Two–Phase Thin Film Equations with Insoluble Surfactant

Von der Fakultät für Mathematik und Physik

der Gottfried Wilhelm Leibniz Universität Hannover

zur Erlangung des Grades

DOKTORIN DER NATURWISSENSCHAFTEN

Dr. rer. nat.

genehmigte Dissertation

von

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geboren am 02.09.1986, in Bielefeld

2015

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Tag der Promotion: 08.05.2015

Für meine Eltern

Above all, I want to express my sincere gratitudes to my supervisors Prof. Dr. Joachim Escher and Prof. Dr. Christoph Walker for their guidance and support during my doctoral studies. I am also deeply indebted to Dr. Bogdan Matioc for fruitful discussions, valuable ideas and suggestions. My thanks go to Prof. Dr. Elmar Schrohe, Prof. Dr. Mats Ehrnström and Dr. Cristina Ivan, who taught me and contributed to my academic development. I am much obliged to all my colleagues from the Institute of Applied Mathematics and the members from the research training group GRK 1463 for discussions and a pleasant working atmosphere. Last but not least, I want to express my deepest gratitude to my family for their endless support and love.

Abstract

This thesis is concerned with the modeling and analysis of a two–phase thin film flow with insoluble surfactant. We consider two immiscible, viscous, incompressible Newtonian fluids on top of each other on a solid substrate with a layer of insoluble surfactant. By cross–sectional averaging and applying lubrication approximation, as in [27, 29], we obtain simplified evolution equations for the two film heights and the surfactant concentration. Depending on the considered driving force, the system of evolution equations is strongly coupled, degenerate and either of second (gravity driven) or of fourth order (capillary driven). Based on recent achievements regarding the local well–posedness and asymptotic stability of steady states for thin film equations with surfactant [19] as well as for two–phase thin film flows [23, 24] we prove analog results for the gravity and capillary driven two–phase thin film flow with insoluble surfactant, respectively. This is done by methods of semigroup theory and the principal of linearized stability. Due to the degeneracy in the evolution equations for the film heights, which may occur when one of the film heights decreases to zero, it is in general not clear whether one can prove global well–posedness results. This leads to the study of the existence of non–negative global weak solutions, which is investigated for the fourth–order system describing the capillary driven two–phase thin film flow with insoluble surfactant. The idea is to construct via Galerkin approximation global weak solutions to a family of regularized systems, which tend in the limit to a global non–negative weak solution of the original system. The proof relays strongly on the existence of an energy functional and combines results from [20, 22, 26]. In [22] the existence of non–negative global weak solutions for a thin–film approximation of a two–phase Stokes problem is studied, whereas [20, 26] investigate the existence of non–negative global weak solutions for a thin film equation with insoluble surfactant driven by capillary forces.

Keywords: Thin film equations, surfactant, degenerate parabolic equations

Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Modellierung und dem Studium von Zwei–Phasen Dünnfilmgleichungen mit unlöslichen Tensiden. Wir betrachten zwei übereinander gelagerte dünne Filme sich nicht mischender, viskoser, inkompressibler Newtonscher Flüssigkeiten auf einem soliden, undurchlässigen Boden, wobei sich auf der Oberfläche der oberen Flüssigkeit eine Schicht von Tensiden befindet. Unter der Annahme, dass der Zwei–Phasen Film in eine horizontale Richtung uniform ist, wenden wir, wie in [27, 29], die Methode der Lubrikationsapproximation an, um vereinfachte Gleichungen zu erhalten, die die Evolution der beiden Filmhöhen und der Tensidkonzentration beschreiben. Das System von Evolutionsgleichungen ist stark gekoppelt, degeneriert und, je nachdem, welche physikalische Kraft als führend betrachtet wird, von vierter oder von zweiter Ordnung. Betrachtet man Gravitation als dominierend, so ist das System von zweiter Ordnung, wohingegen das System unter Berücksichtigung von kapillaren Kräften Terme vierter Ordnung aufweist. Basierend auf jüngsten Resultaten bezüglich der Wohlgestelltheit und asymptotischen Stabilität von Equilibria für Dünnfilmgleichungen mit Tensiden [19] und für Zwei–Phasen Dünnfilmgleichungen [23, 24], zeigen wir ähnliche Ergebnisse für die Zwei–Phasen Dünnfilmgleichungen mit unlöslichen Tensiden, wobei wir sowohl das System mit kapillaren Kräften als auch das mit Gravitation als dominierende Kraft betrachten. Für das Studium der Wohlgestelltheit und der asymptotischen Stabilität von Equilibria verwenden wir Methoden aus der Halbgruppentheorie sowie das Prinzip der linearisierten Stabilität. Aufgrund der Degeneriertheit der Evolutionsgleichungen für die Filmhöhen, ist im Allgemeinen die globale Wohlgestelltheit nicht zu erwarten. Dies führt zum Studium von (nicht negativen) schwachen Lösungen, welche wir für das System vierter untersuchen. Die Idee ist, Galerkin Approximationen für eine Familie von regularisierten Systemen zu konstruieren, die global existieren und im Grenzwert gegen eine nicht negative globale schwache Lösung des ursprünglichen Systems

konvergieren. Der Beweis basiert essenziell auf der Existenz eines Energiefunktionals. Wir kombinieren Resultate aus der Arbeit [22], in dem die Existenz von nicht negativen globalen schwachen Lösungen für eine Dünnfilmapproximation des Zwei–Phasen Stokes Problems untersucht wird, und [20, 26], in denen die Existenz von nicht negativen globalen Lösungen für eine Dünnfilmgleichung mit unlöslichen Tensiden bewiesen wird.
Schlagworte: Dünnfilmgleichungen, Tenside, degenerierte parabolische Gleichungen

Contents

Introduction

The study of thin film equations constitutes a rich and complex area of research with a long list of contributions by physicists, engineers and mathematicians. Of particular fascination for many scientists is the role of surface tension and the influence of surface active agents (short *surfactants*) on the dynamics of thin liquid films since this finds applications in various industrial and biomedical fields. As for instants in surfactant replacement therapy, which is used to treat the immature lungs of premature infants, coating flow technology or film drainage in emulsions and foams. A report on the diverse areas of applications of surfactant can be found in [34] and the references therein.

Surfactants act on the surface of a fluid film by lowering the surface tension and induce a twofold dynamic. On the one hand, the resulting surface gradients influence the dynamics of the fluid film. On the other hand, the surfactants spread along the interface (from low surface tension to high surface tension). The latter effect is named after the Italian physicist Carlo Giuseppe Matteo Marangoni and called *Marangoni effect*. This phenomenon has already been observed, but not explained in the 17th century. Two centuries later, in 1871, Marangoni was the first to publish explaining results for the spreading of a substrate due to surface tension gradients, followed by other physicists like van der Mensbrugghe and Lüdtge. For a survey on the historical investigations of the Marangoni effect we refer to [4, 36] and references therein.

There is a large amount of literature regarding the dynamics of thin films with surfactant in different settings or configurations, such as soluble/insoluble surfactant, presences/negligence of driving forces as intermolecular (van der Waals), gravitational and capillary forces, contact angle of the thin liquid to the (impermeable) bottom, known under non–/partial or complete wetting, to name a few. Pioneering results on the dynamics of a thin fluid with insoluble surfactant are [27, 29, 30], where the approach via *lubrication approximation* for thin liquid films is introduced and first numerical result are presented under consideration of different driving forces. Applying lubrication approximation, the parameters are scaled in accordance to the undisturbed film height and the governing equations for the motion of the fluids and the surfactant spreading are considered in the limit where the film height tends to zero. The resulting equations do not represent the complex mechanisms of the problem completely, but still they preserve many of the main features of its physics. A survey on the theory of modeling thin film flows can be found for instance in [39, 40].

Mathematically the analysis of a thin film flow with insoluble surfactant corresponds to a free boundary problem for the Navier–Stokes equation together with a transport equation on the upper free surface. In virtue of lubrication approximation, the resulting equations for the evolution of the film height is nonlinear, degenerate, of fourth order (in general) and strongly coupled to a second–order, nonlinear transport equation for the surfactant concentration. Depending on the considered driving forces, the evolution equation for the film height is of fourth order (if capillary forces are taken into account) or of second order (if capillary forces are neglected).

Although, during the last decades there has been various modeling and numerical treatment of several aspects of the surfactant induced movement of thin films (see e.g. [6, 9, 10, 14, 27, 29, 30, 35]), only recently analytical investigations have started. An overview regarding the analytical and numerical achievements for thin films with surface tension until the end of the 20th century can be found in [37]. The main feature all models share, derived under different assumptions on the driving forces, is the degeneracy, which occurs in the equations when the variable representing the film thickness decreases to zero. The mathematical analysis is especially concerned with questions regarding the *well–posedness* of a model, that is, whether a solution exists (at least locally in time), if it is unique (depending on the initial data) and if the solution is continuously dependent on the initial data. Due to the possible occurrence of degeneracies, it is in general not clear whether one can prove the existence of global solutions in a classical sense. This leads to the study of so–called *weak solutions*.

Results regarding non–negative global weak solutions of equations modeling the dynamics of a thin liquid film have been studied among others by [5, 7, 8]. Regarding the one–phase problem with insoluble surfactant, several authors contributed to the analysis of well–posedness and existence of global weak solutions for a coupled system of evolution equations describing the dynamics of the interface and the surfactant spreading under certain assumptions on the driving forces. Local existence for a thin film with insoluble surfactant driven only by Marangoni forces (surface tension gradients) has been studied in [42]. In absence of capillary and intermolecular forces but including gravitational forces, the via lubrication approximation derived system in [19] is of second order and local well–posedness as well as asymptotic stability of steady states are proven. In particular, the surfactants in [19] are considered to be soluble, which leads to an additional evolution equation for the surfactant distribution in the bulk. In [21] the existence of global weak solutions is investigated for the system derived in [19] in the case of insoluble surfactant. Taking instead of gravitational forces capillary effects into account, the equation describing the evolution of the thin film is of fourth order and strongly coupled to a second–order transport equation for the surfactant concentration. By using the method of *Galerkin approximation* and compact embeddings, the existence of global non–negative weak solutions have been shown in [20, 26, 47]. In [15] additionally gravitational forces are included and an upper bound for the non–negative weak solution for the surfactant concentration is stated ($\Gamma \leq 1$).

Being interested in the dynamics of a two–phase thin film flow with insoluble surfactant, we resort not only to results for thin film equations with surfactant, but also to the analytical studies of two–phase thin films. As for instance in [24] local well–posedeness and asymptotic stability of a thin–film approximation of the two–phase Stokes problem are investigated by methods of *semigroup theory* and the *principal of linearized stability*. A similar approach has also been successfully applied in [23] to prove local existence and stability results for a strongly coupled fourth–order parabolic degenerate system modeling the motion of two thin fluid films in the presence of gravity and capillary forces. The existence of non–negative weak solutions for a degenerate parabolic system approximating the two–phase stokes problem is investigated in [22] by the method of Galerkin approximation and compact embeddings.

It turns out (see e.g. [15, 19, 20, 21, 22, 23, 24, 26]) that in the mathematical analysis of thin films the existence of an *energy functional* becomes a crucial part studying the stability of steady states and in particular in proving the existence of global weak solutions. In the context of steady states the energy functional determines the set of steady states in [19, 23, 24]. Studying weak solutions of degenerate systems describing the evolution of thin films, commonly regularized systems are considered (cf. e.g. [15, 20, 21, 22, 26]) and the existence of global weak solutions of a family of regularized systems is shown. The energy functional then provides necessary a priori estimates, which allow to extract weakly convergent subsequences tending in the limit to a global weak solution of the original problem.

In this thesis a mathematical model for the evolution of a two–phase flow with insoluble surfactant is presented. The two–phase flow consists of two immiscible, incompressible Newtonian and viscous thin liquid films on top of each other on a solid substrate. We assume that there is no contact angle between the two–phase flow and the bottom, which places the setting in the context of complete wetting. The interface of the upper fluid is endowed with a layer of insoluble surfactant. By cross–sectional averaging and applying lubrication approximation, the resulting system of evolution equations consists of two equations describing the evolution of the two film heights and one equation for the surfactant concentration on a one–dimensional domain. A numerical treatment of the approximation of a two–phase thin film with insoluble surfactant can be found in [6], where the presence of surfactant is assumed to be on both interfaces (liquid–liquid and liquid–gas).

In Chapter 1 the two–phase thin film flow with insoluble surfactant is introduced and lubrication approximation is applied to the governing equations for the motion of viscous fluid films, which is the full Navier–Stokes equation, and an advection–transport equation for the surfactant spreading together with suitable boundary conditions (*no–slip* and *kinematic boundary condition* together with balance equations on the free surfaces). We consider both, gravitational and capillary forces, but neglect intermolecular forces (van der Waaals forces) since they are highly unlikely to be of the same order as gravitational forces (cf. [29]), so that considering both simultaneously appears to be physically not relevant. Equipping the system of evolution equations with Neumann–boundary conditions and initial data, we study in Chapter 2 and 3 the existence and asymptotic stability of the system when either capillary forces (Chapter 2) or gravitational forces (Chapter 3) are neglected. In addition to the local well–posedness result in Chapter 3, we also investigate the existence of non–negative global weak solutions.

Using the functional analytical tool of semigroups we prove in Chapter 2 the local well–posedness for the gravity driven two–phase flow with insoluble surfactant. Similar as in

[19, 24] we show that there exists an *energy functional* for the system of evolution equations. In particular, the energy functional determines (uniquely, in virtue of given initial data) the set of steady states to be of the form when the two thin films are flat and the surfactant is uniquely disturbed. By means of the *principal of linearized stability* we show according to to [19, 24] the asymptotic stability of the steady states. The proof relays essential due to the fact that the surfactant spreading is not only due to Marangoni forces but also due to a strictly positive diffusion coefficient in the transport equation for the surfactant spreading ($D > 0$). This is in accordance to the phenomenon explained in [27], that due to diffusion along the surface, the surfactant spreading not only accelerates but also decreases the film disturbance.

Chapter 3 is devoted to the study of the system derived by lubrication approximation in Chapter 1, when taking capillary forces instead of gravity into account. Since the evolution equations are strongly coupled and of fourth order for the two film heights and additionally strongly coupled to the second–order transport equation for the surfactant concentration, the analysis of local well–posedness is more involved than in the case of the second–order parabolic system considered in Chapter 2. Even though the existence of global weak solutions for the one–phase thin film equation with insoluble surfactant driven by capillary forces is shown in [20, 26], to our knowledge local well–posedness results of this system have not yet been investigated. Orienting on [23], in Section 3.1 local well–posedness is shown for the capillary driven two–phase thin film equation with insoluble surfactant by methods of semigroup theory. Under an additional assumption on the surface tension another local well–posedness result is shown in Appendix **??**, generalizing the result in Section 3.1, if the assumption on the surface tension is satisfied. Proving the existence of an energy functional, the result regarding asymptotic stability is based on [19, 23]. The system considered in [20, 26] can be recovered in Section 3.1 by setting the initial data of the lower fluid to zero. Hence, the well–posedness serves also for the one–phase thin film with insoluble surfactant driven by capillary forces. Section 3.3 is concerned with the existence of non–negative global weak solutions to the capillary driven two–phase flow with insoluble surfactant. The strategy here is to assemble the proof combining results from [20, 22, 26, 47]. We regularize the system and show by Galerkin approximation and an energy functional that the family of regularized systems possesses global weak solutions, which are so far not claimed to be non–negative.

The energy functional provides a priori estimates which imply compact embeddings and allow to extract weakly convergent subsequences, which tend in the limit to non–negative functions being solutions of the original system.

General Notations and Conventions

In this section we introduce some notation, which will be frequently used in the sequel.

Linear Operators. Suppose that $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ are Banach spaces. The space of all linear and bounded operators from E into F is denoted by $\mathcal{L}(E, F)$ and the norm $\|\cdot\|_{\mathcal{L}(E,F)}$ is given by

$$\|A\|_{\mathcal{L}(E,F)} := \sup_{\|x\|_E \leq 1} \|Ax\|_F.$$

For linear and bounded operators from E into itself we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.
A linear operator $A \in \mathcal{L}(E, F)$ is called *compact* if the image of the unit ball is relatively compact in F. We say that a (not necessary bounded) linear operator A from a vector subspace $\mathrm{dom}(A)$ of E, called *domain* of A, into E is *closed* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(A)$ that converges to x in E with $(Ax_n)_{n\in\mathbb{N}}$ converges to y in E one has $x \in \mathrm{dom}(A)$ and $Ax = y$. The *resolvent set* of an operator $A \in \mathcal{L}(E)$ is defined by

$$\varrho(A) := \{\lambda \in \mathbb{C} \mid (A - \lambda)^{-1} \text{ belongs to } \mathcal{L}(E)\}$$

and the *spectrum* of A by

$$\mathrm{spec}(A) := \mathbb{C} \setminus \varrho(A).$$

Further, we say A is *densely defined* exactly when $\overline{\mathrm{dom}(A)} = E$.
Let E be a Banach space over a field $\mathbb{F}$. The set $E' := \mathcal{L}(E, \mathbb{F})$ of all linear, continuous functions from E into $\mathbb{F}$ is called the *dual space* of E. We denote by

$$\langle \cdot, \cdot \rangle_E$$

the *dual pairing* between E' and E.

Embeddings. Let E and F be Banach spaces with $F \subset E$. If the inclusion map $\iota : F \to E$, $x \mapsto x$ is continuous then F is called *continuously embedded* in E and we write

$$F \hookrightarrow E.$$

The subspace F is said to be *compactly embedded* in E if $F \hookrightarrow E$ and the inclusion map $\iota : F \to E$ is compact, we write

$$F \hookrightarrow\hookrightarrow E.$$

Function Spaces. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}$, a measurable function. Then, for $1 \le p \le \infty$ we denote by $L_p(\Omega) := \{f : \Omega \to \mathbb{R}^m$ measurable and $\|f\|_p < \infty\}$ the space of *p–Lebesgue integrable functions*, where the norm $\|\cdot\|_p$ is given by

$$\|f\|_p := \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \text{ess–sup}_{x\in\Omega} |f(x)|, & p = \infty. \end{cases}$$

Note that $L_p(\Omega)$ is an equivalence class with $f \sim g$, if $f = g$ almost everywhere. The space $L_2(\Omega)$ is a Hilbert space and we denote the scalar product in $L_2(\Omega)$ by $(\cdot \mid \cdot)_2$. The *Sobolev space* $H^k(\Omega)$ for $k \in \mathbb{N}$ is defined to consists of functions $f \in L_2(\Omega)$, whose first k weak derivatives again belong to $L_2(\Omega)$ and the norm of a function $f \in H^k(\Omega)$ is given by

$$\|f\|_{H^k} := \left(\sum_{l=0}^{k} \|\partial_x^l f\|_2^2\right)^{\frac{1}{2}}.$$

The *Bessel potential spaces* $H^s(\Omega)$, $s \ge 0$, occur as complex interpolation spaces between two Sobolev spaces

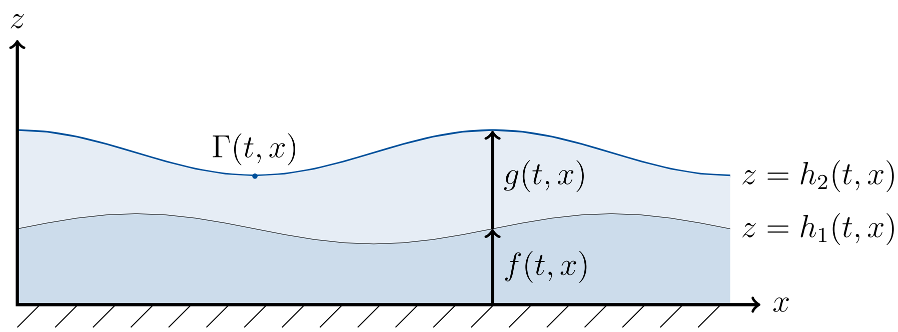
$$H^s(\Omega) = [H^k(\Omega), H^l(\Omega)]_\theta,$$

where $\theta \in [0,1]$ and $s := (1-\theta)k + \theta l$. Moreover, we denote by $C^{1-}(\Omega)$ the space of locally Lipschitz continuous functions on Ω.

General Conventions. If $(x_n)_{n\in\mathbb{N}}$ is a sequence, we understand by taking a subsequence *(not relabeled)*, that we consider without denotation in the sequel $(x_{n_k})_{k\in\mathbb{N}}$. Similar, if we consider two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ and extract a subsequence (not relabeled) of $(x_n)_{n\in\mathbb{N}}$, we consider in the sequel without denotation $(x_{n_k})_{k\in\mathbb{N}}$ and $(y_{n_k})_{k\in\mathbb{N}}$. The variables $t, T \ge 0$ and $x \in \mathbb{R}$ will always denote the time and space, respectively. Further, if $\Omega := (0, L)$, we set $\Omega_T := (0, L) \times (0, T)$. We denote by $c > 0$ various constants, which may differ from occurrence to occurrence. The dependence of c on the free variables is denoted by $c(\cdot, \cdot, \ldots)$. Sometimes we write $c = c(\cdot, \cdot, \ldots)$ in order to emphasize the dependence. Else, c denotes a positive constant, independent of the free variables. Unless otherwise stated, the underlying vector field is $\mathbb{R}$.

1. Physical Model for the Two–Phase Thin Film with Insoluble Surfactant

We consider two viscous, incompressible Newtonian and immiscible thin films on top of each other on a horizontal impermeable bottom at $z = 0$ occupying the regions Ω_1, Ω_2, respectively, with a layer of insoluble surfactant on the surface of the upper fluid. The contact angle between the fluids and the solid ground is assumed to be zero, which corresponds to the frame of *complete wetting*. The surfactant is acting on the interface of the upper fluid by lowering the surface tension. We assume the surface tension on the interface separating the fluids to be independent of external influences and the material outside of the two–phase flow to have no effect on the fluids. In particular, the material outside is assumed to be static and with zero pressure. Let L be the length of the two–phase film and take the undisturbed film height H to be given as small compared to the film length, that is $\frac{H}{L} = \varepsilon$ with $\varepsilon \ll 1$. By cross–sectional averaging we assume the film to be uniform in one horizontal level and let x and z denote the horizontal and vertical direction, respectively. Further, we denote the two film heights by f and g, so that the free surfaces at time $t \geq 0$ and position $x \in (0, L)$ are located at $z = h_1(t, x) := f(t, x)$ and $z = h_2(t, x) := (f + g)(t, x)$. The concentration of surfactant at time $t \geq 0$ and position $x \in (0, L)$ is given by $\Gamma(t, x)$.



As common in the analysis of thin films (see e.g. [27, 29] for pioneering works), we apply a lubrication approximation to the governing equations for the dynamics of the fluids and the surfactant concentration together with suitable boundary conditions, in order obtain simplified evolution equations for the two film heights f, g and the concentration of surfactant Γ on the fluid–gas interface. Set $i = 1, 2$, then the velocity field of the fluid contained in Ω_i will be denoted by $v_i = (u_i, w_i)$, where each particle of the fluid contained in Ω_i is moving with the velocity $u_i(t, x, z)$ in horizontal and $w_i(t, x, z)$ in vertical direction. The velocity and the pressure, given by p_i, are functions of position and time. The gravitational constant is given by $\gamma = (0, G)$. Moreover, assuming the fluid to be incompressible and Newtonian, the density and viscosity of the fluids, denoted by ρ_i and μ_i are material constants.

Governing equations. In fluid dynamics, the governing equations for the motion of a viscous, incompressible and Newtonian fluid occupying $\Omega_i, i = 1, 2$, is given by the *Navier–Stokes equation*

$$\rho_i(\partial_t v_i + (v_i \cdot \nabla)v_i) = \mu_i \Delta v_i - \nabla p_i - \rho_i \gamma. \tag{1.1}$$

Further, conservation of mass for incompressible fluids implies the *continuity equation*

$$\partial_x u_i + \partial_z w_i = 0 \tag{1.2}$$

in Ω_i, $i = 1, 2$. The dynamics of thin liquid films is strongly influenced by surface tension (cf. e.g. [32]). Since surface tension affects only the free surface, it does not appear in the Navier–Stokes equation, but contributes to the motion of a fluid through boundary conditions. The surfactant spreading on the free surface $z = h_2$ is governed by the *advection–transport equation*

$$\partial_t \Gamma + \partial_x(u_2 \Gamma - D\partial_x \Gamma) = 0, \tag{1.3}$$

where $D > 0$ is the surface diffusion coefficient. Note that additionally to the diffusion, the spreading of surfactant is also induced by surface tension gradients, which occur due to the present of surfactant itself (Marangoni effect). This effect will enter into the tangential balance equation (1.13).

Boundary conditions. Besides initial data, we need to implement boundary conditions at the impermeable bottom and on the free surfaces in order to well–pose the two–phase thin film flow with insoluble surfactant.

Since the bottom at $z = 0$ is impermeable, there is no mass transfer across this boundary and the perpendicular velocity at the bottom is zero. There is a long historical and philosophical discussion about whether to assume in fluid dynamics a *no–slip* or a *slip* condition on an impermeable, solid bottom (see e.g [16, 38] and references therein). Still, in the case of a viscous fluid, the no–slip boundary condition is commonly accepted and employed in the theoretical study of fluid flows. An example, where by lubrication approximation a thin film model with insoluble surfactant has been derived using a slip condition on the solid substrate, can be found in [47]. In accordance to e.g. [17, 19, 20, 26, 27], we impose the no–slip boundary condition, so that

$$v_1 = (u_1, w_1) = 0 \qquad \text{on} \quad z = 0. \tag{1.4}$$

We suppose the velocity field to be continuous across the immiscible fluid–fluid interface $z = f$, which means that the velocities at the interface coincide in tangential direction (in analogy to the no–slip boundary condition at the bottom)

$$(v_1 - v_2) \cdot t_1 = 0 \qquad \text{on} \quad z = h_1 \tag{1.5}$$

and since there is no diffusion at the interface, the velocities also coincide in normal direction as a consequence of conservation of mass. Hence

$$(v_1 - v_2) \cdot n_1 = 0 \qquad \text{on} \quad z = h_1. \tag{1.6}$$

Here, $n_i := \frac{1}{\sqrt{1+|\partial_x h_i|^2}}(-\partial_x h_i, 1)$ and $t_i := \frac{1}{\sqrt{1+|\partial_x h_i|^2}}(1, \partial_x h_i)$ denote the unit normal and tangential vector at h_i, $i = 1, 2$ respectively. Note that (1.5), (1.6) already imply that

$$v_1 = v_2 \qquad \text{on} \quad z = h_1. \tag{1.7}$$

Indeed, owing to (1.5), (1.6) we obtain that

$$u_1 - u_2 = -\partial_x h_1 (w_1 - w_2), \tag{1.8}$$

$$w_1 - w_2 = \partial_x h_1 (u_1 - u_2). \tag{1.9}$$

At the points where h_1 is flat (thus $\partial_x h_1$=0) the velocity fields are equal at the interface, that is $v_1 = v_2$. In the case that $\partial_x h_1 \neq 0$, we multiply (1.9) by $-\partial_x h_1$ and deduce that $u_1 - u_2 = -\partial_x h_1 (w_1 - w_2) = -(\partial_x h_1)^2 (u_1 - u_2)$. This is only possible if $u_1 = u_2$, which in turn implies that $w_1 = w_2$, in virtue of $\partial_x h_1 \neq 0$. Hence, $v_1 = v_2$ on $z = h_1$.

Due to interfacial tension, the *stress balance equation* at an interface between two immiscible materials occupying the regions Ω_- and Ω_+ has to be satisfied (cf. e.g. [11])

$$[\Sigma_- - \Sigma_+]n = \sigma\kappa n + \nabla_s\sigma, \tag{1.10}$$

where Σ_-, Σ_+ denote the stress tensor of the fluid in Ω_-, Ω_+, respectively, σ the surface tension coefficient, κ the mean curvature, $\nabla_s\sigma := \nabla\sigma - n(n\cdot\nabla\sigma)$ the gradient of σ in direction of the surface and n the outer normal pointing into Ω_+. The first term on the right hand side of (1.10) represents *capillary* forces and the second term *Marangoni* forces. Since we assume the material outside of the two–phase flow have zero pressure, the stress balance equation takes the form

$$\begin{cases} [\Sigma(v_1,p_1) - \Sigma(v_2,p_2)]n_1 = \sigma_1\kappa_1 n_1 + \nabla_s\sigma_1 & \text{on} \quad z = h_1, \\ \Sigma(v_2,p_2)n_2 = \sigma_2\kappa_2 n_2 + \nabla_s\sigma_2 & \text{on} \quad z = h_2, \end{cases} \tag{1.11}$$

where $\Sigma(v_i,p_i) = \frac{1}{2}\mu_i(\nabla v_i + \nabla v_i^T) - p_i$ denotes the stress tensor, σ_i the surface tension coefficient and $\nabla_s\sigma_i$ the gradient of σ_i in direction of the surface h_i, $i = 1,2$. Multiplying (1.11) by n_i, yields the *normal stress balance equation*

$$\begin{cases} ([\Sigma(v_1,p_1) - \Sigma(v_2,p_2)]n_1)\cdot n_1 = \sigma_1\kappa_1, & z = h_1, \\ (\Sigma(v_2,p_2)n_2)\cdot n_2 = \sigma_2\kappa_2, & z = h_2, \end{cases} \tag{1.12}$$

where the mean curvature κ_i of the interface h_i is given by $\kappa_i = \frac{\partial_x^2 h_i}{(1+|\partial_x h_i|^2)^{\frac{3}{2}}}$, $i = 1,2$. The surface tension coefficient σ_1 on $z = h_1$ is constant, whereas the surface tension coefficient of the free surface of the upper fluid depends non–increasingly on the surfactant concentration $\sigma_2 = \sigma_2(\Gamma)$. Hence $\nabla_s\sigma_1\cdot t_1 = 0$ and $\nabla_s\sigma_2\cdot t_2 = \nabla\sigma_2\cdot t_2 - n_2(n_2\cdot\nabla\sigma_2)\cdot t_2 = \nabla\sigma_2\cdot t_2 = \frac{\partial_x\sigma_2(\Gamma)}{\sqrt{1+|\partial_x^2 h_2|}}$. Thus, multiplying (1.11) by the tangential vector t_i, leads to the *tangential stress balance equation*

$$\begin{cases} ([\Sigma(v_1,p_1) - \Sigma(v_2,p_2)]n_1)\cdot t_1 = 0, & z = h_1, \\ (\Sigma(v_2,p_2)n_2)\cdot t_2 = \dfrac{\partial_x\sigma_2(\Gamma)}{\sqrt{1+|\partial_x h_2|^2}}, & z = h_2. \end{cases} \tag{1.13}$$

Observe that the normal stress balance is controlled by the capillary forces, whereas the Marangoni forces, induced by the surfactant, enter the tangential stress balance equation. We assume that the particles on the interfaces $z = h_i, i = 1,2$ stay there when time evolves. Consider the particle trajectory $(x(t), z(t))$. Then $x'(t) = u(t, x(t), z(t))$ and

$z'(t) = w(t, x(t), z(t))$. Since the particle is supposed to stay on $z = h_i$, we deduce that

$$z(t) = h(t, x(t)). \tag{1.14}$$

By differentiating (1.14), we obtain the so–called *kinematic boundary condition*

$$\partial_t h_i + u_i \partial_x h_i = w_i \qquad \text{on} \quad z = h_i. \tag{1.15}$$

We summarize that the motion of the two–phase thin film flow with insoluble surfactant is described by the Navier–Stokes equation (1.1) together with the continuity equation (1.2), the surfactant spreading equation (1.3) and the boundary conditions (1.4), (1.7), (1.12), (1.13) and (1.15).

1.1. Lubrication Approximation

The method of lubrication approximation [27, 29, 30] enables us to simplify the system of equations by rescaling the parameters and considering the system in the limit, where the quotient $\frac{H}{L} = \varepsilon$ tends to zero. The equations we obtain do not represent the complex mechanisms of the original problem completely, but still preserve the main features. Introducing the dimensionless variables

$$\bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{z}{H}, \quad \bar{t} = \varepsilon^k \tau_0 t$$

and

$$u(t,x,z) = \alpha \bar{u}(\bar{t}, \bar{x}, \bar{z}), \qquad w(t,x,z) = \beta \bar{w}(\bar{t}, \bar{x}, \bar{z}), \qquad p(t,x,z) = \gamma \bar{p}(\bar{t}, \bar{x}, \bar{z})$$

with dimensions $[\alpha] = [\beta] = \frac{m}{s}, [\tau_0] = \frac{1}{s}, [\gamma] = \frac{kg}{ms^2}$, we rescale first the Navier–Stokes equation (1.1). In order to simplify the notation, we suppress the subscript i for now:

$$\begin{cases} \rho(\partial_t u + u\partial_x u + w\partial_z u) &= \mu(\partial_x^2 u + \partial_z^2 u) - \partial_x p, \\ \rho(\partial_t w + u\partial_x w + w\partial_z w) &= \mu(\partial_x^2 w + \partial_z^2 w) - \partial_z p - \rho G. \end{cases}$$

Substituting the variables from above leads to

$$\begin{cases} \rho\left(\alpha\varepsilon^k\tau_0\partial_{\bar{t}}\bar{u} + \frac{\alpha^2}{L}\bar{u}\partial_{\bar{x}}\bar{u} + \frac{\alpha\beta}{H}\bar{w}\partial_{\bar{z}}\bar{u}\right) &= \mu\left(\frac{\alpha}{L^2}\partial_{\bar{x}}^2 u + \frac{\alpha}{H^2}\partial_{\bar{z}}^2\bar{u}\right) - \frac{\gamma}{L}\partial_{\bar{x}}\bar{p}, \\ \rho\left(\beta\varepsilon^k\tau_0\partial_{\bar{t}}\bar{w} + \frac{\alpha\beta}{L}\bar{u}\partial_{\bar{x}}\bar{w} + \frac{\beta^2}{H}\bar{w}\partial_{\bar{z}}\bar{w}\right) &= \mu\left(\frac{\beta}{L^2}\partial_{\bar{x}}^2\bar{w} + \frac{\beta}{H^2}\partial_{\bar{z}}^2\bar{w}\right) - \frac{\gamma}{H}\partial_{\bar{z}}\bar{p} - \rho G. \end{cases} \tag{1.16}$$

For equal order of dimensions we claim that $\varepsilon^k \tau_0 = \frac{\alpha}{L} = \frac{\beta}{H}$, which means

$$\alpha = \varepsilon^k \tau_0 L \quad \text{and} \quad \beta = \varepsilon^{k+1} \tau_0 L. \tag{1.17}$$

Observe that (1.17) implies $\frac{\beta}{\alpha} = \varepsilon$. Hence, physically interpreted, the velocity in horizontal direction is high compared to the velocity in vertical direction. Plugging (1.17) into (1.16), we obtain that

$$\begin{cases} \rho \varepsilon^{2k} \tau_0^2 L \left(\partial_{\bar t} \bar u + \bar u \partial_{\bar x} \bar u + \bar w \partial_{\bar z} \bar u \right) & = \mu \dfrac{\varepsilon^{k-2} \tau_0}{L} \left(\varepsilon^2 \partial_{\bar x}^2 u + \partial_{\bar z}^2 \bar u \right) - \dfrac{\gamma}{L} \partial_{\bar x} \bar p, \\ \rho \varepsilon^{2k+1} \tau_0^2 L \left(\partial_{\bar t} \bar w + \bar u \partial_{\bar x} \bar w + \bar w \partial_{\bar z} \bar w \right) & = \mu \dfrac{\varepsilon^{k-1} \tau_0}{L} \left(\varepsilon^2 \partial_{\bar x}^2 \bar w + \partial_{\bar z}^2 \bar w \right) - \dfrac{\gamma}{H} \partial_{\bar z} \bar p - \rho G, \end{cases}$$

which is equivalent to

$$\begin{cases} \varepsilon^{2k} \dfrac{\rho \tau_0 L^2}{\mu} \left(\partial_{\bar t} \bar u + \bar u \partial_{\bar x} \bar u + \bar w \partial_{\bar z} \bar u \right) & = \varepsilon^{k-2} \left(\varepsilon^2 \partial_{\bar x}^2 u + \partial_{\bar z}^2 \bar u \right) - \dfrac{\gamma}{\mu \tau_0} \partial_{\bar x} \bar p, \\ \varepsilon^{2k+2} \dfrac{\rho \tau_0 L^2}{\mu} \left(\partial_{\bar t} \bar w + \bar u \partial_{\bar x} \bar w + \bar w \partial_{\bar z} \bar w \right) & = \varepsilon^{k-1} \left(\varepsilon^2 \partial_{\bar x}^2 \bar w + \partial_{\bar z}^2 \bar w \right) - \dfrac{\gamma}{\varepsilon \mu \tau_0} \partial_{\bar z} \bar p - \dfrac{\rho L}{\mu \tau_0} G. \end{cases} \tag{1.18}$$

In order to keep the pressure and gravitation term in the second equation of (1.18) and again the pressure term in the first equation of (1.18) set

$$\gamma = \varepsilon \mu \tau_0 \quad \text{and} \quad k = 3.$$

Thus, dividing the first equation of (1.18) by ε yields

$$\begin{cases} \varepsilon^2 \operatorname{Re} \left(\partial_{\bar t} \bar u + \bar u \partial_{\bar x} \bar u + \bar w \partial_{\bar z} \bar u \right) & = \left(\varepsilon^2 \partial_{\bar x}^2 u + \partial_{\bar z}^2 \bar u \right) - \partial_{\bar x} \bar p, \\ \varepsilon^5 \operatorname{Re} \left(\partial_{\bar t} \bar w + \bar u \partial_{\bar x} \bar w + \bar w \partial_{\bar z} \bar w \right) & = \varepsilon^2 \left(\varepsilon^2 \partial_{\bar x}^2 \bar w + \partial_{\bar z}^2 \bar w \right) - \partial_{\bar z} \bar p - \dfrac{\rho L}{\mu \tau_0} G, \end{cases} \tag{1.19}$$

where $\operatorname{Re} := \frac{\rho \tau_0 \alpha L}{\mu}$ is the so–called *Reynold's number*, which is the ratio of inertial forces to viscous forces and characterizes whether the flow is *laminar* (small Reynold's number) or *turbulent* (high Reynold's number). Hence, the film being thin enough, we can assume the flow to be laminar. Letting ε tend to zero in (1.19) and using again the subscript i, we obtain

$$\begin{cases} -\partial_{\bar x} \bar p_i + \partial_{\bar z}^2 \bar u_i & = 0, \\ -\partial_{\bar z} \bar p_i - \dfrac{\rho_i L}{\mu_i \tau_0} G & = 0, \end{cases} \qquad \text{in} \quad \Omega_i, \; i = 1, 2. \tag{1.20}$$

The lubrication approximation does not affect neither the continuity equation (1.2) nor the no–slip boundary conditions (1.4), (1.7) or the kinematic boundary condition (1.14). Indeed, rescaling (1.2) we obtain that

$$\frac{\alpha}{L} \partial_{\bar x} \bar u_i + \frac{\beta}{H} \partial_{\bar z} \bar w_i = 0 \qquad \text{in} \quad \Omega_i, \; i = 1, 2.$$

Using (1.17) and dividing by ε^3 leads to

$$\partial_{\bar{x}}\bar{u}_i + \partial_{\bar{z}}\bar{w}_i = 0, \qquad \text{in} \quad \Omega_i,\ i = 1, 2. \tag{1.21}$$

Note that in the rescaled framework the bottom and accordingly the free surfaces are located at $\bar{z} = 0$ and $\bar{h}_i = \frac{h_i}{H}$ for $i = 1, 2$, respectively. Rescaling the no–slip and the kinematic boundary conditions yields

$$\begin{cases} \bar{v}_1 &= 0 \quad \text{on} \quad \bar{z} = 0, \\ \bar{v}_1 &= \bar{v}_2 \quad \text{on} \quad \bar{z} = \bar{h}_1 \end{cases} \tag{1.22}$$

and

$$\partial_{\bar{t}}\bar{h}_i + \bar{u}_i\partial_{\bar{x}}\bar{h}_i = \bar{w}_i \qquad \text{on} \quad \bar{z} = \bar{h}_i,\ i = 1, 2. \tag{1.23}$$

Rescaling the normal and tangential boundary conditions, given in (1.12) and (1.13), we have to determine how to scale the surface tension and the surfactant concentration. In order to understand the choice of scaling for the surfactant concentration, we illustrate roughly a characteristic of surfactant. Surfactant molecules are amphiphilic and can be imagined to have the form of a head (hydrophilic) with a tale (hydrophobic). Getting into contact with the free surface, the surfactant places its hydrophilic parts into the structure of the molecules on the fluid surface and thus reduces the surface tension. Since the surfactant is assumed to be insoluble it does not diffuse into the bulk but stays on the surface pointing its tale (hydrophobic part) outside of the two–phase flow. If the surfactant concentration reaches a certain critical value Γ_m, the so–called *critical micelle concentration*, the surfactant molecules aggregate such that the heads of the molecules surround the tales (like a spherical, called *micelle*). A further increase of surfactant contributes only to the micelles and does not have any significant additional decreasing effect on the surface tension. We refer to [43] for a detailed chemical background on surfactant.

In accordance to [29], we set $\bar{\Gamma} = \frac{\Gamma}{\Gamma_m}$ and

$$\sigma_2 = \mu_2\tau_0 L\left(\sigma_2^c + \varepsilon^2\bar{\sigma}\right)$$

where σ_2^c is the rescaled surface tension coefficient of the interface when $\bar{\Gamma} = \Gamma_m$ and $\bar{\sigma}$ the part of the surface tension coefficient, which depends on the the surfactant concentration. Recall that $\sigma_1 = \mu_1\tau_L\sigma_1^c$ is constant, since the surface tension coefficient of the interface between the fluids is independent of Γ, for the insoluble surfactant is acting on the surface of the upper fluid only. The equation for the normal stress boundary condition on $z = \bar{h}_1$ is given by (1.12)

$$\left[\mu_1 \begin{pmatrix} 2\partial_x u_1 - \frac{p_1}{\mu_1} & \partial_x w_1 + \partial_z u_1 \\ \partial_x w_1 + \partial_z u_1 & 2\partial_x w_1 - \frac{p_1}{\mu_1} \end{pmatrix} - \mu_2 \begin{pmatrix} 2\partial_x u_2 - \frac{p_2}{\mu_2} & \partial_x w_2 + \partial_z u_2 \\ \partial_x w_2 + \partial_z u_2 & 2\partial_x w_2 - \frac{p_2}{\mu_2} \end{pmatrix}\right] \begin{pmatrix} -\partial_x h_1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\partial_x h_1 \\ 1 \end{pmatrix}$$
$$= \frac{\sigma_1 \partial_x^2 h_1}{\sqrt{1 + |\partial_x h_1|^2}}.$$

This is equivalent to

$$\frac{\sigma_1 \partial_x^2 h_1}{\sqrt{1 + |\partial_x h_1|^2}} = 2|\partial_x h_1|^2(\mu_1 \partial_x u_1 - \mu_2 \partial_x u_2) - (1 + |\partial_x h_1|^2)(p_1 - p_2) - 2\partial_x h_1[\mu_1(\partial_x w_1 + \partial_z u_1)$$
$$- \mu_2(\partial_x w_2 + \partial_z u_2)] + 2(\mu_1 \partial_x w_1 - \mu_2 \partial_x w_2).$$

By rescaling the variables, we obtain that

$$\frac{\sigma_1^c \tau_0 \mu_1 \partial_{\bar{x}}^2 \bar{h}_1 \varepsilon}{\sqrt{1 + |\partial_{\bar{x}} \bar{h}_1|^2 \varepsilon^2}} = 2|\partial_{\bar{x}} \bar{h}_1|^2 \varepsilon^5 \tau_0(\mu_1 \partial_{\bar{x}} \bar{u}_1 - \mu_2 \partial_{\bar{x}} \bar{u}_2) - (1 + |\partial_{\bar{x}} \bar{h}_1|^2 \varepsilon^2)\varepsilon \tau_0(\bar{p}_1 \mu_1 - \bar{p}_2 \mu_2)$$
$$- 2\partial_{\bar{x}} \bar{h}_1 \varepsilon^3 \tau_0[\mu_1(\partial_{\bar{x}} \bar{w}_1 \varepsilon^2 + \partial_{\bar{z}} \bar{u}_1) - \mu_2 \varepsilon^2(\partial_{\bar{x}} \bar{w}_2 \varepsilon^2 + \partial_{\bar{z}} \bar{u}_2)] + 2\varepsilon^4 \tau_0(\mu_1 \partial_{\bar{x}} \bar{w}_1 - \mu_2 \partial_{\bar{x}} \bar{w}_2).$$

Dividing the above equation by ε and then letting ε tend to zero, the normal boundary condition at $z = \bar{h}_1$ reads

$$\sigma_1^c \partial_{\bar{x}}^2 \bar{h}_1 = \bar{p}_1 - \frac{\mu_2}{\mu_1} \bar{p}_2. \tag{1.24}$$

At the interface $z = \bar{h}_2$, the normal boundary condition (1.12) yields

$$\mu_2 \begin{pmatrix} 2\partial_x u_2 - \frac{p_2}{\mu_2} & \partial_x w_2 + \partial_z u_2 \\ \partial_x w_2 + \partial_z u_2 & 2\partial_x w_2 - \frac{p_2}{\mu_2} \end{pmatrix} \begin{pmatrix} -\partial_x h_2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\partial_x h_2 \\ 1 \end{pmatrix} = \frac{\sigma_2 \partial_x^2 h_2}{\sqrt{1 + |\partial_x h_2|^2}},$$

thus

$$\frac{\sigma_2 \partial_x^2 h_2}{\sqrt{1 + |\partial_x h_2|^2}} = \mu_2 \left[2\partial_x u_2 |\partial_x h_2|^2 - \left(1 + |\partial_x h_2|^2\right) \frac{p_2}{\mu_2} - 2\left(\partial_x w_2 + \partial_z u_2\right) \partial_x h_2 + 2\partial_x w_2\right]$$

Substituting the rescaled variables implies

$$\frac{(\sigma_2^c + \varepsilon^2 \bar{\sigma}(\Gamma)))\tau_0 \mu_2 \partial_{\bar{x}}^2 \bar{h}_2 \varepsilon}{\sqrt{1 + |\partial_{\bar{x}} \bar{h}_2|^2 \varepsilon^2}} = \mu_2 \tau_0 \left[2\varepsilon^5 \partial_{\bar{x}} \bar{u}_2 |\partial_{\bar{x}} \bar{h}_2|^2 - \left(1 + |\partial_{\bar{x}} \bar{h}_2|^2 \varepsilon^2\right) \varepsilon \frac{\bar{p}_2}{\mu_2}\right]$$
$$- 2\varepsilon^3 \mu_2 \tau_0 \left(\varepsilon^2 \partial_{\bar{x}} \bar{w}_2 + \partial_{\bar{z}} \bar{u}_2\right) \partial_{\bar{x}} \bar{h}_2 + 2\varepsilon^4 \partial_{\bar{x}} \bar{w}_2.$$

Dividing the above equation by ε and then letting ε tend to zero, the normal boundary condition on $z = \bar{h}_2$ is given by

$$\sigma_2^c \partial_{\bar{x}}^2 \bar{h}_2 = \bar{p}_2. \tag{1.25}$$

Considering the tangential boundary condition (1.13), we need to proceed the approximation again on both interfaces separately. Starting with the bottom layer on $z = \bar{h}_1$, we have

$$\left[\mu_1 \begin{pmatrix} 2\partial_x u_1 - \dfrac{p_1}{\mu_1} & \partial_x w_1 + \partial_z u_1 \\ \partial_x w_1 + \partial_z u_1 & 2\partial_x w_1 - \dfrac{p_1}{\mu_1} \end{pmatrix} - \mu_2 \begin{pmatrix} 2\partial_x u_2 - \dfrac{p_2}{\mu_2} & \partial_x w_2 + \partial_z u_2 \\ \partial_x w_2 + \partial_z u_2 & 2\partial_x w_2 - \dfrac{p_2}{\mu_2} \end{pmatrix}\right] \begin{pmatrix} -\partial_x h_1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \partial_x h_1 \end{pmatrix}$$
$$= 0.$$

This is equivalent to

$$2\mu_1 \partial_x h_1 (\partial_z w_1 - \partial_x u_1) - 2\mu_2 \partial_x h_1 (\partial_z w_2 - \partial_x u_2) + \mu_1 (1 - |\partial_x h_1|^2)(\partial_x w_1 + \partial_z u_1)$$
$$- \mu_2 (1 - |\partial_x h_1|^2)(\partial_x w_2 + \partial_z u_2) = 0.$$

Using the rescaled parameters yields

$$2\mu_1 \partial_{\bar{x}} \bar{h}_1 \varepsilon^4 (\varepsilon \partial_{\bar{z}} \bar{w}_1 - \partial_{\bar{x}} \bar{u}_1) - 2\mu_2 \partial_{\bar{x}} \bar{h}_1 \varepsilon^4 (\varepsilon \partial_{\bar{z}} \bar{w}_2 - \partial_{\bar{x}} \bar{u}_2) + \mu_1 (1 - |\partial_{\bar{x}} \bar{h}_1|^2 \varepsilon^2) \varepsilon^2 (\varepsilon^2 \partial_{\bar{x}} \bar{w}_1 + \partial_{\bar{z}} \bar{u}_1)$$
$$- \mu_2 (1 - |\partial_{\bar{x}} \bar{h}_1|^2 \varepsilon^2) \varepsilon^2 (\varepsilon^2 \partial_{\bar{x}} \bar{w}_2 + \partial_{\bar{z}} \bar{u}_2) = 0.$$

Dividing the above equation by ε^2 and then letting ε tend to zero we obtain the tangential boundary condition at $z = \bar{h}_1$

$$\mu_1 \partial_{\bar{z}} \bar{u}_1 - \mu_2 \partial_{\bar{z}} \bar{u}_2 = 0. \tag{1.26}$$

On $z = \bar{h}_2$, the tangential boundary condition (1.13) reads

$$\mu_2 \begin{pmatrix} 2\partial_x u_2 - \dfrac{p_2}{\mu_2} & \partial_x w_2 + \partial_z u_2 \\ \partial_x w_2 + \partial_z u_2 & 2\partial_x w_2 - \dfrac{p_2}{\mu_2} \end{pmatrix} \begin{pmatrix} -\partial_x h_2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \partial_x h_2 \end{pmatrix} = \frac{\partial_x \sigma(\Gamma)}{\sqrt{1 + |\partial_x h_2|^2}},$$

thus

$$2\mu_2 \partial_x h_2 (\partial_z w_2 - \partial_x u_2) + \mu_2 (1 - |\partial_x h_2|^2)(\partial_x w_2 + \partial_z u_2) = \partial_x \sigma(\Gamma) \sqrt{1 + |\partial_x h_2|^2}.$$

Using the dimensionless variables, we get

$$2\partial_{\bar{x}} \bar{h}_2 \varepsilon^4 (\varepsilon \partial_{\bar{z}} \bar{w}_2 - \partial_{\bar{x}} \bar{u}_2) + (1 - |\partial_{\bar{x}} \bar{h}_2|^2 \varepsilon^2) \varepsilon^2 (\varepsilon^2 \partial_{\bar{x}} \bar{w}_2 + \partial_{\bar{z}} \bar{u}_2) = \varepsilon^2 \partial_{\bar{x}} \bar{\sigma}(\Gamma) \sqrt{1 + |\partial_{\bar{x}} \bar{h}_2|^2 \varepsilon^2}.$$

Dividing by ε^2 and letting then ε tend to zero implies

$$\partial_{\bar{z}}\bar{u}_2 = \partial_{\bar{x}}\bar{\sigma}(\Gamma) \qquad \text{on} \quad \bar{z} = \bar{h}_2. \tag{1.27}$$

The remaining equation to consider is the equation for the surfactant spreading (1.3)

$$\partial_t\Gamma + \partial_x(u_2\Gamma - D\partial_x\Gamma) = 0.$$

We rescale the diffusion coefficient $D > 0$ via $D = d\bar{D}$, with $d = \alpha = L\varepsilon^3\tau_0$. Using the scaled variables above, yields again

$$\partial_{\bar{t}}\bar{\Gamma} + \partial_{\bar{x}}(\bar{u}_2\bar{\Gamma} - \bar{D}\partial_{\bar{x}}\bar{\Gamma}) = 0 \qquad \text{on} \quad \bar{z} = \bar{h}_2. \tag{1.28}$$

Gathering (1.20)–(1.28), we obtain the following simplified and dimensionless equations for the motion of the two–phase thin film with insoluble surfactant

$$\text{Navier–Stokes} \quad \begin{cases} \partial_x p_i + \partial_z^2 u_i &= 0 \\ \partial_z p_i - G_i &= 0 \end{cases} \qquad \text{in } \Omega_i \tag{1.29}$$

$$\text{Incompressibility} \quad \partial_x u_i + \partial_z w_i = 0 \qquad \text{in } \Omega_i \tag{1.30}$$

$$\text{Conservation of mass/no–slip} \quad \begin{cases} w_1 = u_1 = 0 & z = 0 \\ w_1 = w_2, \quad u_1 = u_2 & z = h_1 \end{cases} \tag{1.31}$$

$$\text{Kinematic boundary condition} \quad \partial_t h_i + u_i\partial_x h_i = w_i \qquad z = h_i \tag{1.32}$$

$$\text{Normal boundary condition} \quad \begin{cases} -p_1 + \frac{\mu_2}{\mu_1}p_2 &= \sigma_1^c\partial_x^2 h_1 & z = h_1 \\ -p_2 &= \sigma_2^c\partial_x^2 h_2 & z = h_2 \end{cases} \tag{1.33}$$

$$\text{Tangential boundary condition} \quad \begin{cases} \mu_1\partial_z u_1 &= \mu_2\partial_z u_2 & z = h_1 \\ \partial_z u_2 &= \partial_x\sigma_2 & z = h_2 \end{cases} \tag{1.34}$$

$$\text{Surfactant spreading} \quad \partial_t\Gamma + \partial_x(u_2\Gamma - D\partial_x\Gamma) = 0 \qquad z = h_2 \tag{1.35}$$

where

$$G_i := \frac{\rho_i L}{\mu_i\tau_0}G, \qquad \sigma_i^c := \frac{1}{\tau_0 L}\sigma_i^*, \qquad i = 1, 2 \tag{1.36}$$

are a modified gravitational constant depending on the density and viscosity of the fluid and a modified constant surface tension coefficient, respectively. In addition, we suppressed the bars in order to simplify notation.

1.2. Evolution Equations

Similar as in [19], we use (1.29)–(1.35) in order to derive evolution equations for the two film heights f, g and the concentration of surfactant Γ. Set

$$\mu := \frac{\mu_2}{\mu_1}, \tag{1.37}$$

the relative viscosity between the two fluids.

Evolution equation $f = h_1$. Integrating (1.29) with respect to z and using (1.33) we obtain equations for the pressure in the fluids contained in $\Omega_i, i = 1, 2$,

$$p_1(t,x,z) = G_1(f(t,x) - z) = \mu p_2(t,x,f) - \sigma_1^c \partial_x^2 f(t,x), \tag{1.38}$$

$$p_2(t,x,z) = G_2(f(t,x) + g(t,x) - z) = \sigma_2^c \partial_x^2 (f+g)(t,x). \tag{1.39}$$

Plugging equation (1.39) into (1.38), the pressure within the lower fluid is given by

$$p_1(t,x,z) = G_1(f(t,x) - z) = \mu G_2 g(t,x) - \mu\sigma_2^c \partial_x^2 (f+g)(t,x) - \sigma_1^c \partial_x^2 f(t,x).$$

Differentiating with respect to x and using (1.29) implies

$$-\partial_z^2 u_1(t,x,z) = G_1 \partial_x f(t,x) + \mu G_2 \partial_x g(t,x) - \mu\sigma_2^c \partial_x^3 (f+g)(t,x) - \sigma_1^c \partial_x^3 f(t,x),$$

hence, by (1.34),

$$\begin{aligned}\partial_z u_1(t,x,z) = &- \big(G_1 \partial_x f(t,x) + \mu G_2 \partial_x g(t,x) - \mu\sigma_2^c \partial_x^3 (f+g)(t,x) - \sigma_1^c \partial_x^3 f(t,x)\big)\,(f(t,x) - z) \\ &+ \mu \partial_z u_2(t,x,f).\end{aligned}$$

Integrating with respect to z yields, in view of the no–slip boundary condition (1.31),

$$\begin{aligned}u_1(t,x,z) = &- \big(G_1 \partial_x f(t,x) + \mu G_2 \partial_x g(t,x) - \mu\sigma_2^c \partial_x^3 (f+g)(t,x) - \sigma_1^c \partial_x^3 f(t,x)\big) \\ &\times \left(f(t,x)z - \frac{1}{2}z^2\right) + \mu \partial_z u_2(t,x,f) z.\end{aligned} \tag{1.40}$$

Note that

$$\int_0^{f(t,x)} \partial_x u_1(t,x,z)\, dz = -w_1(t,x,z) = -\partial_t f(t,x) - u_1(t,x,f)\partial_x f(t,x),$$

by (1.30)–(1.32). Thus

$$\partial_t f(t,x) + \partial_x \left(\int_0^{f(t,x)} u_1(t,x,z)\, dz \right) = 0,$$

which is equivalent to

$$\begin{aligned}\partial_t f(t,x) - \partial_x \int_0^{f(t,x)} \Big\{ &\big(G_1 \partial_x f(t,x) + \mu G_2 \partial_x g(t,x) - \mu \sigma_2^c \partial_x^3 (f+g)(t,x) - \sigma_1^c \partial_x^3 f(t,x)\big) \\ &\times \left(f(t,x) z - \frac{1}{2} z^2 \right) - \mu \partial_z u_2(t,x,f) z \Big\}\, dz = 0. \end{aligned} \tag{1.41}$$

In order to obtain an evolution equation for f, which depends only on g, Γ and f itself we need to determine an equation for u_2. Recalling (1.39) and using (1.29), (1.34), we get

$$\partial_z u_2(t,x,z) = -\big(G_2 \partial_x (f+g)(t,x) - \sigma_2^c \partial_x^3 (f+g)(t,x)\big)\,(f+g-z)(t,x) + \partial_x \sigma_2(\Gamma(t,x)). \tag{1.42}$$

Hence, (1.41) and (1.42) imply that

$$\begin{aligned}\partial_t f - \partial_x \Bigg[&\left(G_1 \frac{f^3}{3} + G_2 \mu \frac{f^2 g}{2} \right) \partial_x f + G_2 \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2} \right) \partial_x g - \mu \frac{f^2}{2} \partial_x \sigma(\Gamma) \\ &- \left((\sigma_1^c + \sigma_2^c \mu) \frac{f^3}{3} + \sigma_2^c \mu \frac{f^2 g}{2} \right) \partial_x^3 f - \sigma_2^c \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2} \right) \partial_x^3 g \Bigg] = 0, \end{aligned} \tag{1.43}$$

where f, g and Γ depend on $(t,x) \in (0,\infty) \times (0,L)$.

Evolution equation $g = h_2 - h_1$. Owing to (1.31), which states that $u_1 = u_2$ on $z = h_1$, and (1.40), (1.42), we obtain that

$$\begin{aligned} u_2(t,x,z) = &-\Big(G_2 \partial_x (f+g)(t,x) - \sigma_2^c \partial_x^3 (f+g)(t,x)\Big) \Big[(f+g)(t,x) z - \frac{1}{2} z^2 - \frac{1}{2} f^2(t,x) \\ &- fg(t,x) \Big] + \partial_x \sigma_2(\Gamma(t,x))[z - f(t,x)] \\ &- \Big(G_1 \partial_x f(t,x) + G_2 \mu \partial_x g(t,x) - \mu \sigma_2^c \partial_x^3 (f+g)(t,x) - \sigma_1^c \partial_x^3 f(t,x) \Big) \frac{f^2(t,x)}{2} \\ &- \mu \Big(G_2 \partial_x (f+g)(t,x) - \sigma_2^c \partial_x^3 (f+g)(t,x) \Big) (fg)(t,x) + \mu \partial_x \sigma_2(\Gamma(t,x)) f(t,x). \end{aligned} \tag{1.44}$$

Hence, in virtue of (1.31), (1.32), the evolution equation for g is determined by

$$\partial_t g(t,x) + \partial_x \left(\int_{f(t,x)}^{(f+g)(t,x)} u_2(t,x,z)\, dz \right) = 0$$

and it follows from (1.44) that

$$\begin{aligned}\partial_t g - \partial_x \Bigg[&\left(G_2 \frac{g^3}{3} + G_1 \frac{f^2 g}{2} + G_2 \mu f g^2 \right) \partial_x f + \left(G_2 \frac{g^3}{3} + G_2 \mu \left(\frac{f^2 g}{2} + f g^2 \right) \right) \partial_x g \\ &- \left(\mu f g + \frac{g^2}{2} \right) \partial_x \sigma(\Gamma) - \left(\sigma_2^c \frac{g^3}{3} + (\sigma_1^c + \sigma_2^c \mu) \frac{f^2 g}{2} + \sigma_2^c \mu f g^2 \right) \partial_x^3 f \\ &- \left(\sigma_2^c \frac{g^3}{3} + \sigma_2^c \mu \left(\frac{f^2 g}{2} + f g^2 \right) \right) \partial_x^3 g \Bigg] = 0,\end{aligned} \tag{1.45}$$

where f, g and Γ depend on $(t, x) \in (0, \infty) \times (0, L)$.

Evolution equation for Γ. The equation for surfactant spreading on the layer $z = h_2$ is given by the advection–transport equation (1.35)

$$\partial_t \Gamma + \partial_x (u_2 \Gamma - D \partial_x \Gamma) = 0.$$

In view of (1.44) we obtain the following equation for the evolution of Γ:

$$\begin{aligned}\partial_t \Gamma - \partial_x \Bigg[&\left(G_2 \frac{g^2}{2} + G_1 \frac{f^2}{2} + G_2 \mu f g \right) \Gamma \partial_x f + \left(G_2 \frac{g^2}{2} + G_2 \mu \left(\frac{f^2}{2} + f g \right) \right) \Gamma \partial_x g \\ &- (\mu f + g) \Gamma \partial_x \sigma(\Gamma) + D \partial_x \Gamma - \left(\sigma_2^c \frac{g^2}{2} + (\sigma_1^c + \sigma_2^c \mu) \frac{f^2}{2} + \sigma_2^c \mu f g \right) \Gamma \partial_x^3 f \\ &- \left(\sigma_2^c \frac{g^2}{2} + \sigma_2^c \mu \left(\frac{f^2}{2} + f g \right) \right) \Gamma \partial_x^3 g \Bigg] = 0,\end{aligned} \tag{1.46}$$

where f, g and Γ depend on $(t, x) \in (0, \infty) \times (0, L)$.

System of evolution equations. Recalling (1.43), (1.45), (1.46), the evolution of the film heights of the two–phase flow and the surfactant spreading is given by a *strongly coupled*, *degenerate* system of equations of fourth order:

$$\begin{cases} \partial_t f - \partial_x \Big[\left(G_1 \frac{f^3}{3} + G_2 \mu \frac{f^2 g}{2} \right) \partial_x f + G_2 \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2} \right) \partial_x g - \mu \frac{f^2}{2} \partial_x \sigma(\Gamma) \\ \qquad - \left((\sigma_1^c + \sigma_2^c \mu) \frac{f^3}{3} + \sigma_2^c \mu \frac{f^2 g}{2} \right) \partial_x^3 f - \sigma_2^c \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2} \right) \partial_x^3 g \Big] = 0, \\ \partial_t g - \partial_x \Big[\left(G_2 \frac{g^3}{3} + G_1 \frac{f^2 g}{2} + G_2 \mu f g^2 \right) \partial_x f + \left(G_2 \frac{g^3}{3} + G_2 \mu \left(\frac{f^2 g}{2} + f g^2 \right) \right) \partial_x g \\ \qquad - \left(\mu f g + \frac{g^2}{2} \right) \partial_x \sigma(\Gamma) - \left(\sigma_2^c \frac{g^3}{3} + (\sigma_1^c + \sigma_2^c \mu) \frac{f^2 g}{2} + \sigma_2^c \mu f g^2 \right) \partial_x^3 f \\ \qquad - \left(\sigma_2^c \frac{g^3}{3} + \sigma_2^c \mu \left(\frac{f^2 g}{2} + f g^2 \right) \right) \partial_x^3 g \Big] = 0, \\ \partial_t \Gamma - \partial_x \Big[\left(G_2 \frac{g^2}{2} + G_1 \frac{f^2}{2} + G_2 \mu f g \right) \Gamma \partial_x f + \left(G_2 \frac{g^2}{2} + G_2 \mu \left(\frac{f^2}{2} + f g \right) \right) \Gamma \partial_x g \\ \qquad - (\mu f + g) \Gamma \partial_x \sigma(\Gamma) + D \partial_x \Gamma - \left(\sigma_2^c \frac{g^2}{2} + (\sigma_1^c + \sigma_2^c \mu) \frac{f^2}{2} + \sigma_2^c \mu f g \right) \Gamma \partial_x^3 f \\ \qquad - \left(\sigma_2^c \frac{g^2}{2} + \sigma_2^c \mu \left(\frac{f^2}{2} + f g \right) \right) \Gamma \partial_x^3 g \Big] = 0, \end{cases} \tag{1.47a}$$

for $t > 0$ and $x \in (0, L)$ with initial data at $t = 0$

$$f(0, \cdot) = f_0, \quad g(0, \cdot) = g_0, \quad \Gamma(0, \cdot) = \Gamma_0 \tag{1.47b}$$

and boundary conditions

$$\begin{aligned} \partial_x f = \partial_x g = \partial_x \Gamma = 0, \\ \sigma_1^c \partial_x^3 f = \sigma_2^c \partial_x^3 g = 0 \end{aligned} \tag{1.47c}$$

at $x = 0, L$. The degeneracy occurs in the equations for f and g in the sense that if f or g become zero in the first or second equation of (1.47a), respectively, the highest order terms (to be precise both, the fourth and second order terms) vanish. Hence, the system (1.47a) is not uniformly parabolic. It is said to be strongly coupled, since each equation contains highest order derivatives of all three unknowns. Note here, that the highest order of the surfactant concentration Γ is of second order in contrast to the highest orders of the film heights f and g, which occur as fourth–order derivatives. Observe also that due to the special structure of (1.47a), the boundary conditions (1.47c) guarantee that the mass of the each fluid and the mass of surfactant concentration is preserved.

Setting formally $f = 0$, the system (1.47) describes the evolution of a thin film endowed with insoluble surfactant, which has been derived and studied numerically in [29].
Neglecting constant surface tension, but keeping gravitation, that is

$$\sigma_1^c = \sigma_2^c = 0, \qquad G > 0,$$

(1.47) reduces to a strongly coupled, degenerate system of second order

$$\begin{cases} \partial_t f - \partial_x \left[\left(G_1 \dfrac{f^3}{3} + G_2 \mu \dfrac{f^2 g}{2} \right) \partial_x f + G_2 \mu \left(\dfrac{f^3}{3} + \dfrac{f^2 g}{2} \right) \partial_x g - \mu \dfrac{f^2}{2} \partial_x \sigma(\Gamma) \right] = 0, \\ \partial_t g - \partial_x \left[\left(G_2 \dfrac{g^3}{3} + G_1 \dfrac{f^2 g}{2} + G_2 \mu f g^2 \right) \partial_x f + \left(G_2 \dfrac{g^3}{3} + G_2 \mu \left(\dfrac{f^2 g}{2} + f g^2 \right) \right) \partial_x g \right. \\ \qquad \left. - \left(\mu f g + \dfrac{g^2}{2} \right) \partial_x \sigma(\Gamma) \right] = 0, \\ \partial_t \Gamma - \partial_x \left[\left(G_2 \dfrac{g^2}{2} + G_1 \dfrac{f^2}{2} + G_2 \mu f g \right) \Gamma \partial_x f + \left(G_2 \dfrac{g^2}{2} + G_2 \mu \left(\dfrac{f^2}{2} + f g \right) \right) \Gamma \partial_x g \right. \\ \qquad \left. - (\mu f + g) \Gamma \partial_x \sigma(\Gamma) + D \partial_x \Gamma \right] = 0, \end{cases} \tag{1.48a}$$

for $t > 0$ and $x \in (0, L)$ with initial data at $t = 0$

$$f(0, \cdot) = f_0, \quad g(0, \cdot) = g_0, \quad \Gamma(0, \cdot) = \Gamma_0 \tag{1.48b}$$

and Neumann–boundary conditions

$$\partial_x f = \partial_x g = \partial_x \Gamma = 0 \tag{1.48c}$$

at $x = 0, L$. Formally, setting $f = 0$, the system (1.48) describes the evolution of a thin film endowed with insoluble surfactant, driven by gravity only, where capillary effects are neglected. This system has been studied numerically in [27, 29] and analytically in [19, 21, 26], where in [19] the surfactant are considered to be soluble. Local existence of strong solutions and asymptotic stability of steady states, which are, in virtue of an energy functional, determined by constants, are shown. In the case of a thin film with insoluble surfactant, the authors in [21, 26] investigate the existence global weak solutions. In Chapter 2, the system (1.48) is studied and following the ideas in [19, 24] local existence of strong solutions and an asymptotic stability result are shown for the two–phase thin film equation with insoluble surfactant driven by gravity.

Neglecting the gravitational force and keeping instead the capillary effects, which means that we set

$$\sigma_1^c, \sigma_2^c > 0, \qquad G = 0,$$

(1.47) reduces to a system consisting of two strongly coupled, degenerate equations of fourth order, which are additionally strongly coupled to the diffusion equation for the surfactant concentration, where the highest orders appear as second–order derivatives.

$$\begin{cases} \partial_t f + \partial_x \left[\left((\sigma_1^c + \sigma_2^c \mu) \dfrac{f^3}{3} + \sigma_2^c \mu \dfrac{f^2 g}{2} \right) \partial_x^3 f + \sigma_2^c \mu \left(\dfrac{f^3}{3} + \dfrac{f^2 g}{2} \right) \partial_x^3 g + \mu \dfrac{f^2}{2} \partial_x \sigma(\Gamma) \right] = 0 \\ \partial_t g + \partial_x \left[\left(\sigma_2^c \dfrac{g^3}{3} + (\sigma_1^c + \sigma_2^c \mu) \dfrac{f^2 g}{2} + \sigma_2^c \mu f g^2 \right) \partial_x^3 f + \left(\sigma_2^c \dfrac{g^3}{3} + \sigma_2^c \mu \left(\dfrac{f^2 g}{2} + f g^2 \right) \right) \partial_x^3 g \right. \\ \qquad \left. + \left(\mu f g + \dfrac{g^2}{2} \right) \partial_x \sigma(\Gamma) \right] = 0 \\ \partial_t \Gamma + \partial_x \left[\left(\sigma_2^c \dfrac{g^2}{2} + (\sigma_1^c + \sigma_2^c \mu) \dfrac{f^2}{2} + \sigma_2^c \mu f g \right) \Gamma \partial_x^3 f + \left(\sigma_2^c \dfrac{g^2}{2} + \sigma_2^c \mu \left(\dfrac{f^2}{2} + f g \right) \right) \Gamma \partial_x^3 g \right. \\ \qquad \left. + (\mu f + g) \Gamma \partial_x \sigma(\Gamma) - D \partial_x \Gamma \right] = 0, \end{cases} \tag{1.49a}$$

for $t > 0$ and $x \in (0, L)$ with initial data at $t = 0$

$$f(0, \cdot) = f_0, \quad g(0, \cdot) = g_0, \quad \Gamma(0, \cdot) = \Gamma_0 \tag{1.49b}$$

and boundary conditions

$$\begin{aligned} \partial_x f = \partial_x g = \partial_x \Gamma = 0, \\ \partial_x^3 f = \partial_x^3 g = 0 \end{aligned} \tag{1.49c}$$

at $x = 0, L$. A similar system without surfactant has been studied analytically in [22, 24] as a thin–film approximation of the two–phase Stokes problem. In the cases when formally $f = 0$ in (1.49) (see [15, 21, 26]), the existence of global weak solutions to the one–phase thin film model driven by capillary effects and insoluble surfactant is shown. The authors use the method of Galerkin approximation in order to obtain global weak solutions to a family of regularized systems, which tends in the limit to a global weak solution of the thin film flow with insoluble surfactant. By further regularization [21] receives more regularity of the weak solution for the surfactant concentration, which allows to prove non–negativity of the weak solutions not only for the film heights but also for the surfactant concentration. In Chapter

3, we prove in the first two sections a local existence and asymptotic stability result for (1.49). Even though the system is of mixed order, which requires a more involved analysis, we follow the structure and use ideas from [19, 24]. Combining [21, 22, 26] we investigate in the third section of Chapter 3 the existence of non–negative global weak solutions for the fourth–order two–phase thin film model driven by capillary effects and insoluble surfactant (1.49).

2. Second Order Two–Phase Thin Film Model Driven by Gravity with Insoluble Surfactant

In this section we prove a well–posedness and asymptotic stability result for the two–phase thin film equation with insoluble surfactant, where the motion of the fluids is driven by gravity only (constant surface tension as a governing force is neglected). We apply the tool of analytic semigroups (cf. e.g. [2, 3, 33, 41]) in order to prove local well–posedness, and the *principle of linearized stability* to deduce an asymptotic stability result for the steady state solutions, which are, in view of an *energy functional*, determined to consist of constants. The evolution of the thin–film flow is described by (1.48), which is a degenerate, strongly coupled parabolic system of second order. Following the methods used in [19, 24], where local existence and asymptotic stability of strong solutions for systems modeling the evolution of a thin film with soluble surfactant and for systems describing a thin–film approximation of the two–phase Stokes problem, respectively, is shown, we prove analog results for (1.48). We recall the gravity driven two–phase flow with insoluble surfactant (1.48)

$$\begin{cases} \partial_t f - \partial_x \left[\left(G_1 \frac{f^3}{3} + G_2 \mu \frac{f^2 g}{2} \right) \partial_x f + G_2 \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2} \right) \partial_x g - \mu \frac{f^2}{2} \partial_x \sigma(\Gamma) \right] = 0, \\ \partial_t g - \partial_x \left[\left(G_2 \frac{g^3}{3} + G_1 \frac{f^2 g}{2} + G_2 \mu f g^2 \right) \partial_x f + \left(G_2 \frac{g^3}{3} + G_2 \mu \left(\frac{f^2 g}{2} + f g^2 \right) \right) \partial_x g \right. \\ \qquad \left. - \left(\mu f g + \frac{g^2}{2} \right) \partial_x \sigma(\Gamma) \right] = 0, \\ \partial_t \Gamma - \partial_x \left[\left(G_2 \frac{g^2}{2} + G_1 \frac{f^2}{2} + G_2 \mu f g \right) \Gamma \partial_x f + \left(G_2 \frac{g^2}{2} + G_2 \mu \left(\frac{f^2}{2} + f g \right) \right) \Gamma \partial_x g \right. \\ \qquad \left. - \left(\mu f + g \right) \Gamma \partial_x \sigma(\Gamma) + D \partial_x \Gamma \right] = 0 \end{cases} \tag{2.1a}$$

for $t > 0$ and $x \in (0, L)$ with initial data at $t = 0$

$$f(0, \cdot) = f^0, \quad g(0, \cdot) = g^0, \quad \Gamma(0, \cdot) = \Gamma^0 \tag{2.1b}$$

and Neumann–boundary conditions

$$\partial_x f = \partial_x g = \partial_x \Gamma = 0, \tag{2.1c}$$

at $x = 0, L$. The parameters, which appear in (2.1a) are related to material properties of the fluids, cf. (1.36) and (1.37). We impose the following assumptions:

G1) The density of the fluid on the bottom of the two–phase flow is higher than the density of the fluid on top, that is $\rho_1 > \rho_2$.

Assumption G1) in particular ensures, in view of (1.36) and (1.37), that

$$G_1 > G_2 \mu. \tag{2.2}$$

The surface tension, which depends on the surfactant concentration is assumed to be twice continuous differentiable and non–increasing

S1) $\sigma \in C^2(\mathbb{R})$ and $-\sigma'(s) \geq 0$ for all $s \geq 0$.

Moreover, let Φ be a function, such that

A1) $\Phi \in C^2(\mathbb{R})$ with $\Phi(1) = \Phi'(1) = 0$ and

$$\Phi''(s) = -\frac{\sigma'(s)}{s} \geq 0 \qquad \text{for all } s > 0.$$

In order to study the well–posedness of the system of evolution equations (2.1), we need to find suitable spaces for solutions to work with. For the remainder of this section, we define

$$L_2 := L_2(0, L; \mathbb{R}^3),$$
$$H_N^2 := H_N^2(0, L; \mathbb{R}^3) := \{u \in H^2(0, L; \mathbb{R}^3) \mid \partial_x u(0) = \partial_x u(L) = 0\},$$

where $H^2(0, L; \mathbb{R}^3)$ is the Sobolev space consisting of functions $u \in L_2$, whose first and second distributional derivatives belong again to L_2. The variable u is to be seen as the triple $u := (f, g, \Gamma)$. Observe that we already incorporated the Neumann–boundary condition in the space H_N^2. For $\alpha \in [0, 1]$ we define

$$U^\alpha := H_N^{2\alpha}(0, L; \mathbb{R}^3) \cap C([0, L], (0, \infty)^3),$$

where

$$H_N^{2\alpha} := H_N^{2\alpha}(0,L;\mathbb{R}^3) := \begin{cases} \{u \in H^{2\alpha}(0,L;\mathbb{R}^3) \mid \partial_x u = 0 \text{ at } x = 0, L\}, & \text{if} \quad \alpha > \frac{3}{4}, \\ H^{2\alpha}(0,L;\mathbb{R}^3), & \text{if} \quad \alpha \in [0, \frac{3}{4}], \end{cases}$$

with $H^{2\alpha}(0,L;\mathbb{R}^3) := [L_2, H^2]_\alpha$ being the complex interpolation space between H^2 and L_2, called the *Bessel potential space*. Let $\alpha > \frac{3}{4}$, then (cf. [46, Theorem 4.6.1 e)])

$$H_N^{2\alpha} \subset C^1([0,L];\mathbb{R}^3)$$

and $U^\alpha \subset H_N^{2\alpha}$ is an open subset. For $u = (f, g, \Gamma) \in U^\alpha$ we define the diffusion matrix

$$a_G(u) := \begin{pmatrix} G_1\frac{f^3}{3} + G_2\mu\frac{f^2 g}{2} & G_2\mu\left(\frac{f^3}{3} + \frac{f^2 g}{2}\right) & -\mu\frac{f^2}{2}\sigma'(\Gamma) \\ G_2\frac{g^3}{3} + G_1\frac{f^2 g}{2} + G_2\mu f g^2 & G_2\frac{g^3}{3} + G_2\mu\left(\frac{f^2 g}{2} + f g^2\right) & -\left(\mu f g + \frac{g^2}{2}\right)\sigma'(\Gamma) \\ \left(G_2\frac{g^2}{2} + G_1\frac{f^2}{2} + G_2\mu f g\right)\Gamma & \left(G_2\frac{g^2}{2} + G_2\mu\left(\frac{f^2}{2} + f g\right)\right)\Gamma & -\left(\mu f + g\right)\Gamma\sigma'(\Gamma) + D \end{pmatrix} \tag{2.3}$$

and recast the problem (2.1) as an autonomous quasi–linear equation in the space L^2

$$\partial_t u + A_G(u)u = 0, \qquad t > 0, \qquad u(0) = u^0, \tag{2.4}$$

where the operator $A_G : U^\alpha \to \mathcal{L}(H_N^2, L_2)$ is given by

$$A_G(u)w := -\partial_x(a_G(u)\partial_x w), \qquad u \in U^\alpha, w \in H_N^2 \tag{2.5}$$

and $u^0 = (f^0, g^0, \Gamma^0)$.

2.1. Local Well–Posedness

Studying the operator A_G defined in (2.5), we prove that, assuming G1), S1), A1) and $u^0 \in U^\alpha$, there exists a unique, strictly positive solution on some time interval $[0,T)$, where $T \in (0,\infty)$ depends on the initial datum $u^0 \in U^\alpha$. We claim that for fixed $u \in U^\alpha$, the linear operator $A_G(u) \in \mathcal{L}(H_N^2, L_2)$ is the negative generator of an analytic semigroup. Observe that the principal symbol of the linear operator $A_G(u)$, $u \in U^\alpha$, defined in (2.5) is given by the matrix $a_G(u)$, which has positive eigenvalues in virtue of G1) and S1). Indeed, the eigenvalues of $a_G(u)$ are the roots of

$$\det(a_G(u) - \lambda\,\mathrm{Id}) = \lambda^3 - A\lambda^2 + B\lambda - C \tag{2.6}$$

with

$$\begin{aligned}
A &= -(\mu f + g)\Gamma\sigma'(\Gamma) + D + G_1\tfrac{f^3}{3} + G_2\tfrac{g^3}{3} + G_2\mu\left(f^2g + fg^2\right)\\
B &= \left(G_2\tfrac{f^3g^3}{9} + \tfrac{f^4g^2}{12}G_2\mu\right)[G_1 - G_2\mu] + D\left(G_1\tfrac{f^3}{3} + (G_1 + G_2\mu)\tfrac{f^2g}{2} + G_2\mu\tfrac{fg^2}{2} + G_2\tfrac{g^3}{3}\right)\\
&\quad -\sigma'(\Gamma)\Gamma\left(G_1\mu\tfrac{f^4}{12} + [G_1 - G_2\mu]\mu\tfrac{f^3g}{2} + G_1\tfrac{f^3g}{3} + G_1\tfrac{f^2g^2}{4} + [G_1 - G_2\mu]\tfrac{f^2g^2}{4} + 2G_2\mu\tfrac{fg^3}{2}\right)\\
C &= \left(D\left(G_2\tfrac{f^3g^3}{9} + G_2\mu\tfrac{f^4g^2}{12}\right) - \sigma'(\Gamma)\Gamma\left(G_2\tfrac{f^3g^4}{36} + G_2\mu\tfrac{f^4g^3}{36}\right)\right)[G_1 - G_2\mu].
\end{aligned} \tag{2.7}$$

We refer to Lemma A.1 for a more detailed derivation of (2.7). We want to apply the Hurwitz Lemma, which states that the roots of the cubic polynomial (2.6) are strictly positive if $A, B, C > 0$ and $AB - C > 0$. Since $u \in U^\alpha$ and therefore point–wise positive, we deduce due to Assumption G1) and S1) that $A, B, C > 0$. Observe that

$$AB > (-\sigma'(\Gamma)\Gamma(\mu f + g) + D)\left(G_2\frac{f^3g^3}{9} + \frac{f^4g^2}{12}G_2\mu\right)[G_1 - G_2\mu].$$

Hence

$$\begin{aligned}
AB - C > &\left((-\sigma'(\Gamma)\Gamma(\mu f + g) + D)\left(G_2\frac{f^3g^3}{9} + G_2\mu\frac{f^4g^2}{12}\right)\right.\\
&\left. - D\left(\frac{f^3g^3}{9}G_2 + G_2\mu\frac{f^4g^2}{12}\right) + \sigma'(\Gamma)\Gamma\left(G_2\frac{f^3g^4}{36} + G_2\mu\frac{f^4g^3}{36}\right)\right)[G_1 - G_2\mu]\\
&> -\sigma'(\Gamma)\Gamma\mu f\left(G_2\frac{f^3g^4}{36} + G_2\mu\frac{f^4g^3}{36}\right)[G_1 - G_2\mu] > 0
\end{aligned}$$

and the Hurwitz Lemma implies the strict positivity of all eigenvalues of $a_G(u)$. It follows from [2, Ex. 4.3.e)] that $(A_G(u), B)$ is *normally elliptic*, where $Bw = \partial_x w$ at $x = 0, L$ for $w \in U^\alpha$. Taking into account that the coefficients of the matrix $a_G(u)$ are continuously differentiable and A_G depends smoothly on its coefficients, [2, Theorem 4.1] implies that $-A_G(u) \in \mathcal{H}(H^2_N, L_2)$ and

$$-A_G \in C^{1-}(U^\alpha, \mathcal{H}(H^2_N, L_2)). \tag{2.8}$$

With this, [2, Theorem 12.1] guarantees the following well–posedness result for (2.1):

Theorem 2.1 (Local Existence)**.** *Let $\alpha \in (\frac{3}{4}, 1]$ and $u^0 = (f^0, g^0, \Gamma^0) \in U^\alpha$. Assuming G1) and S1), the problem* (2.4) *admits a unique positive strong solution*

$$u = (f, g, \Gamma) \in C([0,T), U^\alpha) \cap C^\alpha([0,T), L_2) \cap C^1((0,T), L_2) \cap C((0,T), H^2_N)$$

with maximal time of existence $T \in (0, \infty]$. Moreover, u depends in U^α continuously on its initial datum u^0.

Remark that Assumption G1) is crucial in order to obtain the well–posedness result. Hence, studying local strong solutions of (2.1), we need to exclude the case when $\rho_1 = \rho_2$, that is, when both fluids have the same density but may differ in their viscous behavior. If $\rho_1 = \rho_2$, then $G_1 = G_2\mu$ (cf. (1.36) and (1.37)) and it is easy to see that the matrix $a_G(u)$ has an eigenvalue $\lambda = 0$ (note that then $C = 0$ in (2.7)). In this case we can no longer apply the theory in [2].

2.2. Asymptotic Stability

We show that the only steady states of (2.1) are of the form where the films are flat and the surfactant concentration is uniquely disturbed. Under the assumption that the surface tension is strictly decreasing, we obtain that that the steady states are asymptotically stable. Similar as in [21, 24, 23], we prove the existence of an energy functional, which provides together with Assumption G1) that the set of steady states is determined by constants if the surface tension strictly decreasing. Moreover, we show that if $u_* > 0$ is a steady state, then it is asymptotically stable. Considering the system (2.1), it is clear that $u_* = (f_*, g_*, \Gamma_*)$, where f_*, g_* and Γ_* are positive constants, is an equilibrium. In order to determine all steady state solutions of (2.1), we show that $\mathcal{E} : U^\alpha \to [0, \infty)$, defined by

$$\mathcal{E}(u) := \int_0^L \left\{ \frac{1}{2} \left(\frac{G_1 - G_2\mu}{G_2\mu} f^2 + (f+g)^2 \right) + \frac{1}{G_2} \Phi(\Gamma) \right\} dx,$$

where $u = (f, g, \Gamma) \in U^\alpha$ and the function Φ being such that (cf. Assumption A1))

$$\Phi''(s)s = -\sigma'(s) \geq 0, \qquad \text{for} \quad s > 0,$$

is an energy functional for (2.1), that is, $\mathcal{E}$ decreases along solutions u given by Theorem 2.1. Observe that, physically interpreted, the terms $\int_0^L \frac{1}{2} \left(\frac{G_1 - G_2\mu}{G_2\mu} f^2 + (f+g)^2 \right) dx$ and $\int_0^L \frac{1}{G_2} \Phi(\Gamma)\, dx$ represent the kinetic and the free surface energy, respectively.

Proposition 2.2 (Energy Functional). *Let be $\alpha \in (\frac{3}{4}, 1]$ and $u^0 = (f^0, g^0, \Gamma^0) \in U^\alpha$. Then, under the assumption of G1) and S1), the corresponding solution $u = (f, g, \Gamma)$ to* (2.1) *given* *by Theorem 2.1 satisfies*

$$\frac{d}{dt}\int_0^L \left\{\frac{1}{2}\left(\frac{G_1-G_2\mu}{G_2\mu}f^2+(f+g)^2\right)+\frac{1}{G_2}\Phi(\Gamma)\right\}dx$$
$$=-\int_0^L \left\{\left[\frac{f^{\frac{3}{2}}\partial_x(G_1f+G_2\mu g)}{\sqrt{3G_2\mu}}+\frac{\sqrt{3}}{2}\left(\sqrt{G_2\mu f}g\partial_x(f+g)-\frac{\sqrt{\mu f}\partial_x\sigma(\Gamma)}{\sqrt{G_2}}\right)\right]^2\right.$$
$$+\frac{1}{4}\left[\sqrt{G_2\mu f}g\partial_x(f+g)-\frac{\sqrt{\mu f}\partial_x\sigma(\Gamma)}{\sqrt{G_2}}\right]^2+\left[\frac{\sqrt{G_2}g^{\frac{3}{2}}}{\sqrt{3}}\partial_x(f+g)-\frac{\sqrt{3g}}{2\sqrt{G_2}}\partial_x\sigma(\Gamma)\right]^2$$
$$\left.+\frac{g}{4G_2}|\partial_x\sigma(\Gamma)|^2+\frac{D}{G_2}\Phi''|\partial_x\Gamma|^2\right\}dx \tag{2.9}$$

for $t\in(0,T)$.

Proof. Since $u=(f,g,\Gamma)$ satisfies (2.1), we use integration by parts, where the boundary terms vanish due to the Neumann–boundary conditions and the special structure of (2.1), and obtain that

$$\frac{d}{dt}\int_0^L \left\{\frac{1}{2}\left(\frac{G_1-G_2\mu}{G_2\mu}f^2+(f+g)^2\right)+\frac{\Phi(\Gamma)}{G_2}\right\}dx$$
$$=\int_0^L \left\{\frac{G_1-G_2\mu}{G_2\mu}f\partial_tf+(f+g)\partial_t(f+g)+\frac{\Phi'(\Gamma)}{G_2}\partial_t\Gamma\right\}dx$$
$$=-\int_0^L \left\{\frac{G_1-G_2\mu}{G_2\mu}\partial_xf\left[\left(G_1\frac{f^3}{3}+G_2\mu\frac{f^2g}{2}\right)\partial_xf+G_2\mu\left(\frac{f^3}{3}+\frac{f^2g}{2}\right)\partial_xg-\mu\frac{f^2}{2}\partial_x\sigma(\Gamma)\right]\right\}dx$$
$$-\int_0^L \left\{\partial_x(f+g)\left[\left(G_1\frac{f^3}{3}+G_2\mu\frac{f^2g}{2}\right)\partial_xf+G_2\mu\left(\frac{f^3}{3}+\frac{f^2g}{2}\right)\partial_xg-\mu\frac{f^2}{2}\partial_x\sigma(\Gamma)\right.\right.$$
$$+\left(G_2\frac{g^3}{3}+G_1\frac{f^2g}{2}+G_2\mu fg^2\right)\partial_xf+\left(G_2\frac{g^3}{3}+G_2\mu\left(\frac{f^2g}{2}+fg^2\right)\right)\partial_xg$$
$$\left.\left.-\left(\mu fg+\frac{g^2}{2}\right)\partial_x\sigma(\Gamma)\right]\right\}dx$$
$$-\int_0^L \left\{\frac{\Phi''(\Gamma)}{G_2}\partial_x\Gamma\Big[\left(G_2\frac{g^2}{2}+G_1\frac{f^2}{2}+G_2\mu fg\right)\Gamma\partial_xf+\left(G_2\frac{g^2}{2}+G_2\mu\left(\frac{f^2}{2}+fg\right)\right)\Gamma\partial_xg\right.$$
$$\left.-(\mu f+g)\Gamma\partial_x\sigma(\Gamma)+D\partial_x\Gamma\Big]\right\}dx$$
$$=-\int_0^L \left\{\frac{1}{3}\frac{G_1-G_2\mu}{G_2\mu}\partial_xf\partial_x(G_1f+G_2\mu g)f^3+\frac{1}{2}(G_1-G_2\mu)\partial_xf\partial_x(f+g)f^2g\right.$$
$$\left.-\frac{1}{2}\frac{G_1-G_2\mu}{G_2}\partial_xf\partial_x\sigma(\Gamma)f^2\right\}dx$$

$$
\begin{aligned}
&-\int_0^L \Big\{\frac{1}{3}\partial_x(f+g)\partial_x(G_1f+G_2\mu g)f^3+\frac{1}{2}G_2\mu|\partial_x(f+g)|^2f^2g-\frac{1}{2}\mu\partial_x(f+g)\partial_x\sigma(\Gamma)f^2\\
&\quad+\frac{G_2}{3}|\partial_x(f+g)|^2g^3+\frac{1}{2}\partial_x(f+g)\partial_x(G_1f+G_2\mu g)f^2g+G_2\mu|\partial_x(f+g)|^2fg^2\\
&\quad-\frac{1}{2}\partial_x(f+g)\partial_x\sigma(\Gamma)g^2-\mu\partial_x(f+g)\partial_x\sigma(\Gamma)fg\Big\}\,dx\\
&-\int_0^L \Big\{-\left(\frac{g^2}{2}+\frac{G_1}{G_2}\frac{f^2}{2}+\mu fg\right)\partial_x(\Gamma_\varepsilon)\partial_xf-\left(\frac{g^2}{2}+\mu\left(\frac{f^2}{2}+fg\right)\right)\partial_x(\Gamma_\varepsilon)\partial_xg\\
&\qquad+\frac{1}{G_2}\left(\mu f+g\right)|\partial_x\sigma(\Gamma)|^2+\frac{D}{G_2}\Phi''(\Gamma)|\partial_x\Gamma|^2\Big\}\,dx.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\frac{1}{3}\frac{G_1-G_2\mu}{G_2\mu}&\partial_xf\partial_x(G_1f+G_2\mu g)f^3+\frac{1}{3}\partial_x(f+g)\partial_x(G_1f+G_2\mu g)f^3\\
&=\frac{f^3}{3G_2\mu}|\partial_x(G_1f+G_2\mu g)|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}(G_1-G_2\mu)\partial_xf\partial_x(f+g)f^2g+\frac{1}{2}G_2\mu|\partial_x(f+g)|^2f^2g+\frac{1}{2}\partial_x(f+g)\partial_x(G_1f+G_2\mu g)f^2g\\
=f^2g\partial_x(f+g)\partial_x(G_1f+G_2\mu g).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{d}{dt}\int_0^L\left\{\frac{1}{2}\left(\frac{G_1-G_2\mu}{G_2\mu}f^2+(f+g)^2\right)+\frac{1}{G_2}\Phi(\Gamma)\right\}\,dx\\
&\quad=-\int_0^L\Big\{\frac{f^3}{3G_2\mu}|\partial_x(G_1f+G_2\mu g)|^2+f^2g\partial_x(f+g)\partial_x(G_1+G_2\mu)+G_2\mu fg^2|\partial_x(f+g)|^2\\
&\qquad+\frac{G_2}{3}|\partial_x(f+g)|^2g^3-\frac{f^2}{G_2}\partial_x\sigma(\Gamma)\partial_x(G_1f+G_2\mu g)-g^2\partial_x\sigma(\Gamma)\partial_x(f+g)\\
&\qquad-2\mu fg\partial_x\sigma(\Gamma)\partial_x(f+g)+\left(\frac{g}{G_2}+\frac{\mu f}{G_2}\right)|\partial_x\sigma(\Gamma)|^2+\frac{D}{G_2}\Phi''|\partial_x\Gamma|^2\Big\}\,dx\\
&\quad=-\int_0^L\Bigg\{\frac{f^3}{3G_2\mu}|\partial_x(G_1f+G_2\mu g)|^2+\left[\sqrt{G_2\mu f}g\partial_x(f+g)-\frac{\sqrt{\mu f}}{\sqrt{G_2}}\partial_x\sigma(\Gamma)\right]^2\\
&\qquad+\frac{f^{\frac{3}{2}}}{\sqrt{G_2\mu}}\partial_x(G_1f+G_2\mu g)\left(\sqrt{G_2\mu f}g\partial_x(f+g)-\frac{\sqrt{\mu f}}{\sqrt{G_2}}\partial_x\sigma(\Gamma)\right)+\frac{g}{4G_2}|\partial_x\sigma(\Gamma)|^2\\
&\qquad+\left[\frac{\sqrt{G_2}g^{\frac{3}{2}}}{\sqrt{3}}\partial_x(f+g)-\frac{\sqrt{3g}}{2\sqrt{G_2}}\partial_x\sigma(\Gamma)\right]^2+\frac{D}{G_2}\Phi''(\Gamma)|\partial_x\Gamma|^2\Bigg\}\,dx.
\end{aligned}
$$

Finally, we note that the first three terms in the integral above can be written as

$$\left[\frac{f^{\frac{3}{2}}\partial_x(G_1 f + G_2\mu g)}{\sqrt{3G_2\mu}} + \frac{\sqrt{3}}{2}\left(\sqrt{G_2\mu f}g\partial_x(f+g) - \frac{\sqrt{\mu f}\partial_x\sigma(\Gamma)}{\sqrt{G_2}}\right)\right]^2$$
$$+\frac{1}{4}\left[\sqrt{G_2\mu f}g\partial_x(f+g) - \frac{\sqrt{\mu f}\partial_x\sigma(\Gamma)}{\sqrt{G_2}}\right]^2,$$

which yields the assertion. □

Note that all terms on the right–hand side of the energy equality (2.9) are non–positive. Hence, if $u = (f, g, \Gamma)$ is an equilibrium to (2.1), every single term on the right–hand side has to vanish, which implies that $\partial_x\sigma(\Gamma) = \partial_x(f+g) = \partial_x(G_1 f + G_2\mu g) = 0$. If σ is strictly decreasing, we deduce that f, g and Γ are constant, in view of Assumption G1).

Corollary 2.3. *Suppose that $\sigma \in C^2(\mathbb{R})$ is strictly decreasing and Assumption G1) is satisfied. Then, the only positive steady states to* (2.1) *are of the form* $(f_*, g_*, \Gamma_*) \in U^\alpha$ *with constants* $f_*, g_*, \Gamma_* > 0$.

In order to study the stability properties of these equilibria, we observe first, by a simple computation, that the mass of each fluid and the mass of surfactant concentration is preserved by the evolution of the system, which is due to the Neumann–boundary conditions.

Lemma 2.4 (Conservation of mass)**.** *Let* $u = (f, g, \Gamma)$ *be a solution to* (2.1) *as found in Theorem 2.1. Then, the mass of u is preserved with time, that is,*

$$\frac{d}{dt}\int_0^L f(t,x)\,dx = 0 \qquad \textit{and} \qquad \frac{d}{dt}\int_0^L g(t,x)\,dx = 0 \qquad \textit{and} \qquad \frac{d}{dt}\int_0^L \Gamma(t,x)\,dx = 0$$

on $(0, T)$.

The remainder of this section is dedicated to prove that, assuming the averaged initial surfactant concentration to be small, there exists for every initial data being close enough to the steady state a global positive strong solution to (2.1) tending exponentially to the constant steady state.

Set $u_* = (f_*, g_*, \Gamma_*)$ with f_*, g_*, Γ_* being positive constants and denote by

$$\langle h\rangle := \frac{1}{L}\int_0^L h(x)\,dx$$

the average (with respect to space) of a function h. Let $u = (f, g, \Gamma)$ be the unique strong solution to (2.1) corresponding to the initial data $u^0 = (f^0, g^0, \Gamma^0) \in U^\alpha$, satisfying $\langle f^0 \rangle = f_*, \langle g^0 \rangle = g_*$ and $\langle \Gamma^0 \rangle = \Gamma_*$. By conservation of mass (cf. Lemma 2.4), it is clear that then

$$\langle u(t) \rangle = \langle u_* \rangle \qquad \text{for all} \quad t \in [0, T).$$

In order to study the stability property of the equilibrium u_*, we follow the ideas used in [19, 24] and eliminate the non–zero constant functions from the space we work in by introducing the projection $P \in \mathcal{L}(L_2) \cap \mathcal{L}(H^2_N)$, defined by

$$Pu := u - \langle u \rangle = \left(f - \frac{1}{L} \int_0^L f(x)\,dx, g - \frac{1}{L} \int_0^L g(x)\,dx, \Gamma - \frac{1}{L} \int_0^L \Gamma(x)\,dx \right).$$

Clearly, P defines a projection as

$$P^2 u = PPu = P(u - \langle u \rangle) = Pu.$$

By means of the continuous projection we can decompose the spaces

$$L_2 = PL_2 \oplus (1 - P)L_2,$$

$$H^2_N = PH^2_N \oplus (1 - P)H^2_N$$

into direct sums (cf. [44, Theorem 5.16]), where PL_2, PH^2_N contain the non–constant functions and the zero function in L_2, H^2_N and $(1 - P)L_2, (1 - P)H^2_N$ contain the constant functions in L_2, H^2_N, respectively. Due to mass conservation and continuity in $t = 0$, a solution u of (2.1), which satisfies initially $(1 - P)u(0) = u_*$ fulfills $(1 - P)u(t) = u_*$ as long as the solution exists. Hence, we can decompose the solution u with respect to the orthogonal sums:

$$u(t) = z(t) + u_* \in PL_2 \oplus (1 - P)L_2, \qquad t \geq 0,$$

with $z(t) = Pu(t)$. By u being the corresponding solution to the initial data $u^0 \in U^\alpha$, the function $z = u - u_*$ is a solution of

$$\partial_t z + A_G(z + u_*)z = 0, \qquad z(0) = u^0 - u_*.$$

Hence, the stability property for u_* is equivalent to the one for the stationary solution $z = 0$ of

$$\partial_t z + A^*_G z = (A^*_G - A_G(z + u_*)\big|_{PH^2_N})z =: F(z), \tag{2.10}$$

with $A_G^* w := A_G(u_*)w$ for $w \in PH_N^2$. Due to the Neumann–boundary conditions, both operators, A_G^* and $[z \to A_G(z+u_*)z]$, map PH_N^2 into PL_2. Indeed, if $z \in PH_2$, then

$$(1-P)A_G^* z = \langle A_G^* z\rangle = -\frac{1}{L}\int_0^L \partial_x(a_G(u_*)\partial_x z)\,dx = 0$$

and

$$(1-P)A_G(z+u_*)z = \langle A_G(z+u_*)\rangle = -\frac{1}{L}\int_0^L \partial_x(a(z+u_*)\partial_x z)\,dx = 0.$$

Note that, in view of PH_N^2 being continuously embedded into PL_2, the set PH_N^2 is an open neighborhood of zero in PL_2. Furthermore,

$$F \in C^1(PH_N^2, PL_2) \quad \text{with} \quad F(0) = F'(0) = 0, \tag{2.11}$$

where F' denotes the Fréchet derivative of F.

Lemma 2.5. *The operator $A_G^* : PH_N^2 \subset PL_2 \to PL_2$ belongs to $\mathcal{H}(PH_N^2, PL_2)$, that is, $-A_G^*$ is the generator of an analytic semigroup on PL_2.*

Proof. We already know from (2.8) that $-A_G \in C^{1-}(U^\alpha, \mathcal{H}(H_N^2, L_2))$, hence $-A_G(u_*) \in \mathcal{H}(H_N^2, L_2)$. By means of the orthogonal projection P we can represent $-A_G(u_*)$ as a matrix operator

$$-A_G(u_*) = \begin{pmatrix} -A_G(u_*)\big|_{PH_N^2} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{H}(PH_N^2 \oplus (1-P)H_N^2, PL_2 \oplus (1-P)L_2).$$

Because $A_G(u_*)(1-P)w = 0$, the second column of the matrix has zero entries. Moreover $(1-P)A_G(u_*)w = \langle A_G(u_*)w\rangle = -\frac{1}{L}\int_0^L \partial_x(a_G(u_*)\partial_x w) = 0$ for $w \in H_N^2$, which justifies the zero in the first entry of the second row. It follows from [3, Theorem I.1.6.3] that

$$-A_G(u_*)\big|_{PH_N^2} \in \mathcal{H}(PH_N^2, PL_2).$$

□

In order to prove asymptotic stability for the equilibrium $z = 0$ of (2.10), we apply the *principle of linearized stability* (cf. [33, 9.1.1]). For this purpose we state the following lemma:

Lemma 2.6. *Suppose $\sigma \in C^2(\mathbb{R})$ is strictly decreasing and Assumption G1) is satisfied. Then there are numbers $\varepsilon, \omega_0 > 0$ such that the spectrum $\operatorname{spec}(-A_G^*)$ of $-A_G^*$ is contained in the half plane $[\operatorname{Re} z \le -\omega_0]$ provided that $0 \le \Gamma_* < \varepsilon$.*

Proof. Take $w^0 = (f^0, g^0, \Gamma^0) \in PL_2$ arbitrary and let $w(t) := e^{-tA_G^*}w^0$, $t \geq 0$, be the unique strong solution in PL_2 to the linearized problem

$$\partial_t w + A_G^* w = 0, \qquad t > 0, \qquad w(0) = w^0. \tag{2.12}$$

By definition of $A_G^* = A_G(u_*)$, the function $w = (f, g, \Gamma) \in PH_N^2$ satisfies

$$\partial_t \begin{pmatrix} \frac{G_2\mu}{G_1 - G_2\mu}(f+g) \\ f \\ z\Gamma \end{pmatrix} - \partial_x \left(\tilde{a}_G^z(u_*)\partial_x \begin{pmatrix} f+g \\ f \\ \Gamma \end{pmatrix} \right) = 0,$$

where $z > 0$ is a constant and the matrix $\tilde{a}_G^z(u_*)$ is given by

$$\begin{pmatrix} d_1 & G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & -\frac{G_2\mu}{G_1 - G_2\mu}\left(\mu\frac{f_*^2}{2} + \mu f_* g_* + \frac{g_*^2}{2}\right)\sigma'(\Gamma_*) \\ G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & (G_1 - G_2\mu)\frac{f_*^3}{3} & -\mu\frac{f_*^2}{2}\sigma'(\Gamma_*) \\ z\left(G_2\frac{g_*^2}{2} + G_2\mu\left(\frac{f_*^2}{2} + f_* g_*\right)\right)\Gamma_* & z(G_1 - G_2\mu)\frac{f_*^2}{2}\Gamma_* & d_3 \end{pmatrix},$$

where

$$d_1 := \frac{G_2\mu}{G_1 - G_2\mu}\left(G_2\frac{g_*^3}{3} + G_2\mu\left(\frac{f_*^3}{3} + f_*^2 g_* + f_* g_*^2\right)\right),$$

$$d_3 := -z(\mu f_* + g_*)\Gamma_*\sigma'(\Gamma_*) + zD.$$

Introducing the to $\tilde{a}_G^z(u_*)$ corresponding symmetric matrix

$$b_G^z(u_*) := \begin{pmatrix} d_1 & G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & j \\ G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & (G_1 - G_2\mu)\frac{f_*^3}{3} & k \\ j & k & d_3 \end{pmatrix}. \tag{2.13}$$

with

$$j := -\frac{1}{2}\left(\frac{G_2\mu}{G_1 - G_2\mu}\left(\mu\frac{f_*^2}{2} + \mu f_* g_* + \frac{g_*^2}{2}\right)\sigma'(\Gamma_*) - z\left(G_2\frac{g_*^2}{2} + G_2\mu\left(\frac{f_*^2}{2} + f_* g_*\right)\right)\Gamma_*\right),$$

$$k := -\frac{1}{2}\left(\mu\frac{f_*^2}{2}\sigma'(\Gamma_*) - z(G_1 - G_2\mu)\frac{f_*^2}{2}\Gamma_*\right),$$

we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\frac{G_2\mu}{G_1 - G_2\mu}\|f+g\|_2^2 + \|f\|_2^2 + z\|\Gamma\|_2^2\right) + \left(b_G^z(u_*)\partial_x \begin{pmatrix} f+g \\ f \\ \Gamma \end{pmatrix} \middle| \partial_x \begin{pmatrix} f+g \\ f \\ \Gamma \end{pmatrix} \right)_2 = 0.$$

If $\Gamma_* = 0$, the matrix $b_G^z(f_*, g_*, 0)$ is positive definite for some sufficiently large constant $z > 0$, since all principal minors are positive. For a detailed computation we refer to Lemma A.2. Hence, there exists $\varepsilon = \varepsilon(f_*, g_*) > 0$ such that for $0 \le \Gamma_* < \varepsilon$ the matrix $b_G^z(f_*, g_*, \Gamma_*)$ is positive definite and we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\frac{G_2\mu}{G_1 - G_2\mu}\|f+g\|_2^2 + \|f\|_2^2 + z\|\Gamma\|_2^2\right) \le -\eta \left\| \partial_x \begin{pmatrix} f+g \\ f \\ \Gamma \end{pmatrix} \right\|_2^2$$

for some positive constant $\eta > 0$. Recall that the average value of $\tilde{w} := (f+g, f, \Gamma)$ where $(f, g, \Gamma) \in PH_N^2$ is given by $\langle \tilde{w} \rangle = 0$. Hence, there exists, by Poincaré's inequality, a constant $c > 0$ such that $\|\tilde{w}\|_2^2 \le c^{-1}\|\partial_x \tilde{w}\|_2^2$ and it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\frac{G_2\mu}{G_1 - G_2\mu}\|f+g\|_2^2 + \|f\|_2^2 + z\|\Gamma\|_2^2\right) \le -\eta c(\|f+g\|_2^2 + \|f\|_2^2 + \|\Gamma\|_2^2).$$

Set $m := \max\left\{\frac{G_2\mu}{G_1 - G_2\mu}, z, 1\right\}$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{G_2\mu}{G_1 - G_2\mu}\|f+g\|_2^2 + \|f\|_2^2 + z\|\Gamma\|_2^2\right) &\le -\tfrac{\eta c}{m}(m\|f+g\|_2^2 + m\|f\|_2^2 + m\|\Gamma\|_2^2) \\ &\le -\tfrac{\eta c}{m}\left(\frac{G_2\mu}{G_1 - G_2\mu}\|f+g\|_2^2 + \|f\|_2^2 + z\|\Gamma\|_2^2\right).\end{aligned} \tag{2.14}$$

We will show that $\tilde{w} = (f+g, f, \Gamma)$, where (f, g, Γ) is a solution to (2.12), has exponential decay, which implies that also $w = (f, g, \Gamma)$ is exponentially decreasing. Observe that for $\tilde{w} = (f+g, f, \Gamma) \in PL_2$

$$|||\tilde{w}|||_2 := \left(\frac{G_2\mu}{G_1 - G_2\mu}\|f+g\|_2^2 + \|f\|_2^2 + z\|\Gamma\|_2^2\right)^{\frac{1}{2}}$$

defines an equivalent norm on PL_2. In virtue of (2.14), we deduce that $\frac{d}{dt}|||\tilde{w}|||_2^2 \le -C|||\tilde{w}|||_2^2$ with $C := 2\eta\frac{c}{m} > 0$. Hence,

$$|||\tilde{w}|||_2 \le e^{-t\frac{C}{2}}|||\tilde{w}^0|||_2, \qquad t \ge 0, \qquad \tilde{w}^0 = (f^0 + g^0, f^0, \Gamma^0). \tag{2.15}$$

By equivalence of the norms $|||\cdot|||_2$ and $\|\cdot\|_2$ and (2.15), we obtain that $\|\tilde{w}\|_2 \le \tilde{c}e^{-t\frac{C}{2}}\|\tilde{w}^0\|_2$ for some constant $\tilde{c} > 0$,

which means that $\tilde{w}$ has exponential decay. Therefore, also w has exponential decay and

$$\|w(t)\|_2 = \|e^{-tA_*}w^0\|_2 \le Me^{-t\omega_0}\|w^0\|_2$$

for some $M \ge 1$ and $\omega_0 > 0$. We deduce that $\operatorname{spec}(-A_G^*) \subset [\operatorname{Re} z \le -\omega_0]$. □

Combining Lemma 2.5, Lemma 2.6 and (2.11), we can apply [33, Theorem 9.1.2] and arrive at the following asymptotic stability result for steady states of (2.1):

Theorem 2.7 (Asymptotic Stability). *Let $\sigma \in C^2(\mathbb{R})$ be strictly decreasing and Assumption G1) be satisfied. Further let $f_*, g_* > 0$ be arbitrary. Then there exist numbers $\varepsilon = \varepsilon(f_*, g_*) > 0$, $\omega > 0$ and $M \geq 1$, such that for $0 \leq \Gamma_* < \varepsilon$ and any initial data $u^0 = (f^0, g^0, \Gamma^0) \in H^2_N$ with $\langle f^0 \rangle = f_*$, $\langle g^0 \rangle = g_*$ and $\langle \Gamma^0 \rangle = \Gamma_*$ satisfying the smallness condition $\|u^0 - u_*\|_{H^2} \leq \varepsilon$, there exists a* unique global positive solution

$$f, g, \Gamma \in C([0,\infty), U^\alpha) \cap C^\alpha([0,\infty), L_2) \cap C^1((0,\infty), L_2) \cap C((0,\infty), H^2_N)$$

to (2.1). *The solution satisfies*

$$\|u(t) - u_*\|_{H^2} + \|\partial_t u(t)\|_2 \leq M e^{-\omega t} \|u^0 - u_*\|_{H^2} \qquad \text{for} \quad t \geq 0,$$

where $u_* = (f_*, g_*, \Gamma_*)$.

3. Fourth Order Two–Phase Thin Film Model Driven by Capillary Effects with Insoluble Surfactant

This chapter is devoted to study the two–phase thin film equation equipped with insoluble surfactant, where capillary effects serve as the only driving force. Here we neglect the effect of gravitation on the motion of the two–phase flow. Analogously to the previous chapter, we prove in the first and second section a well–posedness and asymptotic stability result. It occurs in particular one major difference in treating the fourth–order system (1.49) with regard to the second–order system studied in Chapter 2. Observe that (1.49) is of fourth order in the evolution equations for the two film heights and only of second order in the evolution equation for the surfactant concentration, which is strongly coupled to the fourth–order equations. Translating (1.49) into an abstract setting, the appearing matrix operator is of mixed order. The strong coupling of evolution equations of different orders courses difficulties in studying the matrix operator. Still, demanding a smallness condition on the surfactant concentration, we are able to show, by a perturbation argument, that the matrix operator is a generator of an analytic semigroup, such that as before [2, Theorem 12.1] implies the well–posedness. We will see that, in contrary to Theorem 2.1, which states the well–posedness for the gravity driven two–phase thin film flow, considering the two–phase thin film with insoluble surfactant, where capillary effects are the only driving force, we do not need any assumption on the density of the two fluids (in Theorem 2.1, we forced the fluid with higher density to be on the bottom in order to establish well–posedness). As before, an energy functional provides that the set of steady states is determined by the solutions of (1.49), which are constant. Studying stability properties of these steady states then is similar to the analysis in the previous chapter. The third section of this chapter is devoted to study the existence of non–negative global weak solutions to (1.49). Following the approach used in [20, 22, 26], we regularize the system (1.49) and prove, by using a

Galerkin approximation, the existence of global weak solutions to the regularized problem. In doing so, we obtain a family of weak solutions to the regularized systems possessing a converging subsequence, whose limit function is non–negative, under the assumption of non–negative initial data, and provides a global weak solution to the original system. Since the weak solutions to (1.49) appear as limit functions of converging subsequences, it is not clear whether they can be proven to be unique. As the main ingredient in the analysis of weak solutions to (1.49) serves an energy functional by providing a priori estimates, which allow to extract converging subsequences of the Galerkin approximation.

Recall the system of evolution equations given in (1.49)

$$\begin{cases} \partial_t f + \partial_x \left[\left((\sigma_1^c + \sigma_2^c \mu) \frac{f^3}{3} + \sigma_2^c \mu \frac{f^2 g}{2} \right) \partial_x^3 f + \sigma_2^c \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2} \right) \partial_x^3 g + \mu \frac{f^2}{2} \partial_x \sigma(\Gamma) \right] = 0 \\ \partial_t g + \partial_x \left[\left(\sigma_2^c \frac{g^3}{3} + (\sigma_1^c + \sigma_2^c \mu) \frac{f^2 g}{2} + \sigma_2^c \mu f g^2 \right) \partial_x^3 f + \left(\sigma_2^c \frac{g^3}{3} + \sigma_2^c \mu \left(\frac{f^2 g}{2} + f g^2 \right) \right) \partial_x^3 g \right. \\ \qquad \left. + \left(\mu f g + \frac{g^2}{2} \right) \partial_x \sigma(\Gamma) \right] = 0 \\ \partial_t \Gamma + \partial_x \left[\left(\sigma_2^c \frac{g^2}{2} + (\sigma_1^c + \sigma_2^c \mu) \frac{f^2}{2} + \sigma_2^c \mu f g \right) \Gamma \partial_x^3 f + \left(\sigma_2^c \frac{g^2}{2} + \sigma_2^c \mu \left(\frac{f^2}{2} + f g \right) \right) \Gamma \partial_x^3 g \right. \\ \qquad \left. + (\mu f + g) \Gamma \partial_x \sigma(\Gamma) - D \partial_x \Gamma \right] = 0, \end{cases} \tag{3.1a}$$

for $t > 0$ and $x \in (0, L)$ with initial data at $t = 0$

$$f(0, \cdot) = f^0, \quad g(0, \cdot) = g^0, \quad \Gamma(0, \cdot) = \Gamma^0 \tag{3.1b}$$

and boundary conditions

$$\begin{aligned} \partial_x f = \partial_x g = \partial_x \Gamma = 0, \\ \partial_x^3 f = \partial_x^3 g = 0 \end{aligned} \tag{3.1c}$$

at $x = 0, L$. We impose the following assumptions: Given the surface tension coefficients $\sigma_1 = \sigma_1^c \geq 0$ and σ_2 of the form

$$\sigma_2(\Gamma) = \sigma_2^c + \sigma(\Gamma),$$

where Γ is the surfactant concentration, we assume that the part of the surface tension, which is independent of the concentration of surfactant, is strictly positive and the part of the surface tension, which depends on Γ, is non–increasing, that is, S1) $\sigma \in C^2(\mathbb{R})$ and $-\sigma'(s) \geq 0$ for all $s \geq 0$,

S2) $\sigma_1^c, \sigma_2^c > 0$.

Moreover, let Φ be a function, such that

A1) $\Phi \in C^2(\mathbb{R})$ with $\Phi(1) = \Phi'(1) = 0$ and

$$\Phi''(s) = -\frac{\sigma'(s)}{s} \geq 0 \qquad \text{for all } s > 0.$$

3.1. Local Well–Posedness

We need to define suitable spaces for the well–posedness of the system of evolution equations (3.1). Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we set in the sequel

$$H_B^k(0, L; \mathbb{R}^n) := \{u \in H^k(0, L; \mathbb{R}^n) \mid \partial_x^{2l+1} u(0) = \partial_x^{2l+1} u(L) = 0 \text{ for all } l \in \mathbb{N} \text{ with } 2l+2 \leq k\}.$$

These spaces are well defined by the Sobolev Embedding Theorem and endowed with the usual Sobolev norms. Since the system we are analyzing features both, second– and fourth–order derivatives, the space $H_B^4(0, L; \mathbb{R}^2) \times H_B^2(0, L; \mathbb{R})$ will play an important role. For $\alpha \in [0, 1]$ and $\varepsilon > 0$ we define

$$O^\alpha := \big(H_B^{4\alpha}(0, L; \mathbb{R}^2) \times H_B^{2\alpha}(0, L; \mathbb{R})\big) \cap C([0, L], (0, \infty)^3),$$

$$O_\varepsilon^\alpha := O^\alpha \cap \{u = (f, g, \Gamma) \in H_B^{4\alpha}(0, L; \mathbb{R}^2) \times H_B^{2\alpha}(0, L; \mathbb{R}) \mid \|\Gamma\|_{H^{2\alpha}} < \varepsilon\},$$

where

$$H_B^s(0, L; \mathbb{R}^n) := \begin{cases} \{u \in H^s(0, L; \mathbb{R}^n) \mid \partial_x u = \partial_x^3 u = 0 \text{ at } x = 0, L\}, & \text{if } \quad s \in (\frac{7}{2}, 4], \\ \{u \in H^s(0, L; \mathbb{R}^n) \mid \partial_x u = 0 \text{ at } x = 0, L\}, & \text{if } \quad s \in (\frac{3}{2}, \frac{7}{2}], \\ H^s(0, L; \mathbb{R}^n), & \text{if } \quad s \in [0, \frac{3}{2}] \end{cases}$$

with $H^s(0, L; \mathbb{R}^n)$ being the *Bessel potential space* for $s \in [0, 4]$. The product space

$$H_B^{4\alpha}(0, L; \mathbb{R}^2) \times H_B^{2\alpha}(0, L; \mathbb{R})$$

is the complex interpolation space $[L_2(0, L; \mathbb{R}^2) \times L_2(0, L; \mathbb{R}), H_B^4(0, L; \mathbb{R}^2) \times H_B^2(0, L; \mathbb{R})]_\alpha$ for $\alpha \in [0, 1]$ between the product spaces $H_B^4(0, L; \mathbb{R}^2) \times H_B^2(0, L; \mathbb{R})$ and $L_2(0, L; \mathbb{R}^2) \times L_2(0, L; \mathbb{R})$. If $\alpha > \frac{7}{8}$, then (cf. [46, Theorem 4.6.1 e)])

$$H_B^{4\alpha}(0, L; \mathbb{R}^2) \times H_B^{2\alpha}(0, L; \mathbb{R}) \subset C^3([0, L]; \mathbb{R}^2) \cap C^1([0, L]; \mathbb{R}).$$

Furthermore, O^α and O^α_ε are open subsets in $H^{4\alpha}_B(0,L;\mathbb{R}^2)\times H^{2\alpha}_B(0,L;\mathbb{R})$. Note that the boundary conditions as well as the positivity are already incorporated into the sets O^α and O^α_ε. For each $u=(f,g,\Gamma)\in O^\alpha$ we define the matrix

$$a_c(u) := \begin{pmatrix} \sigma^c_{12}\frac{f^3}{3}+\sigma^c_2\mu\frac{f^2 g}{2} & \sigma^c_2\mu\left(\frac{f^3}{3}+\frac{f^2 g}{2}\right) & \frac{f^2}{2}\mu\sigma'(\Gamma) \\ \sigma^c_2\frac{g^3}{3}+\sigma^c_{12}\frac{f^2 g}{2}+\sigma^c_2\mu f g^2 & \sigma^c_2\frac{g^3}{3}+\sigma^c_2\mu\left(\frac{f^2 g}{2}+fg^2\right) & \left(\frac{g^2}{2}+\mu f g\right)\sigma'(\Gamma) \\ \left(\sigma^c_2\frac{g^2}{2}+\sigma^c_{12}\frac{f^2}{2}+\sigma^c_2\mu f g\right)\Gamma & \left(\sigma^c_2\frac{g^2}{2}+\sigma^c_2\mu\left(\frac{f^2}{2}+fg\right)\right)\Gamma & (\mu f+g)\Gamma\sigma'(\Gamma)-D \end{pmatrix},$$

where $\sigma^c_{12}:=\sigma^c_1+\sigma^c_2\mu$, and rewrite the problem (3.1) as a quasi–linear equation in the space $L^2(0,L;\mathbb{R}^3)$

$$\partial_t u + A_c(u)u = 0, \qquad t>0, \qquad u(0)=u^0, \tag{3.2}$$

where $u^0=(f^0,g^0,\Gamma^0)$ and the operator $A_c := O^\alpha \to \mathcal{L}(H^4_B(0,L;\mathbb{R}^2)\times H^2_B(0,L;\mathbb{R}), L_2(0,L;\mathbb{R}^3))$ is given by

$$A_c(u)w := \partial_x\left(a_c(u)\begin{pmatrix}\partial_x^3\tilde f\\ \partial_x^3\tilde g\\ \partial_x\tilde\Gamma\end{pmatrix}\right), \qquad \text{for} \quad u\in O^\alpha, w:=(\tilde f,\tilde g,\tilde\Gamma)\in H^4_B(0,L;\mathbb{R}^2)\times H^2_B(0,L;\mathbb{R}).$$

Letting $\alpha\in(\frac{7}{8},1)$, we prove that there exists $\varepsilon>0$, such that starting with an initial data $u^0\in O^\alpha_\varepsilon$ and under the Assumptions S1) and S2), there exists a unique, strong solution on some time interval $[0,T)$, where $T\in(0,\infty]$ depends on the initial datum $u^0\in O^\alpha_\varepsilon$.

Theorem 3.1 (Local Existence)**.** *Let $\alpha\in(\frac{7}{8},1)$, S1) and S2) be satisfied. Then, there exists $\varepsilon>0$, such that given $u^0=(f^0,g^0,\Gamma^0)\in O^\alpha_\varepsilon$, the problem* (3.2) *possesses a unique maximal strong solution*

$$\begin{aligned}(f,g,\Gamma)\in & C([0,T);O^\alpha_\varepsilon)\cap C^\alpha([0,T);L_2(0,L;\mathbb{R}^3))\cap C((0,T);H^4_B(0,L;\mathbb{R}^2)\times H^2_B(0,L;\mathbb{R}))\\ &\cap C^1((0,T);L_2(0,L;\mathbb{R}^3)),\end{aligned}$$

with maximal time of existence $T\in(0,\infty]$. Moreover, $u=(f,g,\Gamma)$ depends in O^α continuously on its initial datum u^0.

Set $E_0:=L_2(0,L;\mathbb{R}^3)$ and $E_1:=H^4_B(0,L;\mathbb{R}^2)\times H^2_B(0,L;\mathbb{R})$. Furthermore let $E_\theta:=[E_1,E_0]_\theta$ be the complex interpolation space between E_1 and E_0 for $\theta\in[0,1]$. With

$$O(\varepsilon):=C([0,L];(0,\infty)^3)\cap\{u=(f,g,\Gamma)\in H^4(0,L;\mathbb{R}^2)\times H^2(0,L;\mathbb{R})\mid \|\Gamma\|_{H^{2\alpha}}<\varepsilon\},$$

we identify $O^\alpha_\varepsilon = O(\varepsilon) \cap E_\alpha$. By taking into account that A_c depends smoothly on its coefficients we obtain that

$$A_c \in C^{1-}(O^\alpha_\varepsilon, \mathcal{L}(E_1, E_0)). \tag{3.3}$$

We show that for fixed $u \in O^\alpha_\varepsilon$, the linear operator $A_c(u) \in \mathcal{L}(E_1, E_0)$ is the negative generator of an analytic semigroup. Then, Theorem 3.1 is a consequence of [2, Theorem 12.1].

Theorem 3.2. *Under Assumption S1) and S2), there exists $\varepsilon > 0$, such that given $u = (f, g, \Gamma) \in O^\alpha_\varepsilon$, the operator $-A_c(u)$ generates an analytic semigroup in $L_2(0, L; \mathbb{R}^3)$, that is*

$$-A_c(u) \in \mathcal{H}(H^4_B(0, L; \mathbb{R}^2) \times H^2_B(0, L; \mathbb{R}); L_2(0, L; \mathbb{R}^3)).$$

Observe, that the linear operator $A_c(u)$, where $u = (f, g, \Gamma) \in O^\alpha_\varepsilon$, can be considered a matrix operator of the form

$$A_c(f, g, \Gamma) := \begin{pmatrix} A_{11}(f,g) & A_{12}(f,g,\Gamma) \\ A_{21}(f,g,\Gamma) & A_{22}(f,g,\Gamma) \end{pmatrix} \in \mathcal{L}(H^4_B(0, L, \mathbb{R}^2) \times H^2_B(0, L; \mathbb{R}), L_2(0, L; \mathbb{R})), \tag{3.4}$$

with

$$(A_{11}(f,g)) \begin{pmatrix} \tilde f \\ \tilde g \end{pmatrix} := \partial_x \left(\begin{pmatrix} \sigma^c_{12}\frac{f^3}{3} + \sigma^c_2\mu\frac{f^2 g}{2} & \sigma^c_2\mu\left(\frac{f^3}{3} + \frac{f^2 g}{2}\right) \\ \sigma^c_2\frac{g^3}{3} + \sigma^c_{12}\frac{f^2 g}{2} + \sigma^c_2\mu f g^2 & \sigma^c_2\frac{g^3}{3} + \sigma^c_2\mu\left(\frac{f^2 g}{2} + f g^2\right) \end{pmatrix} \partial^3_x \begin{pmatrix} \tilde f \\ \tilde g \end{pmatrix} \right), \tag{3.5}$$

$$(A_{12}(f,g,\Gamma))\,\tilde\Gamma := \partial_x \left(\begin{pmatrix} \frac{f^2}{2}\mu\sigma'(\Gamma) \\ \left(\frac{g^2}{2} + \mu f g\right)\sigma'(\Gamma) \end{pmatrix} \partial_x \tilde\Gamma \right),$$

$$(A_{21}(f,g,\Gamma)) \begin{pmatrix} \tilde f \\ \tilde g \end{pmatrix} := \partial_x \left(\left(\left(\sigma^c_2\frac{g^2}{2} + \sigma^c_{12}\frac{f^2}{2} + \sigma^c_2\mu f g\right)\Gamma \quad \left(\sigma^c_2\frac{g^2}{2} + \sigma^c_2\mu\left(\frac{f^2}{2} + fg\right)\right)\Gamma \right) \partial^3_x \begin{pmatrix} \tilde f \\ \tilde g \end{pmatrix} \right),$$

$$(A_{22}(f,g,\Gamma))\,\tilde\Gamma := \partial_x \left(((\mu f + g)\Gamma\sigma'(\Gamma) - D)\,\partial_x \tilde\Gamma \right)$$

for $(f, g, \Gamma) \in O^\alpha_\varepsilon$ and $(\tilde f, \tilde g, \tilde\Gamma) \in H^4_B(0, L; \mathbb{R}^2) \times H^2_B(0, L; \mathbb{R})$. We want to make use of an result from [3], which states a characterization of matrix generators.

Theorem 3.3 ([3], Theorem I.1.6.1, Remark I.1.6.2)**.** *Let $(E_0, E_1), (F_0, F_1)$ be densely injected Banach couples. Then $(E_0 \times F_0, E_1 \times F_1)$ is a densely injected Banach couple as well. Suppose that*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{L}(E_1 \times F_1, E_0 \times F_0)$$

and

$$A_{11} \in \mathcal{H}(E_1, E_0, \kappa_1, \omega_1) \qquad \text{and} \qquad A_{22} \in \mathcal{H}(F_1, F_0, \kappa_2, \omega_2).$$

If one of the following holds, then $A \in \mathcal{H}(E_1 \times F_1, E_0 \times F_0, \frac{\kappa}{1-\kappa r}, \max\{\omega, \frac{\beta}{r}\})$.

i) Put

$$\kappa := \max\{\kappa_1(1 + \kappa_2 \|A_{21}\|_{\mathcal{L}(E_1,F_0)}), \kappa_2\} \qquad \text{and} \qquad \omega := \max\{\omega_1, \omega_2\}$$

and suppose that there are $r \in (0, \frac{1}{\kappa})$ and $\beta \geq 0$ such that

$$\|A_{12} y\|_{E_0} \leq r\|y\|_{F_1} + \beta\|y\|_{F_0}, \qquad y \in F_1. \tag{3.6}$$

ii) Put

$$\kappa := \max\{\kappa_2(1 + \kappa_1 \|A_{12}\|_{\mathcal{L}(F_1,E_0)}), \kappa_1\} \qquad \text{and} \qquad \omega := \max\{\omega_1, \omega_2\}$$

and suppose that there are $r \in (0, \frac{1}{\kappa})$ and $\beta \geq 0$ such that

$$\|A_{21} y\|_{F_0} \leq r\|y\|_{E_1} + \beta\|y\|_{E_0}, \qquad y \in E_1. \tag{3.7}$$

Motivated by the above theorem, we want to prove that for each $u \in O_\varepsilon^\alpha$ the operator $-A_c(u)$ generates an analytic semigroup, by showing first that $-A_{11}(u)$, $-A_{22}(u)$ generate analytic semigroups. Observe that due to an interpolation estimate (cf. [3, Proposition I.2.2.1]) and Young's inequality the norm $\|\partial_x^k h\|_2$ of a function $h \in H^l$, where $k, l \in \mathbb{N}$ and $k \leq l-1$ can be estimated as follows

$$\|\partial_x^k h\|_2^2 \leq \|h\|_{H^k} \leq \|h\|_{H^l}^\theta \|h\|_2^{1-\theta} \leq \varepsilon\|h\|_{H^k}^2 + c(\varepsilon,\theta)\|h\|_2^2 \tag{3.8}$$

for all $\varepsilon > 0$, where $\theta = \frac{k}{l}$ and $c(\varepsilon, \theta) > 0$ is a constant depending on ε and θ. In virtue of such an estimate, we could apply Theorem 3.3 easily, if $A_{12}(u)$ or $A_{21}(u)$ would consist of lower order terms, respectively. However, since $A_{12}(u)$ contains second–order derivatives acting on $H^2(0, L)$ and $A_{12}(u)$ contains fourth–order derivatives acting on $H^4(0, L)$, it is not obvious whether one can obtain an estimate like (3.6) or (3.7).

Since $A_{21}(f,g,0)=0$, it follows immediately from Theorem 3.3 that, if $u=(f,g,\Gamma)\in O^\alpha$, the matrix operator

$$-A_c^0(f,g,\Gamma):=-A_c(f,g,0) \qquad \text{belongs to} \quad \mathcal{H}(H_B^4(0,L,\mathbb{R}^2)\times H_B^2(0,L;\mathbb{R}),L_2(0,L;\mathbb{R}^3))$$

if $-A_{11}(f,g)$, $-A_{22}(f,g,\Gamma)$ generate analytic semigroups. Then, by means of a perturbation argument, we obtain the existence of $\varepsilon>0$, such that $A_c(u)$, $u\in O_\varepsilon^\alpha$, is the negative generator of an analytic semigroup.

Proposition 3.4. *Let $\alpha\in(\frac{7}{8},1)$, S1) and S2) be satisfied. Then*

i) $-A_{11}(f,g)\in\mathcal{H}(H_B^4(0,L,\mathbb{R}^2),L_2(0,L,\mathbb{R}^2))$ *for all* $(f,g)\in\{H_B^{4\alpha}(0,L;\mathbb{R}^2)\mid f,g>0\}$,

ii) $-A_{22}(u)\in\mathcal{H}(H_B^2(0,L;\mathbb{R}),L_2(0,L,\mathbb{R}))$ *for all* $u\in O^\alpha$.

It is already well known, that the strongly elliptic second–order operator $A_{22}(u)$, $u\in O^\alpha$, is the negative generator of an analytic semigroup on $L_2(0,L;\mathbb{R})$ (cf. e.g. [41, Theorem 7.2.7]). Remark that the strong ellipticity of $A_{22}(u)$, $u\in O^\alpha$, is due to Assumption S1) and $D>0$. We are left to show part $i)$ of Proposition 3.4.

Following the lines of the proof of [24, Lemma 4.1], where a similar proof is investigated for the more general case $n\geq 1$ (here $\Omega=(0,L)\subset\mathbb{R}$), we show Proposition 3.4 i) by verifying the *Lopatinskii–Shapiro condition* for the pair $(\mathbb{A},\mathcal{B})$, where $\mathbb{A}:=A_{11}(X)Y=\partial_x(\tilde{a}(X)\partial_x^3Y)$ for $X=(f,g)\in\{X\in H_B^{4\alpha}(0,L;\mathbb{R}^2)\mid X>0\}$ fixed and $Y\in H_B^4(0,L;\mathbb{R}^2)$ with

$$\tilde{a}(X)=\begin{pmatrix}\sigma_{12}^c\frac{f^3}{3}+\sigma_2^c\mu\frac{f^2g}{2} & \sigma_2^c\mu\left(\frac{f^3}{3}+\frac{f^2g}{2}\right)\\ \sigma_2^c\frac{g^3}{3}+\sigma_{12}^c\frac{f^2g}{2}+\sigma_2^c\mu fg^2 & \sigma_2^c\frac{g^3}{3}+\sigma_2^c\mu\left(\frac{f^2g}{2}+fg^2\right)\end{pmatrix} \qquad \text{in } \Omega=(0,L)$$

and $\mathcal{B}$ being the boundary operator $\mathcal{B}:=(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4)$ with

$$\mathcal{B}_1Y=(1,0)\partial_xY,\qquad \mathcal{B}_2Y=(0,1)\partial_xY,\qquad \mathcal{B}_3Y=(1,0)\partial_x^3Y,\qquad \mathcal{B}_4Y=(0,1)\partial_x^3Y$$

on $\partial\Omega=\{0,L\}$ for $Y\in H_B^4(0,L;\mathbb{R}^2)$. The associate principal symbols of $(\mathbb{A},\mathcal{B})$ are given by

$$a_\pi(x,\xi)=\tilde{a}(X(x))|\xi|^4 \qquad \text{for} \quad (x,\xi)\in[0,L]\times\mathbb{R},$$

$$b_\pi(x,\xi)=\big((1,0)\xi,(0,1)\xi,(1,0)\xi^3,(0,1)\xi^3\big) \qquad \text{for} \quad (x,\xi)\in\{0,L\}\times\mathbb{R}.$$

The operator $\mathbb{A}$ is normally elliptic, since

$$\operatorname{spec}(a_\pi(x,\xi)) \subset [\operatorname{Re} z > 0] \qquad \text{for all} \quad (x,\xi) \in [0,L] \times \{\xi \in \mathbb{R} \mid |\xi| = 1\},$$

which can be easily verified by observing that the principal minors of $\tilde{a}$ are positive, which implies that $\tilde{a}$ is positive definite. The boundary operator $\mathcal{B}$ is said to satisfy the *Lopatinskii–Shapiro condition* with respect to $\mathbb{A}$ if for each (x,ξ) belonging to the tangent bundle $T(\partial\Omega)$ and $\lambda \in [Re z \geq 0]$ with $(\xi,\lambda) \neq 0$ the only exponentially decaying solution of the boundary value problem on the half–line

$$[\lambda + a_\pi(x,\xi + i\partial_t)]u = 0, \quad t > 0, \qquad b_\pi(x,\xi + i\partial_t)u(0) = 0 \tag{3.9}$$

is the zero solution. Then, the boundary value problem $(\mathbb{A},\mathcal{B})$ is *normally elliptic* if $\mathbb{A}$ is normally elliptic and $\mathcal{B}$ satisfies (3.9).
Due to [2, Remark 4.2 b)] it is sufficient to verify the Lopatinskii–Shaprio condition (3.9) for $(\mathbb{A},\mathcal{B})$ in order to prove that $\mathbb{A}$ is the negative generator of an analytic semigroup. Since $\Omega = (0,L)$ is a subset of an one–dimensional space, the boundary $\partial\Omega = \{0,L\}$ is of dimension zero, which implies that the tangent space at the boundary is zero. This simplifies the Lopatinskii–Shapiro condition (3.9) in this respect that we are left to to show that for all $\lambda \in [Re\, z \geq 0]$ the only exponentially decaying solution of the boundary value problem on the half–line

$$[\lambda + a_\pi(x,i\partial_t)]u = 0, \quad t > 0, \qquad b_\pi(x,i\partial_t)u(0) = 0 \tag{3.10}$$

is the zero solution. The argumentation in the sequel follows the lines in the proof of [24, Lemma 4.1] setting $\xi = 0$. The boundary value problem (3.10) is equivalent to

$$\begin{cases} \lambda a^{11}u_1 + \lambda a^{12}u_2 + u_1^{(4)} = 0, \\ \lambda a^{21}u_1 + \lambda a^{22}u_2 + u_2^{(4)} = 0, \end{cases} \qquad t > 0 \tag{3.11}$$

with initial conditions

$$u_1'(0) = u_2'(0) = u_1'''(0) = u_2'''(0) = 0, \tag{3.12}$$

where $u_i^{(k)}$ denotes the kth derivative of u_i, $i = 1,2$ and the matrix $(a^{ij})_{1\leq i,j\leq 2}$ the inverse of $\tilde{a}(X)$, which exists by $\tilde{a}(X)$ being positive definite. Since $\lambda \neq 0$, we can express u_2 in virtue of the first equation in (3.11) as

$$u_2 = -\frac{1}{\lambda a^{12}}\left[u_1^{(4)} + \lambda a_{11}u_1\right], \tag{3.13}$$

so that

$$u_2^{(4)} = -\frac{1}{\lambda a^{12}} \left[u_1^{(8)} + \lambda a_{11} u_1^{(4)} \right].$$

By means of the above equations, we obtain from the second equation in (3.11) an 8th–order ordinary differential equation for u_1:

$$u_1^{(8)} + \lambda[a^{11} + a^{22}]u_1^{(4)} + \lambda^2 \left[a^{11}a^{22} - a^{12}a^{21}\right] u_1 = 0, \qquad t > 0 \tag{3.14}$$

with initial conditions

$$u_1'(0) = u_1'''(0) = u_1^{(5)}(0) = u_1^{(7)}(0) = 0. \tag{3.15}$$

A general solution of (3.14) is given by the polynomial

$$u_1(t) = \sum_{k=1}^{8} c_k e^{\Lambda_k} t, \qquad t \geq 0, \tag{3.16}$$

where $\{\Lambda_k \in \mathbb{C} \mid k = 1, \ldots, 8\}$ are the roots of the characteristic polynomial

$$\Lambda^8 + \lambda[a^{11} + a^{22}]\Lambda^4 + \lambda^2 \left[a^{11}a^{22} - a^{12}a^{21}\right] = 0.$$

A solution to the above equation of 8th–order is given via

$$\Lambda_\pm^4 = \frac{\lambda}{2} \left(-[a^{11} + a^{22}] \pm \sqrt{(a^{11} - a^{22})^2 + 4a^{12}a^{21}} \right) =: \lambda E_\pm,$$

with $E_\pm < 0$ and $E_+ \neq E_-$. Hence, the roots Λ_k are given by

$$\Lambda_{1/2} = \pm \frac{1}{\sqrt{2}}(1+i)\sqrt[4]{-E_+}, \qquad \Lambda_{3/4} = \pm \frac{1}{\sqrt{2}}(1-i)\sqrt[4]{-E_+}$$
$$\Lambda_{5/6} = \pm \frac{1}{\sqrt{2}}(1+i)\sqrt[4]{-E_-}, \qquad \Lambda_{7/8} = \pm \frac{1}{\sqrt{2}}(1-i)\sqrt[4]{-E_-}.$$

Recall that u_1 is claimed to have exponential decay, which implies in virtue of $\operatorname{Re}\Lambda_k > 0$ for $k \in \{1, 3, 5, 7\}$, that $c_1, c_3, c_5, c_7 = 0$. In view of (3.15) and (3.16) we deduce that

$$\begin{pmatrix} u_1^{(1)}(0) \\ u_1^{(3)}(0) \\ u_1^{(5)}(0) \\ u_1^{(7)}(0) \end{pmatrix} = \begin{pmatrix} \Lambda_2 & \Lambda_4 & \Lambda_6 & \Lambda_8 \\ \Lambda_2^3 & \Lambda_4^3 & \Lambda_6^3 & \Lambda_8^3 \\ \Lambda_2^5 & \Lambda_4^5 & \Lambda_6^5 & \Lambda_8^5 \\ \Lambda_2^7 & \Lambda_4^7 & \Lambda_6^7 & \Lambda_8^7 \end{pmatrix} \begin{pmatrix} c_2 \\ c_4 \\ c_6 \\ c_8 \end{pmatrix} = 0$$

Due to $E_+ \neq E_-$ it is clear that $\Lambda_2 \neq \Lambda_4 \neq \Lambda_6 \neq \Lambda_8$ and the determinant of the 4×4 matrix above is different from zero. Hence, $c_2, c_4, c_6, c_8 = 0$. This implies that $u_1 = u_2 = 0$ in virtue of (3.13) and (3.16), which completes the proof. Eventually, we conclude that, in virtue of Proposition 3.4, the operator $-A_c^0$ belongs to $\mathcal{H}(H_B^4(0,L,\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}), L_2(0,L;\mathbb{R}^3))$. Introducing the operator $B : O_\varepsilon^\alpha \to \mathcal{L}(H_B^4(0,L,\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}), L_2(0,L;\mathbb{R}^3))$, defined by

$$B(u) := \begin{pmatrix} 0 & 0 \\ A_{21}(u) & \overline{A}_{22}(u) \end{pmatrix}, \qquad \text{for} \quad u \in O_\varepsilon^\alpha,$$

where $\overline{A}_{22}(f,g,\Gamma)\tilde{\Gamma} := \partial_x((\mu f + g)\Gamma\sigma'(\Gamma)\partial_x\tilde{\Gamma})$ for $\tilde{\Gamma} \in H^2(0,L;\mathbb{R})$, we obtain that for all $\tilde{\varepsilon}$ there exists $\varepsilon > 0$, such that if $u = (f,g,\Gamma) \in O_\varepsilon^\alpha$,

$$\|B(u)\|_{\mathcal{L}(H_B^4(0,L,\mathbb{R}^2)\times H_B^2(0,L;\mathbb{R}),L_2(0,L;\mathbb{R}^3))} < \tilde{\varepsilon}.$$

Hence, by means of a perturbation argument (cf. [3, Theorem 1.3.1]), the operator

$$-A_c(u) = -A_c^0(u) - B(u) \qquad \text{belongs to} \quad \mathcal{H}(H_B^4(0,L,\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}), L_2(0,L;\mathbb{R}^3))$$

for all $u \in O_\varepsilon^\alpha$ and Theorem 3.1 is a consequence of (3.3) and [2, Theorem 12.1].

Remark 3.5. *An alternative proof of Proposition 3.4 i) by showing that the operator $A_{11}(X)$ is sectorial, is included in Appendix A.2.*

3.2. Asymptotic Stability

In this section we study the stability properties of equilibrium solutions to (3.1). Following the approach as in [19, 24], the analysis is similar to the one applied in Section 2.2. Observe that formally the functional

$$\mathcal{E}(f,g,\Gamma)(t) := \int_0^L \left\{ \frac{1}{2}|\partial_x(f+g)(t,x)|^2 + \frac{\sigma_1^c}{2\sigma_2^c\mu}|\partial_x f(t,x)|^2 + \frac{1}{\sigma_2^c}\Phi(\Gamma(t,x)) \right\} dx \qquad (3.17)$$

dissipates along solutions of (1.49). Comparing the systems (2.1a) and (3.1a), the evolution equations for the fourth–order system can be recovered from (2.1a) by replacing the appearing third order derivatives ∂_x^3 by (negative) first order derivatives $-\partial_x$ and G_1 by $\sigma_1^c + \sigma_2^c\mu$, G_2 by σ_2^c, respectively. Therefore (after integrating by parts the terms involving f and g

twice), the same computation as in the previous section yields (formally)

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(f,g,\Gamma)(t)\\
&\quad= -\left\|\sqrt{f}\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}f\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f+\sigma_2^c\mu g)+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}g\partial_x^3(f+g)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right)\right]\right\|_2^2\\
&\quad\quad -\frac{1}{4}\left\|\sqrt{f}\left[\sqrt{\sigma_2^c}g\partial_x^3(f+g)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right]\right\|_2^2+\left\|\sqrt{g}\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}g\partial_x^3(f+g)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right]\right\|_2^2\\
&\quad\quad -\frac{1}{4\sigma_2^c}\left\|\sqrt{g}\partial_x\sigma(\Gamma)\right\|_2^2-\frac{D}{\sigma_2^c}\left\|\sqrt{\Phi''(\Gamma)}\partial_x\Gamma\right\|_2^2.
\end{aligned} \tag{3.18}$$

But, the regularity of the local solutions found in Theorem 3.1 is not sufficient in order to differentiate the functional with respect to time. However, like in [24], we may improve the regularity of a solution u of (3.1).

Corollary 3.6. *The local solution u found in Theorem 3.1 admits the regularity*

$$u \in C^{\frac{5}{4}}((0,T); H_B^1(0,L;\mathbb{R}^2)\times H_B^{\frac{1}{2}}(0,L;\mathbb{R})).$$

Proof. We follow the lines in [24, Section 4.1]. Theorem 3.1 provides that

$$u \in C((0,T); H_B^4(0,L;\mathbb{R}^2)\times H_B^2(0,L;\mathbb{R}))\cap C^1((0,T); L_2(0,L;\mathbb{R}^3)).$$

By [3, Proposition II.1.1.2], this implies that

$$u \in C^{1-\theta}((0,T); H_B^{4\theta}(0,L;\mathbb{R}^2)\times H_B^{2\theta}(0,L;\mathbb{R}))$$

for $\theta\in[0,1]$. For $\rho\in\left(\frac{3}{8},1\right)$, the Sobolev Embedding Theorem yields

$$u \in C^{1-\rho}((0,T); C^1([0,L],\mathbb{R}^2)\times C([0,L],\mathbb{R})).$$

Since A_c depends smoothly on its coefficients, we deduce from Theorem 3.2 that

$$-A_c(u)\in C^{1-\rho}((0,T);\mathcal{H}(H_B^4(0,L;\mathbb{R}^2)\times H_B^2(0,L;\mathbb{R}))).$$

Note that $w := u$ solves the linear parabolic problem

$$\partial_t w + A_c(u)w = 0, \qquad w(0)=u(0)=u^0.$$

By [2, Theorem 10.1], the unique solution w profits from the 'regularizing' effect for parabolic equations and we obtain, in view of $w = u$, that

$$u \in C^{1-\rho}((0,T); H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})) \cap C^{2-\rho}((0,T); L_2(0,L;\mathbb{R}^3)). \tag{3.19}$$

Since $\rho \in \left(\frac{3}{8}, 1\right)$, (3.19) yields in particular that $u \in C^{\frac{1}{2}}((0,T); H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})) \cap C^{\frac{3}{2}}((0,T); L_2(0,L;\mathbb{R}^3))$ and, by [33, Proposition 1.1.5],

$$\begin{aligned} u &\in C^{\frac{1}{2}}((0,T); H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})) \cap C^{\frac{3}{2}}((0,T); L_2(0,L;\mathbb{R}^3)) \\ &\subset C^{\frac{3}{2}-\delta}((0,T); H_B^{4\delta}(0,L;\mathbb{R}^2) \times H_B^{2\delta}(0,L;\mathbb{R}), L_2(0,L;\mathbb{R}^2) \times L_2(0,L;\mathbb{R})) \end{aligned}$$

for $\delta \in (0,1)$. Set $\delta = \frac{1}{4}$, then $u \in C^{\frac{5}{4}}((0,T); H_B^1(0,L;\mathbb{R}^2) \times H_B^{\frac{1}{2}}(0,L;\mathbb{R}))$. □

The above Corollary allows us to differentiate (3.17) with respect to time and we find (3.18) satisfied for a solution u given by Theorem 3.1. Since all the terms on the right–hand side of (3.18) are non–positive, if $u_* = (f_*, g_*, \Gamma_*)$ is a steady solution of (1.49), each term on the right–hand side of (3.18) needs to vanish. We deduce that, due to Assumption S2), $\partial_x \Gamma_* = \partial_x^3 f_* = \partial_x^3 g_* = 0$. Hence, $\Gamma_*, \partial_x^2 f_*$ and $\partial_x^2 g_*$ are a constant, which in particular implies that $\partial_x f_*$ and $\partial_x g_*$ are linear functions. In virtue of the boundary conditions, we deduce that $\partial_x f_* = \partial_x g_* = 0$. Thus also f_* and g_* are constant.

Corollary 3.7. *The only positive steady state solutions to* (3.1) *are of the form* (f_*, g_*, Γ_*), *where* f_*, g_* *and* Γ_* *are positive constants.*

Observe, by a simple computation, that the mass of each fluid and the mass of the surfactant concentration is preserved by the evolution of the system, which is due to the boundary conditions.

Lemma 3.8 (Conservation of mass)**.** *Let* $u = (f, g, \Gamma)$ *be a solution to* (3.1) *as found in Theorem 3.1. Then the mass of* u *is preserved with time, that is*

$$\frac{d}{dt}\int_0^L f(t,x)\,dx = 0 \qquad \textit{and} \qquad \frac{d}{dt}\int_0^L g(t,x)\,dx = 0 \qquad \textit{and} \qquad \frac{d}{dt}\int_0^L \Gamma(t,x)\,dx = 0$$

on $(0,T)$.

The last part of this section is devoted to prove that, assuming the averaged initial surfactant concentration to be sufficiently small, there exists for every initial data close enough to the steady state a global positive strong solution to (3.1) tending exponentially to the constant steady state, which is given by the average initial data. As in the previous section we denote by

$$\langle h \rangle := \frac{1}{L}\int_0^L h(x)\,dx$$

the average of a function h with regard to space and introduce the projection

$$P \in \mathcal{L}(L_2(0,L;\mathbb{R}^3)) \cap \mathcal{L}(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})), \qquad Pu := u - \langle u \rangle.$$

Recall that due to mass conservation and continuity in $t = 0$, a solution u of (3.1), which satisfies initially $(1-P)u(0) = u_*$ fulfills $(1-P)u(t) = u_*$ as long as the solution exists. By the projection, we can decompose the spaces $L_2(0,L;\mathbb{R}^3) = PL_2(0,L;\mathbb{R}^3) \oplus (1-P)L_2(0,L;\mathbb{R}^3)$ and $H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}) = P(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})) \oplus (1-P)(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})$ and express a solution u in the terms

$$u(t) = z(t) + u_*$$

with $z(t) = Pu(t) = u(t) - \langle u(t) \rangle = u(t) - u_*$ for all $t \geq 0$. If u is the corresponding solution of (1.49) to the the initial data $u^0 \in H_B^4(0,L,\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})$, then $z = u - u_*$ is a solution of

$$\partial_t z + A_c(z + u_*)z = 0.$$

Hence, the stability property of u_* is equivalent to the one for the stationary solution $z = 0$ of

$$\partial_t z + A_c^* z = \left(A_c^* - A_c(z+u_*)\Big|_{P(H_B^4 \times H_B^2)} \right) z =: F(z), \tag{3.20}$$

where $A_c^* w := A_c(u_*)w$ for $w \in H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})$. Observe that

$$(1-P)(A_c^* z) = 0 \quad \text{and} \quad (1-P)A_c(z+u_*)z = 0 \quad \text{for } z \in P(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}^2)),$$

due to u_* being constant and z satisfying the boundary conditions. Hence,

$$F \in C^1(P(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})), P(L_2(0,L,\mathbb{R}^3))) \quad \text{with} \quad F(0) = F'(0) = 0, \tag{3.21}$$

where F' denotes the Frechét derivative of F. Problem (3.20) is in fact the restriction of (3.2) to the subspace $P(L_2(0,L;\mathbb{R}^3))$ of $L_2(0,L;\mathbb{R}^3)$, where the constant functions (except the zero function) are eliminated. Next we prove a lemma providing the necessary conditions in order to apply the *principle of linearized stability* (cf. [33, Theorem 9.1.2]), which implies the asymptotic stability of the zero solution $z = 0$ of (3.20).

Lemma 3.9. *There exists a constant $\varepsilon_* > 0$, such that if $0 \leq \Gamma_* < \varepsilon_*$ the operator $-A_c^*$ belongs to $\mathcal{H}(P(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}^2), P(L_2(0,L;\mathbb{R}^3)))$, that is, $-A_c^*$ is the generator of an analytic semigroup on $PL_2(0,L;\mathbb{R}^3)$. Further, there exists $\omega_0 > 0$, such that the spectrum* $\mathrm{spec}(-A_c^*)$ *of $-A_c^*$ is contained in the half plane* $[\mathrm{Re}\, z \leq -\omega_0]$.

Proof. We know already from Theorem 3.2, that there exists $\varepsilon_1 > 0$ such that $A_c^* \in \mathcal{H}(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R}^2))$ provided that $0 \leq \Gamma_* < \varepsilon_1$. By the same argument as in Lemma 2.5, we can represent $-A_c^*$ as a matrix operator

$$-A_c^* = \begin{pmatrix} -A_c^*|_{P(H_B^4 \times H_B^2)} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{H}(P(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})), P(L_2(0,L;\mathbb{R}^3))).$$

Now, [3, Theorem I.1.6.3] implies that

$$-A_c^*\Big|_{P(H_B^4 \times H_B^2)} \in \mathcal{H}(P(H_B^4(0,L;\mathbb{R}^2) \times H_B^2(0,L;\mathbb{R})), P(L_2(0,L;\mathbb{R}^3))).$$

In order to study the spectrum of A_c^*, let $w^0 = (f^0, g^0, \Gamma^0) \in P(L_2(0,L;\mathbb{R}^3))$ be arbitrary and $w(t) := e^{-tA_C^*} w^0$ be the unique strong solution in $P(L_2(0,L;\mathbb{R}^3))$ of the linearized problem

$$\partial_t w + A_c^* w = 0, \qquad w(0) = w^0, \tag{3.22}$$

which exists in virtue of A_c^* being the negative generator of an analytic semigroup. Note that if $w = (f, g, \Gamma)$ is a solution to (3.22), then

$$\frac{1}{2}\frac{d}{dt}\left(\|(f+g)\|_2^2 + \frac{\sigma_1^c}{\sigma_2^c \mu}\|f\|_2^2 + z\|\Gamma\|_2^2\right) + \left(b_*(f_*, g_*, \Gamma_*)\begin{pmatrix} \partial_x^2(f+g) \\ \partial_x^2 f \\ \partial_x \Gamma \end{pmatrix} \middle| \begin{pmatrix} \partial_x^2(f+g) \\ \partial_x^2 f \\ \partial_x \Gamma \end{pmatrix}\right)_2 = 0,$$

where $z \in \mathbb{R}$ and b_c^z being the symmetric matrix

$$b_c^z(f_*, g_*, \Gamma_*) := \begin{pmatrix} \sigma_2^c \frac{g_*^3}{3} + \sigma_2^c \mu \left(\frac{f_*^3}{3} + f_*^2 g_* + f_* g_*^2\right) & \sigma_1^c \left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & l \\ \sigma_1^c \left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & \frac{(\sigma_1^c)^2}{\sigma_2^c \mu}\frac{f_*^3}{3} & k \\ l & k & -z(\mu f_* + g_*)\Gamma_* \sigma'(\Gamma_*) + zD \end{pmatrix}$$

with

$$k := \frac{1}{2}\left(-\frac{\sigma_1^c}{\sigma_2^c}\frac{f_*^2}{2}\sigma'(\Gamma_*) + z\left(\sigma_2^c \frac{g_*}{2} + \sigma_2^c \mu \left(\frac{f_*^2}{2} + f_* g_*\right)\right)\Gamma_*\right),$$

$$l := \frac{1}{2}\left(-\left(\mu f_* g_* + \frac{g_*^2}{2}\right)\sigma'(\Gamma_*) + z\sigma_1^c \frac{f_*^2}{2}\Gamma_*\right).$$

Observe that similar to (2.13), for sufficiently large $z \in \mathbb{R}$, the matrix $b_c^z(f_*, g_*, 0)$ is positive definite since, due to Assumption S2), its leading principal minors are positive, when choosing z suitable. Hence, there exists $\varepsilon_2 = \varepsilon_2(f_*, g_*) > 0$ depending on f_* and g_*, such that for $0 \leq \Gamma_* < \varepsilon_2$, the matrix $b_c^z(f_*, g_*, \Gamma_*)$ is positive definite. Therefore, we find a positive constant $\nu > 0$, such that

$$\frac{1}{2}\|\partial_t(f+g)\|_2^2 + \frac{\sigma_1^c}{\sigma_2^c \mu}\|\partial_t f\|_2^2 + z\|\partial_t \Gamma\|_2^2 \leq -\nu \left\| \begin{pmatrix} \partial_x^2(f+g) \\ \partial_x^2 f \\ \partial_x \Gamma \end{pmatrix} \right\|_2^2 .$$

Recall that the average value of $\tilde{w} := (f+g, f, \Gamma) \in P(H_B^4(0,L,\mathbb{R}^2) \times H_B^2(0,L,\mathbb{R}))$ is zero, so that we find, by Poincaré's inequality a constant $c > 0$, such that

$$\frac{1}{2}\|\partial_t(f+g)\|_2^2 + \frac{\sigma_1^c}{\sigma_2^c \mu}\|\partial_t f\|_2^2 + z\|\partial_t \Gamma\|_2^2 \leq -c\left(\|f+g\|_2^2 + \|f\|_2^2 + \|\Gamma\|_2^2\right).$$

Following the argumentation in Section 2, we deduce that w has exponential decay, that is,

$$\|w\|_2^2 \leq M e^{-t\omega_0}\|w^0\|_2^2$$

and $\operatorname{spec}(-A_c^*) \subset [\operatorname{Re} z \leq -\omega_0]$ for some $M \geq 1$ and $\omega_0 > 0$. Setting $\varepsilon_* := \min\{\varepsilon_1, \varepsilon_2\}$ the proof of Lemma 3.9 is complete. □

Combining Lemma 3.9 and (3.21), the *principle of linearized stability* ([33, Theorem 9.1.2]) implies the following theorem:

Theorem 3.10 (Asymptotic Stability). *Let $u_* = (f_*, g_*, \Gamma_*)$ be a positive steady state solution of* (1.49). *Then f_*, g_* and Γ_* are positive constants and there exist numbers $\varepsilon_* = \varepsilon_*(f_*, g_*) > 0, \omega > 0$ and $M \geq 1$, such that for $0 \leq \Gamma_* < \varepsilon_*$ and any initial data $u^0 = (f^0, g^0, \Gamma^0) \in H_B^4(0,L,\mathbb{R}^2) \times H_B^2(0,L,\mathbb{R})$ with $\langle f^0 \rangle = f_*$, $\langle g^0 \rangle = g_*$ and $\langle \Gamma^0 \rangle = \Gamma_*$ satisfying the smallness condition*

$$\|u^0 - u_*\|_{H_B^4 \times H_B^2} < \varepsilon_*,$$

the solution u of (3.1) *found in Theorem 3.1 exists globally and*

$$\|u(t) - u_*\|_{H_B^4 \times H_B^2} + \|\partial_t u(t)\|_2 \leq M e^{-\omega t}\|u^0 - u_*\|_{H_B^4 \times H_B^2} \qquad \text{for all} \quad t \geq 0.$$

3.3. Weak Solutions

Following [20, 22, 26], we prove in this section the existence of a global weak solution to (1.49). The mathematical model describing the evolution of a two–phase flow driven by capillary effects only and endowed with a layer of insoluble surfactant is given by (cf. (1.49))

$$\begin{cases} \partial_t f + \partial_x \left[\sigma_1^c \frac{f^3}{3}\partial_x^3 f + \sigma_2^c \mu \left(\frac{f^3}{3} + \frac{f^2 g}{2}\right)\partial_x^3 (f+g) + \mu \frac{f^2}{2}\partial_x \sigma(\Gamma)\right] = 0, \\ \partial_t g + \partial_x \left[\sigma_1^c \frac{f^2 g}{2}\partial_x^3 f + \left(\sigma_2^c \frac{g^3}{3} + \sigma_2^c \mu \left(\frac{f^2 g}{2} + f g^2\right)\right)\partial_x^3 (f+g) + \left(\mu f g + \frac{g^2}{2}\right)\partial_x \sigma(\Gamma)\right] = 0, \\ \partial_t \Gamma + \partial_x \left[\sigma_1^c \frac{f^2}{2}\Gamma \partial_x^3 f + \left(\sigma_2^c \frac{g^2}{2} + \sigma_2^c \mu \left(\frac{f^2}{2} + f g\right)\right)\Gamma \partial_x^3 (f+g) + (\mu f + g)\,\Gamma \partial_x \sigma(\Gamma) \right. \\ \qquad\qquad \left. - D\partial_x \Gamma \right] = 0, \end{cases} \tag{3.23a}$$

for $t > 0$ and $x \in (0, L)$ with initial conditions at $t = 0$

$$f(0,\cdot) = f^0, \quad g(0,\cdot) = g^0, \quad \Gamma(0,\cdot) = \Gamma^0 \tag{3.23b}$$

and boundary conditions at $x = 0, L$

$$\begin{aligned} \partial_x f &= \partial_x g = \partial_x \Gamma = 0, \\ \partial_x^3 f &= \partial_x^3 g = 0. \end{aligned} \tag{3.23c}$$

We recall the assumptions from the beginning of this chapter and impose additionally Assumption A2) and A3) below: Given the surface tension coefficients $\sigma_1 = \sigma_1^c \geq 0$ and σ_2 of the form

$$\sigma_2(\Gamma) = \sigma_2^c + \sigma(\Gamma),$$

where Γ is the surfactant concentration, we assume that the part of the surface tension, which is independent of the concentration of surfactant, is strictly positive and the part of the surface tension, which depends on Γ, is non–increasing, that is,

S1) $\sigma \in C^2(\mathbb{R})$ and $-\sigma'(s) > 0$ for all $s \geq 0$.

S2) $\sigma_1^c, \sigma_2^c > 0$.

Moreover, let Φ be a function, such that

A1) $\Phi \in C^2(\mathbb{R})$ with $\Phi(1) = \Phi'(1) = 0$ and

$$\Phi''(s) = -\frac{\sigma'(s)}{s} \qquad \text{for all } s \in \mathbb{R}. \tag{3.24}$$

A2) There exists $c_\Phi > 0$ such that $\Phi''(s) \geq c_\Phi$ for all $s \in \mathbb{R}$.

A3) There exists $C_\Phi > 0$ and some $r \in (0,2)$ for which $\Phi''(s) \leq C_\Phi(|s|^r + 1)$ for all $s \in \mathbb{R}$.

In A1)–A3), we suppose the assumptions to hold on the whole real line instead of the physically relevant range $[0,\infty)$. For our purpose, this is needed due to the fact that a–priori it is not clear whether the solution we construct for the surfactant concentration is non–negative. Unfortunately, theses assumptions do not allow to consider surface tension profiles as commonly used and suggested in [27]. In [20] the existence of non–negative weak solutions for the one–phase thin film with insoluble surfactant is shown under less restrictive assumptions on the surface profile, which allows for more general surface tension profiles.
Following the same approach as in [20, 26], where global weak solutions to a one–phase thin film model with insoluble surfactant are proven and [22], where the existence of global weak solutions to a two–phase thin film model is shown, we combine these results and prove the following theorem, which states the existence of global weak solutions for the fourth–order two–phase thin film problem with insoluble surfactant (3.23). Moreover, we show that the solutions corresponding to non–negative initial data stay non–negative almost everywhere, which is again done by the same methods used in [20, 22].

Theorem 3.11 (Global Weak Solutions). *Let $f^0, g^0 \in H^1(0,L)$ and $\Gamma^0 \in L_{2(r+1)}(0,L)$, where $r \in (0,2)$ corresponds to Assumption A3), be non–negative functions. Then, there exists at least one global weak solution (f, g, Γ) of problem (3.23) in the sense that:*

a) the solution has the regularity

$$f, g \in L_\infty(0,T; H^1(0,L)) \cap C([0,T]; C^\alpha([0,L])) \quad \text{for all} \quad \alpha \in \Big[0, \frac{1}{2}\Big),$$

$$\Gamma \in L_\infty(0,T; L_2(0,L)) \cap L_2(0,T; H^1(0,L)) \cap C([0,T]; (H^1(0,L))')$$

for all $T > 0$,

b) $(f,g,\Gamma)(0) = (f^0, g^0, \Gamma^0)$ *and* $f \geq 0$, $g \geq 0$, $\Gamma \geq 0$ *in* Ω_T, *where* $\Gamma(0) = \Gamma^0$ *and* $\Gamma \geq 0$ *is to be understood as almost everywhere,*

c) the mass of the fluids is conserved, that is for almost every $t \geq 0$ *we have*

$$\|f(t)\|_1 = \|f^0\|_1, \qquad \|g(t)\|_1 = \|g^0\|_1, \qquad \|\Gamma(t)\|_1 = \|\Gamma^0\|_1,$$

d) defining for every $T > 0$ *the sets*

$$\mathcal{P}_f := \{(t,x) \in \Omega_T : \ f(t,x) > 0\} \quad \text{and} \quad \mathcal{P}_g := \{(t,x) \in \Omega_T : \ g(t,x) > 0\},$$

we have $\partial_x^3 f, \partial_x^3 g \in L_2(\mathcal{P}_f \cap \mathcal{P}_g)$ *and there exist functions* $H_f, H_g, H_\Gamma \in L_2(\Omega_T)$, *which can be identified on the set* $\mathcal{P}_f \cap \mathcal{P}_g$ *with*

$$H_f = \frac{\sqrt{\sigma_2^c \mu}}{\sqrt{3}} f^2 \left[\frac{f}{\sqrt{3\sigma_2^c \mu}} \partial_x^3((\sigma_1^c + \sigma_2^c \mu) f + \sigma_2^c \mu g) + \frac{\sqrt{3}}{2} \left(\sqrt{\sigma_2^c \mu} g \partial_x^3 (f+g) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}} \partial_x \sigma(\Gamma) \right) \right],$$

$$\begin{aligned} H_g = {} & \frac{\sqrt{3\sigma_2^c \mu}}{2} gf \left[\frac{f}{\sqrt{3\sigma_2^c \mu}} \partial_x^3((\sigma_1^c + \sigma_2^c \mu) f + \sigma_2^c \mu g) + \frac{2}{\sqrt{3}} \left(\sqrt{\sigma_2^c \mu} g \partial_x^3 (f+g) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}} \partial_x \sigma(\Gamma) \right) \right] \\ & + \frac{\sqrt{\sigma_2^c}}{\sqrt{3}} g^2 \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} g \partial_x^3 (f+g) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}} \partial_x \sigma(\Gamma) \right], \end{aligned}$$

$$\begin{aligned} H_\Gamma = {} & \frac{\sqrt{3\sigma_2^c}}{2} \Gamma f \left[\frac{f}{\sqrt{3\sigma_2^c \mu}} \partial_x^3((\sigma_1^c + \sigma_2^c \mu) f + \sigma_2^c \mu g) + \frac{2}{\sqrt{3}} \left(\sqrt{\sigma_2^c \mu} g \partial_x^3 (f+g) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}} \partial_x \sigma(\Gamma) \right) \right] \\ & + \frac{\sqrt{3\sigma_2^c}}{2} \Gamma g \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} g \partial_x^3 (f+g) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}} \partial_x \sigma(\Gamma) \right] + \frac{1}{4} \Gamma g \partial_x \sigma(\Gamma) - D \partial_x \Gamma. \end{aligned}$$

Further,

$$\int_0^T \langle \partial_t f(t), \xi(t) \rangle_{H^1} \, dt = \int_{\Omega_T} H_f \partial_x \xi \, d(x,t), \tag{3.25}$$

$$\int_0^T \langle \partial_t g(t), \xi(t) \rangle_{H^1} \, dt = \int_{\Omega_T} H_g \partial_x \xi \, d(x,t), \tag{3.26}$$

$$\int_0^T \langle \partial_t \Gamma(t), \xi(t) \rangle_{H^1} \, dt = \int_{\Omega_T} H_\Gamma \partial_x \xi \, d(x,t) \tag{3.27}$$

for all $\xi \in L_2(0,T; H^1(0,L))$.

e) the energy inequality

$$\mathcal{E}(f,g,\Gamma)(T) + \mathcal{D}(f,g,\Gamma)(T) \leq \mathcal{E}(f^0, g^0, \Gamma^0)$$

is satisfied for almost all $T \geq 0$, where

$$\mathcal{E}(f,g,\Gamma)(T) := \int_0^L \left\{ \frac{1}{2}|\partial_x(f+g)(T,x)|^2 + \frac{\sigma_1^c}{2\sigma_2^c\mu}|\partial_x f(T,x)|^2 + \frac{1}{\sigma_2^c}\Phi(\Gamma(T,x)) \right\} dx$$

and

$$\begin{aligned}\mathcal{D}(f,g,\Gamma)(T) := \int_{\mathcal{P}_f\cap\mathcal{P}_g} f \Bigg[&\frac{1}{\sqrt{3\sigma_2^c\mu}} f\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f + \sigma_2^c\mu g) \\ &+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}g\partial_x^3(f+g) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right)\Bigg]^2 d(x,t) \\ &+ \int_{\mathcal{P}_f\cap\mathcal{P}_g} \frac{1}{4} f\left[\sqrt{\sigma_2^c}g\partial_x^3(f+g) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right]^2 d(x,t) \\ &+ \int_{\mathcal{P}_f\cap\mathcal{P}_g} g\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}g\partial_x^3(f+g) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right]^2 d(x,t) \\ &+ \int_{\Omega_T}\frac{1}{4\sigma_2^c}g|\partial_x\sigma(\Gamma)|^2\,d(x,t) + \int_{\Omega_T}\frac{D}{\sigma_2^c}\Phi''(\Gamma)|\partial_x\Gamma|^2\,d(x,t).\end{aligned}$$

In order to prove the existence of global weak solutions to (3.23), we construct a family of regularized systems, which tend in the limit to the original system, and prove by using a Galerkin approximation and a–priori estimates that there exist global weak solutions to the regularized problems (Section 3.3.1). In a second step we show that a certain sequence of weak solutions of the regularized problems tends in the limit to a weak solution of the original problem (Section 3.3.2).

3.3.1. The Regularized Systems

Proceeding analogously to what follows, it is also possible to construct Galerkin approximations $(f^n, g^n, \Gamma^n)_{n\in\mathbb{N}}$ directly for the original system (3.23). However, there occur technical problems as for example the lack of an energy functional, which would provide not only the global existence of the Galerkin approximations but also needed a–priori bounds on the Galerkin approximation, which allow to extract subsequences converging to a weak solution of (3.23). This difficulty arises due to the degeneracy in (3.23a), which may appear in the first and second equation if f and g decrease to zero, respectively. To avoid this, we define for every $\varepsilon \in (0,1]$ the function $a_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}_+$ by

$$a_\varepsilon(s) := \varepsilon + \max\{0, s\} \qquad \text{for } \ s \in \mathbb{R}$$

and replace f and g in (3.23a) by $a_\varepsilon(f) > 0$ and $a_\varepsilon(g) > 0$, respectively. In doing so, we may prove Theorem 3.11, by combining the ideas in [22] and [26], apart from the non–negativity of Γ. In [20] a similar result as in [26] is shown, where additionally the non–negativity of Γ is proven. This is done by substituting the terms involving Γ in the evolution equations by a truncation operator, which ensures that the solutions of the regularized problems satisfy the non–negativity of the surfactant concentration. In accordance to [20] we introduce the truncation function

$$\mathcal{T}(s) := \begin{cases} s, & \text{if} \quad s \in (0,1), \\ 2 - s, & \text{if} \quad s \in [1,2], \\ 0, & \text{if} \quad s \geq 2, \end{cases} \qquad\qquad \mathcal{T}(s) = \mathcal{T}(-s), \quad \text{if} \quad s < 0$$

and put $\mathcal{T}_\varepsilon := \varepsilon^{-1}\mathcal{T}(\cdot\,\varepsilon)$ for $\varepsilon^{-1}(0,1]$. Further, we set

$$\sigma_\varepsilon(s) := \int_1^s \mathcal{T}_\varepsilon(\sigma'(\tau))\, d\tau, \qquad \text{for} \quad s \in \mathbb{R}.$$

Note, that by construction and Assumption S2), $\sigma_\varepsilon \in C^1(\mathbb{R})$

$$|\sigma_\varepsilon'(s)| \leq |\sigma'(s)| \qquad \text{for all} \quad s \in \mathbb{R}. \tag{3.28}$$

Associated to σ_ε, we introduce a truncation of the identity

$$\tau_\varepsilon(s) := s\frac{\sigma_\varepsilon'(s)}{\sigma'(s)} \qquad \text{for} \quad s \in \mathbb{R}. \tag{3.29}$$

This is well–defined in view of (3.28). We emphasize that τ_ε has compact support $\operatorname{supp}(\tau_\varepsilon) \subset [-2\varepsilon^{-1}, 2\varepsilon^{-1}]$, is Liptschitz continuous and

$$|\tau_\varepsilon(s)| \leq |s| \qquad \text{for} \quad s \in \mathbb{R}, \tag{3.30}$$

which implies that $\|\tau_\varepsilon\|_\infty \le 2\varepsilon^{-1}$. Using now a_ε for f, g and the truncation τ_ε of the identity for the surfactant concentration Γ, we introduce the regularized problem

$$\begin{cases} \partial_t f_\varepsilon + \partial_x \left[\sigma_1^c \frac{a_\varepsilon(f_\varepsilon)^3}{3}\partial_x^3 f_\varepsilon + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon)^3}{3} + \frac{a_\varepsilon(f_\varepsilon)^2 a_\varepsilon(g_\varepsilon)}{2}\right)\partial_x^3(f_\varepsilon+g_\varepsilon) + \mu\frac{a_\varepsilon(f_\varepsilon)^2}{2}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right] \\ \qquad = 0, \\ \partial_t g_\varepsilon + \partial_x \left[\sigma_1^c \frac{a_\varepsilon(f_\varepsilon)^2 a_\varepsilon(g_\varepsilon)}{2}\partial_x^3 f_\varepsilon + \left(\sigma_2^c\frac{a_\varepsilon(g_\varepsilon)^3}{3} + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon)^2 a_\varepsilon(g_\varepsilon)}{2} + a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon)^2\right)\right)\right. \\ \qquad \left. \times\partial_x^3(f_\varepsilon+g_\varepsilon) + \left(\mu a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon) + \frac{a_\varepsilon(g_\varepsilon)^2}{2}\right)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right] = 0, \\ \partial_t\Gamma_\varepsilon + \partial_x\left[\sigma_1^c\frac{a_\varepsilon(f_\varepsilon)^2}{2}\tau_\varepsilon(\Gamma_\varepsilon)\partial_x^3 f_\varepsilon + \left(\sigma_2^c\frac{a_\varepsilon(g_\varepsilon)^2}{2} + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon)^2}{2} + a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon)\right)\right)\right. \\ \qquad \left. \times\tau_\varepsilon(\Gamma_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon) + \left(\mu a_\varepsilon(f_\varepsilon) + a_\varepsilon(g_\varepsilon)\right)\tau_\varepsilon(\Gamma_\varepsilon)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) - D\partial_x\Gamma_\varepsilon\right] = 0 \end{cases} \tag{3.31}$$

in Ω_T supplemented with initial and boundary conditions (3.23b), (3.23c). Observe that (formally), if ε tends to zero and the limit functions $\lim_{\varepsilon\to 0} f_\varepsilon = f$, $\lim_{\varepsilon\to 0} g_\varepsilon = g$ and $\lim_{\varepsilon\to 0}\Gamma_\varepsilon = \Gamma$ are non–negative, $a_\varepsilon, \tau_\varepsilon$ tend to the identity and the regularized system tends to the original system. The system (3.31) is more regular than (3.23a) in the sense that the coefficients of the fourth–order terms in the equations for f and g are bounded from below by $\varepsilon > 0$. Hence, (3.31) is uniformly parabolic. We show that for any fixed $\varepsilon > 0$ the problem (3.31), supplemented with initial and boundary conditions (3.23b), (3.23c), admits a global weak solution.

Theorem 3.12 (Global Weak Solutions for the Regularized Systems)**.** *Let $\varepsilon \in (0,1]$ be fixed and $(f^0, g^0, \Gamma^0) \in (H^1(0,L))^2 \times L_{2(r+1)}(0,L)$, where $r \in (0,2)$ corresponds to Assumption A3). Then, for any $T > 0$ there exists at least one triple of functions $(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)$ having the regularity*

$$\begin{aligned} & f_\varepsilon, g_\varepsilon \in L_\infty(0,T;H^1(0,L)) \cap L_2(0,T;H^3(0,L)) \cap C([0,T];C^\alpha([0,L])), \quad \alpha \in \left[0,\frac{1}{2}\right), \\ & \Gamma_\varepsilon \in L_\infty(0,T;L_2(0,L)) \cap L_2(0,T;H^1(0,L)) \cap C([0,L],(H^1(0,L))'), \end{aligned} \tag{3.32}$$

$$\partial_t f_\varepsilon, \partial_t g_\varepsilon, \partial_t\Gamma \in L_2(0,T;(H^1(0,L))'),$$

satisfying

$$\int_0^T \langle \partial_t f_\varepsilon(t), \xi(t)\rangle_{H^1}\, dt = \int_{\Omega_T} F_f^\varepsilon \partial_x \xi \, d(x,t), \tag{3.33}$$

$$\int_0^T \langle \partial_t g_\varepsilon(t), \xi(t)\rangle_{H^1}\, dt = \int_{\Omega_T} F_g^\varepsilon \partial_x \xi \, d(x,t), \tag{3.34}$$

$$\int_0^T \langle \partial_t \Gamma_\varepsilon(t), \xi(t)\rangle_{H^1}\, dt = \int_{\Omega_T} F_\Gamma^\varepsilon \partial_x \xi \, d(x,t), \tag{3.35}$$

with

$$F_f^\varepsilon := \left[\sigma_1^c \frac{a_\varepsilon(f_\varepsilon)^3}{3}\partial_x^3 f_\varepsilon + \sigma_2^c \mu \left(\frac{a_\varepsilon(f_\varepsilon)^3}{3} + \frac{a_\varepsilon(f_\varepsilon)^2 a_\varepsilon(g_\varepsilon)}{2}\right)\partial_x^3(f_\varepsilon + g_\varepsilon) + \mu \frac{a_\varepsilon(f_\varepsilon)^2}{2}\partial_x \sigma_\varepsilon(\Gamma_\varepsilon)\right],$$

$$\begin{aligned} F_g^\varepsilon := \Bigg[&\sigma_1^c \frac{a_\varepsilon(f_\varepsilon)^2 a_\varepsilon(g_\varepsilon)}{2}\partial_x^3 f_\varepsilon + \left(\sigma_2^c \frac{a_\varepsilon(g_\varepsilon)^3}{3} + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon)^2 a_\varepsilon(g_\varepsilon)}{2} + a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon)^2\right)\right)\partial_x^3(f_\varepsilon + g_\varepsilon) \\ &+ \left(\mu a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon) + \frac{a_\varepsilon(g_\varepsilon)^2}{2}\right)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\Bigg], \end{aligned}$$

$$\begin{aligned} F_\Gamma^\varepsilon := \Bigg[&\sigma_1^c \frac{a_\varepsilon(f_\varepsilon)^2}{2}\tau_\varepsilon(\Gamma_\varepsilon)\partial_x^3 f_\varepsilon + \left(\sigma_2^c \frac{a_\varepsilon(g_\varepsilon)^2}{2} + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon)^2}{2} + a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon)\right)\right)\tau_\varepsilon(\Gamma_\varepsilon)\partial_x^3(f_\varepsilon + g_\varepsilon) \\ &+ \left(\mu a_\varepsilon(f_\varepsilon) + a_\varepsilon(g_\varepsilon)\right)\tau_\varepsilon(\Gamma_\varepsilon)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) - D\partial_x\Gamma_\varepsilon\Bigg] \end{aligned}$$

for all $\xi \in L_2(0,T;H^1(0,L))$. Further

$$(f_\varepsilon(0,\cdot), g_\varepsilon(0,\cdot), \Gamma_\varepsilon(0,\cdot)) = (f^0, g^0, \Gamma^0) \tag{3.36}$$

and the mass of the fluids and the surfactant concentration is preserved

$$\int_0^L f_\varepsilon(t)\, dx = \|f^0\|_1, \qquad \int_0^L g_\varepsilon(t)\, dx = \|g^0\|_1, \qquad \int_0^L \Gamma_\varepsilon(t)\, dx = \|\Gamma^0\|_1 \tag{3.37}$$

for almost all $t \geq 0$. Moreover, there holds the energy inequality

$$\mathcal{E}(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)(T) + \mathcal{D}_\varepsilon(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)(T) \leq \mathcal{E}(f^0, g^0, \Gamma^0) \tag{3.38}$$

for almost all $T \geq 0$, where

$$\begin{aligned} \mathcal{D}_\varepsilon(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)(T) := \int_{\Omega_T} \Bigg\{ a_\varepsilon(f_\varepsilon)\Bigg[&\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon + \sigma_2^c\mu g_\varepsilon) \\ &+ \frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon + g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\Bigg]^2\Bigg\}\, d(x,t) \end{aligned}$$

$$+\int_{\Omega_T}\frac{1}{4}a_\varepsilon(f_\varepsilon)\left[\sqrt{\sigma_2^c}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]^2 d(x,t)$$

$$+\int_{\Omega_T}a_\varepsilon(g_\varepsilon)\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]^2 d(x,t)$$

$$+\int_{\Omega_T}\frac{1}{4\sigma_2^c}a_\varepsilon(g_\varepsilon)|\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)|^2\,d(x,t)+\int_0^T\int_0^L\frac{D}{\sigma_2^c}\Phi''(\Gamma_\varepsilon)|\partial_x\Gamma_\varepsilon|^2\,d(x,t).$$

Proving Theorem 3.12, we first construct a Galerkin approximation for a weak solution of problem (3.31), (3.23b) and (3.23c).

Approximation of a Weak Solution of (3.31) **by Fourier Series Expansions.** Let $\varepsilon\in(0,1]$ be fixed. Following [22, 26, 47], we construct a solution to (3.31), (3.23b) and (3.23c) by the method of Galerkin approximation. That is, we are seeking for functions $f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n$, such that the problem is satisfied in a weak sense, when testing against functions from an n–dimensional subspace. These solutions are called *Galerkin approximations*.
Note that the normalized eigenvectors of $-\Delta : H^2(0,L)\longrightarrow L_2(0,L)$, which satisfy zero Neumann–boundary conditions are given by

$$\phi_0:=\sqrt{\frac{1}{L}}\qquad\text{and}\qquad\phi_k:=\sqrt{\frac{2}{L}}\cos\left(\frac{k\pi x}{L}\right),\qquad k\geq 1,$$

and form an orthonormal basis in $L_2(0,L)$. It is known that any function belonging to $H^1(0,L)$ can be written as $\sum_{k=0}^\infty\alpha_k\phi_k$, where the series converges in $H^1(0,L)$ and $\alpha_k:=(f\mid\phi_k)_2$ for $k\geq 0$. We refer to e.g. [13] for more details. We take a Galerkin–ansatz for $f_\varepsilon, g_\varepsilon$ and $\Phi'(\Gamma_\varepsilon)$. Since $\Phi'\in C^1(\mathbb{R})$ and $\Phi''>0$, by (3.24), there exists a continuous differentiable inverse function $W:=(\Phi')^{-1}$. Set $v_\varepsilon:=\Phi'(\Gamma_\varepsilon)$, then $\Gamma_\varepsilon=W(v_\varepsilon)$ and

$$\partial_x\sigma_\varepsilon(W(v))=\sigma_\varepsilon'(W(v))W'(v)\partial_x v=-\tau_\varepsilon(W(v))\frac{\sigma'(W(v))}{W(v)}\frac{W(v)}{\sigma'(W(v))}\partial_x v=-\tau_\varepsilon(W(v))\partial_x v,$$

so that the third equation of (3.31) becomes

$$\partial_t W(v_\varepsilon)+\partial_x\left[\sigma_1^c\frac{a_\varepsilon(f_\varepsilon)^2}{2}\tau_\varepsilon(W(v_\varepsilon))\partial_x^3 f_\varepsilon+\left(\sigma_2^c\frac{a_\varepsilon(g_\varepsilon)^2}{2}+\sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon)^2}{2}+a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon)\right)\right)\right.$$
$$\left.\times\,\tau_\varepsilon(W(v_\varepsilon))\partial_x^3(f_\varepsilon+g_\varepsilon)-\left(\mu a_\varepsilon(f_\varepsilon)+a_\varepsilon(g_\varepsilon)\right)\tau_\varepsilon(W(v_\varepsilon))^2\partial_x v_\varepsilon-D\partial_x W(v_\varepsilon)\right]=0.$$

Observe that Assumption A2) implies that

$$|\Phi'(s)|=\left|\int_1^s\Phi''(t)\,dt\right|\leq C_\Phi\left|\int_1^s(|t|^r+1)\,dt\right|=C_\Phi\left(|s-1|+\left|\frac{s^{r+1}}{r+1}-\frac{1}{r+1}\right|\right).\quad(3.39)$$

Hence, $\Phi'(\Gamma^0) \in L_2(0,L)$, by $\Gamma^0 \in L_{2(r+1)}(0,L)$. For $f^0, g^0 \in H^1(0,L)$ and $v^0 := \Phi'(\Gamma^0) \in L_2(0,L)$ there exist sequences $(f_{0k})_{k\in\mathbb{N}}, (g_{0k})_{k\in\mathbb{N}}$ and $(v_{0k})_{k\in\mathbb{N}}$, such that

$$\begin{aligned} f_0^n &:= \sum_{k=0}^n f_{0k}\phi_k \qquad \text{with} \qquad f_0^n \longrightarrow f^0 \quad \text{in } H^1(0,L), \\ g_0^n &:= \sum_{k=0}^n g_{0k}\phi_k \qquad \text{with} \qquad g_0^n \longrightarrow g^0 \quad \text{in } H^1(0,L), \\ v_0^n &:= \sum_{k=0}^n v_{0k}\phi_k \qquad \text{with} \qquad v_0^n \longrightarrow v^0 \quad \text{in } L_2(0,L). \end{aligned}$$

We seek for continuous differentiable functions

$$f_\varepsilon^n(t,x) := \sum_{k=0}^n F_\varepsilon^k(t)\phi_k(x), \quad g_\varepsilon^n(t,x) := \sum_{k=0}^n G_\varepsilon^k(t)\phi_k(x), \quad v_\varepsilon^n(t,x) := \sum_{k=0}^n V_\varepsilon^k(t)\phi_k(x) \quad \text{in} \quad \Omega_T,$$

which solve (3.31) when testing with functions from the linear subspace spanned by $\{\phi_0, \dots, \phi_n\}$ and which satisfy initially

$$f_\varepsilon^n(0,\cdot) = f_0^n, \qquad g_\varepsilon^n(0,\cdot) = g_0^n, \qquad v_\varepsilon^n(0,\cdot) = v_0^n.$$

Set $\Gamma_\varepsilon^n := W(v_\varepsilon^n)$. By construction the functions $f_\varepsilon^n, g_\varepsilon^n, v_\varepsilon^n$ satisfy the boundary condition. Due to $\partial_x v_\varepsilon^n = \Phi''(\Gamma_\varepsilon^n)\partial_x\Gamma_\varepsilon^n = 0$ at $x = 0, L$ and $\Phi'' > 0$ by (3.24), we obtain that also Γ_ε^n satisfies $\partial_x\Gamma_\varepsilon^n = 0$ at $x = 0, L$.

Lemma 3.13. *For $\varepsilon \in (0,1]$ fixed and for any $T > 0$, the problem* (3.31), (3.23b), (3.23c) *admits for every $n \in \mathbb{N}$ a unique global Galerkin approximation $(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)$, where $\Gamma_\varepsilon^n = W(v_\varepsilon^n)$. The approximation has for each $n \in \mathbb{N}$ the regularity*

$$f_\varepsilon^n, g_\varepsilon^n \in C^1([0,T]; C^\infty(0,L)),$$

$$\Gamma_\varepsilon^n \in C^1([0,T]; C^1(0,L))$$

and the boundary conditions

$$\partial_x^3 f_\varepsilon^n = \partial_x^3 g_\varepsilon^n = \partial_x f_\varepsilon^n = \partial_x g_\varepsilon^n = \partial_x \Gamma_\varepsilon^n = 0 \qquad \text{at} \quad x = 0, L$$

are satisfied. Furthermore, conservation of mass

$$\int_0^L f_\varepsilon^n(t)\,dx = \|f^0\|_1, \qquad \int_0^L g_\varepsilon^n(t)\,dx = \|g^0\|_1, \qquad \int_0^L \Gamma_\varepsilon^n(t)\,dx = \|\Gamma^0\|_1 \tag{3.40}$$

for all $t \geq 0$ *and the energy equality*

$$\mathcal{E}(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) + \mathcal{D}_\varepsilon(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) = \mathcal{E}(f_0^n, g_0^n, \Gamma_0^n) \tag{3.41}$$

hold true for all $T \geq 0$.

Proof. We test the equations in (3.31) successively with $\phi_0, \ldots, \phi_n$ and integrate by parts. Due to the boundary conditions and the special structure of the equations in (3.31), the boundary terms vanish and we obtain a system of ordinary differential equations, which due to the Picard–Lindelöf Theorem admits a unique local solution. Testing (3.31) against ϕ_j for some $j \in \{0, \ldots, n\}$ yields

$$(\partial_t f_\varepsilon^n \mid \phi_j)_2 - \left(\sigma_1^c \frac{a_\varepsilon(f_\varepsilon^n)^3}{3} \partial_x^3 f_\varepsilon^n + \sigma_2^c \mu \left(\frac{a_\varepsilon(f_\varepsilon^n)^3}{3} + \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2}\right) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \right.$$
$$\left. - \mu \frac{a_\varepsilon(f_\varepsilon^n)^2}{2} \tau_\varepsilon(W(v_\varepsilon^n)) \partial_x v_\varepsilon^n \,\Big|\, \partial_x \phi_j \right)_2 = 0,$$

$$(\partial_t g_\varepsilon^n \mid \phi_j)_2 - \left(\sigma_1^c \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2} \partial_x^3 f_\varepsilon^n + \left(\sigma_2^c \frac{a_\varepsilon(g_\varepsilon^n)^3}{3} + \sigma_2^c \mu \left(\frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2} + a_\varepsilon(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n)^2\right)\right)\right.$$
$$\left. \times \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) + \left(\mu a_\varepsilon(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n) - \frac{a_\varepsilon(g_\varepsilon^n)^2}{2}\right) \tau_\varepsilon(W(v_\varepsilon^n)) \partial_x v_\varepsilon^n \,\Big|\, \partial_x \phi_j \right)_2 = 0,$$

$$(\partial_t W(v_\varepsilon^n) \mid \phi_j)_2 - \left(\sigma_1^c \frac{a_\varepsilon(f_\varepsilon^n)^2}{2} \tau_\varepsilon(W(v_\varepsilon^n)) \partial_x^3 f_\varepsilon^n + \left(\sigma_2^c \frac{a_\varepsilon(g_\varepsilon^n)^2}{2} + \sigma_2^c \mu \left(\frac{a_\varepsilon(f_\varepsilon^n)^2}{2} + a_\varepsilon(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n)\right)\right)\right.$$
$$\left. \times \tau_\varepsilon(W(v_\varepsilon^n)) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) - (\mu a_\varepsilon(f_\varepsilon^n) + a_\varepsilon(g_\varepsilon^n)) \left(\tau_\varepsilon(W(v_\varepsilon))\right)^2 \partial_x v_\varepsilon^n - D \partial_x W(v_\varepsilon^n) \,\Big|\, \partial_x \phi_j \right)_2 = 0.$$

Define $\Psi := (\Psi_1, \Psi_2, \Psi_3) : \mathbb{R}^{3(n+1)} \longrightarrow \mathbb{R}^{3(n+1)}$ by

$$\Psi_{1,j}(p,q,r) := \sum_{k=1}^n p^k \left(\sigma_1^c \frac{a_\varepsilon(\Theta_f(p))^3}{3} \partial_x^3 \phi_k \Big| \partial_x \phi_j \right)_2$$
$$+ \sum_{k=1}^n (p^k + q^k) \left(\sigma_2^c \mu \left(\frac{a_\varepsilon(\Theta_f(p))^3}{3} + \frac{a_\varepsilon(\Theta_f(p))^2 a_\varepsilon(\Theta_g(q))}{2}\right) \partial_x^3 \phi_k \,\Big|\, \partial_x \phi_j \right)_2$$
$$- \sum_{k=1}^n r^k \left(\mu \frac{a_\varepsilon(\Theta_f(p))^2}{2} \tau_\varepsilon(W(\Theta_v(r))) \partial_x \phi_k \,\Big|\, \partial_x \phi_j \right)_2,$$

$$\Psi_{2,j}(p,q,r) := \sum_{k=1}^n p^k \left(\sigma_1^c \frac{a_\varepsilon(\Theta_f(p))^2 a_\varepsilon(\Theta_g(q))}{2} \partial_x^3 \phi_k \,\Big|\, \partial_x \phi_j \right)_2 + \sum_{k=1}^n (p^k + q^k) \left(\left(\frac{\sigma_2^c a_\varepsilon(\Theta_g(q))^3}{3}\right.\right.$$
$$\left.\left. + \sigma_2^c \mu \left(\frac{a_\varepsilon(\Theta_f(p))^2 a_\varepsilon(\Theta_g(q))}{2} + a_\varepsilon(\Theta_f(p)) a_\varepsilon(\Theta_g(q))^2\right)\right) \partial_x^3 \phi_k \,\Big|\, \partial_x \phi_j \right)_2$$

$$-\sum_{k=1}^{n} r^k \left(\left(\mu a_\varepsilon(\Theta_f(p)) a_\varepsilon(\Theta_g(q)) + \frac{a_\varepsilon(\Theta_g(q))^2}{2} \right) \tau_\varepsilon(W(\Theta_v(r)))\partial_x\phi_k \,\Big|\, \partial_x\phi_j \right)_2$$

and

$$\Psi_{3,j}(p,q,r) := \sum_{k=1}^{n} p^k \left(\sigma_1^c \frac{a_\varepsilon(\Theta_f(p))^2}{2} W(\Theta_v(r))\partial_x^3\phi_k \,\Big|\, \partial_x\phi_j \right)_2 + \sum_{k=1}^{n} (p^k+q^k) \left(\left(\frac{\sigma_2^c a_\varepsilon(\Theta_f(p))^2}{2} \right.\right.$$

$$\left.\left. + \sigma_2^c \left(\frac{a_\varepsilon(\Theta_f(q))^2}{2} + a_\varepsilon(\Theta_f(p)) a_\varepsilon(\Theta_g(q)) \right) \right) W(\Theta_v(r))\partial_x^3\phi_k \,\Big|\, \partial_x\phi_j \right)_2$$

$$-\sum_{k=1}^{n} r^k \left((\mu a_\varepsilon(\Theta_f(p)) + a_\varepsilon(\Theta_g(q))) \left(\tau_\varepsilon(W(\Theta_v(r)))\right)^2 \partial_x\phi_k - DW'(\Theta_v(r))\partial_x\phi_k \,\Big|\, \partial_x\phi_j \right)_2,$$

for $j = 0, \ldots, n$, $(p,q,r) \in \mathbb{R}^{3(n+1)}$ and

$$\Theta_f(p) := \sum_{k=0}^{n} p^k\phi_k, \qquad \Theta_g(q) := \sum_{k=0}^{n} q^k\phi_k, \qquad \Theta_v(r) := \sum_{k=0}^{n} r^k\phi_k.$$

Note that for $(F,G,V) := (F_\varepsilon^0, \ldots, F_\varepsilon^n, G_\varepsilon^0, \ldots, G_\varepsilon^n, V_\varepsilon^0, \ldots, V_\varepsilon^n)$ we obtain the ordinary differential equation

$$\frac{d}{dt}(F,G,V) = \Psi(F,G,V), \qquad (F,G,V)(0) = (f_{00}, \ldots f_{0n}, g_{00}, \ldots g_{0n}, v_{00}, \ldots v_{0n}). \quad (3.42)$$

The function $\Psi = (\Psi_1, \Psi_2, \Psi_3) : \mathbb{R}^{3(n+1)} \to \mathbb{R}^{3(n+1)}$ is locally Lipschitz continuous, since a_ε as well as τ_ε are locally Lipschitz continuous, so that the problem (3.42) admits due to the Picard–Lindelöf Theorem a unique local solution $(F,G,V) \in \left(C^1([0,T_\varepsilon^n), \mathbb{R}^n)\right)^3$, where $[0,T_\varepsilon^n)$ is the maximal time interval of existence. Hence,

$$f_\varepsilon^n, g_\varepsilon^n \in C^1([0,T_\varepsilon^n); C^\infty([0,L])),$$

$$\Gamma_\varepsilon^n \in C^1([0,T_\varepsilon^n); C^1([0,L]))$$

is a local weak solution of (3.31) in the sense that it solves the system by testing against the finite dimensional subspace spanned by $\{\phi_0, \ldots, \phi_n\}$. Note that the regularity of $\Gamma_\varepsilon^n(t)$ for each $t \in [0,T_\varepsilon^n)$ can only be shown to be continuous differentiable, since $\Gamma_\varepsilon^n = W(v_\varepsilon^n)$ and W is assumed to be only once continuous differentiable. In order to prove that the solution is global in time for every $n \in \mathbb{N}$, we use that the functional

$$\mathcal{E}(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n) = \int_0^L \left\{ \frac{1}{2}|\partial_x(f_\varepsilon^n + g_\varepsilon^n)|^2 + \frac{\sigma_1^c}{2\sigma_2^c\mu}|\partial_x f_\varepsilon^n|^2 + \frac{1}{\sigma_2^c}\Phi(\Gamma_\varepsilon^n) \right\} dx$$

decreases along the solution $(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)$ of (3.31).

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n) &= \frac{d}{dt}\int_0^L \left\{\frac{1}{2}|\partial_x(f_\varepsilon^n+g_\varepsilon^n)|^2 + \frac{\sigma_1^c}{2\sigma_2^c\mu}|\partial_x f_\varepsilon^n|^2 + \frac{1}{\sigma_2^c}\Phi(\Gamma_\varepsilon^n)\right\}\,dx \\
&= \int_0^L \left\{\partial_x(f_\varepsilon^n+g_\varepsilon^n)\partial_x\partial_t(f_\varepsilon^n+g_\varepsilon^n) + \frac{\sigma_1^c}{\sigma_2^c\mu}\partial_x f_\varepsilon^n\partial_x\partial_t f_\varepsilon^n + \frac{1}{\sigma_2^c}\Phi'(\Gamma_\varepsilon^n)\partial_t\Gamma_\varepsilon^n\right\}\,dx \\
&= -\int_0^L \left\{\partial_x^2(f_\varepsilon^n+g_\varepsilon^n)\partial_t(f_\varepsilon^n+g_\varepsilon^n) + \frac{\sigma_1^c}{\sigma_2^c\mu}\partial_x^2 f_\varepsilon^n\partial_t f_\varepsilon^n - \frac{1}{\sigma_2^c}v_\varepsilon^n\partial_t\Gamma_\varepsilon^n\right\}\,dx.
\end{aligned}$$

Since $\partial_x(f_\varepsilon^n(t)+g_\varepsilon^n(t))$ as well as $\frac{\sigma_1^c}{\sigma_2^c\mu}\partial_x^2 f_\varepsilon^n(t)$ and $\frac{1}{\sigma_2^c}v_\varepsilon^n(t) = \frac{1}{\sigma_2^c}\Phi'(\Gamma_\varepsilon^n(t))$[1] belong to $\operatorname{span}\{\phi_0,\dots,\phi_n\}$ for all $t\in[0,T_\varepsilon^n)$, we may use them as test functions for the equations (3.31) and obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_0^L \left\{\frac{1}{2}|\partial_x(f_\varepsilon^n+g_\varepsilon^n)|^2 + \frac{\sigma_1^c}{2\sigma_2^c\mu}|\partial_x f_\varepsilon^n|^2 + \frac{1}{\sigma_2^c}\Phi(\Gamma_\varepsilon^n)\right\}\,dx \\
&= -\int_0^L \Bigg\{\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)\Bigg[\sigma_1^c\frac{a_\varepsilon(f_\varepsilon^n)^3}{3}\partial_x^3 f_\varepsilon^n + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon^n)^3}{3} + \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2}\right)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n) \\
&\qquad - \mu\frac{a_\varepsilon(f_\varepsilon^n)^2}{2}\tau_\varepsilon(W(v_\varepsilon^n))\partial_x v + \sigma_1^c\frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2}\partial_x^3 f_\varepsilon^n \\
&\qquad + \left(\sigma_2^c\frac{a_\varepsilon(g_\varepsilon^n)^3}{3} + \sigma_2^c\mu\left(\frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2} + a_\varepsilon(f_\varepsilon^n)a_\varepsilon(g_\varepsilon^n)^2\right)\right)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n) \\
&\qquad - \left(\mu a_\varepsilon(f_\varepsilon^n)a_\varepsilon(g_\varepsilon^n) + \frac{a_\varepsilon(g_\varepsilon^n)^2}{2}\right)\tau_\varepsilon(W(v_\varepsilon^n))\partial_x v_\varepsilon^n\Bigg]\Bigg\}\,dx \\
&\quad - \int_0^L \Bigg\{\partial_x^3 f_\varepsilon^n\Bigg[\frac{(\sigma_1^c)^2}{\sigma_2^c\mu}\frac{a_\varepsilon(f_\varepsilon^n)^3}{3}\partial_x^3 f_\varepsilon^n + \sigma_1^c\left(\frac{a_\varepsilon(f_\varepsilon^n)^3}{3} + \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2}\right)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n) \\
&\qquad - \frac{\sigma_1^c}{\sigma_2^c}\frac{a_\varepsilon(f_\varepsilon^n)^2}{2}\tau_\varepsilon(W(v_\varepsilon^n))\partial_x v_\varepsilon^n\Bigg]\Bigg\}\,dx \\
&\quad - \int_0^L \Bigg\{\partial_x v_\varepsilon^n\Bigg[-\left(\frac{a_\varepsilon(g_\varepsilon^n)^2}{2} + \mu\left(\frac{a_\varepsilon(f_\varepsilon^n)^2}{2} + a_\varepsilon(f_\varepsilon^n)a_\varepsilon(g_\varepsilon^n)\right)\right)\tau_\varepsilon(W(v_\varepsilon^n))\partial_x^3(f_\varepsilon^n+g_\varepsilon^n) \\
&\qquad - \frac{\sigma_1^c}{\sigma_2^c}\frac{a_\varepsilon(f_\varepsilon^n)^2}{2}\tau_\varepsilon(W(v_\varepsilon^n))\partial_x^3 f_\varepsilon^n + \left(\frac{\mu}{\sigma_2^c}a_\varepsilon(f_\varepsilon^n) + \frac{1}{\sigma_2^c}a_\varepsilon(g_\varepsilon^n)\right)(\tau_\varepsilon(W(v_\varepsilon^n)))^2\partial_x v_\varepsilon^n \\
&\qquad + \frac{D}{\sigma_2^c}\partial_x W(v_\varepsilon^n)\Bigg]\Bigg\}\,dx.
\end{aligned}$$

Note that, though we used integration by parts, the boundary terms vanish due to the

[1] Now, it becomes clear why we used a Galerkin–ansatz for $v = \Phi'(\Gamma)$ instead of Γ, which would have been the more natural choice. Assume we took the Galerkin–ansatz for Γ. Since Φ' is nonlinear, it would not be possible to write $\Phi'(\Gamma)$ as a linear combination of ϕ_k, $0\le k\le n$.

boundary conditions. Since $\partial_x \sigma_\varepsilon(W(v)) = -\tau_\varepsilon(W(v))\partial_x v$, we obtain that

$$\frac{d}{dt}\int_0^L \left\{ \frac{1}{2}|\partial_x(f_\varepsilon^n + g_\varepsilon^n)|^2 + \frac{\sigma_1^c}{2\sigma_2^c \mu}|\partial_x f_\varepsilon^n|^2 + \frac{1}{\sigma_2^c}\Phi(\Gamma_\varepsilon^n) \right\} dx$$

$$= -\int_0^L \left\{ \partial_x^3(f_\varepsilon^n + g_\varepsilon^n) \left[\sigma_1^c \frac{a_\varepsilon(f_\varepsilon^n)^3}{3}\partial_x^3 f_\varepsilon^n + \sigma_2^c \mu \left(\frac{a_\varepsilon(f_\varepsilon^n)^3}{3} + \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2} \right) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \right.\right.$$

$$+ \mu \frac{a_\varepsilon(f_\varepsilon^n)^2}{2}\partial_x \sigma_\varepsilon(W(v)) + \sigma_1^c \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2}\partial_x^3 f_\varepsilon^n$$

$$+ \left(\sigma_2^c \frac{a_\varepsilon(g_\varepsilon^n)^3}{3} + \sigma_2^c \mu \left(\frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2} + a_\varepsilon(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n)^2 \right) \right) \partial_x^3(f_\varepsilon^n + g_\varepsilon^n)$$

$$\left.\left. + \left(\mu a_\varepsilon(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n) + \frac{a_\varepsilon(g_\varepsilon^n)^2}{2} \right) \partial_x \sigma_\varepsilon(W(v)) \right] \right\} dx$$

$$- \int_0^L \left\{ \partial_x^3 f_\varepsilon^n \left[\frac{(\sigma_1^c)^2}{\sigma_2^c \mu} \frac{a_\varepsilon(f_\varepsilon^n)^3}{3} \partial_x^3 f_\varepsilon^n + \sigma_1^c \left(\frac{a_\varepsilon(f_\varepsilon^n)^3}{3} + \frac{a_\varepsilon(f_\varepsilon^n)^2 a_\varepsilon(g_\varepsilon^n)}{2} \right) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \right.\right.$$

$$\left.\left. - \frac{\sigma_1^c}{\sigma_2^c} \frac{a_\varepsilon(f_\varepsilon^n)^2}{2} \tau_\varepsilon(W(v_\varepsilon^n))\partial_x v_\varepsilon^n \right] \right\} dx$$

$$- \int_0^L \left\{ \left(\frac{a_\varepsilon(g_\varepsilon^n)^2}{2} + \mu \left(\frac{a_\varepsilon(f_\varepsilon^n)^2}{2} + a_\varepsilon(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n) \right) \right) \partial_x \sigma_\varepsilon(W(v)) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \right.$$

$$+ \frac{\sigma_1^c}{\sigma_2^c} \frac{a_\varepsilon(f_\varepsilon^n)^2}{2} \partial_x \sigma_\varepsilon(W(v)) \partial_x^3 f_\varepsilon^n + \left(\frac{\mu}{\sigma_2^c} a_\varepsilon(f_\varepsilon^n) + \frac{1}{\sigma_2^c} a_\varepsilon(g_\varepsilon^n) \right) |\partial_x \sigma_\varepsilon(W(v))|^2$$

$$\left. + \frac{D}{\sigma_2^c} \partial_x W(v_\varepsilon^n) \right\} dx$$

and an analog computation as in the proof of Proposition 2.2 leads to the claim (3.41)

$$\mathcal{E}(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) + \mathcal{D}_\varepsilon(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) = \mathcal{E}(f_0^n, g_0^n, \Gamma_0^n)$$

for all $T \in [0, T_\varepsilon^n)$. Due to the energy inequality (3.41), $\|\partial_x f_\varepsilon^n(t)\|_2^2$ is bounded by a constant depending on the initial data f_0^n, g_0^n, Γ_0^n for every $t \in [0, T_\varepsilon^n)$, so that

$$\|\partial_x f_\varepsilon^n(t)\|_2^2 = \left(\sum_{k=0}^n F_\varepsilon^k(t)\partial_x \phi_k \mid \sum_{k=0}^n F_\varepsilon^k(t)\partial_x \phi_k \right)_2 = \sum_{k=0}^n \left(F_\varepsilon^k(t)\partial_x\phi_k \mid F_\varepsilon^k(t)\partial_x \phi_k \right)_2$$

$$= \sum_{k=0}^n |F_\varepsilon^k(t)|^2 \|\partial_x \phi_k\|_2^2$$

is bounded, where we use the fact that $(\partial_x \phi_k \mid \partial_x \phi_l)_2 = 0$ for $k \neq l$. Therefore, $F_\varepsilon^k(t)$ is uniformly bounded for all $t \in [0, T_\varepsilon^n)$ and $k \in 0, \ldots, n$. Likewise one shows that $G_\varepsilon^k(t)$ is uniformly bounded for all $t \in [0, T_\varepsilon^n)$ and $k \in 0, \ldots, n$. The energy inequality gives a bound on $(\Phi(\Gamma_\varepsilon^n))_{n\in\mathbb{N}}$ in $L_\infty(0,T_\varepsilon^n;L_1(0,L))$. Using Assumption A2) and $\Phi(1)=\Phi'(1)=0$, we obtain that

$$\Phi(s)=\int_1^s\int_1^t \Phi''(u)\,du\,dt \geq \frac{c_\Phi}{2}(s-1)^2 \qquad \text{for all} \quad s\in\mathbb{R}. \tag{3.43}$$

Hence

$$\begin{aligned}\int_0^L |\Gamma_\varepsilon^n(t)|^2\,dx &= \int_0^L |(\Gamma_\varepsilon^n(t)-1+1)|^2\,dx \leq 2\int_0^L |\Gamma_\varepsilon^n(t)-1|^2+1\,dx \leq \frac{4}{c_\Phi}\int_0^L \Phi(\Gamma_\varepsilon^n(t))\,dx+2L\\ &\leq M^2,\end{aligned}$$

by $(\Phi(\Gamma_\varepsilon^n))_{n\in\mathbb{N}}$ being bounded in $L_\infty(0,T_\varepsilon^n;L_1(0,L))$ and (3.43), where M is a constant independent of $t\in[0,T_\varepsilon^n)$, $n\in\mathbb{N}$ and $\varepsilon\in(0,1]$. Hence

$$\|\Gamma_\varepsilon^n\|_{L_\infty(0,T_\varepsilon^n;L_2(0,L))} \leq M, \qquad \text{for} \quad n\in\mathbb{N}, \varepsilon\in(0,1]. \tag{3.44}$$

We will show that v_ε^n is uniformly bounded on $[0,T_\varepsilon^n)$, which implies the uniform boundedness of Γ_ε^n, by $|v_\varepsilon^n|\geq|c_\phi(\Gamma_\varepsilon^n-1)|$ (cf. Assumption A3) and $v_\varepsilon=\Phi'(\Gamma_\varepsilon^n)$). Recall from (3.39) that

$$|\Phi'(s)|\leq C_\Phi\left(|s-1|+\left|\frac{s^{r+1}}{r+1}-\frac{1}{r+1}\right|\right).$$

Hence $v_\varepsilon^n(t)=\Phi'(\Gamma_\varepsilon^n(t))$ is bounded in $L_p(0,L)$ with $p=\frac{2}{r+1}$. That is,

$$\|v_\varepsilon^n(t)\|_p^p=\int_0^L\left|\sum_{k=0}^n V_\varepsilon^k(t)\phi_k(x)\right|^p\,dx\leq C \qquad \text{for all} \quad t\in[0,T_\varepsilon^n)$$

for some constant $C>0$, which is independent of $n\in\mathbb{N}$ and $\varepsilon\in(0,1]$. Assume that $v_\varepsilon^n(t)$ is not uniformly bounded for every $t\in[0,T_\varepsilon^n)$, then there exists a sequence $(t_N)_{N\in\mathbb{N}}\subset[0,T_\varepsilon^n)$ with $t_N\longrightarrow T_\varepsilon^n$ for $N\longrightarrow\infty$ and $k'\in\{0,\ldots,n\}$ such that $V_\varepsilon^{k'}(t_N)\longrightarrow\infty$ if $t_N\longrightarrow T_\varepsilon^n$. Set $M(t):=\max_{0\leq k\leq n}|V_\varepsilon^k(t)|$, then

$$M(t)^p\int_0^L\left|\sum_{k=0}^n\frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right|^p\,dx\leq C \qquad \text{for all} \quad t\in[0,T_\varepsilon^n). \tag{3.45}$$

Note that $\frac{V_\varepsilon^k(t)}{M(t)}\in[-1,1]$ and there exists $k\in\{0,\ldots,n\}$ such that $\left|\frac{V_\varepsilon^k(t)}{M(t)}\right|=1$ for each $t\in[0,T_\varepsilon^n)$. We suppose that for all $\delta>0$ there exists $t\in[0,T_\varepsilon^n)$, such that

$$\int_0^L\left|\sum_{k=0}^n\frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right|^p\,dx<\delta$$

and prove a contradiction. Observe that $\frac{p}{2} = \frac{1}{r+1} \in (0,1)$ and $\left|\frac{1}{n+1}\sum_{k=0}^{n} \frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right| \leq 1$, so that

$$\frac{1}{(n+1)^2}\left\|\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k\right\|_2^2 = \int_0^L \left|\frac{1}{n+1}\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right|^2 dx \leq \int_0^L \left|\frac{1}{n+1}\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right|^{2\frac{p}{2}} dx$$

$$= \frac{1}{(n+1)^p}\int_0^L \left|\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right|^p dx < \frac{\delta}{(n+1)^p}.$$

We deduce that

$$\left\|\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k\right\|_2^2 = \left(\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k \Big| \sum_{l=0}^{n}\frac{V_\varepsilon^l(t)}{M(t)}\phi_l\right)_2 = \sum_{k=0}^{n}\left|\frac{V_\varepsilon^k(t)}{M(t)}\right|^2 < \delta(n+1)^{2-p},$$

which in turn implies that $\left|\frac{V_\varepsilon^k(t)}{M(t)}\right|^2 < \delta(n+1)^{2-p}$ for each $k \in \{0,\dots,n\}$. This is a contradiction to the definition of $M(t)$. Hence, there exists a constant $m > 0$, such that

$$\int_0^L \left|\sum_{k=0}^{n}\frac{V_\varepsilon^k(t)}{M(t)}\phi_k(x)\right|^p dx > m \qquad \text{for all} \quad t \in [0, T_\varepsilon^n).$$

Since $M(t_N) \longrightarrow \infty$ if t_N tends to T_ε^n, we find $N_0 \geq 0$ such that $M(t_N)^p > \frac{C}{m}$ for all $N \geq N_0$. Therefore

$$M(t_N)^p \int_0^L \left|\sum_{k=0}^{n}\frac{V_\varepsilon^k(t_N)}{M(t_N)}\phi_k(x)\right|^p dx > C \qquad \text{for all} \quad N \geq N_0,$$

which contradicts (3.45) and we have shown that v_ε^n is uniformly bounded for all $t \in [0, T_\varepsilon^n)$. We conclude that the Galerkin approximation $(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)$ exists globally.

Furthermore, the mass of each fluid and the surfactant concentration is preserved by the Galerkin approximation, which is a consequence of testing the equations in (3.31) against the constant function $\phi = 1$, integrating by parts and using that $\partial_x^3 f_\varepsilon^n = \partial_x^3 g_\varepsilon^n = \partial_x \Gamma_\varepsilon^n = 0$ at $x = 0, L$. Hence, also (3.40) is satisfied, which completes the proof. □

The next step is to prove that the Galerkin approximations converge towards weak solutions of (3.31), (3.23b), (3.23c).

Convergence of the Galerkin Approximations. Let $T > 0$ be fixed. In Lemma 3.13 we have seen that for every $n \in \mathbb{N}$, $\varepsilon \in (0,1]$, there exists a unique global Galerkin approximation $f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n$ of the regularized system (3.31), (3.23b), (3.23c) in the sense that (3.31) is satisfied when testing against functions belonging to the finite dimensional subspace spanned by

$\{\phi_0, \dots \phi_n\}$. We show that there exists a weakly converging subsequence of $(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)_{n\in\mathbb{N}}$, such that the accumulation point is a weak solution of the regularized problem in sense of Theorem 3.12. The proof will mainly base on a–priori estimates provided by the energy inequality (3.41) and follows [22, 26, 47]. Next, we collect all bounds satisfied by the Galerkin approximation $(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)_{n\in\mathbb{N}}$, which are uniform in $n \in \mathbb{N}$, $\varepsilon \in (0,1]$, resulting directly from (3.41):

$$\{\partial_x f_\varepsilon^n, \partial_x g_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\} \qquad \text{in } L_\infty(0,T;L_2(0,L)), \tag{3.46}$$

$$\{\Phi(\Gamma_\varepsilon^n) \mid n \in \mathbb{N}, \varepsilon \in (0,1]\} \qquad \text{in } L_\infty(0,T;L_1(0,L)), \tag{3.47}$$

$$\Bigg\{ \sqrt{a_\varepsilon(f_\varepsilon^n)} \Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) + \tfrac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\Bigg] \Bigg|\, n \in \mathbb{N}, \varepsilon \in (0,1] \Bigg\} \quad \text{in } L_2(\Omega_T), \tag{3.48}$$

$$\left\{ \sqrt{a_\varepsilon(f_\varepsilon^n)} \left[\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \right] \Bigg|\, n \in \mathbb{N}, \varepsilon \in (0,1] \right\} \text{ in } L_2(\Omega_T), \tag{3.49}$$

$$\left\{ \sqrt{a_\varepsilon(g_\varepsilon^n)} \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \right] \Bigg|\, n \in \mathbb{N}, \varepsilon \in (0,1] \right\} \text{ in } L_2(\Omega_T), \tag{3.50}$$

$$\left\{ \sqrt{a_\varepsilon(g_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \mid n \in \mathbb{N}, \varepsilon \in (0,1] \right\} \qquad \text{in } L_2(\Omega_T), \tag{3.51}$$

$$\left\{ \sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1] \right\} \qquad \text{in } L_2(\Omega_T). \tag{3.52}$$

Note, that (3.48)–(3.51) also imply bounds of

$$\left\{ \sqrt{a_\varepsilon(f_\varepsilon^n)} \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n+\sigma_2^c\mu g_\varepsilon^n) \right.\right.$$
$$\left.\left. +\tfrac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\,a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right] \middle| n\in\mathbb{N},\varepsilon\in(0,1] \right\} \quad \text{in } L_2(\Omega_T), \tag{3.53}$$

$$\left\{a_\varepsilon(f_\varepsilon^n)^{\frac{3}{2}}\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n+\sigma_2^c\mu g_\varepsilon^n) \mid n\in\mathbb{N},\varepsilon\in(0,1]\right\} \quad \text{in } L_2(\Omega_T), \tag{3.54}$$

$$\left\{a_\varepsilon(g_\varepsilon^n)^{\frac{3}{2}}\partial_x^3(f_\varepsilon^n+g_\varepsilon^n) \mid n\in\mathbb{N},\varepsilon\in(0,1]\right\} \quad \text{in } L_2(\Omega_T). \tag{3.55}$$

Lemma 3.14. *The Galerkin approximation satisfies*

i) $\{f_\varepsilon^n, g_\varepsilon^n \mid n\in\mathbb{N},\varepsilon\in(0,1]\}$ *bounded in* $L_\infty(0,T;H^1(0,L))$,
$\{f_\varepsilon^n, g_\varepsilon^n \mid n\in\mathbb{N}\}$ *bounded in* $L_\infty(0,T;H^1(0,L))\cap L_2(0,T;H^3(0,L))$,

ii) $\{\Gamma_\varepsilon^n \mid n\in\mathbb{N},\varepsilon\in(0,1]\}$ *bounded in* $L_\infty(0,T;L_2(0,L))\cap L_2(0,T;H^1(0,L))$.

We emphasize that Lemma 3.14 ii) implies the bound of

$$\{\Gamma_\varepsilon^n \mid n\in\mathbb{N},\varepsilon\in(0,1]\} \qquad \text{in} \quad L_6(\Omega_T),$$

due to [18, Proposition I.3.2].

Proof of Lemma 3.14. *i)* We know from (3.46) that $\{\partial_x f_\varepsilon^n \mid n\in\mathbb{N},\varepsilon\in(0,1]\}$ is bounded in $L_\infty(0,T;L_2(0,L))$. By the Poincaré–Wirtinger Theorem and conservation of mass, we deduce that

$$\|f_\varepsilon^n(t)\|_2 \le \left\| f_\varepsilon^n(t) - \frac{1}{L}\int_0^L f_\varepsilon^n(t)\,dx \right\|_2 + \left| \frac{1}{\sqrt{L}}\int_0^L f_\varepsilon^n(t)\,dx \right| \le c\|\partial_x f_\varepsilon^n(t)\|_2 + \frac{1}{\sqrt{L}}\|f^0\|_1$$

for some constant $c>0$ independent of $n\in\mathbb{N}$, $\varepsilon\in(0,1]$. It follows that

$$\{f_\varepsilon^n, g_\varepsilon^n \mid n\in\mathbb{N},\varepsilon\in(0,1]\} \quad \text{is bounded in} \quad L_\infty(0,T;H^1(0,L)). \tag{3.56}$$

For $\varepsilon\in(0,1]$ fixed, it follows from (3.54), (3.55) and the definition of a_ε that

$$\varepsilon^{\frac{3}{2}}\|\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n+\sigma_2^c\mu g_\varepsilon^n)\|_2 \le \|a_\varepsilon(f_\varepsilon^n)^{\frac{3}{2}}\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n+\sigma_2^c\mu g_\varepsilon^n)\|_2 < c \tag{3.57}$$

and

$$\varepsilon^{\frac{3}{2}}\|\partial_x^3(f_\varepsilon^n + g_\varepsilon^n)\|_2 \leq \|a_\varepsilon(g_\varepsilon^n)^{\frac{3}{2}}\partial_x^3(f_\varepsilon^n + g_\varepsilon^n)\|_2 < c \tag{3.58}$$

for some constant $c > 0$. Since $\sigma_1^c, \sigma_2^c > 0$ (cf. Assumption S1)), we deduce from (3.57), (3.58) that there exists a constant $c = c(\varepsilon) > 0$, such that

$$\|\partial_x^3 f_\varepsilon^n\|_{L_2(\Omega_T)}, \|\partial_x^3 g_\varepsilon^n\|_{L_2(\Omega_T)} < c(\varepsilon). \tag{3.59}$$

Again by the Poincaré–Wirtinger Theorem

$$\int_0^T \|\partial_x^2 f_\varepsilon^n(t)\|_2^2\,dt \leq \int_0^T \left(c\|\partial_x^3 f_\varepsilon^n(t)\|_2 + \left|\frac{1}{\sqrt{L}}\int_0^L \partial_x^2 f_\varepsilon^n(t)\,dx\right|\right)^2 dt \leq c^2\int_0^T \|\partial_x^3 f_\varepsilon^n(t)\|_2^2\,dt$$

for all $t \geq 0$ and some constant $c > 0$ independent of $t \geq 0$, since by construction $\int_0^L \partial_x^2 f_\varepsilon^n(t)\,dx = \partial_x f_\varepsilon^n(t,L) - \partial_x f_\varepsilon^n(t,0) = 0$. Together with (3.46), (3.56) and (3.59) we have shown that

$$\{f_\varepsilon^n, g_\varepsilon^n \mid n \in \mathbb{N}\} \quad \text{bounded in} \quad L_2(0,T;H^3(0,L)),$$

where the prove for $(g_\varepsilon^n)_{n\in\mathbb{N}}$ works likewise.

ii) We know already from (3.44), that

$$\{\Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\} \quad \text{is bounded in} \quad L_\infty(0,T;L_2(0,L)).$$

It is left to show that $\{\Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\}$ is bounded in $L_2(0,T;H^1(0,L))$.

$$\begin{aligned}\|\Gamma_\varepsilon^n\|_{L_2(0,T;H^1(0,L))}^2 &= \int_0^T \|\Gamma_\varepsilon^n(t)\|_{H^1(0,L)}^2\,dt = \int_0^T \|\Gamma_\varepsilon^n(t)\|_2^2\,dt + \int_0^T \|\partial_x\Gamma_\varepsilon^n(t)\|_2^2\,dt \\ &\leq T\|\Gamma_\varepsilon^n\|_{L_\infty(0,T;L_2(0,L))}^2 + \frac{1}{c_\Phi}\int_0^T c_\Phi\|\partial_x\Gamma_\varepsilon^n(t)\|_2^2\,dt.\end{aligned}$$

We use Assumption A2) in order to estimate the second term on the right–hand side

$$\frac{1}{c_\Phi}\int_0^T c_\Phi\|\partial_x\Gamma_\varepsilon^n(t)\|_2^2\,dt \leq \frac{1}{c_\Phi}\int_0^T \|\sqrt{\Phi''(\Gamma_\varepsilon^n(t))}\partial_x\Gamma_\varepsilon^n(t)\|_2^2\,dt,$$

which is bounded, due to (3.52). Hence, together with (3.44),

$$\{\Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\} \quad \text{is bounded in} \quad L_\infty(0,T;L_1(0,L)) \cap L_2(0,T;H^1(0,L)).$$

□

Let us notice that the bounds $f_\varepsilon^n, g_\varepsilon^n \in L_2(0,T,H^3(0,L))$ are only provided, since we fixed $\varepsilon \in (0,1]$ (cf. (3.57), (3.58)). We loose these bounds in the limit when ε tends to zero. All other bounds we established so far are uniform not only in $n \in \mathbb{N}$, but also in $\varepsilon \in (0,1]$. We make use of the a–priori bounds provided by the energy inequality and the facts that $\{f_\varepsilon^n, g_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\}$ is bounded in $L_\infty(\Omega_T)$ and $\{\Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\}$ is bounded in $L_6(\Omega_T)$ (cf. Lemma 3.14) in order to derive uniform bounds for the time derivatives of the Galerkin approximation. Setting

$$
\begin{aligned}
H_f^{\varepsilon,n} &:= \frac{\sqrt{\sigma_2^c\mu}}{\sqrt{3}} a_\varepsilon(f_\varepsilon^n)^2 \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\qquad \left. + \frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right],\\
H_g^{\varepsilon,n} &:= \frac{\sqrt{3\sigma_2^c\mu}}{2} a_\varepsilon(g_\varepsilon^n)a_\varepsilon(f_\varepsilon^n) \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\qquad \left. + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\\
&\quad + \frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon^2(g_\varepsilon^n)\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right],\\
H_\Gamma^{\varepsilon,n} &:= \frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon^n)a_\varepsilon(f_\varepsilon^n) \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\qquad \left. + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\\
&\quad + \frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon^n)a_\varepsilon(g_\varepsilon^n)\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right]\\
&\quad + \frac{1}{4}\Gamma_\varepsilon^n a_\varepsilon(g_\varepsilon^n)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) - D\partial_x\Gamma_\varepsilon^n,
\end{aligned}
$$

it will be useful in the sequel to rewrite the system (3.31) as

$$
\left\{
\begin{aligned}
\partial_t f_\varepsilon^n &= -\partial_x H_f^{\varepsilon,n},\\
\partial_t g_\varepsilon^n &= -\partial_x H_g^{\varepsilon,n},\\
\partial_t \Gamma_\varepsilon^n &= -\partial_x H_\Gamma^{\varepsilon,n}.
\end{aligned}
\right.
\tag{3.60}
$$

Lemma 3.15. *The time derivatives of the Galerkin approximation satisfy*

$$\{\partial_t f_\varepsilon^n, \partial_t g_\varepsilon^n, \partial_t \Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\} \quad \textit{bounded in} \quad L_2(0,T;(H^1(0,L))').$$

Proof. Observe that $H_f^{\varepsilon,n}, H_g^{\varepsilon,n} \in L_2(\Omega_T)$, since

$$\begin{aligned}
\|H_f^{\varepsilon,n}\|_{L_2(\Omega_T)} &= \Bigg\| \frac{\sqrt{\sigma_2^c\mu}}{\sqrt{3}} a_\varepsilon(f_\varepsilon^n)^2 \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\quad \left. + \frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\Bigg\|_{L_2(\Omega_T)}\\
&\le \left\| \frac{\sqrt{\sigma_2^c\mu}}{\sqrt{3}} a_\varepsilon(f_\varepsilon^n)^{\frac{3}{2}} \right\|_{L_\infty(\Omega_T)} \Bigg\| a_\varepsilon(f_\varepsilon^n)^{\frac{1}{2}} \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\quad \left. + \frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\Bigg\|_{L_2(\Omega_T)} < c,
\end{aligned}$$

by (3.48) and Lemma 3.14 i), and

$$\begin{aligned}
\|H_g^{\varepsilon,n}\|_{L_2(\Omega_T)} &= \Bigg\| \frac{\sqrt{3\sigma_2^c\mu}}{2} a_\varepsilon(g_\varepsilon^n) a_\varepsilon(f_\varepsilon^n) \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\quad \left. + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\\
&\quad + \frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)^2 \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right]\Bigg\|_{L_2(\Omega_T)}\\
&\le \left\| \frac{\sqrt{3\sigma_2^c\mu}}{2} a_\varepsilon(g_\varepsilon^n) a_\varepsilon(f_\varepsilon^n)^{\frac{1}{2}} \right\|_{L_\infty(\Omega_T)} \Bigg(\Bigg\| a_\varepsilon^{\frac{1}{2}}(f_\varepsilon^n) \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\quad \left. + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\Bigg\|_{L_2(\Omega_T)}\\
&\quad + \left\| \frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)^2 \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right]\right\|_{L_2(\Omega_T)} \Bigg) < c,
\end{aligned}$$

by (3.50), (3.53) and Lemma 3.14 i), where $c > 0$ is a constant independent of $n \in \mathbb{N}$, $\varepsilon \in (0,1]$. Regularizing in (3.23) the terms involving Γ by introducing the bounded function τ_ε for $\varepsilon \in (0,1]$ allows us to estimate $H_\Gamma^{\varepsilon,n}$ likewise in $L_2(\Omega_T)$, by

$$\begin{aligned}
\|H_\Gamma^{\varepsilon,n}\|_{L_2(\Omega_T)} &= \Bigg\| \frac{\sqrt{3\sigma_2^c}}{2} \tau_\varepsilon(\Gamma_\varepsilon^n) a_\varepsilon(f_\varepsilon^n) \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n + \sigma_2^c\mu g_\varepsilon^n) \right.\\
&\quad \left. + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\right]\\
&\quad + \frac{\sqrt{3\sigma_2^c}}{2} \Gamma_\varepsilon^n a_\varepsilon(g_\varepsilon^n) \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n + g_\varepsilon^n) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right]
\end{aligned}$$

$$+\frac{1}{4}\tau_\varepsilon(\Gamma_\varepsilon^n)a_\varepsilon(g_\varepsilon^n)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) - D\partial_x\Gamma_\varepsilon^n\Big\|_{L_{\frac{3}{2}}(\Omega_T)}$$

$$\leq \|\tau_\varepsilon(\Gamma_\varepsilon^n)\|_{L_\infty(\Omega_T)}\Bigg(\Bigg\|\frac{\sqrt{3\sigma_2^c}}{2}a_\varepsilon(f_\varepsilon^n)\Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n+\sigma_2^c\mu g_\varepsilon^n)$$

$$+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\Bigg]\Bigg\|_{L_2(\Omega_T)}$$

$$+\Bigg\|\frac{\sqrt{3\sigma_2^c}}{2}a_\varepsilon(g_\varepsilon^n)\Bigg[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\Bigg]\Bigg\|_{L_2(\Omega_T)}$$

$$+\left\|\frac{1}{4}a_\varepsilon(g_\varepsilon^n)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right\|_{L_2(\Omega_T)}\Bigg)+D\,\|\partial_x\Gamma_\varepsilon^n\|_{L_2(\Omega_T)}<c,$$

in virtue of (3.50), (3.51), (3.53) and Lemma 3.14, where $c>0$ is a constant independent of $n\in\mathbb{N}$, $\varepsilon\in(0,1]$. Indeed, due to (3.30) and Lemma 3.14, there exists a constant $c>0$, independent of $n\in\mathbb{N}$, $\varepsilon\in(0,1]$, such that $\|\tau_\varepsilon(\Gamma_\varepsilon^n)\|_{L_\infty(\Omega_T)}\leq\|\Gamma_\varepsilon^n\|_{L_\infty(\Omega_T)}<c^2$.

Note that if $f\in(H^1(0,L)')\cap L_2(0,L)$, the dual pairing between f and a function $g\in H^1(0,L)$ reduces to the scalar product in $L_2(0,L)$, that is

$$\langle f,g\rangle_{H^1}=(f\mid g)_2\,, \tag{3.62}$$

since a function in $L_2(0,L)$ is identified with a functional in $(L_2(0,L))'$ via $f\mapsto(f\mid\cdot)_2$. Given $\xi\in H^1(0,L)$, we define the truncation for each $n\in\mathbb{N}$

$$\xi^n:=\sum_{k=0}^n(\xi\mid\phi_k)_2\,\phi_k\in\operatorname{span}\{\phi_0,\dots,\phi_n\}.$$

The Galerkin approximation $(f_\varepsilon^n,g_\varepsilon^n,\Gamma_\varepsilon^n)$ satisfies (3.60) when testing against functions belonging to $\operatorname{span}\{\phi_0,\dots,\phi_n\}$, so that

$$\langle\partial_t f_\varepsilon^n(t),\xi\rangle_{H^1}=(\partial_t f_\varepsilon^n(t)\mid\xi^n)_2=\left(H_f^{\varepsilon,n}(t)\mid\partial_x\xi^n\right)_2\leq\|H_f^{\varepsilon,n}(t)\|_2\|\partial_x\xi^n\|_2\leq\|H_f^{\varepsilon,n}(t)\|_2\|\xi\|_{H^1(0,L)}$$

for every $t>0$, where the last inequality holds due to Parseval's identity, which implies that

$$\|\partial_x\xi^n\|_2^2=\left(\sum_{l=0}^n(\partial_x\xi\mid\phi_l)_2\,\phi_l\,\Bigg|\,\sum_{k=0}^n(\partial_x\xi\mid\phi_k)_2\,\phi_k\right)_2=\sum_{l,k=0}^n((\partial_x\xi\mid\phi_l)_2\,\phi_l\mid(\partial_x\xi\mid\phi_k)_2\,\phi_k)_2$$

[2] Without regularizing (3.23) by introducing τ_ε, the uniform bound $\{\Gamma_\varepsilon^n\mid n\in\mathbb{N},\varepsilon\in(0,1]\}\subset L_6(\Omega_T)$ (cf. Lemma 3.14) would lead to

$$(H_\Gamma^{\varepsilon,n})_{n\in\mathbb{N}}\quad\text{bounded in}\quad L_{\frac{3}{2}}(\Omega_T), \tag{3.61}$$

which is due to the Hölder inequality.

$$= \sum_{l=0}^{n} \left((\partial_x \xi \mid \phi_l)_2 \, \phi_l \mid (\partial_x \xi \mid \phi_l)_2 \, \phi_l \right)_2 = \sum_{l=0}^{n} (\partial_x \xi \mid \phi_l)_2^2 \le \sum_{l=0}^{\infty} (\partial_x \xi \mid \phi_l)_2^2$$

$$= \|\partial_x \xi\|_2^2 \le \|\xi\|_{H^1(0,L)}^2.$$

Hence, the function $\partial_t f_\varepsilon^n(t)$ belongs to the dual $(H^1(0,L))'$ of $H^1(0,L)$ for all $t > 0$ and integration with respect to time yields

$$\|\partial_t f_\varepsilon^n\|_{L_2(0,T,(H^1(0,L))')}^2 = \int_0^T \|\partial_t f_\varepsilon^n(t)\|_{(H^1(0,L))'}^2 \, dt = \int_0^T \sup_{\|\xi\|_{H^1(0,L)} \le 1} |\left(\partial_t f_\varepsilon^n(t) \mid \xi\right)_2|^2 \, dt$$

$$\le \int_0^T \|H_f^{\varepsilon,n}(t)\|_2^2 \, dt = \|H_f^{\varepsilon,n}\|_{L_2(\Omega_T)}^2.$$

Analogously one shows $\|\partial_t g_\varepsilon^n\|_{L_2(0,T,(H^1(0,L))')}^2 \le \|H_g^{\varepsilon,n}\|_{L_2(\Omega_T)}^2$ and $\|\partial_t \Gamma_\varepsilon^n\|_{L_2(0,T,(H^1(0,L))')}^2 \le \|H_\Gamma^{\varepsilon,n}\|_{L_2(\Omega_T)}^2$[3], so that

$$\{\partial_t f_\varepsilon^n, \partial_t g_\varepsilon^n, \partial_t \Gamma_\varepsilon^n \mid n \in \mathbb{N}, \varepsilon \in (0,1]\} \quad \text{is bounded in} \quad L_2(0,T,(H^1(0,L))').$$

□

Let $\varepsilon \in (0,1]$ be fixed. Lemma 3.14 and 3.15 provide necessary bounds for the Galerkin approximation $(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)$ to extract weakly convergent subsequences. Since

$$H^1(0,L) \hookrightarrow\hookrightarrow C^\alpha([0,L]) \hookrightarrow (H^1(0,L))', \quad H^3(0,L) \hookrightarrow\hookrightarrow C^{2+\alpha}([0,L]) \hookrightarrow (H^1(0,L))'$$

for all $\alpha \in [0, \frac{1}{2})$ by the Rellich–Kondrachov Theorem (cf. [1, Theorem 6.3]), the bounds of

$$\begin{aligned} &\{f_\varepsilon^n, g_\varepsilon^n \mid n \in \mathbb{N}\} && \text{in} \quad L_\infty(0,T;H^1(0,L)) \cap L_2(0,T;H^3(0,L)), \\ &\{\partial_t f_\varepsilon^n, \partial_t g_\varepsilon^n \mid n \in \mathbb{N}\} && \text{in} \quad L_2(0,T;(H^1(0,L))'), \end{aligned}$$

imply in virtue of [45, Corollary 4] that

$$(f_\varepsilon^n)_{n\in\mathbb{N}}, (g_\varepsilon^n)_{n\in\mathbb{N}} \quad \text{are relatively compact in} \quad C([0,T];C^\alpha([0,L])) \cap L_2(0,T;C^{2+\alpha}([0,L])) \tag{3.63}$$

[3] The regularity for $\partial_t \Gamma_\varepsilon^n$ is a consequence of the improved regularity $H_\Gamma^{\varepsilon,n} \in L_2(\Omega_T)$ due to regularizing the terms involving Γ by $\tau_\varepsilon(\Gamma)$. Without regularizing, the time derivative of Γ_ε^n would satisfy

$$\partial_t \Gamma_\varepsilon^n \in L_{\frac{3}{2}}(0,T;(W_3^1(0,L))'),$$

where $(W_3^1(0,L))'$ denotes the dual space of $W_3^1(0,L)$.

for $\alpha \in [0, \frac{1}{2})$. Observe that since $H^1(0,L) \hookrightarrow L_2(0,L)$ (cf. [1, Theorem 6.3]), we deduce that $L_2(0,T) \cong (L_2(0,T))' \hookrightarrow (H^1(0,L))'$ (cf. e.g. [44, Theorem 4.19]). Hence, the bounds of

$$\{\Gamma_\varepsilon^n \mid n \in \mathbb{N}\} \quad \text{in} \quad L_\infty(0,T;L_2(0,L)) \cap L_2(0,T;H^1(0,L)),$$

$$\{\partial_t \Gamma_\varepsilon^n \mid n \in \mathbb{N}\} \quad \text{in} \quad L_2(0,T;(H^1(0,L))')$$

imply that

$$(\Gamma_\varepsilon^n)_{n\in\mathbb{N}} \quad \text{is relatively compact in} \quad C([0,T];(H^1(0,L))') \cap L_2(0,T;C^\alpha([0,L])) \tag{3.64}$$

for $\alpha \in [0, \frac{1}{2})$. The relative compactnesses in (3.63) and (3.64) provide the existence of converging subsequences (not relabeled)

$$f_\varepsilon^n \longrightarrow f_\varepsilon \qquad \text{in} \qquad C([0,T];C^\alpha([0,L])) \cap L_2(0,T;C^{2+\alpha}([0,L])), \tag{3.65}$$

$$g_\varepsilon^n \longrightarrow g_\varepsilon \qquad \text{in} \qquad C([0,T];C^\alpha([0,L])) \cap L_2(0,T;C^{2+\alpha}([0,L])), \tag{3.66}$$

$$\Gamma_\varepsilon^n \longrightarrow \Gamma_\varepsilon \qquad \text{in} \qquad C([0,T];(H^1(0,L))') \cap L_2(0,T;C^\alpha([0,L])). \tag{3.67}$$

Lemma 3.16. *The limit functions $f_\varepsilon, g_\varepsilon$ obtained in* (3.65), (3.66) *belong to*

$$L_\infty(0,T;H^1(0,L)) \cap L_2(0,T;H^3(0,L))$$

and there exists a subsequence (not relabeled), such that

$$\partial_x^k f_\varepsilon^n \rightharpoonup \partial_x^k f_\varepsilon, \qquad \partial_x^k g_\varepsilon^n \rightharpoonup \partial_x^k g_\varepsilon \qquad \textit{in} \quad L_2(\Omega_T) \quad \textit{for} \quad k = 1,2,3. \tag{3.68}$$

Moreover, the time derivatives $\partial_t f_\varepsilon, \partial_t g_\varepsilon$ belong to $L_2(0,T;(H^1(0,L))')$ with

$$\partial_t f_\varepsilon^n \rightharpoonup \partial_t f_\varepsilon, \qquad \partial_t g_\varepsilon^n \rightharpoonup \partial_t g_\varepsilon \qquad \textit{in} \quad L_2(0,T;(H^1(0,L))').$$

Proof. We will prove the statements only for f_ε^n, the proofs for g_ε^n are similar. Owing to Lemma 3.14 i), the sequence $(f_\varepsilon^n)_{n\in N}$ is bounded in $L_2(0,T;H^3(0,L))$. Thus, by Eberlein–Smulyan's theorem, there exists a weakly convergent subsequence (not relabeled), such that

$$f_\varepsilon^n \rightharpoonup \overline{f}_\varepsilon \qquad \text{in} \quad L_2(0,T;H^3(0,L)) \tag{3.69}$$

for some $\overline{f}_\varepsilon \in L_2(0,T;H^3(0,L))$. The weak convergence of $(f_\varepsilon^n)_{n\in\mathbb{N}}$ in (3.69) implies the strong convergence

$$f_\varepsilon^n \to \overline{f}_\varepsilon \qquad \text{in} \quad \mathcal{D}'(\Omega_T).$$

Together with (3.65), we deduce that $f_\varepsilon = \overline{f}_\varepsilon \in L_2(0,T;H^3(0,L))$. Hence, in virtue of (3.69), the claim (3.68) is satisfied. Due to Lemma 3.14 i) there exists $M > 0$ independent of $n \in \mathbb{N}, t \geq 0$, such that

$$(f_\varepsilon^n(t))_{n\in\mathbb{N}} \subset \overline{\mathbb{B}}_{H^1(0,L)}(0,M).$$

Since the unit ball of a Hilbert space is weakly closed, we obtain the existence of a weakly convergent subsequence (not relabeled), such that for almost all $t \geq 0$

$$f_\varepsilon^n(t) \rightharpoonup f_\varepsilon(t) \qquad \text{in} \quad H^1(0,L),$$

where the identification of the limit is again due to (3.65). We conclude that the limit function of $(f_\varepsilon^n)_{n\in\mathbb{N}}$ belongs to $L_\infty(0,T;H^1(0,L)) \cap L_2(0,T;H^3(0,L))$.
In view of Lemma 3.15, the time derivative $(\partial_t f_\varepsilon^n)_{n\in\mathbb{N}}$ is bounded in the Hilbert space $L_2(0,T;(H^1(0,L))')$. Thus, by Eberlein–Smulyan's theorem, there exists a weakly convergent subsequence (not relabeled)

$$\partial_t f_\varepsilon^n \rightharpoonup h \qquad \text{in} \quad L_2(0,T;(H^1(0,L))'),$$

for some limit function $h \in L_2(0,T;(H^1(0,L))')$. That is, for all $\xi \in C_c^\infty(\Omega_T)$, we obtain in virtue of (3.65), that

$$\begin{aligned}\int_0^T \langle \partial_t f_\varepsilon^n(t), \xi(t)\rangle_{H^1}\, dt &= -\int_0^T \langle f_\varepsilon^n(t), \partial_t \xi(t)\rangle_{H^1}\, dt \\ &\to -\int_0^T \langle f_\varepsilon(t), \partial_t \xi(t)\rangle_{H^1}\, dt = \int_0^T \langle \partial_t f_\varepsilon(t), \xi(t)\rangle_{H^1}\, dt.\end{aligned}$$

Hence $h = \partial_t f_\varepsilon$.

□

Remark 3.17. Note that the bounds of $(f_\varepsilon^n)_{n\in\mathbb{N}}, (g_\varepsilon^n)_{n\in\mathbb{N}}$ in $L_\infty(0,T;H^1(0,L))$ and the bounds of $(\partial_t f_\varepsilon^n)_{n\in\mathbb{N}}, (\partial_t g_\varepsilon^n)_{n\in\mathbb{N}}$ in $L_2(0,T;(H^1(0,L)))$ (cf. Lemma 3.14 and Lemma 3.15) are independent of $\varepsilon \in (0,1]$. Therefore, in virtue of [45, Corollary 4] and Lemma 3.16, we obtain that

$$\{f_\varepsilon, g_\varepsilon \mid \varepsilon \in (0,1]\} \quad \text{is bounded in} \quad L_\infty(0,T;H^1(0,L)).$$

This uniform bound will be in particular necessary in the proof of Theorem 3.23, where we show the non–negativity of the family of Galerkin approximations $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$.

In the following lemma we collect some information regarding weak and strong convergences concerning $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ in certain Hilbert spaces. Note that all convergences are independent of $\varepsilon \in (0,1]$.

Lemma 3.18. *The family $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ satisfies*

i) $\Gamma_\varepsilon^n \longrightarrow \Gamma_\varepsilon$ *in* $L_q(\Omega_T) \cap C([0,T];(H^1(0,L))')$ *for all* $q \in [2,6)$,

ii) $\partial_x \Gamma_\varepsilon^n \rightharpoonup \partial_x \Gamma_\varepsilon$ *in* $L_2(\Omega_T)$,

iii) $\partial_t \Gamma_\varepsilon^n \rightharpoonup \partial_t \Gamma_\varepsilon$ *in* $L_2(0,T;(H^1(0,L))')$,

iv) $\Phi(\Gamma_\varepsilon^n) \longrightarrow \Phi(\Gamma_\varepsilon)$ *in* $L_\infty(0,T;L_1(0,L))$,

v) $\Phi''(\Gamma_\varepsilon^n) \longrightarrow \Phi''(\Gamma_\varepsilon)$ *in* $L_q(\Omega_T)$ *for all* $q \in \left[3, \frac{6}{r}\right)$,

vi) $\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x \Gamma_\varepsilon^n \rightharpoonup \sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x \Gamma_\varepsilon$ *in* $L_2(\Omega_T)$,

where the weak convergences are considered as convergent subsequences, which are not relabeled.

Proof. i) Due to (3.67), Γ_ε^n converges strongly to Γ_ε in $L_2(\Omega_T)$ and $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ is bounded in $L_6(\Omega_T)$, by Lemma 3.14. Since $L_6(\Omega_T)$ is a reflexive Banach space, we can extract, by Eberlein–Smulyan's theorem, a weakly convergent subsequence (not relabeled) with

$$\Gamma_\varepsilon^n \rightharpoonup \overline{\Gamma_\varepsilon} \qquad \text{in} \quad L_6(\Omega_T)$$

for some limit function $\overline{\Gamma_\varepsilon} \in L_6(\Omega_T)$. Hence $\Gamma_\varepsilon^n \to \overline{\Gamma_\varepsilon}$ in the dual space of $L_6(\Omega_T)$, which is identified with $L_{\frac{6}{5}}(\Omega_T)$. In virtue of (3.67), we deduce that $\overline{\Gamma_\varepsilon} = \Gamma_\varepsilon$. Using an interpolation estimate (cf. [1, Theorem 2.11]), we obtain

$$\|\Gamma_\varepsilon^n - \Gamma_\varepsilon\|_q \leq \|\Gamma_\varepsilon^n - \Gamma_\varepsilon\|_p^{1-\theta} \|\Gamma_\varepsilon^n - \Gamma_\varepsilon\|_l^{\theta}$$

for $\theta \in [0,1]$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{l}$. Choosing $l = 2$ and $p = 6$ it follows from Lemma 3.14 and (3.67) that

$$\|\Gamma_\varepsilon^n - \Gamma_\varepsilon\|_q \leq (\|\Gamma_\varepsilon^n\|_6 + \|\Gamma_\varepsilon\|_6)^{1-\theta} \|\Gamma_\varepsilon^n - \Gamma_\varepsilon\|_2^{\theta} \longrightarrow 0$$

for $q = \frac{6}{1+2\theta} \in [2,6)$ and $n \longrightarrow \infty$.

ii) and iii) are a consequence of Lemma 3.14 and 3.15 and Eberlein–Smulyan's theorem, where the identification of the limits is due to (3.67).

iv) Since $\Gamma_\varepsilon^n \longrightarrow \Gamma_\varepsilon$ in $L_q(\Omega_T)$, there is a subsequence of $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ (not relabeled) such that $\Gamma_\varepsilon^n \longrightarrow \Gamma_\varepsilon$ point–wise almost everywhere. Hence $\Phi(\Gamma_\varepsilon^n) \longrightarrow \Phi(\Gamma)$ almost everywhere, by Φ being continuous (cf. Assumption A1)) and (3.47) implies that

$$\Phi(\Gamma_\varepsilon^n) \longrightarrow \Phi(\Gamma_\varepsilon) \qquad \text{in} \quad L_\infty(0,T;L_1(0,L)).$$

v) Since Φ'' is continuous by Assumption A1) and $\Gamma_\varepsilon^n \longrightarrow \Gamma_\varepsilon$ in $L_q(\Omega_T)$ for $q \in [2,6)$, we have $\Phi''(\Gamma_\varepsilon^n) \longrightarrow \Phi''(\Gamma_\varepsilon)$ point–wise almost everywhere. By means of Assumption A3), which states that $\Phi''(s) \leq C_\Phi(|s|^r + 1)$ for all $s \in \mathbb{R}$, we deduce that

$$\|\Phi''(\Gamma_\varepsilon^n)\|_p^p \leq C\||\Gamma_\varepsilon^n|^r + 1\|_p^p,$$

which is bounded for $p \in [\frac{2}{r}, \frac{6}{r})$. Since $r \in (0,2)$, it follows that $(\Phi''(\Gamma_\varepsilon^n))_{n\in\mathbb{N}} \subset L_p(\Omega_T)$ for $p \in [3, \frac{6}{r})$ and

$$\Phi''(\Gamma_\varepsilon^n) \longrightarrow \Phi''(\Gamma_\varepsilon) \qquad \text{in} \quad L_p(\Omega_T) \qquad \text{for} \quad p \in \left[3, \frac{6}{r}\right).$$

vi) Owing to (3.52) and Eberlein–Smulyan's theorem, we can extract a weakly convergent subsequence (not relabeled), such that

$$\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n \rightharpoonup v \quad \text{in} \quad L_2(\Omega_T), \tag{3.70}$$

where $v \in L_2(\Omega_T)$ is the limit function, which we show to coincide with $\sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon$. For all $\xi \in C_c^\infty(\Omega_T)$ we have

$$\begin{aligned}\int_{\Omega_T} (\sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon - v)\xi\, d(x,t) &= \int_{\Omega_T} \left(\sqrt{\Phi''(\Gamma_\varepsilon)} - \sqrt{\Phi''(\Gamma_\varepsilon^n)}\right)\partial_x\Gamma_\varepsilon\xi\, d(x,t)\\ &\quad + \int_{\Omega_T} \sqrt{\Phi''(\Gamma_\varepsilon^n)}\xi(\partial_x\Gamma_\varepsilon - \partial_x\Gamma_\varepsilon^n)\, d(x,t)\\ &\quad + \int_{\Omega_T} (\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n - v)\xi\, d(x,t),\end{aligned} \tag{3.71}$$

where the last integral on the right–hand side tends to zero, due to (3.70). Notice that, in view of part iv) and the square root being continuous, $\sqrt{\Phi''(\Gamma_\varepsilon^n)} \longrightarrow \sqrt{\Phi''(\Gamma_\varepsilon)}$ in $L_p(\Omega_T)$ for $p \in [6, \frac{12}{r})$. Hence, it follows from part ii), that also the first two integrals of the right–hand side of (3.71) tend to zero as $n \longrightarrow \infty$. Therefore, the limit v is identified with $\sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon$. □

Lemma 3.19. *The Galerkin approximation* $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ *has a further subsequence (not relabeled), such that*

$$\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \rightharpoonup \partial_x\sigma_\varepsilon(\Gamma_\varepsilon) \quad in \quad L_s(\Omega_T)$$

for $s \in [\frac{6}{5}, \frac{12}{8+r})$.

Proof. Note that in virtue of (3.24) and (3.29) we can write

$$\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) = \sigma_\varepsilon'(\Gamma_\varepsilon^n)\partial_x\Gamma_\varepsilon^n = \frac{\tau_\varepsilon(\Gamma_\varepsilon^n)}{\Gamma_\varepsilon^n}\sigma'(\Gamma_\varepsilon^n)\partial_x\Gamma_\varepsilon^n = -\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)}\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n.$$

Since $\Gamma_\varepsilon^n \to \Gamma_\varepsilon$ in $L_p(\Omega_T)$ for $p \in [2,6)$ (cf. Lemma 3.18 i)), there exists a subsequence (not relabeled), such that the convergence is point–wise almost everywhere. Taking into account that τ_ε is continuous and, by construction, $|\tau_\varepsilon(s)| \leq |s|$ for all $s \in \mathbb{R}$ (cf. (3.30)), we deduce that

$$\tau_\varepsilon(\Gamma_\varepsilon^n) \to \tau_\varepsilon(\Gamma_\varepsilon) \qquad \text{in} \qquad L_p(\Omega_T), \quad \text{for} \quad p \in [2,6). \tag{3.72}$$

The Hölder inequality implies that

$$\|\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)}\|_m \leq \|\tau_\varepsilon(\Gamma_\varepsilon^n)\|_p\|\sqrt{\Phi''(\Gamma_\varepsilon^n)}\|_q$$

for $p \in [2,6)$, $q \in [6, \frac{12}{r})$, by (3.72) and Lemma 3.18, with $\frac{1}{m} = \frac{1}{p} + \frac{1}{q} \in (\frac{2+r}{12}, \frac{2}{3}]$, so that

$$(\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)})_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L_m(\Omega_T) \quad \text{for} \quad m \in \Big[\frac{3}{2}, \frac{12}{2+r}\Big).$$

Thus, owing to (3.72) and $\sqrt{\Phi''(\Gamma_\varepsilon^n)} \longrightarrow \sqrt{\Phi''(\Gamma_\varepsilon)}$ in $L_p(\Omega_T)$ for $p \in [6, \frac{12}{r})$ (cf. Lemma 3.18), we deuce that

$$\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)} \longrightarrow \tau_\varepsilon(\Gamma_\varepsilon)\sqrt{\Phi''(\Gamma_\varepsilon)} \qquad \text{in} \quad L_m(\Omega_T) \quad \text{for} \quad m \in \Big[\frac{3}{2}, \frac{12}{2+r}\Big). \tag{3.73}$$

Recalling that the energy inequality provides the bound of $(\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ in $L_2(\Omega_T)$ (cf. Lemma 3.18), we apply again the Hölder inequality and obtain that

$$\|\partial_x\sigma(\Gamma_\varepsilon^n)\|_s \leq \|\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)}\|_m\|\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n\|_2,$$

where $\frac{1}{s} = \frac{1}{m} + \frac{1}{2} \in (\frac{8+r}{12}, \frac{7}{6}]$. Hence,

$$(\partial_x\sigma(\Gamma_\varepsilon^n))_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L_s(\Omega_T), \qquad \text{for} \quad s \in \Big[\frac{6}{5}, \frac{12}{8+r}\Big).$$

By Eberlein–Smulyan's theorem and $L_s(\Omega_T)$ being reflexive for $s \in [\frac{6}{5}, \frac{12}{8+r})$, there exists a weakly convergent subsequence (not relabeled), such that

$$\partial_x \sigma(\Gamma_\varepsilon^n) = -\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)}\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x \Gamma_\varepsilon^n \rightharpoonup v \quad \text{in} \quad L_s(\Omega_T), \tag{3.74}$$

for some $v \in L_s(\Omega_T)$. In order to identify the limit v with $\partial_x\sigma(\Gamma_\varepsilon)$ let ξ be an arbitrary function belonging $C_c^\infty(\Omega_T)$ and consider

$$\begin{aligned}\int_{\Omega_T} (\partial_x\sigma(\Gamma_\varepsilon) - v)\xi \, d(x,t) &= \int_{\Omega_T} \left(-\tau_\varepsilon(\Gamma_\varepsilon)\sqrt{\Phi''(\Gamma_\varepsilon)}\sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon - v\right)\xi \, d(x,t)\\ &= -\int_{\Omega_T} \tau_\varepsilon(\Gamma_\varepsilon)\sqrt{\Phi''(\Gamma_\varepsilon)}\xi\left(\sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon - \sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n\right) d(x,t)\\ &\quad - \int_{\Omega_T} \sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n\xi\left(\tau_\varepsilon(\Gamma_\varepsilon)\sqrt{\Phi''(\Gamma_\varepsilon)} - \tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)}\right) d(x,t)\\ &\quad - \int_{\Omega_T}\left(\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)}\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n + v\right)\xi \, d(x,t).\end{aligned} \tag{3.75}$$

The last integral on the right–hand side of (3.75) tends to zero by (3.74). Since we have $\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n \rightharpoonup \sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon$ in $L_2(\Omega_T)$, the first integral on the right–hand side of (3.75) tends to zero because $\tau_\varepsilon(\Gamma_\varepsilon)\sqrt{\Phi''(\Gamma_\varepsilon)}\xi \in L_q(\Omega_T)$ with $q > 2$, by (3.73). The second integral on the right–hand side of (3.75) also tends to zero in view of $\tau_\varepsilon(\Gamma_\varepsilon^n)\sqrt{\Phi''(\Gamma_\varepsilon^n)} \longrightarrow \tau_\varepsilon(\Gamma_\varepsilon)\sqrt{\Phi''(\Gamma_\varepsilon)}$ in $L_q(\Omega_T)$ for $q \in [\frac{3}{2}, \frac{12}{2+r})$ with $r \in (0,2)$. Thus, q can be chosen to be greater then 3 and $(\sqrt{\Phi''(\Gamma_\varepsilon^n)}\partial_x\Gamma_\varepsilon^n\xi)_{n\in\mathbb{N}} \subset L_2(\Omega_T) \subset L_{\frac{3}{2}}(\Omega_T)$, where $L_{\frac{3}{2}}(\Omega_T)$ is identified with the dual space $(L_3(\Omega_T))'$. □

Since $(f_\varepsilon^n, g_\varepsilon^n)(t)$ tends towards $(f_\varepsilon, g_\varepsilon)(t)$ in $(C^\alpha([0,L]))^2$, by (3.65), (3.66), for every $t \geq 0$ and $\alpha \in [0, \frac{1}{2})$, the initial data

$$f_\varepsilon(0) = f^0, \qquad g_\varepsilon(0) = g^0$$

are satisfied and

$$\|f_\varepsilon(t)\|_1 = \|f^0\|_1, \qquad \|g_\varepsilon(t)\|_1 = \|g^0\|_1$$

for all $t \geq 0$. In virtue of $\Gamma_\varepsilon^n \longrightarrow \Gamma_\varepsilon$ in $C([0,T];(H^1(0,L))')$ (cf. (3.67)), we obtain $\Gamma_\varepsilon^n(0) \longrightarrow \Gamma_\varepsilon(0)$ in $(H^1(0,L))'$. Recall that by definition and construction of the Galerkin approximation

$$\Gamma_\varepsilon^n(0) = W(v_\varepsilon^n(0)) = W(v_0^n) \to W(v^0) = \Gamma_0$$

in $L_p(\Omega_T)$ for $p = 2(r+1)$. Hence, we deduce that the initial datum $\Gamma_\varepsilon(0) = \Gamma^0$ is satisfied. By (3.67), $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ converges towards Γ_ε in $L_2(0,T;C^\alpha([0,L]))$, which implies the existence of a further subsequence of $(\Gamma_\varepsilon^n)_{n\in\mathbb{N}}$ (not relabeled) such that $\Gamma_\varepsilon^n(t) \longrightarrow \Gamma_\varepsilon(t)$ for almost every $t \geq 0$ in $C^\alpha([0,L])$. Therefore,

$$\|\Gamma_\varepsilon(t)\|_1 = \|\Gamma^0\|_1 \qquad \text{for almost all} \quad t \geq 0.$$

Now we want to prove that the energy inequality still holds for the limit $(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)$ of the Galerkin approximation. Since by construction a_ε and τ_ε are locally Lipschitz (3.65)–(3.67) imply that

$$a_\varepsilon(f_\varepsilon^n) \longrightarrow a_\varepsilon(f_\varepsilon) \qquad \text{in} \quad C(\Omega_T), \tag{3.76}$$

$$a_\varepsilon(g_\varepsilon^n) \longrightarrow a_\varepsilon(g_\varepsilon) \qquad \text{in} \quad C(\Omega_T), \tag{3.77}$$

$$\tau_\varepsilon(\Gamma_\varepsilon^n) \longrightarrow \tau_\varepsilon(\Gamma_\varepsilon) \qquad \text{in} \quad L_2(0,T;C^\alpha([0,L])) \tag{3.78}$$

for $\alpha \in [0, \frac{1}{2})$. The energy inequality implies that $(\sqrt{a_\varepsilon(g_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n))_{n\in\mathbb{N}}$ is bounded in $L_2(\Omega_T)$ (cf.(3.51)). Since $L_2(\Omega_T)$ is a reflexive Banach space, we use Eberlein–Smulyan's theorem and extract a weakly convergent subsequence (not relabeled), so that

$$\sqrt{a_\varepsilon(g_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \rightharpoonup v \qquad \text{in} \quad L_2(\Omega_T) \tag{3.79}$$

for some limit function v in $L_2(\Omega_T)$. In order to identify v with $\sqrt{a_\varepsilon(g_\varepsilon)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)$, consider for arbitrary $\xi \in C^\infty(\overline{\Omega_T})$

$$\begin{aligned}\int_{\Omega_T} (\sqrt{a_\varepsilon(g_\varepsilon)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) - v)\xi \, d(x,t) &= \int_{\Omega_T} \sqrt{a_\varepsilon(g_\varepsilon)}\xi \left(\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) - \partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right) d(x,t) \\ &\quad + \int_{\Omega_T} \partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\xi \left(\sqrt{a_\varepsilon(g_\varepsilon)} - \sqrt{a_\varepsilon(g_\varepsilon^n)}\right) d(x,t) \\ &\quad + \int_{\Omega_T} (\sqrt{a_\varepsilon(g_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) - v)\xi \, d(x,t),\end{aligned} \tag{3.80}$$

where the last term converges to zero by (3.79) and the other terms on the right–hand side tend to zero by Lemma 3.19 and (3.77). Hence,

$$\sqrt{a_\varepsilon(g_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \rightharpoonup \sqrt{a_\varepsilon(g_\varepsilon)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) \qquad \text{in} \quad L_2(\Omega_T). \tag{3.81}$$

Observe that (3.51), (3.76), (3.77), and $a_\varepsilon(s) \geq \varepsilon > 0$ for all $s \in \mathbb{R}$ imply that

$$\left(\sqrt{a_\varepsilon(f_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)_{n\in\mathbb{N}} = \left(\frac{\sqrt{a_\varepsilon(f_\varepsilon^n)}}{\sqrt{a_\varepsilon(g_\varepsilon^n)}}\sqrt{a_\varepsilon(g_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L_2(\Omega_T).$$

Hence, by Eberlein–Smulyan's theorem, there exists a subsequence (not relabeled), such that

$$\sqrt{a_\varepsilon(f_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \rightharpoonup w \qquad \text{in} \quad L_2(\Omega_T)$$

for some limit function $w \in L_2(\Omega_T)$. The identification of w as $\sqrt{a_\varepsilon(f)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)$ is then analog to (3.80), so that

$$\sqrt{a_\varepsilon(f_\varepsilon^n)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n) \rightharpoonup \sqrt{a_\varepsilon(f_\varepsilon)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) \qquad \text{in} \quad L_2(\Omega_T). \tag{3.82}$$

Hence, there exist weakly convergent subsequences (not relabeled) with

$$\begin{aligned}
&\sqrt{a_\varepsilon(f_\varepsilon^n)}\Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon^n)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon^n+\sigma_2^c\mu g_\varepsilon^n)\\
&\quad+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right)\Bigg]\\
&\quad\rightharpoonup \sqrt{a_\varepsilon(f_\varepsilon)}\Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)\\
&\qquad+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\Bigg]
\end{aligned} \quad \text{in } L_2(\Omega_T), \tag{3.83}$$

$$\begin{aligned}
&\sqrt{a_\varepsilon(f_\varepsilon^n)}\left[\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right]\\
&\quad\rightharpoonup \sqrt{a_\varepsilon(f_\varepsilon)}\left[\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]
\end{aligned} \quad \text{in } L_2(\Omega_T), \tag{3.84}$$

$$\begin{aligned}
&\sqrt{a_\varepsilon(g_\varepsilon^n)}\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon^n)\partial_x^3(f_\varepsilon^n+g_\varepsilon^n)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon^n)\right]\\
&\quad\rightharpoonup \sqrt{a_\varepsilon(g_\varepsilon)}\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]
\end{aligned} \quad \text{in } L_2(\Omega_T), \tag{3.85}$$

by means of (3.76), (3.77), (3.81), (3.82) and Lemma 3.16. We deduce that, owing to Lemma 3.18 vi) and (3.81)–(3.85) there exists a subsequence (not relabeled), such that

$$\mathcal{D}_\varepsilon(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) \to \mathcal{D}_\varepsilon(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)(T) \qquad \text{for all} \quad T > 0. \tag{3.86}$$

Since the norm of the limit function of a weakly convergent sequence can be estimated from above, we obtain in virtue of (3.86) that

$$\mathcal{D}_\varepsilon(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)(T) \leq \liminf_{n\to\infty}\mathcal{D}_\varepsilon(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) \qquad \text{for all} \quad T > 0. \tag{3.87}$$

Moreover, $\mathcal{E}(f_\varepsilon^n, g_\varepsilon^n, \Gamma_\varepsilon^n)(T) \longrightarrow \mathcal{E}(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)(T)$ for almost all $T > 0$, by (3.65), (3.66) and Lemma 3.18 iv). Hence, we have shown that, in view of (3.87), the energy inequality (3.38) holds.

To finish the proof of Theorem 3.12, we are only left to show that (3.33)–(3.35) are satisfied. Let $\xi \in L_2(0,T; H^1(0,L))$ be given. For each $n \in \mathbb{N}$ consider the truncation

$$\xi^n(t,\cdot) := \sum_{k=0}^{n} (\xi(t,\cdot) \mid \phi_k)_2 \, \phi_k, \qquad t \in (0,T).$$

Using integration by parts, we find that for every $n \in \mathbb{N}$

$$\int_0^T \langle \partial_t f_\varepsilon^n(t), \xi^n(t) \rangle_{H^1} \, dt = \int_{\Omega_T} H_f^{\varepsilon,n} \partial_x \xi^n \, d(x,t). \tag{3.88}$$

We show that we can pass to the limit $n \to \infty$ in (3.88), after possibly extracting a further subsequence. Observe that (3.76) and (3.83) imply that

$$H_f^{\varepsilon,n} \rightharpoonup H_f^\varepsilon \quad \text{in} \quad L_2(\Omega_T),$$

where H_f^ε is given by

$$H_f^\varepsilon := \frac{\sqrt{\sigma_2^c \mu}}{\sqrt{3}} a_\varepsilon(f_\varepsilon)^2 \left[\frac{1}{\sqrt{3\sigma_2^c \mu}} a_\varepsilon(f_\varepsilon) \partial_x^3((\sigma_1^c + \sigma_2^c \mu) f_\varepsilon + \sigma_2^c \mu g_\varepsilon) \right.$$
$$\left. + \frac{\sqrt{3}}{2} \left(\sqrt{\sigma_2^c \mu} a_\varepsilon(g_\varepsilon) \partial_x^3 (f_\varepsilon + g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}} \partial_x \sigma_\varepsilon(\Gamma_\varepsilon) \right) \right].$$

Due to

$$\int_{\Omega_T} H_f^{\varepsilon,n} \partial_x \xi^n \, d(x,t) = \int_{\Omega_T} (H_f^{\varepsilon,n} - H_f^\varepsilon) \partial_x \xi^n \, dx \, dt + \int_{\Omega_T} H_f^\varepsilon \partial_x (\xi^n - \xi) \, d(x,t) + \int_{\Omega_T} H_f^\varepsilon \partial_x \xi \, d(x,t),$$

the weak convergence of $(H_f^{\varepsilon,n})_{n\in\mathbb{N}}$ towards H_f^ε and the strong convergence $\partial_x \xi^n \longrightarrow \partial_x \xi$ in $L_2(\Omega_T)$, we obtain

$$\int_{\Omega_T} H_f^{\varepsilon,n} \partial_x \xi^n \, dx \, dt \longrightarrow \int_{\Omega_T} H_f^\varepsilon \partial_x \xi \, d(x,t). \tag{3.89}$$

Since, by Lebesgue dominated convergence, $\xi^n \to \xi$ in $L_2(0,T; H^1(0,L))$, it follows from Lemma 3.16 that

$$\langle \partial_t f_\varepsilon^n, \xi^n \rangle_{H^1} \to \langle \partial_t f_\varepsilon, \xi \rangle_{H^1} \qquad \text{in} \quad L_2(\Omega_T) \tag{3.90}$$

and (3.33) is satisfied in virtue of (3.88), (3.89) and (3.90). Using (3.76), (3.77), (3.84) and (3.85) we obtain that

$$H_g^{\varepsilon,n} \rightharpoonup H_g^\varepsilon \quad \text{in} \quad L_2(\Omega_T),$$

where H_g^ε is identified as

$$H_g^\varepsilon := \frac{\sqrt{3\sigma_2^c\mu}}{2} a_\varepsilon(g_\varepsilon)a_\varepsilon(f_\varepsilon)\left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)\right.$$
$$\left. + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\right]$$
$$+\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)^2\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right].$$

Analogously, since $(H_\Gamma^{\varepsilon,n})_{n\in\mathbb{N}}$ is bounded in $L_2(\Omega_T)$, we obtain that

$$H_\Gamma^{\varepsilon,n} \rightharpoonup H_\Gamma^\varepsilon \qquad \text{in} \quad L_2(\Omega_T),$$

where the limit function

$$H_\Gamma^\varepsilon := \frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(f_\varepsilon)\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)\right.$$
$$\left.+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\right]$$
$$+\frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(g_\varepsilon)\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]$$
$$+\frac{1}{4}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(g_\varepsilon^n)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)-D\partial_x\Gamma_\varepsilon,$$

can be identified in view of (3.65)–(3.67), (3.82), (3.83), (3.85) and Lemma 3.18. Passing to the limit as in (3.88), we deduce that (3.34) and (3.35) are satisfied, so that the proof of Theorem 3.12 is complete.

3.3.2. Existence and Non–Negativity of Weak Solutions for the Original System

In this section we prove the main result Theorem 3.11. We use the global weak solutions $(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ of the regularized problem (3.31) to find, in the limit $\varepsilon \searrow 0$, global weak solutions of the original problem (3.23). We emphasize that in the sequel, the initial data f^0, g^0, Γ^0 are non–negative. Following [22] we show that if $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1]$ is such that $\varepsilon_k \searrow 0$ for $k \longrightarrow \infty$ and there exist functions $f, g \in C(\overline{\Omega_T})$ with

$$f_{\varepsilon_k} \to f, \qquad g_{\varepsilon_k} \to g \qquad \text{in} \quad C(\overline{\Omega_T}) \quad \text{for} \quad k\to\infty, \tag{3.91}$$

then the accumulation points f, g are non–negative. Considering the sequence $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$, we use the idea in [20] to prove that already $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]} \geq 0$, so that if there exists a function $\Gamma \in L_2(0,T;C^\alpha([0,L]))$ with $\Gamma_\varepsilon \to \Gamma$ in $L_2(0,T;C^\alpha([0,L]))$ for $\varepsilon \searrow 0$, the almost everywhere non–negativity of the accumulation point Γ will be inherited from $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$.

Non–Negativity of the Accumulation Points of the Galerkin Approximation. Let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1]$ be such that $\varepsilon_k \searrow 0$ for $k \longrightarrow \infty$ and assume there exist functions $f, g \in C(\overline{\Omega_T})$, such that (3.91) is satisfied. In order to show that for non–negative initial data f^0, g^0 the accumulation points (f,g) as in (3.91) satisfy the non–negativity property, we define in analogy to [22] a function $\psi \in C^\infty(\mathbb{R})$, which is non–negative, supported in $[-1,0]$ and satisfies

$$\int_{\mathbb{R}} \psi(x)\,dx = \int_{-1}^{0} \psi(x)\,dx = 1. \tag{3.92}$$

Further, let $\chi_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\chi_1(x) := \int_x^0 \int_s^\infty \psi(\tau)\,d\tau\,ds \qquad \text{for} \quad x \in \mathbb{R}$$

and $(\chi_\delta)_{\delta>0}$ be the associated mollifier

$$\chi_\delta(x) := \delta\chi_1(\frac{x}{\delta}). \tag{3.93}$$

Then, the following properties hold true

Lemma 3.20. *The function* χ_δ *satisfies*

i) $\|\chi_\delta - \max\{-\mathrm{Id}, 0\}\|_\infty \leq \delta$,

ii) $\|\chi_\delta'\|_\infty \leq 1$ *and* $\|\chi_\delta''\|_\infty \leq \delta^{-1}\|\psi\|_\infty$,

iii) $|s\chi_\delta''(s)| \leq K$ *for all* $s \in [-\delta,\delta]$, *where* $K := \|\psi\|_\infty$,

iv) $\chi_\delta''(s) = 0$ *on* $\mathbb{R}\setminus[-\delta,0]$.

Proof. i) $\|\chi_\delta - \max\{-\mathrm{Id},0\}\|_\infty = \sup_{x\in\mathbb{R}} |\delta\chi_1(\frac{x}{\delta}) - \delta\max\{-\frac{x}{\delta},0\}|$. If $x \geq 0$, then $\chi_1(x)$ as well as $\max\{-x,0\}$ are zero. Let $x \leq -1$, then

$$|\chi_1(x) - \max\{-x,0\}| = \left|\int_x^0 \int_s^0 \psi(\tau)\,d\tau - 1\,ds\right|$$

$$\leq \int_x^{-1} \left| \int_s^0 \psi(\tau)\, d\tau - 1 \right| ds + \int_{-1}^0 \left| \int_s^0 \psi(\tau)\, d\tau - 1 \right| ds \leq \int_{-1}^0 \left| \int_s^0 \psi(\tau)\, d\tau - 1 \right| ds,$$

in virtue of (3.92). Since $\int_s^0 \psi(\tau)\, d\tau \in (0,1)$ for all $s \in (0,1)$, we deduce that

$$|\chi_1(x) - \max\{-x, 0\}| \leq 1.$$

Similar, if $x \in [-1, 0]$, we obtain that

$$|\chi_1(x) - \max\{-x, 0\}| \leq \int_x^0 \left| \int_s^0 \psi(\tau)\, d\tau - 1 \right| ds \leq \int_x^0 1\, ds = -x \leq 1.$$

Hence $\|\chi_\delta - \max\{-\operatorname{Id}, 0\}\|_\infty \leq \delta$.

ii) By (3.92), $\chi_\delta'(x) = \chi_1'(\frac{x}{\delta}) = -\int_{\frac{x}{\delta}}^\infty \psi(\tau)\, d\tau$ implies $\|\chi_\delta'\|_\infty \leq 1$. Moreover, $\chi_\delta''(x) = \delta\chi_1''(\frac{x}{\delta}) = \delta^{-1}\psi(\frac{x}{\delta})$. Hence $\|\chi_\delta''\|_\infty \leq \delta^{-1}\|\psi\|_\infty$.

iii) The statement follows directly from ii).

iv) Since $\chi_\delta''(s) = \frac{1}{\delta}\psi(\frac{x}{\delta})$ and $\operatorname{supp}(\psi) \subset [-1, 0]$, we obtain that $\chi_\delta''(s) = 0$ on $\mathbb{R} \setminus [-\delta, 0]$. □

We emphasize that χ_δ is a smooth approximation of $\max\{-\cdot, 0\}$. The following lemma will play the key role in proving the non–negativity of f and g.

Lemma 3.21. *There exists a constant $c > 0$, independent of $\varepsilon \in (0,1]$ and $t \geq 0$, such that the Galerkin approximations f_ε and g_ε satisfy*

$$\left| \int_0^L \chi_{\sqrt{\varepsilon}}(f_\varepsilon(t))\, dx \right| \leq c\sqrt{t}\varepsilon, \qquad \left| \int_0^L \chi_{\sqrt{\varepsilon}}(g_\varepsilon(t))\, dx \right| \leq c\sqrt{t}\varepsilon \tag{3.94}$$

for all $\varepsilon \in (0,1]$ and $t \geq 0$.

Proof. Let $\delta > 0$. The statement is true for $t = 0$, since $\chi_{\sqrt{\varepsilon}}(f^0) = \chi_{\sqrt{\varepsilon}}(g^0) = 0$ for $f^0, g^0 \geq 0$. By [28, Lemma 7.5], the composition $\chi_\delta'(f_\varepsilon)$ belongs to $L_2(0,T;H^1)$. Notice that formally[4]

$$\begin{aligned} \frac{d}{dt}\int_0^L \chi_\delta(f_\varepsilon)(t)\, dx &= \langle \chi_\delta'(f_\varepsilon)(t), \partial_t f_\varepsilon(t) \rangle_{H^1}, \\ \frac{d}{dt}\int_0^L \chi_\delta(g_\varepsilon)(t)\, dx &= \langle \chi_\delta'(g_\varepsilon)(t), \partial_t g_\varepsilon(t) \rangle_{H^1}. \end{aligned} \tag{3.95}$$

[4] $\partial_t f_\varepsilon(t), \partial_t g_\varepsilon(t) \in (H^1(0,L))'$ for almost every $t > 0$, so that the dual pairings in (3.95) exist. However, they only coincide with their left–hand side, respectively, if $\partial_t f_\varepsilon(t), \partial_t g_\varepsilon(t) \in (H^1(0,L))' \cap L_2(0,L)$. Then (cf. (3.62))

$$\frac{d}{dt}\int_0^L \chi_\delta(f_\varepsilon)(t)\, dx = \int_0^L \chi_\delta'(f_\varepsilon)\partial_t f_\varepsilon(t)\, dx = \langle \chi_\delta'(f_\varepsilon)(t), \partial_t f_\varepsilon(t) \rangle_2 = \langle \chi_\delta'(f_\varepsilon)(t), \partial_t f_\varepsilon(t) \rangle_{H^1}.$$

Recalling that $f^0, g^0 \geq 0$, which is why the terms $\int_0^L \chi_\delta(f^0)\,dx = \int_0^L \chi_\delta(g^0)\,dx$ equal zero, (3.95) yields after (formally) taking the integral with respect to time and integration by parts

$$\begin{aligned}\int_0^L \chi_\delta(f_\varepsilon(T))\,dx &= \int_{\Omega_T} H_f^\varepsilon \chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\,d(x,t),\\ \int_0^L \chi_\delta(g_\varepsilon(T))\,dx &= \int_{\Omega_T} H_g^\varepsilon \chi_\delta''(g_\varepsilon)\partial_x g_\varepsilon\,d(x,t)\end{aligned} \tag{3.96}$$

for all $T > 0$. Assume that (3.96) holds true. Since $\chi_\delta'' = 0$ on $\mathbb{R} \setminus [-\delta, 0]$, the Hölder inequality implies that

$$\begin{aligned}\left(\int_0^L \chi_\delta(f_\varepsilon(T))\,dx\right)^2 &\leq \left(\int_{[-\delta\leq f_\varepsilon\leq 0]} H_f^\varepsilon \chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\,d(x,t)\right)^2\\
&\leq \left(\int_{[-\delta\leq f_\varepsilon\leq 0]} \frac{\sqrt{\sigma_2^c\mu}}{\sqrt{3}} a_\varepsilon(f_\varepsilon)^2 \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon + \sigma_2^c\mu g_\varepsilon)\right.\right.\\
&\qquad \left.\left.+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon + g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\right] \chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\,d(x,t)\right)^2\\
&\leq \int_{[-\delta\leq f_\varepsilon\leq 0]} \left|\frac{\sqrt{\sigma_2^c\mu}}{\sqrt{3}}\sqrt{a_\varepsilon(f_\varepsilon)}\left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon + \sigma_2^c\mu g_\varepsilon)\right.\right.\\
&\qquad \left.\left.+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon + g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\right]\right|^2 d(x,t)\\
&\qquad \times \int_{[-\delta\leq f_\varepsilon\leq 0]} a_\varepsilon(f_\varepsilon)^3|\chi_\delta''(f_\varepsilon)|^2|\partial_x f_\varepsilon|^2\,d(x,t).\end{aligned}$$

Choosing now $\delta := \sqrt{\varepsilon}$ and recalling that $a_\varepsilon = \varepsilon$ on $(-\infty, 0]$, the energy inequality (3.38) together with Lemma 3.20 iii) imply the existence of a constant $c > 0$, independent of $\varepsilon \in (0, 1]$, such that

$$\begin{aligned}\left|\int_0^L \chi_{\sqrt{\varepsilon}}(f_\varepsilon(T))\,dx\right| &\leq c\left(\int_{[-\sqrt{\varepsilon}\leq f_\varepsilon\leq 0]} a_\varepsilon^3(f_\varepsilon)|\chi_{\sqrt{\varepsilon}}''(f_\varepsilon)|^2|\partial_x f_\varepsilon|^2\,d(x,t)\right)^{\frac{1}{2}}\\
&\leq c\varepsilon\|\psi\|_\infty\left(\int_{\Omega_T}|\partial_x f_\varepsilon|^2\,d(x,t)\right)^{\frac{1}{2}} \leq C\sqrt{T}\varepsilon,\end{aligned}$$

which is the desired estimate for f_ε in (3.94). Using a similar argument we prove the statement for g_ε. We obtain again by Hölder's inequality that

$$\left(\int_0^L \chi_\delta(g_\varepsilon(T))\,dx\right)^2 \leq \left(\int_{[-\delta\leq g_\varepsilon\leq 0]} H_g^\varepsilon \chi_\delta''(g_\varepsilon)\partial_x g_\varepsilon\,d(x,t)\right)^2$$

$$
\begin{aligned}
&\leq \Bigg(\int_{[-\delta\leq g_\varepsilon\leq 0]} \Bigg(\frac{\sqrt{3\sigma_2^c\mu}}{2} a_\varepsilon(g_\varepsilon)a_\varepsilon(f_\varepsilon) \Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon + \sigma_2^c\mu g_\varepsilon) \\
&\quad + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma_\varepsilon)\right)\Bigg] \\
&\quad + \frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon)^2 \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]\Bigg) \chi_\delta''(g_\varepsilon)\partial_x g_\varepsilon\, d(x,t)\Bigg)^2 \\
&\leq \int_{[-\delta\leq g_\varepsilon\leq 0]} \Bigg| \frac{\sqrt{3\sigma_2^c\mu}}{2}\sqrt{a_\varepsilon(f_\varepsilon)} \Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon + \sigma_2^c\mu g_\varepsilon) \\
&\quad + \frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}\, a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\Bigg]\Bigg|^2 d(x,t) \\
&\quad \times \int_{[-\delta\leq g_\varepsilon\leq 0]} a_\varepsilon(f_\varepsilon)a_\varepsilon(g_\varepsilon)^2|\chi_\delta''(g_\varepsilon)|^2|\partial_x g_\varepsilon|^2\, d(x,t) \\
&+ \int_{[-\delta\leq g_\varepsilon\leq 0]} \Bigg| \frac{\sqrt{\sigma_2^c}}{\sqrt{3}}\sqrt{a_\varepsilon(g_\varepsilon)} \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}} a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma_\varepsilon)\right]\Bigg|^2 d(x,t) \\
&\quad \times \int_{[-\delta\leq g_\varepsilon\leq 0]} a_\varepsilon^3(g_\varepsilon)|\chi_\delta''(g_\varepsilon)|^2|\partial_x f_\varepsilon|^2\, d(x,t).
\end{aligned}
$$

By taking $\delta := \sqrt{\varepsilon}$, using that $a_\varepsilon = \varepsilon$ on $(-\infty, 0]$ and the energy inequality (3.38), we obtain the existence of a constant $c > 0$, independent of $\varepsilon \in (0,1]$, such that

$$
\begin{aligned}
\left|\int_0^L \chi_{\sqrt{\varepsilon}}(g_\varepsilon(T))\,dx\right| &\leq c\left(\int_{\Omega_T} \varepsilon^2\|\psi\|_\infty^2|\partial_x g_\varepsilon|^2\,d(x,t)\right)^{\frac{1}{2}} + c\left(\int_{\Omega_T} \varepsilon^2\|\psi\|_\infty^2|\partial_x g_\varepsilon|^2\,d(x,t)\right)^{\frac{1}{2}} \\
&\leq c\sqrt{T}\varepsilon,
\end{aligned}
$$

which proves the second statement in (3.94).

Now we are left to show that (3.96) holds true. Consider for $t > 0$

$$
\frac{d}{dt}\int_0^L \chi_\delta(f_\varepsilon^n(t))\,dx = \int_0^L \chi_\delta'(f_\varepsilon^n(t))\partial_t f_\varepsilon^n(t)\,dx \tag{3.97}
$$

and remark that the integrals in (3.97) exist in virtue of to the regularity properties of the Galerkin approximation f_ε^n. Since $\chi_\delta'(f_\varepsilon^n(t))$ belongs to $H^1(0,L)$ for all $t > 0$, we can use its Fourier expansion as a test function for $\partial_t f_\varepsilon^n$ and find that

$$
\frac{d}{dt}\int_0^L \chi_\delta(f_\varepsilon^n(t))\,dx = \int_0^L \chi_\delta'(f_\varepsilon^n(t))\partial_t f_\varepsilon^n(t)\,dx = \int_0^L \partial_t f_\varepsilon^n(t)\sum_{k=0}^\infty \left(\chi_\delta'(f_\varepsilon^n(t)) \mid \phi_k\right)_2 \phi_k\,dx
$$

$$= \int_0^L \partial_t f_\varepsilon^n(t) \sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon^n(t)) \mid \phi_k\right)_2 \phi_k \, dx$$

$$= \int_0^L H_f^{\varepsilon,n}(t) \partial_x \left(\sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon^n(t)) \mid \phi_k\right)_2 \phi_k \right) dx.$$

Integrating with respect to time yields

$$\int_0^L \chi_\delta(f_\varepsilon^n(T)) \, dx = \int_0^L \chi_\delta(f^0) \, dx + \int_{\Omega_T} H_f^{\varepsilon,n} \sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon^n(t)) \mid \phi_k\right)_2 \partial_x \phi_k \, d(x,t) \tag{3.98}$$

for all $T > 0$. Since the function χ_δ is continuous and $f_\varepsilon^n(t) \to f_\varepsilon(t)$ point–wise for every $t > 0$, the left–hand side of (3.98) tends to $\int_0^L \chi_\delta(f_\varepsilon(T)) \, dx$. Owing to f^0 being non–negative, the first integral on the right–hand side of (3.98) vanishes and we are only left to prove that we can pass to the limit in the second term of the right–hand side of (3.98). First observe that

$$\chi_\delta''(f_\varepsilon)(t)\partial_x f_\varepsilon - \sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon^n(t)) \mid \phi_k\right)_2 \partial_x \phi_k = \left(\chi_\delta''(f_\varepsilon)(t)\partial_x f_\varepsilon - \sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon(t)) \mid \phi_k\right)_2 \partial_x \phi_k \right)$$
$$+ \sum_{k=0}^n \left((\chi_\delta'(f_\varepsilon)(t) - \chi_\delta'(f_\varepsilon^n)(t)) \mid \phi_k \right)_2 \partial_x \phi_k. \tag{3.99}$$

Note that, since $f_\varepsilon(t) \in H^1(0,L)$, the composition $\chi_\delta'(f_\varepsilon(t))$ belongs to $H^1(0,L)$, due to $\chi_\delta' \in L_\infty(\mathbb{R})$ (cf. [28, Lemma 7.5]). Thus, $\chi_\delta'(f_\varepsilon(t))$ possesses a Fourier expansion and

$$\sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon^n(t)) \mid \phi_k\right)_2 \phi_k \to \chi_\delta'(f_\varepsilon(t)) \qquad \text{in} \qquad H^1(0,L).$$

Hence, the first term of the right–hand side of (3.99) converges to zero in $L_2(\Omega_T)$, since

$$\chi_\delta''(f_\varepsilon(t))\partial_x f_\varepsilon - \sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon(t)) \mid \phi_k\right)_2 \partial_x \phi_k = \partial_x \left(\chi_\delta'(f_\varepsilon(t)) - \sum_{k=0}^n \left(\chi_\delta'(f_\varepsilon(t)) \mid \phi_k\right)_2 \phi_k \right) \to 0$$

in $L_2(0,L)$. Regarding the convergence in $L_2(\Omega_T)$ of the second term in (3.99), note that the sum is the truncation function of the Fourier expansion of $\chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon - \chi_\delta''(f_\varepsilon^n)\partial_x f_\varepsilon^n$ and may be estimated as follows

$$\left\| \sum_{k=0}^n \left((\chi_\delta'(f_\varepsilon) - \chi_\delta'(f_\varepsilon^n)) \mid \phi_k \right)_2 \partial_x \phi_k \right\|_2^2 \leq \| \chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon - \chi_\delta''(f_\varepsilon^n)\partial_x f_\varepsilon^n \|_2^2$$
$$= \| \chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon - \chi_\delta''(f_\varepsilon^n)\partial_x f_\varepsilon + \chi_\delta''(f_\varepsilon^n)\partial_x f_\varepsilon - \chi_\delta''(f_\varepsilon^n)\partial_x f_\varepsilon^n \|_2^2$$

$$\leq 2\left(\|\chi_\delta''(f_\varepsilon)-\chi_\delta''(f_\varepsilon^n)\|_\infty^2\|\partial_x f_\varepsilon\|_2^2+\|\chi_\delta''(f_\varepsilon^n)\|_\infty^2\|\partial_x f_\varepsilon-\partial_x f_\varepsilon^n\|_2^2\right).$$

Since $\chi_\delta''=\delta^{-1}\psi(\frac{\cdot}{\delta})$ and ψ is globally Lipschitz continuous, we deduce, that

$$\|\chi_\delta''(f_\varepsilon)-\chi_\delta''(f_\varepsilon^n)\|_\infty^2\leq c_1(\delta)\|f_\varepsilon-f_\varepsilon^n\|_\infty^2$$

and, in virtue of Lemma 3.20 ii),

$$\|\chi_\delta''(f_\varepsilon^n)\|_\infty^2\leq c_2(\delta),$$

for some constants $c_1(\delta),c_2(\delta)>0$, depending on $\delta>0$. Hence

$$\left\|\sum_{k=0}^n\left((\chi_\delta'(f_\varepsilon)-\chi_\delta'(f_\varepsilon^n))\mid\phi_k\right)_2\partial_x\phi_k\right\|_2^2\leq 2\left(c_1(\delta)\|f_\varepsilon-f_\varepsilon^n\|_\infty^2\|\partial_x f_\varepsilon\|_2^2+c_2(\delta)\|\partial_x f_\varepsilon-\partial_x f_\varepsilon^n\|_2^2\right),$$

which tends to zero if $n\to\infty$, by (3.65) and Lemma 3.20 ii). Hence,

$$\sum_{k=0}^n\left(\chi_\delta'(f_\varepsilon^n)\mid\phi_k\right)_2\partial_x\phi_k\longrightarrow\chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\qquad\text{in}\qquad L_2(\Omega_T).$$

Since $(H_f^{\varepsilon,n})_{n\in\mathbb{N}}$ converges weakly to H_f^ε in $L_2(\Omega_T)$, we deduce that

$$\int_{\Omega_T}H_f^{\varepsilon,n}\sum_{k=0}^n\left(\chi_\delta'(f_\varepsilon^n)\mid\phi_k\right)_2\partial_x\phi_k\,d(x,t)-\int_{\Omega_T}H_f^\varepsilon\chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\,d(x,t)$$

$$=\int_{\Omega_T}(H_f^{\varepsilon,n}-H_f^\varepsilon)\chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\,d(x,t)+\int_{\Omega_T}H_f^{\varepsilon,n}\left(\sum_{k=0}^n\left(\chi_\delta'(f_\varepsilon^n)\mid\phi_k\right)_2\partial_x\phi_k-\chi_\delta''(f_\varepsilon)\partial_x f_\varepsilon\right)d(x,t)$$

tends to zero if $n\to\infty$. Therefore, we can pass to the limit also in the second term of (3.98), which implies the first statement in (3.96). The assertion for g_ε in (3.96) works similarly, such that the proof of the lemma is complete. □

The next corollary shows that an accumulation point (f,g) of the Galerkin approximation $(f_{\varepsilon_k},g_{\varepsilon_k})_{\varepsilon_k}$ as in (3.91) is non–negative.

Corollary 3.22. *Assume that $f^0,g^0\geq 0$, then an accumulation point $(f,g)\in(C(\overline{\Omega_T}))^2$ of the Galerkin approximation $(f_{\varepsilon_k},g_{\varepsilon_k})_{\varepsilon_k}$ as in (3.91) is non–negative.*

Proof. Let $(\varepsilon_k)_{k\in\mathbb{N}}\in(0,1]$ be such that $\varepsilon_k\searrow 0$ for $k\longrightarrow\infty$. Consider

$$\|\chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k})-\max\{-f,0\}\|_\infty\leq\|\chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k})-\chi_{\sqrt{\varepsilon_k}}(f)\|_\infty+\|\chi_{\sqrt{\varepsilon_k}}(f)-\max\{-f,0\}\|_\infty$$

$$\leq \|f_{\varepsilon_k} - f\|_\infty + \sqrt{\varepsilon_k},$$

by Lemma 3.20 i) and ii). Hence, (3.91) and Lemma 3.20 ii) guarantee the convergence $\chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k}) \longrightarrow \max\{-f, 0\}$ in $C(\overline{\Omega_T})$. Recall that in the previous lemma we have shown that

$$\left| \int_0^L \chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k}(T))\, dx \right| \leq c\sqrt{T}\varepsilon_k,$$

where $c > 0$ is a constant independent of $\varepsilon \in (0, 1]$ and $T \geq 0$. Hence, letting k tend to zero, implies that

$$\int_0^L \max\{-f(t), 0\}\, dx = 0$$

for all $t \in [0, T]$, which proves the statement for f. The non–negativity of g follows by the same argumentation[5]. □

Following the idea in [20], we prove in the next theorem that the sequence $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ already admits the almost everywhere non–negativity property.

Theorem 3.23. *Assume that $\Gamma^0 \geq 0$, then the Galerkin approximation Γ_ε, $\varepsilon \in (0, 1]$, is non–negative almost everywhere in Ω_T.*

Proof. Let $\delta > 0$ and χ_δ the function defined in (3.93). Then, $\chi_\delta(\Gamma_\varepsilon^n(t)) \in H^1(0, L)$ for all $t > 0$ and

$$\frac{d}{dt}\int_0^L \chi_\delta(\Gamma_\varepsilon^n(t))\, dx = \int_0^L \chi_\delta'(\Gamma_\varepsilon^n(t))\partial_t\Gamma_\varepsilon^n(t)\, dx = \int_0^L \partial_t\Gamma_\varepsilon^n(t)\sum_{k=0}^n \left(\chi_\delta'(\Gamma_\varepsilon^n(t)) \mid \phi_k\right)_2 \phi_k\, dx,$$

which yields after integration with respect to time

$$\int_0^L \chi_\delta(\Gamma_\varepsilon^n(T))\, dx = \int_{\Omega_T} H_\Gamma^{\varepsilon,n}\sum_{k=0}^n \left(\chi_\delta'(\Gamma_\varepsilon^n(t)) \mid \phi_k\right)_2 \partial_x\phi_k\, d(x,t) \tag{3.100}$$

for each $T > 0$. We can pass to the limit in (3.100), by the same argument as in the proof of Lemma 3.21, and obtain

$$\int_0^L \chi_\delta(\Gamma_\varepsilon(t))\, dx = \int_{\Omega_T} H_\Gamma^\varepsilon \partial_x\chi_\delta'(\Gamma_\varepsilon)\, d(x,t), \tag{3.101}$$

[5] The proof of Corollary 3.22 is essentially due to Lemma 3.21, which provides an estimate depending on ε of the negative part of a function. Remark that we did not claim the non–negativity of $(f_\varepsilon, g_\varepsilon)_{\varepsilon\in(0,1]}$ itself , but only for an accumulation point of this family, when $\varepsilon \searrow 0$.

where H_Γ^ε represents the limit of a weakly convergent subsequence of $H_\Gamma^{\varepsilon,n}$ in $L_2(\Omega_T)$[6]. By construction $\chi_\delta''=0$ on $\mathbb{R}\setminus[-\delta,0]$, so that (3.101) yields

$$\begin{aligned}\int_0^L \chi_\delta(\Gamma_\varepsilon(T))\,dx = \int_{\Omega_\delta}\Bigg\{&\frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(f_\varepsilon)\Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)\\
&+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\Bigg]\\
&+\frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(g_\varepsilon)\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]\\
&+\frac{1}{4}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(g_\varepsilon)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)-D\partial_x\Gamma_\varepsilon\Bigg\}\chi_\delta''(\Gamma_\varepsilon)\partial_x\Gamma_\varepsilon\,d(x,t)\\
\le \int_{\Omega_\delta}\Bigg\{&\frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(f_\varepsilon)\Bigg[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)\\
&+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\Bigg]\\
&+\frac{\sqrt{3\sigma_2^c}}{2}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(g_\varepsilon)\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]\\
&+\frac{1}{4}\tau_\varepsilon(\Gamma_\varepsilon)a_\varepsilon(g_\varepsilon)\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\Bigg\}\chi_\delta''(\Gamma_\varepsilon)\partial_x\Gamma_\varepsilon\,d(x,t),\end{aligned}$$

where $\Omega_\delta:=[-\delta\le\Gamma_\varepsilon\le 0]$ and we used the fact that $\chi_\delta''=\delta^{-1}\psi(\frac{\cdot}{\delta})\ge 0$, which implies $-D\chi_\delta''(\Gamma_\varepsilon)|\partial_x\Gamma_\varepsilon|^2\le 0$. By means of $|\tau_\varepsilon(s)\chi_\delta''(s)|\le|s\chi_\delta''(s)|\le K$ if $|s|\le\delta$ (cf. Lemma 3.20 iii)), we deduce that

$$\begin{aligned}&\int_0^L \chi_\delta(\Gamma_\varepsilon(T))\,dx\\
&\le \frac{\sqrt{3\sigma_2^c}}{2}\left\|\sqrt{a_\varepsilon(f_\varepsilon)}\right\|_\infty K\int_{\Omega_\delta}\Bigg\{\left|\sqrt{a_\varepsilon(f_\varepsilon)}\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)\right.\right.\end{aligned}$$

[6]Recall that without regularizing the terms involving Γ in (3.23) by means of τ_ε, we would obtain that (cf. (3.61))

$$H_\Gamma^{\varepsilon,n}\rightharpoonup H_\Gamma^\varepsilon \qquad\text{in}\qquad L_{\frac{3}{2}}(\Omega_T).$$

Note that then passing to the limit in (3.100) would not have been possible, since $H_\Gamma^{\varepsilon,n}\rightharpoonup H_\Gamma^\varepsilon$ in $L_{\frac{3}{2}}(\Omega_T)$ would have demanded that

$$\sum_{k=0}^n\langle\chi_\delta'(\Gamma_\varepsilon^n),\phi_k\rangle_2\partial_x\phi_k\longrightarrow\chi_\delta''(\Gamma_\varepsilon)\partial_x\Gamma_\varepsilon\qquad\text{in}\quad L_3(\Omega_T),$$

which is out of reach if $\partial_x\Gamma_\varepsilon^n$ in $L_2(\Omega_T)$ only (cf. Lemma 3.18 ii)).

$$+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\Big]\,\partial_x\Gamma_\varepsilon\Big|\Big\}\;d(x,t)$$

$$+\frac{\sqrt{3\sigma_2^c}}{2}\left\|\sqrt{a_\varepsilon(g_\varepsilon)}\right\|_\infty K\int_{\Omega_\delta}\left|\sqrt{a_\varepsilon(g_\varepsilon)}\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]\partial_x\Gamma_\varepsilon\right|\;d(x,t)$$

$$+\frac{1}{4}\left\|\sqrt{a_\varepsilon(g_\varepsilon)}\right\|_\infty K\int_{[-\delta\le\Gamma_\varepsilon\le 0]}\left|\sqrt{a_\varepsilon(g_\varepsilon)}\sigma_\varepsilon'(\Gamma_\varepsilon)\partial_x\Gamma_\varepsilon\right|\;d(x,t).$$

By Hölder's inequality, the estimates implied by the energy inequality (3.38), the bound of $(\partial_x\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ in $L_2(\Omega)$ and the definition of a_ε together with $(f_\varepsilon)_{\varepsilon\in(0,1]},(g_\varepsilon)_{\varepsilon\in(0,1]}$ being bounded in $L_\infty(\Omega_T)$ (cf. Remark 3.17), the above inequality implies that

$$\int_0^L \chi_\delta(\Gamma_\varepsilon(t))\,dx\le c\int_{[-\delta\le\Gamma_\varepsilon\le 0]}|\partial_x\Gamma_\varepsilon|^2\;d(x,t)$$

for some constant $c>0$. It follows from [31, Lemma A.4] that for almost all $t\ge 0$

$$\int_0^L \max\{-\Gamma_\varepsilon(t),0\}\,dx=\lim_{\delta\to 0}\int_0^L\chi_\delta(\Gamma_\varepsilon(t))\,dx\le 0,$$

which completes the proof. □

Existence of Weak Solutions to the Original Problem. Now, we prove that there exists indeed an accumulation point of the Galerkin approximation $(f_\varepsilon,g_\varepsilon,\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ being a global weak solutions to the original problem (3.23). Note that the following bounds, established before, are uniform in $\varepsilon\in(0,1]$:

$$\{f_\varepsilon,g_\varepsilon\mid\varepsilon\in(0,1]\}\qquad\text{in } L_\infty(0,T;H^1(0,L)),\tag{3.102}$$

$$\{\partial_t f_\varepsilon,\partial_t g_\varepsilon\mid\varepsilon\in(0,1]\}\qquad\text{in } L_2(0,T,(H^1(0,L))'),\tag{3.103}$$

$$\{\Gamma_\varepsilon\mid\varepsilon\in(0,1]\}\qquad\text{in } L_q(\Omega_T),\quad q\in[2,6),\tag{3.104}$$

$$\{\partial_t\Gamma_\varepsilon\mid\varepsilon\in(0,1]\}\qquad\text{in } L_{\frac{3}{2}}(0,T;(H^1(0,L))'),\tag{3.105}$$

$$\{\partial_x\Gamma_\varepsilon\mid\varepsilon\in(0,1]\}\qquad\text{in } L_2(\Omega_T),\tag{3.106}$$

$$\{\Phi(\Gamma_\varepsilon)\mid\varepsilon\in(0,1]\}\qquad\text{in } L_\infty(0,T;L_1(0,L)),\tag{3.107}$$

$$\{\sqrt{\Phi''(\Gamma_\varepsilon)}\partial_x\Gamma_\varepsilon\mid\varepsilon\in(0,1]\}\qquad\text{in } L_2(\Omega_T),\tag{3.108}$$

$$\left\{ \sqrt{a_\varepsilon(f_\varepsilon)} \left[\frac{1}{\sqrt{3\sigma_2^c\mu}} a_\varepsilon(f_\varepsilon)\partial_x^3((\sigma_1^c + \sigma_2^c\mu)f_\varepsilon + \sigma_2^c\mu g_\varepsilon) + \frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon^n + g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\right] \,\middle|\, \varepsilon \in (0,1] \right\} \quad \text{in } L_2(\Omega_T), \tag{3.109}$$

$$\left\{ \sqrt{a_\varepsilon(f_\varepsilon)} \left[\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon + g_\varepsilon) + \frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right] \,\middle|\, \varepsilon \in (0,1] \right\} \quad \text{in } L_2(\Omega_T), \tag{3.110}$$

$$\left\{ \sqrt{a_\varepsilon(g_\varepsilon)} \left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon + g_\varepsilon) + \frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right] \,\middle|\, \varepsilon \in (0,1] \right\} \quad \text{in } L_2(\Omega_T), \tag{3.111}$$

$$\{\sqrt{a_\varepsilon(g_\varepsilon)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) \mid \varepsilon \in (0,1]\} \quad \text{in } L_2(\Omega_T). \tag{3.112}$$

We emphasize, that the bounds $\partial_x^3 f_\varepsilon^n$ and $\partial_x^3 g_\varepsilon^n$ in $L_2(\Omega_T)$ have not been uniform in $\varepsilon \in (0,1]$ and we loose these regularities, when passing to the limit $\varepsilon \searrow 0$. However, by the same arguments used before, we find a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \in (0,1]$ with $\varepsilon_k \searrow 0$, such that

$$f_{\varepsilon_k} \longrightarrow f \quad \text{and} \quad g_{\varepsilon_k} \longrightarrow g \quad \text{in} \quad C([0,T], C^\alpha([0,L])), \tag{3.113}$$

$$f_{\varepsilon_k} \rightharpoonup f \quad \text{and} \quad g_{\varepsilon_k} \rightharpoonup g \quad \text{in} \quad L_2(0,T; H^1([0,L])), \tag{3.114}$$

$$\partial_t f_{\varepsilon_k} \rightharpoonup \partial_t f, \quad \text{and} \quad \partial_t g_{\varepsilon_k} \rightharpoonup \partial_t g \quad \text{in} \quad L_2(0,T;(H^1([0,L]))'), \tag{3.115}$$

$$\Gamma_{\varepsilon_k} \longrightarrow \Gamma \quad \text{in} \quad L_2(0,T; C^\alpha([0,L])) \cap C([0,T]; (H^1(0,L))'), \tag{3.116}$$

$$\Gamma_{\varepsilon_k} \rightharpoonup \Gamma \quad \text{in} \quad L_2(0,T; H^1(0,L)), \tag{3.117}$$

$$\partial_t\Gamma_{\varepsilon_k} \rightharpoonup \partial_t\Gamma \quad \text{in} \quad L_2(0,T;(H^1(0,L))'), \tag{3.118}$$

$$\Phi(\Gamma_{\varepsilon_k}) \longrightarrow \Phi(\Gamma) \quad \text{in} \quad L_\infty(0,T; L_1(0,L)), \tag{3.119}$$

for $\alpha \in [0, \frac{1}{2})$. Observe that, as before, (3.102), (3.113) and (3.114) imply

$$\partial_x f_{\varepsilon_k}(t) \rightharpoonup \partial_x f(t) \quad \text{and} \quad \partial_x g_{\varepsilon_k}(t) \rightharpoonup \partial_x g(t) \quad \text{in} \quad L_2(0,L).$$

for almost all $t \in [0,T]$ and

$$f, g \in L_\infty(0,T,H^1(0,L)). \tag{3.120}$$

Further, $\Phi(\Gamma) \in L_\infty(0,T;L_1(0,L))$ implies that

$$\Gamma \in L_\infty(0,T; L_2(0,L)), \tag{3.121}$$

due to Assumption A2). Thus, by (3.113), (3.116), (3.120) and (3.121) we have shown the regularity for f, g and Γ claimed by Theorem 3.11 a). In virtue of Corollary 3.22, the functions f and g are non–negative, whereas $\Gamma \geq 0$ almost everywhere in view of Theorem 3.23 and (3.116). Further, $f(0) = f^0$, $g(0) = g^0$ point–wise and $\Gamma(0) = \Gamma^0$ almost everywhere, by (3.36), (3.113) and (3.116). Therefore claim b) of Theorem 3.11 is satisfied. Due to (3.37), (3.113) and (3.116), the mass conservation property is satisfied for almost every $t \geq 0$, which proves part c) of Theorem 3.11.

Next we establish the identities in Theorem 3.11 d). In order to be able to pass to the limit in (3.33)–(3.35), we investigate, like in [20, Proof of Theorem 3], the convergence of the regularized terms τ_ε and σ_ε, which occur in $H_f^\varepsilon, H_g^\varepsilon$ and H_Γ^ε. Note first that (as in Lemma 3.18), we can prove that $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ is bounded in $L_6(\Omega_T)$ and the convergence $\Gamma_\varepsilon \to \Gamma$ takes place in $L_p(\Omega_T)$ for $p \in [2, 6)$. Moreover, by construction

$$\tau_\varepsilon(s) = s \qquad \text{for} \quad 0 \leq s \leq s_\varepsilon := \left[\left(\frac{1}{\varepsilon c_\Phi}\right)^{\frac{r}{r+1}} - 1\right]^{\frac{1}{r}}, \tag{3.122}$$

which is due to Assumption A3). Indeed, by definition, $\tau_\varepsilon(s) = s$ if $\sigma_\varepsilon'(s) = \mathcal{T}_\varepsilon(\sigma'(s)) = \sigma'(s)$, which in turn is satisfied if

$$|\sigma'(s)| \leq \varepsilon^{-1}. \tag{3.123}$$

Using Assumption A3), it is sufficient to show that

$$\left(|s|^{r+1} + s\right) \leq \frac{1}{\varepsilon c_\Phi} \qquad \text{for} \quad s \geq 0 \tag{3.124}$$

in order to guarantee that (3.123) is satisfied. Assume $s \leq s_\varepsilon$, then

$$s^r + 1 \leq \left(\frac{1}{\varepsilon c_\Phi}\right)^{\frac{r}{r+1}}$$

and we deduce twofold

$$s^{r+1} + s \leq \left(\frac{1}{\varepsilon c_\Phi}\right)^{\frac{r}{r+1}} s \qquad \text{and} \qquad s \leq \left(\frac{1}{\varepsilon c_\Phi}\right)^{\frac{1}{r+1}}. \tag{3.125}$$

Inserting the second inequality in (3.125) into the first one, we obtain (3.124). Thus, $\tau_\varepsilon(s) = s$ for $0 \leq s \leq s_\varepsilon$. In particular,

$$\sigma_\varepsilon'(s) = \sigma'(s) \qquad \text{for all} \quad s \in [0, s_\varepsilon]. \tag{3.126}$$

Lemma 3.24. *There exists a subsequence (not relabeled) of* $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ *satisfying* *i)* $\tau_\varepsilon(\Gamma_\varepsilon) \longrightarrow \Gamma$ *in* $L_q(\Omega_T)$ *for* $q \in [3,6)$,

ii) $\partial_x\sigma_\varepsilon(\Gamma_\varepsilon) \rightharpoonup \partial_x\sigma(\Gamma)$ *in* $L_s(\Omega_T)$ *for* $s \in [1, \frac{6}{r+4})$.

Proof. Recall that $\Gamma_\varepsilon \geq 0$ almost everywhere, due to Theorem 3.23.

i) We show first that

$$\frac{\tau_\varepsilon(\Gamma_\varepsilon)}{\Gamma_\varepsilon} \to 1 \qquad \text{in} \quad L_p(\Omega_T) \quad \text{for any} \quad p \geq 1. \tag{3.127}$$

Then, the statement follows in virtue of

$$\|\tau_\varepsilon(\Gamma_\varepsilon) - \Gamma\|_p \leq \|\tau_\varepsilon(\Gamma_\varepsilon) - \Gamma_\varepsilon\|_p + \|\Gamma_\varepsilon - \Gamma\|_p \leq \|\Gamma_\varepsilon\|_6 \left\|\frac{\tau_\varepsilon(\Gamma_\varepsilon)}{\Gamma_\varepsilon} - 1\right\|_{\frac{p}{6-p}} + \|\Gamma_\varepsilon - \Gamma\|_p\,,$$

$\Gamma_\varepsilon \to \Gamma$ in $L_m(\Omega_T)$ for $m \in [2,6)$ and (3.127). In order to prove (3.127), recall that $\tau_\varepsilon(\Gamma_\varepsilon) = \Gamma_\varepsilon$ if $\Gamma_\varepsilon \leq s_\varepsilon$ (cf. (3.122)). Thus, for any $p \geq 1$

$$\begin{aligned}\int_{\Omega_T}\left|\frac{\tau_\varepsilon(\Gamma_\varepsilon)}{\Gamma_\varepsilon} - 1\right|^p d(x,t) &= \int_{[\Gamma_\varepsilon > s_\varepsilon]}\left|\frac{\tau_\varepsilon(\Gamma_\varepsilon)}{\Gamma_\varepsilon} - 1\right|^p d(x,t) \leq \int_{[\Gamma_\varepsilon > s_\varepsilon]} 2^p\left(\left|\frac{\tau_\varepsilon(\Gamma_\varepsilon)}{\Gamma_\varepsilon}\right|^p - 1\right) d(x,t)\\ &\leq 2^{p+1}\int_{[\Gamma_\varepsilon > s_\varepsilon]} 1\, d(x,t) \leq 2^{p+1}\int_{[\Gamma_\varepsilon > s_\varepsilon]}\frac{\Gamma_\varepsilon^6}{s_\varepsilon^6}\, d(x,t) \leq \frac{C(p,T)}{s_\varepsilon^6},\end{aligned} \tag{3.128}$$

since $|\tau_\varepsilon(s)| \leq |s|$ and $(\Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ being uniformly bounded in $L_6(\Omega_T)$. Letting ε tend to zero, (3.128) implies the assertion in view of $s_\varepsilon \longrightarrow \infty$ if $\varepsilon \searrow 0$.

ii) Given $p \in [1, \frac{6}{r+1})$, $R \geq 1$ and $\varepsilon \in (0,1]$, such that $1 \leq R \leq s_\varepsilon$, we have that

$$\begin{aligned}\int_{\Omega_T} |\sigma_\varepsilon'(\Gamma_\varepsilon) - \sigma'(\Gamma)|^p\, d(x,t) =& \int_{[\max\{\Gamma_\varepsilon,\Gamma\}\leq R]} |\sigma_\varepsilon'(\Gamma_\varepsilon) - \sigma'(\Gamma)|^p\, d(x,t)\\ &+ \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} |\sigma_\varepsilon'(\Gamma_\varepsilon) - \sigma'(\Gamma)|^p\, d(x,t).\end{aligned} \tag{3.129}$$

Estimating the integrals on the right–hand side of (3.129) separately, noting that $\sigma_\varepsilon' = \sigma'$ everywhere in $[\Gamma_\varepsilon \leq R]$ (cf. (3.126)) and since $\sigma' \in C^1(\mathbb{R})$, the Mean Value Theorem implies that the first integral reduces to

$$\begin{aligned}\int_{[\max\{\Gamma_\varepsilon,\Gamma\}\leq R]} |\sigma_\varepsilon'(\Gamma_\varepsilon) - \sigma'(\Gamma)|^p\, d(x,t) &= \int_{[\max\{\Gamma_\varepsilon,\Gamma\}\leq R]} |\sigma'(\Gamma_\varepsilon) - \sigma'(\Gamma)|^p\, d(x,t)\\ &\leq \|\sigma''\|_{L_\infty(0,R)}\int_{[\max\{\Gamma_\varepsilon,\Gamma\}\leq R]} |\Gamma_\varepsilon - \Gamma|^p\, d(x,t),\end{aligned} \tag{3.130}$$

which tends to zero if $\varepsilon \searrow 0$ for any $p \in [1,6)$. The second integral yields in virtue of $|\sigma_\varepsilon'| \le |\sigma'|$ and Assumption A3)

$$\begin{aligned}
\int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} |\sigma_\varepsilon'(\Gamma_\varepsilon) - \sigma'(\Gamma)|^p \, d(x,t) &\le \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} 2^p \left(|\sigma'(\Gamma_\varepsilon)|^p + |\sigma'(\Gamma)|^p \right) d(x,t) \\
&\le 2^p C_\Phi \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} |\Gamma_\varepsilon(\Gamma_\varepsilon^r + 1)|^p + |\Gamma(\Gamma^r + 1)|^p \, d(x,t) \\
&\le 2^{p+1} C_\Phi \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} \left\{ \Gamma_\varepsilon^{p(r+1)} + \Gamma^{p(r+1)} \right\} d(x,t) \\
&\le 2^{p+1} C_\Phi \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} 2 \max\{\Gamma_\varepsilon, \Gamma\}^{p(r+1)} \, d(x,t) \\
&= \frac{2^{p+3} C_\Phi}{R^{6-p(r+1)}} \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} \max\{\Gamma_\varepsilon, \Gamma\}^{p(r+1)} R^{6-p(r+1)} \, d(x,t) \\
&\le \frac{2^{p+2} C_\Phi}{R^{6-p(r+1)}} \int_{[\Gamma_\varepsilon > R]\cup[\Gamma > R]} \Gamma_\varepsilon^6 + \Gamma^6 \, d(x,t).
\end{aligned} \tag{3.131}$$

Again, since $(\Gamma_\varepsilon)_{\varepsilon \in (0,1]}$ is bounded in $L_6(\Omega_T)$ and thus, in virtue of (3.116), also $\Gamma \in L_6(\Omega_T)$, we may let first $\varepsilon \searrow 0$ and then $R \to \infty$ in (3.131). Gathering (3.129)–(3.131), we have shown that

$$\sigma_\varepsilon'(\Gamma_\varepsilon) \longrightarrow \sigma'(\Gamma) \qquad \text{in} \quad L_p(\Omega_T) \quad \text{for} \quad p \in \Big[1, \frac{6}{r+1}\Big). \tag{3.132}$$

Since $r \in (0,2)$, we can choose $p > 2$ in (3.132). Note that $(\partial_x \sigma_\varepsilon(\Gamma_\varepsilon))_{\varepsilon \in (0,1]} = (\sigma_\varepsilon'(\Gamma_\varepsilon)\partial_x \Gamma_\varepsilon)_{\varepsilon \in (0,1]}$ is bounded in $L_s(\Omega_T)$ for $s \in [1, \frac{6}{r+4})$, since $(\partial_x \Gamma_\varepsilon)_{\varepsilon \in (0,1]}$ is bounded in $L_2(\Omega_T)$, $(\sigma_\varepsilon'(\Gamma_\varepsilon))_{\varepsilon \in (0,1]}$ is bounded in $L_p(\Omega_T)$ for $p \in [1, \frac{6}{r+1})$ and Hölder's inequality. Hence, by Eberlein–Smulyan's theorem, there exists a subsequence (not relabeled), such that $\partial_x \sigma_\varepsilon(\Gamma_\varepsilon)$ converges weakly to a function v in $L_s(\Omega_T)$. The identification of the limit v with $\partial_x \sigma(\Gamma)$ follows in virtue of

$$\begin{aligned}
\int_{\Omega_T} (\partial_x \sigma(\Gamma) - v)\xi \, d(x,t) &= \int_{\Omega_T} (\partial_x \sigma(\Gamma) - \partial_x \sigma_\varepsilon(\Gamma_\varepsilon))\xi \, d(x,t) + \int_{\Omega_T} (\partial_x \sigma_\varepsilon(\Gamma_\varepsilon) - v)\xi \, d(x,t) \\
&= \int_{\Omega_T} \sigma'(\Gamma)(\partial_x \Gamma - \partial_x \Gamma_\varepsilon)\xi \, d(x,t) + \int_{\Omega_T} \partial_x \Gamma_\varepsilon (\sigma'(\Gamma) - \sigma_\varepsilon'(\Gamma_\varepsilon))\xi \, d(x,t) \\
&\quad + \int_{\Omega_T} (\partial_x \sigma_\varepsilon'(\Gamma_\varepsilon) - v)\xi \, d(x,t),
\end{aligned}$$

which tends to zero for arbitrary $\xi \in C^\infty(\overline{\Omega_T})$ by $\partial_x \sigma_\varepsilon(\Gamma_\varepsilon) \rightharpoonup v$ in $L_s(\Omega_T)$, (3.117) and (3.132). This proves the statement. □

Let $\xi \in L_2(0,T;H^1(0,L))$ be given and $(f_\varepsilon, g_\varepsilon, \Gamma_\varepsilon)_{\varepsilon\in(0,1]}$ be the Galerkin approximation of the regularized system, which admits a subsequence converging towards (f,g,Γ). We know from (3.33) that

$$\sqrt{a_\varepsilon(f_\varepsilon)}\left[\frac{a_\varepsilon(f_\varepsilon)}{\sqrt{3\sigma_2^c\mu}}\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f_\varepsilon+\sigma_2^c\mu g_\varepsilon)+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right)\right]$$
$$\rightharpoonup \sqrt{f}\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}f\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f+\sigma_2^c\mu g)+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}g\partial_x^3(f+g)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right)\right],$$

$$\frac{1}{4}\sqrt{a_\varepsilon(f_\varepsilon)}\left[\sqrt{\sigma_2^c}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]\rightharpoonup\frac{\sqrt{f}}{4}\left[\sqrt{\sigma_2^c}g\partial_x^3(f+g)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right],$$

$$\sqrt{a_\varepsilon(g_\varepsilon)}\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}a_\varepsilon(g_\varepsilon)\partial_x^3(f_\varepsilon+g_\varepsilon)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\right]\rightharpoonup\sqrt{g}\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}g\partial_x^3(f+g)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right],$$

$$\frac{1}{4\sigma_2^c}\sqrt{a_\varepsilon(g_\varepsilon)}\partial_x\sigma_\varepsilon(\Gamma_\varepsilon)\rightharpoonup\frac{\sqrt{g}}{4\sigma_2^c}\partial_x\sigma(\Gamma)$$

in $L_2(\mathcal{P}_f\cap\mathcal{P}_g)$. By using the same arguments as before, we are now able to identity the limit function H_f on the set $\mathcal{P}_f\cap\mathcal{P}_g$ where f and g are strictly positive as

$$H_f=\frac{\sqrt{\sigma_2^c\mu}}{\sqrt{3}}f^2\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}f\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f+\sigma_2^c\mu g)+\frac{\sqrt{3}}{2}\left(\sqrt{\sigma_2^c\mu}g\partial_x^3(f+g)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right)\right].$$

Analogously we prove (3.26), (3.27), where the limit functions H_g, H_Γ are identified as

$$H_g=\frac{\sqrt{3\sigma_2^c\mu}}{2}gf\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}f\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f+\sigma_2^c\mu g)+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}g\partial_x^3(f+g)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right)\right]$$
$$+\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}g\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}g\partial_x^3(f+g)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right]$$

and

$$H_\Gamma=\frac{\sqrt{3\sigma_2^c}}{2}\Gamma f\left[\frac{1}{\sqrt{3\sigma_2^c\mu}}f\partial_x^3((\sigma_1^c+\sigma_2^c\mu)f+\sigma_2^c\mu g)+\frac{2}{\sqrt{3}}\left(\sqrt{\sigma_2^c\mu}g\partial_x^3(f+g)+\frac{\sqrt{\mu}}{\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right)\right]$$
$$+\frac{\sqrt{3\sigma_2^c}}{2}\Gamma g\left[\frac{\sqrt{\sigma_2^c}}{\sqrt{3}}g\partial_x^3(f+g)+\frac{\sqrt{3}}{2\sqrt{\sigma_2^c}}\partial_x\sigma(\Gamma)\right]+\frac{1}{4}\Gamma g\partial_x\sigma(\Gamma)-D\partial_x\Gamma$$

on $\mathcal{P}_f\cap\mathcal{P}_g$. Finally, similar as before, we pass to the weak limit in the energy inequality (3.38) and obtain claim Theorem 3.11 e), by (3.113), (3.119).

Remark 3.25. Recall that the surfactant concentration is rescaled by $\bar{\Gamma}=\frac{\Gamma}{\Gamma_m}$, where Γ_m is the *critical micelle concentration* (cf. Section 1.1). Hence, from a physical point of view it is expected that the weak solution Γ, constructed above, satisfies not only the non–negativity property, but also $0 \leq \Gamma \leq 1$. In [15] a thin film model with insoluble surfactant taking into account gravitational, capillary and van der Waals forces is studied and the existence of global non–negative weak solutions as well as the upper bound $\Gamma \leq 1$ is investigated.

A. Appendix

A.1. Calculations

Let for each $u \in U^\alpha$ the matrix $a_G(u)$ be given as in (2.3). We prove that all eigenvalues of $a_G(u)$ are strictly positive.

Lemma A.1. *If Assumption G1) and A1) are satisfied, all eigenvalues of $a_G(u)$ are strictly positive for each fixed $u \in U^\alpha$.*

Proof. Set for $u = (f, g, \Gamma) \in U^\alpha$

$$\tilde{a} := G_1 \tfrac{f^3}{3} + G_2 \mu \tfrac{f^2 g}{2}, \qquad \bar{a} := G_2 \mu \left(\tfrac{f^3}{3} + \tfrac{f^2 g}{2} \right)$$

$$b := G_2 \tfrac{g^3}{3} + G_1 \tfrac{f^2 g}{2} + G_2 \mu f g^2, \qquad \bar{b} := G_2 \tfrac{g^3}{3} + G_2 \mu \left(\tfrac{f^2 g}{2} + f g^2 \right)$$

$$c := \left(G_2 \tfrac{g^2}{2} + G_1 \tfrac{f^2}{2} + G_2 \mu f g \right) \Gamma, \qquad \bar{c} := \left(G_2 \tfrac{g^2}{2} + G_2 \mu \left(\tfrac{f^2}{2} + f g \right) \right) \Gamma$$

$$d := -\tfrac{f^2}{2} \mu \sigma'(\Gamma), \qquad e := -\left(\mu f g + \tfrac{g^2}{2} \right) \sigma'(\Gamma)$$

$$h := -(\mu f + g) \Gamma \sigma'(\Gamma) + D.$$

Since we assume σ' to be non–positive (cf. A1)), $\tilde{a}, \bar{a}, b, \bar{b}, c, \bar{c}, h > 0$ and $d, e \geq 0$. The matrix $a_G(u)$ in (3.1) can be written as

$$a_G(u) = \begin{pmatrix} \tilde{a} & \bar{a} & d \\ b & \bar{b} & e \\ c & \bar{c} & h \end{pmatrix}.$$

and the eigenvalues of $a_G(u)$ are the roots of

$$\begin{aligned} \det(a(u) - \lambda \, \mathrm{Id}) = \lambda^3 - \lambda^2 [h + \tilde{a} + \bar{b}] + \lambda [\tilde{a}\bar{b} - \bar{a} b + h(\tilde{a} + \bar{b}) - cd - \bar{c} e] \\ - [h(\tilde{a}\bar{b} - \bar{a} b) + d(b\bar{c} - \bar{b} c) + e(\bar{a} c - \tilde{a} \bar{c})]. \end{aligned} \tag{A.1}$$

Rewriting $(A.1)$ as

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0,$$

where $A := h+\tilde{a}+\bar{b}$, $B := \tilde{a}\bar{b}-\bar{a}b+h(\tilde{a}+\bar{b})-cd-\bar{c}e$ and $C := h(\tilde{a}\bar{b}-\bar{a}b)+d(b\bar{c}-\bar{b}c)+e(\bar{a}c-\tilde{a}\bar{c})$, we can use the Hurwitz Lemma and prove the statement by showing that $A, B, C > 0$ and $AB - C > 0$. We have

$$\begin{aligned}
A &= -(\mu f + g)\Gamma\sigma'(\Gamma) + D + G_1\tfrac{f^3}{3} + G_2\tfrac{g^3}{3} + G_2\mu\left(f^2 g + f g^2\right)\\
B &= \left(G_2\tfrac{f^3g^3}{9} + \tfrac{f^4g^2}{12}G_2\mu\right)[G_1 - G_2\mu] + D\left(G_1\tfrac{f^3}{3} + (G_1 + G_2\mu)\tfrac{f^2g}{2} + G_2\mu\tfrac{fg^2}{2} + G_2\tfrac{g^3}{3}\right)\\
&\quad -\sigma'(\Gamma)\Gamma\left(G_1\mu\tfrac{f^4}{12} + [G_1 - G_2\mu]\mu\tfrac{f^3g}{2} + G_1\tfrac{f^3g}{3} + G_1\tfrac{f^2g^2}{4} + [G_1 - G_2\mu]\tfrac{f^2g^2}{4} + 2G_2\mu\tfrac{fg^3}{2}\right)\\
C &= \left(D\left(G_2\tfrac{f^3g^3}{9} + G_2\mu\tfrac{f^4g^2}{12}\right) - \sigma'(\Gamma)\Gamma\left(G_2\tfrac{f^3g^4}{36} + G_2\mu\tfrac{f^4g^3}{36}\right)\right)[G_1 - G_2\mu].
\end{aligned}$$

Assumption G1) implies that $G_1 - G_2\mu > 0$, thus $A, B, C > 0$. Observe that

$$AB > (D - \sigma'(\Gamma)\Gamma(\mu f + g))\left(G_2\frac{f^3g^3}{9} + \frac{f^4g^2}{12}G_2\mu\right)[G_1 - G_2\mu].$$

Hence

$$\begin{aligned}
AB - C &> \Big((-\sigma'(\Gamma)\Gamma(\mu f + g) + D)\left(G_2\tfrac{f^3g^3}{9} + G_2\mu\tfrac{f^4g^2}{12}\right)\\
&\quad - D\left(\tfrac{f^3g^3}{9}G_2 + G_2\mu\tfrac{f^4g^2}{12}\right) + \sigma'(\Gamma)\Gamma\left(G_2\tfrac{f^3g^4}{36} + G_2\mu\tfrac{f^4g^3}{36}\right)\Big)[G_1 - G_2\mu]\\
&= -\sigma'(\Gamma)\Gamma[G_1 - G_2\mu]\mu f\left(G_2\tfrac{f^3g^4}{36} + G_2\mu\tfrac{f^4g^3}{36}\right) > 0
\end{aligned}$$

Applying the Hurwitz Lemma, we deduce that the spectrum of $a_G(u)$ is for each fixed $u \in U^\alpha$ consists only of strictly positive numbers. □

In the next lemma we show that there exists $z \geq 0$, such that the matrix $b^z_G(f_*, g_*, 0)$, defined in (2.13), where $f_*, g_* > 0$ are constants, is positive definite

Lemma A.2. *Let Assumption G1) and S1) be satisfied. Then, there exists $z > 0$, such that the matrix $b^z_G(f_*, g_*, 0)$ is positive definite.*

Proof. The matrix $b^z_G(f_*, g_*, 0)$ is given by

$$b^z_G(f_*, g_*, 0) := \begin{pmatrix} \frac{G_2\mu}{G_1 - G_2\mu}\left(G_2\frac{g_*^3}{3} + G_2\mu\left(\frac{f_*^3}{3} + f_*^2 g_* + f_* g_*^2\right)\right) & G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & j \\ G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right) & (G_1 - G_2\mu)\frac{f_*^3}{3} & k \\ j & k & +zD \end{pmatrix} \tag{A.2}$$

with

$$j := -\frac{1}{2}\frac{G_2\mu}{G_1 - G_2\mu}\left(\mu\frac{f_*^2}{2} + \mu f_* g_* + \frac{g_*^2}{2}\right)\sigma'(0),$$

$$k := -\frac{1}{2}\mu\frac{f_*^2}{2}\sigma'(0).$$

In order to prove that the matrix (A.2) is positive definite, we show that its leading principal minors are positive. Denote its first, second and third principal minor by M_1, M_2 and M_3, respectively. Clearly $M_1 = \frac{G_2\mu}{G_1-G_2\mu}\left(G_2\frac{g_*^3}{3} + G_2\mu\left(\frac{f_*^3}{3} + f_*^2 g_* + f_* g_*^2\right)\right)$ is positive, by assumption G1). The second principal minor M_2 is given by

$$\begin{aligned} M_2 &= \tfrac{G_2\mu}{G_1-G_2\mu}\left(G_2\tfrac{g_*^3}{3} + G_2\mu\left(\tfrac{f_*^3}{3} + f_*^2 g_* + f_* g_*^2\right)\right)(G_1 - G_2\mu)\tfrac{f_*^3}{3} - (G_2\mu)^2\left(\tfrac{f_*^3}{3} + \tfrac{f_*^2 g_*}{2}\right)^2 \\ &= G_2\mu\left(G_2\tfrac{g_*^3}{9} + G_2\mu\tfrac{f_*^4 g_*^2}{12}\right), \end{aligned}$$

hence $M_2 > 0$. It remains to show that also $M_3 = \det(b(f_*, g_*, 0))$ is positive. We calculate via Leibniz formula

$$\begin{aligned} M_3 = zDM_2 - \frac{1}{4}\sigma'(0)\mu^2 &\left[\frac{f_*^4}{4}\frac{G_2\mu}{G_1 - G_2\mu}\left(G_2\frac{g_*^3}{3} + G_2\mu\left(\frac{f_*^3}{3} + f_*^2 g_* + f_* g_*^2\right)\right)\right. \\ &+ \left(\frac{G^2}{G_1 - G_2^\mu}\right)^2\left(\mu\frac{f^2}{2} + \mu f_* g_* + \frac{g_*^2}{2}\right)^2 (G_1 - G_2\mu)\frac{f_*^3}{3} \\ &\left.-\frac{G_2}{G_1 - G_2\mu}f^2\left(\mu\frac{f^2}{2} + \mu f_* g_* + \frac{g_*^2}{2}\right)G_2\mu\left(\frac{f_*^3}{3} + \frac{f_*^2 g_*}{2}\right)\right]. \end{aligned}$$

Since D and M_2 are strictly positive, there exists $z > 0$, such that $M_3 > 0$ and we have shown, that the matrix $b_G^z(f_*, g_*, 0)$ is positive definite. □

A.2. Alternative proof of Proposition 3.4 i)

We give an alternative proof of Proposition 3.4 i) by showing the sectorial property of $A_{11}(\overline{X})$, which is based on a similar result in [23]. Fix $\overline{X} = (\overline{f}, \overline{g}) \in \{H_B^{4\alpha}(0, L; \mathbb{R}^2) \mid \overline{f}, \overline{g} > 0\}$. Introducing a weighted scalar product, we show that $\mathbb{A} := A_{11}(\overline{X})$ satisfies the conditions of being a densely defined, sectorial operator, which implies that $-\mathbb{A}$ is the generator of an analytic semigroup. Set

$$a_{11} := (\sigma_1^c + \sigma_2^c\mu)\frac{\overline{f}^3}{3} + \sigma_2^c\mu\frac{\overline{f}^2\overline{g}}{2}, \qquad a_{12} := \sigma_2^c\mu\left(\frac{\overline{f}^3}{3} + \frac{\overline{f}^2\overline{g}}{2}\right),$$

$$a_{21} := \sigma_2^c\frac{\overline{g}^3}{3} + (\sigma_1^c + \sigma_2^c\mu)\frac{\overline{f}^2\overline{g}}{2} + \sigma_2^c\mu\overline{f}\overline{g}^2, \qquad a_{22} := \sigma_2^c\frac{\overline{g}^3}{3} + \sigma_2^c\mu\left(\frac{\overline{f}^2\overline{g}}{2} + \overline{f}\overline{g}^2\right).$$

Then, since $\overline{X} \in \{H_B^{4\alpha}(0,L;\mathbb{R}^2) \mid \overline{f},\overline{g} > 0\}$, we can ensure that $a_{ij} > 0, 1 \leq i,j \leq 2$ and belong to $C^2([0,L];\mathbb{R})$. Observe that

$$a_{11}a_{22} > a_{12}a_{21}. \tag{A.3}$$

Using the strict positivity of $\overline{X}$, we define, in accordance to [23], a weighted scalar product on $H_B^k(0,L;\mathbb{R}^2)$, where $k \in \mathbb{N}$, by the relation

$$\left(X \mid \tilde{X}\right)_k := \sum_{i=0}^{k} \int_0^L a_{21}\partial_x^i f \partial_x^i \tilde{f} + a_{12}\partial_x^i g \partial_x^i \tilde{g}\, dx \tag{A.4}$$

for $X = (f,g)$, $\tilde{X} = (\tilde{f},\tilde{g}) \in H_B^k(0,L;\mathbb{R}^2)$. Since a_{12} and a_{21} are both continuous and positive functions, it is clear that

$$\|\cdot\|_k := (\cdot \mid \cdot)_k^{\frac{1}{2}}$$

defines an equivalent norm to the usual Sobolev norm $\|\cdot\|_{H^k}$. In order to avoid any confusion, note that within this paragraph (*Proof of Proposition 3.4 i)*), $\|\cdot\|_2$ and $(\cdot \mid \cdot)_2$ always denote the above introduced norm and scalar product on the Sobolev space $H^2(0,L;\mathbb{R}^2)$. The L_2–norm and scalar product of a function X is then given by $\|X\|_0$ and $(\cdot \mid \cdot)_0$, respectively. The following three lemmata provide estimates, which imply that the operator $\mathbb{A}$ is closed and dissipative.

Lemma A.3. *There exist constants $c_0, \lambda_0 > 0$ such that*

$$\|\mathbb{A}X\|_0^2 \geq c_0\|X\|_4^2 - \lambda_0\|X\|_0^2 \qquad \text{for} \quad X \in H_B^4(0,L;\mathbb{R}^2). \tag{A.5}$$

Proof. By definition of the introduced norm, we have

$$\begin{aligned}
\|\mathbb{A}X\|_0^2 &= \int_0^L \left\{a_{21}\left|\partial_x\left(a_{11}\partial_x^3 f + a_{12}\partial_x^3 g\right)\right|^2 + a_{12}\left|\partial_x\left(a_{21}\partial_x^3 f + a_{22}\partial_x^3 g\right)\right|^2\right\} dx \\
&\geq m\int_0^L \left\{\left|\partial_x a_{11}\partial_x^3 f + a_{11}\partial_x^4 f + \partial_x a_{12}\partial_x^3 g + a_{12}\partial_x^4 g\right|^2\right. \\
&\qquad \left. + \left|\partial_x a_{21}\partial_x^3 f + a_{21}\partial_x^4 f + \partial_x a_{22}\partial_x^3 g + a_{22}\partial_x^4 g\right|^2\right\} dx,
\end{aligned}$$

where $m := \min\{a_{12}, a_{21}\}$. Since $(x+y)^2 \geq \frac{3}{4}x^2 - 3y^2$, we obtain that

$$\begin{aligned}
\|\mathbb{A}X\|_0^2 \geq \frac{3m}{4}\int_0^L &\left\{\left|a_{11}\partial_x^4 f + a_{12}\partial_x^4 g\right|^2 + \left|a_{21}\partial_x^4 f + a_{22}\partial_x^4 g\right|^2\right\} dx \\
&- d\left(\|\partial_x^3 f\|_0^2 + \|\partial_x^3 g\|_0^2\right)
\end{aligned}$$

$$= \frac{3m}{4}\int_0^L \left\{(a_{11}^2+a_{21}^2)|\partial_x^4 f|^2 + 2(a_{11}a_{12}+a_{21}a_{22})\partial_x^4 f \partial_x^4 g + (a_{12}^2+a_{22}^2)|\partial_x^4 g|^2\right\} dx$$

$$- d\left(\|\partial_x^3 f\|_0^2 + \|\partial_x^3 g\|_0^2\right),$$

where $d > 0$ is a constant depending on $a_{ij}, \|\partial_x a_{ij}\|_\infty$, $1 \le i,j \le 2$. Note that, since $a_{11}a_{22} \neq a_{21}a_{12}$ (cf. (A.3)),

$$\begin{aligned}\sqrt{(a_{11}^2+a_{21}^2)(a_{12}^2+a_{22}^2)} &= \sqrt{a_{11}^2a_{12}^2 + a_{11}^2a_{22}^2 + a_{21}^2a_{12}^2 + a_{21}^2a_{22}^2}\\ &= \sqrt{(a_{11}a_{12}+a_{21}a_{22})^2 + (a_{11}a_{22}-a_{21}a_{12})^2}\\ &= (a_{11}a_{12}+a_{21}a_{22}) + c\end{aligned}$$

for some $c > 0$. Thus,

$$(a_{11}a_{12}+a_{21}a_{22}) = \sqrt{(a_{11}^2+a_{21}^2)(a_{12}^2+a_{22}^2)} - c = \sqrt{(a_{11}^2+a_{21}^2)(a_{12}^2+a_{22}^2)}\,(1-k),$$

where $0 < k := \frac{c}{\sqrt{(a_{11}^2+a_{21}^2)(a_{12}^2+a_{22}^2)}} < 1$. We deduce that

$$\begin{aligned}\|\mathbb{A}X\|_0^2 \ge \frac{3m}{4}\int_0^L \Bigg\{ &\left[\sqrt{(1-k)(a_{11}^2+a_{21}^2)}|\partial_x^4 f| + \sqrt{(1-k)(a_{12}^2+a_{22}^2)}|\partial_x^4 g|\right]^2\\ &+ k\left((a_{11}^2+a_{21}^2)|\partial_x^4 f|^2 + (a_{12}^2+a_{22}^2)|\partial_x^4 g|^2\right)\Bigg\} dx - d\left(\|\partial_x^3 f\|_0^2 + \|\partial_x^3 g\|_0^2\right)\\ \ge e\left(\|\partial_x^4 f\|_0^2 + \|\partial_x^4 f\|_0^2\right) &- d\left(\|\partial_x^3 f\|_0^2 + \|\partial_x^3 g\|_0^2\right)\\ \ge e\left(\|f\|_4^2 + \|g\|_4^2\right) &- (e+d)(\|f\|_3^2 + \|g\|_3^2)\end{aligned}$$

for $e > 0$ being a constant. By an interpolation estimate and Young's inequality (cf. (3.8)), we deduce the existence of $c_0, \lambda_0 > 0$, such that (A.5) is satisfied. □

Lemma A.4. *For all $X \in H_B^4(0,L;\mathbb{R}^2)$ there exist constants $c_1, \lambda_1 > 0$, such that*

$$(\mathbb{A}X \mid X)_0 \ge c_1\|X\|_2^2 - \lambda_1\|X\|_0^2. \tag{A.6}$$

Proof. Given $X \in H_B^4(0,L;\mathbb{R}^2)$, we obtain in virtue of two times integration by parts, where the boundary terms vanish due to the boundary conditions,

$$(\mathbb{A}X \mid X)_0 = \int_0^L \left\{a_{21}\partial_x\left(a_{11}\partial_x^3 f + a_{12}\partial_x^3 g\right) f + a_{12}\partial_x\left(a_{21}\partial_x^3 f + a_{22}\partial_x^3 g\right) g\right\} dx$$

$$
\begin{aligned}
&= -\int_0^L \left\{ \left(a_{11}\partial_x^3 f + a_{12}\partial_x^3 g\right) \partial_x(a_{21}f) + \left(a_{21}\partial_x^3 f + a_{22}\partial_x^3 g\right) \partial_x(a_{12}g) \right\} \, dx, \\
&= \int_0^L \left\{ \partial_x^2 f \partial_x(a_{11}\partial_x(a_{21}f)) + \partial_x^2 g \partial_x(a_{12}\partial_x(a_{21}f)) \right. \\
&\qquad \left. + \partial_x^2 f \partial_x(a_{21}\partial_x(a_{12}g)) + \partial_x^2 g \partial_x(a_{22}\partial_x(a_{12}g)) \right\} \, dx, \\
&= \int_0^L \left\{ \partial_x^2 f \partial_x \left(a_{11}a_{21}\partial_x f + a_{11}\partial_x a_{21} f + a_{21}a_{12}\partial_x g + a_{21}\partial_x a_{12} g\right) \right. \\
&\qquad \left. + \partial_x^2 g \partial_x \left(a_{12}a_{21}\partial_x f + a_{11}\partial_x a_{21} f + a_{22}a_{12}\partial_x g + a_{22}\partial_x a_{12} g\right) \right\} \, dx \\
&\geq \int_0^L \left\{ a_{11}a_{21}|\partial_x^2 f|^2 + 2a_{21}a_{12}\partial_x^2 f \partial_x^2 g + a_{12}a_{22}|\partial_x^2 g|^2 \right\} \, dx \\
&\qquad - E\left(\|\partial_x^2 f\|_0 + \|\partial_x^2 g\|_0\right)\left(\|f\|_1 + \|g\|_1\right),
\end{aligned}
$$

where $E > 0$ is a constant depending on $\|a_{ij}\|_\infty, \|\partial_x a_{ij}\|_\infty$ and $\|\partial_x^2 a_{ij}\|_\infty$, $1 \leq i, j \leq 2$. Note that we used the chain rule for Sobolev functions, which states that $a \in W_\infty^k(0, L; \mathbb{R}^2)$ and $X \in H^k(0, L; \mathbb{R}^2)$ imply $aX \in H^k(0, L; \mathbb{R}^2)$ and $\partial_x(aX) = \partial_x a X + a\partial_x X$ (see e.g. [25, Theorem 5.8.4]). The integrand of the first integral in the last estimate above can be written as[1]

$$
\begin{aligned}
&a_{11}a_{21}|\partial_x^2 f|^2 + 2a_{21}a_{12}\partial_x^2 f \partial_x^2 g + a_{12}a_{22}|\partial_x^2 g|^2 \\
&\qquad = \sqrt{\frac{a_{12}a_{21}}{a_{11}a_{22}}} \left[\sqrt{a_{11}a_{21}}\partial_x^4 f + \sqrt{a_{12}a_{22}}\partial_x^4 g\right]^2 + \left(1 - \sqrt{\frac{a_{12}a_{21}}{a_{11}a_{22}}}\right) \left[|\partial_x^4 f|^2 + |\partial_x^4 g|^2\right].
\end{aligned}
$$

Hence, in virtue of $a_{11}a_{22} > a_{12}a_{21}$ (cf. (A.3)), there exist constants $\tilde{c}, \tilde{E} > 0$ with

$$
\begin{aligned}
(\mathbb{A}X \mid X)_0 &\geq \tilde{c}\left(\|\partial_x^2 f\|_0^2 + \|\partial_x^2 g\|_0^2\right) - E\left(\|\partial_x^2 f\|_0 + \|\partial_x^2 g\|_0\right)\left(\|f\|_1 + \|g\|_1\right) \\
&\leq \tilde{c}\left(\|X\|_2^2 - \|X\|_1^2\right) - \tilde{E}\|X\|_2\|X_1\| \\
&\leq c_1\|X\|_2^2 - \lambda_1\|X\|_0^2,
\end{aligned}
$$

by an interpolation estimate and Young's inequality, for some constants $c_1, \lambda_1 > 0$. □

Lemma A.5 (Dissipativity)**.** *There exist constants $c_0, \lambda_* > 0$, such that*

$$
\|(\mathbb{A} + \lambda) X\|_0^2 \geq c_0\|X\|_4 \qquad \textit{for all} \quad \lambda \geq \lambda_* \quad \textit{and} \quad X \in H_B^4(0, L; \mathbb{R}^2).
$$

[1] Here appears the only situation, where we make use of the, by (A.4), introduced norm.

Proof. For $X \in H^4_B(0,L;\mathbb{R}^2)$, we obtain that

$$\|\left(\mathbb{A}+\lambda\right)X\|_0^2 = \|\mathbb{A}X\|_0^2 + 2\lambda\left(\mathbb{A}X \mid X\right)_0 + \lambda^2\|X\|_0^2.$$

Then, Lemma A.3 and A.4 imply that

$$\|\left(\mathbb{A}+\lambda\right)X\|_0^2 \geq c_0\|X\|_4^2 + \left(\lambda^2 - 2\lambda\lambda_1 - \lambda_0\right)\lambda^2\|X\|_0^2.$$

Hence, there exists $\lambda_*, c_0 > 0$, such that the assertion is satisfied for all $\lambda \geq \lambda_*$. □

Note that the estimate given in Lemma A.3 implies that the operator

$$\mathbb{A} : H^4_B(0,L;\mathbb{R}^2) \subset L_2(0,L;\mathbb{R}^2) \to L_2(0,L;\mathbb{R}^2)$$

is closed. To see this, take a sequence $(X_n)_{n\in\mathbb{N}} \subset H^4_B(0,L;\mathbb{R}^2)$, which converges in $L_2(0,L;\mathbb{R}^2)$ towards a limit function $X \in L_2(0,L;\mathbb{R}^2)$ and $\mathbb{A}X_n \to Y$ in $L_2(0,L;\mathbb{R}^2)$. Then, $(\mathbb{A}X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(0,L;\mathbb{R}^2)$. Now, Lemma A.3 warrant that $(X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^4_B(0,L;\mathbb{R}^2)$. We deduce that $X \in H^4_B(0,L;\mathbb{R}^2)$ and

$$\|\mathbb{A}X - Y\|_0 \leq \|\mathbb{A}X - \mathbb{A}X_n\|_0 + \|\mathbb{A}X_n - Y\|_0 \to 0,$$

by $\mathbb{A}$ being continuous on $H^4_B(0,L;\mathbb{R}^2)$ and the assumption $\mathbb{A}X_n \to Y$ in $L_2(0,L;\mathbb{R}^2)$. This indicates that $\mathbb{A}$ is a closed operator.

We show that the operator $\mathbb{A} : H^4_B(0,L;\mathbb{R}^2) \subset L_2(0,L;\mathbb{R}^2) \to L_2(0,L;\mathbb{R}^2)$ is *sectorial*, that is, the spectrum spec$(\mathbb{A})$ of $\mathbb{A}$ is contained in a sector $\sum_\nu := \{z \in \mathbb{C} \mid -\nu \leq \arg(z) \leq \nu\}$, where $\nu \in (0,\pi/2)$ and for all $\lambda \in \sum_\nu^c$, where $\sum_\nu^c$ denotes the complement of $\sum_\nu$, there exists $M \geq 1$, such that

$$\|(\mathbb{A}-\lambda)^{-1}\|_{\mathcal{L}(L_2)} \leq \frac{M}{|\lambda|}.$$

Since $\mathbb{A}$ is densely defined, the operator being sectorial implies that $-\mathbb{A}$ generates an analytic semigroup. We want to make use of [41, Theorem 1.3.9], which implies the sectorial property of $\mathbb{A}$. To this end, we show that there exists $\lambda_* > 0$ such that for all $\lambda \geq \lambda_*$, the operator $\mathbb{A}+\lambda \in \mathcal{L}(H^4_B(0,L;\mathbb{R}^2), L_2(0,L;\mathbb{R}^2))$ is bijective. If $X \in H^4_B(0,L;\mathbb{R}^2)$, Lemma A.5 implies that the operator $\mathbb{A}+\lambda$ is bounded from below in $L_2(0,L;\mathbb{R}^2)$

$$\|(\mathbb{A}+\lambda)X\|_0^2 \geq c\|X\|_4^2 \qquad \text{for all} \quad \lambda \geq \lambda_*. \tag{A.7}$$

Observe, that (A.7) ensures that the linear operator $\mathbb{A}+\lambda$ is injective and closed. It remains to show that $\mathbb{A}+\lambda$ is surjective for all $\lambda \geq \lambda_*$. We define a continuous bilinear form closely related to the operator $\mathbb{A}+\lambda$

$$\mathcal{B}_\lambda : H^2_B(0,L;\mathbb{R}^2) \times H^2_B(0,L;\mathbb{R}) \to \mathbb{R}$$

by

$$\begin{aligned}\mathcal{B}_\lambda[X,\tilde{X}] := &\int_0^L \left\{ \partial_x^2 f \partial_x \left[a_{11}\partial_x(a_{21}\tilde{f}) + a_{21}\partial_x(a_{12}\tilde{g})\right] + \partial_x^2 g \partial_x \left[a_{12}\partial_x(a_{21}\tilde{f}) + a_{22}\partial_x(a_{12}\tilde{g})\right]\right\} dx \\ &+ \lambda \int_0^L a_{21} f\tilde{f} + a_{12} g\tilde{g}\, dx\end{aligned} \tag{A.8}$$

for $X=(f,g)$ and $\tilde{X}=(\tilde{f},\tilde{g}) \in H^2_B(0,L;\mathbb{R}^2)$. Note that if in addition $X \in H^4_B(0,L;\mathbb{R}^2)$, the relation

$$\mathcal{B}_\lambda[X,\tilde{X}] = \Big((\mathbb{A}+\lambda)X \mid \tilde{X}\Big)_0 \tag{A.9}$$

is satisfied. Moreover, if $X \in H^4_B(0,L;\mathbb{R}^2)$, Lemma A.4 ensures that

$$\mathcal{B}_\lambda[X,X] = ((\mathbb{A}+\lambda)X \mid X)_0\, \lambda = (\mathbb{A}X \mid X)_0 + \lambda \|X\|_0^2 \geq c\|X\|_2^2 \tag{A.10}$$

for all $\lambda \geq \lambda_*$ if λ_* sufficiently large. Taking into account that $H^4_B(0,L;\mathbb{R}^2)$ is a dense subset of $H^2_B(0,L;\mathbb{R}^2)$ and $\mathcal{B}_\lambda$ is continuous, we infer that (A.10) is satisfied for all $X \in H^2_B(0,L;\mathbb{R}^2)$. Thus the bilinear form (A.8) is coercive. Observe that for all $F=(F_1,F_2) \in L_2(0,L;\mathbb{R}^2)$

$$F(X) := (F \mid X)_0$$

defines a linear functional on $L_2(0,L;\mathbb{R}^2)$. Since the continuous bilinear form $\mathcal{B}_\lambda$ is coercive on $H^2_B(0,L;\mathbb{R}^2)$, the Lax–Milgram Theorem implies that for every $F \in L_2(0,L;\mathbb{R}^2)$ there exists a unique $X \in H^2_B(0,L;\mathbb{R}^2)$ such that

$$\mathcal{B}_\lambda[X,\tilde{X}] = \Big(F \mid \tilde{X}\Big)_0 \qquad \text{for all} \quad \tilde{X} \in H^2_B(0,L;\mathbb{R}^2). \tag{A.11}$$

We conclude that it remains to show, that indeed X belongs to $H^4_B(0,L;\mathbb{R}^2)$. Then, by $C_c^\infty((0,L);\mathbb{R}^2)$, the space of smooth functions with compact support, being a subset of $H^2_B(0,L;\mathbb{R}^2)$, we infer from (A.9) and (A.11), that for every $F \in L_2(0,L;\mathbb{R}^2)$ there exists a unique $X \in H^4_B(0,L;\mathbb{R}^2)$, such that $(\mathbb{A}+\lambda)X = F$. This means that $A+\lambda$ is surjective.

In order to show that X belongs to $H^4_B(0,L;\mathbb{R}^2)$, observe that (A.11) holds true for all $\tilde{X} \in C^\infty_c((0,L);\mathbb{R}^2)$. We deduce that in particular

$$\int_0^L \left\{ \partial_x^2 f \partial_x [a_{11}\partial_x(a_{21}\tilde{f})] + \partial_x^2 g \partial_x [a_{12}\partial_x(a_{21}\tilde{f})] \right\} dx = \int_0^L (F_1 - \lambda f)\, \tilde{f}\, dx, \tag{A.12}$$

$$\int_0^L \left\{ \partial_x^2 f \partial_x \left[a_{21}\partial_x(a_{12}\tilde{g})\right] + \partial_x^2 g \partial_x \left[a_{22}\partial_x(a_{12}\tilde{g})\right] \right\} dx = \int_0^L (F_2 - \lambda g)\, \tilde{g}\, dx \tag{A.13}$$

for all $\tilde{X} = (\tilde{f},\tilde{g}) \in C^\infty_c((0,L);\mathbb{R}^2)$. Now, (A.12) yields

$$\begin{aligned}
\int_0^L (F_1 - \lambda f)\, \tilde{f}\, dx &= \int_0^L \left\{ \partial_x^2 f \partial_x \left[a_{11}\partial_x(a_{21}\tilde{f})\right] + \partial_x^2 g \partial_x \left[a_{12}\partial_x(a_{21}\tilde{f})\right] \right\} dx \\
&= \int_0^L \left\{ \partial_x^2 f \left[\partial_x a_{11}\partial_x(a_{21}\tilde{f}) + a_{11}\partial_x^2(a_{21}\tilde{f})\right] + \partial_x^2 g \left[\partial_x a_{12}\partial_x(a_{21}\tilde{f}) + a_{12}\partial_x^2(a_{21}\tilde{f})\right] \right\} dx \\
&= \int_0^L \left\{ \left(\partial_x^2 f \partial_x a_{11} + \partial_x^2 g \partial_x a_{12}\right) \partial_x(a_{21}\tilde{f}) + \left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) \partial_x^2(a_{21}\tilde{f}) \right\} dx \\
&= \int_0^L \Big\{ \left(\partial_x^2 f \partial_x a_{11} + \partial_x^2 g \partial_x a_{12}\right) \partial_x a_{21}\tilde{f} + \left(\partial_x^2 f \partial_x a_{11} + \partial_x^2 g \partial_x a_{12}\right) a_{21}\partial_x\tilde{f} \\
&\qquad + \left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) \partial_x^2 a_{21}\tilde{f} + 2\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) \partial_x a_{21}\partial_x\tilde{f} \\
&\qquad + \left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) a_{12}\partial_x^2\tilde{f} \Big\} dx,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\int_0^L \bar{F}_1\tilde{f}\, dx = &\int_0^L \Big\{ \left(\left(\partial_x^2 f \partial_x a_{11} + \partial_x^2 g \partial_x a_{12}\right) a_{21} + 2\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) \partial_x a_{21}\right) \partial_x\tilde{f} \\
&\qquad + \left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) a_{12}\partial_x^2\tilde{f} \Big\} dx,
\end{aligned} \tag{A.14}$$

where

$$\bar{F}_1 := (F_1 - \lambda f) - \left(\left(\partial_x^2 f \partial_x a_{11} + \partial_x^2 g \partial_x a_{12}\right) \partial_x a_{21} + \left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) \partial_x^2 a_{21}\right) \in L_2(0,L;\mathbb{R}). \tag{A.15}$$

Following a similar computation for (A.13) yields analogously

$$\begin{aligned}
\int_0^L \bar{F}_2\tilde{g}\, dx = &\int_0^L \Big\{ \left(\left(\partial_x^2 f \partial_x a_{21} + \partial_x^2 g \partial_x a_{22}\right) a_{12} + 2\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right) \partial_x a_{12}\right) \partial_x\tilde{g} \\
&\qquad + \left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right) a_{21}\partial_x^2\tilde{g} \Big\} dx,
\end{aligned} \tag{A.16}$$

where

$$\bar{F}_2 := (F_2 - \lambda g) - \left(\left(\partial_x^2 f \partial_x a_{21} + \partial_x^2 g \partial_x a_{22}\right) \partial_x a_{12} + \left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right) \partial_x^2 a_{12}\right) \in L_2(0,L;\mathbb{R}). \tag{A.17}$$

Since (A.14), (A.16) hold true for all $(\tilde{f},\tilde{g}) \in C_c^\infty((0,L);\mathbb{R}^2)$, we deduce that

$$\bar{F}_1 = -\partial_x\left[\left(\partial_x^2 f\partial_x a_{11} + \partial_x^2 g\partial_x a_{12}\right)a_{21} + 2\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right)\partial_x a_{21}\right] + \partial_x^2\left[\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right)a_{12}\right],$$
$$\bar{F}_2 = -\partial_x\left[\left(\partial_x^2 f\partial_x a_{21} + \partial_x^2 g\partial_x a_{22}\right)a_{12} + 2\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right)\partial_x a_{12}\right] + \partial_x^2\left[\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right)a_{21}\right] \tag{A.18}$$

in the sense of distributions. In virtue of $\bar{F}_1, \bar{F}_2$ belonging to $L_2(0,L;\mathbb{R})$, the above equations imply that

$$-\partial_x\Big[\left(\partial_x^2 f\partial_x a_{11} + \partial_x^2 g\partial_x a_{12}\right)a_{21} + 2\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right)\partial_x a_{21} - \partial_x\left[\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right)a_{12}\right]\Big],$$
$$-\partial_x\Big[\left(\partial_x^2 f\partial_x a_{21} + \partial_x^2 g\partial_x a_{22}\right)a_{12} + 2\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right)\partial_x a_{12} - \partial_x\left[\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right)a_{21}\right]\Big]$$

belong to $L_2(0,L;\mathbb{R})$, which in turn yields

$$\left(\partial_x^2 f\partial_x a_{11} + \partial_x^2 g\partial_x a_{12}\right)a_{21} + 2\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right)\partial_x a_{21} - \partial_x\left[\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right)a_{12}\right], \tag{A.19}$$

$$\left(\partial_x^2 f\partial_x a_{21} + \partial_x^2 g\partial_x a_{22}\right)a_{12} + 2\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right)\partial_x a_{12} - \partial_x\left[\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right)a_{21}\right] \tag{A.20}$$

are elements in $H^1(0,L;\mathbb{R})$ Since $H^1(0,L;\mathbb{R}) \subset L_2(0,L;\mathbb{R})$ and the first two terms in (A.19), (A.20), respectively, belong to $L_2(0,L;\mathbb{R})$, the last terms in (A.19), (A.20) belong to $L_2(0,L;\mathbb{R})$ as well. In view of the fact that $a_{ij} \in C^2(0,L;\mathbb{R}), 1 \leq i,j \leq 2$ is strictly positive, we deduce that

$$a_{11}\partial_x^2 f + a_{12}\partial_x^2 g \qquad \text{and} \qquad a_{21}\partial_x^2 f + a_{22}\partial_x^2 g \qquad \text{belong to} \quad H^1(0,L;\mathbb{R}). \tag{A.21}$$

The above implies that $\frac{a_{12}}{a_{22}}\left(a_{21}\partial_x^2 f + a_{22}\partial_x^2 g\right) \in H^1(0,L;\mathbb{R})$. Hence

$$a_{11}\partial_x^2 f + a_{12}\partial_x^2 g - \frac{a_{12}}{a_{22}}\left(a_{21}\partial_x^2 f + a_{22}\partial_x^2 g\right) = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{22}}\partial_x^2 f \in H^1(0,L;\mathbb{R}).$$

Due to $a_{ij} > 0$, $1 \leq i,j \leq 2$ and $a_{11}a_{22} > a_{21}a_{12}$ (cf. (A.3)), this yields that $f \in H^3(0,L;\mathbb{R})$, which in turn implies in view of (A.21), that also $g \in H^3(0,L;\mathbb{R})$. Hence, the first two terms in (A.19), (A.20), respectively, belong to $H^1(0,L;\mathbb{R})$, so that the last terms in (A.19), (A.20) belong to $H^1(0,L;\mathbb{R})$ as well. Repeating the same argumentation, we arrive at $X = (f,g) \in H^4(0,L;\mathbb{R}^2)$. Thus, for $F \in L_2(0,L;\mathbb{R})$, there exists $X \in H^4(0,L;\mathbb{R}^2) \cap H_B^2(0,L;\mathbb{R}^2)$, such that (A.11) is satisfied. We need to verify that X satisfies the boundary condition $\partial_x^3 X = 0$ at $x = 0, L$. In view of $X \in H^4(0, L; \mathbb{R}^2)$, (A.15), (A.17) and (A.18), a straight forward computation yields

$$\begin{aligned}(F_1 - \lambda f) &= \bar{F}_1 + \left(\left(\partial_x^2 f a_{11} + \partial_x^2 g a_{12}\right) \partial_x^2 a_{21} + \left(\partial_x^2 f \partial_x a_{11} + \partial_x^2 g \partial_x a_{12}\right) \partial_x a_{21}\right) \\ &= a_{21} \partial_x \left[a_{11} \partial_x^3 f + a_{12} \partial_x^3 g\right],\end{aligned} \tag{A.22}$$

$$\begin{aligned}(F_2 - \lambda g) &= \bar{F}_2 + \left(\left(\partial_x^2 f a_{21} + \partial_x^2 g a_{22}\right) \partial_x^2 a_{12} + \left(\partial_x^2 f \partial_x a_{21} + \partial_x^2 g \partial_x a_{22}\right) \partial_x a_{12}\right) \\ &= a_{12} \partial_x \left[a_{21} \partial_x^3 f + a_{22} \partial_x^3 g\right].\end{aligned} \tag{A.23}$$

By means of (A.11) and the definition of $\mathcal{B}_\lambda$ (A.8), we obtain after integrating by parts twice

$$\begin{aligned}\left(F - \lambda X \mid \tilde{X}\right)_0 &= \mathcal{B}_\lambda[X, \tilde{X}] - \lambda \left(X \mid \tilde{X}\right)_0 \\ &= -\left(\partial_x^3 f \left[a_{11} a_{21} \tilde{f} + a_{21} a_{12} \tilde{g}\right] + \partial_x^3 g \left[a_{12} a_{21} \tilde{f} + a_{22} a_{21} \tilde{g}\right] \Big|_{x=0,L}\right) \\ &\quad + \int_0^L \left\{a_{21} \partial_x \left[a_{11} \partial_x^3 f + a_{12} \partial_x^3 g\right] \tilde{f} + a_{12} \partial_x \left[a_{21} \partial_x^3 f + a_{22} \partial_x^3 g\right] \tilde{g}\right\} dx\end{aligned} \tag{A.24}$$

for all $\tilde{X} \in H_B^2(0, L; \mathbb{R}^2)$. Now, (A.22), (A.23) imply that the boundary term in (A.24) needs to vanish for all $\tilde{X} \in H_B^2(0, L; \mathbb{R}^2)$. We deduce that $\partial_x^3 X = 0$ at $x = 0, L$ and thus, $X \in H_B^4(0, L; \mathbb{R}^2)$. Summarizing, we have shown that the operator $\mathbb{A} + \lambda : H_B^4(0, L; \mathbb{R}^2) \to L_2(0, L; \mathbb{R}^2)$ is an isomorphism.

Given $k \in \mathbb{N}$, denote by $H_{B,c}^k(0, L; \mathbb{R}^2) := \{X_r + iX_c \mid X_r, X_c \in H_B^k(0, L; \mathbb{R}^2)\}$ the complexification of $H_B^k(0, L; \mathbb{R}^2)$, which is again a Hilbert space with scalar product

$$\left(X_r + iX_c \mid \tilde{X}_r + i\tilde{X}_c\right)_{k,c} := \left(X_r \mid \tilde{X}_r\right)_k + \left(X_c \mid \tilde{X}_c\right)_k - i \left(X_r \mid \tilde{X}_c\right)_k + i \left(X_c \mid \tilde{X}_r\right)_k.$$

In the case $k = 0$, we write $L_{2,c}(0, L; \mathbb{R}^2) := H_{B,c}^0(0, L; \mathbb{R}^2)$ and let

$$\mathbb{A}_c : H_{B,c}^4(0, L; \mathbb{R}^2) \to L_{2,c}(0, L; \mathbb{R}^2)$$

be the complexification of $\mathbb{A}$ defined by

$$\mathbb{A}_c(X_r + iX_c) := \mathbb{A}X_r + i\mathbb{A}X_c \quad \text{for} \quad X = X_r + iX_c \in H_{B,c}^4(0, L; \mathbb{R}^2).$$

Note that $\mathbb{A}_c$ is a well–defined operator and $\mathbb{A}_c X = \mathbb{A}X$ for $X \in H_B^4(0, L; \mathbb{R}^2)$. By the same arguments as above, the operator $\mathbb{A}_c + \lambda \in \mathcal{L}(H_{B,c}^4(0, L; \mathbb{R}^2), L_{2,c}(0, L; \mathbb{R}^2))$ is an isomorphism for all $\operatorname{Re}\lambda \geq \lambda_*$, meaning that the non–positive half–axis is contained in the resolvent set of $\mathbb{A}_c + \lambda_*$. Analogously to the analysis in [23], we estimate the *numerical range* S of the operator $\mathbb{A}_c + \lambda_*$, which is defined by

$$S(\mathbb{A}_c + \lambda_*) := \{((\mathbb{A}_c + \lambda_*)X \mid X)_{0,c} \mid X \in H^4_{B,c}(0, L; \mathbb{R}^2)\}.$$

Due to (A.6), the the real part of $S(\mathbb{A}_c + \lambda_*)$ can be estimated as

$$\begin{aligned}\operatorname{Re}((\mathbb{A}_c + \lambda_*)X \mid X)_{0,c} &= ((\mathbb{A} + \lambda_*)X_r \mid X_r)_0 + ((\mathbb{A} + \lambda_*)X_c \mid X_c)_0 \\ &\geq c_1(\|X_r\|_2^2 + \|X_c\|_2^2) = c_1\|X\|_{2,c}^2 \geq 0.\end{aligned}$$

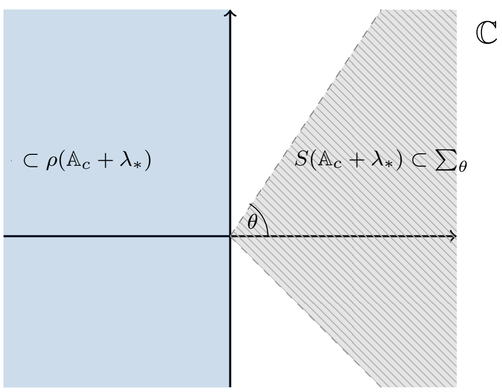
Note that in virtue of applying integration by parts twice, there exists $C > 0$, such that

$$\left|\left((\mathbb{A} + \lambda_*)X \mid \tilde{X}\right)_2\right| \leq (C + \lambda_*)\left(\|X\|_2^2 + \|\tilde{X}\|_2^2\right)$$

for $X, \tilde{X} \in H^4_B(0, L; \mathbb{R}^2)$. Hence, an estimate for the imaginary part of the numerical range is given by

$$\begin{aligned}|\operatorname{Im}((\mathbb{A}_c + \lambda_*)X \mid X)_{0,c}| &\leq |((\mathbb{A} + \lambda_*)X_r \mid X_c)_0| + |((\mathbb{A} + \lambda_*)X_c \mid X_r)_0| \\ &\leq 2(C + \lambda_*)\left(\|X_r\|_2^2 + \|X_c\|_2^2\right) \leq \tilde{c}\|X\|_{2,c}^2,\end{aligned}$$

where $\tilde{c} > 0$ is a constant. We conclude that the numerical range is contained in a sector $\sum_\theta := \{z \in \mathbb{C} \mid -\theta \leq \arg(z) \leq \theta\}$ in the right–half plane, which has angle $\theta < \frac{\pi}{2}$.



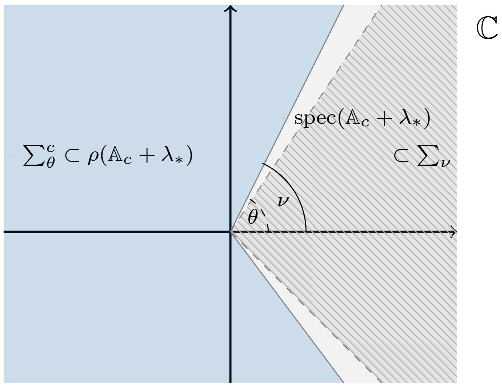
Choosing $\nu \in (\theta, \frac{\pi}{2})$, there exists a constant $C_\theta \in (0, 1)$[2], such that $d(\lambda : S(\mathbb{A}_c + \lambda_*)) \geq C_\theta|\lambda|$ for λ in the complement of $\sum_\nu$, denoted by $\sum_\nu^c$, where $d(\lambda : S(\mathbb{A}_c + \lambda_*))$ indicates the distance

[2] The constant C_θ is given by $C_\theta = \sin(|\nu - \theta|)$.

of λ from $S(\mathbb{A}_c + \lambda_*)$. We infer from [41, Theorem 1.3.9], that the spectrum $\mathrm{spec}(\mathbb{A}_c + \lambda_*)$ of $\mathbb{A}_c + \lambda_*$ is contained in $\sum_\nu$ and that for all $\lambda \in \sum_\nu^c$ the estimate

$$\|((\mathbb{A}_c + \lambda_*) - \lambda)^{-1}\|_{\mathcal{L}(L_2)} \leq \frac{1}{d(\lambda : S(\mathbb{A}_c + \lambda_*))} \leq \frac{1}{C_\theta |\lambda|}$$

holds true.



It follows from [33, Proposition 2.1.11] that $\mathbb{A}_c + \lambda_*$ is sectorial and $-(\mathbb{A}_c + \lambda_*)$ is the generator of an analytic semigroup. Hence, due to the perturbation by a linear, bounded operator and in view of [33, Theorem 2.1.3], $-\mathbb{A}$ is the generator of an analytic semigroup.

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Februar 2015