Euler-Poincaré-Arnold equations on semi-direct products

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Abstract

Using a geometric approach we study in this thesis the well-posedness of the Euler-Poincaré-Arnold equations, in the smooth category, on semi-direct products of the group of orientation-preserving diffeomorphisms of the circle with itself. To achieve this goal we had to extend the results obtained in [15] for the general case of inertia operators of pseudo-differential type. In order to give a rigorous theoretical framework for the so called "geometric method in hydrodynamics" a geodesic spray related to a right-invariant weak Riemannian metric is defined for manifolds modelled on Fréchet spaces. Our unified abstract approach enables us to recover various fundamental facts concerning the geometrical method in hydrodynamics. In the last chapter a comparison with a Nash-Moser approach is made in order to highlight the advantages of the geometric approach.

2010 Mathematics Subject Classification: $58D05 \cdot 58B25 \cdot 22E65 \cdot 35Q35$
Keywords: Euler-Poincaré-Arnold equations, diffeomorphism group of the circle, semi-direct products.

Zusammenfassung

In der vorliegenden Arbeit wird ein geometrischer Zugang zu der Wohlgestelltheit der Euler-Poncaré-Arnold Gleichungen in semidirekten Produkten von der Diffeomorphismengruppe des Einheitskreises mit sich selbst vorgestellt. In diesem Zusammenhang gelingt es die Resultate aus [15] auf den allgemeinen Fall von Pseudodifferentialoperatoren als Trägheitsoperatoren zu erweitern. Im Kontext von Fréchet-Mannigfaltigkeiten definieren wir bezüglich einer rechts-invarianten schwachen Riemannischen Metrik einen geodätischen Spray, um einen rigorosen abstrakten Rahmen, für die so genannte geometrische Methode in der Hydrodynamik zu schaffen. Im letzten Kapitel wird die geometrische Methode mit der Nash-Moser Methode verglichen. Dabei werden die Vorteile der ersten herausgearbeitet.

2010 Mathematics Subject Classification: $58D05 \cdot 58B25 \cdot 22E65 \cdot 35Q35$.
Schlagwörter: Euler-Poincaré-Arnold Gleichung, Diffeomorphismengruppe des Einheitskreises, semidirekte Produkte.

Preface

In an original paper [1] from 1966, Vladimir Arnold observed that the Euler equations of motion of a rigid body, and also the Euler equations of fluid dynamics of an inviscid incompressible fluid, can be regarded as geodesic flows on a possibly infinite dimensional Lie group, endowed with a right-invariant Riemannian metric. D. Ebin and J. Marsden gave in [14] an analytical approach to this idea and since then a lot of nonlinear equations, mostly coming from hydrodynamics, were recast as geodesic flows on infinite dimensional manifolds. Examples include the equations: Camassa-Holm, Korteweg–de Vries, Burgers, Constantin-Lax-Majda, Hunter-Saxton, or Euler-Weil-Petersson. The equation satisfied by the Eulerian velocity of a geodesic of a right-invariant metric on a Lie group is now called the Euler-Poincaré-Arnold equation.

Arnold's geometric view on hydrodynamics has "opened" different research directions. First of all it stimulated the research of infinite dimensional Lie groups in order to obtain new algebraic structures and metrics, but also to provide a rigorous treatment of the topic. Nowadays similar methods are applied in shape analysis when one deals with image processing and pattern recognition. It stimulated the study of the fluid Lagrangian instability and sectional curvatures of diffeomorphism groups. V. Arnold suggested that a negative sectional curvature implies Lagrangian instability. By applying this idea to atmospheric flows he gave a qualitative explanation of unreliability of long-term wheater forecasts. Arnold's contributions in hydrodynamics are also related to: Arnold's stability and Hamiltonian methods in hydrodynamics, the topology of steady flows, the fast dynamo and magnetohydrodynamics, or the asymptotic Hopf invariant.

In this thesis, inspired by the ideas presented in [15], I study the Euler-Poincaré-Arnold equations on a structure which apparently can not be studied in a similar manner: the semi-direct products of the group of orientation-preserving diffeomorphisms of the circle with itself. A surprising isomorphism between the semi-direct structure and the direct structure is leading to the need to extend the results obtained in [15] for the case of inertia operators of pseudo-differential type. This problem is solved in this thesis with a method which possibly can be extended to a more general setting. The adequate theoretical framework for infinite dimensional manifolds is the convenient setting of A. Frölicher and A. Kriegl. It allows us to overcome various theoretical barriers and to define the geometric objects in a practical way. I devoted the first chapter to this subject in order to help the reader to understand some further arguments.

In the second chapter fundamental propositions for the geometrical method in hydrodynamics are presented but in a general and unitary way. Afterwards we introduce the semi-direct structure generated by an arbitrary action and we discover how rich in properties can be the Lie group $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$. When I started to study this topic I was surprised to find that no acceptable spray theory on Fréchet manifolds was available. I tried to give a brief but necessary spray theory in the third chapter with the help of the convenient calculus, considering the Fréchet Lie groups as special cases of the convenient Lie groups in the sense of P. Michor. In the fourth chapter the case of inertia operators of pseudo-differential type is investigated and solved. It will allow us to use the results of the second chapter to study the well-posedness of the Euler-Poincaré-Arnold equations on semi-direct products of $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ with itself.

In the last chapter of this thesis I tried to compare the geometric method in hydrodynamics with its principal competitor: the Nash-Moser theory. It seems that this geometric method is not only mathematically appealing but also efficient since it extends beyond the tame category.

Emanuel Ciprian Cismas, Hannover in July 2015

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I dedicate this thesis to my family and to the memory of my father. I also dedicate it to a few former professors of mine who instilled in me the passion for mathematics: Dorina Mihu, Mircea Cotoc, Dorel Mihet, Viorel Radu†, Petre Preda.

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Emanuel Cismas, Hannover in July 2015

Contents

"Without deviation from the norm, progress is not possible."
Frank Zappa

1 The convenient setting for infinite dimensional Lie groups

In this thesis we study applications of infinite dimensional manifolds modelled on Fréchet spaces. As is well-known beyond Banach spaces a lot of pathologies occur: ordinary differential equations may not have solutions or the solutions may not be unique, there is no genuine inverse function theorem, there is no natural topology for the dual space and none of the candidates is metrizable. All these problems oblige us to handle carefully the geometric objects related to an infinite dimensional Fréchet manifold. For decades there was a common belief between mathematicians that for infinite dimensional calculus each serious application needs its own foundation. But in 1982 A. Frölicher and A. Kriegl presented independently the solution to the question for the right differential calculus in infinite dimensions: *the convenient calculus*. P. Michor and A. Kriegl laid afterwards the foundations of the infinite dimensional differential geometry and brought everything together in their seminal book [48]. The chief aim of this first chapter is to make the reader familiar with the convenient setting for infinite dimensional Lie groups and to prepare the germs of some future arguments.

1.1 Smooth differentiable mappings

To discuss about a smooth structure of a topological manifold we need a notion of differentiability between Fréchet spaces. In Banach spaces we have a notion of differentiability, called *Fréchet differentiability*, which permits us to extend the differential calculus from finite dimension to Banach spaces.

Definition 1.1.1. Let $\mathbb{E}, \mathbb{F}$ be Banach spaces, and U an open subset of $\mathbb{E}$. A mapping f is said to be *differentiable* at a point $x \in U$, if there is an element $A_x \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ such that:

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x(h)\|}{\|h\|} = 0.$$

In this way f can be approximated locally by a linear mapping A_x, usually denoted by $D_x f$. If f is differentiable at every point $x \in U$ then Df can be regarded as a mapping of U into $\mathcal{L}(\mathbb{E}, \mathbb{F})$, and f is a C^1 differentiable mapping if and only if Df is continuous. Since $\mathcal{L}(\mathbb{E}, \mathbb{F})$ is a Banach space a C^k differentiable mapping can be defined inductively.

When $\mathbb{E}, \mathbb{F}$ are non-normable Fréchet spaces we have to cope with another phenomenon, namely the composition:

$$\circ : \mathcal{L}(\mathbb{F}, \mathbb{G}) \times \mathcal{L}(\mathbb{E}, \mathbb{F}) \to \mathcal{L}(\mathbb{E}, \mathbb{G})$$

is not continuous for any locally convex topology which can endow the space of linear mappings, excepting the case when all the spaces are Banach. If we define a concept of differentiability which uses the continuity of the mapping:

$$Df : U \to \mathcal{L}(\mathbb{E}, \mathbb{F})$$

the concept will not be conserved by compositions, i.e. there will be no chain rule. For a discussion on this topic one can consult [7] or [48].

The most used concept of differentiability for infinite dimensional manifolds modelled on locally convex topological vector spaces, see [27], [49], [55] or [56], is avoiding the topology of $\mathcal{L}(\mathbb{E}, \mathbb{F})$ in the following way:

Definition 1.1.2. Let $f : U \subseteq \mathbb{E} \to \mathbb{F}$ be a mapping between Fréchet spaces, where U is an open subset in $\mathbb{E}$. We say that f is Gâteaux differentiable at $x \in U$ in the direction $h \in \mathbb{E}$ if the following limit exists:

$$D_x f(h) := \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

We say that f is C^1-Gâteaux differentiable on U if f is continuous, the limit exists for all $x \in U$ and $h \in \mathbb{E}$, and $Df : U \times \mathbb{E} \to \mathbb{F}$ is continuous relative to the product topology. Inductively we define the C^k-Gâteaux differentiable mappings for $k \geq 2$, and the Gâteaux smooth mappings.

This notion of differentiability is weaker even in the context of Banach spaces, where C^{k+1}-Gâteaux differentiability implies C^k-Fréchet differentiability, but the classes of smooth mappings coincide. There is no need for the spaces to be Fréchet, one can use locally convex topological vector spaces in the above definition, but we are focused on our goal: the smooth Fréchet manifolds. This concept of C^k-Gâteaux differentiable mappings coincides with those of C^1_{MB}-mappings in the sense of Michal-Bastiani [7], [47] or C^k_c-mappings in the sense of Keller [32].

Using this differentiability concept one can introduce a Fréchet manifold, according to [27]:

Definition 1.1.3. A smooth *Fréchet manifold* is a Hausdorff topological space with an atlas of coordinate charts taking their value in Fréchet spaces, such that the coordinate transition functions are all Gâteaux smooth mapings between Fréchet spaces.

Although this definition is the most popular it raises serious barriers when one tries to define some elementary geometric objects, e.g. differential forms. Of course, there are attempts in this field to use a stronger notion of differentiability, see [53] for example, but most of them seem to fail in having serious applications in infinite dimensional differential geometry. To be able to do some decent analysis one has to consider smooth Fréchet manifolds as particular cases of a more general notion: the *smooth convenient manifolds*.

J. Boman had in [10] the idea to test the smoothness along smooth curves: a mapping f from $\mathbb{R}^d$ to $\mathbb{R}$ is smooth if and only if it sends smooth curves $u \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ in smooth curves $f \circ u \in C^\infty(\mathbb{R}, \mathbb{R})$. This concept was extended to mappings between locally convex spaces by A. Frölicher and A. Kriegl and it will agree in the case of Fréchet spaces with most of the smoothness notions already defined there. One can consult [3] or [32] for a comparison between different differentiability concepts for locally convex spaces. Thus, in [23], the authors constructed the so called convenient calculus for locally convex topological vector spaces. In this context the k-fold differentiability is defined directly as well as infinite differentiability and one can avoid the topology of the space $\mathcal{L}(\mathbb{E}, \mathbb{F})$. In the remaining of this section we present the notion of convenient smothness and prove that this notion can substitute the Gâteaux smoothness in the case of Fréchet manifolds.

For locally convex topological vector space $\mathbb{E}$ we call the final topology with respect to all smooth curves $c \in C^\infty(\mathbb{R}, \mathbb{E})$, the *$c^\infty$-topology*:

Definition 1.1.4. A subset $U \subseteq \mathbb{E}$ is called *c^∞-open* iff $c^{-1}(U)$ is open in $\mathbb{R}$ for all $c \in C^\infty(\mathbb{R}, \mathbb{E})$, and we denote by $c^\infty\mathbb{E}$ the space $\mathbb{E}$ equiped with this topology.

In other words the c^∞-topology is the finest topology on $\mathbb{E}$ such that all the smooth curves $c : \mathbb{R} \to \mathbb{E}$ become continuous. If $\mathbb{E}$ is a Fréchet space then the c^∞-topology coincides with the given locally convex topology, according to [48], but in general the c^∞-topology is finer than any locally convex topology with the same bounded sets. The space $c^\infty\mathbb{E}$ is not a topological vector space in general.

In a locally convex space $\mathbb{E}$ a curve $c : \mathbb{R} \to \mathbb{E}$ is called smooth if all derivatives exist and are continuous. The smoothness of the curves does not depend on the topology given on $\mathbb{E}$, in the sense that for all topologies leading to the same dual we have the same family of smooth curves. In fact it depends only on the family of bounded sets, the *bornology* of $\mathbb{E}$.

Definition 1.1.5. Let $\mathbb{E}, \mathbb{F}$ be locally convex spaces, a mapping $f : U \subseteq \mathbb{E} \to \mathbb{F}$ defined on a c^∞-open subset U it is called *convenient smooth* if it maps smooth curves in U to smooth curves in $\mathbb{F}$.

With this concept of smoothness there exist convenient smooth mappings which are not continuous, but all the convenient smooth mappings are continuous relative to the c^∞-topology, according to [23], [48]. The Gâteaux smoothness will imply convenient smoothness but not conversely. Anyway on Fréchet spaces the two notions coincide:

Proposition 1.1.6. *Let $\mathbb{E}, \mathbb{F}$ be Fréchet spaces and $U \subseteq \mathbb{E}$ a c^∞-open subset, then U is open and the mapping $f : U \subseteq \mathbb{E} \to \mathbb{F}$ is Gâteaux smooth if and only if is convenient smooth.*

Proof. As we mentioned before in the case of a Fréchet space $c^\infty\mathbb{E} = \mathbb{E}$ and thus U is open for the given topology on $\mathbb{E}$. If f is Gâteaux smooth then one can easily see that $f \circ c$ will be Gâteaux smooth for all smooth curves $c \in C^\infty(\mathbb{R}, \mathbb{E})$, thus f is convenient smooth. If f is smooth in the convenient sense then by Proposition 1.2.8 the mapping $df : U \to \mathrm{L}(\mathbb{E}, \mathbb{F})$ exists and is smooth. The cartesian closedness property, Proposition 1.2.10, implies $Df := df : U \times \mathbb{E} \to \mathbb{F}$ is smooth, thus continuous relative to the c^∞- topologies, which coincide here with the given topologies on $\mathbb{E}, \mathbb{F}$. □

Remark 1.1.7. In conclusion this notion of convenient smoothness or Boman smoothness can substitute the notion of smoothness most used for Fréchet spaces and implicitly for smooth Fréchet manifolds.

1.2 A glimpse into the convenient calculus

We continue now, following closely the books [23] and [48], to present basic facts about the convenient calculus of A. Frölicher and A. Kriegl. All the proofs of the statements presented below can be found in [48].

In locally convex spaces there is a weaker notion than that of Cauchy sequences namely the *Mackey-Cauchy sequences*.

Definition 1.2.1. A sequence $(x_n)_n$ in $\mathbb{E}$ is called Mackey-Cauchy if there exists a bounded and absolutely convex set B and for every $\varepsilon > 0$ an integer $n_\varepsilon \in \mathbb{N}$ such that:

$$x_n - x_m \in \varepsilon B, \quad \forall n > m > n_\varepsilon.$$

This is equivalent with $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$.

Definition 1.2.2. A *convenient vector space* is a locally convex topological vector space which is Mackey complete (every Mackey-Cauchy sequence converges in $\mathbb{E}$).

Any sequentially complete topological vector space is Mackey-complete.

Proposition 1.2.3. *A locally convex space $\mathbb{E}$ is convenient if one of the following equivalent conditions hold:*

(1) For any $c_1 \in C^\infty(\mathbb{R}, \mathbb{E})$ there is $c_2 \in C^\infty(\mathbb{R}, \mathbb{E})$ such that $c_1 = c_2'$.

(2) For any smooth curve $c \in C^\infty(\mathbb{R}, \mathbb{E})$ the Riemann integral $\int_0^1 c(t)dt$ exists in $\mathbb{E}$.

(3) If $c : \mathbb{R} \to \mathbb{E}$ is a curve such that $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $l \in \mathbb{E}^*$, then c is smooth

(4) If B is a bounded, closed , absolutely convex set then $\mathbb{E}_B$ is a Banach space.

(5) Any continuous linear mapping from a normed space into $\mathbb{E}$ has a continuous extension to the completion of the normed space.

Proof. Theorem 2.14 in [48]. □

Definition 1.2.4. (Bornology) Let X be a set, a *bornology* on X is a collection $\mathcal{B}$ of subsets such that:

(i) $\mathcal{B}$ covers X, i.e. $X = \bigcup\limits_{B \in \mathcal{B}} B$.

(ii) $\mathcal{B}$ is stable under inclusions, if $B \in \mathcal{B}$ and $B_0 \subseteq B$, then $B_0 \in \mathcal{B}$.

(ii) $\mathcal{B}$ is stable under finite unions, if $B1, \ldots, B_n \in \mathcal{B}$, then $\bigcup\limits_{i=1}^{n} B_i \in \mathcal{B}$.

Given a locally convex space $(\mathbb{E}, \tau)$ we obtain a natural bornology on $\mathbb{E}$ (von Neumann bornology) consisting of all bounded sets, i.e. those subsets $B \in \mathbb{E}$ having the property: for every 0-neighbrohood U there exists a scalar α such that $B \in \alpha U$. We call a set $U \subseteq \mathbb{E}$ *bornivorous* if it absorbs every bounded set from its von Neumann bornology.

Definition 1.2.5. A Hausdorff locally convex space $\mathbb{E}$ is called *bornological* if each convex, balanced and bornivorous set in $\mathbb{E}$ is a neighborhood of 0.

An equivalent defintion is: $\mathbb{E}$ is an inductive limit of normed spaces. More specifically let $\mathcal{B}$ denote the collection of bounded absolutely convex sets of $\mathbb{E}$ and for each $B \in \mathcal{B}$ let $\mathbb{E}_B$ denote the linear span of B in $\mathbb{E}$ equiped with the Minkowski-functional $p_B(v) := \inf\{\lambda : v \in \lambda \cdot B\}$. Then (E_B, p_B) is a normed space and:

$$\mathbb{E} = \varinjlim_{B \in \mathcal{B}} \mathbb{E}_B.$$

Definition 1.2.6. Let $(\mathbb{E}, \tau)$ be a locally convex topological vector space, then the collection of all absolutely convex bornivorous subsets forms a locally convex topology τ_{born} called the *bornologification* of the initial topology. The space $\mathbb{E}_{born} := (\mathbb{E}, \tau_{born})$ is called the *attached bornological space* and it is the finest locally convex structure having the same bounded sets as $(\mathbb{E}, \tau)$.

The cornerstone of the calculus in convenient vector spaces, together with the cartesian closedness, is the fact that the two fundamental spaces $\mathrm{C}^\infty(\mathbb{E}, \mathbb{F})$, $\mathrm{L}(\mathbb{E}, \mathbb{F})$ will remain in this category for $\mathbb{E}, \mathbb{F}$ convenient vectors spaces. In this thesis $\mathrm{L}(\mathbb{E}, \mathbb{F})$ denotes the space of linear and bounded mappings and in general do not coincide with the space $\mathcal{L}(\mathbb{E}, \mathbb{F})$ of linear and continuous mappings. The smoothness of a curve $c : \mathbb{R} \to \mathbb{E}$ does not depend on the initial topology on $\mathbb{E}$, it depends only on its bornology. We can substitute the initial locally convex topology with its bornologification and work with bornological locally convex spaces. In this way we can exploit the characteristic property: on bornological spaces a linear mapping is continuous if and only if is bounded.

We equip $\mathrm{C}^\infty(\mathbb{R}, \mathbb{F})$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. The space $\mathrm{C}^\infty(\mathbb{E}, \mathbb{F})$ will be equiped with the bornologification of the initial topology relative to all mappings $c^* : \mathrm{C}^\infty(\mathbb{E}, \mathbb{F}) \to \mathrm{C}^\infty(\mathbb{R}, \mathbb{F})$, $c^*(f) = f \circ c$, for all $c \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{E})$. If a locally convex space $\mathbb{E}$ is Mackey-complete (convenient) then its attached bornological space¸ $\mathbb{E}_{born}$, having the same bounded sets, will be Mackey-complete (Corollary 4.4 in [48]).

Proposition 1.2.7. *For locally convex spaces $\mathbb{E}, \mathbb{F}$ we have:*

(i) If $\mathbb{F}$ is a convenient vector space then $\mathrm{C}^\infty(\mathbb{E}, \mathbb{F})$ is a convenient vector space, for any $\mathbb{E}$. The space $\mathrm{L}(\mathbb{E}, \mathbb{F})$ is a closed linear subspace and it is a convenient vector space endowed with the initial topology relative to the inclusion mapping.

(ii) *If $\mathbb{E}$ is a convenient vector space then a curve $c : \mathbb{R} \to \mathrm{L}(\mathbb{E}, \mathbb{F})$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in $\mathbb{F}$, for all $x \in \mathbb{E}$.*

Proposition 1.2.8. *Let $\mathbb{E}, \mathbb{F}, \mathbb{G}$ be convenient vector spaces, $U \subset \mathbb{E}$, $V \subset \mathbb{F}$ c^∞-open subsets:*

(i) *If the mapping $f : U \to \mathbb{F}$ is convenient smooth, then the mapping $df : U \to \mathrm{L}(\mathbb{E}, \mathbb{F})$ is convenient smooth and bounded linear in the second component, where:*

$$d_x f(h) := \left.\frac{d}{dt}\right|_{t=0} f(x + th).$$

(ii) *The differentiation operator $d : \mathrm{C}^\infty(U, \mathbb{F}) \to \mathrm{C}^\infty(U, \mathrm{L}(\mathbb{E}, \mathbb{F}))$ exists is linear and bounded (smooth) and the chain rule holds:*

$$d_x(f \circ g)(v) = d_{g(x)} f(d_x g(v)).$$

(iii) *Convenient smooth mappings are continuous with respect to the c^∞-topology.*

(iv) *Multilinear mappings are convenient smooth if and only if they are bounded and for the derivative we have the product rule:*

$$d_{(x_1, \ldots x_n)} f(v_1, \ldots, v_n) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, v_i, x_{i+1}, \ldots, x_n).$$

(v) *Let $f : U \subset \mathbb{E} \times \mathbb{R} \to \mathbb{F}$ be convenient smooth on a c^∞-open subset U. Then:*

$$x \xrightarrow{F} \int_0^1 f(x, t)\ dt$$

is convenient smooth on the c^∞-open set $\{x \in \mathbb{E} : \{x\} \times [0, 1] \subseteq U\}$ with values on the completion $\widehat{\mathbb{F}}$ and:

$$d_x F(v) = \int_0^1 d_x(f(\cdot, t))(v)\ dt.$$

(vi) *(Smooth uniform boundedness) A linear mapping $f : \mathbb{E} \to \mathrm{C}^\infty(V, \mathbb{G})$ is convenient smooth (bounded) if and only if $ev_v \circ f : \mathbb{E} \to \mathbb{G}$ is convenient smooth, for each $v \in V \subset \mathbb{F}$, where $\mathrm{ev}_v : \mathrm{C}^\infty(V, \mathbb{G}) \to \mathbb{G}$ denotes the evaluation mapping.*

(vii) *(Smooth detection principle) A mapping $f : U \subset \mathbb{E} \to \mathrm{L}(\mathbb{F}, \mathbb{G})$ is convenient smooth if and only if $\mathrm{ev}_y \circ f : U \to \mathbb{G}$ is convenient smooth for all $y \in \mathbb{F}$.*

(viii) *A mapping $f : U \to \mathrm{L}(\mathbb{F}, \mathbb{G})$ is convenient smooth if and only if $f : U \to \mathrm{C}^\infty(\mathbb{F}, \mathbb{G})$ is convenient smooth, i.e. $\mathrm{L}(\mathbb{F}, \mathbb{G}) \hookrightarrow \mathrm{C}^\infty(\mathbb{F}, \mathbb{G})$ is initial.*

(ix) *Let $[x, x + h] := \{x + sh : s \in [0, 1]\} \subset U$, then Taylor's formula is true at $x \in U$, where the higher derivatives are defined as usual:*

$$f(x + h) = \sum_{i=0}^{n} \frac{1}{i!} d^i f(x) h^{(i)} + \int_0^t \frac{(1 - t)^n}{n!} d^{n+1} f(x + th) h^{(n+1)} dt,$$

for all $n \in \mathbb{N}$.

Remark 1.2.9. Because sometimes we work with locally convex spaces which may not be bornological we have two notions for the dual of a locally convex space $\mathbb{E}$: the bornological dual, denoted $\mathbb{E}^{'}$, i.e. the set of all bounded linear functionals $f : \mathbb{E} \to \mathbb{R}$, and the topological dual, denoted $\mathbb{E}^{*}$, which is the set of all continuous linear functionals. Throughout this thesis we use these two notations.

If X, Y, Z are sets for two mappings $f : X \to Z^{Y}$ and $g : X \times Y \to Z$ one can define the cannonically attached mappings, sometimes called adjoint mappings:

$$f^{\wedge} : X \times Y \to Z, \quad f^{\wedge}(x, y) := f(x)(y), \quad x \in X, y \in Y,$$

$$g^{\vee} : X \to Z^{Y}, \quad g^{\vee}(x) := g(x, \cdot), \quad x \in X.$$

Proposition 1.2.10. *(Cartesian closedness) Let $U_i \subseteq \mathbb{E}_i$, $i = \overline{1,2}$, be two c^{∞}-open subsets in locally convex spaces which need not to be convenient. Then a mapping $f : U_1 \times U_2 \to \mathbb{F}$ is convenient smooth if and only if the cannonically associated mapping $f^{\vee} : U_1 \to \mathrm{C}^{\infty}(U_2, \mathbb{F})$ exists and is convenient smooth:*

$$\mathrm{C}^{\infty}(U_1 \times U_2, \mathbb{F}) = \mathrm{C}^{\infty}(U_1, \mathrm{C}^{\infty}(U_2, \mathbb{F})).$$

As a consequence of the cartesian closedness property let us note that the evaluation mapping:

$$\mathrm{ev} : \mathrm{C}^{\infty}(U, \mathbb{F}) \times U \to \mathbb{F}, \ \ \mathrm{ev}(f, x) := f(x),$$

is convenient smooth. Also the composition mapping:

$$\circ : \mathrm{C}^{\infty}(\mathbb{F}, \mathbb{G}) \times \mathrm{C}^{\infty}(U, \mathbb{F}) \to \mathrm{C}^{\infty}(U, \mathbb{G})$$

is convenient smooth and the insertion mapping:

$$\mathrm{ins} : \mathbb{E} \to \mathrm{C}^{\infty}(\mathbb{F}, \mathbb{E} \times \mathbb{F}), \qquad x \mapsto \mathrm{ins}_x(y) := (x, y).$$

Proposition 1.2.11. *Let $f : \mathbb{E} \to \mathbb{F}$ and $A : \mathbb{E} \to \mathrm{L}(\mathbb{F}, \mathbb{G})$ be convenient smooth mappings, then:*

$$d_x(A(\cdot)f(\cdot))v = d_x A(v)(f(x)) + A(x)(d_x f(v)),$$

for all $x, v \in \mathbb{E}$.

Proof. The evaluation mapping $\mathrm{ev} : \mathrm{C}^{\infty}(\mathbb{F}, \mathbb{G}) \times \mathbb{F} \to \mathbb{G}$ is convenient smooth and the curve $c : \mathbb{R} \to \mathrm{L}(\mathbb{F}, \mathbb{G})$ is smooth iff $c : \mathbb{R} \to \mathrm{C}^{\infty}(\mathbb{F}, \mathbb{G})$ is smooth by Proposition 1.2.8 (viii). Thus $\mathrm{ev} : \mathrm{L}(\mathbb{F}, \mathbb{G}) \times \mathbb{F} \to \mathbb{G}$ is convenient smooth and bilinear. Hence:

$$d_x(A(\cdot)f(\cdot))v = d_x(\mathrm{ev}(A(\cdot), f(\cdot)))(v) = d_{(A(x), f(x))}\mathrm{ev}(d_x A(v), d_x f(v))$$

$$= \mathrm{ev}_{d_x f(v)} A(x) + \mathrm{ev}_{f(x)} d_x A(v) = d_x A(v)(f(x)) + A(x)(d_x f(v)),$$

using Proposition 1.2.8 (iii). ∎

Remark 1.2.12. The identity is also true for $L^k(\mathbb{E}, \mathbb{F})$ instead of $L(\mathbb{E}, \mathbb{F})$.

1.3 Convenient manifolds

Definition 1.3.1. (Convenient manifolds) A chart (U, φ) on a set M is a bijection $\varphi : U \to \varphi(U) \subseteq \mathbb{E}_U$ from a subset $U \subseteq M$ onto a c^∞-open subset of a convenient vector space $\mathbb{E}_U$. For two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on M the mapping:

$$\varphi_{_{\alpha\beta}} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is called the transition mapping. A family $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of charts on M is called an *atlas* for M, if the sets U_α form a cover of M and all transition mappings are defined on c^∞-open subsets.

An atlas for M is called smooth if all transition mappings $\varphi_{\alpha\beta}$ are *convenient smooth*. Two smooth atlases are called smooth-equivalent if their union is again a smooth atlas. An equivalence class of smooth atlases is a *smooth structure* for M. A *smooth convenient manifold* M is a set together with a smooth structure on it.

The isomorphism type of the modeling spaces $\mathbb{E}_\alpha$ is constant on the connected components of the manifold M, since the derivative of the chart changings are linear isomorphisms. The manifold $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$, considered in this thesis, is a connected manifold. Since we are focused only in offering a theoretical background for the Euler-Poincaré equations on it, we are entitled to consider $\mathbb{E}_\alpha = \mathbb{E}$ in some of our reasonings to avoid further technicalities.

The natural topology of a convenient manifold is the final topology with respect to all inverses of chart mappings in some smooth atlas: a subset $U \subseteq M$ is open in M if and only if $\varphi_\alpha(U_\alpha \cap U)$ is c^∞-open in $\mathbb{E}_\alpha$ for all $\alpha \in A$. In the case of manifolds modelled on Fréchet spaces the above definition coincides with the one of Fréchet manifolds from [27], for example.

Definition 1.3.2. A subset S of a convenient manifold M is called a submanifold, if for each $p \in S$ there is a chart $(U_\alpha, \varphi_\alpha)$ of M such that:

$$\varphi_\alpha(U_\alpha \cap S) = \varphi_\alpha(U_\alpha) \cap \mathbb{F}_\alpha,$$

where $\mathbb{F}_\alpha$ is a closed linear subspace of the convenient model space $\mathbb{E}_\alpha$. Then S becomes a manifold with the atlas $(U_\alpha \cap S, \varphi_\alpha|_{U_\alpha \cap S})_{\alpha \in A}$.

Definition 1.3.3. A mapping $f : M \to N$ between convenient smooth manifolds is called *convenient smooth* if for each $p \in M$ and each chart (V_β, ψ_β) on N, with $f(p) \in V_\beta$ there is a chart $(U_\alpha, \varphi_\alpha)$ on M with $p \in U_\alpha$, $f(U_\alpha) \subseteq V_\beta$ and *the local representative* $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is convenient smooth. This is the case if and only if $f \circ c$ is a smooth curve on M for each smooth curve $c : \mathbb{R} \to M$.

Remark 1.3.4. If M, N are convenient smooth manifolds described by the smooth atlases $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$ then the family of charts defined by $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)_{(\alpha,\beta) \in A \times B}$ forms a smooth atlas for the product $M \times N$. The manifold topology of $M \times N$ may be finer then the product topology (Section 27.3 in [48]) but if M, N are metrizable it coincides with it. However the projections $pr_1 : M \times N \to M$ and $pr_2 : M \times N \to N$ are convenient smooth mappings and the universal property holds: for any convenient smooth manifold P and any convenient smooth mappings $f : P \to M, g : P \to N$, the mapping:

$$(f, g) : P \to M \times N, \quad (f, g)(p) := (f(p), g(p))$$

is the unique convenient smooth mapping with:

$$pr_1(f,g) = f, \quad pr_2(f,g) = g.$$

For $u \in \mathbb{E}$ the *kinematic tangent vector* with foot point u is the pair (u, X), $X \in \mathbb{E}$. The space $T_u\mathbb{E} = \mathbb{E}$ of kinematic tangent vectors with foot point u consists of all derivatives $c^{'}(0)$ of the smooth curves $c : \mathbb{R} \to \mathbb{E}$ with $c(0) = u$. For a convenient smooth mapping $f : \mathbb{E} \to \mathbb{F}$ the *kinematic tangent mapping* at u is defined by:

$$T_u f : T_u\mathbb{E} \to T_{f(u)}\mathbb{F}, \;\; T_u f(u, X) := (f(u), d_u f(X)).$$

If M is a convenient smooth manifold on the set:

$$\bigcup_{\alpha \in A} U_\alpha \times \mathbb{E}_\alpha \times \{\alpha\},$$

we consider the equivalence relation:

$$(p, v, \alpha) \sim (q, w, \beta) \iff p = q, \;\; \text{and} \;\; d_{\varphi_\beta(p)}(\varphi_{\alpha\beta})w = v$$

and denote the quotient set by TM, the *kinematic tangent bundle* of M. We define $\pi_M : TM \to M$ by $\pi_M([p, v, \alpha]) = p$ and $TU_\alpha := \pi_M^{-1}(U_\alpha) \subset TM$. The mapping $T\varphi_\alpha : TU_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{E}_\alpha$ defined by:

$$T\varphi_\alpha([p, w, \beta]) = (\varphi_\alpha(p), d_{\varphi_\beta(p)}(\varphi_{\alpha\beta})w)$$

is giving a chart for an atlas $(TU_\alpha, T\varphi_\alpha)_{\alpha \in A}$ of TM.

The set $T_pM := \pi_M^{-1}(p)$ is called the *fiber* over p of the tangent bundle. It carries a canonical convenient vector space structure induced by:

$$T_p\varphi_\alpha := T\varphi_\alpha|_{T_pM} : T_pM \to \{p\} \times \mathbb{E}_\alpha \cong \mathbb{E}_\alpha,$$

for $p \in U_\alpha$. For connected convenient manifolds, e.g. $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$, the fiber of the tangent bundle coincides with the modeling space. The same observation holds, in particular, for the Lie algebra of a connected Lie group.

The kinematic tangent bundle can be also defined as the quotient of the space $\mathrm{C}^\infty(\mathbb{R}, M)$ by the equivalence relation: $c_1 \sim c_2 \iff c_1(0) = c_2(0)$ and in each chart $(U_\alpha, \varphi_\alpha)$ with $c_1(0) = c_2(0) \in U_\alpha$ we have $(\varphi_\alpha \circ c_1)^{'}(0) = (\varphi_\alpha \circ c_2)^{'}(0)$. In this way any curve $c \in \mathrm{C}^\infty(\mathbb{R}, M)$ corresponds to the kinematic tangent vector $[c(0), (\varphi_\alpha \circ c)^{'}(0), \alpha]$. For a convenient smooth mapping $f : M \to N$ the tangent mapping Tf will send the equivalence class $[c]$ in the equivalence class $[f \circ c]$ and its local representative with respect to some charts is the kinematic tangent mapping of the local representative of f.

The partial tangent mappings of a smooth mapping $f : M_1 \times M_2 \to N$ are defined as:

$$T^1_{(p,q)} f := T_p(f \circ \mathrm{ins}^q) : T_pM_1 \to T_{f(p,q)}N,$$
$$T^2_{(p,q)} f := T_q(f \circ \mathrm{ins}_p) : T_qM_2 \to T_{f(p,q)}N,$$

using the insertion mappings $\mathrm{ins}^q(p) := (p, q)$ and $\mathrm{ins}_p(q) := (p, q)$, $p \in M_1$, $q \in M_2$. One has the identity:

$$T_{(p,q)} f(X_p, Y_q) = T^1_{(p,q)} f(X_p) + T^2_{(p,q)} f(Y_q), \;\; X_p \in T_pM_1, Y_q \in T_qM_2,$$

because $T(M_1 \times M_2) = TM_1 \times TM_2$ in a canonical way.

Remark 1.3.5. On convenient vector spaces another kind of tangent vectors are available: the operational tangent vectors, see Section 28.1 in [48]. The two notions will not coincide in general and will give two different tangent bundles of a convenient manifold. This difference causes some headaches and is leading to the existence of 12 different notions of differential forms in the convenient setting. The ”right” notion for a convenient manifold is the kinematic tangent bundle, the other one is not even preserving products or there exist no vertical lifts. Anyway for manifolds modelled on nuclear Fréchet spaces the two notions coincide and one recovers the result from the finite dimensional case: any tangent vector is a derivation.

1.3.1 Vector bundles

Let $p : E \to M$ be a convenient smooth mapping between convenient smooth manifolds. By a *vector bundle chart* on (E, p, M, V) we mean a pair (U, ψ), where U is an open subset in M, and where ψ is a fiber respecting diffeomorphism, and V is a fixed convenient vector space, called the *standard fiber*.

$$\begin{tikzcd}
E|U := p^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times V \arrow[dl, "\mathrm{pr}_1"] \\
& U. &
\end{tikzcd}$$

Two vector bundle charts $(U_\alpha, \psi_\alpha), (U_\beta, \psi_\beta)$ are called *compatible*, if the mapping $\psi_1 \circ \psi_2^{-1}$ is a fiber linear isomorphism:

$$\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, \psi_{\alpha\beta}(x)v),$$

for some mapping $\psi_{\alpha\beta} : U_{\alpha\beta} := U_\alpha \cap U_\beta \to GL(V)$. The mapping is then unique and convenient smooth into $L(V, V)$, and is called the *transition function* between the two vector bundle charts.

Remark 1.3.6. Compare this definition with the definition of a Banach vector bundle, presented in [41]. An extension of a result known for Banach spaces ([41] Proposition III.1.1, [56] Theorem 5.3) holds: if the mapping $f : U \times \mathbb{E} \to \mathbb{F}$ is a convenient smooth and linear in the second argument, then the mapping of U into $L(\mathbb{E}, \mathbb{F})$, $x \mapsto f(x, \cdot)$, is a convenient smooth mapping. The converse also holds, by Proposition 1.2.10, since $L(\mathbb{E}, \mathbb{F}) \subset \mathrm{C}^\infty(\mathbb{E}, \mathbb{F})$ is initial. In [41] the author has omitted the veracity of the result for infinite dimensional Banach spaces and thus the conditions VB1 and VB2, in Chapter III, are enough to define a Banach vector bundle. With VB1 and VB2 in Definition III.1 of [41] we obtain the above formulation for Banach manifolds. It's worth mentioning that the continuity of the mapping f, in the Banach case, implies the local boundedness of $x \mapsto f(x, \cdot)$.

If $(E, p, M, \mathbb{E})$ is a convenient smooth vector bundle with a vector bundle atlas $(\varphi_\alpha, p^{-1}(U_\alpha))_{\alpha \in A}$, then we define the dual vector bundle:

$$E^{'} := \bigcup_{x \in M} E^{'}_x,$$

with the standard fiber the bornological dual $\mathbb{E}^{'}$ and the transition functions:

$$\psi_{\alpha\beta}(x) := (\varphi_{\beta\alpha}(x))^t,$$

naturally obtained using the transpose mapping relative to the bornological duals. For two convenient smooth vector bundles $(E, p_1, M, \mathbb{E})$ and $(F, p_2, M, \mathbb{F})$ with $(\varphi_\alpha, p_1^{-1}(U_\alpha))_{\alpha\in A_1}$, and $(\phi_\alpha, p_2^{-1}(V_\alpha))_{\alpha\in A_2}$ the corresponding vector bundle atlases, one can construct another vector bundle over M, the Hom-bundle:

$$L(E, F) := \bigcup_{x\in M} L(E_x, F_x),$$

having the standard fiber the convenient vector space $L(\mathbb{E}, \mathbb{F})$. The transition functions are:

$$\psi_{\alpha\beta}(x)(T) := \phi_{\alpha\beta}(x) \circ T \circ \varphi_{\alpha\beta}^{-1}(x), \qquad T \in L(E_x, F_x).$$

With this terminology we have $E^{'} = L(E, M \times \mathbb{R})$. We are ready to define now the *kinematic cotangent bundle* $T^{'}M$, having the transition functions:

$$\psi_{\alpha\beta}(x) := T_{\varphi_\alpha(x)}(\varphi_\beta \circ \varphi_\alpha^{-1})^t \in GL(\mathbb{E}^{'}) \subset L(\mathbb{E}^{'}, \mathbb{E}^{'}).$$

If we use the Gâteaux smoothness to define a manifold modelled by locally convex spaces then we can not define differential forms as Gâteaux smooth sections of a vector bundle, see [55] for a discussion. This is the case because for non normable locally convex spaces the evaluation mapping:

$$\mathrm{ev} : \mathbb{E} \times \mathbb{E}^{'} \to \mathbb{R},$$

is not continuous for any linear topology on $\mathbb{E}^{'}$, by a theorem of B. Maissen [44].

In the convenient setting a differential k-form is defined as a convenient smooth section of the vector bundle $L^k_{alt}(TM, M \times \mathbb{R})$:

$$\Omega^k(M) := \mathrm{C}^\infty(L^k_{alt}(TM, M \times \mathbb{R})),$$

with the modeling space $L^k_{alt}(\mathbb{E}, \mathbb{R})$, the space of bounded k-linear alternating mappings, where $\mathbb{E}$ is the modeling space of M. This construction is the only one which is invariant under Lie derivatives, pullbacks or exterior derivatives. There are a lot of other candidates but all have major drawbacks, see Section 33 of [48] for a discussion.

Remark 1.3.7. The reason why this construction is possible is the following: the evaluation mapping $\mathrm{ev} : \mathbb{E} \times \mathbb{E}^{'} \to \mathbb{R}$ is always convenient smooth, thus continuous relative to the c^∞-topology on $c^\infty\ (\mathbb{E} \times \mathbb{E}^{'})$. But $c^\infty\ (\mathbb{E} \times \mathbb{E}^{'})$ is not a topological vector spaces in general (if $\mathbb{E}$ is not normable), and thus ev is not continuous relative to some linear topology on the space $\mathbb{E}^{'}$, to avoid any contradiction with Maissen's theorem.

1.3.2 Regular convenient Lie groups

Definition 1.3.8. A convenient Lie group G is a convenient smooth manifold and a group such that the multiplication: $m_G : G \times G \to G$ and the inversion $i_G : G \to G$ are convenient smooth.

The conjugation mapping $c_g(x) := gxg^{-1}$ generates the adjoint representation of the Lie group G :

$$\mathrm{Ad} : G \to GL(\mathfrak{g}) \subset L(\mathfrak{g}, \mathfrak{g}),$$

considered as a convenient smooth mapping into $L(\mathfrak{g}, \mathfrak{g})$. In this way it makes sense to define the ajoint representation of the Lie algebra $\mathfrak{g}$ as:

$$\mathrm{ad} := T_e \,\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \subset L(\mathfrak{g}, \mathfrak{g}).$$

The right-trivialization $\rho := (\pi_G, \kappa^r) : TG \to G \times \mathfrak{g}$ induces a convenient smooth mapping $\kappa^r : TG \to \mathfrak{g}$, $\kappa^r(V) := pr_2 \circ \rho(V) := R_{g^{-1}}V$, $V \in T_g G$, and a convenient smooth section of the vector bundle $L(TG, G \times \mathfrak{g})$, thus defines a $\mathfrak{g}$-valued 1-form on G by $\kappa^r_g(\xi_g) := R_{g^{-1}}(\xi_g)$, $\xi_g \in T_g G$, called the Maurer-Cartan form.

The Maurer-Cartan form satisfies the Maurer-Cartan equation:

$$d\kappa^r - \frac{1}{2}[\kappa^r, \kappa^r]_\wedge = 0, \tag{1.3.1}$$

where $\frac{1}{2}[\omega, \omega]_\wedge(X, Y) := [\omega(X), \omega(Y)]_\mathfrak{g}$ is a $\mathfrak{g}$-valued 2-form obtained from the $\mathfrak{g}$-valued 1-form ω.

For an infinite dimensional Lie group the Lie exponential mapping may not exist or may not be smooth. An attempt to find a condition which ensures both these properties leaded to the notion of regular Lie groups, introduced by J. Milnor [49]:

Definition 1.3.9. A convenient Lie group is called *regular* if for every curve $u \in \mathrm{C}^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in \mathrm{C}^\infty(\mathbb{R}, G)$ such that:

$$\begin{cases} g(0) = e_{\mathrm{G}}, \\ R_{g(t)^{-1}}\dot{g}(t) = u(t). \end{cases}$$

and the evolution mapping:

$$\mathrm{evol}^r_{\mathrm{G}} : \mathrm{C}^\infty(\mathbb{R}, \mathfrak{g}) \to G, \qquad \mathrm{evol}^r_{\mathrm{G}}(u) := g(1)$$

exists and is convenient smooth.

One also denotes by $\mathrm{Evol}^r_{\mathrm{G}} : \mathrm{C}^\infty(\mathbb{R}, \mathfrak{g}) \to \{g \in \mathrm{C}^\infty(\mathbb{R}, G) : g(0) = e\}$, *the right evolution* of the curve X in G, defined as $\mathrm{Evol}^r_{\mathrm{G}}(X)(t) := g(t)$, for $X \in \mathrm{C}^\infty(\mathbb{R}, \mathfrak{g})$.

Remark 1.3.10. If $g(t)$ is satifying the above initial value problem then $g(t)g_0$ is satisfying the same equation but with $g(0) = g_0$, for an arbitrary $g_0 \in G$.

If G is a regular convenient Lie group then also TG is a regular convenient Lie group. The tangent bundle of the Lie group TG is trivial and it has two trivializations available: the first trivialization is the obvious one $TTG \cong TG Ⓢ T\mathfrak{g}$, and the second trivialization is obtained after we distribute the tangent functor T in $G Ⓢ g$.

By a *right action* of a regular convenient Lie group G on a convenient smooth manifold M we mean a convenient smooth mapping:

$$\alpha : G \times M \to M,$$

such that $\alpha^\vee : G \to \mathrm{Diff}(M)$ is a group anti-homomorphism (in the algebraic sense only). For any $u \in \mathfrak{g}$ we define the *fundamental vector field* $\xi_u \in \mathcal{X}(M)$ by:

$$\xi_u := T_{(e,x)}\alpha(u, 0_x), \qquad u \in \mathfrak{g}.$$

It will satisfy the property:

$$\xi_{[u,v]} = [\xi_u, \xi_v], \qquad u, v \in \mathfrak{g}.$$

An example of regular Lie groups is offered by the strong ILH-Lie groups in the sense of H. Omori [56]:

Proposition 1.3.11. *A topological group G is a strong ILH-Lie group modelled on $\{\mathbb{E}, \mathbb{E}^q, q \geq d\}$ if and only if there exists a system $\{G^q, q \geq d\}$ of topological groups G^q satisfying the following conditions:*

- *(G1) every group G^q is a Hilbert manifold modelled on $\mathbb{E}^q$,*
- *(G2) G^{q+1} is a dense subgroup in G^q, and the embedding $G^{q+1} \subset G^q$ is a mapping of class C^∞,*
- *(G3) $G = \bigcap_{q \geq d} G^q$ with inverse limit topology,*
- *(G4) the group multiplication $m_G : G \times G \to G$ extends to a mapping $G^{q+l} \times G^q \to G^q$ of class C^l,*
- *(G5) the inversion mapping $i_G : G \to G$ extends to a mapping $G^{q+l} \to G^q$ of class C^l,*
- *(G6) for each $\eta \in G^q$ the right translation $r_\eta : G^q \to G^q$ is a mapping of class C^∞,*
- *(G7) let $\mathfrak{g}^q$ be the tangent space of G^q at the identity $e \in G^q$, and let TG^q be the tangent bundle. The mapping $Tr : \mathfrak{g}^{q+l} \times G^q \to TG^q$ defined by $Tr(u, \eta) = Tr_\eta u$ is a mapping of class C^l,*
- *(G8) there exists an open neighborhood U of zero in $\mathfrak{g}^d$ and a diffeomorphism Φ of U onto an open neighborhood $\tilde{U}$ of the unity $e \in G^d$, $\Phi(0) = e$, such that the restriction of Φ to $U \cap \mathfrak{g}^q$ is a diffeomorphism of the open subset $U \cap \mathfrak{g}^q$ from $\mathfrak{g}^q$ onto an open subset $\tilde{U} \cap G^q$ from G^q for any $q \geq d$.*

"In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren, forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun."

Vladimir Arnold

2 The Euler-Poincaré equations on semi-direct products

In his influential article [1] V. Arnold had the idea to analyze the motion of hydrodynamical systems using geodesic flows. Actually he showed that the Euler equations of hydrodynamics can be recast as geodesic equations of a right-invariant Riemannian metric on the group of volume-preserving diffeomorphisms. This approach became the so called geometric method in hydrodynamics (see [20] for more details) and involves the use of geometric arguments to study issues like well-posedness or stability.

For example, as is shown in [42], the Hunter-Saxton equation:

$$u_{txx} = -2u_x u_{xx} - u u_{xxx} \qquad t > 0, \quad x \in \mathbb{R},$$

describes the geodesic flow on the homogeneous space $\mathrm{Rot}(\mathbb{S}^1) \setminus \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ of the infinite dimensional Fréchet -Lie group of orientation-preserving diffeomorphisms of the circle modulo the subgroup of rotations. The right-invariant metric considered is:

$$\langle [u], [v] \rangle = \int_{\mathbb{S}^1} u_x v_x \ \ dx, \qquad [u], [v] \in T_{[id]} \left(\mathrm{Rot}(\mathbb{S}^1) \setminus \mathrm{Diff}_+^\infty(\mathbb{S}^1) \right),$$

and one can observe that the operator $A = -D^2$, called *inertia operator* in this context, generates the inner product.

There is a high flexibility of the method given by the choice of an inertia operator or by the choice of an algebraic structure usually involving diffeomorphism groups. In the last decade different inertia operators were studied starting with differential operators with constant coefficients [13], Hilbert transforms [18], [74] or Fourier multipliers [8], [15]. The last one offers a quite nice generality and covers the other previous cases. Among the algebraic structures studied one should mention homogeneous spaces, the Bott-Virasoro group, semi-direct products between a group and a vector space.

The idea behind this geometric approach, initially developed by D.Ebin and J. Marsden in [14], is to use the right-invariance of the spray to obtain, via a "no gain, no loss" result (see [20] for details) a Cauchy-Lipschitz type theorem on a Fréchet space. Using this method we avoid the Nash-Moser schemes to obtain well-posedness in the smooth category.

In this thesis we study Euler-Arnold-Poincaré equations on semi-direct products of the group of orientation-preserving diffeomorphisms of the circle with itself. We start with a brief introduction of the main ideas behind this geometrical approach in hydrodynamics:

2.1 Euler-Arnold equations on regular Lie groups

To define a Riemannian metric on a regular convenient Lie group G an inner product on the Lie algebra $\mathfrak{g}$ is extended to every tangent space by right translations:

$$\langle u, v\rangle_g = \langle R_{g^{-1}}u, R_{g^{-1}}v\rangle_e, \qquad u, v \in T_gG, \quad g \in G. \tag{2.1.1}$$

If this inner product is generated by an isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$ which is positive-definite and symmetric with respect to the natural pairing $(\cdot, \cdot)$ between elements of $\mathfrak{g}^*$ and $\mathfrak{g}$:

$$\langle u, v\rangle_e^A := (u, Av) = (Au, v), \qquad u, v \in \mathfrak{g}, \tag{2.1.2}$$

then this operator is called the *inertia operator* on G. The natural pairing is actually the evaluation mapping and by Remark 1.3.7 is always convenient smooth if, for example, the topological dual g^* is endowed with the strong topology and g is a convenient vector space. It will never be Gâteaux smooth because Gâteaux smoothness implies continuity.

When working with an infinite dimensional Lie group one can not consider bi-invariant metrics because in this case the Riemannian exponential mapping and the Lie exponential mapping will coincide and the latter one can behave bizarrely. In this thesis the Fréchet-Lie group $G = \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ of orientation-preserving diffeomorphisms of the circle is used, together with its Lie algebra $\mathfrak{g} = \mathrm{C}^\infty(\mathbb{S}^1)$. It is a regular convenient Lie group being a strong ILH-Lie group in the terminology of H. Omori [56] and Fréchet spaces are convenient vector spaces.

In order to maintain the isomorphism property of the inertia operator, described above, we have to restrict $\mathfrak{g}^*$ to its regular dual:

$$\mathfrak{g}^*_{reg} \cong \mathrm{C}^\infty(\mathbb{S}^1),$$

defined as the space of linear functionals of the form:

$$u \to \int_{\mathbb{S}^1} m \cdot u \, dx,$$

for $m \in \mathrm{C}^\infty(\mathbb{S}^1)$, due to [31], [37]. The pairing between the elements of $\mathfrak{g}^*_{reg}$ and $\mathfrak{g}$ will be given by the $L^2(\mathbb{S}^1)$-inner product:

$$(u, v) := \langle u, v\rangle_{L^2(\mathbb{S}^1)} := \int_{\mathbb{S}^1} u \cdot v \; dx. \tag{2.1.3}$$

The topology on $\mathfrak{g}^*_{reg}$ is not the induced one and now the pairing becomes even Gâteaux smooth, which is impossible without the above convention. With this convention the inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*_{reg}$ is called *regular inertia operator*.

Remark 2.1.1. To integrate the above convention on the dual of $C^\infty(\mathbb{S}^1)$ in a general and rigorous theory we make some further observations on $\mathfrak{g}^*_{reg}$. The subset $C^\infty(\mathbb{S}^1)^*_{reg}$ is convex and dense relative to the weak* topology on $C^\infty(\mathbb{S}^1)^*$ (compare with [31]). Since $C^\infty(\mathbb{S}^1)$ is a semi-reflexive Fréchet space it will be dense also relative to the strong topology $\beta(C^\infty(\mathbb{S}^1)^*, C^\infty(\mathbb{S}^1))$ of the dual, as a consequence of the Mackey-Arens theorem. The strong topology of the dual is related to the bounded sets of $C^\infty(\mathbb{S}^1)$ and the smoothness of curves is influenced only by the bornology of $C^\infty(\mathbb{S}^1)$, hence we consider it the natural choice for its dual. $C^\infty(\mathbb{S}^1)^*_\beta$ is a locally convex complete space, thus a convenient vector space. What is even more remarkable is that $C^\infty(\mathbb{S}^1)$ is also a Fréchet-Schwartz space, thus by a result of Komatsu (Theorem 11 in [40]) the inductive dual topology coincides with the strong topology, i.e.:

$$C^\infty(\mathbb{S}^1)^*_\beta = \varinjlim_{k\geq 0} H^k(\mathbb{S}^1)^*.$$

The c^∞-topology on $C^\infty(\mathbb{S}^1)^*$ coincides with the strong topology β (Theorem 4.11 in [48]). As a one can see the space $C^\infty(\mathbb{S}^1)$ is very rich in properties, most of them arise from the property of being a nuclear Fréchet space.

If the adjoint of ad_v relative to the inner product (2.1.2) exists then the geodesics can be determined with the help of this operator. We remind here that a bilinear operator is bounded if and only if is convenient smooth by Theorem 1.2.8. On $C^\infty(\mathbb{S}^1)$ boundedness will be equivalent with continuity, being a bornological space.

Theorem 2.1.2. *(V. Arnold, [1]) If the inner product $\langle \cdot, \cdot \rangle_e : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined in (2.1.2) is bounded and there exists a bounded bilinear operator:*

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

with the property:

$$\langle B(u, v), w\rangle_e = \langle u, \mathrm{ad}_v\, w\rangle_e, \qquad w \in \mathfrak{g},$$

where ad_v is the adjoint representation of $\mathfrak{g}$, then a smooth curve $g(t)$ on the regular convenient Lie group G is a geodesic for the right-invariant metric defined by (2.1.1) if and only if its Eulerian velocity $u(t) = R_{g(t)^{-1}}\dot{g}(t)$ satisfies the first order equation:

$$u_t = -B(u, u).$$

Proof. We give the proof of the theorem adapted to the convenient approach (Section 46.4 in [48]), whereas we considered the original proof of V. Arnold not rigurous enough in the infinite dimensional setting. Let $g : [a, b] \to G$ be a smooth curve. The energy functional of the curve g is:

$$E(g) := \frac{1}{2}\int_a^b \langle \dot{g}(t), \dot{g}(t)\rangle_{g(t)} dt = \frac{1}{2}\int_a^b \langle g^*\kappa^r(\partial_t), g^*\kappa^r(\partial_t)\ \rangle_e dt.$$

We consider now a smooth variation $g(s,t)$ of the curve g, $s \in (-\varepsilon, \varepsilon)$ and $t \in [a,b]$, with fixed endpoints $g(s,a) = g(a)$, $g(s,a) = g(b)$. Let's denote $u(s,t) := R_{g(s,t)^{-1}}\partial_t g(s,t)$ and $v(s,t) := R_{g(s,t)^{-1}}\partial_s g(s,t)$. In particular we have $u_0(t) := u(0,t) : [a,b] \to \mathfrak{g}$ and $v_0(t) := v(0,t) : [a,b] \to \mathfrak{g}$.

$$\partial_s E(g) = \frac{1}{2}\int_a^b 2\langle \partial_s(g^*\kappa^r(\partial_t)), g^*\kappa^r(\partial_t)\ \rangle_e\ dt$$

$$= \int_a^b \langle \partial_t(g^*\kappa^r(\partial_s)) - d(g^*\kappa^r)(\partial_t, \partial_s), g^*\kappa^r(\partial_t)\ \rangle_e\ dt$$

because $[\partial_t, \partial_s] = 0$ by Schwarz's theorem. Further:

$$= \int_a^b -\langle g^*\kappa^r(\partial_s), \partial_t(g^*\kappa^r(\partial_t))\rangle_e - \langle [g^*\kappa^r(\partial_t), g^*\kappa^r(\partial_s)], g^*\kappa^r(\partial_t)\ \rangle_e\ dt$$

$$= -\int_a^b \langle g^*\kappa^r(\partial_s), \partial_t(g^*\kappa^r(\partial_t)) + B(g^*\kappa^r(\partial_t), g^*\kappa^r(\partial_t))\ \rangle_e\ dt,$$

exploiting the fixed endpoints of the variation and applying the right Maurer-Cartan equation.

The curve g is a geodesic for the metric (2.1.1) iff the derivative vanishes at $s = 0$ for all variations $g(s,t)$ of g with fixed endpoints. By Corollary 38.13 in [48] the group $C^\infty(\mathbb{R}, G)$ is a regular convenient Lie group if G is a regular convenient Lie group with the evolution operator:

$$\mathrm{Evol}^r_{C^\infty(\mathbb{R},G)} = C^\infty(\mathbb{R}, \mathrm{Evol}^r_G),$$

and Lie algebra $C^\infty(\mathbb{R}, \mathfrak{g})$ with the bracket $[X,Y](t) := [X(t), Y(t)]_{\mathfrak{g}}$. Thus following the definition of a regular convenient Lie group for every curve:

$$v(s,t) \in C^\infty(\mathbb{R}, C^\infty(\mathbb{R}, \mathfrak{g}))$$

exists a curve:

$$g(s,t) \in C^\infty(\mathbb{R}, C^\infty(\mathbb{R}, G)) = C^\infty(\mathbb{R}\times\mathbb{R}, G),$$

such that:

$$\begin{cases} g(0,t) = g(t) \in C^\infty(\mathbb{R}, G), \\ v(s,t) := R_{g(s,t)^{-1}}\partial_s g(s,t). \end{cases}$$

In particular every smooth curve $v_0 : [a,b] \to \mathfrak{g}$ corresponds to a variation with fixed endpoints of g. Thus for all v_0 :

$$\int_a^b \langle v_0(t), \dot{u}_0(t) + B(u_0(t), u_0(t))\rangle_e\ dt = 0.$$

Applying this identity for the smooth curve $v_0(t) := \dot{u}_0(t) + B(u_0(t), u_0(t))$ we get the conclusion, since the inner product (2.1.2) is positive-definite and smoothness implies continuity for curves.

□

This equation is called the *Euler-Arnold equation* induced by an inertia operator A. In general a Levi-Civita connection related to the Riemannian metric (2.1.1) is not granted, because the metric (2.1.1) is usually leading to a flat mapping $X \mapsto \langle X, \cdot \rangle_g$ which is only injective. If the adjoint ad_u^T exists such a connection also exists. To derive its formula we have to introduce the isomorphism $R : \mathrm{C}^\infty(G, \mathfrak{g}) \to \mathcal{X}^\infty(G)$, given by $R_X(g) = R_g(X(g))$, for every $X \in \mathrm{C}^\infty(G, \mathfrak{g})$, $g \in G$. This isomorphism is induced by the right trivialization $\rho = (\pi_G, \kappa^r) : TG \to G \times \mathfrak{g}$. The idea is to write every element of $\mathcal{X}^\infty(G)$ in a unique way in the form $R_X(g)$, $g \in G$, for some mapping $X \in \mathrm{C}^\infty(G, \mathfrak{g})$.

Proposition 2.1.3. *Assume that for all $u \in \mathfrak{g}$ the adjoint ad_u^T with respect to the bounded inner product $\langle \cdot, \cdot \rangle_e$ exists and that $u \mapsto \mathrm{ad}_u^T$ is bounded. Then the Levi-Civita connection related to the metric (2.1.1) exists and is given by:*

$$\nabla_{R_X} R_Y := R_{\nabla_X Y}, \qquad \nabla_X Y \in \mathrm{C}^\infty(G, \mathfrak{g}),$$

where:

$$(\nabla_X Y)(g) := T_g Y(R_X(g)) - \frac{1}{2} \mathrm{ad}_{X(g)} Y(g) + \frac{1}{2} \mathrm{ad}_{X(g)}^T Y(g) + \frac{1}{2} \mathrm{ad}_{Y(g)}^T X(g),$$

for $X, Y \in \mathrm{C}^\infty(G, \mathfrak{g})$, $g \in G$.

Proof. It is presented in Section 46.5 in [48]. Because the flat mapping is only injective we construct the above candidate and prove that it is the unique Levi-Civita connection related to the right-invariant metric (2.1.1):

$$R_X \langle Y, Z \rangle = \langle TY.R_X, Z \rangle + \langle Y, TZ.R_X \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

The above defined connection is also torsionfree, because:

$$\nabla_X Y - \nabla_Y X = TY.R_X - TX.R_Y - [X, Y]_{\mathfrak{g}},$$

and the next identity holds:

$$[R_X, R_Y] = R_{-[X,Y]_{\mathfrak{g}} + TY(R_X) - TX(R_Y)}, \qquad X, Y \in \mathrm{C}^\infty(G, \mathfrak{g}) \tag{2.1.4}$$

□

Remark 2.1.4. This formula coincides pointwise with the corrected version of the formula given by the authors in [12], for the particular case $G = \mathrm{Diff}_+^\infty(\mathbb{S}^1)$:

$$(\nabla_X Y)_\eta = [X, Y - Y_\eta^R] + \frac{1}{2} \left([X_\eta^R, Y_\eta^R]_\eta + B(X_\eta^R, Y_\eta^R) + B(Y_\eta^R, X_\eta^R) \right),$$

where, for $X \in \mathcal{X}^\infty(\mathrm{Diff}_+^\infty(\mathbb{S}^1))$, the term X_η^R denotes the right-invariant vector field whose value at $\eta \in \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is X_η and $B(u, v) := \mathrm{ad}_v^T u$ was extended to the family of right-invariant vector fields by $B(Z, W)_\eta = R_\eta B(Z_{id}, W_{id})$.

To verify this assumption one has to use the identity (2.1.4), after to write $X_\eta^R = R_{\overline{X}}$, for the constant mapping:

$$\overline{X} := \kappa^r(X(\eta)) \in \mathrm{C}^\infty(\mathrm{Diff}_+^\infty(\mathbb{S}^1), \mathrm{C}^\infty(\mathbb{S}^1)),$$

when $\eta \in \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ is fixed, and to use the transfomations in terms of the isomorphism R for the other members of the identity.

As it was observed in [17] for a geodesic of a connection, i.e. $\nabla_{\dot g(t)}\dot g(t) = 0$, we don't need a connection related to a Riemannian metric. One can define a right-invariant connection:

$$\nabla_{R_X} R_Y := R_{\nabla_X Y}, \qquad \nabla_X Y \in \mathrm{C}^{\infty}(G, \mathfrak{g}) \tag{2.1.5}$$

where:

$$(\nabla_X Y)(g) := T_g Y(R_X(g)) - \frac{1}{2}\,\mathrm{ad}_{X(g)}\, Y(g) + B(X(g), Y(g)),$$

and B is a bounded bilinear operator $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

For right-invariant vector fields ξ_u, ξ_v, $u, v \in \mathfrak{g}$, we have $\xi_u(g) = R_u(g)$, for a constant mapping $u(g) = u \in \mathrm{C}^{\infty}(G, \mathfrak{g})$. Again using the formula (2.1.4) we obtain the same expression like in [17]:

$$\nabla_{\xi_u}\xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v),$$

just because $[\xi_u, \xi_u] = -\xi_{[u,v]}$.

Proposition 2.1.5. *A smooth curve $g(t)$ on the regular convenient Lie group G is a geodesic for the right-invariant linear connection ∇ defined by (2.1.5) if and only if its Eulerian velocity $u(t) = R_{g(t)^{-1}}\dot g(t)$ satisfies the first order equation:*

$$u_t = -B(u, u).$$

Proof. Let $g(t)$ be a curve in G, then one can use the isomorphism R to write $\dot g(t) = R_u(g(t))$, for a mapping $u \in \mathrm{C}^{\infty}(G, \mathfrak{g})$. This mapping induces a curve in $\mathfrak{g}$ which will be denoted with the same letter $u : \mathbb{R} \to \mathfrak{g}$, $u(t) := u(g(t))$. Now $\nabla_{\dot g(t)}\dot g(t) = 0$ iff $(\nabla_u u)(g(t)) = 0$ iff:

$$T_{g(t)}u(\dot g(t)) + B(u, u) = 0.$$

But $u_t := T_{g(t)}u(\dot g(t))$. $\square$

This equation is called the *non-metric Euler-Arnold equation*. In the *metric case* the operator B is given by:

$$B(u, v) = \frac{1}{2}\left(\mathrm{ad}_u^T\, v + \mathrm{ad}_v^T\, u\right),$$

if the adjoint exists. In the sequel only the metric case is considered.

It's worth mentioning that the Riemannian curvature tensor can be expressed in terms of the operator ad_u^T, if the last one exists. The stability of geodesics is determined by the sectional curvature. Since we don't discuss these aspects in the sequel, we present here only the formula obtained via the isomorphism R :

Proposition 2.1.6. *If the adjoint* ad_u^T *exists the Riemannian curvature tensor can be computed by:*

$$\mathcal{R}(R_u, R_v)R_w := R_{\mathcal{R}(u,v)w},$$

$$\mathcal{R}(u,v) := -\frac{1}{4}[\mathrm{ad}_u^T + \mathrm{ad}_u, \mathrm{ad}_v^T + \mathrm{ad}_v] + \frac{1}{4}[\alpha(u), \alpha(v)] + \frac{1}{2}\alpha([u,v]_{\mathfrak{g}}) \quad (2.1.6)$$
$$+[\mathrm{ad}_u^T - \mathrm{ad}_u, \mathrm{ad}_v] + [\mathrm{ad}_u, \mathrm{ad}_v^T - \mathrm{ad}_v]$$

where $\alpha(u)v := \mathrm{ad}_v^T u$.

Proof. see Section 46.6 in [48] for details. □

2.2 The coadjoint representations of a Lie group

One can establish a bridge towards an old idea of H. Poincaré [57] making use of the coadjoint representations of a Lie group.

The coadjoint representation of G on $\mathfrak{g}^*$ is defined as $\mathbb{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$:

$$(\mathbb{Ad}_g^* \alpha, u) = (\alpha, \mathrm{Ad}_{g^{-1}} u) \qquad \alpha \in \mathfrak{g}^*, u \in \mathfrak{g}.$$

The coadjoint representation (action) of $\mathfrak{g}$ on $\mathfrak{g}^*$ is $\mathbb{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$:

$$(\mathbb{ad}_u^* \alpha, v) = -(\alpha, \mathrm{ad}_u v) \qquad \alpha \in \mathfrak{g}^*, u, v \in \mathfrak{g}. \quad (2.2.1)$$

For an inertia operator A we have:

$$\mathrm{ad}_u^T v = -A^{-1} \mathbb{ad}_u^*(Av), \quad (2.2.2)$$

where ad_u^T is the adjoint relative to the inner product induced by A. The inertia operator transforms the Euler-Arnold equation in:

$$m_t = \mathbb{ad}_u^* m, \qquad m = Au,$$

in terms of the coadjoint action of $\mathfrak{g}$, known as the *Euler-Poincaré equation*.

Remark 2.2.1. The following formulae hold for $G = \mathrm{Diff}_+^\infty(\mathbb{S}^1)$:

$$\mathbb{ad}_u^* m = -u \cdot m_x - 2u_x \cdot m, \quad (2.2.3)$$

$$\mathbb{Ad}_g^* m = m \circ g^{-1} \cdot (g^{-1})_x^2. \quad (2.2.4)$$

Observe that the transpose mapping of $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ coincides in this case with the $L^2(\mathbb{S}^1)$-adjoint, due to the identification of the dual of $g = \mathrm{C}^\infty(\mathbb{S}^1)$ with its regular dual $g_{reg}^* \cong \mathrm{C}^\infty(\mathbb{S}^1)$.

2.3 Group actions and semi-direct products

The easiest way to construct a group from two existing groups G, H is to use the *direct product* $G \times H$ and both groups will be isomorphic with normal subgroups of the direct product. Working with *semi-direct products* only one of the groups will be isomorphic with a normal subgroup of the product. It is also possible to contruct a product group, the *knit product*, such that none of the component-groups will be isomorphic with normal subgroups of the product group.

Let us now consider the semi-direct product of the Fréchet-Lie group $G := \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ with itself, generated by a smooth right action constructed in the following way:

$$\varphi.\psi = \alpha(\varphi^{-1})(\psi), \qquad \varphi, \psi \in \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1), \tag{2.3.1}$$

where the mapping $\alpha : \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1) \times \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1) \to \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ is convenient smooth (thus also Gâteaux smooth) and $\alpha^{\vee} : \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1) \to \mathrm{Aut}(\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1))$ is a group homomorphism (only algebraic). Here $\mathrm{Aut}(\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1))$ stands, as usual, for the group of automorphisms of the Lie group $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ i.e. group automorphisms which are also diffeomorphisms. In the sequel by a group homomorphism we understand an homomorphism in the algebraic sense and when we will refer to Lie group homomorphisms this will be clearly specified.

Such an action defines a semi-direct product on $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1) \times \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ by:

$$(\varphi_1, \psi_1) * (\varphi_2, \psi_2) = (\varphi_1 \circ \varphi_2, \alpha(\varphi_2^{-1})(\psi_1) \circ \psi_2). \tag{2.3.2}$$

The inverse of an element (φ, ψ) is given by:

$$(\varphi, \psi)^{-1} = (\varphi^{-1}, \alpha(\varphi)(\psi^{-1})).$$

The notation $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)\circledS\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ will be used for the semi-direct product induced by the action by conjugacy:

$$\varphi.\psi = c_{\varphi^{-1}}(\psi) := \varphi^{-1} \circ \psi \circ \varphi, \qquad \varphi, \psi \in \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1), \tag{2.3.3}$$

The action given by:

$$\varphi.\psi = \mathrm{id}_G(\psi) = \psi, \qquad \forall \ \varphi \in G = \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1),$$

generates the direct product $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1) \times \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$. The semi-direct product induced by an arbitrary smooth right action is denoted $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)\circledS_{\alpha}\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ and its Lie algebra is $\mathrm{C}^{\infty}(\mathbb{S}^1)\circledS_{\beta}\mathrm{C}^{\infty}(\mathbb{S}^1)$ for the mapping:

$$\beta(u)v := T_{id}T_{id}\alpha(v, u), \quad u, v \in \mathrm{C}^{\infty}(\mathbb{S}^1), \tag{2.3.4}$$

where $\varphi \mapsto T_e\alpha(\varphi) : \mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1) \to \mathrm{Aut}(\mathrm{C}^{\infty}(\mathbb{S}^1)) \subset \mathcal{L}(\mathrm{C}^{\infty}(\mathbb{S}^1), \mathrm{C}^{\infty}(\mathbb{S}^1))$ is considered as a convenient smooth mapping into $\mathcal{L}(\mathrm{C}^{\infty}(\mathbb{S}^1), \mathrm{C}^{\infty}(\mathbb{S}^1))$.

The Lie bracket on $\mathrm{C}^{\infty}(\mathbb{S}^1)\circledS_{\beta}\mathrm{C}^{\infty}(\mathbb{S}^1)$ is defined as:

$$\mathrm{ad}_{(u_1,v_1)}(u_2, v_2) = ([u_1, u_2], [v_1, v_2] + \beta(u_1)v_2 - \beta(u_2)v_1) \tag{2.3.5}$$

Proposition 2.3.1. *Semi-direct products of regular convenient Lie groups are regular.*

Proof. see Theorem 38.6 in [48]. □

$\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is a strong ILH-Lie group in the sense of H. Omori [56], and thus a regular convenient Lie group modeled on a Fréchet space. The fundamental result follows: all the Lie groups $\mathrm{Diff}_+^\infty(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ are regular Fréchet-Lie groups.

The algebraic structure of the semi-direct product $\mathrm{Diff}_+^\infty(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is completely determined by the action α. To understand these kind of actions we have to investigate the algebraic homomorphisms between diffeomorphism groups, a subject where fortunately in the last period a significant progress has been made.

2.4 Isomorphisms and homomorphisms between groups of diffeomorphisms

The richness of the algebraic structure of diffeomorphisms groups was a key point in the proof of a remarkable result obtained by R. Filipkiewicz, who used the stabilizer subgroups of the group of diffeomorphisms to obtain a relationship between the topologies of the supporting connected manifolds.

Theorem 2.4.1. *(R. Filipkiewicz, [21]) Let M and N be connected finite dimensional smooth manifolds, without boundary, and let $\mathrm{Diff}^p(M)$, $\mathrm{Diff}^q(N)$, $1 \le p, q \le \infty$, denote the groups of C^p-diffeomorphisms of M, respectively of C^q-diffeomorphisms of N. If:*

$$\Phi : \mathrm{Diff}^p(M) \to \mathrm{Diff}^q(N)$$

is a group isomorphism, then $p = q$ and there exists a C^p-diffeomorphism $g : M \to N$ such that:

$$\Phi(f) = g \circ f \circ g^{-1},$$

for all $f \in \mathrm{Diff}^p(M)$.

If $\dim M \ge 2$ and the group $\mathrm{Diff}^p(M)$ is satisfying the *path transitivity property* i.e. for every smooth path $c : I \to M$ and every open neighborhood U of $Im(c)$ there exists $f \in \mathrm{Diff}^p(M)$ with $\mathrm{supp}(f) \subseteq U$ and $f(c(0)) = c(1)$, then the C^p-diffeomorphism g is unique, due to similar arguments like in the proof of Lemma 2.4.4 below.

This theorem was afterwards extended by A. Banyaga [5], [6] and T.Rybicki [64] for other diffeomorphism groups and for more general settings. Filipkiewicz's theorem is also true for isomorphisms between the identity component $\mathrm{Diff}_c^k(M)_0$, a proof can be found in [29].

Kathryn Mann has obtained in [45] the structure of the algebraic homomorphisms $\Phi : \mathrm{Diff}_c^p(M_1)_0 \to \mathrm{Diff}_c^q(M_2)_0$, when M_1, M_2 are 1-dimensional connected manifolds and $\mathrm{Diff}_c^k(M)_0$, represents the group of compactly supported diffeomorphisms of class C^k on M, isotopic to the identity. Her result constitutes a solution for the problem posed by S. Matsumoto in [46] to determine the endomorphisms of the group $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$. Recently S. Hurtado [30] has extended the result for arbitrary connected manifolds M_1, M_2 of equal dimensions.

Theorem 2.4.2. *(K. Mann, [45]) Let $3 \leq k \leq \infty$ and M_1, M_2 be connected smooth 1-manifolds. Every homomorphism $\Phi : \mathrm{Diff}_c^k(M_1)_0 \to \mathrm{Diff}_c^k(M_2)_0$ has the form:*

$$\Phi(f)(x) = \begin{cases} g_i \circ f \circ g_i^{-1} & \text{if } x \in g_i(M_1) \\ x & \text{otherwise} \end{cases}$$

where $g_1, g_2, g_3, \ldots$ is a possibly infinite collection of C^k embeddings from M_1 to M_2, whose images are pairwise disjoint and contained in some compact subset of M_2.

From now on we will focus on the case $M = \mathbb{S}^1$. Since $\mathbb{S}^1$ is a connected manifold every diffeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ is either orientation-preserving or orientation-reversing. Thus we have two classes Φ_g^+, Φ_g^- of automorphisms on $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$, generated by an orientation-preserving diffeomorphism g, respectively by an orientation-reversing. The next corollary to Mann's theorem is the main ingredient for some of the arguments presented here:

Corollary 2.4.3. *Any endomorphism of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is a Lie group automorphism or is trivial.*

Proof. In the case of homomorphisms between the groups $\mathrm{Diff}_+^k(\mathbb{S}^1)$ in [45] K. Mann is mentioning that the embeddings g_i can be patched together in a global mapping $g : \mathbb{S}^1 \to \mathbb{S}^1$ which is a C^k diffeomorphism but we can also use, for our immediate purpose, the following lemma:

Lemma: Smooth embeddings between compact and connected smooth manifolds of the same dimension are diffeomorphisms.

Proof of lemma: Let $g : M_1 \to M_2$ be a smooth embedding. Then g is in particular an immersion. The mapping $T_x g$ is linear and injective between linear spaces of the same dimension, thus is bijective. Further g is a local diffeomorphism and in conclusion an open mapping. Now $g(M_1)$ is open and compact and so $g(M_1) = M_2$, by connectivity reasons. The mapping g is a bijective local diffeomorphism, thus a diffeomorphism.

Now for every nontrivial endomorphisms Φ of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ we can choose a smooth diffeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Phi(f) = g \circ f \circ g^{-1}$. This mapping is obviously Gâteaux smooth when is defined on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ but when is restricted to $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$, because g can be also orientation-reversing, the Gâteaux smoothness has to be justified. the set $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is the connected component containing the identity of $\mathrm{Diff}^\infty(\mathbb{S}^1)$, thus is open, because Fréchet manifolds are locally path-connected. It is also a submanifold of the Fréchet manifold $\mathrm{Diff}^\infty(\mathbb{S}^1)$ and the inclusion mapping $i_{\mathrm{Diff}_+^\infty(\mathbb{S}^1)}$ is Gâteaux smooth. In conclusion $\Phi : \mathrm{Diff}_+^\infty(\mathbb{S}^1) \to \mathrm{Diff}^\infty(\mathbb{S}^1)$ is Gâteaux smooth as a composition of two Gâteaux smooth mappings. To finish the proof one just has to use, for $S = \mathrm{Diff}_+^\infty(\mathbb{S}^1)$, the next standard lemma:

Lemma: If M, N are Fréchet manifolds and $S \subseteq M$ an open submanifold such that $\Phi : N \to M$ is Gâteaux smooth and $\Phi(N) \subseteq S$, then $\Phi : N \to S$ is a Gâteaux smooth mapping.

$\square$

The next result is well-known for the group of orientation-preserving diffeomorphisms $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$, due to its simplicity, but for $\mathrm{Diff}^\infty(\mathbb{S}^1)$ we couldn't find a reference, thus we prove it. First of all the property holds also in the case $\mathrm{Diff}^p(M)$, when $\dim M \geq 2$, because the path transitivity implies in this case, according to [6], the *n-fold transitivity*: for any two ordered sets of n different points $(x_1, \dots x_n)$ and $(y_1, \dots y_n)$ there is a diffeomorphism $f \in \mathrm{Diff}^p(M)$ such that $f(x_i) = y_i$ for each $i = \overline{1, n}$.

Lemma 2.4.4. *The group* $\mathrm{Diff}^\infty(\mathbb{S}^1)$ *has a trivial center.*

Proof. We start proving the 2-fold transitivity of the group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ using two lemmas:

Lemma: (see [9]) The group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is 2-fold transitive if and only if for each $x \in \mathbb{S}^1$ the group Stab_x acts transitively on $\mathbb{S}^1 \setminus \{x\}$.

The stabilizer group $\mathrm{Stab}_x = \{f \in \mathrm{Diff}^\infty(\mathbb{S}^1) : f(x) = x\}$ acts transitively on $\mathrm{S}^1 \setminus \{x\}$, $x \in \mathrm{S}^1$ fixed, as a direct consequence of a lemma proven by J.Milnor in [50]:

Homogeneity Lemma: Let y, z be arbitrary points of the smooth connected manifold $\mathbb{S}^1 \setminus \{x\}$. Then there exists a diffeomorphism:

$$h : \mathbb{S}^1 \setminus \{x\} \to \mathbb{S}^1 \setminus \{x\},$$

that is smoothly isotopic to the identity and carries y in z.

Now if one has a diffeomorphism $g \neq id$ such that $g \circ f = f \circ g$ for every $f \in \mathrm{Diff}^\infty(\mathbb{S}^1)$ then there exists $x, y \in \mathbb{S}^1$ such that $g(x) = y \neq x$. The double transitivity implies the existence of an element $f_0 \in \mathrm{Diff}^\infty(\mathbb{S}^1)$ such that $f_0(x) = x$ and $f_0(y) = z$, $z \neq x, z \neq y$. Then:

$$g \circ f_0(x) = g(x) = y \neq f_0 \circ g(x),$$

and the result follows by contradiction. $\square$

Corollary 2.4.5. *Any element* Φ *of* $Aut(\mathrm{Diff}^\infty_+(\mathbb{S}^1))$ *is of the form:*

$$\Phi(\varphi) = \Phi_g(\varphi) = g \circ \varphi \circ g^{-1},$$

for a unique element $g \in \mathrm{Diff}^\infty(\mathbb{S}^1)$.

Having everything prepared one can obtain a nice characterization of the right action from (2.3.1), which generates the semi-direct structure:

Proposition 2.4.6. *The convenient smooth right action which defines the general semi-direct product* $\mathrm{Diff}^\infty_+(\mathbb{S}^1) \circledS_\alpha \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ *has the form:*

$$\varphi.\psi := \alpha(\varphi^{-1})(\psi) = g \circ \varphi^{-1} \circ g^{-1} \circ \psi \circ g \circ \varphi \circ g^{-1},$$

for some fixed $g \in \mathrm{Diff}^\infty(\mathbb{S}^1)$*, or:*

$$\varphi.\psi := \alpha(\varphi^{-1})(\psi) = \psi.$$

Proof. First of all the right action satisfies the relation:

$$\alpha(\varphi^{-1}) \in \operatorname{Aut}(\operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1)), \qquad \varphi \in \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1),$$

and thus by Corollary 2.4.5 one obtains $\alpha(\varphi^{-1})(\psi) = \tilde{\alpha}(\varphi^{-1}) \circ \psi \circ \tilde{\alpha}(\varphi)$, where $\tilde{\alpha} : \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1) \to \operatorname{Diff}^{\infty}(\mathbb{S}^1)$ is a group homomorphism, because $\operatorname{Diff}^{\infty}(\mathbb{S}^1)$ has a trivial center.

Now remember that $\operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1)$ is a normal subgroup in $\operatorname{Diff}^{\infty}(\mathbb{S}^1)$ and:

$$\operatorname{Diff}^{\infty}(\mathbb{S}^1) \,/\, \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1) \cong \mathbb{Z}_2.$$

Finally the homomorphism $\tilde{\alpha}$ will induce an homomorphism:

$$\tilde{\alpha}_0 : \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1) \to \operatorname{Diff}^{\infty}(\mathbb{S}^1) \,/\, \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1),$$

by $\tilde{\alpha}_0(\varphi) = [\tilde{\alpha}(\varphi)]$. According to [26] such an homomorphism has to be injective or trivial and in conclusion:

$$\tilde{\alpha}(\operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1)) \subseteq \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1),$$

which is enough for proving the statement of the proposition using Mann's theorem and Corollary 2.4.3. □

As an immediate consequence of the above structure theorem, for the right action needed to define a semi-direct product, one can obtain the adjoint actions generated by the Lie group $\operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1) \circledS_{\alpha} \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1)$:

Remark 2.4.7. The adjoint action of $\operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1) \circledS_{\alpha} \operatorname{Diff}^{\infty}_{+}(\mathbb{S}^1)$ on its Lie algebra $C^{\infty}(\mathbb{S}^1) \circledS_{\beta} C^{\infty}(\mathbb{S}^1)$ is:

$$\operatorname{Ad}_{(\varphi,\psi)}(u, v) = (\operatorname{Ad}_{\varphi} u, \operatorname{Ad}_{c_g(\varphi)\circ\psi} v + \operatorname{Ad}_{c_g(\varphi)\circ\psi} \operatorname{Ad}_g u - \operatorname{Ad}_{c_g(\varphi)} \operatorname{Ad}_g u),$$

excepting the trivial case $\alpha(\varphi^{-1})(\psi) = \psi$, which will be excluded from now on from our reasonings.

The corresponding adoint action of the Lie algebra $C^{\infty}(\mathbb{S}^1) \circledS_{\beta} C^{\infty}(\mathbb{S}^1)$ on itself will be:

$$\operatorname{ad}_{(u_1,v_1)}(u_2, v_2) = ([u_1, u_2], [v_1, v_2] + [\operatorname{Ad}_g u_1, v_2] - [\operatorname{Ad}_g u_2, v_1]),$$

for $[u, v] = u_x \cdot v - v_x \cdot u$, and $g \in \operatorname{Diff}^{\infty}(\mathbb{S}^1)$ the fixed element which defines the action in Proposition 2.4.6. This formula corresponds to (2.3.5) for the mapping $\beta(u)v := [\operatorname{Ad}_g u, v]$.

Remark 2.4.8. One can observe now, directly, the Lie algebra isomorphism between the direct product and the semi-direct product $C^{\infty}(\mathbb{S}^1) \circledS_{\beta} C^{\infty}(\mathbb{S}^1)$. The connection between these two structures is even deeper, since the isomorphism persists also at the group level. This is a consequence of the intriguing results of Filipkiewicz type. We remind here a well-known pathology of the infinite-dimensional Fréchet-Lie groups: the existence of a Lie algebra isomorphism does not imply the existence of an isomorphism between the underlying Lie groups, according to [49].

2.5 Euler-Poincaré equations on $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)Ⓢ_{\alpha}\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$

Now we want to investigate the Euler-Poincaré equations on a general semi-direct product $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)Ⓢ_{\alpha}\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ defined as in the previous section. To work in a general setting we introduce the notion of regular inertia operators of Fourier type:

Definition 2.5.1. A continuous linear operator on the space $\mathrm{C}^{\infty}(\mathbb{S}^1, \mathbb{C})$ which satisfies one of the following equivalent conditions:

i) P commutes with all translations,

ii) [P,D]=0, where $D = \frac{d}{dx}$,

iii) for each $n \in \mathbb{Z}$, there is a $p(n) \in \mathbb{C}$ such that $Pe_n = p(n)e_n$ where $e_n(x) = e^{2\pi inx}, \quad x \in \mathbb{S}^1$,

is called a *Fourier multiplier*.

The sequence $p : \mathbb{Z} \to \mathbb{C}$ is called the *symbol* of P and the notation $P := Op(p)$ can be used for the Fourier multiplier induced by the sequence p. A Fourier multiplier P is $L^2(\mathbb{S}^1)$-symmetric if and only if its symbol p is real. We are interested in Fourier multipliers on $\mathrm{C}^{\infty}(\mathbb{S}^1, \mathbb{C})$ which send real-valued functions on the circle in real-valued functions on the circle, i.e. Fourier multipliers on $\mathrm{C}^{\infty}(\mathbb{S}^1)$. It is not a very restrictive requirement since every Fourier multiplier P on $\mathrm{C}^{\infty}(\mathbb{S}^1, \mathbb{C})$ with a *hermitian symbol*:

$$p(-n) = \overline{p(n)}, \qquad n \in \mathbb{Z}, \tag{2.5.1}$$

has this property and this condition is also necessary.

Definition 2.5.2. A continuous linear operator $P : \mathrm{C}^{\infty}(\mathbb{S}^1) \to \mathrm{C}^{\infty}(\mathbb{S}^1)$ is called a Fourier multiplier if it is a Fourier multiplier on $\mathrm{C}^{\infty}(\mathbb{S}^1, \mathbb{C})$ with a hermitian symbol.

By (2.3.4) the action α induces an action β which generates the corresponding Lie algebra $\mathfrak{g}Ⓢ_{\beta}\mathfrak{g}$ of a semi-direct product $GⓈ_{\alpha}G$.

Definition 2.5.3. If $\mathfrak{g} = \mathrm{C}^{\infty}(\mathbb{S}^1) \times \mathrm{C}^{\infty}(\mathbb{S}^1)$ a linear and continuous operator $\mathbb{A} : \mathfrak{g} \to \mathfrak{g}$ is called *regular inertia operator* if $\mathbb{A} \in Isom(\mathfrak{g}, \mathfrak{g})$ and if it is symmetric and positive definite with respect to the L^2-inner product on $\mathfrak{g}$, given by:

$$\langle (u_1, v_1), (u_2, v_2) \rangle := \int_{\mathbb{S}^1} \langle (u_1, v_1)(x) | (u_2, v_2)(x) \rangle_{\mathbb{R}^2} dx. \tag{2.5.2}$$

Such an operator defines an inner product on $\mathrm{C}^{\infty}(\mathbb{S}^1)Ⓢ_{\beta}\mathrm{C}^{\infty}(\mathbb{S}^1)$ by:

$$\langle (u_1, v_1), (u_2, v_2) \rangle^{\mathbb{A}}_{(\mathrm{id},\mathrm{id})} := \int_{\mathbb{S}^1} \langle (u_1, v_1)(x) | \mathbb{A}(u_2, v_2)(x) \rangle_{\mathbb{R}^2} dx \tag{2.5.3}$$

for $u_1, u_2, v_1, v_2 \in \mathrm{C}^{\infty}(\mathbb{S}^1)$ and this inner product will be extended to a weak right-invariant Riemannian metric on $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)Ⓢ_{\alpha}\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ by right translations:

$$\langle (u_1, v_1), (u_2, v_2) \rangle^{\mathbb{A}}_{(\varphi,\psi)} := \langle R_{(\varphi,\psi)^{-1}}(u_1, v_1), R_{(\varphi,\psi)^{-1}}(u_2, v_2) \rangle^{\mathbb{A}}_{(\mathrm{id},\mathrm{id})}. \tag{2.5.4}$$

This Riemannian metric is called weak because the topology induced by the above inner product is weaker than the existing topology on the tangent space. Further, to exploit the algebraic structure of our Lie group we consider block operators which fulfill the above conditions.

Definition 2.5.4. A regular inertia operator is called of *Fourier type* if:

$$\mathbb{A} := \begin{pmatrix} Op(a) & Op(d) \\ Op(c) & Op(b) \end{pmatrix},$$

for some Fourier multipliers $Op(a), Op(b), Op(c), Op(d)$ on $\mathrm{C}^\infty(\mathbb{S}^1)$.

We can characterize these operators in terms of the symbols:

Proposition 2.5.5. *An operator* $\mathbb{A} : \mathrm{C}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1)$*:*

$$\mathbb{A} := \begin{pmatrix} A & D \\ C & B \end{pmatrix},$$

is a regular inertia operator of Fourier type if and only if:

$$\mathbb{A} := \begin{pmatrix} Op(a) & Op(\overline{c}) \\ Op(c) & Op(b) \end{pmatrix},$$

for some Fourier multipliers on $\mathrm{C}^\infty(\mathbb{S}^1)$*, with their symbols satisfying the conditions:*

$$a(n), b(n) \in \mathbb{R}, \qquad a(n) > 0, \qquad a(n)b(n) > |c(n)|^2, \qquad \forall n \in \mathbb{Z}. \tag{2.5.5}$$

Proof. The proof has its roots in some standard arguments for block matrices, see [75]. We have to prove the equivalence only in one direction, the other one being obvious. If $\mathbb{A}$ is symmetric relative to the inner product:

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{L^2} := \int_{\mathbb{S}^1} \langle (u_1, v_1)(x) | (u_2, v_2)(x) \rangle_{\mathbb{R}^2} dx,$$

then $A = A^*$, $B = B^*$ and $D = C^*$, where C^* means the $L^2(\mathbb{S}^1)$-adjoint of the operator C. For Fourier multipliers on $\mathrm{C}^\infty(\mathbb{S}^1)$ these properties are translated by $a(n), b(n) \in \mathbb{R}$ and $C^* = Op(\overline{c})$.

The operator $\mathbb{A}$ is also positive-definite with respect to the above inner product and hence A will be positive-definite relative to the inner product on $L^2(\mathbb{S}^1)$. Since for any Fourier multiplier A and $u \in \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$ we have the expansion:

$$(Au)(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} a(n) \hat{u}(n), \qquad x \in \mathbb{S}^1,$$

one gets $a(n) > 0$, for every $n \in \mathbb{Z}$. As a consequence A is invertible because $a(n) \neq 0$, $n \in \mathbb{Z}$. If the operator A is invertible one has the following decomposition:

$$\mathbb{A} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B - CA^{-1}C^* \end{pmatrix} \begin{pmatrix} I & A^{-1}C^* \\ 0 & I \end{pmatrix},$$

and then also $B - CA^{-1}C^*$ has to be positive-definite with respect to the inner product on $L^2(\mathbb{S}^1)$, which in terms of the symbols is the last part of the conclusion.

□

Now we introduce the Euler-Poincaré equations on a general semi-direct product $\mathrm{Diff}_+^\infty(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}_+^\infty(\mathbb{S}^1)$:

Proposition 2.5.6. *The Euler-Poincaré equations on the semi-direct product* $\mathrm{Diff}_+^\infty(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}_+^\infty(\mathbb{S}^1)$*, induced by a regular inertia operator, are:*

$$\begin{cases} m_t = \mathbb{ad}_u^* m + (\mathbb{Ad}_{g^{-1}}^* \circ \mathbb{ad}_v^*)n \\ n_t = \mathbb{ad}_{\mathrm{Ad}_g u}^* n + \mathbb{ad}_v^* n \end{cases}, \tag{2.5.6}$$

where $\mathbb{ad}_u^*$ *is the coadjoint action of the Lie algebra* $\mathrm{C}^\infty(\mathbb{S}^1)$ *and:*

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad u, v \in \mathrm{C}^\infty(\mathbb{S}^1).$$

Proof. The inertia operator $\mathbb{A}$ transforms the Euler-Arnold equation into an equation of $\begin{pmatrix} m \\ n \end{pmatrix} = \mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix}$:

$$\begin{pmatrix} m \\ n \end{pmatrix} = \mathbb{ad}_{(u,v)}^*(m,n),$$

so one must prove the relation:

$$\mathbb{ad}_{(u,v)}^*(m,n) = \begin{pmatrix} \mathbb{ad}_u^* m + (\mathbb{Ad}_{g^{-1}}^* \circ \mathbb{ad}_v^*)n \\ \mathbb{ad}_{\mathrm{Ad}_g u}^* n + \mathbb{ad}_v^* n \end{pmatrix}.$$

First of all:

$$\langle \mathrm{ad}_{(u_1,v_1)}^T(u,v),(u_2,v_2)\rangle^{\mathbb{A}} := \langle (X,Y), \mathbb{A}(u_2,v_2)\rangle_{L^2}$$

$$= \langle X, Au_2\rangle_{L^2(\mathbb{S}^1)} + \langle X, C^*v_2\rangle_{L^2(\mathbb{S}^1)} + \langle Y, Cu_2\rangle_{L^2(\mathbb{S}^1)} + \langle Y, Bv_2\rangle_{L^2(\mathbb{S}^1)}$$

$$= (AX + C^*Y, u_2) + (CX + BY, v_2),$$

where $(\cdot,\cdot)$ stands for the pairing between the elements of $\mathfrak{g}_{reg}^* \cong \mathrm{C}^\infty(\mathbb{S}^1)$ and $\mathfrak{g} = \mathrm{C}^\infty(\mathbb{S}^1)$, previously defined in (2.1.3). In the same time:

$$\langle \mathrm{ad}_{(u_1,v_1)}^T(u,v),(u_2,v_2)\rangle^{\mathbb{A}} = \langle (u,v), \mathrm{ad}_{(u_1,v_1)}(u_2,v_2)\rangle^{\mathbb{A}}$$

$$= \langle (u, A[u_1,u_2] + C^*[\mathrm{Ad}_g u_1, v_2] + C^*[v_1, \mathrm{Ad}_g u_2] + C^*[v_1,v_2]\rangle_{L^2(\mathbb{S}^1)} +$$

$$+ \langle (v, C[u_1,u_2] + B[\mathrm{Ad}_g u_1, v_2] + B[v_1, \mathrm{Ad}_g u_2] + B[v_1,v_2]\rangle_{L^2(\mathbb{S}^1)}$$

$$= -(\mathbb{ad}_{u_1}^*(Au + C^*v), u_2) - (\mathbb{ad}_{v_1}^*((Cu+Bv)), \mathrm{Ad}_g u_2) -$$

$$-(\mathbb{ad}_{\mathrm{Ad}_g u_1}^*(Cu+Bv), v_2) - (\mathbb{ad}_{v_1}^*(Cu+Bv), v_2).$$

Thus:

$$AX + C^*Y = -\mathbb{ad}_{u_1}^*(Au + C^*v) - \mathbb{Ad}_{g^{-1}}^* \circ \mathbb{ad}_{v_1}^*(Cu+Bv)$$

$$CX + BY = -\mathbb{ad}_{\mathrm{Ad}_g u_1}^*(Cu+Bv) - \mathbb{ad}_{v_1}^*(Cu+Bv)$$

Using the identity (2.2.2):

$$\mathbb{A}(\mathrm{ad}^T_{(u_1,v_1)}(u,v)) = -\,\mathrm{ad}^*_{(u_1,v_1)}(\mathbb{A}(u,v)),$$

and:

$$\left(\begin{array}{c} m \\ n \end{array}\right) = \left(\begin{array}{c} Au + C^*v \\ Cu + Bv \end{array}\right),$$

to get the conclusion.

□

Remark 2.5.7. A particular case when the action α is given by the action by conjugacy is studied in [16] and the resulted equations are:

$$\begin{cases} m_t = -2mu_x - m_x u - 2nv_x - n_x v \\ n_t = -2n(u_x + v_x) - n_x(u+v) \end{cases},$$

corresponding to the case $g = \mathrm{id}$ after applying the formula (2.2.3) for the coadjoint action and for the inertia operator:

$$\mathbb{A} = (1 - D_x^2)\left(\begin{array}{cc} a & c \\ c & b \end{array}\right), \qquad a, b, c \in \mathbb{R}, \;\; ab - c^2 \neq 0.$$

Remark 2.5.8. If an operator $\mathbb{A}$ satisfies the conditions of Definition 2.5.3 of an inertia operator, then the related Arnold operator ad^T exists and is Gâteaux smooth.

In [15] the authors have imposed some conditions on the symbol, of an inertia operator A of Fourier type, to obtain the well-posedness of an Euler-Poincaré equation, using a geometric approach. A central point of their arguments is the convention $TR_\varphi = R_\varphi$, where R_φ is the right translation on TG. It can be considered in the case of a one-component equation, when the Lie group is $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ but this "symmetry" is broken for a general semi-direct product $\mathrm{Diff}^\infty_+(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}^\infty_+(\mathbb{S}^1)$. Luckily it still persists in the direct product case. Thus one can expect similar results using the following proposition:

Proposition 2.5.9. *The Euler-Poincaré equations on a general semi-direct product $\mathrm{Diff}^\infty_+(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}^\infty_+(\mathbb{S}^1)$, corresponding to a Gâteaux smooth weak right-invariant Riemannian metric $\langle\cdot,\cdot\rangle^{\mathbb{A}}$, are equivalent to the Euler-Poincaré equations on the direct product $\mathrm{Diff}^\infty_+(\mathbb{S}^1)\times\mathrm{Diff}^\infty_+(\mathbb{S}^1)$, corresponding to some Gâteaux smooth weak right-invariant Riemannian metric $\langle\cdot,\cdot\rangle^{\mathbb{B}}$.*

Proof. We exclude the trivial case $\varphi.\psi = \alpha(\varphi^{-1})\psi := \psi$, when there is nothing to prove. In Proposition 2.4.6 has been proved that:

$$\alpha(\varphi) \in Inn(\mathrm{Diff}^\infty_+(\mathbb{S}^1)), \qquad \forall\;\; \varphi \in \mathrm{Diff}^\infty_+(\mathbb{S}^1),$$

and thus applying Proposition A.0.11 from the Appendix, $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ being a simple group, one obtains the isomorphism:

$$\mathrm{Diff}^\infty_+(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}^\infty_+(\mathbb{S}^1) \cong \mathrm{Diff}^\infty_+(\mathbb{S}^1) \times \mathrm{Diff}^\infty_+(\mathbb{S}^1).$$

By Remark A.0.12 the group isomorphism is given by the mapping:

$$\Phi : \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \circledS_{\alpha} \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \to \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \times \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1),$$

defined as:

$$\Phi(f, g) = (f, n(f) \circ g).$$

To prove that this is a Lie group isomorphism it is enough to show the Gâteaux smoothness of the group homomorphism:

$$n : \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \to \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1).$$

This is straightforward from Corollary 2.4.3 and thus the group isomorphism is also a Lie group isomorphism.

If there is a Lie group isomorphism Φ between $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \circledS_{\alpha} \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ and $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \times \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ we know that $\phi := T_{(\mathrm{id},\mathrm{id})}\Phi : \mathfrak{g} \circledS_{\beta} \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ is a linear, continuous and invertible mapping between Frechét spaces, as a consequence of Proposition 1.2.8 (i).

The isomorphism between the corresponding Lie algebras is given by:

$$\phi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Id & 0 \\ N & Id \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $N(v) = T_e n(v)$ is the tangent mapping of n defined above. N is a linear and continuous mapping, thus admits a transpose mapping N^t. Actually $N = \mathrm{Ad}_g$, for some fixed $g \in \mathrm{Diff}^{\infty}(\mathbb{S}^1)$. It is the same element $g \in \mathrm{Diff}^{\infty}(\mathbb{S}^1)$ which appears in the structure of the action α, according to Proposition 2.4.6.

Let's denote whith $\tilde{\mathrm{ad}}_u v$ the bracket in $\mathfrak{g} \circledS_{\beta} \mathfrak{g}$, respectively with $\mathrm{ad}_u v$ the bracket in $\mathfrak{g} \times \mathfrak{g}$. First ϕ is an algebra homomorphism:

$$\phi(\tilde{\mathrm{ad}}_u v) = \mathrm{ad}_{\phi(u)} \phi(v),$$

so:

$$(\phi \circ \tilde{\mathrm{ad}}_u)(v) = (\mathrm{ad}_{\phi(u)} \circ \phi)(v).$$

Again, the continuity of these linear applications implies the existence of the transpose mappings, and:

$$\tilde{\mathrm{ad}}_u^t \circ \phi^t(M) = \phi^t \circ \mathrm{ad}^t_{\phi(u)} M.$$

Taking $m := \phi^t(M) \in \mathfrak{g}^*_{reg} \circledS_{\beta} \mathfrak{g}^*_{reg}$ the connection between the coadjoint actions of the algebras $\mathfrak{g} \circledS_{\beta} \mathfrak{g}$, respectively $\mathfrak{g} \times \mathfrak{g}$ will be:

$$\tilde{\mathrm{ad}}_u^* m = \phi^t \circ \mathrm{ad}^*_{\phi(u)} \circ (\phi^{-1})^t(m). \tag{2.5.7}$$

Since we have restricted the dual of $\mathfrak{g} = \mathrm{C}^{\infty}(\mathbb{S}^1)$ to the regular dual, to prove the continuity of the mapping $(\phi^{-1})^t$ one has just to prove the continuity of the mapping $\mathrm{Ad}^t_g : \mathrm{C}^{\infty}(\mathbb{S}^1) \to \mathrm{C}^{\infty}(\mathbb{S}^1)$, for some fixed $g \in \mathrm{Diff}^{\infty}(\mathbb{S}^1)$. This follows immediately from the inequality:

$$\| \mathrm{Ad}^t_g u \|_{H^n(\mathbb{S}^1)} \leq c \|u\|_{H^n(\mathbb{S}^1)} \cdot \|g_x^2\|_{H^n(\mathbb{S}^1)}, \qquad n \geq 1,$$

since Ad^t_g coincides with the $L^2(\mathbb{S}^1)$-adjoint $\mathrm{Ad}^*_g u = u \circ g \cdot g_x^2$, by (2.2.4).

The Euler equation induced by the inertia operator $\mathbb{A} : \mathfrak{g} \circledS_\beta \mathfrak{g} \to \mathfrak{g}^*_{reg} \circledS_\beta \mathfrak{g}^*_{reg}$:

$$m_t = \tilde{\mathrm{ad}}^*_u m, \qquad m = \mathbb{A}u, \tag{2.5.8}$$

is equivalent with:

$$m_t = \phi^t \circ \mathrm{ad}^*_{\phi(u)} (\phi^{-1})^t (m),$$
$$(\phi^{-1})^t (m_t) = \mathrm{ad}^*_{\phi(u)} (\phi^{-1})^t (m),$$

and making the substitutions $(\phi^{-1})^t (m) = M$, $\phi(u) = U$ because $(\phi^{-1})^t$ is continuous results:

$$M_t = \mathrm{ad}^*_U M, \qquad M = \left(\phi^{-1}\right)^t \circ \mathbb{A} \circ \phi^{-1}(U). \tag{2.5.9}$$

The operator

$$\mathbb{B} = \left(\phi^{-1}\right)^t \circ \mathbb{A} \circ \phi^{-1} : \mathfrak{g} \times \mathfrak{g} \to Im(\mathbb{B}) := \mathfrak{g}^*_{reg} \times \mathfrak{g}^*_{reg}$$

is linear, symmetric and positive definite, thus an inertia operator for the Euler-Poincaré equation (2.5.9). Finally the Euler-Poincaré equation (2.5.8) is transformed in (2.5.9) by a linear smooth transformation.

□

Remark 2.5.10. When we consider regular inertia operators of Fourier type the inertia operator $\mathbb{B}$ is:

$$\mathbb{B} = \begin{pmatrix} Id & -\mathrm{Ad}^*_g \\ 0 & Id \end{pmatrix} \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} Id & 0 \\ -\mathrm{Ad}_g & Id \end{pmatrix}$$

and in general the operator $\mathbb{B}$ will not remain in the class of Fourier multipliers. In the case of the action by conjugacy $\alpha_h = c_h$ the linear mapping $N(v) = v$ extends to a Fourier multiplier $N = Op(n)$ with symbol $n \equiv 1$ and the Euler-Poincaré equation on the semi-direct product $\mathrm{Diff}^\infty_+(\mathbb{S}^1) \circledS \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ induced by the regular Fourier multiplier $\mathbb{A}$ as in Proposition 2.5.6 is equivalent with the Euler-Poincaré equation induced by the regular Fourier multiplier $\mathbb{B}$ on the direct product $\mathrm{Diff}^\infty_+(\mathbb{S}^1) \times \mathrm{Diff}^\infty_+(\mathbb{S}^1)$, where:

$$\mathbb{B} = \begin{pmatrix} A - C - (C^* - B) & C^* - B \\ C - B & B \end{pmatrix}. \tag{2.5.10}$$

This reduction result from Proposition 2.5.9 leads to an **open problem**:

Is it possible to extend the results obtained in [15] for the case of an invertible, elliptic pseudo-differential operator in the Hörmander class $S^r_{1,0}$?

"Each mistake teaches you something new about yourself. There is no failure, remember, except in no longer trying. It is the courage to continue that counts."

Chris Bradford

3 The geodesic spray

An interesting phenomenon concerning the Euler-Poincare equations occurs for some Fréchet-Lie groups: the propagator of the evolution equation which describes the geodesic flow, i.e. the spray equation (Lagrangian coordinates), has better properties than the one corresponding to the Euler-Arnold equation (Eulerian coordinates). For example, under some conditions, it is possible to recast the spray equation as an ODE on suitable Hilbert spaces and to work on the Hilbert approximations of the ILH Lie groups $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$, or $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \times \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$. We exploit this phenomenon in order to obtain the existence of an integral curve of the geodesic spray.

First of all we need to define the spray in the case of Fréchet manifolds. We are interested here only in sprays related to a right-invariant metric on a regular Lie group. There is not a direct way to define a spray vector field on a Fréchet -Lie group using the concept of Gâteaux smoothness. One can take advantage of the additional ILH structure of some regular Lie groups to define the spray as an inverse limit of sprays defined on Hilbert manifolds, as in [60], but this strategy may not succeed every time since some operators involved in its construction may be non-extendable to an inverse limit of smooth operators on the Hilbert approximations. In [15] the authors used this strategy to give a meaning to an expression which naturally corresponds to the geodesic spray but the concept of smoothness used obstructs a rigorous definition of it on Fréchet manifolds. The convenient setting permits us to define a geodesic spray on Fréchet manifolds and afterwards one can use the ILH structure of the Lie group just to prove the existence of an integral curve of it. With this approach we are not interested if the right-invariant metric used extends to a smooth metric on the Hilbert approximations, but this property is obtained anyway as a bonus to the related smoothnes of the twisted operator A_{φ}.

In the case of a Banach manifold M modeled on a Banach space $\mathbb{E}$ to define the spray related to a metric we make use of the flat mapping:

$$\widehat{g} : T_x M \to T_x^* M, \qquad \xi \to g(x)(\xi, \cdot),$$

where g is a smooth Riemannian metric on M. On the cotangent bundle of M we can define the canonical Liouville 1-form by:

$$\Theta_\omega(X) := \omega(T\pi^*(X)), \qquad \omega \in T_x^*M, X \in T_\omega(T^*M),$$

where $\pi^* : T^*M \to M$ is the canonical projection. There is also a canonical symplectic form on T^*M obtained as:

$$\Omega = -d\Theta,$$

where d is the exterior derivative of a 1-form.

We can pull-back the Liouville form by the flat mapping $\widehat{g}$ to obtain a 1-form Θ^g on TM :

$$\Theta^g_\xi(X) := g(x)(\xi, T\pi_M(X)), \qquad \xi \in TM, X \in T_\xi(TM),$$

and further a symplectic form on TM:

$$\Omega^g := -d\Theta^g.$$

If the metric is strong we can associate to every function H on TM a Hamiltonian vector field F_H on TM defined as:

$$d_\xi H(X) := \Omega^g(F_H(\xi), X), \tag{3.0.1}$$

where $\xi \in TM$ and $X \in T_\xi(TM)$. If the metric is weak the flat mapping and the symplectic form Ω^g are only injective and thus given a function H on TM the Hamiltonian vector field corresponding to it may not exist, but if exists it is given by the above relation (3.0.1).

Definition 3.0.11. ([38],[41]) The geodesic spray F associated to a metric g is defined (if exists) as the Hamiltonian vector field of the energy function:

$$E(\xi) := \frac{1}{2}g(x)(\xi, \xi).$$

In a local chart $U_{\mathbb{E}} \times \mathbb{E}$ of TM the Hamiltonian vector field F is:

$$F(x, v) := (x, v, v, S(x, v)),$$

where $S(x, v)$ is defined by:

$$g(x)(S(x, v), u) = \frac{1}{2}D_x g(u)(v, v) - D_x g(v)(v, u),$$

for $x \in U_{\mathbb{E}} \subseteq \mathbb{E}$ and $u, v \in \mathbb{E}$, where $D_x g$ represents the Fréchet derivative of the local representative of the metric. Since the flat mapping $\widehat{g}$ is bijective one gets:

$$S(x, v) = pr_2[\widehat{g}^{-1}(x, \frac{1}{2}D_x g(u)(v, v) - D_x g(v)(v, u))].$$

In the sequel we try to construct a reasonable spray theory for regular convenient Lie groups. A more substantial theory is available for nuclear Fréchet manifolds because such manifolds admit a smooth partition of unity.

Definition 3.0.12. We define a spray S to be a convenient smooth section of both $\pi_{TM} : TTM \to TM$ and $d\pi_M : TTM \to TM$ which satisfies the quadratic condition:

$$S \circ m_\lambda^{TM} = Tm_\lambda^{TM} \circ m_\lambda^{TTM} \circ S,$$

where m_λ^{TM} denotes the fiber scalar multiplication.

Let G be a regular convenient Lie group and g a convenient smooth right-invariant metric defined as in (2.1.1) by a bounded inner product. Generally, a Riemannian metric g on an convenient manifold G can be defined as a convenient smooth section of the vector bundle $L(TG \oplus TG, G \times \mathbb{R})$ which gives a positive definite and symmetric bilinear form $g(p)(\cdot,\cdot)$ on each tangent space T_pG, $p \in G$. Let us consider the following mapping on TTG :

$$\Theta_V^g(X) := g(V, T_V\pi_G(X)) = \langle \kappa^r(V), (\pi_G^*\kappa^r)_V(X)\rangle, \quad X \in T_V TG,$$

where $\langle\cdot,\cdot\rangle$ is the inner product (2.1.2). In the first argument is the mapping κ^r which defines the right Maurer-Cartan form, and in the second is the pullback of the Maurer-Cartan form. The mapping Θ^g defines a 1-form on the kinematic tangent bundle TG, when G is a regular convenient Lie group. Since $(\pi_G^*\kappa^r)_V(X)$ is the pullback by the convenient smooth mapping π_G of the Maurer-Cartan form it will be a 1-form on TG. The smoothness of the inner product and of the mapping $\kappa^r : TG \to \mathfrak{g}$ are sufficient for Θ^g to be a convenient smooth section of the convenient smooth vector bundle $L(TTG, TG \times \mathbb{R})$.

Now one can define a 2-form on TG by:

$$\omega^g(Y, X) := -d\Theta^g(Y, X).$$

To every right-invariant metric on a regular convenient Lie group G one can associate the energy function $E : TG \to \mathbb{R}$:

$$E(V) := \frac{1}{2} g(V, V) = \langle \kappa^r(V), \kappa^r(V)\rangle, \qquad V \in TG.$$

Proposition 3.0.13. *If there exists a vector field S on the kinematic tangent bundle TG satisfying:*

$$i_S\omega^g = dE, \tag{3.0.2}$$

then it is unique and it is a right-invariant spray.

Proof. If $S(x, v) = (x, v, S_1(x, v), S_2(x, v))$ is the local representative of the vector field S then using Remark 1.2.12 the local form of $i_S\omega^g = dE$ is:

$$d_x g(u)(v, S_1(x, v)) + g(x)(w, S_1(x, v)) - d_x g(S_1(x, v))(v, u) - g(x)(S_2(x, v), u)$$
$$= \frac{1}{2} d_x g(u)(v, v) + g(x)(v, w),$$

for $X = (x, v, u, w)$. Choosing $u = 0$ implies $S_1(x, v) = v$. Hence S is a symmetric vector field and $S_2(x, v)$ satisfies:

$$g(x)(S_2(x, v), u) = \frac{1}{2} d_x g(u)(v, v) - d_x g(v)(v, u).$$

Finally $S_2(x, \lambda v) = \lambda^2 S_2(x, v)$ and S_2 is quadratic in v, thus S is a spray.

The vector field (if exists) defined by (3.0.2) has to be unique because g is non-degenerate and one can easily see that S is actually invariant under any isometry of the metric (2.1.1) because E and ω^g are. $\square$

We prove now that when the Arnold operator exists also a spray related to the right-invariant metric exists. The formula is similar to the one obtained in [36] in a more restrictive setting.

Proposition 3.0.14. *If the inner product* (2.1.2) *is bounded and the operator* $ad^T : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *exists and is convenient smooth then the mapping:*

$$S : TG \to TG \times (\mathfrak{g} \times \mathfrak{g}),$$

defined, with respect to the second trivialization of TTG*, by:*

$$S(V) := (V, \kappa^r(V), -ad^T_{\kappa^r(V)}\kappa^r(V)), \qquad V \in TG, \tag{3.0.3}$$

is convenient smooth and satisfies the identity:

$$i_S \omega^g(X) = dE(X), \qquad \forall\, X \in TTG.$$

Proof. Let $\mathbb{E}$ be a convenient vector space, then for a convenient smooth mapping $f : M \to \mathbb{E}$ and a convenient smooth vector field X one obtains a convenient smooth $\mathbb{E}$-valued mapping by $X(f)(p) := T_p f(X(p))$. Using local arguments and Proposition 1.2.8 (iv) the formula holds:

$$X(\langle f(\cdot), g(\cdot)\rangle_{\mathbb{E}})(p) = \langle X(f)(p), g(p)\rangle_{\mathbb{E}} + \langle f(p), X(g)(p))\rangle_{\mathbb{E}}, \quad p \in M,$$

for a bounded inner product $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ on $\mathbb{E}$.

By its definition the vector field S defined above is symmetric:

$$\pi_{TG}(S(V)) = V = T\pi_G(S(V)).$$

Therefore using this property:

$$\begin{aligned}
\omega^g_V(S(V), X(V)) &= -S(\Theta^g(X))(V) + X(\Theta^g(X))(V) + \Theta^g_V([S, X](V)) \\
&= -\langle T_V\kappa^r(S(V)), \pi_G^*\kappa^r(X(V))\rangle - \langle \kappa^r(V), S(\pi_G^*\kappa^r(X))(V)\rangle \\
&\quad +\langle T_V\kappa^r(X(V)), \kappa^r(V)\rangle + \langle \kappa^r(V), X(\pi_G^*\kappa^r(S))(V)\rangle \\
&\quad +\langle \kappa^r(V), \pi_G^*\kappa^r([S, X](V))\rangle \\
&= -\langle T_V\kappa^r(S(V)), \pi_G^*\kappa^r(X(V))\rangle + \langle T_V\kappa^r(X(V)), \kappa^r(V)\rangle \\
&\quad -\langle \kappa^r(V), d(\pi_G^*\kappa^r)(S(V), X(V))\rangle.
\end{aligned}$$

But:

$$\begin{aligned}
d(\pi_G^*\kappa^r)(S, X) &= \pi_G^*\, d\kappa^r(S, X) = \pi_G^*(\frac{1}{2}[\kappa^r, \kappa^r]_{\wedge}(S, X)) \\
&:= [\kappa^r(T\pi_G(S)), \kappa^r(T\pi_G(X))] = \mathrm{ad}_{\kappa^r(T\pi_G(S))}\, \kappa^r(T\pi_G(X)),
\end{aligned}$$

applying the Maurer-Cartan equation. In the same time $T\pi_G(S(V)) = V$, and $T_V\kappa^r(S(V)) = -\,\mathrm{ad}^T_{\kappa^r(V)}\,\kappa^r(V)$, since:

$$T\rho := (T\pi_G, T\kappa^r) : TTG \to TG \times T\mathfrak{g}$$

is giving the second trivialization of the Lie group TG. Now:

$$\omega_V^g(S(V), X(V)) = -\langle -\operatorname{ad}^T_{\kappa^r(V)} \kappa^r(V), \kappa^r(T\pi_G(X(V))\rangle + \langle T_V\kappa^r(X(V)), \kappa^r(V)\rangle$$
$$-\langle \operatorname{ad}^T_{\kappa^r(V)} \kappa^r(V), \kappa^r(T\pi_G(X(V))\rangle = \langle T_V\kappa^r(X(V)), \kappa^r(V)\rangle = dE(X(V)).$$

□

In the case $G := \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ if φ is the flow of a time-dependent vector field u, i.e. $\varphi_t = u \circ \varphi$, then if one denotes $v := \varphi_t$:

$$v_t = u_t \circ \varphi + (u_x u) \circ \varphi, \tag{3.0.4}$$

and this simple identity enables us to compute the spray S. Since only elementary computations on $\mathrm{C}^\infty(\mathbb{S}^1)$ were involved in the derivation of the above relation is appropriate to ask ourselves: What is the geometrical meaning of the above identity ?

Proposition 3.0.15. *The spray given by the formula (3.0.3) can be expressed as:*

$$S(V) = TR_g \circ \overline{S} \circ R_{g^{-1}}(V), \qquad g \in G, V \in T_gG, \tag{3.0.5}$$

for a mapping $\overline{S} : \mathfrak{g} \to TTG\Big|_{\mathfrak{g}}$, *defined as:*

$$\overline{S}(u) := \xi_u(u) - \operatorname{ad}^T_u u, \quad u \in \mathfrak{g},$$

where ξ_u is the fundamental vector field induced by the right action:

$$\gamma : G \times TG \to TG, \qquad \gamma(g, V) := R_g(V), \quad V \in TG.$$

Proof. By its very definition $S(V) := T\rho^{-1}(R_g u, -\operatorname{ad}^T_u u)$, where $V := R_g u$, and $(R_g u, -\operatorname{ad}^T_u u) \in T_{(g,u)}(G \times \mathfrak{g})$.

There will exist two curves $g : \mathbb{R} \to G$ and $u : \mathbb{R} \to \mathfrak{g}$ such that $g(0) = g$, $\dot{g}(0) = R_g u$, respectively $u(0) = u$, $\dot{u}(0) = -\operatorname{ad}^T_u u$, because G is a regular Lie group for example.

$$S = \frac{d}{dt}\Big|_0 \rho^{-1}(g(t), u(t)) = \frac{d}{dt}\Big|_0 R_{g(t)} u(t) = \frac{d}{dt}\Big|_0 R_g(R_{h(t)} u(t)),$$
$$= TR_g \left(\frac{d}{dt}\Big|_0 R_{h(t)} u(t) \right) = TR_g \left(T^1_{(e,u)} \rho^{-1}(u) + T^2_{(e,u)} \rho^{-1}(-\operatorname{ad}^T_u u) \right),$$

for $h(t) = g(t)g^{-1}$, with $h(0) = e$, $\dot{h}(0) = u$.

On the other hand $T^1_{(e,u)}\rho^{-1}(u) := \frac{d}{dt}\Big|_0 R_{h(t)} u = \xi_u(u)$, where ξ_u is the fundamental vector field, defined in Section 1.3.2, induced by the action by right translations on TG. Obviously $\xi_u(u) \in T_u TG$, for all $u \in \mathfrak{g}$. Further:

$$T^2_{(e,u)}\rho^{-1}(-\operatorname{ad}^T_u u) := \frac{d}{dt}\Big|_0 R_e(u(t)) = \frac{d}{dt}\Big|_0 u(t) = -\operatorname{ad}^T_u u \in T_u\mathfrak{g} \subset T_u TG,$$

because by $\mathrm{ad}_u^T u$ we denoted here the corresponding element in $T_u\mathfrak{g} \cong \mathfrak{g}$ for the value $\mathrm{ad}_u^T u$ of the Arnold's operator. $\square$

Proposition 3.0.16. *If the inner product* (2.1.2) *is bounded and the operator* ad^T *exists and is bounded, then a smooth curve* $g : \mathbb{R} \to G$ *is a geodesic of the right-invariant metric* (2.1.1) *if and only if* $\dot{g}(t) : \mathbb{R} \to TG$ *is an integral curve of the right-invariant spray* S *defined by* (3.0.2)*, and we call it the geodesic spray corresponding to the metric* (2.1.1).

Proof. It is a straightforward consequence of Proposition 2.1.2 and of Proposition 3.0.14, since $\kappa^r(\dot{g}(t)) = R_{g(t)^{-1}}\dot{g}(t)$. $\square$

Since we are working on manifolds modelled on non normable locally convex spaces the existence of an integral curve for a smooth vector field is not granted.

3.1 Regular Lie groups modelled on nuclear Fréchet spaces

The model space $\mathrm{C}^\infty(\mathbb{S}^1)$ for the regular convenient Lie group considered in this thesis is a nuclear Fréchet space and for this particular case a substantial local theory is available, following [48]. A normed space is nuclear if and only if it is finite dimensional. It is interesting to observe how facts from the finite dimensional setting are inherited by manifolds modelled on nuclear Fréchet spaces.

According to [65], for example, every paracompact manifold modelled on a locally convex space with a smooth partition of unity has itself a smooth partition of unity. But any regular Fréchet-Lie group is paracompact, according to [39], thus $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ admits a smooth partition of unity. With the same argument or combining Lemma 27.8 and Lemma 27.9, from [48], the manifold $\mathrm{Diff}_+^\infty(\mathbb{S}^1) \times \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ will also admit a smooth partition of unity. Additionally, any vector bundle over $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$, or $\mathrm{Diff}_+^\infty(\mathbb{S}^1) \times \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ will admit a smooth partition of unity:

Proposition 3.1.1. *Let* $\pi : E \to M$ *be a convenient smooth vector bundle with standard fiber* $\mathbb{F}$*, and suppose that* M *and the product of the model space of* M *and* $\mathbb{F}$ *admit a smooth partition of unity. In particular this holds if* M *and* $\mathbb{F}$ *are metrizable and admit a smooth partition of unity. Then the total space* E *admits a smooth partition of unity.*

Proof. Proposition 27.9 in [48]. $\square$

If (U_α, ψ_α) is a vector bundle atlas of $(E, M, \pi, \mathbb{F})$ then with the help of a partition of unity one can construct global convenient smooth sections $S : M \to E$ starting with a convenient smooth mapping $f_\alpha : U_\alpha \to \mathbb{F}$ and the induced local section $x \mapsto \psi_\alpha^{-1}(x, f_\alpha(x))$ on U_α. In conclusion, on $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ for example, the set of second order vector fields is quite big.

On nuclear Fréchet manifolds the operational tangent bundle DM will coincide with the kinematic tangent bundle TM and every tangent vector can be considered as a bounded derivation like in the finite dimensional case, (Theorem 28.7 in [48]). Because in the convenient setting also the differential forms can be defined as sections of a vector bundle, one can expect to recover, on nuclear Fréchet manifolds, some localization arguments from the finite dimensional case.

"In the middle of difficulty lies opportunity."

Albert Einstein

4 Inertia operators of pseudo-differential type

4.1 An overview of the strategy

To prove the existence of an integral curve for the geodesic spray, already defined in the previous section, one possible approach is to exploit the ILH structure of the Lie group. We give a short overview of the method used to overcome the well-known pathology of Fréchet spaces: if $P : U \subseteq \mathbb{E} \to \mathbb{F}$ is a Gâteaux smooth mapping between Fréchet spaces, then differential equations as:

$$x'(t) = P(x(t)), \quad x(0) = x_0 \in U,$$

do not always have solutions and even in the case of their existence, the solutions need not be unique. We will exemplify here for the case $G = \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ but the same arguments hold for $G = \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1) \times \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$.

Let's start considering the set $H^q(\mathbb{S}^1, \mathbb{S}^1)$ of Sobolev H^q-mappings from $\mathbb{S}^1$ to itself, $q > \frac{3}{2}$, and define:

$$\mathcal{D}^q(\mathbb{S}^1) := \{\varphi \in \mathrm{Diff}^1_+(\mathbb{S}^1) : \varphi \in H^q(\mathbb{S}^1, \mathbb{S}^1)\}, \tag{4.1.1}$$

where $\mathrm{Diff}^1_+(\mathbb{S}^1)$ denotes the set of all orientation-preserving C^1 smooth diffeomorphisms of the circle. The above set has the structure of a smooth Hilbert manifold modelled on the space $H^q(\mathbb{S}^1)$, which is defined as the completion of $\mathrm{C}^{\infty}(\mathbb{S}^1)$ under the norm:

$$\|u\|_{H^q(\mathbb{S}^1)} := \left(\sum_{n \in \mathbb{Z}} (1 + n^2)^q |\widehat{u}(n)|^2 \right)^{\frac{1}{2}}.$$

It's worth mentioning that the manifold $\mathcal{D}^q(\mathbb{S}^1)$ has a trivial tangent bundle $T\mathcal{D}^q(\mathbb{S}^1) \cong \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$.

The smooth structure of the Fréchet manifold $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ can be recovered from the smooth structures of the Hilbert manifolds $\mathcal{D}^q(\mathbb{S}^1)$ and we call them *Hilbert approximations* and $\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ an *inverse limit of Hilbert manifolds (ILH)*, see [56] for details.

If A is the inertia operator of order $r \geq 1$, which generates a right-invariant metric like in (2.1.2), then the geodesic spray can be expressed (according [15]) in terms of the twisted operator $A_\varphi(v) := R_\varphi \circ A \circ R_\varphi^{-1}(v)$, where R_φ denotes a right translation by φ on $T\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$. As a consequence the Gâteaux smoothness, on the Hilbert approximations, of the geodesic spray and the right-invariant metric is strongly related to the Gâteaux (Fréchet) smoothness of the mapping:

$$\varphi \mapsto A_\varphi, \;\; \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)).$$

Since the operator:

$$\varphi \mapsto R_\varphi, \;\; \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^q(\mathbb{S}^1)),$$

is not continuous and the above problem is not a trivial one. Due to [15] a necessary and sufficient condition for the Gâteaux smoothness of the twisted operator $\varphi \mapsto A_\varphi$ is the boundedness of each $(n+1)$-linear operator A_n from the product space $\underbrace{H^q(\mathbb{S}^1) \times \ldots \times H^q(\mathbb{S}^1)}_{\text{n times}}$ to $H^{q-r}(\mathbb{S}^1)$. Here by A_n we mean the extension to this space of the operator:

$$A_n : \underbrace{\mathrm{C}^\infty(\mathbb{S}^1) \times \ldots \times \mathrm{C}^\infty(\mathbb{S}^1)}_{\text{n times}} \to \mathrm{C}^\infty(\mathbb{S}^1),$$

given by the recurrence resulted from $\partial_\varphi^n A_\varphi = R_\varphi \circ A_n \circ R_{\varphi^{-1}}$:

$$\begin{aligned} A_0 &= A, \\ A_{n+1}(u_0, u_1, \ldots, u_{n+1}) &= [u_{n+1} D, A_n(\cdot, u_1, \ldots, u_n)] u_0 \\ &\quad - \sum_{s=1}^{n} A_n(u_0, u_1, \ldots u_{n+1} D(u_s), \ldots, u_n), \end{aligned} \tag{4.1.2}$$

where $u_0, u_1, \ldots u_n \in \mathrm{C}^\infty(\mathbb{S}^1)$.

If one can prove the Gâteaux smoothness, on the Hilbert approximations, of the geodesic spray S, then one gets on each $T\mathcal{D}^q(\mathbb{S}^1)$ a solution of the induced ODE. Finally a solution on the Fréchet vector bundle $T\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ is obtained, exploiting the right-invariance of the geodesic spray, via the following lemma (compare with [14], [15]):

Lemma 4.1.1. *(No gain, no loss in spatial regularity) If the initial data $(\varphi_0, v_0) \in T\mathrm{Diff}_{+}^{\infty}(\mathbb{S}^1)$ and for any $q > \frac{3}{2}$ the spray equation:*

$$(\varphi_t, v_t) = S(\varphi, v),$$

$$(\varphi(0), v(0)) = (\varphi_0, v_0)$$

has a solution on $T\mathcal{D}^q(\mathbb{S}^1)$ on some maximal interval of existence $J_q(\varphi_0, v_0)$, then this interval is independent on q.

4.2 A commutator formula

Previously we saw how the boundedness of the multi-linear operator A_n plays a crucial role in the strategy used to construct a solution for the geodesic equation corresponding to a right-invariant metric. Following Proposition 2.5.9 and Remark 2.5.10 we are constrained to study inertia operators of pseudo-differential type and implicitly the boundedness of A_n in this case. In this section we will extend the results from [15] for the case of an invertible elliptic pseudo-differential operator $A : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$, in the Hörmander class $Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$, $r \geq 1$. We try to avoid the para-differential calculus and the symbolic calculus in order to study the boundedness of the multi-linear operator A_n. The trick is to use a similar lemma to Lemma A.6 in [15] and an operatorial formula for A_n.

If A is a Fourier multiplier an elegant formula for the multi-linear operator A_n is available:

Proposition 4.2.1. *The multi-linear operator given by the recurrence (4.1.2), when $A : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$ is a Fourier multiplier, satisfies the identity:*

$$A_n(u_0, u_1, \ldots, u_n) = [u_n, [u_{n-1}, \ldots [u_1, A \circ D^{n-1}]..]D(u_0)$$

for all $u_0, u_1, \ldots, u_n \in \mathrm{C}^\infty(\mathbb{S}^1)$.

This formula appears naturally if we keep in mind the form of the condition imposed in [15] where one has to deal with the expression $\xi^{n-1}p(\xi)$ which in fact is the symbol of $A \circ D^{n-1}$, up to a constant, for $D := D_x$. Working with symbols of pseudo-differential operators can be tedious, thus we prefer an operatorial approach to reduce the problem of boundedness to a simpler problem.

We are going to prove a more general formula, for the case of a pseudo-differential operator A on the 1-torus $\mathbb{S}^1$. If A is a pseudo-differential operator, then $[D, A] \neq 0$ and one can introduce the operator:

$$\operatorname{ad}_D A := [D, A] = DA - AD,$$

with the conventions $\operatorname{ad}_D^0 A := A$ and $\operatorname{ad}_D^n A := \operatorname{ad}_D(\operatorname{ad}_D^{n-1} A)$.

As a consequence of the Jacobi's identity:

$$[A, [B, C]] - [B, [A, C]] = [[A, B], C],$$

the following expression:

$$[u_n, [u_{n-1}, \ldots [u_1, P]..]D(u_0),$$

is symmetric in $u_1, u_2, \ldots u_n$, for every linear operator P. One has also to introduce, for a fixed $k = \overline{0, n}$, the notation:

$$\sum_{\substack{J \subseteq I_n \\ |J| = k}} \prod_{j \in J} u_j \prod_{i \in J^c} \operatorname{ad}_{u_i} P,$$

to denote the sum after all possibile k-couples $(u_j)_{j\in J,|J|=k}$ chosen from the set $\{u_1, u_2, \dots u_n\}$, where $I_n := \{1, 2 \dots n\}$. By $\prod_{i\in J^c} \operatorname{ad}_{u_i} P$ we understand:

$$\operatorname{ad}_{u_{i_1}}(\operatorname{ad}_{u_{i_2}}(\dots \operatorname{ad}_{u_{i_{n-k}}}(P)..), \qquad i_1, i_2, \dots, i_{n-k} \in J^c.$$

Due to the symmetry in $u_{i_1}, u_{i_2}, \dots, u_{i_{n-k}}$ the order inside the brackets is not important. For example:

$$\sum_{\substack{J\subseteq I_3\\|J|=1}} \prod_{j\in J} u_j \prod_{i\in J^c} \operatorname{ad}_{u_i} P := u_3[u_2,[u_1,P]] + u_1[u_3,[u_2,P]] + u_2[u_1,[u_3,P]].$$

Further for $k = 0$ and any $n \geq 1$ the sum will have one term:

$$[u_n,[u_{n-1},\dots[u_1,P]..],$$

and for $k = n$ only the term:

$$u_n \cdot u_{n-1} \cdot \dots \cdot u_1 \cdot P.$$

Proposition 4.2.2. *The multi-linear operator given by the recurrence (4.1.2), when A is a linear operator on $\mathrm{C}^\infty(\mathbb{S}^1)$, satisfies the identity:*

$$A_n(u_0,u_1,..,u_n) = \sum_{J\subsetneq I_n} \left[\prod_{j\in J} u_j \prod_{i\in J^c} \operatorname{ad}_{u_i} \left((\operatorname{ad}_D^{|J|} A)\circ D^{n-|J|-1}\right)\right] D(u_0)$$

$$+u_1 \cdot u_2 \cdot \dots \cdot u_n \operatorname{ad}_D^n A(u_0),$$

for all $u_0, u_1, \dots, u_n \in \mathrm{C}^\infty(\mathbb{S}^1)$.

Of course, for a Fourier multiplier A the terms containing $\operatorname{ad}_D^k A$, $k = |J|$, will vanish, excepting the one for $k = 0$:

$$[u_n,[u_{n-1},\dots[u_1,A\circ D^{n-1}]..]D(u_0),$$

and this expression is the one displayed in Proposition 4.2.1. In the particular case $n = 3$ one obtains:

$$A_3(u_0,u_1,u_2,u_3) = [u_3,[u_2,[u_1,AD^2]]]D(u_0) + u_3[u_2,[u_1,[D,A]D]]D(u_0)$$

$$+u_2[u_3,[u_1,[D,A]D]]D(u_0) + u_1[u_3,[u_2,[D,A]D]]D(u_0)+$$

$$+u_1u_3[u_2,[D,[D,A]]]D(u_0) + u_2u_3[u_1,[D,[D,A]]]D(u_0)$$

$$+u_1u_2[u_3,[D,[D,A]]]D(u_0) + u_1u_2u_3[D,[D,[D,A]]](u_0).$$

Remark 4.2.3. What the above formula is actually saying is that, the study of the boundedness, on Hilbert approximations, of the multi-linear operator A_n can be reduced to a simpler problem, namely the boundedness of the expression:

$$[u_s,[u_{s-1},\dots[u_1,P]..]D(u_0), \qquad P\in Op(S^{r+s-1}(\mathbb{S}^1\times\mathbb{Z})),\ s=\overline{0,n},$$

which can be identified in the general term of each of the above sums. More specific, whenever $A \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ the pseudo-differential operator $\mathrm{ad}_D^k A \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$, for any $k = \overline{0, n}$. Additionally, the expression $\prod_{i \in J^c} \mathrm{ad}_{u_i}(\mathrm{ad}_D^{|J|} A \circ D^{n-|J|-1})$ contains $|J^c| = n - |J|$ brackets, leading to an operator:

$$P := \mathrm{ad}_D^{|J|} A \circ D^{n-|J|-1} \in Op(S^{r+(n-|J|)-1}(\mathbb{S}^1 \times \mathbb{Z})).$$

The proof of the Proposition 4.2.2 is a long journey and we start with a few small steps proving some useful identities and lemmas.

Lemma 4.2.4. *The next identities hold for any linear operators A, B, C and $u \in \mathrm{C}^\infty(\mathbb{S}^1)$:*

$$[A, BC] = [A, B]C + B[A, C], \tag{4.2.1}$$

$$[AB, C] = A[B, C] + [A, C]B, \tag{4.2.2}$$

$$[u, DA] = D[u, A] - D(u) \cdot A, \tag{4.2.3}$$

$$[u, AD] = [u, A]D - A(D(u)\cdot), \tag{4.2.4}$$

$$[uD, A] = [u, A]D + u[D, A], \tag{4.2.5}$$

where the multiplication operator $M_u(v) := uv$ is denoted by $u \cdot$.

Proof. The first two identities are basic properties of the commutator of two linear operators and the next three represent direct consequences. $\square$

Lemma 4.2.5. *(Leibniz type Lemma) For a pseudo-differential operator A and any $u_1, .., u_n \in \mathrm{C}^\infty(\mathbb{S}^1)$ we have the identity:*

$$[u_n, \ldots [u_1, AD]..] = [u_n, \ldots [u_1, A]..]D - \sum_{s=1}^{n} [u_n, \ldots [\widehat{u_s}, \ldots [u_1, A]..](D(u_s)\cdot).$$

Proof. To start an induction over $n \geq 1$ the case $n = 1$ corresponds to the identity (4.2.4) in Lemma 4.2.4. Let's suppose that the identity is true for an $n > 1$, then:

$$\begin{aligned}
&[u_{n+1}, [u_n, [u_{n-1}, \ldots [u_1, A \circ D]..] = M_{u_{n+1}}[u_n, [u_{n-1}, \ldots [u_1, A]..]D \\
&- \sum_{s=1}^{n} M_{u_{n+1}}[u_n, \ldots [\widehat{u_s}, \ldots [u_1, A]..]M_{D(u_s)} - [u_n, [u_{n-1}, \ldots [u_1, A]..]DM_{u_{n+1}} \\
&+ \sum_{s=1}^{n} [u_n, \ldots [\widehat{u_s}, \ldots [u_1, A]..]M_{D(u_s)}M_{u_{n+1}} = [u_{n+1}, [u_n, [u_{n-1}, \ldots [u_1, A]..]D \\
&-[u_{n+1}, [u_n, [u_{n-1}, \ldots [u_1, A]..]M_{D(u_{n+1})} - \sum_{s=1}^{n} [u_{n+1}, [u_n, \ldots [\widehat{u_s}, \ldots [u_1, A]..]M_{D(u_s)} \\
&= [u_{n+1}, \ldots [u_1, A]..]D - \sum_{s=1}^{n+1} [u_{n+1}, \ldots [\widehat{u_s}, \ldots [u_1, A]..](D(u_s)\cdot)
\end{aligned}$$

because $DM_{u_{n+1}} = M_{D(u_{n+1})} + M_{u_{n+1}} D$, and thus the conclusion follows by the principle of mathematical induction.

$\square$

Lemma 4.2.6. *(Commutation Lemma) With the same hypotheses the next identity holds:*

$$D[u_n, [u_{n-1}, \dots [u_1, A]..] = [u_n, [u_{n-1}, \dots [u_1, A]..]D +$$
$$\sum_{s=1}^{n} [u_n, \dots [D(u_s), \dots [u_1, A]..] + [u_n, [u_{n-1}, \dots [u_1, [D, A]]..]$$

Proof. Using again the principle of mathematical induction, for $n = 1$ one just has to substract (4.2.4) from (4.2.3) in Lemma 4.2.4:

$$D[u_1, A] = [u_1, A]D + [D(u_1), A] + [u_1, [D, A]].$$

Further let's suppose that the property is true for $n > 1$ and then:

$$D[u_{n+1}, [u_n, \dots [u_1, A]..]] = D(u_{n+1})[u_n, \dots [u_1, A]..] + u_{n+1} D[u_n, \dots [u_1, A]..]$$
$$-D[u_n, \dots [u_1, A]..](u_{n+1} \cdot).$$

Using the induction hypothesis this equals:

$$D(u_{n+1})[u_n, \dots [u_1, A]..] +$$
$$u_{n+1} \left([u_n, \dots [u_1, A]..]D + \sum_{s=1}^{n} [u_n, \dots [D(u_s), \dots [u_1, A]..] + [u_n, [u_{n-1}, \dots [u_1, [D, A]]..] \right)$$
$$-[u_n, \dots [u_1, A]..]D(u_{n+1} \cdot) - \sum_{s=1}^{n} [u_n, \dots [D(u_s), \dots [u_1, A]..](u_{n+1} \cdot)$$
$$-[u_n, [u_{n-1}, \dots [u_1, [D, A]]..](u_{n+1} \cdot).$$

Inserting $D(u_{n+1} \cdot) = u_{n+1} D + D(u_{n+1}) \cdot$, the first and the second term will be modified to:

$$[D(u_{n+1}), [u_n, \dots [u_1, A]..]] + [u_{n+1}, [u_n, \dots [u_1, A]..]]D$$
$$+u_{n+1} \left(\sum_{s=1}^{n} [u_n, \dots [D(u_s), \dots [u_1, A]..] + [u_n, [u_{n-1}, \dots [u_1, [D, A]]..] \right)$$
$$-\sum_{s=1}^{n} [u_n, \dots [D(u_s), \dots [u_1, A]..](u_{n+1} \cdot) - [u_n, [u_{n-1}, \dots [u_1, [D, A]]..](u_{n+1} \cdot)$$
$$= [u_{n+1}, [u_n, \dots [u_1, A]..]]D + [D(u_{n+1}), [u_n, \dots [u_1, A]..]]$$
$$+\sum_{s=1}^{n} [u_{n+1}, [u_n, \dots [D(u_s), \dots [u_1, A]..]] + [u_{n+1}, [u_n, [u_{n-1}, \dots [u_1, [D, A]]..]]$$
$$= [u_{n+1}, [u_n, \dots [u_1, A]..]]D + \sum_{s=1}^{n+1} [u_{n+1}, [u_n, \dots [D(u_s), \dots [u_1, A]..]]$$
$$+[u_{n+1}, [u_n, [u_{n-1}, \dots [u_1, [D, A]]..]],$$

the estimated expression. $\square$

Corollary 4.2.7. *One also has the identity:*

$$[u_n, \dots [u_1, [D, A]]..] = [D, [u_n, \dots [u_1, A]..]] - \sum_{s=1}^{n} [u_n, \dots [D(u_s), \dots [u_1, A]..]$$

We saw how the operator D is interacting with the expression $[u_n, [u_{n-1}, ..[u_1, A]..]$, which is the bottom line of the whole argument. We can proceed now to the proof of the Proposition 4.2.2:

Proof. For $n = 1$:

$$A_1(u_0, u_1) := [u_1 D, A](u_0) = [u_1, A]D(u_0) + u_1[D, A](u_0),$$

by (4.2.5) in Lemma 4.2.4. For $n = 2$:

$$A_2(u_0, u_1, u_2) := [u_2 D, A_1(\cdot, u_1)](u_0) - A_1(u_0, u_2 D(u_1)) =$$

$$[u_2 D, [u_1, A]D] - [u_2 D(u_1), A]D + [u_2 D, u_1[D, A]] - u_2 D(u_1)[D, A].$$

We split up this sum into two sums:

$$[u_2 D, [u_1, A]D] - [u_2 D(u_1), A]D = [u_2 D, [u_1, AD] + A(D(u_1)\cdot)] - [u_2 D(u_1), A]D,$$

using (4.2.4). At the next step:

$$u_2 D[u_1, AD] - [u_1, AD](u_2 D) + [u_2 D, A(D(u_1)\cdot)] - [u_2 D(u_1), A]D,$$

and the commutation lemma 4.2.6 is necessary:

$$= u_2[u_1, AD]D + u_2[D(u_1), AD] + u_2[u_1, [D, AD]] - [u_1, AD](u_2 D)$$

$$+[u_2 D, A(D(u_1)\cdot)] - [u_2 D(u_1), A]D = [u_2, [u_1, AD]]D + u_2[u_1, [D, A]D]$$

$$+u_2[D(u_1), AD] + [u_2 D, A(D(u_1)\cdot)] - [u_2 D(u_1), A]D.$$

Again using (4.2.4) this sum becomes:

$$[u_2, [u_1, AD]]D + u_2[u_1, [D, A]]D - u_2[D, A](D(u_1)\cdot) + u_2[D(u_1), AD]$$

$$+[u_2 D, A(D(u_1)\cdot)] - [u_2 D(u_1), A]D.$$

Thus the first part of the sum is:

$$[u_2, [u_1, AD]]D + u_2[u_1, [D, A]]D. \tag{4.2.6}$$

This happens because:

$$-u_2[D, A](D(u_1)\cdot) + u_2[D(u_1), AD] + [u_2 D, A(D(u_1)\cdot)] - [u_2 D(u_1), A]D =$$

$$-u_2[D, A](D(u_1)\cdot) + u_2[D(u_1), AD] + u_2 DA(D(u_1)\cdot) - A(D(u_1)u_2 D)$$

$$-u_2 D(u_1)AD + A(u_2 D(u_1)D) = -u_2 DA(D(u_1)\cdot) + u_2 AD(D(u_1)\cdot)$$

$$+u_2 D(u_1)AD - u_2 AD(D(u_1)\cdot) + u_2 DA(D(u_1)\cdot) - u_2 D(u_1)AD = 0.$$

The second part of the initial sum is:

$$[u_2 D, u_1[D, A]] - u_2 D(u_1)[D, A] = u_2 D(u_1)[D, A] + u_2 u_1 D[D, A],$$

$$-u_1[D,A](u_2D) - u_2D(u_1)[D,A] = u_1[u_2D,[D,A]],$$

and:

$$u_1[u_2D,[D,A]] = u_1[u_2[D,A]]D + u_1u_2[D,[D,A]], \tag{4.2.7}$$

after applying the formula (4.2.5) for $[D,A]$ instead of A. Adding (4.2.6) to (4.2.7):

$$A_2(u_0,u_1,u_2) = \tag{4.2.8}$$

$$[u_2,[u_1,AD]]D(u_0) + u_2[u_1,[D,A]]D(u_0) + u_1[u_2[D,A]]D(u_0) + u_1u_2[D,[D,A]](u_0).$$

We suppose that the identity for A_n is true for a number $n > 2$, then for $n+1$ we choose to split up the formula of A_n into $n+1$ different levels. Thus also the formula for A_{n+1} will be split up into $n+1$ levels:

$$A_{n+1}(\bullet, u_1, \dots, u_{n+1}) = \big[u_{n+1}D, [u_n, \dots [u_1, \operatorname{ad}_D^0 A \circ D^{n-1}]..]D\big] \tag{4.2.9}$$

$$-\sum_{s=1}^{n}[u_n, \dots [u_{n+1}D(u_s), \dots [u_1, \operatorname{ad}_D^0 A \circ D^{n-1}]..]D$$

$$+\sum_{\substack{J\subseteq I_n\\|J|=1}}\left[u_{n+1}D, \prod_{j\in J}u_j \prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A \circ D^{n-2})D\right] \tag{4.2.10}$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=1}}\left[\prod_{j\in J}u_{n+1}D(u_j)\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D A\circ D^{n-2})\right]D$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=1}}\left[\sum_{s=1}^{n-1}\prod_{j\in J}u_j\prod_{i\in J^c\setminus\{i_s\}}\operatorname{ad}_{u_i}\operatorname{ad}_{u_{n+1}D(u_{i_s})}(\operatorname{ad}_D A\circ D^{n-2})\right]D$$

....

$$+\sum_{\substack{J\subseteq I_n\\|J|=k}}\left[u_{n+1}D, \prod_{j\in J}u_j \prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A \circ D^{n-k-1})D\right] \tag{4.2.11}$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=k}}\left[\sum_{s=1}^{k}\left(\prod_{j\in J\setminus\{j_s\}}u_j\right)u_{n+1}D(u_{j_s})\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n-k-1})\right]D$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=k}}\left[\sum_{s=1}^{n-k}\prod_{j\in J}u_j\prod_{i\in J^c\setminus\{i_s\}}\operatorname{ad}_{u_i}\operatorname{ad}_{u_{n+1}D(u_{i_s})}(\operatorname{ad}_D^k A\circ D^{n-k-1})\right]D$$

....

$$+\sum_{\substack{J\subseteq I_n\\|J|=n-1}}\left[u_{n+1}D, \prod_{j\in J}u_j \prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A \circ D^0)D\right] \tag{4.2.12}$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=n-1}}\left[\sum_{s=1}^{n-1}\left(\prod_{j\in J\setminus\{j_s\}}u_j\right)u_{n+1}D(u_{j_s})\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^0)\right]D$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=n-1}}\left[\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_{n+1}D(u_i)}(\operatorname{ad}_D^k A\circ D^0)\right]D$$

. . . .

$$[u_{n+1}D,u_n\dots u_2u_1\operatorname{ad}_D^n A]-\sum_{s=1}^{n}u_n\dots(u_{n+1}D(u_s))\dots u_2u_1\operatorname{ad}_D^n A. \quad (4.2.13)$$

Written in this way the expression for A_{n+1} stresses the different levels of A_n. The level zero presented in (4.2.9) is the multiplier part of the formula being the only part which exists in the particular case of a Fourier multiplier A. We start to examine each level studying the outcome and the possible heritage given to the superior level.

The level zero of the previous formula is:

$$\big[u_{n+1}D,[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D\big]$$

$$-\sum_{s=1}^{n}[u_n,\dots[u_{n+1}D(u_s),\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D.$$

Applying the Leibniz type lemma 4.2.5 and the symmetry in $u_n,\dots u_1$, of the second term, one gets:

$$=\big[u_{n+1}D,[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]\big]$$

$$+\sum_{s=1}^{n}\big[u_{n+1}D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)\big]$$

$$-\sum_{s=1}^{n}[u_{n+1}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D.$$

thus:

$$=u_{n+1}D[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]-[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..](u_{n+1}D)$$

$$+\sum_{s=1}^{n}\big[u_{n+1}D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)\big]$$

$$-\sum_{s=1}^{n}[u_{n+1}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D.$$

At this step the commutation lemma 4.2.6 is useful:

$$u_{n+1}[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]D+u_{n+1}\sum_{s=1}^{n}[D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]$$

$$+u_{n+1}[u_n,[u_{n-1},\dots[u_1,\operatorname{ad}_D^1 A\circ D^n]..]-[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..](u_{n+1}D)$$

$$+\sum_{s=1}^{n}\left[u_{n+1}D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)\right]$$

$$-\sum_{s=1}^{n}[u_{n+1}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D.$$

The first, the third and the fourth term give:

$$[u_{n+1},[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]D+u_{n+1}[u_n,[u_{n-1},\dots[u_1,\operatorname{ad}_D^1 A\circ D^n]..]$$

$$+u_{n+1}\sum_{s=1}^{n}[D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]$$

$$+\sum_{s=1}^{n}\left[u_{n+1}D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)\right]$$

$$-\sum_{s=1}^{n}[u_{n+1}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D.$$

To extract a D from the second term we apply the Leibniz type lemma 4.2.5 for it:

$$[u_{n+1},[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]D+u_{n+1}[u_n,[u_{n-1},\dots[u_1,\operatorname{ad}_D^1 A\circ D^{n-1}]..]D$$

$$-u_{n+1}\sum_{s=1}^{n}[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^1 A\circ D^{n-1}]..](D(u_s)\cdot)$$

$$+u_{n+1}\sum_{s=1}^{n}[D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]$$

$$+\sum_{s=1}^{n}\left[u_{n+1}D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)\right]$$

$$-\sum_{s=1}^{n}[u_{n+1}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D.$$

And the result is:

$$[u_{n+1},[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]D+u_{n+1}[u_n,[u_{n-1},\dots[u_1,\operatorname{ad}_D^1 A\circ D^{n-1}]..]D, \tag{4.2.14}$$

which is the entire expected level zero of A_{n+1} and a heritage for the level one of A_{n+1}:

$$u_{n+1}[u_n,[u_{n-1},\dots[u_1,\operatorname{ad}_D^1 A\circ D^{n-1}]..]D.$$

To prove the above result one just has to prove that the remainig terms will vanish. Making use of the Corollary 4.2.7:

$$-u_{n+1}\sum_{s=1}^{n}[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^1 A\circ D^{n-1}]..](D(u_s)\cdot)$$

$$+u_{n+1}\sum_{s=1}^{n}[D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^n]..]$$

$$+\sum_{s=1}^{n}\left[u_{n+1}D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)\right]$$

$$-\sum_{s=1}^{n}[u_{n+1}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D=$$

$$-u_{n+1}\sum_{s=1}^{n}[D,[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)$$

$$+u_{n+1}\sum_{s=1}^{n}\sum_{i\neq s}[u_n,\dots[\widehat{u_s},\dots[D(u_i),\dots[u_1,\operatorname{ad}_D^0\circ D^{n-1}]..](D(u_s)\cdot)$$

$$+u_{n+1}\sum_{s=1}^{n}[D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n}]..]$$

$$+u_{n+1}\sum_{s=1}^{n}D[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..](D(u_s)\cdot)$$

$$-u_{n+1}\sum_{s=1}^{n}D(u_s),[u_n,\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D,$$

where also the last two terms were modified inserting the commutator formula. Immediately:

$$u_{n+1}\sum_{s=1}^{n}[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D(D(u_s)\cdot)$$

$$-u_{n+1}\sum_{s=1}^{n}D(u_s)[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n-1}]..]D$$

$$+u_{n+1}\sum_{s=1}^{n}\sum_{i\neq s}[D(u_s),[u_n,\dots[\widehat{u_i},\dots[u_1,\operatorname{ad}_D^0\circ D^{n-1}]..](D(u_i)\cdot)$$

$$+u_{n+1}\sum_{s=1}^{n}[D(u_s),[u_n,\dots[\widehat{u_s},\dots[u_1,\operatorname{ad}_D^0 A\circ D^{n}]..]=0$$

due to the Leibniz type lemma.

We start now to study the level one of the formula displayed in (4.2.10):

$$\sum_{\substack{J\subseteq I_n\\|J|=1}}\left[u_{n+1}D,\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A\circ D^{n-2})D\right]$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=1}}\left[\prod_{j\in J}u_{n+1}D(u_j)\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D A\circ D^{n-2})\right]D$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=1}}\left[\sum_{s=1}^{n-1}\prod_{j\in J}u_j\prod_{i\in J^c\setminus\{i_s\}}\operatorname{ad}_{u_i}\operatorname{ad}_{u_{n+1}D(u_{i_s})}(\operatorname{ad}_D A\circ D^{n-2})\right]D$$

It is enough to study the effect of the recurrence on one of the C_n^1 terms of the sum:

$$\begin{aligned}&\big[u_{n+1}D, u_n[u_{n-1},\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-2}]..]D\big]\\ &-u_{n+1}D(u_n)[u_{n-1},\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-2}]..]D\\ -\sum_{s=1}^{n-1}&u_n[u_{n-1},\dots[u_{n+1}D(u_s),\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-2}]..]D\end{aligned}$$

chosing the term $u_n[u_{n-1},\dots[u_1, \operatorname{ad}_D A\circ D^{n-2}]..]D$ as the representative term for the n terms in the sum. After the first commutator is restyled, the expression becomes:

$$\begin{aligned}&u_n\,\big[u_{n+1}D,[u_{n-1},\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-2}]..]D\big]\\ -u_n&\sum_{s=1}^{n-1}[u_{n-1},\dots[u_{n+1}D(u_s),\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-2}]..]D\end{aligned}$$

Applying the result already proven at level 0 in the case A_n(available by the principle of mathematical induction) for the operator $\operatorname{ad}_D A$, instead of A, we get:

$$\begin{aligned}&u_n[u_{n+1},[u_{n-1},\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-1}]..]D\\ &+u_nu_{n+1}[u_{n-1},\dots[u_1, \operatorname{ad}_D^2 A\circ D^{n-2}]..]D.\end{aligned}$$

If we extend these computations to every term of whole sum we get:

$$\sum_{\substack{J\subseteq I_n\\|J|=1}}\prod_{j\in J}u_j \operatorname{ad}_{u_{n+1}}\left(\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A\circ D^{n-1})\right)D$$

$$\sum_{\substack{J\subseteq I_n\\|J|=1}}u_{n+1}\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A\circ D^{n-1})D.$$

Again we have additional terms (here are C_n^1 terms) for the next level:

$$\sum_{\substack{J\subseteq I_n\\|J|=1}}u_{n+1}\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A\circ D^{n-1})D.$$

Keeping in mind the heritage given by the level 0, studied previously, we can fill the level 1 of A_{n+1} :

$$\begin{aligned}&\sum_{\substack{J\subseteq I_n\\|J|=1}}\prod_{j\in J}u_j \operatorname{ad}_{u_{n+1}}\left(\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A\circ D^{n-1})\right)D\\ &+u_{n+1}[u_n,[u_{n-1},\dots[u_1, \operatorname{ad}_D^1 A\circ D^{n-1}]..]D=\\ &\sum_{\substack{J\subseteq I_{n+1}\\|J|=1}}\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^1 A\circ D^{n-1})D,\end{aligned}$$

and the total number of terms is $C_n^0+C_n^1=C_{n+1}^1$, as we expected.

Let's proceed at level k:

$$\sum_{\substack{J\subseteq I_n\\|J|=k}}\left[u_{n+1}D,\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n-k-1})D\right]$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=k}}\left[\sum_{s=1}^{k}\left(\prod_{j\in J\setminus\{j_s\}}u_j\right)u_{n+1}D(u_{j_s})\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n-k-1})\right]D$$

$$-\sum_{\substack{J\subseteq I_n\\|J|=k}}\left[\sum_{s=1}^{n-k}\prod_{j\in J}u_j\prod_{i\in J^c\setminus\{i_s\}}\operatorname{ad}_{u_i}\operatorname{ad}_{u_{n+1}D(u_{i_s})}(\operatorname{ad}_D^k A\circ D^{n-k-1})\right]D$$

We start, again, to study the effect of the recurrence only on the term:

$$\Big[u_{n+1}D,u_nu_{n-1}\dots u_{n-k+1}[u_{n-k},\dots[u_1,\operatorname{ad}_D^k A\circ D^{n-k-1}]..]D\Big]$$

$$-\sum_{s=n-k+1}^{n}u_n\dots u_{n+1}D(u_s)\dots u_{n-k+1}[u_{n-k},\dots[u_1,\operatorname{ad}_D^k A\circ D^{n-k-1}]..]D$$

$$-\sum_{s=1}^{n-k}u_nu_{n-1}\dots u_{n-k+1}[u_{n-k},\dots[u_{n+1}D(u_s),\dots[u_1,\operatorname{ad}_D^k A\circ D^{n-k-1}]..]D$$

We restyle the first commutator, in the first part:

$$u_nu_{n-1}\dots u_{n-k+1}\Big[u_{n+1}D,[u_{n-k},\dots[u_1,\operatorname{ad}_D^k A\circ D^{n-k-1}]..]D\Big]$$

$$-u_nu_{n-1}\dots u_{n-k+1}\sum_{s=1}^{n-k}[u_{n-k},\dots[u_{n+1}D(u_s),\dots[u_1,\operatorname{ad}_D^k A\circ D^{n-k-1}]..]D$$

Now everything is prepared to make use of the formula proven at level zero of A_{n-k+1}, for $\operatorname{ad}_D^k A$ instead of A :

$$u_nu_{n-1}\dots u_{n-k+1}[u_{n+1},[u_{n-k},\dots[u_1,\operatorname{ad}_D^k A\circ D^{n-k}]..]D$$

$$+u_{n+1}u_nu_{n-1}\dots u_{n-k+1}[u_{n-k},\dots[u_1,\operatorname{ad}_D^{k+1} A\circ D^{n-k-1}]..]D$$

Using this pattern the entire sum becomes:

$$\sum_{\substack{J\subseteq I_n\\|J|=k}}\prod_{j\in J}u_j\operatorname{ad}_{u_{n+1}}\left(\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n-k})\right)D$$

$$+\sum_{\substack{J\subseteq I_n\\|J|=k}}u_{n+1}\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^{k+1} A\circ D^{n-k-1})D.$$

Thus again C_n^k terms are generated for the level k of A_{n+1} and C_n^{k-1} are given by the previuous level to obtain te necessary amount of terms: C_{n+1}^k. The element u_{n+1} is fixed inside the commutators, for the first C_n^k terms, but the heritage contains C_n^{k-1} terms with it outside the brackets to fill the whole level k of A_{n+1}:

$$\sum_{\substack{J\subseteq I_n\\|J|=k}}\prod_{j\in J}u_j\operatorname{ad}_{u_{n+1}}\left(\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n-k})\right)D$$

$$+\sum_{\substack{J\subseteq I_n\\|J|=k}}u_{n+1}\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n-k})D=$$

$$\sum_{\substack{J\subseteq I_{n+1}\\|J|=k}}\prod_{j\in J}u_j\prod_{i\in J^c}\operatorname{ad}_{u_i}(\operatorname{ad}_D^k A\circ D^{n+1-k-1})D$$

For the level n of the recurrence formula we can apply (4.2.5) to obtain:

$$[u_{n+1}D,u_n\dots u_2u_1\operatorname{ad}_D^n A]-\sum_{s=1}^n u_n\dots(u_{n+1}D(u_s))\dots u_2u_1\operatorname{ad}_D^n A=$$

$$u_n\dots u_2u_1[u_{n+1}D,\operatorname{ad}_D^n A]=u_n\dots u_2u_1[u_{n+1},\operatorname{ad}_D^n A]D$$
$$+u_n\dots u_2u_1u_{n+1}\operatorname{ad}_D^{n+1}A,$$

the additional term being the $(n+1)$-th level of A_{n+1}:

$$u_{n+1}u_n\dots u_2u_1\operatorname{ad}_D^{n+1}A.$$

The proof is now complete by the principle of mathematical induction.

□

4.3 Pseudo-differential operators on the 1-torus

For the clarity of our presentation we will make a short journey into the theory of pseudo-differential operators on the 1-torus $\mathbb{S}^1$. The principal source of information is the book [63] which offers a nice introduction in this topic. All the proofs, for the propositions presented below, and further comments can be found there.

Definition 4.3.1. (Toroidal symbols of operators on $\mathbb{S}^1$) The *toroidal symbol* of a continuous linear operator $P:\mathrm{C}^\infty(\mathbb{S}^1,\mathbb{C})\to\mathrm{C}^\infty(\mathbb{S}^1,\mathbb{C})$ is defined by:

$$\sigma_P(x,m):=\mathcal{F}_{\mathbb{S}^1}(k_P(x))(m),$$

at $x\in\mathbb{S}^1$ and $m\in\mathbb{Z}$, where $k_P(x)(y):=k_P(x,y)$ is the convolution kernel of the operator and is related to the periodic Schwartz distributional kernel K_P by:

$$K_P(x,y):=k_P(x,x-y),$$

i.e. we have, in the sense of distributions:

$$P\varphi(x)=\int_{\mathbb{S}^1}k_P(x,x-y)\varphi(y)dy,\qquad\varphi\in\mathrm{C}^\infty(\mathbb{S}^1,\mathbb{C}).$$

Proposition 4.3.2. *(Quantization of operators on $\mathbb{S}^1$) Let σ_P be the toroidal symbol of a continuous linear operator $P : \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Then:*

$$(Pu)(x) = \sum_{m \in \mathbb{Z}} e^{2\pi i m \cdot x} \sigma_P(x, m) \hat{u}(m). \tag{4.3.1}$$

for every $u \in \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$ and $x \in \mathbb{S}^1$.

Proposition 4.3.3. *(Formula for the toroidal symbol) Let σ_p be the toroidal symbol of a continuous linear operator $P : \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Then for all $x \in \mathbb{S}^1$ and $m \in \mathbb{Z}$ we have:*

$$\sigma_P(x, m) := e^{-2\pi i m \cdot x} (Pe_m)(x) = \overline{e_m(x)} (Pe_m)(x).$$

Definition 4.3.4. (Toroidal symbol class $S^r_{1,0}(\mathbb{S}^1 \times \mathbb{Z})$) Let $r \in \mathbb{R}$, then the *toroidal symbol classs* $S^r_{1,0}(\mathbb{S}^1 \times \mathbb{Z})$ consists of those functions $p(x, m)$ which are smooth in $x \in \mathbb{S}^1$ for all $m \in \mathbb{Z}$, and which satisfy the inequalities:

$$|\Delta^\alpha_m \partial^\beta_x p(x, m)| \leq C_{p,\alpha,\beta,r} \langle m \rangle^{r-\alpha},$$

for every $\alpha, \beta \in \mathbb{N}$ and $x \in \mathbb{S}^1, m \in \mathbb{Z}$, where:

$$\Delta_m p(x, m) := p(x, m+1) - p(x, m), \qquad \Delta^\alpha_m := \Delta_m(\Delta^{\alpha-1}_m),$$

and $\langle m \rangle := (1 + m^2)^{\frac{1}{2}}$.

The class $S^r_{1,0}(\mathbb{S}^1 \times \mathbb{Z})$ will be sometimes denoted by $S^r(\mathbb{S}^1 \times \mathbb{Z})$, and it is enough for our purpose. Let's define further:

$$S^{-\infty}(\mathbb{S}^1 \times \mathbb{Z}) := \bigcap_{r \in \mathbb{R}} S^r(\mathbb{S}^1 \times \mathbb{Z}).$$

Definition 4.3.5. (Toroidal pseudo-differential operators) If $p \in S^r_{1,0}(\mathbb{S}^1 \times \mathbb{Z})$ we denote by $Op(p)$ the corresponding *toroidal pseudo-differential operator* defined by:

$$Op(p)u(x) := \sum_{m \in \mathbb{Z}} e^{2\pi i m x} p(x, m) \widehat{u}(m).$$

The series converges if, for example $u \in \mathrm{C}^\infty(\mathbb{S}^1)$. The set of operators $Op(p)$ with $p \in S^r_{1,0}(\mathbb{S}^1 \times \mathbb{Z})$ will be denoted by $Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$. Of course, if $P \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ and $Q \in Op(S^l(\mathbb{S}^1 \times \mathbb{Z}))$, then $PQ \in Op(S^{r+l}(\mathbb{S}^1 \times \mathbb{Z}))$, for $r, l \in \mathbb{R}$.

We can define a pseudo-differential operator on the torus as a pseudo-differential operator on a 1-dimensional, compact manifold M, i.e. for an order $r \in \mathbb{R}$, $P \in \Psi^r(M)$, but for our goal it is more elegant to work with toroidal symbols and with the class $Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$. Anyway, the two classes are equivalent, as is proven in [63]:

$$Op(S^r(\mathbb{S}^1 \times \mathbb{Z})) = \Psi^r(\mathbb{S}^1).$$

A linear continuous operator $P : \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$ has the property $P(\mathrm{C}^\infty(\mathbb{S}^1)) \subseteq \mathrm{C}^\infty(\mathbb{S}^1)$ iff its toroidal symbol is hermitian:

$$p(x, -m) = \overline{p(x, m)}, \qquad m \in \mathbb{Z},$$

and from now on we consider just operators with hermitian toroidal symbols and pseudo-differential operators corresponding to such a symbol.

Lemma 4.3.6. *Let u_1 be a function in $\mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$ and $p \in S^r(\mathbb{S}^1 \times \mathbb{Z})$, $r \in \mathbb{R}$. Then the commutator $[u_1, P]$ has the symbol:*

$$\sigma_{[u_1,P]}(x,m) = \sum_{m_1 \in \mathbb{Z}} e^{2\pi i x m_1} \hat{u}_1(m_1)[p(x,m) - p(x, m+m_1)].$$

Proof. It is a consequence of the toroidal composition formula which can be found in [63]. The symbol of $M_{u_1} \circ P$ is by definition $u_1(x) \cdot p(x,m)$. We just have to compute the symbol of $P \circ M_{u_1}$:

$$M_{u_1} v(y) = \sum_{m \in \mathbb{Z}} e^{2\pi i m y} u_1(y) \hat{v}(m) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{S}^1} e^{2\pi i m (y-z)} u_1(y) v(z) dz.$$

$$P \circ M_{u_1} v(x) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{S}^1} e^{2\pi i n (x-y)} p(x,n) M_{u_1} v(y) dy$$

$$= \sum_{n \in \mathbb{Z}} \int_{\mathbb{S}^1} \sum_{m \in \mathbb{Z}} \int_{\mathbb{S}^1} e^{2\pi i n (x-y)} e^{2\pi i m (y-z)} p(x,n) u_1(y) v(z) dz dy$$

It can be written in the form:

$$= \sum_{n \in \mathbb{Z}} \int_{\mathbb{S}^1} \sum_{m \in \mathbb{Z}} \int_{\mathbb{S}^1} e^{2\pi i (n-m)(x-y)} e^{2\pi i m (x-z)} p(x,n) u_1(y) v(z) dz dy$$

$$= \sum_{m \in \mathbb{Z}} \int_{\mathbb{S}^1} e^{2\pi i m (x-z)} \left(\sum_{n \in \mathbb{Z}} e^{2\pi i n x} p(x, n+m) \hat{u}_1(n) \right) v(z) dz$$

Thus the symbol of $P \circ M_{u_1}$ is:

$$q(x,m) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} p(x, n+m) \hat{u}_1(n). \tag{4.3.2}$$

□

We focus now on the boundedness from $H^q(\mathbb{S}^1; \mathbb{C}) \times \ldots \times H^q(\mathbb{S}^1; \mathbb{C})$ to $H^{q-r}(\mathbb{S}^1; \mathbb{C})$ of the multi-linear operator:

$$P_n(u_0, u_1, \ldots u_n)(x) := [u_n, [u_{n-1}, \ldots [u_1, P]..] D(u_0)(x),$$

when P is a pseudo-differential operator in the class $Op(S^{r+n-1}(\mathbb{S}^1 \times \mathbb{Z}))$.

Proposition 4.3.7. *Given an operator $P \in Op(S^{r+n-1}(\mathbb{S}^1 \times \mathbb{Z}))$, $r \geq 1$, the following formula holds:*

$$P_n(u_0, u_1, \ldots u_n)(x) =$$

$$\sum_{k \in \mathbb{Z}} e^{2\pi i k \cdot x} \cdot \sum_{m_0 + m_1 + \ldots + m_n = k} \hat{u}_0(m_0) \hat{u}_1(m_1) \ldots \hat{u}_n(m_n) \cdot p_n(x, m_0, m_1, \ldots m_n),$$

where:

$$p_n(x, m_0, m_1, \ldots m_n) = 2\pi i \cdot m_0 \cdot \sum_{k=0}^{n} (-1)^k \sum_{\substack{J \subseteq I_n \\ |J| = k}} p(x, m_0 + \sum_{j \in J} m_j),$$

for p the symbol of P and $u_0, u_1, \ldots u_n \in \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

To prove the above proposition and to study the boundedness of the multilinear operator P_n a few estimates, on p_n and on its Fourier coefficient $\widehat{p_n}$, are necessary:

Lemma 4.3.8. *Let $p \in S^{r+n-1}(\mathbb{S}^1 \times \mathbb{Z})$, $r \geq 1$, then the estimate holds:*

$$|p_n(x, m_0, m_1, \dots m_n)| \leq C_{n,r} \langle m_0 \rangle^r \langle m_1 \rangle^r \dots \langle m_n \rangle^r, \tag{4.3.3}$$

where $C_{n,r}$ depends only on $n \in \mathbb{N}$ and r. Moreover we have the intermediary estimates:

$$|p_s(x, m_0, m_1, \dots m_s)| \leq C_{s,r} \langle m_0 \rangle^{r+(n-s)} \langle m_1 \rangle^{r+(n-s)} \dots \langle m_s \rangle^{r+(n-s)}, \tag{4.3.4}$$

for every $s = \overline{1, n}$. Finally for every $t \in \mathbb{N}$ there is a constant $C_n > 0$ such that:

$$|\widehat{p_n}(m, m_0, m_1 \dots m_n)| \leq C_n \langle m \rangle^{-t} \langle m_0 \rangle^r \langle m_1 \rangle^r \dots \langle m_n \rangle^r, \tag{4.3.5}$$

and C_n is independent on $m, m_0, m_1, \dots m_n \in \mathbb{Z}$.

Proof. The symbol $p \in S^{r+n-1}(\mathbb{S}^1 \times \mathbb{Z})$, $r \geq 1$ is a toroidal symbol for P iff there exists an *Euclidean symbol* $\tilde{p} \in S^{r+n-1}(\mathbb{S}^1 \times \mathbb{R})$ such that $\tilde{p}\mid_{\mathbb{S}^1 \times \mathbb{Z}} = p$ and $\tilde{p}$ is unique modulo $S^{-\infty}(\mathbb{S}^1 \times \mathbb{R})$, according to [63]. Now one can use Lemma A.6 in [15], because $|\partial_\xi^n \tilde{p}(x, \xi)| \leq C_n \langle \xi \rangle^{r-1}$, to obtain the first two estimates.

For the last part of the lemma we use a classical argument, following [63]. The corresponding operator of the symbol $\langle \xi \rangle^2$ is $I - \frac{1}{(2\pi)^2} D_x^2$ and:

$$\left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q (e^{-2\pi i x \cdot m}) = \langle m \rangle^{2q} \cdot e^{-2\pi i x \cdot m},$$

for every $q \in \mathbb{N}$.

$$\begin{aligned}
|\widehat{p_n}(m, m_0, m_1 \dots m_n)| &= \left| \int_{\mathbb{S}^1} e^{-2\pi i x \cdot m} p_n(x, m_0, m_1, \dots m_n) dx \right| \\
&= \langle m \rangle^{-2q} \left| \int_{\mathbb{S}^1} \langle m \rangle^{2q} e^{-2\pi i x \cdot m} p_n(x, m_0, m_1, \dots m_n) dx \right| \\
&= \langle m \rangle^{-2q} \left| \int_{\mathbb{S}^1} \left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q (e^{-2\pi i x \cdot m}) p_n(x, m_0, m_1, \dots m_n) dx \right| \\
&= \langle m \rangle^{-2q} \left| \int_{\mathbb{S}^1} e^{-2\pi i x \cdot m} \left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q p_n(x, m_0, m_1, \dots m_n) dx \right| \\
&\leq \langle m \rangle^{-2q} \int_{\mathbb{S}^1} \left| \left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q p_n(x, m_0, m_1, \dots m_n) \right| dx \\
&\leq C_n \langle m \rangle^{-2q} \langle m_0 \rangle^r \langle m_1 \rangle^r \dots \langle m_n \rangle^r,
\end{aligned}$$

because the estimate on p_n is not affected by a derivative in x. To obtain the estimate for the case $t = 2q + 1, q \in \mathbb{N}$, one has to apply a square root to the product obtained from the estimates for $t = 2q$ and $t = 2q + 2$.

□

Proof. (of Proposition 4.3.7): For the pseudo-differential operator $[u_2,[u_1,P]]$ applying Lemma 4.3.6:

$$\sigma_{[u_2,[u_1,P]]}(x,m) = \sum_{m_2\in\mathbb{Z}} e^{2\pi i m_2\cdot x}\cdot \hat{u}_2(m_2)\cdot\left[\sigma_{[u_1,P]}(x,m)-\sigma_{[u_1,P]}(x,m+m_2)\right]$$

$$= \sum_{m_2\in\mathbb{Z}} e^{2\pi i m_2\cdot x}\cdot \hat{u}_2(m_2)\cdot \sum_{m_1\in\mathbb{Z}} e^{2\pi i m_1\cdot x}\cdot \hat{u}_1(m_1)\cdot \overline{p_2}(x,m,m_1,m_2)$$

for:

$$\overline{p_2}(x,m,m_1,m_2) := \sum_{k=0}^{2}(-1)^k \sum_{\substack{J\subseteq I_2\\ |J|=k}} p(x,m+\sum_{j\in J} m_j).$$

We can use Fubini's theorem in the last series to obtain:

$$\sigma_{[u_2,[u_1,P]]}(x,m) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\cdot x}\cdot \sum_{m_1+m_2=k} \hat{u}_1(m_1)\hat{u}_2(m_2)\cdot \overline{p_2}(x,m,m_1,m_2).$$

The absolute covergence of this series follows by the discrete Hölder's inequality and by similar arguments with those from Corollary A.7 in [15]:

$$\sum_{k\in\mathbb{Z}}\sum_{m_1+m_2=k}\left|e^{2\pi i k\cdot x}\hat{u}_1(m_1)\hat{u}_2(m_2)\cdot\overline{p_2}(x,m,m_1,m_2)\right| \le$$

$$\sum_{k\in\mathbb{Z}}\sum_{m_1+m_2=k}|\hat{u}_1(m_1)|\cdot|\hat{u}_2(m_2)|\cdot|\overline{p_2}(x,m,m_1,m_2)| \le$$

$$\left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{-2q}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2q}\left(\sum_{m_1+m_2=k}|\hat{u}_1(m_1)|\cdot|\hat{u}_2(m_2)|\cdot|\overline{p_2}(x,m,m_1,m_2)|\right)^2\right)^{\frac{1}{2}}$$

$$\lesssim \langle m\rangle^{r-1+(n-2)}\|u_1\|_{H^{q+r+(n-2)}}\|u_2\|_{H^{q+r+(n-2)}},$$

for some $q>\frac{1}{2}$.

The same idea is leading to a formula for the symbol of $[u_n,[u_{n-1},\dots[u_1,P]..]$:

$$\sigma_{[u_n,[u_{n-1},\dots[u_1,P]..]}(x,m) =$$

$$\sum_{k\in\mathbb{Z}} e^{2\pi i k\cdot x}\cdot \sum_{m_1+\ldots+m_n=k}\hat{u}_1(m_1)\dots\hat{u}_n(m_n)\cdot\overline{p_n}(x,m,m_1,\dots m_n),$$

for:

$$\overline{p_n}(x,m,m_1,\dots m_n) := \sum_{k=0}^{n}(-1)^k\sum_{\substack{J\subseteq I_n\\ |J|=k}} p(x,m+\sum_{j\in J} m_j).$$

Because D is a Fourier multiplier with symbol $2\pi i\cdot\xi$ the symbol of $P_n(u_0,u_1,\dots u_n)$ will have exactly the same formula but with $\overline{p_n}$ substituted by p_n. The required formula for $P_n(u_0,u_1,\dots u_n)(x)$ follows using a similar pattern like above. $\square$

For the next proposition we need the discrete Young's inequality, (see [63]):

Lemma 4.3.9. *Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a function which satisfies:*

$$C_1 := \sup_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z}} |h(\eta, \xi)| < \infty,$$

and:

$$C_2 := \sup_{\eta \in \mathbb{Z}} \sum_{\xi \in \mathbb{Z}} |h(\eta, \xi)| < \infty.$$

Then for $1 \leq p \leq \infty$, $f \in \ell^p(\mathbb{Z})$ and:

$$g(\eta) := \sum_{\xi \in \mathbb{Z}} h(\eta, \xi) f(\xi), \quad \eta \in \mathbb{Z},$$

the estimate holds:

$$\|g\|_{\ell^p} \leq C_1^{\frac{1}{p}} \cdot C_2^{\frac{1}{q}} \cdot \|f\|_{\ell^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 4.3.10. *For an operator $P \in Op(S^{r+n-1}(\mathbb{S}^1 \times \mathbb{Z}))$, with $r \geq 1$, we have $P_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1; \mathbb{C}), H^{q-r}(\mathbb{S}^1; \mathbb{C}))$, for any $q > \frac{3}{2}$ and $q - r > \frac{1}{2}$.*

Proof. We can write:

$$P_n(u_0, u_1, \ldots u_n)(x) =$$

$$\sum_{k \in \mathbb{Z}} e_k(x) \sum_{m_0 + m_1 + \ldots + m_n = k} \hat{u}_0(m_0) \hat{u}_1(m_1) \ldots \hat{u}_n(m_n) \cdot p_n(x, m_0, m_1, \ldots m_n) =$$

$$\sum_{k \in \mathbb{Z}} e_k(x) \sum_{m_0 + \ldots + m_n = k} \sum_{m \in \mathbb{Z}} e_m(x) \hat{u}_0(m_0) \ldots \hat{u}_n(m_n) \cdot \widehat{p_n}(m, m_0, \ldots m_n) =$$

$$\sum_{\eta \in \mathbb{Z}} e_\eta(x) \sum_{k \in \mathbb{Z}} \sum_{m_0 + \ldots + m_n = k} \hat{u}_0(m_0) \ldots \hat{u}_n(m_n) \cdot \widehat{p_n}(\eta - k, m_0, \ldots m_n) =$$

Thus:

$$\widehat{P_n(u_0, u_1, \ldots u_n)}(\eta) = \sum_{k \in \mathbb{Z}} \sum_{m_0 + \ldots + m_n = k} \hat{u}_0(m_0) \ldots \hat{u}_n(m_n) \cdot \widehat{p_n}(\eta - k, m_0, \ldots m_n)$$

and:

$$\|P_n(u_0, u_1, \ldots u_n)\|^2_{H^{q-r}(\mathbb{S}^1; \mathbb{C})} :=$$

$$\sum_{\eta \in \mathbb{Z}} \langle \eta \rangle^{2(q-r)} \left| \sum_{k \in \mathbb{Z}} \sum_{m_0 + \ldots + m_n = k} \hat{u}_0(m_0) \ldots \hat{u}_n(m_n) \cdot \widehat{p_n}(\eta - k, m_0, \ldots m_n) \right|^2$$

$$\lesssim \sum_{\eta \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \langle \eta \rangle^{(q-r)} \sum_{m_0 + \ldots + m_n = k} |\hat{u}_0(m_0)| \ldots |\hat{u}_n(m_n)| \cdot |\widehat{p_n}(\eta - k, m_0, \ldots m_n)| \right)^2$$

$$\lesssim \sum_{\eta \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \langle \eta - k \rangle^{(q-r)} \langle k \rangle^{(q-r)} \sum_{m_0 + \ldots + m_n = k} \langle m_0 \rangle^r |\hat{u}_0(m_0)| \ldots \langle m_n \rangle^r |\hat{u}_n(m_n)| \cdot \right.$$

$$\left. \cdot \langle m_0 \rangle^{-r} .. \langle m_n \rangle^{-r} |\widehat{p_n}(\eta - k, m_0, \ldots m_n)| \right)^2$$

using Peetre's inequality and further the Lemma 4.3.8 to obtain:

$$\lesssim \sum_{\eta\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \langle \eta - k\rangle^{(q-r)-t} \langle k\rangle^{(q-r)} \sum_{m_0+\ldots+m_n=k} \langle m_0\rangle^r |\hat{u}_0(m_0)| \ldots \langle m_n\rangle^r |\hat{u}_n(m_n)| \right)^2$$

At this step one can use the discrete Young's inequality for:

$$h(\eta,\xi) = \langle \eta - \xi\rangle^{(q-r)-t},$$

$$f(\xi) = \langle \xi\rangle^{q-r} \sum_{m_0+\ldots+m_n=\xi} \langle m_0\rangle^r |\hat{u}_0(m_0)| \ldots \langle m_n\rangle^r |\hat{u}_n(m_n)|,$$

and $p = q = 2$, to get:

$$\lesssim \sup_{\eta\in\mathbb{Z}} \sum_k \langle \eta - k\rangle^{(q-r)-t} \cdot \sup_{k\in\mathbb{Z}} \sum_\eta \langle \eta - k\rangle^{(q-r)-t} \cdot$$

$$\cdot \sum_{k\in\mathbb{Z}} \langle k\rangle^{2(q-r)} \left(\sum_{m_0+\ldots+m_n=k} \langle m_0\rangle^r |\hat{u}_0(m_0)| \ldots \langle m_n\rangle^r |\hat{u}_n(m_n)| \right)^2$$

$$\lesssim \left(\sum_\eta \langle \eta\rangle^{(q-r)-t} \right)^2 \|u_0\|^2_{H^q(\mathbb{S}^1;\mathbb{C})} \cdot \ldots \cdot \|u_n\|^2_{H^q(\mathbb{S}^1;\mathbb{C})},$$

with similar arguments like in Corollary A.7 in [15]. Choosing $t \in \mathbb{N}$ big enough we get the desired inequality with a constant independent on $u_0, u_1, \ldots u_n$.

□

Definition 4.3.11. A toroidal symbol $p(x,m)$ and the corresponding pseudo-differential operator $P \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ are called *elliptic* of order $r \in \mathbb{R}$, if p satisfies:

$$\forall (x,m) \in \mathbb{S}^1 \times \mathbb{Z} : |m| \geq m_0 \implies |p(x,m)| \geq c_0 \langle m\rangle^r,$$

for some constants $m_0, c_0 > 0$.

Remark 4.3.12. If P is elliptic it can not belong to $Op(S^l(\mathbb{S}^1 \times \mathbb{Z}))$ if $l < r$. Moreover we can assume that $|p(x,m)| \geq c_0 \langle m\rangle^r$ for all $m \in \mathbb{Z}$.

We list below some useful properties of elliptic pseudo-differential operators:

Proposition 4.3.13. *(Properties of elliptic pseudo-differential operators):*

i) If $P \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ then its adjoint P^* belongs to $Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ and P is elliptic iff P^* is elliptic.

ii) An operator $P \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ is elliptic iff there exists and operator $Q \in Op(S^{-r}(\mathbb{S}^1 \times \mathbb{Z}))$ such that $PQ \sim I \sim QP$. (Q is called a parametrix of P)

iii) Let $P \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$ be an elliptic operator, then it is a Fredholm operator $P \in \mathcal{L}(H^s(\mathbb{S}^1), H^{s-r}(\mathbb{S}^1))$, for every $s \in \mathbb{R}$.

Proposition 4.3.14. *Let $A : \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathrm{C}^\infty(\mathbb{S}^1, \mathbb{C})$ be a regular inertia operator which is an invertible elliptic pseudo-differential operator in the class $Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$, $r \geq 1$, with a hermitian symbol. Then for $q - r > \frac{1}{2}$ the geodesic spray:*

$$(\varphi, v) \mapsto S_\varphi(v) = R_\varphi \circ \overline{S} \circ R_{\varphi^{-1}},$$

where:

$$\overline{S}(u) = A^{-1}\{[A, u]D(u) + u[A, D](u) - 2A(u)D(u)\},$$

extends to a smooth mapping on $T\mathcal{D}^q(\mathbb{S}^1) = \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$.

Proof. If an elliptic operator $A : \mathrm{C}^\infty(\mathbb{S}^1; \mathbb{C}) \to \mathrm{C}^\infty(\mathbb{S}^1; \mathbb{C})$ is invertible the inverse equals the parametrix, modulo $Op(S^{-\infty}(\mathbb{S}^1 \times \mathbb{Z}))$, and thus belongs to the class $Op(S^{-r}(\mathbb{S}^1 \times \mathbb{Z}))$ and induces an isomorphism from $H^q(\mathbb{S}^1; \mathbb{C})$ to $H^{q-r}(\mathbb{S}^1; \mathbb{C})$. The proof is similar with the proof of Theorem 3.10 in [15], because:

$$\partial_\varphi A_\varphi(v, v) = [v, A]D(v) - v[D, A](v),$$

and by Proposition 3.0.15 together with (2.2.2) :

$$\overline{S}(u) = A^{-1}(\mathrm{ad}_u^* Au + A(uu_x)) =$$

$$A^{-1}(-2(Au)u_x - (Au)_x u + A(uu_x)) = A^{-1}\{[A, u]u_x + u[A, D](u) - 2A(u)u_x\}.$$

□

Also the right-invariant metric generated by A extends to a Gâteaux smooth mapping on the Hilbert approximations if the twisted operator A_φ does, as is proven in [15].

Proposition 4.3.15. *The Euler-Arnold equation:*

$$u_t = A^{-1}\{u \cdot (Au)_x + 2Au \cdot u_x\},$$

corresponding to a regular inertia operator of pseudo-differential type, satisfying the same conditions as above, has for any initial data $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ a unique non-extendable smooth solution:

$$u \in \mathrm{C}^\infty(J, \mathrm{C}^\infty(\mathbb{S}^1)).$$

The maximal interval of existence J is open and it contains 0.

Proof. It is done using the information given by Proposition 4.3.14 with similar reasonings to those used in [15]. The essential properties used are the smoothness of the twisted operator, on the Hilbert approximations, and the linearity and continuity on $\mathrm{C}^\infty(\mathbb{S}^1)$ of the operator A. The mappings:

$$(\varphi, v) \mapsto R_\varphi(v), \qquad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \to H^q(\mathbb{S}^1)$$

and $\varphi \mapsto \varphi^{-1}$, $\mathcal{D}^q(\mathbb{S}^1) \to \mathcal{D}^q(\mathbb{S}^1)$ are continuous by the definition of a strong ILH Lie group (see [56]), thus by Remark 1.3.6 the mapping:

$$\varphi \mapsto R_{\varphi^{-1}}, \qquad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^q(\mathbb{S}^1))$$

is locally bounded. For the local boundedness of:

$$\varphi \mapsto R_\varphi, \qquad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^{q-r}(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)),$$

one has to use Corollary B.3 in [15]. $\square$

"Equations are just the boring part of mathematics. I attempt to see things in terms of geometry."

Stephen Hawking

5

Well-posedness of the Euler-Arnold equations on semi-direct products $\mathrm{Diff}_+^\infty(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}_+^\infty(\mathbb{S}^1)$

According to the second chapter to study the well-posedness of the Euler-Arnold equations, on semi-direct products of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ with itself, when the inertia operator is of Fourier type, we have to study the similar problem on the direct product $\mathrm{Diff}_+^\infty(\mathbb{S}^1)\times\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ but for an inertia operator of pseudo-differential type. We extended, previously, the results from [15] for the case of an inertia operator of pseudo-differential type and now we want to accomplish our initial task: the well-posedness of the Euler-Poincaré-Arnold equations on semi-direct products of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ with itself. The strategy to follow is the one described in Section 4.1, this time for the strong ILH Lie group $G = \mathrm{Diff}_+^\infty(\mathbb{S}^1)\times\mathrm{Diff}_+^\infty(\mathbb{S}^1)$.

In [15] the authors have posed and solved the following problem:

> **Problem:** Given a Fourier multiplier A, with $a(k) = O(|k|^r)$, under which conditions is the mapping:
>
> $$\varphi \to A_\varphi, \qquad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$
>
> Gâteaux smooth ?

The adequate question seems to be:

> **Problem:** Is the Gâteaux smoothness, on the Hilbert approximations, of the twisted operator A_φ traceable to the inertia operator A or is an algebraic property influenced only by the ILH-Lie group used ?

Since we proved the smoothness for the large class of pseudo-differential operators of Hörmander type $Op(S^r_{1,0}(\mathbb{S}^1\times\mathbb{Z}))$, and there may be possible to extend the results to even more general classes, we think that we are entitled to draw the conclusion: only the algebraic structure is responsible for the smoothness of the twisted operator $\varphi \to A_\varphi$. This conclusion will be argued in the sequel showing how the change of the algebraic structure corrupts the results obtained in the previous chapter. Pursuing this goal we prove, for example:

Proposition 5.0.16. *If for an inertia operator $\mathbb{A}$, of Fourier type, the mapping:*

$$(\psi,\varphi) \to \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}_{(\psi,\varphi)}, \qquad A, B, C \in Op(S^r), r \geq 1,$$

considered relative to the direct product structure, extends to a Gâteaux smooth mapping from $\mathcal{D}^q(\mathbb{S}^1)\times\mathcal{D}^q(\mathbb{S}^1)$ to $\mathcal{L}(H^q(\mathbb{S}^1)\times H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)\times H^{q-r}(\mathbb{S}^1))$, then C has to be a smoothing operator.

We defined a regular inertia operator on the Lie group $\mathrm{Diff}^\infty_+(\mathbb{S}^1)\circledS_\alpha\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ in Definition 2.5.3 of Section 2.4. We work here with the same structure but with operators of pseudo-differential type in the Hörmander class S^r, as they were defined in Section 4.3, instead of Fourier multipliers. Thus, let us consider from this moment:

$$\mathbb{P} := \begin{pmatrix} A & D \\ C & B \end{pmatrix},$$

for pseudo-differential operators $A, B, C, D \in Op(S^r(\mathbb{S}^1\times\mathbb{Z}))$, with hermitian toroidal symbols. If such an operator is a regular inertia operator then is necessary of the form:

$$\mathbb{P} := \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}, \tag{5.0.1}$$

for C^* the $L^2(\mathbb{S}^1)$-adjoint of the operator C. Now let's denote $e_m(x) := e^{2\pi i m x}$, and:

$$\overline{e}_m(x) := \begin{pmatrix} e_m(x) \\ 0 \end{pmatrix}, \qquad \underline{e}_m(x) := \begin{pmatrix} 0 \\ e_m(x) \end{pmatrix}, \qquad m \in \mathbb{Z}.$$

Definition 5.0.17. For a $\mathrm{C}^1(\mathbb{S}^1;\mathbb{C})\times\mathrm{C}^1(\mathbb{S}^1;\mathbb{C})$ function $\begin{pmatrix} u \\ v \end{pmatrix}$ the Fourier series expansion is defined as:

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix}(x) &= \sum_{m\in\mathbb{Z}} \langle (u,v), \overline{e}_m\rangle_{L^2}\cdot\overline{e}_m(x) + \langle (u,v), \underline{e}_m\rangle_{L^2}\cdot\underline{e}_m(x)\\ &= \sum_{m\in\mathbb{Z}} \begin{pmatrix} \hat{u}(m) \\ \hat{v}(m) \end{pmatrix}\cdot e_m(x).\end{aligned}$$

In this way the Fourier coefficient of a function $\overline{u}(x) := \begin{pmatrix} u \\ v \end{pmatrix}(x)$ is defined as:

$$\hat{\overline{u}}(k) := \begin{pmatrix} \hat{u}(k) \\ \hat{v}(k) \end{pmatrix}.$$

The $H^k(\mathbb{S}^1;\mathbb{C})\times H^k(\mathbb{S}^1;\mathbb{C})$ norm induced by the above definition is:

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{H^k(\mathbb{S}^1;\mathbb{C})\times H^k(\mathbb{S}^1;\mathbb{C})} = \left(\|u\|^2_{H^k(\mathbb{S}^1;\mathbb{C})} + \|v\|^2_{H^k(\mathbb{S}^1;\mathbb{C})}\right)^{\frac{1}{2}}.$$

For a regular inertia operator the next identity holds:

$$\mathbb{P}\left(\begin{array}{c} u \\ v \end{array}\right)(x) = \sum_{m\in\mathbb{Z}} e_m(x)\cdot\sigma_{\mathbb{P}}(x,m)\left(\begin{array}{c} \hat{u}(m) \\ \hat{v}(m) \end{array}\right), \tag{5.0.2}$$

where:

$$\sigma_{\mathbb{P}}(x,m) := \left(\begin{array}{cc} a(x,m) & c^*(x,m) \\ c(x,m) & b(x,m) \end{array}\right).$$

We say that the *symbol* $\sigma_{\mathbb{P}}(x,m)$ *is smooth* if it is smooth in $x\in\mathbb{S}^1$ for every $m\in\mathbb{Z}$. An inertia operator is called *hermitian* iff:

$$\sigma_{\mathbb{P}}(x,-m) = \overline{\sigma_{\mathbb{P}}(x,m)}, \qquad m\in\mathbb{Z},$$

and this condition is necessary and sufficient for a linear operator $\mathbb{P}$ to send $\mathrm{C}^\infty(\mathbb{S}^1)\times\mathrm{C}^\infty(\mathbb{S}^1)$ to $\mathrm{C}^\infty(\mathbb{S}^1)\times\mathrm{C}^\infty(\mathbb{S}^1)$. We assume again, if it is not specified, that all the operators have hermitian symbols.

Sometimes it is more effective to write the expansion (5.0.2) as:

$$\mathbb{P}\left(\begin{array}{c} u \\ v \end{array}\right)(x) = \sum_{m\in\mathbb{Z}} e_m(x)\cdot p(x,\overline{m})\cdot\hat{u}(m) + e_m(x)\cdot p(x,\underline{m})\cdot\hat{v}(m),$$

for:

$$p(x,\overline{m}) := \left(\begin{array}{c} a(x,m) \\ c(x,m) \end{array}\right), \qquad p(x,\underline{m}) := \left(\begin{array}{c} c^*(x,m) \\ b(x,m) \end{array}\right).$$

With these notations:

$$\mathbb{P}(\overline{e}_m) = p(x,\overline{m})\cdot e_m, \qquad \mathbb{P}(\underline{e}_m) = p(x,\underline{m})\cdot e_m.$$

Consequently, to manoeuvre multi-linear operators $\mathbb{P}_n$ we have to introduce 2^{n+1} multi-symbols denoted as:

$$p_{n,J}(x,m_0,m_1,\ldots,m_{n-1},m_n), \qquad J\subseteq I_n := \{0,1,\ldots,n\}.$$

These 2^{n+1} multi-symbols will satisfy, for all $m_0,m_1,\ldots m_n\in\mathbb{Z}$, the identity:

$$\mathbb{P}_n(\overline{e}_{m_0},\overline{e}_{m_1},..,\underline{e}_{m_{n-1}},\underline{e}_{m_n}) = p_{n,J}(x,m_0,m_1,\ldots,m_n)\cdot e_{m_0+m_1+\ldots+m_n} \tag{5.0.3}$$

having overline for all the m's with indices situated in J and underline for those $m's$ with indices in J^c. We will try to express the multi-linear operators as:

$$\mathbb{P}_n(\overline{u}_0,\overline{u}_1,\ldots\overline{u}_n)(x) = \tag{5.0.4}$$
$$\sum_{k\in\mathbb{Z}} e_k(x)\sum_{m_0+\ldots+m_n=k}\ \sum_{J\subseteq I_n} p_{n,J}(x,m_0,m_1,\ldots,m_n)\prod_{j\in J}\hat{u}_j(m_j)\prod_{i\in J^c}\hat{v}_i(m_i).$$

5.1 Recurrence relations and multi-symbols

We keep the notation $\overline{u}$ for a 2-component vector $\left(\begin{array}{c} u \\ v \end{array}\right)$ and define the operator:

$$\mathcal{M}_{\overline{u}} := \left(\begin{array}{cc} M_u & 0 \\ 0 & M_v \end{array}\right), \qquad u,v\in\mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C}),$$

where $M_u(\cdot) := u\cdot$ is the multiplication operator. In a similar manner:

$$\mathcal{D} := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

for $D := D_x$, $x \in \mathbb{S}^1$. In this way the next identity holds:

$$\mathcal{D}\mathcal{M}_{\overline{u}} = \mathcal{M}_{\mathcal{D}(\overline{u})} + \mathcal{M}_{\overline{u}}\mathcal{D}. \tag{5.1.1}$$

We define here $R_{(\varphi,\psi)}(u,v) = (u \circ \varphi, v \circ \psi)$. The directional derivative in (φ,ψ) of $R_{(\varphi,\psi)}(u,v)$, in the direction (w_1, w_2), satisfies:

$$\dot{R}_{(\varphi,\psi)}(u,v) = R_{(\varphi,\psi)}\left(\mathcal{M}_{\overline{u}_1}\mathcal{D}(\overline{u})\right), \tag{5.1.2}$$

where $\overline{u}_1 := (u_1, v_1) = R^{-1}_{(\varphi,\psi)}(w_1, w_2)$. Thus, the following identity holds:

$$R^{-1}_{(\varphi,\psi)}\dot{R}_{(\varphi,\psi)}(u,v) = \mathcal{M}_{\overline{u}_1}\mathcal{D}(\overline{u}). \tag{5.1.3}$$

It is also important to observe the identity:

$$\dot{\overline{u}}_i = -R^{-1}_{(\varphi,\psi)}\dot{R}_{(\varphi(s),\psi(s))}(R^{-1}_{(\varphi,\psi)}\overline{w}_i) = -\mathcal{M}_{\overline{u}_{n+1}}\mathcal{D}(\overline{u}_i). \tag{5.1.4}$$

when $\overline{u}_i = R^{-1}_{(\varphi,\psi)}(\overline{w}_i)$ is derivated in the direction $\overline{w}_{n+1} = R_{(\varphi,\psi)}(\overline{u}_{n+1})$.

Proposition 5.1.1. *If $\mathbb{A}$ is a continuous linear operator on $\mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C})\times\mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C})$ and:*

$$\mathbb{A}_{(\varphi,\psi)} = R_{(\varphi,\psi)} \circ \mathbb{A} \circ R^{-1}_{(\varphi,\psi)} \quad ,$$

where $\varphi, \psi \in \mathrm{Diff}^\infty_+(\mathbb{S}^1)$, then the following recurrence formula holds for the $n-th$ directional derivative:

$$\partial^n_{(\varphi,\psi)}\mathbb{A}_{(\varphi,\psi)}(\overline{w}, \overline{w}_1, \ldots \overline{w}_n) = R_{(\varphi,\psi)} \circ \mathbb{A}_n \circ R^{-1}_{(\varphi,\psi)}(\overline{w}, \overline{w}_1, \ldots \overline{w}_n)$$

where $\mathbb{A}_n$ is the $(n+1)$ multi-linear operator defined recursively by $\mathbb{A}_0 = \mathbb{A}$ and:

$$\begin{aligned} \mathbb{A}_{n+1}(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{n+1}) &= \left[\mathcal{M}_{\overline{u}_{n+1}}\mathcal{D}, \mathbb{A}_n(\cdot, \overline{u}_1, \ldots, \overline{u}_n)\right](\overline{u}_0) \\ &- \sum_{k=1}^{n} \mathbb{A}_n(\overline{u}_0, \overline{u}_1, \ldots, \mathcal{M}_{\overline{u}_{n+1}}\mathcal{D}(\overline{u}_k), \ldots, \overline{u}_n) \end{aligned} \tag{5.1.5}$$

Proof. It is straightforward from the proof of Lemma 3.2 in [15]. ∎

All three operators $\mathcal{M}_{\overline{u}}$, $\mathcal{D}$ and $\mathbb{A}$ are linear operators defined on the set $\mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C}) \times \mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C})$. Moreover $\mathcal{M}_{\overline{u}}$ and $\mathcal{D}$ satisfy (5.1.1), also one can consider:

$$\mathrm{ad}_{\overline{u}}\,\mathbb{A} := \mathcal{M}_{\overline{u}}\mathbb{A} - \mathbb{A}\mathcal{M}_{\overline{u}} := [\mathcal{M}_{\overline{u}}, \mathbb{A}].$$

Thus all the formulae from the previous section are true for M_u substituted with $\mathcal{M}_{\overline{u}}$, D substituted with $\mathcal{D}$ and $\mathrm{ad}_u\, A$ with $\mathrm{ad}_{\overline{u}}\,\mathbb{A}$. There is a similar result for the multi-linear operator generated by the above recurrence:

Proposition 5.1.2. *The multi-linear operator given by the recurrence* (5.1.5) *satisfies the identity:*

$$\mathbb{A}_n(\overline{u}_0, \overline{u}_1, .., \overline{u}_n) = \sum_{J \subsetneq I_n} \left[\prod_{j \in J} \mathcal{M}_{\overline{u}_j} \prod_{i \in J^c} \mathrm{ad}_{\overline{u}_i} \left((\mathrm{ad}_{\mathcal{D}}^{|J|} \mathbb{A}) \circ \mathcal{D}^{n-|J|-1} \right) \right] \mathcal{D}(\overline{u}_0)$$

$$+\mathcal{M}_{\overline{u}_1} \circ \mathcal{M}_{\overline{u}_2} \circ \ldots \circ \mathcal{M}_{\overline{u}_n} \mathrm{ad}_{\mathcal{D}}^n \mathbb{A}(\overline{u}_0),$$

for all $\overline{u}_0, \overline{u}_1, .., \overline{u}_n \in \mathrm{C}^\infty(\mathbb{S}^1; \mathbb{C}) \times \mathrm{C}^\infty(\mathbb{S}^1; \mathbb{C})$.

Further, our attention is pointed to the multi-linear operator:

$$\mathbb{P}_n(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n) := [\mathcal{M}_{\overline{u}_n}, [\mathcal{M}_{\overline{u}_{n-1}}, \ldots [\mathcal{M}_{\overline{u}_1}, \mathbb{P}]..]\mathcal{D}(\overline{u}_0). \tag{5.1.6}$$

We have to change the meaning of I_n and from now on it will represent the set $I_n := \{0, 1, \ldots n\}$.

Proposition 5.1.3. *For an operator:*

$$\mathbb{P} = \begin{pmatrix} A & D \\ C & B \end{pmatrix},$$

having A, B, C, D *in the class* $Op(S^{r+n-1})$ *one has the identity:*

$$\mathbb{P}_n(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n)(x) =$$

$$\sum_{k \in \mathbb{Z}} e^{2\pi i k \cdot x} \sum_{m_0 + m_1 + \ldots + m_n = k} \sum_{J \subseteq I_n} p_{n,J}(x, m_0, m_1, \ldots, m_n) \prod_{j \in J} \hat{u}_j(m_j) \prod_{i \in J^c} \hat{v}_i(m_i),$$

where $I_n := \{0, 1, \ldots n\}$ *and:*

$$p_{n,I_n}(x, m_0, m_1, \ldots, m_n) := \begin{pmatrix} a_n(x, m_0, m_1, \ldots, m_n) \\ (-1)^n \cdot 2\pi i m_0 \cdot c(x, m_0 + m_1 + \ldots m_n) \end{pmatrix},$$

$$p_{n,\varnothing}(x, m_0, m_1, \ldots, m_n) := \begin{pmatrix} (-1)^n \cdot 2\pi i m_0 \cdot d(x, m_0 + m_1 + \ldots m_n) \\ b_n(x, m_0, m_1, \ldots, m_n) \end{pmatrix},$$

$$p_{n,J}(x, m_0, m_1, \ldots, m_n) := \begin{pmatrix} 0 \\ (-1)^{|J|-1} \cdot 2\pi i m_0 \cdot c(x, \sum_{j \in J} m_j) \end{pmatrix}, \text{ if } 0 \in J \neq I_n,$$

$$p_{n,J}(x, m_0, m_1, \ldots, m_n) := \begin{pmatrix} (-1)^{|J^c|-1} \cdot 2\pi i m_0 \cdot d(x, \sum_{i \in J^c} m_i) \\ 0 \end{pmatrix}, \text{ if } 0 \in J^c \neq I_n,$$

with $a_n(x, m_0, m_1, \ldots, m_n)$ *and* $b_n(x, m_0, m_1, \ldots, m_n)$ *given in the statement of Proposition 4.3.7.*

Proof. The conclusion follows by Lemma 4.3.6, formula (4.3.2), Proposition 4.3.7 and an induction after n. For example, applying similar arguments like in the proof of Lemma 4.3.6, for $\mathbb{P}_1(\overline{u}_0, \overline{u}_1) = [\mathcal{M}_{\overline{u}_1}, \mathbb{P}]\mathcal{D}(\overline{u}_0)$, one gets:

$$\sum_{k \in \mathbb{Z}} e^{2\pi i k \cdot x} \sum_{m_0 + m_1 = k} \begin{pmatrix} 2\pi i m_0 \cdot [(a(x, m_0) - a(x, m_0 + m_1)] \\ -2\pi i m_0 \cdot c(x, m_0 + m_1) \end{pmatrix} \hat{u}_0(m_0)\hat{u}_1(m_1)$$

$$+\sum_{k\in\mathbb{Z}} e^{2\pi ik\cdot x} \sum_{m_0+m_1=k} \begin{pmatrix} -2\pi im_0 \cdot d(x, m_0+m_1) \\ 2\pi im_0 \cdot [b(x,m_0) - b(x, m_0+m_1)] \end{pmatrix} \hat{v}_0(m_0)\hat{v}_1(m_1)$$

$$+\sum_{k\in\mathbb{Z}} e^{2\pi ik\cdot x} \sum_{m_0+m_1=k} \begin{pmatrix} 2\pi im_0 \cdot d(x, m_0) \\ 0 \end{pmatrix} \hat{u}_1(m_1)\hat{v}_0(m_0)$$

$$+\sum_{k\in\mathbb{Z}} e^{2\pi ik\cdot x} \sum_{m_0+m_1=k} \begin{pmatrix} 0 \\ 2\pi im_0 \cdot c(x, m_0) \end{pmatrix} \hat{u}_0(m_0)\hat{v}_1(m_1)$$

which in terms of our notation for multi-symbols means:

$$p_{1,I_1}(x, m_0, m_1) = \begin{pmatrix} a_1(x, m_0, m_1) \\ -2\pi im_0 \cdot c(x, m_0+m_1) \end{pmatrix}, \quad J = \{0,1\},$$

$$p_{1,\varnothing}(x, m_0, m_1) = \begin{pmatrix} -2\pi im_0 \cdot d(x, m_0+m_1) \\ b_1(x, m_0, m_1) \end{pmatrix}, \quad J^c = \{0,1\},$$

$$p_{1,\{1\}}(x, m_0, m_1) = \begin{pmatrix} 2\pi im_0 \cdot d(x, m_0) \\ 0 \end{pmatrix}, \quad J^c = \{0\},$$

$$p_{1,\{0\}}(x, m_0, m_1) = \begin{pmatrix} 0 \\ 2\pi im_0 \cdot c(x, m_0) \end{pmatrix}, \quad J = \{0\}.$$

To accomplish an induction on n one just has to keep in mind the definition of $\mathcal{M}_{\overline{u}_i}$ and the aforementioned lemma and proposition. □

5.2 Smoothness of the twisted operator and the extended spray

Proposition 5.2.1. *If $\mathbb{P}_n$, $n \geq 0$, is a $(n+1)$-multi-linear operator defined as in (5.0.4) and for every $\beta \in \mathbb{N}$ there is a constant $C_{n,\beta} > 0$, independent of x, or $m_j \in \mathbb{Z}$, such that the multi-symbols have the following polynomial growth:*

$$\left|\mathcal{D}_x^\beta p_{n,J}(x, m_0, m_1, ...m_n)\right| \leq C_{n,\beta} \cdot \langle m_0\rangle^r \langle m_1\rangle^r ... \langle m_n\rangle^r, \tag{5.2.1}$$

for all $m_j \in \mathbb{Z}$, then $\mathbb{P}_n$ extends to a bounded multi-linear operator:

$$\mathbb{P}_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1;\mathbb{C}) \times H^q(\mathbb{S}^1;\mathbb{C}), H^{q-r}(\mathbb{S}^1;\mathbb{C}) \times H^{q-r}(\mathbb{S}^1;\mathbb{C})),$$

for all $q > r + \frac{1}{2}$.

Remark 5.2.2. The condition (5.2.1) is giving in the case $n = 0$ a sufficient condition for a continuous linear operator:

$$\mathbb{A} : \mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C}) \times \mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C}) \to \mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C}) \times \mathrm{C}^\infty(\mathbb{S}^1;\mathbb{C})$$

to extend to a bounded operator from the space $H^q(\mathbb{S}^1;\mathbb{C}) \times H^q(\mathbb{S}^1;\mathbb{C})$ to the space $H^{q-r}(\mathbb{S}^1;\mathbb{C}) \times H^{q-r}(\mathbb{S}^1;\mathbb{C})$. In the case of an inertia operator $\mathbb{A}$ of Fourier type this condition reduces to:

$$\|\sigma_{\mathbb{A}}(x, m)\|_2 \leq C\langle m\rangle^r, \qquad m \in \mathbb{Z},$$

for the Euclidean matrix norm $\|\cdot\|_2$.

To prove the proposition we need the next lemma:

Lemma 5.2.3. *For any complex numbers u_j, v_i, $i, j = \overline{0, n}$ we have the following moduli inequality:*

$$\sum_{J \subseteq I_n} \prod_{j \in J} |u_j| \cdot \prod_{i \in J^c} |v_i| \leq \sqrt{2}^{n+1} |\overline{u}_0| \cdot |\overline{u}_1| \cdot \ldots \cdot |\overline{u}_n| ,$$

for $I_n = \{0, 1, ..., n\}$.

Proof. The left sum contains 2^{n+1} products, because the set $\{0, 1, ..., n\}$ has 2^{n+1} subsets, and each product is of length $n+1$.

$$\left| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right| \cdot \left| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right| \cdot \ldots \cdot \left| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right| = \sqrt{\prod_{i=0}^{n} (u_i^2 + v_i^2)}$$

$$= \sqrt{\sum_{J \subseteq I_n} \prod_{j \in J} |u_j|^2 \cdot \prod_{i \in J^c} |v_i|^2}$$

the inequality becomes:

$$\left(\sum_{J \subseteq I_n} \prod_{j \in J} |u_j| \cdot \prod_{i \in J^c} |v_i| \right)^2 \leq 2^{n+1} \cdot \sum_{J \subseteq I_n} \prod_{j \in J} |u_j|^2 \cdot \prod_{i \in J^c} |v_i|^2$$

and this is just a form of the Cauchy-Schwarz inequality:

$$\left(\sum_{k=1}^{n} x_k \right)^2 \leq n \cdot \sum_{k=1}^{n} x_k^2 \quad .$$

Now the conclusion is obtained with exactly the same arguments like in Lemma 4.3.8 and in Proposition 4.3.10 using the above lemma.

□

Proof of Proposition 5.2.1: The multi-symbols $p_{n,J}(x, m_0, \ldots, m_n)$ have two components and the property (5.2.1) is satisfied componentwise. With exactly the same pattern like in the proof of Lemma 4.3.8 we obtain the existence for every $t \in \mathbb{N}$ of a constant $C_n > 0$ such that:

$$|\widehat{p_{n,J}}(m, m_0, \ldots, m_n)| \leq C_n \langle m \rangle^{-t} \langle m_0 \rangle^r \ldots \langle m_n \rangle^r, \quad m, m_i \in \mathbb{Z}.$$

We run over the proof of Proposition 4.3.10 now:

$$\mathbb{P}_n(\overline{u}_0, \overline{u}_1, \ldots \overline{u}_n)(x) =$$

$$\sum_{k \in \mathbb{Z}} e_k(x) \sum_{m_0 + \ldots + m_n = k} \sum_{J \subseteq I_n} p_{n,J}(x, m_0, m_1, \ldots, m_n) \prod_{j \in J} \hat{u}_j(m_j) \prod_{i \in J^c} \hat{v}_i(m_i) =$$

$$\sum_{k \in \mathbb{Z}} e_k(x) \sum_{m_0 + \ldots + m_n = k} \sum_{J \subseteq I_n} \left(\sum_{m \in \mathbb{Z}} e_m(x) \cdot \widehat{p_{n,J}}(m, m_0, \ldots, m_n) \right) \prod_{j \in J} \hat{u}_j(m_j) \prod_{i \in J^c} \hat{v}_i(m_i)$$

$$= \sum_{\eta \in \mathbb{Z}} e_\eta(x) \sum_{k \in \mathbb{Z}} \sum_{m_0+\ldots+m_n=k} \sum_{J \subseteq I_n} \widehat{p_{n,J}}(\eta - k, m_0, \ldots m_n) \prod_{j \in J} \hat{u}_j(m_j) \prod_{i \in J^c} \hat{v}_i(m_i).$$

Thus:

$$\widehat{\mathbb{P}_n(\overline{u}_0, \overline{u}_1, \ldots \overline{u}_n)}(\eta) = \sum_{k \in \mathbb{Z}} \sum_{m_0+\ldots+m_n=k} \sum_{J \subseteq I_n} \widehat{p_{n,J}}(\eta - k, m_0, \ldots m_n) \prod_{j \in J} \hat{u}_j(m_j) \prod_{i \in J^c} \hat{v}_i(m_i)$$

and:

$$\|\mathbb{P}_n(\overline{u}_0, \overline{u}_1, \ldots \overline{u}_n)\|^2_{H^{q-r}(\mathbb{S}^1)} :=$$

$$\sum_{\eta \in \mathbb{Z}} \langle \eta \rangle^{2(q-r)} \left| \sum_{k \in \mathbb{Z}} \sum_{m_0+\ldots+m_n=k} \sum_{J \subseteq I_n} \widehat{p_{n,J}}(\eta - k, m_0, \ldots m_n) \prod_{j \in J} \hat{u}_j(m_j) \prod_{i \in J^c} \hat{v}_i(m_i) \right|^2 \lesssim$$

$$\sum_{\eta \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \langle \eta \rangle^{(q-r)} \sum_{m_0+\ldots+m_n=k} \sum_{J \subseteq I_n} |\widehat{p_{n,J}}(\eta - k, m_0, \ldots m_n)| \prod_{j \in J} |\hat{u}_j(m_j)| \prod_{i \in J^c} |\hat{v}_i(m_i)| \right)^2$$

From this point everything is identical with the proof of Proposition 4.3.10 after one uses the estimate for the Fourier coefficients of $p_{n,J}$ and Lemma 5.2.3.

Proposition 5.2.4. *For a linear and continuous operator:*

$$\mathbb{A} = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix},$$

with a hermitian symbol, having the properties $A, B \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$, $r \geq 1$, and $C \in Op(S^{-\infty}(\mathbb{S}^1 \times \mathbb{Z}))$, the twisted operator relative to the direct product structure:

$$(\varphi, \psi) \to \mathbb{A}_{(\varphi,\psi)}, \;\; \mathcal{D}^q(\mathbb{S}^1) \times \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)),$$

is Gâteaux smooth when $q > r + \frac{1}{2}$.

Proof. If we plug in the form of $\mathbb{A}$ in the formula given by Proposition 5.1.2 we are addressed to investigate the boundedness of the multi-linear operators $\mathbb{P}_s(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_s)$, $s = \overline{1,n}$ like in (5.1.6) for:

$$\mathbb{P} = \begin{pmatrix} \mathrm{ad}_D^{n-s} A \circ D^{s-1} & \mathrm{ad}_D^{n-s} C^* \circ D^{s-1} \\ \mathrm{ad}_D^{n-s} C \circ D^{s-1} & \mathrm{ad}_D^{n-s} B \circ D^{s-1} \end{pmatrix},$$

because:

$$\mathrm{ad}_{\mathcal{D}} \mathbb{A} = \begin{pmatrix} \mathrm{ad}_D A & \mathrm{ad}_D C^* \\ \mathrm{ad}_D C & \mathrm{ad}_D B \end{pmatrix}.$$

According to Proposition 5.1.3 the multi-symbols are:

$$p_{s,I_s} := \begin{pmatrix} a_s(x, m_0, m_1, \ldots, m_s) \\ (-1)^s (2\pi i)^s m_0 (m_0 + m_1 + \ldots m_s)^{s-1} c(x, m_0 + m_1 + \ldots m_s), \end{pmatrix},$$

$$p_{s,\varnothing} := \begin{pmatrix} (-1)^s (2\pi i)^s m_0 (m_0 + m_1 + \ldots m_s)^{s-1} c^*(x, m_0 + m_1 + \ldots m_s) \\ b_s(x, m_0, m_1, \ldots, m_s) \end{pmatrix},$$

$$p_{s,J} := \begin{pmatrix} 0 \\ (-1)^{|J|-1} (2\pi i)^s m_0 (\sum_{j \in J} m_j)^{s-1} c(x, \sum_{j \in J} m_j) \end{pmatrix}, \quad 0 \in J \neq I_s,$$

$$p_{s,J} := \left(\begin{array}{c} (-1)^{|J^c|-1}(2\pi i)^s m_0(\sum_{i\in J^c} m_i)^{s-1}c^*(x, \sum_{i\in J^c} m_i) \\ 0 \end{array} \right), \ 0 \in J^c \neq I_s,$$

with $c(x,m)$ and $c^*(x,m)$ the symbols of $\mathrm{ad}_D^{n-s} C$, respectively $\mathrm{ad}_D^{n-s} C^*$, and $a_s(x, m_0, m_1, \ldots, m_s)$, $b_s(x, m_0, m_1, \ldots, m_s)$ given in the statement of Proposition 4.3.7, but this time for the symbols:

$$a(x,m) := \sigma_{\mathrm{ad}_D^{n-s} A\circ D^{s-1}}(x,m), \qquad b(x,m) := \sigma_{\mathrm{ad}_D^{n-s} B\circ D^{s-1}}(x,m).$$

The polynomial growth of $a_s(x, m_0, m_1, \ldots, m_s)$, $b_s(x, m_0, m_1, \ldots, m_s)$, generated by $a(x,m) := \sigma_{\mathrm{ad}_D^{n-s} A\circ D^{s-1}}(x,m)$, and $b(x,m) := \sigma_{\mathrm{ad}_D^{n-s} B\circ D^{s-1}}(x,m)$, when $A, B \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z}))$, $r \geq 1$, was discussed in Lemma 4.3.8. The condition (5.2.1) is verified by all the above multi-symbols when A, B, C satisfy the assumptions of the hypothesis.

□

The condition $C \in Op(S^{-\infty}(\mathbb{S}^1 \times \mathbb{Z}))$ is also necessary if $\mathbb{A}$ is of Fourier type, in the sense that an operator:

$$(\varphi, \psi) \mapsto \left(\begin{array}{cc} A & C^* \\ C & B \end{array} \right)_{(\varphi,\psi)},$$

relative to the direct structure is extending to a Gâteaux smooth operator on the aforementioned spaces, then C has to be a smoothing operator. This is the case because writing the boundedness inequality for $\mathbb{A}_n$ and $u_j = e_{m_j}$, with $j \in J$, $0 \in J \neq I_n$ and $u_i = e_{m_i}$ for $i \in J^c$, by (5.0.3) follows:

$$\left| (2\pi i)^n m_0 \left(\sum_{j\in J} m_j \right)^{n-1} \cdot c \left(\sum_{j \in J} m_j \right) \right| \langle m_0 + \ldots m_n \rangle^q \leq C_q \langle m_0 \rangle^{q-r} \ldots \langle m_n \rangle^{q-r}$$

for every $n \in \mathbb{N}$ and $m_0, m_1 \ldots m_n \in \mathbb{Z}$. The operator C will have the property $C \in \mathcal{L}(H^a(\mathbb{S}^1; \mathbb{C}), H^b(\mathbb{S}^1; \mathbb{C}))$, for every $a, b \in \mathbb{R}$. This is equivalent, by Theorem 4.3.1 in [63], with C being a smoothing operator.

By Proposition 3.0.16 and Arnold's theorem 2.1.2 a solution of the Euler-Arnold equation corresponds to an integral curve of the spray equation.

Proposition 5.2.5. *The Euler-Poincaré-Arnold equations* (2.5.6) *on the semi-direct product* $\mathrm{Diff}_+^\infty(\mathbb{S}^1) ⓢ_\alpha \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ *corresponding to an inertia operator:*

$$\mathbb{A} = \left(\begin{array}{cc} A & C^* \\ C & B \end{array} \right),$$

with a hermitian symbol, which satisfies the conditions:

$$A, B, C \in Op(S^r(\mathbb{S}^1 \times \mathbb{Z})), r \geq 1,$$

the operators A *and* $B - CA^{-1}C^*$ *are invertible and elliptic, and:*

$$C = B \circ \mathrm{Ad}_g \ (\mathrm{mod}\ Op(S^{-\infty}(\mathbb{S}^1 \times \mathbb{Z}))),$$

where $g \in \mathrm{Diff}^\infty(\mathbb{S}^1)$ *is defining the action* α, *are locally well-posed in the smooth category.*

Proof. When we switch to the direct product structure using Proposition 2.5.9 the inertia operator becomes:

$$\mathbb{B} = \begin{pmatrix} A - C^* \operatorname{Ad}_g - \operatorname{Ad}_g^* C + \operatorname{Ad}_g^* B \operatorname{Ad}_g & C^* - \operatorname{Ad}_g^* B \\ C - B \operatorname{Ad}_g & B \end{pmatrix},$$

The conditions imposed in Proposition 5.2.4 are fulfilled and the twisted operator $\mathbb{B}_{(\varphi,\psi)}$ extends to a Gâteaux smooth mapping on the Hilbert approximations.

If the operators A and its Schur complement $D := B - CA^{-1}C^*$ in $\mathbb{A}$ are invertible, then $\mathbb{A}$ and $\mathbb{B}$ are invertible and become regular inertia operators. Actually:

$$\mathbb{B}^{-1} =$$

$$\begin{pmatrix} I & 0 \\ \operatorname{Ad}_g & I \end{pmatrix} \begin{pmatrix} I & -A^{-1}C^* \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} I & \operatorname{Ad}_g^* \\ 0 & I \end{pmatrix}$$

Of course $A^{-1}, (B - CA^{-1}C^*)^{-1} \in Op(S^{-r}(\mathbb{S}^1 \times \mathbb{Z}))$, thus the operator:

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & (B - CA^{-1}C^*)^{-1} \end{pmatrix}$$

satisfies the condition imposed by Proposition 5.2.1 and will extend to an operator from $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^{q+r}(\mathbb{S}^1) \times H^{q+r}(\mathbb{S}^1)$. The other operators in the decomposition are extending to operators from $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$. Consequently B extends to a linear isomorphism from $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$, when $q > r + \frac{1}{2}$.

According to (5.1.2) we get $\xi_{\overline{u}}\overline{u} = \mathcal{M}_{\overline{u}}\mathcal{D}(\overline{u})$. Following Proposition 3.0.15 or extending directly the 1-component case we obtain the geodesic spray corresponding to the weak right-invariant Riemannian metric $\langle \cdot, \cdot \rangle^{\mathbb{B}}_{(\varphi,\psi)}$:

$$S_{(\varphi,\psi)}(\overline{v}) = R_{(\varphi,\psi)} \circ \overline{S} \circ R^{-1}_{(\varphi,\psi)}(\overline{v}), \qquad \overline{v} \in T_{(\varphi,\psi)} \left(\operatorname{Diff}^\infty_+(\mathbb{S}^1) \times \operatorname{Diff}^\infty_+(\mathbb{S}^1) \right),$$

$$\overline{S}(\overline{u}) = \mathbb{B}^{-1}\{[\mathbb{B}, \mathcal{M}_{\overline{u}}]\mathcal{D}(\overline{u}) + \mathcal{M}_{\overline{u}}[\mathbb{B}, \mathcal{D}](\overline{u}) - 2\mathcal{M}_{\mathbb{B}(\overline{u})}\mathcal{D}(\overline{u})\},$$

and the arguments are identical with those of Propositions 4.3.14 and 4.3.15. □

“Somebody suggested that I was a prodigy. Another time it was suggested that I should be called “bug brains,” because I had ideas, but they were sort of buggy or not perfectly sound... To some extent, sanity is a form of conformity. And to some extent, people who are insane are non-conformists...”

John Forbes Nash, Jr.

6 A Nash-Moser approach for the Euler-Arnold equations

For a lot of nonlinear equations, mostly coming from hydrodynamics, the well-posedness of the periodic solutions, in the smooth category, can be studied using geodesic flows on infinite dimensional Lie groups. This approach is the main subject of this thesis, it avoids the Nash-Moser techniques and is now called the geometric method in hydrodynamics, following [20]. A natural question occurs:

> **Question:** To what extent can the geometric method substitute a Nash-Moser approach ?

In the present chapter we try to tackle this question and to obtain similar results, regarding the local well-posedness, to those obtained in [15], using a Nash-Moser approach. It seems that the geometric method is a serious competitor for the Nash-Moser alternative since it can be extended beyond the tame category.

As we saw in Section 2.1 if the inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ is invertible, then the Euler-Poincaré equation can be transformed in what we call the *Euler equation* (or Euler-Arnold), because it generalizes the Euler equations in the description of the motion of a rigid body. For the Lie group $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ it has the form:

$$u_t = -A^{-1}\{2(Au) \cdot u_x + (Au)_x \cdot u\}, \quad (t, x) \in \mathbb{R} \times \mathbb{S}^1 \tag{6.0.1}$$

and for $\mathrm{Diff}^{\infty}(M)$ one obtains, according to [51]:

$$u_t = -A^{-1}\{\nabla_u Au + (\mathrm{div}\ u)Au + (\nabla u)^t Au\}, \qquad u \in \mathrm{C}^{\infty}(M). \tag{6.0.2}$$

where $(\nabla u)^t$ is the pointwise adjoint of the operator $v \to \nabla_v u$.

Most of the arguments presented here may also work for the space $C^\infty(M)$, with M a compact, finite dimensional manifold, but for our goal and a more elegant presentation we restrict to the case $M = \mathbb{S}^1$.

We investigate the case of an invertible elliptic pseudo-differential operator $A : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ which is $L^2(\mathbb{S}^1)$-symmetric and positive definite, such that: $A \in OPS^r_{1,0}$ and $A^{-1} \in OPS^{-r}_{1,0}$, with the order $r \geq 1$.

Remark 6.0.6. A Fourier multiplier $A : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$, with positive real symbol $a(k), k \in \mathbb{Z}$, having the properties:

i) A is of order $r \geq 1$, i.e. $a(k) = O(|k|^r)$,

ii) A is invertible , i.e. $a(k) \neq 0, \quad k \in \mathbb{Z}$,

iii) A^{-1} is of order $-r$, i.e. $\frac{1}{a(k)} = O(|k|^{-r})$,

and $a(\xi), \xi \in \mathbb{R}$, satisfying some additional growth conditions, was considered in [15] and if one takes into account the Remark 3.8 of [15], then one can consider the above settings as a natural extension of the case studied there.

Remark 6.0.7. $\Lambda^{2s} = \mathbf{op}\left((1+k^2)^s\right)$ satisfies these relations and corresponds to an inertia operator for the H^s Sobolev metrics, $s \geq \frac{1}{2}$.

6.1 A glimpse into the Nash-Moser theory

Some facts from the Nash-Moser theory for an inverse function theorem in Fréchet spaces are presented in the sequel following closely the way are presented by R. S. Hamilton in [27]. As it was discovered by J. Nash in [54] and then extended by J. Moser in [52], F. Sergeraert in [68], and others, a weaker version of the inverse function theorem can be given in some category of Fréchet spaces. For example the derivative of the mapping must be invertible in a whole neighborhood because the space of invertible linear operators is no longer an open set in $\mathcal{L}(\mathbb{E}, \mathbb{F})$ if $\mathbb{E}$ and $\mathbb{F}$ are Fréchet spaces. During this chapter smoothness will mean Gâteaux smoothness since we are working only with Fréchet spaces.

Definition 6.1.1. A *grading* on a Fréchet space is a collection of seminorms $\{\|\cdot\|_n : n \geq 0\}$ indexed by integers such that:

$$\|u\|_0 \leqslant \|u\|_1 \leqslant \|u\|_2 \leqslant \dots$$

and which define the topology. A *graded Fréchet space* is one with a choice of grading.

Every Fréchet space $\mathbb{E}$ can be seen as a graded Fréchet space taking the seminorms:

$$\|u\|_n = \sum_{i=0}^{n} p_i(u), \qquad u \in \mathbb{E}$$

where $\{p_n\}_n$ is the countable collection of seminorms that defines the Fréchet space topology.

Example. Let $\Sigma(B)$ denote the space of all sequences $\{u_k\}$ of elements in a Banach space B such that:

$$\|\{u_k\}\|_n := \sum_{k=0}^{\infty} e^{nk}\|u_k\|_B < \infty,$$

for all $n \geq 0$. Then the space $\Sigma(B)$ is a graded space with these norms.

A closed subspace of a graded space will be again a graded space with the induced norms and a cartesian product $\mathbb{E} \times \mathbb{F}$ of to graded spaces is a graded space with the norms:

$$\|(u,v)\|_n = \|u\|_n + \|v\|_n.$$

Definition 6.1.2. Two gradings $\{\|\cdot\|_n\}$ and $\{\|\cdot\|_n^{'}\}$ are *tamely equivalent* of degree $r \in \mathbb{N}$ and base b if:

$$\|u\|_n \leqslant C\|u\|_{n+r}^{'}, \qquad \text{and} \qquad \|u\|_n^{'} \leqslant C\|u\|_{n+r},$$

for all $n \geq b$, and a constant C that may depend on n.

If M is a compact manifold then the gradings:

$$\|u\|_n = \|u\|_{C^n(M)} \qquad \text{and} \qquad \|u\|_n^{'} = \|u\|_{H^n(M)}$$

on the Fréchet space $\mathrm{C}^\infty(M)$ are tamely equivalent.

To have a tamely equivalence the natural number r must not depend on n, an example of gradings which are not tamely equivalent is given by $\{\|\cdot\|_n\}$ and $\{\|\cdot\|_n^{'}\}$, where $\|u\|_n^{'} := \|u\|_{2n}$.

Definition 6.1.3. A linear mapping $L : \mathbb{E} \to \mathbb{F}$ between two graded spaces is a *tame linear mapping* if it satisfies a tame estimate for some $r, b \in \mathbb{N}$:

$$\|Lu\|_n \leqslant C\|u\|_{n+r}$$

for all $n \geq b$, and a constant C that may depend on n.

A tame linear mapping is automatically continuous relative to the Fréchet space topologies of $\mathbb{E}$, $\mathbb{F}$.

Remark 6.1.4. A pseudo-differential operator $A : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$, which satisfies the conditions presented at the begining of this section, is a tame linear mapping of degree $[r]+1$ considering the grading induced on $\mathrm{C}^\infty(\mathbb{S}^1)$ by the Hölder norms $|u|_n = \|u\|_{C^n(\mathbb{S}^1)}$, $n \geq 1$, and the operator's behaviour on the Zygmund spaces $C_*^s(\mathbb{S}^1)$, $s \in \mathbb{R}$, (see [71]).

Definition 6.1.5. A linear mapping L is a *tame isomorphism* if L is a linear isomorphism and both L and L^{-1} are linear tame mappings.

Thus two gradings on a space are tamely equivalent if and only if the identity mapping is a tame isomorphism. Moreover the aforementioned pseudo-differential operator A is a tame isomorphism satisfying the estimate:

$$|A^{-1}u|_n \lesssim |u|_{n-[r]+1}, \quad \text{for } \ n \geq [r].$$

Of course a composition of two tame linear mappings is tame linear.

To formulate an inverse function theorem for Fréchet spaces we have to restrict ourselves to the category of tame Fréchet spaces, defined below. M. Poppenberg has actually extended, see [58], [59], the Nash-Moser inverse function theorem beyond the tame category. In principle the results obtained for $H^\infty(R^d)$ in [8], with geometric arguments, may also be obtained with a Poppenberg-Nash-Moser approach. The space $\mathrm{C}^\infty([0,1], H^\infty(R^d))$ is no more a tame space in the sense of R.S. Hamilton, but it satisfies the conditions imposed in [58].

Definition 6.1.6. Let $\mathbb{E}, \mathbb{F}$ be graded spaces. Then $\mathbb{E}$ is a *tame direct summand* of $\mathbb{F}$ if we can find tame linear mappings $L : \mathbb{E} \to \mathbb{F}$ and $M : \mathbb{F} \to \mathbb{E}$ such that the composition $ML : \mathbb{E} \to \mathbb{E}$ is the identity.

Definition 6.1.7. A graded space is *tame* if it is a tame direct summand of the space $\Sigma(B)$ of exponentially decreasing sequences in some Banach space B.

It is important to mention that a tame direct summand of a tame space is tame, this fact being a direct consequence of the definition.

Proposition 6.1.8. *If M is a compact manifold then $C^\infty(M)$ is tame.*

Proof. A proof can be found in [27].

□

Definition 6.1.9. Let $\mathbb{E}$ and $\mathbb{F}$ graded spaces, $P : U \subseteq \mathbb{E} \to \mathbb{F}$ a nonlinear mapping of an open subset U in $\mathbb{E}$ into $\mathbb{F}$ is a *tame mapping* if it is continuous and satisfies a tame estimate of degree r and base b :

$$\|Pu\|_n \leqslant C(1 + \|u\|_{n+r})$$

for all $n \geq b$ and all u in some neighborhood of each point of U. The degree r, base b and constants C can vary from neighborhood to neighborhood and C can depend also on n.

Definition 6.1.10. A mapping $P : U \subseteq \mathbb{E} \to \mathbb{F}$ is a *smooth tame mapping* if P is smooth and all its derivatives $D^k P$ are tame.

Remark 6.1.11. A mapping is a linear tame mapping if and only if is linear and tame. Moreover a linear tame mapping is a smooth tame mapping.

Theorem 6.1.12. *(Nash-Moser Implicit Function Theorem) Let $\mathbb{E}, \mathbb{F}$ and $\mathbb{G}$ be tame spaces, $U \subset \mathbb{E}$ and $V \subset \mathbb{F}$ open subsets and $P : U \times V \to \mathbb{G}$ a smooth tame mapping. For every $(u, v) \in U \times V$ the partial derivative:*

$$D_2 P(u, v) : \mathbb{F} \to \mathbb{G},$$

is invertible with a tame family of inverses $VP : U \times V \times \mathbb{G} \to \mathbb{F}$. If for some $(u_0, v_0) \in U \times V$ the equality $P(u_0, v_0) = 0$ holds, then there are neighborhoods $U_0 \subseteq U$ of u_0 and $V_0 \subseteq V$ of v_0 and a smooth tame mapping $\Psi : U_0 \to V_0$, such that:

$$P^{-1}(0) \cap (U_0 \times V_0) = \{(u, \Psi(u)) : u \in U_0\}.$$

Proof. This version is a mixture of Theorem II.3.1.1 and Theorem III.1.1.1 of [27] applied to the mapping:

$$\Phi : U \times V \to \mathbb{E} \times \mathbb{G}, \qquad (u,v) \xrightarrow{\Phi} (u, P(u,v)).$$

$\square$

6.2 Local well-posedness for solutions of an Euler-Arnold equation

The arguments presented in this section are following the same line as in [61] , [62], which are the principal source of inspiration. Our goal will be to prove the local well-posedness for the smooth periodic solutions of the following class of Euler-Arnold equations:

$$\begin{cases} u_s = -A^{-1}\{2(Au)\cdot u_x + (Au)_x \cdot u\}, & (s,x) \in \mathbb{R} \times \mathbb{S}^1 \\ u(0,x) = u_0 \in \mathrm{C}^\infty(\mathbb{S}^1), \end{cases} \tag{6.2.1}$$

thus for such a solution s can be considered in the interval $[-\varepsilon, \varepsilon]$.

After the transformation:

$$\begin{cases} s = \varepsilon \cdot t, & -1 \leqslant t \leqslant 1, \\ u(\varepsilon \cdot t, x) = u_0(x) + u_\varepsilon(t,x), \end{cases}$$

the above equation becomes:

$$\begin{cases} u_{\varepsilon,t} = -\varepsilon \cdot A^{-1}\{2A(u_0+u_\varepsilon)\cdot(u_0+u_\varepsilon)_x + (A(u_0+u_\varepsilon))_x \cdot (u_0+u_\varepsilon)\}, \\ u_\varepsilon(0,x) = 0, \end{cases}$$

and now $(t,x) \in [-1,1] \times \mathbb{S}^1$.

Notation: The facts presented in Section 6.1 are true up to a tamely equivalence of the chosen gradings and this observation is motivating us to use the symbol $\lesssim$ even if the constant can depend on n in an estimate.

Lemma 6.2.1. *The following exponential law holds:*

$$\mathrm{C}^\infty([-1,1]; \mathrm{C}^\infty(\mathbb{S}^1)) \cong \mathrm{C}^\infty([-1,1] \times \mathbb{S}^1),$$

and the grading:

$$\|u\|_n := \sup_{i=\overline{0,n}} \sup_{t\in[-1,1]} \|u^{(i)}(t,x)\|_{\mathrm{C}^{n-i}(\mathbb{S}^1)},$$

where the i-th derivative is taken with respect to t, endows the first space with a Fréchet space structure.

Proof. The first part follows from a special case of the cartesian closedness which holds in the case of Gâteaux smoothness:

$$\mathrm{C}^\infty(U \times V, \mathbb{F}) = \mathrm{C}^\infty(U, \mathrm{C}^\infty(V, \mathbb{F}),$$

when V is locally compact due to [66] or finite dimensional due to [25]. For the second part one can see that the above-mentioned grading is nothing else then than a rewriting of the natural grading $\|u\|_n := \|u\|_{C^n([-1,1]\times\mathbb{S}^1)}$ corresponding to $\mathrm{C}^\infty([-1,1]\times\mathbb{S}^1)$. $\square$

We prefer to consider the space $\mathrm{C}^\infty([-1,1],\mathrm{C}^\infty(\mathbb{S}^1))$ because we use a grading resulted from the interpretation of $\mathrm{C}^\infty([-1,1]\times\mathbb{S}^1)$ as a space of smooth curves on $\mathrm{C}^\infty(\mathbb{S}^1)$, which fits better with our future estimates. To use the Nash-Moser Implicit Function Theorem we have to define the Fréchet spaces:

$$\mathbb{E} := \mathrm{C}^\infty(\mathbb{S}^1),$$

with the grading $|u|_n := \|u\|_{C^n(\mathbb{S}^1)}$,

$$\mathbb{F}_0 := \{(v(t,x),\varepsilon) \in \mathrm{C}^\infty([-1,1],\mathrm{C}^\infty(\mathbb{S}^1))\times\mathbb{R} : \quad v(0,x)=0\},$$

with $\|(v,\varepsilon)\|_n := \|v\|_n + |\varepsilon|$, where $\|\cdot\|_n$ is the above-mentioned grading from Lemma 6.2.1, and:

$$\mathbb{G} := \mathrm{C}^\infty([-1,1],\mathrm{C}^\infty(\mathbb{S}^1))\times\mathbb{R},$$

with the same grading.

Important tools in our future estimates are the following inequalities:

Proposition 6.2.2. *If $d \leqslant i \leqslant n$ and $i+j=n+d$ we have the following interpolation inequality:*

$$\|u\|_i\,\|v\|_j \lesssim \|u\|_d\,\|v\|_n + \|v\|_d\,\|u\|_n\,, \tag{6.2.2}$$

and a tame estimate of the product holds:

$$\|uv\|_n \lesssim \|u\|_0\,\|v\|_n + \|v\|_0\,\|u\|_n\,, \tag{6.2.3}$$

or more generally:

$$\|u_1\dots u_k\|_n \lesssim \sum_{i=1}^{k}\|u_1\|_0\dots\widehat{\|u_i\|_0}\dots\|u_k\|_0\cdot\|u_i\|_n\,, \tag{6.2.4}$$

for all $u,v\in\mathrm{C}^\infty([0,1],\mathrm{C}^\infty(\mathbb{S}^1))$.

Proof. The first inequality is a direct consequence of the tameness of the Fréchet space used, see Corollary III.1.4.2 in [27]. The last one comes from the second one. For the tame estimate of a product:

$$\|D_t^i(uv)\|_{\mathrm{C}^{n-i}(\mathbb{S}^1)} \lesssim \sum_{a+b=i}\|D_t^a u D_t^b v\|_{\mathrm{C}^{n-i}(\mathbb{S}^1)} \lesssim$$

$$\sum_{a+b=i}\ \sum_{p+q\leq n-i}\ \sup_{x\in\mathbb{S}^1}|D_t^a D_x^p u\cdot D_t^b D_x^q v| \lesssim \sum_{a+b=i}\ \sum_{p+q\leq n-i}\|u\|_{a+p}\|v\|_{b+q} \lesssim$$

$$\sum_{a+b=i}\ \sum_{p+q\leq n-i}(\|u\|_0\|v\|_{a+b+p+q}+\|u\|_{a+b+p+q}\|v\|_0) \lesssim \|u\|_n\|v\|_0+\|u\|_0\|v\|_n,$$

and the conclusion follows one step further. $\square$

Remark 6.2.3. The preceding estimates are also true for the gradings $\{\|\cdot\|_{C^n(\mathbb{S}^1)}\}$ and $\{\|\cdot\|_{H^n(\mathbb{S}^1)}\}$ on $\mathrm{C}^\infty(\mathbb{S}^1)$.

Now it is possible to define an operator $P : \mathbb{E} \times \mathbb{F}_0 \to \mathbb{G}$ by:

$$P(u_0, (v, \varepsilon)) = (v_t + \varepsilon \cdot B(u_0 + v, u_0 + v), \varepsilon),$$

where $B(u_0+v, u_0+v) = A^{-1}\{2A(u_0+v)\cdot(u_0+v)_x + (A(u_0+v))_x \cdot (u_0+v)\}$. Because $P(0,(0,0)) = 0$ the obvious idea is to apply the Nash-Moser Implicit Function Theorem to obtain the following result:

Proposition 6.2.4. *There exist an interval $J = [-T, T]$ and the real numbers $\delta > 0$, and $n_0 = n_0(r)$, such that for each $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ satisfying $\|u_0\|_{\mathrm{C}^{n_0}(\mathbb{S}^1)} < \delta$ there exists a unique solution $u \in \mathrm{C}^\infty(J, \mathrm{C}^\infty(\mathbb{S}^1))$ of (6.2.1) with the initial data $u(0) = u_0$. Moreover, the solution u depends smoothly on (t, u_0) from $J \times \mathrm{C}^\infty(\mathbb{S}^1)$.*

In the same time $P(\tilde{u}_0, (0, 0)) = 0$ holds for every $\tilde{u}_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ and the same result is true for each u_0 in a C^{n_0}- neighborhood, of an arbitrary $\tilde{u}_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$, not only in a C^{n_0}- neighborhood of 0 as above.

In the sequel we start proving the above proposition in a couple of steps:

Proposition 6.2.5. *The mapping P is a smooth tame mapping.*

Proof. The operator P is expressed as:

$$P(u_0, (v, \varepsilon)) = (v_t + \varepsilon \cdot B(v + u_0, v + u_0), \varepsilon),$$

where $B : \mathrm{C}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$, defined by:

$$B(u, v) := A^{-1}\{2Au \cdot v_x + (Au)_x \cdot v\},$$

is bilinear and separately continuous, thus continuous. It has to be proven that P is a smooth mapping and satisfies a tame estimate together with all its derivatives $D^k P$. The operator B is smooth by construction, compare with Remark 2.5.8, and so are $h_1(\varepsilon, u) = \varepsilon \cdot u$ and $h_2(u, v) = u + v$. We obtain the smoothness of P as a composition of smooth mappings. We prove now the tame estimates of the mapping P and of its derivatives.

First, we have to show the estimate:

$$\|P(u_0, (v, \varepsilon))\|_n \lesssim 1 + |u_0|_{n+r} + \|(v, \varepsilon)\|_{n+r}\,, \tag{6.2.5}$$

for some $r \in \mathbb{N}$ and for all $n \geqslant b$ and $(u_0, (v, \varepsilon))$ in a neighborhood of $(0, (0, 0))$ in $\mathbb{E} \times \mathbb{F}_0$. A pseudo-differential operator $A \in OPS^r_{1,0}$, $r \geq 1$ is commuting with the strong derivatives D_t and thus for all $v \in \mathrm{C}^\infty([-1, 1], \mathrm{C}^\infty(\mathbb{S}^1))$ the estimate:

$$\|Av\|_{C^{n+\delta}_*(\mathbb{S}^1)} \lesssim \|v\|_{C^{n+\delta+r}_*(\mathbb{S}^1)}\,, \qquad n \geqslant 0, \delta \in (0, 1),$$

will imply:

$$\|Av\|_n \lesssim \|v\|_{n+[r]+1}\,,$$

where we have denoted with [r] the integer part of the order $r \in \mathbb{R}$.

Thus using the definition of the norm $\|\cdot\|_n$, the above inequality and (6.2.3) we get:

$$\|P(u_0,(v,\varepsilon))\|_n \lesssim \|v\|_{n+1} + |\varepsilon| \, \|w\|_{[r]+1} \, \|w\|_{n+1} + |\varepsilon| \, \|w\|_1 \, \|w\|_{n+[r]+1}$$

$$\lesssim 1 + |u_0|_{n+[r]+1} + \|v\|_{n+[r]+1} + |\varepsilon|,$$

in a neighborhood of the origin $|u_0|_{[r]+1} < c$ and $\|v\|_{[r]+1} < c, |\varepsilon| < c$. Finally for higher derivatives of P the same arguments are used to obtain a tame estimate.

□

Proposition 6.2.6. *The linearized equation:*

$$D_2 P(u_0,(v,\varepsilon))(h,\omega) = (k,\tau),$$

has a unique solution $(h,\tau) \in \mathbb{F}_0$, for every $(u_0,(v,\varepsilon))$ in a neighborhood of the origin in $\mathbb{E} \times \mathbb{F}_0$ and arbitrary $(k,\tau) \in \mathbb{G}$.

Simple computations will lead to:

$$DP(u_0,(v,\varepsilon))(h,\omega) = (h_t + \omega B(v+u_0, v+u_0) + \varepsilon B(h, v+u_0) + \varepsilon B(v+u_0,h), \omega)$$

and one can be observe that $\omega = \tau$ and $h(0,x) = 0$, being from $\mathbb{F}_0$.

To simplify the equation let's introduce the notation $w := v + u_0$. We apply the modified Galerkin's method, see Chapter 16 in [70] for details, to prove the existence of a solution $h \in \mathrm{C}^\infty([-1,1], \mathrm{C}^\infty(\mathbb{S}^1))$ for the linearized equation:

$$h_t = -\tau A^{-1}(2Aw \cdot w_x + (Aw)_x \cdot w) \tag{6.2.6}$$
$$-\varepsilon A^{-1}\{2Ah \cdot w_x + 2Aw \cdot h_x + (Ah)_x \cdot w + (Aw)_x \cdot h\} + k,$$

when $(u_0,(v,\varepsilon))$ is in a neighborhood of the origin in $\mathbb{E} \times \mathbb{F}_0$ and a fixed $(k,\tau) \in \mathrm{C}^\infty([-1,1], \mathrm{C}^\infty(\mathbb{S}^1)) \times \mathbb{R}$. This equation will be written as:

$$m_t = -\tau(2Aw \cdot w_x + (Aw)_x \cdot w) \tag{6.2.7}$$
$$-\varepsilon\{2m \cdot w_x + 2Aw \cdot D_x A^{-1} m + m_x \cdot w + (Aw)_x \cdot A^{-1} m\} + Ak,$$
$$m = Ah.$$

More precisely it will be proven that the solution m_δ of the approximating equation:

$$(m_\delta)_t = -\tau J_\delta(2Aw \cdot w_x + (Aw)_x \cdot w) \tag{6.2.8}$$
$$-\varepsilon J_\delta\{2(J_\delta m_\delta) \cdot w_x + 2Aw \cdot D_x A^{-1}(J_\delta m_\delta) + (J_\delta m_\delta)_x \cdot w + (Aw)_x \cdot A^{-1}(J_\delta m_\delta)\} + J_\delta Ak,$$
$$m_\delta(0) = 0$$

exists for an interval which does not depend on δ and has a limit for $\delta \to 0$ which solves the linearized equation.

Here J_δ represents a Friedrichs mollifier obtained from a nonnegative, even, smooth bump function ρ of total weight 1 and compactly supported:

$$J_\delta u := \rho_\delta * u, \qquad \rho_\delta = \frac{1}{\delta}\rho\left(\frac{x}{\delta}\right).$$

Then J_δ will be a Fourier multiplier on $C^\infty(\mathbb{S}^1)$, symmetric relative to the $L^2(\mathbb{S}^1)$ inner product and a bounded linear mapping from $L^2(\mathbb{S}^1)$ to $H^k(\mathbb{S}^1)$ for all $k \geq 0$, but the estimate depends on δ. Moreover the estimate:

$$\|J_\delta u\|_{H^q} \lesssim \|u\|_{H^q}, \qquad u \in H^q(\mathbb{S}^1)$$

is uniform in δ for $q \geq 0$.

We will also make use of the well-known Kato-Ponce commutator estimate:

$$\|\Lambda^s(uv) - u\Lambda^s(v)\|_{L^2} \lesssim \|u_x\|_{L^\infty}\|\Lambda^{s-1}v\|_{L^2} + \|\Lambda^s u\|_{L^2}\|v\|_{L^\infty}, \tag{6.2.9}$$

for $s > 0$ and $u, v \in H^s(\mathbb{S}^1)$.

Lemma 6.2.7. *Any solution m_δ of the approximating equation (6.2.8) satisfies the a priori estimate:*

$$\|m_\delta(t)\|_{H^q(\mathbb{S}^1)} \leqslant C_q, \qquad q > \frac{1}{2},$$

for every $t \in [-1, 1]$ and C_q independent of δ, when $(u_0, (v, \varepsilon))$ lies in a small enough neighborhood of the origin in $\mathbb{E} \times \mathbb{F}_0$.

Proof. After the Fourier multiplier Λ^q is applied to the equation we multiply it by $\Lambda^q m_\delta$:

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^q m_\delta\|^2_{L^2(\mathbb{S}^1)} = -\tau \int_{\mathbb{S}^1} \Lambda^q J_\delta\{2Aw \cdot w_x + (Aw)_x \cdot w\}\Lambda^q m_\delta dx$$

$$-\varepsilon \int_{\mathbb{S}^1} \Lambda^q J_\delta\{2J_\delta m_\delta \cdot w_x + 2Aw \cdot D_x A^{-1}(J_\delta m_\delta) + (J_\delta m_\delta)_x \cdot w + (Aw)_x \cdot A^{-1} J_\delta m_\delta\}\Lambda^q m_\delta dx$$

$$+ \int_{\mathbb{S}^1} \Lambda^q J_\delta Ak \cdot \Lambda^q m_\delta dx = -\tau I_1 - \varepsilon I_2 + I_3.$$

For the first integral using the $L^2(\mathbb{S}^1)$-symmetry of J_δ, which is sent to m_δ, and the estimate $ab \lesssim a^2 + b^2$ one obtains:

$$|I_1| \lesssim \|Aw \cdot w_x\|^2_{H^q} + \|(Aw)_x \cdot w\|^2_{H^q} + \|J_\delta m_\delta\|^2_{H^q}.$$

For the second integral using again the same $L^2(\mathbb{S}^1)$-symmetry, a couple of times the Kato-Ponce estimate and the fact that the order of the pseudo-differential operator $D_x A^{-1}$ is at most 0 one gets:

$$|I_2| \lesssim \left(1 + \|w\|^2_{H^{q+[r]+2}}\right)\|J_\delta m_\delta\|^2_{H^q}.$$

As an example, for the term:

$$\int_{\mathbb{S}^1} \Lambda^q J_\delta\{w \cdot (J_\delta m_\delta)_x\} \cdot \Lambda^q m_\delta dx =$$

$$\int_{\mathbb{S}^1} \{\Lambda^q(w \cdot (J_\delta m_\delta)_x) - w \cdot \Lambda^q(J_\delta m_\delta)_x\}\Lambda^q(J_\delta m_\delta)dx + \int_{\mathbb{S}^1} w \cdot \Lambda^q(J_\delta m_\delta)_x \cdot \Lambda^q(J_\delta m_\delta)dx$$

$$\lesssim \|w_x\|_{L^\infty}\|\Lambda^{q-1}(J_\delta m_\delta)_x\|_{L^2}\|\Lambda^q(J_\delta m_\delta)\|_{L^2}$$

$$+\|\Lambda^q w_x\|_{L^2}\|(J_\delta m_\delta)_x\|_{L^\infty}\|\Lambda^q(J_\delta m_\delta)\|_{L^2} + \|w_x\|_{C^0}\|\Lambda^q(J_\delta m_\delta)\|^2_{L^2},$$

where to obtain the last term we had to integrate by parts.

The last integral is estimated by:

$$|I_3| \lesssim \|Ak\|_{H^q}^2 + \|J_\delta m_\delta\|_{H^q}^2.$$

Now for $(u_0, (v, \varepsilon))$ in a neighborhood:

$$U \times V = \{(u_0, (v, \varepsilon)) \in \mathbb{E} \times \mathbb{F}_0 : |u_0|_{q+[r]+2} < c, \|v\|_{q+[r]+2} < c, |\varepsilon| < c\},$$

of the origin, adding the above inequalities, one obtains:

$$\left|\frac{d}{dt}\|m_\delta\|_{H^q}^2\right| \lesssim \left(1 + \|w\|_{H^{q+[r]+2}}^2\right) \|m_\delta\|_{H^q}^2 + \|w\|_{H^{q+[r]+2}}^2 + \|k\|_{H^{q+[r]+1}}^2,$$

for $t \in [-1, 1]$. We will make use of the following version of Gronwall's lemma:

Lemma: Let $J \in \mathbb{R}$ be an open interval which contains 0 and $a(t), b(t), \varphi(t)$ continuous and positive functions on J satisfying the inequality:

$$\varphi(t) \leqslant a(t) + \left|\int_0^t b(s)\varphi(s)ds\right|, \qquad \text{for all } t \in J.$$

Then:

$$\varphi(t) \leqslant a(t) + \left|\int_0^t a(s)b(s)e^{\left|\int_0^s b(\tau)d\tau\right|}ds\right|, \quad \text{for all } t \in J.$$

Further, if $t \in [0, 1]$ we keep:

$$\frac{d}{dt}\|m_\delta\|_{H^q}^2 \lesssim \left(1 + \|w\|_{H^{q+[r]+2}}^2\right) \|m_\delta\|_{H^q}^2 + \|w\|_{H^{q+[r]+2}}^2 + \|k\|_{H^{q+[r]+1}}^2,$$

integrate on $[0, t]$ and apply Gronwall's inequality:

$$\|m_\delta(t)\|_{H^q}^2 \lesssim e^{1+\int_0^1 \|w(s)\|_{H^{q+[r]+2}}^2 ds} \left(\|m_\delta(0)\|_{H^q}^2 + \int_0^1 \|w(s)\|_{H^{q+[r]+2}}^2 + \|k(s)\|_{H^{q+[r]+1}}^2 ds\right),$$

inserting $\|\cdot\|_{H^n(\mathbb{S}^1)} \lesssim \|\cdot\|_{C^n(\mathbb{S}^1)}$:

$$\|m_\delta\|_{H^q}^2 \lesssim e^{1+\|w\|_{q+[r]+2}^2}(\|m_\delta(0)\|_{H^q}^2 + \|w\|_{q+[r]+2}^2 + \|k\|_{q+[r]+1}^2).$$

If $t \in [-1, 0]$ we use the inequality:

$$-\frac{d}{dt}\|m_\delta\|_{H^q}^2 \lesssim \left(1 + \|w\|_{H^{q+[r]+2}}^2\right) \|m_\delta\|_{H^q}^2 + \|w\|_{H^{q+[r]+2}}^2 + \|k\|_{H^{q+[r]+1}}^2,$$

to obtain in the neighborhood defined above:

$$-\frac{d}{dt}\|m_\delta\|_{H^q}^2 \lesssim \|m_\delta\|_{H^q}^2 + 1,$$

and to integrate afterwards on $[t, 0]$:

$$\|m_\delta(t)\|_{H^q}^2 \lesssim \|m_\delta(0)\|_{H^q}^2 + \int_t^0 \|m_\delta(s)\|_{H^q}^2 ds + 1.$$

With Gronwall's inequality we obtain:

$$\|m_\delta(t)\|_{H^q} \lesssim \|m_\delta(0)\|^2_{H^q} + 1 + \int_t^0 \left(\|m_\delta(0)\|^2_{H^q} + 1\right) e^{-s} ds.$$

The conclusion follows, for a fixed $(k,\tau) \in \mathbb{G}$:

$$\|m_\delta(t)\|_{H^q} \leqslant C_q,$$

for C_q independent of δ, for all $t \in [-1,1]$. □

Proof of Proposition 6.2.6:
Consider now the aforementioned ODE's:

$$(m)_t = F_\delta(t,m),$$

where:

$$F_\delta : [-1,1] \times H^q(\mathbb{S}^1) \to H^q(\mathbb{S}^1)$$

are continuous, global Lipschitz in the second variable with a Lipschitz constant depending on δ, by the basic properties of J_δ. Applying the global Cauchy-Lipschitz theorem the above ODE's will have a global solution m_δ on the interval $[0,1]$, such that $m_\delta \in C^1([0,1], H^q(\mathbb{S}^1))$. We fix a neighborhood, for a q_0 big enough:

$$U \times V = \{(u_0,(v,\varepsilon)) \in \mathbb{E} \times \mathbb{F}_0 : |u_0|_{q_0+[r]+2} < c, \|v\|_{q_0+[r]+2} < c, |\varepsilon| < c\},$$

If one applies the Banach-Alaoglu theorem for Hilbert spaces and the Rellich-Kondrachov embedding theorem for the bounded sequence:

$$\|m_\delta\|_{H^{q_0}} \leqslant C_{q_0},$$

then there exists a subsequence m_{δ_k} such that:

$$m_{\delta_k} \rightharpoonup m \quad \text{in } \; H^{q_0}, \;\; \delta_k \to 0,$$

and

$$m_{\delta_k} \to m \quad \text{in } \; H^s, s < q_0.$$

By the general version of Arzela-Ascoli's theorem (Lemma 7.2 in [2]) there exists a subsequence also denoted by m_{δ_k}, such that $m_{\delta_k} \to m$ in $\mathrm{C}([-1,1], H^s(\mathbb{S}^1))$. Using Sobolev embeddings also $m_{\delta_k} \to m$ in $\mathrm{C}([-1,1], \mathrm{C}^{s'}(\mathbb{S}^1))$ for some s' big enough. As in [61] or in Chapter 16 of [70], we can conclude that m satisfies the linearized equation (6.2.7) and is unique, as a consequence of Gronwall's inequality. The existence interval for m does not depend on q so we can vary $q \in \mathbb{N}$ to gain a better regularity for m such that $m \in \mathrm{C}([-1,1], \mathrm{C}^q(\mathbb{S}^1))$, for every $q \in \mathbb{N}$. Since the derivatives in time are provided by the equation one obtains the existence of a $\mathrm{C}^\infty([-1,1], \mathrm{C}^\infty(\mathbb{S}^1))$ solution m and then of a unique solution $h \in \mathrm{C}^\infty([-1,1], \mathrm{C}^\infty(\mathbb{S}^1))$ for the initial equation (6.2.6), when $(u_0,(v,\varepsilon)) \in U \times V$, the neighborhood presented above.

Proposition 6.2.8. *The family of inverses $VP : (U \times V \subseteq \mathbb{E} \times \mathbb{F}_0) \times \mathbb{G} \to \mathbb{F}_0$ is a tame map.*

Proof. Here $U \times V$ is the neighborhood defined in the paragraph above. First we prove the tame estimate and for this purpose we define the energy:

$$E_{i,j}(t) := \frac{1}{2}\int_{\mathbb{S}^1} \left(D_t^i D_x^j Ah(t,x)\right)^2 dx$$

Step 1: $\int\limits_{-1}^{1} E_{0,0}(t)\,dt \lesssim 1 + \|w\|^2_{[r]+2} + \|k\|^2_{[r]+2}$.

If the operator A is applied to the linearized equation and after that the equation is multiplied with Ah then:

$$\frac{d}{dt}E_{0,0}(t) = \int_{\mathbb{S}^1} (Ah)_t \cdot Ah\,dx = -2\tau \int_{\mathbb{S}^1} Aw \cdot w_x \cdot Ah\,dx$$

$$-\tau \int_{\mathbb{S}^1} (Aw)_x \cdot w \cdot Ah\,dx - 2\varepsilon \int_{\mathbb{S}^1} Ah \cdot w_x \cdot Ah\,dx$$

$$-2\varepsilon \int_{\mathbb{S}^1} Aw \cdot D_x A^{-1}(Ah) \cdot Ah\,dx - \varepsilon \int_{\mathbb{S}^1} D_x A(h) \cdot w \cdot Ah\,dx$$

$$-\varepsilon \int_{\mathbb{S}^1} (Aw)_x \cdot A^{-1}(Ah) \cdot Ah\,dx + \int_{\mathbb{S}^1} Ak \cdot Ah\,dx$$

Integrating by parts:

$$\int_{\mathbb{S}^1} D_x Ah \cdot w \cdot Ah \quad dx = -\frac{1}{2}\int_{\mathbb{S}^1} (Ah)^2 \cdot w_x \quad dx$$

using Sobolev embeddings , Hölder's inequality, the estimate $ab \lesssim a^2 + b^2$ and the fact that $D_x A^{-1}$ and A^{-1} have orders at most 0, we get:

$$\left|\frac{d}{dt}E_{0,0}(t)\right| \lesssim (\|w_x\|_{C^0(\mathbb{S}^1)} + \|Aw\|_{C^0(\mathbb{S}^1)} + \|(Aw)_x\|_{C^0(\mathbb{S}^1)})E_{0,0}(t)$$

$$\|w\|^2_{H^{[r]+2}(\mathbb{S}^1)} + \|k\|^2_{H^{[r]+1}(\mathbb{S}^1)} + 2E_{0,0}(t), \qquad t \in [-1,1].$$

For $(u_0,(v,\varepsilon)) \in U \times V$ one has $|u_0|_{[r]+2} < c, \|v\|_{[r]+2} < c, |\varepsilon| < c$ and via Gronwall's inequality we obtain like in the proof of Lemma 6.2.7 the estimate:

$$E_{0,0}(t) \lesssim 1 + \|w\|^2_{[r]+2} + \|k\|^2_{[r]+2}$$

and the conclusion integrating on $[-1,1]$.

Step 2: $\int\limits_{-1}^{1} E_{0,j}(t)\,dt \lesssim 1 + \|w\|^2_{j+[r]+2} + \|k\|^2_{j+[r]+2}$, for all $j \in \mathbb{N}$.

We check the estimate for $j = 1$ and we suppose it's true for $k < j$, then for j starting with:

$$\frac{d}{dt}E_{0,j}(t) = \int_{\mathbb{S}^1} (D_x^j Ah)_t \cdot D_x^j Ah\,dx = -2\tau \int_{\mathbb{S}^1} D_x^j(Aw \cdot w_x) \cdot D_x^j Ah\,dx$$

$$-\tau \int_{\mathbb{S}^1} D_x^j((Aw)_x \cdot w) \cdot D_x^j Ah\,dx - 2\varepsilon \int_{\mathbb{S}^1} D_x^j(Ah \cdot w_x) \cdot D_x^j Ah\,dx$$

$$-2\varepsilon\int_{\mathbb{S}^1} D_x^j(Aw\cdot D_xA^{-1}(Ah))\cdot D_x^jAh\,dx-\varepsilon\int_{\mathbb{S}^1} D_x^j((Ah)_x\cdot w)\cdot D_x^jAh\,dx$$
$$-\varepsilon\int_{\mathbb{S}^1} D_x^j((Aw)_x\cdot A^{-1}(Ah))\cdot D_x^jAh\,dx+\int_{\mathbb{S}^1} D_x^jA(k)\cdot D_x^jAh\,dx$$

A problematic term will be:

$$\int_{\mathbb{S}^1} D_x^j((Ah)_x\cdot w)\cdot D_x^jAh\,dx=\int_{\mathbb{S}^1}\left(\sum_{\substack{a+b=j\\ a<j}} D_x^{a+1}Ah\cdot D_x^bw\right)D_x^jAh\,dx+$$

$$\int_{\mathbb{S}^1} w\cdot D_x^{j+1}Ah\cdot D_x^jAh\,dx\lesssim\left\|\sum_{\substack{a+b=j\\ a<j}} D_x^{a+1}Ah\cdot D_x^bw\right\|^2_{L^2(\mathbb{S}^1)}+E_{0,j}(t)+\|w_x\|_{C^0(\mathbb{S}^1)}\cdot E_{0,j}(t)$$

$$\lesssim\left(\|Ah\|_{H^j(\mathbb{S}^1)}\|w\|_{H^1(\mathbb{S}^1)}+\|Ah\|_{H^1(\mathbb{S}^1)}\|w\|_{H^j(\mathbb{S}^1)}\right)^2+\left(1+\|w_x\|_{C^0(\mathbb{S}^1)}\right)E_{0,j}(t),$$

where an interpolation inequality is used. Using now the induction hypothesis one gets:

$$\lesssim\|w\|^2_{H^1}\left(1+\|w\|^2_{j+[r]+1}+\|k\|^2_{j+[r]+1}+E_{0,j}(t)\right)+\|w\|^2_{H^j}\left(1+\|w\|^2_{[r]+3}+\|k\|^2_{[r]+3}\right)$$

$$+\left(1+\|w_x\|_{C^0(\mathbb{S}^1)}\right)E_{0,j}(t)\lesssim\left(1+\|w\|_2^2\right)E_{0,j}(t)+\|w\|_2^2\left(1+\|w\|^2_{j+[r]+1}+\|k\|^2_{j+[r]+1}\right)$$
$$+\|w\|^2_{j+1}\left(1+\|w\|^2_{[r]+3}+\|k\|^2_{[r]+3}\right).$$

The tameness of the operator VP is a local behaviour and it can be obtained in a neighborhood of any point in $U\times V\times\mathbb{G}$. Thus, if we impose the restriction $4\leqslant q_0+[r]+2$, we can chose around every point in $U\times V\times\mathbb{G}$ a neighborhood such that $|u_0|_4\leqslant c$, $\|(v,\varepsilon)\|_4\leqslant c$ and $\|(k,\tau)\|_4\leqslant C$, for every point in that neighborhood, where $c>0$ is the constant from the definition of $U\times V$ and $C>0$ is usually a different value. Combining this fact with the interpolation inequality (6.2.2) gives the estimate:

$$\varepsilon\int_{\mathbb{S}^1} D_x^j((Ah)_x\cdot w)\cdot D_x^jAh\,dx\lesssim E_{0,j}(t)+1+\|w\|^2_{j+[r]+2}+\|k\|^2_{j+[r]+2}$$

Consequently, if we apply similar strategies, as those from step 1 or the above one, we get:

$$\left|\frac{d}{dt}E_{0,j}(t)\right|\lesssim E_{0,j}(t)+1+\|w\|^2_{j+[r]+2}+\|k\|^2_{j+[r]+2},$$

and the conclusion using Gronwall's inequality.

Step 3: $\int\limits_{-1}^{1} E_{i,j}(t)\,dt\lesssim 1+\|w\|^2_{i+j+[r]+2}+\|k\|^2_{i+j+[r]+2}$, for all $i,j\in\mathbb{N}$.

Since is no longer true that $E_{i,j}(0)=0$, if $i\neq 0$, the idea is to prove by induction on i:

$$\int_{-1}^{1} E_{i,j}(t)\,dt\lesssim 1+\|w\|^2_{i+j+[r]+2}+\|k\|^2_{i+j+[r]+2}$$

for any $j \in \mathbb{N}$ when i is fixed.

If $i = 0$ it was proven above that $\int\limits_{-1}^{1} E_{0,j}(t)\, dt \lesssim 1 + \|w\|^2_{j+[r]+2} + \|k\|^2_{j+[r]+2}$ for all $j \in \mathbb{N}$. Let's suppose now that for all $l < i$ and $j \in \mathbb{N}$:

$$\int\limits_{-1}^{1} E_{l,j}(t)\, dt \lesssim 1 + \|w\|^2_{l+j+[r]+2} + \|k\|^2_{l+j+[r]+2}$$

To prove the statement for i one applies $D_x^j D_t^{i-1}$ to the linearized equation (6.2.7) then one squares both parts and integrates on $M = [-1, 1] \times \mathbb{S}^1$.

$$\int_{-1}^{1} E_{i,j}(t)\, dt \lesssim \int_M \{D_x^j D_t^{i-1}(Aw \cdot w_x)\}^2 + \{D_x^j D_t^{i-1}((Aw)_x \cdot w)\}^2\, dxdt$$

$$\int_M \{D_x^j D_t^{i-1}(Ah{\cdot}w_x)\}^2 + \{D_x^j D_t^{i-1}(Aw{\cdot}D_x A^{-1}(Ah))\}^2 + \{D_x^j D_t^{i-1}((Ah)_x{\cdot}w)\}^2\, dxdt$$

$$\int_M \{D_x^j D_t^{i-1}((Aw)_x \cdot A^{-1}(Ah))\}^2\, dx + \int_M \{D_x^j D_t^{i-1} A(k)\}^2\, dxdt$$

the estimate being realized in a neighborhood around an arbitrary point in $U \times V \times \mathbb{G}$ such that $|u_0|_4 \leqslant c$, $\|(v, \varepsilon)\|_4 \leqslant c$ and $\|(k, \tau)\|_4 \leqslant c$.

For every term in the right side we apply the tame estimate of a product (6.2.3) and an interpolation inequality of type (6.2.2) but both for the grading $\|\cdot\|_{H^q(\mathbb{S}^1)}$ on $\mathrm{C}^\infty(\mathbb{S}^1)$:

$$\int_M \{D_x^j D_t^{i-1}(Aw \cdot w_x)\}^2 dxdt \lesssim \int_M \{D_x^j \left(\sum_{a+b=i-1} D_t^a Aw \cdot D_t^b w_x\right)\}^2\, dxdt$$

$$\lesssim \sum_{a+b=i-1} \int_{-1}^{1} \|D_t^a Aw\|^2_{H^j(\mathbb{S}^1)} \left\|D_t^b w_x\right\|^2_{H^1(\mathbb{S}^1)} + \|D_t^a Aw\|^2_{H^1(\mathbb{S}^1)} \left\|D_t^b w_x\right\|^2_{H^j(\mathbb{S}^1)}\, dt$$

$$\lesssim \sum_{a+b=i-1} \int_{-1}^{1} \left(\sum_{k=0}^{j} E_{a,k}(t)\right) \|w\|^2_{b+3} + \left(\sum_{k=0}^{1} E_{a,k}(t)\right) \|w\|^2_{b+j+3}\, dt$$

$$\lesssim \sum_{a+b=i-1} \|w\|^2_{b+3} \sum_{k=0}^{j} \int_{-1}^{1} E_{a,k}(t)\, dt + \sum_{a+b=i-1} \|w\|^2_{b+j+3} \sum_{k=0}^{1} \int_{-1}^{1} E_{a,k}(t)\, dt$$

Since $a \le i - 1$ we can use the induction hypothesis and the interpolation inequality like in the proof of step 2 to get:

$$\int_M \{D_x^j D_t^{i-1}(Aw \cdot w_x)\}^2 dxdt \lesssim 1 + \|w\|^2_{i+j+[r]+2} + \|k\|^2_{i+j+[r]+2}$$

In a similar manner we find estimates for any other term to complete the induction on i.

In the end:

$$\|h\|_n^2 \lesssim \|Ah\|_n^2 \lesssim \sup_{i+j\leqslant n} \int_{-1}^{1} E_{i,j}(t)\, dt \lesssim 1 + \|w\|^2_{n+[r]+2} + \|k\|^2_{n+[r]+2}\,,$$

Following the definition of the norms this implies:

$$\|(h,\tau)\|_n^2 \lesssim 1 + |u_0|_{n+[r]+2} + \|(v,\varepsilon)\|^2_{n+[r]+2} + \|(k,\tau)\|^2_{n+[r]+2} \tag{6.2.10}$$

in a neighborhood, specified above, of an arbitrary point from $U \times V \times \mathbb{G}$. Thus the operator VP is tame.

In the remaining part of the proof we will have to deal with the continuity of the operator $VP : (U \times V \subseteq \mathbb{E} \times \mathbb{F}_0) \times \mathbb{G} \to \mathbb{F}_0$ and to shorten a little bit the proof we use the following lemma which is proven in [15]:

Lemma 6.2.9. *Let $\mathbb{X}, \mathbb{Y}$ Fréchet spaces and G a metric space. Given a mapping $F : G \times \mathbb{X} \to \mathbb{Y}$, assume that $F(g,\cdot) \in \mathcal{L}(\mathbb{X},\mathbb{Y})$ for all $g \in G$ and also $F(\cdot, x) \in C(G,\mathbb{Y})$ for all $x \in \mathbb{X}$. Then $F \in C(G \times \mathbb{X}, \mathbb{Y})$.*

The mapping $VP(u_0,(v,\varepsilon),\cdot)$ is linear and the tame estimate (6.2.10) provides the continuity in 0. So let $(k,\tau) \in \mathbb{G}$ be fixed:

$$A(h_1 - h_2)_t = -\tau \left\{2Aw_1 \cdot (w_1)_x - 2Aw_2 \cdot (w_2)_x + A(w_1)_x \cdot w_1 - A(w_2)_x \cdot w_2\right\}$$

$$-\varepsilon_1 \left\{2Ah_1 \cdot (w_1)_x + 2Aw_1 \cdot D_x A^{-1}(Ah_1) + (Ah_1)_x \cdot w_1 + (Aw_1)_x \cdot A^{-1}(Ah_1)\right\}$$

$$+\varepsilon_2 \left\{2Ah_2 \cdot (w_2)_x + 2Aw_2 \cdot D_x A^{-1}(Ah_2) + (Ah_2)_x \cdot w_2 + (Aw_2)_x \cdot A^{-1}(Ah_2)\right\}$$

With the methods from step 3 and the tame estimates available for Ah_1, Ah_2, working in a small neighborhood of the arbitrary point $(u_1,(v_1,\varepsilon_1)) \in U \times V$, for (k,τ) fixed, the next estimate can be proven:

$$\|A(h_1 - h_2)\|_n^2 \lesssim |\varepsilon_1 - \varepsilon_2|^2 + \|w_1 - w_2\|^2_{n+[r]+2}$$

for all $n \in \mathbb{N}$, the constant obviously depending on n. As a consequence we get the continuity of $A \circ VP(\cdot,(k,\tau))$, for (k,τ) fixed, and finally the conclusion using the above lemma. □

Proof of Proposition 6.2.4: Since now we proved all the technical requirements needed for the Nash-Moser Implicit Function Theorem 6.1.12. Consequently, there exists a C^{n_0}-neighborhood U_0 of 0 in $\mathbb{E} = \mathrm{C}^\infty(\mathbb{S}^1)$ such that for $\|u_0\|_{\mathrm{C}^{n_0}(\mathbb{S}^1)} < \delta$ the equation (6.2.1) has a unique solution $u = v + u_0$ on an interval $J = [-T, T]$, $T := \varepsilon = \mathrm{pr}_2 \circ \Phi(u_0)$ and $v = \mathrm{pr}_1 \circ \Phi(u_0)$, for a smooth tame mapping $\Phi : U_0 \to \mathbb{F}_0$. The "magnitude" n_0 of the neighborhood depends on the tameness degree of the mapping P which is $[r] + 1$, where $r \geq 1$ is the order of the pseudo-differential operator A. Using the exponential law mentioned in the proof of Lemma 6.2.1 one has $\mathrm{pr}_1 \circ \Phi \in \mathrm{C}^\infty(\mathrm{C}^\infty(\mathbb{S}^1) \times J, \mathrm{C}^\infty(\mathbb{S}^1))$ and therefore u depends smoothly on (u_0, t).

Remark 6.2.10. It seems that even with a Nash-Moser approach one still needs the restriction $r \geq 1$, considered in [15].

A

Isomorphisms of semi-direct products

In this section an isomorphim criterion is presented from the theory of semi-direct groups. The idea of the proof was communicated to me by prof. Derek Holt, to whom I am very grateful.

Proposition A.0.11. *Let H, N be arbitrary groups and $\alpha : H \to Aut(N)$ a group homomorphism which generates the semi-direct product $H \circledS_\alpha N$:*

$$(h_1, n_1) * (h_2, n_2) = (h_1 h_2, \alpha(h_2^{-1})(n_1) n_2)$$

If $Im(\alpha) \leq Inn(N)$ and $Z(N) = \{e\}$ then:

$$G = H \circledS_\alpha N \cong H \times N.$$

Proof. Because $Im(\alpha) \leq Inn(N)$ to every $h \in H$ it corresponds an unique $n_h \in N$ that induces the same inner automorphism as h does:

$$\alpha_h(n) = c_{n_h}(n) \qquad n \in N$$

where c is the conjugation map $c_g(n) = gng^{-1}$.

The corespondence $h \in H \to n(h) \in N$ determines a group homomorphism as a consequence of the fact that $Z(N) = \{e\}$. Considering now $\overline{N} = \{e\} \times N$ and $\overline{H} = H \times \{e\}$ the idea is to prove:

$$(h, e) * (e, n_h)^{-1} = (h, n_h^{-1}) \in C_G(\overline{N}) \tag{A.0.1}$$

the last being the centralizer of $\overline{N}$ in G:

$$C_G(\overline{N}) := \{g \in G : g\overline{n} = \overline{n}g, \quad \overline{n} \in \overline{N}\}$$

Thus $(h, n_h^{-1})*(e,n) = (h, \alpha(e^{-1})(n_h^{-1})n) = (h, n_h^{-1}n)$ and $(e,n)*(h,n_h^{-1}) = (h, \alpha(h^{-1})(n)n_h^{-1}) = (h, c_{n_h^{-1}}(n)n_h^{-1}) = (h, n_h^{-1}n)$

By its very definition $G = \overline{H} * \overline{N}$, $\overline{H} \cap \overline{N} = \{(e,e)\}$ and $\overline{N}$ is a normal subgroup in G. From ((A.0.1)) and the above relations results: $G = C_G(\overline{N}) * \overline{N}$.

Further $C_G(\overline{N}) \cong \overline{H}$ due to the fact that $(h, n') \in C_G(\overline{N})$ forces $c_{n'} = \alpha(h^{-1}) = c_{n_h^{-1}}$ and finally $(h, n') = (h, n_h^{-1})$. It is now easy to construct an isomorphism .

The centralizer is by definition a normal subgroup of the normalizer:

$$N_G(\overline{N}) := \{g \in G : g\overline{N} = \overline{N}g\}$$

which is the largest subgroup of G in which $\overline{N}$ is normal i.e. $C_G(\overline{N}) \cong \overline{H}$ is a normal subgroup of G.

Also $C_G(\overline{N}) \cap \overline{N} = \{(e,e)\}$ because the center of N is trivial. In conclusion:

$$G = C_G(\overline{N}) \times \overline{N} \cong \overline{H} \times \overline{N}$$

$\square$

Remark A.0.12. An isomorphism between $G \circledS_\alpha G$ with $Im(\alpha) \leq Inn(G)$ and $G \times G$ is given by the mapping:

$$\varphi : G \circledS_\alpha G \to G \times G, \qquad \varphi(h,n) = (h, n(h) \cdot n)$$

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Curriculum Vitae

Ich, Emanuel-Ciprian Cismas, wurde am 27. Februar 1985 in Ineu, Rumänien, geboren. Im Juli 2003 habe ich Abitur an der Liceul Teoretic Pancota gemacht. Danach habe ich mich an der Universtatea de Vest Timişoara immatrikuliert und dort Mathematik studiert. Seit Juli 2007 bin ich Diplom-Mathematiker. Der Titel meiner Diplomarbeit lautet *Algebraische Polynome.* Im Juli 2009 habe ich das Masterstudium der Mathematik in Timişoara mit der Arbeit *Fixpunktsätze von Krasnoselskii Typ* abgeschlossen. Seit November 2011 bin ich im Rahmen des GRK 1463 "Analysis, Geometry and String Theory" wissenschaftlicher Mitarbeiter an der Gottfried Wilhelm Leibniz Universität Hannover und arbeite unter der Betreuung von Prof. Dr. J. Escher am Institut für Angewandte Mathematik an meinem Promotionsthema *Euler-Poincaré-Arnold Gleichungen auf semidirekten Produkten.* Meine Forschungsinteressen sind unter anderem: Unendliche Differentialgeometrie, Theorie der Frécheträume, Nichtlineare Partielle Differentialgleichungen.