Boutet de Monvel's calculus via groupoid actions

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Abstract

We consider general pseudodifferential boundary value problems on a Lie manifold with boundary. This is accomplished by constructing a suitable generalization of the Boutet de Monvel calculus for boundary value problems. The data consists of a compact manifold with corners M which is endowed with a Lie structure of vector fields $2\mathcal{V}$, a so-called Lie manifold as introduced by Bernd Ammann, Robert Lauter and Victor Nistor. We consider a Lie manifold M which is split into two equal parts X_+ and X_- each of which are Lie manifolds which intersect in an embedded hypersurface $Y \subset X_\pm$. In this setup our goal is to describe a transmission Boutet de Monvel calculus for boundary value problems.

Starting with a suitable Lie structure on a compact manifold with corners we consider the groupoids integrating the Lie structures. The groupoid corresponding to the Lie structure on the boundary hypersurface Y and the groupoid on the double Lie manifold M are used to describe the pseudodifferential operators on the whole manifold and the boundary respectively. First we consider the example of the b-vector fields and the corresponding minimal integrating groupoids. There is a priori no relation between a chosen groupoid corresponding to the Lie structure on M and a groupoid corresponding to the Lie structure on Y. We show that for b-vector fields the two groupoids can be chosen in such a way that we obtain a bimodule structure (a groupoid correspondence) and in mild cases these groupoids are isomorphic inside the category of Lie groupoids (Morita equivalent).

With the help of the bimodule structure and canonically defined manifolds with corners, which are blow-ups in particular cases, we define a class of extended Boutet de Monvel operators. These operators take the form of matrices consisting of the pseudodifferential, potential, trace and singular Green operators. At first we consider these operators as defined via their Schwartz kernels which are conormal distributions. The bimodule structure consists in particular of actions of the groupoids on the canonical blow-up spaces. These actions induce a multiplicative structure and a convolution algebra structure on the extended calculus. Using this convolution structure we establish closedness under composition of the extended calculus. Therefore in the 0-order case we obtain an algebra.

A deep result due to Ammann, Lauter and Nistor establishes a link between a pseudodifferential calculus on an abstract Lie manifold and the pseudodifferential calculus on the integrating groupoid. We generalize this representation of the groupoid pseudodifferential calculus to our algebra of extended operators by first defining the class of represented operators and suitable homomorphisms. Then we prove the generalized representation theorem for extended Boutet de Monvel operators. This result can be viewed as a certain non-commutative completion of a Lie manifold with boundary.

Then we define the restricted transmission Boutet de Monvel calculus by truncation of the extended operators. We define the representation for restricted operators and show closedness under composition. Finally, we analyze the parametrix construction and in the last section state the index problem for boundary value problems on Lie manifolds.

Keywords: Boutet de Monvel's calculus, groupoid, Lie manifold.

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Zusammenfassung

Wir untersuchen allgemeine pseudodifferentielle Randwertprobleme auf einer Lie Mannigfaltigkeit mit Rand. Dies wird erreicht mittels einer passenden Verallgemeinerung des Boutet de Monvel Kalküls für Randwertprobleme. Gegeben ist dabei eine kompakte Mannigfaltigkeit mit Ecken M, welche mit einer Lie Struktur von Vektorfeldern $2\mathcal{V}$ versehen ist, was eine sogenannte Lie-Mannigfaltigkeit ergibt nach Bernd Ammann, Robert Lauter und Victor Nistor. Dabei betrachten wir eine Lie-Mannigfaltigkeit M, welche aus zwei Halbräumen X_+ and X_- besteht, welche ebenfalls Lie-Mannigfaltigkeiten sind, die sich in einer eingebetteten Hyperfläche Y überschneiden. Auf Basis dieser Daten beschreiben wir ein Transmissionskalkül nach Boutet de Monvel.

Von der Lie Struktur ausgehend betrachten wir Gruppoide, welche die Lie Struktur integrieren. Ein Gruppoid, welcher die Lie Struktur auf der Hyperfläche Y integriert und der Gruppoid, welcher die Lie-Struktur von M integriert werden verwendet um jeweils ein Pseudodifferentialkalkül auf der gesamten Mannigfaltigkeit und dem Rand zu definieren. Dabei betrachten wir zunächst für das Beispiel der b-Vektorfelder die korrespondierenden minimalen Gruppoide. Zu beachten ist, dass a priori keine nützliche Beziehung zwischen dem gewählten Gruppoid zur Lie-Struktur auf M und dem Gruppoid zur Lie-Struktur auf Y existiert. Wir zeigen aber, dass für den Fall der b-Vektorfelder die Gruppoide so gewählt werden können, dass sie eine Bimodul-Struktur oder Gruppoid-Korrespondenz definieren und in gutartigen Fällen ein Isomorphismus in der Kategorie der Lie-Gruppoide existiert.

Mithilfe der Bimodul-Struktur und kanonischen Mannigfaltigkeiten mit Ecken, welche in bestimmten Spezialfällen Blow-Ups beschreiben, können wir eine Klasse von fortgesetzen Boutet de Monvel Operatoren beschreiben. Ein Element im fortgesetzten Kalkül setzt sich zusammen aus Pseudodifferentialoperatoren, Potential-, Trace- und singulären Greenoperatoren. Die Operatoren sind charakterisiert durch ihre Schwartz-Kerne, welche konormale Distributionen sind. Mittels der Gruppoid-Wirkungen auf die Blow-ups erhalten wir eine multiplikative Struktur, welche eine Konvolutionsalgebrenstruktur induziert. Dies ermöglicht einen Beweis für die Abgeschlossenheit unter Komposition im fortgesetzten Kalkül. Insbesondere erhalten wir im Ordnung 0 Fall eine Algebra.

Im nächsten Schritt verallgemeinern wir ein wichtiges Resultat von Ammann, Lauter und Nistor über die Darstellung des Gruppoidkalküls. Wir verallgemeinern diesen Satz auf die Algebra von forgesetzten Operatoren vom Boutet de Monvel Typ, indem wir zunächst den Darstellungshomomorphismus definieren. Dies stellt eine Form nichtkommutativer Vervollständigung einer Lie-Mannigfaltigkeit mit Rand dar.

Schließlich betrachten wir die eingeschränkten Boutet de Monvel Operatoren definiert mittels des forgesetzten Kalküls, zeigen die Abgeschlossenheit unter Komposition und definieren die Darstellung dieser Algebra. Wir analysieren die Parametrix-Konstruktion und beschreiben im letzten Abschnitt das verallgemeinerte Indexproblem für Randwertprobleme auf Lie-Mannigfaltigkeiten.

Schlüsselwörter: Boutet de Monvel Kalkül, Gruppoid, Lie Mannigfaltigkeit.

Contents

1. Introduction

The analysis on singular manifolds has a long history, and the subject is to a large degree motivated by the study of partial differential equations (with or without boundary conditions) and generalizations of index theory to the singular setting, e.g. Atiyah-Singer type index theorems. One particular approach is based on the observation first made by A. Connes (cf. [8]) that groupoids are good models for singular spaces. The outlines for a pseudodifferential calculus on singular foliations were made precise for the general case of (longitudinally smooth) groupoids later by B. Monthubert, V. Nistor, A. Weinstein and P. Xu; see e.g. [24]. Such a calculus can be defined on manifolds with singularities of various types in a unified and general setting. A notion of global ellipticity with Fredholm conditions is then also possible. On the other hand it is important for applications in the study of PDE's to also pose boundary conditions and a parametrix for general boundary value problems. It has been a hard problem to incorporate boundary conditions in the singular context. The construction of a refined parametrix, with appropriate Fredholm conditions, in such a setting is particularly difficult for technical reasons (the small calculus is not sufficient). In this work we will enlarge the groupoid pseudodifferential calculus and develop a notion of general pseudodifferential boundary value problems for the groupoid calculus. The most natural approach seems to be a generalization of the Boutet de Monvel calculus.

Boutet de Monvel's calculus (e.g. [6]) was established in 1971. This calculus provides a convenient and general tool to study the classical boundary value problems. At the same time parametrices are contained in the calculus and it is closed under composition of elements. The elements of Boutet de Monvel's algebra consist of matrices of operators which act in a suitable sense as pseudodifferential operators but have non-pseudodifferential components.

It is thus our aim in this paper to first describe the calculus in terms of groupoids and groupoid actions on suitable spaces. We start by recalling a characterization of Boutet de Monvel operators in terms of conormal distributions due to Grubb / Hörmander. Then we recover the characterization by letting groupoids act on suitable spaces.

We apply this understanding to the following problem: Given a manifold with corners and a Lie structure encoding geometric singularities (the boundary *at infinity*) our objective is to pose boundary conditions on *regular* strata which may intersect the *singular* boundary at infinity. A convenient way to do this is to construct a Boutet de Monvel calculus adapted to this situation. Similar problems have been considered in the literature before. The closest approach to our problem is a work by T. Krainer (cf. [15]). There he considers from our view a special case of the problem on a manifold with polycylindrical ends and cusp type singularities. The approach is also considerably different from our own: first a local Boutet de Monvel calculus is constructed using the collar neighborhood structure of the manifold with corners. Then the calculus is obtained iteratively and closedness under composition is proven via induction on the codimension of the manifold. The inductive approach is also employed for manifolds with fibered corners in the construction of a pseudodifferential calculus in [11]. The inductive step in these proofs depends strongly on the particular type of singularities (e.g. fibred or generalized cusps etc.). Therefore this will not be the right approach in the more general setting we are considering here.

In our case we consider the following data: a Lie manifold $(X, \mathcal{V})$ with boundary Y which is an embedded, transversal hypersurface $Y \subset X$ and which is a Lie submanifold of X (cf. [3], [2]). From X we define the double $M = 2X$ at the hypersurface Y which is a Lie manifold $(M, 2\mathcal{V})$. Transversality of Y in relation to M is briefly described by the following equality

$$T_x M = T_x F \oplus T_x Y, \ x \in Y \cap F \tag{1}$$

for each given hyperface $F \subset \partial M$.

Introduce the following notation for interior and boundary: by ∂M we mean the singular, stratified boundary of the manifold with corners M.

$$M_0 := M \setminus \partial M, \ Y_0 := Y \cap M_0, \ X_0 := X \cap M_0 \text{ and } \partial Y := Y \cap \partial M.$$

The hypersurface Y is endowed with a Lie structure as in [2]:

$$\mathcal{W} = \{V_{|Y} : V \in 2\mathcal{V}, V_{|Y} \text{ tangent to } Y\}. \tag{2}$$

The goal is to construct a Boutet de Monvel calculus for general pseudodifferential boundary value problems adapted to this data.
The pseudodifferential calculus on a Lie manifold was constructed in [3] via representations of pseudodifferential operators on a Lie groupoid. This representation also yields closedness under composition. We enlarge the calculus to take boundary conditions into account by first working on the groupoid level.
In this approach at first we proceed as usual, i.e. find a Lie algebroid $A_{2\mathcal{V}} \to M$ such that $\Gamma(\mathcal{A}_{2\mathcal{V}}) \cong 2\mathcal{V}$.
Via integrating the algebroid $\mathcal{A}_{2\mathcal{V}}$ we obtain a Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}_{2\mathcal{V}}$.
On the boundary Lie structure on Y we also obtain a groupoid $\mathcal{G}_\partial \rightrightarrows Y$ in the same fashion with an associated Lie algebroid $\pi_\partial \colon \mathcal{A}_\partial \to Y$.
The other essential ingredient in our construction is a *refined* blow-up $\mathcal{X}$ of $Y \times M$ with regard to the diagonal Δ_Y and $\mathcal{X}^t$ of $M \times Y$ with regard to Δ_Y as manifolds with corners. The singularities at the (codimension 2) intersection of Y with the singular boundary of M essentially are *blown-up* and the boundary problem is posed on the blown-up version of the submanifolds. We require there to be an isomorphism f between these longitudinally smooth spaces $\mathcal{X}$ and $\mathcal{X}^t$. Since the hypersurface Y divides the double $M = 2X$ we denote by $X := X_+$ the *right* half and by X_- the *left* half.
These halves have corresponding Lie structures and hence corresponding groupoids $\mathcal{G}^\pm \rightrightarrows X_\pm$.
On the symbols of pseudodifferential operators from the groupoid calculus we impose a *fiberwise* or $\mathcal{V}$-transmission property with regard to the subgroupoids $\mathcal{G}^+, \mathcal{G}^- \subset \mathcal{G}$.
The compatibility requirements we will state particularly imply that $\mathcal{G}^+, \mathcal{G}^-$ have fiberwise boundaries consisting of the fibers $\mathcal{X}_x$ for $x \in X_\pm$.
The Boutet de Monvel operators are defined and adapted to data given by the tuple $(\mathcal{G}, \mathcal{G}_\partial, \mathcal{G}^\pm, \mathcal{X}, \mathcal{X}^t, \mathrm{f})$.
This tuple merely depends on the initial Lie structure and integrability properties of the corresponding Lie algebroids. In this sense the closedness under composition is ultimately equivalent to the fact that the Lie algebroids occuring in the theory are integrable.
The Boutet de Monvel calculus adapted to this data should then be an algebra $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial)$ (of order 0 and type 0) which is a subalgebra of:

$$\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \subset \operatorname{End}\begin{pmatrix} C_c^\infty(\mathcal{G}^+) \\ \oplus \\ C_c^\infty(\mathcal{G}_\partial) \end{pmatrix}.$$

The first objective of this work is the proof of the following result.

Theorem (Theorem 8.5 in section 8.)**.** *Given a Lie manifold $(X, \mathcal{V})$ with embedded hypersurface $Y \subset X$ yielding a Lie manifold X with boundary Y such that $M = 2X$, the double. Then for a pair of associated groupoids $\mathcal{G} \rightrightarrows M$, $\mathcal{G}_\partial \rightrightarrows Y$ adapted to a boundary structure the equivariant transmission Boutet de Monvel calculus is closed under composition. This means that given the order $m \in \mathbb{Z}$ we have*

$$\mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{G}_\partial) \cdot \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \subseteq \mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{G}_\partial).$$

In the next section we describe a *vector-representation* of our algebra.
Just as in the case of a pseudodifferential operator on a groupoid there is a homomorphism which maps $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial)$ to an algebra $\mathcal{B}_{\mathcal{V}}^{0,0}(X, Y)$.
The first algebra on the left consists of equivariant families on a suitable *boundary structure*. The right hand side is the *realization*.

$\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \dashrightarrow \mathcal{B}_{\mathcal{V}}^{0,0}(X, Y)$.

This latter algebra is defined to consist of matrices of pseudodifferential, trace, potential and singular Green operators. These operators are extensions from the usual operator calculus on the interior manifold with boundary (X_0, Y_0).
Hence we want to define a homomorphism ϱ_{BM} of algebras from

$$\operatorname{End}\begin{pmatrix} C_c^\infty(\mathcal{G}^+) \\ \oplus \\ C_c^\infty(\mathcal{G}_\partial) \end{pmatrix} \supset \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \to \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y) \subset \operatorname{End}\begin{pmatrix} C^\infty(X) \\ \oplus \\ C^\infty(Y) \end{pmatrix}.$$

It is a non-trivial task to prove that in certain particular cases ϱ_{BM} furnishes an isomorphism between these two algebras. Furthermore, as can already be shown by simply viewing the special case of pseudodifferential operators, it is not true in general. Instead we prove an analog of a result due to Ammann, Lauter and Nistor (cf. [3]).

Theorem (Theorem 8.6 in section 8.)**.** *Given the vector representation ϱ_{BM} and a boundary structure we have the following isomorphism*

$$\varrho_{BM}\left(\mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{G}_\partial)\right) \cong \mathcal{B}_{\mathcal{V}}^{m,0}(X,Y).$$

In many ways our second algebra $\mathcal{B}_{\mathcal{V}}^{*,*}$, the vector representations of our first algebra, is more practical. We will demonstrate this aspect in a future work.
In order to limit the size of the paper we restrict most discussions to the order-0 algebras. Future goals include the realization of this algebra by means of a representation on the $\mathcal{V}$-Sobolev spaces, proving basic continuity properties and studying Fredholm conditions. A further future goal is the proof of a topological Index Theorem for the adapted Boutet de Monvel calculus and generalizing the formula given by Boutet de Monvel in [6]. At the end of the paper we make some remarks on this problem. We also provide a statement of the index problem which is independent of the calculus.
The paper is organized as follows. In section 2 we recall the general theory of Lie groupoids, groupoid actions and Lie algebroids. There we also introduce operators defined via their Schwartz kernels and discuss reduced kernels. Then in section 3 we define the general setup for boundary value problems on manifolds with corners and fix the notation. Section 4 is concerned with the notion of boundary structure. We motivate the definition by considering the special case of b-vector fields. We prove that for the special case such a boundary structure or tuple always exists. In section 5 we introduce the extended operators of Boutet de Monvel type which are special instances of the operators defined in section 2. Then we show how to compose these operators in section 6 and in section 7 we prove the first version of the representation theorem for extended operators. Section 8 is concerned with the truncated Boutet de Monvel algebra. The main results include closedness under composition and representation which are derived from the corresponding results for the extended class. Finally, in section 9 we discuss parametrices and the necessary construction of completions to obtain parametrices in the Lie calculus.

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2. Groupoids, actions and algebroids

2.1. **Lie groupoids.**

Definition 2.1. Groupoids are small categories in which every morphism is invertible.

First we will introduce and fix the notation for the rest of this paper. Then we will give the definition of a Lie groupoid. For more details on groupoids we refer the reader for example to the book [25].

Notation 2.2. A groupoid will be denoted $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$. We denote by $\mathcal{G}^{(1)}$ the set of morphisms and by $\mathcal{G}^{(0)}$ the set of objects. By a common abuse of notation we write $\mathcal{G}$ for $\mathcal{G}^{(1)}$.
The set of composable arrows is given as pullback

$$\mathcal{G}^{(2)} := \mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} = \{(\gamma, \eta) \in \mathcal{G} \times \mathcal{G} : r(\eta) = s(\gamma)\}.$$

We have the range / source maps $r,\ s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ such that $\gamma \in \mathcal{G}$ is

$$\gamma\colon s(\gamma) \to r(\gamma).$$

Also denote the inversion

$$i\colon \mathcal{G} \to \mathcal{G},\ \gamma \mapsto \gamma^{-1}$$

and unit map

$$u\colon \mathcal{G}^{(0)} \to \mathcal{G},\ x \mapsto u(x) = \mathrm{id}_x \in \mathcal{G}.$$

Multiplication is denoted by

$$m\colon \mathcal{G}^{(2)} \to \mathcal{G},\ (\gamma, \eta) \mapsto \gamma \cdot \eta.$$

We also set

$$\mathcal{G}_x := s^{-1}(x),\ \mathcal{G}^x := r^{-1}(x),\ \mathcal{G}_x^x := \mathcal{G}_x \cap \mathcal{G}^x$$

for the r and s fibers and their intersection $\mathcal{G}_x^x$. The latter is easily checked to be a group for each $x \in \mathcal{G}^{(0)}$.

Axioms: One can summarize the maps in a sequence

$$\mathcal{G}^{(2)} \xrightarrow{m} \mathcal{G} \xrightarrow{i} \mathcal{G} \overset{r,s}{\rightrightarrows} \mathcal{G}^{(0)} \xrightarrow{u} \mathcal{G}.$$

With the above notation we can give an alternative way of defining groupoids axiomatically as follows.
(i) $(s \circ u)_{|\mathcal{G}^{(0)}} = (r \circ u)_{|\mathcal{G}^{(0)}} = \mathrm{id}_{\mathcal{G}^{(0)}}$.
(ii) For each $\gamma \in \mathcal{G}$

$$(u \circ r)(\gamma) \cdot \gamma = \gamma,\ \gamma \cdot (u \circ s)(\gamma) = \gamma.$$

(iii) $s \circ i = r,\ r \circ i = s$.
(iv) For $(\gamma, \eta) \in \mathcal{G}^{(2)}$ we have

$$r(\gamma \cdot \eta) = r(\gamma),\ s(\gamma \cdot \eta) = s(\eta).$$

(v) For $(\gamma_1, \gamma_2),\ (\gamma_2, \gamma_3) \in \mathcal{G}^{(2)}$ we have

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3).$$

(vi) For each $\gamma \in \mathcal{G}$ we have

$$\gamma^{-1} \cdot \gamma = \mathrm{id}_{s(\gamma)},\ \gamma \cdot \gamma^{-1} = \mathrm{id}_{r(\gamma)}.$$

Definition 2.3. The 7-tuple $(\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, r, s, m, u, i)$ defines a *Lie groupoid* if and only if $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a groupoid, $M := \mathcal{G}^{(0)},\ \mathcal{G}^{(1)}$ are C^∞-manifolds (with corners), all the maps are C^∞ and s is a submersion.

Remark 2.4. We notice that r is automatically a submersion due to the axiom *iii)*. Hence the pullback

$$\begin{tikzcd} \mathcal{G}^{(2)} \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathcal{G} \arrow[d, "s"] \\ \mathcal{G} \arrow[r, "r"] & M \end{tikzcd}$$

exists in the C^∞-category if $\mathcal{G}$ is C^∞ and thus $\mathcal{G}^{(2)}$ is a smooth manifold as well.

2.2. **Groupoid actions.** Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ we introduce spaces $\mathcal{X}$ which are fibred over $\mathcal{G}^{(0)}$ and such that $\mathcal{G}$ acts on $\mathcal{X}$. This notion as well as some of the notation is adapted from the paper [26].

Definition 2.5. Let $(\mathcal{X}, q)$ be a $\mathcal{G}$-space, i.e. $q\colon \mathcal{X} \to M$ is a smooth map and $\mathcal{X}$ is a smooth manifold. Set $\mathcal{X} * \mathcal{G} := \mathcal{X} \times_M \mathcal{G} = \{(z, \gamma) \in \mathcal{X} \times \mathcal{G} : q(z) = r(\gamma)\}$ to be the *composable elements*. We say that $\mathcal{G}$ *acts on* $\mathcal{X}$ from the right iff the following conditions hold:
i) For each $(z, \gamma) \in \mathcal{X} * \mathcal{G}$

$$q(z \cdot \gamma) = s(\gamma).$$

ii) For each $(z, \gamma) \in \mathcal{X} * \mathcal{G}$ and $(\gamma, \eta) \in \mathcal{G}^{(2)}$

$$z \cdot (\gamma \cdot \eta) = (z \cdot \gamma) \cdot \eta.$$

iii) For each $(z, \gamma) \in \mathcal{X} * \mathcal{G}$ we have

$$(z \cdot \gamma) \cdot \gamma^{-1} = z.$$

A left action of $\mathcal{G}$ on a $\mathcal{G}$-space $(\mathcal{X}, p)$ is a right-action in the opposite category $\mathcal{G}^{\mathrm{op}}$.

Notation 2.6. Given two Lie groupoids $\mathcal{G} \rightrightarrows M,\ \mathcal{H} \rightrightarrows N$ let $(\mathcal{X}, p)$ be an $\mathcal{H}$-space and $(\mathcal{X}, q)$ be a $\mathcal{G}$-space.
A left action is denoted by

$$\begin{tikzcd} \mathcal{H} \arrow[r, hook] & \mathcal{X} \arrow[d, "p"] \\ & N \end{tikzcd}$$

and a right action by

$$\begin{tikzcd} \mathcal{X} \arrow[d, "q"'] & \mathcal{G} \arrow[l, hook'] \\ M & \end{tikzcd}$$

Additionally, we fix the following notation for the fibers: $\mathcal{X}^y := p^{-1}(y),\ \mathcal{X}_x := q^{-1}(x),\ y \in N,\ x \in M$.

Remark 2.7. Note that for any $\mathcal{H}$-space $(\mathcal{X}, p)$ the pullback $\mathcal{H} * \mathcal{X}$ exists in the C^∞-category if $\mathcal{H},\ \mathcal{X}$ are C^∞, which follows from the requirement that p be a surjective submersion. And analogously for a $\mathcal{G}$-space.

Consider the following actions of two Lie groupoids $\mathcal{G}, \mathcal{H}$:

$$\begin{tikzcd} \mathcal{H} \arrow[r, hook] & \mathcal{X} \arrow[dl, "p"'] \arrow[dr, "q"] & \mathcal{G} \arrow[l, hook'] \\ \mathcal{H}^{(0)} & & \mathcal{G}^{(0)} \end{tikzcd}$$

We can define a so-called left Haar system on $\mathcal{X}$ induced by the action of $\mathcal{H}$ and analogously a right Haar system induced by the action of $\mathcal{G}$. This enables us to define left- and right-operators coming from the actions.
Let $\{\lambda_x\}_{x \in \mathcal{G}^{(0)}}$ be a Haar system induced on $\mathcal{X}$ by the right action of $\mathcal{G}$, see also [26], p. 6. This is a family of measures such that

- The support is $\operatorname{supp}\lambda_x = \mathcal{X}_x$ for each $x \in \mathcal{G}^{(0)}$.
- The map
$$\mathcal{G}^{(0)} \ni x \mapsto \int_{\mathcal{X}_x} f \, d\lambda_x$$
is C^∞.
- We have the invariance condition
$$\int_{\mathcal{X}_{r(\gamma)}} f(z\cdot\gamma)\, d\lambda_{r(\gamma)}(z) = \int_{\mathcal{X}_{s(\gamma)}} f(w)\, d\lambda_{s(\gamma)}(w). \tag{3}$$

Then also fix the *right-multiplication* for given $\gamma \in \mathcal{G}$
$$r_\gamma \colon \mathcal{X}_{r(\gamma)} \to \mathcal{X}_{s(\gamma)},\ z \mapsto z\cdot\gamma.$$
This is a diffeomorphism.
The induced operators acting on C^∞-functions are given by
$$R_\gamma \colon C_c^\infty(\mathcal{X}_{s(\gamma)}) \to C_c^\infty(\mathcal{X}_{r(\gamma)}),\ (R_\gamma f)(z) := f(z\cdot\gamma),\ z \in \mathcal{X}.$$
These operators $\mathcal{R}_\gamma$ yield $*$-(anti-)homomorphisms since $(R_\gamma)^{-1} = R_{\gamma^{-1}}$ is the inverse and $R_{\gamma\cdot\eta} = R_\eta \circ R_\gamma,\ (\gamma,\eta) \in \mathcal{G}^{(2)}$.

Definition 2.8. *i)* A continuous linear operator $T\colon C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{X})$ is called a *right $\mathcal{X}$-operator* if and only if $T = (T_x)_{x\in\mathcal{G}^{(0)}}$ is a family of continuous linear operators $T_x \colon C_c^\infty(\mathcal{G}_x) \to C_c^\infty(\mathcal{X}_x)$ such that
$$R_{\gamma^{-1}} T_{r(\gamma)} R_\gamma = T_{s(\gamma)},\ \gamma \in \mathcal{G}. \tag{4}$$
This can be expressed alternatively by requiring the following diagram to commute for each $\gamma \in \mathcal{G}$
$$\begin{tikzcd} C_c^\infty(\mathcal{G}_{s(\gamma)}) \arrow[r, "T_{s(\gamma)}"] & C_c^\infty(\mathcal{X}_{s(\gamma)}) \arrow[d, "R_\gamma"'] \\ C_c^\infty(\mathcal{G}_{r(\gamma)}) \arrow[u, "R_{\gamma^{-1}}"'] \arrow[r, "T_{r(\gamma)}"] & C_c^\infty(\mathcal{X}_{r(\gamma)}). \end{tikzcd}$$
ii) By analogy $\tilde{T}\colon C_c^\infty(\mathcal{X}) \to C_c^\infty(\mathcal{H})$ is a *left $\mathcal{X}$-operator* if and only if $\tilde{T} = (\tilde{T}^y)_{y\in\mathcal{H}^{(0)}}$ is a family of continuous linear operators $\tilde{T}^y \colon C_c^\infty(\mathcal{X}^y) \to C_c^\infty(\mathcal{H}^y)$ such that the diagram
$$\begin{tikzcd} C_c^\infty(\mathcal{X}^{s(\gamma)}) \arrow[r, "\tilde{T}^{s(\gamma)}"] & C_c^\infty(\mathcal{H}_{s(\gamma)}) \arrow[d, "L_{\gamma^{-1}}"'] \\ C_c^\infty(\mathcal{X}^{r(\gamma)}) \arrow[u, "L_\gamma"'] \arrow[r, "\tilde{T}^{r(\gamma)}"] & C_c^\infty(\mathcal{H}_{r(\gamma)}) \end{tikzcd}$$
commutes for each $\gamma \in \mathcal{H}$ where L_γ denotes in this case the corresponding left multiplication.

The next Proposition tells us that the family of Schwartz kernels $(k_x)_{x\in\mathcal{G}^{(0)}}$ for a given $\mathcal{X}$-operator can be replaced by a so-called reduced kernel. This is not unlike the situation for groupoids and the pseudodifferential calculus where reduced kernels are used extensively (cf. [24]).

Proposition 2.9. *Given a right-$\mathcal{X}$-operator $T\colon C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{X})$. Then for $u \in C_c^\infty(\mathcal{G}),\ z \in \mathcal{X}$*
$$(Tu)(z) = \int_{\mathcal{G}_{q(z)}} k_T(z\cdot\gamma^{-1}) u(\gamma)\, d\mu_{q(z)}(\gamma)$$
with $k_T(z\cdot\gamma^{-1}) := k_{r(\gamma)}(z,\gamma)$ depending only on $z\cdot\gamma^{-1} \in \mathcal{X}$ for each $(z,\gamma^{-1}) \in \mathcal{X} * \mathcal{G}$.

Proof. First we can write for $z \in \mathcal{X}$

$$(R_{\gamma^{-1}} T_{r(\gamma)} R_\gamma) u(z) = (T_{r(\gamma)} R_\gamma u)(z \cdot \gamma^{-1}) = \int_{\mathcal{G}_{r(\gamma)}} k_{r(\gamma)}(z \cdot \gamma^{-1}, \eta) u(\eta\gamma)\, d\mu_{r(\gamma)}(\eta)$$
$$= \int_{\mathcal{G}_{s(\gamma)}} k_{r(\gamma)}(z \cdot \gamma^{-1}, \tilde{\eta} \cdot \gamma^{-1}) u(\tilde{\eta})\, d\mu_{s(\gamma)}(\tilde{\eta})$$

via the substitution $\tilde{\eta} := \eta\gamma$ and invariance of Haar system. By use of (4) we see that the last integral equals

$$(T_{s(\gamma)} u)(z) = \int_{\mathcal{G}_{s(\gamma)}} k_{s(\gamma)}(z, \eta) u(\eta)\, d\mu_{s(\gamma)}(\eta).$$

This implies the following identity by the uniqueness of the Schwartz kernel for T_x for each $x \in M$

$$\forall_{\gamma \in \mathcal{G}}\ k_{s(\gamma)}(z, \tilde{\eta}) = k_{r(\gamma)}(z \cdot \gamma^{-1}, \tilde{\eta} \cdot \gamma^{-1}). \tag{$*$}$$

To see that k_T is well-defined assume $\beta = z \cdot \gamma^{-1} = \tilde{z} \cdot \tilde{\gamma}^{-1}$ and $\delta = \gamma^{-1} \cdot \tilde{\gamma}$, then

$$\begin{aligned} k_{s(\tilde{\gamma})}(\tilde{z}, \tilde{\gamma}) = k_{s(\delta)}(\tilde{z}, \tilde{\gamma}) &=_{(*)} k_{r(\delta)}(\tilde{z}\delta^{-1}, \tilde{\gamma}\delta^{-1}) \\ &= k_{s(\gamma)}(\beta\tilde{\gamma}\delta^{-1}, \tilde{\gamma}\delta^{-1}) \\ &= k_{s(\gamma)}(z, \gamma). \end{aligned}$$

This completes the proof. □

2.3. **Lie algebroids.** The aim of this section is to give a definition of Lie algebroids and subalgebroids. We restrict ourselves to the bare minimum needed in the following text of the paper. For a more detailed exposition the reader may consult e.g. [20].

Definition 2.10.
- A *Lie algebroid* is a tuple $(E,\ \varrho)$. Here $\pi\colon E \to M$ is a vector bundle over a manifold M and $\varrho\colon E \to TM$ is a vector bundle map such that

  $$\varrho \circ [V, W]_{\Gamma(E)} = [\varrho \circ V, \varrho \circ W]_{\Gamma(TM)}$$

  and

  $$[V, fW]_{\Gamma(E)} = f[V, W]_{\Gamma(E)} + \varrho(V)(f)W,\ f \in C^\infty(M),\ V, W \in \Gamma(E).$$

- Given two Lie algebroids $(\mathcal{A},\ \varrho)$ and $(\tilde{\mathcal{A}},\ \tilde{\varrho})$ over the same manifold M. Then a *Lie algebroid morphism* is a map $\varphi\colon \mathcal{A} \to \tilde{\mathcal{A}}$ making diagram commute

  $$\begin{tikzcd} & TM & \\ \mathcal{A} \arrow[ur, "\varrho"] \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \tilde{\mathcal{A}} \arrow[ul, "\tilde{\varrho}"'] \arrow[dl, "\tilde{\pi}"] \\ & M & \end{tikzcd}$$

  and such that φ preserves Lie bracket: $\varphi[V, W]_{\Gamma(\mathcal{A})} = [\varrho(V), \varrho(W)]_{\Gamma(TM)}$.

We briefly summarize some relevant facts about the construction of Lie algebroids.

- For any given Lie groupoid $\mathcal{G}$ we obtain an associated algebroid $\mathcal{A}(\mathcal{G})$ in a covariantly functorial way. Define $T^s\mathcal{G} := \ker(ds)$ the *s-vertical tangent bundle* as a sub-bundle of $T\mathcal{G}$. Denote by $\Gamma(T^s\mathcal{G})$ the smooth sections and define $\Gamma_R(T^s\mathcal{G})$ as the sections V such that

  $$V(\eta\gamma) = (R_\gamma)_* V_\eta \text{ for } (\eta, \gamma) \in \mathcal{G}^{(2)}.$$

We then define the Lie-algebroid associated to $\mathcal{G}$ via the pullback

$$\begin{array}{ccc} \mathcal{A}(\mathcal{G}) & \xrightarrow{u^*} & T^s\mathcal{G} \\ \downarrow & & \downarrow{\scriptstyle \pi_{|s}} \\ M & \xrightarrow{u} & \mathcal{G}. \end{array}$$

In other words $\mathcal{A}(\mathcal{G}) := \{(V, x) | ds(V) = 0,\ u(x) = 1_x = \pi(V)\}$.

- There is a canonical isomorphism of Lie-algebras $\Gamma_R(T^s\mathcal{G}) \cong \Gamma(\mathcal{A}(\mathcal{G}))$. The set of smooth sections $\Gamma(\mathcal{A}(\mathcal{G}))$ is a $C^\infty(M)$-module with the module operation $f \cdot V = (f \circ r) \cdot V$ with $f \in C^\infty(M)$.
- Let $\mathcal{A}(\mathcal{G})$ be given as above and define $\varrho \colon \mathcal{A}(\mathcal{G}) \to TM$ by $\varrho := dr \circ u^*$. Then $(\mathcal{A}(\mathcal{G}), \varrho)$ so defined furnishes a Lie algebroid.
- A Lie algebroid is said to be *integrable* if we can find an associated (with connected s-fibers) Lie groupoid. Not every Lie algebroid is integrable [9].

3. Lie manifolds with boundary - the setup

3.1. **Manifolds with corners.** In the following we will be concerned with compact manifolds with corners. To this end we will now fix our terminology and recall the main definitions related to manifolds with corners. Such manifolds are locally modelled on sets of the type $[0, \infty)^k \times \mathbb{R}^{n-k}$. We give the following *extrinsic definition* (it depends on the choice of boundary defining functions).

Definition 3.1. A Hausdorff topological space M is a *manifold with embedded corners* if the following conditions hold.
i) The space M is an embedded submanifold of a smooth manifold (without corners) denoted by $\widetilde{M}$. If we denote this embedding by $i \colon M \hookrightarrow \widetilde{M}$ then the smooth functions on M are $C^\infty(M) = i^*C^\infty(\widetilde{M})$.
ii) The boundary defining functions $\{\rho_i\}_{i\in I}$ are fixed as maps $\rho_i \in C^\infty(\widetilde{M}),\ i \in I$ with

$$i(M) = \{y \in \widetilde{M} : \rho_i(y) \geq 0\} = \bigcap_{i\in I} \{\rho_i \geq 0\}.$$

iii) For each $J \subset I, x \in \widetilde{M}$ with $\rho_j(x) = 0$ for each $j \in J$ it follows that $\{d\rho_j(x)\}_{j\in J}$ is linearly independent.

The sets $\{\rho_i = 0\}$ will be called the *closed* hyperfaces of M. A manifold with corners M has a boundary ∂M which is stratified by closed (intersecting) hyperfaces. The category of manifolds with boundary is a subcategory of the category of manifolds with corners. We will not go into further details concerning the categorical structure and the structure preserving maps (the *b-maps* according to R. Melrose), but refer instead to the book [19], see also [16].
Additionally, we will be concerned with *submanifolds* of manifolds with corners. In this case given a compact manifold with corners M we say that $Y \subset M$ is a *submanifold with corners* if Y is a manifold with corners and each hyperface F of Y is a connected component of a set of the form $G \cap Y$, where G is a hyperface of M which intersects Y transversally.
An important concept is submersions between manifolds with corners which are defined as follows.

Definition 3.2. A *submersion* between two manifolds with corners M and N is given by a smooth map $f \colon M \to N$ such that *df* is everywhere surjective and v is an inward pointing tangent vector of M if and only if $df(v)$ is an inward pointing tangent vector of N.

The following result is relevant in the consideration of fibered spaces such as groupoids that are longitudinally smooth (meaning the fibers $\mathcal{G}_x = s^{-1}(x),\ \mathcal{G}^x = r^{-1}(x),\ x \in \mathcal{G}^{(0)}$ are smooth manifolds without corners).

Lemma 3.3. *Let $f\colon M \to N$ be a submersion between manifolds with corners M and N. Then for each $y \in N$ the fiber $f^{-1}(y)$ are smooth manifolds without corners.*

Proof. See [16], p.4. □

For the rest of this paper we fix the following notation and conventions.

Notation 3.4.
- For any compact manifold with corners M we denote by ∂M the (stratified) boundary at infinity and by $M_0 = M \setminus \partial M$ the interior. If $Y \subset M$ is a submanifold with corners we denote $Y_0 = Y \cap M_0$ and $\partial Y = Y \cap \partial M$.
- As stated in the introduction assume that Y is transversal to all faces of M in the sense of (1) on p. 6 and that Y is of codimension one.
- Then, if F is an *open* hyperface in M we denote by $\overline{F}$ the closure in M.
- Denote by
$$\partial_{reg} F = \partial_{reg} \overline{F} = \overline{F} \cap Y$$
the *regular boundary* of F.
- If F is such that $\overline{F} \cap Y = \emptyset$ (we say that F is not *incident* to Y) then the regular boundary is empty.
- In addition the notation $\partial F = \partial \overline{F}$ is the boundary *at infinity* of the hyperface, i.e.
$$\partial F = \overline{F} \cap \partial M.$$
- Denote by $\mathring{\mathcal{F}}_1(M)$, $\mathcal{F}_1(M)$ the set of open and closed hyperfaces of the manifold with corners M, and analogously $\mathring{\mathcal{F}}_1(Y)$, $\mathcal{F}_1(Y)$ the open and closed hyperfaces of Y.
- Also denote by $\mathcal{I}(Y)$ the *incident* faces, i.e.
$$\mathcal{I}(Y) := \{F \in \mathring{\mathcal{F}}_1(M) : Y \cap \overline{F} \neq \emptyset\}.$$
- The boundary defining functions of Y are commonly denoted by $(q_j)_{j\in J}$ with the index set J. Denote by $(p_i)_{i\in I}$ the boundary defining functions of M with the index set I.

3.2. **Integrable algebroids.** Our setup can be put in rather general terms. In this description we take the algebroid as the fundamental object. First we define the notion of a Lie subalgebroid.

Definition 3.5. Given a manifold M and a submanifold $N \subset M$ with algebroid $(\mathcal{A}, \varrho)$ defined over M. Then a Lie algebroid $(\tilde{\mathcal{A}}, \tilde{\varrho})$ over N is a *subalgebroid* of $\mathcal{A}$ iff $\tilde{\mathcal{A}} \subset \mathcal{A}_{|N}$ is a subbundle equipped with a Lie algebroid structure s.t. the inclusion $\tilde{\mathcal{A}} \hookrightarrow \mathcal{A}_{|N}$ is a Lie algebroid morphism.

Consider the following situation. We are given a compact manifold with corners M and a hypersurface Y of codimension 1 which is a transversal submanifold of M as a manifold with corners.
Furthermore, let $(\mathcal{A}, \varrho)$ be a Lie algebroid defined on M which is assumed to be integrable.
Also we assume that to the inclusion $i_Y\colon Y \hookrightarrow M$ there corresponds an inclusion of Lie algebroids $j_\partial\colon \mathcal{A}_\partial \hookrightarrow \mathcal{A}_{|Y}$.
We have then the following well-known result (see e.g. [21], p. 4).

Proposition 3.6. *Let $\mathcal{A}_\partial \hookrightarrow \mathcal{A}_{|Y}$ be a Lie algebroid morphism adapted to the data given above. Then $\mathcal{A}_\partial$ is integrable.*

For the benefit of the reader we briefly sketch the idea of the proof.

Proof. First from the integrability of $\mathcal{A}$ we find and fix a Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$. Via the inclusion $i_Y\colon Y \hookrightarrow M$ and the range map of $\mathcal{G}$ we define the pullback

$$\begin{tikzcd} P = Y \times_M \mathcal{G} \arrow[r] \arrow[d, "\pi_1"'] & \mathcal{G} \arrow[d, "r"] \\ Y \arrow[r, "i_Y"] & M \end{tikzcd}$$

Then the anchor map ϱ of $\mathcal{A}$ and the inclusion of Lie algebroids $\mathcal{A}_\partial \hookrightarrow \mathcal{A}_{|Y}$ defines a foliation $\mathcal{F} \to M$. We can thus consider the monodromy groupoid $\mathrm{Mon}(\mathcal{G}, \mathcal{F})$. Since the action of $\mathcal{G}$ on itself is a principal action it induces an action on $\mathrm{Mon}(\mathcal{G}, \mathcal{F})$. Therefore it makes sense to define the groupoid $\mathcal{H} := \mathrm{Mon}(\mathcal{G}, \mathcal{F})/\mathcal{G}$. Then it can be shown that this groupoid integrates $\mathcal{A}_\partial$. $\square$

In fact one can explicitely write down two choices of groupoids integrating $\mathcal{A}_\partial$ in this setup, see also [21], Thm. 2.3. The quotient of the monodromy groupoid

$$\mathcal{H}_{max} = \mathrm{Mon}(\mathcal{G}, \mathcal{F})/\mathcal{G}$$

and the quotient of the holonomy groupoid

$$\mathcal{H}_{min} = \mathrm{Hol}(\mathcal{G}, \mathcal{F})/\mathcal{G}.$$

The latter is the s-connected cover of the monodromy groupoid.
There are in general several choices of groupoids integrating $\mathcal{A}_\partial$.

Example 3.7. Consider the example of the algebroids $\mathcal{A} = TM$, $\mathcal{A}_\partial = TY$. In each case we could take two different integrating groupoids. Either the pair groupoids $M \times M \rightrightarrows M$, $Y \times Y \rightrightarrows Y$ or alternatively the *path groupoids* (see [16], example 2.9) $\mathcal{P}_M \rightrightarrows M$, $\mathcal{P}_Y \rightrightarrows Y$. This example shows that there is not necessarily any relation between the chosen groupoids $\mathcal{G}$, $\mathcal{G}_\partial$ integrating the algebroids. We will address this issue further in the next section.

The following Theorem from [21], p. 6 is relevant in this context.

Theorem 3.8 (Moerdijk, Mrcun). *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid integrating the algebroid $\mathcal{A}$ over M. Assume $\mathcal{G}_\partial \rightrightarrows Y$ is a subgroupoid of $\mathcal{G}_{|Y}$ which integrates the inclusion $\jmath_\partial \colon \mathcal{A}_\partial \hookrightarrow \mathcal{A}_{|Y}$. Then such a subgroupoid $\mathcal{G}_\partial$ is unique up to isomorphism.*

3.3. **Regular boundary.** In the following we want to consider a special case of the abstract setup of the last section.

Definition 3.9. A *Lie structure* $\mathcal{V}$ on a given compact manifold with corners M is a $C^\infty(M)$-module of vector fields on M which is locally finitely generated, projective and closed under Lie bracket. Furthermore, the vector fields in $\mathcal{V}$ are assumed to be tangent to the hyperfaces of M.

Example 3.10. An example of a Lie structure on a compact manifold with corners M is $\mathcal{V}_b \subset \Gamma(TM)$ which consists of all vector fields tangent to the hyperfaces of M. So in particular any Lie structure is contained in $\mathcal{V}_b$. The reader may consult [3] for more details on Lie manifolds.

We fix now the following data:

- A Lie manifold $(X, \mathcal{V})$ which is made into a Lie manifold with boundary in the following sense.
- We are given an embedded, codimension one hypersurface $Y \hookrightarrow X$ which is a submanifold with corners.
- Denote by $M = 2X$ the double of X at the hypersurface Y which is canonically endowed with a Lie structure $2\mathcal{V}$ s.t.

$$\mathcal{V} = \{V_{|X} : V \in 2\mathcal{V}\}.$$

  This notion of double makes sense as defined in [2]. Fix a Lie algebroid $(\pi \colon \mathcal{A} \to M,\ \varrho_M)$ such that $\Gamma(\mathcal{A}) = 2\mathcal{V}$.
- The hypersurface Y is endowed with the Lie structure $\mathcal{W}$ as defined in (4). Furthermore, fix the vector bundle $(\pi_\partial \colon \mathcal{A}_\partial \to Y,\ \varrho_\partial)$ with $\Gamma(\mathcal{A}_\partial) = \mathcal{W}$.
- And $\mathcal{A}_\partial \subset \mathcal{A}_{|Y}$ being a Lie subalgebroid in the sense of Def. 3.5.

Assumption A. We assume the algebroid $\mathcal{A}$ is such that the condition $\mathcal{A}_{|M_0} \cong TM_0$ holds. As well as $(\mathcal{A}_\partial)_{|Y_0} \cong TY_0$. Under isomorphisms with induced anchor map ϱ and ϱ_∂ respectively.

Remark 3.11. *i)* Any Lie algebroid associated with a Lie manifold is integrable. This fact can be seen as a consequence of results in the seminal work of Crainic and Fernandes, see [9]. Another way to see this is from the assumption A $\mathcal{A}_{|M_0} \cong TM_0$ where M_0 is dense in M. It was shown in [10] that with such a condition the algebroid is always integrable.
ii) To see that $\mathcal{W} \subset \Gamma(TY)$ is in fact a Lie structure, note that $\mathcal{W}$ is closed under Lie bracket and locally finitely generated, projective. Here I_Y denotes the ideal of smooth functions on M vanishing on Y.
iii) The Lie submanifold Y is in particular a submanifold with corners of codimension one. In [2] it is shown that for such an embedding of Lie manifolds the condition

$$\varrho_M(A_p) + T_pY = T_pM, \ p \in \partial Y$$

holds which reduces to the ordinary transversality condition (1).

We fix the singular normal bundle[1] $\mathcal{A}_{|Y}/\mathcal{A}_\partial =: \mathcal{N} \to Y$ from the inclusion of Lie manifolds $Y \hookrightarrow M$ as described in the next Lemma.
These facts are also given in [2] and are repeated here for the convenience of the reader.

Lemma 3.12. i) *Given a face F of M such that $F \cap Y \neq \emptyset$, $Y \cap F \subset F$ is a submanifold with corners and we have*

$$\operatorname{codim}(Y \cap F) = \dim M - \dim Y = 1 \tag{5}$$

computed relative to F.
ii) *If we identify the orthogonal complement $\mathcal{A}_\partial^\perp$ in $\mathcal{A}_{|Y}$ with $\mathcal{N}$ we obtain the decomposition*

$$\mathcal{A}_{|Y} = \mathcal{N} \oplus \mathcal{A}_\partial. \tag{6}$$

For each $y \in Y$ the anchor $\varrho_M \colon \mathcal{A} \to TM$ induces an isomorphism

$$\mathcal{N}_y = A_y/(A_\partial)_y \cong T_yM/T_yY = N_yY. \tag{7}$$

Proof. *i)* This follows from (1).
ii) Consider the metric g on M which is induced from a complete (compatible in the sense of section B) Riemannian metric on $\mathcal{A}$. As stated above $A_\partial \subset \mathcal{A}_{|Y}$ is in particular a sub vector bundle of $\mathcal{A}_{|Y}$. Denote by $q \colon \mathcal{A}_{|Y} \to \mathcal{A}_{|Y}/\mathcal{A}_\partial$ the corresponding quotient mapping. Then we consider the exact sequence

$$\mathcal{A}_\partial \rightarrowtail \mathcal{A}_{|Y} \twoheadrightarrow \mathcal{A}_{|Y}/\mathcal{A}_\partial = \mathcal{N}. \tag{8}$$

The sequence splits as a short exact sequence of vector bundles. Using the Riemannian metric we can obtain a splitting as an isomorphism $\eta \colon \mathcal{N} \to \mathcal{A}_\partial^\perp$ such that $q \circ \eta = \mathrm{id}_{\mathcal{N}}$ (cf. [2]). With this isomorphism the decomposition (6) is then clear.
The second assertion follows immediately from the transversality condition

$$T_yY + \varrho(\mathcal{A}_y) = T_yM, \ y \in \partial Y.$$

□

Ammann, Lauter and Nistor (see [3]) have constructed a pseudodifferential calculus $\Psi^\bullet_{2\mathcal{V}}(M)$ adapted to a Lie manifold $(M, 2\mathcal{V})$. This can be viewed as a suitable extension of the enveloping algebra $\mathrm{Diff}^\bullet_{2\mathcal{V}}(M)$ generated by the $2\mathcal{V}$ vector fields. The calculus is closed under composition. This is proven by a representation on the corresponding groupoid calculus $\Psi^\bullet(\mathcal{G})$. Here $\mathcal{G}$ is some (s-connected) groupoid integrating the Lie algebroid $\mathcal{A}_{2\mathcal{V}}$. A pseudodifferential calculus on the Lie manifold $(Y, \mathcal{W})$ is thus also defined. We summarize in the following Proposition and give some details for the benefit of the reader.

Proposition 3.13. *There is a calculus of pseudodifferential operators $\Psi^\bullet_{\mathcal{W}}(Y)$ defined on the Lie manifold $(Y, \mathcal{W})$ such that $\Psi^\bullet_{\mathcal{W}}(Y)$ is a filtered algebra (it is closed under composition).*

[1]We may also call this a *Lie* normal bundle as it is itself smooth (non-singular) but comes from the Lie structure.

Proof. The Lie structure $\mathcal{W}$ is a finitely generated and projective $C^\infty(Y)$-module. Knowing this the vector bundle $\pi_\partial\colon \mathcal{A}_\partial \to Y$ such that $\Gamma(\mathcal{A}_\partial) \cong \mathcal{W}$ is obtained from the Serre-Swan theorem. We also required that $\mathcal{A}_\partial$ is a Lie subalgebroid of $\mathcal{A}_{|Y}$. Either by the integrability of $\mathcal{A}$ and the general result 3.6 or simply by observing that any Lie structure leads to integrable algebroids (e.g. from [10]) it follows that $\mathcal{A}_\partial$ is also integrable. After making use of the pseudodifferential calculus defined on a corresponding groupoid $\mathcal{G}_\partial$ integrating $\mathcal{A}_\partial$ and the representation Theorem proven in [3] we infer that $\Psi_{\mathcal{W}}^\bullet(Y)$ is closed under composition. $\square$

Remark 3.14. *i)* As mentioned in the above proof operators in the pseudodifferential Lie calculus are representations of operators in the corresponding groupoid calculus. This is proven in [3], Thm. 3.2 and we use the notation $\varrho,\ \varrho_\partial$ for representations corresponding to the groupoids $\mathcal{G},\ \mathcal{G}_\partial$ respectively, where $\mathcal{G}$ and $\mathcal{G}_\partial$ are fixed groupoids integrating the algebroids $\mathcal{A}, \mathcal{A}_\partial$. Then the representation theorem is the statement

$$\varrho \circ \Psi^\bullet(\mathcal{G}) \cong \Psi_{2\mathcal{V}}^\bullet(M),\ \ \varrho_\partial \circ \Psi^\bullet(\mathcal{G}_\partial) \cong \Psi_{\mathcal{W}}^\bullet(Y).$$

We will discuss representation theory including the definition of the maps $\varrho,\ \varrho_\partial$ further in section 7.

ii) It can be checked easily from the definition that $\mathcal{W}$ is closed under Lie bracket. To be precise $\mathcal{W}$ lies inside a quotient of $2\mathcal{V}$. First we have the exact sequence

$$I_Y 2\mathcal{V} \rightarrowtail 2\mathcal{V} \twoheadrightarrow 2\mathcal{V}/2\mathcal{V} I_Y$$

where I_Y is the ideal of functions vanishing on Y. We can apply the acyclic functor Γ (associating to a smooth vector bundle the module of vector fields) to the exact sequence (8). Note that $\Gamma(2\mathcal{V}/2\mathcal{V}I_Y) \cong \mathcal{A}_{|Y}$ which yields the two exact sequences

$$\begin{tikzcd} & & \mathcal{W} \arrow[d, tail] \\ I_Y 2\mathcal{V} \arrow[r, tail] & 2\mathcal{V} \arrow[r, two heads] & 2\mathcal{V}_Y = 2\mathcal{V}/2\mathcal{V}I_Y \arrow[d, two heads] \\ & & \Gamma(\mathcal{N}). \end{tikzcd}$$

4. Boundary structure

4.1. **The b-Groupoid case.** With the given Lie manifold with boundary we want to associate a so-called *boundary structure*. To motivate the definition we start with the special case of b-vector fields. Then we define a boundary structure and verify that at least for b-vector fields a boundary structure always exists. What is necessary in the general case is a certain assumption on the groupoids $\mathcal{G},\ \mathcal{G}_\partial$, namely they ought to define a bimodule structure which we are going to specify. In the case of b-vector fields we verify with certain restrictive conditions on M the groupoids are Morita equivalent.

The boundary structure is in fact a good analogy for blow-ups of the corners which are the intersections of Y with the (singular) boundary at infinity of M. These blow-ups are in our setup canonically defined in terms of $\mathcal{G}$ and $\mathcal{G}_\partial$, the groupoids integrating $\mathcal{A}$ and $\mathcal{A}_\partial$.

Therefore our notion of *boundary structure* requires no further assumptions. It only depends on the Lie structure itself and the assumption that it leads to algebroids which are integrable via compatible groupoids.

We recall first the definition of the b-groupoid from [22] (see also [16]). For this consider the case $\mathcal{V} = \mathcal{V}_b$ where $\mathcal{V}_b$ denotes the module of vector fields which are tangent to all hyperfaces of X.

Definition 4.1. Fix boundary defining functions $(\rho_i)_{i\in I}$ of M, then the groupoid is defined as[2]

$$\Pi(M) := \{(x, y, \lambda = (\lambda_i)_{i\in I}) \in M \times M \times (\mathbb{R}_+^*)^I : \rho_i(x) = \lambda_i \rho_i(y),\ i \in I\}.$$

The composition and inverse is defined as follows:

$$(x, y, \lambda) \circ (y, z, \mu) = (x, z, \lambda \cdot \mu),\ (x, y, (\lambda_i)_{i\in I})^{-1} = (y, x, (\lambda_i^{-1})_{i\in I}).$$

Range and source maps are given by

$$r(x, y, \lambda) = x,\ s(x, y, \lambda) = y.$$

We define the *b-groupoid* $\mathcal{G}(M)$ as the union of the connected components containing the unit of each s-fiber of $\Pi(M)$.

Remark 4.2. *i)* The groupoid of the boundary $\mathcal{G}(Y) = \mathcal{G}_\partial$ is defined analogously. Here we fix the boundary defining functions $(q_j)_{j\in J}$ of Y. We have

$$\Pi(Y) = \{(x, y, \lambda) \in Y \times Y \times (\mathbb{R}_+^*)^J : q_j(x) = \lambda_j q_j(y),\ j \in J\},$$

and $\mathcal{G}(Y)$ is defined as the s-connected envelope of $\Pi(Y)$ which is the union of the s-connected components [3].
ii) Another way to define the b-groupoids is to use so-called *decoupages* as introduced by Monthubert. For this we consider the embedding of the manifold with corners M into a smooth manifold (without corners) $\widetilde{M}$. Inside this manifold $\widetilde{M}$ there is a family of submanifolds $\mathcal{E} = (V_i)_{i\in I}$. Each V_i divides $\widetilde{M}$ into two connected components. The restrictions of the V_i correspond to faces in M. In [22] a smooth groupoid $\mathcal{G}(\mathcal{E}) \rightrightarrows \widetilde{M}$ is defined such that

$$\mathcal{G}(M) = \mathcal{G}^c(\mathcal{E})_M^M.$$

Corresponding decoupages for Y and X are easily obtained defining the groupoids $\mathcal{G}(X)$ and $\mathcal{G}(Y)$. They are given as follows

$$\mathcal{G}(Y) = \mathcal{G}^c(\mathcal{E})_Y^Y,\ \mathcal{G}(X) = \mathcal{G}^c(\mathcal{E})_X^X.$$

These groupoids are therefore closed subgroupoids of $\mathcal{G}(\mathcal{E})$.

In what follows we assume a decoupage $(\widetilde{M}, \mathcal{E})$ is fixed such that $\mathcal{G}(M)$ and $\mathcal{G}(Y)$ are the corresponding b-groupoids.

Definition 4.3. We define the spaces $\mathcal{X} := \mathcal{G}^c(\mathcal{E})_Y^M$ and $\mathcal{X}^t := \mathcal{G}^c(\mathcal{E})_M^Y$. Additionally, on $\mathcal{X}$ and $\mathcal{X}^t$ we fix the canonical groupoid bimodule structure from the actions of the b-groupoids $\mathcal{G}(M)$ and $\mathcal{G}(Y)$.

Lemma 4.4. *The canonical action of the isotropy group $\Gamma := \mathcal{G}(M)_x^x$, for a given $x \in M_0$, on $\mathcal{X} = \mathcal{G}^c(\mathcal{E})_Y^M$ and $\mathcal{X}^t = \mathcal{G}^c(\mathcal{E})_M^Y$ is free and proper.*

Proof. Consider the case of the right-action of Γ on $\mathcal{X}$ for some fixed $x \in M_0$. The assertion for $\mathcal{X}^t$ follows analogously. If $z \cdot g = z$ for a $(z, g) \in \mathcal{X} * \Gamma$ it follows from the definition of the composition in $\mathcal{G}(M)$ that g is the identity. Therefore the action of Γ is free. Secondly, note that the (right or left) action of Γ on $\mathcal{G}$ is proper. From the definition of $\mathcal{X}$, this Γ-action restricts to the right action of Γ on $\mathcal{X}$. We have a proper mapping $\varphi\colon \mathcal{X} * \Gamma \to \mathcal{X} \times \mathcal{X}$, $(z, g) \mapsto (z, z \cdot g)$. Hence we see that the right action of Γ on $\mathcal{X}$ is proper. □

It is important to note that the fibers of the manifolds with corners $\mathcal{X}$ and $\mathcal{X}^t$ are smooth (no corners).

Lemma 4.5. *The spaces $\mathcal{X}$ and $\mathcal{X}^t$ are longitudinally smooth.*

[2] This groupoid is commonly denoted $\Gamma(M)$. To avoid overuse of the symbol Γ we use this notation instead.

[3] Given a Lie groupoid $\mathcal{G}$, the s-connected envelope $\mathcal{G}^c$ is also described as the smallest subgroupoid of $\mathcal{G}$ containing the units $\mathcal{G}^{(0)}$.

Proof. The assertion follows from the longitudinal smoothness of the groupoid $\mathcal{G}$ where $r,\ s$ are surjective submersions in the sense of [24]. We have the actions

$$\begin{tikzcd} \mathcal{G}(Y) \arrow[r, hook] & \mathcal{X} \arrow[dl, "p=r_\partial"'] \arrow[dr, "q=s"] & \mathcal{G}(M) \arrow[l, hook'] \\ Y & & M \end{tikzcd}$$

Since $p,\ q$ are restrictions of $r_\partial,\ s$ we obtain that $\mathcal{X}$ has smooth fibers. The same reasoning applies to $\mathcal{X}^t$. $\square$

The strongest possible relation between the groupoids $\mathcal{G}(M)$ and $\mathcal{G}(Y)$ is that of isomorphism in the category of Lie groupoids. An isomorphism in this category is given by (smooth) Morita equivalence (cf. [27]).
In order to prove the next result we consider the following *model case*: M is such that Y intersects every hyperface of M in exactly one codimension two face.
Consequence: Note that in particular every hyperface of M is incident, i.e. $\mathring{\mathcal{F}}_1(M) = \mathcal{I}(Y)$. In the model case we have the equality $|\mathcal{F}_1(Y)| = |\mathcal{I}(Y)| = |\mathcal{F}_1(M)|$. Since Y is transversal and of codimension one to each face of Y there is exactly one incident hyperface in M so $|\mathcal{F}_1(Y)| \geq |\mathcal{I}(Y)|$. If Y intersects every hyperface of M exactly once we have $|\mathcal{F}_1(Y)| \leq |\mathcal{F}_1(M)| = |\mathcal{I}(Y)|$. Since every hyperface of Y arises from an intersection of Y with M, there will be the same number of boundary defining functions for M and Y. We can therefore write in this case $(p_j)_{j\in J}$ and $(q_j)_{j\in J}$ for the boundary defining functions of M and Y respectively.
Now we can prove the following Theorem.

Theorem 4.6. *In the model case there is a Morita equivalence $\mathcal{G}(M) \sim_{\mathcal{M}} \mathcal{G}(Y)$.*

Proof. Recall $\mathcal{X} = \mathcal{G}^c(\mathcal{E})_Y^M,\ \mathcal{X}^t = \mathcal{G}^c(\mathcal{E})_M^Y$, then since $\mathcal{G}(Y) = \mathcal{X} \cap \mathcal{X}^t$ we obtain a Morita equivalence as follows. We have the canonical left and right actions $\mathcal{G}(Y) \hookrightarrow \mathcal{X} \hookleftarrow \mathcal{G}(M)$ and $\mathcal{G}(M) \hookrightarrow \mathcal{X}^t \hookleftarrow \mathcal{G}(Y)$. It is immediate to see that the actions of $\mathcal{G}(Y)$ and $\mathcal{G}(M)$ commute, i.e.

$$\gamma \cdot (z \cdot \eta) = (\gamma \cdot z) \cdot \eta$$

for each $(\gamma, z) \in \mathcal{G}(Y) * \mathcal{X}$ and $(z, \eta) \in \mathcal{X} * \mathcal{G}(M)$.
Secondly, $\mathcal{G}(Y) \setminus \mathcal{X}$ is in bijective correspondence with $\mathcal{G}^{(0)} = M$ and $\mathcal{X}/\mathcal{G}(M)$ is similarly bijective to $\mathcal{G}(Y)^{(0)} = Y$. For this we first show that p induces the homeomorphism $\mathcal{X}/\mathcal{G}(M) \to Y$. We show that we have $p(z) = p(w)$ for $z,\ w \in \mathcal{X}$ if and only if there is a (necessarily unique) $\eta \in \mathcal{G}(M)$ such that $z \cdot \eta = w$. So let $z,\ w$ be such that $p(z) = p(w)$ with $z = (x', y, (\lambda_i)_{i\in J}),\ w = (x', \tilde{y}, (\mu_i)_{i\in J})$. We set $\eta := (y, \tilde{y}, (\mu_j/\lambda_j)_{j\in J})$ and we only need to verify that η is actually contained in $\mathcal{G}(M)$. In order to see this note that we have two sequences $(x_n', y_n) \in Y_0 \times M_0$ and $(x_n', \tilde{y}_n) \in Y_0 \times M_0$. Where we have convergence in local charts of $x_n' \to x,\ y_n \to y,\ \tilde{y}_n \to \tilde{y}$ such that

$$\begin{aligned} &\frac{q_j(x_n')}{p_j(y_n)} \to \lambda_j,\ j \in J,\ n \to \infty, \\ &\frac{q_j(x_n')}{p_j(\tilde{y}_n)} \to \mu_j,\ j \in J,\ n \to \infty. \end{aligned}$$

Hence in particular we have $y_n \to y,\ \tilde{y}_n \to \tilde{y}$ such that

$$\frac{p_j(y_n)}{p_j(\tilde{y}_n)} = \frac{q_j(x_n')}{p_j(\tilde{y}_n)} \cdot \left(\frac{q_j(x_n')}{p_j(y_n)}\right)^{-1} \to \frac{\mu_j}{\lambda_j},\ j \in J,\ n \to \infty.$$

Similarly, one shows that q induces a homeomorphism $\mathcal{G}_\partial \setminus \mathcal{X} \to M$, i.e. $q(z) = q(w)$ if and only if there is a $\gamma \in \mathcal{G}(Y)$ such that $\gamma \cdot z = w$.
Finally, note that $\mathcal{G}$ acts on itself freely and properly. In particular the right action on $\mathcal{G}^c(\mathcal{E})_Y^M = \mathcal{X}$ is free and proper. $\square$

4.2. **General definition.** In the following we give the axioms necessary to define a boundary structure. At first we state a standing assumption. This assumption will get rid of the ambiguity in choice mentioned in example 3.7.

Assumption B. There is a decoupage $(\widetilde{M}, \mathcal{E})$ such that $\mathcal{G} = \mathcal{G}^c(\mathcal{E})_M^M$, $\mathcal{G}_\partial = \mathcal{G}^c(\mathcal{E})_Y^Y$, $\mathcal{X} = \mathcal{G}^c(\mathcal{E})_Y^M$, $\mathcal{X}^t = \mathcal{G}^c(\mathcal{E})_M^Y$ and a canonical bimodule structure $(\mathcal{G}, \mathcal{G}_\partial)$. The groupoids $\mathcal{G} \rightrightarrows M$, $\mathcal{G}_\partial \rightrightarrows Y$ are such that $\mathcal{A} \cong \mathcal{A}(\mathcal{G})$, $\mathcal{A}_\partial \cong \mathcal{A}(\mathcal{G}_\partial)$.

Definition 4.7. A *boundary structure* is defined as a tuple $(\mathcal{G}, \mathcal{G}_\partial, \mathcal{G}^\pm, \mathcal{X}, \mathcal{X}^t, \mathrm{f})$ consisting of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and two manifolds (possibly with corners) $\mathcal{X}$, $\mathcal{X}^t$ which are diffeomorphic via a *flip diffeomorphism* f and subgroupoids $\mathcal{G}^\pm \rightrightarrows X_\pm$ of $\mathcal{G}$.
We impose the following axioms on this data:
i) $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}_{2\mathcal{V}}$ as well as $\mathcal{A}(\mathcal{G}^\pm) \cong \mathcal{A}_{\mathcal{V}}$ as Lie algebroids.
ii) $\mathcal{X}$ and $\mathcal{X}^t$ are $\mathcal{G}$- and $\mathcal{G}_\partial$-spaces each. We have smooth maps $p\colon \mathcal{X} \to Y$, $q\colon \mathcal{X} \to M$ and $p^t\colon \mathcal{X}^t \to M$, $q^t\colon \mathcal{X}^t \to Y$ such that p and q^t are surjective submersions.
iii) Restricted to the interior we have

$$\mathcal{X}_{|Y_0 \times M_0} = p^{-1}(M_0) \cap q^{-1}(Y_0) \cong Y_0 \times M_0$$
$$\mathcal{X}^t_{|M_0 \times Y_0} = (p^t)^{-1}(M_0) \cap (q^t)^{-1}(Y_0) \cong M_0 \times Y_0.$$

iv) The fibers of $\mathcal{G}^\pm$ are the interiors of smooth manifolds with boundary, namely:

$$\partial_{reg}\mathcal{G}_x^+ = \mathcal{X}_x,\ x \in X_+$$
$$\partial_{reg}\mathcal{G}_x^- = \mathcal{X}_x,\ x \in X_-.$$

Remark 4.8. In the following discussion our goal will be to verify the properties of a boundary structure for the case of b-vector fields. This includes the verification of assumption B in this particular case. They also carry over easily to the case of the c_n vector fields (generalized cusps) as defined in [18]. In the general case for arbitrary $\mathcal{V}$ it is possible to find such a boundary structure with assumption B. First the groupoids $\mathcal{G}^\pm$ exist by integrability of the algebroid corresponding to the Lie structure $\mathcal{V}$. One can check that the actions induce longitudinally smooth manifolds $\mathcal{X}$, $\mathcal{X}^t$ via the general method in [17] as in the proof of Lemma 4.5. While axiom *iv)* is more difficult to see in general. It can be seen for the special case of b-groupoids in the following examples. The essential point is the application of the definition of a Lie manifold with boundary.

Example 4.9. *i)* Consider a compact manifold X with boundary $\partial X = Y$ and interior $\mathring{X} := X \setminus Y$. Then we also fix the double $M = 2X$. In this (trivial) case the spaces are given by $\mathcal{X} := Y \times M$, $\mathcal{X}^t = M \times Y$ with the flip $\mathrm{f}(x', y) = (y, x')$, $(x', y) \in Y \times M$. We have here the pair groupoids $\mathcal{G} = M \times M$, $\mathcal{G}_\partial = Y \times Y$ as well as $\mathcal{G}^+ = \mathring{X}_+ \times \mathring{X}_+$, $\mathcal{G}^- = \mathring{X}_- \times \mathring{X}_-$. Then p, q are just the projections $\pi_1\colon Y \times M \to Y$, $\pi_2\colon Y \times M \to M$.
ii) Consider the example of a manifold with corners X and the b-type vector fields $\mathcal{V} = \mathcal{V}_b$ for the Lie structure and consider a regular embedded codimension one submanifold $Y \subset X$ which is transversal in the sense of (1), p. 6. We can consider the double $M = 2X$ as a Lie manifold with Lie structure $2\mathcal{V}$. In this case since $M \times M$ is a manifold with corners the pair groupoid $M \times M$ is no longer fiberwise smooth. Instead we use the groupoids as given by Monthubert (cf. [22], see also [16]) and defined in the last section.
We can obtain a boundary structure for b-type vector fields by the next result.

Theorem 4.10. *For a Lie manifold with boundary $(X, \mathcal{V})$ where $\mathcal{V} = \mathcal{V}_b$, the b-vector fields, there is an adapted boundary structure.*

Proof. Denote by $\mathcal{F}(M)$, $\mathcal{F}(X)$ the collections of open faces of M and X respectively. Introduce the following notation:

$$\operatorname{codim}(F) := \max\{\operatorname{codim}(x) : x \in F\}$$

where $\operatorname{codim}(x)$ denotes the codimension of a point x in M.

The groupoids and spaces that we defined in Definition 4.1 can be written *as sets* as follows

$$\mathcal{G}(M) = \bigcup_{F \in \mathcal{F}(M)} F \times F \times (\mathbb{R}_+^*)^{\operatorname{codim}(F)},$$

$$\mathcal{G}(X) = \bigcup_{F \in \mathcal{F}(X)} F \times F \times (\mathbb{R}_+^*)^{\operatorname{codim}(F)} = \mathcal{G}(M)_X^X,$$

$$\mathcal{G}(Y) = \bigcup_{F \in \mathcal{F}(M)} (F \cap Y) \times (F \cap Y) \times (\mathbb{R}_+^*)^{\operatorname{codim}(F \cap Y)} = \mathcal{G}(M)_Y^Y,$$

$$\mathcal{X} = \mathcal{G}(M)_M^Y = \bigcup_{F, G \in \mathcal{F}(M)} (F \cap Y) \times G \times (\mathbb{R}_+^*)^{\operatorname{codim}(G)},$$

$$\mathcal{X}^t = \mathcal{G}(M)_Y^M = \bigcup_{F, G \in \mathcal{F}(M)} F \times (G \cap Y) \times (\mathbb{R}_+^*)^{\operatorname{codim}(F)}.$$

For the following argument we recall the notation introduced in the subsection 3.1. We have the fixed boundary defining functions for the hyperfaces of M and denote this family by $(p_j)_{j \in I}$. On Y there are the boundary defining functions (relative to Y), denoted by $(q_j)_{j \in J}$ for some index set J. These are the boundary defining functions of the faces from the intersections of Y with the strata of M. Consider now the topology of $\mathcal{X}$ and $\mathcal{X}^t$. It is defined in local charts by the rule

$$Y_0 \times M_0 \ni (x_n', y_n) \to (x', y, \lambda = (\lambda_j)_{j \in J}) :\Leftrightarrow \frac{q_j(x_n')}{p_i(y_n)} \to \lambda_j, \ n \to \infty, \ i \in I, \ j \in J, \ x_n' \to x', \ y_n \to y.$$

On the interior we only have the pair groupoids. This yields the trivial actions

$$\begin{tikzcd} Y_0 \times Y_0 \arrow[r, hook] & Y_0 \times M_0 \arrow[dl, "p=\pi_1"'] \arrow[dr, "q=\pi_2"] & M_0 \times M_0 \arrow[l, hook'] \\ Y_0 & & M_0 \end{tikzcd}$$

These actions can be extended continuously to the closure of $Y_0 \times M_0$ in $\mathcal{G}$ and also of $M_0 \times Y_0$, and we obtain from the definition of the topology that

$$\mathcal{X} = \overline{Y_0 \times M_0}^{\mathcal{G}}, \ \mathcal{X}^t = \overline{M_0 \times Y_0}^{\mathcal{G}}.$$

The continued actions are defined

$$\begin{tikzcd} \mathcal{G}(Y) \arrow[r, hook] & \mathcal{X} \arrow[dl, "p=r_\partial"'] \arrow[dr, "q=s"] & \mathcal{G}(M) \arrow[l, hook'] \\ Y & & M \end{tikzcd}$$

Axiom *i)* holds because of [22] where it was shown that the given b-groupoids integrate the Lie structure of b-vector fields. Then *ii)* was verified in Lem. 4.5 and *iii)* follows from the definition of the action we just gave. Also note that the flip diffeomorphism $\mathrm{f} \colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}^t$ is defined by

$$\mathrm{f} \colon (x', y, (\lambda_i)_{i \in J}) \mapsto \left(y, x', \left(\frac{1}{\lambda_i}\right)_{i \in J}\right).$$

It remains to verify condition *iv)*. For this we define $\mathcal{G}^{\pm} := \mathcal{G}(X_{\pm})$ and prove that this groupoid has the required property. Thus we want to show that

$$\partial_{reg} \mathcal{G}_x^+ = \mathcal{X}_x, \ x \in X.$$

The boundary is possibly empty (for x not incident to the hypersurface Y). We have to distinguish two cases: x in the interior and x on the boundary of M. The groupoid fiber $\mathcal{G}_x^{\pm}$ for x in the interior M_0 trivializes to the pair groupoids and this case is thus immediate. We need to consider the case of a point on the boundary of M. Assume that $x \in F$ for some open face F

of M which is incident to Y (i.e. shares a hyperface with Y). By the local triviality property of groupoids (see [22]) we have

$$\mathcal{G}_x^+ \cong F \times \mathbb{R}_+^*.$$

The same follows by definition for $\mathcal{X}$, i.e.

$$\mathcal{X}_x \cong F_{ij} \times \mathbb{R}_+^*$$

where F_{ij} denotes the face of F such that $F_{ij} = \overline{F} \cap Y$. Via the definition of the Lie manifold with boundary (cf. [2]) we obtain that the component $F \times \mathbb{R}_+^*$ is the interior of a manifold with boundary. In particular we see that

$$\partial_{reg}(F \times \mathbb{R}_+^*) = \partial_{reg}(\overline{F} \times \mathbb{R}_+^*) \cong F_{ij} \times \mathbb{R}_+^*.$$

In summary, we obtain

$$\begin{aligned} \partial_{reg}\mathcal{G}_x^+ &\cong \begin{cases} Y_0 & \text{for } x \in X_0 \\ F_{ij} \times \mathbb{R}_+^* & \text{for } x \text{ incident to some } F \\ \emptyset & \text{otherwise} \end{cases} \\ &\cong \mathcal{X}_x. \end{aligned}$$

□

5. Operators on groupoids

The next goal is to define potential, trace and singular Green operators on the groupoid level. These operators should be equivariant families of operators on the fibers, similar to the case of pseudodifferential operators on groupoids. The singular Green, trace and potential operators are ordinarily defined so as to act like pseudodifferential operators in the cotangent direction and as convolution operators in the normal direction. This somewhat complicated behaviour is difficult to realize in the groupoid setting. The equivariance condition is especially hard to realize. We start from a different but equivalent definition. Our approach is inspired by the ordinary case of a smooth, compact manifold with boundary as explained in the appendix. Here the trace, potential and singular Green operators are extended to the double of the manifold and can be understood as conormal distributions with rapid decay along the normal direction. In our general setting we would therefore like to consider conormal distributions on $Y \times M$ and $M \times Y$ as well as $M \times M$. Since we are working in the setting of manifolds with corners we will desingularize these manifolds and pull back the integral kernels to the desingularized versions. This is were the previously introduced notion of a boundary structure enters. For the cases of $M \times M$ and $Y \times Y$ this is realized through the groupoids $\mathcal{G}$ and $\mathcal{G}_\partial$ respectively and the pseudodifferential operators on groupoids. We introduce additional blowups $\mathcal{X}$ and $\mathcal{X}^t$ with good properties (fibered over the manifolds Y and M) with regard to $\mathcal{G}$ and $\mathcal{G}_\partial$. Then we define the trace, potential and singular Green operators as distributions on these spaces and $\mathcal{G}$ conormal to the diagonal Δ_Y.

5.1. Actions.

From now on we fix: A boundary structure $(\mathcal{G},\ \mathcal{G}_\partial,\ \mathcal{G}^\pm,\ \mathcal{X},\ \mathcal{X}^t,\ \mathrm{f})$ satisfying assumption B adapted to our Lie manifold $(X,\ \mathcal{V})$ with boundary Y and its double $(M,\ 2\mathcal{V})$. We then fix the groupoid actions which are summarized in the following picture. The first column are the inclusions of the groupoids which both act on the second column.

$$\begin{tikzcd}
\mathcal{G}_\partial \arrow[rr, hook, "i_\partial"] \arrow[dr, hook] & & \mathcal{G} \arrow[dl, hook] \arrow[dr, hook] & & \mathcal{G}_\partial \arrow[ll, hook', "i_\partial"'] \arrow[dl, hook] \\
& \mathcal{X} \arrow[rr, tail, two heads, "\mathrm{f}"] \arrow[dl, "p=r_\partial"'] \arrow[dr, "q=s"] & & \mathcal{X}^t \arrow[dl, "p^t=r"'] \arrow[dr, "q^t=s_\partial"] & \\
Y \arrow[rr, hook, "i_Y"'] & & M & & Y \arrow[ll, hook', "i_Y"]
\end{tikzcd}$$

Fix also Haar systems on the groupoids and fibered spaces as follows.

$$\mathcal{G} : \{\mu_x\}_{x\in M}, \ \mathcal{X} : \{\lambda_x\}_{x\in M},$$
$$\mathcal{G}_\partial : \{\mu_y^\partial\}_{y\in Y}, \ \mathcal{X}^t : \{\lambda_x^t\}_{x\in M}.$$

In each case the system is a (left / right)-Haar system if the corresponding action is a (left / right)-action.

5.2. **Local charts.** In order to define the operators on groupoids and actions as given in the last section we have to introduce the local charts. The charts are given by diffeomorphisms which preserve the s-fibers, see also [26], p. 3.
Fix the dimensions $n = \dim M = \dim M_0, \ n-1 = \dim Y = \dim Y_0$.

- A chart of $\mathcal{G}$ is an open subset $\Omega \subset \mathcal{G}$ which is diffeomorphic to two open subsets of $\mathcal{G}^{(0)} \times \mathbb{R}^n$. Choose two open subsets $V_s \times W_s$ and $V_r \times W_r$. Then choose two diffeomorphisms $\psi_s : \Omega \to V_s \times W_s$ and $\psi_r : \Omega \to V_r \times W_r$. Additionally, we require that these diffeomorphisms are *fiber-preserving* in the sense that $s(\psi_s(x, w)) = x$ for $(x, w) \in V_s \times W_s$ and $r(\psi_r(x, w)) = x$ for $(x, w) \in V_r \times W_r$. Hence we have the following commuting diagrams:

$$\begin{tikzcd}
& r(\Omega)\times W_r \arrow[dl] & & \Omega \arrow[ll, "\psi_r"'] \arrow[rr, "\psi_s"] \arrow[dl] \arrow[dr] & & s(\Omega)\times W_s \arrow[dr] & \\
V_r & & r(\Omega) \arrow[ll] & & s(\Omega) \arrow[rr] & & V_s
\end{tikzcd}$$

- Similarly, the charts for $\mathcal{X}$ are given by the sets of the form $\tilde{\Omega} = \Omega \cap \mathcal{X}$ for charts Ω of $\mathcal{G}$ fitting into the following commuting diagrams:

$$\begin{tikzcd}
& V_q \times W_q \arrow[dl] & & \tilde{\Omega} \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] & & V_{p^t}\times W_{p^t} \arrow[dr] & \\
W_q & & q(\tilde{\Omega}) \arrow[ll] & & p^t(\tilde{\Omega}) \arrow[rr] & & V_{p^t}
\end{tikzcd}$$

- Analogously, $\tilde{\Omega} \subset \mathcal{X}^t$ are charts with the actions reversed and hence in this case we have the commuting diagrams:

$$\begin{tikzcd}
& V_p \times W_p \arrow[dl] & & \tilde{\Omega} \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] & & V_{q^t}\times W_{q^t} \arrow[dr] & \\
W_p & & p(\tilde{\Omega}) \arrow[ll] & & q^t(\tilde{\Omega}) \arrow[rr] & & V_{q^t}
\end{tikzcd}$$

Definition 5.1. *i)* A family $T = (T_x)_{x\in M}$ of operators $T_x : C_c^\infty(\mathcal{G}_x) \to C_c^\infty(\mathcal{X}_x)$ is a *differentiable family of trace type* iff the following holds. Given any chart $\Omega \subset \mathcal{G}$ with fiber preserving diffeomorphism, $s(\Omega) \sim \Omega \times W$ for some $W \subset \mathbb{R}^n$ open. Moreover for $\tilde{\Omega} := \Omega \cap \mathcal{X}$ such that $\tilde{W} := W \cap \mathbb{R}^{n-1}$ we have a fiber-preserving diffeomorphism $q(\tilde{\Omega}) \sim \tilde{\Omega} \times \tilde{W}$, and for each $\varphi \in C_c^\infty(\Omega)$, $\tilde{\varphi} \in C_c^\infty(\tilde{\Omega})$ the operator $\tilde{\varphi} T \varphi$ has a Schwartz kernel

$$k \in I^m(s(\Omega) \times \tilde{W} \times W, \Delta_{\tilde{W}}) \cong C^\infty(s(\Omega)) \hat{\otimes} I^m(\tilde{W} \times W, \Delta_{\tilde{W}}).$$

The operator $\tilde{\varphi} T_x \varphi$ for each $x \in s(\Omega)$ corresponds to the Schwartz kernel k_x via the diffeomorphisms $\mathcal{X}_x \cap \tilde{\Omega} \cong \tilde{W}$ and $\mathcal{G}_x \cap \Omega \cong W$.
ii) Analogously, we define a family $K = (K_x)_{x \in M}$ of operators $K \colon C_c^\infty(\mathcal{X}_x^t) \to C_c^\infty(\mathcal{G}_x)$ with the charts reversed. This is called *differentiable family of potential type.*
iii) A *differentiable family of singular Green type* $(G_x)_{x \in M}$ is a family of operators $G_x \colon C_c^\infty(\mathcal{G}_x) \to C_c^\infty(\mathcal{G}_x)$ defined as follows. Given any chart $\Omega \subset \mathcal{G}$ with fiber preserving diffeomorphism $s(\Omega) \sim \Omega \times W$ for some $W \subset \mathbb{R}^n$ open and $\tilde{W} = W \cap \mathbb{R}^{n-1}$. Then for each $\varphi \in C_c^\infty(\Omega)$ the operator $\varphi G \varphi$ has a Schwartz kernel

$$k \in I^m(s(\Omega) \times W \times W, \Delta_{\tilde{W}}) \cong C^\infty(s(\Omega)) \hat{\otimes} I^m(W \times W, \Delta_{\tilde{W}}).$$

Furthermore, $\varphi G_x \varphi$ for each $x \in s(\Omega)$ corresponds to the Schwartz kernel k_x via the diffeomorphism $\mathcal{G}_x \cap \Omega \cong W$.

Fix the following operations

$$\begin{aligned} &\mu_{\mathcal{G}} \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, (\gamma, \eta) \mapsto \gamma \eta^{-1}, \\ &\mu \colon \mathcal{X} \times \mathcal{G} \to \mathcal{X}, (z, \gamma) \mapsto z \cdot \gamma^{-1}, \\ &\mu^t \colon \mathcal{G} \times \mathcal{X}^t \to \mathcal{X}, (\gamma, z) \mapsto \gamma^{-1} \cdot z \end{aligned}$$

whenever defined.
A trace type family T has a family of Schwartz kernels $(k_x^T)_{x \in M}$. Define the support of T as

$$\operatorname{supp}(T) = \overline{\bigcup_{x \in M} \operatorname{supp}(k_x^T)}.$$

The *reduced support* of T is written

$$\operatorname{supp}_\mu(T) = \mu(\operatorname{supp}(T)).$$

The analogous definitions for potential type operators K and Green type operators G are given by

$$\operatorname{supp}_{\mu^t}(K) = \mu^t(\operatorname{supp}(K)), \ \operatorname{supp}_{\mu_{\mathcal{G}}}(G) = \mu_{\mathcal{G}}(\operatorname{supp}(G)).$$

Definition 5.2.
- An *extended trace operator* is a differentiable family $T = (T_x)_{x \in M}$ of trace type which is a right $\mathcal{X}$-operator (see Definition 2.8, p. 11) such that the reduced support of T is a compact subset of $\mathcal{X}$.
- An *extended potential operator* is a differentiable family $K = (K_x)_{x \in M}$ of potential type which is a left $\mathcal{X}^t$-operator such that the reduced support of K is a compact subset of $\mathcal{X}$.
- An *extended singular Green operator* is a differentiable family $G = (G_x)_{x \in M}$ of singular Green type which is equivariant and such that the reduced support of G is a compact subset of $\mathcal{G}$.

Remark 5.3. *i)* Since we also have a right action of $\mathcal{G}$ on $\mathcal{X}$ and $\mathcal{X}$ is diffeomorphic (via f) to $\mathcal{X}^t$ we obtain that being a left $\mathcal{X}^t$-operator is equivalent to the equivariance condition with regard to the right action of $\mathcal{G}$ on $\mathcal{X}$ given in equation (4) on p. 11. Hence a potential operator is also a right operator with regard to $\mathcal{X}$ in this sense, which furnishes by the proof of Prop. 2.9 a reduced kernel for extended potential operators.
ii) Note that we obtain the reduced kernels for pseudodifferential operators on $\mathcal{G}$ and extended singular Green operators with an argument completely analogous to the proof of Prop. 2.9.

Proposition 5.4. i) *Given an extended trace operator T the reduced kernel k_T (see Proposition (2.9), p. 11) is a compactly supported distribution on $\mathcal{X}$ conormal to Δ_Y.*
ii) *Analogously an extended potential operator K has reduced kernel k_K a compactly supported distribution on $\mathcal{X}^t$ conormal to Δ_Y. Furthermore, K is the adjoint of an extended trace operator.*
iii) *An extended singular Green operator G has a reduced kernel k_G being a compactly supported distribution on $\mathcal{G}$ conormal to Δ_Y.*

Proof. We give a proof of conormality for the case *i)* of extended trace operators. The other cases are the same.
Given a family of Schwartz kernels for $(k_x^T)_{x\in M}$ contained in $I^m(\mathcal{X}_x \times \mathcal{G}_x, \mathcal{X}_x)$ for each $x \in M$. Rewrite this as

$$k_x^T = \mu^*(k_T)_{|\mathcal{X}_x\times\mathcal{G}_x},\ \mathcal{X}_x \subset \mathcal{G}_x \text{ (transversal)}.$$

Here μ is the map $\mathcal{X} * \mathcal{G} \ni (z,\gamma) \mapsto z\cdot\gamma^{-1} \in \mathcal{X}$ and

$$\langle \mu^*(k_T), f\rangle = \left\langle k_T(z),\ \int_{w=z\cdot\gamma} f(w,\gamma)\right\rangle.$$

Then we need to show that: $\operatorname{singsupp}(k_T) \subset Y \cong \Delta_Y$.
To this end let $z \in \mathcal{X}\setminus\Delta_Y$ and $\varphi \in C_c^\infty(\mathcal{X})$ a cutoff function such that φ is equal to 1 in a neighborhood of Δ_Y and equal to 0 in a neighborhood containing z. Then

$$\mu^*((1-\varphi)k_T) = (1-\varphi\circ\mu)\mu^*(k_T)$$

restricted to $\mathcal{X}_x \times \mathcal{G}_x$ yields $(1-\varphi\circ\mu)k_x^T$ and this is C^∞ because $\operatorname{singsupp}(k_x^T) \subset \Delta_x \cong \mathcal{X}_x \subset \mathcal{X}_x \times \mathcal{G}_x$ by definition. Hence $(1-\varphi\circ\mu)\mu^*(k_T)$ is C^∞, but this implies that $(1-\varphi)k_T$ is smooth as well. This proves conormality.
Finally, we show that a trace operator is the adjoint of a potential operator and vice versa. Let $T = (T_x)_{x\in M}$ be an extended trace operator and let $(k_x^T)_{x\in M}$ be the corresponding family of Schwartz kernels. The adjoint $T^* = (T_x^*)_{x\in M}$ is given by $T_x^* \colon C_c^\infty(\mathcal{X}_x) \to C_c^\infty(\mathcal{G}_x)$ such that for $u \in C_c^\infty(\mathcal{X}_x)$ we have

$$(T_x^*u)(\gamma) = \int_{\mathcal{X}_{s(\gamma)}} \overline{k_x^T(z,\gamma)}u(z)\,d\lambda_{s(\gamma)}(z).$$

Define the family of operators $K = (K_x)_{x\in M}$ by $K = T^*$ and $k_x^K(\gamma,z) := \overline{k_x^T(z,\gamma)}$. We obtain a family $(k_x^K)_{x\in M}$ of distributions on $\mathcal{G}_x \times \mathcal{X}_x$ conormal to $\Delta_x \cong \mathcal{X}_x$ for each $x \in M$. In addition K is equivariant with regard to the right action of $\mathcal{G}$ which is by remark 5.3 equivalent to being a left $\mathcal{X}^t$-operator. Hence K is an extended potential operator. The same argument shows that the adjoint of an extended potential operator is an extended trace operator. □

Notation 5.5. We fix the notation for the reduced kernels and denote by $I_c^m(\mathcal{X},\Delta_Y)$ the space of reduced kernels of extended trace operators of order m, by $I_c^m(\mathcal{X}^t,\Delta_Y)$ the reduced kernels of extended potential operators of order m and by $I_c^m(\mathcal{G},\Delta_Y)$ the space of reduced kernels of singular Green operators of order m. For the pseudodifferential operators on $\mathcal{G}$ of order m we use the notation $\Psi^m(\mathcal{G})$ for the space of operators and $I_c^m(\mathcal{G},\Delta_M)$ for the reduced kernels. With the Schwartz kernel theorem it can be proven that the spaces $\Psi^m(\mathcal{G})$ and $I_c^m(\mathcal{G},\Delta_M)$ are isomorphic, see [24], p. 24.

Remark 5.6. We will also use the notation

$$\mathscr{T}^{m,0}(\mathcal{G},\mathcal{G}_\partial) := \mathcal{J}_{tr}\circ I_c^m(\mathcal{X},\Delta_Y),\ \mathcal{K}^{m,0}(\mathcal{G},\mathcal{G}_\partial) := \mathcal{J}_{pot}\circ I_c^m(\mathcal{X}^t,\Delta_Y),$$
$$\mathcal{G}^{m,0}(\mathcal{G},\mathcal{G}_\partial) := \mathcal{J}_{gr}\circ I_c^m(\mathcal{G},\Delta_Y)$$

for these classes of extended trace, potential and singular Green operators, respectively. The $\mathcal{J}_\cdot$ in each case are the appropriate isomorphisms from the Schwartz kernel theorem.
Hence the operators defined previously act as follows. The mapping

$$\mathcal{J}_{tr}\colon I_c^m(\mathcal{X},\Delta_Y) \to \mathscr{T}^{m,0}(\mathcal{G},\mathcal{G}_\partial) \subset \operatorname{Hom}(C_c^\infty(\mathcal{G}), C_c^\infty(\mathcal{X}))$$

is for $z \in \mathcal{X},\ k_T \in I_c^m(\mathcal{X}, \Delta_Y)$ given by

$$(\mathcal{J}_{tr}(k_T)u)(z) = \int_{\mathcal{G}_{q(z)}} k_T(z \cdot \gamma^{-1})u(\gamma)\, d\mu_{q(z)}(\gamma).$$

Analogously for the potential operators we have

$$\mathcal{J}_{pot} \colon I_c^m(\mathcal{X}^t, \Delta_Y) \to \mathcal{K}^{m,0}(\mathcal{G}, \mathcal{G}_\partial) \subset \mathrm{Hom}(C_c^\infty(\mathcal{X}^t), C_c^\infty(\mathcal{G}))$$

which for $\gamma \in \mathcal{G},\ k_K \in I_c^m(\mathcal{X}^t, \Delta_Y)$ is given by

$$(\mathcal{J}_{pot}(k_K)u)(\gamma) = \int_{\mathcal{X}^t_{r(\gamma)}} k_K(\gamma^{-1} \cdot z)u(z)\, d\lambda^t_{r(\gamma)}(z).$$

Lastly, for the singular Green operators

$$\mathcal{J}_{gr} \colon I_c^m(\mathcal{G}, \Delta_Y) \to \mathcal{G}^{m,0}(\mathcal{G}, \mathcal{G}_\partial) \subset \mathrm{Hom}(C_c^\infty(\mathcal{G}), C_c^\infty(\mathcal{G}))$$

we have for $\gamma \in \mathcal{G},\ k_G \in I_c^m(\mathcal{G}, \Delta_Y)$

$$(\mathcal{J}_{gr}(k_G)u)(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} k_G(\gamma\eta^{-1})u(\eta)\, d\mu_{s(\gamma)}(\eta).$$

With any fibered space, longitudinally smooth via an action of a nice enough groupoid one can associate an equivariant calculus of pseudodifferential operators. We want to define such a calculus on $\mathcal{X}$ and $\mathcal{X}^t$. The following definition can in somewhat greater generality also be found in [26].

Definition 5.7. A family of pseudodifferential operators of order m on $\mathcal{X}$ is defined as $S = (S_x)_{x \in M}$ such that
i) each $S_x \colon C^\infty(\mathcal{X}_x) \to C^\infty(\mathcal{X}_x)$ is contained in $\Psi^m(\mathcal{X}_x)$.
ii) For each chart of $\mathcal{X}$ given by $\Omega \sim q(\Omega) \times W$ there is a smooth function $a \colon q(\Omega) \to S^m(T^*W)$ such that for each $x \in q(\Omega)$ we have

$$S_{x|\Omega \cap \mathcal{X}_x} = a_x(y, D_y)$$

via identifying $\Omega \cap \mathcal{X}_x$ with W. Here $a_x(y,\xi) = a(x)(y,\xi)$. We denote by $\Psi^m(\mathcal{X})$ the set of pseudodifferential families on $\mathcal{X}$.

This leads immediately to a definition of equivariant pseudodifferential operators on $\mathcal{X}$ and $\mathcal{X}^t$.

Definition 5.8. The space of equivariant pseudodifferential operators $\Psi^\bullet(\mathcal{X})^{\mathcal{G}}$ on $\mathcal{X}$ consists of elements $S = (S_x)_{x \in M}$ of $\Psi^\bullet(\mathcal{X})$ such that the following equivariance condition holds

$$R_{\gamma^{-1}} S_{r(\gamma)} R_\gamma = S_{s(\gamma)},\ \gamma \in \mathcal{G}.$$

By analogy we define the equivariant pseudodifferential operators ${}^{\mathcal{G}}\Psi^\bullet(\mathcal{X}^t)$ on $\mathcal{X}^t$ coming from the left action of $\mathcal{G}$. The equivariance condition in this case is given as in Definition 2.8, *ii)* on p. 11.

The operators defined here are in each case families parametrized over the double M. We have to clarify what role the pseudodifferential operators defined on $\mathcal{G}_\partial$ play.

Proposition 5.9. *We have the following exact sequence*

$$C_Y^\infty(M)\Psi^\bullet(\mathcal{X})^{\mathcal{G}} \rightarrowtail \Psi^\bullet(\mathcal{X})^{\mathcal{G}} \xrightarrow{\mathcal{R}_Y^{\mathcal{G}}} \Psi^\bullet(\mathcal{G}_\partial)$$

where $\mathcal{R}_Y^{\mathcal{G}}$ is a well-defined restriction of families $(S_x)_{x \in M} \mapsto (S_y)_{y \in Y}$. Here $C_Y^\infty(M)$ are the smooth functions on M that vanish on Y.

Proof. First note that $\mathcal{G}_\partial$ acts (from the left and the right) on itself. Extend this action to the set of families $(S_y)_{y\in Y}$ with $S_y \in \Psi^*((\mathcal{G}_\partial)_y)$. Invariance under this action is just the usual equivariance condition for pseudodifferential operators. Together with the uniform support condition we therefore recover the class of pseudodifferential operators, denoted $\Psi^*(\mathcal{G}_\partial)$.
The exactness of the sequence

$$C_Y^\infty(M)\Psi^\bullet(\mathcal{X}) \rightarrowtail \Psi^\bullet(\mathcal{X}) \xrightarrow{\mathcal{R}_Y} \Psi^\bullet(\mathcal{X}_{|Y})$$

for the restriction operator $\mathcal{R}_Y$ defined by $\mathcal{R}_Y((S_x)_{x\in M}) = (S_y)_{y\in Y}$ is immediate. Here note that

$$\mathcal{X}_{|Y} = q^{-1}(Y) = r^{-1}(Y) \cap s^{-1}(Y) = \mathcal{G}_Y^Y = \mathcal{G}_\partial$$

by assumption.
Note also that by the previous remarks $\Psi^*(\mathcal{X}_{|Y})^{\mathcal{G}_Y^Y} \cong \Psi^*(\mathcal{G}_\partial)$. This furnishes the exact sequence of equivariant pseudodifferential operators with a well-defined restriction map $\mathcal{R}_Y^{\mathcal{G}}$

$$C_Y^\infty(M)\Psi^\bullet(\mathcal{X})^{\mathcal{G}} \rightarrowtail \Psi^\bullet(\mathcal{X})^{\mathcal{G}} \xrightarrow{\mathcal{R}_Y^{\mathcal{G}}} \Psi^\bullet(\mathcal{X}_{|Y})^{\mathcal{G}_Y^Y} \cong \Psi^*(\mathcal{G}_\partial).$$

□

6. Compositions

In order to prove the main Theorem we first establish a Lemma about compositions of conormal distributions.

Lemma 6.1. *The classes of extended Boutet de Monvel operators are closed under compositions induced by groupoid actions and convolution. More precisely we have the following compositions:*

$$*\colon I_c^{m_1}(\mathcal{X}^t, \Delta_Y) \times I_c^{m_2}(\mathcal{X}, \Delta_Y) \to I_c^{m_1+m_2}(\mathcal{G}, \Delta_Y), \tag{9}$$
$$*\colon I_c^{m_1}(\mathcal{X}, \Delta_Y) \times I_c^{m_2}(\mathcal{X}^t, \Delta_Y) \to \Psi^{m_1+m_2}(\mathcal{X})^{\mathcal{G}}, \tag{10}$$
$$*\colon \Psi^{m_1}(\mathcal{X})^{\mathcal{G}} \times I_c^{m_2}(\mathcal{X}, \Delta_Y) \to I_c^{m_1+m_2}(\mathcal{X}, \Delta_Y), \tag{11}$$
$$*\colon \Psi^{m_1}(\mathcal{G}) \times I_c^{m_2}(\mathcal{G}, \Delta_Y) \to I_c^{m_1+m_2}(\mathcal{G}, \Delta_Y), \tag{12}$$
$$*\colon I_c^{m_1}(\mathcal{G}, \Delta_Y) \times \Psi^{m_2}(\mathcal{G}) \to I_c^{m_1+m_2}(\mathcal{G}, \Delta_Y), \tag{13}$$
$$*\colon I_c^{m_1}(\mathcal{G}, \Delta_Y) \times I_c^{m_2}(\mathcal{X}^t, \Delta_Y) \to I_c^{m_1+m_2}(\mathcal{X}^t, \Delta_Y), \tag{14}$$
$$*\colon I_c^{m_1}(\mathcal{X}, \Delta_Y) \times \Psi^{m_2}(\mathcal{G}) \to I_c^{m_1+m_2}(\mathcal{X}, \Delta_Y), \tag{15}$$
$$*\colon \Psi^{m_1}(\mathcal{G}) \times I_c^{m_2}(\mathcal{X}^t, \Delta_Y) \to I_c^{m_1+m_2}(\mathcal{X}^t, \Delta_Y), \tag{16}$$
$$*\colon I_c^{m_1}(\mathcal{X}^t, \Delta_Y) \times {}^{\mathcal{G}}\Psi^{m_2}(\mathcal{X}^t) \to I_c^{m_1+m_2}(\mathcal{X}^t, \Delta_Y), \tag{17}$$
$$*\colon I_c^{m_1}(\mathcal{X}, \Delta_Y) \times I_c^{m_2}(\mathcal{G}, \Delta_Y) \to I_c^{m_1+m_2}(\mathcal{X}, \Delta_Y). \tag{18}$$

Proof. We have the equivalences (16) ⇔ (15), (17) ⇔ (11) and (18) ⇔ (14) by Prop. 5.4, *ii)*. Since the argument in each case goes along the same lines we only treat the first 3 cases of compositions exemplarily.
i) We consider first the case of the composition (9). Consider a family of extended trace operators $T = (T_x)_{x\in M}$ and extended potential operators $K = (K_x)_{x\in M}$. Denote the corresponding family of Schwartz kernels by $k_x^T \in I^{m_1}(\mathcal{X}_x \times \mathcal{G}_x, \mathcal{X}_x)$ as well as $k_x^K \in I^{m_2}(\mathcal{G}_x \times \mathcal{X}_x^t, \mathcal{X}_x^t)$ for $x \in M$.

We make the following computation involving an interchange of integration we still have to justify via a reduction to local coordinates. Let $\gamma \in \mathcal{G}_x$ then

$$
\begin{aligned}
(K_x \cdot T_x)u(\gamma) &= \int_{\mathcal{X}_x^t} k_x^K(\gamma, z)(T_x u)(z)\, d\lambda_x^t(z) \\
&= \int_{\mathcal{X}_x^t} \int_{\mathcal{G}_x} k_x^K(\gamma, z) k_x^T(z, \eta) u(\eta)\, d\mu_x(\eta)\, d\lambda_x^t(z) \\
&= \int_{\mathcal{G}_x} k_x^{K\cdot T}(\gamma, \eta) u(\eta)\, d\mu_x(\eta).
\end{aligned}
$$

The kernels of the composition $k_x^{K\cdot T}$ would therefore take the form

$$
k_x^{K\cdot T}(\gamma, \eta) = \int_{\mathcal{X}_x^t} k_x^K(\gamma, z) k_x^T(z, \eta)\, d\lambda_x^t(z).
$$

This corresponds to the convolution of reduced kernels $k_K * k_T$ which is immediately defined from the actions.
First we check the support condition of the composed operator. The reduced support is compact via the inclusion

$$
\operatorname{supp}_\mu(K \cdot T) \subset \mu_{\mathcal{G}}(\operatorname{supp}_\lambda(K) \times \operatorname{supp}_{\lambda^t}(T)).
$$

Here the inversion of elements in the spaces $\mathcal{X}$ and $\mathcal{X}^t$ is performed inside the groupoid $\mathcal{G}$ where it is always defined.
Fix the projections

$$
p_1 \colon \mathcal{X} \times \mathcal{G} \to \mathcal{X}, \ p_2 \colon \mathcal{X} \times \mathcal{G} \to \mathcal{G}.
$$

Then by the uniform support condition the family $T = (T_x : x \in \mathcal{U})$ is in particular properly supported. This means for compact sets $K_1 \subset \mathcal{G}, K_2 \subset \mathcal{X}$ we have that

$$
p_i^{-1}(K_i) \cap \operatorname{supp}(k_T) \subset \mathcal{X} \times \mathcal{G}, i = 1, 2
$$

is compact. We make use of this property for the following argument.
Next we check the smoothness property of compositions. Let $f \in C_c^\infty(\mathcal{G})$ be given, we will show that $Tf \in C_c^\infty(\mathcal{X})$. Assume that $T = (T_x : x \in M)$ has a Schwartz kernel k_T contained in $I^{-\infty}(\mathcal{X}, \Delta_Y) = \bigcap_m I^m(\mathcal{X}, \Delta_Y)$. Then k_T is C^∞ on the closed subset $\{(z, \gamma) : q(z) = s(\gamma)\}$ of $\mathcal{X} \times \mathcal{G}$ and via fiber preserving diffeomorphisms we obtain a C^∞-atlas. The function Tf yields a smooth function because we integrate the kernels k_x^T which are smooth functions. Hence we can interchange integration and differentiation. Therefore $Tf \in C_c^\infty(\mathcal{X})$ for $k_T \in I^{-\infty}(\mathcal{X}, \Delta_Y)$.
Consider a general extended trace operator T. Let $(\gamma, z) \in \mathcal{G}_x \times \mathcal{X}_x$ and $\Omega \subset \mathcal{G}$ be a chart with fiber preserving diffeomorphism $\Omega \sim s(\Omega) \times W$. We can assume that $W \subset \mathbb{R}^n$ is convex, open, $0 \in W$ and that Ω is a neighborhood of γ such that $(x, 0)$ gets mapped to γ via the diffeomorphism. We also set $\tilde{\Omega} = \Omega \cap \mathcal{X} \subset \mathcal{X}$ and $\tilde{W} = W \cap \mathbb{R}^{n-1}$ with a fiber preserving diffeomorphism $\tilde{\Omega} \sim q(\tilde{\Omega}) \times \tilde{W}$ (recall the fact that $q = s_{|\mathcal{G}_M^Y}$ by assumption). By the previous remarks the family T is properly supported, which implies in particular that each T_x is properly supported for $x \in M$. Hence we obtain that the kernels k_x^T of T satisfy the support estimate

$$
p_1^{-1}\left(q(\tilde{\Omega}) \times \frac{\tilde{W}}{2}\right) \cap p_2^{-1}\left(s(\Omega) \times \frac{W}{2}\right) \cap \overline{\bigcup_x \operatorname{supp}(k_x^T)} \subset \left(q(\tilde{\Omega}) \times \frac{3\tilde{W}}{2}\right) \times \left(s(\Omega) \times \frac{3W}{2}\right)
$$

therefore the fact that $Tf \in C_c^\infty(\mathcal{X})$ reduces to a computation in local coordinates.
Similar reasoning applies to potential operators. Using the same argument as above we deduce that $Kf \in C_c^\infty(\mathcal{G})$ for $f \in C_c^\infty(\mathcal{X})$ if K is smoothing. For a general K we note that each K_x is properly supported for $x \in M$ and hence we obtain that the kernels k_x^K of K satisfy the support estimate

$$
p_1^{-1}\left(q(\tilde{\Omega}) \times \frac{\tilde{W}}{2}\right) \cap p_2^{-1}\left(s(\Omega) \times \frac{W}{2}\right) \cap \overline{\bigcup_x \operatorname{supp}(k_x^K)} \subset \left(q(\tilde{\Omega}) \times \frac{3\tilde{W}}{2}\right) \times \left(s(\Omega) \times \frac{3W}{2}\right).
$$

The smoothness of Kf reduces to a computation in local coordinates. Consider the general composition $K \cdot T$ for T an arbitrary extended trace operator and K an arbitrary extended potential operator. Then make the suitable support estimates as above to show that $K_x \cdot T_x$ are compositions of smooth families of conormal distributions which act on the sets $W \subset \mathbb{R}^n$, $\tilde{W} \subset \mathbb{R}^{n-1}$. It then follows from a general theorem of Hörmander about compositions of conormal distributions, see [14], Thm. 25.2.3, p. 21 that the composition $T \cdot K$ yields a family of operators each of which is a conormal distribution in the sense of Definition 5.1, *iii)*.
We therefore obtain a properly supported family $K \cdot T$ which is by the above argument uniformly supported since K and T are each uniformly supported. Finally, we check the equivariance property for the new family of operators $G = K \cdot T$. Since $(R_\gamma)^{-1} = R_{\gamma^{-1}}$ we can write the equivariance condition from Definition 2.8 in the form

$$T_{r(\gamma)} R_\gamma = R_\gamma T_{s(\gamma)}, \ \forall\, \gamma \in \mathcal{G}.$$

Since K is equivariant with regard to the right action of $\mathcal{G}$ on $\mathcal{X}$ by Remark 5.3, *i)*, the equivariance condition for K reads

$$R_{\gamma^{-1}} K_{r(\gamma)} = K_{s(\gamma)} R_{\gamma^{-1}}, \ \forall\, \gamma \in \mathcal{G}.$$

For $\gamma \in \mathcal{G}$ we calculate

$$\begin{aligned}
&R_{\gamma^{-1}}(K \cdot T)_{r(\gamma)} R_\gamma = R_{\gamma^{-1}}(K_{r(\gamma)} \cdot T_{r(\gamma)}) R_\gamma = R_{\gamma^{-1}} K_{r(\gamma)} R_\gamma T_{s(\gamma)} \\
&= K_{s(\gamma)} R_{\gamma^{-1}} \cdot R_\gamma T_{s(\gamma)} = K_{s(\gamma)}((R_\gamma)^{-1} \cdot R_\gamma) T_{s(\gamma)} = K_{s(\gamma)} \cdot T_{s(\gamma)} \\
&= (K \cdot T)_{s(\gamma)}
\end{aligned}$$

and hence $K \cdot T$ has the required equivariance property with regard to the right action of the groupoid $\mathcal{G}$. We have thus verified all the properties of an extended Green operator.
ii) Consider the next composition $T \cdot K \colon C_c^\infty(\mathcal{X}) \to C_c^\infty(\mathcal{X})$ which is again for $z \in \mathcal{X}$ and $u \in C_c^\infty(\mathcal{X})$ given by

$$\begin{aligned}
(T_x \cdot K_x) u(z) &= \int_{\mathcal{G}_{q(z)}} k_x^T(z, \gamma)(K_x u)(\gamma)\, d\lambda_x(\gamma) \\
&= \int_{\mathcal{G}_x} \int_{\mathcal{X}_x^t} k_x^T(z, \gamma) k_x^K(\gamma, w) u(w)\, d\lambda_x^t(w)\, d\mu_x(\gamma) \\
&= \int_{\mathcal{X}_x^t} k_x^{T \cdot K}(z, w) u(w)\, d\lambda_x^t(w).
\end{aligned}$$

The kernel $k_x^{T \cdot K}$ is written

$$k_x^{T \cdot K}(z, w) = \int_{\mathcal{G}_x} k_x^T(z, \gamma) k_x^K(\gamma, w)\, d\mu_x(\gamma).$$

Now we can argue again analogously to *i)* that the composition has the right support condition and via a reduction to local charts the formal computation can be made precise. We therefore obtain a family of kernels $k_x^{T \cdot K} \in I^{m_1 + m_2}(\mathcal{X}_x^t \times \mathcal{X}_x, \Delta_{\mathcal{X}_x})$.
iii) The third composition $S \cdot T \colon C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{X})$ gives a family of extended trace operators. We obtain for $z \in \mathcal{X}, u \in C_c^\infty(\mathcal{G})$

$$\begin{aligned}
(S_x \cdot T_x) u(z) &= \int_{\mathcal{X}_x} k_x^S(z, w)(T_x u)(w)\, d\lambda_x(w) \\
&= \int_{\mathcal{X}_x} \int_{\mathcal{G}_x} k_x^S(z, w) k_x^T(w, \gamma) u(\gamma)\, d\mu_x(\gamma)\, d\lambda_x(w) \\
&= \int_{\mathcal{G}_x} k_x^{S \cdot T}(z, \gamma) u(\gamma)\, d\mu_x(\gamma).
\end{aligned}$$

We obtain the kernel

$$k_x^{S \cdot T}(z, \gamma) = \int_{\mathcal{X}_x} k_x^S(z, w) k_x^T(w, \gamma)\, d\lambda_x(w).$$

We proceed by making the analogous argument as in *i), ii)*. The right support condition holds on $\Psi(\mathcal{X})^{\mathcal{G}}$ e.g. via the identification from 5.9. The rest of the reasoning then yields a family of kernels $k_x^{S\cdot T} \in I^{m_1+m_2}(\mathcal{X}_x \times \mathcal{G}_x, \Delta_{\mathcal{X}_x})$ with the correct support condition. $\square$

One can also obtain an equivalent definition of these operator classes based on explicit quantization for conormal distributions. We recall this definition with the help of the normal bundles and fibrations introduced in the next section. Hence the reader should compare the notation used here to the beginning of section 7 as well as Remark 7.2.
Let us consider the extended trace operator exemplarily and fix the normal bundle $\mathcal{N}^{\mathcal{X}}\Delta_Y \to \Delta_Y$ to the inclusion $\Delta_Y \hookrightarrow \mathcal{X}$.
With the normal fibration we obtain a map Ψ which is a diffeomorphism from open neighborhoods $\Delta_Y \subset U \subset \mathcal{X}$ to open neighborhoods of the zero section in $\mathcal{N}^{\mathcal{X}}\Delta_Y$.
With the normal fibration of the embedding of Δ_Y in $\mathcal{X}$ the open neighborhoods can be chosen such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\Psi"] \arrow[d, "r_\partial = p"'] & \mathcal{N}^{\mathcal{X}}\Delta_Y \arrow[d] \\ \Delta_Y \arrow[r] & \Delta_Y. \end{tikzcd}$$

Moreover the condition $\Psi(z) = 0 \Leftrightarrow z \in \Delta_Y \cong Y = \mathcal{G}_\partial^{(0)}$ holds.
Then we have the following result.

Proposition 6.2. *The space of extended trace operators on $\mathcal{X}$ identifies with the space of distributional kernels $k_T \in I_c^m(\mathcal{X}, \Delta_Y)$ which are given by the integral*

$$k_T(z) = \int_{(\mathcal{A}_\partial^*)_{p(z)}} \int_{\mathbb{R}} f_{-\xi}(z) t(p(z), \xi', \xi_n)\, d\xi_n\, d\xi' \tag{19}$$

with the Fourier modes

$$f_{-\xi}(z) = e^{i\langle \Psi(z), \xi\rangle} \chi(z)$$

and a fixed cutoff function $\chi \in C_c^\infty(\mathcal{X})$. Here the symbol t is contained in the Hörmander symbol class $S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$.

Proof. The representation follows from Hörmander [13], section 18.2 where the distributional kernel $k_T \in I_c^m(\mathcal{X}, \Delta_Y)$ takes the form

$$k_T(z) = \int_{(\mathcal{N}^{\mathcal{X}}\Delta_Y)^*_{p(z)}} \chi(z) e^{i\langle \Psi(z), \xi\rangle} t(p(z), \xi)\, d\xi$$

for a symbol t contained in $S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$.
From 7.1 we have an isomorphism $\mathcal{N}^{\mathcal{X}}\Delta_Y \cong \mathcal{A}_\partial \oplus \mathcal{N}$. Additionally, for a given $z \in \mathcal{X}$ we can trivialize the normal bundle $\mathcal{N}_{p(z)} \cong \mathbb{R}$. This yields the desired form (19). $\square$

Remark 6.3. With the above form of distributional kernels a trace operator is therefore with $u \in C_c^\infty(\mathcal{X})$ written

$$\begin{aligned} (Tu)(z) &= \int_{\mathcal{G}_{q(z)}} k_T(z \cdot \gamma^{-1}) u(\gamma)\, d\lambda_{q(z)}(\gamma) \\ &= \int_{\mathcal{G}_{q(z)}} \int_{(\mathcal{A}_\partial^*)_{p(z)}} \int_{\mathbb{R}} \chi(z\gamma^{-1}) e^{i\langle \Psi(z\gamma^{-1}), \xi\rangle} t(p(z), \xi) u(\gamma)\, d\xi\, d\lambda_{q(z)}(\gamma). \end{aligned}$$

Using some more notation from section 7 we can write the potential operators in this form as well.

For the potential operators we have for an open neighborhood $\Delta_Y \subset U \subset \mathcal{X}^t$ the analogous commuting square

$$\begin{tikzcd} U \arrow[r, "\Psi^t"] \arrow[d, "s_\partial = p^t"'] & \mathcal{N}^{\mathcal{X}^t}\Delta_Y \arrow[d] \\ \Delta_Y \arrow[r] & \Delta_Y. \end{tikzcd}$$

And the representation of distributional kernels is given by

$$k_K(z) = \int_{(\mathcal{N}^{\mathcal{X}^t}\Delta_Y)^*_{q^t(z)}} e^{i\langle \Psi^t(z), \xi\rangle} \chi^t(z) k(q^t(z), \xi)\, d\xi.$$

Example 6.4. We return to the special case of a compact manifold X with boundary $Y = \partial X$ as well as the double $M = 2X$. Then we note that the equivariant pseudodifferential operators with regard to the pair groupoid $Y \times Y$ identify as follows

$$\Psi^*(Y \times M)^{M\times M} \cong \Psi^*(Y) \cong \Psi^*(Y \times Y)^{Y\times Y}.$$

This follows because the equivariance implies that the families parametrized over $x \in M$ on the left and over $y \in Y$ on the right are simply constant families.
By use of the previous representation of extended operators we can calculate the composition of a pseudodifferential operators on the boundary with a trace operator using only the groupoid action. Hence fix the distributional kernels $k_T \in I_c^{m_1}(Y \times M, \Delta_Y)$ and $k_S \in I_c^{m_2}(Y \times Y, \Delta_Y)$. Since the definition is invariant under changes of coordinates we loosely identify $M = Y \times \mathbb{R}$ in the following calculation[4].
Write the kernel of a trace operator for $w = (w', w_n) \in Y \times \mathbb{R}$

$$k_T(z', w) = \int_{T^*_{x'}Y} \int_{N^*_{x'}Y} e^{i(z'-w')\theta' + iw_n\theta_n} t(z', \theta', \theta_n)\, d\theta_n\, d\theta'.$$

We trivialize the normal bundle $N_{x'}Y \cong \mathbb{R}$.
Consider the following definition

$$k_T(z', w) = \int_{\mathbb{R}} k_{\tilde{S}}(z', w'; \theta_n) e^{iw_n\theta_n}\, d\theta_n. \tag{20}$$

This involves an interchange of integration which we justify by the rapid decay property along the normal direction (compare appendix A). Here $k_{\tilde{S}}(-,-;\theta_n)$ is given for a fixed $\theta_n \in \mathbb{R}$ by

$$k_{\tilde{S}}(z', w'; \theta_n) = \int_{T^*_{z'}Y} e^{i(z'-w')\theta'} t(z', \theta', \theta_n)\, d\theta'. \tag{21}$$

We keep in mind that $I_c^*(Y \times Y, \Delta_Y)$ is a filtered algebra with regard to convolution, and use the action of $Y \times Y$ on $Y \times M$ to calculate for $z' = (z', w)$ with fixed cutoff $\chi \in C_c^\infty(Y \times Y)$, $\tilde{\chi} \in C_c^\infty(Y \times M)$

$$\begin{aligned}(k_S * k_T)(z', w) &= \int_Y \int_{T^*_{z'}Y} \int_{T^*_{z'}Y} \int_{\mathbb{R}} \chi(z', y')\tilde{\chi}(y', w) e^{i(z'-w')\theta' + iw_n\theta_n} e^{i(y'-z')\xi'} t(y', \theta', \theta_n) b(y', \xi')\, d\theta_n\, d\theta'\, d\xi'\, dy' \\ &= \int_{\mathbb{R}} \int_Y \int_{T^*_{z'}Y} \int_{T^*_{z'}Y} \chi(z', y')\tilde{\chi}(y', w) e^{i(z'-w')\theta' + iw_n\theta_n} e^{i(y'-z')\xi'} t(y', \theta', \theta_n) b(y', \xi')\, d\theta'\, d\xi'\, dy'\, d\theta_n \\ &= \int_{\mathbb{R}} [k_S * k_{\tilde{S}}(-,-;\theta_n)](z', w') e^{i\theta_n w_n}\, d\theta_n.\end{aligned}$$

For θ_n fixed we see from this convolution that $k_S * k_{\tilde{S}}(-,-,\theta_n) \in I_c^{m_1+m_2}(Y \times Y, \Delta_Y)$. This example is a reflection of the fact that a trace operator acts in the tangential direction like

[4] In the global context this is the total space of the normal fibration from the tubular neighborhood theorem.

a pseudodifferential operator on the boundary. With the additional rapid decay in the normal direction we can therefore reduce the composition $S \cdot T$ to the form of a composition of pseudodifferential operators on the boundary.

7. Representation

Let us denote by $\mathcal{B}_{prop}^{m,0}(M_0, Y_0)$ the properly supported extended Boutet de Monvel operators of order m, defined on the interior. In this section we introduce an algebra of extended operators $\mathcal{B}_{2\mathcal{V}}^{0,0}(M, Y)$ on the double Lie manifold M. This is defined by extending the distributional kernels in $\mathcal{B}_{prop}^{0,0}(M_0, Y_0)$ to take the Lie structure into account. Then we show that this class of operators are in a certain sense the representations of operators in the groupoid calculus adapted to the boundary structure from 4.2. We also fix the actions with corresponding notation from 5.1.
Next we fix a small tubular neighborhood $Y \subset \mathcal{U} \subset M$ and partition the manifold accordingly

$$M = (X_+ \setminus \mathcal{U}) \cup \mathcal{U} \cup (X_- \setminus \mathcal{U}). \tag{22}$$

Introduce the singular normal bundles for the inclusions $\Delta_Y \hookrightarrow \mathcal{X},\ \Delta_Y \hookrightarrow \mathcal{X}^t$ as well as $\Delta_Y \hookrightarrow \mathcal{G}$.
The singular normal bundles are denoted by

$$\mathcal{N}^{\mathcal{X}}\Delta_Y \to Y,\ \mathcal{N}^{\mathcal{X}^t}\Delta_Y \to Y,\ \mathcal{N}^{\mathcal{G}}\Delta_Y \to Y$$

respectively.
Restricted to the interior we have by axiom *iii)* in Def. 4.7 the isomorphisms

$$\mathcal{N}^{\mathcal{X}}\Delta_Y|_{Y_0} \cong N^{Y_0\times M_0}\Delta_{Y_0},\ \mathcal{N}^{\mathcal{X}^t}\Delta_Y|_{Y_0} \cong N^{M_0\times Y_0}\Delta_{Y_0},\ \mathcal{N}^{\mathcal{G}}\Delta_Y|_{Y_0} \cong N^{M_0\times M_0}\Delta_{Y_0}. \tag{23}$$

Here we denote by $N^{Y_0\times M_0}\Delta_{Y_0}$ the normal bundle to the inclusion $\Delta_{Y_0} \hookrightarrow Y_0 \times M_0$ and the same for the others. Recall the identifications from A of the normal bundles, see also [12], p.227. We can identify the normal bundles on the interior as subsets of $T^*(Y_0 \times M_0)$, $T^*(M_0 \times Y_0)$ and $T^*(M_0 \times M_0)$ respectively as follows.
Let $j \colon T^*(Y_0 \times M_0)_{|\Delta_{Y_0}} \hookrightarrow T^*\Delta_{Y_0}$ be the adjoint to the injection $T\Delta_{Y_0} \subset T(Y_0 \times M_0)_{|\Delta_{Y_0}}$. Denote by by $\mathrm{id} \times (-j)$ the mapping $\{\xi, \eta\} \mapsto \{\xi, -j(\eta)\}$ then we write

$$N^{Y_0\times M_0}\Delta_{Y_0} \cong (\mathrm{id} \times (-j))^{-1}\mathrm{diag}(T^*\Delta_{Y_0} \times T^*\Delta_{Y_0}) \subset T^*(Y_0 \times M_0).$$

Analogously:

$$N^{M_0\times Y_0}\Delta_{Y_0} \cong (j \times (-\,\mathrm{id}))^{-1}\mathrm{diag}(T^*\Delta_{Y_0} \times T^*\Delta_{Y_0}) \subset T^*(M_0 \times Y_0),$$
$$N^{M_0\times M_0}\Delta_{Y_0} \cong (j \times (-j))^{-1}\mathrm{diag}(T^*\Delta_{Y_0} \times T^*\Delta_{Y_0}) \subset T^*(M_0 \times M_0).$$

It is not hard to see that $\mathcal{N}^{\mathcal{X}}\Delta_Y$ can be identified with $\mathcal{A}_{|Y}$ which is isomorphic to $A_\partial \oplus \mathcal{N}$. Hence we can summarize.

Proposition 7.1. *There are (non-canonical) isomorphisms*

$$\mathcal{N}^{\mathcal{X}}\Delta_Y \cong \mathcal{A}_\partial \times \mathcal{N},\ \mathcal{N}^{\mathcal{X}^t}\Delta_Y \cong \mathcal{N} \times \mathcal{A}_\partial \text{ and } \mathcal{N}^{\mathcal{G}}\Delta_Y \cong \mathcal{A}_{|Y} \times \mathcal{N}.$$

Remark 7.2. *i)* On the singular normal bundles we define the Hörmander symbols spaces $S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*) \subset C^\infty(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$ such that for $U \subset Y$ open with

$$\mathcal{N}^{\mathcal{X}}\Delta_Y^*|_U \cong U \times \mathbb{R}^{n-1} \times \mathbb{R},\ K \subset U \text{ compact}.$$

We have the estimates for $t \in S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$

$$|D_{x'}^\alpha D_{\xi',\xi_n}^\beta t(x', \xi)| \leq C_{K,\alpha,\beta}\langle\xi\rangle^{m-|\beta|},\ (x, \xi) \in K \times \mathbb{R}^{n-1} \times \mathbb{R}$$

for each $\alpha \in \mathbb{N}_0^{n-1},\ \beta \in \mathbb{N}_0^n$.

Note that we have by Hörmander's results a correspondence between the spaces of symbols on the normal bundle to a smooth manifold and conormal distributions on the space (at least in the smooth case, cf. [13], Thm 18.2.11):

$$I^L(\mathcal{X}, \Delta_Y)/I^{-\infty}(\mathcal{X}, \Delta_Y) \cong S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)/S^{-\infty}(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$$

where L is the obligatory correction of order

$$m = L - \frac{1}{4}\dim \mathcal{X} + \frac{1}{2}\dim \Delta_Y.$$

We will ignore this order convention in the following discussions. Note that our earlier definition of smooth families of operators defined as conormal distributions suggests immediately a quantization which we state next.
We can require additionally the (local) rapid decay property stated earlier and in the appendix, then we use the notation $S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$ for these symbol spaces.
And analogously the spaces

$$S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}^t}\Delta_Y^*) \subset C^\infty(\mathcal{N}^{\mathcal{X}^t}\Delta_Y^*),\ S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{G}}\Delta_Y^*) \subset C^\infty(\mathcal{N}^{\mathcal{G}}\Delta_Y^*).$$

ii) A second definition we will need is that of conormal distributions on the normal bundles themselves. First given the normal and conormal bundles

$$\pi \colon \mathcal{N}^{\mathcal{X}}\Delta_Y \to Y,\ \overline{\pi} \colon \mathcal{N}^{\mathcal{X}^t}\Delta_Y^* \to Y,$$

(and analogously for $\mathcal{N}^{\mathcal{X}^t}$, $\mathcal{N}^{\mathcal{G}}$) define the *fiberwise Fourier transform* $\mathcal{F}_{\mathrm{f}} \colon S(\mathcal{N}^{\mathcal{X}}\Delta_Y) \to S(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$

$$\mathcal{F}_{\mathrm{f}}(\varphi)(\xi) := \int_{\overline{\pi}(\zeta)=\pi(\xi)} e^{-i\langle \xi, \zeta\rangle}\varphi(\zeta)\, d\zeta.$$

The inverse is given by duality

$$\mathcal{F}_{\mathrm{f}}^{-1}(\varphi)(\zeta) = \int_{\overline{\pi}(\zeta)=\pi(\xi)} e^{i\langle \xi, \zeta\rangle}\varphi(\xi)\, d\xi,\ \varphi \in S(\mathcal{N}^{\mathcal{X}}\Delta_Y^*).$$

Here we use the notation $S(\mathcal{N}^{\mathcal{X}}\Delta_Y),\ S(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$ for the spaces of rapidly decreasing functions on the normal and conormal bundle respectively, see also [29], chapter 1.5.
Then the spaces of conormal distributions are defined as:

$$I^m(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y) := \mathcal{F}_{\mathrm{f}}^{-1} S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$$

and $I^m(\mathcal{N}^{\mathcal{X}^t}\Delta_Y, Y),\ I^m(\mathcal{N}^{\mathcal{G}}\Delta_Y, Y)$ analogously.

As a final preparation we fix the notation for the next type of Fourier and partial Fourier transform we need to consider. Fix the projections $\pi_0 \colon TY_0 \to Y_0,\ \overline{\pi}_0 \colon T^*Y_0 \to Y_0$. First given a cutoff function $\chi \in C_c^\infty(\mathcal{A}_\partial)$ with a function $u \in C_c^\infty(M_0)$ then we set

$$u^\chi(v', x_n) := \chi(v')u(\exp^\partial_{x'}(v'), x_n).$$

Above $\exp^\partial$ denotes the exponential on $\mathcal{A}_\partial$ (by restriction on TY_0). Then the (partial) Fourier transform is defined by

$$(\mathcal{F}_{\xi' \to v'} u)^\chi(\xi', x_n) = \int_{T^*_{\overline{\pi}_0(\xi')}Y_0} e^{-i\langle v', \xi'\rangle} u^\chi(v', x_n)\, dv'.$$

For the definition of the quantization rule we need some further notation. Let $0 < r \leq r_0$ where r_0 is the (positive) injectivity radius of M.

- First for the case of trace operators. We set

  $$(\mathcal{N}^{\mathcal{X}}\Delta_Y)_r = \{v \in \mathcal{N}^{\mathcal{X}}\Delta_Y : \|v\| < r\}$$

  as well as

  $$I^m_{(r)}(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y) = I^m((\mathcal{N}^{\mathcal{X}}\Delta_Y)_r, \Delta_Y).$$

Fix the restriction

$$\mathcal{R}\colon I^m_{(r)}(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y) \to I^m_{(r)}(N^{Y_0\times M_0}\Delta_{Y_0}, \Delta_{Y_0}).$$

We denote by Ψ the normal fibration of the inclusion $\Delta_{Y_0} \hookrightarrow Y_0 \times M_0$ such that Ψ is the local diffeomorphism mapping an open neighborhood of the zero section $O_{Y_0} \subset V \subset N^{Y_0\times M_0}\Delta_{Y_0}$ onto an open neighborhood $\Delta_{Y_0} \subset U \subset Y_0 \times M_0$ (cf. [29], Thm. 4.1.1). Then we have the induced map on conormal distributions

$$\Psi_*\colon I^m_{(r)}(N^{Y_0\times M_0}\Delta_{Y_0}, \Delta_{Y_0}) \to I^m(Y_0 \times M_0, \Delta_{Y_0}).$$

Also let $\chi \in C^\infty_c(\mathcal{N}^{\mathcal{X}}\Delta_Y)$ be a cutoff function which acts by multiplication

$$I^m(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y) \to I^m_{(r)}(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y).$$

- For potential operators we use the analogous notation:

$$\mathcal{R}^t,\ \Psi^t,\ \mathcal{F}^t_f,\ \chi^t.$$

- Finally, in the singular Green case we have the induced normal fibration

$$\Phi_*\colon I^m_{(r)}(N^{M_0\times M_0}\Delta_{Y_0}, \Delta_{Y_0}) \to I^m(M_0 \times M_0, \Delta_{Y_0}).$$

and the fiberwise Fourier transform

$$\mathcal{F}^{\mathcal{G}}_f\colon I^m(\mathcal{N}^{\mathcal{G}}\Delta_Y, \Delta_Y) \to S^m(\mathcal{N}^{\mathcal{G}}\Delta_Y^*).$$

The restriction and cutoff is denoted by

$$\mathcal{R}^{\mathcal{G}}\colon I^m_{(r)}(\mathcal{N}^{\mathcal{G}}\Delta_Y, \Delta_Y) \to I^m_{(r)}(N^{M_0\times M_0}\Delta_{Y_0}, \Delta_{Y_0})$$

and

$$\varphi\colon I^m(\mathcal{N}^{\mathcal{G}}\Delta_Y, \Delta_Y) \to I^m_{(r)}(\mathcal{N}^{\mathcal{G}}\Delta_Y, \Delta_Y).$$

Definition 7.3 (Quantization)**.** *i)* Define

$$q_{T,\chi}\colon S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*) \to \mathscr{T}^{m,0}(M, Y)$$

such that for $t \in S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$ we have

$$q_{T,\chi}(t) = \mathcal{J}_{tr} \circ q_{\Psi,\chi}(t)$$

where

$$q_{\Psi,\chi}(t) = \Psi_*(\mathcal{R}(\chi\mathcal{F}_{\mathrm{f}}^{-1}(t))).$$

ii) Define

$$q_{K,\chi^t}\colon S^m(\mathcal{N}^{\mathcal{X}^t}\Delta_Y^*) \to \mathcal{K}^m(M, Y)$$

such that for $k \in S^m(\mathcal{N}^{\mathcal{X}^t}\Delta_Y^*)$ we have

$$q_{k,\chi^t}(k) = \mathcal{J}_{pot} \circ q_{\Psi^t,\chi^t}(k).$$

iii) Define

$$q_{G,\varphi}\colon S^m(\mathcal{N}^{\mathcal{G}}\Delta_Y^*) \to \mathcal{G}^{m,0}(M, Y)$$

such that for $g \in S^m(\mathcal{N}^{\mathcal{G}}\Delta_Y^*)$ we have

$$q_{G,\varphi}(g) = \mathcal{J}_{pot} \circ q_{\Phi,\varphi}(g).$$

Proposition 7.4. *The fibrations $q_{\Psi,\chi}$, q_{Ψ^t,χ^t} and $q_{\Phi,\varphi}$ define properly supported Schwartz kernels.*

Proof. Consider exemplarily the trace operators. Since $\chi\mathcal{R}(t)$ is properly supported we find that $q_{T,\chi}(t)$ defines a properly supported operator. It is clear from the definition that $q_{T,\chi}(t)\colon C^\infty_c(M_0) \to C^\infty_c(Y_0)$ has the Schwartz kernel $q_{\Psi,\chi}(t)$. $\square$

Instead of the cutoff function $\chi \in C_c^\infty(\mathcal{N}^{\mathcal{X}}\Delta_Y)$ we may choose a cutoff function $\chi \in C_c^\infty(\mathcal{A}_\partial)$ and require rapid decay on part of the normal bundle. It is immediate to see that these conditions are interchangeable. The quantization rule for the calculus is described close to the boundary in the tubular neighborhood $\mathcal{U}$ as follows.
i) Let $t \in S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}}\Delta_Y^*)$ and $\chi \in C_c^\infty(\mathcal{A}_\partial)$ a cutoff function. Then the quantization for extended trace operators is given as follows

$$q_{T,\chi}(t)u(x') = \int_{T^*_{x'}Y_0} \int_{N_{x'}Y_0} \int_{N^*_{x'}Y_0} e^{i\langle x_n, \xi_n\rangle} t(x', \xi', \xi_n)(\mathcal{F}_{v'\to\xi'}u)^\chi(\xi', x_n)\, d\xi_n\, dx_n\, d\xi'.$$

ii) Let $k \in S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}^t}\Delta_Y^*)$ and $\chi \in C_c^\infty(\mathcal{A}_\partial)$ a cutoff function. Then quantization for extended potential operators is given as follows

$$q_{K,\chi}(k)u(x', x_n) = \int_{T^*_{x'}Y_0} \int_{N^*_{x'}Y_0} e^{i\langle x_n, \xi_n\rangle} k(x', \xi', \xi_n)(\mathcal{F}_{v'\to\xi'}u)^\chi(\xi')\, d\xi_n\, d\xi'.$$

iii) Let $g \in S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{G}}\Delta_Y^*)$ and $\chi \in C_c^\infty(\mathcal{A}_\partial)$ a cutoff function. The quantization for extended singular Green operators is then denoted by

$$q_{G,\chi}(g)u(x) = \int_{T^*_{x'}Y_0} \int_{N^*_{x'}Y_0} \int_{N^*_{x'}Y_0} e^{i\langle x_n, \xi_n\rangle} g(x', \xi_n, \eta_n, \xi')\chi(x', \tau(x', y'))(\mathcal{F}_{v'\to\xi'}u)^\chi(\xi', x_n)\, d\xi_n\, d\eta_n\, d\xi'.$$

Remark 7.5. *i)* We remark that a Boutet de Monvel calculus consisting of the classes

$$\Psi^m(M),\ \mathscr{T}^{m,0}(M,Y),\ \mathcal{K}^m(M,Y),\ \mathcal{G}^{m,0}(M,Y)$$

will not form an algebra. This fails already in the case of only pseudodifferential operators. The basic reason is that two different Lie structures can yield the same metric on M_0 (cf. [3]). Hence we need to introduce special smoothing terms to obtain an algebra.
ii) The above quantization can be rewritten as follows. First trivialize the normal and conormal bundles $N_{x'}Y_0 \cong \mathbb{R},\ N^*_{x'}Y_0 \cong \mathbb{R}$. Then a substitution yields for a fixed $\chi \in C_c^\infty(\mathcal{A}_\partial)$

$$\begin{aligned} q_{T,\chi}u(x') &= \int_{T^*_{x'}Y_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{Y_0} e^{i\langle \tau(x',y'), \xi'\rangle + iy_n\xi_n} u(y', y_n)\chi(\tau(x', y'), x')t(x', \xi', \xi_n)\, dy'\, dy_n\, d\xi_n\, d\xi' \\ &= \int_{Y_0\times\mathbb{R}} K_\chi(x', y)u(y)\, dy \ = \int_{\mathcal{U}_0} K_\chi(x', y)u(y)\, dy \end{aligned}$$

where $K_\chi(x', y) := K_{q_{\chi,T}}(x', y)$ is the integral kernel on $Y_0 \times \mathcal{U}_0$. Hence this kernel has the form for $(x', y) \in Y_0 \times \mathcal{U}_0$

$$K_\chi(x', y) = \int_{T^*_{x'}Y_0} \int_{\mathbb{R}} e^{i\langle \tau(x',y), \xi'\rangle + iy_n\xi_n} \chi(\tau(x', y'), x')t(x', \xi', \xi_n)\, d\xi_n\, d\xi'.$$

iii) Let D_n be the differentiation in the normal direction, i.e. $D_n = \partial$ for some fixed $\partial \in \mathrm{Diff}^1_{\mathcal{V}}(M)$ with support in our fixed tubular neighboorhood, which is differentiation in the normal direction close to Y. Then note the formal similarity between our quantization and the quantization of boundary pseudodifferential operators:

$$\begin{aligned} q_{T,\chi}(t) &= q_{\partial,\chi}(t)(-,-,D_n), \\ q_{K,\chi}(k) &= q_{\partial,\chi}(k)(-,-,D_n), \\ q_{G,\chi}(g) &= q_{\partial,\chi}(g)(-,-,D_n). \end{aligned}$$

In fact this can also be shown for the restricted calculus, but we will not do so here.

Proposition 7.6. *The quantizations $q_{T,\chi},\ q_{K,\chi},\ q_{G,\chi}$ are in each case independent of the choice of cutoff functions up to smoothing errors.*

Proof. We show this exemplarily for the trace operators. Let $t \in S^m_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}}\Delta^*_Y)$ and $\chi,\ \tilde{\chi} \in C^\infty_c(\mathcal{A}_\partial)$. Then the kernel of $q_\chi(t) - q_{\tilde{\chi}}(t)$ can be written:

$$K(v', x_n) = \int_{T^*_{\pi_0(v)}Y_0} \int_{N^*_{\pi_0(v)}Y_0} (\chi(v') - \tilde{\chi}(v')) e^{-i\langle \xi', v'\rangle + i\langle \xi_n, x_n\rangle} t(x', \xi)\, d\xi'\, d\xi_n.$$

Since the behaviour in the normal direction is taken care of by the rapid decay condition we only need to regularize the integral in the cotangent direction. Note that the phase function $T^*Y_0 \ni \xi' \mapsto \xi'(v')$ has only critical points for $v' = 0$. For $v' \neq 0$ we can therefore find a vector field L such that $Le^{i\langle v', \cdot\rangle} = e^{i\langle v', \cdot\rangle}$.
Hence write for each k such that $m + k < -n + 1$

$$K(v', x_n) = \int e^{-i\langle \xi', v'\rangle + i\xi_n x_n} (L^t)^k t(x', \xi)\, d\xi$$

This shows $q_{\chi,T} - q_{\tilde{\chi},T}$ is smoothing. $\square$

From the compactness of M and Y we can associate to each vector field in $2\mathcal{V}$ respectively $\mathcal{W}$ a *global flow*

$$2\mathcal{V} \ni V \mapsto \Phi_V \colon \mathbb{R} \times M \to M,$$
$$\mathcal{W} \ni W \mapsto \Psi_W \colon \mathbb{R} \times Y \to Y.$$

Then consider the diffeomorphisms evaluated at time $t = 1$

$$\Phi(1, -)\colon M \to M \text{ and } \Psi(1, -)\colon Y \to Y$$

and fix the corresponding group actions on functions which we denote by

$$2\mathcal{V} \ni V \mapsto \varphi_V \colon C^\infty(M) \to C^\infty(M),$$
$$\mathcal{W} \ni W \mapsto \psi_W \colon C^\infty(Y) \to C^\infty(Y).$$

The upshot of this is a definition of the suitable smoothing terms for our calculus which we state next.

Definition 7.7. *i)* The class of $\mathcal{V}$-trace operators is defined as

$$\mathscr{T}^{m,0}_{2\mathcal{V}}(M, Y) := \mathscr{T}^{m,0}(M, Y) + \mathscr{T}^{-\infty,0}_{2\mathcal{V}}(M, Y).$$

Here $\mathscr{T}^{m,0}(M, Y)$ consists of the extended operators from the previous definition. The residual class is defined as follows

$$\mathscr{T}^{-\infty,0}_{2\mathcal{V}}(M, Y) := \operatorname{span}\{q_{\chi,T}(t)\varphi_{V_1} \cdots \varphi_{V_k} : V_j \in 2\mathcal{V},\ \chi \in C^\infty_c(\mathcal{A}_\partial),\ t \in S^{-\infty}_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}}\Delta^*_Y)\}.$$

ii) The class of $\mathcal{V}$-potential operators is defined in the same fashion

$$\mathcal{K}^m_{2\mathcal{V}}(M, Y) := \mathcal{K}^m(M, Y) + \mathcal{K}^{-\infty}_{2\mathcal{V}}(M, Y)$$

with residual class

$$\mathcal{K}^{-\infty}_{2\mathcal{V}}(M, Y) := \operatorname{span}\{q_{\chi,K}(k)\psi_{W_1} \cdots \psi_{W_k} : W_j \in \mathcal{W},\ \chi \in C^\infty_c(\mathcal{A}_\partial),\ k \in S^{-\infty}_{\mathcal{N}}(\mathcal{N}^{\mathcal{X}^t}\Delta^*_Y)\}.$$

iii) Lastly, the class of $\mathcal{V}$-singular Green operators is defined as

$$\mathcal{G}^{m,0}_{2\mathcal{V}}(M, Y) := \mathcal{G}^{m,0}(M, Y) + \mathcal{G}^{-\infty,0}_{2\mathcal{V}}(M, Y)$$

with residual class

$$\mathcal{G}^{-\infty,0}_{2\mathcal{V}}(M, Y) := \operatorname{span}\{q_{\chi,G}(g)\psi_{V_1} \cdots \psi_{V_k} : V_j \in 2\mathcal{V},\ \chi \in C^\infty_c(\mathcal{A}_\partial),\ g \in S^{-\infty}_{\mathcal{N}}(N^{\mathcal{G}}\Delta^*_Y\}.$$

iv) Then the calculus $\mathcal{B}^{m,0}_{2\mathcal{V}}(M, Y)$ of extended operators consists of matrices of the form

$$A = \begin{pmatrix} P + G & K \\ T & S \end{pmatrix}$$

for $P \in \Psi^m_{2\mathcal{V}}(M), S \in \Psi^m_{\mathcal{W}}(Y)$ and G an extended singular Green operator, K extended potential and T extended trace operator.

This extended calculus is related to our algebra defined in the boundary structure in a very strong sense. In fact in many cases there is a canonical isomorphism which is furnished by the so-called *vector representation*.

Remark 7.8. *i)* Recall the definition of the vector representations ϱ associated to the groupoid $\mathcal{G}$ and ϱ_∂ associated to the groupoid $\mathcal{G}_\partial$. These are homomorphisms of (filtered) algebras (cf. [3])

$$\varrho\colon \mathrm{End}(C_c^\infty(\mathcal{G})) \supset \Psi^m(\mathcal{G}) \to \Psi^m_{2\mathcal{V}}(M) \subset \mathrm{End}(C^\infty(M))$$

and

$$\varrho_\partial\colon \mathrm{End}(C_c^\infty(\mathcal{G}_\partial)) \supset \Psi^m(\mathcal{G}_\partial) \to \Psi^m_{\mathcal{W}}(Y) \subset \mathrm{End}(C_c^\infty(Y)).$$

These can be viewed as suitable extensions of the anchor maps $\mathcal{A} \to TM$ and $\mathcal{A}_\partial \to TY$ (abusing notation by using the same symbols).
The vector representations are uniquely characterized by the equations

$$P(\varphi \circ r) = (\varrho(P)\varphi) \circ r$$

for each $\varphi \in C^\infty(M),\ P \in \Psi^m(\mathcal{G})$.
As well as

$$S(\psi \circ r_\partial) = (\varrho_\partial(S)\psi) \circ r_\partial$$

for each $\psi \in C^\infty(Y)$ and $S \in \Psi^m(\mathcal{G}_\partial)$.
ii) We want to define the following vector representation

$$\varrho_{BM}\colon \mathcal{B}^{m,0}(\mathcal{G},\mathcal{X}) \to \mathcal{B}^{m,0}_{2\mathcal{V}}(M,Y) \subset \mathrm{End}\begin{pmatrix} C^\infty(M) \\ \oplus \\ C^\infty(Y)\end{pmatrix}.$$

which will turn out to be a well-defined surjective map and a homomorphism of algebras for $m = 0$.
The homomorphism ϱ_{BM} is represented as a matrix

$$\varrho_{BM} := \begin{pmatrix} \varrho & \varrho_{pot} \\ \varrho_{tr} & \varrho_\partial \end{pmatrix}.$$

It is uniquely determined by the defining property

$$A\begin{pmatrix} \varphi \circ r \\ \psi \circ r_\partial \end{pmatrix} = \left(\varrho_{BM}(A)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right) \circ \begin{pmatrix} r \\ r_\partial\end{pmatrix}$$

for each $\varphi \in C^\infty(M), \psi \in C^\infty(Y)$ and $A \in \mathcal{B}^{m,0}(\mathcal{G},\mathcal{X})$.
From this we can define linear mappings which represent trace, potential and singular Green operators individually.
So given a trace operator T, a potential operator K and a singular Green operator G (equivariant families) we define

$$\varrho_{tr}(T) := \varrho_{BM}\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

$$\varrho_{pot}(K) := \varrho_{BM}\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}.$$

$$\varrho(G) := \varrho_{BM}\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}.$$

For the pseudodifferential operator on the boundary we simply recover the definition of the representation ϱ_∂.

Theorem 7.9. *Given a $\mathcal{V}$-boundary structure the previously defined vector representation ϱ_{BM} furnishes the isomorphism*

$$\varrho_{BM} \circ \mathcal{B}^{m,0}(\mathcal{G},\mathcal{X}) \cong \mathcal{B}^{m,0}_{2\mathcal{V}}(M,Y).$$

Proof. We have to check two inclusions for each matrix component of the homomorphism ϱ_{BM}. For pseudodifferential operators we can refer to the result in [3], Thm. 3.2. The case of singular Green operators $I_c^m(\mathcal{G}, \Delta_Y)$ follows the same lines of reasoning (in fact, it is an easier case than the trace and potential operators). The case of trace operators $I_c^m(\mathcal{X}, \Delta_Y) = \mathscr{T}^{m,0}(\mathcal{G}, \mathcal{X})$ is equivalent to the case of potential operators $I_c^m(\mathcal{X}^t, \Delta_Y) = \mathcal{K}^m(\mathcal{G}, \mathcal{X})$ by duality 5.4, *ii)* (each component of ϱ_{BM} is adjoint preserving).
Therefore we will establish the assertion for the trace operators. We need to show that (setting $\varrho := \varrho_{tr}$)

$$\varrho \circ I_c^m(\mathcal{X}, \Delta_Y) = \mathscr{T}_{2\mathcal{V}}^{m,0}(M, Y). \tag{24}$$

First we will establish the commutativity of the following diagram:

$$\begin{tikzcd}
I^m(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y) \arrow[r, "\chi"] & I^m_{(r)}(\mathcal{N}^{\mathcal{X}}\Delta_Y, \Delta_Y) \arrow[r, "\mathcal{R}"] \arrow[d, "\alpha_*"'] & I^m_{(r)}(N^{Y_0 \times M_0}\Delta_{Y_0}, \Delta_{Y_0}) \arrow[d, "\Psi_*"'] \\
\mathscr{T}^{m,0}(\mathcal{G}, \mathcal{X}) \arrow[r, tail, two heads] \arrow[d, "e_x"'] & I^m_c(\mathcal{X}, \Delta_Y) \arrow[r, "l_*"] \arrow[d, "\mu^*"'] & I^m(Y_0 \times M_0, \Delta_{Y_0}) \arrow[d, equal] \\
\mathscr{T}^{m,0}_{prop}(\mathcal{G}_x; \mathcal{X}_x)^\Gamma \arrow[r, tail, two heads] & I^m_{prop}(\mathcal{X}_x \times \mathcal{G}_x, \Delta_x)^\Gamma \arrow[r, "\widetilde{(p \otimes r)}_*"] \arrow[d, "\mathcal{J}"'] & I^m(Y_0 \times M_0, \Delta_{Y_0}) \arrow[d, "\mathcal{J}"'] \\
 & \mathscr{T}^{m,0}_{prop}(\mathcal{G}_x; \mathcal{X}_x)^\Gamma \arrow[r] & \mathscr{T}^{m,0}_{prop}(M_0, Y_0)
\end{tikzcd}$$

Each of the maps in this diagram are defined as follows.
Fix $r > 0$ which is stricly smaller than the injectivity radius of the manifold M. Then we have by the tubular neighborhood theorem (see [29], p. 53) an open embedding

$$\alpha \colon (\mathcal{N}^{\mathcal{X}}\Delta_Y)_r \to \mathcal{X}$$

which induces the map on conormal distributions

$$\alpha_* \colon I^m_{(r)}(\mathcal{N}^{\mathcal{X}}\Delta_Y,\ \Delta_Y) \to I^m_c(\mathcal{X}, \Delta_Y).$$

The function χ is a fixed cutoff function and the arrow means a multiplication by this function and $\mathcal{R}$ denotes restriction. Furthermore, we fix a local diffeomorphism Ψ which maps diffeomorphically a neighborhood $O_Y \subset U \subset \mathcal{N}^{\mathcal{X}}\Delta_Y$ of the zero section onto a neighborhood of the diagonal $\Delta_Y \subset V \subset \mathcal{X}$. Restricted to the interior we have by the previous identifications of the normal bundles as subbundles of $T^*(Y_0 \times M_0)$ the definition of the exponential map $\Psi \colon T^*(Y_0 \times M_0)_r \to (Y_0 \times M_0)^2$. Then we recover the previously mentioned normal fibration for the inclusion on the interior $\Delta_{Y_0} \hookrightarrow Y_0 \times M_0$. Hence we then denote by Ψ_* the induced (local) mapping on conormal distributions. The map $\mathcal{F}_{\mathrm{f}}$ denotes the already defined fiberwise Fourier transform, $\mathcal{J}$ is always the suitable isomorphism from the Schwartz kernel theorem, and μ^* is the induced mapping coming from the multiplication $\mu \colon (z, \gamma) \mapsto z \cdot \gamma^{-1}$.

The illucidation of the mappings l_* and $\widetilde{(p \otimes r)}_*$ requires a little more care.
For the remainder of the argument we fix a $x \in M_0$ and set $\Gamma := \mathcal{G}_x^x$ for the isotropy group. On $\mathcal{G}_x$ we have the induced metric g_x from the compatible metric g on $\mathcal{A}$. Hence we consider on $\mathcal{X}_x$ the Riemannian metric induced from g_x on $\mathcal{G}_x$.
We have that $r \colon \mathcal{G}_x \to M_0$ and $p \colon \mathcal{X}_x \to Y_0$ are surjective submersions. Furthermore, $dp \colon T\mathcal{X}_x \to TY_0$ and $dr \colon T\mathcal{G}_x \to TM_0$ are isometries with regard to the metric g_x. It follows that r, p are local diffeomorphisms. Additionally, the discrete group Γ acts freely on $\mathcal{G}_x$ from both sides and on $\mathcal{X}_x$ from the right. Hence we consider the right action of Γ on $\mathcal{X}_x \times \mathcal{G}_x$. We have that $\mathcal{G}_x/\Gamma \cong M_0$ and $\mathcal{X}_x/\Gamma \cong Y_0$. Thus r, p are each covering maps with covering group Γ. Consider the Γ-invariant functions $C^\infty(\mathcal{G}_x)^\Gamma \cong C^\infty(M_0)$ and $C^\infty(\mathcal{X}_x)^\Gamma \cong C^\infty(Y_0)$. For a given

$\varphi \in C_c^\infty(M_0)$ and $\psi \in C_c^\infty(Y_0)$ we have that $\varphi \circ r$ is a Γ-invariant function on $\mathcal{G}_x$ and $\psi \circ p$ is a Γ-invariant function on $\mathcal{X}_x$. Hence for a given $\varphi \in C_c^\infty(M_0)$ the function $T(\varphi \circ r)$ is defined since T is properly supported and there is a $\psi \in C_c^\infty(Y_0)$ such that

$$T(\varphi \circ r) = \psi \circ p.$$

The operator $(p \otimes r)_*$ is then given by $(p \otimes r)_*(T)\varphi = \psi$. We can therefore rewrite the vector representation $\varrho = \varrho_{tr}$ as

$$\varrho(T) = (p \otimes r)_*(e_x(T))$$

and by the commutativity of the left-most rectangle obtain

$$\varrho \circ \mathcal{J} = (p \otimes r)_* \circ e_x \circ \mathcal{J} = (p \otimes r)_* \circ \mathcal{J} \circ \mu^*.$$

Thus the operator $\widetilde{(p \otimes r)}_*$ is completely determined by the commutativity of the bottom rectangle. The map l_* in the diagram is defined by

$$l_* := \widetilde{(p \otimes r)}_* \circ \mu^*. \tag{25}$$

It remains to prove that

$$l_* \circ \alpha_* = \Psi_* \circ \mathcal{R}. \tag{26}$$

For this we need an alternative description of l_* which is given as follows. The space $I_{pr}^m(\mathcal{X}_x \times \mathcal{G}_x, \Delta_x)^\Gamma$ denotes the space of Γ-invariant conormal distributions on $\mathcal{X}_x \times \mathcal{G}_x$. Since $\mathcal{G}_x$ is a covering of M_0 and $\mathcal{X}_x$ is a covering of Y_0 with group Γ we have the identifications

$$I_{pr}^m(\mathcal{X}_x \times \mathcal{G}_x, \Delta_x)^\Gamma \cong I_{pr}^m((\mathcal{X}_x \times \mathcal{G}_x)^\Gamma, \Delta_x^\Gamma).$$

Denote by τ the map $\mathcal{X}_x \times \mathcal{G}_x/\Gamma \to Y_0 \times M_0$ which is then also a covering map. This makes it possible to identify a distribution with small support in $\mathcal{X}_x \times \mathcal{G}_x/\Gamma$ with a distribution with support in a small subset of $Y_0 \times M_0$. The identifications extend by summation along the fibers of $\tau \colon \mathcal{X}_x \times \mathcal{G}_x/\Gamma \to Y_0 \times M_0$. To any distribution u on $(\mathcal{X}_x \times \mathcal{G}_x)/\Gamma$ whose support intersects only finitely many components of $\tau^{-1}(U)$ for any locally trivializing open set $U \subset Y_0 \times M_0$ we associate a distribution $\tau_*(u) \in \mathcal{D}'(Y_0 \times M_0) = C_c^\infty(Y_0 \times M_0)'$. The morphism $\widetilde{(p \otimes r)}_*$ identifies then with τ_*. We observe that

$$\tau(z, \gamma) = (p(z), r(\gamma)) = (p \circ \mu(z, \gamma), s \circ \mu(z, \gamma)). \tag{27}$$

Then define $\tilde{l}_*$ as follows: restrict a distribution $u \in \mathcal{D}'(\mathcal{X}_x \times \mathcal{G}_x/\Gamma)$ to $W_0 := p^{-1}(Y_0) \times r^{-1}(M_0) \subset \mathcal{X}_x \times \mathcal{G}_x$ and apply the pushforward given by $(p, r) \colon p^{-1}(Y_0) \times r^{-1}(M_0) \to Y_0 \times M_0$. Since $r^{-1}(M_0) = s^{-1}(M_0)$ by assumption A on p. 15 it follows

$$\tilde{l}_* = \tau_* \circ \mu^* = \widetilde{(p \otimes r)}_* \circ \mu^*. \tag{28}$$

Hence $\tilde{l}_* = l_*$ and we have an alternative description of l_*.
This establishes (26) since (p, r) is injective on $\alpha((\mathcal{N}^{\mathcal{X}} \Delta_Y)_r)$ and $r \colon \mathcal{G}_x \to M_0$, $p \colon \mathcal{X}_x \to Y_0$ are isometric coverings which preserve the exponential maps.
The equality stated in (24) which we left open can now be established via the commutativity of the diagram.
Consider the first inclusion $\mathscr{T}_{2\mathcal{V}}^{m,0}(M, Y) \subset \varrho(\mathscr{T}^{m,0}(\mathcal{G}, \mathcal{X}))$. We have by commutativity

$$\begin{aligned} q_{\chi,T}(t) &= \mathcal{J} \circ \Psi_* \circ \mathcal{R} \circ \chi \circ \mathcal{F}_f^{-1}(t) \\ &= \varrho_{tr} \circ \mathcal{J} \circ \alpha_* \circ \chi \circ \mathcal{F}_f^{-1}(t) \\ &= \varrho(\tilde{T}). \end{aligned}$$

Where $\tilde{T} = \mathcal{J} \circ \alpha_* \circ \chi \circ \mathcal{F}_f^{-1}$ and $t \in S^m(\mathcal{N}^{\mathcal{X}} \Delta_Y^*)$. Hence every operator of the form $q_{\chi,T}(t)$ is in the range of ϱ. We have thus shown that $\mathscr{T}^{m,0}(M, Y) \subset \varrho(\mathscr{T}^{m,0}(\mathcal{G}, \mathcal{X}))$.
Now let $V \in 2\mathcal{V}$ be a vector field and lift it to a vector field $\tilde{V}$ on $\mathcal{G}$. By integrating this vector field let $\psi_{\tilde{V}}$ be the family of diffeomorphisms of each $\mathcal{G}_x$.

We obtain

$$\varrho_{tr}(Q\psi_{\tilde{V}_1}\cdots\psi_{\tilde{V}_k}) = q_{\chi,T}(t)\psi_{V_1}\cdots\psi_{V_k} \in \mathscr{T}^{-\infty,0}_{2\mathcal{V}}(M,Y)$$

where

$$Q = \mathcal{J}\circ\alpha_*\circ\chi\circ\mathcal{F}_f^{-1}(t)$$

and $Q\psi_{\tilde{V}_1}\cdots\psi_{\tilde{V}_k} \in \mathscr{T}^{-\infty,0}(\mathcal{G},\mathcal{G}_\partial) \cong I^{-\infty}(\mathcal{X},\Delta_Y)$, since a regularizing distribution multiplied with an operator induced by a diffeomorphism is regularizing.
Hence the first inclusion follows.
Consider the opposite inclusion $\varrho(\mathscr{T}^{m,0}(\mathcal{G},\mathcal{X})) \subset \mathscr{T}^{m,0}_{2\mathcal{V}}(M,Y)$: Let $T \in \mathscr{T}^{m,0}(\mathcal{G},\mathcal{X})$ be given and set $t = \mathcal{J}^{-1}(T)$. Fix a cutoff function χ_0 supported in $\alpha((\mathcal{N}^{\mathcal{X}}\Delta_Y)_r) \subset \mathcal{X}$ and let χ be such that $\chi = 1$ on the support of $\chi_0\circ\alpha$. Setting $t_0 := \chi_0 t$ we write

$$\mathcal{J}(t - t_0) = \sum_{j=1}^{l}\mathcal{J}(t_j)\tilde{\psi}_{V_{j_1}}\cdots\tilde{\psi}_{V_{j_k}}$$

for $t_j \in \chi I^{-\infty}_{(r)}(\mathcal{N}^{\mathcal{X}}\Delta_Y,\Delta_Y),\ V_{jk} \in 2\mathcal{V}$ where

$$t_j = \alpha_*\circ\chi\circ\mathcal{F}_f^{-1}(t_j),\ t_0 \in S^m(\mathcal{N}^{\mathcal{X}}\Delta_Y^*),\ t_j \in S^{-\infty}(\mathcal{N}^{\mathcal{X}}\Delta_Y^*),\ j > 0.$$

Then we obtain

$$\varrho_{tr}(T) = q_{\chi,T}(t_0) + \sum_{j=1}^{l} q_{\chi,T}(t_j)\psi_{V_{j_1}}\cdots\psi_{V_{j_k}} \in \mathscr{T}^{m,0}_{2\mathcal{V}}(M,Y).$$

This concludes the proof. □

8. The restricted calculus

From the extended calculus one can easily define the corresponding restricted Boutet de Monvel operators. For this we introduce the truncation operators on the manifold level and the groupoid level.
The restriction r^+ to the interior $\mathring{X}_0 := X_0 \setminus Y_0$ and the extension by zero operator e^+ are given on the manifold level by

$$L^2(M_0) \underset{e^+}{\overset{r^+}{\rightleftarrows}} L^2(\mathring{X}_0)$$

with $r^+e^+ = \mathrm{id}_{L^2(\mathring{X}_0)}$ and e^+r^+ being a projection onto a subspace of $L^2(M_0)$.
On the groupoid level we use the same symbols since it will be clear from context which is meant. So we define the operators

$$L^2(\mathcal{G}) \underset{e^+}{\overset{r^+}{\rightleftarrows}} L^2(\mathcal{G}^+)$$

with $r^+e^+ = \mathrm{id}_{L^2(\mathcal{G}^+)}$ and e^+r^+ being a projection onto a subspace of $L^2(\mathcal{G})$.
First we are going to introduce an important propery of the pseudodifferential operators on the groupoid $\Psi^*(\mathcal{G})$. Namely we require that each element of the family $P = (P_x)_{x\in M}$ has the *transmission property* with regard to the the boundary $\mathcal{X}_x$.

Definition 8.1. The operator $P \in \Psi^m(\mathcal{G})$ has the transmission property if the symbol $a \in S^m_{tr}(\mathcal{A}^*)$. Here the class of Hörmander symbols $a \in S^m_{tr}(\mathcal{A}^*)$ consists of families $a = (a_x)_{x\in M}$ such that each symbol a_x has the transmission property with regard to $\mathcal{X}_x \subset \mathcal{G}_x$. In particular the operators $(r^+Pe^+)_x$ map functions smooth up to the boundary $\mathcal{X}_x$ to functions which have the same property.

Example 8.2.

- Notice first that if $x \in M_0$ is an interior point we have that $\mathcal{G}_x \cong M_0$ and we recover the transmission property on the interior manifold X_0 with boundary Y_0.

- In our trivial case $\mathcal{G} = M \times M$ and $M = 2X$, X a compact manifold with boundary $\partial X = Y$ we recover the transmission property meaning $\Psi_{tr}^m(M) \cong \Psi_{tr}^m(\mathcal{G})$.

Notation 8.3. The operation of *truncation* itself is given as a linear operator.
i) On the groupoid calculus this operator is given by

$$\mathrm{End}\begin{pmatrix} C_c^\infty(\mathcal{G}) \\ C_c^\infty(\mathcal{X}) \end{pmatrix} \supset \mathcal{B}^{m,0}(\mathcal{G}, \mathcal{X}) \ni A = \begin{pmatrix} P+G & K \\ T & S \end{pmatrix} \mapsto \tilde{\mathcal{C}}(A) = \begin{pmatrix} r^+(P+G)e^+ & r^+K \\ Te^+ & S \end{pmatrix} \in \mathrm{End}\begin{pmatrix} C_c^\infty(\mathcal{G}^+) \\ C_c^\infty(\mathcal{X}) \end{pmatrix}.$$

ii) On the extended calculus we define

$$\mathrm{End}\begin{pmatrix} C_c^\infty(M_0) \\ C_c^\infty(Y_0) \end{pmatrix} \supset \mathcal{B}_{2\mathcal{V}}^{m,0}(M, Y) \ni A = \begin{pmatrix} P+G & K \\ T & S \end{pmatrix} \mapsto \mathcal{C}(A) = \begin{pmatrix} r^+(P+G)e^+ & r^+K \\ Te^+ & S \end{pmatrix} \in \mathrm{End}\begin{pmatrix} C_c^\infty(X_0) \\ C_c^\infty(Y_0) \end{pmatrix}.$$

Now we will give a definition of the *restricted calculus* defined on the boundary structure as well as the *representable operators* on the Lie manifold with boundary (X, Y).

Definition 8.4. *i)* The restricted calculus on the boundary structure is defined as the set of operators for $m \leq 0$

$$\mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{X}) := \tilde{\mathcal{C}} \circ \mathcal{B}^{m,0}(\mathcal{G}, \mathcal{X}).$$

ii) The class of *representable operators* is for $m \leq 0$ defined as

$$\mathcal{B}_{\mathcal{V}}^{m,0}(X, Y) := \mathcal{C} \circ \mathcal{B}_{2\mathcal{V}}^{m,0}(M, Y).$$

The *vector representation* $\tilde{\varrho}_{BM}$ is defined by

$$A \begin{pmatrix} \varphi \circ r \\ \psi \circ r_\partial \end{pmatrix} = \left(\tilde{\varrho}_{BM}(A) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right) \circ \begin{pmatrix} r \\ r_\partial \end{pmatrix}$$

for $A \in \mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{X})$ and $\varphi \in C_c^\infty(X_0), \psi \in C_c^\infty(Y_0)$ such that $\tilde{\varrho}_{BM}$ makes the following diagram (as linear operators) is commutative

$$\begin{tikzcd}
\mathcal{B}_{\mathcal{V}}^{m,0}(X, Y) & \mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{X}) \arrow[l, "\tilde{\varrho}_{BM}"'] \\
\mathcal{B}_{2\mathcal{V}}^{m,0}(M, Y) \arrow[u, two heads, "\mathcal{C}"] & \mathcal{B}^{m,0}(\mathcal{G}, \mathcal{X}). \arrow[l, "\varrho_{BM}"] \arrow[u, two heads, "\tilde{\mathcal{C}}"']
\end{tikzcd}$$

The next result is now a consequence of our previous preparations.

Theorem 8.5. *The restricted operators $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X})$ are closed under composition.*

Proof. This is an immediate consequence of the proof of Lemma 6.1. The truncated operators are *honest* Boutet de Monvel operators on a smooth manifold. The support estimates in the proof of Lemma 6.1 therefore reduce the calculations to the ordinary known case of properly supported operators. □

Theorem 8.6. *The vector representation induces an isomorphism for $m \leq 0$*

$$\tilde{\varrho}_{BM} \circ \mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{X}) \cong \mathcal{B}_{\mathcal{V}}^{m,0}(X, Y).$$

In particular $\mathcal{B}_{\mathcal{V}}^{0,0}(X, Y)$ is closed under composition.

Proof. This follows from the commutativity of the diagram in Definition 8.4 combined with Theorem 7.9. Since $\mathcal{C}, \tilde{\mathcal{C}}$ and ϱ_{BM} are surjective we obtain the surjectivity of $\tilde{\varrho}_{BM}$ as follows. Let $B \in \mathcal{B}_{\mathcal{V}}^{m,0}(X, Y)$ then by surjectivity of ϱ_{BM} and $\mathcal{C}$ we lift this to an element $\tilde{B} \in \mathcal{B}^{m,0}(\mathcal{G}, \mathcal{X})$. Then $A := \tilde{\mathcal{C}}(\tilde{B})$ is the required preimage. By commutativity we have

$$\tilde{\varrho}_{BM}(A) = (\tilde{\varrho}_{BM} \circ \tilde{\mathcal{C}})(\tilde{B}) = (\mathcal{C} \circ \varrho_{BM})(\tilde{B}) = B.$$

Hence in this case ϱ_{BM} is surjective. It is also immediate that it is a well-defined homomorphism of algebras. This yields the closedness under composition. □

Finally, we are going to show that the operators in our algebra are continuous on L^2-spaces. For higher orders this result can be easily generalized to Sobolev spaces defined on the groupoids or Lie manifolds as they were introduced e.g. in [2].

Theorem 8.7. *Let $A \in \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X})$ such that*

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix}$$

where $P \in \Psi^0_{tr}(\mathcal{G})$ a pseudodifferential operator with transmission property, T is a restriced trace operator and K a restricted potential operator. G is a restricted singular Green operator, and $S \in \Psi^0(\mathcal{X})^{\mathcal{G}}$ is a pseudodifferential operator on the boundary.
Then we have the following continuous extensions

$$\begin{aligned} &P_+ \colon L^2(\mathcal{G}^+) \to L^2(\mathcal{G}^+), \\ &T \colon L^2(\mathcal{G}^+) \to L^2(\mathcal{X}), \\ &K \colon L^2(\mathcal{X}) \to L^2(\mathcal{G}^+), \\ &G \colon L^2(\mathcal{G}^+) \to L^2(\mathcal{G}^+). \end{aligned}$$

Proof. i) The continuous extension property of P_+ follows from the continuity on each fiber $(r^+ P e^+)_x$ and the fiberwise transmission property. The continuity of S is also clear.
ii) Consider a trace operator T and denote by $K = T^*$ the L^2-adjoint which is a potential operator (cf. 5.4). By Theorem 8.5 we have $T \cdot K$ is a pseudodifferential operator in $\Psi^0(\mathcal{X})^{\mathcal{G}}$. Since $T \cdot K$ is of order zero $T \cdot K \colon L^2(\mathcal{X}) \to L^2(\mathcal{X})$ continuously (via f).
Hence

$$\langle T^* u, T^* v\rangle_{L^2(\mathcal{G})} = \langle u, TT^* u\rangle \leq C\|u\|$$

and thus $K = T^* \colon L^2(\mathcal{X}) \to L^2(\mathcal{G})$ is continuous. The same way we show continuity of the trace operators.
iii) Since the truncated operator G is a family of honest singular Green operators we know that we can write these operators in the form $K \cdot T$. Where K is a potential operator of order 0 and T a trace operator of order and type 0. Then the continuity of G is a consequence of the continuity of T and K. ☐

Vector bundles
Up until now we have only considered scalar operators of Boutet de Monvel type. It does only require minor modifications to consider operators acting on smooth sections of smooth vector bundles, see also [24]. To this effect let $E_1, E_2 \to X$ be smooth vector bundles on X and $J_\pm \to Y$ smooth vector bundles on Y. We can pull back these bundles to $\mathcal{G}^+$ via $\tilde{E}_i := r^* E_i \to \mathcal{G}^+$. Similarly, the actions allow us to pull back the bundles $J_\pm$ to $\mathcal{X}$ and obtain $\tilde{J}_\pm \to \mathcal{X}$.
It is not difficult to modify our construction for operators acting on the smooth sections such that $A \in B^{0,0}_{\mathcal{V}}(X, Y; E_1, E_2, J_\pm)$ is a continuous linear operator

$$A \colon \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(Y, J_+) \end{matrix} \to \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(Y, J_-) \end{matrix}.$$

Similarly, $A \in \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X}; \tilde{E}_i, \tilde{J}_\pm)$ is a continuous linear operator

$$A \colon \begin{matrix} C_c^\infty(\mathcal{G}^+, \tilde{E}_1) \\ \oplus \\ C_c^\infty(\mathcal{X}, \tilde{J}_+) \end{matrix} \to \begin{matrix} C_c^\infty(\mathcal{G}^+, \tilde{E}_2) \\ \oplus \\ C_c^\infty(\mathcal{X}, \tilde{J}_-) \end{matrix}.$$

9.1. **Guillemin completion.** In this section we make preparations for the parametrix constructions for elliptic elements in our algebra. For this we will define a suitable notion of ellipticity in the next section. At the outset it is required to consider an enlarged groupoid calculus of pseudodifferential operators. We present one approach based on non-canonical completions. This ensures that inverses and parametrices are contained in the calculus. The material of this section is mostly combined from [23] and [4]. We will nevertheless provide some details for the benefit of the reader.
First we make a common assumption on the groupoid.

Assumption C. The groupoid $\mathcal{G} \rightrightarrows M$ is Hausdorff.

This assumption has several simplifying consequences.

Lemma 9.1 ([23], p. 11)**.** *The vector representation ϱ_M yields an isomorphism*

$$\Psi^m(\mathcal{G}) = \Psi^m_{\mathcal{V}}(M).$$

Proof. The injectivity of ϱ_M is a consequence of the Hausdorff condition on $\mathcal{G}$. We consider for a fixed $z \in M_0$ the evaluation morphism $e_z \colon \Psi^m(\mathcal{G}) \to \Psi^m(\mathcal{G}_z)$ given by $P = (P_x)_{x \in M} \mapsto P_z$.
The evaluation morphism $e_z \colon \Psi^m(\mathcal{G}) \to \Psi^m(\mathcal{G}_z)$ is injective for $z \in M_0$. In order to prove this we first establish the *claim*: Let $P = (P_x)_{x \in M} \in \Psi^m(\mathcal{G})$, then for any $\varphi \in C_c^{\infty}(\mathcal{G})$ the map $x \mapsto \|P_x \varphi_x\|$ is continuous.
Proof of claim: Let $\varphi \in C_c^{\infty}(\mathcal{G})$, then there is a $\psi \in C_c^{\infty}(\mathcal{G})$ such that $\psi_x = P_x \varphi_x$. Then $x \mapsto \|\psi_x\|$ by a choice of Haar system (using the smoothness property of Haar systems) and the Hausdorff property of $\mathcal{G}$. This proves the claim.
To see that e_z is injective for a fixed $z \in M_0$ let $P_z = 0$. We need to show that then $P_w = 0$ for each $w \in M$, i.e. $P = 0$. By right invariance and the fact that $\mathcal{G}_{M_0} \cong M_0 \times M_0$ it follows that $P_w = 0$ for each $w \in M_0$. Let $w \in M$ be arbitrary, then we show that $P_w \psi = 0$ for any $\psi \in C_c^{\infty}(\mathcal{G}_w)$. Let $\varphi \in C_c^{\infty}(\mathcal{G})$ be such that $\varphi_w = \psi$ which is possible since $\mathcal{G}_w \subset \mathcal{G}$ is closed in the locally compact Hausdorff space $\mathcal{G}$. Then by the claim $w \mapsto \|P_w \varphi_w\|$ is continuous and on $w \in M_0$ the function vanishes. Since M_0 is dense in M it follows $P_w \varphi_w = 0$ for each $w \in M$. Hence e_z is indeed injective.
Using the canonical diffeomorphism $\mathcal{G}_z \cong M_0$ we obtain a bijection $j \colon \Psi^m(\mathcal{G}_z) \to \Psi^m_{\mathcal{V}}(M)$. The composition of e_z with j coincides with the representation ϱ_M. Since ϱ_M is known to be surjective by the representation theorem (compare Remark 3.14, *i)*) this proves that ϱ_M is an isomorphism. □

Remark 9.2. *i)* Under certain additional conditions on the Lie manifold (considering so-called CM-manifolds) we can in fact show that Assumption C implies that the induced representation $\varrho_M \colon C^*(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}))$ is also injective. Therefore in this case $\mathcal{G}$ is automatically amenable. We refer to [23] for the details.
ii) Note that the Lemma implies also that $\tilde{\varrho}_{BM}$ furnishes an isomorphism $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X}) \cong \mathcal{B}^{0,0}_{\mathcal{V}}(X, Y)$ under this assumption.

Given $\xi \in \mathcal{A}^*$ we set

$$r(\xi) := \langle \xi \rangle := (1 + \|\xi\|^2)^{\frac{1}{2}}$$

such that $r \in S^1_{cl}(\mathcal{A}^*)$.
We define the order reducing operators

$$P_s := \mathrm{op}(r(\xi)^s), \ s > 0$$

and

$$P_{-s} := P_s^{-1}, \ s < 0.$$

For each f in the closure of the domain of the operator P_s define

$$\|f\|_s := \|P_s f\|_2^2,\ s \in \mathbb{R}.$$

This norm gives rise to Sobolev spaces henceforth denoted by H^s. It is equivalent to the Sobolev spaces introduced in [2].
These order reductions are insufficient even though the symbol is clearly elliptic for the simple reason that the parametrix is not contained in the calculus. To get to the actual order reduction we need to do some more work.
The Guillemin completion is now defined as

$$\overline{\Psi_{\mathcal{V}}^m}(M) := \Psi_{\mathcal{V}}^m(M) + \overline{\Psi_{\mathcal{V}}^{-\infty}(M)} \tag{29}$$

where $\Psi_{\mathcal{V}}^{-\infty}$ is completed with regard to the seminorms

$$\|\cdot\|_n := \|\cdot\|_{\mathcal{L}(H^{-n},H^n)}.$$

Secondly, we set

$$\Psi_{\mathbb{C}}^{-\infty} := \overline{\Psi_{\mathcal{V}}^{-\infty}} + \mathbb{C}I. \tag{30}$$

Let $\Omega \subset \mathbb{C}$ be a domain and denote by $\mathcal{O}(\Omega, E)$ the algebra of holomorphic functions on Ω with values in a Fréchet space E. From [4] we cite the following Lemma on holomorphic families.

Lemma 9.3. i) *Let $A \in \mathcal{O}(\Omega, \overline{\Psi_{\mathcal{V}}^\mu})$ a holomorphic family of elliptic operators. Then there is a $B \in \mathcal{O}(\Omega, \overline{\Psi_{\mathcal{V}}^{-\mu}})$ such that*

$$AB - I,\ BA - I \in \mathcal{O}(\Omega, \overline{\Psi_{\mathcal{V}}^{-\infty}}).$$

ii) *There is a $b \in \mathcal{O}(\mathbb{C}, S_{cl}^0)$ with $b(0) = 1$ and*

$$q(r^z)q(r^{-z}b(z)) - I \in \overline{\Psi_{\mathcal{V}}^{-\infty}},$$
$$q(r^{-z}b(z))q(r^z) - I \in \overline{\Psi_{\mathcal{V}}^{-\infty}}.$$

Proof. i) The construction of B via asymptotic expansion is standard if we have shown that holomorphic families are compatible with asymptotic expansions in the sense of the proof of Prop. 4.1 in [4].
ii) Apply *i)* to the family $A(z) = q(r^z)q(r^{-z})$. □

This puts us in a position to show that parametrices are contained in the Guillemin completion and construct elliptic invertible elements (order reductions). Additionally, the following theorem contains the basic algebraic properties: (Ψ^*)-algebra and spectral invariance.

Theorem 9.4 (cf. [4], [23])**.** i) *$\overline{\Psi_{\mathcal{V}}^{-\infty}}$ is a Fréchet algebra and $\Psi_{\mathbb{C}}^{-\infty}$ is a (Ψ^*)-algebra.*
ii) *For a given $m \in \mathbb{R}$ there is a $Q \in \overline{\Psi_{\mathcal{V}}^m}(M)$ such that $Q^{-1} \in \overline{\Psi_{\mathcal{V}}^{-m}}(M)$.*
ii) *Let $P \in \overline{\Psi_{\mathcal{V}}^0}$ and $f \in \mathcal{O}(\sigma(P))$ then $f(P) \in \overline{\Psi_{\mathcal{V}}^0}$.*
iii) *For $P \in \overline{\Psi_{\mathcal{V}}^m}(M)$ with $m \geq 0$ which is elliptic and invertible, as a possibly unbounded operator on L^2, we have $P^{-1} \in \overline{\Psi_{\mathcal{V}}^{-m}}$.*

Proof. i) We check that the family of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ as defined above is submultiplicative. Since they generate a Fréchet topology this proves that $\overline{\Psi_{\mathcal{V}}^{-\infty}}$ is a Fréchet algebra. Let $P_1,\ P_2 \in \overline{\Psi_{\mathcal{V}}^{-\infty}}$ and $n \in \mathbb{N}$ then

$$\|P_1P_2\|_n \leq \|P_1\|_{\mathcal{L}(H^0,H^n)}\|P_2\|_{\mathcal{L}(H^{-n},H^n)} \leq \|P_1\|_n\|P_2\|_n.$$

For the Ψ^*-property we refer to the general result [4], Prop. A.1.
ii) Make use of the Lemma and set

$$R(z) := q(r^z)q(r^{-z}b(z)) - I \in \overline{\Psi_{\mathcal{V}}^{-\infty}}.$$

Hence $R \in \mathcal{O}(\mathbb{C}, \overline{\Psi_{\mathcal{V}}^{-\infty}}),\ R(0)=0$. For $|z|<\epsilon$ with $\epsilon>0$ sufficiently small $R(z)$ is invertible on $L^2(M_0)$. Thence $q(r^z)^{-1} = q(r^{-z}b(z))(I+R(z))^{-1} \in \overline{\Psi_{\mathcal{V}}^{-z}}$ for $|z|<\epsilon$ sufficiently small. Choose k such that $|m|<k\epsilon$. This yields that $Q := [q(r^{\frac{m}{k}})^{-1}]^k$ has the required property.
iii) The operator $P \in \Psi_{\mathbb{C}}^{-\infty}$ is invertible if $P \in \operatorname{inv}\mathcal{L}(L^2_{\mathcal{V}})$. Then since $\operatorname{inv}(\Psi_{\mathbb{C}}^{-\infty})$ is open it follows that inversion is continuous. The integral

$$f(P) = \frac{1}{2\pi i}\int_\gamma f(z)(z-P)^{-1}\,dz$$

exists and hence $f(P) \in \Psi_{\mathbb{C}}^{-\infty}$.
iii) α) Let $Q \in \overline{\Psi_{\mathcal{V}}^{-m}}$ be such that $Q^{-1} \in \overline{\Psi_{\mathcal{V}}^{m}}$ which is possible by *i)*. Then $\tilde{P} := PQ \in \overline{\Psi_{\mathcal{V}}^{0}}$ is elliptic, injective and bounded. From the ellipticity of Q^{-1} we obtain that the range of Q is H^m and by ellipticity of P we obtain that the domain of P is also H^m. Thus $\tilde{P} = PQ\colon H^0 \to H^0$ is also surjective and hence invertible on $L^2 = H^0$. Therefore we can consider $P = \tilde{P} \in \overline{\Psi_{\mathcal{V}}^{0}}$ as an L^2-bounded operator.
β) By the previous argument we can wlog assume that $m=0$ and that $P \in \overline{\Psi_{\mathcal{V}}^{0}}(M)$ is invertible as an operator on L^2.
We show first the *claim:* There exists (Q_n) such that $Q_n \to P^{-1}$ with regard to $\|\cdot\|_{\mathcal{L}(L^2)}$.
Clearly PP^* is bounded and invertible. By continuity of the functional calculus, we obtain a sequence (p_n) of polynomials such that

$$p_n(PP^*) \to (PP^*)^{-1} \text{ in } \|\cdot\|_{\mathcal{L}(L^2)}.$$

Set $Q_n := P^* p_n(PP^*)$, then we have

$$\begin{aligned}\|P^{-1}-Q_n\|_{\mathcal{L}(L^2)} &= \|P^{-1} - P^*p_n(PP^*)\|_{\mathcal{L}(L^2)} \\ &\le \|P^*\|_{\mathcal{L}(L^2)}\|(PP^*)^{-1} - p_n(PP^*)\|_{\mathcal{L}(L^2)} \to 0,\ n\to\infty.\end{aligned}$$

This proves the claim.
Set $R := PQ - I \in \overline{\Psi_{\mathcal{V}}^{-\infty}}$. With the claim it follows that

$$P^{-1} = Q - P^{-1}R = \lim_n (Q - Q_nR).$$

Set $Q' := Q - Q_nR$. For n sufficiently large this is a parametrix with

$$PQ' = I + R'$$

This yields $(I+R')^{-1} - I \in \overline{\Psi_{\mathcal{V}}^{-\infty}}$ and $P^{-1} = Q'(I+R')^{-1} \in \overline{\Psi_{\mathcal{V}}^{0}}$. □

9.2. **Parametrices.** In this section we will introduce the principal and principal boundary symbol of an operator in our calculus. We will define the notion of ellipticity and show that a parametrix exists under the previously stated conditions on the calculus. A major technical problem is that in the Lie calculus already inverses of invertible operators are not necessarily contained. This makes a parametrix construction difficult and we state here a version of such a result. Presently, this relies on the strongest assumption made in this work. There are at least two approaches to overcome the assumption: *(1)* using a larger calculus of pseudodifferential operators (with asymptotics) or *(2)* completing the algebra of pseudodifferential operators (non-canonically) such that inverses are contained. In the last section we outlined the second approach. We will henceforth assume that our Boutet de Monvel calculus is built from pseudodifferential operators which are completed in this way.
Fix the smooth, hermitian vector bundles $E_1,\ E_2 \to X,\ J_\pm \to Y$ and recall the notation for the boundary algebroid and its co-bundle $\pi_\partial\colon \mathcal{A}_\partial \to Y,\ \overline{\pi}_\partial\colon \mathcal{A}_\partial^* \to Y$.
We define next the principal symbol and principal boundary symbol on $\mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{X}; \tilde{E}_1, \tilde{E}_2, \tilde{J}_\pm)$ for $m \le 0$.

- Set $T^q\mathcal{X} := \ker dq$ for the vertical tangent bundle over $\mathcal{X}$.
- Let $A = (A_x)_{x\in X} \in \mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{X})$ be a C^∞-family of Boutet de Monvel operators.
- Then for each $x \in X$ we have a Boutet de Monvel operator $A_x \in \mathcal{B}^{m,0}_{prop}(\mathcal{G}^+_x, \mathcal{X}_x)$.

- Hence the principal symbol $\sigma(A_x)$ and the principal boundary symbol $\sigma_\partial(A_x)$ are defined invariantly on $T^*\mathcal{G}_x^+$ and $T^*\mathcal{X}_x$ respectively.
- By right-invariance of the family A these symbols descend to a principal symbol $\sigma(A)$ and principal boundary symbol $\sigma_\partial(A)$ defined invariantly on $\mathcal{A}_+^*$ and $(T^q\mathcal{X})^*$ respectively.

By noting that $T^q_{|Y}\mathcal{X} = \mathcal{A}_\partial$ we make the following definition for principal and principal boundary symbol on the represented algebra $\mathcal{B}_{\mathcal{V}}^{m,0}(X,Y)$ for $m \leq 0$.

Definition 9.5. To an element $A \in \mathcal{B}_{\mathcal{V}}^{m,0}(X,Y;E_1,E_2,J_\pm)$ we associate the two principal symbols.
i) The *principal boundary symbol* $\sigma_\partial^{\mathcal{V}}(A)$ is defined as

$$\sigma_\partial^{\mathcal{V}}(A) := \sigma_{\partial|Y}(A).$$

This yields a section of the infinite dimensional bundle

$$C^\infty(\mathcal{A}_\partial^*, \mathrm{Hom}(\overline{\pi}_\partial^* E_{1|Y} \otimes \mathcal{S}_{\mathcal{V}}, \overline{\pi}_\partial^* E_{2|Y} \otimes \mathcal{S}_{\mathcal{V}})).$$

Here $\mathcal{S}_{\mathcal{V}} \to \mathcal{A}_\partial^*$ is a bundle with fiber $S(\overline{\mathbb{R}}_+)$ on the inward pointing normal direction.
In particular the restriction of $\sigma_\partial^{\mathcal{V}}$ to the interior (X_0, Y_0) agrees with the principal boundary symbol on the interior.
ii) The *principal symbol* $\sigma^{\mathcal{V}}(A)$ which is the principal symbol of the pseudodifferential operator in the upper left corner of the matrix A. This yields a section in

$$C^\infty(\mathcal{A}_+^*, \mathrm{Hom}(E_1, E_2)).$$

iii) Define by $\Sigma_{\mathcal{V}}^{m,0}(X,Y;E_1,E_2,J_\pm)$ the space consisting of pairs of principal symbols (a, a_∂). These are homogenous or κ-homogenous sections of the bundles $\mathcal{A}_+^*, \mathcal{A}_\partial^*$, respectively, with canonical compatibility condition.

Assumption D. Given an invertible Boutet de Monvel operator $A \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y;E_1,E_2,J_\pm)$ the inverse $(\sigma \oplus \sigma_\partial)(A)^{-1}$ is defined and contained in $\Sigma_{\mathcal{V}}^{0,0}(X,Y;E_1,E_2,J_\pm)$.
Assumption E. The calculus $\mathcal{B}_{\mathcal{V}}^{m,0}(X,Y;E_1,E_2,J_\pm)$ is asymptotically complete. That means given a sequence of operators $(A_i)_{i \in \mathbb{N}_0}$ with $A_i \in \mathcal{B}_{\mathcal{V}}^{-i,0}(X,Y)$ there is a $A \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y)$ such that

$$A - \sum_{i=0}^N A_i \in \mathcal{B}_{\mathcal{V}}^{m-N-1,0}(X,Y), \text{ for each } N \in \mathbb{N}_0.$$

We write briefly $A \sim \sum_{i=0}^\infty A_i$ for this.

Lemma 9.6. *Given $A, B \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y;E_1,E_2,J_\pm)$ we have*

$$\sigma^{\mathcal{V}}(A \cdot B) = \sigma^{\mathcal{V}}(A) \cdot \sigma^{\mathcal{V}}(B), \ \sigma_\partial^{\mathcal{V}}(A \cdot B) = \sigma_\partial^{\mathcal{V}}(A) \cdot \sigma_\partial^{\mathcal{V}}(B).$$

Proof. As mentioned in section 9.1 the vector representation furnishes an isomorphism

$$\mathcal{B}_{\mathcal{V}}^{0,0}(X,Y;E_1,E_2,J_\pm) \cong \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X}; \tilde{E}_1, \tilde{E}_2, \tilde{J}_\pm).$$

Since the principal and principal boundary symbol are each defined invariantly on the bundles $\mathcal{A}_+^*,\ \mathcal{A}_\partial^*$ the computation reduces to the equivariant families of Boutet de Monvel operators. Hence given $A = (A_x)_{x \in X},\ B = (B_x)_{x \in X}$ in $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X}; \tilde{E}_1, \tilde{E}_2, \tilde{J}_\pm)$ we have

$$\begin{aligned}\sigma_\partial(A \cdot B) &= \sigma_\partial((A_x \cdot B_x)_{x \in X}) = (\sigma_\partial(A_x \cdot B_x))_{x \in X} \\ &= (\sigma_\partial(A_x) \cdot \sigma_\partial(B_x))_{x \in X} = \sigma_\partial(A) \cdot \sigma_\partial(B).\end{aligned}$$

In the same way we obtain multiplicativity of the principal symbol. □

Since we will in the following only be concerned with represented operators we will simply write σ and σ_∂ for $\sigma^{\mathcal{V}}$ and $\sigma_\partial^{\mathcal{V}}$.
For the following result and proof in the standard case see e. g. [28].

Theorem 9.7. *The following sequence is exact*

$$\mathcal{B}_{\mathcal{V}}^{-1,0}(X,Y;E_1,E_2,J_\pm) \rightarrowtail \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y;E_1,E_2,J_\pm) \xrightarrow{\sigma\oplus\sigma_\partial} \Sigma_{\mathcal{V}}^{0,0}(X,Y;E_1,E_2,J_\pm).$$

Proof. i) The same exact sequence holds for the interior calculus. Since the principal symbols are extensions of the interior we immediately obtain that $\ker\sigma\oplus\sigma_\partial = \ker\sigma \cap \ker\sigma_\partial = \mathcal{B}_{\mathcal{V}}^{-1,0}(X,Y)$.
ii) To prove surjectivity let $(a,a_\partial) \in \Sigma_{\mathcal{V}}^{0,0}$.
Since we also have an analogous exact sequence for the class of pseudodifferential operators $\Psi_{tr,2\mathcal{V}}^m$ it suffices to find singular Green, trace and potential operators in the preimage.
In a fixed small tubular neighborhood $\mathcal{U} \cong Y_{(\epsilon)}$ trivialize the singular normal bundle $\mathcal{N}$. Let $\{U_i\}$ be a normal covering of Y (assumed finite by compactness of Y) and let $\{\varphi_i\}$ be a subordinate partition of unity. We also fix a boundary defining function $\rho_Y\colon M \to \mathbb{R}$ such that $\{\rho_Y = 0\} = Y$. This is defined by the tubular neighborhood theorem for Lie manifolds (cf. [2]). Hence for the diffeomorphism of tubular neighborhoods $\nu\colon Y\times(-\epsilon,\epsilon) \to \mathcal{U} \subset M$ we have

$$(\rho_Y\circ\nu)(x',x_n) = x_n,\ x' \in Y,\ x_n \in (-\epsilon,\epsilon).$$

Let $\varphi \in C^\infty(\overline{\mathbb{R}}_+)$ be a cutoff function such that $\varphi(x_n) = 0$ close to 0. Hence we can locally on each trivialization $\mathcal{N}_{|U_i}^+ \cong U_i \times \overline{\mathbb{R}}_+$ construct singular Green, potential and trace operators. Via our partition of unity we obtain corresponding global symbols.
This furnishes a global right inverse morphism $\mathrm{Op}_{\mathcal{V}}\colon \Sigma_{\mathcal{V}}^{0,0} \to \mathcal{B}_{\mathcal{V}}^{0,0}$. □

The notion of ellipticity we introduce here is the usual condition of Shapiro-Lopatinski type. It is sufficient to obtain a parametrix.

Definition 9.8. An operator $A \in \mathcal{B}_{\mathcal{V}}^{m,0}(X,Y)$ is called *$\mathcal{V}$-elliptic* iff $(\sigma^m \oplus \sigma_\partial)(A)$ is pointwise bijective.

Theorem 9.9. *Let $A \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y)$ be $\mathcal{V}$-elliptic, then there is a parametrix $B \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y)$ which means that*

$$I - AB \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y),\ I - BA \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y).$$

Proof. i) From the exact sequence given in Theorem 9.7 there is a $B \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y)$ such that

$$(\sigma\oplus\sigma_\partial)(B) = (\sigma\oplus\sigma_\partial)(A)^{-1}.$$

Then by the multiplicativity of the principal symbol and the principal boundary symbol it follows that

$$(\sigma\oplus\sigma_\partial)(I - AB) = 0.$$

Applying the exact sequence once more we have that $R := I - AB \in \mathcal{B}_{\mathcal{V}}^{-1,0}(X,Y)$. Hence B is a right parametrix of A of order 1. Setting $B_k = B(I + R + \dots + R^{k-1})$ we obtain

$$AB_k = (I - R)(I + R + \dots + R^{k-1}) = I - R^k$$

with $R^k \in \mathcal{B}_{\mathcal{V}}^{-k,0}(X,Y)$. Thus B_k is a right parametrix of A of order k.
ii) Let $(B_k)_{k\in\mathbb{N}_0}$ be a sequence of right parametrices of A of orders $k \in \mathbb{N}_0$. Using assumption E we can find $B \sim \sum_i B_i$ such that $AB - I \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y)$. Hence we have found a right parametrix up to residual terms.
iii) Fix a right parametrix up to residual terms B_1 of A. Analogously to *i)* and *ii)* we can find a left parametrix B_2 of A such that

$$I - AB_1 = R_1 \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y) \text{ and } I - B_2A = R_2 \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y).$$

Rewrite the operator AB_2AB_1 as follows

$$AB_2AB_1 = AB_2(I - R_1) = A(I - R_2)B_1 = AB_1 - AR_2B_1 = (I - R_1) - AR_2B_1$$

hence $AB_2AB_1 + AR_2B_1 = I - R_1$ so that

$$(AB_2A + AR_2)B_1 = I - R_1.$$

The left hand side of the previous equation equals

$$AB_2(I - R_1) + AR_2B_1$$

and note that this is $\equiv\ AB_2 \mod \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y)$.
Hence B_2 is also a right parametrix equal up to residual terms to $I - R_1$. Analogously one proves that B_1 is also a left parametrix equal up to residual terms to $I - R_2$. It follows that $B_1 \equiv B_2 \mod \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y)$ and therefore there is an operator $B \in \mathcal{B}_{\mathcal{V}}^{0,0}(X,Y)$ equal to B_1 mod $\mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y)$ and B_2 mod $\mathcal{B}_{\mathcal{V}}^{-\infty}(X,Y)$ such that

$$I - AB \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y), \text{ and } I - BA \in \mathcal{B}_{\mathcal{V}}^{-\infty,0}(X,Y).$$

□

10. The index problem

We briefly indicate how to state an index problem for Lie manifolds with boundary. Although we assume here that we are given a boundary structure, we state the problem independently of the calculus. We expect that the Boutet de Monvel calculus or more precisely a suitable completion thereof will play a crucial role in the resolution of the problem. The following approach is a generalization of the statement of the index problem for compact manifolds with boundary from [5]. This is based on the tangent groupoid of A. Connes, cf. [8].
First we introduce the *semi-algebroid* as follows. We fix an invariant connection ∇^+ on the algebroid $(\mathcal{A}^+ \to X, \varrho^+)$ of the Lie manifold with boundary $(X, \mathcal{V})$. Define the associated exponential map $\exp = \exp^{\nabla^+}$ and the *semi-algebroid*

$$\tilde{\mathcal{A}} = \{(x,v) \in \mathcal{A}^+ : \exp_x(-tv) \in X,\ t \text{ small}\}.$$

Then construct a semi-groupoid $\tilde{\mathcal{G}} \rightrightarrows X \times I,\ I := [0,1]$ which is the analog of Connes's tangent groupoid. Here we glue the groupoid $\mathcal{G}^+ \times (0,1]$ to the semi-algebroid $\tilde{\mathcal{A}}$. As a set this is written

$$\tilde{\mathcal{G}} = \mathcal{G}^+ \times (0,1] \cup \tilde{\mathcal{A}} \times \{0\}.$$

Note that restricted to the interior we get (since $\mathcal{G}^+_{|X_0} \cong X_0 \times X_0$ by assumption)

$$X_0 \times X_0 \times (0,1] \cup TX_0 \times \{0\}$$

which recovers the tangent groupoid which is also the adiabatic groupoid $(X_0 \times X_0)^{ad}$.
In general we consider the reduced C^*-algebra $C_r^*(\tilde{\mathcal{G}})$ and define the evaluations at 0 and 1

$$\begin{aligned} e_0 &\colon C_r^*(\tilde{\mathcal{G}}) \to C_r^*(\tilde{\mathcal{A}}), \\ e_1 &\colon C_r^*(\tilde{\mathcal{G}}) \to C_r^*(\mathcal{G}^+) \cong C^*(\mathcal{G}^+). \end{aligned}$$

We make the assumption that $\mathcal{G}^+$ is amenable (see also Remark 9.2, *i)*), hence $C_r^*(\mathcal{G}^+) \cong C^*(\mathcal{G})$. Then we obtain the short exact sequence

$$C_0(0,1] \otimes C^*(\mathcal{G}^+) \rightarrowtail C_r^*(\tilde{\mathcal{G}}) \xrightarrow{e_0} C_r^*(\tilde{\mathcal{A}}).$$

Note that $C_0(0,1] \otimes C^*(\mathcal{G}^+)$ is contractible. Then apply the six-term exact sequence in K-theory. We deduce from contractibility that the induced maps in K-theory $(e_0)_*$ are isomorphisms

$$K_j(C_r^*(\tilde{\mathcal{G}})) \cong K_j(C_r^*(\tilde{\mathcal{A}})),\ j = 0,1.$$

Therefore we can define the analytic index as the map $\operatorname{ind}_a^+ \colon K_0(C_r^*(\tilde{\mathcal{A}})) \to K_0(C^*(\mathcal{G}^+))$ making the following diagram commute

$$\begin{tikzcd} K_1(C_r^*(\tilde{\mathcal{A}}^*)) \arrow[r, "(e_0)_*^{-1}"] \arrow[d, "\operatorname{ind}_a^+"'] & K_0(C_r^*(\tilde{\mathcal{G}})) \arrow[dl, "(e_1)_*"] \\ K_0(C^*(\mathcal{G}^+)). & \end{tikzcd}$$

We also obtain an analytic index on the interior pair groupoid with the same technique.
It remains to prove that ind_a^+ generalizes the Fredholm index for boundary value problems. Moreover, an appropriate topological interpretation needs to be found. In particular one should be able to relate the K-theory of $C_r^*(\tilde{\mathcal{A}})$ with the K-theory of $C_0(T^*X_0)$. In the special case of a compact manifold with boundary the K-theories are isomorphic as was shown in [5].

Appendix A. Conormal distributions

We recall a definition of the Boutet de Monvel calculus on which all of the preceding discussions are based. This particular Theorem can be found in the standard reference [12], p.227.
First we set up the necessary notation. Here we are in the setting of a compact manifold with boundary. This means our discussion here can be seen as an elaboration of Example 4.9, *i)* of a boundary structure in the simplest case.
Let X be a compact manifold with boundary $\partial X = Y$ and denote by $M = 2X$ the double.
Since all the following calculations are local we identify $X = \overline{\mathbb{R}}_+^n,\ Y = \mathbb{R}^{n-1},\ M = \mathbb{R}^n$.
Then we have the setup $\mathcal{G} = M \times M,\ \mathcal{G}_\partial = Y \times Y,\ \mathcal{G}_\pm = X \times X$ as pair groupoids, $\mathcal{X} := Y \times M,\ \mathcal{X}^t := M \times Y$.
We first recall the symbol calculus for Boutet de Monvel operators. Let $g,\ t,\ k$ be the symbols of singular Green, trace and potential operators of order m and type 0 respectively. To each symbol we assign a continuous linear operator

$$\begin{aligned}
&g \mapsto \mathrm{op}_G(g) = G\colon C^\infty(X) \to C^\infty(X),\\
&t \mapsto \mathrm{op}_T(t) = T\colon C^\infty(X) \to C^\infty(Y),\\
&k \mapsto \mathrm{op}_K(k) = K\colon C^\infty(Y) \to C^\infty(X).
\end{aligned}$$

Since trace operators and potential operators are adjoints of each other, we restrict discussion to trace operators and singular Green operators. It will be more convenient to describe the estimates for the associated *symbol kernels* $\tilde{g},\ \tilde{t}$. These are given by

$$\begin{aligned}
&\overline{\mathcal{F}}^{-1}_{x_n \to \xi_n} t(x', \xi_n, \xi') = \tilde{t}(x', x_n, \xi'),\\
&\mathcal{F}^{-1}_{\xi_n \to x_n} \overline{\mathcal{F}}^{-1}_{y_n \to \eta_n} g(x', \eta_n, \xi_n, \xi') = \tilde{g}(x', x_n, y_n, \xi').
\end{aligned}$$

The symbol kernels then are required to satisfy the L^2-estimates

$$\|D_{x'}^\beta x_n^k D_{x_n}^{k'} y_n^l D_{y_n}^{l'} D_{\xi'}^\alpha \tilde{g}\|_{L^2_{x_n,y_n}} \leq C\langle \xi' \rangle^{m-k+k'-l+l'-|\alpha|} \tag{31}$$

$$\tag{32}$$

for each $\alpha,\ \beta \in \mathbb{N}^{n-1}$ and $k,\ k',\ l,\ l' \in \mathbb{N}$ and

$$\|x_n^l D_{x_n}^k D_{\xi'}^\alpha D_{x'}^\beta \tilde{t}(x', \xi', x_n)\|_{L^2_{x_n}} \leq C\langle \xi' \rangle^{m-|\alpha|-l+k} \tag{33}$$

for each $l, k \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^{n-1}$.
We are interested in the Boutet de Monvel operators as described by their Schwartz kernels. These are defined as follows for $(x, y) \in X \times X$

$$\mathcal{K}_G(x, y) = \int e^{i(x'-y')\xi' + ix_n\xi_n - iy_n\eta_n} g(x', \xi, \eta_n)\, d\xi\, d\eta_n \tag{34}$$

and for $(x', y) \in \mathcal{X}$ we have

$$\mathcal{K}_T(x', y) = \int e^{i(x'-y')\xi' + iy_n\xi_n} t(x', \xi', \xi_n)\, d\xi_n\, d\xi'. \tag{35}$$

Furthermore, we define the sets

$$\begin{aligned}
&\mathcal{Y}_T := \{(x', y) \in \mathcal{X} : x' = y', x_n = y_n = 0\} \cong \Delta_Y,\\
&\mathcal{Z}_T := \{(x', y) \in \mathcal{X} : x_n = y_n = 0\} \cong Y^2.
\end{aligned}$$

for the trace operators. For the potential operators

$$\mathcal{Y}_K := \{(x,y') \in \mathcal{X}^t : x' = y', x_n = y_n = 0\} \cong \Delta_Y,$$
$$\mathcal{Z}_K := \{(x,y') \in \mathcal{X}^t : x_n = y_n = 0\} \cong Y^2.$$

Similarly, we have the sets for singular Green operators

$$\mathcal{Y}_G := \{(x,y) \in \mathcal{G} : x' = y', x_n = y_n = 0\} \cong \Delta_Y,$$
$$\mathcal{Z}_G := \{(x,y) \in \mathcal{G} : x_n = y_n = 0\} \cong Y^2.$$

Then we have the following result

Lemma A.1. *(Grubb, [12], p.227)* i) *Given a trace operator $T \in \mathscr{T}^{m,0}(X,Y)$ of order m and type* 0 *we can extend the integral kernel $\mathcal{K}_T$ to the double $2X = M$. Call this extended operator and kernel T_+ and $\mathcal{K}_{T_+}$. Then $\mathcal{K}_{T_+}$ is a conormal distribution in $I^{m-\frac{3}{4}}(Y \times M, \Delta_Y)$ which has rapid decay along the normal direction.*
ii) *Given a potential operator $K \in \mathcal{K}^m(X,Y)$ of order m the same procedure gives a conormal distribution $\mathcal{K}_{K_+}$ in $I^{m-\frac{1}{4}}(M \times Y, \Delta_Y)$. Such that $\mathcal{K}_{K_+}$ has rapid decay along the normal direction.*
iii) *A Green operator $G \in \mathcal{G}^{m,0}(X,Y)$ of order m and type zero can be extended to a yield a kernel $\mathcal{K}_{G_+}$ in $I^{m-\frac{1}{4}}(M \times M, \Delta_Y)$. With rapid decay along the normal direction.*

Although the proof can be found in Grubb's book, we include some arguments for the benefit of the reader.

Proof. i) First as in [12] we Seeley extend the integral kernel of the trace operator to the double M and so obtain the extended trace operator $T_+ : C^\infty(M) \to C^\infty(Y)$. Write this as

$$(T_+u)(x') = (2\pi)^{-n-1} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x'-y')\xi' + iy_n\xi_n} t_+(x',\xi',\xi_n) u(y',y_n)\, dy_n\, d\xi'\, d\xi_n\, dy' \tag{36}$$

with integral kernel

$$\mathcal{K}_{T_+}(x,y) = \int e^{i(x'-y')\xi' + ix_n\xi_n} t_+(x',\xi',\xi_n)\, d\xi'\, d\xi_n. \tag{37}$$

Using Plancherel's Theorem in (33) as well as the elementary inequality $\sup_t |\varphi(t)| \le \|\varphi\|_2 \|D_t\varphi\|_2$

$$|D^\beta_{x'} D^l_{\xi_n} D^\alpha_{\xi'} t_+| \le C\langle\xi'\rangle^{m-l-|\alpha|-1} \left(\frac{\langle\xi'\rangle}{\langle\xi',\xi_n\rangle}\right)^k. \tag{38}$$

This simplifies to

$$|D^\beta_{x'} D^\theta_{\xi',\xi_n} t_+| \le C\langle\xi'\rangle^{m-1-|\theta|} \left(\frac{\langle\xi'\rangle}{\langle\xi',\xi_n\rangle}\right)^N \tag{39}$$

for $\beta \in \mathbb{N}^{n-1}, \theta \in \mathbb{N}^n, N \in \mathbb{N}$.
Therefore t_+ belongs to $S^{m-1}_{1,0}(\mathbb{R} \times \mathbb{R}^{2n})$ by [13], Thm. 18.2.11.

Then we also obtain that $\mathcal{K}_{T_+} \in I^{m-\frac{3}{4}}(\mathcal{X}, \mathcal{Y}_T)$. The rapid decay property as $|\xi_n| \to \infty$ is immediate from the last estimate.
ii) Given a potential operator K of order m.
This furnishes by the same analysis an extended kernel $\mathcal{K}_{K_+}$ contained in the space of conormal distributions $I^{m-\frac{1}{4}}(\mathcal{X}^t, \mathcal{Y}_K)$.
iii) We extend the Schwartz kernel $\mathcal{K}_G$ by applying the Seeley extension operator in the $\overline{\mathbb{R}}_+$ direction. We have the new kernel

$$\mathcal{K}_{G_+}(x,y) = \int e^{i(x'-y')\xi' + ix_n\xi_n - iy_n\eta_n} g_+(x',\xi,\eta_n)\, d\xi\, d\eta_n$$

with $\mathcal{K}_G = \mathcal{K}_{G_+|\mathbb{R}^2_+}$. Then (31) corresponds to

$$\|D^\beta_{x'} D^k_\xi \xi_n^{k'} D^l_{\eta_n} \eta_n^{l'} D^\alpha_{\xi'} g_+(x', \xi, \eta_n)\|_{L^2_{\xi_n}} \le C\langle \xi' \rangle^{m-k+k'-l+l'-|\alpha|}.$$

Applying the Plancherel theorem and the elementary inequality the same way again we obtain the estimate

$$|D^\beta_{x'} D^k_{\xi_n} D^l_{\eta_n} D^\alpha_{\xi'} g_+| \le C\langle \xi' \rangle^{m-k-l-|\alpha|-1} \left(\frac{\xi'}{\langle \xi', \xi_n \rangle} \right)^{k'} \left(\frac{\langle \xi' \rangle}{\langle \xi', \eta_n \rangle} \right)^{l'}$$

for each $\alpha,\ \beta \in \mathbb{N}^{n-1},\ k,\ k',\ l,\ l' \in \mathbb{N}$.
By use of the elementary inequality

$$\langle \xi + \eta \rangle^s \le 2^{|s|} \langle \xi \rangle^s \langle \eta \rangle^s,\ s \in \mathbb{R}$$

we obtain from the last estimate

$$|D^\beta_{x'} D^\theta_{\xi, \xi_n} g_+| \le C\langle \xi' \rangle^{m-1-|\theta|} \left(\frac{\langle \xi' \rangle}{(1 + |\xi'|^2 + \xi_n^2 + \eta_n^2)}^{\frac{1}{2}} \right)^N$$

for each $\beta \in \mathbb{N}^{n-1},\ \theta \in \mathbb{N}^n,\ N \in \mathbb{N}$ and a constant $C = C(\beta, \theta, N)$. Then from [13], Thm. 18.2.11 we obtain that $\mathcal{K}_{G_+}$ is a conormal distribution in $I^{m-\frac{1}{4}}(\mathcal{G}, \mathcal{Y}_G)$, and for $|\xi_n| \to \infty,\ |\eta_n| \to \infty$ the kernel decays rapidly. $\square$

With this we can state the main Theorem. First fix the restriction and extension by zero operators r^+ and e^+

$$L^2(M) \underset{e^+}{\overset{r^+}{\rightleftarrows}} L^2(\mathring{X})$$

with $r^+ e^+ = \mathrm{id}_{L^2(X)}$ and $e^+ r^+$ being a projection onto a subspace of $L^2(M)$.

Theorem A.2. i) *An element in the Boutet de Monvel calculus $A \in \mathcal{B}^{m,0}(X, Y)$ is given by*

$$A = \begin{pmatrix} r^+ G_+ e^+ + r^+ P e^+ & r^+ K_+ e^+ \\ r^+ T_+ e^+ & S \end{pmatrix}.$$

Here the operators are entirely described by their Schwartz kernels which are conormal distributions; $\mathcal{K}_P \in I^{m-\frac{1}{4}}(M \times M, \Delta_M)$ and $\mathcal{K}_S \in I^{m-\frac{1}{4}}(Y \times Y, \Delta_Y)$ for the pseudodifferential operators on M and the boundary Y, respectively. Additionally, P is required to fulfill the transmission property *(mapping functions smooth up to Y to functions also smooth up to Y), and $\mathcal{K}_{G_+} \in I^{m-\frac{1}{4}}(M \times M, \Delta_Y)$ and rapidly decreasing along $\mathcal{N}\mathcal{Y}_G \cap \mathcal{N}\mathcal{Z}_G$. Furthermore, $\mathcal{K}_{T_+} \in I^{m-\frac{1}{4}}(Y \times M, \Delta_Y)$ is rapidly decreasing along $\mathcal{N}\mathcal{Y}_T \cap \mathcal{N}\mathcal{Z}_T$, and $\mathcal{K}_{K_+} \in I^{m-\frac{3}{4}}(M \times Y, \Delta_Y)$ is rapidly decreasing along $\mathcal{N}\mathcal{Y}_K \cap \mathcal{N}\mathcal{Z}_T$.*
ii) *The zero calculus $\mathcal{B}^{0,0}(M, Y)$ consisting of extended operators*

$$A_+ = \begin{pmatrix} P + G_+ & K_+ \\ T_+ & S \end{pmatrix}$$

is closed under composition. Furthermore, the truncation

$$\mathcal{B}^{0,0}(M, Y) \ni A_+ = \begin{pmatrix} P + G_+ & K_+ \\ T_+ & S \end{pmatrix} \mapsto \mathcal{C}(A_+) = \begin{pmatrix} r^+ P e^+ + r^+ G_+ e^+ & r^+ K_+ e^+ \\ r^+ T_+ e^+ & S \end{pmatrix} \in \mathcal{B}^{0,0}(X, Y)$$

is a linear, surjective operator.

Proof. i) This is the characterization of Boutet de Monvel's operators in terms of conormal distributions given in the previous Lemma.

ii) The compositions in the extended calculus are furnished by convolutions of the integral kernels which are conormal distributions. One can apply the trivial actions on the pair groupoids and the corresponding convolution product

$$\begin{tikzcd} Y \times Y \arrow[r, hook] & Y \times M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & M \times M \arrow[l, hook'] \\ Y & & M \end{tikzcd}$$

as well as

$$\begin{tikzcd} M \times M \arrow[r, hook] & M \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & Y \times Y \arrow[l, hook'] \\ M & & Y \end{tikzcd}$$

A direct calcululation then shows that $\mathcal{K}_S * \mathcal{K}_T$ yields again a conormal distribution which describes a trace operator. The other cases can be handled by direct calculation as well. Alternatively, one may invoke the general result due to Hörmander [14], Thm. 25.2.3. The surjectivity of $\mathcal{C}$ is implied in the proof of Lemma A.1, since we can always Seeley extend each operator in $\mathcal{B}^{0,0}(X,Y)$ which gives a preimage in $\mathcal{B}^{0,0}(M,Y)$. □

Appendix B. Tubular neighborhood

In the main part of the paper we have made use of the definition of *Lie manifold with boundary* as well as the tubular neighborhood theorem. For completeness sake we briefly recall these notions.

Let $(M, \mathcal{V})$ be a Lie manifold, i.e. a compact manifold with corners M endowed with a Lie structure (a $C^\infty(M)$-module of vector fields, locally finitely generated, projective). By the Serre-Swan theorem there is a vector bundle $\mathcal{A} \to M$ such that $\Gamma(\mathcal{A}) \cong \mathcal{V}$. This carries the structure of a Lie algebroid (cf. [3]).

A Riemannian metric on M is called *compatible* if g is a suitable restriction of a metric defined on the algebroid $\mathcal{A}$ such that the following holds: for each $x \in M$ there is a neighborhood U_x in M and a local basis of vector fields $\{V_i\}$ from $\mathcal{V}$ such that $\{V_i\}$ forms an orthonormal frame with regard to the metric g restricted to U_x.

We summarize some geometric facts about the category of Lie manifolds (see e.g. [2]).

- (M_0, g_0) is of bounded geometry for a compatible metric g_0 on M_0. This means M_0 has bounded, positive injectivity radius and bounded covariant derivatives of the curvature tensor.
- (M_0, g_0) is of infinite volume and complete.

Next fix the exponential map $\exp \colon \mathcal{A} \to M$ coming from the invariant connection defined by the metric on $\mathcal{A}$. Notice that from the the splitting in (6) and the identification $\mathcal{N} \cong \mathcal{A}_\partial^\perp$ we in particular obtain the normal exponential map

$$\exp^\nu \colon \mathcal{N} \to Y.$$

As was also shown in [2], section 2.3 with the compatible metric g_0 on M_0 we can find a neighborhood $Y \subset U \subset M$ such that on

$$U \cap M_0 \cong (\partial Y_0) \times (-\epsilon_0, \epsilon_0)$$

the metric g_0 restricts to a product-type metric. This is just a tubular neighborhood theorem for Lie manifolds. Such a Lie submanifold with this property is called *tame* in the above cited reference and there it is also shown that a submanifold of codimension 1 is always tame.

Hence the following result holds:

Theorem B.1. *([2], Thm. 2.7) Let Y be a codimension 1 Lie submanifold of M. For an $\epsilon > 0$ sufficiently small the normal exponential induces a diffeomorphism*

$$\mathcal{U} := Y \times (-\epsilon, \epsilon) \xrightarrow{\sim} \{V \in \mathcal{N} : \|V\| < \epsilon\} =: \mathcal{N}_\epsilon.$$

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