mu_D^-(\mu_D(N)) = N$.

We are interested in mutating torsion pairs by mutating the torsion part and the torsion-free part in the sense of Definition 4.2.8. First, we observe a useful fact. Let $X \subseteq T$ be a subcategory. Then the subcategory $X \cap (\Sigma^{-1}X)^{\perp}$ of T (and thus any of its subcategories) is automatically rigid: For $x, y \in X \cap (\Sigma^{-1}X)^{\perp}$ we have

$$\mathrm{Ext}^1_T(x, y) = \mathrm{Hom}_T(x, \Sigma y) \cong \mathrm{Hom}_T(\Sigma^{-1}x, y) = 0,$$

since $x \in X$ and $y \in (\Sigma^{-1}X)^{\perp}$. Thus mutation in T in the sense of Definition 4.2.8 is defined with respect to any subcategory of $X \cap (\Sigma^{-1}X)^{\perp}$. Note further, that the shift ΣD of any rigid subcategory $D \subseteq T$ is again rigid, since for all objects $x, y \in T$ we have $\mathrm{Ext}^1_T(\Sigma x, \Sigma y) \cong \mathrm{Ext}^1_T(x, y)$. Assume that T has Auslander-Reiten triangles and Auslander-Reiten translation τ. Zhou and Zhu prove the following.

Theorem 4.2.13 ([ZZ2, Theorem 3.8])**.** *Let (X, Y) be a torsion pair in T and let $D \subseteq X \cap (\Sigma^{-1}X)^{\perp} \subseteq T$ be a functorially finite subcategory satisfying $\tau D = \Sigma D$. Then the pairs of subcategories $\mu_D(X, Y) := (\mu_D(X), \mu_{\Sigma D}(Y))$ and $\mu_D^-(X, Y) := (\mu_D^-(X), \mu_{\Sigma D}^-(Y))$ are torsion pairs in T as well.*

Remark 4.2.14. The examples in which we study mutation of torsion pairs combinatorially (namely discrete cluster categories of Dynkin type A in Section 4.3 and cluster categories of finite Dynkin type D in Section 4.4) are all 2-Calabi-Yau (see Section 3.3.2 for the former and Remark 3.2.1 for the latter case). If a triangulated category T is 2-Calabi-Yau with Auslander-Reiten translation τ, then by Remark 3.2.1 every subcategory $D \subseteq T$ satisfies $\tau D = \Sigma D$. Thus if T is 2-Calabi-Yau, we can consider mutation of a torsion pair (X, Y) in T with respect to any functorially finite subcategory $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$.

Example 4.2.15. Consider the cluster category $\mathcal{C}_{kA_n}$ of Dynkin type A_n and its combinatorial model via arcs of $\mathcal{Z}$, where $\mathcal{Z} \subseteq S^1$ is a set of marked points on the boundary of the closed disc with $|\mathcal{Z}| = n+3$. Recall from Example 4.2.7, that a subcategory $X \subseteq \mathcal{C}_{kA_n}$ is the torsion part of a torsion pair (X, Y) if and only if the corresponding set of arcs $\mathscr{X}$ of $\mathcal{Z}$ is a Ptolemy diagram of $\mathcal{Z}$. Consider a rigid subcategory $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$. Because $\mathcal{C}_{kA_n}$ is Hom-finite with finitely many isomorphism classes of indecomposable objects, functorial finiteness is automatic and, as noted above, since $\mathcal{C}_{kA_n}$ is 2-Calabi-Yau, we have $\tau D = \Sigma D$. Consider the set of arcs $\mathscr{D}$ of $\mathcal{Z}$ corresponding to D. The fact that D is rigid translates to $\mathscr{D}$ consisting of pairwise non-crossing arcs of $\mathcal{Z}$. The subcategory $X \cap (\Sigma^{-1}X)^{\perp}$ corresponds to the set of arcs $\mathscr{X} \cap \operatorname{nc} \mathscr{X}$, thus $\mathscr{D} \subseteq \mathscr{X} \cap \operatorname{nc} \mathscr{X}$.

Zhou and Zhu [ZZ2] defined mutation of a Ptolemy diagram $\mathscr{X}$ of $\mathcal{Z}$ with respect to such subdiagrams $\mathscr{D} \subseteq \mathscr{X} \cap \operatorname{nc} \mathscr{X}$ consisting of mutually non-crossing arcs and showed that it provides a combinatorial model for mutation of torsion pairs. Geometrically, it can be interpreted as a generalization of diagonal flips. This is analogous to the way in which mutation in triangulated categories can be seen as a generalization of mutation of cluster tilting subcategories, cf. Example 4.2.10. Zhou and Zhu also provide a combinatorial realization of mutation of torsion pairs in the cluster category $\mathcal{C}_{A_\infty}$ of infinite Dynkin type A_∞. In Section 4.3 we provide a generalization of these results via a combinatorial model for mutation of torsion pairs in discrete cluster categories of Dynkin type A.

4.3 Torsion pairs and their mutation in discrete cluster categories of Dynkin type A

Work in progress by Holm and Jørgensen provides a combinatorial classification of torsion pairs in discrete cluster categories of Dynkin type A (cf. Section 3.3.2). This generalizes both the classification of torsion pairs in the cluster category $\mathcal{C}_{A_\infty}$ of infinite Dynkin type A_∞ by Ng [Ng] and in cluster categories of finite Dynkin type A by Holm, Jørgensen and Rubey [HJR1]. In this section, we provide an interpretation of mutation of torsion pairs in discrete cluster categories of Dynkin type A via a combinatorial model: We define mutation of Ptolemy diagrams of an admissible subset $\mathcal{Z} \subseteq S^1$ and prove that it corresponds to mutation of torsion pairs in the discrete cluster category $\mathcal{C}(\mathcal{Z})$ associated to $\mathcal{Z}$. This in turn generalizes work by Zhou and Zhu [ZZ2], who provided a combinatorial model for mutation of torsion pairs in the category $\mathcal{C}_{A_\infty}$ and in the cluster categories of finite Dynkin type A, as mentioned in Example 4.2.15.

4.3.1 Torsion pairs in discrete cluster categories of Dynkin type A

Throughout the rest of Section 4.3 let $\mathcal{Z} \subseteq S^1$ be an admissible subset (see Definition 3.3.1) and denote its set of edges by $\mathcal{E}(\mathcal{Z})$. We call any set $\mathscr{X}$ of internal arcs of $\mathcal{Z}$ a *diagram of* $\mathcal{Z}$, and a subset of $\mathscr{X}$ will be called a *subdiagram of* $\mathscr{X}$. Recall that the indecomposable objects of the discrete cluster category $\mathcal{C}(\mathcal{Z})$ are labelled by the internal arcs of $\mathcal{Z}$. Thus to any subcategory of $\mathcal{C}(\mathcal{Z})$ we can associate a diagram of $\mathcal{Z}$ consisting of the indecomposable objects of $\mathcal{Z}$.

Let X be a subcategory of the discrete cluster category $\mathcal{C}(\mathcal{Z})$ associated to $\mathcal{Z}$ and let $\mathscr{X}$ be the associated diagram of $\mathcal{Z}$. Recall from Proposition 4.2.6 that X is the torsion part of a torsion pair if and only if it is contravariantly finite and $X = {}^\perp(X^\perp)$. These conditions can be nicely translated into the combinatorial model.

Theorem 4.3.1 (Holm-Jørgensen)**.** *The subcategory X is contravariantly finite if and only if $\mathscr{X}$ satisfies the following condition:*

CF1 For every sequence $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs in $\mathscr{X}$ such that $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to $a \in S^1$ from the right and $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to $b \in S^1$ from the left or from the right, there is a sequence $\{a'_i, b'_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs in $\mathscr{X}$ such that $\{a'_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to a from the left and $\{b'_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to b from the left.

Dually one can show that the subcategory X is covariantly finite if and only if $\mathscr{X}$ satisfies the following condition:

CF2 For every sequence $\{a_i, b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs in $\mathscr{X}$ such that $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to $a \in S^1$ from the left and $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to $b \in S^1$ from the left or from the right, there is a sequence $\{a'_i, b'_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs in $\mathscr{X}$ such that $\{a'_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to a from the right and $\{b'_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges to b from the right.

It remains to translate the condition $X = {}^\perp(X^\perp)$ into the combinatorial model. Consider the diagram

$$\operatorname{nc}\mathscr{X} = \{\alpha \text{ an internal arc of } \mathcal{Z} \mid \alpha \text{ does not cross any arc in } \mathscr{X}\}$$

of $\mathcal{Z}$. Holm and Jørgensen showed that $X = {}^\perp(X^\perp)$ if and only if $\mathscr{X} = \operatorname{nc}(\operatorname{nc}\mathscr{X})$ and that this holds if and only if $\mathscr{X}$ satisfies the following two conditions:

(Ptolemy) If two arcs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ in $\mathscr{X}$ cross, then the arcs $\{x_0, y_0\}$, $\{y_0, x_1\}$, $\{x_1, y_1\}$ and $\{y_1, x_0\}$ of $\mathcal{Z}$ lie in $\mathscr{X} \cup \mathcal{E}(\mathcal{Z})$.

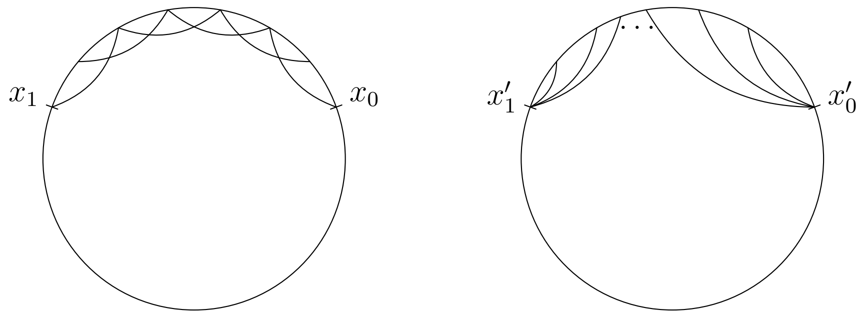


Figure 4.1: The diagram $\mathscr{X}$ on the left does not satisfy the Ptolemy condition, but satisfies CF1 and the diagram $\mathscr{X}'$ on the right does not satisfy CF1, but satisfies the Ptolemy condition. Neither satisfies the blocking condition: Every element in $(x_0, x_1) \cap \mathcal{Z}$ is $\{x_0, x_1\}$-blocked by $\mathscr{X}$, but $\{x_0, x_1\} \notin \mathscr{X}$ and every element in $(x'_0, x'_1) \cap \mathcal{Z}$ is $\{x'_0, x'_1\}$-blocked by $\mathscr{X}'$, but $\{x'_0, x'_1\} \notin \mathscr{X}'$.

(Blocking) Let $\{x_0, x_1\}$ be an internal arc of $\mathcal{Z}$ such that for every $z \in (x_0, x_1) \cap \mathcal{Z}$ there is an arc $\{y_0, y_1\} \in \mathscr{X}$ with $z \in (y_0, y_1) \subseteq (x_0, x_1)$. Then we have $\{x_0, x_1\} \in \mathscr{X}$.

If $\mathscr{X}$ satisfies the Ptolemy condition, then $\mathscr{X}$ is called a *Ptolemy diagram of* $\mathcal{Z}$. If $\{x_0, x_1\}$ is an arc of $\mathcal{Z}$ and for a $z \in (x_0, x_1) \cap \mathcal{Z}$ there is a $\{y_0, y_1\} \in \mathscr{X}$ with $z \in (y_0, y_1) \subseteq (x_0, x_1)$, the marked point z is called $\{x_0, x_1\}$*-blocked by* $\mathscr{X}$.

Theorem 4.3.2 (Holm-Jørgensen)**.** *The subcategory $X \subseteq \mathcal{C}(\mathcal{Z})$ is the torsion part of a torsion pair if and only if $\mathscr{X}$ satisfies conditions CF1, Ptolemy and Blocking.*

Not all diagrams of $\mathcal{Z}$ satisfy the blocking condition, cf. Figure 4.1. However, as the following lemma shows, we can omit the blocking condition from the assumptions in Theorem 4.3.2.

Lemma 4.3.3. *If $\mathscr{X}$ is a Ptolemy diagram of $\mathcal{Z}$ satisfying CF1, then $\mathscr{X}$ satisfies the blocking condition.*

To prove Lemma 4.3.3, we use the following helpful result. We observe that the cyclic order on S^1 induces a total order on any proper interval $[a, b] \subsetneq S^1$, and for $x, y \in [a, b]$ we write $x \leq_{[a,b]} y$ if and only if $[a, x] \subseteq [a, y]$.

Lemma 4.3.4. *If $\mathscr{X}$ is a diagram such that every right fountain in $\mathscr{X}$ is a subsequence of a fountain in $\mathscr{X}$, then for every arc $\{x_0, x_1\}$ of $\mathcal{Z}$ the extremum*

$$x_{max} = \max\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_0, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\}$$

with respect to the order $\leq_{[x_0,x_1]}$ exists. In particular, if $\mathscr{X}$ satisfies CF1, then every right fountain in $\mathscr{X}$ is a subsequence of a fountain in $\mathscr{X}$. If $\mathscr{X}$ is a diagram such that every left fountain in $\mathscr{X}$ is a subsequence of a fountain in $\mathscr{X}$, then for every arc $\{x_0, x_1\}$ of $\mathcal{Z}$ the extremum

$$x_{min} = \min\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_1, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\}$$

with respect to the order $\leq_{[x_0,x_1]}$ *exists. In particular, if* $\mathscr{X}$ *satisfies CF2, then every left fountain in* $\mathscr{X}$ *is a subsequence of a fountain in* $\mathscr{X}$.

Proof. First note that if CF1 holds for a diagram $\mathscr{X}$, then taking the sequence $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in CF1 to be constant yields that every right fountain in $\mathscr{X}$ must be a subsequence of a fountain in $\mathscr{X}$. Dually, if $\mathscr{X}$ satisfies CF2 then every left fountain in $\mathscr{X}$ must be a subsequence of a fountain in $\mathscr{X}$. We prove the existence of x_{max} under the assumption that every right fountain in $\mathscr{X}$ is the subsequence of a fountain, the existence of x_{min} under the dual assumption can be proved analogously.

The set $\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_0, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\}$ is non-empty, since it contains the successor $s(x_0)$ of x_0 (cf. Section 3.3.2 for terminology). If it is finite, it contains a maximal element, which proves the claim in this case. Else, if it is infinite, there is a left or a right fountain in $\mathscr{X}$ at x_0 converging to some $a \in (x_0, x_1)$. By assumption, for every right fountain in $\mathscr{X}$ at x_0 converging to a there is a left fountain in $\mathscr{X}$ at x_0 converging to a, so we have a left fountain at x_0 converging to a. Consider the set of such left fountains:

$$\mathcal{F} = \{b \in [x_0, x_1] \mid \text{there is a left fountain in } \mathscr{X} \text{ at } x_0 \text{ converging to } b\}.$$

The set $\mathcal{F}$ is closed as a subset of S^1: For every $x \in S^1 \setminus \mathcal{F}$ there is an open neighbourhood $x \in U \subseteq S^1$ with $U \cap \mathcal{F} = \emptyset$, since otherwise we would have $x \in [x_0, x_1]$ and there would be infinitely many arcs of the form $\{x_0, b_i\}$ with endpoints b_i in every open neighbourhood of x. Thus there would be a left fountain, or a right fountain and consequently also a left fountain, at x_0 converging to x and we would get $x \in \mathcal{F}$, which contradicts the assumption.

Thus $\mathcal{F}$ is non-empty, closed and bounded in S^1 and therefore has a maximum d. We have $d \neq x_1$, since $x_1 \in \mathcal{Z}$ and so, because $\mathcal{Z}$ is discrete (cf. Definition 3.3.1), x_1 cannot be the limit point of a non-constant sequence in $\mathcal{Z}$, i.e. there cannot be a fountain at x_0 converging to x_1. Consider now the set

$$\mathcal{M} = \{l \in \mathcal{Z} \cap (d, x_1] \mid \{x_0, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\}.$$

It is non-empty, since there is a left fountain in $\mathscr{X}$ at x_0 converging to d. Furthermore, it has a maximum: Else, because it is bounded, there would be a right fountain at x_0 in $\mathscr{X}$ converging to its supremum $\tilde{d} \in (d, x_1]$, and by assumption this would be a subsequence of a fountain in $\mathscr{X}$ converging to $\tilde{d}$, contradicting the maximality of d. Thus $\mathcal{M}$ contains a maximal element and we get $x_{max} = \max\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_0, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\} = \max \mathcal{M}$. □

We can now prove Lemma 4.3.3.

Proof. Let $\{x_0, x_1\}$ be an internal arc of $\mathcal{Z}$ such that every $z \in (x_0, x_1) \cap \mathcal{Z}$ is $\{x_0, x_1\}$-blocked by $\mathscr{X}$. Consider the set $\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_0, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\}$. By Lemma

4.3.4, it has a maximum a. Assume for a contradiction that $a \neq x_1$. Then $a \in (x_0, x_1)$ and by assumption it is $\{x_0, x_1\}$-blocked by $\mathscr{X}$. Thus there exists a $\{y_0, y_1\} \in \mathscr{X}$ with $a \in (y_0, y_1) \subseteq (x_0, x_1)$. Since $a \leq_{[x_0,x_1]} y_1$, by maximality of a we must have $y_0 \neq x_0$. However, in that case we have $y_0 \in (x_0, a)$ and $y_1 \in (a, x_1]$, so the arcs $\{x_0, a\}$ and $\{y_0, y_1\}$ cross. By the Ptolemy condition we have $\{x_0, y_1\} \in \mathscr{X}$ which contradicts the maximality of a. Therefore, we must have $x_1 = \max\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_0, l\} \in \mathscr{X} \cup \mathcal{E}(\mathcal{Z})\}$ and thus in particular $\{x_0, x_1\} \in \mathscr{X}$. $\square$

4.3.2 Mutation of torsion pairs in discrete cluster categories of Dynkin type A

In this section, we define mutation of diagrams of $\mathcal{Z}$ and prove that mutation of Ptolemy diagrams of $\mathcal{Z}$ provides a combinatorial model for mutation of torsion pairs in the discrete cluster category $\mathcal{C}(\mathcal{Z})$ associated to $\mathcal{Z}$. Let (X, Y) be a torsion pair in $\mathcal{C}(\mathcal{Z})$ and let $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$ be a functorially finite subcategory. Since $\mathcal{C}(\mathcal{Z})$ is 2-Calabi-Yau (cf. Section 3.3.2) we have $\tau D = \Sigma D$ (cf. Remark 4.2.14) and D, as a subcategory of the rigid subcategory $X \cap (\Sigma^{-1}X)^{\perp}$, is rigid. So, according to Theorem 4.2.13, mutating the torsion pair (X, Y) with respect to D yields the torsion pairs $(\mu_D(X), \mu_{\Sigma D}(Y))$ and $(\mu^-_D(X), \mu^-_{\Sigma D}(Y))$.

In order to obtain a combinatorial approach to mutation of torsion pairs we translate the situation into the combinatorial model: Let $\mathscr{X}$ be the Ptolemy diagram of $\mathcal{Z}$ corresponding to the torsion part X and let $\mathscr{D}$ be the diagram corresponding to the subcategory D. The fact that D is rigid translates to $\mathscr{D}$ being a *non-crossing diagram*, i.e. the arcs in $\mathscr{D}$ are pairwise non-crossing. Since the Ext^1-space between two objects vanishes if and only if the two corresponding arcs do not cross, the subcategory $(\Sigma^{-1}X)^{\perp} = {}^{\perp}(\Sigma X)$ corresponds to the diagram $\operatorname{nc}\mathscr{X}$ of $\mathcal{Z}$. Thus the fact that D is a subcategory of $X \cap (\Sigma^{-1}X)^{\perp}$ translates to $\mathscr{D}$ being a subdiagram of $\mathscr{X} \cap \operatorname{nc}\mathscr{X}$. Because D is functorially finite, the diagram $\mathscr{D}$ satisfies conditions CF1 and CF2 by Theorem 4.3.1 and the paragraph thereafter. In particular, by Lemma 4.3.4, it satisfies the following condition:

FF For every right (respectively left) fountain $\{a, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $\mathscr{D}$ at $a \in \mathcal{Z}$ converging to $b \in S^1$ there is a left (respectively right) fountain $\{a, b'_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $\mathscr{D}$ at a converging to b.

Let $\mathscr{D}$ be any non-crossing diagram of $\mathcal{Z}$ satisfying condition FF. In the following, we define mutation of a subdiagram $\mathscr{X} \subseteq \operatorname{nc}\mathscr{D}$ with respect to $\mathscr{D}$. Note that by Lemma 4.3.4 for every internal arc $\{x_0, x_1\}$ of $\mathcal{Z}$ the extrema

$$x_1^- = \min\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_1, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\}$$
$$x_1^+ = \max\{l \in \mathcal{Z} \cap [x_1, x_0] \mid \{x_1, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\}$$

exist.

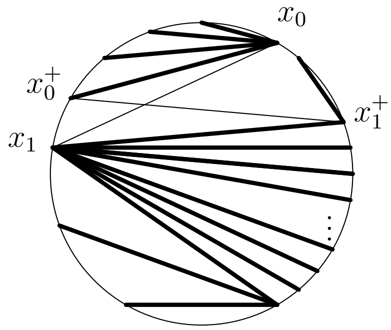


Figure 4.2: The arcs in the non-crossing diagram $\mathscr{D}$ are marked with thick lines. We picture the image $\mu_{\mathscr{D}}(\{x_0, x_1\}) = \{x_0^+, x_1^+\}$ of the element $\{x_0, x_1\}$ in $\operatorname{nc}\mathscr{D}$ under the map $\mu_{\mathscr{D}}$.

Definition 4.3.5. Let $\mathscr{D}$ be a non-crossing diagram of $\mathcal{Z}$ satisfying condition FF. We define the map

$$\mu_{\mathscr{D}}\colon \operatorname{nc}\mathscr{D} \to \{\text{arcs of } \mathcal{Z}\}$$

by setting

$$\mu_{\mathscr{D}}(\{x_0, x_1\}) = \{x_0^+, x_1^+\}.$$

Dually, we define the map

$$\mu_{\mathscr{D}}^-\colon \operatorname{nc}\mathscr{D} \to \{\text{arcs of } \mathcal{Z}\}$$

by setting

$$\mu_{\mathscr{D}}^-(\{x_0, x_1\}) = \{x_0^-, x_1^-\}.$$

Figure 4.2 provides an example for the map μ_D.

Remark 4.3.6. Note that $\mu_{\mathscr{D}}$ leaves any arc in $\mathscr{D}$ invariant, since for $\{x_0, x_1\} \in \mathscr{D}$ we have $x_0^+ = x_1$ and $x_1^+ = x_0$. Dually, the map $\mu_{\mathscr{D}}^-$ leaves any arcs in $\mathscr{D}$ invariant.

Lemma 4.3.7. *The maps $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^-$ from Definition 4.3.5 are mutually inverse bijections on* $\operatorname{nc}\mathscr{D}$*.*

Proof. We first that show that the image of $\mu_{\mathscr{D}}$ lies in $\operatorname{nc}\mathscr{D}$. Let $\{x_0, x_1\}$ be in $\operatorname{nc}\mathscr{D}$. If $\{x_0, x_1\} \in \mathscr{D}$, then by Remark 4.3.6, $\mu_{\mathscr{D}}(\{x_0, x_1\}) = \{x_0, x_1\} \in \mathscr{D} \subseteq \operatorname{nc}\mathscr{D}$. Assume $\{x_0, x_1\} \in (\operatorname{nc}\mathscr{D}) \setminus \mathscr{D}$ with $\mu_{\mathscr{D}}(\{x_0, x_1\}) = \{x_0^+, x_1^+\}$. Let $\{y_0, y_1\}$ be an arc of $\mathcal{Z}$ crossing $\{x_0^+, x_1^+\}$ with

$$y_0 \in (x_0^+, x_1^+) = (x_0^+, x_1) \cup \{x_1\} \cup (x_1, x_1^+)$$
$$y_1 \in (x_1^+, x_0^+) = (x_1^+, x_0) \cup \{x_0\} \cup (x_0, x_0^+).$$

We want to show that $\{y_0, y_1\} \notin \mathscr{D}$. If $y_0 \in (x_1, x_1^+)$, respectively $y_1 \in (x_0, x_0^+)$ (see Figure 4.3) then the arc $\{y_0, y_1\}$ crosses $\{x_1, x_1^+\} \in \mathscr{D}$, respectively $\{x_0, x_0^+\} \in \mathscr{D}$ and hence cannot lie in the non-crossing diagram $\mathscr{D}$. If $y_0 \in (x_0^+, x_1)$ and $y_1 \in (x_1^+, x_0)$, then $\{y_0, y_1\}$ crosses $\{x_0, x_1\} \in \operatorname{nc}\mathscr{D}$ and hence cannot lie in $\mathscr{D}$, cf. Figure 4.4. If $y_0 \in (x_0^+, x_1)$ and $y_1 = x_0$, then $\{y_0, y_1\}$ cannot lie in $\mathscr{D}$ since this would contradict the maximality of

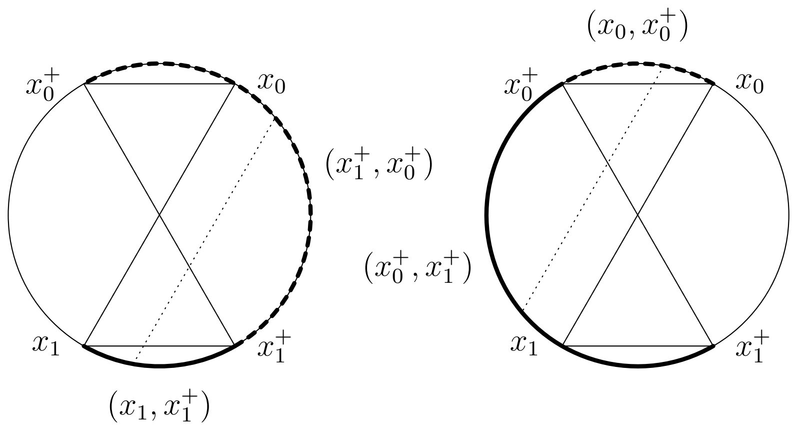


Figure 4.3: If $y_0 \in (x_1, x_1^+)$, respectively $y_1 \in (x_0, x_0^+)$, then the arc $\{y_0, y_1\}$, marked by a dotted line, crosses $\{x_1, x_1^+\}$, respectively $\{x_0, x_0^+\}$.

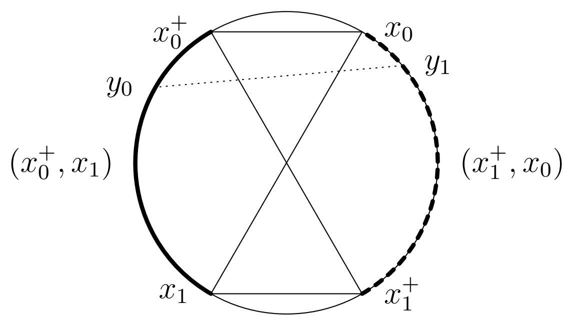


Figure 4.4: If $y_0 \in (x_0^+, x_1)$ and $y_1 \in (x_1^+, x_0)$, then $\{y_0, y_1\}$ crosses $\{x_0, x_1\}$.

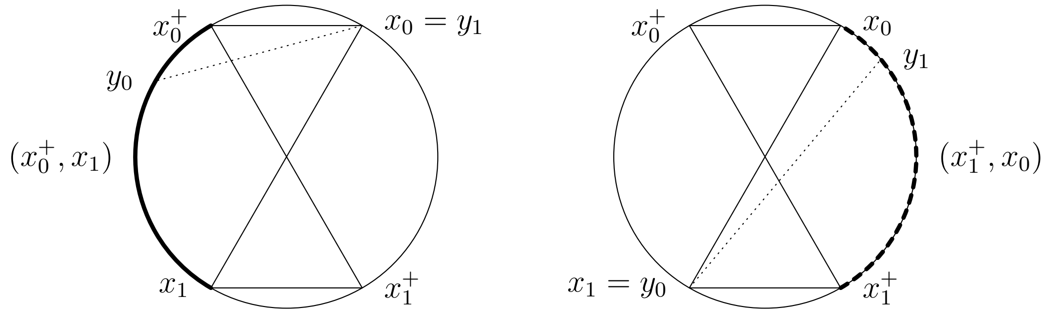


Figure 4.5: Illustration of the case where $y_0 \in (x_0^+, x_1)$ and $y_1 = x_0$ and the case where $y_0 = x_1$ and $y_1 \in (x_1^+, x_0)$

x_0^+ in $\{l \in \mathcal{Z} \cap [x_0, x_1] \mid \{x_0, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\}$, cf. Figure 4.5. Similarly, if $y_0 = x_1$ and $y_1 \in (x_1^+, x_0)$, then $\{y_0, y_1\} \notin \mathscr{D}$ because of the maximality of x_1^+ in $\{l \in \mathcal{Z} \cap [x_1, x_0] \mid \{x_1, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\}$. Finally, if $y_0 = x_1$ and $y_1 = x_0$, then $\{y_0, y_1\} = \{x_0, x_1\} \notin \mathscr{D}$.

Thus any arc crossing $\{x_0^+, x_1^+\}$ cannot lie in $\mathscr{D}$, and we get $\{x_0^+, x_1^+\} \in \operatorname{nc} \mathscr{D}$, so the image of $\mu_{\mathscr{D}}$ lies in $\operatorname{nc} \mathscr{D}$. Analogously, it can be shown that the image of $\mu_{\mathscr{D}}^-$ lies in $\operatorname{nc} \mathscr{D}$.

We now show that $\mu_{\mathscr{D}} \circ \mu_{\mathscr{D}}^-$ is the identity on $\operatorname{nc} \mathscr{D}$. The fact that $\mu_{\mathscr{D}}^- \circ \mu_{\mathscr{D}}$ is the identity can be shown analogously. If $\{x_0, x_1\} \in \mathscr{D}$, then both $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^-$ leave $\{x_0, x_1\}$ invariant. On the other hand given $\{x_0, x_1\} \in (\operatorname{nc} \mathscr{D}) \setminus \mathscr{D}$ with $\mu_{\mathscr{D}}^-(\{x_0, x_1\}) = \{x_0^-, x_1^-\}$ and $\mu_{\mathscr{D}}(\{x_0^-, x_1^-\}) = \{(x_0^-)^+, (x_1^-)^+\}$, we have

$$\begin{aligned}(x_1^-)^+ &= \max\{l \in \mathcal{Z} \cap [x_1^-, x_0^-] \mid \{x_1^-, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\} \\ &= \max\{l \in \mathcal{Z} \cap (x_1^-, x_0^-) \mid \{x_1^-, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\},\end{aligned}$$

where the second equality holds because $\{x_0^-, x_1^-\} \notin \mathscr{D}$. By definition of x_0^- and x_1^- we have $x_1 \in \{l \in \mathcal{Z} \cap (x_1^-, x_0^-) \mid \{x_1^-, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\}$. We show that x_1 is maximal in this set: Let $m \in (x_1^-, x_0^-)$ with $m >_{[x_1^-, x_0^-]} x_1$ and consider the arc $\{x_1^-, m\}$ of $\mathcal{Z}$. Since $m \in (x_1, x_0^-) \subseteq (x_1, x_0)$ and $x_1^- \in (x_0, x_1)$ the arc $\{m, x_1^-\}$ crosses the arc $\{x_0, x_1\}$ and thus cannot lie in $\mathscr{D} \cup \mathcal{E}(\mathcal{Z})$. Thus m cannot lie in $\{l \in \mathcal{Z} \cap (x_1^-, x_0^-) \mid \{x_1^-, l\} \in \mathscr{D} \cup \mathcal{E}(\mathcal{Z})\}$ showing that x_1 is maximal in this set. This proves $(x_1^-)^+ = x_1$.

The fact $(x_0^-)^+ = x_0$ is equivalent and we thus obtain $\mu_{\mathscr{D}}^-(\mu_{\mathscr{D}}(\{x_0, x_1\})) = \{x_0, x_1\}$. Thus $\mu_{\mathscr{D}}^- \circ \mu_{\mathscr{D}}$ is the identity on $\operatorname{nc} \mathscr{D}$. □

Mirroring the definition of mutation pairs in triangulated categories we define the following.

Definition 4.3.8. Let $\mathscr{D}$ be a non-crossing diagram satisfying FF. We call a pair $(\mathscr{X}, \mathscr{X}')$ of subdiagrams $\mathscr{X}, \mathscr{X}' \subseteq \operatorname{nc} \mathscr{D}$ a $\mathscr{D}$-mutation pair if $\mathscr{D} \subseteq \mathscr{X}' \subseteq \mu_{\mathscr{D}}^-(\mathscr{X})$ and $\mathscr{D} \subseteq \mathscr{X} \subseteq \mu_{\mathscr{D}}(\mathscr{X}')$.

Remark 4.3.9. Note that since $\mu_{\mathscr{D}}$ is a bijection with inverse $\mu_{\mathscr{D}}^-$, for any $\mathscr{D}$-mutation pair $(\mathscr{X}, \mathscr{X}')$ we have $\mathscr{X}' = \mu_{\mathscr{D}}^-(\mathscr{X})$ and $\mathscr{X} = \mu_{\mathscr{D}}(\mathscr{X}')$.

We can now state our main theorem for this section.

Theorem 4.3.10. *Let $\mathscr{X}$ be a Ptolemy diagram of $\mathcal{Z}$ satisfying condition CF1, i.e. corresponding to the torsion part of a torsion pair (X, Y) in $\mathcal{C}(\mathcal{Z})$. Let $\mathscr{D} \subseteq \mathscr{X} \cap \operatorname{nc} \mathscr{X}$ be a subdiagram satisfying conditions CF1 and CF2, i.e. corresponding to a functorially finite subcategory $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$. Then the mutation $\mu_{\mathscr{D}}(\mathscr{X})$, respectively $\mu^{-}_{\mathscr{D}}(\mathscr{X})$, corresponds to the torsion part of the torsion pair $(\mu_D(X), \mu_{\Sigma D}(Y))$, respectively $(\mu^{-}_{D}(X), \mu^{-}_{\Sigma D}(Y))$.*

Theorem 4.3.10 is a direct consequence of the following, more general, result:

Theorem 4.3.11. *Let $\mathscr{D}$ be a non-crossing diagram of $\mathcal{Z}$ satisfying condition FF and corresponding to the subcategory D of $\mathcal{C}(\mathcal{Z})$. Let $\mathscr{X}, \mathscr{X}' \subseteq \operatorname{nc} \mathscr{D}$ be diagrams of $\mathcal{Z}$, with $\mathscr{X}$ corresponding to the subcategory $X \subseteq {}^{\perp}(\Sigma D)$ and $\mathscr{X}'$ corresponding to the subcategory $X' \subseteq {}^{\perp}(\Sigma D)$. Then $(\mathscr{X}, \mathscr{X}')$ is a $\mathscr{D}$-mutation pair if and only if (X, X') is a D-mutation pair.*

Using Theorem 4.3.11, we can prove Theorem 4.3.10 as follows.

Proof. Let the notation be as in Theorem 4.3.10. Because $\mathscr{D}$ is a subdiagram of $\mathscr{X} \cap \operatorname{nc} \mathscr{X}$ it is non-crossing and we have $\mathscr{X} \subseteq \operatorname{nc} \mathscr{D}$. Because it satisfies conditions CF1 and CF2 it satisfies condition FF by Lemma 4.3.4. Applying Theorem 4.4.22 to the $\mathscr{D}$-mutation pair $(\mathscr{X}, \mu^{-}_{\mathscr{D}}(\mathscr{X}))$, respectively to the $\mathscr{D}$-mutation pair $(\mu_{\mathscr{D}}(\mathscr{X}), \mathscr{X})$, yields the result. □

For the proof of Theorem 4.3.11, we need to compute some distinguished triangles. For this, we make use of the cluster structure given by those triangulations of the closed disc with marked points $\mathcal{Z}$ which satisfy FF (cf. Section 3.3.2). More precisely we use the existence of the exchange triangles according to Definition 3.2.7, CS2. This relies on [IT3, Proposition 2.4.9] which states that for any triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ and any arc $\{x_0, x_1\} \in \mathcal{T}$ there is a unique arc $\{y_0, y_1\}$ of $\mathcal{Z}$ which crosses $\{x_0, x_1\}$ with $y_0 \in (x_1, x_0)$ and $y_1 \in (x_0, x_1)$ and such that all of $\{x_0, y_1\}$, $\{y_1, x_1\}$, $\{x_1, y_0\}$ and $\{y_0, x_0\}$ are in $(\mathcal{T} \setminus \{x_0, x_1\}) \cup \mathcal{E}(\mathcal{Z})$. Further, as elaborated in [IT3] we get a distinguished triangle

$$\{x_0, x_1\} \to \{x_0, y_0\} \oplus \{x_1, y_1\} \to \{y_0, y_1\} \to \Sigma\{x_0, x_1\},$$

where the first morphism is a left $(\mathcal{C}(\mathcal{Z}) \setminus \{\{x_0, x_1\}\})$-approximation and where we associate a zero-object to any edge of $\mathcal{Z}$. Using this we can prove Theorem 4.3.11.

Proof. Assume first that $(\mathscr{X}, \mathscr{X}')$ is a $\mathscr{D}$-mutation pair. We have $\mathscr{D} \subseteq \mathscr{X}, \mathscr{X}'$ and thus $D \subseteq X, X'$. By Lemma 4.2.11 we further have $D \subseteq \mu_D(X'), \mu^{-}_{D}(X)$. It remains to show that we also have $x \in \mu_D(X')$ for those objects $x \in X$ with $x \notin D$ and $x' \in \mu^{-}_{D}(X)$ for those objects $x' \in X'$ with $x' \notin D$.

The indecomposable objects in X but not in D are labelled by the arcs in $\mathscr{X} \setminus \mathscr{D}$ and the indecomposable objects in X' but not in D are labelled by the arcs in $\mathscr{X}' \setminus \mathscr{D} = \mu^{-}_{\mathscr{D}}(\mathscr{X}) \setminus \mathscr{D}$. Consider an arc $\{x_0, x_1\} \in \mathscr{X} \setminus \mathscr{D}$ and its mutation $\{x_0^-, x_1^-\} = \mu^{-}_{\mathscr{D}}\{x_0, x_1\} \in$

$\mathscr{X}'$. It suffices to show that as objects in $\mathcal{C}(\mathcal{Z})$ we have $\{x_0, x_1\} \in \mu_D(X')$ and $\{x_0^-, x_1^-\} \in \mu_D^-(X)$. By assumption, both arcs $\{x_0, x_1\} \in \mathscr{X}$ and $\{x_0^-, x_1^-\} \in \mathscr{X}'$ lie in nc $\mathscr{D}$, so $\{x_0, x_1\}$ and $\{x_0^-, x_1^-\}$ label indecomposable objects in ${}^{\perp}(\Sigma D) = (\Sigma^{-1} D)^{\perp}$. Furthermore, we can complete the pairwise non-crossing arcs $\{x_0, x_1\}$, $\{x_0, x_1^-\}$, $\{x_1^-, x_1\}$, $\{x_1, x_0^-\}$ and $\{x_0^-, x_0\}$ to a triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$. So by [IT3, Proposition 2.4.9] there exists a distinguished triangle of the form

$$\{x_0, x_1\} \longrightarrow \{x_1^-, x_1\} \oplus \{x_0^-, x_0\} \longrightarrow \{x_0^-, x_1^-\} \longrightarrow \Sigma\{x_0, x_1\}\ , \tag{4.1}$$

and $\{x_1^-, x_1\} \oplus \{x_0^-, x_0\}$ lies in D. Therefore we have $\{x_0^-, x_1^-\} \in \mu_D^-(X)$ and $\{x_0, x_1\} \in \mu_D(X')$.

On the other hand let (X, X') be a D-mutation pair and let $\tilde{X}$ be the subcategory corresponding to the diagram $\mu_{\mathscr{D}}^-(\mathscr{X})$. Then because $(\mathscr{X}, \mu_{\mathscr{D}}^-(\mathscr{X}))$ is a $\mathscr{D}$-mutation pair, the pair $(X, \tilde{X})$ is a D-mutation pair and therefore by Remark 4.2.12 we have $\tilde{X} = \mu_D^-(X) = X'$. So the diagram $\mu_{\mathscr{D}}^-(\mathscr{X})$ corresponds to X' and we get that $\mu_{\mathscr{D}}^-(\mathscr{X}) = \mathscr{X}'$ and $(\mathscr{X}, \mathscr{X}')$ is a $\mathscr{D}$-mutation pair. $\square$

4.4 Torsion pairs and their mutation in cluster categories of finite Dynkin type D

In this section, we provide a combinatorial model for mutation of torsion pairs in cluster categories of finite Dynkin type D. The situation is more complicated than in Dynkin type A, since we have to deal with the indecomposable objects coming from the exceptional vertices of Dynkin diagrams of type D. For the Dynkin diagram D_n with $n \geq 4$ (throughout this Chapter, when we refer to the Dynkin diagram of type D_n or related combinatorial concepts, we always assume $n \geq 4$) these are the vertices $(n-1)_-$ and $(n-1)_+$ in the following picture:

$$\begin{tikzcd}[arrows=dash]
 & & & & & (n-1)_+ \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-2) \arrow[ur] \arrow[dr] & \\
 & & & & & (n-1)_-
\end{tikzcd}$$

Recall from Section 3.2.1, that every two orientations Q and Q' of D_n give rise to equivalent cluster categories $\mathcal{C}_{kQ} \cong \mathcal{C}_{kQ'}$. Therefore it is justified to talk about *the* cluster category of Dynkin type D_n. In Section 3.2.1, the cluster category $\mathcal{C}_{kD_n}$ of Dynkin type D_n was discussed as an example. It is a triangulated, k-linear, Hom-finite, Krull-Schmidt category and thus satisfies all prerequisites we had on our ambient category in the discussion of torsion pairs and mutation in triangulated categories in Section 4.2. Furthermore, as discussed in Section 3.2.1, it is 2-Calabi-Yau, which facilitates some considerations when discussing mutation of torsion pairs.

4.4.1 Cluster categories of finite Dynkin type D: a combinatorial model

The combinatorial model for Dynkin type D_n introduced by Fomin and Zelevinsky in [FZ3] offers a geometric interpretation of the cluster category of Dynkin type D_n. Isomorphism classes of indecomposable objects are represented by rotation-invariant pairs of arcs and diameters in a regular $2n$-gon. As a useful property, this combinatorial model allows for an easy way to determine the dimension of the extension space between two indecomposable objects by counting the number of times the corresponding pairs of arcs or diameters in the regular $2n$-gon cross.

Consider the regular $2n$-gon $\mathcal{P}_{2n}$ with vertices labelled consecutively in an anticlockwise direction by $0, 1, \ldots, 2n-1$. Throughout we will calculate modulo $2n$.

Definition 4.4.1. An *arc* of $\mathcal{P}_{2n}$ is an unordered pair of vertices $\{a, b\}$ of $\mathcal{P}_{2n}$ with $a \neq b$. An arc of the form $\{a, a+1\}$, for $a = 0, \ldots, 2n-1$, is called an *edge* of $\mathcal{P}_{2n}$. An *internal arc* of $\mathcal{P}_{2n}$ is an arc of $\mathcal{P}_{2n}$ that is not an edge. Each arc $\{a, b\}$ of $\mathcal{P}_{2n}$ has a partner $\{a+n, b+n\}$ which is obtained from $\{a, b\}$ by rotation by π. An arc of $\mathcal{P}_{2n}$ is called a *diameter* if it is π-rotation invariant, i.e. if it is of the form $\{a, a+n\}$. A non-diameter arc $\{a, b\}$ together with its partner $\{a+n, b+n\}$ is called a *pair of arcs* and denoted by $\overline{\{a, b\}}$. For each diameter $\{a, a+n\}$ we introduce two coloured diameters: A red one $\overline{\{a, a+n\}}_r$ and a green one $\overline{\{a, a+n\}}_g$. By abuse of notation we sometimes omit the index and just write $\overline{\{a, a+n\}}$ for a coloured diameter, which could be either red or green. If we omit the overline and simply write $\{a, a+n\}$, we refer to the diameter $\{a, a+n\}$ without a colour. We set

$$\mathcal{E}(\mathcal{P}_{2n}) := \{\text{pairs of edges of } \mathcal{P}_{2n}\}$$

and

$$\mathcal{A}(\mathcal{P}_{2n}) := \{\text{coloured diameters and pairs of arcs of } \mathcal{P}_{2n}\} \setminus \mathcal{E}(\mathcal{P}_{2n}).$$

Figure 4.8 features some examples of pairs of arcs and coloured diameters.

Denote by $\Gamma(\mathcal{C}_{kD_n})_0$ the set of vertices of the Auslander–Reiten quiver $\Gamma(\mathcal{C}_{kD_n})$, which correspond to isomorphism classes of indecomposable objects in $\mathcal{C}_{kD_n}$, and recall the labelling of $\Gamma(\mathcal{C}_{kD_n})_0$ from Figure 3.2. We define a bijection $\varphi\colon \Gamma(\mathcal{C}_{kD_n})_0 \to \mathcal{A}(\mathcal{P}_{2n})$ by sending, for $0 \leq a \leq n-1$,

$$\varphi\colon \begin{cases} (a,b) \mapsto \overline{\{a, a+b+1\}} \text{ for } 1 \leq b < n-1, \\ (a,(n-1)_+) \mapsto \begin{cases} \overline{\{a, a+n\}}_g \text{ if } a = 0 \mod 2, \\ \overline{\{a, a+n\}}_r \text{ if } a = 1 \mod 2, \end{cases} \\ (a,(n-1)_-) \mapsto \begin{cases} \overline{\{a, a+n\}}_r \text{ if } a = 0 \mod 2, \\ \overline{\{a, a+n\}}_g \text{ if } a = 1 \mod 2. \end{cases} \end{cases}$$

This map provides the connection between the cluster category $\mathcal{C}_{kD_n}$ and the combinatorial model for Dynkin type D_n. It sends vertices without a sign in $\Gamma(\mathcal{C}_{kD_n})$ to pairs

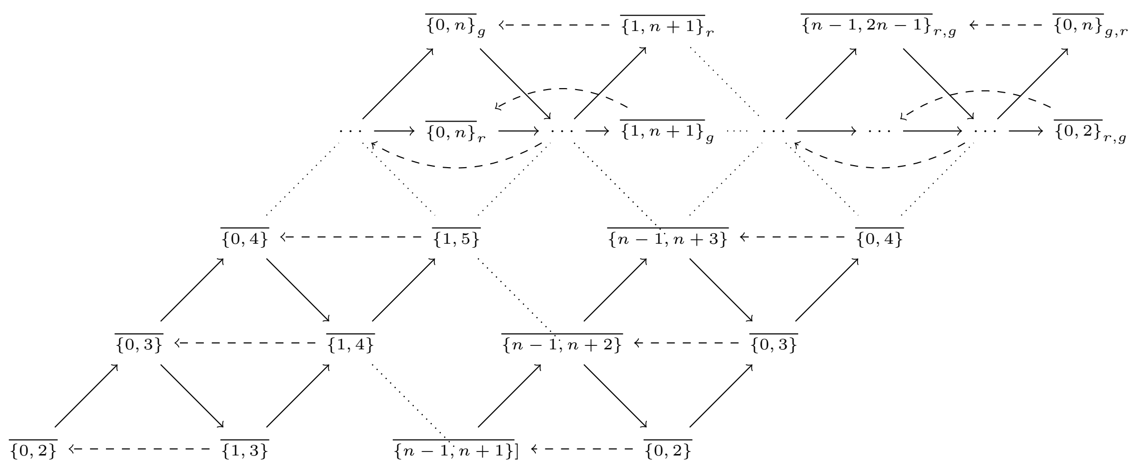


Figure 4.6: The vertices of the Auslander-Reiten quiver $\Gamma(\mathcal{C}_{kD_n})$ are labelled by pairs of arcs and coloured diameters of $\mathcal{P}_{2n}$. The action of the Auslander-Reiten translation τ is marked by dashed arrows.

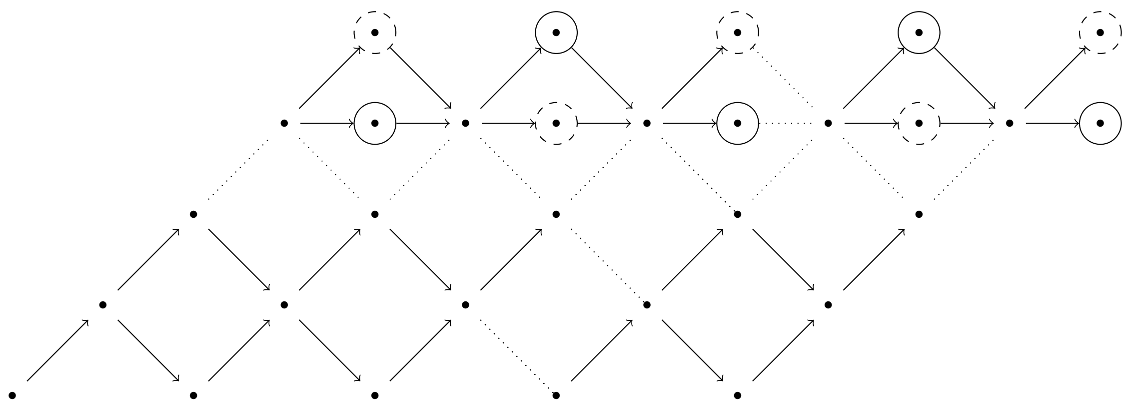


Figure 4.7: The vertices in $\Gamma(\mathcal{C}_{kD_n})$ marked with a dashed circle get matched to green diameters, those marked with a continuous circle get matched to red diameters.

of internal arcs of $\mathcal{P}_{2n}$ and alternatingly matches the vertices $(a,(n-1)_\pm)$ to red and green diameters. The isomorphism classes of indecomposable objects in $\mathcal{C}_{kD_n}$ are thus labelled by $\mathcal{A}(\mathcal{P}_{2n})$. Figure 4.6 shows the Auslander-Reiten quiver $\Gamma(\mathcal{C}_{kD_n})$ with the new labelling. Note that the indexing diameters alternate in colour along both the top and the second to top horizontal levels of the Auslander-Reiten quiver. In other words, the Auslander-Reiten translation τ changes the colours of the diameters, i.e. we have $\tau(\{a,a+n\}_r) = \{a-1,a-1+n\}_g$ and $\tau(\{a,a+n\}_g) = \{a-1,a-1+n\}_r$ for all $0 \leq a \leq n-1$. We have drawn in the Auslander-Reiten translation in Figure 4.6 as an illustration. Figure 4.7 provides an illustration of the alternating matching of colours to the vertices associated to diameters.

Subcategories of $\mathcal{C}_{kD_n}$ correspond to subsets of $\mathcal{A}(\mathcal{P}_{2n})$ and we call any subset $\mathscr{X} \subseteq \mathcal{A}(\mathcal{P}_{2n})$ a *diagram of Dynkin type D_n*, and if the context is clear sometimes just a *diagram* for short. A subset $\mathscr{D} \subseteq \mathscr{X}$ of a diagram $\mathscr{X}$ is called a *subdiagram of $\mathscr{X}$*. Any diagram

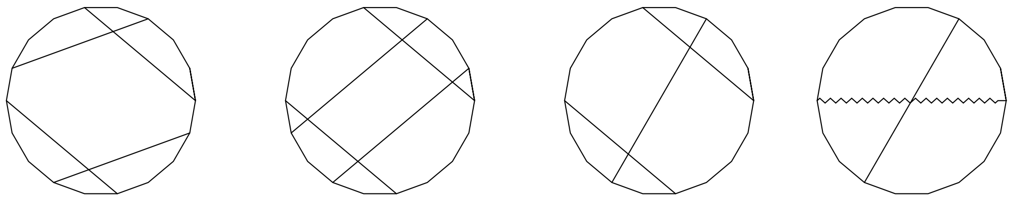


Figure 4.8: The pictures illustrate from left to right: Two pairs of arcs crossing once, two pairs of arcs crossing twice, a diameter crossing a pair of arcs and two diameters of different colour crossing.

of Dynkin type D_n is invariant under rotation by π.

Definition 4.4.2. Two arcs $\{a, b\}$ and $\{x, y\}$ of $\mathcal{P}_{2n}$ are said to *cross*, if a, b, x and y are pairwise distinct and they lie on the boundary of $\mathcal{P}_{2n}$ in the order a, x, b, y or x, a, y, b when moving in an anticlockwise direction. We have the following notion of elements of $\mathcal{A}(\mathcal{P}_{2n})$ crossing:

- Two pairs of arcs $\overline{\{a, b\}}$ and $\overline{\{x, y\}}$ are said to *cross once*, if the arc $\{a, b\}$ crosses either $\{x, y\}$ or $\{x+n, y+n\}$. They are said to *cross twice*, if the arc $\{a, b\}$ crosses both $\{x, y\}$ and $\{x+n, y+n\}$.
- A coloured diameter $\overline{\{a, a+n\}}$ and a pair of arcs $\overline{\{x, y\}}$ are said to *cross once*, if the arc $\{a, a+n\}$ crosses $\{x, y\}$.
- Two coloured diameters $\overline{\{a, a+n\}}_r$ and $\overline{\{x, x+n\}}_g$ of different colours are said to *cross once* if the arcs $\{a, a+n\}$ and $\{x, x+n\}$ cross. Two diameters of the same colour do not cross.

Figure 4.8 illustrates the crossing of arcs. Throughout this paper, when drawing diagrams we will draw green diameters as wriggly lines and red diameters as straight lines.

The combinatorial model for the cluster category of Dynkin type D_n via $\mathcal{A}(\mathcal{P}_{2n})$ is closely related to the model introduced by Schiffler [Sch], where triangulations of the punctured disc were used to first combinatorially describe the cluster category of Dynkin type D_n. In particular, [Sch, Proposition 1.3] translates directly to this model and can be stated as follows:

Proposition 4.4.3 ([Sch, Proposition 1.3])**.** *Let $\overline{\{a, b\}}$ and $\overline{\{x, y\}}$ be elements of $\mathcal{A}(\mathcal{P}_{2n})$ and consider the corresponding indecomposable objects in $\mathcal{C}_{kD_n}$. Then the number of times $\overline{\{a, b\}}$ and $\overline{\{x, y\}}$ cross is equal to the dimension $\dim \mathrm{Ext}^1_{\mathcal{C}_{kD_n}}(\overline{\{a, b\}}, \overline{\{x, y\}})$ of the Ext^1-space between them.*

For the sake of clarity let us explicitly note that the above proposition determines all extensions between such objects: Because $\mathcal{C}_{kD_n}$ is 2-Calabi-Yau we have

$$\dim(\mathrm{Ext}^1(\overline{\{a, b\}}, \overline{\{x, y\}})) = \dim(\mathrm{Ext}^1(\overline{\{x, y\}}, \overline{\{a, b\}})).$$

4.4.2 Torsion pairs in cluster categories of finite Dynkin type D

Torsion pairs in the cluster category of Dynkin type D_n have been classified by Holm, Jørgensen and Rubey in [HJR2] via the combinatorial model described in Section 4.4.1.

Recall that by Proposition 4.2.6 a subcategory $X \subseteq \mathcal{C}_{kD_n}$ is the torsion part of a torsion pair if and only if it is contravariantly finite and ${}^{\perp}(X^{\perp}) = X$. Since there are only finitely many indecomposable objects in $\mathcal{C}_{kD_n}$, every subcategory is functorially finite, so the aim is to classify all subcategories X satisfying ${}^{\perp}(X^{\perp}) = X$. As in the example for discrete cluster categories of Dynkin type A (cf. Section 4.3.1), this condition translates nicely to the combinatorial model. Let $X \subseteq \mathcal{C}_{kD_n}$ be a subcategory corresponding to the diagram $\mathscr{X}$ of Dynkin type D_n. Set

$$\operatorname{nc}\mathscr{X} = \{\overline{\{a,b\}} \in \mathcal{A}(\mathcal{P}_{2n}) \mid \overline{\{a,b\}} \text{ crosses no element of } \mathscr{X}\}.$$

By [HJR2, Proposition 3.5], the pair $(X, X^{\perp})$ is a torsion pair in $\mathcal{C}_{kD_n}$ if and only if $\mathscr{X} = \operatorname{nc}(\operatorname{nc}\mathscr{X})$. In light of this result, the problem of classifying torsion pairs in $\mathcal{C}_{kD_n}$ boils down to finding a combinatorial description for diagrams $\mathscr{X}$ of Dynkin type D_n satisfying $\mathscr{X} = \operatorname{nc}(\operatorname{nc}\mathscr{X})$. Diagrams satisfying this condition can be described combinatorially by considering all their crossing elements, as in the following definition.

Definition 4.4.4 ([HJR2, Definition 4.1])**.** Let $\mathscr{X}$ be a diagram of Dynkin type D_n. It is called a *Ptolemy diagram of Dynkin type* D_n if for any two crossing elements $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ of $\mathscr{X}$ the following hold:

Pt1 If $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ are pairs of arcs, then each of $\overline{\{a,x\}}$, $\overline{\{x,b\}}$, $\overline{\{b,y\}}$ and $\overline{\{y,a\}}$ lies in $\mathscr{X} \cup \mathcal{E}(\mathcal{P}_{2n})$. If any of $\{a,x\}$, $\{x,b\}$, $\{b,y\}$ or $\{y,a\}$ is a diameter, then both the red and the green copy of that diameter lie in $\mathscr{X}$.

Pt2 If $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ are diameters of different colour, then $\overline{\{a,x\}} = \overline{\{b,y\}}$ and $\overline{\{x,b\}} = \overline{\{y,a\}}$ lie in $\mathscr{X} \cup \mathcal{E}(\mathcal{P}_{2n})$.

Pt3 If $\overline{\{a,b\}}$ is a diameter and $\overline{\{x,y\}}$ is a pair of arcs, then those of $\overline{\{a,x\}}$, $\overline{\{x,b\}}$, $\overline{\{b,y\}}$ and $\overline{\{y,a\}}$ which do not cross $\overline{\{x,y\}}$ lie in $\mathscr{X} \cup \mathcal{E}(\mathcal{P}_{2n})$. Furthermore, the diameters $\overline{\{x,x+n\}}$ and $\overline{\{y,y+n\}}$ of the same colour as $\overline{\{a,b\}}$ also lie in $\mathscr{X}$.

Figure 4.9 illustrates the axioms for a Ptolemy diagram.

Holm, Jørgensen and Rubey present the following useful classification of torsion pairs in the cluster category $\mathcal{C}_{kD_n}$ of Dynkin type D_n.

Theorem 4.4.5 ([HJR2, Theorem 1.1])**.** *Let X be a subcategory of $\mathcal{C}_{kD_n}$ and let $\mathscr{X}$ be the corresponding diagram of Dynkin type D_n. Then the following are equivalent.*

- *The pair $(X, X^{\perp})$ is a torsion pair in $\mathcal{C}_{kD_n}$.*
- *The diagram $\mathscr{X}$ is a Ptolemy diagram of Dynkin type D_n.*

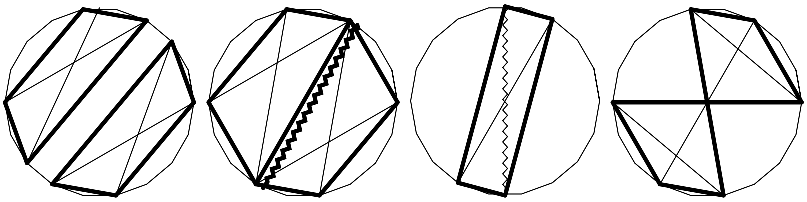


Figure 4.9: The axioms for a Ptolemy diagram of Dynkin type D_n illustrated: Whenever two elements, drawn with thin lines, of a Ptolemy diagram $\mathscr{X}$ of Dynkin type D_n cross, then the thick lines must be contained in $\mathscr{X} \cup \mathcal{E}(\mathcal{P}_{2n})$.

4.4.3 Non-crossing diagrams of Dynkin type D and mutation

Throughout this section, let $\mathscr{D}$ denote a *non-crossing diagram of Dynkin type* D_n, i.e. a diagram of Dynkin type D_n with pairwise non-crossing elements. We define mutation of subdiagrams of nc$\mathscr{D}$ with respect to $\mathscr{D}$ using the concept of $\mathscr{D}$-cells of Dynkin type D_n. Informally speaking, $\mathscr{D}$-cells of Dynkin type D_n are convex polygons with edges in $\mathscr{D} \cup \mathcal{E}(\mathcal{P}_{2n})$ whose diagonals do not lie in $\mathscr{D}$. However, the presence of diameters means that we have to be careful with the definition. The idea of $\mathscr{D}$-cells of Dynkin type D_n is inspired by Zhou and Zhu [ZZ2], who introduced $\mathcal{D}$-cells for non-crossing subdiagrams $\mathcal{D}$ of Ptolemy diagrams of $\mathcal{Z}$, where $\mathcal{Z} \subseteq S^1$ is a finite subset or an admissible subset with exactly one limit point (i.e. the combinatorial model corresponding to cluster categories of finite Dynkin type A and the cluster category $\mathcal{C}_{A_\infty}$ of infinite Dynkin type A_∞ respectively).

For a formal definition of $\mathscr{D}$-cells, it is useful to replace some of the diameters in $\mathcal{A}(\mathcal{P}_{2n})$ by pairs of radii.

Definition 4.4.6. We introduce an additional central vertex c, which is placed at the centre of $\mathcal{P}_{2n}$ and additional arcs $\{a, c\}$ for $a \in \{0, \ldots, 2n-1\}$, which we call *radii*. The π-rotation $\{a+n, c\}$ of a radius $\{a, c\}$ is again a radius and together they form the *pair of radii* $\overline{\{a, c\}}$. For each pair of radii $\overline{\{a, c\}}$ we introduce a copy $\overline{\{a, c\}}_r$ of the colour red and a copy $\overline{\{a, c\}}_g$ of the colour green.

We define the *replacement map* $r_{\mathscr{D}}$ as follows. If $\mathscr{D}$ has no diameters, then we set $r_{\mathscr{D}} : \mathcal{A}(\mathcal{P}_{2n}) \cup \mathcal{E}(\mathcal{P}_{2n}) \to \mathcal{A}(\mathcal{P}_{2n}) \cup \mathcal{E}(\mathcal{P}_{2n})$ to be the identity. Otherwise we define

$$r_{\mathscr{D}} \colon \mathcal{A}(\mathcal{P}_{2n}) \cup \mathcal{E}(\mathcal{P}_{2n}) \to \mathcal{A}(\mathcal{P}_{2n}) \cup \mathcal{E}(\mathcal{P}_{2n}) \cup \{\overline{\{a, c\}} | a \in \{0, \ldots, 2n-1\}\}$$

by $r_{\mathscr{D}}(\overline{\{a, b\}}) = \overline{\{a, b\}}$ for every pair of arcs $\overline{\{a, b\}} \in \mathcal{A}(\mathcal{P}_{2n}) \cup \mathcal{E}(\mathcal{P}_{2n})$ and for every $0 \leq a < n$ we define

$$r_{\mathscr{D}}(\overline{\{a, a+n\}}_{r,g}) = \begin{cases} \overline{\{a, a+n\}}_{r,g} \text{ if } \overline{\{a, a+n\}}_{g,r} \in \mathscr{D} \\ \overline{\{a, c\}}_{r,g} \text{ if } \overline{\{a, a+n\}}_{g,r} \notin \mathscr{D}. \end{cases}$$

that is, a red diameter $\{a, a+n\}_r$ gets sent to a pair of radii if and only if the green diameter $\{a, a+n\}_g$ does not lie in $\mathscr{D}$ and a green diameter $\{x, x+n\}_g$ gets sent to a pair of radii if and only if the red diameter $\{x, x+n\}_r$ does not lie in $\mathscr{D}$.

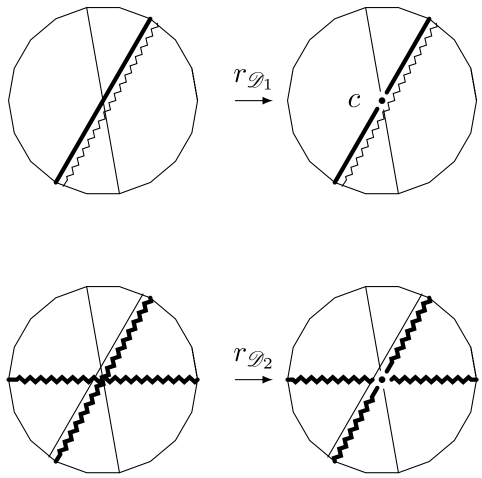


Figure 4.10: The diameters lying in the non-crossing subdiagrams $\mathscr{D}_1$ respectively $\mathscr{D}_2$ are marked by thick lines. The picture shows how some diameters in $\mathcal{A}(\mathcal{P}_{2n})$ get replaced by pairs of radii and some stay as diameters under the replacement maps $r_{\mathscr{D}_1}$ and $r_{\mathscr{D}_2}$.

Figure 4.10 illustrates how the replacement map acts on diameters for different non-crossing diagrams.

A pair of radii $\overline{\{x,c\}}_{r,g}$ is said to cross a diameter or pair of arcs $\overline{\{a,b\}}$ if and only if the corresponding diameter $\overline{\{x,x+n\}}_{r,g}$ crosses $\overline{\{a,b\}}$. Two pairs of radii $\overline{\{x,c\}}_{r,g}$ and $\overline{\{a,c\}}_{g,r}$ are said to cross if and only if the corresponding diameters $\overline{\{x,x+n\}}_{r,g}$ and $\overline{\{a,a+n\}}_{g,r}$ cross. Furthermore, two radii $\{a,c\}$ and $\{x,c\}$ do not cross for any $a,x \in \{0,\dots,2n-1\}$.

One neat effect of replacing diameters with pairs of radii according to the replacement map, is that the image $r_{\mathscr{D}}(\mathscr{D})$ of the non-crossing diagram $\mathscr{D}$ consists of pairwise geometrically non-crossing arcs, i.e. where in $\mathscr{D}$ we had the possibility of diameters of the same colour crossing as arcs, we cannot have that situation in $r_{\mathscr{D}}(\mathscr{D})$, as the following lemma shows.

Lemma 4.4.7. *If $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ lie in $r_{\mathscr{D}}(\mathscr{D})$ then the arcs $\{a,b\}$ and $\{x,y\}$ do not cross.*

Proof. Assume for a contradiction that the arcs $\{a,b\}$ and $\{x,y\}$ cross and that $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ lie in $r_{\mathscr{D}}(\mathscr{D})$. Because $\mathscr{D}$ is non-crossing, the elements $\overline{\{a,b\}} \neq \overline{\{x,y\}}$ must be diameters of the same colour in $\mathscr{D}$, without loss of generality $\overline{\{a,b\}} = \overline{\{a,a+n\}}_r$ and $\overline{\{x,y\}} = \overline{\{x,x+n\}}_r$. Because $\overline{\{a,a+n\}}_r$ and $\overline{\{x,x+n\}}_r$ are in $\mathscr{D}$, both $\overline{\{x,x+n\}}_g$, which crosses $\overline{\{a,a+n\}}_r$ and $\overline{\{a,a+n\}}_g$, which crosses $\overline{\{x,x+n\}}_r$, cannot lie in $\mathscr{D}$. Therefore $r_{\mathscr{D}}(\overline{\{a,a+n\}}_r) = \overline{\{a,c\}}_r$ and $r_{\mathscr{D}}(\overline{\{x,x+n\}}_r) = \overline{\{x,c\}}_r$, which contradicts the assumption that $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ lie in $r_{\mathscr{D}}(\mathscr{D})$. $\square$

When talking about angles between arcs, we adhere to the following convention.

Notation 4.4.8. For two arcs $\{x,y\}$ and $\{y,z\}$ we denote by $\sphericalangle(x,y,z)$ the angle covered when rotating $\{x,y\}$ to $\{y,z\}$ in a clockwise direction. We assume $0 \leq \sphericalangle(x,y,z) < 2\pi$.

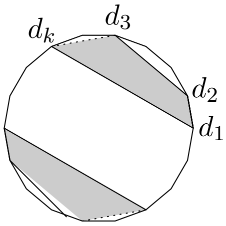


Figure 4.11: Example for a non-central pair of $\mathscr{D}$-cells

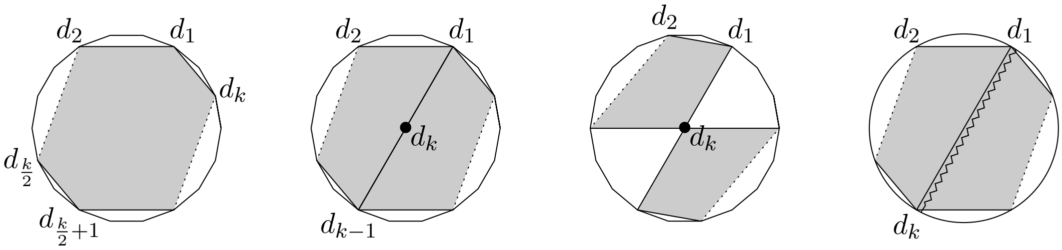


Figure 4.12: Examples for central pairs of $\mathscr{D}$-cells. From left to right: When $\mathscr{D}$ contains no diameters; when $\mathscr{D}$ contains one diameter; when $\mathscr{D}$ contains more than one diameter and all are of the same colour; when $\mathscr{D}$ contains two diameters of different colour

We can now give a rigorous definition of $\mathscr{D}$-cells of Dynkin type D_n.

Definition 4.4.9. Let $\mathscr{D}$ be a non-crossing diagram of Dynkin type D_n. A *$\mathscr{D}$-cell of Dynkin type D_n*, or *$\mathscr{D}$-cell* for short, is a polygon $\langle d_1, \ldots, d_k\rangle$ with $k \geq 3$ with vertices $d_1, \ldots, d_k \in \{0, \ldots, 2n-1\} \cup \{c\}$, ordered in an anticlockwise fashion, such that for $i = 1, \ldots, k$ we have

$$\overline{\{d_i, d_{i+1}\}} \in r_{\mathscr{D}}(\mathscr{D}) \cup \ \mathcal{E}(\mathcal{P}_{2n}),$$

where we calculate modulo k in the indices. Furthermore, for any $\overline{\{d_i, v\}} \in r_{\mathscr{D}}(\mathscr{D})$ with $\sphericalangle(d_{i-1}, d_i, v) \neq 0$ we have

$$0 < \sphericalangle(d_{i-1}, d_i, d_{i+1}) \leq \sphericalangle(d_{i-1}, d_i, v).$$

We call a $\mathscr{D}$-cell $\langle d_1, \ldots, d_k\rangle$ of Dynkin type D_n together with its π-rotation $\langle d_1 + n, \ldots, d_k + n\rangle$ (where we set $c + n = c$) a *pair of $\mathscr{D}$-cells* and denote it by $\overline{\langle d_1, \ldots, d_k\rangle}$. We call a pair of $\mathscr{D}$-cells *central*, if they contain the centre c of the polygon $\mathcal{P}_{2n}$.

Figure 4.11 shows an example of a non-central pair of $\mathscr{D}$-cells, i.e. a pair of $\mathscr{D}$-cells which does not contain the centre of $\mathcal{P}_{2n}$. Figure 4.12 shows examples of central pairs of $\mathscr{D}$-cells.

The examples of $\mathscr{D}$-cells we provided are convex. In fact, all of them are.

Lemma 4.4.10. *Every $\mathscr{D}$-cell of Dynkin type D_n is convex.*

Proof. Let $\langle d_1, \ldots, d_k\rangle$ be a $\mathscr{D}$-cell of Dynkin type D_n. The interior angles are the angles $\sphericalangle(d_{i-1}, d_i, d_{i+1})$ for $i \in \{1, \ldots, k\}$ (calculating modulo k). If $d_i \in \{0, \ldots, 2n-1\}$,

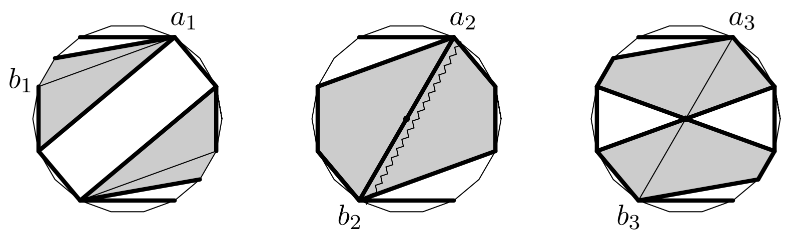


Figure 4.13: The pictures illustrate three different non-crossing diagrams $\mathscr{D}_i$ (for $i = 1, 2, 3$ labelled from left to right) marked by thick lines, with an element $\overline{\{a_i, b_i\}}$ of $(\mathrm{nc}\,\mathscr{D}_i) \setminus \mathscr{D}_i$ and the pair of $\mathscr{D}_i$-cells it is contained in. The pair of $\mathscr{D}_i$-cells containing $\overline{\{a_i, b_i\}} \in \mathrm{nc}\,\mathscr{D}_i \setminus \mathscr{D}_i$ is marked in grey.

then $\sphericalangle(d_{i-1}, d_i, d_{i+1}) \leq \sphericalangle(d_{i-1}, d_i, d_i + 1) < \pi$. If $d_i = c$, then $\sphericalangle(d_{i-1}, d_i, d_{i+1}) \leq \sphericalangle(d_{i-1}, d_i, d_{i-1} + n) = \pi$. Thus all interior angles are less or equal to π, and $\langle d_1, \ldots, d_k \rangle$ is convex. □

We can use $\mathscr{D}$-cells to define mutation of subdiagrams of $\mathrm{nc}\,\mathscr{D}$ with respect to $\mathscr{D}$. We start by showing (in Lemma 4.4.12) that each element of $(\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ is contained in a unique pair of $\mathscr{D}$-cells, by which we mean the following.

Definition 4.4.11. We say that a diameter or a pair of arcs $\overline{\{a, b\}} \in \mathcal{A}(\mathcal{P}_{2n})$ is *contained in a pair of $\mathscr{D}$-cells* $\overline{\langle d_1, \ldots, d_k \rangle}$ if $r_{\mathscr{D}}(\overline{\{a, b\}}) = \overline{\{d_i, d_j\}}$ for some $i, j \in \{1, \ldots, k\}$ with $j \notin \{i - 1, i, i + 1\}$.

For a pair of arcs $\overline{\{a, b\}}$ to be contained in a pair of $\mathscr{D}$-cells $\overline{\langle d_1, \ldots, d_k \rangle}$ means that either the arc $\{a, b\}$ is a diagonal in $\langle d_1, \ldots, d_k \rangle$ and $\{a + n, b + n\}$ is a diagonal in $\langle d_1 + n, \ldots, d_k + n \rangle$ or vice versa. For a coloured diameter $\overline{\{a, a + n\}}$ to be contained in a pair of $\mathscr{D}$-cells $\overline{\langle d_1, \ldots, d_k \rangle}$ can mean any of the following two: Either $r_{\mathscr{D}}(\overline{\{a, a + n\}})$ is a diameter and it is a diagonal in $\langle d_1, \ldots, d_k \rangle$ and $\langle d_1 + n, \ldots, d_k + n \rangle$ or it is a pair of radii and $\{a, c\}$ is a diagonal in $\langle d_1, \ldots, d_k \rangle$ and $\{a + n, c\}$ is a diagonal in $\langle d_1 + n, \ldots, d_k + n \rangle$ or vice versa.

Lemma 4.4.12. *Every element $\overline{\{a, b\}} \in \mathcal{A}(\mathcal{P}_{2n})$ which is contained in a pair of $\mathscr{D}$-cells $\overline{\langle d_1, \ldots, d_k \rangle}$ lies in $(\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$. On the other hand, every element $\overline{\{a, b\}} \in (\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ is contained in a unique pair of $\mathscr{D}$-cells.*

Figure 4.13 shows, for different non-crossing diagrams $\mathscr{D}_i$ of Dynkin type D_n, examples of elements of $\mathrm{nc}\,\mathscr{D}_i \setminus \mathscr{D}_i$ and the pairs of $\mathscr{D}_i$-cells they are contained in. The proof of Lemma 4.4.12 uses the following useful fact.

Lemma 4.4.13. *Let $\overline{\{x, y\}}, \overline{\{y, z\}}, \overline{\{y, z'\}} \in r_{\mathscr{D}}(\mathscr{D})$. Then we have $\sphericalangle(x, y, z) = \sphericalangle(x, y, z')$ if and only if $z = z'$.*

Proof. Assume that $\sphericalangle(x, y, z) = \sphericalangle(x, y, z')$. If both z and z' lie in $\{0, \ldots, 2n-1\}$ it follows from the regularity of $\mathcal{P}_{2n}$ that $z = z'$. Otherwise, if $c \in \{z, z'\}$ then $z \neq z'$ would imply

$\{z, z'\} = \{c, y+n\}$. Without loss of generality we may assume $z = c$ and $z' = y+n$, and $\overline{\{y,z\}} = \overline{\{y,c\}}_r \in r_{\mathscr{D}}(\mathscr{D})$ is of the colour red, so $r_{\mathscr{D}}(\overline{\{y,y+n\}}_r) = \overline{\{y,c\}}_r \in r_{\mathscr{D}}(\mathscr{D})$ and $\overline{\{y,y+n\}}_r \in \mathscr{D}$. Then, by definition of the replacement map, $r_{\mathscr{D}}(\overline{\{y,y+n\}}_g) = \overline{\{y,c\}}_g$. Therefore, both diameters $\overline{\{y,y+n\}}_r$ and $\overline{\{y,y+n\}}_g$ get replaced by pairs of radii under the replacement map $r_{\mathscr{D}}$, which contradicts the assumption that $\overline{\{y,z'\}} = \overline{\{y,y+n\}} \in r_{\mathscr{D}}(\mathscr{D})$. □

We can now prove Lemma 4.4.12.

Proof. Assume that $\overline{\{a,b\}} \in \mathcal{A}(\mathcal{P}_{2n})$ is contained in a pair of $\mathscr{D}$-cells $\overline{\langle d_1,\dots,d_k\rangle}$. Assume for a contradiction that $\overline{\{a,b\}}$ lies in $\mathscr{D}$. Then for some $1 \le i,j \le k$ with $j \notin \{i-1,i,i+1\}$, we have $r_{\mathscr{D}}(\overline{\{a,b\}}) = \overline{\{d_i,d_j\}} \in r_{\mathscr{D}}(\mathscr{D})$. By Lemma 4.4.10 the polygon $\langle d_1,\dots,d_k\rangle$ is convex, so we have $\sphericalangle(d_{i-1},d_i,d_j) \le \sphericalangle(d_{i-1},d_i,d_{i+1})$, where we cannot have equality by Lemma 4.4.13. By minimality of $\sphericalangle(d_{i-1},d_i,d_{i+1})$ we must have $\sphericalangle(d_{i-1},d_i,d_j) = 0$, so either $d_j = c$ or $d_{i-1} = c$ and both $r_{\mathscr{D}}^{-1}(\overline{\{d_i,d_j\}})$ and $r_{\mathscr{D}}^{-1}(\overline{\{d_i,d_{i-1}\}})$ are coloured diameters in $\mathscr{D}$ with underlying non-coloured diameter $\{d_i, d_i+n\}$. Because $\overline{\{d_i,d_j\}} \neq \overline{\{d_i,d_{i-1}\}}$ they must be of different colour. This contradicts the definition of the replacement map $r_{\mathscr{D}}$ and the fact that we have $d_{i-1} = c$ or $d_j = c$. So we have $\overline{\{a,b\}} \notin \mathscr{D}$. Assume now that $\overline{\{x,y\}} \in \mathcal{A}(\mathcal{P}_{2n})$ crosses $\overline{\{a,b\}}$. Then either it crosses a pair of sides of $\overline{\langle d_1,\dots,d_k\rangle}$ or it is also contained in $\overline{\langle d_1,\dots,d_k\rangle}$. In particular, the diameter or pair of arcs $\overline{\{x,y\}}$ cannot lie in $\mathscr{D}$ and thus $\overline{\{a,b\}} \in (\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$. This proves the first statement of Lemma 4.4.12.

Let $\overline{\{a,b\}} \in (\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ with $r_{\mathscr{D}}(\overline{\{a,b\}}) = \overline{\{a',b'\}}$. We first show the existence of a $\mathscr{D}$-cell of Dynkin type D_n containing $\overline{\{a,b\}}$. Construct a sequence of vertices by setting

$$\begin{aligned} d_1 &= a' \\ d_2 &= \min_{\sphericalangle(b',a',v)>0} \{v \in \{0,\dots,2n-1\} \cup \{c\} \mid \overline{\{a',v\}} \in r_{\mathscr{D}}(\mathscr{D}) \cup \mathcal{E}(\mathcal{P}_{2n})\} \end{aligned}$$

and for $i \ge 2$:

$$d_{i+1} = \min_{\sphericalangle(d_{i-1},d_i,v)>0} \{v \in \{0,\dots,2n-1\} \cup \{c\} \mid \overline{\{d_i,v\}} \in r_{\mathscr{D}}(\mathscr{D}) \cup \mathcal{E}(\mathcal{P}_{2n})\}.$$

We show that there exists a $k \in \mathbb{Z}$, such that $\langle d_1,\dots,d_k\rangle$ is a polygon. Since the set of vertices $\{0,\dots,2n-1\} \cup \{c\}$ is finite, there is a $k \in \mathbb{Z}_{\ge 3}$, such that $d_{k+1} = d_i$ for some $i < k$. Choose k to be minimal with this property, i.e. let k be such that there is a $1 \le i < k$ with $d_{k+1} = d_i$ and such that $d_1,\dots,d_k$ are pairwise distinct. Since by Lemma 4.4.7 the arcs $\{d_j, d_{j+1}\}$ for $1 \le j \le k$ are pairwise non-crossing, $\langle d_i,\dots,d_k\rangle$ is a polygon. Assume for a contradiction that $i \ge 2$. Then $\sphericalangle(d_k,d_i,d_{i+1}) \neq 0$. Otherwise, by Lemma 4.4.13 we get $d_k = d_{i+1}$ and thus $d_i = d_{k+1} = d_{i+2}$, which contradicts the condition $\sphericalangle(d_i,d_{i+1},d_{i+2}) \neq 0$. So we have, by the definition of d_i,

$$\sphericalangle(d_{i-1},d_i,d_{i+1}) \le \sphericalangle(d_k,d_i,d_{i+1})$$

and thus, since $\{d_{i-1}, d_i\}$ does not intersect any side of the polygon $\langle d_i, \dots, d_k\rangle$, it is a diagonal or a side in $\langle d_i, \dots, d_k\rangle$. Therefore we have $d_{i-1} = d_l$ for some $i < l \leq k$. This contradicts the assumption that $d_1, \dots, d_k$ are pairwise distinct. Therefore $d_i = d_1$ and $\langle d_1, \dots, d_k\rangle$ is a polygon.

By definition, for $1 < i \leq k$ the angles $\sphericalangle(d_{i-1}, d_i, d_{i+1})$ satisfy the minimality condition for angles in a $\mathscr{D}$-cell of Dynkin type D_n. Furthermore, if $\overline{\{d_1, v\}} \in r_{\mathscr{D}}(\mathcal{A}(\mathcal{P}_{2n}))$ is such that $0 < \sphericalangle(d_k, d_1, v) < \sphericalangle(d_k, d_1, d_2)$, then either it is contained in $\overline{\langle d_1, \dots, d_k\rangle}$ or it intersects one of its pairs of sides. Thus by the first part of the proof it cannot be an element of $\mathscr{D}$. So $\sphericalangle(d_k, d_1, d_2)$ satisfies the minimality condition for angles in a $\mathscr{D}$-cell of Dynkin type D_n and the polygon $\langle d_1, \dots, d_k\rangle$ is a $\mathscr{D}$-cell of Dynkin type D_n.

Because $r_{\mathscr{D}}(\overline{\{a, b\}}) = \overline{\{a', b'\}}$ does not intersect any of the pairs of sides $\overline{\{d_i, d_{i+1}\}}$ of $\overline{\langle d_1, \dots, d_k\rangle}$ and because by definition of d_2 we have $\sphericalangle(b', a', d_2) \leq \sphericalangle(d_k, d_1, d_2)$, the arc $\{a', b'\}$ is a diagonal in the polygon $\langle d_1, \dots, d_k\rangle$ and thus $\overline{\{a, b\}}$ is contained in $\overline{\langle d_1, \dots, d_k\rangle}$.

It remains to show uniqueness. Let $\overline{\{a, b\}} \in (\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ with $r_{\mathscr{D}}(\overline{\{a, b\}}) = \overline{\{a', b'\}}$ and assume it is contained in the pairs of $\mathscr{D}$-cells $\overline{\langle d_1, \dots, d_k\rangle}$ and $\overline{\langle d'_1, \dots, d'_{k'}\rangle}$. Without loss of generality we may assume $a' = d_i = d'_{i'}$ and $b' = d_j = d'_{j'}$ for some $1 \leq i, j \leq k$ and $1 \leq i', j' \leq k'$. First assume $\sphericalangle(d_{i-1}, d_i, d'_{i'-1}) = 0$. By Lemma 4.4.13 we have $d_{i-1} = d'_{i'-1}$ and inductively, using minimality of the exterior angles of both $\mathscr{D}$-cells, we obtain $\langle d_1, \dots, d_k\rangle = \langle d'_1, \dots, d'_{k'}\rangle$. Consider now the case where $\sphericalangle(d_{i-1}, d_i, d'_{i'-1}) > 0$. Without loss of generality we may assume $\sphericalangle(d_{i-1}, d_i, d'_{i'-1}) \leq \pi$, otherwise interchange the roles of $\langle d_1, \dots, d_k\rangle$ and $\langle d'_1, \dots, d'_{k'}\rangle$. By Lemma 4.4.10, both $\langle d_1, \dots, d_k\rangle$ and $\langle d'_1, \dots, d'_{k'}\rangle$ are convex. We have

$$\sphericalangle(d_{i-1}, d_i, d_j) = \sphericalangle(d_{i-1}, d_i, d'_{i'-1}) + \sphericalangle(d'_{i'-1}, d_i, d_j)$$

and by convexity of $\langle d'_1, \dots, d'_{k'}\rangle$ we have $\sphericalangle(d'_{i'-1}, d_i, d_j) = \sphericalangle(d'_{i'-1}, d'_{i'}, d'_{j'}) \leq \pi$. Thus both angles on the right hand side are smaller or equal to π and we obtain

$$\sphericalangle(d_{i-1}, d_i, d'_{i'-1}) \leq \sphericalangle(d_{i-1}, d_i, d_j) \leq \sphericalangle(d_{i-1}, d_i, d_{i+1}),$$

where the last inequality holds by convexity of $\langle d_1, \dots, d_k\rangle$. On the other hand, minimality of the exterior angles of $\langle d_1, \dots, d_k\rangle$ ensures that $\sphericalangle(d_{i-1}, d_i, d_{i+1}) \leq \sphericalangle(d_{i-1}, d_i, d'_{i'-1})$. Thus we have equality and by Lemma 4.4.13 we have $d'_{i'-1} = d_{i+1}$. By convexity of the $\mathscr{D}$-cells we have $0 \leq \sphericalangle(d_j, d_i, d_{i+1}) \leq \pi$ and thus either we have

$$0 = \sphericalangle(d_{i+1}, d_i, d_j) = \sphericalangle(d_{i'-1}, d_i, d_j) \text{ or}$$

$$\pi \leq \sphericalangle(d_{i+1}, d_i, d_j) = \sphericalangle(d_{i'-1}, d_i, d_j) \leq \pi.$$

The second case implies

$$\sphericalangle(d_{i-1}, d_i, d_j) = \sphericalangle(d_{i-1}, d_i, d'_{i'-1}) + \sphericalangle(d'_{i'-1}, d_i, d_j) = \sphericalangle(d_{i-1}, d_i, d'_{i'-1}) + \pi,$$

which contradicts the fact that $\sphericalangle(d_{i-1}, d_i, d_j) \leq \pi$ and $0 < \sphericalangle(d_{i-1}, d_i, d'_{i'-1}) \leq \pi$. Thus we must have $0 = \sphericalangle(d_{i+1}, d_i, d_j)$ and both $r_{\mathscr{D}}^{-1}(\overline{\{d_i, d_{i+1}\}})$ and $r_{\mathscr{D}}^{-1}(\overline{\{d_i, d_j\}}) = \overline{\{a, b\}}$ are diameters with $d_{i+1} = c$ or $d_j = c$. Without loss of generality we may assume that the diameter $r_{\mathscr{D}}^{-1}(\overline{\{d_i, d_{i+1}\}}) \in \mathscr{D}$ is red. Then $\overline{\{a, b\}}$ must be green and by definition of the map $r_{\mathscr{D}}$ we get that $r_{\mathscr{D}}(\overline{\{a, b\}}_g) = \overline{\{d_i, d_j\}}_g$ is a diameter and $d_j \neq c$. Thus we have $d_{i+1} = d'_{i'-1} = c$ and because none of the elements of $\mathscr{D}$ is crossed by $\overline{\{d_i, d_i + n\}}_g = \overline{\{a, b\}}_g \in \operatorname{nc} \mathscr{D}$ or $\overline{\{d_i, d_i + n\}}_r \in \mathscr{D}$ the diameter $\overline{\{d_i, d_i + n\}}_r$ is the only diameter in $\mathscr{D}$ and it follows that $d_{i+2} = d'_{i'-2} = d_i + n$. By rotation invariance of $\mathscr{D}$ we get $\overline{\langle d_1, \ldots, d_k\rangle} = \overline{\langle d'_1, \ldots, d'_{k'}\rangle}$.

□

A special situation, which is worth considering in more detail, is the case where we have a π-rotation invariant $\mathscr{D}$-cell, such as in the left-most picture of Figure 4.12. This is only possible for certain diagrams $\mathscr{D}$. The following result summarizes the most important facts about the case where we have a π-rotation invariant $\mathscr{D}$-cell.

Lemma 4.4.14. *There exists a π-rotation invariant $\mathscr{D}$-cell*

$$\langle d_1, \ldots, d_k\rangle = \langle d_1 + n, \ldots, d_k + n\rangle$$

of Dynkin type D_n if and only if $\mathscr{D}$ contains no diameters. In this case, the π-rotation invariant $\mathscr{D}$-cell of Dynkin type D_n is unique, central and it contains all diameters in $(\operatorname{nc} \mathscr{D}) \setminus \mathscr{D}$. *Furthermore, if $\langle d_1, \ldots, d_k\rangle$ is a π-rotation invariant $\mathscr{D}$-cell of Dynkin type D_n, then $\{d_i, d_j\}$ is a diameter if and only if $\{d_{i-1}, d_{j-1}\}$ is a diameter.*

Proof. Let $\langle d_1, \ldots, d_k\rangle$ be a π-rotation invariant $\mathscr{D}$-cell of Dynkin type D_n. It is of the form

$$\langle d_1, \ldots, d_k\rangle = \langle d_1, \ldots, d_{\frac{k}{2}}, d_{\frac{k}{2}+1} = d_1 + n, \ldots, d_k = d_{\frac{k}{2}} + n\rangle,$$

with $k \geq 4$ even. In particular, $\overline{\{d_i, d_j\}}$ is a diameter if and only if $j = \frac{k}{2} + i$, which is the case if and only if $\overline{\{d_{i-1}, d_{j-1}\}}$ is a diameter. Furthermore with the diameters $\overline{\{d_i, d_i + n\}} = \overline{\{d_i, d_{\frac{k}{2}+i}\}}$, the $\mathscr{D}$-cell $\langle d_1, \ldots, d_k\rangle$ of Dynkin type D_n contains the central vertex c. It is thus a central $\mathscr{D}$-cell of Dynkin type D_n and c lies in its interior. Any diameter $\overline{\{a, a + n\}} \in \mathcal{A}(\mathcal{P}_{2n})$ contains the vertex c and is therefore either contained in $\langle d_1, \ldots, d_k\rangle$ or crosses one of the pairs of arcs $\overline{\{d_i, d_{i+1}\}}$. So any π-rotation invariant $\mathscr{D}$-cell of Dynkin type D_n contains all diameters in $(\operatorname{nc} \mathscr{D}) \setminus \mathscr{D}$ and if there is a π-rotation invariant $\mathscr{D}$-cell $\langle d_1, \ldots, d_k\rangle$ of Dynkin type D_n, there is at least one diameter in $(\operatorname{nc} \mathscr{D}) \setminus \mathscr{D}$, e.g. $\{d_1, d_1 + n\}$. By Lemma 4.4.12 the π-rotation invariant $\mathscr{D}$-cell of Dynkin type D_n is thus unique if it exists and in this case, also by Lemma 4.4.12, $\mathscr{D}$ contains no diameters.

Suppose, on the other hand, that $\mathscr{D}$ contains no diameters. By [HJR2, Lemma 5.1], there exists a diameter $\overline{\{a, a + n\}}$ in $(\operatorname{nc} \mathscr{D}) \setminus \mathscr{D}$. By rotation invariance of $\mathscr{D}$ and because $\mathscr{D}$ contains no diameters, the diameter $\overline{\{a, a + n\}}$ is contained in a π-rotation invariant $\mathscr{D}$-cell. Thus in particular such a $\mathscr{D}$-cell exists. □

We now have all the necessary machinery ready to define mutation with respect to $\mathscr{D}$. Informally speaking, one can think of the mutation $\mu_{\mathscr{D}}$, respectively $\mu_{\mathscr{D}}^{-}$, as rotating the interior of each $\mathscr{D}$-cell of Dynkin type D_n in an anticlockwise, respectively clockwise, direction.

Definition 4.4.15. For every non-crossing diagram $\mathscr{D}$ of Dynkin type D_n we define the mutation maps

$$\mu_{\mathscr{D}}\colon \operatorname{nc}\mathscr{D} \to \operatorname{nc}\mathscr{D} \quad \text{and} \quad \mu_{\mathscr{D}}^{-}\colon \operatorname{nc}\mathscr{D} \to \operatorname{nc}\mathscr{D}$$

as follows.

- The maps $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^{-}$ leave $\mathscr{D} \subseteq \operatorname{nc}\mathscr{D}$ invariant:

$$\mu_{\mathscr{D}}\Big|_{\mathscr{D}} = \mu_{\mathscr{D}}^{-}\Big|_{\mathscr{D}} = \operatorname{id}_{\mathscr{D}}.$$

- Suppose $\overline{\{a,b\}} \in (\operatorname{nc}\mathscr{D}) \setminus \mathscr{D}$. By Lemma 4.4.12, the element $\overline{\{a,b\}}$ is contained in a unique pair of $\mathscr{D}$-cells $\overline{\langle d_1, \ldots, d_k\rangle}$ and thus $r_{\mathscr{D}}(\overline{\{a,b\}}) = \overline{\{d_i, d_j\}}$ for some $i, j \in \{1, \ldots, k\}$. We set

$$\mu_{\mathscr{D}}(\overline{\{a,b\}}) = r_{\mathscr{D}}^{-1}(\overline{\{d_{i+1}, d_{j+1}\}})$$

and

$$\mu_{\mathscr{D}}^{-}(\overline{\{a,b\}}) = r_{\mathscr{D}}^{-1}(\overline{\{d_{i-1}, d_{j-1}\}}),$$

where the colour of $\mu_{\mathscr{D}}(\overline{\{a,b\}})$, respectively $\mu_{\mathscr{D}}^{-}(\overline{\{a,b\}})$, if it is a diameter, is specified as follows.

  - If $\mathscr{D}$ contains no diameters, then by Lemma 4.4.14 all diameters are contained in the unique central pair of $\mathscr{D}$-cells $\overline{\langle d_1, \ldots, d_n\rangle}$ and only diameters get mutated to diameters. We define both $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^{-}$ to change their colour. So if $\overline{\{a,b\}}$ is a red diameter, both $\mu_{\mathscr{D}}(\overline{\{a,b\}})$ and $\mu_{\mathscr{D}}^{-}(\overline{\{a,b\}})$ are set to be green and vice versa.
  - If $\mathscr{D}$ contains more than one diameter of the same colour, then all diameters in $\mathscr{D}$ are of the same colour. Those of $\mu_{\mathscr{D}}(\overline{\{a,b\}})$ and $\mu_{\mathscr{D}}^{-}(\overline{\{a,b\}})$ which are diameters are set to be of the same colour as all the diameters in $\mathscr{D}$.
  - If $\mathscr{D}$ contains exactly one diameter $\overline{\{x, x+n\}}$, then if $\mu_{\mathscr{D}}(\overline{\{a,b\}}) = \overline{\{a', b'\}}$, respectively $\mu_{\mathscr{D}}^{-}(\overline{\{a,b\}}) = \overline{\{a'', b''\}}$, is a diameter, it is set to be of different colour than $\overline{\{x, x+n\}}$ if and only if $\{a', b'\} = \{x, x+n\}$, respectively $\{a'', b''\} = \{x, x+n\}$. In all other cases, if $\mu_{\mathscr{D}}(\overline{\{a,b\}})$, respectively $\mu_{\mathscr{D}}^{-}(\overline{\{a,b\}})$, is a diameter, it is set to be of the same colour as $\overline{\{x, x+n\}}$.

  Note that in the case where $\mathscr{D}$ contains two diameters of different colour, the diagram $(\operatorname{nc}\mathscr{D}) \setminus \mathscr{D}$ of Dynkin type D_n does not contain any diameters.

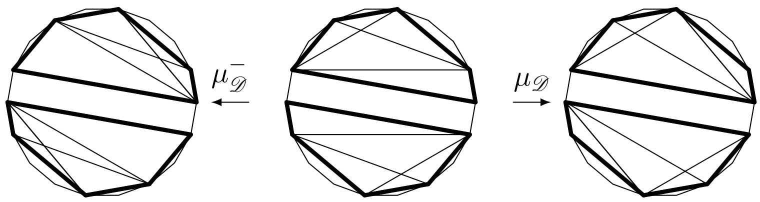


Figure 4.14: This picture illustrates the mutation with respect to a non-crossing diagram $\mathscr{D}$ (marked by thick lines) of some elements in $\operatorname{nc}\mathscr{D}$ contained in a non-central pair of $\mathscr{D}$-cells. As we will see later (Theorem 4.4.22), combinatorial mutation of diagrams of Dynkin type D_n corresponds to mutation in the triangulated category $\mathcal{C}_{kD_n}$. In this picture we see nicely that the mutations $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}}^{-}$ give rise to different diagrams of Dynkin type D_n, which correspond to different subcategories of $\mathcal{C}_{kD_n}$.

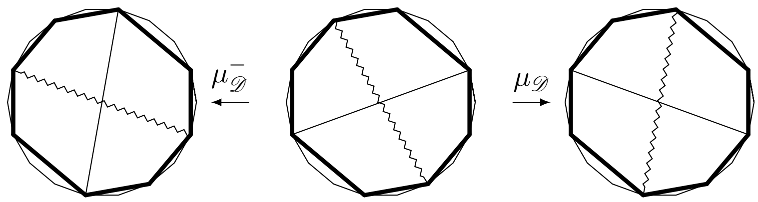


Figure 4.15: This figure illustrates mutation with respect to a non-crossing diagram $\mathscr{D}$ (marked by thick lines) which contains no diameters. Only diameters get mutated to diameters and mutation changes their colour.

Figures 4.14 provides an example of mutation of elements contained in a non-central pair of $\mathscr{D}$-cells. Figures 4.15 to 4.17 provide some examples of mutation of elements contained in central pairs of $\mathscr{D}$-cells for different possible configurations of the non-crossing diagram $\mathscr{D}$, where we cover the case where $\mathscr{D}$ contains no diameters, where it contains more than one diameters of the same colour and where it contains exactly one diameter.

Remark 4.4.16. For any non-crossing diagram $\mathscr{D}$ of Dynkin type D_n, the map $\mu_{\mathscr{D}}\colon \operatorname{nc}\mathscr{D} \to \operatorname{nc}\mathscr{D}$ is a bijection with inverse $\mu_{\mathscr{D}}^{-}$.

The following definition mirrors the definition of D-mutation pairs in triangulated categories by Iyama and Yoshino [IY], cf Definition 4.2.8.

Definition 4.4.17. Let $\mathscr{X}$ and $\mathscr{X}'$ be subdiagrams of $\operatorname{nc}\mathscr{D}$. We call the pair of diagrams $(\mathscr{X}, \mathscr{X}')$ of Dynkin type D_n a *$\mathscr{D}$-mutation pair* if $\mathscr{D} \subseteq \mathscr{X}' \subseteq \mu_{\mathscr{D}}^{-}(\mathscr{X})$ and $\mathscr{D} \subseteq \mathscr{X} \subseteq \mu_{\mathscr{D}}(\mathscr{X}')$.

Remark 4.4.18. Since $\mu_{\mathscr{D}}$ is a bijection on $\operatorname{nc}\mathscr{D}$ with inverse $\mu_{\mathscr{D}}^{-}$, for any $\mathscr{D}$-mutation pair $(\mathscr{X}, \mathscr{X}')$ we have $\mathscr{X} = \mu_{\mathscr{D}}(\mathscr{X}')$ and $\mathscr{X}' = \mu_{\mathscr{D}}^{-}(\mathscr{X})$.

According to Definition 4.4.15, the colour we assign to a diameter we obtain by mutation with respect to $\mathscr{D}$ depends on what the diagram $\mathscr{D}$ looks like. It is useful to note that the following always holds.

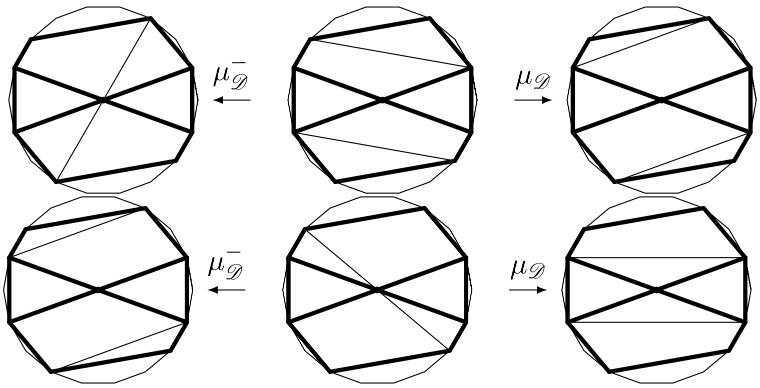


Figure 4.16: This picture illustrates the mutation of some elements in $\operatorname{nc}\mathscr{D}$ with respect to a non-crossing diagram $\mathscr{D}$ (marked by thick lines) which contains more than one diameter of the same colour. Pairs of arcs can get glued together to diameters and diameters can get split up into pairs of arcs when mutating.

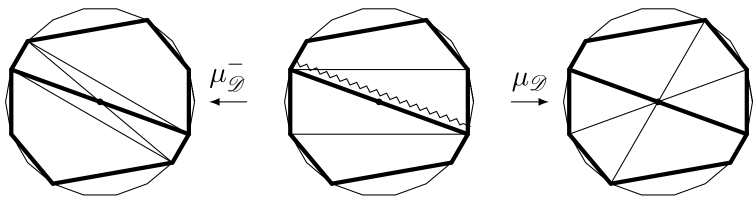


Figure 4.17: This picture illustrates the mutation of some elements in $\operatorname{nc}\mathscr{D}$ with respect to a non-crossing diagram $\mathscr{D}$ (marked by thick lines) which contains exactly one diameter.

Lemma 4.4.19. *Mutation changes the colour of diameters: Let $\overline{\{a,b\}}$ be a diameter in $(\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ such that $\mu^-_{\mathscr{D}}(\overline{\{a,b\}})$ (respectively $\mu_{\mathscr{D}}(\overline{\{a,b\}})$) is also a diameter. Then $\mu^-_{\mathscr{D}}(\overline{\{a,b\}})$ (respectively $\mu_{\mathscr{D}}(\overline{\{a,b\}})$ is of different colour than $\overline{\{a,b\}}$.*

Proof. If $\mathscr{D}$ contains no diameters this is explicitly stated in Definition 4.4.15. If $\mathscr{D}$ contains two diameters of different colour, then $(\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ contains no diameters, so the statement is trivial. Thus we only have to consider the case where $\mathscr{D}$ contains at least one diameter and all its diameters are of the same colour.

We first note that if $r_{\mathscr{D}}(\overline{\{a,b\}})$ is a pair of radii then $r_{\mathscr{D}}(\mu_{\mathscr{D}}(\overline{\{a,b\}}))$ is not: Since the ending vertices of $r_{\mathscr{D}}(\overline{\{a,b\}})$ and $r_{\mathscr{D}}(\mu_{\mathscr{D}}(\overline{\{a,b\}}))$ are pairwise distinct, at most one of them can be the central vertex c.

If $\mathscr{D}$ contains more than one diameter of the same colour, then every diameter in $(\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ gets mapped to a pair of radii under the map $r_{\mathscr{D}}$. Thus, in this case, diameters do not get mutated to diameters.

It remains to check the case where $\mathscr{D}$ contains exactly one diameter. Assume that both $\overline{\{a,b\}} \in (\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D}$ and $\mu_{\mathscr{D}}(\overline{\{a,b\}})$ (respectively $\mu^-_{\mathscr{D}}(\overline{\{a,b\}})$) are diameters. Since $r_{\mathscr{D}}(\overline{\{a,b\}})$ and $r_{\mathscr{D}}(\mu_{\mathscr{D}}(\overline{\{a,b\}}))$ (respectively $r_{\mathscr{D}}(\mu^-_{\mathscr{D}}(\overline{\{a,b\}}))$) cannot both be pairs of radii, one of them has to be a diameter. However, there is only one diameter in $r_{\mathscr{D}}((\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D})$ and it is of different colour to all the pairs of radii in $r_{\mathscr{D}}((\mathrm{nc}\,\mathscr{D}) \setminus \mathscr{D})$. Therefore mutation changes colour. □

Before we start explaining how the combinatorial concept of mutation from Definition 4.4.15 encodes mutation in the triangulated category $\mathcal{C}_{kD_n}$, we note one last useful fact about mutation of diameters.

Lemma 4.4.20. *Consider a diameter $\overline{\{a,a+n\}} \in \mathrm{nc}\,\mathscr{D}$. If $\mu_{\mathscr{D}}(\overline{\{a,a+n\}})$ is a pair of arcs, or if $\mu^-_{\mathscr{D}}(\overline{\{a,a+n\}})$ is a pair of arcs, then $r_{\mathscr{D}}(\overline{\{a,a+n\}}) = \overline{\{a,c\}}$ is a pair of radii.*

The way we may think about Lemma 4.4.20 is that only pairs of radii may get split up into pairs of arcs and diameters have to stay "whole" (at least for one mutation step).

Proof. Let $\overline{\{a,a+n\}} = r^{-1}_{\mathscr{D}}(\overline{\{d_i d_j\}})$. If $\mu^{(-)}_{\mathscr{D}}(\overline{\{a,a+n\}}) = r^{-1}_{\mathscr{D}}\overline{\{d_{i\pm1}, d_{j\pm1}\}}$ is a pair of arcs, then the four vertices $d_{i\pm1}, d_{j\pm1}, (d_{i\pm1}+n)$ and $(d_{j\pm1}+n)$ are pairwise distinct. By rotation invariance of $\mathscr{D}$, the element $\overline{\{d_i, d_j\}} = r_{\mathscr{D}}(\overline{\{a,a+n\}}) \in r_{\mathscr{D}}(\mathscr{D})$ cannot be a diameter. □

4.4.4 Mutation of torsion pairs in cluster categories of finite Dynkin type D

Our goal is to give a combinatorial interpretation for mutation of torsion pairs in the cluster category $\mathcal{C}_{kD_n}$via mutation of Ptolemy diagrams of Dynkin type D_n. Since $\mathcal{C}_{kD_n}$ is 2-Calabi-Yau, every subcategory satisfies $\tau D = \Sigma D$ (cf. Remark 4.2.14) and since it contains only finitely many indecomposable objects (up to isomorphism), any subcategory is functorially finite. By Theorem 4.2.13 mutation of a torsion pair (X,Y) in $\mathcal{C}_{kD_n}$ is thus defined with respect to every subcategory $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$. We now want to translate this into our combinatorial model: The torsion part X of the torsion pair (X,Y) corresponds to a Ptolemy diagram $\mathscr{X}$ of Dynkin type D_n and the subcategory $X \cap (\Sigma^{-1}X)^{\perp}$ corresponds to the diagram $\mathscr{X} \cap \operatorname{nc}\mathscr{X}$ of those arcs in $\mathscr{X}$ that do not cross any other arcs in $\mathscr{X}$. A subcategory $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$ corresponds to a subdiagram $\mathscr{D} \subseteq \mathscr{X} \cap \operatorname{nc}\mathscr{X}$. Any subdiagram of $\mathscr{X} \cap \operatorname{nc}\mathscr{X}$ is a non-crossing diagram of Dynkin type D_n. This corresponds to the fact, that any subcategory of $X \cap (\Sigma^{-1}X)^{\perp}$ is rigid, since rigid subcategories correspond to non-crossing diagrams – recall from Proposition 4.4.3 that crossings count dimensions of Ext^1-spaces between indecomposable objects.

Mutation of Ptolemy diagrams of Dynkin type D_n provides a combinatorial model for mutation of torsion pairs in the cluster category of Dynkin type D_n. This is formalized in the following result.

Theorem 4.4.21. *Let $\mathscr{X}$ be a Ptolemy diagram of Dynkin type D_n corresponding to the torsion part of a torsion pair (X,Y) in $\mathcal{C}_{kD_n}$ and let $\mathscr{D} \subset \mathscr{X} \cap \operatorname{nc}\mathscr{X}$ be a subdiagram corresponding to a subcategory $D \subseteq X \cap (\Sigma^{-1}X)^{\perp}$. The mutation $\mu_{\mathscr{D}}(\mathscr{X})$, respectively $\mu^-_{\mathscr{D}}(\mathscr{X})$ corresponds to the torsion part of the torsion pair $(\mu_D(X), \mu_{\Sigma D}(Y))$, respectively of $(\mu^-_D(X), \mu^-_{\Sigma D}(Y))$.*

Theorem 4.4.21 is a direct corollary of the following, more general result.

Theorem 4.4.22. *Let $\mathscr{D}$ be a non-crossing diagram of Dynkin type D_n corresponding to a rigid subcategory D of $\mathcal{C}_{kD_n}$. Let $\mathscr{X}$ and $\mathscr{X}'$ be subdiagrams of $\operatorname{nc}\mathscr{D}$ with $\mathscr{X}$ corresponding to the subcategory X and $\mathscr{X}'$ corresponding to the subcategory X' of $\mathcal{C}_{kD_n}$. Then $(\mathscr{X}, \mathscr{X}')$ is a $\mathscr{D}$-mutation pair if and only if (X, X') is a D-mutation pair.*

The proof of Theorem 4.4.21 follows from Theorem 4.4.22 as follows.

Proof. Because $\mathscr{D}$ is a subdiagram of $\mathscr{X} \cap \operatorname{nc}\mathscr{X}$ it is non-crossing and we have $\mathscr{X} \subseteq \operatorname{nc}\mathscr{D}$. Applying Theorem 4.4.22 to the $\mathscr{D}$-mutation pairs $(\mathscr{X}, \mu^-_{\mathscr{D}}(\mathscr{X}))$ and $(\mu_{\mathscr{D}}(\mathscr{X}), \mathscr{X})$ yields the result. □

In order to understand how the combinatorial and the categorical mutations agree, we want to calculate middle terms of extensions between certain indecomposable objects in the cluster category $\mathcal{C}_{kD_n}$. The proof of 4.4.22 relies on the following lemma.

Lemma 4.4.23. *Let $\mathscr{D}$ be a non-crossing diagram of Dynkin type D_n corresponding to the rigid subcategory D of $\mathcal{C}_{kD_n}$. Consider the indecomposable objects in $\mathcal{C}_{kD_n}$ corresponding to $\overline{\{x,y\}} \in (\operatorname{nc}\mathscr{D}) \setminus \mathscr{D}$ and to its mutation $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$. Then there exists a distinguished triangle*

$$\overline{\{x,y\}} \longrightarrow d \longrightarrow \mu^-_{\mathscr{D}}(\overline{\{x,y\}}) \longrightarrow \Sigma\overline{\{x,y\}}$$

in $\mathcal{C}_{kD_n}$ with $d \in D$.

Using this result, we can prove Theorem 4.4.22.

Proof. Assume that $(\mathscr{X}, \mathscr{X}')$ is a $\mathscr{D}$-mutation pair. We show that (X, X') is a D-mutation pair, i.e. that $D \subseteq X \subseteq \mu_D(X')$ and $D \subseteq X' \subseteq \mu_D^-(X)$ (cf. Definition 4.2.8). Because $\mathscr{D} \subseteq \mathscr{X}, \mathscr{X}'$ we have $D \subseteq X, X'$. By Lemma 4.2.11 we have $D = \mu_D(D) \subseteq \mu_D(X')$ and $D = \mu_D^-(D) \subseteq \mu_D^-(X)$.

It remains to show that every object m' in X' which is not an object in D is contained in $\mu_D^-(X)$ and that every object m in X which is not an object in D is contained in $\mu_D(X')$. The indecomposable objects in X but not in D are labelled by arcs $\overline{\{x,y\}} \in \mathscr{X} \setminus \mathscr{D}$. The indecomposable objects in X' but not in D are labelled by arcs in $\mathscr{X}' \setminus \mathscr{D} = \mu_{\mathscr{D}}^-(\mathscr{X} \setminus \mathscr{D})$, which are of the form $\mu_{\mathscr{D}}^-(\overline{\{x,y\}})$ for some $\overline{\{x,y\}} \in \mathscr{X} \setminus \mathscr{D}$.

Since both $\overline{\{x,y\}}$ and $\mu_{\mathscr{D}}^-(\overline{\{x,y\}})$ lie in $\operatorname{nc}\mathscr{D}$, they both label objects in ${}^{\perp}(\Sigma D) = (\Sigma^{-1}D)^{\perp}$. Furthermore, by Lemma 4.4.23, for every element $\overline{\{x,y\}} \in (\operatorname{nc}\mathscr{D}) \setminus \mathscr{D}$ there is a distinguished triangle

$$\overline{\{x,y\}} \longrightarrow d \longrightarrow \mu_{\mathscr{D}}^-(\overline{\{x,y\}}) \longrightarrow \Sigma\overline{\{x,y\}}\,, \tag{4.2}$$

with $d \in D$. So $\overline{\{x,y\}} \in \mu_D(X')$ and $\mu_{\mathscr{D}}^-(\overline{\{x,y\}}) \in \mu_D^-(X)$ and thus all indecomposable objects in X' lie in $\mu_D^-(X)$ and all indecomposable objects in X lie in $\mu_D(X')$. Since all subcategories are assumed to be closed under finite direct sums and isomorphisms, this proves the first direction of the claim.

On the other hand, suppose (X, X') is a D-mutation pair and let $\tilde{X}$ be the subcategory corresponding to the diagram $\mu_{\mathscr{D}}^-(\mathscr{X})$. Then, because $(\mathscr{X}, \mu_{\mathscr{D}}^-(\mathscr{X}))$ is a $\mathscr{D}$-mutation pair, the pair $(X, \tilde{X})$ is a D-mutation pair, therefore $\tilde{X} = \mu_D^-(X) = X'$. So the diagram $\mu_{\mathscr{D}}^-(\mathscr{X})$ corresponds to X' and we get that $\mu_{\mathscr{D}}^-(\mathscr{X}) = \mathscr{X}'$ and $(\mathscr{X}, \mathscr{X}')$ is a $\mathscr{D}$-mutation pair. □

The rest of this section is devoted to the proof of Lemma 4.4.23. It can be worked out using methods introduced by Buan, Marsh, Reineke, Reiten and Todorov [BMRRT]. They used graphical calculus to calculate short exact sequences in the module category $\operatorname{mod} kD_n$, which induce distinguished triangles in $\mathcal{C}_{kD_n}$. This technique works when we have one-dimensional Ext^1-spaces. This is the case in our setting, as the following lemma indicates.

Lemma 4.4.24. *Let $\mathscr{D}$ be a non-crossing diagram of Dynkin type D_n. Consider the indecomposable object in $\mathcal{C}_{kD_n}$ corresponding to $\overline{\{a,b\}} \in \operatorname{nc}\mathscr{D}$ and the one corresponding to its mutation $\mu_{\mathscr{D}}(\{a,b\})$. We have*

$$\dim(\operatorname{Ext}^1_{\mathcal{C}_{kD_n}}(\overline{\{a,b\}}, \mu_{\mathscr{D}}^-(\overline{\{a,b\}})) = 1.$$

Proof. By Proposition 4.4.3, the dimension of the extension space between two indecomposable objects is equal to the number of times the corresponding pairs of arcs cross (cf. Definition 4.4.2). Let $\overline{\langle d_1, \ldots, d_k\rangle}$ be the pair of $\mathscr{D}$-cells containing $\overline{\{a,b\}}$ and $\mu_{\mathscr{D}}^-(\overline{\{a,b\}})$ with $\overline{\{a,b\}} = r_{\mathscr{D}}^{-1}(\overline{\{d_i,d_j\}})$ and $\mu_{\mathscr{D}}^-(\overline{\{a,b\}}) = r_{\mathscr{D}}^{-1}(\overline{\{d_{i-1},d_{j-1}\}})$. The vertices d_i, d_{j-1}, d_j

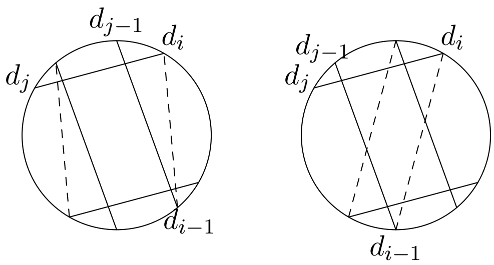


Figure 4.18: This figure illustrates the contradiction we obtain when we assume that $\overline{\{d_i, d_j\}}$ and $\mu^-_{\mathscr{D}}(\overline{\{d_i, d_j\}}) = \overline{\{d_{i-1}, d_{j-1}\}}$ cross twice.

and d_{i-1} appear in this order in an anticlockwise direction on the boundary of $\langle d_1, \ldots, d_k \rangle$. The two arcs $\{d_i, d_j\}$ and $\{d_{i-1}, d_{j-1}\}$ thus cross. We distinguish the following cases.

- If both $\overline{\{a, b\}}$ and $\mu^-_{\mathscr{D}}(\overline{\{a, b\}})$ are diameters then by Lemma 4.4.19 they are of different colour, so they cross once.
- If one of $\overline{\{a, b\}}$ and $\mu^-_{\mathscr{D}}(\overline{\{a, b\}})$ is a diameter and the other one is a pair of arcs it follows directly from Definition 4.4.2 that they cross once.
- Now consider the case where both $\overline{\{a, b\}}$ and $\mu^-_{\mathscr{D}}(\overline{\{a, b\}})$ are pairs of arcs. We show that $\overline{\{d_i, d_j\}}$ and $\overline{\{d_{i-1}, d_{j-1}\}}$ cannot cross twice, i.e. we show that if the arc $\{d_i, d_j\}$ crosses $\{d_{i-1}, d_{j-1}\}$ it cannot cross its partner $\{d_{i-1} + n, d_{j-1} + n\}$. Assume, for a contradiction, that it does and without loss of generality assume $d_j < d_i + n$. Then both d_{j-1} and $d_{i-1} + n$ would lie between d_i and d_j in a clockwise direction, cf. Figure 4.18. This would imply that $\overline{\{d_i, d_{i-1}\}}$ and $\overline{\{d_i, d_j\}}$ cross. However, $\overline{\{d_i, d_{i-1}\}}$ lies in $r_{\mathscr{D}}(\mathscr{D})$ and $\overline{\{d_i, d_j\}}$ is a pair of arcs in $\mathscr{X}$, so this contradicts the fact that $\mathscr{D} \subseteq \operatorname{nc} \mathscr{X}$.

□

By [BMRRT, Proposition 1.6] the indecomposable objects in the cluster category $\mathcal{C}_{kD_n}$ are either induced by indecomposable kD_n-modules or by shifts of indecomposable projective kD_n-modules. We label the (up to isomorphism) unique module which induces an indecomposable object $\overline{\{a, b\}} \in \mathcal{C}_{kD_n}$ by $m_{\overline{\{a,b\}}}$, see Figure 4.19 for the Auslander-Reiten quiver $\Gamma(\operatorname{mod} kD_n)$ of the module category with this labelling. The projective modules are the modules of the form $m_{\overline{\{0,j\}}}$ for $2 \leq j \leq n$.

To compute the middle term of extensions in $\operatorname{mod} kD_n$, the notion of starting and ending frame in $\operatorname{mod} kD_n$ is a useful concept. Defined in [BMRRT, Definition 8.4] for a vertex in the Auslander-Reiten quiver $\Gamma(\operatorname{mod} kD_n)$, we formulate the starting and ending frame for representatives of the isomorphism classes of indecomposable modules in $\operatorname{mod} kD_n$, which is equivalent, as vertices in the Auslander-Reiten quiver are labelled by isomorphism classes of indecomposable objects. Let

$$\mathcal{A}(\operatorname{mod} kD_n) = \{m_{\overline{\{a,b\}}} \mid \overline{\{a, b\}} \in \mathcal{A}(\mathcal{P}_{2n})\}$$

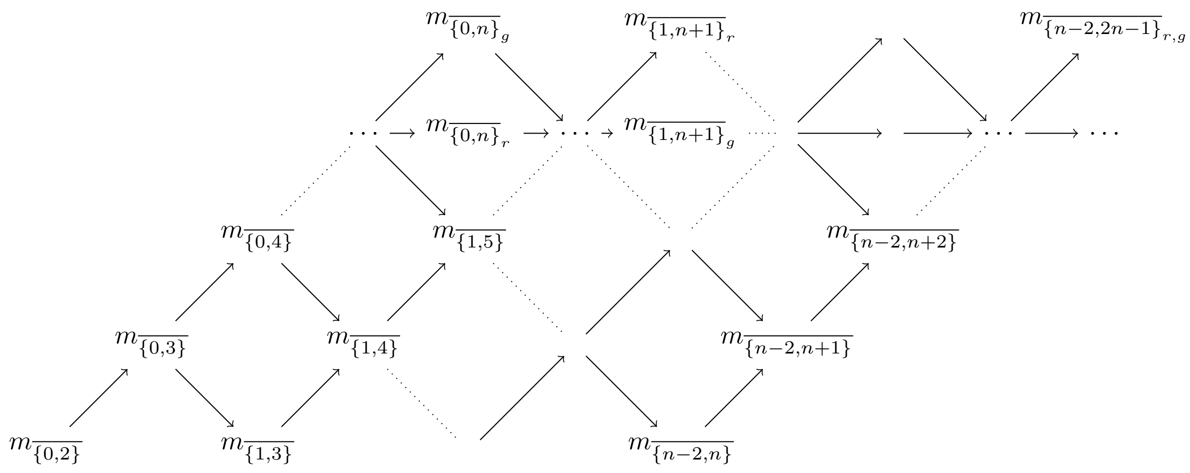


Figure 4.19: The Auslander-Reiten quiver $\Gamma(\operatorname{mod} kD_n)$ of the module category $\operatorname{mod} kD_n$: The colours of the indexing diameters alternate along both the top and the second to top horizontal levels.

be the set of indecomposable modules in $\operatorname{mod} kD_n$ that label the vertices of the Auslander-Reiten quiver according to our labelling. There is one representative in $\mathcal{A}(\operatorname{mod} kD_n)$ for each isomorphism class of indecomposable modules of $\operatorname{mod} kD_n$.

Definition 4.4.25 ([BMRRT, Definition 8.4])**.** Let $m_{\overline{\{a,b\}}} \in \mathcal{A}(\operatorname{mod} kD_n)$. Its *starting frame* $F_s(m_{\overline{\{a,b\}}})$ and *ending frame* $F_e(m_{\overline{\{a,b\}}})$) are defined as follows:

$$F_s(m_{\overline{\{a,b\}}}) = \left\{ m \in \mathcal{A}(\operatorname{mod} kD_n) \;\middle|\; \begin{array}{c} \operatorname{Hom}_{kD_n}(m_{\overline{\{a,b\}}}, m) \neq 0 \text{ and} \\ \operatorname{Hom}_{kD_n}(m_{\overline{\{a,b\}}}, \tau m) = 0 \end{array} \right\}$$

$$F_e(m_{\overline{\{a,b\}}}) = \left\{ m \in \mathcal{A}(\operatorname{mod} kD_n) \;\middle|\; \begin{array}{c} \operatorname{Hom}_{kD_n}(m, m_{\overline{\{a,b\}}}) \neq 0 \text{ and} \\ \operatorname{Hom}_{kD_n}(\tau^{-1} m, m_{\overline{\{a,b\}}}) = 0 \end{array} \right\}$$

Before we explicitly state what the starting and ending frames of modules in $\operatorname{mod} kD_n$ look like, we observe that [BMRRT, Corollary 8.5] allows us to work out middle terms occuring in extensions between two modules in $\operatorname{mod} kD_n$, if we know their starting and ending frame: Let $m_{\overline{\{a,b\}}}$ and $m_{\overline{\{x,y\}}}$ be indecomposable objects in $\operatorname{mod} kD_n$ such that $\operatorname{Ext}^1_{kD_n}(m_{\overline{\{a,b\}}}, m_{\overline{\{x,y\}}})$ is one-dimensional. Then the (up to isomorphism) unique middle term $\tilde{m}$ occuring in each non-trivial extension of $m_{\overline{\{a,b\}}}$ by $m_{\overline{\{x,y\}}}$ is the direct sum of one copy of each indecomposable object in the intersection $F_s(m_{\overline{\{x,y\}}}) \cap F_e(m_{\overline{\{a,b\}}})$:

$$\tilde{m} \cong \bigoplus_{m \in F_s(m_{\overline{\{x,y\}}}) \cap F_e(m_{\overline{\{a,b\}}})} m.$$

The starting and ending frames can be worked out using the tables in Bongartz's paper [B, Section 1.3], see also [BMRRT, Section 8]. For an indecomposable module

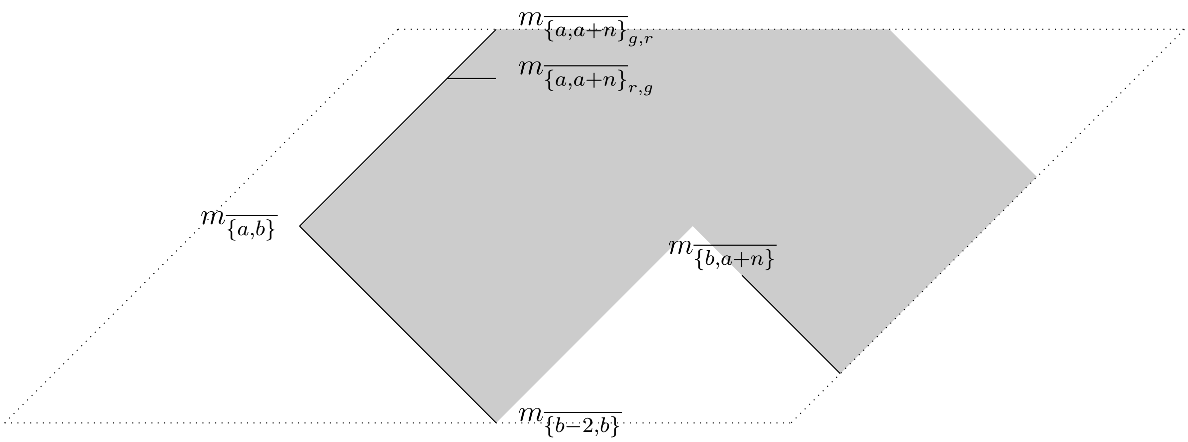


Figure 4.20: The lines mark the starting frame of the module $m_{\overline{\{a,b\}}}$ in $\Gamma(\operatorname{mod} kD_n)$. The area into which there are non-trivial morphisms from $m_{\overline{\{a,b\}}}$ is marked in grey.

$m_{\overline{\{a,b\}}} \in \mathcal{A}(\operatorname{mod} kD_n)$ with $0 \leq a < b < a+n$, they are given by:

$$F_s(m_{\overline{\{a,b\}}}) = \begin{array}{l} \{m_{\overline{\{a,y\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid b \leq y \leq a+n-1\} \\ \cup \{m_{\overline{\{a,a+n\}}_g}, m_{\overline{\{a,a+n\}}_r}\} \\ \cup \{m_{\overline{\{x,b\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid a \leq x \leq b-2\} \\ \cup \{m_{\overline{\{x,a+n\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid b \leq x \leq n-2\} \end{array}$$

and

$$F_e(m_{\overline{\{a,b\}}}) = \begin{array}{l} \{m_{\overline{\{a,y\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid a+2 \leq y \leq b\} \\ \cup \{m_{\overline{\{x,b\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid b-n+1 \leq x \leq a\} \\ \cup \{m_{\overline{\{b-n,b\}}_r}, m_{\overline{\{b-n,b\}}_g}\} \\ \cup \{m_{\overline{\{b-n,y\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid b-n+2 \leq y \leq a\}, \end{array}$$

cf. Figures 4.20 and 4.21. For a module $m_{\overline{\{a,a+n\}}_r} \in \mathcal{A}(\operatorname{mod} kD_n)$ with $0 \leq a \leq n-2$ associated to a red diameter the starting and ending frames are given by

$$F_s(m_{\overline{\{a,a+n\}}_r}) = \begin{array}{l} \{m_{\overline{\{x,a+n\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid a < x \leq n-2\} \\ \cup \{m_{\overline{\{x,x+n\}}_r} \mid a \leq x \leq n-2\} \end{array}$$

and

$$F_e(m_{\overline{\{a,a+n\}}_r}) = \begin{array}{l} \{m_{\overline{\{a,y\}}} \in \mathcal{A}(\operatorname{mod} kD_n) \mid a+2 \leq y < a+n\} \\ \cup \{m_{\overline{\{x,x+n\}}_r} \mid a \geq x \geq 0\}, \end{array}$$

as picture in Figures 4.22 and 4.23. The starting and ending frame for modules corresponding to green diameters is given analogously.

Using these ideas, we now provide a proof for Lemma 4.4.23. First we observe how we can calculate extensions in $\mathcal{C}_{kD_n}$ via the module category $\operatorname{mod} kD_n$: Consider two indecomposable objects $\overline{\{a,b\}}, \overline{\{x,y\}} \in \mathcal{C}_{kD_n}$ that are both induced by indecomposable

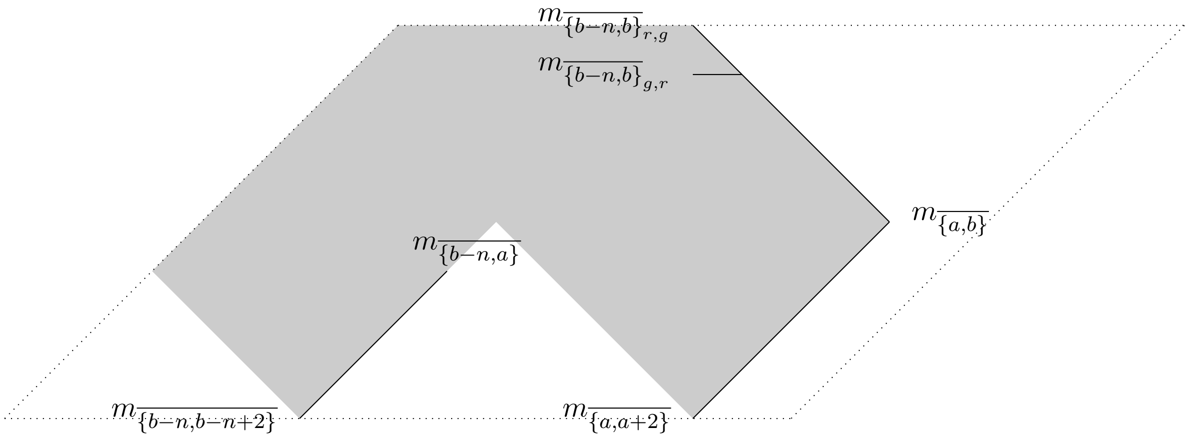


Figure 4.21: The lines mark the ending frame of $m_{\overline{\{a,b\}}}$ in the Auslander-Reiten quiver $\Gamma(\operatorname{mod} kD_n)$. The area from which there are non-trivial morphisms into $m_{\overline{\{a,b\}}}$ is marked in grey.

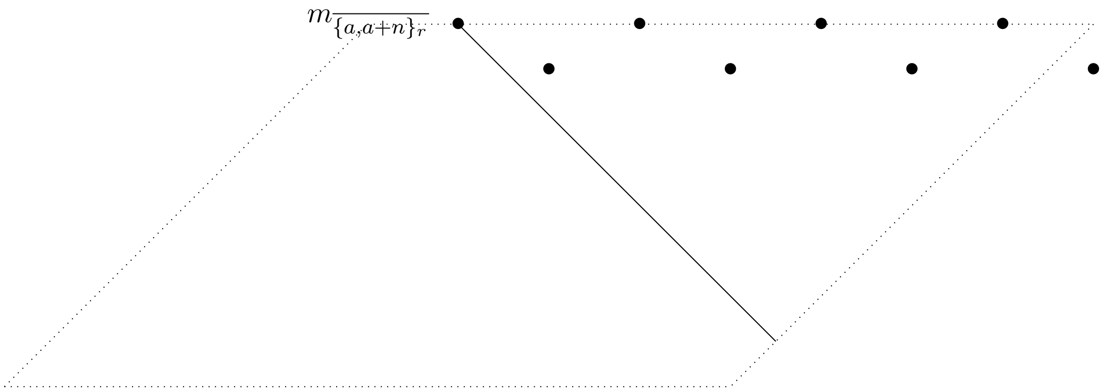


Figure 4.22: The line and the bullets mark the starting frame of $m_{\overline{\{a,a+n\}_r}}$ in $\Gamma(\operatorname{mod} kD_n)$. The bullets correspond to all vertices of the form $\overline{\{x,x+n\}_r}$ with $a \leq x \leq n-2$.

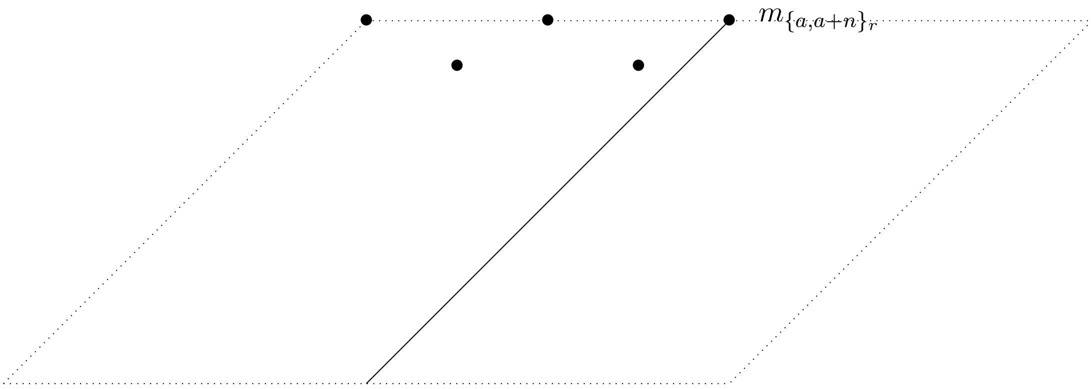


Figure 4.23: The line and the bullets mark the ending frame of $m_{\overline{\{a,a+n\}_r}}$ in $\Gamma(\operatorname{mod} kD_n)$. The bullets correspond to all vertices of the form $\overline{\{x,x+n\}_r}$ with $1 \leq x \leq a$.

modules $m_{\overline{\{a,b\}}}$, respectively $m_{\overline{\{x,y\}}}$ in $\operatorname{mod} kD_n$. Assume that the object $m_{\overline{\{a,b\}}}$ is a projective kD_n-module. Then by [BMRRT, Proposition 1.7(d)] we have

$$\operatorname{Ext}^1_{\mathcal{C}_{kD_n}}(\overline{\{a,b\}}, \overline{\{x,y\}}) \cong \operatorname{Ext}^1_{kD_n}(m_{\overline{\{a,b\}}}, m_{\overline{\{x,y\}}}) \oplus \operatorname{Ext}^1_{kD_n}(m_{\overline{\{x,y\}}}, m_{\overline{\{a,b\}}}).$$

Because $m_{\overline{\{a,b\}}}$ is assumed to be projective, we have $\operatorname{Ext}^1_{kD_n}(m_{\overline{\{a,b\}}}, m_{\overline{\{x,y\}}}) = 0$ and thus

$$\operatorname{Ext}^1_{\mathcal{C}_{kD_n}}(\overline{\{a,b\}}, \overline{\{x,y\}}) = \operatorname{Ext}^1_{kD_n}(m_{\overline{\{x,y\}}}, m_{\overline{\{a,b\}}}).$$

Whenever the elements $\overline{\{a,b\}}$ and $\overline{\{x,y\}}$ in $\mathcal{A}(\mathcal{P}_{2n})$ intersect exactly once, this extension space is one-dimensional and we can apply [BMRRT, Corollary 8.5] to calculate short exact sequences in $\operatorname{mod} kD_n$, which induce distinguished triangles in $\mathcal{C}_{kD_n}$. For simplicity, it will be convenient to not distinguish between pairs of edges and other pairs of arcs. We associate to a pair of edges $\overline{\{a,a+1\}} \in \mathcal{A}(\mathcal{P}_{2n})$ with $0 \le a \le 2n-1$ a zero-object in $\mathcal{C}_{kD_n}$, denoted $\overline{\{a,a+1\}} \cong 0$ and the zero-module $m_{\overline{\{a,a+1\}}} \cong 0$ in $\operatorname{mod} kD_n$. We can now prove Lemma 4.4.23.

Proof. Let $\overline{\{x,y\}} \in (\operatorname{nc}\mathscr{D}) \setminus \mathscr{D}$ and let it and its mutation $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ be contained in the pair of $\mathscr{D}$-cells $\overline{\langle d_1,\dots,d_k\rangle}$ with $\overline{\{x,y\}} = r^{-1}_{\mathscr{D}}(\{d_i,d_j\})$ and $\mu^-_{\mathscr{D}}\overline{\{x,y\}} = r^{-1}_{\mathscr{D}}(\{d_{i-1},d_{j-1}\})$ for some $i,j \in \{1,\dots,k\}$. The vertices $d_i, d_{j-1}, d_j, d_{i-1}$ appear in this order in an anti-clockwise direction on the boundary of the $\mathscr{D}$-cell $\langle d_1,\dots,d_k\rangle$. Without loss of generality, we may assume that $\overline{\{x,y\}}$ is induced by a projective module $m_{\overline{\{x,y\}}}$, and thus assume $d_i = 0$ and $d_j \in \{2,\dots,n\} \cup \{c\}$. Otherwise we obtain the desired distinguished triangle by shifting.

First consider the case where $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ is induced by the shift of a projective module, i.e. $d_{i-1} = 2n-1$ and $d_{j-1} \in \{1,\dots,d_j-1\} \cup \{c\}$. In particular, $\overline{\{d_{i-1},d_i\}}$ is a pair of edges.

- Suppose that $\overline{\{x,y\}} = \overline{\{d_i,d_j\}}$ is a pair of arcs. Then we have $0 = d_i < d_{j-1} < d_j < n$. In particular, $\overline{\{d_{j-1},d_j\}}$ is a pair of arcs. If it is a pair of edges, then as objects in $\mathcal{C}_{kD_n}$ we have $\mu^-_{\mathscr{D}}(\overline{\{x,y\}}) = \Sigma\overline{\{x,y\}}$ and we obtain the desired distinguished triangle

$$\overline{\{x,y\}} \longrightarrow 0 \longrightarrow \mu^-_{\mathscr{D}}(\overline{\{x,y\}}) = \Sigma\overline{\{x,y\}}$$

with middle term $0 \in D$. Otherwise, if $\overline{\{d_{j-1},d_j\}}$ is a pair of internal arcs, then $\overline{\{0,d_{j-1}+1\}}$ and $\overline{\{d_{j-1},d_j\}}$ cross precisely once. Intersecting the starting frame of $m_{\overline{\{0,d_{j-1}+1\}}}$ with the ending frame of $m_{\overline{\{d_{j-1},d_j\}}}$ yields the short exact sequence

$$0 \longrightarrow m_{\overline{\{0,d_{j-1}+1\}}} \longrightarrow m_{\overline{\{0,d_j\}}} \longrightarrow m_{\overline{\{d_{j-1},d_j\}}} \longrightarrow 0,$$

which, since $\Sigma\overline{\{0,d_{j-1}+1\}} = \overline{\{2n-1,d_{j-1}\}} = \mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ and $\overline{\{0,d_j\}} = \overline{\{x,y\}}$, induces the distinguished triangle

$$\overline{\{x,y\}} \longrightarrow \overline{\{d_{j-1},d_j\}} \longrightarrow \mu^-_{\mathscr{D}}(\overline{\{x,y\}}) \longrightarrow \Sigma\overline{\{x,y\}}$$

in $\mathcal{C}_{kD_n}$, whose middle term lies in D.

- Now suppose that $\overline{\{x,y\}} = \overline{\{0,n\}}$ is a coloured diameter. If $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ is also a diameter, then
$$\mu^-_{\mathscr{D}}(\overline{\{x,y\}}) = \overline{\{2n-1, n-1\}} = \Sigma\overline{\{x,y\}}$$
which yields the distinguished triangle
$$\overline{\{x,y\}} \longrightarrow 0 \longrightarrow \mu^-_{\mathscr{D}}(\overline{\{x,y\}}) = \Sigma\overline{\{x,y\}},$$
with middle term $0 \in D$. If on the other hand $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ is a pair of arcs, the pair of arcs $\overline{\{0, d_{j-1}+1\}}$ crosses the diameter $\overline{\{d_{j-1}, d_{j-1}+n\}}$ once and intersecting the starting frame of $m_{\overline{\{0,d_{j-1}+1\}}}$ with the ending frame of $m_{\overline{\{d_{j-1},d_{j-1}+n\}}_{r,g}}$ in $\Gamma(\operatorname{mod} kD_n)$ yields the short exact sequence
$$0 \to m_{\overline{\{0,d_{j-1}+1\}}} \to m_{\overline{\{0,n\}}_{r,g}} \to m_{\overline{\{d_{j-1},d_{j-1}+n\}}_{r,g}} \to 0,$$
where $\overline{\{0,n\}}_{r,g} = \overline{\{x,y\}}_{r,g}$ has the same colour as $\overline{\{d_{j-1}, d_{j-1}+n\}}_{r,g}$. Because the diameter $\overline{\{x,y\}}$ was mutated to a pair of arcs $\overline{\{d_{i-1}, d_{j-1}\}}$ we have $d_j = c$ by Lemma 4.4.20. Thus we have $\overline{\{d_{j-1}, d_{j-1}+n\}}_{r,g} = r^{-1}_{\mathscr{D}}(\overline{\{d_{j-1}, d_j\}}_{r,g})$ and the short exact sequence above gives rise to the distinguished triangle
$$\overline{\{x,y\}}_{r,g} \to r^{-1}_{\mathscr{D}}(\overline{\{d_{j-1}, d_j\}}_{r,g}) \to \overline{\{2n-1, d_{j-1}\}} = \mu^-_{\mathscr{D}}(\overline{\{x,y\}}) \to \Sigma\overline{\{x,y\}}_{r,g},$$
in $\mathcal{C}_{kD_n}$, where $\overline{\{x,y\}}_{r,g}$ is of the same colour as $\overline{\{d_{j-1}, d_j\}}_{r,g}$. Hence the middle term lies in D.

Now consider the case, where the indecomposable object $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ is induced by a kD_n-module. By Lemma 4.4.24, the elements $\overline{\{x,y\}}$ and $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ of $\mathcal{A}(\mathcal{P}_{2n})$ cross exactly once, so we can apply [BMRRT, Corollary 8.5] to calculate middle terms of extensions of $m_{\mu^-_{\mathscr{D}}(\overline{\{x,y\}})}$ by $m_{\overline{\{x,y\}}}$ in $\operatorname{mod} kD_n$.

- Suppose both $\overline{\{x,y\}}$ and $\mu^-_{\mathscr{D}}(\overline{\{x,y\}})$ are pairs of arcs. We have $0 = d_i < d_{j-1} < d_j < n < d_{i-1} < 2n-1$. Because $\overline{\{d_i, d_j\}}$ and $\overline{\{d_{i-1}, d_{j-1}\}}$ intersect exactly once, and since we have $d_i < d_{j-1} < d_j$ we cannot have $d_i + n < d_{i-1} < d_j + n$. Thus either $d_j + n \leq d_{i-1} < 2n-1$ or $d_j < d_{i-1} \leq n$. In both cases, if $d_{i-1} \neq n$, intersecting the starting frame of the module $m_{\overline{\{0,d_j\}}}$ with the ending frame of the module $m_{\overline{\{d_{j-1},d_{i-1}\}}}$ yields the short exact sequence
$$0 \to m_{\overline{\{0,d_j\}}} \to m_{\overline{\{d_{j-1},d_j\}}} \oplus m_{\overline{\{d_i,d_{i-1}\}}} \to m_{\overline{\{d_{j-1},d_{i-1}\}}} \to 0,$$
in $\operatorname{mod} kD_n$ which induces the distinguished triangle
$$\overline{\{x,y\}} \to r^{-1}_{\mathscr{D}}(\overline{\{d_{i-1}, d_i\}}) \oplus r^{-1}_{\mathscr{D}}(\overline{\{d_{j-1}, d_j\}}) \to \Sigma\overline{\{x,y\}}$$
with middle term in D. If on the other hand $d_{i-1} = n$, then we obtain the short exact sequence
$$0 \to m_{\overline{\{0,d_j\}}} \to m_{\overline{\{d_{j-1},d_j\}}} \oplus m_{\overline{\{0,n\}}_r} \oplus m_{\overline{\{0,n\}}_g} \to m_{\overline{\{d_{j-1},n\}}} \to 0.$$

This induces the distinguished triangle

$$\overline{\{x,y\}} \to r_{\mathscr{D}}^{-1}(\overline{\{d_{j-1},d_j\}}) \oplus r_{\mathscr{D}}^{-1}(\overline{\{0,n\}}_r) \oplus r_{\mathscr{D}}^{-1}(\overline{\{0,n\}}_g) \to \Sigma\overline{\{x,y\}}.$$

Because $d_i = 0$ and $d_{i-1} = n$ are neighbouring vertices of the $\mathscr{D}$-cell $\langle d_1,\ldots,d_k\rangle$, the red diameter $\overline{\{0,n\}}_r$ or the green diameter $\overline{\{0,n\}}_g$ have to lie in $\mathscr{D}$. Without loss of generality assume that $\overline{\{0,n\}}_r \in \mathscr{D}$. Then by definition of the replacement map $r_{\mathscr{D}}$, the green diameter $\overline{\{0,n\}}_g$ is also contained in $\mathscr{D}$. Thus the middle term of the distinguished triangle is an object in D.

- Suppose now $\overline{\{x,y\}} = \overline{\{0,n\}}$ is a diameter and $\mu_{\mathscr{D}}^-(\overline{\{x,y\}})$ is a pair of arcs. By Lemma 4.4.20 we have $d_j = c$. Since the arcs $\{0,c\}$ and $\{d_{i-1},d_{j-1}\}$ cross (both are diagonals in the $\mathscr{D}$-cell $\langle d_1,\ldots,d_k\rangle$) we have $0 < d_{j-1} < d_{i-1}+n < n$ and thus $0 < d_{i-1}+n < d_{j-1}+n < d_{i-1}$. Intersecting the starting frame of $m_{\overline{\{0,n\}}_{r,g}}$ with the ending frame of $m_{\overline{\{d_{i-1}+n,d_{j-1}+n\}}}$ yields the short exact sequence

$$0 \to m_{\overline{\{0,n\}}_{r,g}} \to m_{\overline{\{d_{i-1}+n,n\}}} \oplus m_{\overline{\{d_{j-1},d_{j-1}+n\}}_{r,g}} \to m_{\overline{\{d_{i-1}+n,d_{j-1}+n\}}} \to 0,$$

where $\overline{\{d_{j-1},d_{j-1}+n\}}_{r,g}$ has the same colour as $\overline{\{0,n\}}_{r,g}$. This induces the distinguished triangle

$$m_{\overline{\{x,y\}}_{r,g}} \to m_{r_{\mathscr{D}}^{-1}(\overline{\{d_{j-1},d_j\}}_{r,g})} \oplus m_{\overline{\{d_i,d_{i-1}\}}} \to m_{\mu_{\mathscr{D}}^-\overline{\{x,y\}}} \to \Sigma m_{\overline{\{x,y\}}},$$

where $\overline{\{d_{j-1},d_j\}}_{r,g}$ is of the same colour as $\overline{\{x,y\}}_{r,g}$. The middle term thus lies in D. Dual considerations show, that if $\overline{\{x,y\}}$ is a pair of arcs and $\mu_{\mathscr{D}}^-(\overline{\{x,y\}})$ is a diameter, we can find the desired distinguished triangle.

- Finally, suppose both $\overline{\{x,y\}}$ and $\mu_{\mathscr{D}}^-(\overline{\{x,y\}})$ are diameters. By Lemma 4.4.19 they are of different colour. Without loss of generality let $\overline{\{x,y\}} = \overline{\{0,n\}}_r$ and $\mu_{\mathscr{D}}(\overline{\{x,y\}})) = r_{\mathscr{D}}^{-1}\overline{\{d_{i-1},d_{j-1}\}}_g$.

  If $d_{i-1} = c$ then, because the vertices $d_i, d_j, d_{i-1}, d_{j-1}$ are pairwise distinct and $d_i = 0$, we have $d_j = n$ and $0 < d_{j-1} < n$. The intersection of the starting frame of the module $m_{\overline{\{0,n\}}_r}$ with the ending frame of the module $m_{\overline{\{d_{j-1},d_{j-1}+n\}}_g}$ yields the short exact sequence

$$0 \to m_{\overline{\{0,n\}}_r} \to m_{\overline{\{d_{j-1},n\}}} \to m_{\overline{\{d_{j-1},d_{j-1}+n\}}_g} \to 0.$$

  Because $\overline{\{d_{j-1},n\}} = \overline{\{d_{j-1},d_j\}}$ we get the induced distinguished triangle

$$\overline{\{x,y\}} \to \overline{\{d_{j-1},d_j\}} \to \mu_{\mathscr{D}}^-\overline{\{x,y\}} \to \Sigma\overline{\{x,y\}}$$

  with $\overline{\{d_{j-1},d_j\}} \in D$.

  If on the other hand $d_{i-1} \neq c$, then $n < d_{i-1} \leq 2n-1$ and intersecting the starting frame of the module $m_{\overline{\{0,n\}}_r}$ with the ending frame of the module $m_{\overline{\{d_{i-1}+n,d_{i-1}\}}_g}$ yields the short exact sequence

$$0 \to m_{\overline{\{0,n\}}_r} \to m_{\overline{\{d_{i-1}+n,n\}}} \to m_{\overline{\{d_{i-1}+n,d_{i-1}\}}_g} \to 0.$$

Because $d_i = 0$ we have $\overline{\{d_{i-1}+n, n\}} = \overline{\{d_{i-1}, d_i\}}$ and we get the induced distinguished triangle

$$\overline{\{x,y\}} \to \overline{\{d_{i-1}, d_i\}} \to \mu^-_{\mathscr{D}}\overline{\{x,y\}} \to \Sigma\overline{\{x,y\}}$$

with $\overline{\{d_{i-1}, d_i\}} \in D$.

$\square$

Chapter 5

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Appendices

Appendix A

Wissenschaftlicher Werdegang

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Mai 2012 – Oktober 2015

Eingestellt an der Leibniz Universität Hannover für das DFG Projekt HO 1880/5-1 im Rahmen des SPP1388 “Darstellungstheorie”.
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Doktorandin an der Leibniz Universität Hannover.
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Oktober 2011

Master of Science ETH in Mathematik mit Auszeichnung, ETH Zürich.
Abschlussarbeit: *Cluster algebras of infinite rank.*
Betreuerin an der ETH Zürich: Karin Baur.
Betreuer an der University of Oxford: Jan Grabowski.

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Bachelor of Science ETH in Mathematik mit Auszeichnung, ETH Zürich.
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