hp-BEM for Contact Problems and Extended Ms-FEM in Linear Elasticity

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades
Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation

von

Dipl.- Math. Abderrahman Issaoui

geboren am 03.02.1977 in Metlaoui (Tunesien)

2014

Referent:	Prof. Dr. Ernst P. Stephan, Leibniz Universität Hannover
Korreferent:	Prof. Dr. Joachim Gwinner, Universität der Bundeswehr, München
Tag der Promotion:	05.06.2014

Abstract

We consider a contact problem between an elastic body and a rigid foundation with Tresca's friction law, present a hp-discretization technique of a mixed formulation based on biorthogonal basis functions and solve the corresponding discrete system with the semi-smooth Newton method. A-posteriori error estimates using an error indicator in an hp-adaptive refinement algorithm are derived. We show convergence of the hp-version of BEM and present some numerical experiments. Furthermore we consider a mixed boundary element formulation, which is stabilized following ideas of P.Hild, Y. Renard and V.Lleras for the FEM. A mesh-dependent stabilization term is added to the discrete mixed formulation, in order to avoid the discrete inf-sup condition. Existence and uniqueness of the solution of the discrete problem are shown together with a priori error and a posteriori error estimates. Numerical experiments are listed for the stabilized and the non-stabilized cases. A stochastic contact problem with a boundary integral formulation is analyzed. We show that the stochastic mixed formulation is well-posed, study the deformation of an elastic homogeneous material in which Young's modulus (parameter that characterizes the material properties) is a random variable. An extended multiscale finite element method EMsFEM is derived for the analysis of linear elastic heterogeneous materials. The main idea is to construct numerically finite element basis functions that captures the small-scale information (the fine mesh) within each coarse element. The construction of the basis functions is done separately for each coarse element with piecewise linear functions. The boundary conditions for the construction of the multiscale basis functions have a big influence on capturing the small-scale information. We analyse a corresponding FEM/BEM coupling and derive an a priori error and a-posteriori error estimate. Next we present finite element implementations for nonperiodic case.

Keywords. Tresca frictional contact problem, biorthogonal basis functions, stabilized hp-BEM, semi-smooth Newton, Stochastic BEM, Multiscale-FEM.

Zusammenfassung

Wir betrachten ein Kontaktproblem mit Trescareibung zwischen einem elastischen Körper und einem starren Untergrund. Für die entsprechende gemischte Formulierung präsentieren wir eine hp-Diskretisierungstechnik basierend auf biorthogonalen Basisfunktionen und lösen das entsprechende diskrete System mit einem halbglatten Newton-Verfahren. A posteriori Fehlerabschätzungen erlauben die Definition eines Fehlerindikators und eine hp-adaptive Verfeinerungsstrategie. Wir zeigen die Konvergenz einer hp-Version der BEM und präsentieren numerische Experimente. Darüber hinaus betrachten wir eine gemischte Randelementformulierung, die entsprechend den Ideen von P. Hild, V. Lleras and Y. Renard für FEM stabilisiert wird. Ein gitter-abhängiger Stabilisierungsterm wird zur diskreten gemischten Formulierung addiert, um die diskrete inf-sup-Bedingung zu vermeiden. Existenz und Eindeutigkeit der Lösung des diskreten Problems werden zusammen mit a priori und a posteriori Fehlerabschätzungen gezeigt. Numerische Experimente für den stabilisierten und den nicht stabilisierten Fall bestätigen die theoretischen Ergebnisse. Zudem wird ein stochastisches Kontaktproblem in einer Randintegralformulierung analysiert. Wir zeigen, dass die stochastische gemischte Formulierung wohlgestellt ist, und studieren die Verformung eines homogenen elastischen Materials, in dem das Young-Modul (das die Materialeigenschaften charakterisiert) eine Zufallsvariable ist. Eine erweiterte Multiskalen-Finite-Element-Methode EMsFEM wird für die Analyse von heterogenen linearen Materialien hergeleitet. Die Hauptidee besteht darin, numerisch Finite-Element-Basisfunktionen zu konstruieren, die die Mikrostruktur in jedem Grobelement erfassen. Die Konstruktion der Basisfunktionen wird für jedes Grobelement mit stckweise linearen funktionen durchgeführt. Die Randbedingungen für die Konstruktion der Mehrskalen-Basisfunktionen haben nämlich einen großen Einfluss auf die Erfassung der Mikrostruktur. Wir analysieren eine entsprechende FEM/BEM-Kopplung sowie a priori und a posteriori Fehlerabschtzungen. Weiter präsentieren wir Finite-Element-Implementierungen für den nicht periodischen Fall.

Schlagworte: Tresca Reibungskontakt, biorthogonale Basisfunktionen, stabilisiertes hp-BEM, halbglattes Newton-Verfahren, stochastische BEM, Mehrskalen-FEM

Acknowledgments

First of all, I would like to express my sincere gratitude to my advisor

Prof. Dr. Ernst P. Stephan for the continuous support of my Ph.D study and research, for his patience, motivation, enthusiasm, and immense knowledge.

I have furthermore to thank my co-referee Prof. Dr. Joachim Gwinner for his readiness to examine my thesis.

My special thanks go to Dr. Lothar Banz, Dr. Heiko Gimperlein, and my friend Zouhair Nezhi for their support and help in all stages of this thesis.

I also thank all of the members of the Graduiertenkolleg for the various support.

I wish to extend my thanks to the whole staff at the Institute for Applied Mathematics. Particularly to Mrs. Carmen Gatzen and Mrs. Ulla Fleischhauer for their kindness and their support on technical issues.

I am also very grateful to my brothers Chokri, Imed, Taoufik, and to my sisters Zohra and Hanen, and my most heartfelt thanks to my brother Adel Aissaoui, my belle-soeur noussa. Most especially, I would like to thank my wife Imen for her support and patience during my work in this thesis.

I thank my brother Adel for having been the heart of the family and taking such good care of everybody.

Last but not least, I would like to thank my parents for their spiritual care and protection, and for their amazing love and support.

This thesis was supported by IRTG 1627 (International Research and Training Group, granted by DFG).

Hannover, den 09/04/2014 Abderrahman Issaoui

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1 Introduction

Frictional contact problems in linear elasticity play an important role in engineering and structural mechanics. In most cases, contact problems are reformulated in terms of variational inequality problems, see Glowinski et al.[28], Kikuchi and Oden [43]. The approximation of contact problems can be realized by the mixed finite element method in which the non-penetration condition and the friction law are only weakly enforced by a variational inequality. The theory of the mixed finite element method is proposed by Haslinger et al.[33],[34] and developed by Babuška [6], [7] and Brezzi [14]. The key of this approach is the inf-sup condition. The finite element spaces for the primal variables and the multipliers have to fullfil inf-sup condition, which is needed in the convergence analysis. However, a discrete inf-sup condition is difficult to obtain for the hp-BEM. Solving frictional contact problems with the mixed hp-BEM boundary element method is a challenging task in mechanics.

To circuvent this difficulty a stabilization technique is used, by adding supplementary terms in the weak formulation, this method has been introduced and analyzed by Barbosa and Hughes in [11] and [12]. This approach was taken in [36, 37, 45] in the context of h-FEM and is extended to hp-BEM in the present work. The great advantage of this approach is that the stabilized scheme is stable for arbitrary finite element discretizations, since for the Lagrange multiplier space any discretization can be chosen. The stabilized method for low-order finite element discretizations is based on linear H^1-conform functions for the displacement and piecewise linear (or constant) functions for the Lagrange multipliers, see [36, 37, 45]. The use of higher-order polynomials leads to a certain non-conformity in the discretization which requires attention in the convergence analysis.

Many multiscale finite element methods MsFEM have been developed and studied for the analysis of linear elastic heterogeneous materials. The MsFEM has been originally indroduced by Babuška et al. [4, 5] and was developed further by Hou et al. [38, 39] for solving second order elliptic boundary value problems.

In this thesis, an extended multiscale finite element method EMsFEM is derived for the analysis of linear elastic heterogeneous materials. We analyse a FEM/BEM coupling for EMsFEM and derive a priori and a posteriori error estimates .

The thesis is organized as follows. In **Chapter** 1, frictional contact problems in linear elasticity with Tresca friction in 2D are presented and analyzed. A higher-order hp-BEM discretization thechnique of a mixed formulation based an biorthogonal basis functions is introduced. The use of the biorthogonality allows a componentwise decoupling of the inequality constraints. On the other hand the decoupled contact conditions can be represented by the problem of finding the root of a non-linear complementarity function, and therefore the so-called semi-smooth Newton methods can be applied. An a priori error analysis is carried out in the case, where we assume the discrete inf-sup condition.

A posteriori error estimates using an error indicator in the hp-adaptive refinement algorithm are derived. We show convergence of the hp-version of BEM and present numerical experiments that illustrate and confirm our theoretical results.

In **Chapter** 2, we consider a hp-BEM stabilized mixed boundary element formulation for frictional contact problems in linear elasticity in 2D. Here a mesh dependent stabilization term is added to the discrete mixed formulation, in order to avoid the inf-sup condition. The contact constraints are imposed on the discrete global set of affinely transformed Gauss-Lobatto points on the elements. Furthermore, a priori error estimates are presented and a posteriori estimates are derived for higher-order boundary element methods. Numerical experiments are listed for the stabilized and the non-stabilized cases.

In **Chapter** 3, a stochastic contact problem with boundary integral formulation is analyzed. A stochastic mixed formulation is shown to be well-posed. This problem is transformed into an equivalent deterministic one by using a Karhunen-Loève spectral decomposition. The biorthogonality is adapted in the context of a stochastic hp-BEM . The componentwise decoupling of the weak constraints allows to use an Uzawa algorithm to solve the discrete problem. Here the Uzawa algorithm uses a pointwise projection of the Lagrange multiplier. The deterministic formulation allows us to obtain a residual a posteriori error estimator.

Finally in the last chapter, an extended multiscale finite element method EMsFEM is presented for the analysis of linear elastic heterogeneous materials. The idea of the method is to construct numerically the multiscale basis functions to capture the fine scale features of the coarse elements in the multiscale finite element analysis. The construction is done separately for each coarse element by solving a subgrid problem together with suitable boundary conditions.

We introduce a FEM/BEM coupling for the EMsFEM with standard BEM and derive an a posteriori error estimate. Finally, we give for the EMsFEM some numerical experiments.

2 hp-BEM For Frictional Contact Problem in Linear Elasticity

In this chapter, we consider a contact problem between an elastic body and a rigid foundation with Tresca's friction law and present a hp-discretization technique of a mixed formulation based on biorthogonal basis functions. For standard h-version BEM and Signorini conditions see [32]. We solve the corresponding discrete system with the semi-smooth Newton method, we derive a posteriori error estimates and use the error indicator in an hp-adaptive refinement algorithm. We show convergence for hp-version of BEM and present some numerical experiments. Related a posteriori error estimates for friction problems have been considered, for example, in [27, 30, 36, 42, 46, 50].

2.1 Mixed Formulation for Frictional Contact Problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with the boundary $\Gamma := \partial\Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_C$ be decomposed into non-intersecting Neumann Γ_N, Dirichlet Γ_D and contact boundaries Γ_C, where Γ_C can come in contact with the rigid foundation. The problem then consists in finding the displacement fields $\mathbf{u} : \Omega \longrightarrow \mathbb{R}$ such that

$$-\operatorname{div} \sigma(\mathbf{u}) = 0 \qquad \text{in } \Omega \tag{2.1a}$$

$$\sigma(\mathbf{u}) = \mathcal{C} : \epsilon(\mathbf{u}) \qquad \text{in } \Omega \tag{2.1b}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma_D \tag{2.1c}$$

$$\sigma(\mathbf{u}) \cdot n = \mathbf{t} \qquad \text{on } \Gamma_N \tag{2.1d}$$

$$\sigma_n \leq 0,\ u_n \leq g,\ \sigma_n(u_n - g) = 0 \qquad \text{on } \Gamma_C \tag{2.1e}$$

$$|\sigma_t| \leq \mathcal{F}, \quad \sigma_t u_t + \mathcal{F}|u_t| = 0 \qquad \text{on } \Gamma_C \tag{2.1f}$$

Let G(x,y) be the fundamental solution of the Lamé equation

$$G(x,y) := \begin{cases} \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}\left\{\log\frac{1}{|x-y|}\mathbf{I} + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(x-y)\otimes(x-y)}{|x-y|^2}\right\}, & \text{for} \quad d = 2, \\ \\ \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\left\{\frac{1}{|x-y|}\mathbf{I} + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(x-y)\otimes(x-y)}{|x-y|^3}\right\}, & \text{for} \quad d = 3, \end{cases}$$

We consider a homogenous, isotropic, linear Saint Venant-Kirchhoff material , where the stress tensor is given in term of Hooke's tensor by

$$\sigma(\mathbf{u}) = \mathcal{C} : \varepsilon(\mathbf{u}) := \lambda tr\varepsilon(\mathbf{u})\mathbf{I} + 2\mu\varepsilon(\mathbf{u})$$

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

where λ,μ are the Lamé constants, moreover, tr denotes the matrix trace operator and $\mathbf{I}$ the identity in $\mathbb{R}^d$, $d = 2, 3$

The scalar normal and tangential boundary stresses

$$\sigma_n := \mathbf{n} \cdot \sigma(\mathbf{u}) \cdot \mathbf{n} \quad \text{and} \quad \sigma_t := \mathbf{t} \cdot \sigma(\mathbf{u}) \cdot \mathbf{t}$$

are well defined, where $\mathbf{n}$ denotes the unit normal exterior to the contact boundary Γ_C.

Let us denote by $\mathcal{F} > 0$ the given friction threshold on Γ_C, we assume that $\mathcal{F}$ is a constant for the sake of simplicity.

The static Tresca friction condition reads as follows

$$\begin{cases} |\sigma_t| \leq \mathcal{F}, \quad \text{a.e} \quad \text{on} \quad \Gamma_C \\ \text{if} \quad |\sigma_t| < \mathcal{F} \quad \text{then} \quad u_t = 0 \\ \text{if} \quad |\sigma_t| = \mathcal{F} \quad \text{then there exists} \quad \nu \geqslant 0 \quad \text{such that} \quad u_t = -\nu\sigma_t \end{cases}$$

We introduice the single layer potential V, the double layer potential K, the adjoint double layer potential $K^{'}$ and the hypersingular integral operator W for $x \in \Gamma$ by

$$\begin{aligned} V\boldsymbol{\phi}(x) &:= \int_\Gamma G(x,y)\boldsymbol{\phi}(y)\, ds_y, \\ K\mathbf{u}(x) &:= \int_\Gamma (\mathcal{T}_{n_y} G(x,y)^T)\mathbf{u}(y)\, ds_y, \\ K^{'}\boldsymbol{\phi}(x) &:= \mathcal{T}_{n_x} \int_\Gamma G(x,y)\boldsymbol{\phi}(y)\, ds_y, \\ W\mathbf{u}(x) &:= -\mathcal{T}_{n_x} \int_\Gamma (\mathcal{T}_{n_y} G(x,y)^T)\mathbf{u}(y)\, ds_y \end{aligned}$$

where $\mathcal{T}_n$ is the boundary traction operator given by

$$\mathcal{T}_n(\mathbf{u}) := \sigma(\mathbf{u})|_\Gamma \cdot \mathbf{n} \tag{2.2}$$

Lemma 2.1. *[22, 23] Let $\Gamma := \partial\Omega$ be the boundary of a Lipschitz domain Ω. Then the integral operators*

$$\begin{aligned} V \quad &: \boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma) \to \boldsymbol{H}^{\frac{1}{2}+s}(\Gamma), \quad K \quad : \boldsymbol{H}^{\frac{1}{2}+s}(\Gamma) \to \boldsymbol{H}^{\frac{1}{2}+s}(\Gamma) \\ K^{'} \quad &: \boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma) \to \boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma), \quad W \quad : \boldsymbol{H}^{\frac{1}{2}+s}(\Gamma) \to \boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma). \end{aligned}$$

are bounded for all $s \in [-\frac{1}{2}, \frac{1}{2}]$*,i.e. there exists constants* C_V,C_K,$C_{K'}$,$C_W > 0$ *such that*

$$\|V\boldsymbol{\phi}\|_{\boldsymbol{H}^{\frac{1}{2}+s}(\Gamma)} \leq C_V\|\boldsymbol{\phi}\|_{\boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma)}, \quad \|K'\boldsymbol{\phi}\|_{\boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma)} \leq C_{K'}\|\boldsymbol{\phi}\|_{\boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma)},$$
$$\|K\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+s}(\Gamma)} \leq C_K\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+s}(\Gamma)}, \quad \|W\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac{1}{2}+s}(\Gamma)} \leq C_W\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+s}(\Gamma)}.$$

Lemma 2.2. *[51, 19] Let* $\Gamma := \partial\Omega \subset \mathbb{R}^d$ *be boundary of a Lipschitz domain* Ω*.Let* $cap(\Omega) < 1$ *in case* $d = 2$*. Then the single layer potential* V *is* $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$*-elliptic,i.e. there exists a constant* $c_V > 0$*, such that*

$$\langle V\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_\Gamma \geq c_V\|\boldsymbol{\phi}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}, \quad \forall\boldsymbol{\phi} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \tag{2.3}$$

Moreover its inverse operator $V^{-1} : \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \to \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ *exists and*

$$\|V^{-1}\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq c_V^{-1}\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}, \quad \forall\boldsymbol{u} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \tag{2.4}$$

where c_V *is the ellipticity constant of* V*.*

The Steklov-Poincaré operator S is defined by

$$S := W + (K' + \frac{1}{2})V^{-1}(K + \frac{1}{2}).$$

Lemma 2.3. *[19] Let* $\Gamma_0 \subset \Gamma$*. Then the Steklov-Poincaré operator* $S : \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ *is continuous and* $\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Gamma_0)$*-elliptic, i.e. there exists* c_S, $C_S > 0$ *such that*

$$\|S\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C_S\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \quad \forall\boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \tag{2.5}$$

$$\langle S\boldsymbol{u}, \boldsymbol{u}\rangle_\Gamma \geq c_S\|\boldsymbol{u}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Gamma_0)} \quad \forall\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Gamma_0) \tag{2.6}$$

For any displacement $\mathbf{u}$ and for any boundary traction $\sigma(\mathbf{u})\mathbf{n}$ defined on $\partial\Omega$ the following notations are frequently used in this chapter.

$$\mathbf{u} = u_n\mathbf{n} + u_t\mathbf{t} \quad \text{and} \quad \sigma(\mathbf{u})\mathbf{n} = \sigma_n(\mathbf{u})\mathbf{n} + \sigma_t(\mathbf{u})\mathbf{t} \tag{2.7}$$

We need the following function spaces

$$\mathcal{V} := [\tilde{H}^{\frac{1}{2}}(\Sigma)]^d = \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) := \{u \in \mathbf{H}^{\frac{1}{2}}(\Gamma);\ \mathrm{supp}(u) \subset \Sigma\}$$

$$\mathbf{V} := \{\mathbf{u} \in \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) : \mathbf{u} = 0 \quad \text{on} \quad \Gamma_D\}$$
$$\mathcal{W} := [H^{\frac{1}{2}}(\Gamma_C)]^d = \mathbf{H}^{\frac{1}{2}}(\Gamma_C)$$
$$\mathcal{M} := [H^{-\frac{1}{2}}(\Gamma_C)]^d = \mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$$

where

$$\Sigma := \overline{\Gamma_C \cup \Gamma_N}, \quad \Gamma_C \cap \Gamma_N = \emptyset$$

Let $\mathbf{K}$ be a closed convex and nonempty subset of $\mathbf{V}$, to model the nonpenetration condition on the contact boundary Γ_C, where $g \geq 0$ is a given gap function.

$$\mathbf{K} := \{\mathbf{u} \in \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) : u_n \leq g \quad \text{on} \quad \Gamma_C\}$$

We assume that $\mathcal{F} \in L^2(\Gamma_C)$, $\mathbf{t} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ and $g \in H^{\frac{1}{2}}(\Gamma_C)$.

We define the D_tN (Dirichlet-to-Neumann) mapping $\mathbf{u}_\Gamma \to \sigma(\mathbf{u})\cdot \mathbf{n}$, Since S is the D_tN mapping, there holds

$$\sigma_n \equiv S\mathbf{u} \cdot \mathbf{n}|_{\Gamma_C}, \qquad \sigma_t \equiv S\mathbf{u} \cdot \mathbf{t}|_{\Gamma_C} \tag{2.8}$$

where $\mathbf{u}$, σ_n, σ_t solve (2.1) in the distributional sense.

We define the space of Lagrange multipliers by

$$\mathbf{M} = M_n \times M_t \tag{2.9}$$

where

$$M_n := \{\mu_n \in \tilde{H}^{-\frac{1}{2}}(\Gamma_C) : \ \langle \mu_n, v_n \rangle_{\Gamma_C} \leq 0, \ \forall \mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_C) \text{ with } v_n \leq 0 \ a.e \text{ on } \Gamma_C\}$$

and

$$M_t(\mathcal{F}) := \{\mu_t \in L_2(\Gamma_C) : \ |\mu_t| \leq \mathcal{F} \ a.e \text{ on } \Gamma_C\}$$

are the sets of normal and tangential Lagrange multipliers.

The classical formulation (2.1) can be rewritten in a weak sence as a saddle point problem as follows:

Find $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbf{V} \times \mathbf{M}$ such that

$$\langle S\mathbf{u}, \mathbf{v} \rangle_\Sigma + b(\boldsymbol{\lambda}, \mathbf{v}) = \langle \mathbf{t}, \mathbf{v} \rangle_{\Gamma_N} \qquad \forall \mathbf{v} \in \mathbf{V} \tag{2.10a}$$
$$b(\boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{u}) \leqslant \langle g, \mu_n - \lambda_n \rangle_{\Gamma_C} \quad \forall\, \boldsymbol{\mu} \in \mathbf{M}. \tag{2.10b}$$

Here

$$b(\boldsymbol{\mu}, \mathbf{v}) := \langle \mu_n, v_n \rangle_{\Gamma_C} + \langle \mu_t, v_t \rangle_{\Gamma_C} \tag{2.11}$$

where the notation $\langle \cdot, \cdot \rangle_{\Gamma_C}$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma_C)$ and $\tilde{H}^{\frac{1}{2}}(\Gamma_C)$. Note $\boldsymbol{\lambda} = -\sigma(\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_C} \geq 0$ since $\sigma(\mathbf{u}) \cdot \mathbf{n} \leq 0$ on Γ_C in (2.1).

Another classical weak formulation of the problem (2.1) is the primal variational inequality:

Find $\mathbf{u} \in \mathbf{K}$ such that

$$\langle S\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle_\Sigma + j(\mathbf{v}) - j(\mathbf{u}) \geq \langle \mathbf{t}, \mathbf{v} - \mathbf{u} \rangle_{\Gamma_N} \quad \forall \mathbf{v} \in \mathbf{K} \tag{2.12}$$

where

$$j(\mathbf{u}) = \int_{\Gamma_C} \mathcal{F}|u_t|\, ds \tag{2.13}$$

is the friction functional.

As a minimization problem, it reads

$$\mathbf{J}(\mathbf{u}) \le \mathbf{J}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{K} \tag{2.14}$$

where

$$\mathbf{J}(\mathbf{v}) = \frac{1}{2}\langle S\mathbf{v}, \mathbf{v}\rangle_\Sigma + j(\mathbf{v}) - \langle \mathbf{t}, \mathbf{v}\rangle_{\Gamma_N} \tag{2.15}$$

2.2 Existence and uniqueness of the solution

In this section, we study the existence and uniqueness of the solution of the mixed formulation

Theorem 2.1. *There exists a unique solution of problem* (2.10).

Proof. We know from [19] that the minimization problem (2.14) and the variational problem (2.12) are equivalent. According to [24] , $\mathbf{K}$ is a closed, convex set, and $\mathbf{J} : \mathbf{K} \longrightarrow \mathbb{R}$ is continuous, convex and coercive, i.e.

$$\mathbf{J}(\mathbf{v}) \longrightarrow \infty, \text{for} \quad \mathbf{v} \in \mathbf{K} \quad \text{and} \quad \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \longrightarrow \infty.$$

This implies the existence of a minimizer $\mathbf{u}$ of (2.14). Furthermore from [24] $\mathbf{J}$ is strictly convex. Therefore the minimizer $\mathbf{u}$ is unique.

Note that

$$j(\mathbf{v}) = \int_{\Gamma_C} \mathcal{F}|v_t|\, ds = \sup_{\mu_t \in M_t(\mathcal{F})} \langle \mu_t, v_t\rangle_{\Gamma_C}. \tag{2.16}$$

We define the Lagrange functional

$$\mathcal{L}(\mathbf{v}, \mu_n, \mu_t) := \frac{1}{2}\langle S\mathbf{v}, \mathbf{v}\rangle - L(\mathbf{v}) + \langle \mu_n, v_n - g\rangle_{\Gamma_C} + \langle \mu_t, v_t\rangle_{\Gamma_C} \tag{2.17}$$

Note that by (2.16) we get

$$\mathbf{J}(\mathbf{u}) = \inf_{\mathbf{v}\in\mathbf{V}} \sup_{(\mu_n,\mu_t)\in M_n\times M_t(\mathcal{F})} \mathcal{L}(\mathbf{v}, \mu_n, \mu_t) = \mathcal{L}(\mathbf{v}, \lambda_n, \lambda_t). \tag{2.18}$$

Thus, for any saddle point $(\mathbf{u}, \lambda_n, \lambda_t) \in \mathbf{V} \times M_n \times M_t(\mathcal{F})$ of $\mathcal{L}$, $\mathbf{u}$ is a minimizer of $\mathbf{J}$.

Since $M_t(\mathcal{F})$ is bounded, the existence of a unique saddle point is guaranteed, if there exists a constant $\beta > 0$ such that

$$\beta \|\mu_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq \sup_{\mathbf{v}\in\mathbf{V}, \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}=1} \langle \mu_n, \gamma_n^c(\mathbf{v})\rangle \tag{2.19}$$

where

$$\gamma_n^c(\mathbf{v}) = v_n|_{\Gamma_C}$$

The above condition (inf-sup condition) follows from the closed range theorem and the surjectivity of γ_n^c, see [50] .

We conclude that there exists a unique saddle point $(\mathbf{u}, \lambda_n, \lambda_t) \in \mathbf{V} \times M_n \times M_t(\mathcal{F})$ of $\mathcal{L}$, which is equivalently characterized by the mixed variational formulation (2.10). $\square$

2.3 Discretization of the Lamé problem

Let $\mathcal{T}_{hp}$ denote a partition of $\bar{\Gamma}_C \cup \bar{\Gamma}_N$ into disjoint straight line segments I, such that all corners of $\bar{\Gamma}_C \cup \bar{\Gamma}_N$ and all end points $\bar{\Gamma}_C \cap \bar{\Gamma}_N$, $\bar{\Gamma}_D \cap \bar{\Gamma}_N$ are nodes of $\mathcal{T}_{hp}$. For simplicity we assume $meas(\Gamma_C) > 0$ and $\bar{\Gamma}_C \cap \bar{\Gamma}_D = \emptyset$. Furthermore we define the set of Gauss-Lobatto points $G_{I,hp}$ on each element $I \in \mathcal{T}_{hp}$ of corresponding polynomial degree p_I and set $G_{hp} := \cup_{I\in\mathcal{T}_{hp}} G_{I,hp}$, where $p = (p_I)_{I\in\mathcal{T}_{hp}}$ associates to each element of $\mathcal{T}_{hp}$ a polynomial degree $p_I \geq 1$.

We introduce the space of continuous piecewise polynomials for the discretization of $\mathbf{u}$:

$$\mathbf{V}_{hp} := \{\mathbf{u}^{hp} \in \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) : \forall I \in \mathcal{T}_{hp}, \mathbf{u}^{hp}|_I \in [\mathcal{P}_{p_I}(I)]^2, \mathbf{u}^{hp} = 0 \quad \text{on} \quad \Gamma_D\} \subset \mathbf{H}^{\frac{1}{2}}(\Gamma)$$

and the space of piecewise polynomial discrete boundary tractions

$$\mathbf{W}_{hp} := \{\boldsymbol{\phi} \in \mathbf{H}^{-\frac{1}{2}}(\Sigma) : \forall I \in \mathcal{T}_h, \boldsymbol{\phi}|_I \in [\mathcal{P}_{p_I-1}(I)]^2\} \subset \mathbf{H}^{-\frac{1}{2}}(\Gamma)$$

The space $\mathbf{V}_{hp}$ is spanned by the 2-d nodal basis $\{\phi_i \mathbf{e}_k, i = 1, ..., N_{\mathbf{V}}, k = 1, 2\}$, where $\mathbf{e}_k$ denotes the k-th unit vector, ϕ_i the scalar Gauss-Lobatto Lagrange basis function associated with the node i and $N_{\mathbf{V}}$ the total number of the nodes. We denote by N_C the set of all nodes on the contact boundary Γ_C. Furthermore we define the space of discrete vectorial Lagrange multipliers by $\mathbf{M}_{hp}$, which is spanned by $\{\psi_i \mathbf{e}_k, i = 1, ..., N_C, k = 1, 2\}$.

The dual or biorthogonal basis functions ψ_i satisfy the orthogonality relations

$$\int_{\Gamma_C} \psi_i \phi_j ds = \delta_{ij} \int_{\Gamma_C} \phi_j ds. \tag{2.20}$$

In order to obtain a well-defined normal we introduce the averaged normal at the node in $\Gamma_C \cap G_{hp}$ as follows.

Let us define by $\mathcal{E}_i$ for $i \in N_C$ all surface elements of the mesh containing node i .

$$\mathcal{E}_i = \{e_i \in \mathcal{T}_{hp} : e_i \in \Gamma_C, i \in N_C\} \tag{2.21}$$

Then the normal $\mathbf{n}_i$ can be defined via

$$\mathbf{n}_i := \frac{1}{\sum_{e_i \in \mathcal{E}_i} \mathbf{n}_{e_i}} \sum_{e_i \in \mathcal{E}_i} \mathbf{n}_{e_i}, \quad i \in N_C \tag{2.22}$$

where $\mathbf{n}_{e_i}$ is the unit normal vector of the surface edge e_i.

We can then express the discrete function $\mathbf{u}^{hp} \in \mathbf{V}_{hp}$ for $1 \leq i \leq \dim \mathbf{V}_{hp}$ by

$$\mathbf{u}^{hp} := \sum_i (u_{in}\mathbf{n}_i + \mathbf{u}_{it})\phi_i. \tag{2.23}$$

The normal and tangential part of the discrete function $\mathbf{u}^{hp}$ are given by

$$u_n^{hp} := \sum_i u_{in}\phi_i, \quad u_t^{hp} = \sum_i \mathbf{u}_{it}\phi_i, \tag{2.24}$$

where u_{in} and $\mathbf{u}_{it}$ are the normal and tangential component of the nodal value $\mathbf{u}_i$, given by

$$u_{in} := \mathbf{u}_i\mathbf{n}_i, \quad u_{it} := \mathbf{u}_i - u_{in}\mathbf{n}_i. \tag{2.25}$$

We define in the same way the discrete Lagrange multiplier for $1 \leq i \leq \dim \mathbf{M}_{hp}(\mathcal{F})$ as

$$\boldsymbol{\lambda}^{hp} := \sum_i (\lambda_{in}\mathbf{n}_i + \boldsymbol{\lambda}_{it})\psi_i \tag{2.26}$$

For the frictionless contact problem the tangential component λ_{it} equals 0.

We introduce the subset

$$V_{hp}^- := \{\mathbf{v}^{hp} \in \mathbf{V}_{hp} : v_n^{hp} \leq 0\}. \tag{2.27}$$

As in [41] we define the space for the discrete Lagrange multiplier by

$$\mathbf{M}_{hp}(\mathcal{F}) := \{\boldsymbol{\mu}^{hp} \in span\{\psi_i\}_{i=1}^{N_C} : \langle \boldsymbol{\mu}^{hp}, \mathbf{v}^{hp} \rangle \leq \langle \mathcal{F}, |v_t^{hp}|_h \rangle, \mathbf{v}^{hp} \in V_{hp}^-\}, \tag{2.28}$$

where the absolute value of the function v_t^{hp} is given for $1 \leq i \leq \dim \mathbf{M}_{hp}(\mathcal{F})$ and $x \in \Gamma_C$ by (see.[41])

$$|v_t^{hp}|_h := \sum_i |v_{it}|\phi_i(x). \tag{2.29}$$

We define the weighted gap vector for $1 \leq i \leq \dim \mathbf{M}_{hp}(\mathcal{F})$ by

$$g_i := \frac{1}{D_i} \int_{\Gamma_C} g\psi_i(x)\, ds, \tag{2.30}$$

where

$$D_i = \int_{\Gamma_C} \phi_i(x)\, ds.$$

The spaces V_{hp}^- and $\mathbf{M}_{hp}(\mathcal{F})$ can be rewritten as follows; cf. [[41], Lemma 2.3] for the h-version, i.e. $p = 1$.

Lemma 2.4 ([8], Lemma 6.1)**.** *The space V_{hp}^{-} in* (2.27) *can be rewritten as*

$$V_{hp}^{-} := \{\boldsymbol{v}^{hp} = \sum_{i \in N_C} \boldsymbol{v}_i \phi_i \in \boldsymbol{V}_{hp} : v_{in} \leq 0, i \in N_C\} \tag{2.31}$$

and the space in (2.28) *is equivalent to*

$$\boldsymbol{M}_{hp}(\mathcal{F}) := \{\boldsymbol{\mu}^{hp} = \sum_{i \in N_C} \boldsymbol{\mu}_i \psi_i : \mu_{in} \geq 0, |\mu_{it}| \leq \mathcal{F}, i \in N_C\} \tag{2.32}$$

Since an explicit representation of V^{-1} is not known, we need to approximate the Steklov-Poincaré operator.

Let $i_{hp} : \mathbf{W}_{hp} \hookrightarrow \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $j_{hp} : \mathbf{V}_{hp} \hookrightarrow \mathbf{H}^{\frac{1}{2}}(\Gamma)$ denote the canonical imbedding the dual i^*_{hp} and j^*_{hp}

The approximation S_{hp} of the Poincaré-Steklov operator is given by

$$S_{hp} := W + (K' + \frac{1}{2}) i_{hp} (i^*_{hp} V i_{hp})^{-1} i^*_{hp} (K + \frac{1}{2}) \tag{2.33}$$

We can rewrite the space $\mathbf{M}_{hp}(\mathcal{F})$ as $\mathbf{M}_{hp}(\mathcal{F}) := M_{n,hp} \times M_{t,hp}(\mathcal{F})$, where $M_{n,hp}$ and $M_{t,hp}$ are the sets of normal and tangential discrete Lagrange multiplier, respectively.

The discretized version of (2.10) reads as:

Find $\mathbf{u}^{hp} \in \mathbf{V}_{hp}$ and $\boldsymbol{\lambda}^{hp} = (\lambda_n^{hp}, \lambda_t^{hp}) \in \mathbf{M}_{hp}(\mathcal{F}) := M_{n,hp} \times M_{t,hp}(\mathcal{F})$

$$\langle S_{hp} \mathbf{u}^{hp}, \mathbf{v}^{hp} \rangle_{\Sigma} + b(\boldsymbol{\lambda}^{hp}, \mathbf{v}^{hp}) = L(\mathbf{v}^{hp}) \qquad \forall \mathbf{v}^{hp} \in \mathbf{V}_{hp} \tag{2.34a}$$

$$b(\boldsymbol{\mu}^{hp} - \boldsymbol{\lambda}^{hp}, \mathbf{u}^{hp}) \leqslant \left\langle g, \mu_n^{hp} - \lambda_n^{hp} \right\rangle_{\Gamma_C} \quad \forall\, \boldsymbol{\mu}^{hp} \in \mathbf{M}_{hp}(\mathcal{F}). \tag{2.34b}$$

Lemma 2.5. *[19] Let $\Gamma := \partial\Omega \subset \mathbb{R}^d$ be the boundary of a Lipschitz domain Ω and arbitrary $\Gamma_0 \subset\subset \Gamma$. Then the approximation of the Steklov-Poincaré operator $S_{hp} : \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ is continuous and $\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Gamma_0)$-elliptic, i.e. there exists $c_{S_{hp}}$, $C_{S_{hp}} > 0$ such that*

$$\|S_{hp}\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C_{S_{hp}} \|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \tag{2.35}$$

$$\langle S_{hp}\boldsymbol{u}, \boldsymbol{u} \rangle_{\Gamma} \geq c_{S_{hp}} \|\boldsymbol{u}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Gamma_0)} \quad \forall \boldsymbol{u} \in \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Gamma_0). \tag{2.36}$$

We define the operator $E_{hp} = S - S_{hp}$ which represents the error in the approximation of the Steklov-Poincaré operator.

Lemma 2.6. *[19] The operator E_{hp} is bounded, i.e. there exists $C_{E_{hp}} > 0$ such that*

$$\|E_{hp}\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C_{E_{hp}} \|u\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \tag{2.37}$$

and there exists a constant $C_0 > 0$, such that

$$\|E_{hp}\boldsymbol{u}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C_0 \inf_{\Phi \in \boldsymbol{W}_{hp}} \|V^{-}(K+\frac{1}{2})\boldsymbol{u} - \Phi\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma). \tag{2.38}$$

Extending the approach in [8] for the scalar case to the vector case we obtain the following result.

Theorem 2.2. *There exists exactly one solution to the discrete mixed formulation* (2.34).

Proof. **Uniqueness**: Let $(\mathbf{u}_1^{hp}, \boldsymbol{\lambda}_1^{hp})$ and $(\mathbf{u}_2^{hp}, \boldsymbol{\lambda}_2^{hp})$ two solutions of the discrete mixed formulation (2.34). Then we have

$$\langle S_{hp}(\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}), \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp} \rangle_\Sigma + b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) = 0 \tag{2.39}$$

Choosing $\boldsymbol{\mu}_1^{hp} = \boldsymbol{\lambda}_2^{hp}$ and $\boldsymbol{\mu}_2^{hp} = \boldsymbol{\lambda}_1^{hp}$ in (2.34b) we get

$$b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) \geq 0 \tag{2.40}$$

Using Lemma 2.5 we obtain

$$c_{S_{hp}} \|\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \leq \langle S_{hp}(\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}), \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp} \rangle_\Sigma + b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) = 0. \tag{2.41}$$

Consequently the first argument $\mathbf{u}^{hp}$ is unique.

Since $\mathbf{u}^{hp}$ is unique we have for all $\mathbf{v}^{hp} \in \mathbf{V}_{hp}$

$$0 = b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{v}^{hp}). \tag{2.42}$$

Using the linear combinations of $\boldsymbol{\lambda}_1^{hp}$,$\boldsymbol{\lambda}_2^{hp}$ and $\mathbf{v}^{hp}$ and the biorthogonality, we obtain the relation

$$0 = \sum_{p \in \Gamma_C \cap G_{hp}} [(\lambda_{1,pn} - \lambda_{2,pn})v_{pn} + (\boldsymbol{\lambda}_{1,pt} - \boldsymbol{\lambda}_{2,pt})\mathbf{v}_{pt}]D_p. \tag{2.43}$$

where

$$D_p = \int_{\Gamma_C} \phi_p(x)\, ds.$$

Due to the arbitrary choice of v_{pn} and $\mathbf{v}_{pt}$, choosing $\mathbf{v}_{pt} = 0$ in (2.43) we get

$$(\lambda_{1,pn} - \lambda_{2,pn})v_{pn} = 0 \quad \text{and} \quad \lambda_{1,pn} = \lambda_{2,pn}$$

We choose now $v_{pn} = 0$, we obtain $\boldsymbol{\lambda}_{1,pt} = \boldsymbol{\lambda}_{2,pt}$, consequently the second argument $\boldsymbol{\lambda}^{hp}$ is unique .

Existence: The inequality (2.34b) is obviously equivalent to the following conditions:

$$\lambda_n^{hp} \in M_{n,hp}, \quad \langle \mu_n^{hp} - \lambda_n^{hp}, u_n^{hp} - g \rangle \leq 0, \quad \forall \mu_n^{hp} \in M_{n,hp}, \tag{2.44}$$

$$\lambda_t^{hp} \in M_{t,hp}, \quad \langle \mu_t^{hp} - \lambda_t^{hp}, u_t^{hp} \rangle \leq 0, \quad \forall \mu_t^{hp} \in M_{t,hp}. \tag{2.45}$$

The inequalities can be replaced by projections $P_{M_{n,hp}}$ and $P_{M_{t,hp}}$ respectively, where

$$\lambda_n^{hp} = P_{M_{n,hp}}(\lambda_n^{hp} + r(u_n^{hp} - g)), \quad \lambda_t^{hp} = P_{M_{t,hp}}(\lambda_t^{hp} + r u_t^{hp}) \tag{2.46}$$

Here the maps $P_{M_{n,hp}}$ and $P_{M_{t,hp}}$ stand for the L_2 projections onto $M_{n,hp}$ and $M_{t,hp}$, respectively, and $r > 0$ is an arbitrary parameter. The discrete mixed formulation leads to a fixed points formulation

$$\begin{aligned} T : M_{n,hp} \times M_{t,hp}(\mathcal{F}) &\longrightarrow M_{n,hp} \times M_{t,hp}(\mathcal{F}) \\ (\lambda_n^{hp}, \lambda_t^{hp}) &\longmapsto \left(P_{M_{n,hp}}(\lambda_n^{hp} + r(u_n^{hp} - g)), P_{M_{t,hp}}(\lambda_t^{hp} + r u_t^{hp})\right) \end{aligned} \tag{2.47}$$

where T is the fixed point operator ,which defines the fixed point iteration

$$\left((\lambda_n^{hp})^{k+1}, (\lambda_t^{hp})^{k+1}\right) = T\left((\lambda_n^{hp})^{k}, (\lambda_t^{hp})^{k}\right). \tag{2.48}$$

From Lemma 2.5 we have

$$-\|\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} \gtrsim -\|\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \geq -c^{-1}_{S_{hp}} \langle S_{hp}(\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}), \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\rangle_\Sigma \tag{2.49}$$

Using the notation $\delta\mathbf{u}^{hp} = \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}$ and $\delta\boldsymbol{\lambda}^{hp} = \boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}$

$$\begin{aligned} \|T(\boldsymbol{\lambda}_1^{hp}) - T(\boldsymbol{\lambda}_2^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)} &\leq \|\delta\lambda_n^{hp} + r\delta u_n^{hp}|_{\Gamma_C}\|^2_{\mathbf{L}_2(\Gamma_C)} + \|\delta\lambda_t^{hp} + r\delta u_t|^{hp}_{\Gamma_C}\|^2_{\mathbf{L}_2(\Gamma_C)} \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} + 2rb(\delta\boldsymbol{\lambda}^{hp}, \delta\mathbf{u}^{hp}) + r^2\|\delta\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} - 2r\langle S_{hp}\delta\mathbf{u}^{hp}, \delta\mathbf{u}^{hp}\rangle_\Sigma + r^2\|\delta\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} - 2rc_{S_{hp}}\|\delta\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} + r^2\|\delta\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)} \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)}(1 - 2rc_{S_{hp}}\beta^2 + r^2\beta^2) \end{aligned}$$

where $\beta = \frac{\|\delta\boldsymbol{\lambda}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)}}{\|\delta\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(\Gamma_C)}}$ and T is strict contraction for $0 < r < 2c_{S_{hp}}$. By the Banach fixed point theorem exists a $\boldsymbol{\lambda}^{hp} = (\lambda_n^{hp}, \lambda_t^{hp})$ which satisfies (2.47). For any given $\boldsymbol{\lambda}^{hp}$, problem (2.34a) reduces to a linear, finite dimentional problem. Hence, the uniqueness result of $\mathbf{u}^{hp}$ implies the existence of a $\mathbf{u}^{hp}(\boldsymbol{\lambda}^{hp})$. □

2.4 Semi-smooth Newton method and algebraic representation

For the solution of the discrete mixed formulation (2.34) we describe a semi-smooth Newton approach which is equivalent to an active set strategy.

Lemma 2.7. *The variational inequality constraint* (2.34b) *is equivalent to the decoupled pointwise non-penetration conditions*

$$u_{in} \leq g_i, \quad \lambda_{in} \geq 0, \quad \lambda_{in}(u_{in} - g_i) = 0 \tag{2.50}$$

and the friction conditions

$$\begin{aligned} &|\boldsymbol{\lambda}_{it}| \leq \mathcal{F}_i \\ &|\boldsymbol{\lambda}_{it}| < \mathcal{F}_i \Rightarrow u_{it} = 0 \\ &|\boldsymbol{\lambda}_{it}| = \mathcal{F}_i \Rightarrow \exists \alpha \in \mathbb{R} : \lambda_{it} = \alpha u_{it} \end{aligned} \tag{2.51}$$

for all $1 \leq i \leq N_C$ *where* $\mathcal{F}_i$ *is the friction at the node defined by*

$$\mathcal{F}_i := \int_{\Gamma_C} \mathcal{F} \phi_i \, ds \tag{2.52}$$

Proof. see [41] and [8] □

The penalized Fischer-Burmeister non linear complementarity function presented in [18] is defined by

$$\begin{aligned} \varphi_\mu(\mathbf{u}^{hp}|_{\Gamma_C}, \boldsymbol{\lambda}^{hp}) = \mu &\left(\lambda_{in} + (g_i - u_{in}) - \sqrt{\lambda_{in}^2 + (g_i - u_{in})^2} \right) \\ &+ (1-\mu)\max\{0, \lambda_{in}\}\max\{0, (g_i - u_{in})\} \end{aligned} \tag{2.53}$$

with $\mu \in (0,1]$ and $1 \leq i \leq N_C$.

We define the nonlinear complementarity function (NCF) for Tresca friction as

$$C_T(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{hp}) = \max\{\mathcal{F}_i, \|\boldsymbol{\lambda}_{it} + \alpha \mathbf{u}_{it}\|\}\boldsymbol{\lambda}_{it} - \mathcal{F}_i(\boldsymbol{\lambda}_{it} + \alpha\, \mathbf{u}_{it}) \tag{2.54}$$

for any positive parameter $\alpha > 0$.

Theorem 2.3. *The pair* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp})$ *satisfies the frictional contact conditions* (2.51) *if and only if*

$$C_T(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp}) = 0. \tag{2.55}$$

The decoupled pointwise non penetration condition is equivalent to

$$\varphi_\mu(\boldsymbol{u}^{hp}|_{\Gamma_C}, \boldsymbol{\lambda}^{hp}) = 0. \tag{2.56}$$

Proof. For the first equivalence see[[41],theoerem5.1]. For the second equivalence we know that the NCF-function satisfies

$$\varphi_\mu(\mathbf{u}^{hp}|_{\Gamma_C}, \boldsymbol{\lambda}^{hp}) = 0 \Leftrightarrow u_{in} \leq g_i, \quad \lambda_{in} \geq 0, \quad \lambda_{in}(u_{in} - g_i) = 0. \tag{2.57}$$

□

Lemma 2.8. *The discrete problem* (2.34) *is equivalent to solving the nonlinear system*

$$0 = F(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp}) = \begin{pmatrix} A\boldsymbol{u}^{hp} - D\boldsymbol{\lambda}^{hp} - f \\ \varphi_\mu(\boldsymbol{u}^{hp}|_{\Gamma_C}, \boldsymbol{\lambda}^{hp}) \\ C_T(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp}) \end{pmatrix}. \tag{2.58}$$

Proof. $A\mathbf{u}^{hp} - D\boldsymbol{\lambda}^{hp} - f = 0$ is the matrix representation of the first equation of the discrete problem. By Theorem (2.3) the conditions $\varphi_\mu(\mathbf{u}^{hp}|_{\Gamma_C}, \boldsymbol{\lambda}^{hp}) = 0$ and $C_T(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{hp}) = 0$ are equivalent to the decoupled pointwise non-penetration condition (2.50) and the frictional condition (2.51), respectively, which are equivalent to (2.34b) by lemma 2.7. □

The generalized Newton's method for the solution of the nonlinear system (2.58) can be defined as

$$(\mathbf{u}^k, \boldsymbol{\lambda}^k) = (\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}) - F'(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1})^{-1} F(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}), \quad k = 1, 2, \dots \tag{2.59}$$

Since F is not differentiable everywhere, and therefore the Jacobian matrix $F'(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1})$ does not exist everywhere, one has to choose a suitable approximation

$H_{k-1} \in F'(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1})$ and solve the equation

$$(\mathbf{u}^k, \boldsymbol{\lambda}^k) = (\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}) - H_{k-1}^{-1} F(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}), \quad k = 1, 2, \dots \tag{2.60}$$

which is equivalent to

$$H_{k-1} \begin{pmatrix} \delta\mathbf{u}^k \\ \delta\boldsymbol{\lambda}^k \end{pmatrix} = -F(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}). \tag{2.61}$$

Here H_{k-1} is the Clarke subdifferential of F at $(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1})$ defined by

$$H_k = \begin{pmatrix} A & -D \\ \frac{\partial\varphi_\mu(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\mathbf{u}} & \frac{\partial\varphi_\mu(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\boldsymbol{\lambda}} \\ \frac{\partial C_T(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\mathbf{u}} & \frac{\partial C_T(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\boldsymbol{\lambda}} \end{pmatrix}. \tag{2.62}$$

We obtain $(\delta\mathbf{u}^k, \delta\boldsymbol{\lambda}^k)$ by solving the system

$$\begin{pmatrix} A & -D \\ \frac{\partial\varphi_\mu(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\mathbf{u}} & \frac{\partial\varphi_\mu(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\boldsymbol{\lambda}} \\ \frac{\partial C_T(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\mathbf{u}} & \frac{\partial C_T(\mathbf{u}^k, \boldsymbol{\lambda}^k)}{\partial\boldsymbol{\lambda}} \end{pmatrix} \begin{pmatrix} \delta\mathbf{u}^k \\ \delta\boldsymbol{\lambda}^k \end{pmatrix} = - \begin{pmatrix} F_1(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}) \\ F_2(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}) \\ F_3(\mathbf{u}^{k-1}, \boldsymbol{\lambda}^{k-1}) \end{pmatrix} \tag{2.63}$$

where

$$\begin{aligned} F_1(\mathbf{u}, \boldsymbol{\lambda}) &= A\mathbf{u} - D\boldsymbol{\lambda} - f \\ F_2(\mathbf{u}, \boldsymbol{\lambda}) &= \varphi_\mu(\mathbf{u}, \boldsymbol{\lambda}) \\ F_3(\mathbf{u}, \boldsymbol{\lambda}) &= C_T(\mathbf{u}, \boldsymbol{\lambda}). \end{aligned}$$

Let $\mathcal{S}$ be the set of all nodes on Γ_C, and $\mathcal{N}$ all remaining nodes . Now, the algebraic representation is given by the linear system

$$\begin{pmatrix} A_{\mathcal{N}\mathcal{N}} & A_{\mathcal{N}\mathcal{S}} & 0 \\ A_{\mathcal{S}\mathcal{N}} & A_{\mathcal{S}\mathcal{S}} & D_{\mathcal{S}} \\ 0 & \frac{\partial \varphi_\mu(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1})}{\partial \mathbf{u}_{\mathcal{S}}} & \frac{\partial \varphi_\mu(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1})}{\partial \boldsymbol{\lambda}_{\mathcal{S}}} \\ 0 & \frac{\partial C_T(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1})}{\partial \mathbf{u}_{\mathcal{S}}} & \frac{\partial C_T(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1})}{\partial \boldsymbol{\lambda}_{\mathcal{S}}} \end{pmatrix} \begin{pmatrix} \delta \mathbf{u}^k_{\mathcal{N}} \\ \delta \mathbf{u}^k_{\mathcal{S}} \\ \delta \boldsymbol{\lambda}^k_{\mathcal{S}} \end{pmatrix} = - \begin{pmatrix} F_{1,\mathcal{N}}(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \\ F_{1,\mathcal{S}}(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \\ F_2(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \\ F_3(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \end{pmatrix}. \quad (2.64)$$

We remark that the zero block in the coupling matrix D refers to the lines associated with the nodes in set $\mathcal{N}$.

We have

$$\varphi_\mu(\mathbf{u},\boldsymbol{\lambda}) = \varphi_\mu(\mathbf{N}.\mathbf{u}|_{\Gamma_C}, \mathbf{N}.\lambda) = \varphi_\mu(u_{n,\mathcal{S}}, \lambda_{n,\mathcal{S}}) \quad (2.65)$$

and

$$C_T(\mathbf{u},\boldsymbol{\lambda}) = C_T(\mathbf{T}.\mathbf{u}|_{\Gamma_C}, \mathbf{T}.\boldsymbol{\lambda}) = C_T(u_{t,\mathcal{S}}, \lambda_{t,\mathcal{S}}), \quad (2.66)$$

where **N**, **T** are the algebraic representations of the normal,respectively, tangential vector. Altogether,

$$\begin{pmatrix} A_{\mathcal{N}\mathcal{N}} & A_{\mathcal{N}\mathcal{S}} & 0 \\ A_{\mathcal{S}\mathcal{N}} & A_{\mathcal{S}\mathcal{S}} & D_{\mathcal{S}} \\ 0 & \frac{\partial \varphi_\mu(u^{k-1}_{n,\mathcal{S}},\lambda^{k-1}_{n,\mathcal{S}})}{\partial u_{n,\mathcal{S}}} & \frac{\partial \varphi_\mu(u^{k-1}_{n,\mathcal{S}},\lambda^{k-1}_{n,\mathcal{S}})}{\partial \lambda_{n,\mathcal{S}}} \\ 0 & \frac{\partial C_T(u^{k-1}_{t,\mathcal{S}},\lambda^{k-1}_{t,\mathcal{S}})}{\partial u_{t,\mathcal{S}}} & \frac{\partial C_T(u^{k-1}_{t,\mathcal{S}},\lambda^{k-1}_{t,\mathcal{S}})}{\partial \lambda_{t,\mathcal{S}}} \end{pmatrix} \begin{pmatrix} \delta \mathbf{u}^k_{\mathcal{N}} \\ \delta \mathbf{u}^k_{\mathcal{S}} \\ \delta \boldsymbol{\lambda}^k_{\mathcal{S}} \end{pmatrix} = - \begin{pmatrix} F_{1,\mathcal{N}}(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \\ F_{1,\mathcal{S}}(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \\ F_2(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \\ F_3(\mathbf{u}^{k-1},\boldsymbol{\lambda}^{k-1}) \end{pmatrix} \quad (2.67)$$

with

$$\frac{\partial \varphi_\mu}{\partial \mathbf{u}}(\mathbf{u},\boldsymbol{\lambda}) = \begin{cases} \mu & \text{, if } \lambda_n = u_n - g = 0 \\ \mu\left(1 - \frac{u_n - g}{\sqrt{\lambda^2 + (u_n-g)^2}}\right) + (1-\mu)\lambda_n & \text{, if } \lambda_n > 0 \text{ and } u_n > g \\ \mu\left(1 - \frac{u_n - g}{\sqrt{\lambda_n^2 + (u_n-g)^2}}\right) & \text{, otherwise} \end{cases}$$

$$\frac{\partial \varphi_\mu}{\partial \boldsymbol{\lambda}}(\mathbf{u},\boldsymbol{\lambda}) = \begin{cases} \mu & \text{, if } \lambda_n = u_n - g = 0 \\ \mu\left(1 - \frac{\lambda_n}{\sqrt{\lambda_n^2 + (u_n-g)^2}}\right) + (1-\mu)(u_n - g) & \text{, if } \lambda_n > 0 \text{ and } u_n > g \\ \mu\left(1 - \frac{\lambda_n}{\sqrt{\lambda_n^2 + (u_n-g)^2}}\right) & \text{, otherwise} \end{cases}$$

$$\frac{\partial C_T}{\partial \mathbf{u}}(\mathbf{u},\boldsymbol{\lambda}) = \begin{cases} -\mathcal{F}\,\alpha & \text{, if } \|\lambda_t + \alpha u_t\| \leq \mathcal{F} \\ \alpha(\lambda_t - \mathcal{F}) & \text{, if } \|\lambda_t + \alpha u_t\| > \mathcal{F} \\ -\alpha(\lambda_t - \mathcal{F}) & \text{, otherwise} \end{cases}$$

$$\frac{\partial C_T}{\partial \boldsymbol{\lambda}}(\mathbf{u},\boldsymbol{\lambda}) = \begin{cases} 0 & \text{, if } \|\lambda_t + \alpha u_t\| \leq \mathcal{F} \\ \lambda_t + \|\lambda_t + \alpha u_t\| - \mathcal{F} & \text{, if } \|\lambda_t + \alpha u_t\| > \mathcal{F} \\ -\lambda_t + \|\lambda_t + \alpha u_t\| - \mathcal{F} & \text{, otherwise.} \end{cases}$$

If a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and $\Psi(x) = 0$ if and only if x solves the NCF, then $\Psi(x)$ is called a merit function of the NCF-function. Hence finding a solution of the NCF is equivalent to finding a global minimum of the unconstrained minimization $\min_{x\in\mathbb{R}^n} \Psi(x)$ with optimal value zero .

We define the nonnegative merit function

$$\Psi(\mathbf{u},\boldsymbol{\lambda}) = \frac{1}{2}F(\mathbf{u},\boldsymbol{\lambda})^2 \tag{2.68}$$

Algorithm 2.1. *(Semi-smooth Newton algorithm)*

---

1. *Initialisation: Choose initial solution* $\boldsymbol{u}^0, \boldsymbol{\lambda}^0 \in \mathbb{R}^n$, $\rho > 0$, $\beta \in (0,1)$, $\sigma \in (0, \frac{1}{2})$, $p > 2$
2. *For* $k = 0, 1, 2, \ldots$ *do*
   a) *Termination Criterion*

      If $\|\nabla\Psi(\boldsymbol{u}^k, \boldsymbol{\lambda}^k)\| < tol$ *or* $\|\Psi(\boldsymbol{u}^k, \boldsymbol{\lambda}^k)\| < tol$ *then stop*

   b) *Search Direction Calculation*

      Compute subdifferential $H_k \in \partial F(\boldsymbol{u}^k, \boldsymbol{\lambda}^k)$ *and find* $d^k = (d^k_{\boldsymbol{u}}, d_{\boldsymbol{\lambda}})^k \in \mathbb{R}^{2n}$ *s.t*

$$H_k d^k = -F(\boldsymbol{u}^k, \boldsymbol{\lambda}^k). \tag{2.69}$$

If (2.69) *not solvable or if the descent condition*

$$\nabla\Psi(\boldsymbol{u}^k, \boldsymbol{\lambda}^k) d^k \leq -\rho \|d^k\|^p \tag{2.70}$$

is not satisfied, set $d^k := -\nabla\Psi(\boldsymbol{u}^k, \boldsymbol{\lambda}^k)$.

(c) Line Search

Compute search length $t_k := max \beta^l : l = 0, 1, 2, ...$ *s.t*

$$\Psi(\boldsymbol{u}^k + t_k d^k_{\boldsymbol{u}}, \boldsymbol{\lambda}^k + t_k d^k_{\boldsymbol{\lambda}}) \leq \Psi(\boldsymbol{u}^k, \boldsymbol{\lambda}^k) + \sigma t_k \nabla\Psi(\boldsymbol{u}^k, \boldsymbol{\lambda}^k) d^k.$$

(d) Update

Update the solution vectors and goto step 2.

$$\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + t_k d^k_{\boldsymbol{u}}, \quad \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + t_k d^k_{\boldsymbol{\lambda}}.$$

Theorem 2.4. *The semi-smooth Newton algorithm* (2.71) *converges locally superlinear, i.e.* $\lim_{k\to\infty} \frac{||e^{k+1}||}{||e^k||} = 0$.

$$\left(u^{k+1}, \lambda^{k+1}\right)^T = \left(u^k, \lambda^k\right)^T - H_k^{-1} F(u^k, \lambda^k) \tag{2.71}$$

with $H_k \in \partial F(u^k, \lambda^k)$ *a subgradient of* F *at* $\left(u^k, \lambda^k\right)^T$.

In the frictionless case, i.e. $\mathcal{F} = 0$ *and* $C_T(u, \lambda) := \lambda$, *SSN converges locally Q-quadratic, i.e.* $\lim_{k\to\infty} \frac{||e^{k+1}||}{||e^k||^2} = const.$

Proof. See[8],[10],[53] □

Lemma 2.9. *(Galerkin orthogonality) Let* $\mathbf{u} \in \boldsymbol{V}$ *be an exact solution of the continuous problem* (2.10) *and* $\boldsymbol{u}^{hp} \in \boldsymbol{V}_{hp}$ *be the solution of the discrete problem* (2.34). *There holds*

$$\langle S\boldsymbol{u} - S_{hp}\boldsymbol{u}^{hp}, \boldsymbol{v}^{hp}\rangle_\Sigma + b(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \boldsymbol{v}^{hp}) = 0 \quad \forall \boldsymbol{v}^{hp} \in \boldsymbol{V}_{hp} \tag{2.72}$$

Proof. We choose $\mathbf{v}^{hp} \in \mathbf{V}_{hp} \subset \mathbf{V}$ in (2.10) and subtract (2.10) from the discrete formulation (2.34). □

Let $\mathbf{u} \in \mathbf{V}$ and $\mathbf{u}^{hp} \in \mathbf{V}_{hp}$. From [15], we define the following notation

$$\begin{aligned}
\psi &:= V^{-1}(K + \frac{1}{2})\mathbf{u} \\
\psi^{hp} &:= i_{hp} V_{hp}^{-1} i^*_{hp}(K + \frac{1}{2})\mathbf{u}^{hp} \\
\psi^*_{hp} &:= V^{-1}(K + \frac{1}{2})\mathbf{u}^{hp}.
\end{aligned} \tag{2.73}$$

Lemma 2.10. *[15] Let $\boldsymbol{u} \in \boldsymbol{V}$ be an exact solution of the continuous problem* (2.10) *and $\boldsymbol{u}^{hp} \in \boldsymbol{V}_{hp}$ be the solution of the discrete problem. There holds*

$$\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_W^2 + \|\psi - \psi_{hp}\|_V^2 = \langle S\boldsymbol{u} - S_{hp}\boldsymbol{u}^{hp}, \boldsymbol{u} - \boldsymbol{u}^{hp}\rangle + \langle V(\psi_h^* - \psi_{hp}), \psi - \psi_{hp}\rangle \quad (2.74)$$

where

$$\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_W := \langle W(\boldsymbol{u} - \boldsymbol{u}^{hp}), \boldsymbol{u} - \boldsymbol{u}^{hp}\rangle^{\frac{1}{2}}$$

$$\|\psi - \psi_{hp}\|_V := \langle V(\psi - \psi_{hp}), \psi - \psi_{hp}\rangle^{\frac{1}{2}}$$

Lemma 2.11. *[13, 52, 19] Assume that $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)$ with $\nu \in [0, \frac{1}{2}]$ and $\psi \in \boldsymbol{H}^{\nu}(\Gamma)$. There exists a constant $C > 0$ such that the following approximation properties hold:*

$$\inf_{\boldsymbol{v}^{hp} \in \boldsymbol{V}_{hp}} \|\boldsymbol{u} - \boldsymbol{v}^{hp}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \leq C \left(\frac{h}{p}\right)^{\frac{1}{2}+\nu} \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)}$$

$$\inf_{\boldsymbol{\phi} \in \boldsymbol{W}_{hp}} \|\psi - \boldsymbol{\phi}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} \leq C \left(\frac{h}{p}\right)^{\frac{1}{2}+\nu} \|\psi\|_{\tilde{\boldsymbol{H}}^{\nu}(\Gamma)}$$

2.5 A priori error estimate

The a priori error estimate is based on the use of the discrete inf-sup condition.

In this section we assume that the discrete inf-sup condition is valid.

Assumption 2.1. *There exists a constant $\beta_{hp} > 0$ depending on h and p, such that*

$$\beta_{hp} \leq \inf_{\boldsymbol{\mu}^{hp} \in \boldsymbol{M}_{hp}(\mathcal{F})} \sup_{\boldsymbol{v}^{hp} \in \boldsymbol{V}_{hp}} \frac{b(\boldsymbol{\mu}^{hp}, \boldsymbol{v}^{hp})}{\|\boldsymbol{v}^{hp}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \|\boldsymbol{\mu}^{hp}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)}} \quad (2.75)$$

Lemma 2.12. *Let $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{M}$ be the solution of the mixed formulation* (2.10)*and $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp}) \in \boldsymbol{V}_{hp} \times \boldsymbol{M}_{hp}(\mathcal{F})$ be the solution of the discrete problem* (2.34)*. Then there exists a constant $C > 0$ independent of h and p such that for all $\boldsymbol{\mu}^{hp} \in \boldsymbol{M}_{hp}(\mathcal{F})$ there holds*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} \leq C \left(\frac{(C_S + C_{E_{hp}})}{\beta_{hp}} \|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \frac{C_0}{\beta_{hp}} \|\psi - \boldsymbol{\phi}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} + (1 + \frac{1}{\beta_{hp}}) \|\boldsymbol{\lambda} - \boldsymbol{\mu}^{hp}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_C)} \right) \quad (2.76)$$

Proof. In this proof we use the notation in[19].

We use the following identity

$$S\mathbf{u} - S_{hp}\mathbf{u}^{hp} = S(\mathbf{u} - \mathbf{u}^{hp}) + E_{hp}\mathbf{u}^{hp} = S(\mathbf{u} - \mathbf{u}^{hp}) + E_{hp}(\mathbf{u}^{hp} - \mathbf{u}) + E_{hp}\mathbf{u}$$

where

$$E_{hp} = S - S_{hp}.$$

From Lemma 2.5 and Lemma 2.6 S_{hp} and E_{hp} are continuous. Therefore for all $\boldsymbol{\phi} \in \mathbf{W}_{hp}$

$$\begin{aligned}\langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v}^{hp}\rangle &\leq \left((C_S + C_{E_{hp}})\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \|E_{hp}\mathbf{u}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Sigma)}\right)\|\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &\leq \left((C_S + C_{E_{hp}})\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_0\|\psi - \boldsymbol{\phi}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\right)\|\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}.\end{aligned} \tag{2.77}$$

Using the Galerkin orthogonality lemma 2.9 and (2.77) we get

$$\begin{aligned}\langle \boldsymbol{\lambda}^{hp} - \boldsymbol{\mu}^{hp}, \mathbf{V}_{hp}\rangle &= \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v}^{hp}\rangle + \langle \boldsymbol{\lambda} - \boldsymbol{\mu}^{hp}, \mathbf{V}_{hp}\rangle \\ &\leq \left((C_S + C_{E_{hp}})\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_0\|\psi - \boldsymbol{\phi}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\right)\|\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &+ \|\boldsymbol{\lambda} - \boldsymbol{\mu}^{hp}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)}\|\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}.\end{aligned} \tag{2.78}$$

Using the discrete inf-sup condition (2.75) we obtain

$$\begin{aligned}\|\boldsymbol{\lambda}^{hp} - \boldsymbol{\mu}^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C)} \leq C(\beta_{hp})^{-1}\Big((C_S + C_{E_{hp}})\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_0\|\psi - \boldsymbol{\phi}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \\ + \|\boldsymbol{\lambda} - \boldsymbol{\mu}^{hp}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)}\Big)\end{aligned} \tag{2.79}$$

The triangle inequality and (2.79) yield the assertion. □

Theorem 2.5. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{M}$ *be the solution of mixed formulation* (2.10) *such that* $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)$ *with* $\nu \in [0, \frac{1}{2}]$ *and* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp}) \in \boldsymbol{V}_{hp} \times \boldsymbol{M}_{hp}(\mathcal{F})$ *be the solution of the discrete problem* (2.34)*. We assume that* $\boldsymbol{t} - S\boldsymbol{u} \in \boldsymbol{L}_2(\Gamma)$ *and*

$$\|\lambda_n\|_{H^\nu(\Gamma_C)} + \|\lambda_t\|_{H^\nu(\Gamma_C)} + \|\mathcal{F}\|_{L_2(\Gamma_C)} \leq \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\nu+1}(\Sigma)} \tag{2.80}$$

then there exists $C > 0$ *independent of the polynomial degrees* p *and of the mesh size* h *such that with* (2.73)

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} \\ \leq C\beta_{hp}^{-1}\left(\frac{h}{p}\right)^{\min\{\frac{1}{4},\nu\}}\|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)}\end{aligned} \tag{2.81}$$

We have in particular for $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{\frac{3}{2}}(\Sigma)$

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} \leq C\beta_{hp}^{-1}h^{\frac{1}{4}}p^{-\frac{1}{4}}\|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{3}{2}}(\Sigma)} \tag{2.82}$$

Proof. We choose

$$\mathbf{K}_{hp} := \{\mathbf{u}^{hp} \in \mathbf{V}_{hp} : u_n^{hp}(x) \le g(x), \forall x \in G_{hp} \cap \Gamma_C\}$$

which is a convex, closed subset of $\mathbf{V}_{hp}$, and we define the Lagrange interpolation operator $\mathcal{I}_{hp}$ on the set of Gauss-Lobatto points . Let $\mathbf{v}^{hp} := \mathcal{I}_{hp}\mathbf{u} \in \mathbf{K}_{hp}$. With [[13],Theorems 4.2 and 4.5] we have the following approximation properties: There exist constants C, $C_1 > 0$ independent of $\mathbf{u}$ and h, p such that

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}^{hp}\|_{\mathbf{L}_2(\Gamma)} &\le C h^{\nu} p^{-\nu} \|\mathbf{u}\|_{\mathbf{H}^{\nu}(\Gamma)}, \quad \nu > \frac{1}{2} \\ \|\mathbf{u} - \mathbf{v}^{hp}\|_{\mathbf{H}^1(\Gamma)} &\le C_1 h^{\nu-1} p^{1-\nu} \|\mathbf{u}\|_{\mathbf{H}^{\nu}(\Gamma)}, \quad \nu > 1 \end{aligned} \tag{2.83}$$

From Lemma 2.10 we obtain

$$\|\mathbf{u} - \mathbf{u}^{hp}\|_W^2 + \|\psi - \psi_{hp}\|_V^2 = \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}\rangle + \langle V(\psi_{hp}^* - \psi^{hp}), \psi - \psi^{hp}\rangle.$$

Since we have

$$S\mathbf{u} - S_{hp}\mathbf{u}^{hp} = S(\mathbf{u} - \mathbf{u}^{hp}) + E_{hp}\mathbf{u}^{hp} = S(\mathbf{u} - \mathbf{u}^{hp}) + E_{hp}(\mathbf{u}^{hp} - \mathbf{u}) + E_{hp}\mathbf{u},$$

we obtain

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^{hp}\|_W^2 + \|\psi - \psi_{hp}\|_V^2 &= \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}\rangle + \langle V(\psi_{hp}^* - \psi^{hp}), \psi - \psi^{hp}\rangle \\ &= \langle S(\mathbf{u} - \mathbf{u}^{hp}), \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle S\mathbf{u}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle + \langle S\mathbf{u}, \mathbf{v}^{hp} - \mathbf{u}^{hp}\rangle \\ &+ \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}\rangle + \langle V(\psi_{hp}^* - \psi^{hp}), \psi - \psi^{hp}\rangle. \end{aligned} \tag{2.84}$$

Using (2.34) and the definition of S_h we get

$$\langle S\mathbf{u}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle = \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle - b(\boldsymbol{\lambda}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}) + \langle \mathbf{t}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle \tag{2.85}$$

Using (2.10), (2.84) and (2.85), we obtain

$$\begin{aligned} C_W\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}^2 + C_V\|\psi - \psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}^2 &\le \langle S(\mathbf{u} - \mathbf{u}^{hp}), \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle S\mathbf{u}, \mathbf{v}^{hp} - \mathbf{u}^{hp}\rangle \\ &+ \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{v}^{hp}\rangle - b(\boldsymbol{\lambda}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}) \\ &+ \langle \mathbf{t}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle - \langle S\mathbf{u}, \mathbf{u} - \mathbf{v}\rangle - b(\boldsymbol{\lambda}, \mathbf{u} - \mathbf{v}) \\ &+ \langle \mathbf{t}, \mathbf{u} - \mathbf{v}\rangle + \langle V(\psi_{hp}^* - \psi^{hp}), \psi - \psi^{hp}\rangle \\ &\le \langle S(\mathbf{u} - \mathbf{u}^{hp}), \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{v}^{hp}\rangle \\ &+ \langle \mathbf{t} - S\mathbf{u}, \mathbf{u}^{hp} - \mathbf{v}\rangle + \langle \mathbf{t} - S\mathbf{u}, \mathbf{u} - \mathbf{v}^{hp}\rangle \\ &+ b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{u}^{hp}) - b(\boldsymbol{\lambda}^{hp}, \mathbf{u} - \mathbf{v}^{hp}) \\ &+ \langle V(\psi_{hp}^* - \psi^{hp}), \psi - \psi^{hp}\rangle \\ &+ \langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{u} - \mathbf{u}^{hp}\rangle. \end{aligned} \tag{2.86}$$

For every $\mathbf{v} \in \mathbf{K}$, where C_W and C_V are the ellipticity constants of V, W.

As in [47], choosing $\mathbf{v}$ such that

$$\begin{aligned} v_n|\Gamma_C &:= g + \inf(u_n^{hp} - g_{hp}, 0) \\ v_n|\Gamma_N &:= u_n^{hp}|\Gamma_N \\ v_t|\Sigma &:= u_t^{hp}|\Sigma \end{aligned} \tag{2.87}$$

where g_{hp} is the interpolation of g, we can write

$$\mathbf{v} - \mathbf{u}^{hp} = \begin{cases} 0 & \text{on} \quad \Gamma_N \\ \mathbf{n} \cdot (g - g_{hp} + \delta_u) & \text{on} \quad \Gamma_C \end{cases}$$

with $\delta_u := \inf(0, g_{hp} - u_n^{hp})$.

Then, we have

$$\|\mathbf{v} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(\Gamma)} \leq \|g - g_{hp}\|_{L_2(\Gamma_C)} + \|\delta_u\|_{L_2(\Gamma_C)} \tag{2.88}$$

Due to $\mathbf{u}^{hp} \in \mathbf{K}_{hp}$ we have $I_{hp} \inf(g - u_n^{hp}, 0) = 0$ on Γ_C, and with [[13],Theorems 4.2] we have

$$\begin{aligned} \|\delta_u - 0\|_{L_2(\Gamma_C)} &\leq \|g_{hp} - u_n^{hp}\|_{L_2(\Gamma_C)} \\ &\leq C_1 h^1 p^{-1} \|\delta_u - 0\|_{H^1(\Gamma_C)} \\ &\leq C_1 h^1 p^{-1} \|g_{hp} - u_n^{hp}\|_{H^1(\Gamma_C)}. \end{aligned}$$

By interpolation we have

$$\|\delta_u - 0\|_{L_2(\Gamma_C)} \leq C_1 h^{\frac{1}{2}} p^{-\frac{1}{2}} \|g_{hp} - u_n^{hp}\|_{H^{\frac{1}{2}}(\Gamma_C)}. \tag{2.89}$$

For the first term in (2.88) we have

$$\|g - g_{hp}\|_{L_2(\Gamma_C)} \leq C_1 h^{\frac{1}{2}} p^{-\frac{1}{2}} \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}. \tag{2.90}$$

From [47] we know that there exists a constant $C_2 > 0$, independent of h,p, such that

$$\|\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_C)} \leq C_2 \|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_C)}. \tag{2.91}$$

Therefore we have

$$\|\mathbf{v} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(\Gamma)} \leq C_3 h^{\frac{1}{2}} p^{-\frac{1}{2}} \left(\|g\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_C)} \right). \tag{2.92}$$

Finally, there exist a constant C such that

$$\|\mathbf{v} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(\Gamma)} \leq C h^{\frac{1}{2}} p^{-\frac{1}{2}} \|\mathbf{u}\|_{\mathbf{H}^{1+\nu}(\Gamma)} \tag{2.93}$$

with $\nu \in [0, \frac{1}{2}]$.

For the first term in (2.86), employing Cauchy-Schwarz and Young's inequality ,we obtain

$$\begin{aligned}\langle S(\mathbf{u}-\mathbf{u}^{hp}), \mathbf{u}-\mathbf{v}^{hp}\rangle &\leq \frac{1}{2\epsilon}\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \frac{\epsilon}{2}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &\leq \frac{1}{2\epsilon}\frac{h^{2\nu+1}}{p^{2\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} + \frac{\epsilon}{2}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}.\end{aligned} \tag{2.94}$$

Now we estimate the second term in (2.86).

According to Lemma2.11and Lemma2.6, we get

$$\begin{aligned}\langle E_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle &= \langle E_{hp}(\mathbf{u}^{hp}-\mathbf{u}+\mathbf{u}, \mathbf{u}-\mathbf{v}^{hp}\rangle \\ &\leq \frac{1}{2\epsilon}\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \frac{\epsilon}{2}\|E_{hp}(\mathbf{u}^{hp}-\mathbf{u})\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Sigma)} \\ &+ \frac{1}{2}\|E_{hp}(\mathbf{u})\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Sigma)} + \frac{1}{2}\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &\lesssim \epsilon\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \frac{h^{2\nu+1}}{p^{2\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} + \frac{h^{2\nu+1}}{p^{2\nu+1}}\|\psi\|^2_{\tilde{\mathbf{H}}^{\nu}(\Gamma)}\end{aligned} \tag{2.95}$$

We can easily estimate the third term in (2.86). From (2.83) and (2.93), we obtain

$$\begin{aligned}\langle \mathbf{t}-S\mathbf{u}, \mathbf{u}^{hp}-\mathbf{v}\rangle + \langle \mathbf{t}-S\mathbf{u}, \mathbf{u}-\mathbf{v}^{hp}\rangle &\leq \|\mathbf{t}-S\mathbf{u}\|_{\mathbf{L}_2(\Sigma)}\Big(\|\mathbf{u}^{hp}-\mathbf{v}\|_{\mathbf{L}_2(\Sigma)} + \|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(\Sigma)}\Big) \\ &\leq Ch^{\frac{1}{2}}p^{-\frac{1}{2}}\|\mathbf{u}\|_{\mathbf{H}^{1+\nu}(\Gamma)}.\end{aligned} \tag{2.96}$$

We estimate now the term

$$b(\boldsymbol{\lambda}, \mathbf{v}-\mathbf{u}^{hp}) = \langle \lambda_n, v_n - u_n^{hp}\rangle + \langle \lambda_t, v_t - u_t^{hp}\rangle \tag{2.97}$$

We start with the first term in (2.97) ,using (2.93), we get

$$\begin{aligned}\langle \lambda_n, v_n - u_n^{hp}\rangle = \int_{\Gamma_C} \lambda_n(v_n - u_n^{hp})\, ds &\leq \|\lambda_n\|_{L_2(\Gamma_C)}\|\mathbf{v}-\mathbf{u}^{hp}\|_{\mathbf{L}_2(\Gamma)} \\ &\leq Ch^{\frac{1}{2}}p^{-\frac{1}{2}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)}\end{aligned} \tag{2.98}$$

Due to the approximation (2.87) the second term in (2.97) is

$$\langle \lambda_t, v_t - u_t^{hp}\rangle = 0. \tag{2.99}$$

We estimate the term

$$b(\boldsymbol{\lambda}^{hp}, \mathbf{u}-\mathbf{v}^{hp}) = \langle \lambda_n^{hp}, u_n - v_n^{hp}\rangle + \langle \lambda_t^{hp}, u_t - v_t^{hp}\rangle \tag{2.100}$$

Employing Cauchy Schwarz and Young's inequality, we have

$$\begin{aligned}\langle\lambda_n^{hp}, u_n - v_n^{hp}\rangle &= \int_{\Gamma_C}(\lambda_n^{hp}-\lambda_n)(u_n - v_n^{hp})\,ds + \int_{\Gamma_C}\lambda_n(u_n - v_n^{hp})\,ds\\ &\le \|\lambda_n^{hp}-\lambda_n\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|u_n - v_n^{hp}\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n\|_{L_2(\Gamma_C)}\|u_n - v_n^{hp}\|_{L_2(\Gamma_C)}\\ &\le \frac{\epsilon}{2}\|\lambda_n^{hp}-\lambda_n\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{2\epsilon}\frac{h^{2\nu+1}}{p^{2\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} + \frac{h^{\nu+1}}{p^{\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)}\end{aligned} \tag{2.101}$$

Similarly to (2.101), we get

$$\langle\lambda_t^{hp}, u_t - v_t^{hp}\rangle \le \frac{\epsilon}{2}\|\lambda_t^{hp}-\lambda_t\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{2\epsilon}\frac{h^{2\nu+1}}{p^{2\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} + \frac{h^{\nu+1}}{p^{\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} \tag{2.102}$$

Combining (2.101) and (3.57), we obtain

$$b(\boldsymbol{\lambda}^{hp}, \mathbf{u}-\mathbf{v}^{hp}) \le \frac{\epsilon}{2}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{2\epsilon}\frac{h^{2\nu+1}}{p^{2\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} + \frac{h^{\nu+1}}{p^{\nu+1}}\|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} \tag{2.103}$$

and we have, see[19]

$$\begin{aligned}\langle V(\psi^*_{hp}-\psi^{hp}), \psi-\psi^{hp}\rangle &\le (C_K+\frac{1}{2})\|\mathbf{u}-\mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\|\psi-\boldsymbol{\phi}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\\ &\quad + C_V\|\psi-\psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\|\psi-\boldsymbol{\phi}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\\ &\le \frac{1}{2\epsilon}(C_K+\frac{1}{2})^2\|\psi-\boldsymbol{\phi}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} + \frac{\epsilon}{2}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\\ &\quad + \frac{1}{2\epsilon}C_V^2\|\psi-\boldsymbol{\phi}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} + \frac{\epsilon}{2}\|\psi-\psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \quad \forall\boldsymbol{\phi}\in\mathbf{W}_{hp}\end{aligned} \tag{2.104}$$

Finally we estimate the term

$$\langle\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp}, \mathbf{u}-\mathbf{u}^h\rangle = \langle\lambda_n-\lambda_n^{hp}, u_n-u_n^{hp}\rangle + \langle\lambda_t-\lambda_t^{hp}, u_t-u_t^{hp}\rangle. \tag{2.105}$$

Choosing $\mu_t = \lambda_t$ and $\mu_n = 0$, $\mu_n = 2\lambda_n$ in the inequality (2.10b) and $\mu_t^{hp} = \lambda_t^{hp}$ and $\mu_n^{hp} = 0$, $\mu_n^{hp} = 2\lambda_n^{hp}$ in the discrete inequality (2.34b), we obtain the complementary conditions

$$\int_{\Gamma_C}\lambda_n(u_n-g)\,ds = \int_{\Gamma_C}\lambda_n^{hp}(u_n^{hp}-g)\,ds = 0 \tag{2.106}$$

Let $\pi_{\mathbf{M}_{hp}}$ be the $\mathbf{L}_2$-projection operator mapping $\mathbf{M}_{hp}$ defined by

$$\pi_{\mathbf{M}_{hp}} = \begin{cases}\pi_{M_{n,hp}}: L_2(\Gamma_C)\longrightarrow M_{n,hp}\\ \pi_{M_{t,hp}}: L_2(\Gamma_C)\longrightarrow M_{t,hp}\end{cases} \tag{2.107}$$

which satisfies

$$\int_{\Gamma_C}(\lambda_n-\pi_{M_{n,hp}}\boldsymbol{\lambda})\mu_n\,ds = 0, \quad \int_{\Gamma_C}(\lambda_t-\pi_{M_{t,hp}}\boldsymbol{\lambda})\mu_t = 0\,ds \quad \forall\boldsymbol{\mu}=(\mu_n,\mu_t)\in\mathbf{M}_{hp} \tag{2.108}$$

and

$$\|\lambda_n - \pi_{M_{n,hp}}\boldsymbol{\lambda}\|_{\mathbf{L}_2(\Gamma_C)} \leq C\left(\frac{h}{p}\right)^{\nu}\|\lambda_n\|_{H^{\nu}(\Gamma_C)} \tag{2.109}$$

$$\|\lambda_t - \pi_{M_{t,hp}}\boldsymbol{\lambda}\|_{\mathbf{L}_2(\Gamma_C)} \leq C\left(\frac{h}{p}\right)^{\nu}\|\lambda_t\|_{H^{\nu}(\Gamma_C)} \tag{2.110}$$

for any real number $\nu \geq 0$.

We obtain

$$\begin{aligned}\|\pi_{M_{n,hp}}\boldsymbol{\lambda} - \lambda_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} &= \sup_{0\neq v_n \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \pi_{M_{n,hp}}\boldsymbol{\lambda} - \lambda_n, v_n - \pi_{M_{t,hp}}\mathbf{v}\rangle}{\|v_n\|_{H^{\frac{1}{2}}(\Gamma_C)}} \\ &\leq \sup_{0\neq v_n \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\|\pi_{M_{n,hp}}\boldsymbol{\lambda} - \lambda_n\|_{L_2(\Gamma_C)}\|v_n - \pi_{M_{n,hp}}\mathbf{v}\|_{L_2(\Gamma_C)}}{\|v_n\|_{H^{\frac{1}{2}}(\Gamma_C)}} \\ &\leq C\left(\frac{h}{p}\right)^{\frac{1}{2}}\|\pi_{M_{n,hp}}\boldsymbol{\lambda} - \lambda_n\|_{L_2(\Gamma_C)}\end{aligned} \tag{2.111}$$

and

$$\|\pi_{M_{t,hp}}\boldsymbol{\lambda} - \lambda_t\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\left(\frac{h}{p}\right)^{\frac{1}{2}}\|\pi_{M_{t,hp}}\boldsymbol{\lambda} - \lambda_t\|_{L_2(\Gamma_C)} \tag{2.112}$$

According to the continuity of the Direchlet-to-Neumann operator, we have

$$\|\lambda_n\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_t\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)}. \tag{2.113}$$

We choose $\mu_n^{hp} = \pi_{M_{n,hp}}\boldsymbol{\lambda}$ and $\mu_t^{hp} = \pi_{M_{t,hp}}\boldsymbol{\lambda}$, as a consequence we get

$$\begin{aligned}\inf_{\mu_n \in M_n}\|\lambda_n - \mu_n^{hp}\|_{H^{-\frac{1}{2}}(\Gamma_C)} &\leq C\left(\frac{h}{p}\right)^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)} \\ \inf_{\mu_t \in M_t(\mathcal{F})}\|\lambda_t - \mu_t^{hp}\|_{H^{-\frac{1}{2}}(\Gamma_C)} &\leq C\left(\frac{h}{p}\right)^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)}\end{aligned} \tag{2.114}$$

We start with the first term in (2.105). According to (2.106), we have

$$\langle \lambda_n - \lambda_n^{hp}, u_n^{hp} - u_n\rangle = \int_{\Gamma_C}\lambda_n(u_n^{hp} - g)\, ds + \int_{\Gamma_C}\lambda_n^{hp}(u_n - g)\, ds \tag{2.115}$$

The normal component can be written as

$$u_n^{hp} := \sum_{i\in N_C}\alpha_{in}\phi_i, \quad \mu_n^{hp} := \sum_{i\in N_C}\beta_{in}\psi_i, \quad \lambda_n^{hp} := \sum_{i\in N_C}\lambda_{in}\psi_i. \tag{2.116}$$

For $\mu_n^{hp} \in M_{n,hp}$ we have

$$\begin{aligned}\int_{\Gamma_C} \mu_n^{hp}(u_n^{hp} - g)\, ds &= \sum_i \alpha_{in}\mu_{in}D_i - \langle \mu_n^{hp}, g\rangle \\ &= \sum_i \alpha_{in}\mu_{in}D_i - \sum_i \mu_{in}\int_{\Gamma_C} g\psi_i\, ds \\ &= \sum_i \alpha_{in}\mu_{in}D_i - \sum_i \mu_{in} g_i D_i\, ds \\ &= \sum_i (\alpha_{in} - g_i)\mu_{in}D_i \le 0 \end{aligned} \tag{2.117}$$

Choosing $\mu_n^{hp} = \pi_{M_{n,hp}}\lambda_n$, from (2.117) we get

$$\begin{aligned}\int_{\Gamma_C} \lambda_n(u_n^{hp} - g)\, ds &= \int_{\Gamma_C} (\lambda_n - \mu_n^{hp})(u_n^{hp} - g)\, ds + \int_{\Gamma_C} \mu_n^{hp}(u_n^{hp} - g)\, ds \\ &\le \int_{\Gamma_C} (\lambda_n - \mu_n^{hp})(u_n^{hp} - g)\, ds \\ &\le \int_{\Gamma_C} (\lambda_n - \pi_{M_{n,hp}}\lambda_n)(u_n^{hp} - g)\, ds \\ &= \int_{\Gamma_C} (\lambda_n - \pi_{M_{n,hp}}\lambda_n)((u_n^{hp} - g) - (u_n - g))\, ds \\ &+ \int_{\Gamma_C} (\lambda_n - \pi_{M_{n,hp}}\lambda_n)(u_n - g - \pi_{M_{n,hp}}(u_n - g))\, ds \\ &= \int_{\Gamma_C} (\lambda_n - \pi_{M_{n,hp}}\lambda_n)(u_n^{hp} - u_n)\, ds \\ &+ \int_{\Gamma_C} (\lambda_n - \pi_{M_{n,hp}}\lambda_n)(u_n - \pi_{M_{n,hp}}u_n + \pi_{M_{n,hp}}g - g)\, ds \end{aligned} \tag{2.118}$$

and we obtain

$$\begin{aligned}\int_{\Gamma_C} \lambda_n(u_n^{hp} - g)\, ds &\leq \|\pi_{M_{n,hp}}\lambda_n - \lambda_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|u_n^{hp} - u_n\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &+ \|\pi_{M_{n,hp}}\lambda_n - \lambda_n\|_{L_2(\Gamma_C)} \|u_n - \pi_{M_{n,hp}} u_n\|_{L_2(\Gamma_C)} \\ &+ \|\pi_{M_{n,hp}}\lambda_n - \lambda_n\|_{L_2(\Gamma_C)} \|g - \pi_{M_{n,hp}} g\|_{L_2(\Gamma_C)} \\ &\leq C \left(\left(\frac{h}{p}\right)^{\frac{1}{2}+\nu} \|\lambda_n\|_{H^\nu(\Gamma_C)} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \right. \\ &\left. + \left(\frac{h}{p}\right)^{2\nu} \|\lambda_n\|_{H^\nu(\Gamma_C)} \|\mathbf{u}\|_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \left(\frac{h}{p}\right)^{2\nu} \|\lambda_n\|_{H^\nu(\Gamma_C)} \|g\|_{H^\nu(\Gamma_C)} \right) \\ &\leq \epsilon \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C \left(\frac{h}{p}\right)^{2\nu} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}.\end{aligned} \tag{2.119}$$

We consider the hp-Lagrange interpolation operator $\mathcal{I}_{hp}$ defined on the Gauss-Lobatto points mapping onto $\mathbf{V}_{hp}$.

The linear combination of $\mathcal{I}_{hp} u_n$ can be written as

$$\mathcal{I}_{hp} u_n := \sum_i a_{in} \phi_i \quad \text{with} \quad a_{in} \leq g_i \tag{2.120}$$

where

$$g_i := \frac{1}{D_i} \int_{\Gamma_C} g \psi_i \, ds, \quad D_i = \int_{\Gamma_C} \phi_i \, ds > 0 \tag{2.121}$$

Using the linear combination in (2.116) and the biorthogonality condition, we get

$$\begin{aligned}\int_{\Gamma_C} \lambda_n^{hp} (\mathcal{I}_{hp} u_n - g)\, ds &= \sum_i a_{in} \lambda_{in} D_i - \langle \lambda_n^{hp}, g \rangle \\ &= \sum_i a_{in} \lambda_{in} D_i - \sum_i \lambda_{in} \int_{\Gamma_C} g \psi_i \, ds \\ &= \sum_i a_{pn} \lambda_{in} D_i - \sum_i \lambda_{in} g_i D_i \, ds \\ &= \sum_i (a_{in} - g_i) \lambda_{in} D_i \leq 0.\end{aligned} \tag{2.122}$$

According to (2.122)

$$\begin{aligned}\int_{\Gamma_C} \lambda_n^{hp}(u_n - g)\, ds &= \int_{\Gamma_C} \lambda_n^{hp}((u_n - g) - (\mathcal{I}_{hp}u_n - g))\, ds + \int_{\Gamma_C} \lambda_n^{hp}(\mathcal{I}_{hp}u_n - g)\, ds \\ &\leq \int_{\Gamma_C} \lambda_n^{hp}(u_n - \mathcal{I}_{hp}u_n)\, ds \\ &= \int_{\Gamma_C} (\lambda_n^{hp} - \lambda_n)(u_n - \mathcal{I}_{hp}u_n)\, ds + \int_{\Gamma_C} \lambda_n(u_n - \mathcal{I}_{hp}u_n)\, ds \\ &\leq \|\lambda_n^{hp} - \lambda_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|u_n - \mathcal{I}_{hp}u_n\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &+ \|\lambda_n\|_{L_2(\Gamma_C)} \|u_n - \mathcal{I}_{hp}u_n\|_{L_2(\Gamma_C)} \\ &\leq \epsilon \|\lambda_n^{hp} - \lambda_n\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + C\frac{h^{\nu+1}}{p^{\nu+1}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)} \end{aligned} \tag{2.123}$$

We now estimate the term the second term in (2.105)

From (2.34b), we have

$$\lambda_t^{hp} \in M_{t,hp}, \quad \langle \mu_t^{hp} - \lambda_t^{hp}, u_t^{hp} \rangle \leq 0, \quad \forall \mu_t^{hp} \in M_{t,hp} \tag{2.124}$$

Choosing $\mu_t^{hp} = \pi_{M_{t,hp}} \boldsymbol{\lambda}$, from (2.124) we get

$$\begin{aligned}\langle \lambda_t - \lambda_t^{hp}, u_t^{hp} - u_t \rangle &= \langle \lambda_t - \mu_t^{hp}, u_t^{hp} - u_t \rangle + \langle \mu_t^{hp} - \lambda_t^{hp}, u_t^{hp} - u_t \rangle \\ &\leq \langle \lambda_t - \mu_t^{hp}, u_t^{hp} - u_t \rangle - \langle \mu_t^{hp} - \lambda_t^{hp}, u_t \rangle \\ &\leq \langle \lambda_t - \mu_t^{hp}, u_t^{hp} - u_t \rangle + \langle \lambda_t - \mu_t^{hp}, u_t \rangle + \langle \lambda_t^{hp} - \lambda_t, u_t \rangle. \end{aligned} \tag{2.125}$$

The estimate of the first term in (2.125) gives

$$\begin{aligned}\langle \lambda_t - \mu_t^{hp}, u_t^{hp} - u_t \rangle &= \int_{\Gamma_C} (\lambda_t - \mu_t^{hp})(u_t^{hp} - u_t)\, ds \\ &\leq \frac{1}{2\epsilon} \|\lambda_t - \mu_t^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \frac{\epsilon}{2} \|u_t^{hp} - u_t\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \\ &\leq C\frac{h^{\frac{1}{2}+\nu}}{p^{\frac{1}{2}+\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)} + \frac{\epsilon}{2} \|u_t^{hp} - u_t\|^2_{H^{\frac{1}{2}}(\Gamma_C)}. \end{aligned} \tag{2.126}$$

We now estimate the second term in (2.125)

$$\begin{aligned}\langle \lambda_t - \mu_t^{hp}, u_t\rangle &= \int_{\Gamma_C} (\lambda_t - \mu_t^{hp}) u_t \; ds \\ &\leq \|\lambda_t - \mu_t^{hp}\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|u_t\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &\leq C \frac{h^{\frac{1}{2}+\nu}}{p^{\frac{1}{2}+\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)}.\end{aligned} \tag{2.127}$$

The linear combination of $\mathcal{I}_{hp} u_t$ and λ_t^{hp} can be written as

$$\mathcal{I}_{hp} u_t := \sum_{i \in N_C} a_{it} \phi_i, \quad \lambda_t^{hp} := \sum_{i \in N_C} \lambda_{it} \psi_i. \tag{2.128}$$

Using the linear combination in (2.128), the biorthogonality condition and the fact that $|\lambda_{it}| \leq \mathcal{F}$, we get

$$\begin{aligned}\int_{\Gamma_C} \lambda_t^{hp} \mathcal{I}_{hp} u_t \; ds = \sum_i a_{it} \lambda_{it} D_i &\leq \mathcal{F} \sum_i a_{it} D_i \\ &\leq \mathcal{F} \sum_i a_{it} \int_{\Gamma_C} \phi_i \; ds \\ &\leq \mathcal{F} \int_{\Gamma_C} |\mathcal{I}_{hp} u_t| \; ds.\end{aligned} \tag{2.129}$$

We estimate now the last term in (2.125), using the condion $\lambda_t u_t - \mathcal{F}|u_t| = 0$ and (3.108):

$$\begin{aligned}\int_{\Gamma_C} (\lambda_t^{hp} - \lambda_t) u_t \; ds &= \int_{\Gamma_C} (\lambda_t^{hp} - \lambda_t)(u_t - \mathcal{I}_{hp} u_t) \; ds + \int_{\Gamma_C} (\lambda_t^{hp} - \lambda_t) \mathcal{I}_{hp} u_t \; ds \\ &+ \int_{\Gamma_C} \lambda_t u_t \; ds - \mathcal{F} \int_{\Gamma_C} |u_t| \; ds \\ &\leq \int_{\Gamma_C} (\lambda_t^{hp} - \lambda_t)(u_t - \mathcal{I}_{hp} u_t) \; ds + \int_{\Gamma_C} \lambda_t (u_t - \mathcal{I}_{hp} u_t) \; ds \\ &+ \mathcal{F} \int_{\Gamma_C} (|\mathcal{I}_{hp} u_t| - |u_t|) \; ds \\ &\leq \int_{\Gamma_C} (\lambda_t^{hp} - \lambda_t)(u_t - \mathcal{I}_{hp} u_t) \; ds + \int_{\Gamma_C} \lambda_t (u_t - \mathcal{I}_{hp} u_t) \; ds \\ &+ \mathcal{F} \int_{\Gamma_C} (|\mathcal{I}_{hp} u_t - u_t|) \; ds \\ &\leq \|\lambda_t^{hp} - \lambda_t\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|u_t - \mathcal{I}_{hp} u_t\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &+ \|\lambda_t\|_{L_2(\Gamma_C)} \|u_t - \mathcal{I}_{hp} u_t\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &+ \|\mathcal{F}\|_{L_2(\Gamma_C)} \|u_t - \mathcal{I}_{hp} u_t\|_{L_2(\Gamma_C)} \\ &\leq \epsilon \|\lambda_t^{hp} - \lambda_t\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \frac{h^{\nu+\frac{1}{2}}}{p^{\nu+\frac{1}{2}}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)}\end{aligned} \tag{2.130}$$

Now combining (2.126) ,(2.127),(2.130) we get the estimate of the second term in(2.105)

$$\langle \lambda_t - \lambda_t^{hp}, u_t^{hp} - u_t \rangle \leq \epsilon \|\lambda_t^{hp} - \lambda_t\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \epsilon \|u_t^{hp} - u_t\|^2_{H^{\frac{1}{2}}(\Gamma_C)} + \frac{h^{\nu+\frac{1}{2}}}{p^{\nu+\frac{1}{2}}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)} \tag{2.131}$$

Finally, the estimate follows immediately by using Lemma2.12, Lemma(2.11), (2.131) , (2.94), (2.95), (2.96), (2.104), (2.98), (3.58), (2.119), (2.123).

□

2.6 A posteriori error estimates for contact with friction

Let $(\mathbf{u}, \boldsymbol{\lambda})$ be the exact solution of the continuous problem (2.10) and let $(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{hp})$ be the solution of the discrete boundary element problem (2.34). We now give a computable upper bound for $|||\mathbf{u} - \mathbf{u}^{hp}|||^2$, where

$$|||\mathbf{u} - \mathbf{u}^{hp}|||^2 := \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C)} \tag{2.132}$$

Theorem 2.6. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *be the exact solution of the boundary problem* (2.10) *and* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp})$ *be the solution of the discrete boundary problem* (2.34). *Then there holds the estimate :*

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi_{hp}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} \lesssim & \sum_{I \in \mathcal{T}_{hp}} \eta^2_{hp}(I) + \langle (\lambda_n^{hp})^+, (g - u_n^{hp})^+ \rangle_{\Gamma_C} \\
& + \|(\lambda_n^{hp})^-\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{4\epsilon} \|(u_n^{hp} - g)^+\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \\
& + \epsilon \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} + \|(|\lambda_t^{hp}| - \mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \\
& + \int_{\Gamma_C} \left(|(|\lambda_t^{hp}| - \mathcal{F})^-||u_t^{hp}| + 2(\lambda_t^{hp} u_t^{hp})^- \right) ds,
\end{aligned} \tag{2.133}$$

where

$$\begin{aligned}
\eta^2_{hp}(I) = \frac{h_I}{p_I} \|\boldsymbol{t} - S_{hp} \boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I \cap \Gamma_N)} + & \frac{h_I}{p_I} \|(-\boldsymbol{\lambda}^{hp}) - S_{hp} \boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I \cap \Gamma_C)} \\
& + h_I \|\frac{\partial}{\partial s}(V(\psi^*_{hp} - \psi^{hp}))\|^2_{\boldsymbol{L}_2(I)}
\end{aligned}$$

Proof. Using Lemma 2.10, since $\langle W \cdot, \cdot \rangle$, $\langle V \cdot, \cdot \rangle$ are positive definite, there exist constants $C_W, C_V > 0$ such that

$$\begin{aligned}
C_W \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_V \|\psi - \psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \leq & \langle S\mathbf{u} - S_{hp} \mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp} \rangle \\
& + \langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp} \rangle
\end{aligned}$$

Using Galerkin orthogonality, we obtain

$$\begin{aligned}
C_W\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_V\|\psi-\psi^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma)} &\leq \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{u}^{hp}\rangle \\
&+ \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{u}^{hp}-\mathbf{v}^{hp}\rangle_\Sigma + b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp},\mathbf{u}^{hp}-\mathbf{v}^{hp}) \\
&+ \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle \\
&\leq \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle + \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle \\
&+ b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp},\mathbf{u}^{hp}-\mathbf{v}^{hp}) \\
&\leq \langle \mathbf{t},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N} - b(\boldsymbol{\lambda},\mathbf{u}-\mathbf{v}^{hp}) - \langle S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle \\
&+ \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle + b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp},\mathbf{u}^{hp}-\mathbf{v}^{hp}) \\
&\leq \langle \mathbf{t}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N} + \langle(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_C} \\
&+ b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp},\mathbf{u}^{hp}-\mathbf{u}) + \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle \\
&\leq A+B+C+D
\end{aligned} \tag{2.134}$$

We estimate the terms A and B, employing the Cauchy-Schwarz inequality. We obtain:

$$\begin{aligned}
A+B \leq &\sum_{I\in\mathcal{T}_{hp}\cap\Gamma_C} \|(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \\
+ &\sum_{I\in\mathcal{T}_{hp}\cap\Gamma_N} \|\mathbf{t}-S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(I)}
\end{aligned} \tag{2.135}$$

Let Π_{hp} be the hp-Clément interpolation operator mapping onto $\mathbf{V}_{hp}$ [48]. Then there holds

$$\|\mathbf{u}-\Pi_{hp}\mathbf{u}\|_{\mathbf{L}_2(I)} \leq C\left(\frac{h}{p}\right)^{\frac{1}{2}}\|\mathbf{u}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\omega(I))}, \tag{2.136}$$

where $\omega(I)$ consists of all neighboring elements of I.

We choose in (2.135) $\mathbf{v}^{hp}=\mathbf{u}^{hp}+\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})$ we obtain the following approximation:

$$\|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}^2(I)} \leq C\left(\frac{h_I}{p_I}\right)^{\frac{1}{2}}\|\mathbf{u}-\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\omega(I))} \tag{2.137}$$

We apply a result from [[15],Theorem 5.1] for the term D, to obtain

$$\begin{aligned}
D := \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle &\leq \|V(\psi^*_{hp}-\psi^{hp})\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\|\psi^{hp}-\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\
&\leq c\left(\sum_{I\in\mathcal{T}_{hp}} h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp}-\psi^{hp}))\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}}\|\psi^{hp}-\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}
\end{aligned} \tag{2.138}$$

Finally we estimate the term C. In order to obtain an a posteriori error estimate, we have to estimate the term:

$$C = b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp},\mathbf{u}^{hp}-\mathbf{u}) = \langle\lambda_n-\lambda^{hp}_n,u^{hp}_n-u_n\rangle + \langle\lambda_t-\lambda^{hp}_t,u^{hp}_t-u_t\rangle. \tag{2.139}$$

Using the condition $\langle \lambda_n, u_n - g\rangle = 0$ and $\langle (\lambda_n^{hp})^+, u_n - g\rangle \leq 0$ where $v^+ = \max\{0, v\}$ and $v^- = \min\{0, v\}$, i.e. $v = v^+ + v^-$.

$$\begin{aligned}
\langle \lambda_n - \lambda_n^{hp}, u_n^{hp} - u_n\rangle &= \langle \lambda_n - (\lambda_n^{hp})^+, u_n^{hp} - u_n\rangle - \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&= \langle \lambda_n - (\lambda_n^{hp})^+, u_n^{hp} - g + g - u_n\rangle - \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&= \langle \lambda_n - (\lambda_n^{hp})^+, u_n^{hp} - g\rangle + \langle \lambda_n, g - u_n\rangle \\
&- \langle (\lambda_n^{hp})^+, g - u_n\rangle - \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&\leq \langle \lambda_n - (\lambda_n^{hp})^+, u_n^{hp} - g\rangle - \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&= \langle (\lambda_n^{hp})^+, g - u_n^{hp}\rangle - \langle \lambda_n, (g - u_n^{hp})^+ + (g - u_n^{hp})^-\rangle \\
&- \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&\leq \langle (\lambda_n^{hp})^+, g - u_n^{hp}\rangle + \langle \lambda_n^{hp} - \lambda_n - (\lambda_n^{hp})^+ - (\lambda_n^{hp})^-, (g - u_n^{hp})^-\rangle \\
&- \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&= \langle (\lambda_n^{hp})^+, (g - u_n^{hp})^+\rangle + \langle \lambda_n^{hp} - \lambda_n, (g - u_n^{hp})^-\rangle \\
&- \langle (\lambda_n^{hp})^-, (g - u_n^{hp})^-\rangle - \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&= \langle (\lambda_n^{hp})^+, (g - u_n^{hp})^+\rangle + \langle \lambda_n^{hp} - \lambda_n, (g - u_n^{hp})^-\rangle \\
&- \langle (\lambda_n^{hp})^-, u_n^{hp} - u_n\rangle \\
&\leq \|(\lambda_n^{hp})^-\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \|u_n^{hp} - u_n\|_{\tilde{H}^{\frac{1}{2}}(\Sigma)} \\
&+ \|\lambda_n^{hp} - \lambda_n\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \|(g - u_n^{hp})^-\|_{H^{\frac{1}{2}}(\Gamma_C)} + \langle (\lambda_n^{hp})^+, (g - u_n^{hp})^+\rangle_{\Gamma_C}.
\end{aligned} \tag{2.140}$$

Using the condion $-\lambda_t u_t + \mathcal{F}|u_t| = 0$ and $\lambda_t = \xi\mathcal{F}$ with $|\xi| \leq 1$:

$$\begin{aligned}
\langle \lambda_t - \lambda_t^{hp}, u_t^{hp} - u_t\rangle &= -\lambda_t u_t + \lambda_t^{hp} u_t + \lambda_t u_t^{hp} - \lambda_t^{hp} u_t^{hp} \\
&= -\mathcal{F}|u_t| + \lambda_t^{hp} u_t + \xi\mathcal{F} u_t^{hp} - \lambda_t^{hp} u_t^{hp} \\
&\leq -\mathcal{F}|u_t| + \lambda_t^{hp} u_t + \mathcal{F}|u_t^{hp}| - \lambda_t^{hp} u_t^{hp} \\
&\leq (|\lambda_t^{hp}| - \mathcal{F})^+ |u_t| + \mathcal{F}|u_t^{hp}| - \lambda_t^{hp} u_t^{hp} \\
&\leq (|\lambda_t^{hp}| - \mathcal{F})^+ |u_t - u_t^{hp}| + (|\lambda_t^{hp}| - \mathcal{F})^+ |u_t^{hp}| + \mathcal{F}|u_t^{hp}| - \lambda_t^{hp} u_t^{hp} \\
&\leq \|(|\lambda_t^{hp}| - \mathcal{F})^+\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \|u_t - u_t^{hp}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_C)} \\
&+ [(|\lambda_t^{hp}| - \mathcal{F})^+ - (|\lambda_t^{hp}| - \mathcal{F})] |u_t^{hp}| - \lambda_t^{hp} u_t^{hp} + |\lambda_t^{hp}||u_t^{hp}| \\
&\leq \|(|\lambda_t^{hp}| - \mathcal{F})^+\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \|u_t - u_t^{hp}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_C)} \\
&+ |(|\lambda_t^{hp}| - \mathcal{F})^-| |u_t^{hp}| + 2(\lambda_t^{hp} u_t^{hp})^-
\end{aligned} \tag{2.141}$$

Using Young's inequality we throw the term $\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}$ to the left hand side, and we obtain the estimate of the theorem. □

Now we look for an upper bound of the discretization error $\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}$.

Lemma 2.13. *Let $(\boldsymbol{u}, \boldsymbol{\lambda})$ solve the saddle point problem (2.10), and let $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp})$ be the solution of the discrete problem (2.34). Then there holds*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_C)} \leq C' \left(\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} \right) + C'' \sum_{I \subset \mathcal{T}_{hp}} \xi^2_{hp}(I) \tag{2.142}$$

Where

$$\xi^2_{hp}(I) = \frac{h_I}{p_I} \|\boldsymbol{t} - S_{hp}\boldsymbol{u}^{hp}\|^2_{\mathbf{L}_2(I \cap \Gamma_N)} + \frac{h_I}{p_I} \|(-\boldsymbol{\lambda}^{hp}) - S_{hp}\boldsymbol{u}^{hp}\|^2_{\mathbf{L}_2(I \cap \Gamma_C)} \tag{2.143}$$

Proof. For $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}^{hp} \in \mathbf{V}_{hp}$ we have

$$\langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{v} \rangle = \langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle + \langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{v}^{hp} \rangle$$

Using Galerkin orthogonality and the formulation (2.10a), we obtain

$$\begin{aligned}
\langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{v} \rangle &= \langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle - \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v}^{hp} \rangle_\Sigma \\
&= L(\mathbf{v} - \mathbf{v}^{hp}) - \langle S\mathbf{u}, \mathbf{v} - \mathbf{v}^{hp} \rangle - \langle \boldsymbol{\lambda}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle \\
&\quad - \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v} \rangle_\Sigma - \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v}^{hp} - \mathbf{v} \rangle_\Sigma \\
&= \langle \mathbf{t} - S_{hp}\mathbf{u}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle_{\Gamma_N} + \langle (-\boldsymbol{\lambda}^{hp}) - S_{hp}\mathbf{u}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle_{\Gamma_C} \\
&\quad - \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v} \rangle_\Sigma \\
&= A + B + C
\end{aligned} \tag{2.144}$$

$$A = \langle \mathbf{t} - S_{hp}\mathbf{u}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle_{\Gamma_N} \leq \sum_{I \in \mathcal{T}_{hp} \cap \Gamma_N} \|\mathbf{t} - S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)} \|\mathbf{u} - \mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \tag{2.145}$$

$$B = \langle (-\boldsymbol{\lambda}^{hp}) - S_{hp}\mathbf{u}^{hp}, \mathbf{v} - \mathbf{v}^{hp} \rangle_{\Gamma_C} \leq \sum_{I \in \mathcal{T}_{hp} \cap \Gamma_C} \|(-\boldsymbol{\lambda}^{hp}) - S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)} \|\mathbf{u} - \mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \tag{2.146}$$

$$\begin{aligned}
C = \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{v} \rangle_\Sigma &\leq (C_s + C_{E_{hp}}) \|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\
&\quad + C_0 \|\psi - \psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}
\end{aligned} \tag{2.147}$$

We choose $\mathbf{v}^{hp} = \Pi_{hp}\mathbf{v}$ in A and B, we get

$$\begin{aligned}
\langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{hp}, \mathbf{v} \rangle &\leq \left((C_s + C_{E_{hp}}) \|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_0 \|\psi - \psi_{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \right) \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\
&\quad + C \left(\sum_{I \in \mathcal{T}_{hp} \cap \Gamma_N} \left(\frac{h_I}{p_I} \right)^{\frac{1}{2}} \|\mathbf{t} - S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)} \right) \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\
&\quad + C \left(\sum_{I \in \mathcal{T}_h \cap \Gamma_N} \left(\frac{h_I}{p_I} \right)^{\frac{1}{2}} \|(-\boldsymbol{\lambda}^{hp}) - S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)} \right) \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}
\end{aligned} \tag{2.148}$$

Using the definition of the dual norm and $(a+b)^2 \leq 2a^2+2b^2$, the assertion immediately follows.

□

Theorem 2.7. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *be the exact solution of the boundary problem* (2.10) *and* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp})$ *be the solution of the discrete boundary problem* (2.34)*, then there holds the a posteriori estimate :*

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)}+\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} \lesssim & \sum_{I\in\mathcal{T}_{hp}} \eta^2_{hp}(I) + \langle(\lambda_n^{hp})^+, (g-u_n^{hp})^+\rangle_{\Gamma_C} \\ & + \|(\lambda_n^{hp})^-\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} + \|(u_n^{hp}-g)^+\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \\ & + \|(|\lambda_t^{hp}|-\mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \\ & + \int_{\Gamma_C} \left(|(|\lambda_t^{hp}|-\mathcal{F})^-||u_t^h| + 2(\lambda_t^{hp}u_t^{hp})^-\right) ds \end{aligned} \tag{2.149}$$

where

$$\begin{aligned}\eta^2_{hp}(I) = \frac{h_I}{p_I}\|\boldsymbol{t}-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_N)} + \frac{h_I}{p_I}\|(-\boldsymbol{\lambda}^{hp})-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\ + h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp}-\psi^{hp}))\|^2_{\boldsymbol{L}_2(I)}\end{aligned} \tag{2.150}$$

Proof. Follows immediately from Lemma 2.13 and Theorem 2.6. □

2.7 Numerical Experiments

Numerical results are presented with the MATLAB package of L.Banz for the contact of the two-dimentional elastic body $\Omega = [-0.5, 0.5]^2$ with a rigid straight line. The contact boundary $\Gamma_C = [-0.5, 0.5] \times -0.5$ comes in contact with rigid obstacle which occupies the half space $y \leq -0.5$. The Young's modulus and the Poisson's ratio are $E = 200$, $\nu = 0.3$ respectively. The gap is assumed to be zero, i.e $g = 0$ and the given friction function $\mathcal{F} = 0.3$.

The applied Neumann boundary forces on the top, the left and the right side of the domain are give by see [42]

$$\mathbf{t}_{\text{side}} = \begin{pmatrix} -400\,\text{sign}(x)(y+\frac{1}{2})(\frac{1}{2}-y)exp(-10(y+\frac{3}{10})^2) \\ 10(y+\frac{1}{2})(\frac{1}{2}-y) \end{pmatrix}$$

$$\mathbf{t}_{\text{top}} = \begin{pmatrix} 0 \\ -500(\frac{1}{2}-x)^2(\frac{1}{2}+x)^2 \end{pmatrix}$$

In our numerical experiments we use the semi-smooth Newton Algorithm 2.1 to solve the discrete problem. The initial mesh is uniform and consists of 16 elements. We introduce an hp-adaptive algorithm based on locally testing for smoothness as done in [40] to decide of whether to perform h-or-p-refinement. The details of the procedure are described in Algorithm 2.2 .

Algorithm 2.2. *(Mesh refinement strategy)*

---

1. *Generate an initial mesh* $\mathcal{T}_{hp,0}$*, discrete spaces* $\boldsymbol{V}_{hp,0}$*,* $\boldsymbol{W}_{hp,0}$*, set* $l = 0$
2. *Choose a tolerence* $TOL > 0$ *and steering parameter* $0 \leq \gamma \leq 1$
3. *For* $l = 0, 1, 2, ...$
   a) *Solve the discrete problem, for* $(\boldsymbol{u}_l^{hp}, \boldsymbol{\lambda}_l^{hp})$

      based on the hp-mesh $\mathcal{T}_{hp,l}$
   b) *Compute indicators* η_I *for all segments* $I \in \mathcal{T}_{hp,l}$

      and the global error estimator $\eta = \left(\sum_{I \in \mathcal{T}_{hp,l}} \eta_I^2\right)^{\frac{1}{2}}$
   c) *Stop if* $\left(\sum_{I \in \mathcal{T}_{hp,l}} \eta_I^2\right)^{\frac{1}{2}} \leq TOL$
   d) *Compute the local approximation* Ξ_I *of* η_I *and the global approximation* $\sum_{I \in \mathcal{T}_{hp,l}} \Xi_I^2$ *of* η
   e) *For* $(\theta \in (0,1))$*, mark all elements in* N

$$N = \operatorname{argmin} \left\{ \left| \left\{ \hat{N} \subset \mathcal{T}_{hp,l} : \sum_{I \in \hat{N}} \Xi_I^2 \geq \theta \sum_{I \in \mathcal{T}_{hp,l}} \Xi_I^2 \right\} \right| \right\}$$

   f) *Estimate analyticity [40]* $(\delta \in (0,1))$
      i. *Compute Legendre coefficients of* $u_{hp}|_I$ *with*

$$u_{hp}(x)|_I = \sum_{i=0}^{p_I} a_i L_i(x), \quad a_i = \frac{2i+1}{2} \int_I u_{hp}(x) L_i(x)$$

      ii. *Compute the slope* $m_I \in \mathbb{R}$ *and* $b_I \in \mathbb{R}$*, with*

$$m_I, b_I \in \mathbb{R} : \sum_i (i \cdot m_I + b_I - |\log |a_i||)^2 \to \min$$

      iii. *if* $exp(-m_I) \leq \delta$ *increase* p_I *by one, else bisect* $I \in N$ *and keep the polynomial degree equal to* p_I *on the resulting sub-elements.*
   g) *Compute the new hp-mesh* $\mathcal{T}_{hp,l+1}$

h) Generate the discrete spaces , $\boldsymbol{V}_{hp,l+1}$, $\boldsymbol{W}_{hp,l+1}$ *based on the mesh* $\mathcal{T}_{hp,l+1}$

i) Set $l = l + 1$ *, go to (a)*

In Figure2.1 we show the deformed configuration. Figure 2.2 shows the estimated errors for the h-uniform, h-adaptive and hp-adaptive methods with ($\theta = 0.5$, $\delta = 0.6$). Figure2.3 shows the normal and the tangential component of the Lagrange multiplier. The refined meshes and polynomial degrees obtained with our approach are shown in Figure 2.4 and Figure 2.5. The first figure shows the adaptivety generated meshes and polynomial degrees after 11 and 13 refinement steps using η_{bub} (the bubble indicator) as error indicator see [47]. The second figure shows adaptivety refined meshes and polynomial degrees obtained after 9 and 11 steps using the residual indicator .

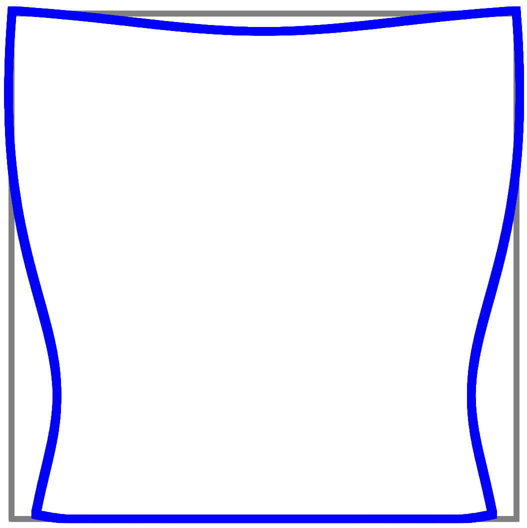


Figure 2.1: Deformed geometry

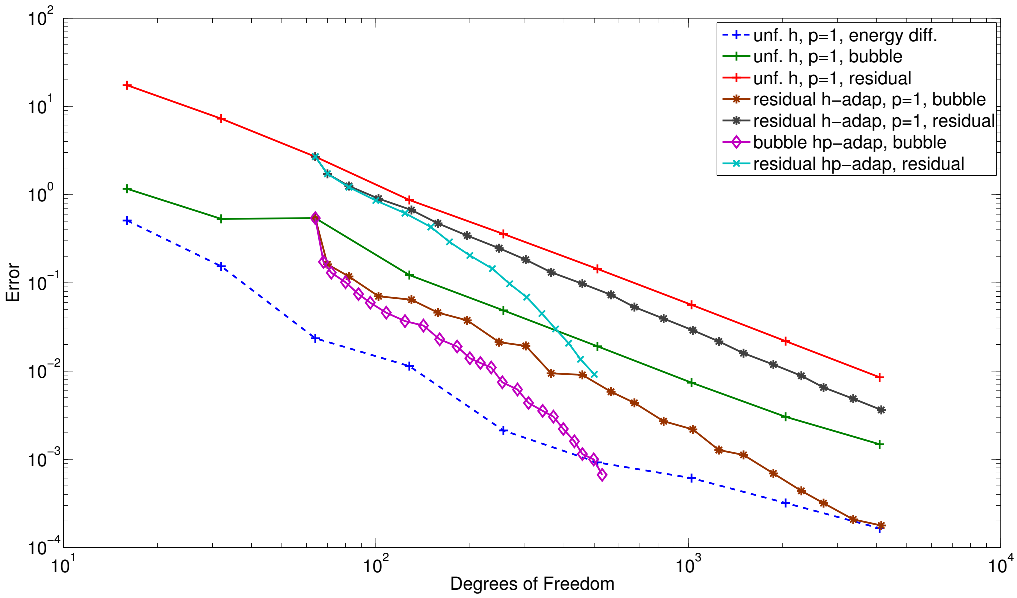
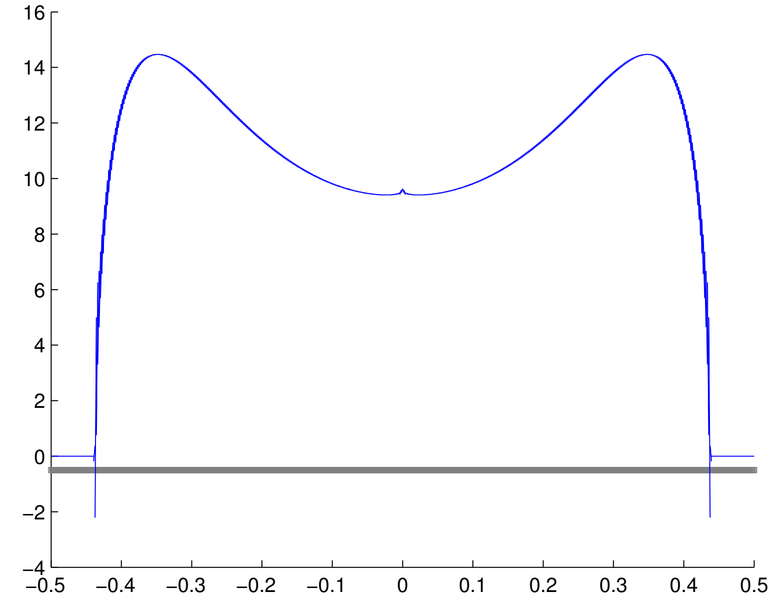
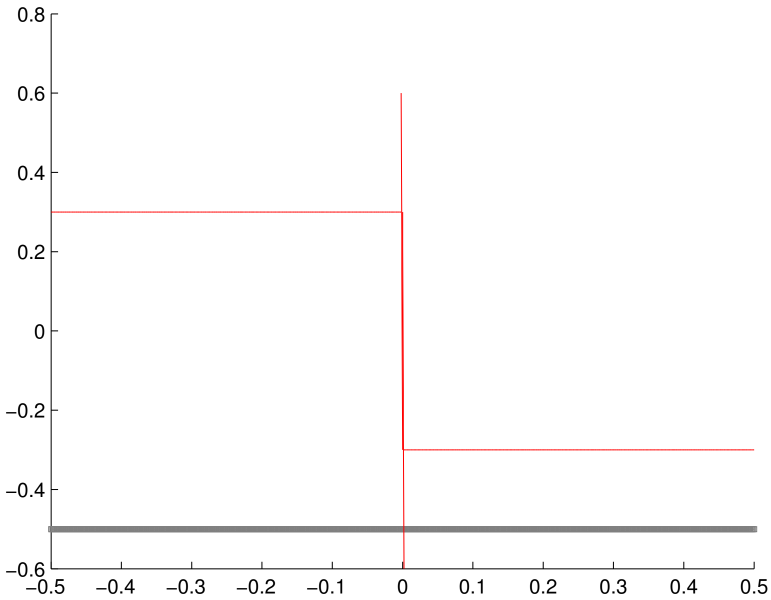


Figure 2.2: Estimated errors

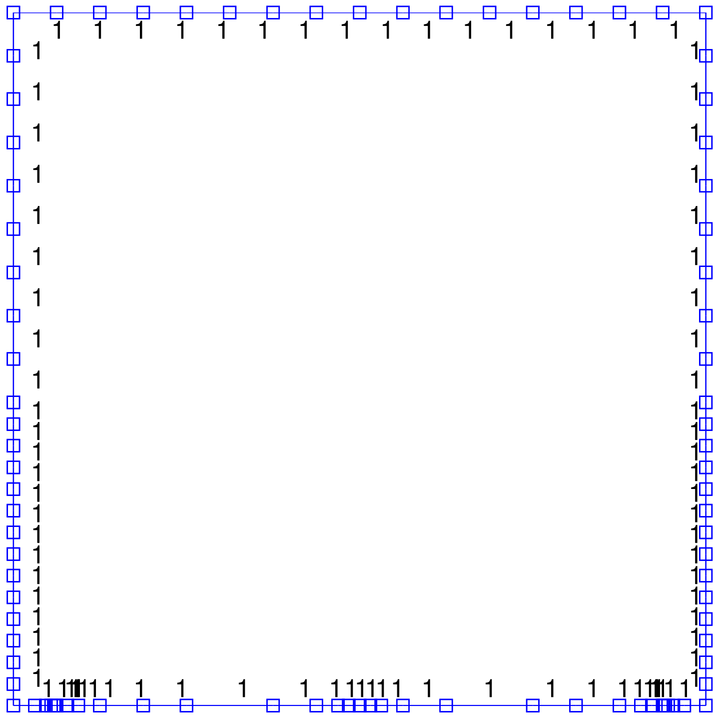


(a) normal component λ_n

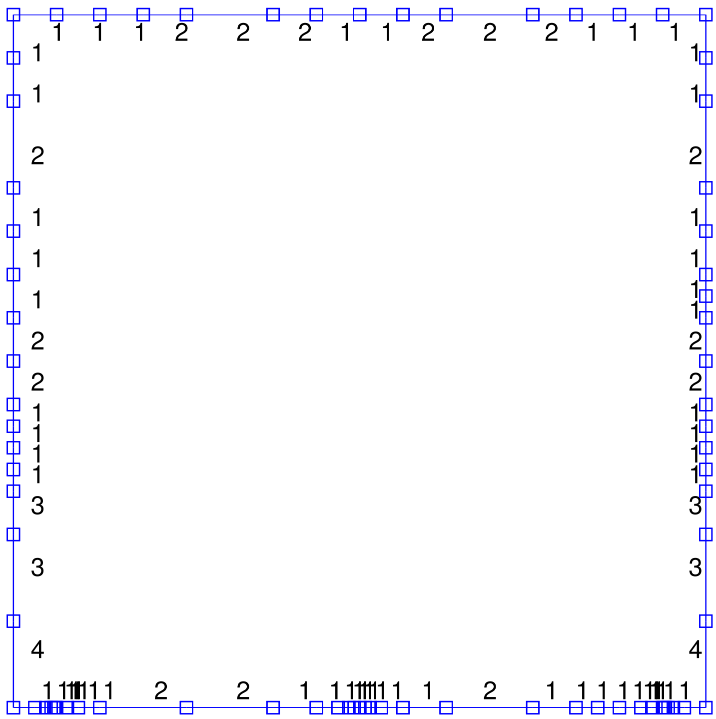


(b) tangential component λ_t

Figure 2.3: Visualization of Lagrange multiplier

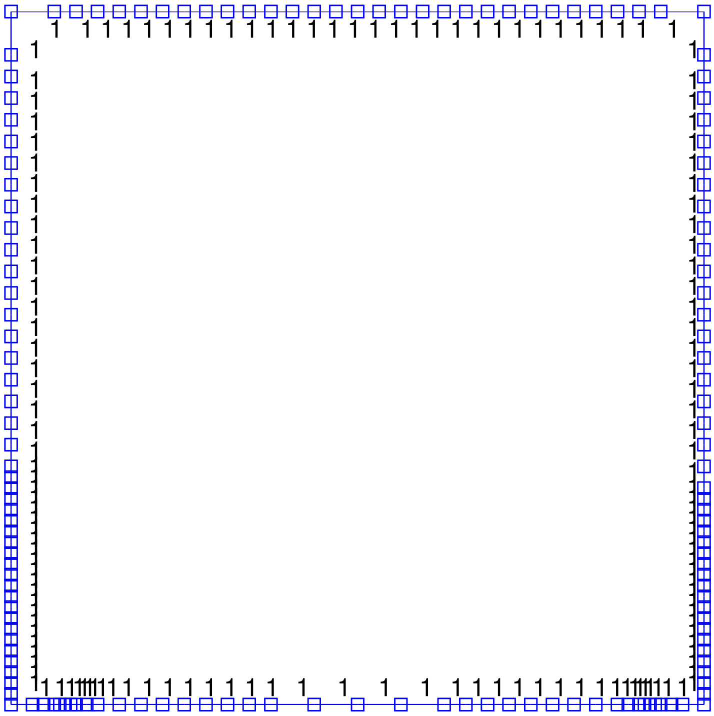


(a) h-adaptivity, $\theta = 0.5$, $\delta = 0.6$

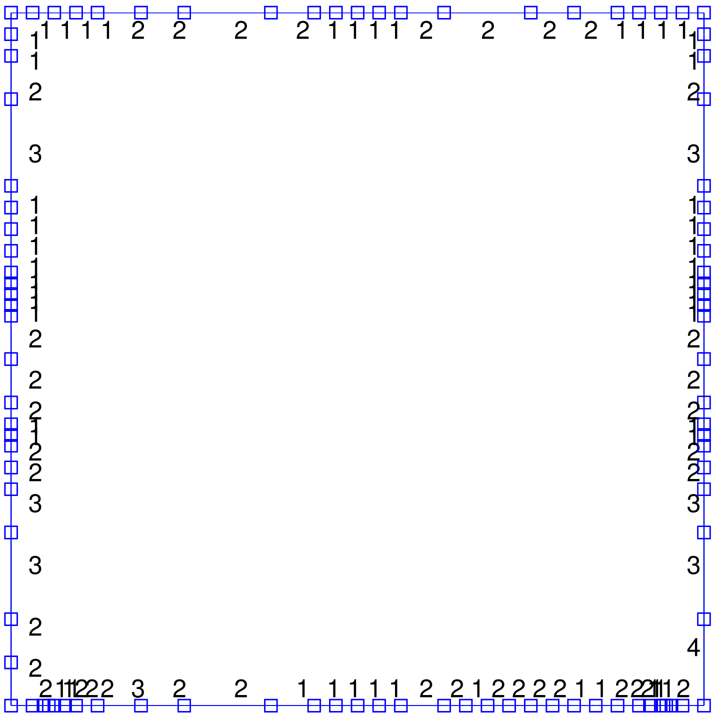


(b) hp-adaptivity, $\theta = 0.5$, $\delta = 0.6$

Figure 2.4: Adaptivety generated meshes for Lamé-BEM (bubble indicator)



(a) h-adaptivity, $\theta = 0.5, \delta = 0.6$



(b) hp-adaptivity, $\theta = 0.5, \delta = 0.6$

Figure 2.5: Adaptivety generated meshes for Lamé-BEM (residual indicator)

3 Stabilized mixed hp-BEM in Linear Elasticity

In this chapter, we consider a contact problem in $2D$ elasticity with Tresca friction. We consider a mixed boundary integral formulation, which is stabilized following ideas of P.Hild, Y. Renard and V.Lleras [36],[37],[45] for the FEM. Here a mesh-dependent stabilization term is added to the discrete mixed formulation, in order to avoid the discrete inf-sup condition.. First we study the existence and uniqueness of the solution of the discrete problem. A subsection is devoted to a priori error and a posteriori error estimates. Finally, we present some numerical experiments, which compare the stabilized and the non-stabilized cases.

3.1 The mixed formulation

The notations in Chapter 2 are used in this chapter. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with the boundary $\Gamma := \partial\Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_C$ be decomposed into the non-intersecting Neumann conditioned segment Γ_N, the Dirichlet conditioned segment Γ_D and the contact conditioned segment Γ_C which potentially can come in contact with the rigid foundation with $\bar{\Gamma}_C \cap \bar{\Gamma}_D = \emptyset$ for simplicity. The problem then consists in finding the displacement field $\mathbf{u} : \Omega \longrightarrow \mathbb{R}$ such that

$$-\operatorname{div}\sigma(\mathbf{u}) = 0 \qquad \text{in } \Omega \tag{3.1a}$$

$$\sigma(\mathbf{u}) = \mathcal{C} : \epsilon(\mathbf{u}) \qquad \text{in } \Omega \tag{3.1b}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma_D \tag{3.1c}$$

$$\sigma(\mathbf{u}) \cdot n = t \qquad \text{on } \Gamma_N \tag{3.1d}$$

$$\sigma_n \leq 0,\ u_n \leq g,\ \sigma_n(u_n - g) = 0 \qquad \text{on } \Gamma_C \tag{3.1e}$$

$$|\sigma_t| \leq \mathcal{F}, \quad \sigma_t u_t + \mathcal{F}|u_t| = 0 \qquad \text{on } \Gamma_C \tag{3.1f}$$

Recall that the scalar normal and tangential boundary stresses are defined as

$$\sigma_n := \mathbf{n} \cdot \sigma(\mathbf{u}) \cdot \mathbf{n} \quad \text{and} \quad \sigma_t := \mathbf{t} \cdot \sigma(\mathbf{u}) \cdot \mathbf{n},$$

and for any displacement $\mathbf{u}$ and for surface stresses defined on $\partial\Omega$ we adopt the following notation

$$\mathbf{u} = u_n\mathbf{n} + u_t\mathbf{t} \quad \text{and} \quad \sigma(\mathbf{u})\mathbf{n} = \sigma_n(\mathbf{u})\mathbf{n} + \sigma_t(\mathbf{u})\mathbf{t} \tag{3.2}$$

As in Chapter2, we introduce the function spaces

$$\mathcal{V} := [\tilde{H}^{\frac{1}{2}}(\Sigma)]^d = \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) := \{u \in \mathbf{H}^{\frac{1}{2}}(\Sigma);\ \mathrm{supp}(u) \subset \Sigma\}$$

$$\mathbf{V} := \{\mathbf{u} \in \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) : \mathbf{u} = 0 \quad \text{on} \quad \Gamma_D\}$$
$$\mathcal{W} := [H^{\frac{1}{2}}(\Gamma_C)]^d = \mathbf{H}^{\frac{1}{2}}(\Gamma_C)$$
$$\mathcal{M} := [H^{-\frac{1}{2}}(\Gamma_C)]^d = \mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$$

where

$$\bar{\Sigma} := \bar{\Gamma}_C \cup \bar{\Gamma}_N$$

and we assume that $\mathcal{F} \in L^2(\Gamma_C)$, $\mathbf{t} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ and $g \in H^{\frac{1}{2}}(\Gamma_C)$.

We define the D_tN (Dirichlet-to-Neumann) mapping $\mathbf{u}|_\Gamma \to \sigma(\mathbf{u})\cdot \mathbf{n}$. There holds

$$\sigma_n(\mathbf{u}) \equiv S\mathbf{u} \cdot \mathbf{n}|_{\Gamma_C}, \qquad \sigma_t(\mathbf{u}) \equiv S\mathbf{u} \cdot \mathbf{t}|_{\Gamma_C}. \tag{3.3}$$

The Steklov-Poincaré operator S is defined by

$$S := W + (K' + \frac{1}{2})V^{-1}(K + \frac{1}{2}).$$

We define the space for the Lagrange multiplier λ by

$$\mathbf{M} = M_n \times M_t$$

where

$$M_n := \{\mu_n \in \tilde{H}^{-\frac{1}{2}}(\Gamma_C) :\ \langle \mu_n, v_n\rangle_{\Gamma_C} \leq 0,\ \forall \mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_C) \text{ with } v_n \leq 0\ a.e \text{ on } \Gamma_C\}$$

and

$$M_t(\mathcal{F}) := \{\mu_t \in L_2(\Gamma_C) :\ |\mu_t| \leq \mathcal{F}\ a.e \text{ on } \Gamma_C\}.$$

are the sets of normal and tangential Lagrange multipliers.

The classical formulation (3.1) can be rewritten in a weak sense as a saddle point problem as follows:

Find $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbf{V} \times \mathbf{M}$ such that

$$\langle S\mathbf{u}, \mathbf{v}\rangle_\Sigma + b(\boldsymbol{\lambda}, \mathbf{v}) = \langle \mathbf{t}, \mathbf{v}\rangle_{\Gamma_N} \qquad \forall \mathbf{v} \in \mathbf{V} \tag{3.4a}$$
$$b(\boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{u}) \leqslant \langle g, \mu_n - \lambda_n\rangle_{\Gamma_C} \quad \forall\, \boldsymbol{\mu} \in \mathbf{M} \tag{3.4b}$$

with the functional

$$b(\boldsymbol{\mu}, \mathbf{v}) := \langle \mu_n, v_n\rangle_{\Gamma_C} + \langle \mu_t, v_t\rangle_{\Gamma_C}. \tag{3.5}$$

Here the notation $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{\frac{1}{2}}(\Gamma_C)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma_C)$.

3.2 The stabilized mixed hp-BEM formulation

Let $\mathcal{T}_{hp}$ be a subdivision of $\bar{\Gamma}_C \cup \bar{\Gamma}_N$ into straight line segments I. We associate each element of $\mathcal{T}_{hp}$ with a polynomial degree $p_I \geq 1$ and set $p = (p_I)_{I \in \mathcal{T}_{hp}}$. Furthermore we define the set of Gauss-Lobatto points $G_{I,hp}$ on each element $I \in \mathcal{T}_{hp}$ of corresponding polynomial degree p_I as the affine mapping of the Gauss-Lobatto points on $[-1, 1]$ onto I and set $G_{hp} := \cup_{I \in \mathcal{T}_{hp}} G_{I,hp.}$.

We introduce the space of continuous piecewise polynomials for the discretization of $\mathbf{u}$:

$$\mathbf{V}_{hp} := \{\mathbf{u}^{hp} \in C^0(\Sigma) : \forall I \in \mathcal{T}_{hp}, \mathbf{u}^{hp}|_I \in [\mathcal{P}_{p_I}(I)]^2, \mathbf{u}^{hp} = 0 \quad \text{on} \quad \Gamma_D\} \subset \mathbf{H}^{\frac{1}{2}}(\Gamma)$$

and the space of piecewise polynomials for the discrete tractions

$$\mathbf{W}_{hp} := \{\mathbf{L}_2(\Sigma) : \forall I \in \mathcal{T}_h, \boldsymbol{\phi}|_I \in [\mathcal{P}_{p_I - 1}(I)]^2\} \subset \mathbf{H}^{-\frac{1}{2}}(\Gamma).$$

An explicit representation of V^{-1} is not known, which causes additional difficulties in the numerical treatment. To resolve this problem we need to approximate the Steklov-Poincaré operators.

Let $i_{hp} : \mathbf{W}_{hp} \hookrightarrow \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $j_{hp} : \mathbf{V}_{hp} \hookrightarrow \mathbf{H}^{\frac{1}{2}}(\Gamma)$ denote the canonical imbeddings with dual maps i^*_{hp} and j^*_{hp}.

The approximation S_{hp} of the Poincaré-Steklov operators is given by

$$S_{hp} := W + (K' + \frac{1}{2}) i_{hp} (i^*_{hp} V i_{hp})^{-1} i^*_{hp} (K + \frac{1}{2}) \tag{3.6}$$

Recall that the operator $E_{hp} = S - S_{hp}$ represents the error in the approximation of the Steklov-Poincaré operator (see Chapter2).

Let $\mathcal{T}_{Hq}$ denote an additional partition of Γ_C, which needs not to coincide with $\mathcal{T}_{hp}|_{\Gamma_C}$. We define the discrete version of the space $\mathbf{M}$ for the Lagrange multiplier as (cf [19])

$$\mathbf{M}_{Hq}(\mathcal{F}) := M_{n,Hq} \times M_{t,Hq}(\mathcal{F}),$$

where

$$M_{n,Hq} := \{\lambda_n^{Hq} \in W_{Hq} : \lambda_n^{Hq}(x) \geq 0 \quad \forall x \in \Gamma_C \cap G_{Hq}\}$$

$$M_{t,Hq} := \{\lambda_t^{Hq} \in W_{Hq} : |\lambda_t^{Hq}(x)| \leq \mathcal{F} \quad \forall x \in \Gamma_C \cap G_{Hq}\}$$

and

$$W_{Hq} := \{\lambda^{Hq} \in L_2(\Gamma_C) : \forall J \in \mathcal{T}_{Hq}, \lambda^{Hq}|_J \in \mathcal{P}_q(J)\}$$

Remark 3.1. *It follows from* (3.4) *that* $\boldsymbol{\lambda} = -\sigma(\boldsymbol{u}) \cdot \boldsymbol{n}$ *in a weak sense. Therefore,* λ_n *has an interpretation as the negative normal contact traction. We consider approximations* λ_n^{Hq} *and* $-\sigma_n^h(\boldsymbol{u}^{hp})$ *(resp.* λ_t^{Hq} *and* $-\sigma_t^h(\boldsymbol{u}^{hp})$*) of* $\lambda_n = -\sigma_n(\boldsymbol{u})$ *(resp.*$\lambda_t = -\sigma_t(\boldsymbol{u})$*).*

where

$$\sigma_n^h(\mathbf{u}^{hp}) := S_{hp}\mathbf{u}^{hp} \cdot \mathbf{n}|_{\Gamma_C}, \qquad \sigma_t^h(\mathbf{u}^{hp}) := S_{hp}\mathbf{u}^{hp} \cdot \mathbf{t}|_{\Gamma_C}. \tag{3.7}$$

The discretized version of (3.4) with stabilized Lagrange multiplier reads as:

Find $\mathbf{u}^{hp} \in \mathbf{V}_{hp}$ and $\boldsymbol{\lambda}^{Hq} = (\lambda_n^{Hq}, \lambda_t^{Hq}) \in \mathbf{M}_{Hq}(\mathcal{F}) := M_{n,Hq} \times M_{t,Hq}(\mathcal{F})$:

$$\begin{aligned}&\langle S_{hp}\mathbf{u}^{hp}, \mathbf{v}^{hp}\rangle_\Sigma + b(\boldsymbol{\lambda}^{Hq}, \mathbf{v}^{hp}) + \int_{\Gamma_C} \gamma(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{v}^{hp})\, ds \\ &+ \int_{\Gamma_C} \gamma(-\lambda_t^{Hq} - \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{v}^{hp})\, ds = L(\mathbf{v}^{hp}) \qquad \forall \mathbf{v}^{hp} \in \mathbf{V}_{hp}\end{aligned} \tag{3.8a}$$

$$\begin{aligned}&b(\boldsymbol{\mu}^{Hq} - \boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp}) + \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp}))\, ds \\ &+ \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})(-\lambda_t^{Hq} - \sigma_t^h(\mathbf{u}^{hp}))\, ds \leqslant \langle g, \mu_n^{Hq} - \lambda_n^{Hq}\rangle_{\Gamma_C} \qquad \forall\, \boldsymbol{\mu}^{Hq} \in \mathbf{M}^H(\mathcal{F}).\end{aligned} \tag{3.8b}$$

Here γ is defined on each element I as the constant $\gamma = \gamma_0 \frac{h_I}{p_I^2}$, with $\gamma_0 > 0$ independent of $\frac{h}{p}$.

Note that the additional stabilization term vanishes for the solution of the continuous problem as $\boldsymbol{\lambda}^{Hq} \to \boldsymbol{\lambda}$ and $S_{hp}\mathbf{u}^{hp} \to S\mathbf{u}$.

3.2.1 Existence and uniqueness of the solution

In this section we show the existence and uniqueness of a solution to the stabilized formulation. We follow ideas of V.Lleras [45] for the h-version of stabilized FEM.

Lemma 3.1. *[9][Coercivity] For* γ_0 *sufficiently small, there exists a constant* $C > 0$ *independent of* h*,* p*,* H *and* q*, such that*

$$\langle S_{hp}\boldsymbol{v}^{hp}, \boldsymbol{v}^{hp}\rangle_\Sigma - \int_{\Gamma_C} \gamma(\sigma_n^h(\boldsymbol{v}^{hp}))^2\, ds - \int_{\Gamma_C} \gamma(\sigma_t^h(\boldsymbol{v}^{hp}))^2\, ds \geqslant C\|\boldsymbol{v}^{hp}\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Sigma)}, \quad \forall \boldsymbol{v}^{hp} \in \boldsymbol{V}_{hp} \tag{3.9}$$

Lemma 3.2. *For* γ_0 *small enough, Problem* (3.8) *admits a unique solution.*

Proof. Problem (3.8) is equivalent to finding a saddle-point $(\mathbf{u}^{hp}, \lambda_n^{Hq}, \lambda_t^{Hq}) \in \mathbf{V}_{hp} \times \mathbf{M}_{Hq}(\mathcal{F})$ which satisfies

$$\mathcal{L}_\gamma(\mathbf{u}^{hp}, \boldsymbol{\nu}^{Hq}) \le \mathcal{L}_\gamma(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{Hq}) \le \mathcal{L}_\gamma(\mathbf{v}^{hp}, \boldsymbol{\lambda}^{Hq}) \quad \forall \mathbf{v}^{hp} \in \mathbf{V}_{hp}, \quad \forall \boldsymbol{\nu}^{Hq} \in \mathbf{M}^H(\mathcal{F}), \tag{3.10}$$

with

$$\mathcal{L}_\gamma(\mathbf{v}^{hp},\boldsymbol{\nu}^{Hq}) = \frac{1}{2}\langle S_{hp}\mathbf{v}^{hp},\mathbf{v}^{hp}\rangle_\Sigma - L(\mathbf{v}^{hp}) + b(\boldsymbol{\nu}^{Hq},\mathbf{v}^{hp}) - \frac{1}{2}\int_{\Gamma_C}\gamma(\nu_n^{Hq}+\sigma_n^h(\mathbf{v}^{hp}))^2\,ds - \frac{1}{2}\int_{\Gamma_C}\gamma(\nu_t^{Hq}+\sigma_t^h(\mathbf{v}^{hp}))^2\,ds. \quad (3.11)$$

Taking $\boldsymbol{\nu}^{Hq}=0$ in (3.11) and using Lemma3.1 with γ_0 small enough , we obtain

$$\begin{aligned}\mathcal{L}_\gamma(\mathbf{v}^{hp},0) &= \frac{1}{2}\langle S_{hp}\mathbf{v}^{hp},\mathbf{v}^{hp}\rangle_\Sigma - L(\mathbf{v}^{hp}) + b(0,\mathbf{v}^{hp})\\ &\quad - \frac{1}{2}\int_{\Gamma_C}\gamma(\sigma_n^h(\mathbf{v}^{hp}))^2\,ds - \frac{1}{2}\int_{\Gamma_C}\gamma(\sigma_t^h(\mathbf{v}^{hp}))^2\,ds\\ &\geq \frac{C}{2}\|\mathbf{v}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} - \|L\|\|\mathbf{v}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Sigma)}\end{aligned} \quad (3.12)$$

That yields

$$\lim_{\|\mathbf{v}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Sigma)}\to\infty}\mathcal{L}_\gamma(\mathbf{v}^{hp},0) = +\infty \quad (3.13)$$

Choosing $\mathbf{v}^{hp}=0$ in (3.11) we obtain

$$\mathcal{L}_\gamma(0,\boldsymbol{\nu}^{Hq}) = -\frac{1}{2}\int_{\Gamma_C}\gamma(\nu_n^{Hq})^2\,ds - \frac{1}{2}\int_{\Gamma_C}\gamma(\nu_t^{Hq})^2\,ds.$$

and

$$\lim_{\|\boldsymbol{\nu}^{Hq}\|_{L_2(\Gamma_C)}\to\infty}\mathcal{L}_\gamma(0,\boldsymbol{\nu}^{Hq}) = -\infty \quad (3.14)$$

Then, due to (3.13) (3.14), $\mathcal{L}_\gamma$ is stricly convex in $\mathbf{v}^{hp}$ and stricly concave in $\boldsymbol{\nu}^{Hq}$.

The existence of the solution to problem (3.8) follows from the fact that $\mathbf{V}_{hp}$ and $\mathbf{M}^H(\mathcal{F})$ are two nonempty closed convex sets, $\mathcal{L}_\gamma(\cdot,\cdot)$ is continuous on $\mathbf{V}_{hp}\times\mathbf{M}^H(\mathcal{F})$, $\mathcal{L}_\gamma(\mathbf{v}^{hp},\cdot)$ (resp. $\mathcal{L}_\gamma(\cdot,\boldsymbol{\nu}^H)$) is stricly concave (resp. stricly convex) for any $\mathbf{v}^{hp}\in\mathbf{V}_{hp}$ (resp. for any $\boldsymbol{\nu}^{Hq}\in\mathbf{M}^H(\mathcal{F})$) (see [35]).

Let $(\boldsymbol{\lambda}_1^{Hq},\mathbf{u}_1^{hp})$ and $(\boldsymbol{\lambda}_2^{Hq},\mathbf{u}_2^{hp})$ be two solution of (3.8). Then, choosing $\boldsymbol{\mu}_1^{Hq}=\boldsymbol{\lambda}_2^{Hq}$ and $\boldsymbol{\mu}_2^{Hq}=\boldsymbol{\lambda}_1^{Hq}$ in (3.8b), we get

$$\begin{aligned}b(\boldsymbol{\lambda}_2^{Hq}-\boldsymbol{\lambda}_1^{Hq},\mathbf{u}_1^{hp}) &+ \int_{\Gamma_C}\gamma(\lambda_{2,n}^{Hq}-\lambda_{1,n}^{Hq})(-\lambda_{1,n}^{Hq}-\sigma_n^h(\mathbf{u}_1^{hp}))\,ds\\ &+ \int_{\Gamma_C}\gamma(\lambda_{2,t}^{Hq}-\lambda_{1,t}^{Hq})(-\lambda_{1,t}^{Hq}-\sigma_t^h(\mathbf{u}_1^{hp}))\,ds \leqslant \left\langle g,\lambda_{2,n}^{Hq}-\lambda_{1,n}^{Hq}\right\rangle_{\Gamma_C}\end{aligned} \quad (3.15)$$

$$\begin{aligned}b(\boldsymbol{\lambda}_1^{Hq}-\boldsymbol{\lambda}_2^{Hq},\mathbf{u}_2^{hp}) &+ \int_{\Gamma_C}\gamma(\lambda_{1,n}^{Hq}-\lambda_{2,n}^{Hq})(-\lambda_{2,n}^{Hq}-\sigma_n^h(\mathbf{u}_2^{hp}))\,ds\\ &+ \int_{\Gamma_C}\gamma(\lambda_{1,t}^{Hq}-\lambda_{2,t}^{Hq})(-\lambda_{2,t}^{Hq}-\sigma_t^h(\mathbf{u}_2^{hp}))\,ds \leqslant \left\langle g,\lambda_{1,n}^{Hq}-\lambda_{2,n}^{Hq}\right\rangle_{\Gamma_C}\end{aligned} \quad (3.16)$$

Adding the last two inequalities, we obtain

$$b(\boldsymbol{\lambda}_1^{Hq}-\boldsymbol{\lambda}_2^{Hq},\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})-\int_{\Gamma_C}\gamma(\lambda_{1,n}^{Hq}-\lambda_{2,n}^{Hq})(\lambda_{1,n}^{Hq}-\lambda_{2,n}^{Hq}+\sigma_n^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}))\,ds$$
$$-\int_{\Gamma_C}\gamma(\lambda_{1,t}^{Hq}-\lambda_{2,t}^{Hq})(\lambda_{1,t}^{Hq}-\lambda_{2,t}^{Hq}+\sigma_t^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}))\,ds\geq 0 \tag{3.17}$$

Futhermore, subtracting the two (3.8a) with $\mathbf{v}^{hp}=\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}$ implies

$$\begin{aligned}
0&=\langle S_{hp}(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}),\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}\rangle_\Sigma+b(\boldsymbol{\lambda}_1^{Hq}-\boldsymbol{\lambda}_2^{Hq},\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\\
&-\int_{\Gamma_C}\gamma(\lambda_{1,n}^{Hq}-\lambda_{2,n}^{Hq}+\sigma_n^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}))\sigma_n^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\,ds\\
&-\int_{\Gamma_C}\gamma(\lambda_{1,t}^{Hq}-\lambda_{2,t}^{Hq}+\sigma_t^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}))\sigma_t^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\,ds\\
&\geq\langle S_{hp}(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}),\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}\rangle_\Sigma-\int_{\Gamma_C}\gamma\sigma_n^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\sigma_n^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\,ds\\
&-\int_{\Gamma_C}\gamma\sigma_t^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\sigma_t^h(\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp})\,ds+\int_{\Gamma_C}\gamma(\boldsymbol{\lambda}_1^{Hq}-\boldsymbol{\lambda}_2^{Hq})^2\,ds\\
&\geq C\|\mathbf{u}_1^{hp}-\mathbf{u}_2^{hp}\|^2_{\mathbf{H}^{\frac12}(\Sigma)}+\|\gamma^{\frac12}(\boldsymbol{\lambda}_1^{Hq}-\boldsymbol{\lambda}_2^{Hq})\|^2_{\mathbf{L}^2(\Gamma_C)}
\end{aligned} \tag{3.18}$$

The conformity in the primal variable implies the uniqueness of the solution of problem (3.8). □

Let $\mathbf{u}\in\mathbf{V}$ and $\mathbf{u}^{hp}\in\mathbf{V}_{hp}$. As in [15], we define the following vectors

$$\begin{aligned}
\psi&:=V^{-1}(K+\frac12)\mathbf{u}\\
\psi^{hp}&:=i_{hp}V_{hp}^{-1}i_{hp}^*(K+\frac12)\mathbf{u}^{hp}\\
\psi_{hp}^*&:=V^{-1}(K+\frac12)\mathbf{u}^{hp}.
\end{aligned} \tag{3.19}$$

Lemma 3.3. *For ψ_{hp}^*, ψ^{hp} defined in* (3.19) *there holds*

$$\langle V(\psi_{hp}^*-\psi^{hp}),\phi\rangle=0\quad\forall\phi\in\boldsymbol{W}_{hp} \tag{3.20}$$

Lemma 3.4. *(Galerkin orthogonality) Let $\boldsymbol{u}\in\boldsymbol{V}$ be the solution of the continuous problem* (3.4) *and $\boldsymbol{u}^{hp}\in\boldsymbol{V}_{hp}$ the solution of the discrete problem. There holds*

$$\langle S\boldsymbol{u}-S_{hp}\boldsymbol{u}^{hp},\boldsymbol{v}^{hp}\rangle_\Sigma+b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq},\boldsymbol{v}^{hp})-\int_{\Gamma_C}\gamma(-\lambda_n^{Hq}-\sigma_n^h(\boldsymbol{u}^{hp}))\sigma_n^h(\boldsymbol{v}^{hp})\,ds$$
$$-\int_{\Gamma_C}\gamma(-\lambda_t^{Hq}-\sigma_t^h(\boldsymbol{u}^{hp}))\sigma_t^h(\boldsymbol{v}^{hp})\,ds=0\qquad\forall\boldsymbol{v}^{hp}\in\boldsymbol{V}_{hp}. \tag{3.21}$$

Proof. We choose $\mathbf{v}\in\mathbf{V}_{hp}\subset\mathbf{V}$ in (3.4a) and subtract (3.4a) from the discrete formulation (3.8a). □

3.3 A priori error analysis for frictional contact problem

Lemma 3.5. *Let $\boldsymbol{u} \in \boldsymbol{V}$, $\boldsymbol{\lambda} \in \boldsymbol{M}$ solve the saddle point problem* (3.4) *and, let $\boldsymbol{u}^{hp} \in \boldsymbol{V}_{hp}$ and $\boldsymbol{\lambda}^{Hq} \in \boldsymbol{M}_{hp}(\mathcal{F})$ be the solution of the discrete problem* (3.8)*, we assume that $\boldsymbol{\lambda} \in \boldsymbol{L}_2(\Gamma_C)$. Then there holds*

$$\begin{aligned}\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} \leq & \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n) u_n \, ds \\ & + \int_{\Gamma_C} (\lambda_n - \mu_n^{Hq})(u_n^{hp} + \gamma(-\lambda_n^{Hq} - \sigma_n(\boldsymbol{u}^{hp}))\, ds \\ & - \int_{\Gamma_C} \gamma(\lambda_n - \lambda_n^{Hq})\sigma_n(\boldsymbol{u} - \boldsymbol{u}^{hp})\, ds - \langle \lambda_n - \lambda_n^{Hq}, u_n^{hp} - u_n \rangle \\ & + \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t) u_t \, ds + \int_{\Gamma_C} (\lambda_t - \mu_t^{Hq})(u_t^{hp} + \gamma(-\lambda_t^{Hq} - \sigma_t(\boldsymbol{u}^{hp}))\, ds \\ & - \int_{\Gamma_C} \gamma(\lambda_t - \lambda_t^{Hq})\sigma_t(\boldsymbol{u} - \boldsymbol{u}^{hp})\, ds - \langle \lambda_t - \lambda_t^{Hq}, u_t^{hp} - u_t \rangle \\ & - \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq}) E_h^n(\boldsymbol{u}^{hp})\, ds - \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq}) E_h^t(\boldsymbol{u}^{hp})\, ds \end{aligned} \tag{3.22}$$

where

$$E_h^n(\boldsymbol{u}^{hp}) = (S - S_{hp})\boldsymbol{u}^{hp} \cdot \boldsymbol{n}|_{\Gamma_C}, \qquad E_h^t(\boldsymbol{u}^{hp}) = (S - S_{hp})\boldsymbol{u}^{hp} \cdot \boldsymbol{t}|_{\Gamma_C} \tag{3.23}$$

Proof. Recall that $\lambda_n = -\sigma_n(\mathbf{u})$ and $\lambda_t = -\sigma_t(\mathbf{u})$.

The inequality in (3.8b) is equivalent to the following conditions:

$$\langle \mu_n^{Hq} - \lambda_n^{Hq}, u_n^{hp} \rangle + \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp}))\, ds \leq 0 \tag{3.24}$$

$$\langle \mu_t^{Hq} - \lambda_t^{Hq}, u_t^{hp} \rangle + \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})(-\lambda_t^{Hq} - \sigma_t^h(\mathbf{u}^{hp}))\, ds \leq 0. \tag{3.25}$$

Note that

$$\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} = \int_{\Gamma_C} \gamma \lambda_n^2 \, ds - 2\int_{\Gamma_C} \gamma \lambda_n \lambda_n^{Hq}\, ds + \int_{\Gamma_C} \gamma (\lambda_n^{Hq})^2\, ds \tag{3.26}$$

$$\|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} = \int_{\Gamma_C} \gamma \lambda_t^2 \, ds - 2\int_{\Gamma_C} \gamma \lambda_t \lambda_t^{Hq}\, ds + \int_{\Gamma_C} \gamma (\lambda_t^{Hq})^2\, ds \tag{3.27}$$

Using (3.24) and (3.25) we get

$$\begin{aligned}\int_{\Gamma_C} \gamma(\lambda_n^{Hq})^2\, ds \leq & \int_{\Gamma_C} \gamma \lambda_n^{Hq} \mu_n^{Hq}\, ds - \int_{\Gamma_C} (\mu_n^{Hq} - \lambda_n^{Hq}) u_n^{hp}\, ds \\ & + \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})\sigma_n^h(\mathbf{u}^{hp})\, ds \\ \int_{\Gamma_C} \gamma(\lambda_t^{Hq})^2\, ds \leq & \int_{\Gamma_C} \gamma \lambda_t^{Hq} \mu_t^{Hq}\, ds - \int_{\Gamma_C} (\mu_t^{Hq} - \lambda_t^{Hq}) u_t^{hp}\, ds \\ & + \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})\sigma_t^h(\mathbf{u}^{hp})\, ds. \end{aligned}$$

From the second equation of the continuous problem (3.4b), we have for all $(\lambda_n, \lambda_t) \in M_n \times M_t(\mathcal{F})$:

$$\begin{aligned}
\int_{\Gamma_C} \gamma(\lambda_n)^2\, ds &\leq \int_{\Gamma_C} \gamma\lambda_n\mu_n\, ds - \int_{\Gamma_C} (\mu_n - \lambda_n)u_n\, ds + \int_{\Gamma_C} \gamma(\mu_n - \lambda_n)\sigma_n(\mathbf{u})\, ds \\
\int_{\Gamma_C} \gamma(\lambda_t)^2\, ds &\leq \int_{\Gamma_C} \gamma\lambda_t\mu_t\, ds - \int_{\Gamma_C} (\mu_t - \lambda_t)u_t\, ds + \int_{\Gamma_C} \gamma(\mu_t - \lambda_t)\sigma_t(\mathbf{u})\, ds.
\end{aligned}$$

This gives

$$\begin{aligned}
&\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} \leq \\
&\int_{\Gamma_C} \gamma(\mu_n - \lambda_n^{Hq})\lambda_n\, ds + \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n)\lambda_n^{Hq}\, ds + \int_{\Gamma_C} (\lambda_n - \mu_n)u_n\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_n - \lambda_n)\sigma_n(\mathbf{u})\, ds + \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n^{Hq})u_n^{hp}\, ds + \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})\sigma_n^h(\mathbf{u}^{hp})\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_t - \lambda_t^{Hq})\lambda_t\, ds + \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t)\lambda_t^{Hq}\, ds + \int_{\Gamma_C} (\lambda_t - \mu_t)u_t\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_t - \lambda_t)\sigma_t(\mathbf{u})\, ds + \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t^{Hq})u_t^{hp}\, ds + \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})\sigma_t^h(\mathbf{u}^{hp})\, ds
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Gamma_C} \gamma(\mu_n - \lambda_n^{Hq})\lambda_n\, ds + \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n)\lambda_n^{Hq}\, ds + \int_{\Gamma_C} (\lambda_n - \mu_n)u_n\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_n - \lambda_n)\sigma_n(\mathbf{u})\, ds + \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n^{Hq})u_n^{hp}\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})\sigma_n(\mathbf{u}^{hp})\, ds - \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})E_h^n(\mathbf{u}^{hp})\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_t - \lambda_t^{Hq})\lambda_t\, ds + \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t)\lambda_t^{Hq}\, ds + \int_{\Gamma_C} (\lambda_t - \mu_t)u_t\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_t - \lambda_t)\sigma_t(\mathbf{u})\, ds + \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t^{Hq})u_t^{hp}\, ds \\
&+ \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})\sigma_t(\mathbf{u}^{hp})\, ds - \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})E_h^t(\mathbf{u}^{hp})\, ds
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n)u_n\, ds + \int_{\Gamma_C} (\lambda_n - \mu_n^{Hq})(u_n^{hp} + \gamma(-\lambda_n^{Hq} - \sigma_n(\mathbf{u}^{hp}))\, ds \\
&- \int_{\Gamma_C} \gamma(\lambda_n - \lambda_n^{Hq})\sigma_n(\mathbf{u} - \mathbf{u}^{hp})\, ds - \langle \lambda_n - \lambda_n^{Hq}, u_n^{hp} - u_n \rangle + \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t)u_t\, ds \\
&+ \int_{\Gamma_C} (\lambda_t - \mu_t^{Hq})(u_t^{hp} + \gamma(-\lambda_t^{Hq} - \sigma_t(\mathbf{u}^{hp}))\, ds - \int_{\Gamma_C} \gamma(\lambda_t - \lambda_t^{Hq})\sigma_t(\mathbf{u} - \mathbf{u}^{hp})\, ds \\
&- \langle \lambda_t - \lambda_t^{Hq}, u_t^{hp} - u_t \rangle - \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n^{Hq})E_h^n(\mathbf{u}^{hp})\, ds - \int_{\Gamma_C} \gamma(\mu_t^{Hq} - \lambda_t^{Hq})E_h^t(\mathbf{u}^{hp})\, ds.
\end{aligned}$$

□

Lemma 3.6. *There exists a constant $C > 0$, independent of h, p, such that*

$$\|\boldsymbol{u}^{hp}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Sigma)} \leq C \tag{3.28}$$

In particular, for $\mathbf{u} \neq 0$, there exists a constant C_u such that

$$\|\boldsymbol{u}^{hp}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Sigma)} \leq C_u \|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Sigma)} \tag{3.29}$$

Proof. Choosing $\mathbf{v}^{hp} = \mathbf{u}^{hp}$ in (3.8a), we obtain

$$\begin{aligned}\langle S_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp}\rangle_\Sigma &+ \int_{\Gamma_C} \lambda_n^{Hq} u_n^{hp}\, ds + \int_{\Gamma_C} \gamma(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp})\, ds \\ &+ \int_{\Gamma_C} \lambda_t^{Hq} u_t^{hp}\, ds + \int_{\Gamma_C} \gamma(-\lambda_t^{Hq} - \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp})\, ds = L(\mathbf{u}^{hp})\end{aligned} \tag{3.30}$$

Choosing $\mu_n^{Hq} = 0$ and $\mu_t^{Hq} = \lambda_t^{Hq}$ in (3.8b), we get

$$-\int_{\Gamma_C} \lambda_n^{Hq} u_n^{hp}\, ds - \int_{\Gamma_C} \lambda_n^{Hq}(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp})) \leq 0 \tag{3.31}$$

Now we choose $\mu_n^{Hq} = 2\lambda_n^{Hq}$ and $\mu_t^{Hq} = \lambda_t^{Hq}$ in (3.8b), we obtain

$$\int_{\Gamma_C} \lambda_n^{Hq} u_n^{hp}\, ds + \int_{\Gamma_C} \lambda_n^{Hq}(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp})) \leq 0 \tag{3.32}$$

Combining (3.31) and (3.32), we obtain

$$\int_{\Gamma_C} \lambda_n^{Hq} u_n^{hp}\, ds + \int_{\Gamma_C} \gamma\lambda_n^{Hq}(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp})) = 0 \tag{3.33}$$

Similarly, we get

$$\int_{\Gamma_C} \lambda_t^{Hq} u_t^{hp}\, ds + \int_{\Gamma_C} \gamma\lambda_t^{Hq}(-\lambda_t^{Hq} - \sigma_t^h(\mathbf{u}^{hp})) = 0 \tag{3.34}$$

Using (3.30), (3.33), and (3.34), we have

$$\begin{aligned}L(\mathbf{u}^{hp}) &= \langle S_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp}\rangle_\Sigma + \int_{\Gamma_C} \lambda_n^{Hq} u_n^{hp}\, ds + \int_{\Gamma_C} \gamma(-\lambda_n^{Hq} - \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp})\, ds \\ &+ \int_{\Gamma_C} \lambda_t^{Hq} u_t^{hp}\, ds + \int_{\Gamma_C} \gamma(-\lambda_t^{Hq} - \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp})\, ds \\ &= \langle S_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp}\rangle_\Sigma + \int_{\Gamma_C} \gamma(\lambda_n^{Hq})^2\, ds - \int_{\Gamma_C} \gamma(\sigma_n^h(\mathbf{u}^{hp}))^2\, ds \\ &+ \int_{\Gamma_C} \gamma(\lambda_t^{Hq})^2\, ds - \int_{\Gamma_C} \gamma(\sigma_t^h(\mathbf{u}^{hp}))^2\, ds\end{aligned} \tag{3.35}$$

From Lemma 3.1 and (3.35), we obtain

$$\begin{aligned}C\|\mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} \leq \langle S_{hp}\mathbf{u}, \mathbf{u}\rangle_\Gamma - \int_{\Gamma_C} \gamma(\sigma_n^h(\mathbf{u}^{hp}))^2\, ds - \int_{\Gamma_C} \gamma(\sigma_t^h(\mathbf{u}^{hp}))^2\, ds \leq L(\mathbf{u}^{hp}) \\ \leq \|\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Sigma)}\end{aligned} \tag{3.36}$$

□

Theorem 3.1. *Let* $(\boldsymbol{u},\boldsymbol{\lambda}) \in \boldsymbol{V}\times\boldsymbol{M}$ *be a solution of the problem* (3.4) *such that* $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^1(\Sigma)$ *and* $\boldsymbol{\lambda} \in \boldsymbol{L}_2(\Gamma_C)$. *Let* $\boldsymbol{t} \in \boldsymbol{L}_2(\Sigma)$. *Let* $(\boldsymbol{u}^{hp},\boldsymbol{\lambda}^{Hq})$ *be the solution of the discrete problem* (3.8). *Then there exists a constant* $C>0$ *independent of h, H, p and q, such that*

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac12}(\Sigma)} + \|\psi-\psi^{hp}\|^2_{\tilde{\boldsymbol{H}}^{-\frac12}(\Gamma)} + \|\gamma^{\frac12}(\lambda_n-\lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \|\gamma^{\frac12}(\lambda_t-\lambda_t^{Hq})\|^2_{L_2(\Gamma_C)}\\
&\leq C\Bigg[\inf_{\boldsymbol{v}^{hp}\in\boldsymbol{V}_{hp}}\Big(\|\boldsymbol{u}-\boldsymbol{v}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac12}(\Sigma)} + \|\gamma^{\frac12}\sigma_n(\boldsymbol{u}-\boldsymbol{v}^{hp})\|^2_{L_2(\Gamma_C)}\\
&+b(\boldsymbol{\lambda}^{Hq},\boldsymbol{u}-\boldsymbol{v}^{hp}) + \|\boldsymbol{u}-\boldsymbol{v}^{hp}\|_{\boldsymbol{L}_2(\Sigma)}\\
&+\int_{\Gamma_C}\gamma(\lambda_n+\sigma_n(\boldsymbol{u}^{hp}))\sigma_n(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds + \int_{\Gamma_C}\gamma(\lambda_t+\sigma_t(\boldsymbol{u}^{hp}))\sigma_t(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds\\
&-\int_{\Gamma_C}\gamma(\lambda_n^{Hq}+\sigma_n(\boldsymbol{u}^{hp}))E_n^h(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds - \int_{\Gamma_C}\gamma(\lambda_t^{Hq}+\sigma_t(\boldsymbol{u}^{hp}))E_t^h(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds\\
&+\int_{\Gamma_C}\gamma E_n^h(\boldsymbol{u}^{hp})\sigma_n(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds + \int_{\Gamma_C}\gamma E_t^h(\boldsymbol{u}^{hp})\sigma_t(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds\\
&+\int_{\Gamma_C}\gamma E_n^h(\boldsymbol{u}^{hp})E_n^h(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds + \int_{\Gamma_C}\gamma E_t^h(\boldsymbol{u}^{hp})E_t^h(\boldsymbol{u}^{hp}-\boldsymbol{v}^{hp})\,ds\Big)\\
&+\inf_{\boldsymbol{v}\in\boldsymbol{V}}\Big(b(\boldsymbol{\lambda},\boldsymbol{v}-\boldsymbol{u}^{hp}) + \|\boldsymbol{u}^{hp}-\boldsymbol{v}\|_{\boldsymbol{L}_2(\Sigma)}\Big)\\
&+\inf_{\phi\in\boldsymbol{W}_{hp}}\|\psi-\phi\|^2_{\tilde{\boldsymbol{H}}^{-\frac12}(\Gamma)} + \inf_{\mu_n\in M_n}\int_{\Gamma_C}(\lambda_n^{Hq}-\mu_n)u_n\,ds\\
&+\inf_{\mu_n^{Hq}\in M_{n,Hq}}\int_{\Gamma_C}(\mu_n^{Hq}-\lambda_n)(u_n^{hp}+\gamma(-\lambda_n^{Hq}-\sigma_n(\boldsymbol{u}^{hp}))\,ds\\
&+\inf_{\mu_t\in M_t(\mathcal{F})}\int_{\Gamma_C}(\lambda_t^{Hq}-\mu_t)u_t\,ds + \inf_{\mu_t^{Hq}\in M_{t,Hq}(\mathcal{F})}\int_{\Gamma_C}(\mu_t^{Hq}-\lambda_t)(u_t^{hp}+\gamma(-\lambda_t^{Hq}-\sigma_t(\boldsymbol{u}^{hp}))\,ds\\
&+\inf_{\mu_n^{Hq}\in M_{n,Hq}}\int_{\Gamma_C}\gamma(\lambda_n^{Hq}-\mu_n^{Hq})E_h^n(\boldsymbol{u}^{hp})\,ds + \inf_{\mu_t^{Hq}\in M_{t,Hq}(\mathcal{F})}\int_{\Gamma_C}\gamma(\lambda_t^{Hq}-\mu_t^{Hq})E_h^t(\boldsymbol{u}^{hp})\,ds\Bigg]
\end{aligned}
$$

Proof. From Lemma 2.10 we obtain

$$\|\mathbf{u}-\mathbf{u}^{hp}\|^2_W + \|\psi-\psi_{hp}\|^2_V = \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{u}^{hp}\rangle + \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle.$$

Since we have with $E_{hp}=S-S_{hp}$

$$S\mathbf{u}-S_{hp}\mathbf{u}^{hp} = S(\mathbf{u}-\mathbf{u}^{hp}) + E_{hp}\mathbf{u}^{hp} = S(\mathbf{u}-\mathbf{u}^{hp}) + E_{hp}(\mathbf{u}^{hp}-\mathbf{u}) + E_{hp}\mathbf{u}, \quad (3.37)$$

we obtain

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}^{hp}\|^2_W + \|\psi-\psi_{hp}\|^2_V &= \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{u}^{hp}\rangle + \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle\\
&= \langle S(\mathbf{u}-\mathbf{u}^{hp}),\mathbf{u}-\mathbf{v}^{hp}\rangle + \langle S\mathbf{u}^{hp},\mathbf{u}^{hp}-\mathbf{v}^{hp}\rangle + \langle S\mathbf{u},\mathbf{v}^{hp}-\mathbf{u}^{hp}\rangle\\
&+ \langle E_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{u}^{hp}\rangle + \langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle.
\end{aligned}
\quad (3.38)
$$

Using (3.8a) and the definition of S_{hp}, we get

$$\langle S\mathbf{u}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle = \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp} - \mathbf{v}^{hp}) + \langle \mathbf{t}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle$$
$$+ \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \tag{3.39}$$

Using (3.4), (3.38) and (3.39), we obtain

$$C_W\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_V\|\psi - \psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \leq \langle S(\mathbf{u} - \mathbf{u}^{hp}), \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle S\mathbf{u}, \mathbf{v}^{hp} - \mathbf{u}^{hp}\rangle$$

$$+ \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}\rangle + \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp} - \mathbf{v}^{hp}) + \langle \mathbf{t}, \mathbf{u}^{hp} - \mathbf{v}^{hp}\rangle$$

$$- \langle S\mathbf{u}, \mathbf{u} - \mathbf{v}\rangle - b(\boldsymbol{\lambda}, \mathbf{u} - \mathbf{v}) + \langle \mathbf{t}, \mathbf{u} - \mathbf{v}\rangle + \langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\rangle$$

$$+ \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n(\mathbf{u}^{hp}))\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t(\mathbf{u}^{hp}))\sigma_t(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds$$

$$\leq \langle S(\mathbf{u} - \mathbf{u}^{hp}), \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle \mathbf{t} - S\mathbf{u}, \mathbf{u}^{hp} - \mathbf{v}\rangle + \langle \mathbf{t} - S\mathbf{u}, \mathbf{u} - \mathbf{v}^{hp}\rangle$$

$$+ b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{u}) - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp} - \mathbf{v}^{hp}) + \langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\rangle$$

$$+ \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \tag{3.40}$$

We have

$$b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{u}) - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp} - \mathbf{v}^{hp}) - b(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp} - \mathbf{u}) = b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{u}^{hp}) - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u} - \mathbf{v}^{hp}) \tag{3.41}$$

From (3.40) and (3.41), we obtain

$$C_W\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_V\|\psi - \psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} - \langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp} - \mathbf{u}\rangle \leq$$

$$\langle S(\mathbf{u} - \mathbf{u}^{hp}), \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{v}^{hp}\rangle + \langle \mathbf{t} - S\mathbf{u}, \mathbf{u}^{hp} - \mathbf{v}\rangle + \langle \mathbf{t} - S\mathbf{u}, \mathbf{u} - \mathbf{v}^{hp}\rangle$$

$$+ b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{u}^{hp}) - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u} - \mathbf{v}^{hp}) + \langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\rangle$$

$$+ \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \tag{3.42}$$

Using Lemma 3.5 and (3.42) we have

$$
\begin{aligned}
&C_W\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}+C_V\|\psi-\psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}+\|\gamma^{\frac{1}{2}}(\lambda_n-\lambda_n^{Hq})\|^2_{L_2(\Gamma_C)}+\|\gamma^{\frac{1}{2}}(\lambda_t-\lambda_t^{Hq})\|^2_{L_2(\Gamma_C)}\\
&\leq \langle S(\mathbf{u}-\mathbf{u}^{hp}),\mathbf{u}-\mathbf{v}^{hp}\rangle+\langle E_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle+\langle \mathbf{t}-S\mathbf{u},\mathbf{u}^{hp}-\mathbf{v}\rangle+\langle \mathbf{t}-S\mathbf{u},\mathbf{u}-\mathbf{v}^{hp}\rangle\\
&+b(\boldsymbol{\lambda},\mathbf{v}-\mathbf{u}^{hp})-b(\boldsymbol{\lambda}^{Hq},\mathbf{u}-\mathbf{v}^{hp})+\langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle\\
&+\int_{\Gamma_C}\gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds+\int_{\Gamma_C}\gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds\\
&+\int_{\Gamma_C}(\lambda_n^{Hq}-\mu_n)u_n\,ds+\int_{\Gamma_C}(\lambda_n-\mu_n^{Hq})(u_n^{hp}+\gamma(-\lambda_n^{Hq}-\sigma_n(\mathbf{u}^{hp}))\,ds\\
&-\int_{\Gamma_C}\gamma(\lambda_n-\lambda_n^{Hq})\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\,ds-\int_{\Gamma_C}\gamma(\lambda_t-\lambda_t^{Hq})\sigma_t(\mathbf{u}-\mathbf{u}^{hp})\,ds\\
&+\int_{\Gamma_C}(\lambda_t^{Hq}-\mu_t)u_t\,ds+\int_{\Gamma_C}(\lambda_t-\mu_t^{Hq})(u_t^{hp}+\gamma(-\lambda_t^{Hq}-\sigma_t(\mathbf{u}^{hp}))\,ds\\
&+\int_{\Gamma_C}\gamma(\lambda_n^{Hq}-\mu_n^{Hq})E_h^n(\mathbf{u}^{hp})\,ds+\int_{\Gamma_C}\gamma(\lambda_t^{Hq}-\mu_t^{Hq})E_h^t(\mathbf{u}^{hp})\,ds
\end{aligned}
\tag{3.43}
$$

Therefore

$$
\begin{aligned}
&C_W\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}+C_V\|\psi-\psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}+\|\gamma^{\frac{1}{2}}(\lambda_n-\lambda_n^{Hq})\|^2_{L_2(\Gamma_C)}+\|\gamma^{\frac{1}{2}}(\lambda_t-\lambda_t^{Hq})\|^2_{L_2(\Gamma_C)}\\
&\leq \langle S(\mathbf{u}-\mathbf{u}^{hp}),\mathbf{u}-\mathbf{v}^{hp}\rangle+\langle E_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle+\langle \mathbf{t}-S\mathbf{u},\mathbf{u}^{hp}-\mathbf{v}\rangle+\langle \mathbf{t}-S\mathbf{u},\mathbf{u}-\mathbf{v}^{hp}\rangle\\
&+b(\boldsymbol{\lambda},\mathbf{v}-\mathbf{u}^{hp})-b(\boldsymbol{\lambda}^{Hq},\mathbf{u}-\mathbf{v}^{hp})+\langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle\\
&\int_{\Gamma_C}\gamma(\lambda_n+\sigma_n(\mathbf{u}^{hp}))\sigma_n(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds+\int_{\Gamma_C}\gamma(\lambda_t+\sigma_t(\mathbf{u}^{hp}))\sigma_t(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds\\
&+\int_{\Gamma_C}(\lambda_n^{Hq}-\mu_n)u_n\,ds+\int_{\Gamma_C}(\lambda_n-\mu_n^{Hq})(u_n^{hp}+\gamma(-\lambda_n^{Hq}-\sigma_n(\mathbf{u}^{hp}))\,ds\\
&-\int_{\Gamma_C}\gamma(\lambda_n-\lambda_n^{Hq})\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\,ds-\int_{\Gamma_C}\gamma(\lambda_t-\lambda_t^{Hq})\sigma_t(\mathbf{u}-\mathbf{v}^{hp})\,ds\\
&+\int_{\Gamma_C}(\lambda_t^{Hq}-\mu_t)u_t\,ds+\int_{\Gamma_C}(\lambda_t-\mu_t^{Hq})(u_t^{hp}+\gamma(-\lambda_t^{Hq}-\sigma_t(\mathbf{u}^{hp}))\,ds\\
&-\int_{\Gamma_C}\gamma(\lambda_n^{Hq}+\sigma_n(\mathbf{u}^{hp}))E_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds-\int_{\Gamma_C}\gamma(\lambda_t^{Hq}+\sigma_t(\mathbf{u}^{hp}))E_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds\\
&-\int_{\Gamma_C}\gamma E_n^h(\mathbf{u}^{hp})\sigma_n(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds-\int_{\Gamma_C}\gamma E_t^h(\mathbf{u}^{hp})\sigma_t(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds\\
&+\int_{\Gamma_C}\gamma E_n^h(\mathbf{u}^{hp})E_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds+\int_{\Gamma_C}\gamma E_t^h(\mathbf{u}^{hp})E_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds\\
&+\int_{\Gamma_C}\gamma(\lambda_n^{Hq}-\mu_n^{Hq})E_n^h(\mathbf{u}^{hp})\,ds+\int_{\Gamma_C}\gamma(\lambda_t^{Hq}-\mu_t^{Hq})E_h^t(\mathbf{u}^{hp})\,ds
\end{aligned}
\tag{3.44}
$$

Using (3.37), Cauchy Schwarz inequality, and the continuity of S_{hp} and E_{hp}, there holds for all $\phi\in\mathbf{W}_{hp}$

$$
A=\langle S(\mathbf{u}-\mathbf{u}^{hp}),\mathbf{u}-\mathbf{v}^{hp}\rangle\leq C_S\|\mathbf{u}-\mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}
\tag{3.45}
$$

Using Lemma2.6and Cauchy Schwarz inequality, we have

$$\begin{aligned} B = \langle E_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle = \langle E_{hp}(\mathbf{u}^{hp}+\mathbf{u}-\mathbf{u}), \mathbf{u}-\mathbf{v}^{hp}\rangle \leq \\ \|E_{hp}\mathbf{u}\|_{\mathbf{H}^{-\frac{1}{2}}(\Sigma)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_{E_{hp}}\|\mathbf{u}-\mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ \leq C_0\|\psi-\phi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_{E_{hp}}\|\mathbf{u}-\mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \end{aligned} \tag{3.46}$$

for $\phi^{hp} \in \mathbf{W}_{hp}$.

From Lemma3.3 and (3.19) follows with $\phi^{hp} \in \mathbf{W}_{hp}$ that

$$\begin{aligned} C &= \langle V(\psi_h^* - \psi^{hp}), \psi - \psi^{hp}\rangle \\ &= \langle V(\psi_h^* - \psi), \psi - \phi^{hp}\rangle + \langle V(\psi - \psi^{hp}), \psi - \phi^{hp}\rangle \\ &\leq (C_K + \frac{1}{2})\|\mathbf{u}-\mathbf{u}^h\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\|\psi-\phi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \\ &+ C_V\|\psi-\psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\|\psi-\phi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \end{aligned} \tag{3.47}$$

Employing Young's inequality, we obtain

$$A \leq \frac{1}{2\epsilon}C_S^2\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \frac{\epsilon}{2}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \tag{3.48}$$

$$\begin{aligned} B &\leq \frac{1}{2\epsilon}C_{E_{hp}}^2\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \frac{\epsilon}{2}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &+ \frac{C_0}{2}\|\psi-\phi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} + \frac{C_0}{2}\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \end{aligned} \tag{3.49}$$

$$\begin{aligned} C &\leq \frac{1}{2\epsilon}(C_K + \frac{1}{2})^2\|\psi-\phi\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} + \frac{\epsilon}{2}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &+ \frac{1}{2\epsilon}C_V^2\|\psi-\phi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} + \frac{\epsilon}{2}\|\psi-\psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \end{aligned} \tag{3.50}$$

Since $\mathbf{t} - S\mathbf{u} \in \mathbf{L}_2(\Sigma)$, we obtain

$$\begin{aligned} D &= \langle \mathbf{t} - S\mathbf{u}, \mathbf{u}^{hp} - \mathbf{v}\rangle + \langle \mathbf{t} - S\mathbf{u}, \mathbf{u} - \mathbf{v}^{hp}\rangle \\ &\leq \|\mathbf{t} - S\mathbf{u}\|_{\mathbf{L}_2(\Sigma)}(\|\mathbf{u}^{hp}-\mathbf{v}\|_{\mathbf{L}_2(\Sigma)} + \|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(\Sigma)}) \end{aligned} \tag{3.51}$$

Employing Cauchy Schwarz and Young's inequality, we obtain

$$\begin{aligned} E_n = \int_{\Gamma_C}\gamma(\lambda_n - \lambda_n^{Hq})\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\,ds \leq \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|_{L_2(\Gamma_C)}\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\|_{L_2(\Gamma_C)} \\ \leq \frac{\epsilon}{2}\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \frac{1}{2\epsilon}\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} \end{aligned} \tag{3.52}$$

$$E_t = \int_{\Gamma_C} \gamma(\lambda_t - \lambda_t^{Hq})\sigma_t(\mathbf{u} - \mathbf{v}^{hp})\, ds \leq \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|_{L_2(\Gamma_C)} \|\gamma^{\frac{1}{2}}\sigma_t(\mathbf{u} - \mathbf{v}^{hp})\|_{L_2(\Gamma_C)}$$
$$\leq \frac{\epsilon}{2}\|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} + \frac{1}{2\epsilon}\|\gamma^{\frac{1}{2}}\sigma_t(\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} \tag{3.53}$$

Using (3.44), and (3.48)-(3.53), we get

$$\alpha_1\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \alpha_2\|\psi - \psi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}$$

$$+ \alpha_3\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \alpha_4\|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)}$$

$$\leq \alpha_5\|\mathbf{u} - \mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \alpha_6\|\psi - \phi^{hp}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}$$

$$+ \frac{1}{\epsilon}\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} + \frac{1}{\epsilon}\|\gamma^{\frac{1}{2}}\sigma_t(\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)}$$

$$+ \|\mathbf{t} - S\mathbf{u}\|_{\mathbf{L}_2(\Sigma)}(\|\mathbf{u}^{hp} - \mathbf{v}\|_{\mathbf{L}_2(\Sigma)} + \|\mathbf{u} - \mathbf{v}^{hp}\|_{\mathbf{L}_2(\Sigma)}) + b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{u}^{hp}) - b(\boldsymbol{\lambda}^{Hq}, \mathbf{u} - \mathbf{v}^{hp})$$

$$+ \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n)u_n\, ds - \int_{\Gamma_C} (\lambda_n - \mu_n^{Hq})(u_n^{hp} + \gamma(-\lambda_n^{Hq} - \sigma_n(\mathbf{u}^{hp}))\, ds$$

$$+ \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t)u_t\, ds - \int_{\Gamma_C} (\lambda_t - \mu_t^{Hq})(u_t^{hp} + \gamma(-\lambda_t^{Hq} - \sigma_t(\mathbf{u}^{hp}))\, ds$$

$$+ \int_{\Gamma_C} \gamma(\lambda_n + \sigma_n(\mathbf{u}^{hp}))\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma(\lambda_t + \sigma_t(\mathbf{u}^{hp}))\sigma_t(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds$$

$$- \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n(\mathbf{u}^{hp}))E^n_{hp}(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds - \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t(\mathbf{u}^{hp}))E^t_{hp}(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds$$

$$+ \int_{\Gamma_C} \gamma E^n_{hp}(\mathbf{u}^{hp})\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma E^t_{hp}(\mathbf{u}^{hp})\sigma_t(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds$$

$$+ \int_{\Gamma_C} \gamma E^n_{hp}(\mathbf{u}^{hp})E^n_{hp}(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma E^t_{hp}(\mathbf{u}^{hp})E^t_{hp}(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds$$

$$+ \int_{\Gamma_C} \gamma(\lambda_n^{Hq} - \mu_n^{Hq})E^n_{hp}(\mathbf{u}^{hp})\, ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq} - \mu_t^{Hq})E^t_{hp}(\mathbf{u}^{hp})\, ds,$$

where the constants

$$\alpha_1 = 2C_W - 3\epsilon$$
$$\alpha_2 = 2C_V - \epsilon$$
$$\alpha_3 = 2 - \epsilon$$
$$\alpha_4 = 2 - \epsilon$$
$$\alpha_5 = \frac{C_S^2}{\epsilon} + \frac{C_{E_{hp}}^2}{\epsilon} + C_0$$
$$\alpha_6 = C_0 + \frac{1}{\epsilon}(C_K + \frac{1}{2})^2 + \frac{1}{\epsilon}C_V^2$$

are independent of h, H, p and q, α_1, α_2, α_3 and α_4 are positive if ϵ is small enough. The estimate of the theorem follows immediately. □

Theorem 3.2. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{M}$ *be the solution of the problem* (3.4) *and* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{Hq})$ *the solution of the discrete problem* (3.8) *with* $\boldsymbol{t} \in L_2(\Gamma_N)$ *and* $g = 0$. *Assume that* $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)$, $\boldsymbol{\lambda} \in \boldsymbol{H}^{\nu}(\Gamma_C)$ *for some* $\nu \in [0, \frac{1}{2}]$. *Suppose that*

$$\|\lambda_n\|_{H^\nu(\Gamma_C)} + \|\lambda_t\|_{H^\nu(\Gamma_C)} + \|\mathcal{F}\|_{L_2(\Gamma_C)} \leq \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)}.$$

Then there exists a constant $C > 0$ *independent of h and p, such that*

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} + \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|_{L_2(\Gamma_C)} + \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|_{L_2(\Gamma_C)} \\
&\leq C \left(\frac{H^{\nu+\frac{1}{2}}}{q^{\nu+\frac{1}{2}}} + \frac{h^{\frac{1}{2}} H^{\nu}}{p q^{\nu}} \right) \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)} \\
&+ \inf_{\mu_n \in M_n} \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n) u_n \, ds + \inf_{\mu_t \in M_t(\mathcal{F})} \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t) u_t \, ds
\end{aligned} \tag{3.54}$$

Proof. The a priori estimate follows from the estimate in Theorem (3.1) with $\mathbf{v} = \mathbf{u}^{hp}$.

We estimate the term

$$b(\boldsymbol{\lambda}^{Hq}, \mathbf{u} - \mathbf{v}^{hp}) = \langle \lambda_n^{Hq}, u_n - v_n^{hp} \rangle + \langle \lambda_t^{Hq}, u_t - v_t^{hp} \rangle \tag{3.55}$$

Employing Cauchy Schwarz and Young's inequality, we have

$$\begin{aligned}
\langle \lambda_n^{Hq}, u_n - v_n^{hp} \rangle &= \int_{\Gamma_C} (\lambda_n^{Hq} - \lambda_n)(u_n - v_n^{hp}) \, ds + \int_{\Gamma_C} \lambda_n (u_n - v_n^{hp}) \, ds \\
&\leq \|\lambda_n^{Hq} - \lambda_n\|_{L_2(\Gamma_C)} \|u_n - v_n^{hp}\|_{L_2(\Gamma_C)} + \|\lambda_n\|_{L_2(\Gamma_C)} \|u_n - v_n^{hp}\|_{L_2(\Gamma_C)} \\
&\leq \frac{\epsilon}{2} \|\gamma^{\frac{1}{2}}(\lambda_n^{Hq} - \lambda_n)\|_{L_2(\Gamma_C)}^2 + \frac{1}{2\epsilon\gamma_0} \frac{p^2}{h} \|u_n - v_n^{hp}\|_{L_2(\Gamma_C)}^2 + \|\lambda_n\|_{L_2(\Gamma_C)} \|u_n - v_n^{hp}\|_{L_2(\Gamma_C)} \\
&\leq \frac{\epsilon}{2} \|\gamma^{\frac{1}{2}}(\lambda_n^{Hq} - \lambda_n)\|_{L_2(\Gamma_C)}^2 + \frac{1}{2\epsilon} \frac{h^{2\nu+1}}{p^{2\nu}} \|\mathbf{u}\|_{\mathbf{H}^{1+\nu}(\Gamma)}^2 + \frac{h^{\nu+1}}{p^{\nu+1}} \|\mathbf{u}\|_{\mathbf{H}^{1+\nu}(\Gamma)}^2
\end{aligned} \tag{3.56}$$

Similarly to (2.101), we get

$$\langle \lambda_t^{Hq}, u_t - v_t^{hp} \rangle \leq \leq \frac{\epsilon}{2} \|\gamma^{\frac{1}{2}}(\lambda_t^{Hq} - \lambda_t)\|_{L_2(\Gamma_C)}^2 + \frac{1}{2\epsilon} \frac{h^{2\nu+1}}{p^{2\nu}} \|\mathbf{u}\|_{\mathbf{H}^{1+\nu}(\Gamma)}^2 + \frac{h^{\nu+1}}{p^{\nu+1}} \|\mathbf{u}\|_{\mathbf{H}^{1+\nu}(\Gamma)}^2 \tag{3.57}$$

Combining (2.101) and (3.57), we obtain

$$b(\boldsymbol{\lambda}^{Hq}, \mathbf{u} - \mathbf{v}^{hp}) \leq \frac{\epsilon}{2} \|\gamma^{\frac{1}{2}} (\lambda_n^{Hq} - \lambda_n)\|^2_{L_2(\Gamma_C)} + \frac{\epsilon}{2} \|\gamma^{\frac{1}{2}} (\lambda_t^{Hq} - \lambda_t)\|^2_{L_2(\Gamma_C)} + \frac{1}{2\epsilon} \frac{h^{2\nu+1}}{p^{2\nu}} \|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} + \frac{h^{\nu+1}}{p^{\nu+1}} \|\mathbf{u}\|^2_{\mathbf{H}^{1+\nu}(\Gamma)} \tag{3.58}$$

Now we estimate the terms

$$A_2 = \|\mathbf{u} - \mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \quad \text{and} \quad A_3 = \|\gamma^{\frac{1}{2}} \sigma_n (\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)}$$

We choose $\mathbf{v}^{hp} = \mathcal{I}_{hp}\mathbf{u}$ and use the aproximation property of the Lagrange interpolation operator:

$$A_2 = \|\mathbf{u} - \mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} = \|\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \leq C \frac{h^{1+2\nu}}{p^{1+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.59}$$

we have

$$\|\mathbf{u} - \mathbf{v}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} = \|\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^1(\Gamma)} \leq C \left(\frac{h}{p}\right)^{2\nu} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.60}$$

Employing the continuity condition of the Dirichlet-to-Neumann operator and (3.60) we obtain

$$\begin{aligned} A_3 = \|\gamma^{\frac{1}{2}} \sigma_n (\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} &\leq \alpha \|\gamma^{\frac{1}{2}} (\mathbf{u} - \mathbf{v}^{hp})\|^2_{\mathbf{H}^1(\Gamma)} \\ &= \alpha\gamma_0 \frac{h}{p^2} \|\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^1(\Gamma)} \\ &= C\alpha\gamma_0 \frac{h^{1+2\nu}}{p^{2+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}. \end{aligned} \tag{3.61}$$

We now estimate the term

$$A_{4,n} = \int_{\Gamma_C} \gamma(\sigma_n(\mathbf{u}^{hp}) + \lambda_n) \sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp}) \, ds.$$

Recall that $\lambda_n = -\sigma_n(\mathbf{u})$. Since $\mathbf{u}^{hp} - \mathbf{v}^{hp} \in \mathbf{V}_{hp}$, we can apply the hp-inverse inequality

$$\begin{aligned} \frac{h}{p^2} \|\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} &\leq \alpha \frac{h}{p^2} \|\mathbf{u}^{hp} - \mathbf{v}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \\ &\leq \alpha \|\mathbf{u}^{hp} - \mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \end{aligned} \tag{3.62}$$

Using Cauchy Schwarz, Young's inequality, (3.59),(3.61) and (3.62), we obtain

$$
\begin{aligned}
A_{4,n} &\leq \frac{1}{2}\gamma_0\frac{h}{p^2}\|\sigma_n(\mathbf{u}^{hp}-\mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)}+\frac{1}{2}\gamma_0\frac{h}{p^2}\|\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{L_2(\Gamma_C)}\\
&\leq \gamma_0\frac{h}{p^2}\|\sigma_n(\mathbf{u}^{hp}-\mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)}+\frac{1}{2}\gamma_0\frac{h}{p^2}\|\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)}\\
&\leq C\alpha\gamma_0\frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+2\alpha\gamma_0\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}+2\alpha\gamma_0\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\\
&\leq \check{C}\left(\frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+\gamma_0\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\right). \qquad (3.63)
\end{aligned}
$$

Similar to (3.63), we get

$$
\begin{aligned}
A_{4,t} &= \int_{\Gamma_C}\gamma(\sigma_t(\mathbf{u}^{hp})+\lambda_t)\sigma_t(\mathbf{u}^{hp}-\mathbf{w}^{hp})\,ds\\
&\leq \check{C}\left(\frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+\left(\frac{h}{p}\right)^{1+2\nu}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+\gamma_0\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\right) \qquad (3.64)
\end{aligned}
$$

We now estimate the term

$$
A_{5,n} = \inf_{\mu_n^{Hq}\in M_{n,Hq}}\int_{\Gamma_C}(\mu_n^{Hq}-\lambda_n)(u_n^{hp}+\gamma(-\lambda_n^{Hq}-\sigma_n(\mathbf{u}^{hp}))\,ds \qquad (3.65)
$$

As in Chapter2, let $\pi_{\mathbf{M}_{hp}}$ be the $\mathbf{L}_2$-projection operator mapping $\mathbf{M}_{Hq}$ defined by

$$
\pi_{\mathbf{M}_{Hq}} = \begin{cases}\pi_{M_{n,Hq}} : L_2(\Gamma_C) \longrightarrow M_{n,Hq}\\ \pi_{M_{t,Hq}} : L_2(\Gamma_C) \longrightarrow M_{t,Hq}.\end{cases} \qquad (3.66)
$$

Choosing $\mu_n^{Hq} = \pi_{M_{n,Hq}}\lambda_n$, we obtain

$$
A_{5,n} \leq \int_{\Gamma_C}(\pi_{M_{n,Hq}}\lambda_n-\lambda_n)u_n^{hp}\,ds+\int_{\Gamma_C}\gamma(\pi_{M_{n,Hq}}\lambda_n-\lambda_n)(-\lambda_n^{Hq}-\sigma_n(\mathbf{u}^{hp}))\,ds. \qquad (3.67)
$$

The estimate of the first term in (3.67) gives, using (2.112):

$$
\begin{aligned}
A_{5,1,n} &= \int_{\Gamma_C}(\pi_{M_{n,Hq}}\lambda_n-\lambda_n)u_n^{hp}\,ds\\
&= \int_{\Gamma_C}(\pi_{M_{n,Hq}}\lambda_n-\lambda_n)(u_n^{hp}-u_n)\,ds+\int_{\Gamma_C}(\pi_{M_{n,Hq}}\lambda_n-\lambda_n)(u_n-\pi_{M_{n,Hq}}u_n)\,ds\\
&\leq \|\pi_{M_{n,Hq}}\lambda_n-\lambda_n\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|u_n^{hp}-u_n\|_{H^{\frac{1}{2}}(\Gamma_C)}\\
&+\|\pi_{M_{n,Hq}}\lambda_n-\lambda_n\|_{L_2(\Gamma_C)}\|u_n-\pi_{M_{n,Hq}}u_n\|_{L_2(\Gamma_C)}\\
&\leq C\left(\frac{H^{\frac{1}{2}+\nu}}{q^{\frac{1}{2}+\nu}}\|\lambda_n\|_{H^\nu(\Gamma_C)}\|\mathbf{u}-\mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}+\frac{H^{1+2\nu}}{q^{1+2\nu}}\|\lambda_n\|_{H^\nu(\Gamma_C)}\|\mathbf{u}\|_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}\right)
\end{aligned} \qquad (3.68)
$$

We employ Young's inequality to obtain

$$A_{5,1,n} \le C \left(\epsilon \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \frac{H^{2\nu+1}}{q^{2\nu+1}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \right) \tag{3.69}$$

Now we estimate the second integral term

$$A_{5,2,n} = \inf_{\mu_n^{Hq} \in M_{n,Hq}} \int_{\Gamma_C} \gamma(\mu_n^{Hq} - \lambda_n)(-\lambda_n^{Hq} - \sigma_n(\mathbf{u}^{hp}))\, ds. \tag{3.70}$$

We have

$$\begin{aligned} A_{5,2,n} \le& \int_{\Gamma_C} \gamma(\pi_{M_{n,Hq}} \lambda_n - \lambda_n)(-\lambda_n^{Hq} + \lambda_n)\, ds \\ &+ \int_{\Gamma_C} \gamma(\pi_{M_{n,Hq}} \lambda_n - \lambda_n)(\sigma_n(\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}))\, ds \\ &+ \int_{\Gamma_C} \gamma(\pi_{M_{n,Hq}} \lambda_n - \lambda_n)(\sigma_n(\mathcal{I}_{hp}\mathbf{u} - \mathbf{u}^{hp}))\, ds \\ \le& C\gamma_0^{\frac{1}{2}} \left(\frac{h}{p^2}\right)^{\frac{1}{2}} \frac{H^\nu}{q^\nu} \|\lambda_n\|_{H^\nu(\Gamma_C)} \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|_{L_2(\Gamma_C)} \\ &+ C\gamma_0^{\frac{1}{2}} \left(\frac{h}{p^2}\right)^{\frac{1}{2}} \frac{H^\nu}{q^\nu} \|\lambda_n\|_{H^\nu(\Gamma_C)} \|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathcal{I}_{hp}\mathbf{u})\|_{L_2(\Gamma_C)} \\ &+ C\gamma_0^{\frac{1}{2}} \left(\frac{h}{p^2}\right)^{\frac{1}{2}} \frac{H^\nu}{q^\nu} \|\lambda_n\|_{H^\nu(\Gamma_C)} \|\gamma^{\frac{1}{2}}\sigma_n(\mathcal{I}_{hp}\mathbf{u} - \mathbf{u}^{hp})\|_{L_2(\Gamma_C)}. \end{aligned} \tag{3.71}$$

Using Young's inequality, (3.61), (3.60) and (3.63), we obtain

$$\begin{aligned} A_{5,2,n} \le C \Bigg(& \frac{hH^{2\nu}}{p^2 q^{2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \epsilon \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} \\ & + \frac{h^{1+2\nu}}{p^{1+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \alpha\gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \Bigg). \end{aligned} \tag{3.72}$$

Finally we have

$$\begin{aligned} A_{5,n} \le C \Bigg(& \epsilon \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ & + \frac{H^{2\nu+1}}{q^{2\nu+1}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{hH^{2\nu}}{p^2 q^{2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{h^{2\nu+1}}{p^{2\nu+1}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \Bigg). \end{aligned} \tag{3.73}$$

Similarly to (3.73), we obtain

$$\begin{aligned} A_{5,t} =& \inf_{\mu_t^{Hq} \in M_{t,Hq}(\mathcal{F})} \int_{\Gamma_C} (\mu_t^{Hq} - \lambda_t)(u_t^{hp} + \gamma(-\lambda_t^{Hq} - \sigma_t(\mathbf{u}^{hp}))\, ds \\ \le& C \Bigg(\epsilon \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} + \gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ & + \frac{H^{2\nu+1}}{q^{2\nu+1}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{hH^{2\nu}}{p^2 q^{2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{h^{2\nu+1}}{p^{2\nu+1}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \Bigg). \end{aligned} \tag{3.74}$$

We now estimate the term

$$A_{6,n} = \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}} \int_{\Gamma_C} \gamma(-\lambda_n^{Hq} - \sigma_n(\mathbf{u}^{hp}))E_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds, \tag{3.75}$$

which we write as

$$A_{6,n} = \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}} \left(\int_{\Gamma_C} \gamma(\lambda_n - \lambda_n^{Hq})E_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma\sigma_n(\mathbf{u} - \mathbf{u}^{hp}))E_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \right) \tag{3.76}$$

From a computation as in(3.63), we have

$$\gamma_0 \frac{h}{p^2} \|\sigma_n(\mathbf{u} - \mathbf{u}^{hp})\|^2_{L_2(\Gamma_C)} \leq C \left(\frac{h^{1+2\nu}}{p^{2+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{h^{1+2\nu}}{p^{1+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \right) \tag{3.77}$$

and with continuity of $E_{hp} = S - S_{hp}$

$$\gamma_0 \frac{h}{p^2} \|E_n^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} \leq C \left(\frac{h^{1+2\nu}}{p^{1+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \right). \tag{3.78}$$

We obtain

$$A_{6,n} \leq C \left(\frac{h^{1+2\nu}}{p^{1+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \epsilon \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} \right) \tag{3.79}$$

Similarly to (3.79), we get

$$\begin{aligned} A_{6,t} &= \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}} \int_{\Gamma_C} \gamma(-\lambda_t^{Hq} - \sigma_t(\mathbf{u}^{hp}))E_t^h(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \\ &\leq C \left(\frac{h^{1+2\nu}}{p^{1+2\nu}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \gamma_0 \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \epsilon \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} \right) \end{aligned} \tag{3.80}$$

Consider now

$$A_{7,n} = \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}} \int_{\Gamma_C} \gamma E_n^h(\mathbf{u}^{hp})\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \tag{3.81}$$

We have

$$A_{7,n} \leq \int_{\Gamma_C} \gamma E_n^h(\mathbf{u}^{hp} - \mathbf{u})\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds + \int_{\Gamma_C} \gamma E_n^h(\mathbf{u})\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds \tag{3.82}$$

Using Cauchy Schwarz, Young's inequality, we have

$$\begin{aligned} A_{7,n} \leq\ & \frac{\epsilon}{2} \|E_n^h(\mathbf{u}^{hp} - \mathbf{u})\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{2\epsilon} \|\gamma\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \\ & + \frac{1}{2} \|E_n^h(\mathbf{u})\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{2} \|\gamma\sigma_n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \end{aligned} \tag{3.83}$$

Employing the continuity condition of the Dirichlet-to-Neumann operator and the inverse inequality, we get

$$\begin{aligned}
\|\gamma\sigma_n(\mathbf{u}^{hp}-\mathbf{v}^{hp})\|^2_{H^{\frac{1}{2}}(\Gamma_C)} &\leq c\gamma^2\|\mathbf{u}^{hp}-\mathbf{v}^{hp}\|^2_{H^{\frac{3}{2}}(\Gamma_C)} \\
&\leq c\gamma_0^2\|\mathbf{u}^{hp}-\mathbf{v}^{hp}\|^2_{H^{\frac{1}{2}}(\Sigma)} \\
&\leq c\gamma_0^2\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+c\gamma_0^2\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}
\end{aligned} \tag{3.84}$$

and with the continuity of E_{hp} (Lemma2.6)

$$\frac{\epsilon}{2}\|E_n^{hp}(\mathbf{u}^{hp}-\mathbf{u})\|^2_{H^{-\frac{1}{2}}(\Gamma_C)}\leq\frac{\epsilon}{2}C_{E_{hp}}\|\mathbf{u}^{hp}-\mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} \tag{3.85}$$

From Lemma2.6 and Lemma2.11, we have

$$\|E_n^h(\mathbf{u})\|^2_{H^{-\frac{1}{2}}(\Gamma_C)}\leq\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\psi\|^2_{\tilde{\mathbf{H}}^{\nu}(\Gamma)}\leq C\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.86}$$

Using (3.84)-(3.86), we obtain

$$A_{7,n}\leq C'\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+[c\gamma_0^2(\frac{1}{2\epsilon}+\frac{1}{2})+\frac{\epsilon}{2}C_{E_{hp}}]\|\mathbf{u}^{hp}-\mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} \tag{3.87}$$

Similarly, we obtain

$$\begin{aligned}
A_{7,t} &= \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}}\int_{\Gamma_C}\gamma E^t_{hp}(\mathbf{u}^{hp})\sigma_t(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\
&\leq C'\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+[c\gamma_0^2(\frac{1}{2\epsilon}+\frac{1}{2})+\frac{\epsilon}{2}C_{E_{hp}}]\|\mathbf{u}^{hp}-\mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} \\
A_{8,n} &= \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}}\int_{\Gamma_C}\gamma E^{hp}_n(\mathbf{u}^{hp})E^h_n(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\
&\leq C'\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+[c\gamma_0^2(\frac{1}{2\epsilon}+\frac{1}{2})+\frac{\epsilon}{2}C_{E_{hp}}]\|\mathbf{u}^{hp}-\mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} \\
A_{8,t} &= \inf_{\mathbf{v}^{hp}\in\mathbf{V}_{hp}}\int_{\Gamma_C}\gamma E^{hp}_t(\mathbf{u}^{hp})E^h_t(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\
&\leq C'\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}+[c\gamma_0^2(\frac{1}{2\epsilon}+\frac{1}{2})+\frac{\epsilon}{2}C_{E_{hp}}]\|\mathbf{u}^{hp}-\mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)}
\end{aligned} \tag{3.88}$$

We estimate now the term

$$A_{9,n}=\inf_{\mu_n^{Hq}\in M_{n,Hq}}\int_{\Gamma_C}\gamma(\lambda_n^{Hq}-\mu_n^{Hq})E^n_{hp}(\mathbf{u}^{hp})\,ds \tag{3.89}$$

We have

$$A_{9,n}\leq\int_{\Gamma_C}\gamma(\lambda_n^{Hq}-\lambda_n)E^n_h(\mathbf{u}^{hp})\,ds+\int_{\Gamma_C}\gamma(\lambda_n-\mu_n^{Hq})E^n_h(\mathbf{u}^{hp})\,ds \tag{3.90}$$

We have to estimate the first term in (3.90)

$$A_{9,1,n} = \int_{\Gamma_C} \gamma(\lambda_n^{Hq} - \lambda_n) E_h^n(\mathbf{u}^{hp})\, ds \tag{3.91}$$

$$A_{9,1,n} = \int_{\Gamma_C} \gamma(\lambda_n^{Hq} - \lambda_n) E_h^n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\, ds - \int_{\Gamma_C} \gamma(\lambda_n^{Hq} - \lambda_n) E_h^n(\mathbf{u} - \mathbf{v}^{hp})\, ds$$
$$+ \int_{\Gamma_C} \gamma(\lambda_n^{Hq} - \lambda_n) E_h^n(\mathbf{u})\, ds \tag{3.92}$$

Employing Cauchy Schwarz and Young's inequality, we have

$$A_{9,1,n} \leq \frac{3\epsilon}{2}\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \frac{\gamma_0 h}{\epsilon p^2}\|E_h^n(\mathbf{u}^{hp} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} + \frac{\gamma_0 h}{\epsilon p^2}\|E_h^n(\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)}$$
$$+ \frac{\gamma_0 h}{\epsilon p^2}\|E_h^n \mathbf{u}\|^2_{L_2(\Gamma_C)} \tag{3.93}$$

with the continuity of E_{hp}, we have

$$\|E_h^n \mathbf{u}\|^2_{L_2(\Gamma_C)} \leq \alpha \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}, \quad \text{for} \quad \nu = 0, 1 \tag{3.94}$$

and

$$\begin{aligned} \|\gamma^{\frac{1}{2}} E_h^n(\mathbf{u} - \mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} &\leq \alpha \|\gamma^{\frac{1}{2}}(\mathbf{u} - \mathbf{v}^{hp})\|^2_{\mathbf{H}^1(\Gamma)} \\ &= \alpha\gamma_0 \frac{h}{p^2}\|\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^1(\Gamma)} \\ &= C\alpha\gamma_0 \frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}. \end{aligned} \tag{3.95}$$

Using (3.78), (3.94) and (3.95), we obtain

$$A_{9,1,n} \leq \frac{3\epsilon}{2}\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \frac{\gamma_0}{\epsilon}\frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{\gamma_0}{\epsilon}\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}$$
$$+ \frac{\gamma_0}{\epsilon}\|\mathbf{u}^{hp} - \mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} + \alpha\frac{\gamma_0}{\epsilon}\frac{h}{p^2}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.96}$$

We estimate now the second term in (3.90)

$$A_{9,2,n} = \int_{\Gamma_C} \gamma(\lambda_n - \mu_n^{Hq}) E_h^n(\mathbf{u}^{hp})\, ds \tag{3.97}$$

The continuity of E_h^n and Lemma3.6, we get

$$\frac{h}{p^2}\|E_h^n \mathbf{u}^{hp}\|^2_{L_2(\Gamma_C)} \leq \alpha\frac{h}{p^2}\|\mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Sigma)} \leq \alpha\|\mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} \leq \alpha\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.98}$$

Using Cauchy Schwarz and (3.98), we obtain

$$\begin{aligned} A_{9,2,n} &\leq \gamma_0 \frac{h^{\frac{1}{2}}}{p}\|\lambda_n - \mu_n^{Hq}\|_{L_2(\Gamma_C)}\frac{h^{\frac{1}{2}}}{p}\|E_h^n \mathbf{u}^{hp}\|^2_{L_2(\Gamma_C)} \\ &\leq \gamma_0 \frac{h^{\frac{1}{2}}}{p}\frac{H^\nu}{q^\nu}\|\lambda_n\|_{H^\nu(\Gamma_C)}\|\mathbf{u}\|_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \\ &\leq \gamma_0 \frac{h^{\frac{1}{2}}}{p}\frac{H^\nu}{q^\nu}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \end{aligned} \tag{3.99}$$

Finally we obtain

$$A_{9,n} \leq \frac{3\epsilon}{2}\|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{L_2(\Gamma_C)} + \frac{\gamma_0}{\epsilon}\frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{\gamma_0}{\epsilon}\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}$$
$$+ \frac{\gamma_0}{\epsilon}\|\mathbf{u}^{hp} - \mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} + \alpha\frac{\gamma_0}{\epsilon}\frac{h}{p^2}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \gamma_0\frac{h^{\frac{1}{2}}}{p}\frac{H^\nu}{q^\nu}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.100}$$

Similarly to (3.100), we obtain

$$A_{9,t} = \inf_{\mu_t^{Hq} \in M_{t,Hq}} \int_{\Gamma_C} \gamma(\lambda_t^{Hq} - \mu_t^{Hq}) E_h^t(\mathbf{u}^{hp})\, ds$$
$$\leq \frac{3\epsilon}{2}\|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)} + \frac{\gamma_0}{\epsilon}\frac{h^{1+2\nu}}{p^{2+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \frac{\gamma_0}{\epsilon}\frac{h^{1+2\nu}}{p^{1+2\nu}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)}$$
$$+ \frac{\gamma_0}{\epsilon}\|\mathbf{u}^{hp} - \mathbf{u}\|^2_{H^{\frac{1}{2}}(\Sigma)} + \alpha\frac{\gamma_0}{\epsilon}\frac{h}{p^2}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} + \gamma_0\frac{h^{\frac{1}{2}}}{p}\frac{H^\nu}{q^\nu}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{1+\nu}(\Sigma)} \tag{3.101}$$

Note that γ_0 is small enough. Moring the terms $\gamma_0\|\mathbf{u}-\mathbf{u}^h\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}$, $\epsilon\|\gamma^{\frac{1}{2}}(\lambda_n-\lambda_n^{Hq})\|^2_{L_2(\Gamma_C)}$ and $\epsilon\|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^{Hq})\|^2_{L_2(\Gamma_C)}$ to the left hand side, we obtain the a priori error estimate of the theorem. □

We consider the terms

$$A_{1,n} = \inf_{\mu_n \in M_n} \int_{\Gamma_C} (\lambda_n^{Hq} - \mu_n) u_n\, ds \tag{3.102}$$

and

$$A_{1,t} = \inf_{\mu_n \in M_t(\mathcal{F})} \int_{\Gamma_C} (\lambda_t^{Hq} - \mu_t) u_t\, ds \tag{3.103}$$

Remark 3.2. *The estimation of the terms $A_{1,n}$ and $A_{1,t}$ seems to be problematic, due to the nonconformity of our approach. Here the positivity condition is enforced only on the discrete set of the Gauss Lobatto points.*

Remark 3.3. *The estimate in Theorem3.3 is of order $h^{\frac{1}{4}}p^{-\frac{1}{4}}$ (when $H = h$ and $p = q$). In the nonstabilized case (see.Chapter2), for vanishing gap function $g = 0$, we obtain a convergence rate of order $h^{\frac{1}{4}}p^{-\frac{1}{4}}$ similar to the stabilized case, when we assume the inf-sup condiction to hold.*

For the completeness of the convergence analysis we also consider the h-version for $p = 1$ and $q = 1$.

In order to estimate the terms $A_{1,n}$ and $A_{1,t}$, we have to define the space of the discrete Lagrange multiplier introduced in [36, 37, 45] for the FEM.

$$M_{n,H} := \{\lambda_n^H \in W_H : \lambda_n^H \geq 0 \quad \text{on } \Gamma_C\}$$

$$M_{t,H} := \{\lambda_t^H \in W_H : |\lambda_t^H| \leq \mathcal{F} \quad \text{on } \Gamma_C\}$$

where

$$W_H := \{\mu^H \in C(\Gamma_C) : \forall J \in \mathcal{T}_H, \mu^H|_J \in \mathcal{P}_1(J)\}$$

It is a conforming discretization on multiplier as $M_{n,H} \subset M_n$ and $M_{t,H} \subset M_t$.

We obtain the following a priori error estimate which proposes a convergence rate of $O(h^{\frac{1}{4}})$ (when $H = h$)

Corollary 3.3. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{M}$ *be the solution of the problem* (3.4) *and* $(\boldsymbol{u}^h, \boldsymbol{\lambda}^H)$ *the solution of the discrete problem* (3.8)*. Assume that* $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)$*,* $\boldsymbol{\lambda} \in \boldsymbol{H}^{\nu}(\Gamma_C)$ *for some* $\nu \in [0, \frac{1}{2}]$*. Suppose that* $\|\lambda_n\|_{H^\nu(\Gamma_C)} + \|\lambda_t\|_{H^\nu(\Gamma_C)} + \|\mathcal{F}\|_{L_2(\Gamma_C)} \leq \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)}$*. Then there exists a constant* $C > 0$ *independent of h and H, such that*

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}^h\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^h\|_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} + \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^H)\|_{L_2(\Gamma_C)} + \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^H)\|_{L_2(\Gamma_C)} \\ \leq C\left(h^{\frac{1}{4}} + H^{\nu+\frac{1}{2}} + h^{\frac{1}{2}} H^{\nu}\right) \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{1+\nu}(\Sigma)} \end{aligned} \tag{3.104}$$

Proof. It is sufficient to estimate the terms $A_{1,n}$ and $A_{1,t}$.

We have to estimate

$$A_{1,n} = \inf_{\mu_n \in M_n} \int_{\Gamma_C} (\lambda_n^H - \mu_n) u_n \, ds \tag{3.105}$$

Setting $\mu_n = 0$, since $u_n \leq 0$ and $\lambda_n^H \geq 0$ on Γ_C, we have

$$A_{1,n} = \inf_{\mu_n \in M_n} \int_{\Gamma_C} (\lambda_n^H - \mu_n) u_n \, ds \leq \int_{\Gamma_C} \lambda_n^H u_n \, ds \leq 0 \tag{3.106}$$

We now estimate the term

$$A_{1,t} = \inf_{\mu_n \in M_t(\mathcal{F})} \int_{\Gamma_C} (\lambda_t^H - \mu_t) u_t \, ds \tag{3.107}$$

We consider the Lagrange interpolation operator $\mathcal{I}_h$ defined on the Gauss-Lobatto points mapping onto $\mathbf{V}_h$, where $\mathbf{V}_h$ is the space of continuous piecewise polynomials for the discretization of $\mathbf{u}$ for $p = 1$.

We have

$$\int_{\Gamma_C} \lambda_t^H \mathcal{I}_h u_t \, ds \leq \mathcal{F} \int_{\Gamma_C} |\mathcal{I}_h u_t| \, ds \tag{3.108}$$

choosing $\mu_t = \lambda_t$ and use the condion $\lambda_t u_t - \mathcal{F}|u_t| = 0$ we get

$$\int_{\Gamma_C} (\lambda_t^H - \lambda_t) u_t \, ds =$$

$$\int_{\Gamma_C} (\lambda_t^H - \lambda_t)(u_t - \mathcal{I}_h u_t) \, ds + \int_{\Gamma_C} (\lambda_t^H - \lambda_t)\mathcal{I}_h u_t \, ds + \int_{\Gamma_C} \lambda_t u_t \, ds - \mathcal{F} \int_{\Gamma_C} |u_t| \, ds$$

$$\leq \int_{\Gamma_C} (\lambda_t^H - \lambda_t)(u_t - \mathcal{I}_h u_t) \, ds + \int_{\Gamma_C} \lambda_t (u_t - \mathcal{I}_h u_t) \, ds + \mathcal{F} \int_{\Gamma_C} (|\mathcal{I}_h u_t| - |u_t|) \, ds$$

$$\leq \int_{\Gamma_C} (\lambda_t^H - \lambda_t)(u_t - \mathcal{I}_h u_t) \, ds + \int_{\Gamma_C} \lambda_t (u_t - \mathcal{I}_h u_t) \, ds + \mathcal{F} \int_{\Gamma_C} (|\mathcal{I}_h u_t - u_t|) \, ds \tag{3.109}$$

and we obtain

$$\int_{\Gamma_C} (\lambda_t^H - \lambda_t) u_t \, ds$$
$$\leq \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^H)\|_{L_2(\Gamma_C)} \gamma^{-\frac{1}{2}} \|u_t - \mathcal{I}_h u_t\|_{L_2(\Gamma_C)} + \|\lambda_t\|_{L_2(\Gamma_C)} \|u_t - \mathcal{I}_h u_t\|_{L_2(\Gamma_C)}$$

$$+ \|\mathcal{F}\|_{L_2(\Gamma_C)} \|u_t - \mathcal{I}_h u_t\|_{L_2(\Gamma_C)}$$

$$\leq \epsilon \|\gamma^{\frac{1}{2}}(\lambda_t - \lambda_t^H)\|^2_{L_2(\Gamma_C)} + \frac{C}{\epsilon \gamma_0} h^{2\nu+1} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)} + C h^{\nu+1} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\nu+1}(\Sigma)} \tag{3.110}$$

The corollary is established by employing the estimation of the terms in Theorem3.3. □

3.4 Reliable and efficient a posteriori error estimates for stabilized hp-BEM for frictional contact problems

3.4.1 Reliability of the BEM a posteriori error estimate

Let $(\mathbf{u}, \boldsymbol{\lambda})$ be the solution of the continuous problem (3.4) and $(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{Hq})$ the solution of the discrete problem (3.8). We now derive an upper bound for $|||\mathbf{u} - \mathbf{u}^{hp}|||^2$,

where

$$|||\mathbf{u} - \mathbf{u}^{hp}|||^2 := \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Sigma)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|^2_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C)} \tag{3.111}$$

Lemma 3.7. *([19].Lemma 3.2.9). There exists an operator* $\Pi_{hp} : \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma) \to \boldsymbol{V}_{hp}$*, which is stable in the* $\tilde{\boldsymbol{H}}^{\frac{1}{2}}$*-norm and has the quasioptimal approximation proprietes in the* $\boldsymbol{L}_2$*-norm,ie. there exists a constant* C *, independent of* h *and* p *such that for all* $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)$ *there holds*

$$\|\Pi_{hp}\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \leq C\|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \tag{3.112}$$

$$\|\boldsymbol{u} - \Pi_{hp}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Sigma)} \leq C\left(\frac{h}{p}\right)^{\frac{(1-\epsilon)}{2}} \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \tag{3.113}$$

with arbitrary small $\epsilon \in (0; \frac{1}{2})$

Theorem 3.4. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *be the solution of the problem* (3.4) *and* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{Hq})$ *the solution of the discrete problem* (3.8)*. Then there holds the estimate for* $\epsilon_1 > 0$ *arbitrary:*

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Sigma)} \lesssim & \sum_{I \in \mathcal{T}_{hp}} \eta_h^2(I) + \langle(\lambda_n^{Hq})^+, (g - u_n^{hp})^+\rangle_{\Gamma_C} \\
& + \|(\lambda_n^{Hq})^-\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} + \frac{1}{4\epsilon_1}\|(u_n^{hp} - g)^+\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \\
& + \epsilon_1\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} + \|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \\
& + \int_{\Gamma_C} \left(|(|\lambda_t^{Hq}| - \mathcal{F})^-||u_t^{hp}| + 2(\lambda_t^{Hq}u_t^{hp})^-\right) ds
\end{aligned}$$

where for an arbitrary segment I

$$\begin{aligned}
\eta_h^2(I) = & \left(\frac{h_I}{p_I}\right)^{1-\epsilon} \|\boldsymbol{t} - S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_N)} + \left(\frac{h_I}{p_I}\right)^{1-\epsilon} \|(-\boldsymbol{\lambda}^{Hq}) - S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
& + \frac{h_I}{p_I^2}\|\lambda_n^{Hq} + \sigma_n^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} + \frac{h_I}{p_I^2}\|\lambda_t^{Hq} + \sigma_t^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
& + h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp} - \psi^{hp}))\|^2_{\boldsymbol{L}_2(I)}.
\end{aligned}$$

Proof. Since $\langle S\cdot, \cdot\rangle$, $\langle V\cdot, \cdot\rangle$ are positive definite, there exist constants $c_s, c_v > 0$ such that

$$\begin{aligned}
c_s\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + c_v\|\psi - \psi^{hp}\|_{\mathbf{H}^{-\frac{1}{2}}(\Sigma)} \leq & \langle S\mathbf{u} - S_{hp}\mathbf{u}^{hp}, \mathbf{u} - \mathbf{u}^{hp}\rangle \\
& + \langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\rangle.
\end{aligned} \tag{3.114}$$

The Galerkin orthogonality property (Lemma 3.4) shows that

$$\begin{aligned}
& c_s\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + c_v\|\psi-\psi^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} \leq \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{u}^{hp}\rangle_\Sigma \\
& + \langle V(\psi_p^*-\psi^{hp}), \psi-\psi^{hp}\rangle_\Sigma + \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp}, \mathbf{u}^{hp}-\mathbf{v}^{hp}\rangle_\Sigma + b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp}-\mathbf{v}^{hp}) \\
& + \int_{\Gamma_C} \gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\
& \leq \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle + \langle V(\psi_{hp}^*-\psi^{hp}), \psi-\psi^{hp}\rangle + b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp}-\mathbf{v}^{hp}) \\
& + \int_{\Gamma_C} \gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\
& \leq \langle \mathbf{t}, \mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N} - b(\boldsymbol{\lambda}, \mathbf{u}-\mathbf{v}^{hp}) - \langle S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle \\
& + \langle V(\psi_{hp}^*-\psi^{hp}), \psi-\psi^{hp}\rangle + b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp}-\mathbf{v}^{hp}) \\
& + \int_{\Gamma_C} \gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\
& \leq \langle \mathbf{t}-S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N} + \langle(-\boldsymbol{\lambda}^{Hq})-S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_C} \\
& + b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp}-\mathbf{u}) + \langle V(\psi_{hp}^*-\psi^{hp}), \psi-\psi^{hp}\rangle \\
& + \int_{\Gamma_C} \gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds + \int_{\Gamma_C} \gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds
\end{aligned} \tag{3.115}$$

We estimate the first and the second terms, employing the Cauchy-Schwarz inequality:

$$A = \langle \mathbf{t}-S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N} \leq \sum_{I\in\mathcal{T}_{hp}\cap\Gamma_N} \|\mathbf{t}-S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \tag{3.116}$$

$$B = \langle(-\boldsymbol{\lambda}^{Hq})-S_{hp}\mathbf{u}^{hp}, \mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_C} \leq \sum_{I\in\mathcal{T}_{hp}\cap\Gamma_C} \|(-\boldsymbol{\lambda}^{Hq})-S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)}\|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \tag{3.117}$$

We apply a result from [15], we obtain

$$\begin{aligned}
D &:= \langle V(\psi_{hp}^*-\psi^{hp}), \psi-\psi^{hp}\rangle \leq \|V(\psi_{hp}^*-\psi^{hp})\|_{\mathbf{H}^{\frac{1}{2}}(\Sigma)}\|\psi^{hp}-\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Sigma)} \\
&\leq c\left(\sum_{I\in\mathcal{T}_{hp}} h_I\|\frac{\partial}{\partial s}(V(\psi_{hp}^*-\psi^{hp}))\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}} \|\psi^{hp}-\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Sigma)}.
\end{aligned} \tag{3.118}$$

Using lemma 3.7, we choose $\mathbf{v}^{hp} = \mathbf{u}^{hp} + \Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})$ to obtain the following estimate:

$$\|\mathbf{u}-\mathbf{v}^{hp}\|_{\mathbf{L}^2(I)} \leq C\left(\frac{h_I}{p_I}\right)^{\frac{(1-\epsilon)}{2}} \|\mathbf{u}-\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\omega(I))}. \tag{3.119}$$

Here $\omega(I)$ is a neighbourhood of I.

The continuity of S_{hp} and Young's inequality imply

$$\begin{aligned} p^{-1}h^{\frac{1}{2}}\|\sigma_n^h(\mathbf{v}^{hp}-\mathbf{u}^{hp})\|_{L_2(\Gamma)} &= p^{-1}h^{\frac{1}{2}}\|\sigma_n^h(\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp}))\|_{L_2(\Gamma)} \\ &\leq p^{-1}h^{\frac{1}{2}}\tilde{C}\|\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})\|_{\mathbf{H}^1(\Gamma)} \\ &\leq \tilde{C}\|\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &\leq \tilde{C}C\|\mathbf{u}-\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \end{aligned} \tag{3.120}$$

Since $\mathbf{v}^{hp} = \mathbf{u}^{hp} + \Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})$, we get

$$\begin{aligned} E_n &= \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\ &= \sum_{I\in\Gamma_C}\int_I \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\Pi_{hp}(\mathbf{u}^{hp}-\mathbf{u}))\,ds \\ &\leq \sum_{I\in\Gamma_C}\int_I \gamma_0\left(\frac{h_I^{\frac{1}{2}}}{p_I}\right)(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\left(\frac{h_I^{\frac{1}{2}}}{p_I}\right)\sigma_n^h(\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp}))\,ds \\ &\leq \gamma_0\left(\sum_{I\in\Gamma_C}\frac{h_I}{p_I^2}\|\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}}\left(\sum_{I\in\Gamma_C}\frac{h_I}{p_I^2}\|\sigma_n^h(\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp}))\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}} \\ &\leq \hat{C}\left(\sum_{I\in\Gamma_C}\frac{h_I}{p_I^2}\|\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}}\|\mathbf{u}-\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{3.121}$$

Similarly to (3.121), we obtain

$$\begin{aligned} E_t &= \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{u}^{hp}-\mathbf{v}^{hp})\,ds \\ &\leq \hat{C}\left(\sum_{I\in\Gamma_C}\frac{h_I}{p_I^2}\|\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}}\|\mathbf{u}-\mathbf{u}^{hp}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{3.122}$$

Finally we estimate the term

$$C = b(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}, \mathbf{u}^{hp}-\mathbf{u}) = \langle \lambda_n - \lambda_n^{Hq}, u_n^{hp} - u_n\rangle + \langle \lambda_t - \lambda_t^{Hq}, u_t^{hp} - u_t\rangle. \tag{3.123}$$

Using the condition $\langle \lambda_n, u_n - g\rangle = 0$ and $\langle (\lambda_n^{Hq})^+, u_n - g\rangle \leq 0$ where $v^+ = \max\{0, v\}$

and $v^- = \min\{0, v\}$, i.e. $v = v^+ + v^-$.

$$
\begin{aligned}
\langle \lambda_n - \lambda_n^{Hq}, u_n^{hp} - u_n \rangle &= \langle \lambda_n - (\lambda_n^{Hq})^+, u_n^{hp} - u_n \rangle - \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&= \langle \lambda_n - (\lambda_n^{Hq})^+, u_n^{hp} - g + g - u_n \rangle - \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&= \langle \lambda_n - (\lambda_n^{Hq})^+, u_n^{hp} - g \rangle + \langle \lambda_n, g - u_n \rangle \\
&- \langle (\lambda_n^{Hq})^+, g - u_n \rangle - \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&\le \langle \lambda_n - (\lambda_n^{Hq})^+, u_n^{hp} - g \rangle - \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&= \langle (\lambda_n^{Hq})^+, g - u_n^{hp} \rangle - \langle \lambda_n, (g - u_n^{hp})^+ + (g - u_n^{hp})^- \rangle \\
&- \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&\le \langle (\lambda_n^{Hq})^+, g - u_n^{hp} \rangle + \langle \lambda_n^{Hq} - \lambda_n - (\lambda_n^{Hq})^+ - (\lambda_n^{Hq})^-, (g - u_n^{hp})^- \rangle \\
&- \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&= \langle (\lambda_n^{Hq})^+, (g - u_n^{hp})^+ \rangle + \langle \lambda_n^{Hq} - \lambda_n, (g - u_n^{hp})^- \rangle \\
&- \langle (\lambda_n^{Hq})^-, (g - u_n^{hp})^- \rangle - \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&= \langle (\lambda_n^{Hq})^+, (g - u_n^{hp})^+ \rangle + \langle \lambda_n^{Hq} - \lambda_n, (g - u_n^{hp})^- \rangle \\
&- \langle (\lambda_n^{Hq})^-, u_n^{hp} - u_n \rangle \\
&\le \| (\lambda_n^{Hq})^- \|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \| u_n^{hp} - u_n \|_{\tilde{H}^{\frac{1}{2}}(\Sigma)} \\
&+ \| \lambda_n^{Hq} - \lambda_n \|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \| (g - u_n^{hp})^- \|_{H^{\frac{1}{2}}(\Gamma_C)} + \langle (\lambda_n^{Hq})^+, (g - u_n^{hp})^+ \rangle_{\Gamma_C}.
\end{aligned}
\tag{3.124}
$$

By the contact condition, $-\lambda_t u_t + \mathcal{F}|u_t| = 0$. Setting $\lambda_t = \xi \mathcal{F}$ with $|\xi| \le 1$

$$
\begin{aligned}
\langle \lambda_t - \lambda_t^{Hq}, u_t^{hp} - u_t \rangle &= -\lambda_t u_t + \lambda_t^{Hq} u_t + \lambda_t u_t^{hp} - \lambda_t^{Hq} u_t^{hp} \\
&= -\mathcal{F}|u_t| + \lambda_t^{Hq} u_t + \xi \mathcal{F} u_t^{hp} - \lambda_t^{Hq} u_t^{hp} \\
&\le -\mathcal{F}|u_t| + \lambda_t^{Hq} u_t + \mathcal{F}|u_t^{hp}| - \lambda_t^{Hq} u_t^{hp} \\
&\le (|\lambda_t^{Hq}| - \mathcal{F})^+ |u_t| + \mathcal{F}|u_t^{hp}| - \lambda_t^{Hq} u_t^{hp} \\
&\le (|\lambda_t^{Hq}| - \mathcal{F})^+ |u_t - u_t^{hp}| + (|\lambda_t^{Hq}| - \mathcal{F})^+ |u_t^{hp}| + \mathcal{F}|u_t^{hp}| - \lambda_t^{Hq} u_t^{hp} \\
&\le \| (|\lambda_t^{Hq}| - \mathcal{F})^+ \|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \| u_t - u_t^{hp} \|_{H^{\frac{1}{2}}(\Gamma_C)} \\
&+ [(|\lambda_t^{Hq}| - \mathcal{F})^+ - (|\lambda_t^{Hq}| - \mathcal{F})] |u_t^{hp}| - \lambda_t^{Hq} u_t^{hp} + |\lambda_t^{Hq}| |u_t^{hp}| \\
&\le \| (|\lambda_t^{Hq}| - \mathcal{F})^+ \|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \| u_t - u_t^{hp} \|_{H^{\frac{1}{2}}(\Gamma_C)} \\
&+ |(|\lambda_t^{Hq}| - \mathcal{F})^-| |u_t^{hp}| + 2(\lambda_t^{Hq} u_t^{hp})^-.
\end{aligned}
\tag{3.125}
$$

Using Young's inequality we move the term $\|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}$ to the left hand side. The estimate of the theorem follows. □

Now we find an upper bound on the discretization error $\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}$.

Lemma 3.8. *Let* $\boldsymbol{\lambda}$ *solve the saddle point problem* (3.4)*, and* $\boldsymbol{\lambda}^{Hq}$ *the solution of the* *discrete problem* (3.8). *Then there holds*

$$\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_C)} \leq C' \left(\|\boldsymbol{u}-\boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi-\psi^{hp}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma)} \right) + C'' \sum_{I\subset\mathcal{T}_{hp}} \xi_h^2(I). \tag{3.126}$$

where

$$\xi_h^2(I) = \left(\frac{h_I}{p_I}\right)^{1-\epsilon} \|\boldsymbol{t}-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_N)} + \left(\frac{h_I}{p_I}\right)^{1-\epsilon} \|(-\boldsymbol{\lambda}^{Hq})-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}$$
$$+ \frac{h_I}{p_I^2}\|\lambda_n^{Hq}+\sigma_n^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} + \frac{h_I}{p_I^2}\|\lambda_t^{Hq}+\sigma_t^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \tag{3.127}$$

Proof. Let $\mathbf{v}\in\mathbf{V}$ and $\mathbf{v}^{hp}:=\Pi_{hp}\mathbf{v}\in\mathbf{V}_{hp}$ we have

$$\langle\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq},\mathbf{v}\rangle = \langle\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq},\mathbf{v}-\mathbf{v}^{hp}\rangle + \langle\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq},\mathbf{v}^{hp}\rangle$$

Using Galerkin orthogonality and the formulation (3.4) we obtain

$$\begin{aligned}
\langle\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq},\mathbf{v}\rangle &= \langle\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq},\mathbf{v}-\mathbf{v}^{hp}\rangle - \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{v}^{hp}\rangle_\Sigma \\
&- \int_{\Gamma_C}\gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{v}^{hp})\,ds - \int_{\Gamma_C}\gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{v}^{hp})\,ds \\
&= \langle\mathbf{t},\mathbf{v}-\mathbf{v}^{hp}\rangle - \langle S\mathbf{u},\mathbf{v}-\mathbf{v}^{hp}\rangle - \langle\boldsymbol{\lambda}^{Hq},\mathbf{v}-\mathbf{v}^{hp}\rangle \\
&- \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{v}\rangle_\Sigma - \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{v}^{hp}-\mathbf{v}\rangle_\Sigma \\
&- \int_{\Gamma_C}\gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{v}^{hp})\,ds - \int_{\Gamma_C}\gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{v}^{hp})\,ds \\
&= \langle\mathbf{t}-S_{hp}\mathbf{u}^{hp},\mathbf{v}-\mathbf{v}^{hp}\rangle_{\Gamma_N} + \langle(-\boldsymbol{\lambda}^{H})-S_{hp}\mathbf{u}^{hp},\mathbf{v}-\mathbf{v}^{hp}\rangle_{\Gamma_C} \\
&- \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{v}\rangle_\Sigma \\
&- \int_{\Gamma_C}\gamma(\lambda_n^{Hq}+\sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{v}^{hp})\,ds - \int_{\Gamma_C}\gamma(\lambda_t^{Hq}+\sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{v}^{hp})\,ds
\end{aligned}$$

We estimate the terms on the right hand side:

$$A = \langle\mathbf{t}-S_{hp}\mathbf{u}^{hp},\mathbf{v}-\mathbf{v}^{hp}\rangle_{\Gamma_N} \leq \sum_{I\in\mathcal{T}_{hp}\cap\Gamma_N} \|\mathbf{t}-S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)}\|\mathbf{v}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \tag{3.128}$$

$$B = \langle(-\boldsymbol{\lambda}^{Hq})-S_{hp}\mathbf{u}^{hp},\mathbf{v}-\mathbf{v}^{hp}\rangle_{\Gamma_C} \leq \sum_{I\in\mathcal{T}_{hp}\cap\Gamma_C} \|(-\boldsymbol{\lambda}^{Hq})-S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)}\|\mathbf{v}-\mathbf{v}^{hp}\|_{\mathbf{L}_2(I)} \tag{3.129}$$

$$\begin{aligned}
C = \langle S\mathbf{u}-S_{hp}\mathbf{u}^{hp},\mathbf{v}\rangle_\Sigma &\leq (C_s+C_{E_{hp}})\|\mathbf{u}-\mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}\|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\
&+ C_0\|\psi-\psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)}\|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)}.
\end{aligned} \tag{3.130}$$

Using the inverse inequality as well as Lemma 3.7:

$$\begin{aligned} D_n &= \int_{\Gamma_C} \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\mathbf{v}^{hp})\, ds \le \sum_{I \in \mathcal{T}_{hp}} \int_I \gamma(\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp}))\sigma_n^h(\Pi_{hp}\mathbf{v}) \\ &= \sum_{I \in \mathcal{T}_{hp}} \int_I \gamma_0 \left(\frac{h_I^{\frac{1}{2}}}{p_I}\right) (\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp})) \left(\frac{h_I^{\frac{1}{2}}}{p_I}\right) \sigma_n^h(\Pi_{hp}\mathbf{v}) \\ &\le \hat{C} \left(\sum_{I \in \Gamma_C} \frac{h_I}{p_I^2} \|\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)} \right)^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{3.131}$$

Similarly to (3.131), we get

$$D_t = \int_{\Gamma_C} \gamma(\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp}))\sigma_t^h(\mathbf{v}^{hp})\, ds \le \hat{C} \left(\sum_{I \in \Gamma_C} \frac{h_I}{p_I^2} \|\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)} \right)^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{3.132}$$

Recalling $\mathbf{v}^{hp} = \Pi_{hp}\mathbf{v}$ in A and B and Lemma 3.7 we obtain

$$\begin{aligned} \langle \boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}, \mathbf{v} \rangle \le &\left((C_s + C_{E_{hp}}) \|\mathbf{u} - \mathbf{u}^{hp}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} + C_0 \|\psi - \psi^{hp}\|_{\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma)} \right) \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &+ C \left(\sum_{I \in \mathcal{T}_{hp} \cap \Gamma_N} \left(\frac{h_I}{p_I}\right)^{\frac{1-\epsilon}{2}} \|\mathbf{t} - S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)} \right) \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &+ C \left(\sum_{I \in \mathcal{T}_{hp} \cap \Gamma_N} \left(\frac{h_I}{p_I}\right)^{\frac{1-\epsilon}{2}} \|(-\boldsymbol{\lambda}^{Hq}) - S_{hp}\mathbf{u}^{hp}\|_{\mathbf{L}_2(I)} \right) \|\mathbf{v}\|_{\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)} \\ &+ \hat{C} \left(\sum_{I \in \Gamma_C} \frac{h_I^{\frac{1}{2}}}{p_I} \|\lambda_n^{Hq} + \sigma_n^h(\mathbf{u}^{hp})\|_{\mathbf{L}_2(I)} \right) \|\mathbf{v}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &+ \hat{C} \left(\sum_{I \in \Gamma_C} \frac{h_I^{\frac{1}{2}}}{p_I} \|\lambda_t^{Hq} + \sigma_t^h(\mathbf{u}^{hp})\|_{\mathbf{L}_2(I)} \right) \|\mathbf{v}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \end{aligned} \tag{3.133}$$

By definition of the dual norm and $(a+b)^2 \le 2a^2 + 2b^2$, the assertion (3.126) follows. □

Theorem 3.5. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *be the solution of problem* (3.4) *and* $(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{Hq})$ *the solution* *of the discrete problem* (3.8)*. Then the following a posteriori estimate holds:*

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}^{hp}\|^2_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} + \|\psi-\psi^{hp}\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Sigma)} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}\|^2_{\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C)} \lesssim & \\
\sum_{I\in\mathcal{T}_{hp}} \eta_h^2(I) + \langle(\lambda_n^{Hq})^+, (g-u_n^{hp})^+\rangle_{\Gamma_C} & \\
+ \|(\lambda_n^{Hq})^-\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} + \|(u_n^{hp}-g)^+\|^2_{H^{\frac{1}{2}}(\Gamma_C)} & \\
+ \|(|\lambda_t^{Hq}|-\mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} & \\
+ \int_{\Gamma_C} \left(|(|\lambda_t^{Hq}|-\mathcal{F})^-||u_t^{hp}| + 2(\lambda_t^{Hq}u_t^{hp})^-\right) ds &
\end{aligned}
\tag{3.134}
$$

where

$$
\begin{aligned}
\eta_h^2(I) = & \left(\frac{h_I}{p_I}\right)^{1-\epsilon} \|\boldsymbol{t}-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_N)} + \left(\frac{h_I}{p_I}\right)^{1-\epsilon} \|(-\boldsymbol{\lambda}^{Hq})-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
& + \frac{h_I}{p_I^2}\|\lambda_n^{Hq}+\sigma_n^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} + \frac{h_I}{p_I^2}\|\lambda_t^{Hq}+\sigma_t^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
& + h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp}-\psi^{hp}))\|^2_{\boldsymbol{L}_2(I)}
\end{aligned}
$$

Proof. The estimate follows immediately from Lemma 3.8 and Theorem 3.4 □

3.4.2 Efficiency of the BEM a posteriori error estimate

In this section we derive an efficiency of the a posteriori error estimates. We follow ideas of [15], [16], [19].

Assumption 3.1. *There exists a constant $C>0$ such that*

$$
\forall I, J \in \mathcal{T}_{hp} \quad \frac{|I|}{|J|} < C, \quad \frac{p_I}{p_J} < C \tag{3.135}
$$

and C is independent of I,J . Let

$$
h_{max} := \max_{I\in\mathcal{T}_{hp}} |I|, \qquad h_{min} := \min_{I\in\mathcal{T}_{hp}} |I|
$$

$$
p_{max} := \max_{I\in\mathcal{T}_{hp}} p_I, \qquad p_{min} := \min_{I\in\mathcal{T}_{hp}} p_I
$$

For the simplicity of the the presentation, we assume that the gap function $g=0$.

Lemma 3.9. *There exists a constant $c>0$ such that for any element $I\in\mathcal{T}_{hp}$ the local* *error indicator* $\eta_h(I)$ *satisfies*

$$\begin{aligned}
c\gamma_0\eta_h^2(I) \leq & \frac{h_I}{p_I}\|W(\boldsymbol{u}-\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Sigma)}+\frac{h_I}{p_I}\|(K'+\frac{1}{2})(\psi-\psi^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Sigma)} \\
&+ h_I\|\frac{\partial}{\partial s}(V(\psi-\psi^{hp}))\|^2_{\boldsymbol{L}_2(I)}+h_I\|\frac{\partial}{\partial s}(K+\frac{1}{2})(\boldsymbol{u}-\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I)} \\
&+ p_I\|\gamma^{\frac{1}{2}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}+\|\gamma^{\frac{1}{2}}\sigma_n(\boldsymbol{u}-\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
&+ \|\gamma^{\frac{1}{2}}\sigma_t(\boldsymbol{u}-\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}
\end{aligned} \tag{3.136}$$

the stabilization parameter γ *is defined on each element* I *as* $\gamma=\gamma_0\frac{h_I}{p_I^2}$*, where* $\gamma_0>0$ *is independent of* h *and* p *and* γ_0 *is chosen small enough, where*

$$\begin{aligned}
\eta_h^2 = & \frac{h_I}{p_I}\|\boldsymbol{t}-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_N)}+\frac{h_I}{p_I}\|(-\boldsymbol{\lambda}^{Hq})-S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
&+ \frac{h_I}{p_I^2}\|\lambda_n^{Hq}+\sigma_n(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}+\frac{h_I}{p_I^2}\|\lambda_t^{Hq}+\sigma_t(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
&+ h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp}-\psi^{hp}))\|^2_{\boldsymbol{L}_2(I)}+\frac{h_I}{p_I^2}\|E_n^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}+\frac{h_I}{p_I^2}\|E_t^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}
\end{aligned}$$

Proof. Noting that $\mathbf{t}=S\mathbf{u}|_{\Gamma_N}$ for the exact solution $\mathbf{u}$, we obtain for $I\subseteq\Gamma_N$:

$$\begin{aligned}
\gamma_0\frac{h_I}{p_I}\|\mathbf{t}-S_{hp}\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(I\cap\Gamma_N)} &= \gamma p_I\|S\mathbf{u}-S_{hp}\mathbf{u}^h\|^2_{\mathbf{L}_2(I\cap\Gamma_N)} \\
&= \gamma p_I\|W(\mathbf{u}-\mathbf{u}^{hp})+(K'+\frac{1}{2})(\psi-\psi^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_N)} \\
&\leq 2\gamma p_I\|W(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_N)} \\
&+ 2\gamma p_I\|(K'+\frac{1}{2})(\psi-\psi^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_N)}.
\end{aligned}$$

Note that $\boldsymbol{\lambda}=-S\mathbf{u}|_{\Gamma_C}$. Then we obtain for $I\subseteq\Gamma_C$

$$\begin{aligned}
\gamma_0\frac{h_I}{p_I}\|(-\boldsymbol{\lambda}^{Hq})-S_{hp}\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(I)} &\leq 2\gamma p_I\left(\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq}\|^2_{\mathbf{L}_2(I)}+\|S\mathbf{u}-S_{hp}\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(I)}\right) \\
&\leq 4\gamma p_I\|W(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)} \\
&+ 4\gamma p_I\|(K'+\frac{1}{2})(\psi-\psi^{hp})\|^2_{\mathbf{L}_2(I)} \\
&+ 2p_I\|\gamma^{\frac{1}{2}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}^{Hq})\|^2_{\mathbf{L}_2(I)}
\end{aligned}$$

and for any $I\subseteq\Gamma$ we have

$$\begin{aligned}
\gamma_0 h_I\|\frac{\partial}{\partial s}(V(\psi^{hp}-\psi^*_{hp}))\|^2_{\mathbf{L}_2(I)} &= \gamma_0 h_I\|\frac{\partial}{\partial s}(V(\psi^{hp}-(K+\frac{1}{2})\mathbf{u}^{hp}))\|^2_{\mathbf{L}_2(I)} \\
&\leq 2\gamma_0 h_I\|\frac{\partial}{\partial s}(V(\psi-\psi^{hp}))\|^2_{\mathbf{L}_2(I)} \\
&+ 2\gamma_0 h_I\|\frac{\partial}{\partial s}(K+\frac{1}{2})(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I)}.
\end{aligned}$$

$$\gamma_0 \frac{h_I}{p_I^2} \|\lambda_n^{Hq} + \sigma_n(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} = \|\gamma^{\frac{1}{2}}(\lambda_n - \lambda_n^{Hq})\|^2_{\mathbf{L}_2(I)} + \|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)}$$
$$\leq \|\gamma^{\frac{1}{2}}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq})\|^2_{\mathbf{L}_2(I)} + \|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \tag{3.137}$$

Similarly to (3.137), we get

$$\gamma_0 \frac{h_I}{p_I^2} \|\lambda_t^{Hq} + \sigma_t(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \leq \|\gamma^{\frac{1}{2}}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq})\|^2_{\mathbf{L}_2(I)} + \|\gamma^{\frac{1}{2}}\sigma_t(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \tag{3.138}$$

$$\begin{aligned}
\gamma_0 \frac{h_I}{p_I^2} \|E_n^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} &= \gamma_0 \frac{h_I}{p_I^2} \|(S - S_{hp})\mathbf{u}^{hp}\cdot\mathbf{n}\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \\
&\leq \gamma_0 \frac{h_I}{p_I^2} \|S\mathbf{u} - S_{hp}\mathbf{u}^{hp}\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} + \gamma_0 \frac{h_I}{p_I^2} \|S(\mathbf{u}-\mathbf{u}^{hp})\cdot\mathbf{n}\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \\
&\leq \gamma_0 \frac{h_I}{p_I^2} \left(\|W(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} + \|(K' + \frac{1}{2})(\psi - \psi^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \right) \\
&+ \|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)}
\end{aligned} \tag{3.139}$$

Finally we have

$$\begin{aligned}
\gamma_0 \frac{h_I}{p_I^2} \|E_t^h(\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} &\leq \gamma_0 \frac{h_I}{p_I^2} \left(\|W(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} + \|(K' + \frac{1}{2})(\psi - \psi^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)} \right) \\
&+ \|\gamma^{\frac{1}{2}}\sigma_t(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(I\cap\Gamma_C)}.
\end{aligned} \tag{3.140}$$

□

Lemma 3.10. *Let $\mathcal{P}_{hp}$ be the $\boldsymbol{L}_2(\Gamma)$-projection operator onto $\boldsymbol{W}_{hp}$, i.e $\mathcal{P}_{hp} : \boldsymbol{L}_2(\Gamma) \longrightarrow \boldsymbol{W}_{hp}$ such that*

$$\langle \mathcal{P}_{hp}\psi - \psi, \Phi \rangle = 0, \forall \Phi \in \boldsymbol{W}_{hp} \tag{3.141}$$

and for any real numbers $\mu \geq 0$ there exists a constant C such that

$$\forall \psi \in \boldsymbol{H}^{\mu}(\Gamma) \quad \|\psi - \mathcal{P}_{hp}\psi\|_{\boldsymbol{L}_2(\Gamma)} \leq C \left(\frac{h}{p}\right)^{\mu} \|\psi\|_{\boldsymbol{H}^{\mu}(\Gamma)}. \tag{3.142}$$

Then there holds

$$\|\psi - \mathcal{P}_{hp}\psi\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C \left(\frac{h}{p}\right) \|\psi\|^2_{\boldsymbol{L}_2(\Gamma)}. \tag{3.143}$$

In particular

$$\|\psi - \mathcal{P}_{hp}\psi\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C \left(\frac{h}{p}\right) \|\psi - \mathcal{P}_{hp}\psi\|^2_{\boldsymbol{L}_2(\Gamma)}. \tag{3.144}$$

Proof. We observe that

$$
\begin{aligned}
\|\psi - \mathcal{P}_{hp}\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} &= \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\langle \psi - \mathcal{P}_{hp}\psi, \phi\rangle}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&= \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\langle \psi - \mathcal{P}_{hp}\psi, \phi - \mathcal{P}_{hp}\phi\rangle}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&= \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\langle \psi, \phi - \mathcal{P}_{hp}\phi\rangle}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&\leq \|\psi\|_{\mathbf{L}_2(\Gamma)} \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\|\phi - \mathcal{P}_{hp}\phi\|_{\mathbf{L}_2(\Gamma)}}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&\leq C\left(\frac{h}{p}\right)^{\frac{1}{2}} \|\psi\|_{\mathbf{L}_2(\Gamma)} \qquad (3.145)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\psi - \mathcal{P}_{hp}\psi\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} &= \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\langle \psi - \mathcal{P}_{hp}\psi, \phi\rangle}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&= \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\langle \psi - \mathcal{P}_{hp}\psi, \phi - \mathcal{P}_{hp}\phi\rangle}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&\leq \sup_{\phi\in\mathbf{H}^{\frac{1}{2}}(\Gamma)} \frac{\|\psi - \mathcal{P}_{hp}\psi\|_{\mathbf{L}_2(\Gamma)}\|\phi - \mathcal{P}_{hp}\phi\|_{\mathbf{L}_2(\Gamma)}}{\|\phi\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}} \\
&= C\left(\frac{h}{p}\right)^{\frac{1}{2}} \|\psi - \mathcal{P}_{hp}\psi\|_{\mathbf{L}_2(\Gamma)} \qquad (3.146)
\end{aligned}
$$

□

Theorem 3.6. *Let $(\boldsymbol{u}, \boldsymbol{\lambda})$ be the solution of problem (3.4) and $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{Hq})$ the solution of the discrete problem (3.8). Assume that $\boldsymbol{\lambda} \in \boldsymbol{L}_2(\Gamma_C)$. Then there exists a constant $c(\gamma_0)$ independent of h, p, H and q such that*

$$
\begin{aligned}
c(\gamma_0) \sum_{I\in\mathcal{T}_{hp}} \eta_h^2(I) \leq C p_{max}^2 &\left(\|\boldsymbol{u} - \boldsymbol{u}^{hp}\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\gamma^{\frac{1}{2}}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq})\|^2_{\boldsymbol{L}_2(\Gamma_C)} \right) \\
&+ h_{max}^2 \left(\|\boldsymbol{u}\|^2_{\tilde{\boldsymbol{H}}^{\frac{3}{2}}(\Sigma)} + \|\psi\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \right) \qquad (3.147)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_h^2(I) = \frac{h_I}{p_I}\|\boldsymbol{t} - S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_N)} &+ \frac{h_I}{p_I}\|(-\boldsymbol{\lambda}^{Hq}) - S_{hp}\boldsymbol{u}^{hp}\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
+ \frac{h_I}{p_I^2}\|\lambda_n^{Hq} + \sigma_n(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} &+ \frac{h_I}{p_I^2}\|\lambda_t^{Hq} + \sigma_t(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} \\
+ h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp} - \psi^{hp}))\|^2_{\boldsymbol{L}_2(I)} &+ \frac{h_I}{p_I^2}\|E_n^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)} + \frac{h_I}{p_I^2}\|E_t^h(\boldsymbol{u}^{hp})\|^2_{\boldsymbol{L}_2(I\cap\Gamma_C)}
\end{aligned}
$$

and we have

$$\int_{\Gamma_C} (\lambda_n^{Hq})^+ (u_n^{hp})^+ ds \leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\boldsymbol{L}_2(\Gamma_C)} \|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\boldsymbol{L}_2(\Gamma_C)}$$

$$+ \|\boldsymbol{\lambda}\|_{\boldsymbol{L}_2(\Gamma_C)} \|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\boldsymbol{L}_2(\Gamma_C)}$$

$$\|(\lambda_n^{Hq})^-\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|^2_{\boldsymbol{L}_2(\Gamma_C)}$$

$$\int_{\Gamma_C} \Big(|(|\lambda_t^{Hq}| - \mathcal{F})^-||u_t^{hp}| + 2(\lambda_t^{Hq} u_t^{hp})^- \Big)\, ds \lesssim \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\boldsymbol{L}_2(\Gamma_C)} \|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\boldsymbol{L}_2(\Gamma_C)}$$

$$+ \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\boldsymbol{L}_2(\Gamma_C)} \|\boldsymbol{u}\|_{\boldsymbol{L}_2(\Gamma_C)}$$

$$+ \|\boldsymbol{\lambda}\|_{\boldsymbol{L}_2(\Gamma_C)} \|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\boldsymbol{L}_2(\Gamma_C)}$$

$$\|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\boldsymbol{L}_2(\Gamma_C)}$$

$$\|(u_n^{hp})^+\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \leq \frac{p_{max}^2}{h_{min}} \|\boldsymbol{u} - \boldsymbol{u}^{hp}\|_{\boldsymbol{L}_2(\Gamma_C)}. \tag{3.148}$$

Proof. Using the continuity of W and K and Lemma3.9 we obtain

$$\gamma_0 \frac{h_{max}}{p_{min}} \|W(\mathbf{u} - \mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Sigma)} \leq C\gamma_0 \frac{h_{max}}{p_{min}} \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \tag{3.149}$$

$$\gamma_0 h_{max} \|\frac{\partial}{\partial s}(K + \frac{1}{2})(\mathbf{u} - \mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Gamma)} \leq \gamma_0 h_{max} \|(K + \frac{1}{2})(\mathbf{u} - \mathbf{u}^{hp})\|^2_{\mathbf{H}^1(\Gamma)}$$

$$\leq C\gamma_0 h_{max} \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \tag{3.150}$$

$$\gamma_0 \frac{h_{max}}{p_{min}} \|(K' + \frac{1}{2})(\psi - \psi^{hp})\|^2_{\mathbf{L}_2(\Gamma)} \leq \gamma_0 \frac{h_{max}}{p_{min}} \|\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} \tag{3.151}$$

$$\gamma_0 h_{max} \|\frac{\partial}{\partial s}(V(\psi - \psi^{hp}))\|^2_{\mathbf{L}_2(\Gamma)} \leq \gamma_0 h_{max} \|V(\psi - \psi^{hp})\|^2_{\mathbf{H}^1(\Gamma)}$$

$$\leq \gamma_0 h_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} \tag{3.152}$$

$$\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)} \leq \gamma_0\frac{h_{max}}{p^2_{min}}\|\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)}$$
$$\leq \gamma_0\frac{h_{max}}{p^2_{min}}\left(\|\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)}+\|\sigma_n(\mathbf{v}^{hp}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)}\right) \tag{3.153}$$

The continuity condition of the Dirichlet-to-Neumann operator , Young's inequality and we choose $\mathbf{v}^{hp}=\mathbf{u}^{hp}+\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})$ we obtain

$$\begin{aligned}\gamma_0\frac{h_{max}}{p^2_{min}}\|\sigma_n(\mathbf{v}^{hp}-\mathbf{u}^{hp})\|^2_{L_2(\Gamma)} &= \gamma_0\frac{h_{max}}{p^2_{min}}\|\sigma_n(\Pi_{hp}(\mathbf{u}-\mathbf{u}^h))\|^2_{L_2(\Gamma)}\\ &\leq \gamma_0\frac{h_{max}}{p^2_{min}}C\|\Pi_{hp}(\mathbf{u}-\mathbf{u}^h)\|^2_{\mathbf{H}^1(\Gamma)}\\ &\leq C\gamma_0\|\Pi_{hp}(\mathbf{u}-\mathbf{u}^h)\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\\ &\leq C\gamma_0\|\mathbf{u}-\mathbf{u}^h\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\end{aligned} \tag{3.154}$$

We employ the continuity condition of the Dirichlet-to-Neumann operator and choosing $v^{hp}=\mathcal{I}_{hp}\mathbf{u}$ we obtain

$$\begin{aligned}\gamma_0\frac{h_{max}}{p^2_{min}}\|\sigma_n(\mathbf{u}-\mathbf{v}^{hp})\|^2_{L_2(\Gamma_C)} &\leq C\gamma_0\frac{h_{max}}{p^2_{min}}\|\mathbf{u}-\mathbf{v}^{hp}\|^2_{\mathbf{H}^1(\Gamma)}\\ &\leq C\gamma_0\frac{h_{max}}{p^2_{min}}\|\mathbf{u}-\mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^1(\Gamma)}\\ &\leq C\gamma_0\frac{h^2_{max}}{p^3_{min}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)}\end{aligned} \tag{3.155}$$

we obtain

$$\gamma_0\frac{h_{max}}{p^2_{min}}\|\sigma_n(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)}= C\gamma_0\left(\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}+\frac{h^2_{max}}{p^3_{min}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)}\right) \tag{3.156}$$

Similarly to (3.157), we get

$$\gamma_0\frac{h_{max}}{p^2_{min}}\|\sigma_t(\mathbf{u}-\mathbf{u}^{hp})\|^2_{\mathbf{L}_2(\Gamma_C)}= C\gamma_0\left(\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}+\frac{h^2_{max}}{p^3_{min}}\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)}\right) \tag{3.157}$$

$$\gamma_0\frac{h_{max}}{p_{min}}\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \leq \gamma_0\frac{h_{max}}{p_{min}}\|\mathbf{u}-\mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^1(\Gamma)} \quad +\gamma_0\frac{h_{max}}{p_{min}}\|\mathcal{I}_{hp}\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \tag{3.158}$$

We apply the inverse inequality for the second term we obtain

$$\begin{aligned}\gamma_0\frac{h_{max}}{p_{min}}\|\mathcal{I}_{hp}\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} &\leq \gamma_0\frac{h_{max}}{p_{min}}\frac{p^2_{max}}{h_{min}}\|\mathcal{I}_{hp}\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\\ &\leq C\gamma_0 p_{max}\left(\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)}+\|\mathcal{I}_{hp}\mathbf{u}-\mathbf{u}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)}\right)\end{aligned} \tag{3.159}$$

we have

$$\gamma_0 \frac{h_{max}}{p_{min}} \|\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^1(\Gamma)} \leq \gamma_0 \frac{h^2_{max}}{p^2_{min}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)} \tag{3.160}$$

and

$$\gamma_0 p_{max} \|\mathbf{u} - \mathcal{I}_{hp}\mathbf{u}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq \gamma_0 \frac{h^2_{max}}{p_{min}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)} \tag{3.161}$$

there holds

$$\gamma_0 \frac{h_{max}}{p_{min}} \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \leq C\gamma_0 \left(p_{max} \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} + \frac{h^2_{max}}{p_{min}} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)} \right) \tag{3.162}$$

and we have

$$\gamma_0 h_{max} \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^1(\Gamma)} \leq \gamma_0 p^2_{max} \|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} + \gamma_0 h^2_{max} \|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)} \tag{3.163}$$

$$h_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} \leq h_{max} \|\psi - \mathcal{P}_h\psi\|^2_{\mathbf{L}_2(\Gamma)} + h_{max} \|\mathcal{P}_h\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} \tag{3.164}$$

using the inverse inequality and lemma(3.10) for the second term

$$\begin{aligned} h_{max} \|\mathcal{P}_h\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} &\leq Cp^2_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} + Cp^2_{max} \|\mathcal{P}_h\psi - \psi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\leq Cp^2_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} + Ch^2_{max} \|\psi\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \end{aligned} \tag{3.165}$$

Employing lemma(3.10) for the first term

$$h_{max} \|\psi - \mathcal{P}_h\psi\|^2_{\mathbf{L}_2(\Gamma)} \leq C \frac{h^2_{max}}{p_{min}} \|\psi\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{3.166}$$

we obtain

$$\gamma_0 h_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} \leq C\gamma_0 \left(p^2_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} + h^2_{max} \|\psi\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \right) \tag{3.167}$$

and

$$\gamma_0 \frac{h_{max}}{p_{min}} \|\psi - \psi^{hp}\|^2_{\mathbf{L}_2(\Gamma)} \leq C\gamma_0 \left(p_{max} \|\psi - \psi^{hp}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} + \frac{h^2_{max}}{p_{min}} \|\psi\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \right) \tag{3.168}$$

Finally we get

$$\begin{aligned} c(\gamma_0) \sum_{I \in \mathcal{T}_{hp}} \eta^2_h(I) \leq Cp^2_{max} &\left(\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} + \|\psi - \psi^{hp}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} + \|\gamma^{\frac{1}{2}}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq})\|^2_{\mathbf{L}_2(\Gamma_C)} \right) \\ &+ h^2_{max} \left(\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{\frac{3}{2}}(\Sigma)} + \|\psi\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \right) \end{aligned} \tag{3.169}$$

Since $u_n \leq 0$, we have

$$0 \leq (u_n^{hp})^+ \leq |u_n^{hp} - u_n| \quad \text{on} \quad \Gamma_C, \tag{3.170}$$

and

$$0 \leq \|(u_n^{hp})^+\|_{\mathbf{L}_2(\Gamma_C \cap I)} \leq \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(\Gamma_C \cap I)}. \tag{3.171}$$

If $I \in \mathcal{T}_{hp}$, let $J \subset I$ be the part of the edge where $(\lambda_n^{Hq})^+ = \lambda_n^{Hq}$

$$\begin{aligned}\int_{\Gamma_C \cap I} (\lambda_n^{Hq})^+ (u_n^{hp})^+ ds &= \int_{\Gamma_C \cap J} \lambda_n^{Hq} (u_n^{hp})^+ ds \\ &\leq \int_{\Gamma_C \cap J} |\lambda_n^{Hq} - \lambda_n| |(u_n^{hp})^+| \, ds + \int_{\Gamma_C \cap J} |\lambda_n| |(u_n^{hp})^+| \, ds \\ &\leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \\ &+ \|\boldsymbol{\lambda}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \end{aligned} \tag{3.172}$$

Noting that $\lambda_n \geq 0$ on Γ_C then we have $0 \leq (\lambda_n^{Hq})^- \leq |\lambda_n - \lambda_n^{Hq}|$ on Γ_C and we obtain

$$\|(\lambda_n^{Hq})^-\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \leq \|(\lambda_n^{Hq})^-\|^2_{L_2(\Gamma_C)} \leq \|\lambda_n - \lambda_n^{Hq}\|^2_{L_2(\Gamma_C)} \leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\mathbf{L}_2(\Gamma_C)} \tag{3.173}$$

We now estimate the term

$$\int_{\Gamma_C} \Big(|(|\lambda_t^{Hq}| - \mathcal{F})^-| |u_t^{hp}| + 2(\lambda_t^{Hq} u_t^{hp})^- \Big) \, ds \tag{3.174}$$

Since $\lambda_t u_t \geq 0$ on Γ_C we have

$$\begin{aligned} 0 \leq (\lambda_t^{Hq} u_t^{hp})^- &\leq |\lambda_t u_t - \lambda_t^{Hq} u_t^{hp}| \\ &\leq |\lambda_t (u_t - u_t^{hp}) + (\lambda_t - \lambda_t^{Hq})(u_t - u_t^{hp}) + (\lambda_t - \lambda_t^{Hq}) u_t| \end{aligned} \tag{3.175}$$

we obtain

$$\begin{aligned}\int_{I \cap \Gamma_C} (\lambda_t^{Hq} u_t^{hp})^- \, ds &\leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \\ &+ \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u}\|_{\mathbf{L}_2(I \cap \Gamma_C)} + \|\boldsymbol{\lambda}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(I \cap \Gamma_C)}. \end{aligned} \tag{3.176}$$

We estimate the second term in (3.174), If $I \in \mathcal{T}_{hp}$, let $J \subset I$ be the part of the edge where

$$|(|\lambda_t^{Hq}| - \mathcal{F})^-| = (|\lambda_t^{Hq}| - \mathcal{F})$$

Since $|\lambda_t| \leq \mathcal{F}$ we obtain

$$\begin{aligned}
\int_{\Gamma_C \cap I} |(|\lambda_t^{Hq}| - \mathcal{F})^-||u_t^{hp}|\, ds &= \int_{\Gamma_C \cap J} (|\lambda_t^{Hq}| - \mathcal{F})|u_t^{hp}|\, ds \\
&= \int_{\Gamma_C \cap J} (|\lambda_t^{Hq}| - |\lambda_t| - \mathcal{F} + |\lambda_t|)|u_t^{hp}|\, ds \\
&\leq \int_{\Gamma_C \cap J} (|\lambda_t^{Hq}| - |\lambda_t|)(|u_t^{hp}| - |u_t|)\, ds + \\
&+ \int_{\Gamma_C \cap J} (|\lambda_t^{Hq}| - |\lambda_t|)|u_t|\, ds \\
&\leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \\
&+ \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq}\|_{\mathbf{L}_2(I \cap \Gamma_C)} \|\mathbf{u}\|_{\mathbf{L}_2(I \cap \Gamma_C)}
\end{aligned} \tag{3.177}$$

We now estimate the term

$$\|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \tag{3.178}$$

we have

$$\|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \leq \|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{L_2(\Gamma_C)} \tag{3.179}$$

Since $|\lambda_t| \leq \mathcal{F}$ we get

$$(|\lambda_t^{Hq}| - \mathcal{F})^+ \leq ||\lambda_t^{Hq}| - |\lambda_t| - \mathcal{F} + \mathcal{F}| \leq |\lambda_t^{Hq} - \lambda_t|. \tag{3.180}$$

Finaly we obtain

$$\|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma_C)} \leq \|(|\lambda_t^{Hq}| - \mathcal{F})^+\|^2_{L_2(\Gamma_C)} \leq \|(\boldsymbol{\lambda} - \boldsymbol{\lambda}^{Hq})\|_{\mathbf{L}_2(\Gamma_C)} \tag{3.181}$$

Using the inverse inequality, we obtain

$$\|(u_n^{hp})^+\|^2_{H^{\frac{1}{2}}(\Gamma_C \cap I)} \leq \frac{p_I^2}{h_I} \|(u_n^{hp})^+\|^2_{L_2(\Gamma_C \cap I)}. \tag{3.182}$$

From (3.171), we have

$$\|(u_n^{hp})^+\|^2_{H^{\frac{1}{2}}(\Gamma_C \cap I)} \leq \frac{p_I^2}{h_I} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{L}_2(\Gamma_C \cap I)}. \tag{3.183}$$

□

3.5 Numerical Experiments

Numerical results are presented with the MATLAB package of L.Banz for the contact of the two-dimentional elastic body $\Omega = [-0.5, 0.5]^2$ with a rigid obstacle. $\Gamma_C =$

$[-0.5, 0.5] \times \{-0.5\}$ and $\Gamma_N = \partial\Omega \setminus (\Gamma_C \cup \Gamma_D)$. The Young's modulus and the Poisson ratio are $E = 1000$, $\nu = 0.3$ respectively. The Neumann force is $\mathbf{t} = (0, 0)^T$, the gap $g = \frac{|x|}{10} - \frac{5}{100}$,the stabilization parameter $\gamma = 10^{-}3\frac{h}{p^2}$ and the give friction coefficient $\mathcal{F} = 0.3$. In Figure3.1 we show the initial and the deformed configuration. Figure 3.2 shows the estimated errors for the h-uniform for the stabilized and the non-stabilized problems, and hp-adaptive methods for the non-stabilized problem.

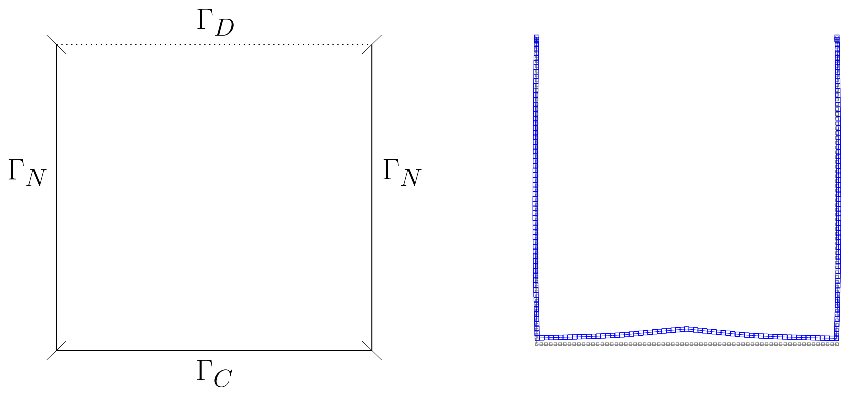


Figure 3.1: Initial(left) and Deformed (right) geometry

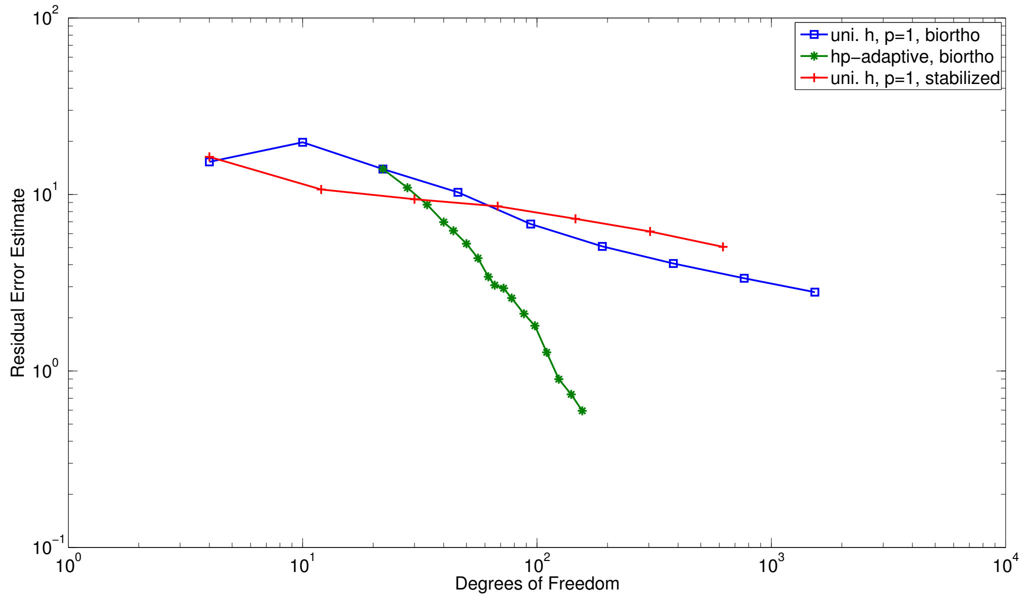


Figure 3.2: Convergence for stabilized and non-stabilized problems

4 hp-BEM for Stochastic Contact Problems in Linear Elasticity

Stochastic methods are, as their name suggests, based on the use of probability to model uncertainty. The objective is to study the effects on the output uncertainty on the input parameters, considered given, mechanical models. The knowledge about the uncertainties of the input data can be modeled as:

Random variables, i.e functions depending only on the hazard, so deterministic in space and time.

Stochastic fields, i.e functions depending on both the hazard and space.

Stochastic process, i.e functions depending on the hazard and time.

In this chapter we present a stochastic mixed BEM formulation for contact problems with Tresca friction.For the theoretical treatment of random variational inequalities see [29], [31] We show that the stochastic mixed formulation is well-posed. We study the deformation of an elastic homogeneous material in which Young's modulus (parameter that characterize the material properties) is a random variable. Similary the surface force $\mathbf{t} = \mathbf{t}(x, \omega)$ and the gap function g are assumed to be random as well.

4.1 Mixed Formulation for Stochastic Contact Problem

Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_C$ decomposed into the non-intersecting Neumann segment Γ_N, the Dirichlet segment Γ_D and the contact segment Γ_C which potentially can come in contact with the rigid foundation. Further let $(\Omega, \mathcal{B}, P)$ be a probability space with the set of outcomes, $\mathcal{B} \subset 2^\Omega$ the algebra of events and $P : \mathcal{B} \longrightarrow [0, 1]$ a probability measure.

Let $\sigma(\mathbf{u})$, $\epsilon(\mathbf{u})$ and $\mathcal{C}$ denote the stress tensor, the linearized strain tensor and the elasticity tensor, respectively. Further on $\mathbf{u}$ and n will denote the displacement field and the outward unit normal.

We define $\kappa(x)$ for each $x \in \mathcal{D}$ as a random variable $\kappa : \mathcal{D} \times \Omega \longrightarrow \mathbb{R}$ on the probability space $(\Omega, \mathcal{B}, P)$. As a consequence $\kappa : \mathcal{D} \times \Omega \longrightarrow \mathbb{R}$ is a random field and one obtains the real number $\kappa(x, \omega)$ for each realization $\omega \in \Omega$.

Let the outcome of the elasticity tensor be a random field $\mathcal{C}(x,.)$, $x \in \mathcal{D}$, defined on a probability space $(\Omega, \mathcal{B}, P)$.

The problem then consists in finding the displacement fields $\mathbf{u} : \mathcal{D} \times \Omega \longrightarrow \mathbb{R}$ such that P-a.e in Ω

$$-\operatorname{div} \sigma(\mathbf{u}(x,\omega)) = f(x,\omega) \quad \text{in } \mathcal{D} \tag{4.1a}$$

$$\sigma(\mathbf{u}(x,\omega)) = \mathcal{C}(x,\omega) : \epsilon(\mathbf{u}(x,\omega)) \quad \text{in } \mathcal{D} \tag{4.1b}$$

$$\mathbf{u}(x,\omega) = 0 \quad \text{on } \Gamma_D \tag{4.1c}$$

$$\sigma(\mathbf{u}(x,\omega)) \cdot n = \mathbf{t}(\omega, x) \quad \text{on } \Gamma_N \tag{4.1d}$$

$$\sigma_n \leq 0, \quad u_n(x,\omega) \leq g(x,\omega), \quad \sigma_n(u_n(x,\omega) - g(x,\omega)) = 0 \quad \text{in } \Gamma_C \tag{4.1e}$$

$$|\sigma_t| \leq \mathcal{F}, \quad \sigma_t u_t(x,\omega) + \mathcal{F}|u_t(x,\omega)| = 0 \quad \text{on } \Gamma_C \tag{4.1f}$$

Bochner spaces are a generalization of the concept of L^p spaces to functions whose values lie in a Banach space which is not necessarily the space $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers.

Let a measure space (T, Σ, μ) with Σ a sigma algebra over T and μ a measure on Σ be given, then for a Banach space $(X, \|\cdot\|_X)$, the Bochner space $L^p(T;X)$ is defined as the space of all measurable functions $u : T \to X$ such that the norm

$$\|u\|_{L^p(T;X)} = (\int_T \|u(t)\|_X^p \, d\mu(t))^{\frac{1}{p}} < +\infty \quad \text{for } 1 \leq p \leq \infty \tag{4.2}$$

$$\|u\|_{L^\infty(T;X)} := \text{ess sup}_{t \in T} \|u(t)\|_X < +\infty \quad \text{if } p = +\infty \tag{4.3}$$

is finite.

In the case where $p = 2$ and X is a separable Hilbert space, we have by Fubini that:

$$L^2_\mu(T;X) \cong L^2(T) \otimes X \tag{4.4}$$

where $\otimes$ denotes the tensor product between Hilbert spaces

Remark 4.1. *In particular $L^2_\mu(T;X)$ is itself again a Hilbert space*

Using the definition of Bochner spaces, let $L^2_P(\Omega; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))$ be the tensor Hilbert space of the second-order random variables defined on the probability space $(\Omega, \mathcal{B}, P)$ with values in $\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)$. Let $\mathbf{K}^P_\omega$ be the non empty closed convex subset of second-order random variables which satisfy the non-penetration condition

$$u_n \leq g \quad P_\omega - \text{almost surely.} \tag{4.5}$$

$$\mathbf{K}^P_\omega := \{\mathbf{u} \in L^2_P(\Omega; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)) : u_n \leq g \quad P_\omega - a.s \text{ on } \Gamma_C \times \Omega\}$$

We introduce some stochastic terminology: The Karhunen-Loève spectral decomposition consists in decomposing the random fields κ with eigenvalues λ_m in the form:

$$\kappa(x,\omega) = \kappa_0(x) + \sum_{m=1}^{\infty} \sqrt{\lambda_m} b_m(x)\xi_m(\omega), \tag{4.6}$$

where κ_0 is the mean of a random field κ at the point $x \in \mathcal{D}$ defined by

$$\kappa_0(x) := \langle \kappa(x,\cdot)\rangle := \int_\Omega \kappa(x,\omega) dP(\omega). \tag{4.7}$$

We approximate the random fields by a truncated Karhunen-Loève expansion:

$$\kappa(x,\omega) \approx \kappa_0(x) + \sum_{m=1}^{M} \sqrt{\lambda_m} b_m(x)\xi_m(\omega), \tag{4.8}$$

This determines a finite number M of independent random variable $\{\xi_m\}_{m=1}^M$ with mean zero and unit variance.The number of random variables will be denoted M or the number of the random dimensions. These random variables are determined by

$$\xi_m(\omega) = \frac{1}{\sqrt{\lambda_m}} \int_{\mathcal{D}} (\kappa(x,\omega) - \kappa_0(x)) b_m(x)\, dx \tag{4.9}$$

As a consequence, the stochastic variation of the random fields is now only through its dependence on the random variables $\xi_1, \cdots, \xi_M$. An assumption of finite dimensional noise must be made for each random input $R(\omega)$:

$$R(\omega) = R(x, \xi_1(\omega), \cdots ; \xi_M(\omega)) =: R(x, \xi(\omega)).$$

We denote the range space of each variable by $\Theta_m = \xi_m(\Omega)$ and the product range space

$$\Theta = \prod_{m=1}^{M} \Theta_m \quad \Theta \equiv \Theta_1 \times \Theta_2 \times \cdots \times \Theta_M.$$

Since we have assumed the $\{\xi_m\}_{m=1}^M$ are independent and continuous, they have a joint density function $\rho : \Theta \to \mathbb{R}$ and $\rho(y) = \rho_1(y_1)\rho_2(y_2)\cdots\rho_M(y_M)$ where the ρ_m are the corresponding density functions of the ξ_m, $y_m \in \Theta_m$ and $y = (y_1, \cdots, y_M)$. . With this assumption by the Doob-Dynkin Lemma, the displacement $\mathbf{u}$ can be also described by a finite number of random variables

$$\mathbf{u}(x,\omega) = \mathbf{u}(x, \xi_1(\omega), \cdots, \xi_M(\omega)). \tag{4.10}$$

The goal of the numerical methods is to seek the solution $\mathbf{u}(x,\xi)$.

We can now replace $L^2_P(\Omega; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))$ by $L^2_\rho(\Theta; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))$ and have

$$\mathcal{K}^\rho := \{\mathbf{u} \in L^2_\rho(\Theta; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)) : u_n \leq g \quad (P) - a.s \text{ on } \Gamma_C \times \Theta\}.$$

The variational inequality of the second kind then consists in finding a random displacement $u \in \mathcal{K}^\rho$ such that $\forall v \in \mathcal{K}^\rho$

$$A(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \mathbf{j}(\mathbf{v}) - \mathbf{j}(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K}^\rho \tag{4.11}$$

with the bilinear form

$$A(\mathbf{u}, \mathbf{v}) := \int_\Theta \langle S_y \mathbf{u}, \mathbf{v} \rangle_\Sigma \, \rho \, dy, \tag{4.12}$$

where

$$S_y := W_y + ({K_y}' + \frac{1}{2}) V_y^{-1} (K_y + \frac{1}{2}).$$

with V_y, ${K_y}'$, K_y and W_y as in in Chapter2 for

$$G(x_1, x_2) := \frac{1}{E(y)} \{ -\beta_1 \log \frac{1}{|x_1 - x_2|} \mathbf{I} + \beta_2 \frac{(x_1 - x_2) \otimes (x_1 - x_2)}{|x_1 - x_2|^2} \}$$

where

$$\beta_1 = \frac{(3 - 4\nu)(1 + \nu)}{2\pi(1 - \nu)}, \quad \beta_2 = \frac{1 + \nu}{4\pi(1 - \nu)}$$

with Poisson's ratio ν and Young's modulus E.

The linear form

$$L(\mathbf{v}) := \int_\Theta \langle \mathbf{t}, \mathbf{v} \rangle_{\Gamma_N} \, \rho \, dy. \tag{4.13}$$

The friction functional is given by

$$\mathbf{j}(\mathbf{v}) = \int_\Theta \int_{\Gamma_C} \mathcal{F} |u_t| \rho \, ds \, dy \tag{4.14}$$

Define the energy functional $J : L^2_\rho(\Theta; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)) \to \mathbb{R}$ as

$$J(\mathbf{v}) := \frac{1}{2} A(\mathbf{u}, \mathbf{v}) + \mathbf{j}(\mathbf{v}) - L(\mathbf{v}). \tag{4.15}$$

Remark 4.2. *We denote by $\{\mathcal{C}(x, \cdot), x \in \mathcal{D}\}$, the random elasticity tensor field. The bilinear form is continuous and elliptic, if $\mathcal{C}(x, \cdot)$ is uniformly bounded from below [2],[3],[1]*

Lemma 4.1. *The bilinear form $A(\cdot, \cdot)$ is a continuous elliptic bilinear form on*

$L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)) \times L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))$, i.e. there exist constants $C_A > 0$ and $c_A > 0$

such that for all $\boldsymbol{u}, \boldsymbol{v} \in L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))$

$$A(\boldsymbol{u}, \boldsymbol{v}) \leq C_A \|\boldsymbol{u}\|_{L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))} \|\boldsymbol{v}\|_{L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))} \tag{4.16}$$

$$A(\boldsymbol{u}, \boldsymbol{v}) \geq c_A \|\boldsymbol{u}\|^2_{L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))} \tag{4.17}$$

Proof. Follows from the fact that the Steklov-Poincaré operator S is continuous and $\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)$-coercive and if the random elasticity tensor field is uniformely bounded from below and above [2],[3],[1]. □

We can formulate the stochastic classical formulation as a saddle point problem equivalent to the stochastic variational inequality.

The stochastic admissible space for Lagrange multiplier $\boldsymbol{\lambda}$ is given by

$$\mathbf{L}(\mathcal{F}) = \left\{ \; \boldsymbol{\mu} \in L^2_\rho(\Theta; \tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C)) : \int_\Theta \langle \boldsymbol{\mu}, \mathbf{v} \rangle_{\Gamma_C} \rho \, dy \leq \int_\Theta \langle \mathcal{F}, \|v_t\| \rangle_{\Gamma_C} \rho \, dy \;, \forall \, v \in V^- \; \right\}$$

where

$$V^- = \{ \mathbf{v} \in L^2_\rho(\Theta; \mathbf{H}^{\frac{1}{2}}(\Gamma_C)), v_n \leqslant 0 \} \tag{4.18}$$

The mixed variational formulation of the stochastic contact problem with friction is given equivalently in the deterministic form by (cf.[8]):

Find $(\mathbf{u}, \boldsymbol{\lambda}) \in L^2_\rho(\Theta; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)) \times \mathbf{L}(\mathcal{F})$ such that

$$A(\mathbf{u}, \mathbf{v}) + b(\boldsymbol{\lambda}, \mathbf{v}) = L(\mathbf{v}) \qquad \forall \, \mathbf{v} \in L^2_\rho(\Theta; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma)) \tag{4.19a}$$

$$b(\boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{v}) \leqslant \int_\Theta \langle g, \mu_n - \lambda_n \rangle_{\Gamma_C} \rho \, dy \quad \forall \, \boldsymbol{\mu} \in \mathbf{L}(\mathcal{F}) \tag{4.19b}$$

with the functional

$$b(\boldsymbol{\lambda}, \mathbf{u}) = \int_\Theta \langle u_n, \lambda_n \rangle_{\Gamma_C} \rho \, dy + \int_\Theta \langle u_t, \lambda_t \rangle_{\Gamma_C} \rho \, dy$$

We define the strong pointwise non penetration condition on Γ_C by

$$u_n(x,y) \leq g(x,y), \quad \lambda_n(x,y) \geq 0, \quad \lambda_n(x,y)(u(x,y)_n - g(x,y)) = 0 \tag{4.20}$$

and the strong pointwise friction condition reads as

$$\begin{aligned} &|\lambda_t(x,y)| \leq \mathcal{F} \\ &|\lambda_t(x,y)| < \mathcal{F} \Rightarrow u_t(x,y) = 0 \\ &|\lambda_t(x,y)| = \mathcal{F} \Rightarrow \exists \alpha \in \mathbb{R} : \lambda_t(x,y) = \alpha^2 u_t(x,y) \end{aligned} \tag{4.21}$$

4.2 Discretization for Stochastic Contact Problem

Let $\mathcal{T}_{hp}$ denote a partition of $\bar{\Gamma}_C \cup \bar{\Gamma}_N$, such that all corners of $\bar{\Gamma}_C \cup \bar{\Gamma}_N$ and all end points $\bar{\Gamma}_C \cap \bar{\Gamma}_N$,$\bar{\Gamma}_D \cap \bar{\Gamma}_N$ are nodes of $\mathcal{T}_{hp}$. For simplicity we assume $meas(\Gamma_C) > 0$ and

$\bar{\Gamma}_C \cap \bar{\Gamma}_D = \emptyset$. $\mathcal{T}_{kq}$ is the mesh of Θ. Furthermore we define the set of Gauss-Lobatto points $G_{I,hp}$ on each element $I \in \mathcal{T}_{hp}$ of corresponding polynomial degree p_I and set $G_{hp} := \cup_{I\in\mathcal{T}_{hp}} G_{I,hp.}$. Analogously, let G_{kq} be the affinely transformed Gauss-Lobatto points in the stochastic domain, with $G_{kq} := \cup_{J\in\mathcal{T}_{kq}} G_{J,kq.}$

For $p = (p_I)_{I\in\mathcal{T}_{hp}}$ associate each element of $\mathcal{T}_{hp}$ with a polynomial degree $p_I \geq 1$. Similary, for $q = (q_J)_{J\in\mathcal{T}_{kq}}$ associate each element of $\mathcal{T}_{kq}$ with a polynomial degree $q_J \geq 0$. In particular, the finite element discretization of the associated deterministic problem can be choosen completely independently of the stochastic discretization, The space $\mathbf{V}_{hp}$ can be spanned by the 2-d nodal basis $\{\phi_i \mathbf{e}_k, i = 1, ..., N_{\mathbf{V}}, k = 1, 2\}$, where $\mathbf{e}_k$ denotes the i-th unit vector, ϕ_i the scalar Gauss-Lobatto Lagrange basis function associated with the node i and $N_{\mathbf{V}}$ the total number of the nodes.

We introduce (cf.[8]) the continuous piecewise polynomial space for the discretization of $\mathbf{u}$,

$$\mathbf{V}_{hp} := \{\mathbf{u}^{hp} \in \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma) : \forall I \in \mathcal{T}_{hp}, \mathbf{u}^{hp}|_I \in [\mathcal{P}_{p_I}(I)]^2, \mathbf{u}^{hp} = 0 \quad \text{on} \quad \Gamma_D\},$$

with

$$\mathbf{V}_{hp} \subset \text{span}\{\phi_i\}_{i=1}^{dim\mathbf{V}_{hp}}$$

and the piecewise polynomial space of discrete tractions ,

$$\mathbf{V}_{hp}^T := \{\boldsymbol{\phi} \in \mathbf{H}^{-\frac{1}{2}}(\Sigma) : \forall I \in \mathcal{T}_h, \boldsymbol{\phi}|_I \in [\mathcal{P}_{p_I-1}(I)]^2\} \subset \text{span}\{\phi_i^T\}_{i=1}^{dim\mathbf{V}_{hp}^T}.$$

We next introduce a finite dimensional subspace

$$\mathbf{W}_{kq} := \left\{v \in L_y^2(\Theta) : v_{kq}|_J \in \mathcal{P}_{p_J}(J) \ \forall \ J \in \mathcal{T}_{kq}\right\} \subset span\{\zeta_i\}_{i=1}^{dim\mathbf{W}_{kq}}$$

of the stochastic parameter space and approximate the tensor product space $L_\rho^2(\Theta; \tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))$ by the tensor product

$$\mathbf{V}_{hp} \otimes \mathbf{W}_{kq} \subset \text{span}\{\phi_i\zeta_j : \phi_i \in \mathbf{V}_{hp}, \zeta_j \in \mathbf{W}_{kq}\}$$

We define the discrete set of admissible displacements as

$$\mathbf{K}_{hp,kq} := \left\{\mathbf{v} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq} : (\mathbf{v}^{hp})_{ij} \cdot \mathbf{n}_i \leq g_{ij}\right\},$$

where g_{ij} is a suitable approximation of g. We note in general that $\mathbf{K}_{hp,kq}$ is not a subset of $\mathbf{K}$. The weighted gap vector is given by

$$g_{ij} := \frac{1}{D_i D_j^{stoch}} \int_\Theta \int_{\Gamma_C} g(x, y)\psi_i(x)\zeta_j(y) \ dx \ \rho \ dy, \tag{4.22}$$

where

$$D_i = \int_{\Gamma_C} \phi_i \ dx, \quad D_j^{stoch} = \int_\Theta \rho\vartheta_j \ dy.$$

Furthermore we define the dual Lagrange space

$$\mathbf{M}_{hp} = (\mathbf{V}_{hp}|_{\Gamma_C})' = \text{span}\,\{\psi_j\}_{j=1}^{\dim \mathbf{V}_{hp}|_{\Gamma_C}}$$

and the stochastic dual Lagrange space

$$\mathbf{T}_{kq} := \text{span}\,\{\zeta_j\}_{j=1}^{\dim \mathbf{W}_{kq}}\,.$$

The discrete Lagrange multiplier space is given by

$$\mathbf{L}_{hp,kq} := \left\{ \boldsymbol{\mu}^{hp} \in \mathbf{M}_{hp} \otimes \mathbf{T}_{kq} : \int_\Theta \langle \boldsymbol{\mu}^{hp}, \mathbf{v}^{hp} \rangle \rho\, dy \leq \int_\Theta \langle \mathcal{F}, |v_t^{hp}|_h \rangle \rho\, dy, \forall \mathbf{v}^{hp} \in V_{hp,kq}^- \right\}, \tag{4.23}$$

where

$$V_{hp,kq}^- := \{\mathbf{v} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq}, v_{n,ij} \leq 0\}.$$

The dual or biorthogonal basis functions ψ_i, ζ_j satisfy the orthogonality relations

$$\int_{\Gamma_C} \psi_i \phi_j ds = \delta_{ij} \int_{\Gamma_C} \phi_j ds, \quad \int_\Theta \zeta_i \vartheta_j \rho\, dy = \delta_{ij} \int_\Theta \vartheta_j \rho\, dy. \tag{4.24}$$

The discrete function $\mathbf{u}^{hp} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq}$ then is of the form (for $1 \leq i \leq \dim \mathbf{V}_{hp}$ and $1 \leq j \leq \dim \mathbf{W}_{hp}$)

$$\begin{aligned} \mathbf{u}^{hp} &:= \sum_{ij} u_{ij} \phi_i(x) \vartheta_j(y) \\ &:= \sum_{ij} (u_{n,ij} \mathbf{n}_i + u_{t,ij}) \phi_i(x) \vartheta_j(y). \end{aligned} \tag{4.25}$$

The normal and tangential part of the discrete function $\mathbf{u}^{hp} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq}$ are given by

$$u_n^{hp} := \sum_{ij} u_{n,ij} \phi_i(x) \vartheta_j(y), \tag{4.26}$$

$$u_t^{hp} := \sum_{ij} u_{t,ij} \phi_i(x) \vartheta_j(y). \tag{4.27}$$

The discrete absolute value on Γ_C of the tangential component is

$$|u_t^{hp}|_h := \sum_{ij} |u_{t,ij}| \phi_i(x) \vartheta_j(y).$$

Similary, the discrete Lagrange multiplier is given by (for $1 \leq i \leq \dim \mathbf{M}_{hp}$ and $1 \leq j \leq \dim \mathbf{T}_{hp}$)

$$\boldsymbol{\lambda}^{hp} := \sum_{ij} \lambda_{ij} \psi_i(x) \xi_j(y) \tag{4.28}$$

$$:= \sum_{ij} (\lambda_{n,ij} \mathbf{n}_i + \lambda_{t,ij}) \psi_i(x) \xi_j(y). \tag{4.29}$$

We can express the normal and the tangential part of $\boldsymbol{\lambda}^{hp}$ as follows:

$$\lambda_n^{hp} := \sum_{ij} \lambda_{n,ij}\psi_i(x)\xi_j(y), \tag{4.30}$$

$$\lambda_t^{hp} := \sum_{ij} \lambda_{t,ij}\psi_i(x)\xi_j(y). \tag{4.31}$$

Lemma 4.2. *The space* $\boldsymbol{L}_{hp,kq}$ (4.23) *is equivalent to*

$$\boldsymbol{L}_{hp,kq} := \left\{ \boldsymbol{\mu}^{hp} := \sum_{ij} \mu_{ij}\psi_i\xi_j,\ \mu_{n,ij} \geq 0,\ |\mu_{t,ij}| \leq \mathcal{F}, 1 \leq i \leq \dim \boldsymbol{M}_{hp}, 1 \leq j \leq \dim \boldsymbol{T}_{hp} \right\} \tag{4.32}$$

Proof. " $\Rightarrow$ " By means of the biorthogonality relations

$$\int_{\Gamma_C} \phi_i\psi_j ds = \delta_{ij}D_i, \qquad D_i = \int_{\Gamma_C} \phi_i ds > 0, \tag{4.33}$$

$$\int_{\Theta} \vartheta_i\xi_j\ \rho(y)dy = \delta_{ij}D_i^{stoch}, \qquad D_i^{stoch} = \int_{\Theta} \vartheta_i\ \rho(y)dy > 0. \tag{4.34}$$

$\boldsymbol{\mu}^{hp} \in \mathbf{L}_{hp,kq}$ and $\mathbf{v}^{hp} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq}$ can be written as

$$\mathbf{v}^{hp} := \sum_{ij}(v_{n,ij}\mathbf{n}_i + v_{t,ij})\phi_i(x)\vartheta_j(y), \tag{4.35}$$

$$\boldsymbol{\mu}^{hp} := \sum_{ij}(\mu_{n,ij}\mathbf{n}_i + \mu_{t,ij})\psi_i(x)\xi_j(y). \tag{4.36}$$

Similary

$$\mu_n^{hp} := \sum_{ij} \mu_{n,ij}\psi_i(x)\xi_j(y),$$

and the norm of the tangential component of $\mathbf{v}^{hp}$ is

$$|v_t^{hp}|_h := \sum_{ij}|v_{t,ij}|\phi_i(x)\vartheta_j(y)$$

Inserting $\boldsymbol{\mu}^{hp}$ and $\mathbf{v}^{hp}$ in the inequality

$$\int_{\Theta}\langle\boldsymbol{\mu}^{hp}, \mathbf{v}^{hp}\rangle\rho\ dy \leq \int_{\Theta}\langle\mathcal{F}, |v_t^{hp}|_h\rangle\rho\ dy,$$

we get

$$\int_{\Theta}\langle\boldsymbol{\mu}^{hp}, \mathbf{v}^{hp}\rangle\rho\ dy = \sum_{ij}(\mu_{n,ij}v_{n,ij} + \mu_{t,ij}v_{t,ij})D_iD_j^{stoch}$$

and

$$\int_{\Theta}\langle\mathcal{F}, |v_t^{hp}|_h\rangle\rho\ dy = \mathcal{F}\sum_{ij}|v_{t,ij}|D_iD_j^{stoch}.$$

Since v_{ij} and $v_{t,ij}$ are arbitrary and $D_i > 0$, $D_j^{stoch} > 0$, we have

$$\mu_{n,ij}v_{n,ij} + \mu_{t,ij}v_{t,ij} \leq \mathcal{F}|v_{t,ij}|. \tag{4.37}$$

Since $v_n^{hp} \leq 0$, we get $v_{n,ij} \leq 0$. Choosing $v_{t,ij} = 0$ in (4.37)

$$\mu_{n,ij} v_{n,ij} \leq 0,$$

and we get

$$\mu_{n,ij} \geq 0$$

Choosing now $v_{t,ij} = \mu_{t,ij}$ and $v_{n,ij} = 0$ in (4.37) , we obtain

$$|\mu_{t,ij}| \leq \mathcal{F}.$$

" $\Leftarrow$ " We have

$$\int_\Theta \langle \boldsymbol{\mu}^{hp}, \mathbf{v}^{hp} \rangle \rho \, dy = \sum_{ij} (\mu_{n,ij} v_{n,ij} + \mu_{t,ij} v_{t,ij}) D_i D_j^{stoch} \tag{4.38}$$

and

$$\int_\Theta \langle \mathcal{F}, |v_t^{hp}|_h \rangle \rho \, dy = \mathcal{F} \sum_{ij} |v_{t,ij}| D_i D_j^{stoch}. \tag{4.39}$$

Since $v_{n,ij} \leq 0$, we get for each $\boldsymbol{\mu}^{hp} \in \mathbf{L}_{hp,kq}$

$$\begin{aligned} \mu_{n,ij} v_{n,ij} + \mu_{t,ij} v_{t,ij} &\leq |\mu_{t,ij}| |v_{t,ij}| \\ &\leq \mathcal{F} |v_{t,ij}|. \end{aligned}$$

Summing over all points, we obtain the assertion, (see[[41],Lemma2.3]). □

The approximation S_{hp} of the Poincaré-Steklov operator is given by

$$S_{hp} := W + (K' + \frac{1}{2}) i_{hp} (i_{hp}^* V i_{hp})^{-1} i_{hp}^* (K + \frac{1}{2}). \tag{4.40}$$

The discrete variational inequality reads: Find $\mathbf{u}^{hp} \in \mathbf{K}_{hp,kq}$ such that

$$A_h(\mathbf{u}^{hp}, \mathbf{v}^{hp}) + \mathbf{j}(\mathbf{v}^{hp}) - \mathbf{j}(\mathbf{u}^{hp}) \geq L(\mathbf{v}^{hp} - \mathbf{u}^{hp}) \quad \forall \mathbf{v}^{hp} \in \mathbf{K}_{hp,kq}, \tag{4.41a}$$

where

$$\mathbf{j}(\mathbf{v}^{hp}) = \int_\Theta \int_{\Gamma_C} \mathcal{F} |u_t^{hp}(x,y)| \rho \, ds \, dy.$$

The discrete bilinear form is

$$A_h(\mathbf{u}^{hp}, \mathbf{v}^{hp}) := \int_\Theta \langle S_{hp} \mathbf{u}^{hp}, \mathbf{v}^{hp} \rangle_\Sigma \, \rho \, dy = \int_\Theta \left\langle W \mathbf{u}^{hp} + (K' + \frac{1}{2}) \Psi_{hp}, \mathbf{v}^{hp} \right\rangle_\Sigma \rho \, dy, \tag{4.42}$$

and $\Psi_{hp} \in \mathbf{V}_{hp}^T$ solves for P-a.e. $y \in \Theta$ the auxiliary problem

$$\left\langle V \Psi_{hp}, \mathbf{v}^{hp} \right\rangle_\Sigma = \left\langle (K + \frac{1}{2}) \mathbf{u}^{hp}, \mathbf{v}^{hp} \right\rangle_\Sigma \quad \forall \, \mathbf{v}^{hp} \in \mathbf{V}_{hp}^T. \tag{4.43}$$

Lemma 4.3. *The discrete bilinear form* $A_h(\boldsymbol{u}^{hp}, \boldsymbol{v}^{hp})$ *is continuous and* $L^2_\rho(\Theta; \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))$*-coercive.*

The discrete mixed stochastic problem in deterministic formulation reads : Find $\mathbf{u}^{hp} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq}$ and $\boldsymbol{\lambda}^{hp} \in \mathbf{L}_{hp,kq}$

$$A_h(\mathbf{u}^{hp}, \mathbf{v}^{hp}) + b(\boldsymbol{\lambda}^{hp}, \mathbf{v}^{hp}) = L(\mathbf{v}^{hp}) \qquad \forall\, \mathbf{v}^{hp} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq} \tag{4.44a}$$

$$b(\boldsymbol{\mu}^{hp} - \boldsymbol{\lambda}^{hp}, \mathbf{u}^{hp}) \leqslant \int_\Theta \left\langle g, \mu_n^{hp} - \lambda_n^{hp} \right\rangle_{\Gamma_C} \rho\, dy \qquad \forall\, \boldsymbol{\mu}^{hp} \in \mathbf{L}_{hp,kq} \tag{4.44b}$$

with the functional

$$b(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{hp}) = \int_\Theta \left\langle u_n^{hp}, \lambda_n^{hp} \right\rangle_{\Gamma_C} \rho\, dy + \int_\Theta \left\langle u_t^{hp}, \lambda_t^{hp} \right\rangle_{\Gamma_C} \rho\, dy.$$

Theorem 4.1. *There exists exactly one solution to the discrete mixed formulation* (4.44).

Proof. **Uniquiness**: We assume that $(\mathbf{u}_1^{hp}, \boldsymbol{\lambda}_1^{hp})$ and $(\mathbf{u}_2^{hp}, \boldsymbol{\lambda}_2^{hp})$ solve the discrete mixed formulation (4.44). Then we have

$$A_h(\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}), \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) + b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) = 0, \tag{4.45}$$

where

$$\begin{aligned} A_h(\mathbf{u}^{hp}, \mathbf{v}^{hp}) &:= \int_\Theta \langle S_{hp}\mathbf{u}^{hp}, \mathbf{v}^{hp} \rangle_\Sigma\, \rho\, dy \\ b(\mathbf{u}^{hp}, \boldsymbol{\lambda}^{hp}) &= \int_\Theta \left\langle u_n^{hp}, \lambda_n^{hp} \right\rangle_{\Gamma_C} \rho\, dy + \int_\Theta \left\langle u_t^{hp}, \lambda_t^{hp} \right\rangle_{\Gamma_C} \rho\, dy. \end{aligned} \tag{4.46}$$

Choosing $\boldsymbol{\mu}_1^{hp} = \boldsymbol{\lambda}_2^{hp}$ and $\boldsymbol{\mu}_2^{hp} = \boldsymbol{\lambda}_1^{hp}$ in (4.44b) we get

$$b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) \geq 0. \tag{4.47}$$

Using Lemma4.3 we obtain

$$c\|\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))} \leq A_h(\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}), \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) + b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}) = 0 \tag{4.48}$$

Consequently the first argument $\mathbf{u}^{hp}$ is unique.

Since $\mathbf{u}^{hp}$ is unique we have for all $\mathbf{v}^{hp} \in \mathbf{V}_{hp} \otimes \mathbf{W}_{kq}$

$$0 = b(\boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}, \mathbf{v}^{hp}). \tag{4.49}$$

Using $\boldsymbol{\lambda}_1^{hp}$, $\boldsymbol{\lambda}_2^{hp}$ and $\mathbf{v}^{hp}$ and the biorthogonality , we obtain the relation

$$0 = \sum_{ij} [(\lambda_{1,n,ij} - \lambda_{2,n,ij}) v_{n,ij} + (\lambda_{1,t,ij} - \lambda_{2,t,ij}) v_{t,ij}] D_i D_j^{stoch} \tag{4.50}$$

Choosing $v_{t,ij} = 0$ in (4.50) we get

$$(\lambda_{1,n,ij} - \lambda_{2,n,ij})v_{n,ij} = 0 \quad \text{and} \quad \lambda_{1,n,ij} = \lambda_{2,n,ij}$$

because $D_i D_j^{stoch} > 0$.

We choose now $v_{n,ij} = 0$, and obtain $\lambda_{1,t,ij} = \lambda_{2,t,ij}$. Consequently the second argument $\boldsymbol{\lambda}^{hp}$ is unique.

Existence: We know that the inequality (4.44b) can be written as a projection equation [44]:

$$\boldsymbol{\lambda}^{hp} = P_{\mathbf{L}_{hp,kq}} \left(\boldsymbol{\lambda}^{hp} + r \begin{pmatrix} u_n^{hp} - g \\ u_t^{hp} \end{pmatrix} \right), \tag{4.51}$$

where the map $P_{\mathbf{L}_{hp,kq}}$ stand for the orthogonal projection onto $\mathbf{L}_{hp,kq}$ and where $r > 0$ is an arbitrary parameter. The fixed point operator T is defined as follows [44] :

$$\begin{aligned} T : \mathbf{L}_{hp,kq} &\longrightarrow \mathbf{L}_{hp,kq} \\ \boldsymbol{\lambda}^{hp} &\longmapsto P_{\mathbf{L}_{hp,kq}} \left(\lambda^{hp} + r \begin{pmatrix} u_n^{hp} - g \\ u_t^{hp} \end{pmatrix} \right). \end{aligned} \tag{4.52}$$

From Lemma 4.3 we have

$$\begin{aligned} -\|\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\|^2_{L^2_\rho(\Theta;\mathbf{L}_2(\Gamma_C))} &\geq -\|\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))} \\ &\geq -c^{-1} \int_\Theta \langle S_{hp}(\mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}), \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp} \rangle_\Sigma \, \rho \, dy \end{aligned} \tag{4.53}$$

Using the notation $\delta\mathbf{u}^{hp} = \mathbf{u}_1^{hp} - \mathbf{u}_2^{hp}$, $\delta\boldsymbol{\lambda}^{hp} = \boldsymbol{\lambda}_1^{hp} - \boldsymbol{\lambda}_2^{hp}$, $\|\cdot\| = \|\cdot\|_{L^2_\rho(\Theta;\mathbf{L}_2(\Gamma_C))}$, we compute:

$$\begin{aligned} \|T(\boldsymbol{\lambda}_1^{hp}) - T(\boldsymbol{\lambda}_2^{hp})\|^2 &\leq \|P_{\mathbf{L}_{hp,kq}} \left(\boldsymbol{\lambda}_1^{hp} + r \begin{pmatrix} u_{n,1}^{hp} - g \\ u_{t,1}^{hp} \end{pmatrix} \right) - P_{\mathbf{L}_{hp,kq}} \left(\boldsymbol{\lambda}_2^{hp} + r \begin{pmatrix} u_{n,2}^{hp} - g \\ u_{t,2}^{hp} \end{pmatrix} \right) \|^2 \\ &\leq \|\delta\boldsymbol{\lambda}^{hp} + r \begin{pmatrix} \delta u_n^{hp} \\ \delta u_t^{hp} \end{pmatrix} \|^2 \\ &\leq \|\delta\boldsymbol{\lambda}^{hp} + r\delta\mathbf{u}^{hp}\|^2 \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2 + 2rb(\delta\boldsymbol{\lambda}^{hp}, \delta\mathbf{u}^{hp}) + r^2\|\delta\mathbf{u}^{hp}\|^2 \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2 - 2r \int_\Theta \langle S_{hp}\delta\mathbf{u}^{hp}, \delta\mathbf{u}^{hp} \rangle_\Sigma \, \rho \, dy + r^2\|\delta\mathbf{u}^{hp}\|^2 \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2 - 2rc\|\delta\mathbf{u}^{hp}\|^2 + r^2\|\delta\mathbf{u}^{hp}\|^2 \\ &\leq \|\delta\boldsymbol{\lambda}^{hp}\|^2(1 - 2rc\beta^2 + r^2\beta^2) \end{aligned}$$

where $\beta = \frac{\|\delta\boldsymbol{\lambda}^{hp}\|^2}{\|\delta\mathbf{u}^{hp}\|^2}$. It follows that T is strict contraction for $0 < r < 2c$. By the Banach fixed point theorem there exists a $\boldsymbol{\lambda}^{hp} = (\lambda_n^{hp}, \lambda_t^{hp})$ which satisfies (4.52). For any given $\boldsymbol{\lambda}^{hp}$, problem (4.44a) reduces to a linear, finite dimentional problem. Hence, the uniqueness result of $\mathbf{u}^{hp}$ implies the existence of a $\mathbf{u}^{hp}(\boldsymbol{\lambda}^{hp})$.

□

Remark 4.3. *Theorem 4.1 is the system case corresponding to the scalar problem considered in [8]. But Theorem 4.1 needs a new proof.*

Lemma 4.4. *The contact constraints* (4.44b) *are equivalent to the pointwise conditions*

$$u_{n,p_{sp},p_{st}} \leq g_{p_{sp},p_{st}}, \quad \lambda_{n,p_{sp},p_{st}} \geq 0, \quad \lambda_{n,p_{sp},p_{st}}(u_{n,p_{sp},p_{st}} - g_{p_sp,p_{st}}) = 0 \tag{4.54}$$

$$\begin{aligned} &|\lambda_{t,p_{sp},p_{st}}| \leq \mathcal{F}\lambda_{n,p_s,p_{st}} \\ &|\lambda_{t,p_{sp},p_{st}}| < \mathcal{F}\lambda_{n,p_s,p_{st}} \Rightarrow u_{n,p_{sp},p_{st}} = 0 \\ &|\lambda_{t,p_{sp},p_{st}}| = \mathcal{F}\lambda_{n,p_s,p_{st}} \Rightarrow \alpha \in \mathbb{R}^2 : \lambda_{t,p_{sp},p_{st}} = \alpha^2 u_{n,p_{sp},p_{st}} \end{aligned} \tag{4.55}$$

$\forall\, (p_{sp}, p_{st}) \in G_{hp} \cap \Gamma_C \times G_{kq}$.

Proof. see [41], [10] and [8]

□

For Tresca's friction law the friction coefficient is given by a function $\mathcal{F}(\cdot\,) : \Gamma_C \to \mathbb{R}$. We define $\mathcal{F}_{p_{sp}p_{st}}$ associated with the nodes $(p_{sp}, p_{st}) \in G_{hp} \cap \Gamma_C \times G_{kq} \cap \Theta$ by

$$\mathcal{F}_{p_{sp}p_{st}} = \int_\Theta \int_{\Gamma_C^s} \mathcal{F}\ \phi_{p_{sp}}(x)\vartheta_{p_{st}}(y)\ \rho\ dy\ ds, \tag{4.56}$$

If $\mathcal{F}$ is a constant function we have

$$\mathcal{F}_{p_{sp}p_{st}} = \mathcal{F}\ D_{p_{sp}} D_{p_{st}}. \tag{4.57}$$

We will now formulate the classical Uzawa algorithm for problem (4.44). As in [8],[44], we introduce the following equivalent formulation:

Lemma 4.5. *The pair* $(\lambda_{n,p_{sp},p_{st}}, u_{n,p_{sp},p_{st}})$ *satisfies the pointwise non-penetration condition* (4.54) *if and only if it satisfies the pointwise condition*

$$\lambda_{n,p_{sp},p_{st}} = \max\{0, \lambda_{n,p_{sp},p_{st}} + c(u_{n,p_{sp},p_{st}} - g_{p_{sp},p_{st}})\} \tag{4.58}$$

Proof. see [[41],Theorem 4.1]

□

Lemma 4.6. *The pair* $(\lambda_{t,p_{sp},p_{st}}, u_{t,p_{sp},p_{st}})$ *satisfies the friction contact condition* (4.55) *if and only if it satisfies the pointwise condition*

$$\lambda_{t,p_{sp},p_{st}} = \mathcal{F}_{ij} \frac{\lambda_{t,p_{sp},p_{st}} + c\ u_{t,p_{sp},p_{st}}}{\max\{\mathcal{F}_{p_s,p_{st}}, |\lambda_{t,p_{sp},p_{st}} + c u_{t,p_s,p_{st}}|\}} \tag{4.59}$$

Proof. see [[41],Theorem 5.1] □

The Uzawa algorithm corresponds to iterations of the fixed point operator. In this algorithm, we use the pointwise condition (4.58) and (4.59) . The Uzawa algorithm can be written as follows:

Algorithm 4.1. *(Uzawa algorithm)*

---

1. *Initialisation: Choose initial solution* $\boldsymbol{\lambda}^0$, $\quad c > 0, \quad tol > 0$
2. *For* $k = 0, 1, 2, ...$ *do*
   a) *For given* $\boldsymbol{\lambda}^k$ *, find* $\boldsymbol{u}^k$ *by solving* (4.44a) *and* (4.44b)
   b) *Update Lagrange multiplier by using*

$$\begin{aligned}
\lambda_{n,p_{sp},p_{st}} &= \max\{0, \lambda_{n,p_{sp},p_{st}} + c(u_{n,p_{sp},p_{st}} - g_{p_{sp},p_{st}})\} \\
\lambda_{t,p_{sp},p_{st}} &= \mathcal{F} \frac{\lambda_{t,p_{sp},p_{st}} + c\; u_{t,p_{sp},p_{st}}}{\max\{\mathcal{F}_{p_s,p_{st}}, |\lambda_{t,p_{sp},p_{st}} + c u_{t,p_s,p_{st}}|\}}
\end{aligned}$$

   c) *Stop if* $\|\boldsymbol{\lambda}^{k-1} - \boldsymbol{\lambda}^k\| < tol\|\boldsymbol{\lambda}^{k-1}\|$ *or* $\|\boldsymbol{\lambda}^{k-1} - \boldsymbol{\lambda}^k\| < tol$*, else go to step* 2

4.3 A posteriori error estimates for stochastic contact with friction

Lemma 4.7. *[[19], Lemma* 3.2.9*] There exists an operator* $\Pi_{hp} : \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma) \to \boldsymbol{V}_{hp}$*, which is stable in the* $\tilde{\boldsymbol{H}}^{\frac{1}{2}}$*-norm and has quasioptimal approximation properties in the* $\boldsymbol{L}_2$*-norm. More precisely, there exists a constant C, independent of h and p, such that for all* $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)$ *there holds*

$$\|\Pi_{hp}\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \leq C\|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \tag{4.60}$$

$$\|\boldsymbol{u} - \Pi_{hp}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Sigma)} \leq C\left(\frac{h}{p}\right)^{\frac{(1-\epsilon)}{2}} \|\boldsymbol{u}\|_{\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma)} \tag{4.61}$$

with arbitrarily small $\epsilon \in (0; \frac{1}{2})$.

Theorem 4.2. *Let* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *be the exact solution of the boundary problem* (4.19) *and* $(\boldsymbol{u}^{hp}, \boldsymbol{\lambda}^{hp})$ *be the solution of the discrete boundary problem* (4.41)*, then there holds the* *a posteriori estimate :*

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\boldsymbol{H}}^{\frac{1}{2}}(\Sigma))} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}^{hp}\|_{L^2_\rho(\Theta;\tilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma_C))} &\lesssim \sum_{I\in\mathcal{T}_{hp}}\sum_{J\in\mathcal{T}_{kq}} \eta^2_{hp}(I,J) \\
&+ \int_\Theta \langle(\lambda_n^{hp})^+,(g-u_n^{hp})^+\rangle_{\Gamma_C}\,\rho\,dy + \|(\lambda_n^{hp})^-\|^2_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \\
&+ \|(u_n^{hp}-g)^+\|^2_{L^2_\rho(\Theta;H^{\frac{1}{2}}(\Gamma_C))} + \|(|\lambda_t^{hp}|-\mathcal{F})^+\|^2_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \\
&+ \int_\Theta\int_{\Gamma_C}\Big(|(|\lambda_t^{hp}|-\mathcal{F})^-||u_t^h| + 2(\lambda_t^{hp}u_t^{hp})^-\Big)\,ds\,\rho\,dy \qquad (4.62)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta^2_{hp} = \left(1+\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right)\Big(\|\boldsymbol{t}-S_{hp}\boldsymbol{u}^{hp}\|^2_{L^2_\rho(J;L_2(I\cap\Gamma_N))} + \|(-\boldsymbol{\lambda}^{hp})-S_{hp}\boldsymbol{u}^{hp}\|^2_{L^2_\rho(J;L_2(I\cap\Gamma_C))}\Big) \\
+ h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp}-\psi^{hp}))\|^2_{L^2_\rho(J;L_2(I))}, \qquad (4.63)
\end{aligned}
$$

with arbitrarily small $\epsilon\in(0;\frac{1}{2})$.

Proof. Using the same arguments as in Chapter 2.

From (2.134) we have

$$
\begin{aligned}
C_W\|\mathbf{u}-\mathbf{u}^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))} + C_V\|\psi-\psi^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma))} &\le \int_\Theta\langle\mathbf{t}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N}\,\rho\,dy \\
&+ \int_\Theta\langle(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_C}\,\rho\,dy \\
&+ \int_\Theta\langle\lambda_n-\lambda_n^{hp},u_n^{hp}-u_n\rangle\,\rho\,dy + \int_\Theta\langle\lambda_t-\lambda_t^{hp},u_t^{hp}-u_t\rangle\,\rho\,dy \\
&+ \int_\Theta\langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle\,\rho\,dy \\
&+ A+B+C \qquad (4.64)
\end{aligned}
$$

We estimate the first and the second terms in (4.64), employing the Cauchy-Schwarz inequality. We obtain

$$
\begin{aligned}
A = \int_\Theta\langle\mathbf{t}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N}\,\rho\,dy + \int_\Theta\langle(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_C}\,\rho\,dy \le \\
\sum_{J\in\mathcal{T}_{kq}}\sum_{I\in\mathcal{T}_{hp}\cap\Gamma_C}\|(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp}\|_{L^2_\rho(J;L_2(I))}\|\mathbf{u}-\mathbf{v}^{hp}\|_{L^2_\rho(J;L_2(I))} \\
+\sum_{J\in\mathcal{T}_{kq}}\sum_{I\in\mathcal{T}_{hp}\cap\Gamma_N}\|\mathbf{t}-S_{hp}\mathbf{u}^{hp}\|_{L^2_\rho(J;L_2(I))}\|\mathbf{u}-\mathbf{v}^{hp}\|_{L^2_\rho(J;L_2(I))} \qquad (4.65)
\end{aligned}
$$

Let π_{kq} the L^2_ρ-projection onto $\mathbf{W}_{kq}$, which satisfies

$$
\int_\Theta(\pi_{kq}w-w)\,v\,\rho\,dy. \qquad (4.66)
$$

Note that π_{kq} is L^2_ρ-stable, using Lemma4.7, we obtain

$$\begin{aligned}\|w-\pi_{kq}\Pi_{hp}w\|^2_{L^2_\rho(\Theta;\mathbf{L}^2(I))} &\leq \|w-\Pi_{hp}w\|^2_{L^2_\rho(\Theta;\mathbf{L}^2(I))}+\|\Pi_{hp}w-\pi_{kq}\Pi_{hp}w\|^2_{L^2_\rho(\Theta;\mathbf{L}^2(I))}\\ &\leq C\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\|w\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\omega(I)))}+\|\Pi_{hp}w\|^2_{L^2_\rho(\Theta;\mathbf{L}^2(I))}\\ &\leq C\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\|w\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\omega(I)))}+\|w\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\omega(I)))}\\ &\leq C\left(1+\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right)\|w\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\omega(I)))},\end{aligned}\tag{4.67}$$

where we used the $\tilde{\mathbf{H}}^{\frac{1}{2}}$-stability of Π_{hp}.

We have

$$\|w-\pi_{kq}\Pi_{hp}w\|^2_{L^2_\rho(\Theta;\mathbf{L}^2(I))}\leq C\left(1+\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right)\|w\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\omega(I)))},\tag{4.68}$$

Choosing $\mathbf{v}^{hp}=\mathbf{u}^{hp}+\pi_{kq}\Pi_{hp}(\mathbf{u}-\mathbf{u}^{hp})$, we get

$$\begin{aligned}A=&\int_\Theta\langle\mathbf{t}-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_N}\,\rho\,dy+\int_\Theta\langle(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp},\mathbf{u}-\mathbf{v}^{hp}\rangle_{\Gamma_C}\,\rho\,dy\leq\\ &\sum_{J\in\mathcal{T}_{kq}}\sum_{I\in\mathcal{T}_{hp}\cap\Gamma_C}\left(1+\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right)^{\frac{1}{2}}\|(-\boldsymbol{\lambda}^{hp})-S_{hp}\mathbf{u}^{hp}\|_{L^2_\rho(J;L_2(I))}\|\mathbf{u}-\mathbf{u}^{hp}\|_{L^2_\rho(J;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))}\\ &+\sum_{J\in\mathcal{T}_{kq}}\sum_{I\in\mathcal{T}_{hp}\cap\Gamma_N}\left(1+\left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right)^{\frac{1}{2}}\|\mathbf{t}-S_{hp}\mathbf{u}^{hp}\|_{L^2_\rho(J;L_2(I))}\|\mathbf{u}-\mathbf{u}^{hp}\|_{L^2_\rho(J;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))}\end{aligned}\tag{4.69}$$

As in (2.138) there holds for the last term C

$$\begin{aligned}&\int_\Theta\langle V(\psi^*_{hp}-\psi^{hp}),\psi-\psi^{hp}\rangle\,\rho\,dy\leq\|V(\psi^*_{hp}-\psi^{hp})\|_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))}\|\psi^{hp}-\psi\|_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Sigma))}\\ &\leq c\left(\int_\Theta\sum_{I\in\mathcal{T}_{hp}}h_I\|\frac{\partial}{\partial s}(V(\psi^*_{hp}-\psi^{hp}))\|^2_{\mathbf{L}_2(I)}\right)^{\frac{1}{2}}\|\psi^{hp}-\psi\|_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Sigma))}\end{aligned}\tag{4.70}$$

From (2.140) and (2.141), we obtain

$$\begin{aligned}
B &= \int_\Theta \langle \lambda_n - \lambda_n^{hp}, u_n^{hp} - u_n \rangle \, \rho \, dy + \int_\Theta \langle \lambda_t - \lambda_t^{hp}, u_t^{hp} - u_t \rangle \, \rho \, dy \\
&\leq \|(\lambda_n^{hp})^-\|_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \|u_n^{hp} - u_n\|_{L^2_\rho(\Theta;H^{\frac{1}{2}}(\Sigma))} \\
&+ \|\lambda_n^{hp} - \lambda_n\|_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \|(u_n^{hp} - g)^+\|_{L^2_\rho(\Theta;H^{\frac{1}{2}}(\Gamma_C))} \\
&+ \int_\Theta \langle (\lambda_n^{hp})^+, (g - u_n^{hp})^+ \rangle_{\Gamma_C} \, \rho \, dy \\
&+ \|(|\lambda_t^{hp}| - \mathcal{F})^+\|_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \|u_t^{hp} - u_t\|_{L^2_\rho(\Theta;H^{\frac{1}{2}}(\Sigma))} \\
&+ \int_\Theta \int_{\Gamma_C} \left(|(|\lambda_t^{hp}| - \mathcal{F})^-||u_t^h| + 2(\lambda_t^{hp} u_t^{hp})^- \right) ds \, \rho \, dy. \qquad (4.71)
\end{aligned}$$

Now we combine estimates (4.69), (4.70), and (4.71), use the Cauchy-Schwarz inequality, and Young's inequality, we obtain

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))} + \|\psi - \psi^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma))} &\lesssim \sum_{I \in \mathcal{T}_{hp}} \sum_{J \in \mathcal{T}_{kq}} \eta^2_{hp}(I, J) \\
&+ \int_\Theta \langle (\lambda_n^{hp})^+, (g - u_n^{hp})^+ \rangle_{\Gamma_C} \, \rho \, dy + \|(\lambda_n^{hp})^-\|^2_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \\
&+ \|(u_n^{hp} - g)^+\|^2_{L^2_\rho(\Theta;H^{\frac{1}{2}}(\Gamma_C))} + \|(|\lambda_t^{hp}| - \mathcal{F})^+\|^2_{L^2_\rho(\Theta;\tilde{H}^{-\frac{1}{2}}(\Gamma_C))} \\
&+ \int_\Theta \int_{\Gamma_C} \left(|(|\lambda_t^{hp}| - \mathcal{F})^-||u_t^h| + 2(\lambda_t^{hp} u_t^{hp})^- \right) ds \, \rho \, dy \\
&+ \epsilon \|\boldsymbol{\lambda}^{hp} - \boldsymbol{\lambda}\|_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C))}, \qquad (4.72)
\end{aligned}$$

where

$$\begin{aligned}
\eta^2_{hp} &= \left(1 + \left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right) \left(\|\mathbf{t} - S_{hp}\mathbf{u}^{hp}\|^2_{L^2_\rho(J;L_2(I\cap\Gamma_N))} + \|(-\boldsymbol{\lambda}^{hp}) - S_{hp}\mathbf{u}^{hp}\|^2_{L^2_\rho(J;L_2(I\cap\Gamma_C))} \right) \\
&+ h_I \|\frac{\partial}{\partial s}(V(\psi^*_{hp} - \psi^{hp}))\|^2_{L^2_\rho(J;L_2(I))} \qquad (4.73)
\end{aligned}$$

As in Lemma2.13, using estimate (4.68), we obtain

$$\begin{aligned}
\|\boldsymbol{\lambda}^{hp} - \boldsymbol{\lambda}\|_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma_C))} &\leq C' \left(\|\mathbf{u} - \mathbf{u}^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{\frac{1}{2}}(\Sigma))} + \|\psi - \psi^{hp}\|^2_{L^2_\rho(\Theta;\tilde{\mathbf{H}}^{-\frac{1}{2}}(\Gamma))} \right) \\
&+ C'' \sum_{I \in \mathcal{T}_{hp}} \sum_{J \in \mathcal{T}_{kq}} \xi^2_{hp}(I, J), \qquad (4.74)
\end{aligned}$$

where

$$\xi^2_{hp} = \left(1 + \left(\frac{h_I}{p_I}\right)^{1-\epsilon}\right) \left(\|\mathbf{t} - S_{hp}\mathbf{u}^{hp}\|^2_{L^2_\rho(J;L_2(I\cap\Gamma_N))} + \|(-\boldsymbol{\lambda}^{hp}) - S_{hp}\mathbf{u}^{hp}\|^2_{L^2_\rho(J;L_2(I\cap\Gamma_C))} \right). \qquad (4.75)$$

Combining (4.72) and (4.74) yields the aposteriori error estimate. □

4.4 Numerical Experiments

In this section, we consider a random position-independent Young's modulus E and a deterministic position-independent Poisson ratio $\nu = 0.3$. This enables the tensor product of matrices, the global matrix can be written as tensor product of the matrix representation of the Steklov-Poicaré operator $\tilde{S}$ and the stochastic mass matrix. Where $\tilde{S}$ is the Steklov-Poincré operator for $E = 1$ and

$$S := W + (K' + \frac{1}{2})V^{-1}(K + \frac{1}{2}) = E\tilde{S}$$

Numerical results are presented for the contact of the two-dimensional elastic body $\mathcal{D} = [-0.5, 0.5]^2$, with a rigid obstacle.The rigid obstacle occupies the half space $x_2 < -0.5$, $\Gamma_C = [-0.5, 0.5] \times \{-0.5\}$. We use the uniform distribution, which corresponds to the density $\rho \equiv \frac{1}{2\sqrt{3}}$ and the stochastic domain $\Theta = \left[-\sqrt{3}, \sqrt{3}\right]$. The Neumann force is $\mathbf{t} = (0, -1)^T$, the gap $g = -0.2$ and the give friction coefficient $\mathcal{F} = 0.3$. Mean values of the Young modulus E was adopted as $2000Pa$. The random Young's modulus is modeled by a unifom random variable with values in the interval between 1800 and 2200.

In Figure 4.2 we present the error in the energy $\mathbf{J}(\mathbf{u}) := \frac{1}{2}\langle \mathbf{u}, S\mathbf{u}\rangle - \langle \mathbf{t}, \mathbf{u}\rangle$ and in the energy norm $\tilde{\mathbf{J}}(\mathbf{u}) := \langle \mathbf{u}, S\mathbf{u}\rangle$ with respect to the number of the degrees of freedom, for h-uniform and p-version. The exact value of the potential $\mathbf{J}(\mathbf{u}) \approx 28.6199$, $\tilde{\mathbf{J}}(\mathbf{u}) \approx 57.2398$ are obtained from extrapolation of the potential values of the lowest order h-version for $p = 1$ and $q = 0$. In Figure4.1 we show the deformed configuration for the mean value of Young modulus E. To solve the nonlinear equation, we require more than 300 Uzawa iterations to achieve a tolerance of 10^{-10}.

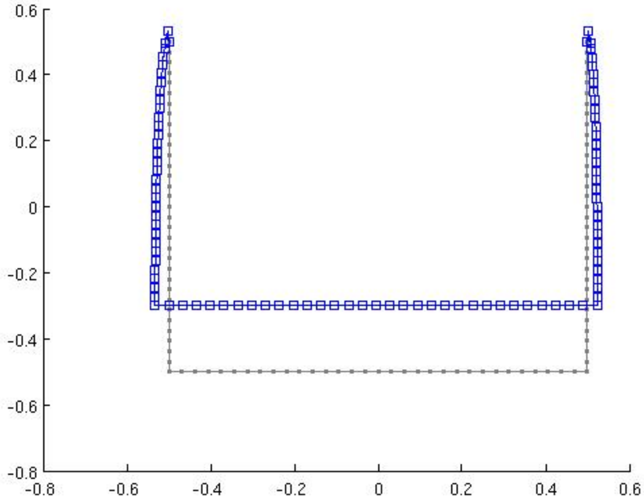


Figure 4.1: Deformed geometry with the mean value of Young modulus E

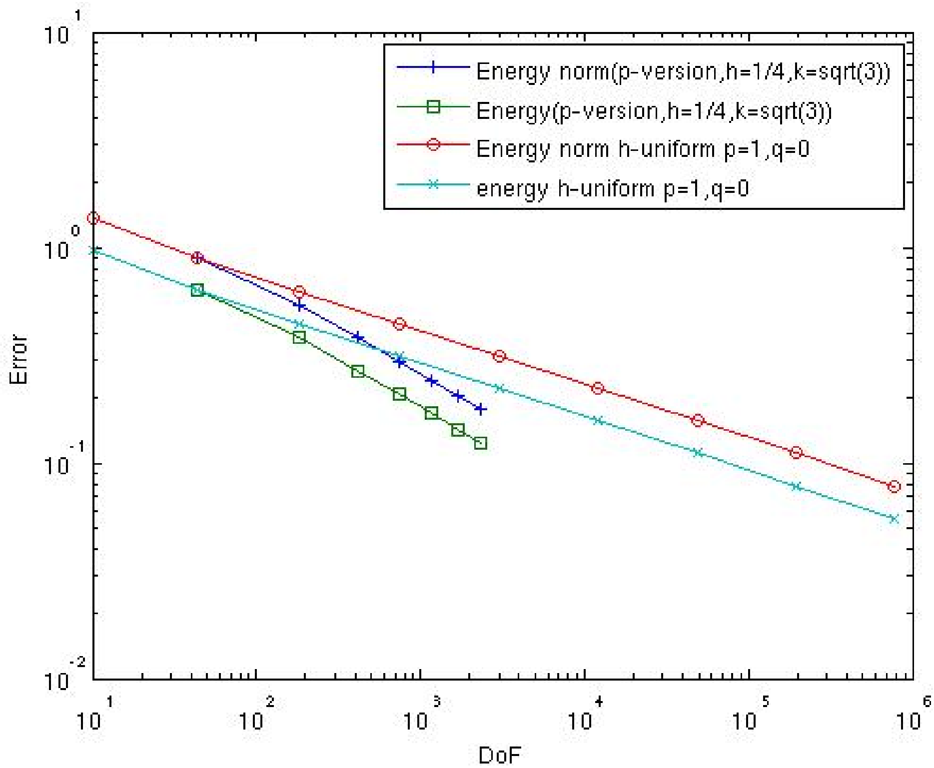


Figure 4.2: Convergence of Stochastic Contact problem

5 Extended Multiscale Finite Element Method in Linear Elasticity

In this chapter, an extended multiscale finite element method EMsFEM is derived for the analysis of linear elastic heterogenous materials. The main idea is to construct numerically a finite element basis functions that captures the small-scale information (the fine mesh) within each coarse element [26], [54]. The construction of the basis functions is done separately for each coarse element with a linear boundary condition. The boundary conditions for the construction of the multiscale basis functions have a big influence on capturing the smale-scale information. We analyse a corresponding FEM/BEM coupling and derive an a priori error and a-posteriori error estimate. Next we present finite element implementations for nonperiodic case.

5.1 The equation of linear elasticity

In this section we consider a two-dimensional plan strain deterministic problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$ in which every point is represented by cartesian coordinate $x = (x_1, x_2)^T$. We consider a solid body in Ω deformed under the influence of a volume force f and a tension force $\mathbf{t}$.

The displacement field $\mathbf{u}$ of the body is governed by the linear elasticity system:

$$-\operatorname{div}\sigma(\mathbf{u}) = f \qquad \text{in } \Omega \tag{5.1a}$$

$$\sigma(\mathbf{u}) = \mathcal{C} : \varepsilon(\mathbf{u}) \qquad \text{in } \Omega, \tag{5.1b}$$

where σ is the stress tensor, the strain tensor ε is given by the symmetric part of the deformation gradient

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \tag{5.2}$$

$\mathcal{C} = \mathcal{C}(x), x \in \Omega$ is the 4-th order elasticity tensor, it describes the elastic stiffness of the material under load.

The system given in equation (5.1) follows the boundary conditions

$$\mathbf{u} = g \qquad \text{on } \Gamma_D \tag{5.3a}$$

$$\sigma(u) \cdot \mathbf{n} = \mathbf{t} \qquad \text{on } \Gamma_N, \tag{5.3b}$$

where $\mathbf{n}$ is the unit outer normal vector on $\partial\Omega$.

The boundary $\Gamma = \partial\Omega$ be decomposed into two disjoint subsets Γ_D and Γ_N, such that $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$ and $\text{meas}(\Gamma_D) > 0$.

Definition 5.1. *Define the Sobolev space for vector fields in* $\mathbb{R}^2$ *by*

$$\boldsymbol{V} = \boldsymbol{H}^1(\Omega) = [H^1(\Omega)]^2$$

where $\mathbf{u}(x) \in \mathbf{H}^1(\Omega)$ means that $u_i(x) \in H^1(\Omega)$ for any $i = 1, 2$,

and

$$H^1(\Omega) = \{f \in L^2(\mathcal{D}) : \partial^2 f \in L^2(\Omega)\}$$

equiped with the norm

$$\|u\|_{H^1(\Omega)} = \left(\sum_{s \leq 1} \int_\Omega |\partial^2 u| dx \right)^{\frac{1}{2}}$$

where $s = (s_1, s_2) \in \mathbb{N}^2$ is a multiindex with $|s| = s_1 + s_2$.

We define the following continuous functions spaces

$$\mathbf{V}_0 = \{\mathbf{u} \in \mathbf{H}^1(\Omega); \mathbf{u} = (u_1, u_2) : \mathbf{u} = 0 \quad \text{on} \quad \Gamma_D\} \subset \mathbf{V}$$

we define the bilinear form

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \varepsilon(\mathbf{u}) : \mathcal{C} : \varepsilon(\mathbf{v}) \, dx. \tag{5.4}$$

This form is symmetric, continuous and coercive, the coercivity i.e

$$\exists \, C_0 > 0 \ : a(\mathbf{u}, \mathbf{v}) \geq C_0 \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \quad \forall \mathbf{u} \in \mathbf{V}$$

The variational formulation is given by

$$a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0, \tag{5.5}$$

where

$$F(\mathbf{v}) = \int_\Omega f \ \mathbf{v} \, dx + \int_{\Gamma_N} \mathbf{t} \ \mathbf{v} \, ds \tag{5.6}$$

5.2 The finite element discretization

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of $\Omega \subset \mathbb{R}^2$, with the mesh parameter h and let $\mathcal{N}_h$ be the set of vertices of $\mathcal{T}_h$ contained in $\overline{\Omega}$, we denote the number of grid points in $\mathcal{N}_h$ by n_p.

Let $\{\varphi_j\}_{j=1}^{n_p}$ be the system of piecewise linear basis functions on the triangulation $\mathcal{T}_h$ of Ω such that $\varphi_i(x_j) = \delta_{ij}\ \forall x_j \in \mathcal{N}_h$.

Space $\mathbf{V}_0$ is then replaced by their discrete approximation $\mathbf{V}_0^h$, the discrete solution u_h is given by:

$$\mathbf{u}_h = (u_{1h}, u_{2h}) \in \mathbf{V}_0^h$$

where

$$u_{\alpha h} = \sum_{j=1}^{n_p} u_\alpha^j\, \varphi_j \quad \text{and} \quad \mathbf{u}_h = \sum_{j=1}^{n_p} u_\alpha^j\, \varphi_j\, e^\alpha \quad \alpha = 1, 2 \tag{5.7}$$

Then we define the basis function

$$\phi_{j(\alpha)} = \varphi_j\, e^\alpha : \Omega \longrightarrow \mathbb{R}^2 \tag{5.8}$$

of $\mathbf{V}^h$ as a vector field with a scalar nodal function in one of their components and zero in the others.

We set

$$U = (u_1^1, u_2^1, u_1^2, u_2^2, \ldots, u_1^{n_p}, u_2^{n_p})$$

where u_α^j are nodal values of $\mathbf{u}_h$ i.e $u_{\alpha h}(x_j) = u_\alpha^j$.

In the two-dimentional problem $n_d = 2n_p$ denotes the total number of degrees of freedom of $\mathbf{V}_0^h$.

The discretization form is : Find $\mathbf{u}_h \in \mathbf{V}^h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}^h \tag{5.9}$$

where

$$a(\mathbf{u}_h, \mathbf{v}_h) = \int_\Omega \varepsilon(\mathbf{u}_h) : \mathcal{C} : \varepsilon(\mathbf{v}_h)\, dx \tag{5.10}$$

and

$$F(\mathbf{v}_h) = \int_\Omega f\ \mathbf{v}_h\, dx + \int_{\Gamma_N} \mathbf{t}\ \mathbf{v}_h\, ds \tag{5.11}$$

Integrals in (5.9) are computed as sums of integrals over all element T using the fact that $\overline{\Omega} = \cup_{T \in \mathcal{T}_h} T$

$$\sum_{T \in \mathcal{T}_h} \int_T \varepsilon(\mathbf{u}_h) : \mathcal{C} : \varepsilon(\mathbf{v}_h)\, dx = \sum_{T \in \mathcal{T}_h} \int_T f\ \mathbf{v}_h\, dx \sum_{E \in \Gamma_N} \mathbf{t}\ \mathbf{v}_h\, ds$$

We write

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

The equation (5.9) can be written as

$$\int_\Omega \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \end{pmatrix} \mathcal{C} \begin{pmatrix} \frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \end{pmatrix} = \int_\Omega (f_1, f_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \int_{\Gamma_N} (t_1, t_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The corresponding linear algebra equation serie is given by $AU = b$.

We introduice the matrix B_i related to a node x_i by

$$B_i = \begin{pmatrix} \frac{\partial \varphi_i}{\partial x_1} & 0 \\ 0 & \frac{\partial \varphi_i}{\partial x_2} \\ \frac{\partial \varphi_i}{\partial x_1} & \frac{\partial \varphi_i}{\partial x_2} \end{pmatrix}$$

The bilinear form in equation (5.9) applied to the basis function of $\mathbf{V}^h$ reads

$$a(\phi_{i(\alpha)}, \phi_{j(\beta)}) = \int_\Omega \varepsilon(\varphi_i e^\alpha)^\top \mathcal{C} \varepsilon(\varphi_j e^\beta)\, dx \tag{5.12}$$

$$= \int_\Omega \sum_{r,l=1}^{2} \mathcal{C}_{\alpha r \beta l} \partial_l \varphi_j \partial_r \varphi_i \, dx \tag{5.13}$$

and we define the stiffness matrix $A = (a_{ij})_{i,j=1}^{n_d = 2n_p} \in \mathbb{R}^{n_d \times n_d}$

It holds

$$\varepsilon(\phi_{i(\alpha)}) = \varepsilon(\varphi_i e^\alpha) = B_i e^\alpha$$

where

$$\phi_{i(\alpha)} = B_i e^\alpha$$

We can write

$$A_{i(\alpha)j(\beta)} = (e^\alpha)^\top \int_\Omega B_i^\top \, \mathcal{C} \, B_j \, dx \, e^\beta \quad \alpha, \beta = 1, 2 \tag{5.14}$$

For each element $T \in \mathcal{T}_h$ we define the stiffness matrix

$$A_T = \int_T B_T^\top \, \mathcal{C} \, B_T \, dx \tag{5.15}$$

where the matrix B_T contains the nodal matrices $B_{T_i}, \quad i = 1, 2, 3, 4$ corresponding to the 4 vertices of T

$$B_T = [B_{T_1}, B_{T_2}, B_{T_3}, B_{T_4}] \tag{5.16}$$

Elementary calculations provide on element T

$$B_T = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & 0 & \frac{\partial \varphi_2}{\partial x_1} & 0 & \frac{\partial \varphi_3}{\partial x_1} & 0 & \frac{\partial \varphi_4}{\partial x_1} & 0 \\ 0 & \frac{\partial \varphi_1}{\partial x_2} & 0 & \frac{\partial \varphi_2}{\partial x_2} & 0 & \frac{\partial \varphi_3}{\partial x_2} & 0 & \frac{\partial \varphi_4}{\partial x_2} \\ \frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_3}{\partial x_2} & \frac{\partial \varphi_3}{\partial x_1} & \frac{\partial \varphi_4}{\partial x_2} & \frac{\partial \varphi_4}{\partial x_1} \end{pmatrix} \in \mathbb{R}^{3 \times 8}$$

5.3 Extended multiscale finite element method for the analysis of linear elastic heterogeneous materials

We present an approach where, instead of approximating in $\mathbf{V}_H$, we use a better space of multiscale functions $\mathbf{V}_H^{Ms} \subset \mathbf{H}_0^1(\Omega)$. The multiscale finite element solution found from solving the finite element problem using $\mathbf{V}_H^{Ms}$.

The space $\mathbf{V}_H^{Ms}$ is the span of the set of multiscale basis fonction $\{\Phi_i^{Ms}\}$ which are defined for each node of a coarse mesh $\mathcal{T}_H(\Omega)$ .The idea of the method is to construct numerically the multiscale basis functions to capture the fine scale features of the coarse elements in the multiscale finite element analysis.

In this section, we also give the definitions of the multiscale basis and the multiscale coarse space .

We consider a two dimentional plan stain problem, let Ω be a bounded domaine in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega = \Gamma$ where Γ_D and Γ_N are the segments on which we prescribe homogeneous Dirichlet and Neumann boundary conditions and $\mathbf{u} = [u_x, u_y]^T$ ist the displacement field, $\mathbf{f}$ and $\mathbf{t}$ are the body forces and tractions respectively and $\mathcal{C}$ is the symmetric elasticity tensor and $\mathbf{n}$ is the outward normal on $\partial\Omega$, it is assumed that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{t} \in \mathbf{L}^2(\Gamma_N)$.

The weak formulation then consists in finding $\mathbf{u} \in \mathbf{V}$, where

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H}^1(\Omega) = [H^1(\Omega)]^2 : \mathbf{v}|_{\Gamma_D} = 0\}$$

such that

$$a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V} \tag{5.17}$$

where $a : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}$ and $F : \mathbf{V} \longrightarrow \mathbb{R}$ are bilinear and linear form on $\mathbf{V}$ defined as

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \sigma(\mathbf{u}) : \varepsilon(\mathbf{v})\, dx \quad \text{and} \quad F(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v}\, ds \tag{5.18}$$

The solution $\mathbf{u} \in \mathbf{V}$ is called the weak solution to the boundary value problem and Lax-Milgran lemma ensures its existence and uniqueness.

Let $\mathcal{T}_H$ be the coarse triangulation of Ω, here we assume again that each coarse element T consists of union of fine elements $\tau \in \mathcal{T}_h$ of the fine triangulation. The coarse elements $T \in \mathcal{T}_H$ are constructed by agglomeration of the fine elements, we construct a set of agglomerated elements $\{T\} = \mathcal{T}_H$ such that each $T = \cup_{\tau \in \mathcal{T}_h} \tau$ is a simply connected union of fine grid elements, let Σ_H the coarse grid points in $\overline{\Omega}$.

For each coarse node $x^p \in \Sigma_H$ we denote the k-th coarse degree of freedom for $k \in \{1, 2\}$ related to this node by $p(k)$, we denote $w_p = \{T \in \mathcal{T}_H : x^p \in T\}$ the union of quadrilaterals connected to the node x^p and $H_{w_p} = \text{diam}(w_p)$, we enummerate its four vertices by $p = 1, 2, 3, 4$ and for every node x^p on the Dirichlet boundary, we also denote $\Gamma_{p,D} = w_P \cap \Gamma_D$ and for every element T in $\mathcal{T}_H$ we denote by w_T the patch of quadrilaterals containing T that is $w_T = \{\cup T^{'} : T \cup T^{'} \neq \emptyset\} \cup T$ and set $H_{w_T} = \text{diam}(w_T)$, it holds that $|T|^{\frac{1}{2}} \sim H_T$, $|w_p|^{\frac{1}{2}} \sim H_{w_p}$ and $|w_T|^{\frac{1}{2}} \sim H_{w_T}$.

The shape regularity of the mesh $\mathcal{T}_H$ imposed, $H_E = |E| \sim H_T$ for every edge E of T, whereas the local quasi-uniformity implies in particularity that $H_T \sim H_{w_p} \sim H_{w_T}$.

5.3.1 The construction of the basis functions

We construct a vector valued multiscale basis function $\Phi^{Ms}_{p(k)} : w_p \longrightarrow \mathbb{R}^2$ where $\Phi^{Ms}_{p(k)}$ is the basis associed with the node p and supported on w_p. The construction is done separatly for each element $T \in \mathcal{T}_H$, $\Phi^{Ms}_{p(k)}$ is used for the displacement in the direction k.

We denote the scalar coarse nodal basis function corresponding to x^p by $\phi^{lin}_p : w_p \longrightarrow \mathbb{R}$, ϕ^{lin}_p is linear in T and $\phi^{lin}_p(x^q) = \delta_{pq}$, with $x^q \in \Sigma_H$ the basis functions $\Phi^{Ms}_{p(k)}$ whose restriction $\Phi^{Ms}_{p(k),T}$ to T must solve the following subgrid problem:

Find $\Phi^{Ms}_{p(k),T} \in \mathbf{H}^1(T)$ such that

$$a_T(\Phi^{Ms}_{p(k)}, \mathbf{v}) = 0 \quad \text{for all} \quad \mathbf{v} \in \mathbf{H}^1(T), \quad T \subset w_p, \tag{5.19}$$

subject to a suitable boundary condition

$$\Phi^{Ms}_{p(k),T} = \phi^{lin}_{p,T} \cdot e^k \quad \text{on} \quad \partial T; \quad T \subset w_p, \tag{5.20}$$

where

$$\sum_{x^p \in \Sigma_H(T)} \Phi^{Ms}_{p(k)} = 1_{\overline{\Omega}} e^k \quad \text{on} \quad \partial T. \tag{5.21}$$

On ∂T, linear boundary conditions are imposed in the k-th component of the vector-field and zero boundary conditions in the $\{1,2\} \setminus \{k\}$.

The local boundary conditions will be constructed so that they are continuous across element edges, that is

$$\Phi^{Ms}_{p(k),T}(x) = \Phi^{Ms}_{p(k),T'}(x) = \phi^{lin}_{p,T} \cdot e^k$$

for $x \in \partial T \cap \partial T^{'}$. For the two-dimensional problem two kinds of basis functions are constructed one is used for the x-axis direction and the other is used for the y-axis direction. Firstly, let us consider the construction of the basis function $\Phi_{p_1}(x) = \Phi^x_{p_1 x}$ on node 1 of the coarse element Figure 5.1 (left). The displacement at all boundary nodes are not constraint in y-axis direction except node 3, for which the displacement are fixed to zero in both coordinate directions in order to avoid rigid displacement, a unit displacement is applied on node 1 in the positive x-direction Figure 5.1 (left). The displacement at nodes $2, 3$ and 4 are fixed to zero in x-direction, the values vary linearly

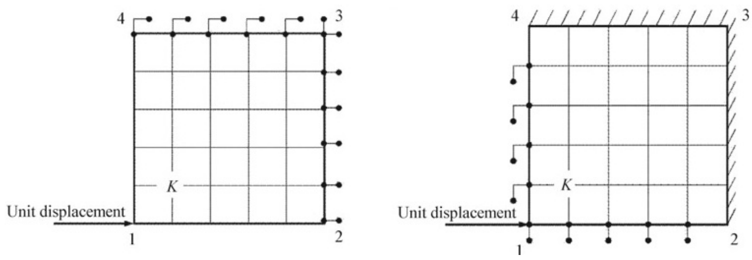


Figure 5.1: [54]The construction of the numerical basis functions

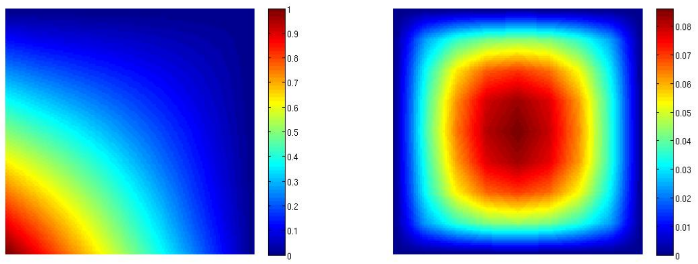


Figure 5.2: Displacement field of T for the basis function $\Phi^x_{p_1x}$ (left) and $\Phi^x_{p_1y}$ (right)

along each side. We ignore the displacement values in y-axis direction and remaining displacement values in x-axis direction to establish the basis function $\Phi_{p_1}(x) = \Phi^x_{p_1x}$ The construction of $\Phi_{p_1}(y) = \Phi^y_{p_1y}$ is similar to that of $\Phi_{p_1}(x)$, we have obtained the basis functions $\Phi_{p_1}(x)$ and $\Phi_{p_1}(y)$ Figure 5.2, the rest of the basis functions of the coarse element can be constructed in the similar way.

A modification of the construction of the basis functions is developed in [55] and used for small deformation elasto-plastic analysis of periodic truss materials. So that the multiscale basis functions for the displacement fields can be constructed in a more accurate way. The most key point is that the additional coupling terms of the basis functions are introduced in the improved construction method. For the element T Figure 5.1 (right), the displacements at all boundary nodes are constraint in y direction, at the same time the nodes on boundary 34 and boundary 23 are constraint in x-direction, a unit displacement is applied on node 1 in the positive x-direction and the values vary linearly along boundaries 12 and 14,the internal displacement field of the element can be obtained directly by standard finite element analysis in fine scale mesh and the basis functions $\Phi^x_{p_1}(x)$ and $\Phi^x_{p_1}(y)$ can be obtained, thus the basis functions $\Phi^x_{p_1} = \{\Phi^x_{p_1x}, \Phi^x_{p_1y}\}$.

Here $\Phi^x_{p_iy}$ is a coupled additional term and means that the displacement field in y-direction within the element induced by unit displacement of node i in the x-direction.

The rest of the basis functions can be constructed in the similar way and the final basis functions ie.$\Phi^x_{p_ix}, \Phi^x_{p_iy}, \Phi^y_{p_iy}$ and $\Phi^y_{p_ix}$ for $i = 1, 2, 3, 4$ obtained by this way

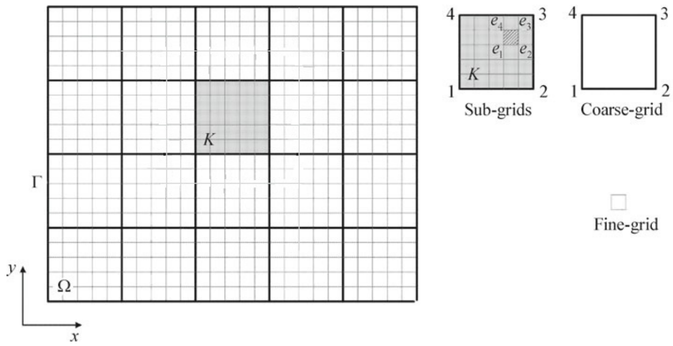


Figure 5.3: [54] Schematic description of the EMsFEM

5.3.2 Stiffness matrix on the coarse mesh

The displacement vector of the nodes in the fine scale omits the representation

$$\mathbf{u}_h = \mathbf{R}\mathbf{u}_H \tag{5.22}$$

where $\mathbf{u}_H$ is the displacement vector of nodes in the coarse mesh and $\mathbf{R}$ is the basis function matrix which contains the coefficient vectors, representing a coarse basis function in terms of the fine scale basis .

$$\mathbf{R} = [\mathbf{R}_1^T \mathbf{R}_2^T \cdots \mathbf{R}_n^T]^T \tag{5.23}$$

$$\mathbf{R}_i = \begin{bmatrix} \Phi^x_{p_1x} & \Phi^x_{p_1y} & \Phi^x_{p_2x} & \Phi^x_{p_2y} & \Phi^x_{p_3x} & \Phi^x_{p_3y} & \Phi^x_{p_iy} & \Phi^x_{p_iy} \\ \Phi^y_{p_1x} & \Phi^y_{p_1y} & \Phi^y_{p_2x} & \Phi^y_{p_2y} & \Phi^y_{p_3x} & \Phi^y_{p_3y} & \Phi^y_{p_iy} & \Phi^y_{p_iy} \end{bmatrix} \tag{5.24}$$

$$i = 1, 2 \ldots, n$$

where n is the total number of the the fine scale mesh within the sub-grids, using (5.22) and (5.24), we have

$$\mathbf{u}_h^e = \mathbf{G}_e \mathbf{u}_H^E \tag{5.25}$$

where

$$\mathbf{G}_e = \begin{bmatrix} \mathbf{R}_{e_1} \\ \mathbf{R}_{e_2} \\ \mathbf{R}_{e_3} \\ \mathbf{R}_{e_4} \end{bmatrix} \tag{5.26}$$

$$\tag{5.27}$$

is the transformation matrix between the displacement vectors of micro-scale nodes and macro-scale nodes and e is an arbitrary fine-scale element within the coarse element show Figure 5.3.

The stiffness matrix of the coarse element is given by

$$\mathbf{K}_E^H = \sum_{e=1}^{N} \tilde{\mathbf{K}}_e, \quad \tilde{\mathbf{K}}_e = \mathbf{G}_e^T \mathbf{K}_e \mathbf{G}_e \tag{5.28}$$

where $\mathbf{K}_e$ is the element stiffness matrix.

The global stiffness matrix on the coarse mesh is obtained as follows

$$\mathbf{K} = \mathbf{A}_{i=1}^{M} \mathbf{K}_E^H(i) \tag{5.29}$$

where $\mathbf{A}_{i=1}^{M}$ is a matrix assembled operator and M is the total number of the coarse element.

The corresponding multiscale coarse space is conforming , we define the coarse space

$$\mathbf{V}_H^{Ms} := \{\Phi_{p(k)}^{Ms}, x^p \in \bar{\Sigma}_H, k = 1, 2\} \subseteq \mathbf{H}_0^1(\Omega) \tag{5.30}$$

Using the defined space of multiscale functions $\mathbf{V}_H^{Ms} \subseteq \mathbf{H}_0^1(\Omega)$ in the finite element problem gives a multiscale finite element approximation $\mathbf{u}_H^{Ms}$ which satisfies for all $\mathbf{v}_H^{Ms} \in \mathbf{V}_H^{Ms}$

$$\begin{aligned} a_\Omega(\mathbf{u}_H^{Ms}, \mathbf{v}_H^{Ms}) &= \int_\Omega \sigma(\mathbf{u}_H^{Ms}) : \varepsilon(\mathbf{v}_H^{Ms})\, dx \\ &= \int_\Omega F \cdot \mathbf{v}_H^{Ms}\, dx \end{aligned} \tag{5.31}$$

It holds that

$$\int_\Omega \sigma(\mathbf{u}_H^{Ms}) : \varepsilon(\mathbf{v}_H^{Ms})\, dx = \sum_{T \in \mathcal{T}_H} \int_T \sigma(\mathbf{u}_H^{Ms}) : \varepsilon(\mathbf{v}_H^{Ms})\, dx$$

5.4 Convergence of the multiscale finite element method

In this section, the convergence of the MsFEM is presented. The MsFEM is defined for non-periodic problem but the convergence analysis is made in the periodic case. As in [26],[39], we restrict ourselves to a periodic case.

We consider the following elasticity problem

$$\begin{aligned} L_\epsilon \mathbf{u}_\epsilon &= f \quad \text{in} \quad \Omega \\ \mathbf{u}_\epsilon &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{5.32}$$

L_ϵ is the elasticity operator, $L_\epsilon = \mathrm{div}(\mathcal{C}(\frac{x}{\epsilon}) : \varepsilon(\cdot))$, and $\mathcal{C}(x) = (\mathcal{C}_{ijkl}(\frac{x}{\epsilon}))$ is the fourth order elasticity tensor, which satisfies symmetry and positiv definiteness. There exists $\alpha > 0$ such that

$$\mathcal{C}_{ijkl}(y)\varepsilon_{ij}\varepsilon_{kl} \geq \alpha\varepsilon_{ij}\varepsilon_{ij} \quad \forall\, y = \frac{x}{\epsilon}, \quad \forall\, \varepsilon_{ij} \text{ symmetric}$$

The multiscale basis functions satisfy

$$L_\epsilon \Phi_{j(k)}^{Ms} = 0 \quad \text{in} \quad T \in \mathcal{T}_H.$$

Recall that H is the coarse mesh size.

We refer to [39] for the convergence analysis of MsFEM, where the convergence for $H < \epsilon$ and $H > \epsilon$ cases are presented for a Dirichlet problem.

The convergence rate of MsFEM contains a term $\frac{\epsilon}{H}$ (see [39]), the error becomes large when the two scale are close, when $H \approx \epsilon$ the multiscale method attains a large error in $\mathbf{H}^1$ and $\mathbf{L}^2$ norms (see [26]).

The multiscale basis functions are smooth if $H < \epsilon$ and can be well approximated by the standard continuous linear (bilinear) basis functions, we apply the traditional finite element method analysis to the multiscale method. When $H > \epsilon$ the multiscale basis functions contains a smooth part and an oscillatory part, which cannot be approximated by linear (bilinear) functions, the MsFEM gives a convergence result uniform in ϵ as ϵ tends to zero, while the traditional FEM with piecewise polynomial basis functions does not.

Theorem 5.1. *[39](Convergence for $H < \epsilon$) Let $\boldsymbol{u}$ and $\boldsymbol{u}_H^{Ms}$ be the solutions of* (5.32) *and* (5.31)*, respectively. Then there exists a constant C, independent of H and ϵ, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_H^{Ms}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\frac{H}{\epsilon})\|f\|_{\boldsymbol{L}^2(\Omega)} \tag{5.33}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_H^{Ms}\|_{\boldsymbol{L}^2(\Omega)} \leq C(\frac{H}{\epsilon})^2\|f\|_{\boldsymbol{L}^2(\Omega)} \tag{5.34}$$

Remark 5.1. *Details of the proof for a Dirichlet problem can be found in [26],[39]. The proof for the transmission problem similarly follows from a convergence analysis of the standard finite element / boundary element coupling, using from [26],[39] that the multiscale basis functions do not differ significantly from piecewise linear hat functions.*

Remark 5.2. *For $H < \epsilon$ the multiscale method gives the same rate of convergence as the linear finite element method , but this estimate is insufficient for practical applications, the estimates* (5.33) *and* (5.34) *blows up as $\epsilon \to 0$.*

Let $\mathcal{I}_H$ be the standard interpolation operator , and $\mathcal{I}_H^{Ms} : C(\overline{\Omega}) \longrightarrow V_H^{Ms}$ be the nodal interpolation operator defined in [20] by

$$\mathcal{I}_H^{Ms}\mathbf{u} = \sum_j \mathbf{u}(x_j)\Phi_j^{Ms} \tag{5.35}$$

From the definition of the multiscale basis functions, we have

$$L_\epsilon(\mathcal{I}_H^{Ms}\mathbf{u}) = 0 \quad \text{in} \quad T, \quad \mathcal{I}_H^{Ms}\mathbf{u} = \mathcal{I}_H\mathbf{u} \quad \text{on} \quad \partial T \tag{5.36}$$

From the approximation theory, we have

$$\|\mathbf{u} - \mathcal{I}_H\mathbf{u}\|_{\mathbf{L}^2(T)} \leq C_1 H|\mathbf{u}|_{\mathbf{H}^1(T)} \tag{5.37}$$

$$|\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} \leq C_2|\mathbf{u}|_{\mathbf{H}^1(T)} \tag{5.38}$$

Lemma 5.1. *Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ be the solution of* (5.32). *There exist constants C_1 and $C_2 > 0$ independent of H, such that*

$$\|\boldsymbol{u} - \mathcal{I}_H^{Ms}\boldsymbol{u}\|_{\boldsymbol{L}^2(T)} = C_1(H|\boldsymbol{u}|_{\boldsymbol{H}^1(T)} + H^2\|f\|_{\boldsymbol{L}^2(T)}) \tag{5.39}$$

$$|\boldsymbol{u} - \mathcal{I}_H^{Ms}\boldsymbol{u}|_{\boldsymbol{H}^1(T)} = C_2(|\boldsymbol{u}|_{\boldsymbol{H}^1(T)} + H\|f\|_{\boldsymbol{L}^2(T)}) \tag{5.40}$$

Proof. We refer to [26],[39], since $\mathcal{I}_H^{Ms}\mathbf{u} = \mathcal{I}_H\mathbf{u}$ on ∂T, we have $\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u} \in \mathbf{H}_0^1(T)$, and by the Poincaré-Friedrichs inequality we obtain

$$\|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}\|_{\mathbf{L}^2(T)} \leq CH|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)}$$

From (5.36), we have

$$\int_T \mathcal{C}(\frac{x}{\epsilon})\varepsilon(\mathcal{I}_H^{Ms}\mathbf{u}^{MS}) : \varepsilon(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u})\,dx = 0, \tag{5.41}$$

and similar to the proof of Lemma 6.3 in [26] we obtain

$$\begin{aligned}
|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|^2_{\mathbf{H}^1(T)} &= \int_T \mathcal{C}(\frac{x}{\epsilon})\varepsilon(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}) : \varepsilon(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u})\,dx \\
&= \int_T \mathcal{C}(\frac{x}{\epsilon})\varepsilon(\mathbf{u} - \mathcal{I}_H\mathbf{u}) : \varepsilon(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u})\,dx - \int_T \mathcal{C}(\frac{x}{\epsilon})\varepsilon(\mathbf{u}) : \varepsilon(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u})\,dx \\
&= \int_T \mathcal{C}(\frac{x}{\epsilon})\varepsilon(\mathbf{u} - \mathcal{I}_H\mathbf{u}) : \varepsilon(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u})\,dx - \int_T \mathcal{C}(\frac{x}{\epsilon})f(\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u})\,dx \\
&\leq C|\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)}|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} + \|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}\|_{\mathbf{L}^2(T)}\|f\|_{\mathbf{L}^2(T)} \\
&\leq C|\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)}|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} + H|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)}\|f\|_{\mathbf{L}^2(T)} \\
&\leq C|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)}(|\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} + H\|f\|_{\mathbf{L}^2(T)})
\end{aligned} \tag{5.42}$$

We obtain

$$|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} \leq C(|\mathbf{u}|_{\mathbf{H}^1(T)} + H\|f\|_{\mathbf{L}^2(T)}) \tag{5.43}$$

$$\|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}\|_{\mathbf{L}^2(T)} \leq C(H|\mathbf{u}|_{\mathbf{H}^1(T)} + H^2\|f\|_{\mathbf{L}^2(T)}) \tag{5.44}$$

Using (5.37), (5.38) ,(5.43) and (5.44), we get

$$\begin{aligned}
|\mathbf{u} - \mathcal{I}_H^{Ms}\mathbf{u}|_{\mathbf{H}^1(T)} &\leq |\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} + |\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}|_{\mathbf{H}^1(T)} \\
&\leq C_2(|\mathbf{u}|_{\mathbf{H}^1(T)} + H\|f\|_{\mathbf{L}^2(T)}) \\
\|\mathbf{u} - \mathcal{I}_H^{Ms}\mathbf{u}\|_{\mathbf{L}^2(T)} &\leq \|\mathbf{u} - \mathcal{I}_H\mathbf{u}\|_{\mathbf{L}^2(T)} + \|\mathcal{I}_H^{Ms}\mathbf{u} - \mathcal{I}_H\mathbf{u}\|_{\mathbf{L}^2(T)} \\
&\leq C_1(H|\mathbf{u}|_{\mathbf{H}^1(T)} + H^2\|f\|_{\mathbf{L}^2(T)})
\end{aligned}$$

□

Now we discuss the convergence analysis for $H > \epsilon$ by exploring the asymptotic behavior of both $\mathbf{u}$ and Φ_j^{Ms}.

We consider the expansion of $\mathbf{u}$:

$$\mathbf{u} = \mathbf{u}^0 + \epsilon\mathbf{u}^1 + \dots \tag{5.45}$$

where $\mathbf{u}^0$ is the solution of the homogenized equation For more details (see [26],[39]).

Let $\mathbf{u}_I$ be the interpolant of $\mathbf{u}^0$, using the multiscale basis functions Φ_j^{Ms}, note that $\mathbf{u}_I$ is different from the definition of $\mathcal{I}_H^{Ms}\mathbf{u}$. In the literature the following homogenization estimates are only available for the Dirichlet [39] and Neumann [49] problems, not the transmission problem.

Lemma 5.2. *Let $\boldsymbol{u}$ be the solution of* (5.32) *and $\boldsymbol{u}_I$ the interpolant of the homogenized solution $\boldsymbol{u}^0$, using the multiscale basis functions Φ_j^{Ms}. Then there exist constants C_1 and C_2, independent of ϵ and H, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\boldsymbol{H}^1(\Omega)} \leq C_1 H\|f\|_{\boldsymbol{L}^2(\Omega)} + C_2\left(\frac{\epsilon}{H}\right)^{\frac{1}{2}} \tag{5.46}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\boldsymbol{L}^2(\Omega)} \leq C_1 H^2\|f\|_{\boldsymbol{L}^2(\Omega)} + C_2\left(\frac{\epsilon}{H}\right)^{\frac{1}{2}} \tag{5.47}$$

Theorem 5.2. *(Convergence for $H > \epsilon$) Let $\boldsymbol{u}$ and $\boldsymbol{u}_H^{Ms}$ be the solutions of* (5.32) *and* (5.31)*, respectively. Then there exist constants C_1 and C_2, independent of ϵ and H, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_H^{Ms}\|_{\boldsymbol{H}^1(\Omega)} \leq C_1 H\|f\|_{\boldsymbol{L}^2(\Omega)} + C_2\left(\frac{\epsilon}{H}\right)^{\frac{1}{2}} \tag{5.48}$$

(5.49)

It is shown in [39] that

$$\|\mathbf{u} - \mathbf{u}_H^{Ms}\|_{\mathbf{L}^2(\Omega)} = O\left(H^2 + \frac{\epsilon}{H}\right) \tag{5.50}$$

Remark 5.3. *While we expect these results to hold for the transmission problem, an analysis of the homogenized transmission problem has not been reported. We leave the necessary analysis as an open problem.*

Remark 5.4. *For $H > \epsilon$ the multiscale method converges to the homogenized solution $\boldsymbol{u}^0$ in the limit as $\epsilon \to 0$.*

5.5 A posteriori error for multiscale finite element method

We consider the problem

$$L_\epsilon \mathbf{u}_\epsilon = f \quad \text{in} \quad \Omega \tag{5.51}$$

Lemma 5.3. *Let $\boldsymbol{u}$ be the solution of* (5.51) *and $\boldsymbol{u}_H^{Ms}$ be the solution of the discrete problem* (5.31)*. Then there holds the estimate:*

$$\begin{aligned} a(e_H^{Ms}, \boldsymbol{v}) = & \sum_{T\in\mathcal{T}_H} \int_T f(\boldsymbol{v} - \boldsymbol{v}_H^{Ms})\, dx \\ & + \sum_{T\in\mathcal{T}_H} \sum_{l=1}^{4} \int_{E_l\subset\partial T} [\sigma(\boldsymbol{u}_H^{Ms})\cdot \boldsymbol{n}]_{E_l}(\boldsymbol{v} - \boldsymbol{v}_H^{Ms})\, ds \end{aligned} \tag{5.52}$$

where $e_H^{Ms} = \boldsymbol{u} - \boldsymbol{u}_H^{Ms}$.

Proof. We consider the quantity $e_H^{Ms} = \mathbf{u} - \mathbf{u}_H^{Ms}$, called multiscale discretization error, in the following we derive and analyze an upper bound for the quantity

$$\|e_H^{Ms}\|_\Omega := \sqrt{a(e_H^{Ms}, e_H^{Ms})} \tag{5.53}$$

measured in the energy norm.

Given the multiscale solution $\mathbf{u}_H^{Ms} \in \mathbf{V}_H^{Ms}$ of the discrete problem (5.31) , we obtain the so called error residual equation

$$a(e_H^{Ms}, \mathbf{v}) = a(\mathbf{u} - \mathbf{u}_H^{Ms}, \mathbf{v}) = F(\mathbf{v}) - a(\mathbf{u}_H^{Ms}, \mathbf{v}) \tag{5.54}$$

Recalling to

$$a_T(\Phi_{p(k)}^{Ms}, \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(T), \quad T \subset w_p \tag{5.55}$$

and

$$\mathbf{V}_H^{Ms} := \{\Phi_{p(k)}^{Ms}, x^p \in \bar{\Sigma}_H, k = 1, 2\} \subseteq \mathbf{H}_0^1(\Omega)$$

we obtain

$$a_T(\mathbf{u}_H^{Ms}, \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(T) \tag{5.56}$$

Let $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_H^{Ms} \in \mathbf{V}_H^{Ms}$, using this we rewrite the discrete problem (5.31) as follows

$$0 = F(\mathbf{v}_H^{Ms}) - a(\mathbf{u}_H^{Ms}, \mathbf{v}_H^{Ms}) \tag{5.57}$$

with the discretization error $e_H^{Ms} = \mathbf{u} - \mathbf{u}_H^{Ms}$ the Galerkin-method for finite sub-domains yields the important Galerkin orthogonality

$$a(e_H^{Ms}, \mathbf{v}_H^{Ms}) = 0 \quad \forall \mathbf{v}_H^{Ms} \in \mathbf{V}_H^{Ms} \tag{5.58}$$

Subtracting (5.54) from (5.57) we obtain

$$\begin{aligned} a(e_H^{Ms}, \mathbf{v}) = & \sum_{T \in \mathcal{T}_H} \left(\int_T f(\mathbf{v} - \mathbf{v}_H^{Ms})\, dx + \int_{\Gamma_N \cap \partial T} \mathbf{t}(\mathbf{v} - \mathbf{v}_H^{Ms})\, ds \right) \\ & - \sum_{T \in \mathcal{T}_H} \int_T \sigma(\mathbf{u}_H^{Ms}) : \varepsilon(\mathbf{v} - \mathbf{v}_H^{Ms})\, dx \quad \forall \quad \mathbf{v} \in \mathbf{V} \end{aligned} \tag{5.59}$$

integration by parts of the term $\int_T \sigma(\mathbf{u}_H^{Ms}) : \varepsilon(\mathbf{v} - \mathbf{v}^{Ms})\, dx$ in (5.59) we obtain

$$\begin{aligned} a(e_H^{Ms}, \mathbf{v}) = & \sum_{T \in \mathcal{T}_H} \left(\int_T f(\mathbf{v} - \mathbf{v}_H^{Ms})\, dx + \int_{\Gamma_N \cap \partial T} \mathbf{t}(\mathbf{v} - \mathbf{v}_H^{Ms})\, ds \right) \\ & + \sum_{T \in \mathcal{T}_H} \left(\int_T \operatorname{div} \sigma(\mathbf{u}_H^{Ms})(\mathbf{v} - \mathbf{v}_H^{Ms})\, dx - \oint_{\partial T} \sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_H^{Ms})\, ds \right) \end{aligned} \tag{5.60}$$

For fixed $T \in \mathcal{T}_H$ let E_l be an edge of T with $l = 1, 2, 3, 4$, we introduce the jump of tractions at the element boundaries

$$[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]_{E_l} := \begin{cases} \frac{1}{2} \left(\sigma^+(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}_{E_l}^+ + \sigma^-(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}_{E_l}^- \right) & \text{if} \quad E_l \subset \Gamma_T = \partial T \\ \mathbf{t} - \sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}_{E_l} & \text{if} \quad E_l \subset \Gamma_{T,N} = \partial T \cap \Gamma_N \\ 0 & \text{if} \quad E_l \subset \Gamma_{T,D} = \partial T \cap \Gamma_D \end{cases} \tag{5.61}$$

here $\mathbf{n}_{E_l}$ is the normal on E_l

Recall that by contruction of the multiscale basis functions, we have $\operatorname{div}\sigma(\mathbf{u}_H^{Ms}) = 0$ in T, we can now regroup the boundary terms in(5.60) for every fixed T so that (5.60) becomes

$$\begin{aligned} a(e_H^{Ms}, \mathbf{v}) = & \sum_{T \in \mathcal{T}_H} \int_T f(\mathbf{v} - \mathbf{v}_H^{Ms})\, dx \\ & + \sum_{T \in \mathcal{T}_H} \sum_{l=1}^{4} \int_{E_l \subset \partial T} [\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]_{E_l} (\mathbf{v} - \mathbf{v}_H^{Ms})\, ds \end{aligned} \tag{5.62}$$

□

In the following, we derive a-posteriori error estimates for $H < \epsilon$ and $H > \epsilon$.

Theorem 5.3. *(A posteriori error estimate for $H < \epsilon$)*

Let $\boldsymbol{u}$ be the solution of (5.32) *and $\boldsymbol{u}_H^{Ms}$ be the solution of the discrete problem* (5.31)*. Then there holds the estimate:*

$$a(e_H^{Ms}, e_H^{Ms}) \lesssim \sum_{T_h \subseteq \Omega} \eta_{T_h}^2 + \sum_{T_H \subseteq \Omega} \eta_{T_H}^2 + \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta_{E_h}^2 + \sum_{E_H \subseteq \mathring{\Omega}} \eta_{E_H}^2 \tag{5.63}$$

where

$$\begin{aligned}
\sum_{T_h \subseteq \Omega} \eta_{T_h}^2 &= \sum_{T_h \subseteq \Omega} H^2(T_h) \int_{T_h} |f - \overline{f}|^2 \, dx \\
\sum_{T_H \subseteq \Omega} \eta_{T_H}^2 &= \sum_{T_H \subseteq \Omega} H^4 |\overline{f}|^2 \\
\sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta_{E_h}^2 &= \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} H(E_h) \int_{E_h} ([\sigma(\boldsymbol{u}_H^{Ms}) \cdot \boldsymbol{n}] - \overline{[\sigma(\boldsymbol{u}_H^{Ms}) \cdot \boldsymbol{n}]})^2 \\
\sum_{E_H \subseteq \mathring{\Omega}} \eta_{E_H}^2 &= \sum_{E_H \subseteq \mathring{\Omega}} H^2 (\overline{[\sigma(\boldsymbol{u}_H^{Ms}) \cdot \boldsymbol{n}]})^2 \\
\overline{f}(x) &= \frac{1}{|T_H|} \int_{T_H} f \, dx, \quad \overline{[\sigma(\boldsymbol{u}_H^{Ms}) \cdot \boldsymbol{n}]}(x) = \frac{1}{|E_H|} \int_{E_H} [\sigma(\boldsymbol{u}_H^{Ms}) \cdot \boldsymbol{n}] \, dx
\end{aligned}$$

Proof. From Lemma 5.3, we have

$$\begin{aligned}
a(e_H^{Ms}, \mathbf{v}) = &\sum_{T \in \mathcal{T}_H} \int_T f(\mathbf{v} - \mathbf{v}_H^{Ms}) \, dx \\
&+ \sum_{T \in \mathcal{T}_H} \sum_{l=1}^{4} \int_{E_l \subset \partial T} [\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]_{E_l} (\mathbf{v} - \mathbf{v}_H^{Ms}) \, ds
\end{aligned} \tag{5.64}$$

First, we estimate the term

$$A_1 := \sum_{T \in \mathcal{T}_H} \int_T f(\mathbf{v} - \mathbf{v}_H^{Ms}) \, dx \tag{5.65}$$

Let $\overline{f}(x) = \frac{1}{|T_H|} \int_{T_H} f$, if $x \in T_H$ then

$$\int_\Omega |f|^2 = \int_\Omega (f - \overline{f} + \overline{f})^2 = \int_\Omega (f - \overline{f})^2 + 2 \sum_{T_H \in \mathcal{T}_H} \overline{f} \int_{T_H} (f - \overline{f}) + \int_\Omega (\overline{f})^2$$

However $\int_{T_H} (f - \overline{f}) = 0$

$$\int_\Omega |f|^2 = \int_\Omega (f - \overline{f})^2 + \int_\Omega (\overline{f})^2 \tag{5.66}$$

Using (5.39), choosing $\mathbf{v} = e_H^{Ms}$ and $\mathbf{v}_H^{Ms} = \mathcal{I}_H^{Ms} e_H^{Ms}$, we obtain

$$\begin{aligned} A_1 &\leq \sum_{T_H \subseteq \Omega} \|f\|_{\mathbf{L}^2(T_H)} \|e_H^{Ms} - \mathcal{I}_H^{Ms} e_H^{Ms}\|_{\mathbf{L}^2(T_H)} \\ &\leq C \sum_{T_H \subseteq \Omega} \|f\|_{\mathbf{L}^2(T_H)} (H |e_H^{Ms}|_{\mathbf{H}^1(T_H)} + H^2 \|f\|_{\mathbf{L}^2(T_H)}) \\ &\leq C(\sum_{T_H \subseteq \Omega} H^2 \|f\|^2_{\mathbf{L}^2(T_H)} + \sum_{T_H \subseteq \Omega} H \|f\|_{\mathbf{L}^2(T_H)} |e_H^{Ms}|_{\mathbf{H}^1(T_H)} \\ &\leq C' \sum_{T_H \subseteq \Omega} H^2 \|f\|^2_{\mathbf{L}^2(T_H)} + \frac{\epsilon_1}{2} |e_H^{Ms}|^2_{\mathbf{H}^1(T_H)} \end{aligned} \tag{5.67}$$

Now we estimate the term $\sum_{T_H \subseteq \Omega} H^2 \|f\|^2_{\mathbf{L}^2(T_H)}$, employing (5.66), we get

$$\begin{aligned} \sum_{T_H \subseteq \Omega} H^2 \|f\|^2_{\mathbf{L}^2(T_H)} &= \sum_{T_H \subseteq \Omega} H^2 (\sum_{T_h \subseteq T_H} \int_{T_h} |f|^2 \, dx) \\ &= \sum_{T_H \subseteq \Omega} H^2 \{ \sum_{T_h \subseteq T_H} \int_{T_h} |f - \overline{f}|^2 \, dx + \sum_{T_h \subseteq T_H} \int_{T_h} |\overline{f}|^2 \, dx \} \\ &= \sum_{T_H \subseteq \Omega} H^2 \{ \sum_{T_h \subseteq T_H} \int_{T_h} |f - \overline{f}|^2 \, dx + \sum_{T_h \subseteq T_H} |T_h| |\overline{f}|^2 \} \\ &= \sum_{T_H \subseteq \Omega} H^2 \{ \sum_{T_h \subseteq T_H} \int_{T_h} |f - \overline{f}|^2 \, dx + |T_H| |\overline{f}|^2 \} \\ &= \sum_{T_h \subseteq \Omega} H^2(T_h) \int_{T_h} |f - \overline{f}|^2 \, dx + \sum_{T_H \subseteq \Omega} H^4 |\overline{f}|^2 \\ &= \sum_{T_h \subseteq \Omega} \eta^2_{T_h} + \sum_{T_H \subseteq \Omega} \eta^2_{T_H}. \end{aligned} \tag{5.68}$$

Here $\overline{f}|_{T_H}$ is a constant on T_H, and $|T_H| \sim H^2$.

We obtain

$$A_1 \lesssim \sum_{T_h \subseteq \Omega} \eta^2_{T_h} + \sum_{T_H \subseteq \Omega} \eta^2_{T_H} + \frac{\epsilon_1}{2} |e_H^{Ms}|^2_{\mathbf{H}^1(T_H)} \tag{5.69}$$

Now we estimate the term

$$A_2 := \sum_{E_H \subseteq \mathring{\Omega}} \int_{E_H} [\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]_{E_H} (\mathbf{v} - \mathbf{v}_H^{Ms}) \, ds. \tag{5.70}$$

where E_H is an interior edge of T.

Choosing $\mathbf{v} = e_H^{Ms}$ and $\mathbf{v}_H^{Ms} = \mathcal{I}_H^{Ms} e_H^{MS}$, since $\mathcal{I}_H^{Ms} e_H^{Ms} = \mathcal{I}_H e_H^{Ms}$ on ∂T, we get

$$\|\mathbf{v} - \mathbf{v}_H^{Ms}\|_{\mathbf{L}^2(E_H)} = \|e_H^{Ms} - \mathcal{I}_H^{Ms} e_H^{Ms}\|_{\mathbf{L}^2(E_H)} = \|e_H^{Ms} - \mathcal{I}_H e_H^{Ms}\|_{\mathbf{L}^2(E_H)}. \tag{5.71}$$

From [[21],Lemma 4], for every edge E_H of T_H and $\mathbf{u} \in \mathbf{H}^1(T_H)$, we have

$$H \|\mathbf{u}\|^2_{\mathbf{L}^2(E_H)} \leq C(\|\mathbf{u}\|^2_{\mathbf{L}^2(T_H)} + H^2 |\mathbf{u}|^2_{\mathbf{H}^1(T_H)}) \tag{5.72}$$

Using (5.72), we get

$$
\begin{aligned}
\|\mathbf{v}-\mathbf{v}_H^{Ms}\|^2_{\mathbf{L}^2(E_H)} &= \|e_H^{Ms}-\mathcal{I}_H^{Ms}e_H^{Ms}\|^2_{\mathbf{L}^2(E_H)} \\
&\le \frac{C}{H}\left(\|e_H^{Ms}-\mathcal{I}_He_H^{Ms}\|^2_{\mathbf{L}^2(T_H)}+H^2|e_H^{Ms}-\mathcal{I}_He_H^{Ms}|^2_{\mathbf{H}^1(T_H)}\right) \\
&\le \frac{C}{H}\left(H^2|e_H^{Ms}|^2_{\mathbf{H}^1(T_H)}+H^2|e_H^{Ms}|^2_{\mathbf{H}^1(T_H)}\right) \\
&\le C'H|e_H^{Ms}|^2_{\mathbf{H}^1(T_H)}. \qquad (5.73)
\end{aligned}
$$

Using (5.73), we obtain with Young's inequality

$$
\begin{aligned}
A_2 &\le \sum_{E_H\subseteq\mathring{\Omega}} \|[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]\|_{\mathbf{L}^2(E_H)}\|\mathbf{v}-\mathbf{v}_H^{Ms}\|_{\mathbf{L}^2(E_H)} \\
&\le |e_H^{Ms}|_{\mathbf{H}^1(\Omega)} \sum_{E_H\subseteq\mathring{\Omega}} H^{\frac{1}{2}}\|[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]\|_{\mathbf{L}^2(E_H)} \\
&\le C\sum_{E_H\subseteq\mathring{\Omega}} H\int_{E_H}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}])^2\,ds+\frac{\epsilon_1}{2}|e_H^{Ms}|^2_{\mathbf{H}^1(\Omega)}. \qquad (5.74)
\end{aligned}
$$

Now we estimate the first term in (5.74), with

$$
\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]}(x)=\frac{1}{|E_H|}\int_{E_H}[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}] \qquad (5.75)
$$

we have

$$
\begin{aligned}
A_{22} := \sum_{E_H\subseteq\mathring{\Omega}} H\int_{E_H}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}])^2 &= \sum_{E_H\subseteq\mathring{\Omega}} H\sum_{E_h\subseteq E_H}\int_{E_h}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}])^2 \\
&= \sum_{E_H\subseteq\mathring{\Omega}} H\left(\sum_{E_h\subseteq E_H}\int_{E_h}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]-\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2+\sum_{E_h\subseteq E_H}\int_{E_h}(\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2\right) \\
&= \sum_{E_H\subseteq\mathring{\Omega}} H\left(\sum_{E_h\subseteq E_H}\int_{E_h}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]-\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2+\sum_{E_h\subseteq E_H}|E_h|(\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2\right) \\
&= \sum_{E_H\subseteq\mathring{\Omega}} H\left(\sum_{E_h\subseteq E_H}\int_{E_h}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]-\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2+|E_H|(\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2\right) \\
&= \sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}} H(E_h)\int_{E_h}([\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]-\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2+\sum_{E_H\subseteq\mathring{\Omega}} H^2(\overline{[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]})^2 \\
&= \sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}} \eta^2_{E_h}+\sum_{E_H\subseteq\mathring{\Omega}}\eta^2_{E_H}. \qquad (5.76)
\end{aligned}
$$

where $|E_H|\sim H$.

We obtain

$$
A_2 \lesssim \sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}} \eta^2_{E_h}+\sum_{E_H\subseteq\mathring{\Omega}}\eta^2_{E_H}+\frac{\epsilon_1}{2}|e_H^{Ms}|^2_{\mathbf{H}^1(\Omega)}. \qquad (5.77)
$$

We throw the term $\frac{\epsilon_1}{2}|e_H^{Ms}|^2_{\mathbf{H}^1(\Omega)}$ to the left hand side, the estimate of the Theorem follows immediately. □

Theorem 5.4. *(A posteriori error estimate for $H > \epsilon$)*

Let $\boldsymbol{u}$ be the solution of (5.32) *and $\boldsymbol{u}_H^{Ms}$ be the solution of the discrete problem* (5.31)*. Then there holds the estimate:*

$$a(e_H^{Ms}, e_H^{Ms}) \lesssim \sum_{T_h\subseteq\Omega} \eta_{T_h}^2 + \sum_{T_H\subseteq\Omega} \eta_{T_H}^2 + \sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}} \eta_{E_h}^2 + \sum_{E_H\subseteq\mathring{\Omega}} \eta_{E_H}^2 + \left(\sum_{T_H\subseteq\Omega} \|f\|^2_{\boldsymbol{L}^2(T_H)}(\frac{\epsilon}{H})\right)^{\frac{1}{2}} \tag{5.78}$$

Proof. To prove the theorem, we first denote

$$\mathbf{u}_I(x) = \mathcal{I}_H^{Ms}\mathbf{u}_0(x) = \sum_j \mathbf{u}_0(x_j)\Phi_j^{Ms}, \tag{5.79}$$

where $\mathbf{u}_0$ is the solution of the homogenized equation and $\mathbf{u}_I$ is the interpolant of $\mathbf{u}_0$ using the multiscale basis functions Φ_j^{Ms}, note that $\mathbf{u}_I$ is different from the definition of $\mathcal{I}_H^{Ms}\mathbf{u}$.

We know that

$$L_\epsilon(\mathbf{u}_I) = 0 \quad \text{in} \quad T, \quad \mathbf{u}_I = \mathcal{I}_H\mathbf{u}_0 \quad \text{on} \quad \partial T \tag{5.80}$$

First, we estimate the term

$$A_1 = \sum_{T\in\mathcal{T}_H} \int_T f(\mathbf{v} - \mathbf{v}_H^{Ms})\, dx \tag{5.81}$$

Choosing $\mathbf{v} = e_H^{Ms}$ and $\mathbf{v}_H^{Ms} = \mathbf{u}_I - \mathbf{u}_H^{Ms}$, using (5.47) and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} A_1 &\le \sum_{T_H\subseteq\Omega} \|f\|_{\mathbf{L}^2(T_H)}\|\mathbf{u} - \mathbf{u}_I\|_{\mathbf{L}^2(T_H)} \\ &\le \sum_{T_H\subseteq\Omega} \|f\|_{\mathbf{L}^2(T_H)}(C_1H^2\|f\|_{\mathbf{L}^2(\Omega)} + C_2(\frac{\epsilon}{H})^{\frac{1}{2}}) \\ &\le \sum_{T_H\subseteq\Omega} C_1H^2\|f\|^2_{\mathbf{L}^2(T_H)} + \sum_{T_H\subseteq\Omega} C_2\|f\|_{\mathbf{L}^2(T_H)}(\frac{\epsilon}{H})^{\frac{1}{2}} \end{aligned} \tag{5.82}$$

From (5.68), we obtain

$$A_1 \lesssim \sum_{T_h\subseteq\Omega} \eta_{T_h}^2 + \sum_{T_H\subseteq\Omega} \eta_{T_H}^2 + \left(\sum_{T_H\subseteq\Omega} \|f\|^2_{\mathbf{L}^2(T_H)}(\frac{\epsilon}{H})\right)^{\frac{1}{2}} \tag{5.83}$$

Now we estimate the term

$$A_2 = \sum_{E_H \subseteq \mathring{\Omega}} \int_{E_H} [\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]_{E_H} (\mathbf{v} - \mathbf{v}_H^{Ms})\, ds. \tag{5.84}$$

Choosing $\mathbf{v} = e_H^{Ms}$ and $\mathbf{v}_H^{Ms} = \mathcal{I}_H(\mathbf{u} - \mathbf{u}_H^{Ms})$ on ∂T , we obtain (see (5.72)-(5.77))

$$A_2 \lesssim \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta_{E_h}^2 + \sum_{E_H \subseteq \mathring{\Omega}} \eta_{E_H}^2 + \frac{\epsilon_1}{2} |e_H^{Ms}|^2_{\mathbf{H}^1(\Omega)}. \tag{5.85}$$

We throw the term $\frac{\epsilon_1}{2} |e_H^{Ms}|^2_{\mathbf{H}^1(\Omega)}$ to the left hand side, the estimate of the theorem follows immediatly. □

5.6 Coupling FEM-BEM

Let Ω be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$ and exterior domain $\Omega_c := \mathbb{R}^2 \setminus \overline{\Omega}$. The displacement field $\mathbf{u}$ of the body satisfies the elasticity material behaviour

$$\sigma(\mathbf{u}) = \mathcal{C} : \varepsilon(\mathbf{u}), \tag{5.86}$$

where σ is the stress tensor, the strain tensor ε is given by the symmetric part of the deformation gradient.

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T). \tag{5.87}$$

Given a volume force f the equilibrium equation reads

$$\operatorname{div} \sigma(\mathbf{u}) + f = 0 \qquad \text{in } \Omega. \tag{5.88}$$

The exterior problem consists of the Navier-Lamé equation

$$0 = -\triangle^* \mathbf{u}_c := -\mu_2 \triangle \mathbf{u}_c - (\lambda_2 + \mu_2) \operatorname{grad} \operatorname{div} \mathbf{u} \qquad \text{in } \Omega_c, \tag{5.89}$$

and a radiation condition of the form

$$D^\alpha(\mathbf{u}_c - a)(x) = O(|x|^{-1-\alpha}), \quad \alpha = 0, 1, \quad (|x| \to \infty), \tag{5.90}$$

where $D = \partial/\partial x_j$ and $a \in \mathbb{R}^2$ is a constant vector.

The transmission problem has the data $f \in \mathbf{L}_2(\Omega)$, $\mathbf{u}_0 \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, and $\mathbf{t}_0 \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and consists in finding $(\mathbf{u}, \mathbf{u}_c) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1_{loc}(\Omega_c)$ satisfying (5.88), (5.89), (5.90), and the interface conditions:

$$\mathbf{u} - \mathbf{u}_c = \mathbf{u}_0 \qquad \text{on} \quad \Gamma \tag{5.91}$$

$$\sigma(\mathbf{u}) \cdot \mathbf{n} = T^*(\mathbf{u}_c) + \mathbf{t}_0 \qquad \text{on} \quad \Gamma \tag{5.92}$$

where T^* is the conormal derivative related to the Lamé operator $\triangle^*$.

$$T^*(\mathbf{u}_c) := 2\mu_2\partial_{\mathbf{n}}\mathbf{u}_c + \lambda_2\mathbf{n}\,\mathrm{div}\,\mathbf{u}_c + \mu_2\mathbf{n}\times\mathrm{curl}\mathbf{u}_c \tag{5.93}$$

Here, $\partial_{\mathbf{n}}$ is the normal derivative and $\mathbf{n}$ is the unit normal vector on Γ pointing from Ω into Ω_c.

The Cauchy data of $\mathbf{u}_c \in \mathbf{H}^1_{loc}(\Omega_c)$ with $\triangle^*\mathbf{u}_c = 0$ satisfy

$$(\xi, \phi) := (\mathbf{u}_c|_\Gamma, T^*(\mathbf{u}_c)|_\Gamma) \in \mathbf{H}^{\frac{1}{2}}(\Gamma)\times\mathbf{H}^{-\frac{1}{2}}(\Gamma) \tag{5.94}$$

The Steklov-Poincaré operator for the exterior Lamé problem is given as

$$\mathcal{S} := (\mathcal{W} + (\mathcal{K}' - 1)\mathcal{V}^{-1}(\mathcal{K} - 1))/2, \tag{5.95}$$

where $\mathcal{W} = 2W$, $\mathcal{K}' = 2K'$, $\mathcal{V} = 2V$, $\mathcal{K} = 2K$ in Chapter2

which satisfies

$$T^*\mathbf{u}_c|_\Gamma = \mathcal{S}\mathbf{u}_c|_\Gamma \tag{5.96}$$

5.6.1 The weak formulation of the Model Problem

The weak form of the interface problem (5.88)-(5.92) reads as:

Find $\mathbf{u} \in \mathbf{H}^1(\Omega)$, such that $\forall\ \mathbf{v} \in \mathbf{H}^1(\Omega)$

$$\langle\sigma(\mathbf{u}), \varepsilon(\mathbf{v})\rangle + \langle\mathcal{S}\mathbf{u}|_\Gamma, \mathbf{v}|_\Gamma\rangle = \langle f, \mathbf{v}\rangle + \langle\mathbf{t}_0 + \mathcal{S}\mathbf{u}_0, \mathbf{v}|_\Gamma\rangle \tag{5.97}$$

Let

$$\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R} := \{\phi \in \mathbf{H}^{\frac{1}{2}}(\Gamma), \langle 1, \phi\rangle = 0\} \tag{5.98}$$

We consider the following weak formulation for a multiscale problem (cf. [17]):

Find $(\mathbf{u}, \xi, \phi) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\Gamma)\times\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}$, such that

$$\langle\mathcal{C}_\epsilon\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})\rangle - \langle\phi, \mathbf{v}\rangle = \langle f, \mathbf{v}\rangle + \langle\mathbf{t}_0, \mathbf{v}\rangle \qquad \forall\mathbf{v} \in \mathbf{H}^1(\Omega) \tag{5.99a}$$

$$-\langle\mathbf{u}, \psi\rangle - \frac{1}{2}\langle\mathcal{V}\phi, \psi\rangle + \frac{1}{2}\langle(\mathcal{K}+1)\xi, \psi\rangle = -\langle\mathbf{u}_0, \psi\rangle \qquad \forall\psi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma) \tag{5.99b}$$

$$\frac{1}{2}\langle(\mathcal{K}'+1)\phi, \theta\rangle + \langle\mathcal{W}\xi, \theta\rangle = 0 \qquad \forall\theta \in \mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R} \tag{5.99c}$$

With the solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$, we define $\phi := \mathcal{S}(\mathbf{u}_0 - \mathbf{u}|_\Gamma)$ and $\xi := \mathbf{u}|_\Gamma - \mathbf{u}_0$.

We rewrite (5.99) as follows:

Find $(\mathbf{u}, \xi, \phi) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\Gamma)\times\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}$, such that

$$B(\mathbf{u}, \xi, \phi; \mathbf{v}, \psi, \theta) = L(\mathbf{v}, \psi) \qquad \forall(\mathbf{v}, \psi, \theta) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\Gamma)\times\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R} \tag{5.100}$$

5.6.2 The discrete problem for Ms-FEM-BEM

Let $\mathbf{V}_H^{Ms} \times \mathbf{H}_H^{-\frac{1}{2}} \times \mathbf{H}_H^{\frac{1}{2}}$ be a family of finite dimensional subspaces of $\mathbf{V} \times \mathbf{H}^{-\frac{1}{2}} \times \mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}$, where $\mathbf{V}_H^{Ms}$ is the defined space of multiscale functions.

The discretized version of (5.100) reads as:

$$B(\mathbf{u}_H^{Ms}, \phi_H, \xi_H; \mathbf{v}_H^{Ms}, \psi_H, \theta_H) = L(\mathbf{v}_H^{Ms}, \psi_H) \qquad \forall (\mathbf{v}_H^{Ms}, \psi_H, \theta_H) \in \mathbf{V}_H^{Ms} \times \mathbf{H}_H^{-\frac{1}{2}} \times \mathbf{H}_H^{\frac{1}{2}} \tag{5.101}$$

As in [17], we consider the relation:

$$\begin{aligned}\langle \phi - \phi_H, \mathbf{u} - \mathbf{u}_H^{Ms} \rangle_\Gamma = & -\frac{1}{2}\langle \mathcal{W}(\xi - \xi_H), \xi - \xi_H \rangle - \frac{1}{2}\langle \mathcal{V}(\phi - \phi_H), \phi - \phi_H \rangle \\ & + \langle \mathbf{u} - \mathbf{u}_H^{Ms}, \phi - \tilde{\phi}_H \rangle + \frac{1}{2}\langle \mathcal{V}(\phi - \phi_H), \phi - \tilde{\phi}_H \rangle \\ & - \frac{1}{2}\langle (\mathcal{K} + 1)(\xi - \xi_H), \phi - \tilde{\phi}_H \rangle + \frac{1}{2}\langle (\mathcal{K}' + 1)(\phi - \phi_H), \xi - \tilde{\xi}_H \rangle \\ & + \frac{1}{2}\langle \mathcal{W}(\xi - \xi_H), \xi - \tilde{\xi}_H \rangle \end{aligned} \tag{5.102}$$

for all $\tilde{\phi}_H \in \mathbf{H}_H^{-\frac{1}{2}}$, $\tilde{\xi}_H \in \mathbf{H}_H^{\frac{1}{2}}$

5.6.3 A priori error estimate

Using the relation (5.102) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & B(\mathbf{u} - \mathbf{u}_H^{Ms}, \phi - \phi_H, \xi - \xi_H; \mathbf{u} - \mathbf{u}_H^{Ms}, \phi - \phi_H, \xi - \xi_H) = \langle \mathcal{C}_\epsilon \varepsilon(\mathbf{u} - \mathbf{u}_H^{Ms}), \varepsilon(\mathbf{u} - \mathbf{u}_H^{Ms}) \rangle \\ & - \langle \phi - \phi_H, \mathbf{u} - \mathbf{u}_H^{Ms} \rangle_\Gamma - \langle \mathbf{u} - \mathbf{u}_H^{Ms}, \phi - \phi_H \rangle - \frac{1}{2}\langle \mathcal{V}(\phi - \phi_H), \phi - \phi_H \rangle \\ & + \frac{1}{2}\langle (\mathcal{K} + 1)(\xi - \xi_H), \phi - \phi_H \rangle - \frac{1}{2}\langle (\mathcal{K}' + 1)(\phi - \phi_H), \xi - \xi_H \rangle - \frac{1}{2}\langle \mathcal{W}(\xi - \xi_H), \xi - \xi_H \rangle \\ & = \langle \mathcal{C}_\epsilon \varepsilon(\mathbf{u} - \mathbf{u}_H^{Ms}), \varepsilon(\mathbf{u} - \mathbf{u}_H^{Ms}) \rangle + \frac{1}{2}\langle \mathcal{W}(\xi - \xi_H), \xi - \xi_H \rangle + \frac{1}{2}\langle \mathcal{V}(\phi - \phi_H), \phi - \phi_H \rangle - 2A \\ & \gtrsim \|\mathcal{C}_\epsilon^{\frac{1}{2}} \varepsilon(\mathbf{u} - \mathbf{u}_H^{Ms})\|^2_{L_2(\Omega)} + \|\xi - \xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi - \phi_H\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} - 2A \end{aligned} \tag{5.103}$$

where

$$\begin{aligned} A := & -\frac{1}{2}\langle \mathcal{W}(\xi - \xi_H), \xi - \xi_H \rangle - \frac{1}{2}\langle \mathcal{V}(\phi - \phi_H), \phi - \phi_H \rangle \\ & + \langle \mathbf{u} - \mathbf{u}_H^{Ms}, \phi - \tilde{\phi}_H \rangle + \frac{1}{2}\langle \mathcal{V}(\phi - \phi_H), \phi - \tilde{\phi}_H \rangle \\ & - \frac{1}{2}\langle (\mathcal{K} + 1)(\xi - \xi_H), \phi - \tilde{\phi}_H \rangle + \frac{1}{2}\langle (\mathcal{K}' + 1)(\phi - \phi_H), \xi - \tilde{\xi}_H \rangle \\ & + \frac{1}{2}\langle \mathcal{W}(\xi - \xi_H), \xi - \tilde{\xi}_H \rangle \end{aligned} \tag{5.104}$$

On the other hand , for all $(\mathbf{v}_H^{Ms}, \psi_H, \theta_H) \in \mathbf{V}_H^{Ms} \times \mathbf{H}_H^{-\frac{1}{2}} \times \mathbf{H}_H^{\frac{1}{2}}$, we have

$$\begin{aligned} &B(\mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\phi_H, \xi-\xi_H; \mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\phi_H, \xi-\xi_H) = \\ &B(\mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\phi_H, \xi-\xi_H; \mathbf{u}-\mathbf{v}_H^{Ms}, \phi-\psi_H, \xi-\theta_H) \\ &+ B(\mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\phi_H, \xi-\xi_H; \mathbf{v}_H^{Ms}-\mathbf{u}_H^{Ms}, \psi_H-\phi_H, \theta_H-\xi_H) \end{aligned} \tag{5.105}$$

Since $B(\mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\phi_H, \xi-\xi_H; \mathbf{v}_H^{Ms}-\mathbf{u}_H^{Ms}, \psi_H-\phi_H, \theta_H-\xi_H) = 0$, we obtain

$$\begin{aligned} &\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms})\|_{L_2(\Omega)}^2 + \|\xi-\xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi-\phi_H\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim B(\mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\phi_H, \xi-\xi_H; \mathbf{u}-\mathbf{v}_H^{Ms}, \phi-\psi_H, \xi-\theta_H) + 2A \\ &= \langle \mathcal{C}_\epsilon\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms}), \varepsilon(\mathbf{u}-\mathbf{v}_H^{Ms})\rangle - \langle \phi-\phi_H, \mathbf{u}-\mathbf{v}_H^{Ms}\rangle_\Gamma \\ &- \langle \mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\psi_H\rangle - \frac{1}{2}\langle \mathcal{V}(\phi-\phi_H), \phi-\psi_H\rangle \\ &+ \frac{1}{2}\langle (\mathcal{K}+1)(\xi-\xi_H), \phi-\psi_H\rangle - \frac{1}{2}\langle (\mathcal{K}'+1)(\phi-\phi_H), \xi-\theta_H\rangle - \frac{1}{2}\langle \mathcal{W}(\xi-\xi_H), \xi-\theta_H\rangle \\ &+ 2\langle \mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\tilde{\phi}_H\rangle + \langle \mathcal{V}(\phi-\phi_H), \phi-\tilde{\phi}_H\rangle \\ &- \langle (\mathcal{K}+1)(\xi-\xi_H), \phi-\tilde{\phi}_H\rangle + \langle (\mathcal{K}'+1)(\phi-\phi_H), \xi-\tilde{\xi}_H\rangle + \langle \mathcal{W}(\xi-\xi_H), \xi-\tilde{\xi}_H\rangle \\ &= \langle \mathcal{C}_\epsilon\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms}), \varepsilon(\mathbf{u}-\mathbf{v}_H^{Ms})\rangle - \langle \phi-\phi_H, \mathbf{u}-\mathbf{v}_H^{Ms}\rangle_\Gamma \\ &+ \langle \mathbf{u}-\mathbf{u}_H^{Ms}, \phi-\psi_H-2\tilde{\phi}_H\rangle + \langle \mathcal{V}(\phi-\phi_H), \frac{1}{2}\phi-\frac{1}{2}\psi_H-\tilde{\phi}_H\rangle \\ &+ \langle (\mathcal{K}+1)(\xi-\xi_H), \tilde{\phi}_H-\frac{1}{2}\psi_H-\frac{1}{2}\phi\rangle + \langle (\mathcal{K}'+1)(\phi-\phi_H), \frac{1}{2}\xi-\frac{1}{2}\theta_H-\tilde{\xi}_H\rangle \\ &+ \langle \mathcal{W}(\xi-\xi_H), \frac{1}{2}\xi+\frac{1}{2}\theta_H-\tilde{\xi}_H\rangle \\ &\lesssim \left(\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms})\|_{L_2(\Omega)} + \|\xi-\xi_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi-\phi_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}\right) \cdot \\ &\left(\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{v}_H^{Ms})\|_{L_2(\Omega)} + \|\xi-\theta_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi-\psi_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}\right) \end{aligned} \tag{5.106}$$

After choosing $\theta_H = \tilde{\xi}_H$, $\psi_H = \tilde{\phi}_H$, we get the following theorem.

Theorem 5.5.

$$\begin{aligned} &\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\boldsymbol{u}-\boldsymbol{u}_H^{Ms})\|_{L_2(\Omega)}^2 + \|\xi-\xi_H\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi-\phi_H\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim \inf_{(\boldsymbol{v}_H^{Ms}, \psi_H, \theta_H) \in \boldsymbol{V}_H^{Ms} \times \boldsymbol{H}_H^{-\frac{1}{2}} \times \boldsymbol{H}_H^{\frac{1}{2}}} \left\{ \|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\boldsymbol{u}-\boldsymbol{v}_H^{Ms})\|_{L_2(\Omega)} + \|\xi-\theta_H\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi-\psi_H\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right\} \end{aligned} \tag{5.107}$$

5.6.4 A posteriori error

In this section, we derive a-posteriori error estimates for $H < \epsilon$ and $H > \epsilon$.

Theorem 5.6. *(A posteriori error estimate for $H < \epsilon$)*

Let $(\boldsymbol{u}, \phi, \xi)$ be the solution of (5.100) *and $(\boldsymbol{u}_H^{Ms}, \phi_H, \xi_H)$ be the solution of the discrete problem* (5.101). *Then there holds the estimate:*

$$\begin{aligned}
&\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\boldsymbol{u}-\boldsymbol{u}_H^{Ms})\|^2_{L_2(\Omega)}+\|\xi-\xi_H\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}}+\|\phi-\phi_H\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\\
&\lesssim \sum_{E_H\subseteq\Gamma} H\|\sigma(\boldsymbol{u}_H^{Ms})\cdot\boldsymbol{n}-\boldsymbol{t}_0-\phi_H\|^2_{\boldsymbol{L}^2(E_H)}\\
&+\sum_{E_H\subseteq\Gamma} H\|\frac{\partial}{\partial s}(\boldsymbol{u}_0-\boldsymbol{u}_H^{Ms}+\frac{1}{2}V\phi_H-\frac{1}{2}(K+1)\xi_H)\|^2_{\boldsymbol{L}^2(E_H)}\\
&+\sum_{E_H\subseteq\Gamma} H\|(K^{'}+1)\phi_H+W\xi_H\|^2_{\boldsymbol{L}^2(E_H)}\\
&+\sum_{T_h\subseteq\Omega}\eta^2_{T_h}+\sum_{T_H\subseteq\Omega}\eta^2_{T_H}+\sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}}\eta^2_{E_h}+\sum_{E_H\subseteq\mathring{\Omega}}\eta^2_{E_H}
\end{aligned}$$

$$\begin{aligned}
\sum_{T_h\subseteq\Omega}\eta^2_{T_h} &= \sum_{T_h\subseteq\Omega} H^2(T_h)\int_{T_h}|f-\overline{f}|^2\,dx, \quad \sum_{T_H\subseteq\Omega}\eta^2_{T_H}=\sum_{T_H\subseteq\Omega}H^4|\overline{f}|^2\\
\sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}}\eta^2_{E_h} &= \sum_{E_h\subseteq E_H\subseteq\mathring{\Omega}} H(E_h)\int_{E_h}([\sigma(\boldsymbol{u}_H^{Ms})\cdot\boldsymbol{n}]-\overline{[\sigma(\boldsymbol{u}_H^{Ms})\cdot\boldsymbol{n}]})^2\\
\sum_{E_H\subseteq\mathring{\Omega}}\eta^2_{E_H} &= \sum_{E_H\subseteq\mathring{\Omega}} H^2(\overline{[\sigma(\boldsymbol{u}_H^{Ms})\cdot\boldsymbol{n}]})^2\\
\overline{f}(x) &= \frac{1}{|T_H|}\int_{T_H} f\,dx, \quad \overline{[\sigma(\boldsymbol{u}_H^{Ms})\cdot\boldsymbol{n}]}(x)=\frac{1}{|E_H|}\int_{E_H}[\sigma(\boldsymbol{u}_H^{Ms})\cdot\boldsymbol{n}]\,dx
\end{aligned}$$

Proof. Recall that

$$\begin{aligned}
&\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms})\|^2_{L_2(\Omega)}+\|\xi-\xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}}+\|\phi-\phi_H\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}\\
&\lesssim B(\mathbf{u}-\mathbf{u}_H^{Ms},\phi-\phi_H,\xi-\xi_H;\mathbf{u}-\mathbf{v}_H^{Ms},\phi-\psi_H,\xi-\theta_H)+2A\\
&=L(\mathbf{u}-\mathbf{v}_H^{Ms},\phi-\psi_H)-B(\mathbf{u}^{Ms},\phi_H,\xi_H;\mathbf{u}-\mathbf{v}_H^{Ms},\phi-\psi_H,\xi-\theta_H)\\
&+2\langle\mathbf{u}_0-\mathbf{u}_H^{Ms}+\frac{1}{2}\mathcal{V}(\phi-\phi_H)-\frac{1}{2}(\mathcal{K}+1)(\xi-\xi_H),\phi-\tilde{\phi}_H\rangle\\
&+\langle(\mathcal{K}^{'}+1)(\phi-\phi_H)+\mathcal{W}(\xi-\xi_H),\xi-\tilde{\xi}_H\rangle\\
&=\int_\Omega f(\mathbf{u}-\mathbf{v}_H^{Ms})\,dx+\langle\mathbf{t}_0,\mathbf{u}-\mathbf{v}_H^{Ms}\rangle-\langle\mathbf{u}_0,\phi-\psi_H\rangle\\
&-\langle\mathcal{C}_\epsilon\varepsilon(\mathbf{u}_H^{Ms}),\varepsilon(\mathbf{u}-\mathbf{v}_H^{Ms})\rangle+\langle\phi_H,\mathbf{u}-\mathbf{v}_H^{Ms}\rangle+\langle\mathbf{u}_H^{Ms},\phi-\psi_H\rangle\\
&+\frac{1}{2}\langle\mathcal{V}\phi_H,\phi-\psi_H\rangle-\frac{1}{2}\langle(\mathcal{K}+1)\xi_H,\phi-\psi_H\rangle\\
&+\frac{1}{2}\langle(\mathcal{K}^{'}+1)\phi_H,\xi-\theta_H\rangle+\frac{1}{2}\langle\mathcal{W}\xi_H,\xi-\theta_H\rangle\\
&+2\langle\mathbf{u}_0-\mathbf{u}_H^{Ms}+\frac{1}{2}\mathcal{V}(\phi-\phi_H)-\frac{1}{2}(\mathcal{K}+1)(\xi-\xi_H),\phi-\tilde{\phi}_H\rangle\\
&+\langle(\mathcal{K}^{'}+1)(\phi-\phi_H)+\mathcal{W}(\xi-\xi_H),\xi-\tilde{\xi}_H\rangle \qquad (5.108)
\end{aligned}$$

We estimate the last term in (5.108)

$$A_1 := \langle (\mathcal{K}^{'}+1)(\phi-\phi_H) + \mathcal{W}(\xi-\xi_H), \xi-\tilde{\xi}_H\rangle \tag{5.109}$$

Using (5.99c), Cauchy Schwarz and the inverse inequality, we get

$$\begin{aligned} A_1 &\leq \|(\mathcal{K}^{'}+1)\phi_H + \mathcal{W}\xi_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \|\xi-\tilde{\xi}_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &\lesssim H^{\frac{1}{2}} \|(\mathcal{K}^{'}+1)\phi_H + \mathcal{W}\xi_H\|_{\mathbf{L}^2(\Gamma)} \inf_{\tilde{\xi}_H} \|\xi-\tilde{\xi}_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \end{aligned} \tag{5.110}$$

Similarly we have

$$\begin{aligned} A_2 &:= \frac{1}{2}\langle (\mathcal{K}^{'}+1)\phi_H, \xi-\theta_H\rangle + \frac{1}{2}\langle \mathcal{W}\xi_H, \xi-\theta_H\rangle \\ &\lesssim H^{\frac{1}{2}} \|(\mathcal{K}^{'}+1)\phi_H + \mathcal{W}\xi_H\|_{\mathbf{L}^2(\Gamma)} \inf_{\theta_H} \|\xi-\theta_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}, \end{aligned} \tag{5.111}$$

Using (5.99b), Cauchy Schwarz and the inverse inequality, we get

$$\begin{aligned} A_3 &:= \langle \mathbf{u}_0 - \mathbf{u}_H^{Ms} + \frac{1}{2}\mathcal{V}(\phi-\phi_H) - \frac{1}{2}(\mathcal{K}+1)(\xi-\xi_H), \phi-\tilde{\phi}_H\rangle \\ &\lesssim \|\mathbf{u}_0 - \mathbf{u}_H^{Ms} - \frac{1}{2}\mathcal{V}\phi_H + \frac{1}{2}(\mathcal{K}+1)\xi_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \inf_{\tilde{\phi}_H} \|\phi-\tilde{\phi}_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\leq H^{\frac{1}{2}} \|\frac{\partial}{\partial s}(\mathbf{u}_0 - \mathbf{u}_H^{Ms} + \frac{1}{2}\mathcal{V}\phi_H - \frac{1}{2}(\mathcal{K}+1)\xi_H)\|_{\mathbf{L}^2(\Gamma)} \inf_{\tilde{\phi}_H} \|\phi-\tilde{\phi}_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \end{aligned} \tag{5.112}$$

Similarly we have

$$\begin{aligned} A_4 &:= \langle \mathbf{u}_H^{Ms}, \phi-\psi_H\rangle - \langle \mathbf{u}_0, \phi-\psi_H\rangle + \frac{1}{2}\langle \mathcal{V}\phi_H, \phi-\psi_H\rangle - \frac{1}{2}\langle (\mathcal{K}+1)\xi_H, \phi-\psi_H\rangle \\ &\leq H^{\frac{1}{2}} \|\frac{\partial}{\partial s}(\mathbf{u}_0 - \mathbf{u}_H^{Ms} - \frac{1}{2}\mathcal{V}\phi_H + \frac{1}{2}(\mathcal{K}+1)\xi_H)\|_{\mathbf{L}^2(\Gamma)} \inf_{\psi_H} \|\phi-\psi_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \end{aligned} \tag{5.113}$$

We estimate the interior terms, emploing the integration by part, we obtain

$$\begin{aligned} A_5 &= \int_\Omega f(\mathbf{u}-\mathbf{v}_H^{Ms})\,dx - \langle \mathcal{C}_\epsilon \varepsilon(\mathbf{u}_H^{Ms}), \varepsilon(\mathbf{u}-\mathbf{v}_H^{Ms})\rangle \\ &= \sum_{T_H\subseteq\Omega} \int_{T_H} \{f + \mathrm{div}\sigma(\mathbf{u}_H^{Ms})\}(\mathbf{u}-\mathbf{v}_H^{Ms}) - \sum_{E_H\subseteq\mathring{\Omega}} \int_{E_H} [\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]_{E_H}(\mathbf{u}-\mathbf{v}_H^{Ms})\,ds \\ &- \sum_{E_H\subseteq\Gamma} \int_{E_h} \sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}|_{E_H}(\mathbf{u}-\mathbf{v}_H^{Ms})\,ds. \end{aligned} \tag{5.114}$$

Recall that by contruction of the multiscale basis functions, we have $\mathrm{div}\sigma(\mathbf{u}_H^{Ms}) = 0$ in T.

Using (5.109)-(5.114), we obtain

$$\begin{aligned}
&\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms})\|^2_{L_2(\Omega)}+\|\xi-\xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}}+\|\phi-\phi_H\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}\\
&\lesssim \int_\Omega f(\mathbf{u}-\mathbf{v}_H^{Ms})\,dx+\sum_{E_H\subseteq\mathring{\Omega}}\int_{E_H}[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]_{E_H}(\mathbf{u}-\mathbf{v}_H^{Ms})\,ds\\
&-\langle\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}-\mathbf{t}_0-\phi_H,\mathbf{u}-\mathbf{v}_H^{Ms}\rangle\\
&+H^{\frac{1}{2}}\|(\mathcal{K}'+1)\phi_H+\mathcal{W}\xi_H\|_{\mathbf{L}^2(\Gamma)}\inf_{\tilde{\xi}_H}\|\xi-\tilde{\xi}_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\\
&+H^{\frac{1}{2}}\|\frac{\partial}{\partial s}(\mathbf{u}_0-\mathbf{u}_H^{Ms}+\frac{1}{2}\mathcal{V}\phi_H-\frac{1}{2}(\mathcal{K}+1)\xi_H)\|_{\mathbf{L}^2(\Gamma)}\inf_{\tilde{\phi}_H}\|\phi-\tilde{\phi}_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}
\end{aligned} \tag{5.115}$$

First, we estimate the term

$$B_1 := \int_\Omega f(\mathbf{u}-\mathbf{v}_H^{Ms})\,dx \tag{5.116}$$

Let $\overline{f}(x)=\frac{1}{|T_H|}\int_{T_H} f$, if $x\in T_H$ then

$$\int_\Omega |f|^2=\int_\Omega (f-\overline{f}+\overline{f})^2=\int_\Omega (f-\overline{f})^2+2\sum_{T_H\in\mathcal{T}_H}\overline{f}\int_{T_H}(f-\overline{f})+\int_\Omega(\overline{f})^2$$

However $\int_{T_H}(f-\overline{f})=0$

$$\int_\Omega |f|^2=\int_\Omega (f-\overline{f})^2+\int_\Omega(\overline{f})^2 \tag{5.117}$$

Using (5.39), choosing $\mathbf{v}_H^{Ms}=\mathbf{u}_H^{Ms}-\mathcal{I}_H^{Ms}(\mathbf{u}-\mathbf{u}_H^{Ms})$, with $\mathbf{e}:=\mathbf{u}-\mathbf{u}_H^{Ms}$, we obtain

$$\begin{aligned}
B_1 &\leq \sum_{T_H\subseteq\Omega}\|f\|_{\mathbf{L}^2(T_H)}\|\mathbf{u}-\mathbf{v}_H^{Ms}\|_{\mathbf{L}^2(T_H)}\\
&\leq C\sum_{T_H\subseteq\Omega}\|f\|_{\mathbf{L}^2(T_H)}(H|\mathbf{e}|_{\mathbf{H}^1(T_H)}+H^2\|f\|_{\mathbf{L}^2(T_H)})\\
&\leq C(\sum_{T_H\subseteq\Omega}H^2\|f\|^2_{\mathbf{L}^2(T_H)}+\sum_{T_H\subseteq\Omega}H\|f\|_{\mathbf{L}^2(T_H)}|\mathbf{e}|_{\mathbf{H}^1(T_H)}\\
&\leq C'\sum_{T_H\subseteq\Omega}H^2\|f\|^2_{\mathbf{L}^2(T_H)}+\frac{\epsilon}{2}|\mathbf{e}|^2_{\mathbf{H}^1(T_H)}
\end{aligned} \tag{5.118}$$

Now we estimate the term $\sum_{T_H \subseteq \Omega} H^2 \|f\|^2_{\mathbf{L}^2(T_H)}$, employing (5.117), we get

$$\begin{aligned}
\sum_{T_H \subseteq \Omega} H^2 \|f\|^2_{\mathbf{L}^2(T_H)} &= \sum_{T_H \subseteq \Omega} H^2 \Big(\sum_{T_h \subseteq T_H} \int_{T_h} |f|^2 \, dx \Big) \\
&= \sum_{T_H \subseteq \Omega} H^2 \Big\{ \sum_{T_h \subseteq T_H} \int_{T_h} |f - \overline{f}|^2 \, dx + \sum_{T_h \subseteq T_H} \int_{T_h} |\overline{f}|^2 \, dx \Big\} \\
&= \sum_{T_H \subseteq \Omega} H^2 \Big\{ \sum_{T_h \subseteq T_H} \int_{T_h} |f - \overline{f}|^2 \, dx + \sum_{T_h \subseteq T_H} |T_h| |\overline{f}|^2 \Big\} \\
&= \sum_{T_H \subseteq \Omega} H^2 \Big\{ \sum_{T_h \subseteq T_H} \int_{T_h} |f - \overline{f}|^2 \, dx + |T_H| |\overline{f}|^2 \Big\} \\
&= \sum_{T_h \subseteq \Omega} H^2(T_h) \int_{T_h} |f - \overline{f}|^2 \, dx + \sum_{T_H \subseteq \Omega} H^4 |\overline{f}|^2 \\
&= \sum_{T_h \subseteq \Omega} \eta^2_{T_h} + \sum_{T_H \subseteq \Omega} \eta^2_{T_H}. \qquad (5.119)
\end{aligned}$$

Hier $\overline{f}|_{T_H}$ is a constant on T_H, and $|T_H| \sim H^2$.

We obtain

$$B_1 \lesssim \sum_{T_h \subseteq \Omega} \eta^2_{T_h} + \sum_{T_H \subseteq \Omega} \eta^2_{T_H} + \frac{\epsilon}{2} |\mathbf{e}|^2_{\mathbf{H}^1(T_H)} \qquad (5.120)$$

Now we estimate the term

$$B_2 = \sum_{E_H \subseteq \mathring{\Omega}} \int_{E_H} [\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]_{E_H} (\mathbf{u} - \mathbf{v}_H^{Ms}) \, ds. \qquad (5.121)$$

Choosing $\mathbf{v}_H^{Ms} = \mathbf{u}_H^{Ms} - \mathcal{I}_H^{Ms}(\mathbf{u} - \mathbf{u}_H^{Ms})$, since $\mathcal{I}_H^{Ms}\mathbf{u} = \mathcal{I}_H \mathbf{u}$ on ∂T, we get

$$\|\mathbf{u} - \mathbf{v}_H^{Ms}\|_{\mathbf{L}^2(E_H)} = \|\mathbf{e} - \mathcal{I}_H^{Ms}\mathbf{e}\|_{\mathbf{L}^2(E_H)} = \|\mathbf{e} - \mathcal{I}_H \mathbf{e}\|_{\mathbf{L}^2(E_H)}. \qquad (5.122)$$

From [[21],Lemma 4], for every edge E_H of T_H and $\mathbf{u} \in \mathbf{H}^1(T_H)$, we have

$$H \|\mathbf{u}\|^2_{\mathbf{L}^2(E_H)} \leq C (\|\mathbf{u}\|^2_{\mathbf{L}^2(T_H)} + H^2 |\mathbf{u}|^2_{\mathbf{H}^1(T_H)}) \qquad (5.123)$$

Using (5.123), we get

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}_H^{Ms}\|^2_{\mathbf{L}^2(E_H)} = \|\mathbf{e} - \mathcal{I}_H^{Ms}\mathbf{e}\|^2_{\mathbf{L}^2(E_H)} &\leq \frac{C}{H} \left(\|\mathbf{e} - \mathcal{I}_H \mathbf{e}\|^2_{\mathbf{L}^2(T_H)} + H^2 |\mathbf{e} - \mathcal{I}_H \mathbf{e}|^2_{\mathbf{H}^1(T_H)} \right) \\
&\leq \frac{C}{H} \left(H^2 |\mathbf{e}|^2_{\mathbf{H}^1(T_H)} + H^2 |\mathbf{e}|^2_{\mathbf{H}^1(T_H)} \right) \\
&\leq C' H |\mathbf{e}|^2_{\mathbf{H}^1(T_H)}. \qquad (5.124)
\end{aligned}$$

Using (5.124), we obtain

$$
\begin{aligned}
B_2 \leq & \sum_{E_H \subseteq \mathring{\Omega}} \|[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]\|_{\mathbf{L}^2(E_H)} \|\mathbf{u} - \mathbf{v}_H^{Ms}\|_{\mathbf{L}^2(E_H)} \\
\leq & |\mathbf{e}|_{\mathbf{H}^1(\Omega)} \sum_{E_H \subseteq \mathring{\Omega}} H^{\frac{1}{2}} \|[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]\|_{\mathbf{L}^2(E_H)} \\
\leq & \, C \sum_{E_H \subseteq \mathring{\Omega}} H \int_{E_H} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}])^2 \, ds + \frac{\epsilon}{2} |\mathbf{e}|^2_{\mathbf{H}^1(\Omega)}.
\end{aligned} \tag{5.125}
$$

Now we estimate the first term in (5.125), with

$$
\overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]}(x) = \frac{1}{|E_H|} \int_{E_H} [\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}] \tag{5.126}
$$

we have

$$
\begin{aligned}
B_{22} = & \sum_{E_H \subseteq \mathring{\Omega}} H \int_{E_H} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}])^2 = \sum_{E_H \subseteq \mathring{\Omega}} H \sum_{E_h \subseteq E_H} \int_{E_h} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}])^2 \\
= & \sum_{E_H \subseteq \mathring{\Omega}} H \left(\sum_{E_h \subseteq E_H} \int_{E_h} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}] - \overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 + \sum_{E_h \subseteq E_H} \int_{E_h} (\overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 \right) \\
= & \sum_{E_H \subseteq \mathring{\Omega}} H \left(\sum_{E_h \subseteq E_H} \int_{E_h} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}] - \overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 + \sum_{E_h \subseteq E_H} |E_h| (\overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 \right) \\
= & \sum_{E_H \subseteq \mathring{\Omega}} H \left(\sum_{E_h \subseteq E_H} \int_{E_h} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}] - \overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 + |E_H| (\overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 \right) \\
= & \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} H(E_h) \int_{E_h} ([\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}] - \overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 + \sum_{E_H \subseteq \mathring{\Omega}} H^2 (\overline{[\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n}]})^2 \\
= & \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta^2_{E_h} + \sum_{E_H \subseteq \mathring{\Omega}} \eta^2_{E_H} + \frac{\epsilon}{2} |\mathbf{e}|^2_{\mathbf{H}^1(\Omega)}.
\end{aligned} \tag{5.127}
$$

where $|E_H| \sim H$.

We obtain

$$
B_2 \lesssim \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta^2_{E_h} + \sum_{E_H \subseteq \mathring{\Omega}} \eta^2_{E_H} + \frac{\epsilon}{2} |\mathbf{e}|^2_{\mathbf{H}^1(\Omega)}. \tag{5.128}
$$

Now we estimate the term

$$
B_3 := \langle \sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} - \mathbf{t}_0 - \phi_H, \mathbf{u} - \mathbf{v}_H^{Ms} \rangle \tag{5.129}
$$

Choosing $\mathbf{v}_H^{Ms} = \mathbf{u}_H^{Ms} - \mathcal{I}_H^{Ms}(\mathbf{u} - \mathbf{u}_H^{Ms})$,

$$\begin{aligned} B_3 &\leq \|\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} - \mathbf{t}_0 - \phi_H\|_{\mathbf{L}^2(\Gamma)} \|\mathbf{u} - \mathbf{v}_H^{Ms}\|_{\mathbf{L}^2(\Gamma)} \\ &\leq C|\mathbf{e}|_{\mathbf{H}^1(\Omega)} H^{\frac{1}{2}} \|\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} - \mathbf{t}_0 - \phi_H\|_{\mathbf{L}^2(\Gamma)} \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \|\sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} - \mathbf{t}_0 - \phi_H\|^2_{\mathbf{L}^2(\Gamma)} + \frac{\epsilon}{2} |\mathbf{e}|^2_{\mathbf{H}^1(\Omega)} \end{aligned} \tag{5.130}$$

Using Young's inequality, we get

$$\begin{aligned} B_4 &:= H^{\frac{1}{2}} \|(\mathcal{K}' + 1)\phi_H + \mathcal{W}\xi_H\|_{\mathbf{L}^2(\Gamma)} \inf_{\tilde{\xi}_H} \|\xi - \tilde{\xi}_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \|(\mathcal{K}' + 1)\phi_H + \mathcal{W}\xi_H\|^2_{\mathbf{L}^2(E_H)} + \frac{\epsilon}{2} \|\xi - \xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}, \end{aligned} \tag{5.131}$$

and

$$\begin{aligned} B_5 &:= H^{\frac{1}{2}} \|\frac{\partial}{\partial s}(\mathbf{u}_0 - \mathbf{u}_H^{Ms} + \frac{1}{2}\mathcal{V}\phi_H - \frac{1}{2}(\mathcal{K} + 1)\xi_H)\|_{\mathbf{L}^2(\Gamma)} \inf_{\tilde{\phi}_H} \|\phi - \tilde{\phi}_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \|\frac{\partial}{\partial s}(\mathbf{u}_0 - \mathbf{u}_H^{Ms} + \frac{1}{2}\mathcal{V}\phi_H - \frac{1}{2}(\mathcal{K} + 1)\xi_H)\|^2_{\mathbf{L}^2(E_H)} + \frac{\epsilon}{2} \|\phi - \phi_H\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \end{aligned} \tag{5.132}$$

We throw the terms $|\mathbf{e}|^2_{\mathbf{H}^1(\Omega)}$, $\|\xi - \xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}$ and $\|\phi - \phi_H\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}$ to the left hand side, and we obtain the estimate of the theorem. □

Theorem 5.7. *(A posteriori error estimate for $H > \epsilon$) Assume that Lemma 5.2 and Theorem 5.2 hold for the transmission problem.*

Let $(\boldsymbol{u}, \phi, \xi)$ be the solution of (5.100) *and $(\boldsymbol{u}_H^{Ms}, \phi_H, \xi_H)$ be the solution of the discrete problem* (5.101). *Then there holds the estimate:*

$$\begin{aligned} &\|\mathcal{C}_\epsilon^{\frac{1}{2}} \varepsilon(\boldsymbol{u} - \boldsymbol{u}_H^{Ms})\|^2_{L_2(\Omega)} + \|\xi - \xi_H\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}} + \|\phi - \phi_H\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \|\sigma(\boldsymbol{u}_H^{Ms}) \cdot \boldsymbol{n} - \boldsymbol{t}_0 - \phi_H\|^2_{\boldsymbol{L}^2(E_H)} \\ &+ \sum_{E_H \subseteq \Gamma} H \|\frac{\partial}{\partial s}(\boldsymbol{u}_0 - \boldsymbol{u}_H^{Ms} + \frac{1}{2}\mathcal{V}\phi_H - \frac{1}{2}(\mathcal{K} + 1)\xi_H)\|^2_{\boldsymbol{L}^2(E_H)} \\ &+ \sum_{E_H \subseteq \Gamma} H \|(\mathcal{K}' + 1)\phi_H + \mathcal{W}\xi_H\|^2_{\boldsymbol{L}^2(E_H)} \\ &+ \sum_{T_h \subseteq \Omega} \eta^2_{T_h} + \sum_{T_H \subseteq \Omega} \eta^2_{T_H} + \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta^2_{E_h} + \sum_{E_H \subseteq \mathring{\Omega}} \eta^2_{E_H} + \left(\sum_{T_H \subseteq \Omega} \|f\|^2_{\boldsymbol{L}^2(T_H)} (\frac{\epsilon}{H}) \right)^{\frac{1}{2}} \end{aligned}$$

Proof. To prove the theorem, we first denote

$$\mathbf{u}_I(x) = \mathcal{I}_H^{Ms} \mathbf{u}^0(x) = \sum_j \mathbf{u}^0(x_j) \Phi_j^{Ms}, \tag{5.133}$$

where $\mathbf{u}^0$ is the solution of the homogenized equation and $\mathbf{u}_I$ is the interpolant of $\mathbf{u}_0$,using the multiscale basis functions Φ_j^{Ms}, note that $\mathbf{u}_I$ is different from the definition of $\mathcal{I}_H^{Ms}\mathbf{u}$.

We know that

$$L_\epsilon(\mathbf{u}_I) = 0 \quad \text{in} \quad T, \quad \mathbf{u}_I = \mathcal{I}_H \mathbf{u}^0 \quad \text{on} \quad \partial T \tag{5.134}$$

From (5.115), we have

$$\begin{aligned}
&\|\mathcal{C}_\epsilon^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_H^{Ms})\|^2_{L_2(\Omega)}+\|\xi-\xi_H\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)/\mathbb{R}}+\|\phi-\phi_h\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}\\
&\lesssim \int_\Omega f(\mathbf{u}-\mathbf{v}_H^{Ms})\,dx + \sum_{E_H\subseteq\mathring{\Omega}}\int_{E_H}[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]_{E_H}(\mathbf{u}-\mathbf{v}_H^{Ms})\,ds\\
&-\langle\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}-\mathbf{t}_0-\phi_H,\mathbf{u}-\mathbf{v}_H^{Ms}\rangle\\
&+H^{\frac{1}{2}}\|(\mathcal{K}'+1)\phi_H+\mathcal{W}\xi_H\|_{\mathbf{L}^2(\Gamma)}\inf_{\tilde{\xi}_H}\|\xi-\tilde{\xi}_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\\
&+H^{\frac{1}{2}}\|\frac{\partial}{\partial s}(\mathbf{u}_0-\mathbf{u}_H^{Ms}+\frac{1}{2}\mathcal{V}\phi_H-\frac{1}{2}(\mathcal{K}+1)\xi_H)\|_{\mathbf{L}^2(\Gamma)}\inf_{\tilde{\phi}_H}\|\phi-\tilde{\phi}_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}
\end{aligned} \tag{5.135}$$

First, we estimate the term

$$B_1 := \int_\Omega f(\mathbf{u}-\mathbf{v}_H^{Ms})\,dx \tag{5.136}$$

We choose $\mathbf{v}_H^{Ms} = \mathbf{u}_I$, using (5.47) and Cauchy Schwarz inequality, we obtain

$$\begin{aligned}
B_1 &\le \sum_{T_H\subseteq\Omega}\|f\|_{\mathbf{L}^2(T_H)}\|\mathbf{u}-\mathbf{u}_I\|_{\mathbf{L}^2(T_H)}\\
&\le \sum_{T_H\subseteq\Omega}\|f\|_{\mathbf{L}^2(T_H)}(C_1H^2\|f\|_{\mathbf{L}^2(\Omega)}+C_2(\frac{\epsilon}{H})^{\frac{1}{2}})\\
&\le \sum_{T_H\subseteq\Omega}C_1H^2\|f\|^2_{\mathbf{L}^2(T_H)}+\sum_{T_H\subseteq\Omega}C_2\|f\|_{\mathbf{L}^2(T_H)}(\frac{\epsilon}{H})^{\frac{1}{2}}
\end{aligned} \tag{5.137}$$

From (5.119), we obtain

$$B_1 \lesssim \sum_{T_h\subseteq\Omega}\eta^2_{T_h}+\sum_{T_H\subseteq\Omega}\eta^2_{T_H}+\left(\sum_{T_H\subseteq\Omega}\|f\|^2_{\mathbf{L}^2(T_H)}(\frac{\epsilon}{H})\right)^{\frac{1}{2}} \tag{5.138}$$

Now we estimate the term

$$B_2 := \sum_{E_H\subseteq\mathring{\Omega}}\int_{E_H}[\sigma(\mathbf{u}_H^{Ms})\cdot\mathbf{n}]_{E_H}(\mathbf{u}-\mathbf{v}_H^{Ms})\,ds. \tag{5.139}$$

Using $\mathbf{v}_H^{Ms} = \mathcal{I}_H \mathbf{u}$ on ∂T , we obtain (see (5.122)-(5.128))

$$B_2 \lesssim \sum_{E_h \subseteq E_H \subseteq \mathring{\Omega}} \eta_{E_h}^2 + \sum_{E_H \subseteq \mathring{\Omega}} \eta_{E_H}^2 + \frac{\epsilon}{2} |\mathbf{e}|_{\mathbf{H}^1(\Omega)}^2. \tag{5.140}$$

Similarly to Theorem 5.6, we obtain

$$\begin{aligned} B_3 &:= \langle \sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} - \mathbf{t}_0 - \phi_H, \mathbf{u} - \mathbf{v}_H^{Ms} \rangle \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \| \sigma(\mathbf{u}_H^{Ms}) \cdot \mathbf{n} - \mathbf{t}_0 - \phi_H \|_{\mathbf{L}^2(\Gamma)}^2 + \frac{\epsilon}{2} |\mathbf{e}|_{\mathbf{H}^1(\Omega)}^2 \end{aligned} \tag{5.141}$$

$$\begin{aligned} B_4 &:= H^{\frac{1}{2}} \| (\mathcal{K}' + 1)\phi_H + \mathcal{W}\xi_H \|_{\mathbf{L}^2(\Gamma)} \inf_{\tilde{\xi}_H} \| \xi - \tilde{\xi}_H \|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \| (\mathcal{K}' + 1)\phi_H + \mathcal{W}\xi_H \|_{\mathbf{L}^2(E_H)}^2 + \frac{\epsilon}{2} \| \xi - \xi_H \|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^2, \end{aligned} \tag{5.142}$$

and

$$\begin{aligned} B_5 &:= H^{\frac{1}{2}} \| \frac{\partial}{\partial s} (\mathbf{u}_0 - \mathbf{u}_H^{Ms} + \frac{1}{2}\mathcal{V}\phi_H - \frac{1}{2}(\mathcal{K}+1)\xi_H) \|_{\mathbf{L}^2(\Gamma)} \inf_{\tilde{\phi}_H} \| \phi - \tilde{\phi}_H \|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim \sum_{E_H \subseteq \Gamma} H \| \frac{\partial}{\partial s} (\mathbf{u}_0 - \mathbf{u}_H^{Ms} + \frac{1}{2}\mathcal{V}\phi_H - \frac{1}{2}(\mathcal{K}+1)\xi_H) \|_{\mathbf{L}^2(E_H)}^2 + \frac{\epsilon}{2} \| \phi - \phi_H \|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^2 \end{aligned} \tag{5.143}$$

We throw the terms $|\mathbf{e}|_{\mathbf{H}^1(\Omega)}^2$, $\|\xi - \xi_H\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^2$ and $\|\phi - \phi_H\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^2$ to the left hand side, the estimate of the Theorem follows immediatly. □

5.7 Numerical Experiments

Example 1: In this example we show the convergence of the EMsFEM for a homogeneous problem. The results obtained by the traditional FEM can be regarded as reference values. We choose Young's modulus $E = 2000$ and Poisson's ratio $\nu = 0.3$. The number of coarse elements is $M_c \times N_c$, where M_c and N_c denote the numbers of the elements in the x and y directions, respectively. We consider a number of coarse elements $M_c \times N_c$ between 12×2 and 96×16. There are $M_f \times N_f$ fine scale elements within each coarse element. In Figure 5.4-5.8 we show the distributions of the microscopic von-Mises stress obtained by the extended multiscale finite element method EMsFEM and the traditional finite element method FEM.

The difference between the EMsFEM and the traditional FEM can be observed for $M_c \times N_c = 12 \times 2$. The results obtained by EMsFEM converge to the reference values as the number of elements $M_c \times N_c$ increases as shown in Figure 5.4-5.7

Example 2. In this example we assume that the Poisson's ratio is a constant, $\nu = 0.3$, and Young's modulus varies in the range of $2.0 - 3.0 \times 10^3$ according to a uniform distribution. Young's modulus and the Poisson ratio are given as shown in Figure 5.9. The length and the width of the model are given by L_x, $L_y = L$ respectively where $L_x = N_x L_y$ and $N_x \in \mathbb{N}$. Here we choose $L = 1$ and $N_x = 6$. The small scale of the heterogeneity is denoted by L_h, and we assume that Young's modulus is constant in each element $L_h \times L_h$. The number of coarse elements is $M_c \times N_c$, where M_c and N_c denote the numbers of coarse elements in the x and y directions, respectively. In Figure 5.10 we show the deformed configuration for $H = 0.062500$. In this example we use a discretization of the size of the heterogeneity, $L_h = h$. For the FEM, good results are obtained only if the mesh size is smaller than the size of the heterogeneities L_h. The results obtained by the FEM can be regarded as reference values for $L_h = 8h$. The results obtained by the traditional FEM have relative large errors if $L_h \leq h$ [54]. The EMsFEM has higher accuracy than the traditional FEM. The numerically constructed basis functions capture the micro scale heterogeneities within each coarse element as shown in Figure 5.11.

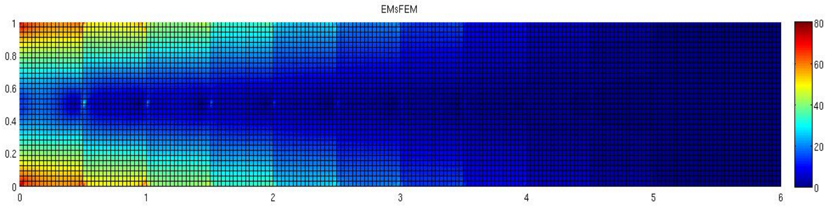


Figure 5.4: The distributions of the Mises stress obtained by the EMsFEM with $M_c \times N_c = 12 \times 2$, $M_f \times N_f = 16 \times 16$, $H = 0.5$, $h = 0.031250$

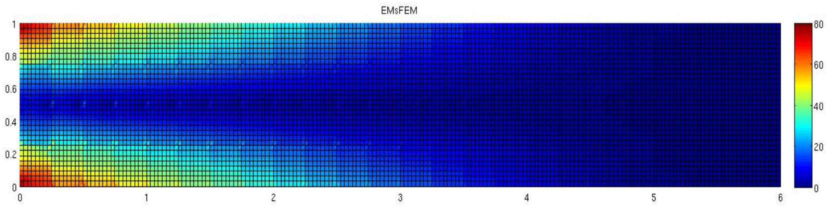


Figure 5.5: The distributions of the Mises stress obtained by the EMsFEM with $M_c \times N_c = 24 \times 4$, $M_f \times N_f = 8 \times 8$, $H = 0.25$, $h = 0.031250$

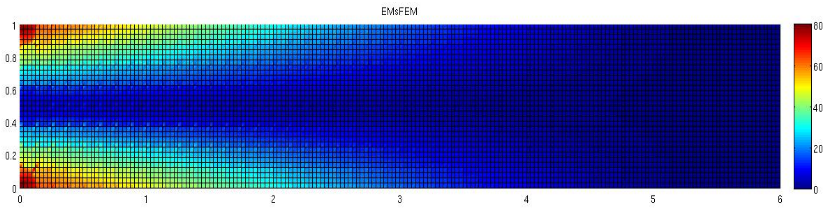


Figure 5.6: The distributions of the Mises stress obtained by the EMsFEM with $M_c \times N_c = 48 \times 8$, $M_f \times N_f = 4 \times 4$, $H = 0.125$, $h = 0.031250$

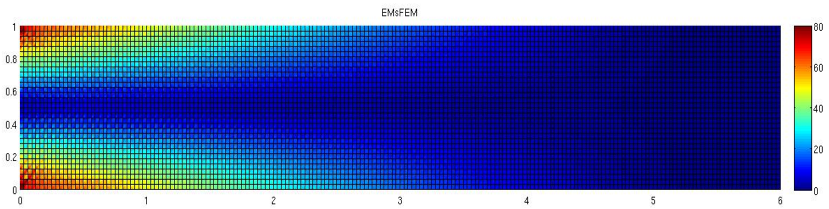


Figure 5.7: The distributions of the Mises stress obtained by the EMsFEM with $M_c \times N_c = 96 \times 16$, $M_f \times N_f = 2 \times 2$, $H = 0.062500$, $h = 0.031250$

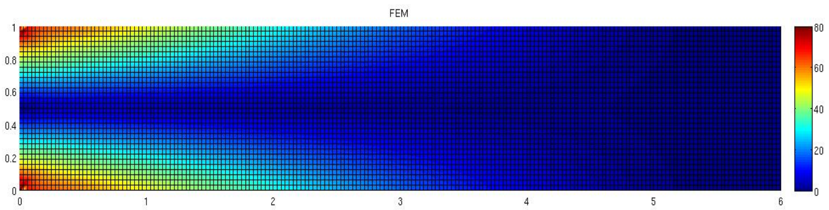


Figure 5.8: The distributions of the Mises stress obtained by FEM with $h = 0.031250$

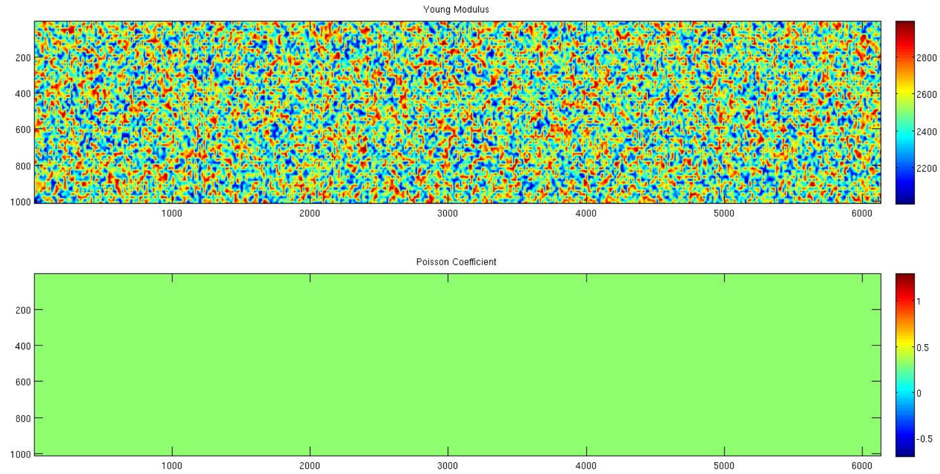


Figure 5.9: The Young's modulus and the Poisson's ratio for the heterogeneous model

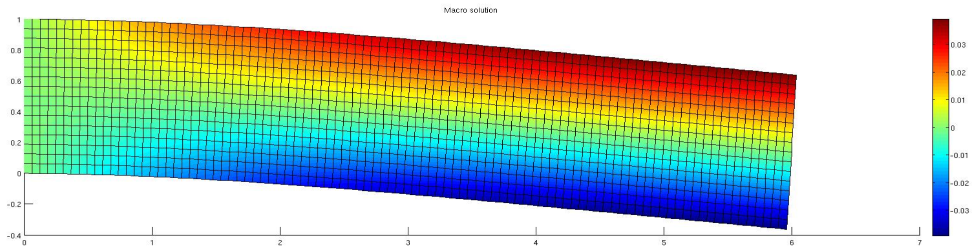


Figure 5.10: Deformed geometry

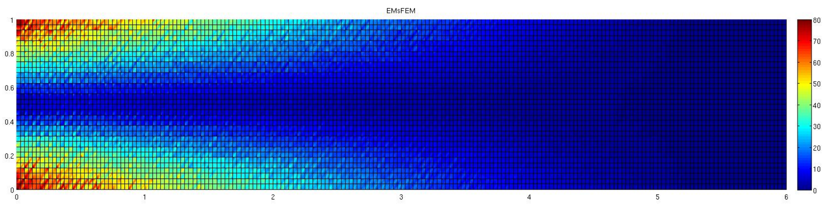


Figure 5.11: The distributions of the Mises stress obtained by the EMsFEM with $M_c \times N_c = 96 \times 16$, $H = 0.062500$, $L_h = h = 0.031250$

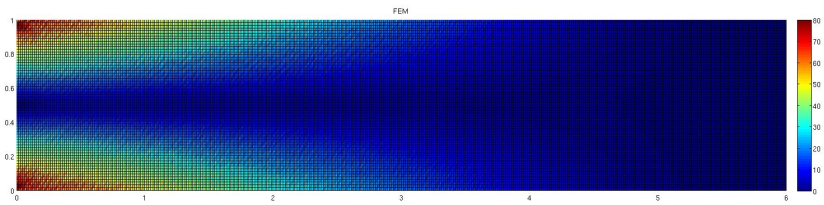


Figure 5.12: The distributions of the Mises stress obtained by FEM with $L_h = 0.031250$ and $h = 0.0039$ ($L_h = 8h$)

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Curriculum Vitae

Personal Data:

Name	Abderrahman Issaoui
Born	03.02.1977 in Metlaoui, Tunisia
Citizenship	Tunisian, German
Family Status	Married, one child

Education and Employment:

1996	Baccalaureat (Abitur) at secondary school, Metlaoui
10/1996 – 05/2001	Student of Mathematics and Physics Faculte des sciences de Bizert (Tunisia)
05/2001	Maitrise in Mathematics
2001 – 2003	Teacher at a private school
2003–2005	German language courses
10/2005 – 03/2010	Start of studies of Mathematics with minor subject Physics at the Leibniz Universität Hannover
03/2010	Diploma in Mathematics, topic of the Diploma thesis: *Mathematische Analysis des Skin Effekts*
since 04/2010	Research assistant and Ph.D. student at the Institute for Applied Mathematics (IfAM) / IRTG 1627 at the Leibniz Universität Hannover