Spectral Triples on Carnot Manifolds

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Dipl.Math. Stefan Hasselmann

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Referent: Prof. Dr. Christian Bär
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Abstract

We analyze whether one can construct a spectral triple for a Carnot manifold M, which detects its Carnot-Carathéodory metric and its graded dimension. Therefore we construct self-adjoint horizontal Dirac operators D^H and show that each horizontal Dirac operator detects the metric via Connes' formula, but we also find that in no case these operators are hypoelliptic, which means they fail to have a compact resolvent.

First we consider an example on compact Carnot nilmanifolds in detail, where we present a construction for a horizontal Dirac operator arising via pullback from the Dirac operator on the torus. Following an approach by Christian Bär to decompose the horizontal Clifford bundle, we detect that this operator has an infinite dimensional kernel. But in spite of this, in the case of Heisenberg nilmanifolds we will be able to discover the graded dimension from the asymptotic behavior of the eigenvalues of this horizontal Dirac operator. Afterwards we turn to the general case, showing that any horizontal Dirac operator fails to be hypoelliptic. Doing this, we develop a criterion from which hypoellipticity of certain graded differential operators can be excluded by considering the situation on a Heisenberg manifold, for which a complete characterization of hypoellipticity in known by the Rockland condition.

Finally, we show how spectral triples can be constructed from horizontal Laplacians via the Heisenberg pseudodifferential calculus developed by Richard Beals and Peter Greiner. We suggest a few of these constructions, and discuss under which assumptions it may be possible to get an equivalent metric to the Carnot-Carathéodory metric from these operators. In addition, we mention a formula by which the Carnot-Carathéodory metric can be detected from arbitrary horizontal Laplacians.

Keywords: Spectral triple, Carnot-Carathéodory metric, Hypoellipticity.

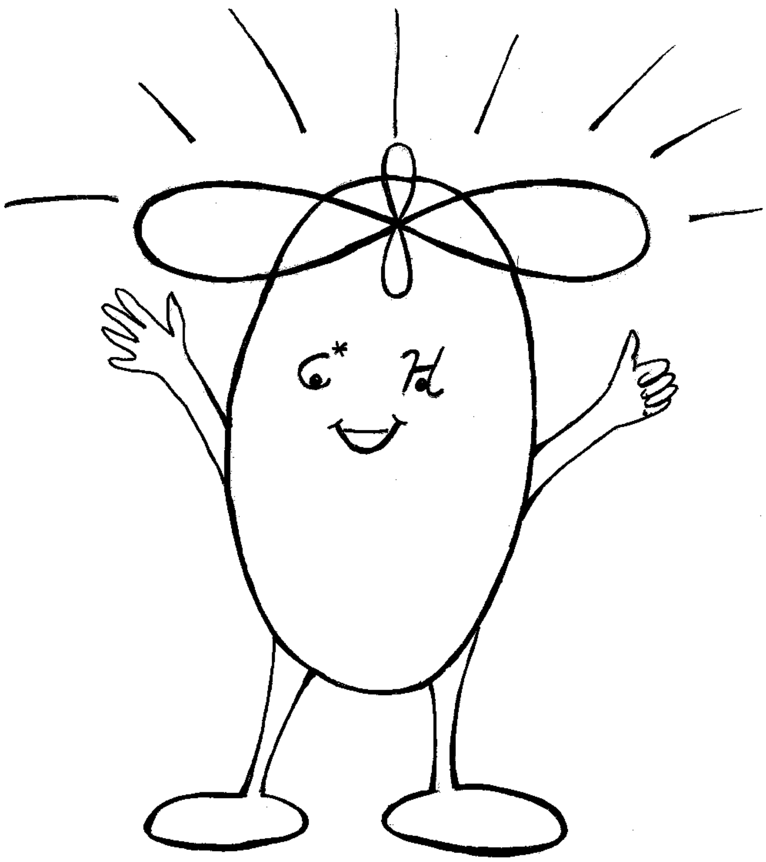
Zusammenfassung

Wir untersuchen, inwiefern man auf einer Carnot-Mannigfaltigkeit M ein spektrales Tripel konstruieren kann, welches die Carnot-Carathéodory Metrik und die gradierte Dimension von M erkennen soll. Zu diesem Zweck konstruieren wir selbst-adjungierte horizontale Dirac Operatoren D^H und zeigen, dass zwar jeder horizontale Dirac Operator über Connes' Formel die Metrik erkennt, allerdings in keinem Fall hypoelliptisch ist und somit keine kompakte Resolvente besitzen kann.

Zunächst betrachten wir ein Beispiel auf kompakten Carnot Nilmannigfaltigkeit detailliert, wobei wir eine Konstruktion für einen horizontalen Dirac Operator über den Pull-back des Dirac Operators auf dem Torus durchführen. Einer Methode von Christian Bär folgend können wir das horizontale Clifford Bündel dieses Operators zerlegen und erkennen, dass der Operator einen unendlich dimensionalen Kern besitzt. Dennoch können wir im Fall von Heisenberg Nilmannigfaltigkeiten die gradierte Dimension aus dem asymptotischen Verhalten der Eigenwerte dieses horizontalen Dirac-Operators erkennen. Anschließend wenden wir uns dem allgemeinen Fall zu, indem wir zeigen dass ein beliebiger horizontaler Dirac Operator nicht hypoelliptisch ist. Dazu entwickeln wir ein Kriterium mit dem man die Hypoelliptizität von bestimmten gradierten Differentialoperatoren ausschließen kann indem man die Situation auf einer Heisenberg Mannigfaltigkeit betrachtet, für welche eine vollständige Charakterisierung der Hypoelliptizität durch die Rockland-Bedingung gegeben ist.

Schließlich zeigen wir wie man spektrale Tripel aus horizontalen Laplace Operatoren mit Hilfe des Heisenberg Pseudodifferentialkalküls, das von Richard Beals und Peter Greiner entwickelt wurde, konstruieren kann. Wir stellen ein paar explizite Konstruktionen vor und diskutieren, unter welchen Voraussetzungen es möglich sein kann aus diesen Operatoren eine zu der Carnot-Carathéodory Metrik äquivalente Metrik zu erhalten. Zusätzlich erwähnen wir eine Formel, mit der die Carnot-Carathéodory Metrik aus beliebigen horizontalen Laplace Operatoren erkannt werden kann.

Schlüsselwörter: Spektrales Tripel, Carnot-Carathéodory Metrik, Hypoelliptizität.



A spectral triple. By S. Wiencierz.

Du sollst dich zu einer Stadt begeben, die den Namen Tsnips-Eg'N-Rih trägt!

(Walter Moers)

Introduction

In the 1980s, Alain Connes presented the concept of non-commutative geometry as an extension of the usual notion of a topological space (see e.g. [Con94], [Con85], [Con89]). The idea goes back to the 1940s, when Israel Gelfand and Mark Naimark showed that every commutative C^*-Algebra is isomorphic to the C^*-algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X vanishing at infinity (see [GN43]). From this starting point, one sees that a lot of properties of the space X can be translated into properties of its C^*-algebra $C_0(X)$. Motivated by this, in non-commutative geometry one considers a general C^*-algebra as a "non-commutative space".

To describe geometry on a non-commutative space, Connes introduced so-called *spectral triples*. The definition of a spectral triple is suggested by the fact that many geometric properties of a compact connected Riemannian spin manifold M without boundary can be obtained from the Dirac operator D acting on a Clifford bundle ΣM over M. For example, one can reproduce the dimension of M by the asymptotic growth of the eigenvalues of D via

$$\dim M = \inf \left\{ p \in \mathbb{R} : \left(D^2 + I\right)^{-\frac{p}{2}} \text{ is trace class} \right\},$$

and one can detect the geodesic distance on M by the formula

$$d_{geo}(x, y) = \sup \left\{ |f(x) - f(y)| : f \in C^\infty(M), \|[D, f]\|_{L^2(\Sigma M)} \leq 1 \right\}.$$

These properties can be transported to the picture of C^*-algebras: A spectral triple is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is a C^*-algebra, $\mathcal{H}$ is a Hilbert space carrying a faithful action of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$ and $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$ such that $[\mathcal{D}, a]$ is bounded for a belonging to a dense sub-algebra of $\mathcal{A}$ and such that the resolvent $(\mathcal{D}^2 + I)^{-1/2}$ of $\mathcal{D}$ is compact. For a general spectral triple, one can define notions of dimension and metric in analogy to the formulas above. Hence on a closed Riemannian spin manifold with Dirac operator D, a spectral triple which detects the dimension and the geodesic metric on M is given by the triple $(C(M), L^2(\Sigma M), D)$.

During the last decades there have been several approaches to construct spectral triples for more examples than the one of a closed Riemannian spin manifold. In particular there are some constructions for certain fractals which have a non-integer Hausdorff dimension, done for example by Erik Christensen, Christina Ivan and their collaborators (see e.g. [CIL08], [CIS12]): For some fractals, it is possible to detect the Hausdorff dimension as well as the geodesic distance of the space from a spectral triple. A more general approach has been suggested by Ian Palmer ([Pal10]) and by John Pearson and Jean Bellissard [PB09]). Palmer shows that under mild conditions the Hausdorff dimension of every compact metric space can be discovered by a spectral triple. For these triples one can also find an estimate from above for the metric. But the constructions Palmer is considering lead away from the geometry of the space, since these constructions make use of an approximation of the space by a discrete subset.

From the aspect of dimension, it is also interesting to note that there has been a construction by Erik Christensen and Christina Ivan showing that to any given positive real number s, one can construct a spectral triple of dimension s, where $\mathcal{A}$ is a limit of finite-dimensional C^*-algebras and $\mathcal{D}$ is a limit of finite-dimensional operators ([CI06]). Hence at least theoretically it is possible to define spectral triples of arbitrary dimension on certain spaces.

The question we are dealing with in this thesis is whether it is possible to define two different spectral triples on one space, which both give reasonable geometries (in terms of dimension and metric). Therefore we consider so-called *sub-Riemannian manifolds* (or in a more specialized setting *Carnot manifolds*), which are Riemannian manifolds M equipped with a bracket-generating horizontal sub-bundle HM of their tangent bundle TM. It has been detected by Wei-Liang Chow ([Cho39]) in 1939 that in this case and if M is connected, any two points on M can be connected by a curve which is tangent to the horizontal distribution HM. This means that we obtain a metric on M via the formula

$$d_{CC}(x,y) := \inf \left\{ \int_0^1 \|\dot{\gamma}(t)\|\, dt : \gamma \text{ horizontal path with } \gamma(0) = x \text{ and } \gamma(1) = y \right\},$$

which differs from the metric induced by the geodesic distance on M. In addition, the Hausdorff dimension of the metric space (M, d_{CC}) turns out to be strictly greater than the Hausdorff dimension of the metric space (M, d_{geo}). This result is due to John Mitchell ([Mit85]) and is also known as *Mitchell's Measure Theorem*.

The most important example for such a sub-Riemannian manifold is the $(2m+1)$-dimensional Heisenberg group $\mathbb{H}^{2m+1}$. It can be represented as the matrix group consisting of matrices of the form

$$\mathbb{H}^{2m+1} = \left\{ \begin{pmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}^m, z \in \mathbb{R} \right\},$$

where the group composition is given by matrix multiplication. Note that as a point set $\mathbb{H}^{2m+1}$ is isomorphic to $\mathbb{R}^{2m+1}$, and that $\mathbb{H}^{2m+1}$ has the structure of a graded nilpotent Lie group. The grading is induced by the Lie algebra $\mathfrak{h}_{2m+1}$ of $\mathbb{H}^{2m+1}$, which is of the form $\mathfrak{h}_{2m+1} = V_1 \oplus V_2$ such that $\dim V_1 = 2m$, $\dim V_2 = 1$ and $[V_1, V_1] = V_2$. More generally, we will consider Carnot groups. A Carnot group is a nilpotent Lie group $\mathbb{G}$ whose Lie algebra is carrying a grading such that $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$ with $[V_S, V_1] = V_{S+1}$ for $S \leq R-1$ and $[V_S, V_R] = 0$ for all $1 \leq S \leq R$. For our work, we will consider Carnot manifolds which will be defined to be Riemannian manifolds which carry such a grading structure on their tangent bundle.

In this thesis, we will use the geometric approach by Dirac operators to discuss whether one can construct spectral triples on compact Carnot manifolds, which furnish the Hausdorff dimension of (M, d_{CC}) and the Carnot-Carathéodory metric of d_{CC}. We will indeed be able to construct so-called *horizontal Dirac operators* D^H acting on a horizontal Clifford bundle $\Sigma^H M$ over a Carnot manifold M in analogy to classical Dirac operators in a quite general setting. In a frame $\{X_1, \dots, X_d\}$ of the horizontal distribution of a Carnot manifold such an operator will have the form

$$D^H = \sum_{j=1}^{d} c^H(X_j)\nabla^{\Sigma^H}_{X_j} + \gamma,$$

where c^H denotes the Clifford action of HM and γ is an endomorphism on the bundle. Then indeed the Carnot-Carathéodory metric can be detected from D^H, as we will show in detail, and it does not matter if one uses the Lipschitz functions with respect to the Carnot-Carathéodory metric or the smooth functions for the dense sub-algebra of $C(M)$ appearing in Connes' metric formula. But we will see that these operators are not hypoelliptic, which means that they do not have a compact resolvent (arising from pseudodifferential calculus). Hence they do not furnish a spectral triple, providing us the unexpected result that the theory of spectral triples does not apply to the Carnot manifolds (M, d_{CC}) in the way one would expect.

After introducing horizontal Dirac operators and proving the metric formula, we will construct a class of examples. To this end we consider local homogeneous spaces of Carnot groups, which arise from the left-action of a lattice sub-group Γ of a Carnot group $\mathbb{G}$. We call these closed Carnot manifold $M = \Gamma\backslash\mathbb{G}$ compact Carnot nilmanifolds. Then the idea is to consider the submersion

$$\pi : \Gamma\backslash M \to \mathbb{T}^d$$

and to define a horizontal Clifford module and a horizontal Dirac operator D^H by pulling back the spinor bundle and the Dirac operator from a spin structure on $\mathbb{T}^d$. For the case where $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^n$, we will be able to calculate the spectrum of this horizontal Dirac operator completely. To do this, we adapt an argument by Christian Bär to our situation([Bae91]): Bär decomposes the spinor bundle belonging to a classical Dirac operator on a Heisenberg nilmanifold $\Gamma\backslash\mathbb{H}^{2m+1}$ into its irreducible components under which the operator is invariant. From these irreducible components, he is able to calculate the spectrum of the Dirac operators by means of the representation theory of the Heisenberg group.

In our case, the horizontal Clifford bundle decomposes in the same way. We will present these calculations in detail, and afterwards we will deduce that our horizontal pull-back Dirac operator D^H on a compact Carnot nilmanifold possesses at least one infinite dimensional eigenspace, hence it cannot have a compact resolvent. In spite of that we will be able to detect the Hausdorff dimension of (M, d_{CC}) in the Heisenberg group case from the asymptotic behavior of the non-degenerate eigenvalues of D^H. We will also extend this approach to the setting of the compact nilmanifold of a general Carnot group $\mathbb{G}$, where we will show that in the spectral decomposition of such a group there is at least one infinite dimensional eigenspace. The reason for this is that in the spectral decomposition of D^H there is at least one subspace isomorphic to the space of L^2-sections on a horizontal Clifford bundle belonging to a group of nilpotency step 2. On this subspace, we will be able to detect an infinite dimensional eigenspace of the horizontal pull-back Dirac operator from the Heisenberg case.

Further we will see that the problem we detected in this concrete example is not due to a bad choice of the horizontal Dirac operator. In fact it is a general phenomenon: We will argue for that using techniques from pseudodifferential calculus. There is a calculus invented by Richard Beals and Peter Greiner (see [BG84]) on Heisenberg manifolds, from which it can be derived that hypoellipticity of a (self-adjoint) operator of positive order implies that it has a compact resolvent and that furthermore the Hausdorff dimension of the Heisenberg manifold can be detected by the eigenvalue asymptotics of this operator. We will prove in this thesis that any horizontal Dirac operator on an arbitrary Carnot manifold cannot be hypoelliptic. This is a big difference to the classical case, where any Dirac operator is elliptic.

On the way, we will develop a criterion for the non-hypoellipticity of an arbitrary graded differential operator of the from

$$D = D(X_1, \dots, X_n) \in \mathcal{U}(\mathfrak{g}),$$

where $\{X_1, \dots, X_n\}$ is a frame of $\mathbb{R}^n$ which is forming a graded nilpotent Lie algebra $\mathfrak{g}$. As soon as there is a graded differential operator

$$\tilde{D} = \tilde{D}(\tilde{X}_1, \dots \tilde{X}_m) \in \mathcal{U}(\tilde{\mathfrak{g}}) \text{ for } m < n,$$

induced by a projection $\mathrm{pr} : X_j \mapsto \tilde{X}_j$ of $\mathfrak{g}$ onto a lower dimensional Lie algebra $\tilde{\mathfrak{g}}$, which is not hypoelliptic, then D cannot be hypoelliptic. This criterion will serve us well, since we will be able to reduce the problem of showing general non-hypoellipticity of a horizontal Dirac operator to the Heisenberg case. In the Heisenberg case we have a complete characterization of hypoellipticity of horizontal Laplacians arising from the representation theory of the Heisenberg group, from which we will be able to exclude that a horizontal Dirac operator is hypoelliptic.

Finally, once having introduced the Heisenberg pseudodifferential calculus, we make use of this calculus and show how hypoelliptic Heisenberg pseudodifferential operators furnishing a spectral triple and detecting in addition the Hausdorff dimension of the Heisenberg manifold can be constructed. We will suggest a few concrete operators, but it remains unclear whether one can detect or at least estimate the Carnot-Carathéodory metric from them. But we will show that the Carnot-Carathéodory metric can be detected by horizontal Laplacians instead of horizontal Dirac operators via the formula

$$d_{CC}(x,y) = \sup\left\{ |f(x) - f(y)| : f \in C^\infty(M), \left\| \frac{1}{2} \left[\left[\Delta^{\mathrm{hor}}, f \right], f \right] \right\| \le 1 \right\}.$$

Hence, maybe in this case the second order operators are the right operators to look at when we want to do non-commutative geometry on Carnot manifolds.

We conclude the thesis with some approaches from which one may be able to estimate or to approximate the Connes metric by first order Heisenberg pseudodifferential operators, and we present some criteria from which such an estimate would follow.

The structure of this thesis will be the following.

- In Chapter 1, we introduce the notion of spectral triples, their metric dimension and the Connes metric, and state a few well-known examples. Then we will turn to the more general approach by Mark Rieffel of compact quantum order-unit spaces and prove a few criteria to detect convergence of a family of spectral triples to such a compact quantum metric space (which does not necessarily have to be a spectral triple).

- In Chapter 2, we give an overview of sub-Riemannian geometry and Carnot manifolds. We introduce the Carnot-Carathéodory metric and state some important theorems in this context. The Sections 2.3 and 2.4 serve to introduce the concept of a Levi form and of certain submersions between Carnot groups which will be of importance later.

- In Chapter 3, we construct the horizontal Dirac operator. We start by analyzing horizontal connections, which will be the connections we want a horizontal Clifford bundle to be compatible with. Then we construct a self-adjoint horizontal Dirac operator on these bundles. In preparation of what we need later we will calculate its square locally and we will state a proposition about the eigenvalues of certain sums of Clifford matrices. Finally, in Section 3.3, we will show that any horizontal Dirac operator on a Carnot manifold M detects the Carnot-Carathéodory metric via Connes' metric formula, since the norm of the commutator $[D^H, f]$ coincides with the Lip-norm of f with respect to the Carnot-Carathéodory metric. We will also show that the metric is already detected by the smooth functions on M.

- In Chapter 4, we treat in detail the example of nilmanifolds $M = \Gamma\backslash\mathbb{G}$ from Carnot groups, arising from the left action of the standard lattice. First we construct a horizontal Dirac operator on M by pulling back the spinor bundle of the horizontal torus. Then it is our aim to show that this horizontal Dirac operator possesses infinite dimensional eigenspaces. Therefore we use an approach which was presented by Christian Bär and Bernd Amman ([Bae91], [AB98]) for the case of the classical Dirac operator on Heisenberg nilmanifolds to find a spectral decomposition of the horizontal Clifford bundle $\Sigma^H M$ which is invariant under D^H. For the case of a general Carnot group, we find that one part of this decomposition is isomorphic to the horizontal Clifford bundle belonging to a horizontal pull-back operator of a Carnot nilmanifold of lower commutator step. For the case of Heisenberg nilmanifold (where the horizontal distribution is of co-dimension 1) we will be able to calculate all eigenvalues of D^H from the approach by Bär and Ammann, and we will detect that D^H has an infinite-dimensional kernel. But in spite of that we will show that the Hausdorff dimension of M can still be detected by these eigenvalues for the case $\mathbb{G} \cong \mathbb{H}^{2m+1}$. In section 4.4 we will put all these results together to show that on any Carnot nilmanifold arising from the left-action of the standard lattice sub-group of $\mathbb{G}$ the horizontal Dirac operator we constructed has an infinite dimensional kernel.

- In Chapter 5, we introduce the Heisenberg pseudodifferential calculus while referring to Richard Beals and Peter Greiner ([BG84]) and to Raphaël Ponge. We will discuss symbol classes, composition of operators, parametrices and the role of hypoellipticity. Finally we mention the well-known results that hypoellipticity in the Heisenberg calculus implies the existence of complex powers and certain eigenvalue asymptotics.

- In Chapter 6, we show that the absence of a compact resolvent detected in Chapter 4 for a specific class of examples is of general nature. In particular we show that any horizontal Dirac operator is not hypoelliptic. Therefore we first review some classical hypoellipticity theorems with special attention to the case of Heisenberg manifolds. Then in Section 6.2 we prove a theorem from which hypoellipticity can be excluded for certain graded differential operators by going back to a graded differential operator acting on a lower dimensional Carnot group. We will present some consequences which arise from going back to the Heisenberg case via this reduction: This reduction criterion provides us with the possibility to exclude hypoellipticity of an arbitrary graded differential operator by looking at the Levi form of its underlying graded Lie algebra. Finally, we will apply the reduction criterion to show that an arbitrary horizontal Dirac operator cannot be hypoelliptic.

- In Chapter 7, we discuss the possibility of constructing spectral triples by taking square roots of horizontal Laplacians and the question if one can get any metric information from these spectral triples. First we show how the application of Heisenberg calculus furnishes spectral triples, and we suggest a few operators being not too far away from the horizontal Dirac operator which produce spectral triples. In Section 7.2, we show that the Carnot-Carathéodory metric can be detected by any horizontal Laplacian, while the business is much more difficult if we want to detect the metric from an arbitrary first-order hypoelliptic and self-adjoint Heisenberg operator. Unfortunately, all we can present concerning the last question are some criteria and some ideas which might approximate the Carnot-Carathéodory metric by metrics arising from spectral triples, but so far we have not been able to prove such an approximation or estimate completely.

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Contents

Chapter 1

Non-commutative Metric Spaces

The intention of the first chapter of this thesis is to introduce the concepts of non-commutative geometry we want to consider. In Section 1.1 we will present the concept of a spectral triple introduced by Alain Connes, which describes the geometry of a space in non-commutative geometry. This will only be a rough introduction covering only the notion of dimension and metric arising from a spectral triple, since these are the objects we are interested in. After defining these objects, we will give a few classical examples for commutative spectral triples to motivate the work of this thesis. The Carnot-Carathéodory spaces which we want to examine from the point of view of non-commutative geometry will be treated in Chapter 2.

Since it will turn out that we do not get spectral triples in the sense of Connes from our constructions, we will state a modified definition of a degenerate spectral triple, to denote an object being a spectral triple except for a degeneration of certain eigenspaces of its Dirac operator. Later in Chapter 4 we will see that this definition fits in our situation. Besides that, in Section 1.2 we will present the more general concept of compact quantum metric spaces, which was introduced by Mark Rieffel and which only provides metric information in the abstract setting of Lip-norms. A Lip-norm can arise from an operator which does not provide a spectral triple, as it will be the case for our situation.

We will close Section 1.2 by some simple observations we made leading to a sufficient criterion for the convergence of a sequence of compact quantum metric spaces to a certain given space. In detail, we show that the metrics ρ_θ arising from a family of Lip-norms L_θ converge to a given metric ρ, if the corresponding Lip-norms converge to the Lip-norm L corresponding to ρ in a uniform way. We will refer to these observations in the final chapter of this thesis, where we will be in the situation that we have a family of spectral triples providing metrics close to the metric we want to discover. But the desired metric itself will be detected by an operator which does not furnish a spectral triple.

1.1 Spectral Triples

We will give a rough overview about the theory of spectral triples now, mentioning only the concepts we will use in this thesis. Most of the definitions and examples we give are well known and can be found in any textbook on non-commutative geometry, see e.g. [Con94], [GVF01] or [Lan97]. First of all we define what a spectral triple is.

Definition 1.1.1
Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space which carries an injective unitary representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Furthermore let $\mathcal{D}$ be an unbounded self-adjoint operator on $\mathcal{H}$ such that

(i) The algebra

$$\mathcal{A}' := \{a \in \mathcal{A} : [\mathcal{D}, \pi(a)] \text{ is densely defined and bounded } \} \tag{1.1}$$

is a dense sub-algebra of $\mathcal{A}$.

(ii) For any number $\lambda \notin \operatorname{spec}(\mathcal{D})$ the operator $(\mathcal{D} - \lambda I)^{-1}$ is compact.

Then the triple $(\mathcal{A}, \mathcal{H}, D)$ is called a spectral triple. ◁

Remark: Considering condition (ii), it follows from the Hilbert identity that the property of $(\mathcal{D} - \lambda I)^{-1}$ being compact for one $\lambda \notin \operatorname{spec}(\mathcal{D})$ implies that $(\mathcal{D} - \lambda I)^{-1}$ is compact for every $\lambda \notin \operatorname{spec}(\mathcal{D})$. Hence one often reformulates condition (ii) in the sense that

$$\left(\mathcal{D}^2 + I\right)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H}),$$

or by simply demanding that $\mathcal{D}$ *has a compact resolvent.* ◁

For a spectral triple we can define a notion of dimension.

Definition 1.1.2
A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called s-summable for some $s \in \mathbb{R}$, if we have

$$\left(\mathcal{D}^2 + I\right)^{-\frac{s}{2}} \in \mathcal{L}_1(\mathcal{H}), \tag{1.2}$$

where $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ denotes the ideal of trace-class operators on $\mathcal{H}$. The number

$$s_0 = \inf \left\{ s \in \mathbb{R} : \left(\mathcal{D}^2 + I\right)^{-\frac{s}{2}} \in \mathcal{L}_1(\mathcal{H}) \right\} \tag{1.3}$$

is called the metric dimension *of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.* ◁

Remark: Like in Definition 1.1.1, one can replace the operator $(\mathcal{D}^2 + I)^{-s/2}$ in (1.2) and (1.3) by any operator of the form $(\mathcal{D} - \lambda I)^{-s}$ for $\lambda \notin \operatorname{spec}(\mathcal{D})$. ◁

One can also define a metric from a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on the state space $\mathcal{S}(\mathcal{A})$ of its C^*-algebra $\mathcal{A}$, equipped with the weak *-topology. It is not hard to check that the expression

$$d_{\mathcal{D}}(\phi, \psi) := \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{A}', \|[\mathcal{D}, \pi(a)]\| \leq 1\}, \tag{1.4}$$

where $\mathcal{A}'$ is the dense sub-algebra (1.1) from condition (i) of the definition of a spectral triple, gives a metric on $\mathcal{S}(\mathcal{A})$.

Definition 1.1.3
Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple, $\mathcal{S}(\mathcal{A})$ be the state space of its C^*-algebra $\mathcal{A}$. Then the metric $d_{\mathcal{D}}$ on $\mathcal{S}(\mathcal{A})$ from (1.4) *is called the* Connes metric *arising from $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.* ◁

It turns out that it is sufficient to consider the expression (1.4) only for the positive elements of $\mathcal{A}$. We cite this proposition here, for a proof we refer to [IKM01], Section 2, Lemma 1.

Proposition 1.1.4
Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple, and let $\mathcal{A}_+$ denote the subset of positive elements of the C^*-algebra $\mathcal{A}$. Then the Connes metric (1.4) *on the state space $\mathcal{S}(\mathcal{A})$ is given by*

$$d_{\mathcal{D}}(\phi, \psi) = \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{A}_+, \|[\mathcal{D}, \pi(a)]\| \leq 1\}$$

for all $\phi, \psi \in \mathcal{S}(\mathcal{A})$. □

Spectral triples can be defined on arbitrary C^*-algebras. In non-commutative geometry, the notion of a C^*-algebra replaces in a way the notion of a space. This is motivated by the Gelfand-Naimark theory: The famous theorem of Israel Gelfand and Mark Naimark (see [GN43]) states that every unital commutative C^*-algebra $\mathcal{A}$ is isometrically isomorphic to the C^*-algebra $C(X)$ of the continuous functions on some compact Hausdorff space X. In this case the state space of $\mathcal{A}$ is exactly the space X, which turns (1.4) into the expression

$$d_{\mathcal{D}}(x, y) := \sup\{|f(x) - f(y)| : f \in \mathcal{A}' \subset C(X), \|[\mathcal{D}, \pi(a)]\| \leq 1\}, \tag{1.5}$$

defining a metric on X. In this thesis, we will only consider this commutative situation.

We will now state the classical (and most important) example of a commutative spectral triple, where X is a compact Riemannian manifold. The central point of this thesis is to construct another commutative spectral triple on this manifold X which describes a different geometry, meaning that one gets a different dimension and a different metric from it.

Example 1.1.5
Let M be an n-dimensional closed Riemannian manifold equipped with a Clifford bundle ΣM and a Dirac operator D. We further consider the algebra $C(M)$ of continuous functions on M and the Hilbert space $L^2(\Sigma M)$ of L^2-sections in the spinor bundle, on which $C(M)$ has a representation by left multiplication. Then we have the following (see e.g. [GVF01], [Lan97] or [Con94]):

- The triple $(C(M), L^2(\Sigma M), D)$ is a spectral triple. A dense sub-algebra of $C(M)$, which furnishes a bounded commutator with D, is given by $C^\infty(M)$ or also by the Lipschitz functions $\mathrm{Lip}(M)$ on M. And since D is an elliptic and self-adjoint differential operator of order 1 with discrete spectrum, it has a compact resolvent.

- The metric dimension of this spectral triple is exactly the dimension n of the manifold: By Weyl asymptotics we have $\lambda_k \sim k^{1/n}$ for the eigenvalues of D (since it is an elliptic and self-adjoint differential operator of order 1), which implies that for $\lambda \notin \mathrm{spec}(D)$ the operator $(D - \lambda I)^{-s}$ is trace class if and only if $s > n$.

- The geodesic distance d_{geo} on M can be detected by the formula

  $$d_{geo}(x, y) = d_D(x, y) = \sup\left\{|f(x) - f(y)| : f \in \mathrm{Lip}(M), \|[D, \pi(f)]\| \leq 1\right\}.$$

  This is the case, since $\|[D, \pi(f)]\|$ is exactly the essential supremum of the gradient of f: One can show

  $$|f(x) - f(y)| = \int_\gamma df \leq \operatorname{ess\,sup} \|df\| \int_0^1 \|\dot{\gamma}(t)\|\, dt$$

  for every geodesic curve connecting x and y, which gives $d_{geo}(x, y) \geq d_D(x, y)$. On the other hand we have $d_D(x, y) \geq d_{geo}(x, y)$ since for a fixed $y \in M$ the essential supremum norm of the gradient of the function $g(x) := d_{geo}(x, y)$ is bounded by 1.

$\triangleleft$

In the commutative world, one can detect further examples for spectral triples which give a meaningful geometry by considering the Hausdorff dimension of a metric space. We briefly sketch its construction.

Definition 1.1.6
Let (X, d) be a metric space, $\Omega \subset X$.

(i) Let $s > 0$. For $\varepsilon > 0$ we set

$$\mu^s_\varepsilon(\Omega) := \inf\left\{\sum_\alpha (\mathrm{diam}(U_\alpha))^s : \mathcal{U} = \{U_\alpha\} \textit{ open cover of } \Omega,\ \sup_\alpha (\mathrm{diam}(U_\alpha)) < \varepsilon\right\}.$$

*Then the s-*dimensional Hausdorff measure *of Ω is given by the number*

$$\mu^s(\Omega) = \lim_{\varepsilon \to 0} \mu^s_\varepsilon(\Omega).$$

(ii) The unique number $0 \leq \dim_H(\Omega) \leq \infty$ such that $\mu^s(\Omega) = \infty$ for all $s < \dim_H(\Omega)$ and $\mu^s(\Omega) = 0$ for all $s > \dim_H(\Omega)$ is called the Hausdorff dimension *of Ω.*

$\triangleleft$

The Hausdorff dimension is in particular used in fractal geometry, where one gets metric spaces of non-integer Hausdorff dimension. One classical example for a spectral triple by Alain Connes is a spectral triple for the Cantor set (see [Con94], Chapter 3.4.ε), whose metric dimension coincides with the Hausdorff dimension of this set. During the last years, spectral triples were constructed for fractals, detecting the Hausdorff dimension and the geodesic distance on this fractal. We refer for example to the work of Erik Christensen, Christina Ivan and their collaborators for that (see e.g. [CIL08] or [CIS12]).

Under mild conditions, it is indeed possible to construct a spectral triple for an arbitrary compact metric space X. The idea is to approximate the space by finite sets of points. Since X is assumed to be compact, one can find a sequence of open covers of X whose diameter decreases to zero. Then one chooses two points in any open set belonging to one of the covers. Using this sequence, it is possible to define a Dirac operator of a spectral triple on the Hilbert space of l^2-sequences indexed by the sets belonging to the covers with values in $\mathbb{C}^2$. This construction can be found in detail in the PhD Thesis of Ian Christian Palmer (see [Pal10]) and has also been used by John Pearson and Jean Bellissard (see e.g. [PB09]). It is shown that there exists such a sequence such that the corresponding spectral triple gives back the Hausdorff dimension of the space ([Pal10], Theorem 4.2.2), and that the Connes metric provided by this spectral triple dominates the original metric on X ([Pal10], Proposition 5.2.1).

In this thesis, we will consider compact Riemannian manifolds which carry a second metric besides the geodesic one: These so-called *Carnot manifolds* will be introduced in Chapter 2. Indeed we will be able to construct a differential operator detecting the second metric, but it will turn out that this operator does not provide a spectral triple. But we can modify the definition of a spectral triple a little bit, such that it will fit to our situation.

Definition 1.1.7
Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space which carries an injective unitary representation $\pi_{\mathcal{A}} \to \mathcal{B}(\mathcal{H})$. Furthermore let $\mathcal{D}$ be an unbounded self-adjoint operator on $\mathcal{H}$ such that

(i) The algebra

$$\mathcal{A}' := \{a \in \mathcal{A} : [\mathcal{D}, \pi(a)] \text{ is densely defined and bounded } \}$$

is a dense sub-algebra of $\mathcal{A}$.

(ii) The spectrum of the operator $\mathcal{D}$ is discrete. If in addition Λ denotes the set of all eigenvalues λ of $\mathcal{D}$ which have an infinite dimensional eigenspace $(\lambda) \subset \mathcal{H}$, then for any number $\mu \notin \operatorname{spec}(D)$ the operator

$$\left((D - \mu I)\Big|_{\left(\bigoplus_{\lambda\in\Lambda} E(\lambda)\right)^{\perp}} \right)^{-1}$$

is compact.

Then we call the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a degenerate spectral triple. $\triangleleft$

1.2 Compact Quantum Order Unit Spaces

In this section we consider a more general version of a non-commutative metric space. The idea is that for any compact metric space (X, d), on can associate a Lipschitz semi-norm to the space $C(X)$ of continuous functions on X via

$$L(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Now since a non-commutative space can be viewed as a generalization of the (commutative) C^*-algebra $C(X)$, the idea is to introduce the notion of a Lip-norm on a general unital C^*-algebra, from which one gets a metric on the state space of $\mathcal{A}$. This approach has been introduced by Mark Rieffel in [Rie98] and [Rie99]. In his approach, instead of C^*-algebras, Rieffel works in the more abstract setting of *order-unit spaces* (see [Rie04]).

Definition 1.2.1
An order-unit space *is a real partially ordered vector space* A *with a distinguished element* e *(the* order unit*), which satisfies*

(i) For each $a \in A$ *there is an* $r \in \mathbb{R}$ *such that* $a \leq re$.

(ii) If $a \in A$ *and if* $a \leq re$ *for all* $r \in \mathbb{R}$ *with* $r > 0$*, then* $a \leq 0$.

Furthermore, the norm of an order-unit space is given by

$$\|a\| := \inf \{r \in \mathbb{R} : -re \leq a \leq re\}.$$

$\triangleleft$

Note that the self-adjoint elements of a unital C^*-algebra $\mathcal{A}$ form an order unit space, such that the non-commutative spaces in the sense of Alain Connes are included in this setting. Now we introduce a semi-norm on these spaces.

Definition 1.2.2
Let (A, e) *be an order-unit space. Then a* Lip-norm *on* A *is a semi-norm* L *on* A *with the following properties.*

(i) We have $L(a) = 0 \Leftrightarrow a \in \mathbb{R}e$.

(ii) The topology on the state space $\mathcal{S}(A)$ *of* A *from the metric*

$$\rho_L(\phi, \psi) := \sup \{|\phi(a) - \psi(a)| : L(a) \leq 1\} \tag{1.6}$$

is the w^**-topology.*

We call a pair (A, L), consisting of an order-unit space A and a Lip-norm L on A a compact quantum metric space. $\triangleleft$

We recognize that the metric (1.6) looks similar to the metric (1.4) arising from a spectral triple. Indeed, if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, where the dense sub-algebra of $\mathcal{A}$ for which $[\mathcal{D}, a]$ is bounded is denoted by $\mathcal{A}'$, then the term

$$L(a) := \|[\mathcal{D}, a]\|$$

gives a Lip-norm on the order unit space A consisting of self-adjoint elements of $\mathcal{A}'$. By Proposition 1.1.4, the Connes metric (1.4) does only depend on the positive elements of $\mathcal{A}'$, and hence it is in particular determined by the self-adjoint elements of $\mathcal{A}'$ forming the order-unit space A.

Remark 1.2.3
It is shown in [Rie99], Section 11, that for any essentially self-adjoint operator D on a Hilbert space $\mathcal{H}$, which carries a faithful representation π of an order-unit space A such that $[D, \pi(a)]$ is bounded for a dense subspace A' of A, furnishes a Lip-norm via

$$L(a) := \|[D, \pi(a)]\|.$$

On the other hand, in [Rie04], Appendix 2, it is shown that for any lower semi-continuous Lip-norm L on an order-unit space A one can define such an operator D describing L via $L(f) = \|[D, f]\|$ on the space $C(\mathcal{S}(A))$ of continuous functions on the state space of A.

Note that in both cases the operator D does not need to have a compact resolvent. In particular, the operator D constructed to a given semi-norm L is in general far away from having a compact resolvent. $\triangleleft$

We will see that for the compact spaces we study in this thesis, the metric will be detected by a Dirac operator which does not have a compact resolvent. But it will fit into the setting of a compact quantum metric space. And in addition, we will detect that we can repair the lack of not being a spectral triple by disturbing the operator a little bit. Then the question arises whether the metrics from the disturbed spectral triples are equivalent to the original metric or whether they even converge towards this metric as the perturbation goes to zero. Sadly we have not been able to give a satisfying answer to this in the later chapters of this thesis. But nonetheless we have made some simple observations, which give criteria for the convergence of a series of compact quantum metric spaces, which we will present now.

We will see that the metric convergence of a family of compact quantum metric spaces will follow from the condition that the corresponding family of Lip-norms is uniformly continuous. Our version of uniform continuity of Lip-norms is characterized by the following definition:

Definition 1.2.4
Let A be a commutative order unit space (which means that it can be realized as the set of continuous real-valued functions on a compact metric space), and for $\theta \in [0,1]$ let L_θ be a family of Lip-norms on A. We set

$$\Sigma_0 := \{f \in A : L_0(f) = 1\}\,. \tag{1.7}$$

Then we call the family $\theta \mapsto (A, L_\theta)$ of compact quantum metric spaces uniformly continuous *towards (A, L_0), if*

$$\forall\varepsilon > 0\ \exists\delta > 0 :\ 0 < \theta < \delta\ \Rightarrow\ |L_\theta(f) - L_0(f)| < \varepsilon\ \forall f \in \Sigma_0. \tag{1.8}$$

◁

We will see now that the property of uniform continuity of a family of Lip-norms implies the fact that the metrics ρ_θ corresponding to (A, L_θ) converge to ρ_0, and that they are equivalent in case θ is small enough. Remember that ρ_θ is defined on the state space $\mathcal{S}(A)$ of A, which is isometrically isomorphic to a compact metric space X with $A \cong C(X)$. In this case the metric is given via the formula

$$\rho_\theta(x, y) = \sup\{|f(x) - f(y)| : L_\theta(f) \le 1\}\,.$$

Then we have the following proposition.

Proposition 1.2.5
Let A be a commutative order unit space and for $\theta \in [0,1]$ let L_θ be a family of Lip-norms on A, such that the family $\theta \mapsto (A, L_\theta)$ of compact quantum metric spaces is uniformly continuous in the sense of (1.8).
Then for every $\varepsilon > 0$ with $\varepsilon < 1$ there is a $\delta > 0$ such that

$$(1-\varepsilon)\,\rho_\theta(x,y) \le \rho_0(x,y) \le (1+\varepsilon)\,\rho_\theta(x,y) \quad \forall x, y \in \mathcal{S}(A) \tag{1.9}$$

for every $0 < \theta < \delta$.

Proof: Since A is commutative, we consider

$$A = \{f \in C(X) : f = f^*\} \subset C(X)$$

for a compact metric space X such that $\mathcal{S}(A) \cong X$. Note that we can forget about the constant functions when calculating ρ_θ since in this case we have $|f(x) - f(y)| = 0$ for any two points x and y.

Now assume $\theta \mapsto (A, L_\theta)$ is uniformly continuous, and let $\varepsilon > 0$. Then we find a $\delta > 0$ such that

$$(1-\varepsilon) \le L_\theta(\tilde{f}) \le (1+\varepsilon)\ \ \forall\theta < \delta$$

for every $\tilde{f} \in A$ with $L_0(\tilde{f}) = 1$. For an arbitrary function f which is not constant on X we have $L_0(f) > 0$. Therefore we can set

$$\tilde{f} := \frac{f}{L_0(f)} \in \Sigma_0,$$

such that $L_\theta(\tilde{f}) = L_\theta(f)/L_0(f)$ and the above estimate becomes

$$L_0(f)\,(1-\varepsilon) \le L_\theta(f) \le (1+\varepsilon)\,L_0(f) \tag{1.10}$$

uniformly for all $f \in A$ which are not constant.
Now for given points $x, y \in M$, (1.10) leads to the estimates

$$\begin{aligned} \rho_\theta(x,y) &= \sup\{|f(x)-f(y)| : L_\theta(f) \le 1\} \\ &\begin{cases} \le \sup\{|f(x)-f(y)| : (1-\varepsilon)L_0(f) \le 1\} \\ \ge \sup\{|f(x)-f(y)| : (1+\varepsilon)L_0(f) \le 1\}\,. \end{cases} \end{aligned}$$

Since for every constant $C := 1 \pm \varepsilon > 0$ one has

$$\begin{aligned} \sup\{|f(x)-f(y)| : CL_0(f) \le 1\} &= \sup\left\{\frac{|f(x)-f(y)|}{C} : L_0(f) \le 1\right\} \\ &= \frac{\rho_0(x,y)}{C}, \end{aligned}$$

this shows (1.9), and therefore the proposition is proved. □

We note a similar observation, which gives equivalence of the metrics in case one can estimate the difference of two Lip-norms against one of them.

Proposition 1.2.6
Let A be a commutative order unit space and let L_1, L_0 be two Lip-norms on A. Assume that there is a $C < 1$ such that we have an estimate

$$|L_0(f) - L_1(f)| \le CL_0(f)$$

for all $f \in A$. Then the metrics ρ_0 and ρ_1, arising from L_0 and L_1, are equivalent.
Proof: On the one hand, we have the estimate

$$L_1(f) \ge L_0(f) - |L_0(f) - L_1(f)| \ge (1-C)L_0(f),$$

and on the other hand we have

$$L_1(f) \le L_0(f) + |L_1(f) - L_0(f)| \le (1+C)L_0(f).$$

Now the equivalence of the metrics follows by an argument analogous to the argument given in the proof of Proposition 1.2.5, using C instead of ε in (1.10). □

Now one can keep on playing the game and show that the statement of Proposition 1.2.5 (and therefore the uniform continuity of the family of Lip-norms) implies that the sequence of compact quantum metric spaces $(C(X), L_\theta)$ converges to the compact quantum metric space $(C(X), L_0)$ in the so-called *quantum Gromov-Hausdorff convergence*. The concept of quantum Gromov-Hausdorff convergence was introduced by Mark Rieffel (see [Rie04]) as an quantum analogy to the Gromov-Hausdorff convergence of classical metric spaces. To prove this, one can adopt the arguments given by Rieffel in [Rie04], Section 11, where this convergence is proved for a field of Lip-norms on finite-dimensional vector spaces.

In our situation, we have to assume the uniform continuity condition from Definition 1.2.4, which can be shown to be fulfilled in the finite dimensional case. But since this does not affect the questions we are considering in this thesis, we will not write down the argumentation here.

Chapter 2

A Review of Sub-Riemannian Geometry

This chapter is devoted to the introduction of the spaces for which we want to construct spectral triples. On some Riemannian manifolds M it is possible to establish another metric besides the geodesic one. One can define the so-called Carnot-Carathéodory distance, determined by shortest paths tangent to a certain sub-bundle of the tangent bundle. Whenever this distance gives a metric on M (meaning that any two points of M can be connected by such a path), the Hausdorff dimension of this metric space differs from the usual dimension of M.

In the first section we will introduce these sub-Riemannian spaces and review the properties mentioned above. Then, we will turn to the example of Carnot groups, which are nilpotent Lie groups carrying a certain grading inside their Lie algebras, and define Carnot manifolds. These will be the central objects to construct spectral triples on in this thesis; and they are fundamental within sub-Riemannian geometry since the tangent space of Riemannian geometry is generalized to a Carnot group in the sub-Riemannian case. The last two sections of this chapter are meant to provide some techniques we will take advantage of later: Section 2.3 deals with the case of Heisenberg manifolds, which have a horizontal distribution of co-dimension 1 and occur at different points in mathematics and physics, and introduces the tool of the Levi form to describe the Lie group structure. Finally, in Section 2.4, we will construct submersions between Carnot groups which will allow us to reduce certain problems for general Carnot groups to lower dimensional cases.

Throughout this section, (M, g) will denote a Riemannian manifold with tangent bundle TM. Since most parts of this chapter are intended to sum up things which are already known, we will refer to the basic literature, which includes in our case the books and monographs by Richard Montgomery [Mon02], by Luca Capogna, Donatella Danielli, Scott D. Pauls and Jeremy T. Tyson [CDPT07] and by Mikhael Gromov [Gro96]. In addition, we refer to books by Ovidiu Calin and Der-Chen Chang [CC09] and by A. Bonfiglioli, E. Lanconelli, F. Uguzzoni [BLU07].

2.1 Sub-Riemannian Manifolds

We start this chapter by giving the definition of a horizontal distribution, on which a sub-Riemannian geometry is based on.

Definition 2.1.1
Let M be a Riemannian manifold with a Riemannian metric $g \in \Gamma^\infty(T^*M \otimes T^*M)$.

(i) A horizontal distribution *of the tangent bundle TM is given by a sub-bundle $HM \subset TM$ together with a fiber wise inner product $\langle \cdot, \cdot \rangle_H$, such that at each point $x_0 \in M$ we have*

$$g(X,Y)\big|_{x_0} = \langle X(x_0), Y(x_0)\rangle_H$$

for all $X, Y \in HM$.

A vector field $X \in TM$ is called horizontal *if $X \in HM$.*

(ii) A (smooth) path $\gamma : [0,1] \to M$ is called horizontal *if at any point $t \in [0,1]$ we have $\dot{\gamma}(t) \in H_{\gamma(t)}M$.*

(iii) If, for $d \in \mathbb{N}$, $HM = \mathrm{span}\{X_1, \dots, X_d\}$ and $X_1, \dots, X_d$ are linearly independent at each point $x \in M$, the integer d is called the Rank *of HM. In this case we call $\{X_1, \dots, X_d\}$* a frame *for the horizontal distribution HM.*

(iv) Let $\{X_1, \dots, X_d\}$ be a frame for HM. Then for $f \in C^1(M)$, the vector field

$$\mathrm{grad}^H f := \sum_{j=1}^{d} X_j(f) X_j$$

is called the horizontal gradient *of f.*

(v) A horizontal distribution HM is called involutive *if $[X,Y] \in HM$ for any $X, Y \in HM$.*

(vi) A horizontal distribution HM is called bracket-generating *if the Lie hull of HM, which is the collection of all vector fields of HM and their (multi-step) commutators, generates the tangent bundle TM. The smallest number $R \in \mathbb{N}$, such that HM together with all its R-step commutators generates TM, is called the* step *of a bracket generating distribution.*

Remark: A horizontal distribution can also be defined by a set of 1-forms: If $\{X_1, \dots, X_n\}$ is a frame of TM such that for $d < n$ $\{X_1, \dots, X_d\}$ spans HM, we denote by $\{d\omega^1, \dots, d\omega^n\}$ the corresponding dual frame of T^*M. Then we have

$$HM = \mathrm{Ker}\left(\mathrm{span}\{d\omega^{d+1}, \dots, d\omega^n\}\right).$$

◁

The Frobenius theorem (see for example [CC09], Theorem 1.3.1) asserts that HM is involutive if and only if it is integrable, which means that the set of all horizontal paths through a fixed point $x \in M$ sweeps out a smooth immersed sub-manifold N of M with $\dim N = \text{Rank } HM$. In this thesis we are interested in non-involutive, but bracket generating horizontal distributions: One can define a distance between two points $x, y \in M$ by considering the length of the shortest horizontal part connecting these points.

Definition 2.1.2
Let (M, g) be a Riemannian manifold which is equipped with a horizontal distribution $HM = \text{span}\{X_1, \ldots, X_d\}$.

(i) The horizontal length *of a smooth horizontal path $\gamma : [0,1] \to M$ is given by the number*

$$L_{CC}(\gamma) := \int_0^1 \left(\sum_{j=1}^{d} g \left\langle \gamma'(t), X_j(\gamma(t)) \right\rangle_H \right)^{\frac{1}{2}} dt.$$

Here, $\langle \cdot, \cdot \rangle_H$ denotes the fiber wise inner product of HM induced by the Riemannian metric of M, see Definition 2.1.1.

(ii) The Carnot-Carathéodory distance *between to points $x, y \in M$ is given by the (not necessarily finite) number*

$$d_{CC}(x,y) := \inf \left\{ L_{CC}(\gamma) : \gamma \text{ horizontal path with } \gamma(0) = x \text{ and } \gamma(1) = y \right\}.$$

(iii) If $d_{CC}(x,y)$ is finite for any $x, y \in M$, we call (M, d_{CC}) a sub-Riemannian manifold.

In general, this distance needs of course not to be finite, but a famous theorem by Chow says that this distance is indeed finite for arbitrary $x, y \in M$ if HM is bracket-generating (see also [Mon02], Theorem 1.17):

Theorem 2.1.3
If HM is a bracket-generating horizontal distribution on a connected manifold M, then any two points $x, y \in M$ can be joined by a horizontal path. □

In particular this means that if the horizontal distribution HM is bracket-generating, then (M, d_{CC}) is a metric space. This leads us to the question whether one can compare the metric spaces (M, d_{CC}) and (M, d_{geo}), where d_{geo} denotes the usual geodesic distance. First of all, it is obvious that for any $x, y \in M$ we have $d_{geo}(x,y) \le d_{CC}(x,y)$, but in general the metrics are not equivalent. But they do indeed induce the same topologies on M, as the following theorem shows (see [Mon02], Theorem 2.3).

Theorem 2.1.4
If HM is a bracket-generating horizontal distribution on M, then the topology on M induced by the Carnot-Carathéodory distance d_{CC} is the same as the usual manifold topology induced by d_{geo}. □

An unexpected observation is that the metric spaces (M, d_{CC}) and (M, d_{geo}) have a different Hausdorff dimension (see Definition 1.1.6 for the construction of this measure theoretic dimension). While the Hausdorff dimension of (M, d_{geo}) coincides with the (usual) topological dimension n of M, the Hausdorff dimension of (M, d_{CC}) is in general strictly greater than n: We will see that the Hausdorff dimension of (M, d_{CC}) is exactly the so-called *graded dimension* of the horizontal distribution.

To define the graded dimension of a sub-Riemannian manifold, we need a little bit of preparation. Since the horizontal distribution HM is bracket generating of step R, we have a sequence of vector bundles

$$HM \subset H^2M \subset \ldots \subset H^RM = TM,$$

where

$$H^{S+1}M := H^SM + [HM, H^SM]$$

with $[HM, H^SM] := \operatorname{span}\left\{[X,Y] : X \in HM, Y \in H^SM\right\}$ for $1 \leq S \leq R-1$ (using the convention that $H^1M = HM$). Now we assume that for a given $1 \leq S \leq R$ the dimension of the space H^S_xM is the same for every point $x \in M$ (this will be an assumption throughout this thesis). In this situation, we are able to make the following definition.

Definition 2.1.5
Let M be a sub-Riemannian manifold with horizontal distribution HM of rank d and step R as above. For $1 \leq S \leq R-1$ we denote by $d_1 := d :=$ Rank HM and by

$$d_{S+1} := \text{Rank } H^{S+1}M - \text{Rank } H^SM$$

the ranks of the spaces $H^{S+1}M\big/H^SM$. Then the graded dimension *(or* homogeneous dimension*) of M is the number*

$$\dim_G(M) := \sum_{S=1}^{R} S \cdot d_S.$$

Remark: Note that the topological dimension of M is given by the number $\sum_{S=1}^{R} d_S$ in this context. ◁

Now we are ready to formulate the so-called *Mitchell's Measure Theorem*, from which it follows that the Hausdorff dimension of (M, d_{CC}) differs from its topological dimension (see also [Mon02], Theorem 2.17).

Theorem 2.1.6
The Hausdorff dimension of a sub-Riemannian manifold M is equal to its graded dimension, i.e. under the notations of Definition 2.1.5 we have

$$\dim_H(M) = \sum_{S=1}^{R} S \cdot d_S.$$

This means in particular, that in general the Hausdorff dimension of the metric space (M, d_{CC}) is strictly greater than the Hausdorff dimension of the metric space (M, d_{geo}). $\square$

We now finish this basic section by giving the most important example for a sub-Riemannian manifold: The $(2m+1)$-dimensional Heisenberg group. In dimension 3 this is the easiest example where we have one space on which two different (with respect to metric and dimension) geometries can be established, and it will also be an example for a Carnot group, which we will introduce in the next section.

Example 2.1.7
For $m \in \mathbb{N}$, we consider the space $\mathbb{R}^{2m+1}$ equipped with a Riemannian metric g such that the vector fields

$$X_j = \partial_{x_j} - \frac{1}{2} x_{m+j} \partial_{x_{2m+1}}, \quad X_{m+1} = \partial_{x_{m+j}} + \frac{1}{2} x_j \partial_{x_{2m+1}}, \quad X_{2m+1} = \partial_{x_{2m+1}}$$

for $1 \leq j \leq m$ form an orthonormal frame of $T\mathbb{R}^{2m+1}$. Note that $[X_j, X_{m+j}] = X_{2m+1}$, such that $H\mathbb{R}^{2m+1} = \operatorname{span}\{X_1, \ldots, X_{2m}\}$ forms a non-involutive, 2-step bracket generating horizontal sub-bundle of $T\mathbb{R}^{2m+1}$. Therefore $(\mathbb{R}^{2m+1}, d_{CC})$ is a metric space of Hausdorff dimension $2m+2 = 2m \cdot 1 + 1 \cdot 2$.

Since at each point, the tangent space generated by $\{X_1, \ldots, X_{2m+1}\}$ has the structure of a 2-step nilpotent Lie algebra $\mathfrak{h}_{2m+1}$ the exponential mapping furnishes a simply-connected nilpotent Lie group $\mathbb{H}^{2m+1} = \exp \mathfrak{h}_{2m+1}$. $\mathbb{H}^{2m+1}$ is called the $2m+1$-*dimensional Heisenberg group*. It is a well known fact that $\mathbb{H}^{2m+1}$ can be realized as the space of upper triangle $((m+2) \times (m+2))$-matrices which have the form

$$\mathbb{H}^{2m+1} = \left\{ \begin{pmatrix} 1 & x^t & z \\ 0 & 1_{(m \times m)} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}^m; z \in \mathbb{R} \right\},$$

where the group rule is given by matrix multiplication. On $\mathbb{R}^{2m+1}$, this multiplication can be written as

$$(x, y, z) \cdot (\tilde{x}, \tilde{y}, \tilde{z}) = (x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + \sum_{j=1}^{m} x_j \tilde{y}_j). \tag{2.1}$$

The Lie algebra $\mathfrak{h}_{2m+1}$ of $\mathbb{H}^{2m+1}$ can be realized by strictly upper triangle matrices

$$\mathfrak{h}_{2m+1} = \left\{ \begin{pmatrix} 0 & x^t & z \\ 0 & 0_{(m \times m)} & y \\ 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R}^m; z \in \mathbb{R} \right\},$$

where the algebra multiplication is once again the matrix multiplication. $\triangleleft$

Remark: The composition rule (2.1) of the Heisenberg group is written down in the so-called *polarized coordinates*. These are exactly the coordinates derived from the matrix model of this Lie group. On the other hand, one can describe points of $\mathbb{H}^{2m+1}$ by their *exponential coordinates*, which are the coordinates of the corresponding Lie algebra having a one-to-one correspondence with the coordinates of $\mathbb{H}^{2m+1}$ by the exponential mapping. We will introduce both kinds of coordinates in the next section in the context of Carnot groups. ◁

2.2 Carnot Groups and Carnot Manifolds

The Heisenberg group $\mathbb{H}^{2m+1}$ introduced in Example 2.1.7 is an example for a greater class of nilpotent Lie groups, which are called Carnot groups.

Definition 2.2.1
A Carnot group *of step $R \in \mathbb{N}$ is a simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ has a stratification (or grading)*

$$\mathfrak{g} = \bigoplus_{S=1}^{R} V_S,$$

such that $V_1, \ldots, V_R$ are vector spaces satisfying the conditions

(i) $[V_1, V_S] = V_{S+1}$ *for* $S = 1, \ldots, R-1$

(ii) $[V_S, V_R] = 0$ *for* $S = 1, \ldots, R$.

We call the number $d_1 := \dim V_1$ the bracket-generating dimension *of $\mathbb{G}$ and the number $\dim \mathbb{G} - d_1$ the* bracket-generating co-dimension *of $\mathbb{G}$.* ◁

Carnot groups are the canonical generalization of the Euclidean space $\mathbb{R}^n$ in sub-Riemannian geometry, since the tangent space (or better the tangent cone) of a Carnot manifold has the structure of a Carnot group. This can be seen as a generalization of the Riemannian case, where the tangent space at any point is isomorphic to the (1-step nilpotent) Carnot group $\mathbb{R}^n$. Without going into detail, the situation is as follows: Let X be any metric space, $x_0 \in X$. Then we define the *tangent cone* as the pointed Gromov-Hausdorff limit of the family $(\lambda X, x_0)$ of pointed metric spaces for $\lambda \to \infty$, if it exists. It has been proved by John Mitchell [Mit85] that in the case where (M, d_{CC}) is a Carnot manifold, this limit exists and has the structure of a Carnot group (see also [Mon02], Theorem 8.8):

Theorem 2.2.2
Let x be a regular point of a Carnot manifold (M, d_{CC}). Then, at every $x \in M$, the tangent cone exists and is a Carnot group, which is arising from the nilpotentization of the horizontal distribution at x. □

Remark: We will not explain the details of the process of nilpotentization here and refer to [Mon02] or [Bel96] instead, since the technical details will not affect this thesis. ◁

Now getting back to the objects we are considering, we finally define a Carnot manifold as a sub-Riemannian manifold, which has globally a Carnot group structure on its tangent bundle.

Definition 2.2.3
Let (M, g) be an n-dimensional Riemannian manifold, whose tangent bundle TM carries a grading

$$TM = \bigoplus_{S=1}^{R} V_S M \tag{2.2}$$

such that $V_1 M, \dots, V_R M$ are vector bundles satisfying the conditions

(i) $[V_1 M, V_S M] = V_{S+1} M$ *for* $S = 1, \dots, R-1$

(ii) $[V_R M, V_R M] = 0$ *for* $S = 1, \dots, R.$

Then we call M a Carnot manifold *of step $R \in \mathbb{N}$. We further call the number $d_1 =$ Rank $V_1 M$ the* bracket-generating dimension *of M and the number $n - d_1$ the* bracket-generating co-dimension *of M.* ◁

Remark: It is clear that every Carnot group is a Carnot manifold. It is also clear by the conditions (i) and (ii) on the brackets of the vector fields that every Carnot manifold contains the structure of a sub-Riemannian manifold, where the horizontal distribution is given by $HM = V_1 M$. It should be mentioned that every Carnot manifold has locally the structure of a Carnot group, while for a general sub-Riemannian manifold this is a bit more involved (see Theorem 2.2.2).

Note that we have not fixed the geometry we are considering in the above definition. So by Section 2.1 we can establish at least two different geometries on a Carnot manifold M: The Riemannian one (equipped with the geodesic distance d_{geo}) and the sub-Riemannian one (equipped with the Carnot-Carathéodory distance d_{CC}). ◁

When working on a Carnot manifold, we will always assume that we have a Riemannian metric g on M such that the spaces $V_S M$ appearing in the grading (2.2) are mutually orthogonal. For $S = 1, \dots, R$ we set $d_S :=$ Rank $V_S M$ and denote by $\{X_{S,j} : j = 1, \dots, d_S\}$ an orthonormal frame of $V_S M$. Hence, a frame of TM is given by

$$\{X_{1,1}, \dots, X_{1,d_1}, X_{2,1}, \dots, X_{2,d_2}, \dots, X_{R,1}, \dots, X_{R,d_R}\}.$$

We will sometimes use the abbreviation $X^{(S)} = (X_{S,1}, \dots, X_{S,d_S})$ to denote the frame of $V_S M$.

From the grading of its Lie algebra $\mathfrak{g}$, one can introduce coordinates on a Carnot group $\mathbb{G}$. Thereby we make use of the fact that because of the nilpotency the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism. We present two different types of coordinates here, where it will be depending on the situation which type is more comfortable to work with.

Definition 2.2.4
Let $\mathbb{G}$ be a Carnot group with grading $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$ of its Lie algebra, such that for $1 \leq S \leq R$, $X^{(S)}$ denotes a basis of V_S.

(a) The coordinates

$$x = \left(x^{(1)}, \ldots, x^{(R)}\right) = \left(x_{1,1}, \ldots, x_{1,d_1}, \ldots, x_{R,1}, \ldots, x_{R,d_R}\right),$$

given by

$$x \leftrightarrow \exp\left(\sum_{S=1}^{R}\sum_{j=1}^{d_S} x_{S,j} X_{S,j}\right), \tag{2.3}$$

are called exponential coordinates *or* canonical coordinates of the first kind *of $\mathbb{G}$.*

(b) The coordinates

$$y = \left(y^{(1)}, \ldots, y^{(R)}\right) = \left(y_{1,1}, \ldots, y_{1,d_1}, \ldots, y_{R,1}, \ldots, y_{R,d_R}\right),$$

given by

$$y \leftrightarrow \prod_{j=1}^{d_1} \exp\left(y_{1,j} X_{1,j}\right) . \prod_{j=1}^{d_2} \exp\left(y_{2,j} X_{2,j}\right) . \ \ldots\ . \prod_{j=1}^{d_R} \exp\left(y_{R,j} X_{R,j}\right) \tag{2.4}$$

are called polarized coordinates *or* canonical coordinates of the second kind *of $\mathbb{G}$.*

Thereby, $x.y$ denotes the group composition on $\mathbb{G}$, and the products are taken with respect to this group composition. ◁

Remark: It is known that there is an isomorphism between the exponential and the polarized coordinates of a Carnot group. For example, in the case of the $(2m+1)$-dimensional Heisenberg group $\mathbb{H}^{2m+1}$ this isomorphism is given via

$$\phi(x_1, \ldots, x_{2m}, x_{2m+1}) = \left(x_1, \ldots, x_{2m}, x_{2m+1} + \frac{1}{2}\sum_{j=1}^{m} x_j x_{m+j}\right),$$

where $(x_1, \ldots, x_{2m}, x_{2m+1})$ denote the exponential coordinates (see [Fol89], Section 1.2). ◁

When calculating a group composition on $\mathbb{G}$, one uses the Baker-Campbell-Hausdorff formula. This formula is given on $\mathfrak{g}$ by

$$\exp X . \exp X = \exp\left(X + Y + B(X,Y)\right), \tag{2.5}$$

where $B(X,Y)$ is a sum of multi-step commutators of order $2, 3, \ldots, R$, see [Kna05]. Therefore, $B(X,Y)$ is a polynomial of degree smaller or equal to the step of $\mathbb{G}$, which does not depend on vectors belonging to V_R since V_R commutes with every $X \in \mathfrak{g}$. In the case of a 2-step Carnot group, we have

$$\exp X . \exp Y = \exp\left(X + Y + \frac{1}{2}[X,Y] \right).$$

Using this expression on the Lie algebra, one can derive the composition rule in exponential or polarized coordinates on the Carnot group $\mathbb{G}$.

Maybe the most important property of a Carnot group is its homogeneity, which is expressed by a weighted dilation of it. Once again there is a one-to-one relation between the dilation on a Carnot group and the corresponding weighted dilation on its graded Lie algebra, given by the exponential map.

Definition 2.2.5

(i) Let $\mathbb{G}$ be a Carnot group with coordinates $x = (x^{(1)}, \ldots, x^{(R)})$ (exponential or polarized), $\lambda > 0$. Then we define by

$$\delta_\lambda : \mathbb{G} \to \mathbb{G}, \quad x = \left(x^{(1)}, \ldots, x^{(R)}\right) \mapsto \lambda . x := \left(\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^R x^{(R)}\right) \tag{2.6}$$

the *(weighted) dilation* on $\mathbb{G}$ by λ.

A function $f : \mathbb{G} \to \mathbb{R}$ is called *homogeneous* of degree $\mu \in \mathbb{R}$ with respect to δ_λ if we have

$$f(\delta_\lambda(x)) = \lambda^\mu \cdot f(x)$$

for all $x \in \mathbb{G}$, $\lambda > 0$.

(ii) Let $\mathfrak{g}$ be a graded nilpotent Lie algebra with grading $\mathfrak{g} \cong V_1 \oplus \ldots \oplus V_R$, $\lambda > 0$. Then we define by

$$\hat{\delta}_\lambda : \mathfrak{g} \to \mathfrak{g}, \quad X = \sum_{S=1}^{R} \sum_{j=1}^{d_S} x_{S,j} X_{S,j} \mapsto \hat{\delta}_\lambda . X := \sum_{S=1}^{R} \sum_{j=1}^{d_S} \lambda^S x_{S,j} X_{S,j} \tag{2.7}$$

the *(weighted) dilation* on $\mathfrak{g}$ by λ.

A function $f : \mathfrak{g} \to \mathbb{R}$ is called *homogeneous* of degree $\mu \in \mathbb{R}$ with respect to $\hat{\delta}_\lambda$ of degree μ if we have

$$f(\hat{\delta}_\lambda(X)) = \lambda^\mu \cdot f(X)$$

for all $X \in \mathfrak{g}$, $\lambda > 0$.

(iii) Let M be a Carnot manifold, δ_λ and $\hat{\delta}_\lambda$ the weighted dilations by $\lambda > 0$ defined point-wise on TM. A vector field $X \in \Gamma(TM)$ is called *homogeneous* of degree $\mu \in \mathbb{R}$ if we have

$$\hat{\delta}_\lambda X = \lambda^\mu \cdot X \tag{2.8}$$

for every $\lambda > 0$.

Remark: Note that if the grading of TM is given by $TM \cong V_1 M \oplus \ldots \oplus V_R M$, we have that $X \in TM$ is homogeneous of degree S if and only if $X \in V_S M$. $\lhd$

We remark that the weight S of δ_λ (or $\hat{\delta}_\lambda$) belonging to certain coordinates reflects exactly the commutator step of the space V_S of $\mathfrak{g}$ which belongs to $x^{(S)}$ (or $X^{(S)}$). These weighted dilations play an important role when one wants to describe a symbol calculus with respect to a Carnot group structure: Asymptotic expansions are given in terms of homogeneous functions with respect to weighted dilations. Furthermore one can use weighted dilations to define a Carnot group, see for example [BLU07], Definition 2.2.1, where a (homogeneous) Carnot group is defined to be a Lie group structure on $\mathbb{R}^n$ which respects these dilations and has a bracket generating structure. In [BLU07] it is also shown that these two definitions are equivalent (up to isomorphisms).

Let us mention now how every Carnot group can be realized as a certain Lie group structure on $\mathbb{R}^n$, using certain vector fields to represent the basis of its Lie algebra. We have already seen this in Example 2.1.7 for the case of the Heisenberg group. The idea is to find a frame of vector fields $X_1, \ldots X_n$ which satisfies the grading conditions of Definition 2.2.1 and which has the property that $X_j(0) = \partial_{x_j}\big|_{x=0}$ for each j. This is indeed possible, and the requested vector fields X_j have polynomial coefficients, as the following proposition shows (see [BLU07], Remark 1.4.6).

Proposition 2.2.6
Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$, where the grading of $\mathfrak{g}$ is given by $\mathfrak{g} = \bigoplus_{S=1}^R V_S$ with $d_S = \dim V_S$. Then for each V_S there exists a frame of vector fields $\{X_{S,1}, \ldots, X_{S,d_S}\}$ such that

$$X_{S,j} = \partial_{x_{S,j}} + \sum_{L=S+1}^{R} \sum_{k=1}^{d_L} p_{j,k}^{(S,L)} \left(x^{(1)}, \ldots, x^{(L-S)}\right) \partial_{x_{L,k}}. \tag{2.9}$$

Here, $p_{j,k}^{(S,L)}$ is a polynomial which is homogeneous with respect to the dilations δ_λ from (2.6) of degree $L - S$, which means

$$p_{j,k}^{(S,L)} \left(\delta_\lambda \left(x^{(1)}, \ldots, x^{(L-S)}\right)\right) = \lambda^{L-S} p_{j,k}^{(S,L)} \left(x^{(1)}, \ldots, x^{(L-S)}\right).$$

□

Now one can consider the homogeneous vector fields from (2.9) as homogeneous differential operators. Thus any polynomial of these vector fields is a differential operator. These so-called *graded differential operators* will play a big role in this thesis: The horizontal Dirac operators we will construct in Chapter 3 fall into this category, and the horizontal Laplacians of (homogeneous) degree 2 will also turn out to be very important.

Definition 2.2.7
Let $\mathfrak{g} \cong V_1 \oplus \ldots \oplus V_R$ be a graded Lie algebra with $d_S = \text{Rank } V_S$, which is represented by vector fields on $\mathbb{R}^{d_1+\ldots+d_R}$ as in Proposition 2.2.6. A frame for V_S shall consist of the vector fields $\{X_{S,1}, \ldots, X_{S,d_S}\}$, where we write $\{X_1, \ldots, X_d\}$ for the frame of V_1 (with $d = d_1$).
Then a graded differential operator *is a differential operator of the from*

$$L = p\left(X_1, \ldots, X_d, X_{2,1}, \ldots, X_{R,d_R}\right),$$

where p is a polynomial with matrix-valued C^∞ coefficients. If p is homogeneous of degree $\mu \in \mathbb{R}$ we call L homogeneous *of degree μ.* ◁

In particular, a horizontal Laplacian *is a graded differential operator of order 2, which means it is an operator of the form*

$$\Delta^{\text{hor}} = -\sum_{j=1}^{d} X_j^2 + \sum_{j=1}^{d_2} b_{2,j} X_{2,j} + \sum_{j=1}^{d} b_{1,j} X_j + b_0,$$

where all the $b_{S,j}$ and b_0 are smooth (matrix valued) functions on $\mathbb{C}^n$. ◁

The above definition suggests to consider a graded differential operator L as an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the graded Lie algebra $\mathfrak{g}$ generated by $\{X_1, \ldots, X_d\}$. This interpretation will be used in Chapter 6 when we analyze graded differential operators using the representation theory of their underlying Lie algebras $\mathfrak{g}$.

We return once again to the dilations from Definition 2.2.5: For our purposes we should mention an additional another property of δ_λ: For arbitrary points $x, y \in \mathbb{G}$ we have

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda \cdot d_{CC}(x, y),$$

so the Carnot-Carathéodory metric on $\mathbb{G}$ respects these dilations. Now we will introduce a quasi-norm (or gauge norm) $\|\cdot\|_{\mathbb{G}}$ on a Carnot group $\mathbb{G}$, which is homogeneous with respect to these dilations and provides us with a quasi-metric on $\mathbb{G}$ which is equivalent to d_{CC}. It will rather be a quasi-norm than a norm since the triangle inequality on $\mathbb{G}$ must be replaced by the condition

$$\|x.y\|_{\mathbb{G}} \leq C\, \|x\|_{\mathbb{G}}\, \|y\|_{\mathbb{G}}\,.$$

This quasi-norm will play an important role when we define a symbol calculus according to Carnot groups.

Definition 2.2.8

Let $\mathbb{G}$ be a Carnot group of step R. The quasi-norm $\|\cdot\|_{\mathbb{G}}$ on $\mathbb{G}$, defined by

$$\|x\|_{\mathbb{G}}^{2R!} := \sum_{S=1}^{R} \sum_{j=1}^{d_S} |x_{S,j}|^{\frac{2R!}{S}}\,, \tag{2.10}$$

is called the Koranyi gauge *of $\mathbb{G}$. The quasi-metric*

$$d_{\mathbb{G}}(x, y) := \left\|y^{-1}.x\right\|_{\mathbb{G}}$$

arising from the Koranyi gauge will be called the Koranyi (quasi-)metric *on $\mathbb{G}$.* ◁

Remark: Note that in the case $\mathbb{G} = \mathbb{H}^{2m+1}$ the Koranyi gauge is given by the formula

$$\|x\|_{\mathbb{H}^{2m+1}} = \left(\sum_{j=1}^{2m} |x_j|^4 + |x_{2m+1}|^2\right)^{\frac{1}{4}}. \tag{2.11}$$

We also note that there are several definitions of the Koranyi gauge on a Carnot group which are equivalent, for example in [Ste93] the Koranyi gauge on $\mathbb{H}^{2m+1}$ is defined via

$$\rho(x) := \max\left\{\|(x_1,\ldots,x_{2m})\|, |x_{2m+1}|^{1/2}\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm of the corresponding vector. ◁

As noted in [CDPT07], the Koranyi quasi-metric on a Carnot group $\mathbb{G}$ is equivalent to the Carnot-Carathéodory metric on $\mathbb{G}$.

Proposition 2.2.9
Let $\mathbb{G}$ be a Carnot group, and let $d_{\mathbb{G}}$ denote the Koranyi quasi-metric and d_{CC} the Carnot-Carathéodory metric on $\mathbb{G}$. Then there are constants $c > 0$ and $C > 0$ such that for all $x, y \in \mathbb{G}$ we have

$$c \cdot d_{CC}(x,y) \le d_{\mathbb{G}}(x,y) \le C \cdot d_{CC}(x,y).$$

□

Now having finished a rough review about the theory on Carnot groups and Carnot manifolds, we want to finish this section by giving some simple examples for compact Carnot manifolds. We consider the local homogeneous space of a discrete lattice subgroup of a Carnot group $\mathbb{G}$. In a way, this is the analogous object to the torus $\mathbb{T}^n$, which arises as the quotient by $\mathbb{Z}^n$ on $\mathbb{R}^n$. Since it is quite comfortable to do calculations on these objects, they will serve as the main example for the considerations of this thesis.

Example 2.2.10
Let $\mathbb{G}$ be a Carnot group, equipped with a (left-invariant) Riemannian metric, with grading $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$, where $\{X_1,\ldots,X_d\}$ is a frame for V_1. We consider the discrete subgroup Γ of $\mathbb{G}$ generated by the basis vectors of V_1, that is

$$\Gamma := \langle\ \{\gamma_j = \exp(X_j) : 1 \le j \le d\}\ \rangle_{\mathbb{G}}. \tag{2.12}$$

Γ acts on $\mathbb{G}$ via the left multiplication $(\gamma, x) \mapsto \gamma.x$ for all $x \in \mathbb{G}$. It follows easily that Γ is a lattice in $\mathbb{G}$ (see [Mon02], Section 9.3), which we will call the *standard lattice* of $\mathbb{G}$.

Since Γ is a lattice, the local homogeneous space $\Gamma\backslash\mathbb{G}$, consisting of the orbits of this group action, is a compact Riemannian manifold which is locally isometric to $\mathbb{G}$. Hence $M := \Gamma\backslash\mathbb{G}$ is a compact Carnot manifold, where the grading of the tangent bundle TM comes from the grading of $\mathfrak{g}$. We will call $M := \Gamma\backslash\mathbb{G}$ the (standard) *compact Carnot nilmanifold* of $\mathbb{G}$. ◁

Remark 2.2.11
The group Γ defined via (2.12) can also be viewed as a discrete nilpotent group. From the grading of $\mathbb{G}$ one can detect a so-called *central descending series*

$$0 = \Gamma^{R+1} \subset \Gamma^R \subset \ldots \subset \Gamma^2 \subset \Gamma^1 = \Gamma,$$

which means we have $\Gamma^{S+1} = [\Gamma, \Gamma^S]$, where Γ^S is the standard lattice of the Carnot group $\exp(V_S \oplus \ldots \oplus V_R)$. One can check that for the graded dimension (and therefore by Theorem 2.1.6 also for the Hausdorff dimension) of $\mathbb{G}$ we have the identity

$$\dim_G(\mathbb{G}) = \sum_{S=1}^{R} S \cdot \text{Rank } \left(\Gamma^S / \Gamma^{S+1}\right). \tag{2.13}$$

Now there is a famous theorem by Bass, Milnor and Wolf (see [Mon02], Theorem 9.3) which states that the right hand side of (2.13) is equal to the polynomial growth of Γ. So, altogether, the Hausdorff dimension of $\mathbb{G}$ coincides with the polynomial growth of its standard lattice subgroup. See [Mon02], Sections 9.2 and 9.3, for details on this. $\triangleleft$

2.3 Heisenberg Manifolds and Levi Forms

We will now pay attention to the most important class of Carnot manifolds: It is the case where the graded co-dimension is equal to 1. Those manifolds are also known as Heisenberg manifolds. In the context of non-commutative geometry, they were treated in details in the work of Raphaël Ponge (see for example [Pon08] and the references there), and recently they have also been a big research area in index theory, for which we refer to Erik van Erp and his collaborators (see e.g. [BE11]). For most of the following definitions and proposition, we refer to Ponge ([Pon08]).

Definition 2.3.1
A Heisenberg manifold is a smooth Riemannian manifold (M, g) *of dimension* $n = d + 1$ *equipped with a hyperplane bundle* $HM \subset TM$ *of rank d, which is bracket-generating.* $\triangleleft$

Remark: One can formulate the definition of a Heisenberg manifold a little bit more general by dropping the assumption that HM has to be bracket generating, see e.g. [Pon08]. In this case, objects like contact manifolds or CR manifolds are included in the class of Heisenberg manifolds. But since we are following more or less theoretical aspects and want to consider manifolds on which we have the Carnot-Carathéodory geometry, we make this further assumption. $\triangleleft$

An important tool to handle Heisenberg manifolds is the so called *Levi form*. In case M is a Carnot manifold of graded co-dimension 1, this is the 1-form describing the Lie algebra structure of the (graded) tangent bundle.

Definition 2.3.2
Let M *be a Carnot manifold of co-dimension* 1*, where the grading of its tangent bundle is given by* $TM = V_1M \oplus V_2M$*, with Rank* $V_2M = 1$ *and* $V_2M = [V_1M, V_1M]$*. Let*

$\{X_1, \dots, X_d\}$ be a frame of V_1M and let $\{X_{d+1}\}$ be a frame of V_2M. Then the Levi form *$\mathcal{L}$ of M is the (antisymmetric) bilinear form*

$$\mathcal{L} : V_1M \times V_1M \to V_2M, \quad (Y_1, Y_2) \mapsto [Y_1, Y_2] \mod V_1M. \tag{2.14}$$

For $\mathcal{L}(X_j, X_k) = L_{ik}X_{d+1}$ with $L_{ik} \in \mathbb{R}$ for $j, k \in \{1, \dots d\}$, we denote by $L = (L_{jk})$ the antisymmetric $(d \times d)$-matrix describing $\mathcal{L}$ according to the basis $\{X_1, \dots, X_d\}$.

Now one can define a Levi form on any Heisenberg manifold M the same way: On can see rather easily that from the Lie bracket of vector fields on HM one gets a 2-form

$$\mathcal{L} : HM \times HM \to TM/HM,$$

such that for any section $X, Y \in HM$ we have

$$\mathcal{L}_{x_0}(X(x_0), Y(x_0)) = [X, Y](x_0) \mod H_{x_0}M$$

near a point $x_0 \in M$ (see [Pon08]). But this allows us to define a bundle of graded 2-step nilpotent Lie algebras $\mathfrak{g}M \cong HM \oplus (TM/HM)$, which gives rise to a bundle of Carnot groups $\mathbb{G}M$ of step 2 and graded co-dimension 1 over M via the exponential mapping. This bundle $\mathbb{G}M$ is also called the *tangent Lie group bundle* of M.

We can even say how this tangent Lie group bundle looks like:

Proposition 2.3.3
Let (M, g) be a Heisenberg manifold. Then for any point $x_0 \in M$, the point-wise Levi form $\mathcal{L}_{x_0}$ has rank $2m$ if and only if $\mathbb{G}_{x_0}M \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$. In especially, this means $\mathcal{L}$ has constant rank $2m$ if and only if $\mathbb{G}M$ is a fiber bundle with typical fiber $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$.
In addition, if Rank $\mathcal{L} = 2m$, it is always possible to find an orthonormal basis $\{X_1, \dots, X_d\}$ of V_1M such that the matrix representation $L = (L_{jk})$ of $\mathcal{L}$ becomes

$$L = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.15}$$

where $D \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ is a diagonal matrix carrying the absolute values $\lambda_1, \dots, \lambda_m$ of the non-zero eigenvalues on its diagonal.

Proof: The first statement is exactly the statement of [Pon08], Proposition 2.1.6. The second statement follows by linear algebra, since L is a skew-symmetric matrix, so it is known to have the non-zero eigenvalues $\pm i\lambda_1, \dots, \pm i\lambda_m$. Then for any orthonormal frame of HM the form (2.15) can be reached by an orthonormal basis transformation at every point of HM. □

Remark: By our assumption of HM being bracket-generating, we always have Rank $\mathcal{L} \geq 2$ and therefore $m \geq 1$. Of course the case $m = 0$ also fits into the theory: In this case the hyperplane bundle HM induces a foliation on M. ◁

We can further introduce Levi forms on arbitrary Carnot manifolds: Since $[V_1M, V_1M] = V_2M$ for the first two summands appearing in the grading of the tangent space of any Carnot manifold, the following definition is well-defined.

Definition 2.3.4
Let M be a Carnot manifold with grading $TM = \bigoplus_{S=1}^{R} V_S M$ of its tangent bundle. We denote by $\{X_{1,1}, \dots, X_{1,d}\}$ an orthonormal frame of $V_1 M$ and by $\{X_{2,1}, \dots, X_{2,d_2}\}$ an orthonormal frame of $V_2 M$. Then for $\nu \in \{1, \dots, d_2\}$, the ν-Levi form *of M is given by the bilinear form*

$$\mathcal{L}_\nu : V_1 \times V_1 \to \operatorname{span}\{X_{2,\nu}\}, \quad (Y_1, Y_2) \mapsto [Y_1, Y_2] \mod (\operatorname{span}\{X_{2,\nu}\})^\perp .$$

For $\mathcal{L}_\nu(X_j, X_k) = L_{ik}^{(\nu)} X_\nu$ with $L_{ik}^{(\nu)} \in \mathbb{R}$, we denote by $L^{(\nu)} = (L_{jk})$ the antisymmetric matrix describing $\mathcal{L}_\nu$. $\triangleleft$

In other words, the collection of ν-Levi forms $\mathcal{L}_\nu$ describes the structure of the first step commutators of a Carnot manifold. We will use this notation later when we look for structures of Heisenberg manifolds inside a Carnot manifold.

2.4 Submersions of Carnot Groups

In the final section of this chapter we will introduce a technique which will play an important role in our later considerations: We will show how one gets a submersion from a given Carnot group $\mathbb{G}_1$ onto a lower dimensional Carnot group $\mathbb{G}_2$. This will provide us the possibility to pull back objects defined on $\mathbb{G}_2$ to objects on $\mathbb{G}_1$, such that statements about $\mathbb{G}_2$ can be transported to $\mathbb{G}_1$.
Once again we consider a Carnot group $\mathbb{G}$ whose Lie algebra has the grading $\mathfrak{g} = \bigoplus_{S=1}^{R} V_S$. Let $\tilde{V} \subset \mathfrak{g}$ be a linear subspace of (the vector space) $\mathfrak{g}$ which has the structure

$$\tilde{V} := \bigoplus_{S=1}^{M-1} V_S \oplus \tilde{V}_M, \quad \text{where } \tilde{V}_M \subset V_M \text{ is a linear subspace,} \tag{2.16}$$

for some $1 \leq M \leq R$. We take a look at the canonical orthogonal projection

$$\operatorname{pr} : \mathfrak{g} \to \tilde{V}, \quad v \mapsto v \mod \tilde{V}^\perp. \tag{2.17}$$

We show now that pr induces a homomorphism of Lie algebras, which gives rise to a submersion (and also to a homomorphism) of Lie groups.

Proposition 2.4.1
Let $\mathbb{G}$, $\mathfrak{g}$, $\tilde{V}$ and pr be as above. Then the vector space $\tilde{\mathfrak{g}} := \operatorname{pr}(\mathfrak{g}) \cong \tilde{V}$ can be equipped with a Lie algebra structure via

$$[X, Y]_{\operatorname{pr}} := \operatorname{pr}([X, Y]) \quad \forall X, Y \in \tilde{V} \tag{2.18}$$

such that $\mathrm{pr} : (\mathfrak{g}, [\cdot,\cdot]) \to (\tilde{\mathfrak{g}}, [\cdot,\cdot]_{\mathrm{pr}})$ *is an homomorphism of graded Lie algebras.*
Further, if we set $\mathfrak{n} := \mathit{Ker}(\mathrm{pr})$, *the spaces* $N := \exp(\mathfrak{n})$ *and* $\tilde{\mathbb{G}} := \exp(\tilde{\mathfrak{g}})$ *are Carnot groups such that* $\tilde{\mathbb{G}} \cong \mathbb{G}/N$ *and hence* $\mathbb{G} \cong \tilde{\mathbb{G}} \times N$. *The resulting map*

$$\psi := \exp_{\tilde{\mathbb{G}}} \circ \mathrm{pr} \ \circ \exp_{\mathbb{G}^{-1}} : \mathbb{G} \to \tilde{\mathbb{G}} \tag{2.19}$$

is a homomorphism of Carnot groups and a submersion of Riemannian manifolds.

Proof: First of all, one checks that the Lie brackets defined via (2.18) are indeed Lie brackets: The bilinearity of $[\cdot,\cdot]_{\mathrm{pr}}$ is obvious because of the bilinearity of $[\cdot,\cdot]$ and the linearity of pr; and the Jacobi identity also follows from these properties in connection with the Jacobi identity of $[\cdot,\cdot]$.
To check that pr is a homomorphism of Lie algebras, one has to check that

$$[\mathrm{pr}(X), \mathrm{pr}(Y)]_{\mathrm{pr}} = \mathrm{pr}\left([X, Y]\right) \quad \forall\ X, Y \in \mathfrak{g}. \tag{2.20}$$

This is by definition true for all $X, Y \in \tilde{V}$. Now let (without loss of generality) $X \in \tilde{V}^{\perp}$. In this case, $\mathrm{pr}(X) = 0$ and therefore the left hand side of (2.20) is zero. But the right hand side is also zero, since $X \in \tilde{V}^{\perp} \subset \bigoplus_{S=M}^{R} V_S$ and therefore $[X, Y] \in \bigoplus_{S=M+1}^{R} V_S$ because of the graded structure of $\mathfrak{g}$. Therefore (2.20) is true, which shows that the linear map pr is a Lie algebra homomorphism. But this means that $\mathfrak{n} := \mathrm{Ker}(\mathrm{pr})$ is an ideal in $\mathfrak{g}$ and therefore also a graded Lie algebra. The grading structures of $\tilde{g}$ follows immediately from the grading structure of $\mathfrak{g}$ since

$$\tilde{\mathfrak{g}} \cong \bigoplus_{S=1}^{M-1} V_S \oplus \tilde{V}_M.$$

Thus $\tilde{\mathfrak{g}}$ is obviously a graded Lie algebra of step M.
Now we consider the map ψ from (2.19). Because of the nilpotency the exponential maps from the Lie algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ to their Lie groups $\tilde{\mathbb{G}}$ and $\mathbb{G}$ are diffeomorphisms, and since pr is a surjective linear map (which is smooth because of its linearity) we have the result that ψ is a submersion. To check that it is a group homomorphism, we calculate, using the Baker-Campbell-Hausdorff formula (2.5):

$$\begin{aligned}
\psi(x._{\mathbb{G}}y) &= \exp_{\tilde{\mathbb{G}}} \circ \mathrm{pr}\left(\exp_{\mathbb{G}}^{-1}(\exp_{\mathbb{G}}\left(\sum_{S=1}^{R}\sum_{j=1}^{d_S} x_{S,j}X_{S,j}\right) ._{\mathbb{G}} \exp_{\mathbb{G}}\left(\sum_{S=1}^{R}\sum_{j=1}^{d_S} y_{S,j}X_{S,j}\right))\right) \\
&= \exp_{\tilde{\mathbb{G}}} \circ \mathrm{pr}\left(\sum_{S,j}(x_{S,j} + y_{S,j})X_{S,j} + B(\sum_{S,j} x_{S,j}X_{S,j}, \sum_{S,j} y_{S,j}X_{S,j})\right) \\
&= \exp_{\tilde{\mathbb{G}}}\left(\sum_{S,j} x_{S,j}\mathrm{pr}(X_{S,j}) + \sum_{S,j} y_{S,j}\mathrm{pr}(X_{S,j}) + \tilde{B}(\sum_{S,j} x_{S,j}\mathrm{pr}(X_{S,j}), \sum_{S,j} y_{S,j}\mathrm{pr}(X_{S,j}))\right) \\
&= \exp_{\tilde{\mathbb{G}}}\left(\sum_{S,j} x_{S,j}\mathrm{pr}(X_{S,j})\right) ._{\tilde{\mathbb{G}}} \exp_{\tilde{\mathbb{G}}}\left(\sum_{S,j} y_{S,j}\mathrm{pr}(X_{S,j})\right) \\
&= \psi(x)._{\tilde{\mathbb{G}}}\psi(y).
\end{aligned}$$

In this calculation we have used the fact that pr is a Lie algebra homomorphism. In particular, the third equation is true since B is a sum of (multi-step) commutators of the vector fields $X_{S,j}$, such that applying pr to B furnishes the polynomial $\tilde{B}$ in the Baker-Campbell-Hausdorff formula on $\tilde{\mathbb{G}}$.

Finally we have

$$N = \exp \mathfrak{n} = \mathrm{Ker}\psi,$$

such that N is a normal Lie subgroup of $\mathbb{G}$. But this shows in addition that $\tilde{\mathbb{G}} \cong \mathbb{G}/N$ and that $\mathbb{G} \cong \tilde{\mathbb{G}} \times N$. Now every statement of the proposition is proved. □

We now show briefly that such a submersion can be lifted to the compact nilmanifolds arising from $\mathbb{G}$ from Example 2.2.10. Let Γ be generated by the images of the basis vector fields of V_1 under the exponential mapping, and let $M = \Gamma\backslash\mathbb{G}$ be the local homogeneous space of the left action of Γ on $\mathbb{G}$. Since the submersion ψ from (2.19) is a Lie group homomorphism, the image $\psi(\Gamma)$ under ψ is a discrete subgroup of $\psi(\mathbb{G})$, and by the definition of ψ it is clear that $\psi(\Gamma)$ is generated by the image of the basis vectors of V_1 under the corresponding projection pr : $\exp_{\mathbb{G}}^{-1}(\mathbb{G}) \to \exp_{\psi(\mathbb{G})}^{-1}(\psi(\mathbb{G}))$ of the Lie algebras. Therefore, we get a compact Carnot nilmanifold $\tilde{M} := \psi(\Gamma)\backslash\psi(\mathbb{G})$.

For the action of $\psi(\Gamma)$ on $\psi(\mathbb{G})$ we have

$$\psi(\gamma.x) = \psi(\gamma).\psi(x)$$

for any $\gamma \in \Gamma$ and $x \in \mathbb{G}$, therefore any orbit of the action of Γ on $\mathbb{G}$ is mapped to a orbit of the action of $\psi(\Gamma)$ on $\psi(\mathbb{G})$. Let $[x]_\Gamma$ denote the orbit belonging to an element $x \in \mathbb{G}$ under the action of Γ. Because ψ is a submersion by Proposition 2.4.1, this means that the mapping

$$\pi : M = \Gamma\backslash\mathbb{G} \to \tilde{M} = \psi(\Gamma)\backslash\psi(\mathbb{G}), \quad [x]_\Gamma \mapsto [\psi(x)]_{\psi(\Gamma)} \tag{2.21}$$

is also a submersion. We summarize the above argumentations in the following corollary:

Corollary 2.4.2

In the above situation, the map π from (2.21) *is a submersion of Riemannian manifolds. Locally, π coincides with the submersion $\psi : \mathbb{G} \to \psi(\mathbb{G})$ from* (2.19) *of the corresponding Carnot groups.*

Proof: The fact that π is a submersion has already been deduced in the discussion ahead of this corollary. Since the nilmanifolds $\Gamma\backslash\mathbb{G}$ and $\psi(\Gamma)\backslash\psi(\mathbb{G})$ are locally isometric to the Carnot groups $\mathbb{G}$ and $\tilde{\mathbb{G}}$ (see Example 2.2.10), the second statement is obvious by the construction of π from ψ. □

Chapter 3

Horizontal Dirac-Operators

The intention of this chapter is to find a first order differential operator which detects the Carnot-Carathéodory metric via Connes' formula. Therefore we follow an approach analogous to the standard example for a spectral triple on a Riemannian manifold: We construct a so-called *horizontal Dirac operator*, arising from the Clifford action of a horizontal distribution of a Carnot manifold.

To really cover the horizontal geometry, we want our horizontal Dirac operator to be compatible with the horizontal part of the Levi-Civita connection on a Carnot manifold M. We thus define and discuss a (partial) connection ∇^H according to the sub-bundle HM of TM in Section 7.1 from the Levi-Civita connection on M. Then in section 7.2, we introduce horizontal Clifford bundles arising from the Clifford action of the bundle HM, which carry a connection compatible with ∇^H. On these bundles, we will be able to define horizontal Dirac operators D^H in a general sense (in analogy to classical Dirac operators on Clifford bundles), and we will be able to modify these operators such that they are self-adjoint. In the end of Section 7.2, we calculate the square of D^H and proof a technical proposition concerning the eigenvalues of a sum of certain Clifford matrices in preparation for future arguments.

Finally in Section 3.3, we will show that the horizontal Dirac operators we constructed are indeed the right operators to detect the Carnot-Carathéodory metric on a Carnot manifold M: We show that the norm of $[D^H, f]$ coincides with the Lipschitz norm of f with respect to the Carnot-Carathéodory metric, such that we can apply Connes' metric formula to the sub-algebra of $C(M)$ consisting of these Lipschitz functions. In addition we show that this metric is already detected by the sub-algebra of C^∞-functions. In the following chapters, we will seen that in spite of all these good properties this operator does not define a spectral triple.

Throughout this section, M will be a compact Carnot of step R with Carnot-Carathéodory metric d_{CC} of dimension n. We will use the notation $\{X_1, \ldots, X_d\}$ for a frame of the horizontal distribution HM of M, and if not stated otherwise the Riemannian metric g on M will be chosen such that this frame is orthonormal. The horizontal Dirac operator will be defined on closed (meaning compact without boundary) Carnot manifolds.

3.1 Horizontal Connections

To define a horizontal Dirac operator using the Clifford module arising from the horizontal distribution HM we need to introduce a connection on HM. This happens straight forward and can be found at various parts of the literature (see e.g. [DGN07], but a horizontal connection is also mentioned in the textbooks [CDPT07] and [CC09]). The idea is simply to start with the Levi-Civita connection on M and to project it onto the horizontal bundle. This approach is justified by the following proposition (see [CDPT07], Proposition 4.2; not that the case of Carnot manifolds follows directly from the case of Carnot groups formulated there).

Proposition 3.1.1
Let M be a Carnot manifold with horizontal distribution $HM \subset TM$ which carries a smoothly varying inner product $\langle\cdot,\cdot\rangle_H$. Let $VM \subset TM$ be a sub-bundle which is complementary to HM. If g_1 and g_2 are Riemannian metrics which make VM and HM orthogonal such that for $j = 1, 2$

$$g_j(X,Y)\big|_{x_0} = \langle X(x_0), Y(x_0)\rangle_H \quad \forall X, Y \in HM \ \forall x_0 \in M,$$

then the associated Levi-Civita connections ∇_1 and ∇_2 coincide when projected to HM: We have

$$g_1\left(\nabla_{1X} Y, Z\right) = g_2\left(\nabla_{2X} Y, Z\right) \tag{3.1}$$

for all sections X, Y, Z of HM. □

This proposition allows us to define a connection only depending on the horizontal bundle, and it shows that it is in fact well defined when constructing it from any Levi-Civita connection of the extension of the horizontal inner product. We follow [CDPT07], where this is done for the Heisenberg group.

Definition 3.1.2
Let M be a Carnot manifold with horizontal distribution $HM \subset TM$, which carries a smoothly varying inner product $\langle\cdot,\cdot\rangle_H$. Let $\pi_H : \Gamma^\infty(TM) \to \Gamma^\infty(HM)$ denote the projection of a tangent vector field onto its horizontal component. We then define

$$\nabla^H : \Gamma^\infty(HM) \times \Gamma^\infty(HM) \to \Gamma^\infty(HM), \quad (X,Y) \mapsto \nabla^H_X(Y) := \pi^H \nabla_X(Y), \tag{3.2}$$

where ∇ denotes the Levi-Civita connection of any extension of the horizontal inner product. ∇^H is called the *horizontal (Levi-Civita-)connection* of M. ◁

Proposition 3.1.3
The map ∇^H defined in Definition 3.1.2 is indeed a (partial) connection defined on the horizontal bundle, which means we have

(i) For all $X, X', Y \in \Gamma^\infty(HM)$ *and for all* $f, g \in C^\infty(M)$*:*

$$\nabla^H_{fX+gX'}Y = f \cdot \nabla^H_X Y + g \cdot \nabla^H_{X'} Y.$$

(ii) For all $X, Y, Y' \in \Gamma^\infty(HM)$ *and for all* $f, g \in C^\infty(M)$*:*

$$\nabla^H_X(fY + gY') = f \cdot \nabla^H_X Y + g \cdot \nabla^H_X Y' + (Xf) \cdot Y + (Xg) \cdot Y'.$$

∇^H *is* metric *with respect to the point-wise inner product* $\langle \cdot, \cdot \rangle_H$ *of* HM*, which means that we have*

(iii) For all $X, Y, Y' \in \Gamma^\infty(HM)$*:*

$$X \langle Y, Y' \rangle_H = \left\langle \nabla^H_X Y, Y' \right\rangle_H + \left\langle Y, \nabla^H_X Y' \right\rangle_H.$$

∇^H *is* torsion free *in the horizontal direction, which means that we have*

(iv) for all $X, Y \in \Gamma(HM)$*:*

$$\pi^H\left(\nabla^H_X Y - \nabla^H_Y X\right) = \pi^H([X, Y]).$$

Proof: Let ∇ be the Levi-Civita connection corresponding to any Riemannian metric g on TM which is an extension of $\langle \cdot, \cdot \rangle_H$. The conditions (i) and (ii) which show that ∇^H is indeed a connection follow immediately from the corresponding properties of the Levi-Civita connection after projection onto the horizontal distribution. The metric property (iii) follows immediately from the metric property of ∇ since $\langle \cdot, \cdot \rangle_H$ is just a restriction of the Riemannian metric g.

Finally, the torsion freeness into the horizontal direction follows because

$$\begin{aligned}
\pi^H\left(\nabla^H_X Y - \nabla^H_Y X\right) &= \pi^H\left(\pi^H\left(\nabla_X Y - \nabla_Y X\right)\right) \\
&= \pi^H\left(\nabla_X Y - \nabla_Y X\right) \\
&= \pi^H\left([X, Y]\right)
\end{aligned}$$

for all $X, Y \in \Gamma^\infty(HM)$, since $\pi_H^2 = \pi_H$ because π_H is a projection and since the Levi-Civita connection ∇ is torsion free. Hence the proposition is proved. □

Remark: We note that we cannot expect to have torsion freeness in the sense that $\nabla^H_X Y - \nabla^H_Y X = [X, Y]$ for all $X, Y \in \Gamma^\infty(HM)$, as it is the case for the Levi-Civita connection: Since HM is not involutive, we have $[X, Y] \notin HM$ for some $X, Y \in HM$, but by definition of the horizontal connection the vector field $\nabla^H_X Y - \nabla^H_Y X$ must be horizontal again. ◁

Remark 3.1.4
We have only defined our horizontal connection to be a partial connection, which means we only allow differentiation into horizontal directions. There are several possibilities to extend ∇^H to a connection

$$\tilde{\nabla}^H : \Gamma^\infty(TM) \times \Gamma^\infty(HM) \to \Gamma^\infty(HM) :$$

For instance, the expression (3.2) makes sense for any $X \in \Gamma^\infty(TM)$, and Proposition 3.1.3 would also hold for this case with the same proof. Another possibility of defining a connection is given by setting $\nabla^H_X := 0$ for all $X \in VM$. ◁

Remark 3.1.5
We cannot get rid of the vertical directions completely as soon as we want to apply horizontal covariant derivatives more than one time: By the tensorial property (ii) of Proposition 3.1.3 we have for any $X_1, X_2, Y \in HM$ and for any $f \in C^\infty(M)$:

$$\begin{aligned}\left(\nabla^H_{X_1}\nabla^H_{X_2} - \nabla^H_{X_2}\nabla^H_{X_1}\right) f \cdot Y &= \nabla^H_{X_1}\left(f \cdot \nabla^H_{X_2} Y + (X_2 f) \cdot Y\right) \\ &\quad -\nabla^H_{X_2}\left(f \cdot \nabla^H_{X_1} Y + (X_1 f) \cdot Y\right) \\ &= f \cdot \nabla^H_{X_1}\nabla^H_{X_2} Y + (X_1 f) \cdot \nabla^H_{X_2} Y + (X_2 f) \cdot \nabla^H_{X_1} Y \\ &\quad + (X_1 X_2 f) \cdot Y - f \cdot \nabla^H_{X_2}\nabla^H_{X_1} Y - (X_2 f) \cdot \nabla^H_{X_1} Y \\ &\quad - (X_1 f) \cdot \nabla^H_{X_2} Y - (X_2 X_1 f) \cdot Y \\ &= f \cdot \left(\nabla^H_{X_1}\nabla^H_{X_2} - \nabla^H_{X_2}\nabla^H_{X_1}\right) Y + ([X_1, X_2] f) \cdot Y,\end{aligned}$$

and since HM is not involutive we have $[X_1, X_2] \notin \Gamma^\infty(HM)$.
This has consequences when one wants to define some kind of curvature belonging to the horizontal connection: It turns out that this curvature shows into the transversal direction of TM. But we will not discuss this aspect any further and refer to [CC09] and [Mon02] instead. ◁

We close this short section with a proposition which shows us how the calculate a horizontal covariant derivative in local coordinates respecting the grading structure of our Carnot manifold M. This is more or less trivial, since after calculating the covariant derivative with respect to the Levi-Civita connection we simply project onto the horizontal distribution.

Proposition 3.1.6
Let M be a Carnot manifold with a horizontal distribution HM of rank d. Let $\{X_1, \dots, X_n\}$ denote an orthonormal frame for TM such that for $d \le n$ $\{X_1, \dots, X_d\}$ is an orthonormal frame for HM. Then we have locally

$$\nabla^H_{X_j} X_k = \sum_{l=1}^{d} \Gamma^l_{jk} X_l,$$

where the Γ^l_{jk} are the Christoffel symbols of the Levi-Civita connection ∇ of TM with respect to the frame $\{X_1, \dots, X_n\}$. □

3.2 Construction of Horizontal Dirac Operators

We want to construct a horizontal Dirac operator in analogy to the (classical) Dirac operator. In the classical case this construction is outlined for example in [Roe98] or [LM89].

We start with a review about the definition of a Clifford algebra and Clifford action. Remember that for each vector space V equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ there exists an (up to isomorphisms) unique *Clifford algebra* $A = \mathrm{Cl}(V)$ which is a unital algebra, equipped with a map $\varphi : V \to A$ such that

$$\varphi(v)^2 = -\langle v, v \rangle \cdot 1. \tag{3.3}$$

In addition, φ is supposed to fulfill the universal property in the sense that for any other unital algebra A' equipped with map $\varphi' : V \to A'$ satisfying (3.3), there is a unique algebra homomorphism $\alpha : A \to A'$ such that $\varphi' = \alpha \circ \varphi$ (see [Roe98], Definition 3.1 and Proposition 3.2). Multiplication in this unique algebra is determined by the rule

$$\varphi(v_1)\varphi(v_2) + \varphi(v_2)\varphi(v_1) = -2\langle v_1, v_2 \rangle, \tag{3.4}$$

and we know that for $\dim V = n$ we have $\dim \mathrm{Cl}(V) = 2^n$. Using this map φ to define multiplication, we can construct a left module over the complex algebra $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ which will be called a *Clifford module*. In other words, a *Clifford module* S for a real inner product space V is a complex vector space S equipped with an $\mathbb{R}$-linear map

$$c : V \to \mathrm{End}_{\mathbb{C}}(S) \tag{3.5}$$

such that $c(v)^2 = -\langle v, v \rangle \cdot 1$ for all $v \in V$. The map c is called the *Clifford action*.

Now the question arises what the minimal possible dimension of such a Clifford module S is, depending on the dimension of V; in other words, we are looking for irreducible representations of A. Such an irreducible representation is given by the so-called *spin representation* Σ, and it is known that for $n = \dim V$ we have

$$\dim \Sigma = 2^{\left[\frac{n}{2}\right]},$$

where $[n/2]$ denotes the Gaussian bracket which gives the greatest integer smaller or equal then $m/2$ (see [Roe98], Chapter 4, or [LM89]).

The idea for our situation is now to take a look at the Clifford algebras generated by the fibers of the horizontal distribution $HM = \mathrm{span}\{X_1, \ldots, X_d\}$. Since the Riemannian metric on M is chosen in a way that $\{X_1, \ldots, X_d\}$ forms an orthonormal frame at each point, we immediately get the fundamental properties of this horizontal Clifford action, which we will denote by c^H, from the above discussion.

Proposition 3.2.1
For $x \in M$, let S_x be a Clifford module for H_xM with (horizontal) Clifford action $c^H : H_xM \to \mathrm{End}_{\mathbb{C}}(S_x)$. Then we have:

(i) $c^H(X_j)^2 = -\mathrm{Id}$ *for all* $j \in \{1, \dots, d\}$.

(ii) $c^H(X_j)c^H(X_k) + c^H(X_k)c^H(X_j) = 0$ *for all* $j, k \in \{1, \dots, d\}$, $j \neq k$. □

Now let $S^H M$ be a bundle of Clifford modules for the horizontal distribution HM. We need to equip $S^H M$ with an point-wise (hermitian) inner product and with a connection, for which we claim certain compatibility conditions. For a classical Dirac operator, we ask the connection on SM to be compatible with the Levi-Civita connection ∇ on TM, see e.g. [Roe98], Definition 3.4. Hence to define a suitable bundle and connection where a horizontal Dirac operator can act on, we would like to have compatibility with the horizontal connection ∇^H on HM defined in Section 3.1 via

$$\nabla^H : \Gamma^\infty(HM) \times \Gamma^\infty(HM) \to \Gamma^\infty(HM), \quad (X, Y) \mapsto \nabla^H_X Y := \pi_H \nabla_X Y, \tag{3.6}$$

where π_H is the orthogonal projection onto the horizontal distribution. For this horizontal connection we formulate the compatibility conditions in the following way:

Definition 3.2.2
Let $S^H M$ *be a bundle of horizontal Clifford moduls over* M *which is equipped with a fiber-wise Hermitian metric* $(\cdot, \cdot)_H$ *and a metric connection* ∇^{S^H}. $S^H M$ *is called a* horizontal Clifford bundle *if*

(i) For each $x \in M$ *we have*

$$\big(c^H(X_x)\sigma_1(x), \sigma_2(x)\big)_H + \big(\sigma_1(x), c^H(X_x)\sigma_2(x)\big)_H = 0$$

for all $X_x \in H_x M$, $\sigma_1, \sigma_2 \in \Gamma^\infty(S^H M)$.

(ii) If ∇^H *is the horizontal Levi-Civita connection* (3.6) *on* HM, *we have*

$$\nabla^{S^H}_X (c^H(Y)\sigma) = c^H(\nabla^H_X Y)\sigma + c^H(Y)\nabla^{S^H}_X \sigma$$

for all $X, Y \in \Gamma^\infty(HM)$ *and for all sections* $\sigma \in \Gamma^\infty(S^H M)$.

◁

Now, still in analogy to the classical Dirac operator, we can define a horizontal Dirac operator. Later we will see that this operator has to be modified by adding an endomorphism in order to be symmetric.

Definition 3.2.3
The formal horizontal Dirac operator $\tilde{D}^H$ *of* $S^H M$ *is the first order differential operator on* $\Gamma^\infty(S^H M)$ *defined by the composition*

$$\Gamma^\infty(S^H M) \to \Gamma^\infty(H^* M \otimes S^H M) \to \Gamma^\infty(HM \otimes S^H M) \to \Gamma^\infty(S^H M).$$

Here the first arrow is given by the connection ∇^{S^H}, the second arrow is given by the identification of H^*M and HM via the horizontal metric, and the third arrow is given by the Clifford action.
If we choose a local orthonormal frame $\{X_1, \ldots, X_d\}$ of sections of HM, we can write

$$\tilde{D}^H \sigma = \sum_{j=1}^{d} c^H(X_j) \nabla^{S^H}_{X_j} \sigma. \tag{3.7}$$

◁

Before we go on with our construction to get a self-adjoint operator from $\tilde{D}^H$, we have a look at the most natural example.

Example 3.2.4
Consider the exterior bundle $\Omega M := \bigwedge T^*M \otimes \mathbb{C}$ on a compact Riemannian manifold M without boundary equipped with its natural metric and connection. It is well known (see for example [Roe98], 49ff.) that (in the classical sense) this is a Clifford bundle with multiplication given by the wedge product and Clifford action of $e \in T^*M \cong TM$ given by

$$c(e)\sigma = e \wedge \sigma + e \lrcorner \sigma.$$

The Dirac operator of this Clifford bundle is given by $d + d^*$. Remember that the interior product is defined by

$$e \lrcorner \sigma = (-1)^{nk+n+1} * (e \wedge *\sigma)$$

via the Hodge star operator $*\sigma$, which (for a k-form σ) is defined to be the unique $(n-k)$-form α such that

$$(\sigma, \alpha)\mathrm{vol} = \alpha \wedge *\sigma.$$

Using this Hodge star operator, the operator d^* is defined by

$$d^*\sigma = (-1)^{nk+n+1} * d * \sigma.$$

Now a horizontal Clifford bundle can be constructed analogously. We assume M to be a Carnot manifold with horizontal distribution $HM = \mathrm{span}\{X_1, \ldots, X_d\}$ such that the frame $\{X_1, \ldots, X_d, X_{d+1}, \ldots, X_n\}$ is orthonormal, Hence we can identify this frame with its dual frame $\{d\omega^1, \ldots, d\omega^n\}$. We further identify H^*M with the sub-bundle of T^*M annihilating $(HM)^\perp$, which means $H^*M = \mathrm{span}\{d\omega^1, \ldots, d\omega^d\}$. Then $\Omega^H M := \bigwedge H^*M \otimes \mathbb{C}$ is a horizontal Clifford bundle: The Clifford action of $e \in H^*M \cong HM$ is again given by

$$c^H(e)\sigma := e \wedge \sigma + e \lrcorner \sigma;$$

note that in case $\sigma \in S^H M$ and $e \in H^*M$ the right side of this equation lies still in $S^H M$. If we use the horizontal exterior derivative

$$d^H(\sigma) := \pi_H(d\sigma),$$

where π_H is the orthonormal projection onto H^*M, we get a connection which is compatible with the horizontal Levi-Civita connection is defined on H^*M. Using this, we can define the formal horizontal Dirac operator on S^HM and see that it is given by

$$D^H\sigma = \sum_{j=1}^{d} d^H\omega^j \wedge \sigma + d^H\omega^j \lrcorner\, \omega = d^H\sigma + \left(d^H\right)^* \sigma.$$

◁

Our approach of constructing a horizontal Clifford bundle and a horizontal Dirac operator can be found in the literature in a greater generality. Igor Prokhorenkov and Ken Richardson recently introduced a class of so-called *transversally Dirac operators* in [PR11] by considering a distribution $QM \subset TM$ of the tangent bundle (which may be integrable or not), which furnishes a $\mathrm{Cl}(QM)$-module structure on a complex Hermitian vector bundle EM over M with Clifford action $c : QM \to \mathrm{End}_{\mathbb{C}}(EM)$ and a connection ∇^E on EM fulfilling the requirements of Definition 3.2.2. Hence our construction of a horizontal Clifford bundle fits into this setting.

It turns out that the gap of this construction so far is that the operator $\tilde{D}^H$ is not symmetric: When one wants to calculate the L^2-adjoint of $\tilde{D}^H$, Stokes' theorem causes an additional term of mean curvature into the vertical direction, projected to HM via the horizontal connection. Hence to get a symmetric operator, we have to add the (horizontal) Clifford action by the mean curvature of the orthogonal complement of the horizontal distribution. This was shown by Prokhorenkov and Richardson for the general case of a distribution of the tangent bundle ([PR11]), such that we refer to their work formulating the following theorem. It shows in addition that the resulting operator is essential self-adjoint, which follows by a theorem of Paul Chernoff about the self-adjointness of certain differential operators ([Che73]).

Altogether, we have the following theorem (see [PR11], Theorem 3.1), which will finally provide us with the possibility to define (essential self-adjoint) horizontal Dirac operators.

Theorem 3.2.5
Let $\tilde{D}^H$ be a formal horizontal Dirac operator, acting on the smooth sections of a horizontal Clifford bundle S^HM over a compact Carnot manifold M without boundary. Then we have

(i) The formal L^2-adjoint of $\tilde{D}^H$ is given by

$$\left(\tilde{D}^H\right)^* = \tilde{D}^H - c^H\left(\sum_{j=d+1}^{n} \pi^H \nabla_{X_j} X_j\right), \tag{3.8}$$

where $\pi^H : TM \to HM$ is the orthogonal projection onto the horizontal distribution and ∇ is the Levi-Civita connection of TM.

(ii) The operator

$$D^H := \sum_{j=1}^{k} c^H(X_j)\nabla^{S^H}_{X_j} - \frac{1}{2}c^H\left(\sum_{j=d+1}^{n} \pi^H \nabla_{X_j} X_j\right) \tag{3.9}$$

is essentially self adjoint on $L^2(S^H M)$.

□

Definition 3.2.6
Let $S^H M$ *be a horizontal Clifford bundle over a closed Carnot manifold* M*. Then the operator defined by* (3.9)*, acting on* $\Gamma^\infty(S^H M)$*, is called the* horizontal Dirac operator *of* $S^H M$*.* ◁

At this point, we want to state another example which is due to [PR11]. It states that on any (classical) Clifford bundle EM over a closed Carnot manifold M one can implement the structure of a horizontal Clifford bundle by adjusting the bundle connection. We refer to [PR11], Section 2, for the following proposition.

Proposition 3.2.7
Let M *be a Carnot manifold with horizontal distribution* HM*, such that* $\{X_1, \ldots, X_d\}$ *is an orthonormal frame for* HM *and* $\{X_{d+1}, \ldots, X_n\}$ *is an orthonormal frame for* $HM^\perp$*. Let in addition* EM *be a Clifford bundle over* M *with Clifford action* c *and bundle connection* ∇^E*, and let* c^H *denote the restriction of* c *to the horizontal distribution* HM*.*
Then there is a connection $\tilde{\nabla}^E$ *on* E *such that* E *equipped with* c^H *and* $\tilde{\nabla}^E$ *is a horizontal Clifford module. This connection* $\tilde{\nabla}^E$ *is given by*

$$\tilde{\nabla}^E_X = \nabla^E_X + \frac{1}{2}\sum_{j=d+1}^{n} c^H\left(\pi^H \nabla_X X_j\right) c(X_j), \tag{3.10}$$

where π^H *denotes the orthogonal projection of* TM *onto* HM *and* ∇ *denotes the Levi-Civita connection on* TM*.* □

In Chapter 4, we will consider another example in detail: On the local homogeneous space of a Carnot group $\mathbb{G}$ of rank d under the action of a lattice subgroup, there is a submersion $\pi : \Gamma\backslash\mathbb{G} \to \mathbb{T}^d$ onto the d-dimensional torus by Section 2.4. Then one can define a horizontal Clifford bundle and a horizontal Dirac operator by pulling back the objects from $\mathbb{T}^d$, which does not depend on the non-horizontal directions of TM. We will discuss this example extensively in Chapter 4, since it serves well as a toy model to the question whether one can define spectral triples from horizontal Dirac operators.

For further considerations it will be important to work with the square of the horizontal Dirac operator. Especially it will follow in Section 6.3 that this operator, considered as an operator on $L^2(S^HM)$, is not hypoelliptic, and we will conclude from that that a horizontal Dirac operator in not hypoelliptic. Hence we calculate $\left(D^H\right)^2$, whose principal term will be a horizontal Laplacian, in a local expression.

Proposition 3.2.8
Let M be a Carnot manifold, and let $\{X_1,\dots,X_d\}$ be an orthonormal frame for its horizontal distribution HM. Let D^H be a horizontal Dirac operator acting on a horizontal Clifford bundle S^HM over M. Then we have locally

$$\left(D^H\right)^2 = -\sum_{j=1}^{d} X_j^2 + \sum_{j<k} c^H(X_j)c^H(X_k)\,[X_j,X_k] + O_H(1), \tag{3.11}$$

where X_j is to be understood as a component-wise directional derivative in a local chart and $O_H(1)$ denotes a graded differential operator of order smaller or equal to 1 (which only depends on the differential operators $X_1,\dots,X_d$ and endomorphisms on the bundle).

Proof: We work with the local expression of D^H, given by (3.9). From this we get for $\sigma \in S^HM$

$$\begin{aligned}\left(D^H\right)^2\sigma =& \sum_{j=1}^{d} c^H(X_j)\nabla^{S^H}_{X_j}\left(\sum_{k=1}^{d} c^H(X_k)\nabla^{S^H}_{X_k}\sigma - \frac{1}{2}c^H(Z)\,\sigma\right)\\ &- \frac{1}{2}c^H(Z)\left(\sum_{k=1}^{d} c^H(X_k)\nabla^{S^H}_{X_k}\sigma - \frac{1}{2}c^H(Z)\sigma\right)\end{aligned} \tag{3.12}$$

where

$$Z = \sum_{j=d+1}^{n} \pi^H\nabla_{X_j}X_j \in HM$$

is the mean curvature of the bundle $HM^\perp$ like in Theorem 3.2.5. Note that we have $c^H(Z) \in \mathrm{End}_{\mathbb{C}}(S^HM)$. We observe that the second summand on the right hand side of (3.12) consists only of differential operators of graded order 1 or 0 applied to σ, so it is contained in the expression $O_H(1)$. Further we have

$$\nabla^{S^H}_{X_j}\left(c^H(Z)\,\sigma\right) = c^H\left(\nabla^H_{X_j}Z\right)\sigma + c^H(Z)\,\nabla^{S^H}_{X_j}\sigma$$

by Definition 3.2.2, since S^HM is a horizontal Clifford bundle, so this expression is also contained in $O_H(1)$.

Now we cannot assume $\nabla^H_{X_j}X_k = 0$ as one does when calculating the square of an ordinary Dirac operator in this case, since our frame is fixed and therefore cannot be chosen to be synchronous at a point $x \in M$. We calculate for $j,k \in \{1,\dots,d\}$, using once again the properties of a horizontal Clifford bundle:

$$\begin{aligned} c^H(X_j)\nabla^{S^H}_{X_j}\left(c^H(X_k)\nabla^{S^H}_{X_k}\sigma\right) &= c^H(X_j)\left(c^H\left(\nabla^H_{X_j}X_k\right)\nabla^{S^H}_{X_k}\sigma + c^H(X_k)\nabla^{S^H}_{X_j}\nabla^{S^H}_{X_k}\sigma\right)\\ &= c^H(X_j)c^H(X_k)\nabla^{S^H}_{X_j}\nabla^{S^H}_{X_k}\sigma + O_H(1)\sigma. \end{aligned}$$

Since we are working in local charts, $\nabla^{S^H}_{X_j}$ is locally given by the expression $X_j + \Gamma_j$ with $\Gamma_j \in \mathrm{End}(S^H M)$. Applying this to the last expression, we find

$$\begin{aligned}\nabla^{S^H}_{X_j}\nabla^{S^H}_{X_k} &= (X_j + \Gamma_j)(X_k + \Gamma_k)\,\sigma \\ &= X_j X_k(\sigma) + X_j(\Gamma_k\sigma) + \Gamma_j X_k(\sigma) + \Gamma_j\Gamma_k\sigma \\ &= X_j X_k(\sigma) + O_H(1)\sigma\end{aligned}$$

and therefore

$$\begin{aligned}&\sum_{j=1}^{d}\sum_{k=1}^{d} c^H(X_j)\nabla^{S^H}_{X_j}\left(c^H(X_k)\nabla^{S^H}_{X_k}\sigma\right) \\ =\;& \sum_{j=1}^{d}\sum_{k=1}^{d} c^H(X_j)c^H(X_k)X_jX_k(\sigma) + O_H(1)\sigma \\ =\;& \sum_{j=k} -X_j^2(\sigma) + \sum_{j\neq k} c^H(X_j)c^H(X_k)X_jX_k(\sigma) + O_H(1)\sigma \\ =\;& -\sum_{j=1}^{d} X_j^2(\sigma) + \sum_{j<k} c^H(X_j)c^H(X_k)\left(X_jX_k - X_kX_j\right)(\sigma) + O_H(1)\sigma,\end{aligned}$$

since $c^H(X_j)^2 = -1$ and $c^H(X_j)c^H(X_k) = -c^H(X_k)c^H(X_j)$ by the characterization of the Clifford algebra. Finally we plug this into (3.12) and find together with the discussion above

$$\begin{aligned}\left(D^H\right)^2\sigma &= \sum_{j=1}^{d} c^H(X_j)\nabla^{S^H}_{X_j}\left(\sum_{k=1}^{d} c^H(X_k)\nabla^{S^H}_{X_k}\sigma\right) + O_H(1)\sigma \\ &= -\sum_{j=1}^{d} X_j^2(\sigma) + \sum_{j<k} c^H(X_j)c^H(X_k)\,[X_j, X_k]\,(\sigma) + O_H(1)\sigma.\end{aligned}$$

This proves the proposition. □

Remark: The proof of this proposition shows that the leading term

$$-\sum_{j=1}^{d} X_j^2 + \sum_{j<k} c^H(X_j)c^H(X_k)\,[X_j, X_k]$$

in the local expression (3.11) of $(D^H)^2$ will not change if we modify D^H by an term of graded order zero. For an operator which is locally of the type

$$\tilde{D}^H = D^H + \gamma(x)$$

with $\gamma \in C^\infty(M, \mathrm{End}_{\mathbb{C}}(S^H M))$ we still have

$$\left(\tilde{D}^H\right)^2 = -\sum_{j=1}^{d} X_j^2 + \sum_{j<k} c^H(X_j)c^H(X_k)\,[X_j, X_k] + O_H(1).$$

In particular this shows that for a modified connection

$$\tilde{\nabla}^{S^H}_{X_j} = \nabla^{S^H}_{X_j} + \gamma_0(x)$$

with $\gamma_0 \in C^\infty(M, \mathrm{End}_{\mathbb{C}}(S^H M))$ the leading term of the local expression of $(D^H)^2$ according to this connection will not change. We will refer to this remark when we check D^H for hypoellipticity in Section 6.3, since the property of being hypoelliptic only depends of the leading term of a horizontal Laplacian. $\lhd$

Before we conclude this section, we want to prove a rather technical lemma about the eigenvalues a certain combination of Clifford matrices can have. This will be needed in the following chapters, and since it will be needed more than one time we decided to put it here. Its statement is quite general, since it can be applied to the Clifford action arising from any vector bundle, horizontal or not.

Proposition 3.2.9
Let V be a vector space of dimension d, and let $A = \mathrm{Cl}(V) \otimes \mathbb{C}$ denote its (complexified) Clifford algebra, which is represented by matrices on a complex vector space S via (3.5). We assume that we have an $m \in \mathbb{N}$ such that $2m \leq d$ and such that $\{e_1, \ldots, e_{2m}, \ldots, e_d\}$ is a basis of V.
Then we have the following:

(i) For any $k \neq l$ with $1 \leq k, l \leq d$, all the eigenvalues of the matrix $c(e_k)c(e_l)$ are given by the numbers $\pm i$, where both eigenvalues have the same multiplicity.

(ii) The eigenvalues of the matrix

$$\sum_{j=1}^{m} c(e_j)c(e_{m+j}) \tag{3.13}$$

are exactly the numbers

$$\mu_l = i\,(-m + 2l) \quad \text{with } l = 0, \ldots, m.$$

In the case where $d = 2m$ and the Clifford action of V on S is irreducible, which means we have $\dim S = 2^m$, each eigenvalue μ_l has the multiplicity $\binom{m}{l}$.

(iii) For $\lambda_j \in \mathbb{R}$ with $\lambda_j > 0$ (with $j \in \{1, \ldots, m\}$), all the eigenvalues of the matrix

$$\sum_{j=1}^{m} \lambda_j c(e_j)c(e_{m+j}). \tag{3.14}$$

are included in the interval

$$\left[-i\sum_{j=1}^{m} \lambda_j, i\sum_{j=1}^{m} \lambda_j\right] \subset i\mathbb{R}$$

on the imaginary line. Thereby, the numbers $i\sum_{j=1}^{m} \lambda_j$ and $-i\sum_{j=1}^{m} \lambda_j$ are eigenvalues of the matrix (3.14).

Remark: Although this proposition has been formulated for vector spaces, it can be transferred to Clifford bundles immediately, since point-wise we have the eigenvalues noted above. In particular, the situation of the proposition occurs for the horizontal Clifford action on a Clifford bundle on a Carnot manifold M, where M is locally diffeomorph to a Carnot group of the type $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$. In this case we will identify the horizontal frame $\{X_1, \ldots, X_d\}$ at each point $x \in M$ with the vector space spanned by $\{e_1, \ldots, e_d\}$, for which the proposition is formulated. $\lhd$

Proof: First of all, since

$$\begin{aligned} (c(e_k)c(e_l))^2 &= c(e_k)c(e_l)c(e_k)c(e_l) \\ &= -\left(c(e_k)\right)^2 \left(c(e_l)\right)^2 \\ &= -\mathrm{id} \quad \forall 1 \leq k, l \leq d, \end{aligned}$$

each of the products $c(e_k)c(e_l)$ has eigenvalues i and $-i$. Furthermore, the matrices $c(e_k)c(e_l)$ and $c(e_l)c(e_k)$ commute since

$$c(e_k)c(e_l)c(e_l)c(e_k) = c(e_l)c(e_k)c(e_k)c(e_l) = \mathrm{id},$$

and are thus simultaneously diagonalizable. Because of the relation $c(e_k)c(e_l) = -c(e_l)c(e_k)$ this means that we have the same number of i- and $-i$-eigenvalues, and therefore statement (i) is proved.

The next observation is that for $j \neq k$ and $1 \leq j, k \leq m$, the matrices $c(e_j)c(e_{m+j})$ and $c(e_k)c(e_{m+k})$ commute: We have

$$\begin{aligned} & [c(e_j)c(e_{m+j}), c(e_k)c(e_{m+k})] \\ = \; & c(e_j)c(e_{m+j})c(e_k)c(e_{m+k}) - c(e_k)c(e_{m+k})c(e_j)c(e_{m+j}) \\ = \; & c(e_k)c(e_{m+k})c(e_j)c(e_{m+j}) - c(e_k)c(e_{m+k})c(e_j)c(e_{m+j}) \\ = \; & 0 \end{aligned}$$

by the commutation rules of the horizontal Clifford action, see Proposition 3.2.1. But this means that all the summands of the matrix $\sum_{j=1}^{m} \lambda_j c(e_j)c(e_{m+j})$ are simultaneously diagonalizable. Let S be a matrix diagonalizing these summands simultaneously, this means its diagonal matrix is given by

$$S^{-1} \left(\sum_{j=1}^{m} \lambda_j c(e_j)c(e_{m+j}) \right) S = \sum_{j=1}^{m} \lambda_j S^{-1} c(e_j)c(e_{m+j}) S. \tag{3.15}$$

Using this, we can prove the rest of the statements by induction over m.

From this point on, we will assume for a moment that the representation of our Clifford algebra is irreducible. This means that we have a map

$$c : V \to \mathrm{End}_{\mathbb{C}}(S)$$

for a vector space S with $\dim S = 2^{[d/2]}$, $[\cdot]$ denoting the Gaussian bracket, such that for every $j \in \{1, \dots, d\}$ we have

$$c(e_j) \in \mathrm{Mat}_{2^{[d/2]} \times 2^{[d/2]}}(\mathbb{C})$$

(see the discussion in the beginning of this section for this). We further denote the (unique) complexified Clifford algebra arising from a (complex) vector space of dimension n by $\mathrm{Cl}_{\mathbb{C}}(n)$. For the proof by induction, we will use the isomorphism

$$\mathrm{Cl}_{\mathbb{C}}(n+2) \cong \mathrm{Cl}_{\mathbb{C}}(n) \otimes \mathrm{Cl}_{\mathbb{C}}(2), \tag{3.16}$$

see [LM89], Theorem I.4.3.

Now we start with the induction argument: For the case $m = 1$ the statements (ii) and (iii) already follow from statement (i). Now we assume the statements to be true for $m-1$. In detail, this means the following: For a vector space $\tilde{V} \cong \mathbb{R}^{d-2}$ with basis $\{\tilde{e}_1, \dots, \tilde{e}_{d-2}\}$, we have a Clifford algebra $\mathrm{Cl}_{m-1}(\tilde{V}) \otimes \mathbb{C} \cong \mathrm{Cl}_{\mathbb{C}}(d-2)$. Since we assumed irreducibility of the Clifford action, the elements $c_{m-1}(\tilde{e}_k)$ are given by $(2^{[d/2]-1} \times 2^{[d/2]-1})$-matrices, such that the statements (ii) and (iii) are true for sums of the form

$$\sum_{j=1}^{m-1} \lambda_j c(\tilde{e}_j) c(\tilde{e}_{m-1+j}),$$

considering $m-1$ instead of m.

Now for a vector space $V \cong \mathbb{R}^d$ we have $\mathrm{Cl}_d(V) \otimes \mathbb{C} \cong \mathrm{Cl}_{\mathbb{C}}(d)$ and consider a basis $\{e_1, \dots, e_d\}$. Hence after using the isomorphism from (3.16), we can work with the following representations of the elements $e_1, \dots, e_d$:

$$c_m(e_j) = \begin{cases} c_{m-1}(\tilde{e}_j) \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } 1 \leq j \leq m-1 \\ \mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \text{for } j = m \\ c_{m-1}(\tilde{e}_{j-1}) \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } m+1 \leq j \leq 2m-1 \\ \mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } j = 2m \\ c_{m-1}(\tilde{e}_{j-2}) \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } 2m+1 \leq j \leq d, \end{cases} \tag{3.17}$$

where c_{m-1} denotes the Clifford action by $\mathrm{Cl}_{m-1}(\tilde{V}) \otimes \mathbb{C}$ and c_m denotes the Clifford action by $\mathrm{Cl}_m(V) \otimes \mathbb{C}$ on the corresponding vector spaces. Note that the matrices

$$E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

together with the (2×2)-unit matrix, form a basis for the Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(2)$. For the representations from (3.17) we get by the rules of the tensor product for matrices:

$$\begin{aligned}
&\sum_{j=1}^{m} \lambda_j c_m(e_j) c_m(e_{m+j}) \\
=\ &\sum_{j=1}^{m-1} \lambda_j c_m(e_j) c_m(e_{m+j}) + \lambda_m c_m(e_m) c_m(e_{2m}) \\
=\ &\sum_{j=1}^{m-1} \lambda_j \left(c_{m-1}(\tilde{e}_j) \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \cdot \left(c_{m-1}(\tilde{e}_{m-1+j}) \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \\
&+\lambda_m \left(\mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) \cdot \left(\mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\
=\ &\sum_{j=1}^{m-1} \lambda_j \left(c_{m-1}(\tilde{e}_j) c_{m-1}(\tilde{e}_{m-1+j}) \otimes \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) + \lambda_m \cdot \mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
=\ &\left(\sum_{j=1}^{m-1} \lambda_j c_{m-1}(\tilde{e}_j) c_{m-1}(\tilde{e}_{m-1+j}) \right) \otimes \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_m \cdot \mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{aligned}$$

Now let $\tilde{S}$ be the matrix diagonalizing the matrices $c_{m-1}(\tilde{e}_j)c_{m-1}(\tilde{e}_{m+j})$ simultaneously, such that we have (3.15) in this situation. But then we see from the above calculation that the matrix

$$S := \tilde{S} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

diagonalizes $\sum_{j=1}^{m} \lambda_j c_m(e_j) c_m(e_{m+j})$ simultaneously, which means we have because of the above calculation

$$\begin{aligned}
&S^{-1} \sum_{j=1}^{m} \lambda_j c_m(e_j) c_m(e_{m+j}) S \\
=&\left(\tilde{S}^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \left(\sum_{j=1}^{m} \lambda_j c_m(e_j) c_m(e_{m+j}) \right) \cdot \left(\tilde{S} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
=&\tilde{S}^{-1} \left(\sum_{j=1}^{m-1} \lambda_j c_{m-1}(\tilde{e}_j) c_{m-1}(\tilde{e}_{m-1+j}) \right) \tilde{S} \otimes \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_m \cdot \mathrm{id}_{2^{[d/2]-1}} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{aligned} \tag{3.18}$$

Since all these matrices are diagonal matrices, we immediately see that for each eigenvalue $\tilde{\mu}_l$, $l \in \{0, \dots, m-1\}$, of $\sum_{j=1}^{m-1} \lambda_j c_{m-1}(\tilde{e}_j) c_{m-1}(\tilde{e}_{m-1+j})$ with multiplicity $\tilde{\nu}_l$ the numbers

$$\mu_l^+ := -\tilde{\mu}_l + i\lambda_m \quad \text{and} \quad \mu_l^- := -\tilde{\mu}_l - i\lambda_m \tag{3.19}$$

are eigenvalues of $\sum_{j=1}^{m} \lambda_j c_m(e_j) c_m(e_{m+j})$. But from (3.19) we can prove the statements (ii) and (iii) by induction:

- Statement (iii) can be seen immediately from (3.19). If $\tilde{\mu}_0 := i\sum_{j=1}^{m-1}\lambda_j$ is the greatest and $\tilde{\mu}_{m-1} := -i\sum_{j=1}^{m-1}\lambda_j$ is the lowest eigenvalue of $\sum_{j=1}^{m-1}\lambda_j c(\tilde{e}_j)c(\tilde{e}_{m-1+j})$, then the greatest and lowest eigenvalue of (3.14) are given by the numbers

$$\mu_{m+1}^+ = \tilde{\mu}_{m-1} + i\lambda_m \quad \text{and} \quad \mu_0^- = \tilde{\mu}_0 - i\lambda_m,$$

  which proves both statements of (iii).

- Statement (ii) follows after choosing $\lambda_1 = \ldots = \lambda_m = 1$ in (3.19). By the assumption that (ii) is true for $m-1$, we have

$$\mu_l^+ = i((m-1)+1-2l) = i(m-2l)$$

  and

$$\mu_l^- = i((m-1)-1-2l) = i(m-2(l+1))$$

  for $0 \leq l \leq m-1$, where each $\mu_l^\pm$ has multiplicity $\binom{m-1}{l}$. We observe for $l \leq m-2$ that

$$\mu_{m-l+1} := \mu_{l+1}^+ = \mu_l^-,$$

  hence each of these eigenvalues has multiplicity $\binom{m-1}{l} + \binom{m-1}{l+1} = \binom{m}{l+1}$. In addition we have the eigenvalues $\mu_0 := \mu_{m-1}^- = -im$ and $\mu_m := \mu_0^+ = im$, which are both of multiplicity 1.

We finally drop the restriction that the representation of the Clifford algebra is irreducible: We do not get any new eigenvalues, because every (reducible) representation is a direct sum of irreducible ones, and hence (ii) and (iii) are also true for this case. But we cannot make a general statement about the multiplicity of the eigenvalues in the reducible case.

Altogether, every statement of this proposition is proved. □

3.3 Detection of the Carnot-Carathéodory Metric

Throughout this section we will assume that M is a compact Carnot manifold without boundary, such that the algebra $C(M)$ is a unital C^*-algebra which can be represented on $L^2(M)$ via left multiplication. Our intention is to show that the operator D^H from the previous section detects the Carnot-Carathéodory metric via the Connes metric formula

$$d_{CC}(x,y) = \sup\left\{|f(x)-f(y) : f \in \mathcal{A}', \left\|[D^H,f]\right\| \leq 1\right\}, \tag{3.20}$$

where $\mathcal{A}'$ is a dense sub-algebra of $C(M)$. This means that although D^H does not furnish a spectral triple (which we will see in general in Chapter 6), we can consider the triple $(C(M), L^2(M), D^H)$ as a compact quantum metric space in the sense of Mark Rieffel, see Definition 1.2.2, with the corresponding Lip-norm $L(f) := \left\|[D^H,f]\right\|$, whose metric is exactly the Carnot-Carathéodory metric on M.

The key observation is that, in analogy to the classical case from the standard example for a spectral triple, the commutator $[D^H, f]$ acts as Clifford action by the horizontal gradient of a function f. Remember that on a Carnot manifold M the horizontal gradient of a function $f \in C^1(M)$ is given by the vector field

$$\mathrm{grad}^H(f) = \sum_{j=1}^{d} X_j(f) \cdot X_j,$$

where, like before, $HM = \mathrm{span}\{X_1, \ldots, X_d\} \subset TM$ is a horizontal distribution of M.

Proposition 3.3.1
Let D^H be a horizontal Dirac operator acting on a horizontal Clifford bundle $S^H M$ with horizontal Clifford action c^H over a closed Carnot manifold M. Then, for any function $f \in C^1(M)$, we have

$$\left[D^H, f\right] = c^H(\mathrm{grad}^H f).$$

Proof: Let us first show that the horizontal Dirac operator fulfills the Leibniz rule. This is just an easy calculation using the properties of the connection and the Clifford action: For any $\sigma \in \Gamma^\infty(S^H M)$ we have

$$\begin{aligned}
D^H(f \cdot \sigma) &= \sum_{j=1}^{d} c^H(X_j)\, \nabla^{S^H}_{X_j}(f \cdot \sigma) \\
&= \sum_{j=1}^{d} c^H(X_j) \left(f \cdot \nabla^{S^H}_{X_j} \sigma + X_j(f) \cdot \sigma \right) \\
&= f \cdot D^H \sigma + \left(\sum_{j=1}^{d} c^H(X_j)\, X_j(f) \right) \sigma \\
&= f \cdot D^H \sigma + c^H \left(\sum_{j=1}^{d} X_j(f) \cdot X_j \right) \sigma \\
&= f \cdot D^H \sigma + c^H \left(\mathrm{grad}^H f \right) \sigma.
\end{aligned}$$

Now the statement follows immediately:

$$\left[D^H, f\right] \sigma = D^H(f \cdot \sigma) - f \cdot D^H \sigma = c^H \left(\mathrm{grad}^H f \right) \sigma,$$

and the proposition is proved. □

Using this proposition, we have to show that the Lip-norm defined by the Connes metric coming from D^H coincides with the supremum of the horizontal gradient of a function f. But yet it is not clear how the sub-algebra $\mathcal{A}'$ of $C(M)$ has to look like such that the Connes metric formula (3.20) is true. In the classical case, this is exactly the algebra of Lipschitz functions, and it will turn out that in the Carnot case this will be the algebra of functions which are Lipschitz with respect to the Carnot-Carathéodory metric.

Definition 3.3.2
Let (M, d_{CC}) be a Carnot manifold, $f \in C(M)$. Then we call the number

$$\mathrm{Lip}_{CC}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d_{CC}(x, y)} : x, y \in M, x \neq y \right\}$$

the Carnot-Carathéodory-Lipschitz constant *of f. If $\mathrm{Lip}_{CC}(f)$ is finite, we call f a* Carnot-Carathéodory-Lipschitz function*.*

We denote the algebra of all Carnot-Carathéodory-Lipschitz functions on M, equipped with the semi-norm $\mathrm{Lip}_{CC}(f)$, by $\mathrm{Lip}_{CC}(M)$. $\triangleleft$

The proof that the horizontal Dirac-operator D^H detects the Carnot-Carathéodory metric works similar to the classical one that a Dirac operator detects the geodesic metric on a Riemannian spin manifold (see e.g. [Con94], [GVF01] or [Lan97]): If we assume that for $f \in \mathrm{Lip}_{CC}(M)$ the horizontal gradient $\mathrm{grad}^H f(x)$ exists almost everywhere, we show that the number $\mathrm{Lip}_{CC}(f)$ coincides with the essential supremum of $\mathrm{grad}^H f$. After that we will show that the Carnot-Carathéodory metric can be described via the Carnot-Carathéodory-Lipschitz constant, such that we can apply Proposition 3.3.1 to get the result.

We need to show that the assumption that $\mathrm{grad}^H f(x)$ exists almost everywhere for $f \in \mathrm{Lip}_{CC}(M)$. Note that the analogous statement for classical Lipschitz functions is well known (see e.g. [Fed69]). The horizontal case is shown in [CDPT07] for the case of the Heisenberg group (see [CDPT07], Proposition 6.12), and this result can be generalized easily to arbitrary Carnot manifolds as we show in the following proposition. In a greater generality, this is also a consequence of the Pansu-Rademacher theorem ([Pan89]; see also for example [CDPT07], Theorem 6.4), which states that a Lipschitz map between two Carnot groups, the so-called *Pansu differential* (see [Pan89]), exists.

Proposition 3.3.3
Let M be a closed Carnot manifold with horizontal distribution $HM = \mathrm{span}\{X_1, \ldots, X_d\}$ and let $f : M \to \mathbb{R}$ be a Carnot-Carathéodory-Lipschitz function. Then the horizontal gradient

$$\mathrm{grad}^H f = \sum_{j=1}^{d} X_j(f) \cdot X_j$$

exists almost everywhere on M.

Proof: We consider the case where $M = \Omega \subset \mathbb{G}$ is an open subset of a Carnot group $\mathbb{G}$; then the general case follows after restricting ourselves to local coordinates (on which we have a Carnot group structure) since the coordinate changes are smooth and therefore do not affect the regularity of f. Thus we choose an arbitrary point $x_0 \in \Omega$. We fix a $j \in \{1, \ldots, d\}$. After an affine change of coordinates on Ω (such that $0 \in \Omega$) we can assume

$$x_0 \in \left\{ \left((x_{1,1}, \ldots, x_{1,d}, x^{(2)}, \ldots, x^{(R)} \right) \in \Omega : x_{1,j} = 0 \right\} =: \Omega_{0,j}, \tag{3.21}$$

where the coordinates on $\Omega \subset \mathbb{G}$ are meant to be either exponential or polarized coordinates, see Section 2.2.

The idea is to consider the integral curves arising from x_0 into the direction of the vector field X_j via the exponential map, which means we have a curve

$$\gamma_{x_0} : [0, a] \to \mathbb{G}, \quad t \mapsto x_0. \exp tX_j \tag{3.22}$$

for any $a > 0$. We denote by $L_{x_0} := \gamma_{x_0}([0, a]) \cap \Omega$ the path of γ_{x_0}. Since this path is horizontal, the map

$$f_{x_0} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f\,(x_0 \cdot \exp tX_j)$$

is a (classical) Lipschitz function on $\mathbb{R}$ because of the Carnot-Carathéodory Lipschitz property of f. Since f_{x_0} is Lipschitz, it is differentiable almost everywhere (see e.g. [Fed69]), and its derivative is given by

$$f'_{x_0}(t) = X_j f\,(x_0 \cdot \exp tX_j)\,.$$

But this shows that $X_j f$ exists almost everywhere on L_{x_0}. This is true for any starting point $x_0 \in \Omega_{0,j}$, such that we can conclude that $X_j f$ exists almost everywhere on Ω because any point of a Carnot group $\mathbb{G}$ can be reached from the hyperplane $\Omega_{0,j}$ by a horizontal curve of the type (3.22).

Now the above argument is true for any $j \in \{1, \dots, d\}$, and hence any horizontal partial derivative $X_j f$ and therefore also the horizontal gradient exists almost everywhere on Ω. $\square$

To show that the Carnot-Carathéodory-Lipschitz constant coincides with the supremum norm of the horizontal gradient, we work with the corresponding object from the cotangent bundle of M. We assume we have a Riemannian metric g on M such that $\{X_1, \dots, X_n\}$ is an orthonormal frame with respect to g, which respects the grading structure of TM. Remember from Section 2.1 that we can choose a basis $\{d\omega^1, \dots, d\omega^n\}$ of T^*M such that

$$HM = \text{span}\{X_1, \dots, X_d\} = \text{Ker}\left(\text{span}\left\{d\omega^{d+1}, \dots, d\omega^n\right\}\right). \tag{3.23}$$

According to this basis, the horizontal differential of f is given by

$$d^H f = \sum_{j=1}^{d} X_j(f) d\omega^j.$$

Obviously we have $\sup_{x\in M} \|\text{grad}^H f(x)\| = \sup_{x\in M} \|d^H f(x)\|$ for any function $f \in C^1(M)$, where $\|\cdot\|$ denotes the (Euclidean) norm of a horizontal tangent (or cotangent) vector on M, coming from the Riemannian metric g.

Lemma 3.3.4
Let M be a closed Carnot manifold with Riemannian metric g as above, and let $f \in \mathrm{Lip}_{CC}(M)$ such that, by Proposition 3.3.3, the horizontal gradient $\mathrm{grad}^H f$ of f exists almost everywhere. Then we have

$$\mathrm{Lip}_{CC}(f) = \operatorname*{ess\,sup}_{x\in M} \left\|\mathrm{grad}^H f(x)\right\|.$$

Proof: For $f \in \mathrm{Lip}_{CC}(M)$ we show that $\mathrm{Lip}_{CC}(f) \leq \operatorname{ess\,sup}_{x\in M} \left\|\mathrm{grad}^H f(x)\right\|$ and that $\operatorname{ess\,sup}_{x\in M} \left\|\mathrm{grad}^H f(x)\right\| \leq \mathrm{Lip}_{CC}(f)$.
Let $x, y \in M$, and let $\gamma : [0,1] \to M$ be a smooth horizontal curve connecting x and y, which means $\gamma(0) = x$, $\gamma(1) = y$ and $\dot\gamma(t) \in H_{\gamma(t)}M$ for all $t \in [0,1]$. Note that because of (3.23) and the characterization of the horizontal differential this means $df(\dot\gamma(t)) = d^H f(\dot\gamma(t))$ for all t, and we have for $f \in C^1(M)$

$$\begin{aligned}
f(x) - f(y) &= f(\gamma(1)) - f(\gamma(0)) \\
&= \int_0^1 \frac{d}{dt} f(\gamma(t))dt \\
&= \int_0^1 df\,(\dot\gamma(t))\,dt \\
&= \int_0^1 d^H f\,(\dot\gamma(t))\,dt \\
&= \int_0^1 g\left(\mathrm{grad}^H f(\gamma(t)), \dot\gamma(t)\right) dt. \\
&\leq \int_0^1 \left\|\mathrm{grad}^H f(\gamma(t))\right\| \cdot \|\dot\gamma(t)\|\,dt \\
&\leq \sup_{x\in M} \left\|\mathrm{grad}^H f(x)\right\| \cdot \int_0^1 \|\dot\gamma(t)\|\,dt,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Taking the infimum over all horizontal curves connecting x and y, we find

$$|f(x) - f(y)| \leq \sup_{x\in M} \left\|\mathrm{grad}^H f(x)\right\| \cdot d_{CC}(x,y). \tag{3.24}$$

Now the above calculation holds not only for C^1-functions f, but for all functions which have a horizontal gradient almost everywhere, such that $\mathrm{grad}^H f$ is defined as an essentially bounded vector field on M. Hence for every f which fulfills the assumptions of the Lemma, (3.24) becomes

$$|f(x) - f(y)| \leq \operatorname*{ess\,sup}_{x\in M} \left\|\mathrm{grad}^H f(x)\right\| \cdot d_{CC}(x,y),$$

and since x and y can be chosen arbitrarily we see

$$\mathrm{Lip}_{CC}(f) = \sup_{\{x,y\in M:\ x\neq y\}} \frac{|f(x) - f(y)|}{d_{CC}(x,y)} \leq \operatorname*{ess\,sup}_{x\in M} \left\|\mathrm{grad}^H f(x)\right\|. \tag{3.25}$$

On the other hand, we choose an $x_0 \in M$ such that $\mathrm{grad}^H f(x_0)$ exists and consider the integral curve along the vector field $\mathrm{grad}^H f(x_0)$ arising from the exponential map $\exp_{x_0} : T_{x_0}M \to M$ from Riemannian geometry, that is

$$\gamma_{x_0}(t) := x_0.\exp\left(t \cdot \mathrm{grad}^H f(x_0)\right).$$

For a small $\varepsilon > 0$ we set $x := \gamma_{x_0}(\varepsilon)$ and denote the length of a horizontal curve $\gamma_{x_0}([0,\varepsilon])$ which is connecting x_0 and x by $L_{\gamma_{x_0}}(x, x_0)$. We observe that we have

$$\begin{aligned} \lim_{x\to x_0}\left|\frac{f(x)-f(x_0)}{L_{\gamma_{x_0}}(x,x_0)}\right| &= \lim_{\varepsilon\to 0}\left|\frac{f\left(x_0.\exp\left(\varepsilon\cdot \mathrm{grad}^H f(x_0)\right)\right)-f(x_0)}{\varepsilon}\right| \\ &= \left\|\mathrm{grad}^H f(x_0)\right\|. \end{aligned} \tag{3.26}$$

Now, since γ_{x_0} is a horizontal curve connecting x and x_0 we have $d_{CC}(x_0, x) \le L_{\gamma_{x_0}}(x, x_0)$, and hence we get from (3.26)

$$\begin{aligned} \left\|\mathrm{grad}^H f(x_0)\right\| &\le \lim_{x\to x_0}\left|\frac{f(x)-f(x_0)}{d_{CC}(x,x_0)}\right| \\ &\le \lim_{x\to x_0}\frac{\mathrm{Lip}_{CC}(f)\cdot d_{CC}(x,x_0)}{d_{CC}(x,x_0)} = \mathrm{Lip}_{CC}(f) \end{aligned}$$

by definition of the Carnot-Carathéodory-Lipschitz constant. But since this works for every $x_0 \in M$ where $\mathrm{grad}^H f(x_0)$ exists (which is almost every $x_0 \in M$), we have

$$\operatorname*{ess\,sup}_{x\in M}\left\|\mathrm{grad}^H f(x)\right\| \le \mathrm{Lip}_{CC}(f). \tag{3.27}$$

Altogether (3.25) and (3.27) prove the statement of the lemma. □

To make use of the above Lemma, one has to show that the Carnot-Carathéodory distance can be expressed using the Carnot-Carathéodory-Lipschitz constant. This should be obvious, since it is just the Lipschitz semi-norm belonging to the compact metric space (M, d_{CC}), but for completeness we write down the proof. From our point of view it is important that the function describing the Carnot-Carathéodory distance from a fixed point $x_0 \in M$ is a Carnot-Carathéodory-Lipschitz function, which has Lip-norm bounded by 1 and is differentiable into horizontal directions almost everywhere. But note that this is not a C^1-function.

Lemma 3.3.5
On a closed Carnot manifold M the Carnot-Carathéodory distance between two points $x, y \in M$ is given by

$$d_{CC}(x,y) = \sup\left\{|f(x)-f(y)| : f \in \mathrm{Lip}_{CC}(M), \mathrm{Lip}_{CC}(f) \le 1\right\}.$$

Proof: For every $f \in C(M)$ such that $\mathrm{Lip}_{CC}(f) \leq 1$ we have for all $x, y \in M$:

$$|f(x) - f(y)| \leq \mathrm{Lip}_{CC}(f) \cdot d_{CC}(x, y) \leq d_{CC}(x, y). \tag{3.28}$$

On the other hand, we define a function $h : M \to \mathbb{R}$ via $h(y) := d_{CC}(x, y)$ for a given $x \in M$. Obviously h is continuous, and since d_{CC} is a metric on M we have for any $z \in M$

$$|h(y) - h(z)| = |d_{CC}(x, y) - d_{CC}(x, z)| \leq d_{CC}(y, z).$$

This shows $\mathrm{Lip}_{CC}(h) \leq 1$, and since $|h(x) - h(y)| = d_{CC}(x, y)$ we get

$$d_{CC}(x, y) \leq \sup \{|f(x) - f(y)| : \mathrm{Lip}_{CC}(f) \leq 1\}. \tag{3.29}$$

From (3.28) and (3.29) the statement of the lemma follows. □

Remark: The fact that the function h appearing in the proof is differentiable almost everywhere can also be deduced from the fact that any two points $x, y \in M$ can be joint by a so-called minimizing geodesic, which is a part from x to y realizing the Carnot-Carathéodory distance; fulfilling the additional property that it has a derivative for almost all t whose components are measurable functions (see [Mon02], Theorem 1.19). We do not need this argument, since the proof above shows that h is a Carnot-Carathéodory-Lipschitz function which has the property mentioned above by Proposition 3.3.3. ◁

Now we simply have to put everything together to get the identity of the metrics.

Theorem 3.3.6
Let M be a closed Carnot manifold and D^H the horizontal Dirac operator acting on a horizontal Clifford bundle $S^H M$ over M. Then the Carnot-Carathéodory metric of M can be detected via the formula

$$d_{CC}(x, y) = \sup \left\{|f(x) - f(y)| : f \in \mathrm{Lip}_{CC}(M),\ \left\|[D^H, f]\right\| \leq 1\right\}. \tag{3.30}$$

Proof: By Lemma 3.3.4 and Lemma 3.3.5 we have

$$d_{CC}(x, y) = \sup \left\{|f(x) - f(y)| : f \in \mathrm{Lip}_{CC}(M), \operatorname*{ess\,sup}_{x \in M} \left\|\mathrm{grad}^H f(x)\right\| \leq 1\right\}.$$

Now Proposition 3.3.1 tells us that

$$[D^H, f] = c^H(\mathrm{grad}^H f) \tag{3.31}$$

for every $f \in C^1(M)$, where $c^H : HM \to \mathrm{End}_{\mathbb{C}}(S^H M)$ denotes the horizontal Clifford action, and the norm of the operator $c^H(\mathrm{grad}^H f)$ is given by

$$\left\|c^H\left(\mathrm{grad}^H f\right)\right\| = \sup_{x \in M} \left\|c^H\left(\mathrm{grad}^H f(x)\right)\right\|.$$

But since because of $(c^H)^2 = -\mathrm{id}$ the map $c^H : H_xM \to S^H_xM$ is an isometry for any $x \in M$, this shows

$$\left\| c^H \left(\mathrm{grad}^H f\right) \right\| = \sup_{x\in M} \left\| \mathrm{grad}^H f(x) \right\|.$$

Hence together with (3.31) we see that the identity

$$\sup_{x\in M} \left\| \mathrm{grad}^H f(x) \right\| = \left\| \left[D^H, f \right] \right\| \tag{3.32}$$

is true for any $f \in C^1(M)$. Since for $f \in \mathrm{Lip}_{CC}(M)$ the horizontal gradient $\mathrm{grad}^H f$ exists almost everywhere by Proposition 3.3.3, (3.32) implies

$$\operatorname*{ess\,sup}_{x\in M} \left\| \mathrm{grad}^H f(x) \right\| = \left\| \left[D^H, f \right] \right\| \quad \forall f \in \mathrm{Lip}_{CC}(M),$$

and therefore the theorem is proved. □

In Theorem 3.3.6 we have seen that any horizontal Dirac operator detects the Carnot-Carathéodory metric via Connes metric formula, where the supremum is taken over the Carnot-Carathéodory-Lipschitz functions on M. We will see now that it suffices to take the supremum over all C^∞-functions, since each $f \in \mathrm{Lip}_{CC}(M)$ can be approximated by functions $f_\varepsilon \in C^\infty(M)$ with smaller Lip_{CC}-norm by a standard approximation argument.

Corollary 3.3.7
Let M be a closed Carnot manifold and D^H the horizontal Dirac operator acting on a horizontal Clifford bundle S^HM over M. Then the Carnot-Carathéodory metric of M can be detected via the formula

$$d_{CC}(x,y) = \sup \left\{ |f(x) - f(y)| : f \in C^\infty(M),\ \left\| [D^H, f] \right\| \le 1 \right\}. \tag{3.33}$$

Proof: Let $\mathbb{G}$ be the tangent Carnot group $\mathbb{G}$ of the Carnot manifold M, where like before R is the nilpotency step of $\mathbb{G}$ and for $1 \le S \le R$ the number d_S denotes the dimension of the vector space V_S belonging to the grading $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$ of $\mathbb{G}$. We show that each function $f \in \mathrm{Lip}_{CC}(\mathbb{G})$ can be approximated uniformly by a sequence of C^∞-functions f_ε such that $\mathrm{Lip}_{CC}(f_\varepsilon) \le \mathrm{Lip}_{CC}(f)$ for all ε. Then the statement for Lip_{CC}-functions on the manifold M follows, since M is compact, by considering local charts, and hence the corollary follows immediately from Theorem 3.3.6.
Let $f \in \mathrm{Lip}_{CC}(\mathbb{G})$. We use the Koranyi gauge

$$\|x\|_{\mathbb{G}} = \left(\sum_{S=1}^{R} \sum_{j=1}^{d_S} |x_{S,j}|^{\frac{2R!}{S}} \right)^{\frac{1}{2R!}},$$

see Definition 2.2.8, and consider the unit ball

$$B_{\mathbb{G}}(0,1) = \{x \in \mathbb{G} : \|x\|_{\mathbb{G}} \le 1\}$$

with respect to this semi-norm. Then we can consider the smooth function

$$u:\mathbb{R}\to\mathbb{R},\quad u(t)=\begin{cases}e^{-\frac{1}{t}}, & t>0\\ 0 & \text{otherwise}\end{cases}.$$

and set

$$\varphi(x):=c\cdot u\left(1-\|x\|_{\mathbb{G}}^{2R!}\right),$$

where we choose the constant $c>0$ such that $\int_{\mathbb{G}}\varphi(x)dx=\int_{B_{\mathbb{G}}(0,1)}\varphi(x)dx=1$. But this means that we have for all $\varepsilon>0$

$$\int_{\mathbb{G}}\varphi\left(\delta_{\varepsilon^{-1}}(x)\right)dx=\left(\varepsilon^{\dim_H(\mathbb{G})}\right)^{-1}, \tag{3.34}$$

where $\delta_{\varepsilon^{-1}}$ denotes the weighted dilation on $\mathbb{G}$ by ε^{-1} (see Definition 2.2.5), and

$$\dim_H(\mathbb{G})=\sum_{S=1}^{R}S\cdot\dim V_S$$

is the Hausdorff dimension of $\mathbb{G}$ (see Theorem 2.1.6). Note that because of the smoothness of u the function φ is a smooth function on $\mathbb{G}$, which is compactly supported in the unit ball with respect to the Koranyi gauge on $\mathbb{G}$.

We finally define

$$\varphi_\varepsilon(x):=\varepsilon^{\dim_H(\mathbb{G})}\varphi\left(\delta_{\varepsilon^{-1}}(x)\right), \tag{3.35}$$

which provides us a family of functions $\varphi_\varepsilon\in C_c^\infty(\mathbb{G})$ with $\varphi_\varepsilon\geq 0$ such that $\int_{\mathbb{G}}\varphi_\varepsilon(x)dx=1$ for all $\varepsilon>0$ (because of (3.34)) and $\lim_{\varepsilon\to 0}\varphi_\varepsilon=\delta_0$ in the sense of distributions. Because of the equivalence of the Koranyi gauge and the Carnot-Carathéodory metric (see Proposition 2.2.9) there is a $C>0$ such that each of these functions φ_ε is supported in $C\cdot B_{CC}(0,\varepsilon)$, where

$$B_{CC}(0,\varepsilon):=\{x\in\mathbb{G}:d_{CC}(0,x)<\varepsilon\}$$

denotes the Carnot-Carathéodory ball with radius ε.

Using the compactly supported smooth functions φ_ε from 3.35, we consider

$$f_\varepsilon(x):=f*_{\mathbb{G}}\varphi_\varepsilon(x):=\int_{\mathbb{G}}f(y^{-1}.x)\varphi_\varepsilon(y)dy,$$

where . denotes the composition on $\mathbb{G}$. It is clear from the rules of convolution that we have $f_\varepsilon\in C^\infty(\mathbb{G})$. Further, since f in continuous (with respect to the Carnot-Carathéodory metric on $\mathbb{G}$), we know that for any compact subset $K\subset\mathbb{G}$ and any $\delta>0$ there is an $\varepsilon'>0$ such that $|f(x)-f(z)|<\delta$ for all $x\in K$ and for all $z\in C\cdot B_{CC}(x,\varepsilon')$, where

$$B_{CC}(x,\varepsilon'):=\{y\in\mathbb{G}:d_{CC}(x,y)<\varepsilon'\}$$

and the constant $C > 0$ is chosen such that $\varphi_{\varepsilon'}$ is supported in $C \cdot B_{CC}(0, \varepsilon')$. But this shows that for all $\varepsilon < \varepsilon'$ we have because of the translation invariance of the Carnot-Carathéodory metric with respect to the composition on $\mathbb{G}$ (see [CDPT07]) and the properties of φ_ε

$$\begin{aligned} |f(x) - f_\varepsilon(x)| &= \left| C \cdot \int_{B_{CC}(0,\varepsilon)} \big(f(x) - f(y^{-1}.x)\big) \varphi_\varepsilon(y) dy \right| \\ &\leq \int_{C \cdot B_{CC}(0,\varepsilon)} \big|f(x) - f(y^{-1}.x)\big| \varphi_\varepsilon(y) dy \\ &< \delta \int_{C \cdot B_{CC}(0,\varepsilon)} \varphi_\varepsilon(y) dy \\ &= \delta, \end{aligned}$$

for all $x \in K$. Since M is a compact manifold (hence we can restrict ourselves to compact subsets in any chart), this shows that any Lip_{CC}-function f on M can be approximated by these C^∞-functions f_ε in the supremum norm.

Finally, we show that for any $\varepsilon > 0$ we have $\mathrm{Lip}_{CC}(f_\varepsilon) \leq \mathrm{Lip}_{CC}(f)$. This follows once again because the Carnot-Carathéodory metric is translation invariant with respect to the composition on the Carnot group $\mathbb{G}$ (see e.g. [CDPT07]). From the definition of the Carnot-Carathéodory-Lipschitz constant this invariance in connection with the properties of φ_ε leads to the estimate

$$\begin{aligned} \mathrm{Lip}_{CC}(f_\varepsilon) &= \sup_{x_1 \neq x_2} \left| \frac{f_\varepsilon(x_1) - f_\varepsilon(x_2)}{d_{CC}(x_1, x_2)} \right| \\ &= \sup_{x_1 \neq x_2} \left| \int_{\mathbb{G}} \frac{f(y^{-1}.x_1)\varphi_\varepsilon(y) - f(y^{-1}.x_2)\varphi_\varepsilon(y)}{d_{CC}(x_1, x_2)} dy \right| \\ &\leq \sup_{x_1 \neq x_2} \int_{\mathbb{G}} \left| \frac{f(y^{-1}.x_1) - f(x_2.y^{-1})}{d_{CC}(y^{-1}.x_1, y^{-1}.x_2)} \right| \varphi_\varepsilon(y) dy \\ &\leq \sup_{x_1 \neq x_2} \int_{\mathbb{G}} \mathrm{Lip}_{CC}(f) \varphi_\varepsilon(y) dy \\ &= \mathrm{Lip}_{CC}(f). \end{aligned}$$

Altogether we have proved that any function $f \in \mathrm{Lip}_{CC}(\mathbb{R}^n)$ can be approximated uniformly by a sequence of C^∞-functions f_ε such that $\mathrm{Lip}_{CC}(f_\varepsilon) \leq \mathrm{Lip}_{CC}(f)$ for all ε, and the statement of the corollary follows. □

Since $\mathrm{Lip}_{CC}(M)$ (or $C^\infty(M)$) is a dense sub-algebra of $C(M)$, Proposition 3.3.1 shows that $[D^H, f]$ is bounded for a dense sub-algebra of $C(M)$ and therefore D^H fulfills condition (i) for a spectral triple (see Definition 1.1.1. But as we have already mentioned (and will see in the next chapter for a concrete class of examples and in Chapter 6 in general), D^H fails to have a compact resolvent, and therefore $(C(M), L^2(S^H M), D^H)$ is not a spectral triple. But on the other hand, Theorem 3.3.6 and the preceding lemmas suggest that

D^H seems to be the logical candidate for a first order differential operator to detect the Carnot-Carathéodory metric.

We will now show that at least $(C(M), L^2(\Sigma M), D^H)$ gives rise to a compact quantum metric space in the sense of Mark Rieffel (see Definition 1.2.2) if we consider the real-valued functions of $C(M)$ as an order-unit space.

Corollary 3.3.8
The pair (A, L), where

$$A := \{f \in C(M) : f^* = f\} \quad \text{and} \quad L(f) := \left\|[D^H, f]\right\|,$$

is a compact quantum metric space which detects the Carnot-Carathéodory metric on a closed Carnot manifold M.

Proof: It is obvious that A is an order-unit space with norm

$$\|f\| := \sup_{x \in M} |f(x)|.$$

Now for calculating the Connes metric from L it suffices to consider only the self-adjoint elements of $C(M)$ (see Proposition 1.1.4, which is exactly the space A. We have to show that L is a Lip-norm on A, i.e.

(i) For every $f \in A$ we have $L(f) = 0 \Leftrightarrow f \in \mathbb{R} \cdot 1$.

(ii) The topology on the state space $\mathcal{S}(A)$ of A from the Connes metric defined by L is the w^*-topology.

The non-trivial part of condition (i) follows from Proposition 3.3.1: Since

$$[D^H, f] = c^H(\left(\text{grad}^H f\right),$$

$[D^H, f] = 0$ implies $\text{grad}^H f = 0$ almost everywhere, which implies $X_j(f) = 0$ for all $j \in \{1, \ldots, d\}$ almost everywhere. But this also means $X_k(f) = 0$ almost everywhere for every $k \in \{d+1, \ldots, n\}$ because every vector field of the X_k's can be written as a commutator consisting of the X_j's. Since $\{X_1, \ldots, X_n\}$ spans the tangent space of M and f is continuous by assumption, this implies that f must have been a constant. Therefore (i) is proved.

For condition (ii), note that $\mathcal{S}(A) = \mathcal{S}(C(M)) \cong M$ by Gelfand-Naimark theory, where the w^*-topology on M is exactly the usual manifold topology. Now, by the sub-Riemannian theorem on topologies (see Theorem 2.1.4), this topology coincides with the topology induced by the Carnot-Carathéodory metric d_{CC}. But by Theorem 3.3.6,

$$d_{CC}(x, y) = \rho_L(x, y) := \sup\left\{|f(x) - f(y)| : L(f) \leq 1\right\},$$

which shows condition (ii).

Hence (A, L) is a compact quantum metric space, and the statement about the metric is just the statement of Theorem 3.3.6. □

Remark: Note that the Carnot-Carathéodory-Lipschitz constant L also provides a Lip-norm on A. As we have seen in Lemma 3.3.5, this is exactly the Lip-norm which belongs to the Carnot-Carathéodory metric d_{CC}. In this context, Corollary 3.3.8 shows that the compact quantum metric spaces $(A, \mathrm{Lip}_{CC}(\cdot))$ and $\big(A, \big\|[D^H, \cdot]\big\|\big)$ are identical. ◁

Chapter 4

Degenerate Spectral Triples on Nilmanifolds

In the previous chapter we constructed horizontal Dirac operators more or less analogous to classical Dirac operators and we saw that they detect the Carnot-Carathéodory metric. Therefore they seem to be a natural candidate to construct a spectral triple which covers the horizontal geometry of a Carnot manifold.

In this chapter we will do a concrete construction of horizontal Dirac operators D^H on local homogeneous spaces of Carnot groups which arise from the action of a lattice subgroup, namely compact Carnot nilmanifolds $M = \Gamma\backslash\mathbb{G}$. This can be seen as a generalization of the torus in the non-abelian case. We will make use of the spin structures of the horizontal torus, arising as the image of a submersion from M, and observe that we obtain a horizontal Clifford structure via pullback where the representation of the horizontal Clifford algebra is irreducible. Afterwards we will use an approach developed by Christian Bär and Bernd Ammann (see [Bae91] and [AB98]) to get information about the spectrum of D^H. It will follow that our horizontal Dirac operator has an infinite dimensional eigenspace. In particular its resolvent in not compact and thus it does not furnish a spectral triple. The strategy is to decompose the L^2-space of horizontal Clifford sections for the case where the horizontal distribution has co-dimension 1: In this case we can use well-known results from the representation theory of Heisenberg groups to calculate the eigenvalues of D^H. Using this, we will be able to deduce the statement for the general case. These calculations on Carnot nilmanifolds can be seen as an example of a more general statement: In later chapters, we will use this idea of transferring the problem to the co-dimension 1 case to show that on any Carnot manifold a horizontal Dirac operator is not hypoelliptic.

Despite this lack, as an additional result we are still able to extract the Hausdorff dimension of (M, d_{CC}) from the asymptotic behavior of the non-degenerate eigenvalues of D^H in the case where $\mathbb{G} = \mathbb{H}^{2m+1}$ is a Heisenberg group.

We will use the notion of a compact Carnot nilmanifold introduced in Example 2.2.10. Throughout this chapter, we assume that Γ is the standard lattice of a Carnot group $\mathbb{G}$. We denote the resulting local homogeneous space by $M = \Gamma\backslash\mathbb{G}$.

4.1 The Pullback Construction

Let $\mathbb{G}$ be a Carnot group with horizontal distribution of rank d and nilpotency step R, and let $\Gamma \lhd \mathbb{G}$ be its standard lattice given by

$$\Gamma := \langle\ \{\gamma_j = \exp(X_j) : 1 \le j \le d\}\ \rangle_{\mathbb{G}}\,,$$

see Example 2.2.10. We consider the local homogeneous space $M = \Gamma\backslash\mathbb{G}$, where the action of Γ on $\mathbb{G}$ is given by the group composition from the left. We equip M with a Riemannian metric g such that the vector bundles $V_1M \oplus \ldots \oplus V_RM$, forming the grading of TM, are pairwise orthogonal at each point.

To construct an example for a horizontal Clifford connection on M, we start by calculating the Christoffel symbols for the horizontal connection on M arising from the Levi-Civita connection (see Section 3.1).

Proposition 4.1.1
Let ∇^H be the horizontal connection arising from the Levi-Civita connection ∇ on the compact Carnot nilmanifold $M = \Gamma\backslash\mathbb{G}$. If $\{X_1, \ldots, X_d\}$ is an orthonormal frame for the (bracket generating) horizontal distribution $HM = V_1M$, then we have

$$\Gamma^l_{jk} = 0$$

for all Christoffel symbols Γ^l_{jk} of ∇^H, $j,k,l \in \{1,\ldots,d\}$, corresponding to this basis.
In addition, if we extend the horizontal frame $\{X_1,\ldots,X_d\}$ to an orthonormal tangent frame $\{X_1,\ldots,X_n\}$ of M which respects the grading of TM, then all Christoffel symbols of ∇ belonging to this frame satisfy

$$\Gamma^l_{jj} = 0$$

for $j = d+1,\ldots,n$ and $l = 1,\ldots,d$.

Proof: By the construction of the horizontal connection the horizontal Christoffel symbols are exactly the horizontal Christoffel symbols of the Levi-Civita connection on M, see Proposition 3.1.6. If g is a Riemannian metric on M such that $\{X_1,\ldots,X_d\}$ forms an orthonormal frame at every point, they can be calculated using the properties of the Levi-Civita connection. We have locally $\nabla_{X_j}X_K = \sum_{l=1}^n \Gamma^l_{jk}X_l$ (with $n = \dim M$), and hence we get by the Koszul formula for the Levi-Civita connection because of the orthonormality of the frame $\{X_1,\ldots,X_d\}$

$$\begin{aligned}\Gamma^l_{jk} =& g(\nabla_{X_j}X_k, X_l)\\ =& \frac{1}{2}\big(\partial_{X_j}g(X_k,X_l) + \partial_{X_k}g(X_l,X_j) - \partial_{X_l}g(X_j,X_k)\big)\\ &+ \frac{1}{2}\left(-g(X_k,[X_j,X_l]) - g(X_l,[X_k,X_j]) + g(X_j,[X_k,X_l])\right)\\ =& \frac{1}{2}\left(-g(X_k,[X_j,X_l]) - g(X_l,[X_k,X_j]) + g(X_j,[X_k,X_l])\right)\end{aligned} \tag{4.1}$$

for all $1 \leq j,k,l \leq d$. Since for any choice of vector fields $X, Y, Z \in V_1M$ the vector fields $X \in V_1M$ and $[Y,Z] \in V_2M$ are orthogonal with respect to g, the right hand side of (4.1) vanishes. Therefore all the horizontal Christoffel symbols are 0.

The fact that $\Gamma^l_{jj} = 0$ for all $d+1 \leq j \leq n$ and for all $1 \leq l \leq d$ can also be seen immediately from (4.1): If $X_j \in V_SM$ for $S \geq 2$, we have $[X_j, X_j] = 0$ and $[X_j, X_l] \in V_{S+1}M$ for $l = 1, \ldots, d$ (which means $X_l \in V_1M$) by the grading structure of TM. But this means that we also have $g(X_j, [X_j, X_l]) = 0$ since $X_j \perp V_{S+1}$. Hence every term on the right hand side of (4.1) is 0, and the additional statement of the proposition follows. □

Remark: Using formula (4.1), we can also calculate all the other Christoffel symbols belonging to the Levi-Civita connection of M as soon as we know the commutator relations of the Lie algebra generated by the vector fields $\{X_1, \ldots, X_d\}$. ◁

We now construct an irreducible horizontal Clifford bundle over M only involving the horizontal distribution of M. The idea is to consider the submersion $\psi : \mathbb{G} \to \mathbb{R}^d$ from Section 2.4, which is given in exponential coordinates via

$$\psi : \mathbb{G} \to \mathbb{R}^d, \quad \psi\left(x^{(1)}, \ldots, x^{(R)}\right) = x^{(1)}.$$

We have seen in Corollary 2.4.2 that ψ can be lifted to the nilmanifold given by the action of Γ on $\mathbb{G}$. This means we have a submersion

$$\pi : M \to \mathbb{T}^d \cong \mathbb{Z}^d \backslash \mathbb{R}^d \tag{4.2}$$

of M onto the d-dimensional torus, which (by Corollary 2.4.2) coincides locally with the submersion ψ of the Carnot groups. If $\{\tilde{X}_1, \ldots, \tilde{X}_d\}$ is a local frame for $T\mathbb{T}^d$ such that we have $\psi(\exp_{\mathbb{G}}(X_j)) = \exp_{\tilde{\mathbb{R}}^d}(\tilde{X}_j)$ locally on the Carnot groups, then the differential of the submersion π applied to our orthonormal frame $\{X_1, \ldots, X_d, \ldots, X_n\}$ is given by

$$D\pi : TM \to T\mathbb{T}^d, \quad D\pi(X_j) = \begin{cases} \tilde{X}_j, & 1 \leq j \leq d \\ 0 & \text{otherwise} \end{cases}. \tag{4.3}$$

We choose the Riemannian metric $g^{\mathbb{T}^d}$ on $\mathbb{T}^d$ such that $\{\tilde{X}_1, \ldots, \tilde{X}_d\}$ forms an orthonormal frame.

The idea for our construction of a horizontal Dirac operator is now to exploit the fact that $\mathbb{T}^d$ is a spin manifold. The starting point for our constructions is the following theorem, which summarizes the well known facts about the spin structures on $\mathbb{T}^d$ and the realization of their corresponding spinor bundles.

Theorem 4.1.2
There are 2^d different spin structures $\Sigma^{\mathbb{T}^d}_{\delta}$ on $\mathbb{T}^d$, which are in one-to-one correspondence to the group homomorphisms

$$\varepsilon : \mathbb{Z}^d \to \mathbb{Z}/2\mathbb{Z}. \tag{4.4}$$

They are indexed by

$$\delta = (\delta_1, \ldots, \delta_d) \in (\mathbb{Z}/2\mathbb{Z})^d ,$$

where δ_j *is the image of the generator* e_j *of* $\mathbb{Z}^d$ *under* ε.
Furthermore, for each spin structure the spinor bundle $\Sigma_\delta \mathbb{T}^d$ *is a (complex) vector bundle of rank* $2^{[d/2]}$, *where* $[\cdot]$ *denotes the Gaussian bracket, and the space of sections of* $\Gamma^\infty(\Sigma_\delta \mathbb{T}^d)$ *can be identified with functions* $\tilde{\sigma} \in C^\infty(\mathbb{R}^d, \mathbb{C}^{2^{[d/2]}})$ *such that*

$$\tilde{\sigma}(a + x) = \varepsilon(a)\tilde{\sigma}(x) \quad \text{for all } x \in \mathbb{R}^d, a \in \mathbb{Z}^d. \tag{4.5}$$

Proof: The spin structures of a connected Riemannian manifold M (if they exist) are characterized by group homomorphisms from the fundamental group of M to $\mathbb{Z}/2\mathbb{Z}$ (see e.g. [LM89], Theorem II.2.1). In the case of the d-dimensional torus $\mathbb{T}^d$, the fundamental group is $\mathbb{Z}^d$, and since a group homomorphism $\varepsilon : \mathbb{Z}^d \to \mathbb{Z}/2\mathbb{Z}$ is uniquely determined by the images $\delta_j = \varepsilon(e_j)$ of the d generators e_j of $\mathbb{Z}^d$, there are 2^d possibilities for such a homomorphism.
The second statement of the theorem follows from the construction of the spinor bundle corresponding to a spin structure (see e.g. [LM89] or [Roe98]). □
Remark: Note that (4.5) is also true for $\sigma \in L^2(\Sigma_\delta \mathbb{T}^d$ to be an L^2-spinor, since $\Gamma^\infty(\Sigma_\delta \mathbb{T}^d)$ is dense in $L^2(\Sigma_\delta \mathbb{T}^d)$.

For a given spin structure $\Sigma_\delta^{\mathbb{T}^d}$ on the torus $\mathbb{T}^d$, we have a corresponding spinor bundle $\Sigma_\delta \mathbb{T}^d$, equipped with a spinor connection $\nabla^{\Sigma_\delta \mathbb{T}^d}$. Since π is a submersion, there exist unique pullbacks of these objects on M (see e.g. [AH11]). In detail, we have the following:

- The sections of the pullback $\pi^* \Sigma_\delta \mathbb{T}^d$ of the spinor bundle have the form

$$\pi^* \Sigma_\delta \mathbb{T}^d = \left\{ \sum_{j=1}^{2^{[d/2]}} f_j \pi^* \varphi_j : \; f_j \in C^\infty(M), \; \varphi_1, \ldots, \varphi_{2^{[d/2]}} \text{ basis sections of } \Sigma_\delta \mathbb{T}^d \right\}. \tag{4.6}$$

- If $\langle \cdot, \cdot \rangle_{\Sigma_\delta \mathbb{T}^d}$ is the bundle metric on $\Sigma_\delta \mathbb{T}^d$, a metric on $\pi^* \Sigma_\delta \mathbb{T}^d$ is given via the pullback

$$\langle \pi^* \varphi_1, \pi^* \varphi_2 \rangle_{\pi^* \Sigma_\delta \mathbb{T}^d} := \langle \varphi_1, \varphi_2 \rangle_{\Sigma_\delta \mathbb{T}^d} . \tag{4.7}$$

- The pullback $\pi^* \nabla^{\Sigma_\delta \mathbb{T}^d}$ of the spinor connection to $\pi^* \Sigma_\delta \mathbb{T}^d$ has the form

$$\pi^* \nabla^{\Sigma_\delta \mathbb{T}^d}_{X_j} (\pi^* \varphi) = \pi^* \left(\nabla^{\Sigma_\delta \mathbb{T}^d}_{D X_j} \varphi \right) = \begin{cases} \pi^* \left(\nabla^{\Sigma_\delta \mathbb{T}^d}_{\tilde{X}_j} \varphi \right) & \text{for } 1 \leq j \leq d \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

  on the pull-backs of sections of $\Sigma_\delta \mathbb{T}^d$. For an arbitrary element of $\Gamma^\infty(\pi^* \Sigma_\delta \mathbb{T}^d)$ described by (4.6) $\pi^* \nabla^{\Sigma_\delta \mathbb{T}^d}$ is defined by using the linear and tensorial behavior of a connection.

Considering the Clifford action $c^{\mathbb{T}} : T\mathbb{T}^d \to \mathrm{End}_{\mathbb{C}}(\Sigma_\delta \mathbb{T}^d)$ on $\Sigma_\delta \mathbb{T}^d$, we can use the identification $HM \cong \pi^* T\mathbb{T}^d$ given by the differential $D\pi$ from (4.3) to define a horizontal Clifford action on $\pi^*\Sigma_\delta \mathbb{T}^d$. This is done by the pull-back of the endomorphism bundle: For $X \in HM$ we define

$$c^H(X) := \pi^* \left(c^{\mathbb{T}^d}(D\pi(X)) \right),$$

which is an endomorphism on $\pi^*\Sigma_\delta \mathbb{T}^d$ since $c^{\mathbb{T}^d}$ is an endomorphism on $\Sigma_\delta \mathbb{T}^d$. In detail, point-wise we have by the definition of the pull-back of an endomorphism

$$c^H(X)(\pi^*\varphi) = \pi^* \left(c^{\mathbb{T}^d}(D\pi(X)) \right) (\pi^*\varphi) = \pi^* \left(c^{\mathbb{T}^d}(D\pi(X))\varphi \right) \tag{4.9}$$

for any basis section φ of $\Sigma_\delta \mathbb{T}^d$, which extends to the whole bundle via linearity. From the identification $HM \cong \pi^* T\mathbb{T}^d$ via $D\pi$ we can conclude that the restriction of the Riemannian metric g on M to HM is exactly the pull-back of the Riemannian metric $g^{\mathbb{T}^d}$ on $\mathbb{T}^d$, which was chosen such that $\{D\pi(X_1), \ldots, D\pi(X_d)\}$ forms an orthonormal frame for $T\mathbb{T}^d$.

By writing

$$\Sigma_\delta^H M := \pi^*\Sigma_\delta \mathbb{T}^d \quad \text{and} \quad \nabla^{\Sigma_\delta^H} := \pi^* \nabla^{\Sigma_\delta \mathbb{T}^d},$$

we will now show that this structure indeed gives a horizontal Clifford bundle. Furthermore, we can write down the horizontal Dirac operator for this bundle.

Theorem 4.1.3

Let $M = \Gamma \backslash \mathbb{G}$ be the nilmanifold of a Carnot group $\mathbb{G}$, and let $\Sigma_\delta^H M$, $\nabla^{\Sigma_\delta^H}$ and c^H be as above. Then $\Sigma_\delta^H M$ equipped with the connection $\nabla^{\Sigma_\delta^H}$ and the horizontal Clifford multiplication c^H is a horizontal Clifford bundle over M.

L^2-sections $\sigma \in L^2(\Sigma_\delta^H M)$ can be identified with $\mathbb{C}^{2^{[d/2]}}$-valued functions σ on $\mathbb{G}$ such that

$$\sigma\left(\left(a^{(1)}, \ldots, a^{(R)}\right) .x \right) = \varepsilon\left(a^{(1)}\right) \sigma(x) \tag{4.10}$$

for all $a \in \Gamma$ and $x \in \mathbb{G}$, where ε is the group homomorphism (4.4) describing the spin structure of $\mathbb{T}^d$ and $a.x$ denotes the group operation on $\mathbb{G}$.

The horizontal Dirac operator D^H acting on $\Gamma^\infty(\Sigma_\delta^H M)$ is given in local coordinates by

$$D^H \sigma = \sum_{j=1}^{d} c^H(X_j) \partial_{X_j}, \tag{4.11}$$

where ∂_{X_j} denotes the partial derivative belonging to a local coordinate chart of M.

Proof: First of all, it is clear that the action of HM on $\Sigma_\delta^H M$ via c^H furnishes a Clifford module over M: Since $c^{\mathbb{T}^d}$ furnishes a Clifford module structure on $\Sigma_\delta^{\mathbb{T}^d} M$, we have for any $X \in HM$

$$\begin{aligned} \left(c^H(X)\right)^2 (\pi^*\varphi) &= \pi^* \left(c^{\mathbb{T}^d}(D\pi(X)) \right)^2 (\varphi) \\ &= -g^{\mathbb{T}^d}(D\pi(X), D\pi(X)) \pi^*\varphi \\ &= -g(X, X)\pi^*\varphi \end{aligned}$$

for the basis sections $\pi^*\varphi$ of Σ_δ^H by (4.9). Note thereby that the restriction of the Riemannian metric g on M to the horizontal distribution HM is exactly the pull-back of the Riemannian metric $g^{\mathbb{T}^d}$ we chose above.

Analogously, the condition (i) and (ii) of Definition 3.2.2 follow because the (classical) Clifford bundle $\Sigma_\delta\mathbb{T}^d$ on the torus fulfills these conditions: For the metric (4.7) on Σ_δ^H we have point-wise for any basis sections $\pi^*\varphi_1$, $\pi^*\varphi_2$ and for any horizontal vector field $X \in HM$

$$\begin{aligned}\left\langle c^H(X)\pi^*\varphi_1, \pi^*\varphi_2\right\rangle_{\Sigma_\delta^H M} &= \left\langle c^{\mathbb{T}^d}(D\pi(X))\varphi_1, \varphi_2\right\rangle_{\Sigma_\delta\mathbb{T}^d} \\ &= \left\langle \varphi_1, c^{\mathbb{T}^d}(D\pi(X))\varphi_2\right\rangle_{\Sigma_\delta\mathbb{T}^d} \\ &= \left\langle \pi^*\varphi_1, c^H(X)\pi^*\varphi_2\right\rangle_{\Sigma_\delta^H M}\end{aligned}$$

by (4.7) and (4.9), and hence the metric compatibility (i) follows.

To show the compatibility of $\nabla^{\Sigma_\delta^H}$ with the horizontal connection ∇^H, we first calculate this for any basis section $\pi^*\varphi$ of $\Sigma_\delta^H M$. For any $X, Y \in HM$ we have

$$\begin{aligned}\nabla_X^{\Sigma_\delta^H}(c^H(Y)\pi^*\varphi) &= \nabla_X^{\Sigma_\delta^H}(\pi^* c^{\mathbb{T}^d}(D\pi(Y))\varphi) \\ &= \pi^*\nabla_{D\pi(X)}^{\Sigma_\delta\mathbb{T}^d}(c^{\mathbb{T}^d}(D\pi(Y))\varphi) \\ &= \pi^*\left(c^{\mathbb{T}^d}\left(\nabla_{D\pi(X)}^{\mathbb{T}^d} D\pi(Y)\right)\varphi + c^{\mathbb{T}^d}(D\pi(Y))\nabla_{D\pi(X)}^{\Sigma_\delta\mathbb{T}^d}\varphi\right) \\ &= c^H\left(\nabla_X^H Y\right)\pi^*\varphi + \pi^*\left(c^{\mathbb{T}^d}(D\pi(Y))\nabla_{D\pi(X)}^{\Sigma_\delta\mathbb{T}^d}\varphi\right) \\ &= c^H\left(\nabla_X^H Y\right)\pi^*\varphi + c^H(Y)\pi^*\left(\nabla_{D\pi(X)}^{\Sigma_\delta\mathbb{T}^d}\varphi\right) \\ &= c^H\left(\nabla_X^H Y\right)\pi^*\varphi + c^H(Y)\nabla_X^{\Sigma_\delta^H}\pi^*\varphi\end{aligned}$$

by (4.8), (4.9) and since the compatibility condition is true on the spinor bundle $\Sigma_\delta\mathbb{T}^d$, using the Levi-Civita connection $\nabla^{\mathbb{T}^d}$ over $\mathbb{T}^d$. Thereby the fourth equation needs a bit more explanation. The crucial point is that for all $X, Y \in HM$ we have

$$\nabla_X^H Y = \pi^*\nabla_{D\pi(X)} D\pi(Y) \tag{4.12}$$

under the identification $HM \cong \pi^* T\mathbb{T}^d$ via $D\pi$ from (4.3): If we write both sides of (4.12) in local coordinates, we see that they coincide since the Christoffel symbols of $\mathbb{T}^d$ with respect to frame $\{D\pi(X_1), \ldots, D\pi(X_d)\}$ vanish as well as the Christoffel symbols of the horizontal Levi-Civita connection on M with respect to the frame $\{X_1, \ldots, X_d\}$ do. But this means by (4.9)

$$\pi^*\left(c^{\mathbb{T}^d}\left(\nabla_{D\pi(X)}^{\mathbb{T}^d} D\pi(Y)\right)\varphi\right) = c^H\left(\nabla_X^H Y\right)\pi^*\varphi$$

for all $\varphi \in \Gamma^\infty(\Sigma_\delta\mathbb{T}^d)$ by linearity of $c^{\mathbb{T}^d}$ and c^H and by (4.9).

We still need to show the compatibility with the connection for an arbitrary element $\sigma \in \Gamma^\infty(\Sigma_\delta^H M)$. By (4.6), such a σ can be written in the form

$$\sigma(x) = \sum_{j=1}^{d} f_j(x)\pi^*\varphi_j(x), \tag{4.13}$$

where $\varphi_1, \ldots, \varphi_d$ is a local spinor basis for $\Sigma_\delta \mathbb{T}^d$ and $f_j \in C^\infty(M)$ for $1 \le j \le d$. But then the general compatibility follows from the above calculation and by the linearity and the tensorial behavior of the connection ∇^{Σ^H}, since for each j we have

$$\begin{aligned}
\nabla_X^{\Sigma_\delta^H}(c^H(Y)f_j\pi^*\varphi_j) &= X(f_j)c^H(Y)\pi^*\varphi_j + f_j\nabla_X^{\Sigma_\delta^H}c^H(Y)\pi^*\varphi_j \\
&= X(f_j)c^H(Y)\pi^*\varphi_j + f_jc^H(\nabla_X^H Y)\pi^*\varphi_j + f_jc^H(Y)\nabla_X^{\Sigma_\delta^H}\pi^*\varphi_j \\
&= f_jc^H(\nabla_X^H Y)\pi^*\varphi_j + c^H(Y)\nabla_X^{\Sigma_\delta^H} f_j\pi^*\varphi_j
\end{aligned}$$

Altogether we have shown that $\Sigma_\delta^H M$ is a horizontal Clifford bundle over M.

Turning to the representation (4.10) of our horizontal spinors, we use (4.13) to describe a general section $\sigma \in \Gamma^\infty(\Sigma_\delta^H M)$. Since $M = \Gamma\backslash\mathbb{G}$ is a local homogeneous space, each $f_j \in C^\infty(M)$ can be considered as a function $f \in C^\infty(\mathbb{G})$ such that $f(a.x) = f(x)$ for all $a \in \Gamma$ and $x \in \mathbb{G}$. And since the submersion π is a Lie group homomorphism according to the group operations on $\mathbb{G}$ and $\mathbb{R}^d$, we have for each $\pi^*\varphi_j$

$$\begin{aligned}
\pi^*\varphi_j(a.x) &= \varphi_j(\pi(a.x)) = \varphi_j(a^{(1)} + x^{(1)}) = \varepsilon\left(a^{(1)}\right)\varphi_j(x^{(1)}) \\
&= \varepsilon\left(a^{(1)}\right)\pi^*\varphi_j(x)
\end{aligned}$$

by Equation (4.5) from Theorem 4.1.2. From this, (4.10) follows for sections $\sigma \in \Gamma^\infty(\Sigma_\delta^H M)$, and since $\Gamma^\infty(\Sigma_\delta^H M)$ is dense in the Hilbert space $L^2(\Sigma_\delta^H M)$ of L^2-sections of this vector bundle, the second statement of the theorem is proved.

Finally we prove the local description (4.11) of the horizontal Dirac operator from this Clifford module. Let $\tilde{\Gamma}_{jk}^l$ denote the Christoffel symbols of $T\mathbb{T}^d$ according to the frame $\{\tilde{X}_1, \ldots, \tilde{X}_d\} = \{D\pi(X_1), \ldots, D\pi(X_d)\}$, which are all zero. Then the formula (4.11) follows, because it is known that we have locally

$$\nabla_{\tilde{X}_j}^{\Sigma_\delta\mathbb{T}^d}\sigma = \partial_{\tilde{X}_j}\sigma - \frac{1}{4}\sum_{k,l=1}^{d}\tilde{\Gamma}_{jk}^l c^{\mathbb{T}^d}(\tilde{X}_k)c^{\mathbb{T}^d}(\tilde{X}_l)\sigma = \partial_{\tilde{X}_j}\sigma \tag{4.14}$$

for the spinor connection in the torus (see e.g. [Bae91], Lemma 4.1): Using once again the representation (4.13) for elements of $\Gamma(\Sigma_\delta^H M)$, we get from the definition (4.8) of $\nabla^{\Sigma_\delta^H}$ for every $j \in \{1, \ldots, d\}$:

$$\begin{aligned}
\nabla^{\Sigma^H_\delta}_{X_j}\left(\sum_{j=1}^{d} f_j \pi^* \varphi\right) &= \sum_{j=1}^{d}\left(X_j(f)\cdot \pi^*\varphi + f_j \cdot \pi^*\left(\nabla^{\Sigma_\delta \mathbb{T}^d}_{\tilde{X}_j}\varphi_j\right)\right)\\
&= \sum_{j=1}^{d}\left(X_j(f)\cdot \pi^*\varphi + f_j \cdot \pi^*\left(\partial_{\tilde{X}_j}\varphi_j\right)\right)\\
&= \sum_{j=1}^{d}\left(X_j(f)\cdot \pi^*\varphi + f_j \cdot \partial_{X_j}\pi^*\varphi_j\right)\\
&= \partial_{X_j}\left(\sum_{j=1}^{d} f_j \pi^* \varphi\right).
\end{aligned}$$

Now by Definition 3.2.6, the horizontal Dirac operator is given locally via

$$D^H = \sum_{j=1}^{d} c^H(X_j)\nabla^{\Sigma^H_\delta}_{X_j} - \frac{1}{2}c^H\left(\sum_{j=d+1}^{n}\pi^H\nabla_{X_j}X_j\right), \tag{4.15}$$

where ∇ is the Levi-Civita connection on TM and π^H denotes the orthonormal projection of TM onto the horizontal distribution HM. For the first term on the right hand side of (4.15) we can plug in the result from the above calculation, and for the second term we can use the additional statement of Proposition 4.1.1 which states that all the Christoffel symbols Γ^l_{jj} for $j = d+1, \ldots, n$ and $l = 1, \ldots, d$ are zero: By definition of the Levi-Civita connection this implies that we have locally

$$\pi^H\nabla_{X_j}X_j = \pi^H\left(\sum_{l=1}^{n}\Gamma^l_{jj}X_l\right) = \sum_{l=1}^{d}\Gamma^l_{jj}X_l = 0$$

for any $j \in \{d+1, \ldots, n\}$. Altogether, the local expression (4.11) for the horizontal Dirac operator we constructed follows. □

Definition 4.1.4
Let $M = \Gamma\backslash\mathbb{G}$ be a compact Carnot nilmanifold, equipped with the objects $\Sigma^H_\delta M$, $\nabla^{\Sigma^H_\delta}$, c^H and D^H from Theorem 4.1.3. Then we call

$$D^H : \Gamma^\infty(\Sigma^H_\delta M) \to \Gamma^\infty(\Sigma^H_\delta M)$$

the horizontal pull-back Dirac operator *on M. Further, we call $\Sigma^H_\delta M$ the* horizontal spinor bundle *and $\nabla^{\Sigma^H_\delta}$ the* horizontal spinor connection *on M.* ◁

Let us summarize what we have done in this section: We have constructed a horizontal Clifford bundle together with a horizontal Dirac operator on an arbitrary compact Carnot nilmanifold $M = \Gamma\backslash\mathbb{G}$ only depend on the horizontal distribution of this manifold. For dimensional reasons this representation of the (bundle of) Clifford algebras over HM is irreducible, since our Clifford bundle $\Sigma^H M$ has rank $2^{[d/2]}$ whenever d is the rank of HM. Therefore we can claim we have constructed a natural candidate for a horizontal Dirac operator on M.

We remark that one can also construct horizontal Dirac operators from already existing (classical) Clifford or spinor bundles of M. For this we take a look at Proposition 3.2.7 and note that the horizontal connection in this case depends on the Clifford action of the whole tangent bundle TM. But nonetheless, whenever we choose an orthonormal frame $\{X_1, \ldots, X_n\}$ of TM we will get a local expression similar to (4.11) for the resulting horizontal Dirac operator, which only differs by a matrix term arising from the Clifford action. Therefore, methods similar to the ones described in the following sections can also be applied to this situation, and we expect similar results.

4.2 Spectral Decomposition from the Center

Our aim is to show that the horizontal pull-back Dirac operator constructed in the preceding section, which detects the Carnot-Carathéodory metric by Section 3.3, does not have a compact resolvent. This will be the case if we find an eigenvalue of D^H which possesses an infinite dimensional eigenspace. Therefore we are interested in getting information about the spectrum of this operator.

We intend to use the local expression (4.11) of the horizontal pull-back operator D^H we constructed in the last section. From this expression, we will be able to use techniques involving the representation theory of its underlying Carnot group: The following proposition shows that D^H can be expressed using the right regular representation of $\mathbb{G}$ (see also [Bae91] for the case of the classical Dirac operator).

Proposition 4.2.1
Let $M = \Gamma\backslash\mathbb{G}$ be a compact Carnot nilmanifold, and let D^H be the horizontal pull-back Dirac operator defined on the horizontal spinor bundle $\Sigma^H_\delta M$, arising from a spin structure $\Sigma^{\mathbb{T}^d}_\delta$ on $\mathbb{T}^d$.

We denote by $R : \mathbb{G} \to L^2(\Sigma^H_\delta M)$ the right regular representation of the Carnot group $\mathbb{G}$ on the Hilbert space $L^2(\Sigma^H_\delta M)$, which is defined by

$$(R(x_0)\sigma)(x) := \sigma(x.x_0) \tag{4.16}$$

for all $x_0 \in \mathbb{G}$. Then D^H can be expressed locally using R via

$$D^H\sigma(x) = \sum_{j=1}^{d} c^H(X_j)\, \frac{d}{dt}\left(R(\exp tX_j)\sigma\right)(x)\Big|_{t=0} \tag{4.17}$$

for any $\sigma \in \Gamma^\infty(\Sigma^H_\delta M)$.

As a consequence, a closed subspace $\mathcal{H}' \subset L^2(\Sigma_\delta^H M)$ is invariant under D^H if it is invariant under R and under the Clifford action of HM.

Remark: The expression (4.16) is well-defined since elements of $L^2(\Sigma_\delta^H M)$ can be viewed as periodic functions on $\mathbb{G}$ by Theorem 4.1.3. ◁

Remark: For an element $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Carnot group $\mathbb{G}$, we can define

$$R_*(X)\sigma(x) := \frac{d}{dt}\left(R(\exp tX)\sigma\right)(x)\bigg|_{t=0}.$$

This is the so-called *right-regular representation* of the Lie algebra $\mathfrak{g}$ on $L^2(\Sigma_\delta^H M)$ adopted from R. Using this representation R_*, we can rewrite (4.17) in the form

$$D^H = \sum_{j=1}^{d} R_*(X_j) \otimes c^H(X_j).$$

◁

Proof: The local expression of the horizontal pull-back Dirac operator follows immediately from the expression of the directional differentiation along a vector field X on a Lie group, which is

$$\partial_X\sigma(x) = \frac{d}{dt}\sigma\left(x.\exp(tX)\right)\bigg|_{t=0}.$$

The statement about the invariance is straight forward. □

The idea, which has been used by Christian Bär and Bernd Amman for the classical Dirac operator on compact nilmanifolds of Heisenberg groups and which we will adopt to our situation, is now to find a direct sum decomposition of the horizontal Clifford bundle $L^2(\Sigma_\delta^H M)$, which is invariant under the horizontal Dirac operator. Thus the determination of the spectrum of D^H splits into parts which are easier to handle. Remember that by (4.10) from Theorem 4.1.3 we can consider elements of $L^2(\Sigma_\delta^H M)$ as functions on $\mathbb{G}$ which have certain periodicity properties. We intend to use these properties to decompose $L^2(\Sigma_\delta^H M)$. In the language of representation theory and in view of Proposition 4.2.1 this is just the decomposition of the unitary right-regular representation R, acting on the Hilbert space $L^2(\Sigma_\delta^H M)$, into its irreducible components.

Let $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$ be the grading of the Lie algebra belonging to $\mathbb{G}$. We start with the decomposition of $L^2(\Sigma_\delta^H M)$ using the periodicities arising from the subgroup $\exp_{\mathbb{G}}(V_R)$ of $\mathbb{G}$.

Definition 4.2.2
For the grading $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$ of the Carnot group $\mathbb{G} = \exp_{\mathbb{G}} \mathfrak{g}$, we call

$$Z(\mathbb{G}) := \exp_{\mathbb{G}}(V_R) \subset \mathbb{G}$$

the Carnot center *of* $\mathbb{G}$. ◁

Remark: Note that the center of a general Lie group G is the maximal (normal) subgroup of G commuting with every element of G. In the case $\mathbb{G}$ is a Carnot group, the center of $\mathbb{G}$ needs not to coincide with the Carnot center of $\mathbb{G}$ defined above. Consider for example the Carnot group $\mathbb{G} = \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ for $d > 2m$, where the components belonging to $\mathbb{R}^{d-2m}$ belong to the center but not to the Carnot center of $\mathbb{G}$. $\lhd$

Now the decomposition of $L^2(\Sigma_\delta^H M)$ from the Carnot center works as follows. In addition, we will detect that the horizontal spinor space of a compact Carnot nilmanifold of lower step can be found in this decomposition.

Theorem 4.2.3
Let $M = \Gamma\backslash\mathbb{G}$ be the compact nilmanifold of a Carnot group $\mathbb{G}$ with center $Z(\mathbb{G}) \cong \mathbb{R}^{d_R}$, and let $\Sigma_\delta^H M$ be the horizontal Clifford bundle arising via pull-back from a spin structure Σ_δ^H on $\mathbb{T}^d$.
Then there is a decomposition

$$L^2\left(\Sigma_\delta^H M\right) = \bigoplus_{\tau\in\mathbb{Z}^{d_R}} \mathcal{H}_\tau \tag{4.18}$$

into Hilbert spaces $\mathcal{H}_\tau$, which are invariant under the pull-back horizontal Dirac operator D^H acting on $\Sigma_\delta^H M$. The elements $\sigma \in \mathcal{H}_\tau$ are exactly those elements $\sigma \in L^2(\Sigma_\delta^H M)$ fulfilling

$$\sigma\left(x^{(1)}, \dots, x^{(R)}\right) = e^{2\pi i\langle\tau, x^{(R)}\rangle} \cdot \sigma\left(x^{(1)}, \dots, x^{(R-1)}, 0\right). \tag{4.19}$$

In addition, the space $\mathcal{H}_0$ is isomorphic to the space $L^2(\Sigma_\delta^H \tilde{M})$, where

$$\tilde{M} = (\Gamma/Z(\Gamma))\backslash(\mathbb{G}/Z(\mathbb{G}))$$

is the compact nilmanifold of the Carnot group $\tilde{\mathbb{G}} := \mathbb{G}/Z(\mathbb{G})$ of step $R-1$. Under this isomorphism, the restriction of the pull-back horizontal Dirac operator D^H to $\mathcal{H}_0$ can be identified with the pull-back horizontal Dirac operator $\tilde{D}^H$ acting on $\Sigma_\delta^H \tilde{M}$.

Proof: Throughout this proof we will use the characterization of a horizontal spinor $\sigma \in L^2(\Sigma_\delta^H M)$ as a map $\sigma : \mathbb{G} \to \mathbb{C}^{2^{[d/2]}}$ such that

$$\sigma\left(\left(a^{(1)}, \dots, a^{(R)}\right).x\right) = \varepsilon\left(a^{(1)}\right)\sigma(x) \tag{4.20}$$

for every $a = (a^{(1)}, \dots, a^{(R)}) \in \Gamma$ given by Theorem 4.1.3. Note that this σ, considered as a periodic function on $\mathbb{G}$, is an L^2-function on any fundamental domain for the action of Γ on $\mathbb{G}$ by left translation.

We will work with the projection $\pi_{\tilde{\mathbb{G}}} : \mathbb{G} \to \mathbb{G}/Z(\mathbb{G})$. The Carnot center of $\mathbb{G}$ can be described via (exponential or polarized) coordinates by

$$Z(\mathbb{G}) = \exp(V_R) = \left\{\left(0, \dots, 0, x^{(R)}\right) \in \mathbb{G} : x^{(R)} \in \mathbb{R}^{d_R}\right\} \cong \mathbb{R}^{d_R},$$

where $\mathfrak{g} \cong V_1 \oplus \ldots \oplus V_R$ is the grading of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ with $d_R = \dim V_R$. The image $\tilde{\mathbb{G}} \cong \mathbb{G}/Z(\mathbb{G})$ of $\pi_{\tilde{\mathbb{G}}}$ has the structure of a Carnot group of step $R-1$: This follows from Proposition 2.4.1, using the inherited group composition from $\mathbb{G}$. We can realize $\tilde{\mathbb{G}}$ the subset

$$\tilde{\mathbb{G}} = \left\{ \left(x^{(1)}, \ldots, x^{(R)}\right) \in \mathbb{G} : x^{(R)} = 0 \right\} \tag{4.21}$$

of $\mathbb{G}$, which leads to the coordinate expression

$$\pi_{\tilde{\mathbb{G}}} : \mathbb{G} \to \tilde{\mathbb{G}}, \quad \left(x^{(1)}, \ldots, x^{(R-1)}, x^{(R)}\right) \mapsto \left(x^{(1)}, \ldots, x^{(R-1)}, 0\right) \tag{4.22}$$

of $\pi_{\tilde{\mathbb{G}}}$. In this description, the group composition on $\tilde{\mathbb{G}}$ is given by executing the group composition on $\mathbb{G}$ first and projecting the result to $\tilde{\mathbb{G}}$ afterwards, i.e.

$$x._{\tilde{\mathbb{G}}} y := \pi_{\tilde{\mathbb{G}}} \left(x._{\mathbb{G}} y\right).$$

For the rest of the proof we will work with this coordinate description.
In addition, the Carnot center of the standard lattice $\Gamma \lhd \mathbb{G}$ of $\mathbb{G}$ is given by

$$Z(\Gamma) := Z(\mathbb{G}) \cap \Gamma \cong \mathbb{Z}^{d_R}.$$

Hence, and since the generators of the Lie algebra of $\mathbb{G}$ can be identified with the generators of the Lie algebra of $\tilde{\mathbb{G}}$ via $\pi_{\tilde{\mathbb{G}}}$, the discrete Carnot group $\tilde{\Gamma} := \Gamma/Z(\Gamma)$ is the standard lattice of $\tilde{\mathbb{G}} := \mathbb{G}/Z(\mathbb{G})$. This means that we can form the local homogeneous space $\tilde{M} = \tilde{\Gamma}\backslash\tilde{\mathbb{G}}$, which is a compact Carnot nilmanifold of $\tilde{\mathbb{G}}$. From the projection $\pi_{\tilde{\mathbb{G}}} : \mathbb{G} \to \mathbb{G}/Z(\mathbb{G})$ we see that M has the structure of a principle $\mathbb{T}^{d_R}$-bundle over $\tilde{M}$, since $\mathbb{T}^{d_R}$ is the local homogeneous space of $\mathbb{R}^{d_R} \cong Z(\mathbb{G})$ under the action of $\mathbb{Z}^{d_R} \cong Z(\Gamma)$.
After this preparation, we can finally prove the statements of the theorem. We fix a point $y = (y^{(1)}, \ldots, y^{(R)}) \in \mathbb{G}$. Then for any $\sigma \in L^2(\Sigma^H_\delta M)$, we can define a map φ_y via

$$\varphi_y : Z(\mathbb{G}) \cong \mathbb{R}^{d_R} \to \mathbb{C}^{2^{[d/2]}}, \quad z \mapsto \sigma\left(y._{\mathbb{G}}(0, \ldots, 0, z)\right). \tag{4.23}$$

As one can calculate using the Baker-Campbell-Hausdorff formula, we have for each $z \in \mathbb{R}^{d_R}$

$$\left(y^{(1)}, \ldots, y^{(R)}\right) ._{\mathbb{G}} (0, \ldots, 0, z) = \left(y^{(1)}, \ldots, y^{(R-1)}, y^{(R)} + z\right)$$

since $(0, \ldots, 0, z)$ lies in the Carnot center of $\mathbb{G}$, which means it commutes with any other element of $\mathbb{G}$. Thus, since $\varepsilon(0) = 1$ for any homomorphism ε characterizing a spin structure Σ^H_δ on $\mathbb{T}^d$ (see Theorem 4.1.2), (4.20) provides us the periodicity

$$\begin{aligned} \varphi_y(z + a) &= \sigma\left(y.(0, \ldots, 0, z).(0, \ldots, 0, a)\right) \\ &= \varepsilon(0)\sigma\left(y.(0, \ldots, 0, z)\right) \\ &= \varphi_y(z) \end{aligned} \tag{4.24}$$

for each $a \in \mathbb{Z}^{d_R}$. Furthermore, φ_y is an L^2-section on $\mathbb{T}^{d_R} \cong \mathbb{Z}^{d_R}\backslash\mathbb{R}^{d_R}$ (or on a corresponding fundamental domain of $\mathbb{Z}^{d_R}$ on $\mathbb{R}^{d_R}$, which is any cube of edge length 1): This follows from the fact that M is a $\mathbb{T}^{d_R}$-principle bundle over $\tilde{M}$ and hence integration over M is integration over the fiber $\mathbb{T}^{d_R}$ followed by integration over the base space $\tilde{M}$. But since σ is an L^2 section on M, it must also be an L^2-section on $\mathbb{T}^{d_R}$. Together with the periodicity (4.24) this means that we can develop φ_y into a Fourier series

$$\varphi_y(z) = \sum_{\tau \in \mathbb{Z}^{d_R}} \varphi_\tau(y) \cdot e^{2\pi i \langle \tau, z \rangle} \tag{4.25}$$

with Fourier coefficients φ_τ. If we choose $z = 0$ in (4.23) and (4.25), we find

$$\sigma(y) = \varphi_y(0) = \sum_{\tau \in \mathbb{Z}^{d_R}} \varphi_\tau(y) \tag{4.26}$$

for any $y \in \mathbb{G}$, which proves the decomposition (4.18).

To find an explicit description of the elements of the spaces $\mathcal{H}_\tau$, we use the expression of the Fourier coefficients from (4.25) via the integral

$$\varphi_\tau(y) = \int_{[0,1]^{d_R}} \sigma\left(y.(0, \ldots, 0, t)\right) e^{-2\pi i \langle \tau, t \rangle} đ^{d_R} t. \tag{4.27}$$

Assume $\sigma \in \mathcal{H}_\tau$, which means $\sigma(y) = \varphi_\tau(y)$ by (4.26). Let the coordinates of $\mathbb{G}$ belonging to the Carnot center be denoted by $z \in \mathbb{R}^{d_R}$, which means we set $z := y^{(R)}$. Since we have

$$(y^{(1)}, \ldots, y^{(R-1)}, z).(0, \ldots, 0, t) = (y^{(1)}, \ldots, y^{(R-1)}, 0).(0, \ldots, 0, z + t)$$

from the group rule in $\mathbb{G}$, we have the following calculation using the substitution $u := t + z$ in the integral and the periodicity (4.20) of σ:

$$\begin{aligned}
\varphi_\tau(y) &= \int_{[0,1]^{d_R}} \sigma\left((y^{(1)}, \ldots, y^{(R-1)}, 0).(0, \ldots, 0, z+t)\right) e^{-2\pi i \langle \tau, t \rangle} đ^{d_R} t \\
&= e^{2\pi i \langle \tau, z \rangle} \int_{[z_1, z_1+1] \times \ldots \times [z_{d_R}, z_{d_R}+1]} \sigma\left((y^{(1)}, \ldots, y^{(R-1)}, 0).(0, \ldots, 0, u)\right) e^{-2\pi i \langle \tau, u \rangle} đ^{d_R} u \\
&= e^{2\pi i \langle \tau, z \rangle} \int_{[0,1]^{d_R}} \sigma\left((y^{(1)}, \ldots, y^{(R-1)}, 0).(0, \ldots, 0, u)\right) e^{-2\pi i \langle \tau, u \rangle} đ^{d_R} u \\
&= e^{2\pi i \langle \tau, z \rangle} \cdot \sigma\left(y^{(1)}, \ldots, y^{(R-1)}, 0\right),
\end{aligned}$$

where we used the identity (4.27) once again in the last line. But from this we get the description (4.19) for any element $\sigma \in \mathcal{H}_\tau$.

On the other hand, for any $\sigma \in L^2(\Sigma^H_\delta M)$ fulfilling

$$\sigma\left(x^{(1)}, \ldots, x^{(R)}\right) = e^{2\pi i \left\langle \tau, x^{(R)} \right\rangle} \cdot \sigma\left(x^{(1)}, \ldots, x^{(R-1)}, 0\right)$$

we fix $y \in \mathbb{G}$ and write down the Fourier series of the corresponding function $\varphi_y(z)$ from (4.23). For any $\tau' \in \mathbb{Z}^{d_R}$ we calculate the Fourier coefficients $\varphi_{\tau'}(y)$ from (4.27):

$$\begin{aligned}
\varphi_{\tau'}(y) &= \int_{[0,1]^{d_R}} \sigma\left(y.(0,\ldots,0,t)\right) e^{-2\pi i\langle \tau',t\rangle} đ^{d_R}t \\
&= \int_{[0,1]^{d_R}} e^{2\pi i\left\langle \tau, y^{(R)}+t\right\rangle} \cdot \sigma\left(y^{(1)},\ldots,y^{(R-1)},0\right) e^{-2\pi i\langle \tau',t\rangle} đ^{d_R}t \\
&= e^{2\pi i\left\langle \tau, y^{(R)}\right\rangle} \cdot \sigma\left(y^{(1)},\ldots,y^{(R-1)},0\right) \int_{[0,1]^{d_R}} e^{2\pi i\langle \tau-\tau',t\rangle} đ^{d_R}t \\
&= \begin{cases} \sigma(y) & \text{for } \tau=\tau' \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

This shows immediately that we must have $\sigma \in \mathcal{H}_\tau$.

Using this description, we also see the invariance under D^H: By Proposition 4.2.1 we have to check the invariance of $\sigma \in \mathcal{H}_\tau$ under the Clifford action of HM and the right-regular representation R. While the first invariance is trivial, the invariance under R follows from a small calculation: Let $\sigma \in \mathcal{H}_\tau$. For an $x_0 \in \mathbb{G}$ we set

$$\tilde{\sigma}(x) := (R(x_0)\sigma)(x) = \sigma(x.x_0).$$

Because of the Baker-Campbell-Hausdorff formula we have in (exponential or polarized) coordinates

$$(x.x_0)^{(R)} = x^{(R)} + x_0^{(R)} + B(x,x_0), \tag{4.28}$$

where $B(x,x_0)$ is a polynomial in the coordinates of $x^{(1)},\ldots,x^{(R-1)}$ and $x_0^{(1)},\ldots,x_0^{(R-1)}$ (see Section 2.2). Then we have because of (4.19) and (4.28)

$$\begin{aligned}
\tilde{\sigma}(x) &= e^{2\pi i\left\langle \tau,(x.x_0)^{(R)}\right\rangle} \sigma\left((x.x_0)^{(1)},\ldots,(x.x_0)^{(R-1)},0\right) \\
&= e^{2\pi i\left\langle \tau,x^{(R)}\right\rangle} \sigma\left((x.x_0)^{(1)},\ldots,(x.x_0)^{(R-1)},x_0^{(R)}+B(x,x_0)\right) \\
&= e^{2\pi i\left\langle \tau,x^{(R)}\right\rangle} \sigma\left((x^{(1)},\ldots,x^{(R-1)},0).(x_0^{(1)},\ldots,x_0^{(R-1)},x_0^{(R)})\right) \\
&= e^{2\pi i\left\langle \tau,x^{(R)}\right\rangle} \tilde{\sigma}(x^{(1)},\ldots,x^{(R-1)},0),
\end{aligned}$$

since $B(x,x_0)$ does not depend on the components $x^{(R)}$ and $x_0^{(R)}$. This shows $\tilde{\sigma} \in \mathcal{H}_\tau$ and therefore the invariance of $\mathcal{H}_\tau$ under R.

The next step is to show that $\mathcal{H}_0 \cong L^2(\Sigma_\delta^H \tilde{M})$: First of all, the horizontal distributions of the compact Carnot nilmanifolds $M = \Gamma\backslash\mathbb{G}$ and $\tilde{M} = (\Gamma/Z(\Gamma))\backslash(\mathbb{G}/Z(\mathbb{G}))$ can be identified using the Carnot group homomorphism $\pi_{\tilde{\mathbb{G}}} : \mathbb{G} \to \mathbb{G}/Z(\mathbb{G})$ from (4.22), which means that for a given spin structure $\Sigma_\delta^{\mathbb{T}^d}$ on the horizontal torus $\mathbb{T}^d$ we find horizontal spinor bundles $\Sigma_\delta^H M$ and $\Sigma_\delta^H \tilde{M}$ on both manifolds. Now for any $\tilde{\sigma} \in L^2(\Sigma_\delta^H \tilde{M})$ we find can define

$$\sigma(x) := \tilde{\sigma}(\pi_{\tilde{\mathbb{G}}}(x)),$$

which is an element of $\mathcal{H}_0$: If we consider $\pi_{\tilde{\mathbb{G}}}(\mathbb{G})$ as the subset (4.21) of $\mathbb{G}$, we have $\sigma(\pi_{\tilde{\mathbb{G}}}(x)) = \sigma(x)$, which is exactly the characterization of the space $\mathcal{H}_0$ from (4.19). And since elements of $\mathcal{H}_0$ are uniquely determined by its values on $\pi_{\tilde{\mathbb{G}}}(\mathbb{G})$, the map

$$\varphi : L^2(\Sigma^H_\delta \tilde{M}) \to \mathcal{H}_0, \quad \tilde{\sigma} \mapsto \sigma = \tilde{\sigma} \circ \pi_{\tilde{\mathbb{G}}} \tag{4.29}$$

is an isomorphism.

Finally, we show that the restriction of the pull-back horizontal Dirac operator D^H on $L^2(\Sigma^H_\delta M)$ to $\mathcal{H}_0$ can be identified with the pull-back horizontal Dirac operator $\tilde{D}^H$ on $L^2(\Sigma^H_\delta \tilde{M})$, coming from the same spin structure $\Sigma^{\mathbb{T}^d}_\delta$ on the horizontal torus $\mathbb{T}^d$. Note that by the characterization (4.21) of $\tilde{\mathbb{G}}$ as a subset of $\mathbb{G}$, a horizontal frame $\{X_1, \ldots, X_d\}$ of HM is also a horizontal frame of $H\tilde{M}$, and since the horizontal spinor bundles on M and $\tilde{M}$ are constructed from the same horizontal torus $\mathbb{T}^d$ the horizontal Clifford action from elements of this frame coincides in both cases. Then the identification of the operators follows immediately from the isomorphism (4.29) when we use the local expression (4.17) of the pull-back horizontal Dirac operator from Proposition 4.2.1: We have for every $\sigma \in \mathcal{H}_0$

$$\begin{aligned} D^H\sigma(x) &= \sum_{j=1}^{d} c^H(X_d)\, \frac{d}{dt}\sigma\left(x.\exp_{\mathbb{G}}(tX_j)\right)\Big|_{t=0} \\ &= \sum_{j=1}^{d} c^H(X_d)\, \frac{d}{dt}\tilde{\sigma}\left(\pi_{\tilde{\mathbb{G}}}\left(x.\exp_{\mathbb{G}}(tX_j)\right)\right)\Big|_{t=0} \\ &= \sum_{j=1}^{d} c^H(X_d)\, \frac{d}{dt}\tilde{\sigma}\left(\pi_{\tilde{\mathbb{G}}}(x).\exp_{\tilde{\mathbb{G}}}(tX_j)\right)\Big|_{t=0} \\ &= \tilde{D}^H\tilde{\sigma}\left(\pi_{\tilde{\mathbb{G}}}(x)\right), \end{aligned}$$

and since $\mathcal{H}_0$ is invariant under D^H this shows $\varphi(\tilde{D}^H\tilde{\sigma}) = D^H\sigma$ for the isomorphism φ from (4.29).

Altogether every statement of the theorem is proved. □

Remark: Note that the coordinate expressions we used in the proof do not depend on whether we choose exponential or polarized coordinates on the Carnot group $\mathbb{G}$. This is the case since only actions by the Carnot center are involved. In the next section, it will turn our to be more comfortable to work with polarized coordinates. ◁

The above theorem reduces the problem of showing that the horizontal Dirac operator has an infinite dimensional eigenspace to the corresponding horizontal Dirac operator on a Carnot group of a lower step. We are interested in using this theorem inductively until $\tilde{M}$ is the compact nilmanifold of a Carnot group of step 2. As we will see in the next section, for a compact Heisenberg nilmanifold $\tilde{M}$ we will be able to do a complete spectral decomposition of $L^2(\tilde{M})$ and detect infinite dimensional eigenspaces from this. Then in Section 4.4, we will use Theorem 4.2.3 to lift these infinite dimensional eigenspace to compact nilmanifolds of general Carnot groups.

4.3 The Case of Compact Heisenberg Nilmanifolds

We will now do the complete spectral decomposition of the horizontal pull-back Dirac operator on compact nilmanifolds $M = \Gamma\backslash\mathbb{G}$, where $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ is a Carnot group of step 2 and rank d for some integer $1 \leq m \leq d/2$. Hence the horizontal distribution is of co-dimension 1.

For our spectral decomposition we will follow an argument of Christian Bär (see Section II.2 of his PhD thesis [Bae91]), which was used to determine the spectrum of Dirac operators on nilmanifolds from Heisenberg groups. This method was also used by Christian Bär and Bernd Ammann in [AB98] for the case of the Dirac operator on the 3-dimensional Heisenberg group. Although the situation in case of our horizontal Dirac operator differs a bit, the space $L^2(\Sigma^H_\delta M)$ can be decomposed in exactly the same way since our local expression of D^H from (4.11) is quite similar to the local expression of the Dirac operator Bär is considering. In this thesis, we have to consider a slightly more general case where $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$, but the commutative part of the group will not effect the general strategy too much. After doing the decomposition, we will be able to detect infinite-dimensional eigenspaces of D^H; and as an additional result we will show that the asymptotic behaviors of the non-degenerate eigenvalues gives back the homogeneous dimension of M.

We start by introducing some notation. Let the grading of TM be given by $TM = V_1M \oplus V_2M$, where $\{X_1, \ldots, X_d\}$ is an orthonormal frame of V_1M such that the Levi form according to this frame is given by

$$L = \begin{pmatrix} 0 & D & \\ -D & 0 & \\ & & 0_{d-2m} \end{pmatrix}, \quad \text{where } D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} \text{ with} \lambda_j > 0.$$

Note that this can always be achieved by Proposition 2.3.3 and that the numbers $\lambda_1, \ldots, \lambda_m$ are exactly the absolute values of the non-zero eigenvalues $\pm i\lambda_j$ of the Levi form of M. If V_2M is spanned by X_{d+1}, this means that we have the commutator relations

$$[X_j, X_k] = \begin{cases} \lambda_j X_{d+1} & \text{for } 1 \leq j \leq m,\ k = m+j \\ -\lambda_j X_{d+1} & \text{for } 1 \leq k \leq m,\ j = k+m\ . \\ 0 & \text{otherwise} \end{cases} \tag{4.30}$$

In what follows, it will be more comfortable to use the polarized coordinates instead of the exponential coordinates of $\mathbb{G}$ (see Definition 2.2.4). We will use polarized coordinates with respect to the frame $\{\tilde{X}_1, \ldots, \tilde{X}_{d+1}\}$ of TM, where

$$\tilde{X}_j = \frac{1}{\sqrt{\lambda_j}} X_j \quad \text{and} \quad \tilde{X}_{m+j} = \frac{1}{\sqrt{\lambda_j}} X_{m+j} \quad \text{for } 1 \leq j \leq m \tag{4.31}$$

and $\tilde{X}_k = X_k$ otherwise. We denote these polarized coordinates on $\mathbb{G}$ by (x, y, z, t) with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{d-2m}$ and $t \in \mathbb{R}$, where

$$(x, y, z, t) = \prod_{j=1}^{m} \exp(x_j X_j)\ .\ \prod_{j=1}^{m} \exp(y_j X_{m+j})\ .\ \prod_{k=1}^{d-2m} \exp(z_k X_{2m+k})\ .\ \exp(t X_{d+1}).$$

Obviously, $\prod$ denotes the product according to the group composition . of $\mathbb{G}$. In other words, the coordinates (x, y, t) describe the $\mathbb{H}^{2m+1}$-part of $\mathbb{G}$ and z describes the $\mathbb{R}^{d-2m}$-part of $\mathbb{G}$.

Now let us calculate the composition rule on $\mathbb{G}$ in these coordinates. In exponential coordinates corresponding to the frame $\{\tilde{X}_1, \ldots, \tilde{X}_{d+1}\}$ we set for abbreviation

$$V := \sum_{j=1}^{m} a_j \tilde{X}_j + \sum_{j=1}^{m} b_j \tilde{X}_{m+j} + \sum_{j=1}^{d-2m} c_j \tilde{X}_{2m+j} + s\tilde{X}_{d+1}$$

and

$$W := \sum_{j=1}^{m} x_j \tilde{X}_j + \sum_{j=1}^{m} y_j \tilde{X}_{m+j} + \sum_{j=1}^{d-2m} z_j \tilde{X}_{2m+j} + t\tilde{X}_{d+1},$$

such that we have because of (4.30)

$$\begin{aligned} [V, W] &= \sum_{j=1}^{m} \left(a_j y_j \cdot \frac{1}{\lambda_j}[X_j, X_{m+j}] + b_j x_j \cdot \frac{1}{\lambda_j}[X_{m+j}, X_j] \right) \\ &= \left(\sum_{j=1}^{m} a_j y_j - b_j x_j \right) \tilde{X}_{d+1}. \end{aligned}$$

By the Baker-Campbell-Hausdorff formula (see Section 2.2), we calculate

$$\begin{aligned} (a, b, c, s).(x, y, z, t) &= \exp V. \exp W \\ &= \exp \left(V + W + \frac{1}{2}[V, W] \right) \\ &= \exp \left(\sum_{j=1}^{m} (a_j + x_j)\tilde{X}_j + \sum_{j=1}^{m} (b_j + y_j)\tilde{X}_{m+j} + \sum_{j=1}^{d-2m} (c_j + z_j)\tilde{X}_{2m+j} \times \right. \\ &\qquad \left. \times + \frac{1}{2} \left(s + t + \sum_{k=1}^{m} (a_k y_k - b_k x_k) \right) X_{d+1} \right) \\ &= \left(a + x,\ b + y,\ c + z,\ s + t + \frac{1}{2} \sum_{k=1}^{m} (a_k y_k - b_k x_k) \right) \end{aligned}$$

for the composition rule on $\mathbb{G}$ in exponential coordinates. We can now use the isomorphism between the exponential and the polarized coordinates on a Heisenberg group (see the remark after Definition 2.2.4) and conclude that for the composition rule in coordinates according to the frame $\{\tilde{X}_1, \ldots, \tilde{X}_{d+1}\}$ we have

$$(a, b, c, s).(x, y, z, t) = \left(a + x, b + y, c + z, s + t + \sum_{j=1}^{m} a_j y_{j+m} \right). \tag{4.32}$$

Using this composition rule, we can start with the spectral decomposition of $L^2(\Sigma_\delta^H M)$. The first decomposition of $L^2(\Sigma_\delta^H M)$ is already given by Theorem 4.2.3. In the situation $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ we have

$$L^2(\Sigma_\delta^H M) \cong \bigoplus_{\tau \in \mathbb{Z}} \mathcal{H}_\tau, \tag{4.33}$$

which is invariant under D^H according to the theorem, where elements of $\mathcal{H}_\tau$ are identified by the relation

$$\sigma(x, y, z, t) = e^{2\pi i \tau t} \cdot \sigma(x, y, z, 0). \tag{4.34}$$

We set

$$f_\sigma(x, y, z) := \sigma(x, y, z, 0), \tag{4.35}$$

considered as a function on $\mathbb{R}^d$. Note that f_σ must be an L^2-function on the torus $\mathbb{T}^d$ since σ is an L^2-function on M, and that any $\sigma \in \mathcal{H}_\tau$ is fully determined by f_σ.

The strategy is to look for periodicities of f_σ in order to find a further decomposition of the spaces $\mathcal{H}_\tau$. These periodicities can be detected from the lemma below.

Lemma 4.3.1

Let $\varepsilon : \mathbb{Z}^d \to \mathbb{Z}/2\mathbb{Z}$ be the group homomorphism characterizing the spin structure $\Sigma_\delta^{\mathbb{T}^d}$ on $\mathbb{T}^d$ from Theorem 4.1.2, which induces the horizontal spinor bundle $\Sigma_\delta^H M$ on the compact Heisenberg nilmanifold M. For $\tau \in \mathbb{Z}$, we assume $\sigma \in \mathcal{H}_\tau$ according to the decomposition (4.33) of $L^2(\Sigma_\delta^H M)$. Then for any $c \in \mathbb{Z}^d$ we have

$$f_\sigma\left((x, y, z)\right) = \varepsilon(c) e^{2\pi i \left\langle \sum_{j=1}^m c_j y_j, \tau \right\rangle} \cdot f_\sigma\left((x, y, z) + c\right). \tag{4.36}$$

Proof: We use the periodicities of σ from (4.10) of Theorem 4.1.3 and the composition rule (4.32) to calculate for every $c \in \mathbb{Z}^d$:

$$\begin{aligned} f_\sigma(x, y, z) &= \varepsilon(c) \cdot \sigma((c, 0).(x, y, z, 0)) \\ &= \varepsilon(c)\sigma(\left((x, y, z) + c, \sum_{j=1}^m c_j y_j\right)) \\ &= \varepsilon(c) e^{2\pi i \tau \left(\sum_{j=1}^m c_j y_j\right)} \cdot \sigma((x, y, z) + c, 0) \\ &= \varepsilon(c) e^{2\pi i \tau \left(\sum_{j=1}^m c_j y_j\right)} \cdot f_\sigma((x, y, z) + c). \end{aligned}$$

Thereby, in the third equation we have used the characterization (4.34) of $\sigma \in H_\tau$. □

From this lemma, we can detect further decompositions of the spaces H_τ. We start with the case $\tau = 0$:

Lemma 4.3.2
For the space $\mathcal{H}_0$ from the decomposition (4.33) of $L^2(\Sigma^H_\delta M)$ we have the further decomposition

$$\mathcal{H}_0 \cong \bigoplus_{\left\{\alpha\in\frac{1}{2}\mathbb{Z}^d:\ e^{2\pi i\alpha_j}=\delta_j\right\}} \mathcal{H}^\alpha_0$$

with $\mathcal{H}^\alpha_0 \cong \mathbb{C}^{2^{[d/2]}}$ for every α, where each of the spaces $\mathcal{H}^\alpha_0$ is invariant under the horizontal pull-back Dirac operator D^H on M.

Proof: Let $\sigma \in \mathcal{H}_0$, which means we have by (4.34) and (4.35)

$$\sigma(x, y, z, t) = f_\sigma(x, y, z)$$

with f_σ as in Lemma 4.3.1. If we set $\tau = 0$ in (4.36), we immediately see that f_σ is $2\mathbb{Z}^d$-periodic, and therefore can be developed into a Fourier series

$$f_\sigma(x, y, z) = \sum_{\alpha\in\frac{1}{2}\mathbb{Z}^d} a_\alpha e^{2\pi i\langle\alpha,(x,y,z)\rangle}$$

with $a_\alpha \in \mathbb{C}^{2^{[d/2]}}$. Now the homomorphism $\varepsilon : \mathbb{Z}^d \to \mathbb{Z}/2\mathbb{Z}$ from (4.36) is given by

$$\varepsilon(c) = \delta_1^{c_1} \cdot \ldots \cdot \delta_d^{c_d},$$

see Theorem 4.1.2, such that Lemma 4.3.1 provides us further

$$\begin{aligned}\sum_{\alpha\in\frac{1}{2}\mathbb{Z}^d} a_\alpha e^{2\pi i\langle\alpha,(x,y,z)\rangle} &= \varepsilon(c) \sum_{\alpha\in\frac{1}{2}\mathbb{Z}^d} a_\alpha e^{2\pi i\langle\alpha,(x,y,z)+c\rangle} \\ &= \sum_{\alpha\in\frac{1}{2}\mathbb{Z}^d} a_\alpha e^{2\pi i\langle\alpha,(x,y,z)\rangle} \cdot \delta_1^{c_1} \ldots \delta_d^{c_d} e^{2\pi i\langle\alpha,c\rangle}\end{aligned}$$

for every $c \in \mathbb{Z}^d$. But this leads to the constraint that for every $j \in \{1, \ldots, d\}$ we must have $\delta_j e^{2\pi i\alpha_j} = 1$ and therefore $e^{2\pi i\alpha_j} = \delta_j$. This means that we have

$$\alpha_j \in \mathbb{Z} \ \Leftrightarrow\ \delta_j = 1 \quad \text{and} \quad \alpha_j \in \mathbb{Z} + \frac{1}{2} \ \Leftrightarrow\ \delta_j = -1,$$

and so the Fourier series of f_σ becomes

$$f_\sigma(x, y, z) = \sum_{\left\{\alpha\in\frac{1}{2}\mathbb{Z}^d:\ e^{2\pi i\alpha_j}=\delta_j\right\}} a_\alpha e^{2\pi i\langle\alpha,(x,y,z)\rangle}, \tag{4.37}$$

with $a_\alpha \in \mathbb{C}^{[d/2]}$. Therefore we have the decomposition

$$\mathcal{H}_0 \cong \bigoplus_{\left\{\alpha\in\frac{1}{2}\mathbb{Z}^d:\ e^{2\pi i\alpha_j}=\delta_j\right\}} \mathcal{H}^\alpha_0, \tag{4.38}$$

where each $\mathcal{H}_0^\alpha$ is spanned by $e^{2\pi i\langle\alpha,(x,y,z)\rangle}$ and hence isomorphic to $\mathbb{C}^{2^{[d/2]}}$.

Now we immediately see that each $\mathcal{H}_0^\alpha$ is invariant under the Clifford action of any $X_j \in HM$ (which only acts on the coefficient a_α) and also under the right regular representation of $\mathbb{G}$: Since $\sigma \in \mathcal{H}_0^\alpha$, we have for every given $(x_0, y_0, z_0, t_0) \in \mathbb{G}$

$$\begin{aligned}\sigma\left((x,y,z,0).(x_0,y_0.z_0,t_0)\right) &= \sigma\left(x+x_0, y+y_0, z+z_0, 0\right)\\ &= a_\alpha e^{2\pi i\langle\alpha,(x_0,y_0,z_0,0)\rangle}\cdot e^{2\pi i\langle\alpha,(x,y,z,0)\rangle},\end{aligned}$$

with $a_\alpha \in \mathbb{C}^{[d/2]}$. But this shows that $R(x_0,y_0,z_0,0)\sigma \in \mathcal{H}_0^\alpha$, and altogether every statement of the lemma is proved. □

Remark: Since $\mathbb{G}/Z(\mathbb{G}) \cong \mathbb{R}^d$, we already know by Theorem 4.2.3 that the space $\mathcal{H}_0$ is isomorphic to the space $L^2(\Sigma_\delta^H\mathbb{T}^d)$, which is by construction exactly the space $L^2(\Sigma_\delta\mathbb{T}^d)$, and the horizontal pull-back Dirac operator acting on $L^2(\Sigma_\delta\mathbb{T}^d)$ is exactly the classical Dirac operator $D^{\mathbb{T}^d}$ on the torus. Since the spectral decomposition of the Dirac operator on the torus is well-known, we could have deduced the statement of this lemma directly from Theorem 4.2.3. ◁

For the cases $\tau \neq 0$ the spectral decomposition is a bit more involved.

Lemma 4.3.3
Let $\tau \neq 0$. Then for the space $\mathcal{H}_\tau$ from the decomposition (4.33) of $L^2(\Sigma_\delta^H M)$ we have the decomposition

$$\mathcal{H}_\tau \cong \bigoplus_{\left\{\gamma\in\frac{1}{2}\mathbb{Z}^{d-2m}:\ e^{2\pi i\gamma_j}=\delta_{2m+j}\right\}} \bigoplus_{J=1}^{|\tau|^m} \mathcal{H}_{\tau,\gamma}^J, \tag{4.39}$$

with $\mathcal{H}_{\tau,\gamma}^J \cong L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$, where each of the spaces $\mathcal{H}_{\tau,\gamma}^J$ is invariant under the horizontal pull-back Dirac operator D^H on M.

Proof: Once again we use the identity (4.36) for f_σ from Lemma 4.3.1, looking for periodicities in the case $\tau \neq 0$. We write $b = (\xi, \beta, \gamma) \in \mathbb{Z}^d$ with $\xi \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}^m$ and $\gamma \in \mathbb{Z}^{d-2m}$ to distinguish the periodicities which belong to x, y and z. Remember that we have $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ for our Carnot group $\mathbb{G}$, such that the coordinates x and y (together with the coordinate from the center) belong to the Heisenberg part and z forms the (commutative) $\mathbb{R}^{d-2m}$-part of $\mathbb{G}$.

From (4.36) we see immediately that for any $\gamma \in \mathbb{Z}^{d-2m}$ we have

$$f_\sigma(x,y,z) = \varepsilon(0,0,\gamma)\cdot f_\sigma(x,y,z+\gamma),$$

with $\varepsilon : \mathbb{Z}^d \to \mathbb{Z}/2\mathbb{Z}$ describing the spin structure of $\mathbb{T}^d$ from which D^H is constructed. Therefore f_σ is $2\mathbb{Z}^{d-2m}$-periodic in z and after fixing x and y we can develop f_σ in a Fourier series

$$f_\sigma(x,y,z) = \sum_{\gamma\in\frac{1}{2}\mathbb{Z}^{d-2m}} a_\gamma^\sigma(x,y)e^{2\pi i\langle\gamma,z\rangle}, \tag{4.40}$$

where $a_\gamma^\sigma(x,y)$ is a $\mathbb{C}^{2^{[d/2]}}$-valued function. We note that a_γ^σ must be an L^2-function on $\mathbb{T}^{2m}$, since f_σ is an L^2-function on $\mathbb{T}^d \cong \mathbb{T}^{2m} \times \mathbb{T}^{d-2m}$.

As we did before we can put further conditions on the indices γ over which we sum up in (4.40): For any $c \in \mathbb{Z}^{d-2m}$ we have

$$\varepsilon(0,0,c) = \delta_{2m+1}^{c_1} \cdot \ldots \cdot \delta_d^{c_{d-2m}},$$

and thus from Lemma 4.3.1 we can deduce the restriction

$$\begin{aligned}\sum_{\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}} a_\gamma^\sigma(x,y) e^{2\pi i \langle \gamma, z\rangle} &= \varepsilon(0,0,\gamma) \sum_{\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}} a_\gamma^\sigma(x,y) e^{2\pi i \langle \gamma, z+c\rangle} \\ &= \sum_{\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}} a_\gamma(x,y)^\sigma e^{2\pi i \langle \gamma, z\rangle} \cdot \delta_{2m+1}^{c_1} \ldots \delta_d^{c_{d-2m}} e^{2\pi i \langle \gamma, c\rangle}\end{aligned}$$

for every $c \in \mathbb{Z}^{d-2m}$ by (4.36), which leads to the constraint that for all $j \in \{1, \ldots, d-2m\}$ we must have $\delta_{2m+j} e^{2\pi i \gamma_j} = 1$ and therefore $e^{2\pi i \gamma_j} = \delta_{2m+j}$. But this means that we have

$$\gamma_j \in \mathbb{Z} \iff \delta_{2m+j} = 1 \quad \text{and} \quad \gamma_j \in \mathbb{Z} + \frac{1}{2} \iff \delta_{2m+j} = -1,$$

and so the Fourier series of f_σ becomes

$$f_\sigma(x,y,z) = \sum_{\left\{\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}:\ e^{2\pi i \gamma_j} = \delta_{2m+j}\right\}} a_\gamma^\sigma(x,y) e^{2\pi i \langle \gamma, z\rangle}. \tag{4.41}$$

In Equation (4.36) we also see that we have the same periodicity properties for f_σ in the variable y (since the β-components of $b = (\xi, \beta, \gamma) \in \mathbb{Z}^d$ do not appear in the exponent of e), so we can repeat the above argument to show that

$$f_\sigma(x,y,z) = \varepsilon(0,\beta,0) f_\sigma(x, y+\beta, z),$$

which gives us

$$a_\gamma^\sigma(x,y) = \varepsilon(0,\beta,0) a_\gamma^\sigma(x, y+\beta)$$

together with (4.41). But therefore we can develop a_γ^σ into a Fourier series with respect to y, and after taking care of the restrictions arising from the components $\delta_{m+1}, \ldots, \delta_{2m}$ of the spin structure similarly to the way we did above we find that

$$a_\gamma^\sigma(x,y) = \sum_{\left\{\beta \in \frac{1}{2}\mathbb{Z}^{m}:\ e^{2\pi i \beta_j} = \delta_{m+j}\right\}} b_{\gamma,\beta}^\sigma(x) e^{2\pi i \langle \beta, y\rangle}$$

for a $\mathbb{C}^{2^{[d/2]}}$-valued function $b_{\gamma,\beta}^\sigma$ on $\mathbb{R}^m$. Plugging this into (4.41), we can write

$$f_\sigma(x,y,z) = \sum_{\left\{\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}:\ e^{2\pi i \gamma_j} = \delta_{2m+j}\right\}} \sum_{\left\{\beta \in \frac{1}{2}\mathbb{Z}^{m}:\ e^{2\pi i \beta_j} = \delta_{m+j}\right\}} b_{\gamma,\beta}^\sigma(x) e^{2\pi i \langle \beta, y\rangle} e^{2\pi i \langle \gamma, z\rangle}. \tag{4.42}$$

We continue to find periodicities for f_σ by considering dilations by $\xi \in \mathbb{Z}^m$ of the x variable. This will lead to a further characterization of the functions $b^\sigma_{\gamma,\beta}$ appearing in (4.42). Let $\sigma \in H_\tau$. We use the the composition rule (4.32) on $\mathbb{G}$, the characterization (4.34) of elements belonging to H_τ and the Fourier series development (4.42) to calculate for any $\xi \in \mathbb{Z}^m$

$$\begin{aligned}\sigma((\xi,0,0,0).(x,y,z,t)) &= \sigma(\left(\xi + x, y, z, t + \sum_{j=1}^m \xi_j y_j\right))\\ &= e^{2\pi i\tau\left(t+\sum_{j=1}^m \xi_j y_j\right)} \cdot f_\sigma(\xi + x, y, z)\\ &= e^{2\pi i\tau(t+\langle\xi,y\rangle)} \cdot \sum_\gamma \sum_\beta b^\sigma_{\gamma,\beta}(x+\xi) e^{2\pi i\langle\beta,y\rangle} e^{2\pi i\langle\gamma,z\rangle}\\ &= e^{2\pi i\tau t} \cdot \sum_\gamma \sum_\beta b^\sigma_{\gamma,\beta}(x+\xi) e^{2\pi i\langle\beta+\tau\xi,y\rangle} e^{2\pi i\langle\gamma,z\rangle},\end{aligned}$$

where the summation over γ and β is given as in (4.42). Now, since $\tau\xi \in \mathbb{Z}^m$ and therefore $e^{2\pi i\beta} = e^{2\pi i(\beta+\tau\xi)}$, we can use the substitution $\eta := \beta + \tau\xi$ in the last equation of the above calculation and find

$$\sigma((\xi,0,0,0).(x,y,z,t)) = e^{2\pi i\tau t} \cdot \sum_\gamma \sum_{\left\{\eta\in\frac{1}{2}\mathbb{Z}^m:\ e^{2\pi i\eta_j}=\delta_{m+j}\right\}} b^\sigma_{\gamma,\eta-\tau\xi}(x+\xi) e^{2\pi i\langle\eta,y\rangle} e^{2\pi i\langle\gamma,z\rangle}.$$

Because of $\sigma((\xi,0,0,0).(x,y,z,t)) = \varepsilon(\xi,0,0)\sigma(x,y,z,t)$ for every element $\sigma \in L^2(\Sigma^H_\delta M)$ this means

$$\varepsilon(\xi,0,0)\sigma(x,y,z,t) = e^{2\pi i\tau t} \cdot \sum_\gamma \sum_{\left\{\eta\in\frac{1}{2}\mathbb{Z}^m:\ e^{2\pi i\eta_j}=\delta_{m+j}\right\}} b^\sigma_{\gamma,\eta-\tau\xi}(x+\xi) e^{2\pi i\langle\eta,y\rangle} e^{2\pi i\langle\gamma,z\rangle}. \quad (4.43)$$

On the other hand we have because of (4.34) and (4.42)

$$\begin{aligned}\sigma(x,y,z,t) &= e^{2\pi i\tau t} f_\sigma(x,y,z)\\ &= e^{2\pi i\tau t} \sum_\gamma \sum_{\left\{\beta\in\frac{1}{2}\mathbb{Z}^m:\ e^{2\pi i\beta_j}=\delta_{m+j}\right\}} b^\sigma_{\gamma,\beta}(x) e^{2\pi i\langle\beta,y\rangle} e^{2\pi i\langle\gamma,z\rangle},\end{aligned} \quad (4.44)$$

and together with (4.43) this leads to the identity $b^\sigma_{\gamma,\beta}(x) = \varepsilon(\xi,0,0) \cdot b^\sigma_{\gamma,\beta-\tau\xi}(x+\xi)$ or, equivalently,

$$b^\sigma_{\gamma,\beta}(x+\xi) = \varepsilon(\xi,0,0) b^\sigma_{\gamma,\beta+\tau\xi}(x) \quad (4.45)$$

for all $\xi \in \mathbb{Z}^m$. But this means that for a given γ we have altogether $|\tau|^m$ independent functions $b^\sigma_{\gamma,\beta}$ which characterize H_τ.

Now we can choose $|\tau|^m$ independent functions which determine $\sigma \in \mathcal{H}_\tau$ completely. After fixing a $\beta_1 \in \{\beta \in \frac{1}{2}\mathbb{Z}^m :\ e^{2\pi i\beta_j} = \delta_{m+j}\}$, we consider the set

$$B := \left\{\beta_1 + \sum_{j=1}^m b_j e_j : b_j \in \{0,\ldots,|\tau|-1\}\right\},$$

where $e_1, \ldots, e_m$ are the generators of $\mathbb{Z}^m$. Obviously we have $\#B = |\tau|^m$, and after enumerating the elements of B by $\beta_1, \ldots, \beta_{|\tau|^m}$ we have the identity

$$\left\{\beta \in \frac{1}{2}\mathbb{Z}^m : \; e^{2\pi i \beta_j} = \delta_{m+j}\right\} = \dot{\bigcup}_{J=1}^{|\tau|^m} \left\{\beta_J + \tau\xi : \xi \in \mathbb{Z}^d\right\}$$

for the index set over which the β's are summated. Note that the union on the right hand side of this identity is disjoint, and after using the property (4.45) we see that for every $\xi \in \mathbb{Z}^m$ we have

$$b^\sigma_{\gamma,\beta_J+\tau\xi}(x) = \varepsilon(\xi,0,0) b^\sigma_{\gamma,\beta_J}(x+\xi).$$

Plugging this into the expression (4.44) for elements $\sigma \in \mathcal{H}_\tau$, we see that every $\sigma \in \mathcal{H}_\tau$ is uniquely determined by the functions $b^\sigma_{\gamma,\beta_1}, \ldots, b^\sigma_{\gamma,\beta_{|\tau|^m}}$ via

$$\begin{aligned}\sigma(x,y,z,t) &= e^{2\pi i\tau t} \cdot \sum_\gamma a^\sigma_\gamma(x,y) e^{2\pi i\langle\gamma,z\rangle} \\ &= e^{2\pi i\tau t} \cdot \sum_\gamma \sum_{J=1}^{|\tau|^m} \sum_{\xi\in\mathbb{Z}^m} \varepsilon(\xi,0,0) b^\sigma_{\gamma,\beta_J}(x+\xi) e^{2\pi i\langle\beta_J+\tau\xi,y\rangle} e^{2\pi i\langle\gamma,z\rangle}\end{aligned} \tag{4.46}$$

We further make use of the fact that for any γ $a^\sigma_\gamma(x,y)$ is an L^2-function on $\mathbb{T}^{2m}$ to derive that $b^\sigma_{\gamma,\beta_J} \in L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$. We can argue using the expression (4.46) as follows: For any γ we have

$$\begin{aligned}\int_{[0,1]^{2m}} \left|a^\sigma_\gamma(x,y)\right|^2 dydx &= \int_{[0,1]^m}\int_{[0,1]^m} \left|\sum_{J=1}^{|\tau|^m}\sum_{\xi\in\mathbb{Z}^m} \varepsilon(\xi,0,0) e^{2\pi i\langle\beta_J+\tau\xi,y\rangle}\right|^2 dydx \\ &= \int_{[0,1]^m}\int_{[0,1]^m} \sum_{J=1}^{|\tau|^m}\sum_{\xi\in\mathbb{Z}^m} \left|b^\sigma_{\gamma,\beta_J}(x+\xi)\right|^2 dydx \\ &= \sum_{J=1}^{|\tau|^m}\int_{[0,1]^m}\sum_{\xi\in\mathbb{Z}^m} \left|b^\sigma_{\gamma,\beta_J}(x+\xi)\right|^2 dx \\ &= \sum_{J=1}^{|\tau|^m}\int_{\mathbb{R}^m} \left|b^\sigma_{\gamma,\beta_J}(x)\right|^2 dx,\end{aligned}$$

and from the last line we can conclude $b^\sigma_{\gamma,\beta_J} \in L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$.
Since elements of $\mathcal{H}_\tau$ are fully characterized by the values of

$$\sigma(x,y,z,0) = \sum_\gamma \sum_{J=1}^{|\tau|^m} \sum_{\xi\in\mathbb{Z}^m} \varepsilon(\xi,0,0) b^\sigma_{\gamma,\beta_J}(x+\xi) e^{2\pi i\langle\beta_J+\tau\xi,y\rangle} e^{2\pi i\langle\gamma,z\rangle},$$

from the above calculation it furthermore follows that there is an isometry

$$H_\tau \cong \bigoplus_{\left\{\gamma\in\frac{1}{2}\mathbb{Z}^{d-2m},\ e^{2\pi i\gamma_j}=\delta_{2m+j}\right\}} \bigoplus_{J=1}^{|\tau|^m} L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}}).$$

But this shows the decomposition (4.39) stated in the theorem.

To finish the proof, we have to show that this decomposition is invariant under D^H. While once again the invariance under the Clifford action is trivial, the invariance under the right regular representation R follows after a simple calculation. Remember that in the polarized coordinates we are using on $\mathbb{G}$, for a given point $\bar{x}_0 = (x_0, y_0, z_0, t_0) \in \mathbb{G}$ the right regular action on $L^2(\Sigma^H_\delta M)$ is given via

$$(R(\bar{x}_0)\sigma)(x, y, z, t) = \sigma((x + x_0, y + y_0, z + z_0, t + t_0 + \langle x, y_0\rangle)),$$

see (2.1). Now since $\sigma \in \mathcal{H}_\tau$, we can use the characterization (4.46) of σ and find that

$$\begin{aligned}
(R(\bar{x}_0)\sigma)(x, y, z, t) &= e^{2\pi i\tau(t+t_0+\langle x,y_0\rangle)} \cdot \times \\
&\quad \times \sum_\gamma \sum_{J=1}^{|\tau|} \sum_{\xi\in\mathbb{Z}^m} \varepsilon(\xi, 0, 0) b^\sigma_{\gamma,\beta_J}(x + x_0 + \xi) e^{2\pi i\langle\beta,y+y_0\rangle} e^{2\pi i\langle\gamma,z+z_0\rangle} \\
&= C(x_0, y_0, z_0, t_0) e^{2\pi i\tau t} \cdot \times \\
&\quad \times \sum_\gamma \sum_{J=1}^{|\tau|} \sum_{\xi\in\mathbb{Z}^m} \varepsilon(\xi, 0, 0) e^{2\pi i\tau\langle x,y_0\rangle} b^\sigma_{\gamma,\beta_J}(x + x_0 + \xi) e^{2\pi i\langle\beta,y\rangle} e^{2\pi i\langle\gamma,z\rangle}.
\end{aligned}$$

But since for $b^\sigma_{\beta,\gamma_J} \in L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$ we also have $e^{2\pi i\tau\langle\cdot,y_0\rangle} b^\sigma_{\gamma,\beta_J}(\cdot + x_0) \in L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$, this proves the invariance of this decomposition (4.39) under R, which means that it is also invariant under D^H. Hence the lemma is proved. □

The arguments for the decomposition of $L^2(\Sigma^H_\delta M)$ used so far (except for the small modification of including the commutative part $\mathbb{R}^{d-2m}$) are exactly the arguments Bär uses to decompose the (not horizontal) spinor bundle on M in [Bae91]. But now we are ready to make some interesting observation for our situation by looking at the eigenvalues resulting from this decomposition. We can calculate all eigenvalues of D^H, and first of all we will see that D^H does not have a compact resolvent, and therefore does not provide a spectral triple.

Theorem 4.3.4

Let $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ be a Carnot group of step 2, rank d and co-rank 1 of its horizontal distribution with compact nilmanifold $M = \Gamma\backslash\mathbb{G}$. The non-zero eigenvalues of the Levi form of $\mathbb{G}$ shall be given by the numbers $\pm\lambda_j$ with $\lambda_j > 0$ for $1 \le j \le m$. Let D^H be a horizontal pull-back Dirac operator on M arising via pullback from a spin structure $\Sigma^{\mathbb{T}^d}_\delta$ with $\delta = (\delta_1, \dots, \delta_d) \in (\mathbb{Z}/2\mathbb{Z})^d$ on the torus $\mathbb{T}^d$.

Then we have the following statements about the spectrum of D^H:

(a) *In any case, the spectrum of D^H is discrete. The absolute values of the eigenvalues of D^H are of the types*

(i) $|\mu_{0,\alpha}| = 2\pi\sqrt{\sum_{j=1}^{m}\lambda_j(\alpha_j^2+\alpha_{m+j}^2)+\sum_{j=2m+1}^{d}\alpha_j^2}$ *of multiplicity* $2^{[d/2]}$ *for* $\alpha = (\alpha_1,\ldots,\alpha_d) \in \frac{1}{2}\mathbb{Z}^d$ *such that* $e^{2\pi i\alpha_j} = \delta_j$.

(ii) $|\mu_{\tau,\gamma,\kappa}| \sim \sqrt{2\pi|\tau|\sum_{j=1}^{m}\lambda_j(2\kappa_j+1)+4\pi^2\sum_{j=2m+1}^{d}\gamma_j^2}$ *of multiplicity* $2^{[d/2]}|\tau|^m$ *for* $\kappa \in \mathbb{N}^m$, $\tau \in \mathbb{Z}\setminus\{0\}$ *and* $\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}$ *such that* $e^{2\pi i\gamma_j} = \delta_{2m+j}$.

(b) *In the case $2m = d$, D^H has an infinite dimensional kernel, and any eigenspace belonging to an eigenvalue $\mu \neq 0$ of D^H is finite dimensional.*

(c) *In the case $2m < d$, there are infinitely many eigenvalues of D^H which have an infinite dimensional eigenspace.*

In particular, since D^H has in any case an infinite-dimensional eigenspace, it does not have a compact resolvent.

Proof: The strategy is to use the decomposition of $L^2(\Sigma_\delta^H M)$ we get from Theorem 4.2.3, Lemma 4.3.2 and Lemma 4.3.3, which is invariant under D^H. Then we can use Proposition 4.2.1 to write D^H locally in the form

$$D^H\sigma(x) = \sum_{j=1}^{d} c^H(X_d)\,\frac{d}{ds}\left(R(\exp sX_j)\sigma\right)(x)\bigg|_{s=0}. \tag{4.47}$$

To shorten notation in what follows, we set

$$\tilde\lambda_j = \tilde\lambda_{m+j} := \sqrt{\lambda_j} \quad \text{for } 1 \le j \le m \qquad \text{and} \qquad \tilde\lambda_j = 1 \quad \text{for } 2m+1 \le j \le d+1. \tag{4.48}$$

We will keep on working in the polarized coordinates according to the frame $\{\tilde X_1,\ldots,\tilde X_{d+1}\}$ from (4.31), such that the group composition on $\mathbb{G}$ is given by (2.1). By (4.48) we have $X_j = \tilde\lambda_j\tilde X_j$ for all $1 \le j \le d+1$, and the representation of D^H with respect to the frame $\{\tilde X_1,\ldots,\tilde X_{d+1}\}$ becomes

$$D^H\sigma(x) = \sum_{j=1}^{d} c^H(X_d)\,\frac{d}{ds}\left(R(\exp s\tilde\lambda_j\tilde X_j)\sigma\right)(x)\bigg|_{s=0}. \tag{4.49}$$

Since each of the spaces $\mathcal{H}_0^\alpha$ from Lemma 4.3.2 and $\mathcal{H}_{\tau,\gamma}^J$ from Lemma 4.3.3 is invariant under the right regular representation R, this decomposition also provides a decomposition of R into its irreducible parts. For $\tau = 0$ these will be the trivial representations, which can be calculated directly, and for $\tau \neq 0$ we can make use of the fact that the representation theory for the Heisenberg group is fully known to detect the corresponding infinite dimensional representations.

For $\sigma \in \mathcal{H}_0^\alpha$, which means by (4.37)

$$\sigma(x,y,z,t) = a_\alpha e^{2\pi i\langle\alpha,(x,y,z)\rangle}$$

with $a_\alpha \in \mathbb{C}^{2^{[d/2]}}$, we calculate from (4.49) (using the notation $(x_1, \ldots, x_{d+1})$ instead of (x, y, z, t) for the coordinates of $\mathbb{G}$) by (2.1)

$$\begin{aligned} D^H\sigma(x_1, \ldots, x_{d+1}) &= \sum_{j=1}^{d} c^H(X_j) \left.\frac{d}{ds} a_\alpha e^{2\pi i \langle \alpha, (x_1, \ldots, x_j + \tilde{\lambda}_j s, \ldots, x_d)\rangle}\right|_{s=0} \\ &= \sum_{j=1}^{d} c^H(X_j) 2\pi i \alpha_j \tilde{\lambda}_j \sigma(x_1, \ldots, x_{d+1}). \end{aligned} \tag{4.50}$$

This shows that the spectrum of D^H restricted to each $\mathcal{H}_0^\alpha$ consists of the eigenvalues of the matrices $\sum_{j=1}^{d} 2\pi i \alpha_j \tilde{\lambda}_j c^H(X_j)$. To calculate the absolute values of these eigenvalues, we calculate the eigenvalues of $(D^H)^2$ restricted to $\mathcal{H}_0^\alpha$:

$$\begin{aligned} \left(D^H\right)^2 \sigma &= \left(\sum_{j=1}^{d} c^H(X_j) 2\pi i \alpha_j \tilde{\lambda}_j\right)^2 \sigma \\ &= \sum_{j=1}^{d}\sum_{k=1}^{d} -4\pi^2 \alpha_j \alpha_k \tilde{\lambda}_j \tilde{\lambda}_k c^H(X_j) c^H(X_k)\sigma \\ &= \sum_{j=1}^{d} 4\pi^2 \alpha_j^2 \tilde{\lambda}_j^2 \sigma, \end{aligned}$$

since $c^H(X_j)c^H(X_k) + c^H(X_k)c^H(X_j) = 0$ for $j \neq k$ and $(c^H(X_j))^2 = -\mathrm{id}$ by the rules of the Clifford action. But this shows that on each $\mathcal{H}_0^\alpha$ the absolute values of the eigenvalues of D^H are given by

$$|\mu_{0,\alpha}| = 2\pi \sqrt{\sum_{j=1}^{d} \tilde{\lambda}_j^2 \alpha_j^2} = 2\pi \sqrt{\sum_{j=1}^{m} \lambda_j \left(\alpha_j^2 + \alpha_{m+j}^2\right) + \sum_{j=2m+1}^{d} \alpha_j^2}, \tag{4.51}$$

each one with multiplicity $2^{[d/2]}$ because $\Sigma_\delta^H M$ is a vector bundle of rank $2^{[d/2]}$. Note that this also shows that the part of the spectrum belonging to $\mathcal{H}_0$ coincides with the spectrum of the Dirac operator $D^{\mathbb{T}^d}$ on the torus $\mathbb{T}^d$ (equipped with a Riemannian metric such that $\{\tilde{X}_1, \ldots, \tilde{X}_d\}$ is orthonormal), from which D^H was constructed. This can also be deduced from Theorem 4.2.3.

To determinate the spectrum on the spaces $\mathcal{H}_{\tau,\gamma}^J \cong L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$, we use results from the representation theory of the Heisenberg group. It is known that $L^2(\mathbb{R}^m)$ is the representation space of irreducible unitary representations of $\mathbb{H}^{2m+1}$, and what we have done so far is to decompose the right regular representation $R : \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m} \to L^2(\Sigma_\delta^H M)$ into its irreducible components. Thereby, the frame $\{\tilde{X}_1, \ldots, \tilde{X}_{d+1}\}$ exactly corresponds to the Carnot group $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$, while the frame $\{X_1, \ldots, X_{d+1}\}$ only corresponds to a Carnot group isomorphic to $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$.

Now it is well known by the theorem of Stone and von Neumann how they have to look like on $L^2(\mathbb{R}^m)$ (see e.g. [Fol89], Theorem (1.50)): In the polarized coordinates we are using, the infinite dimensional unitary irreducible (Schrödinger) representations of $\mathbb{H}^{2m+1}$ are given by

$$\pi_r : \mathbb{H}^{2m+1} \to \mathcal{U}(L^2(M)), \quad \pi_r(x,y,t)f(u) = e^{2\pi i r(t+\langle u,y\rangle)} f(x+u) \tag{4.52}$$

for any $r \in \mathbb{R} \setminus \{0\}$. In our case, for $\sigma \in \mathcal{H}^J_{\tau,\gamma} \cong L^2(\mathbb{R}^m) \otimes \mathbb{C}^{2^{[d/2]}}$, we have $r = \tau$. From now on, we write f instead of σ whenever we consider an element of $L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$. Hence using the expression (4.49) of the horizontal Dirac operator via the right-regular representation R we have, after plugging in (4.52),

$$\begin{aligned}\frac{d}{ds} R(\exp s\tilde{\lambda}_j \tilde{X}_j) f(u)\Big|_{s=0} &= \begin{cases} \frac{d}{ds} f(u + s\tilde{\lambda}_j e_j)\Big|_{s=0} & j \in \{1,\ldots,m\} \\ \frac{d}{ds} e^{2\pi i\tau\langle u, s\tilde{\lambda}_{j-m} e_{j-m}\rangle} f(u)\Big|_{s=0} & j \in \{m+1,\ldots,2m\} \end{cases} \\ &= \begin{cases} \tilde{\lambda}_j \frac{\partial}{\partial u_j} f(u) & j \in \{1,\ldots,m\} \\ 2\pi i\tau u_{j-m} \tilde{\lambda}_{j-m} f(u) & j \in \{m+1,\ldots,2m\} \end{cases}.\end{aligned}$$

We still need to express the action of $X_{2m+j} = \tilde{X}_{2m+j}$ for $1 \le j \le d-2m$ on $\mathcal{H}^J_{\tau,\gamma}$. But this is just multiplication by $2\pi i\gamma_j$, since $\sigma \in \mathcal{H}_\tau$ is characterized via

$$\sigma(x,y,z,t) = e^{2\pi i t\tau} \sum_\gamma \sum_\beta b^\sigma_{\gamma,\beta}(x) e^{2\pi i\langle\beta,y\rangle} e^{2\pi i\langle\gamma,z\rangle}$$

(see (4.3.4) in the proof of Lemma 4.3.3), and therefore we get

$$\frac{d}{ds} R(\exp s\tilde{X}_{2m+j}) \sigma(x,y,z,t)\Big|_{s=0} = 2\pi i\gamma_j \cdot \sigma(x,y,z,t).$$

Plugging everything into the expression (4.49) we see that the action of D^H on every $\mathcal{H}^J_{\tau,\gamma} \cong L^2(\mathbb{R}^m) \otimes \mathbb{C}^{2^{[d/2]}}$ is given by

$$\begin{aligned} D^H f(u) &= \sum_{j=1}^m \tilde{\lambda}_j \left(c^H(X_j) \frac{\partial}{\partial u_j} + 2\pi i\tau u_j c^H(X_{m+j}) \right) f(u) + \sum_{k=2m+1}^d 2\pi i\gamma_k c^H(X_k) \cdot f(u) \\ &= \sum_{j=1}^m \sqrt{\lambda_j} \left(c^H(X_j) \frac{\partial}{\partial u_j} + 2\pi i\tau u_j c^H(X_{m+j}) \right) f(u) + \sum_{k=2m+1}^d 2\pi i\gamma_k c^H(X_k) \cdot f(u). \end{aligned} \tag{4.53}$$

To calculate the absolute values of the eigenvalues of D^H restricted to $\mathcal{H}^J_{\tau,\gamma}$, we once again consider the square of the operator. From (4.53) we get

$$\begin{aligned}\left(D^H\right)^2 f(u) = &\left(\sum_{j=1}^{m}\sqrt{\lambda_j}\left(c^H(X_j)\frac{\partial}{\partial u_j} + 2\pi i\tau u_j c^H(X_{m+j})\right)\right)^2 f(u)\\ &+\left(\sum_{k=2m+1}^{d} 2\pi i\gamma_k c^H(X_k)\right)^2 f(u)\\ &+\sum_{j=1}^{m}\sqrt{\lambda_j}\left(c^H(X_j)\frac{\partial}{\partial u_j} + 2\pi i\tau u_j c^H(X_{m+j})\right)\cdot\sum_{k=2m+1}^{d} 2\pi i\gamma_k c^H(X_k)\\ &+\sum_{k=2m+1}^{d} 2\pi i\gamma_k c^H(X_k)\cdot\sum_{j=1}^{m}\sqrt{\lambda_j}\left(c^H(X_j)\frac{\partial}{\partial u_j} + 2\pi i\tau u_j c^H(X_{m+j})\right),\end{aligned} \tag{4.54}$$

and because of the Clifford relation $c^H(X_l)c^H(X_{l'}) + c^H(X_{l'})c^H(X_l) = 0$ for any $l \neq l'$, the sum of the third and the forth term on the right hand side of (4.54) vanish. For the same reason, and since $c(X_l)^2 = -\mathrm{id}$ for any l, we get

$$\left(\sum_{k=2m+1}^{d} 2\pi i\gamma_k c^H(X_k)\right)^2 = 4\pi^2\sum_{k=2m+1}^{d}\gamma_k^2$$

for the second term. Finally, using again the rules of Clifford action and the Leibniz rule for differentiation, we calculate

$$\begin{aligned}&\sum_{j=1}^{m}\sqrt{\lambda_j}c^H(X_j)\frac{\partial}{\partial u_j}\sum_{k=1}^{m}\sqrt{\lambda_k}\left(c^H(X_k)\frac{\partial}{\partial u_k}f(u) + 2\pi i\tau u_k c^H(X_{m+k})f(u)\right)\\ = &\sum_{j=1}^{m}\sum_{k=1}^{m}\sqrt{\lambda_j\lambda_k}c^H(X_j)c^H(X_k)\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_k}f(u) + \times\\ \times &\sum_{j=1}^{m}\sum_{k=1}^{m}2\pi i\tau\sqrt{\lambda_j\lambda_k}c^H(X_j)c^H(X_{m+k})\frac{\partial}{\partial u_j}\left(u_k f(u)\right)\\ = &-\sum_{j=1}^{m}\lambda_j\frac{\partial^2}{\partial u_j^2}f(u) + \times\\ \times\ 2\pi i\tau&\left(\sum_{j=1}^{m}\sum_{k=1}^{m}u_k\sqrt{\lambda_j\lambda_k}c^H(X_j)c^H(X_{m+k})\frac{\partial}{\partial u_j}f(u) + \sum_{j=1}^{m}\lambda_j c^H(X_j)c^H(X_{m+j})f(u)\right)\end{aligned}$$

and

$$\begin{aligned}
& 2\pi i\tau \sum_{j=1}^{m} u_j \sqrt{\lambda_j} c^H(X_{m+j}) \sum_{k=1}^{m} \sqrt{\lambda_k} \left(c^H(X_k) \frac{\partial}{\partial u_k} f(u) + 2\pi i\tau u_k c^H(X_{m+k}) f(u) \right) \\
= \;& 2\pi i\tau \sum_{j=1}^{m}\sum_{k=1}^{m} u_j \sqrt{\lambda_j\lambda_k} c^H(X_{m+j}) c^H(X_k) \frac{\partial}{\partial u_k} f(u) \times \\
& \times \; - 4\pi^2\tau^2 \sum_{j=1}^{m}\sum_{k=1}^{m} u_j u_k \sqrt{\lambda_j\lambda_k} c^H(X_{m+j}) c^H(X_{m+k}) f(u) \\
= \;& 2\pi i\tau \sum_{j=1}^{m}\sum_{k=1}^{m} u_j \sqrt{\lambda_j\lambda_k} c^H(X_{m+j}) c^H(X_k) \frac{\partial}{\partial u_k} f(u) + 4\pi^2\tau^2 \sum_{j=1}^{m} \lambda_j u_j^2 f(u).
\end{aligned}$$

Because of the identity $c^H(X_k)c^H(X_{m+j}) + c^H(X_{m+j})c^H(X_k) = 0$ for any $k, j \in \{1, \ldots, m\}$, we have

$$\sum_{j=1}^{m}\sum_{k=1}^{m} u_k \sqrt{\lambda_j\lambda_k} c^H(X_j) c^H(X_{m+k}) \frac{\partial}{\partial u_j} f + \sum_{j=1}^{m}\sum_{k=1}^{m} u_j \sqrt{\lambda_j\lambda_k} c^H(X_{m+j}) c^H(X_k) \frac{\partial}{\partial u_k} f = 0,$$

and thus get for the first term of (4.54) from these two calculations

$$\begin{aligned}
& \left(\sum_{j=1}^{m} \sqrt{\lambda_j} \left(c^H(X_j) \frac{\partial}{\partial u_j} + 2\pi i\tau u_j c^H(X_{m+j}) \right) \right)^2 f(u) \\
= \;& -\sum_{j=1}^{m} \lambda_j \frac{\partial^2}{\partial u_j^2} f(u) + 4\pi^2\tau^2 \sum_{j=1}^{m} \lambda_j u_j^2 f(u) + 2\pi i\tau \sum_{j=1}^{m} \lambda_j c^H(X_j) c^H(X_{m+j}) f(u) \\
= \;& \sum_{j=1}^{m} \lambda_j \left(-\frac{\partial^2}{\partial u_j^2} + 4\pi^2\tau^2 u_j^2 + 2\pi i\tau c^H(X_j) c^H(X_{m+j}) \right) f(u).
\end{aligned}$$

Now after plugging everything into the expression (4.54) for the restriction of $(D^H)^2$ onto the space $\mathcal{H}^J_{\tau,\gamma} \cong L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$ we see that

$$\left(D^H\right)^2\Big|_{\mathcal{H}^J_{\tau,\gamma}} = \sum_{j=1}^{m} \lambda_j \left(-\frac{\partial^2}{\partial u_j^2} + 4\pi^2\tau^2 u_j^2 + 2\pi i\tau c^H(X_j) c^H(X_{m+j}) \right) + 4\pi^2 \sum_{k=2m+1}^{d} \gamma_k^2 \quad (4.55)$$

on these spaces.

From (4.55) we observe that we can calculate the eigenvalues of D^H from the eigenvalues of the harmonic oscillator. It is well known that the operator

$$\sum_{j=1}^{m} \lambda_j \left(-\frac{\partial^2}{\partial u_j^2} + 4\pi^2\tau^2 u_j^2 \right)$$

possesses the eigenvalues $\eta_{\tau,\kappa} = 2\pi|\tau| \sum_{j=1}^{m} \lambda_j(2\kappa_j + 1)$, where $\kappa = (\kappa_1, \dots, \kappa_m) \in \mathbb{N}^m$ (see e.g. [Fol89], Section 1.7). Since this operator is acting on the L^2-space of $\mathbb{C}^{2^{[d/2]}}$-valued functions, each of these eigenvalues has the multiplicity $2^{[d/2]}$. In addition, for each $\kappa \in \mathbb{N}^m$ we have to add the eigenvalues of the operator

$$-\sum_{j=1}^{m} 2\pi i\tau\lambda_j c^H(X_j)c^H(X_{m+j}).$$

By Proposition 3.2.9 from Section 3.2, every eigenvalue $\eta_{\tau,\kappa,l}$, $l \in \{1, \dots, 2^{[d/2]}\}$ of this $(2^{[d/2]} \times 2^{[d/2]})$-matrix is included in the set

$$\{-2\pi|\tau|\sum_{j=1}^{m}\lambda_j, \dots, 2\pi|\tau|\sum_{j=1}^{m}\lambda_j\}.$$

The third term of (4.55) is just an additive constant (since γ is fixed by choosing a space $\mathcal{H}^J_{\tau,\gamma}$), and hence on each $\mathcal{H}^J_{\tau,\gamma}$ the operator $(D^H)^2$ from (4.55) has the eigenvalues

$$\tilde{\mu}_{\tau,\gamma,\kappa,l} = 2\pi|\tau| \left(\sum_{j=1}^{m} \lambda_j(2\kappa_j + 1) + \tilde{\eta}_{\kappa,l} \right) + 4\pi^2 \sum_{k=2m+1}^{d} \gamma_k^2 \tag{4.56}$$

with $\kappa \in \mathbb{N}^m$ and $-\sum_{j=1}^{m} \lambda_j \le \tilde{\eta}_{\kappa,l} \le \sum_{j=1}^{m} \lambda_j$ for $l \in \{1, \dots, 2^{[d/2]}\}$. But since each number $\tilde{\eta}_{\kappa,l}$ is bounded by constants depending only on the constants $\lambda_1, \dots, \lambda_m$ which are given by the Levi form on M, they do not change the asymptotic behavior of the eigenvalues from (4.56). Therefore we can say that

$$(\mu_{\tau,\gamma,\kappa})^2 \sim 2\pi|\tau| \sum_{j=1}^{m} \lambda_j(2\kappa_j + 1) + 4\pi^2 \sum_{k=2m+1}^{d} \gamma_k^2, \tag{4.57}$$

where each $(\mu_{\tau,\gamma,\kappa})^2$ has the multiplicity $2^{[d/2]}$, for the eigenvalues of $(D^H)^2$ restricted to $\mathcal{H}^J_{\tau,\gamma}$.

From (4.57) we get for the absolute values of the eigenvalues of D^H

$$|\mu_{\tau,\gamma,\kappa}| \sim \sqrt{2\pi|\tau| \sum_{j=1}^{m} \lambda_j(2\kappa_j + 1) + 4\pi^2 \sum_{k=2m+1}^{d} \gamma_k^2} \tag{4.58}$$

for $\tau \in \mathbb{Z} \setminus \{0\}$, $\gamma \in \frac{1}{2}\mathbb{Z}^{d-2m}$ such that $e^{2\pi\gamma_j} = \delta_{2m+j}$ and $\kappa \in \mathbb{N}^m$. Each of these (asymptotic) eigenvalues has the multiplicity $2^{[d/2]}|\tau|^m$, since there are $|\tau|^m$ copies of the spaces $\mathcal{H}^J_{\tau,\gamma} \cong L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$. Thus we have proved statement (a).

We still have to show the statements (b) and (c) about the degeneracy of D^H. This will be done by showing that for each $\tau \in \mathbb{Z}\setminus\{0\}$, the first term of the operator $(D^H)^2$ from (4.55) has at least one 0-eigenvalue on $\mathcal{H}^J_{\tau,\gamma}$: From that it will follow that for every $\gamma \in \mathbb{Z}^{d-2m}$ the number

$$4\pi^2 \sum_{k=2m+1}^{d} \gamma_k^2$$

appears as an eigenvalue on each space $\mathcal{H}^J_{\tau,\gamma}$, and is thus an eigenvalue of infinite multiplicity. Furthermore all the degenerate eigenvalues of D^H are of this form, since every other eigenvalue on $H^J_{\tau,\gamma}$ depends on τ and is of finite multiplicity on this space. This shows statement (c), and since for $\mathbb{G} \cong \mathbb{H}^{2m+1}$ the term

$$\sum_{k=2m+1}^{d} 2\pi i \gamma_k c^H(X_k)$$

does not appear in the expression (4.53) of D^H on $H^J_{\tau,\gamma}$ the statement (b) also follows. Indeed the first term

$$\sum_{j=1}^{m} \lambda_j \left(-\frac{\partial^2}{\partial u_j^2} + 4\pi^2\tau^2 u_j^2 \right) - \sum_{j=1}^{m} 2\pi i \tau \lambda_j c^H(X_j) c^H(X_{m+j}) \tag{4.59}$$

of (4.55) has at least one 0-eigenvalue. As we already noted, the eigenvalues of the harmonic oscillator are the numbers

$$\tilde{\mu}_\kappa = 2\pi|\tau| \sum_{j=1}^{m} \lambda_j (2\kappa_j + 1),$$

where $\kappa = (\kappa_1, \dots, \kappa_m) \in \mathbb{N}^m$, and the eigenvalues of the matrix

$$-\sum_{j=1}^{m} 2\pi i \tau \lambda_j c^H(X_j) c^H(X_{m+j})$$

belong to the set

$$\left\{ -2\pi|\tau| \sum_{j=1}^{m} \lambda_j, \dots, 2\pi|\tau| \sum_{j=1}^{m} \lambda_j \right\}.$$

Hence, the only chance for the operator (4.59) to have a 0-eigenvalue is that $\kappa = 0$ and that the number $-2\pi|\tau| \sum_{j=1}^{m} \lambda_j$ is indeed an eigenvalue of $-\sum_{j=1}^{m} 2\pi i|\tau|\lambda_j c^H(X_j)c^H(X_{m+j})$. But the last statement is true by Proposition 3.2.9 from Section 3.2, and hence the above argumentation proves the statements (b) and (c) of the theorem. □

As mentioned before, the results of Theorem 4.3.4 imply that D^H does not furnish a spectral triple on $M = \Gamma\backslash\mathbb{G}$. But it fits into our definition of a degenerate spectral triple and it detects the Carnot-Carathéodory metric of our Carnot manifold M, which we write down in the following corollary.

Corollary 4.3.5
Let $M = \Gamma\backslash\mathbb{G}$ be the compact nilmanifold of a Carnot group $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$. Let D^H be a horizontal Dirac operator on M arising via pull-back from a spin structure $\Sigma_\delta^{\mathbb{T}^d}$ on the torus $\mathbb{T}^d$. Then the triple $(C(M), L^2(\Sigma_\delta^H M), D^H)$ is a degenerate spectral triple, which detects the Carnot-Carathéodory metric via Connes' metric formula.

Proof: Since D^H is a horizontal Dirac operator in the sense of Section 3.2, it has been shown in Section 3.3 that it detects the Carnot-Carathéodory metric on M via Connes metric formula. In particular it has also been shown that

$$\|[D^H, f]\| = \operatorname*{ess\,sup}_{x\in M} \left\|\mathrm{grad}^H f\right\|$$

for every $f \in \mathrm{Lip}_{CC}(M)$, and since the number on the right hand side is bounded and $\mathrm{Lip}_{CC}(M)$ is a dense sub-algebra of $C(M)$, condition (i) for a spectral triple (see Definition 1.1.1) is fulfilled.

Now from Theorem 4.3.4 we know that the spectrum of D^H is discrete. If we exclude the eigenvalues of infinite multiplicity, then for a given number $\Lambda \in \mathbb{R}$ there are only finitely many eigenvalues in the spectrum of D^H which are smaller than Λ. This shows us that D^H has a compact resolvent if we restrict it to the orthonormal complement of the degenerate eigenspaces, which means we have a degenerate spectral triple according to Definition 1.1.7.
□

As a further consequence of Theorem 4.3.4, we show that besides the Carnot-Carathéodory metric we can also detect the Hausdorff dimension of the metric space (M, d_{CC}) from the degenerate spectral triple $(C(M), L^2(\Sigma_\delta^H M), D^H)$. Therefore we consider the asymptotic behavior of the non-degenerate eigenvalues of D^H.

To detect the asymptotic behavior in our case, we will make use of the following proposition, which shows the equivalence about the growth function of the number of eigenvalues within a certain radius and their asymptotic growth. We write down the statement in a very general matter, since for its proof it is not important what exactly the elements of the sets whose growth rate we analyze describe. For a proof, we refer to [Shu01], Proposition 13.1.

Proposition 4.3.6
Let $\Lambda := \{\lambda_0, \lambda_1, \lambda_2, \ldots\} \subset \mathbb{R}^+$ be a discrete ordered set of positive real numbers, which means we have $\lambda_0 \leq \lambda_1 \leq \ldots$. For $t \in \mathbb{R}^+$, we denote by

$$N_\Lambda(t) := \sum_{\{k\in\mathbb{N}:\lambda_k\leq t\}} 1 = \#\{\lambda_k \in \Lambda : \lambda_k \leq t\}$$

the number of all elements of Λ which are bounded by t. In addition, we consider numbers $V_0 \in \mathbb{R}$ and $n \in \mathbb{N}^+$. Then the following statements are equivalent:

(i) $N_\Lambda(t) \sim V_0 t^n$ *as* $t \to \infty$.

(ii) $\lambda_k \sim V_0^{-1/n} \cdot k^{1/n}$ *as* $k \to \infty$. □

Using this proposition, we can show that the Hausdorff dimension on (M, d_{CC}) coincides with the metric dimension of the degenerate spectral triple $(C(M), L^2(\Sigma_\delta^H M), D^H)$.

Corollary 4.3.7
Let $\mathbb{G} \cong \mathbb{H}^{2m+1}$ *be isomorphic to the* $2m+1$*-dimensional Heisenberg group, and let* $M = \Gamma\backslash\mathbb{G}$ *be its compact nilmanifold. Let* D^H *be a horizontal Dirac operator on* M *arising via pull-back from a spin structure* $\Sigma_\delta^{\mathbb{T}^d}$ *on the torus* $\mathbb{T}^d$ *with* $d = 2m$.
Then the metric dimension of the degenerate spectral triple

$$\big(C(M), L^2(\Sigma_\delta^H M), D^H\big)$$

is $d+2$, *i.e. it coincides with the Hausdorff dimension of the metric space* (M, d_{CC}).
Proof: By Theorem 4.3.4, the only degenerate eigenvalue of D^H on $M = \Gamma\backslash\mathbb{H}^{2m+1}$ is 0. Hence let Λ denote the set of all eigenvalues of D^H which are not 0. We have to show by Definition 1.1.2:

$$\sum_{\mu\in\Lambda} \frac{1}{|\mu|^p} < \infty \quad \Leftrightarrow \quad p > 2m+2. \tag{4.60}$$

Now we can decompose Λ into two disjoint sets

$$\Lambda = \Lambda_1 \dot{\cup} \Lambda_2, \tag{4.61}$$

where Λ_1 contains all the eigenvalues of D^H listed under (i) and Λ_2 contains all the eigenvalues of D^H listed under (ii) in Theorem 4.3.4 which are not 0. The set Λ_1 contains exactly the eigenvalues of the classical Dirac operator, acting on the spinor bundle $\Sigma_\delta\mathbb{T}^d$ of the torus $\mathbb{T}^d$ (with respect to a Riemannian metric on $\mathbb{T}^d$ such that the vector fields $\tilde{X}_1, \ldots, \tilde{X}_d$ from (4.31) used in the proof of the theorem form an orthonormal frame for $\mathbb{T}^d$), and since this classical Dirac operator is elliptic it is known that they grow proportional to the function k^d by Weyl asymptotic. But this shows

$$\sum_{\mu\in\Lambda_1} \frac{1}{|\mu|^p} < \infty \quad \Leftrightarrow \quad p > d = 2m. \tag{4.62}$$

Therefore the crucial point of the asymptotic behavior of our eigenvalues lies in the set Λ_2. Applying Theorem 4.3.4 to our situation where $\mathbb{G} \cong \mathbb{H}^{2m+1}$, we see that the eigenvalues belonging to Λ_2 are given asymptotically by

$$|\mu_{\tau,\kappa}| \sim \sqrt{2\pi|\tau| \sum_{j=1}^{m} \lambda_j(2\kappa_j+1)} \sim \sqrt{|\tau|}\sqrt{\sum_{j=1}^{m} \kappa_j} \tag{4.63}$$

for $\tau \in \mathbb{Z}\setminus\{0\}$ and $\kappa \in \mathbb{N}^m \setminus \{0\}$. We make a further (disjoint) decomposition

$$\Lambda_2 = \dot{\bigcup}_{\tau\in\mathbb{Z}\setminus\{0\}} \Lambda_{2,\tau}$$

with $\Lambda_{2,\tau}$ containing exactly the eigenvalues $\mu_{\tau,\kappa}$ with $\kappa \in \mathbb{N}^m$. Note that each of these eigenvalues occurs with the multiplicity $2^{[d/2]}|\tau|^m$. For a fixed $\tau \in \mathbb{Z} \setminus \{0\}$ we denote the absolute values of these elements belonging to $\Lambda_{2,\tau}$ by

$$\tilde{\mu}_{\tau,0} \leq \tilde{\mu}_{\tau,1} \leq \dots, \tag{4.64}$$

such that we can work with the number $N_{\Lambda_{2,\tau}}(t)$ from Proposition 4.3.6. It is well known that for any $C \in \mathbb{R}^+$ we have

$$\#\left\{\kappa \in \mathbb{N}^m : \sum_{j=1}^m \kappa_j \leq C\right\} \sim \#\left\{\kappa \in \mathbb{N}^m : \sqrt{\sum_{j=1}^m \kappa_j^2} \leq C\right\} \sim C^m$$

(one can consider these numbers as the eigenvalues of the elliptic Dirac operator on the m-dimensional torus, and then this statement follows from Weyl asymptotic), which means

$$\#\left\{\kappa \in \mathbb{N}^m : \sqrt{\sum_{j=1}^m \kappa_j} \leq C\right\} \sim C^{2m},$$

and hence for every $\tau \in \mathbb{Z} \setminus \{0\}$ it follows from (4.63) that

$$\begin{aligned} N_{\Lambda_{2,\tau}}(t) &\sim \#\left\{\kappa \in \mathbb{N}^m : \sqrt{|\tau|}\sqrt{\sum_{j=1}^m \kappa_j} \leq t\right\} \\ &= \#\left\{\kappa \in \mathbb{N}^m : \sqrt{\sum_{j=1}^m \kappa_j} \leq \frac{t}{\sqrt{|\tau|}}\right\} \\ &\sim \left(\frac{t}{\sqrt{|\tau|}}\right)^{2m} = \frac{1}{|\tau|^m} t^{2m}. \end{aligned} \tag{4.65}$$

At this point we can apply Proposition 4.3.6, which tells us that (4.65) is equivalent to the fact that

$$\tilde{\mu}_{\tau,k} \sim |\tau|^{\frac{1}{2}} k^{\frac{1}{2m}} \tag{4.66}$$

for the elements of $\Lambda_{2,\tau}$ denoted by (4.64), where we do not care about the multiplicity $2^{[d/2]}|\tau|^m$ of these numbers as eigenvalues of D^H for a moment.

With these results, we can check (4.60) for the set Λ_2 which will prove the corollary: Because of our decomposition of Λ_2 and (4.66) we have

$$\begin{aligned} \sum_{\mu \in \Lambda_2} \frac{1}{|\mu|^p} &= \sum_{\tau \in \mathbb{Z}\setminus\{0\}} \sum_{\mu \in \Lambda_{2,\tau}} \frac{1}{|\mu|^p} \\ &\sim \sum_{\tau \in \mathbb{Z}\setminus\{0\}} \sum_{k \in \mathbb{N}} 2^{\left[\frac{d}{2}\right]} |\tau|^m \cdot \frac{1}{\left(|\tau|^{\frac{1}{2}} k^{\frac{1}{2m}}\right)^p} \\ &= \sum_{\tau \in \mathbb{Z}\setminus\{0\}} 2^{\left[\frac{d}{2}\right]} \frac{1}{|\tau|^{\frac{p}{2}-m}} \cdot \sum_{k \in \mathbb{N}} \frac{1}{k^{\frac{p}{2m}}}, \end{aligned}$$

where we have taken care of the multiplicity $2^{[d/2]}|\tau|^m$ of every eigenvalue $\tilde{\mu}_{\tau,k}$ in the second equation. Now the second geometric series in the last line of this calculation converges if and only if $p > 2m$, and the first geometric series in this line converges if and only if $p > 2m+2$. But this, together with (4.62) tells us that the series

$$\sum_{\mu\in\Lambda}\frac{1}{|\mu|^p} = \sum_{\mu\in\Lambda_1}\frac{1}{|\mu|^p} + \sum_{\mu\in\Lambda_2}\frac{1}{|\mu|^p}$$

converges if and only if $p > 2m+2$, which shows that (4.60) is true and hence the corollary is proved. □

Remark: The same statement should also be true for the case $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$, but in this case the argument is a bit more involved. ◁

Remark 4.3.8

From the argumentation of Theorem 4.3.4 it is also possible to calculate the eigenvalues of the horizontal pull-back Dirac operator exactly for given numbers $m \in \mathbb{N}$ and $d \in \mathbb{N}$ (after determining the matrices $c^H(X_j)$ describing the Clifford action). The idea is to use the orthonormal basis of $L^2(\mathbb{R}^m)$ consisting of the Hermite functions

$$h_k(v) = e^{\|v\|^2/2}\frac{\partial^{k_1+\ldots+k_m}}{\partial v_1^{k_1}\ldots v_m^{k_m}}e^{-\|v\|^2}, \tag{4.67}$$

which allows us to decompose $\mathcal{H}^J_{\tau,\gamma}$ into finite dimensional subspaces which are invariant under D^H. On these subspaces the determination of the spectrum is done by calculating eigenvalues of matrices.

In the case $m = 1$ where $\mathbb{G} \cong \mathbb{H}^3$ is the 3-dimensional compact Heisenberg nilmanifold, this has been done in [Bae91] and [AB98] for an ordinary Dirac operator. The argument used there can be transfered to our case. After these calculations, we find that the eigenvalues of D^H are exactly the numbers

(i) $\mu^{\pm}_{\alpha,\beta} = \pm 2\pi\sqrt{\alpha^2+\beta^2}$ for $\alpha, \beta \in \mathbb{Z}$ such that $e^{2\pi i\alpha} = \delta_1$, $e^{2\pi i\beta} = \delta_2$ of multiplicity 1,

(ii) $\mu^{\pm}_{\tau,\kappa} = \pm 2\sqrt{\kappa\pi|\tau|}$ for $\kappa \in \mathbb{Z}^+$, $\tau \in \mathbb{Z}\setminus\{0\}$ of multiplicity $|\tau|$,

(iii) $\mu_{\tau,0} = 0$ for $\tau \in \mathbb{Z}\setminus\{0\}$ of multiplicity $|\tau|$.

Considering the inverses of the absolute values of these eigenvalues which are not zero, we find (for the set Λ_2 from (4.61) in the proof of Theorem 4.3.4) that

$$\sum_{\tau\in\mathbb{Z}\setminus\{0\}}\sum_{\kappa\in\mathbb{N}}\frac{1}{\left|\mu^{\pm}_{\tau,\kappa}\right|^p} \leq \infty \quad \Leftrightarrow \quad p > 4,$$

which shows the statement of Corollary 4.3.7 for this example.

Using the Hermite functions from (4.67), it is possible to describe the kernel of D^H on the sub-spaces $L^2(\mathbb{R}^m, \mathbb{C}^{2^{[d/2]}})$ of the decomposition (4.39) of each space $\mathcal{H}_\tau$ from Lemma 4.3.3. In the case $m = 1$ and $\mathbb{G} \cong \mathbb{H}^3$, one finds that on each space $L^2(\mathbb{R}, \mathbb{C}^2)$ belonging to $\mathcal{H}_\tau$ the kernel of D^H is spanned by the $\mathbb{C}^2$-valued functions

$$f_0(t) = h_0(\sqrt{2\pi|\tau|}t)\begin{pmatrix}1\\0\end{pmatrix} = e^{-\pi|\tau|t^2}\begin{pmatrix}1\\0\end{pmatrix}$$

for $\tau > 0$ and

$$f_0(t) = h_0(\sqrt{2\pi|\tau|}t)\begin{pmatrix}0\\1\end{pmatrix} = e^{-\pi|\tau|t^2}\begin{pmatrix}0\\1\end{pmatrix}$$

for $\tau < 0$. In particular we observe that these functions are smooth.

This observation leads to the idea that D^H may be an example for a so-called *weakly hypoelliptic operator*, in the sense that $D^H\varphi = 0$ implies that φ is smooth. Recently, these weakly hypoelliptic operators have been considered by Christian Bär in [Bae12]. $\lhd$

To close this section, let us once again summarize the results we achieved so far: We have seen that for the case $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$, the horizontal pull-back Dirac operator on $\Gamma\backslash\mathbb{G}$ does not have a compact resolvent. Hence it only furnishes a degenerate spectral triple, but from this degenerate spectral triple the most important ingredients of the Carnot-Carathéodory geometry on M, which are the Carnot-Carathéodory metric and the Hausdorff dimension, can be detected.

In the next section, we will deduce from this co-dimension 1 case that the absence of a compact resolvent of D^H occurs on any compact Carnot nilmanifold.

4.4 Degeneracy of D^H in the General Case

This section will provide the most general result of this chapter concerning compact Carnot nilmanifolds: On an arbitrary compact Carnot nilmanifold the horizontal pull-back Dirac operator has (at least one) infinite dimensional eigenspace and does therefore not have a compact resolvent. Later we will prove that any horizontal Dirac operator on an arbitrary Carnot manifold fails to be hypoelliptic, which can be seen as a generalization of this statement.

The starting point is Theorem 4.2.3, which allows us to reduce the problem from an arbitrary Carnot group to a 2-step nilpotent one. In the last section we have made a detailed treatment of compact Heisenberg nilmanifolds, which are exactly those 2-step nilmanifolds whose horizontal distribution has co-dimension 1, and seen that we have a degenerate eigenspace in this case. So the only step missing is to get from an arbitrary space of step 2 to a compact Heisenberg nilmanifold.

To do this, we consider a submersion of the type introduced in Section 2.4. Let $M_2 = \Gamma_2\backslash\mathbb{G}_2$ be the compact nilmanifold of a Carnot group $\mathbb{G}_2$ of rank 2. We assume that the Lie algebra

$\mathfrak{g}_2$ of $\mathbb{G}$ has the grading $\mathfrak{g}_2 = V_1 \oplus V_2$, where $\{X_{1,1}, \ldots, X_{1,d_1}\}$ is an orthonormal frame for V_1 and $\{X_{2,1}, \ldots, X_{2,d_2}\}$ is an orthonormal frame for V_2. For a $\nu \in \{1, \ldots, d_2\}$ we consider the orthonormal projection

$$\begin{aligned} \mathrm{pr}_\nu : \ & \mathfrak{g}_2 \to \mathfrak{g}_{2,\nu} \simeq V_1 \oplus \mathrm{span}\{X_{2,\nu}\}, \\ & v \mapsto v \quad \mathrm{mod}\ \mathrm{span}\left(\{X_{2,1}, \ldots, X_{2,d_2}\} \setminus \{X_{2,\nu}\}\right). \end{aligned} \tag{4.68}$$

We have seen in Section 2.4 that the vector space $\mathfrak{g}_{2,\nu}$ can be canonically equipped with a Lie bracket, such that it is a graded nilpotent Lie algebra of rank 2, where V_1 is bracket generating of step 2 and co-dimension 1 for $\mathfrak{g}_{2,\nu}$. We denote the Carnot group arising from $\mathfrak{g}_{2,\nu}$ by $\mathbb{G}_{2,\nu}$, and the Lie group homomorphism arising from pr_ν by

$$\psi_\nu := \exp_{\mathbb{G}_{2,\nu}} \circ \mathrm{pr}_\nu \circ \exp_{\mathbb{G}_2}^{-1} : \mathbb{G}_2 \to \mathbb{G}_{2,\nu}. \tag{4.69}$$

For $\Gamma_{2,\nu} := \psi_\nu(\Gamma_2)$, we further define the compact Carnot nilmanifold $M_{2,\nu} := \Gamma_{2,\nu} \backslash \mathbb{G}_{2,\nu}$ over $\mathbb{G}_{2,\nu}$.

Assume that a horizontal Clifford bundle $\Sigma^H_\delta M_2$ and a horizontal pull-back Dirac operator $D^H_{M_2}$ on $\Gamma_2 \backslash \mathbb{G}_2$, arising from a spin structure $\Sigma^{\mathbb{T}^d}_\delta$ on $\mathbb{T}^d$, are given. Now $\Sigma^{\mathbb{T}^d}_\delta$ also defines a horizontal Clifford bundle $\Sigma^H_\delta M_{2,\nu}$ on $M_{2,\nu}$, which is a vector bundle of the same rank as $\Sigma^H_\delta M_2$; and since the horizontal distributions of M_2 and $M_{2,\nu}$ can both be identified with $T\mathbb{T}^d$ we have a horizontal Clifford action on $M_{2,\nu}$ which coincides with the horizontal Clifford action on M_2, meaning $c^H_{M_2}(X_{1,j}) = c^H_{M_{2,\nu}}(\mathrm{pr}_\nu(X_{1,j}))$ for all $1 \leq 1 \leq d$ as endomorphisms on the vector bundles of rank $2^{[d/2]}$. Hence we have a horizontal pull-back Dirac operator

$$D^H_{M_{2,\nu}} := \sum_{j=1}^{d_1} c^H_{M_{2,\nu}}(\mathrm{pr}_\nu(X_{1,j}))\partial_{\mathrm{pr}_\nu(X_{1,j})} \tag{4.70}$$

on $M_{2,\nu}$. But this operator allows us to prove the following lemma.

Lemma 4.4.1
Let $D^H_{M_2}$ be the horizontal pull-back Dirac operator on the compact Carnot nilmanifold $M_2 = \Gamma_2 \backslash \mathbb{G}_2$, where $\mathbb{G}_2$ is a Carnot group of nilpotency step 2, and for a $\nu \in \{1, \ldots, d_2\}$ let $D^H_{M_{2,\nu}}$ be the horizontal pull-back Dirac operator from (4.70) *on the compact Heisenberg nilmanifold $M_{2,\nu} = \Gamma_{2,\nu} \backslash \mathbb{G}_{2,\nu}$ constructed above.*

Then if there is a $\nu \in \{1, \ldots, d_2\}$ such that the section $\sigma_\nu \in \Gamma^\infty(\Sigma^H_\delta M_{2,\nu})$ lies in the kernel of $D^H_{M_{2,\nu}}$, the section

$$\sigma := \sigma_\nu \circ \psi_\nu$$

is an element of the kernel of $D^H_{M_2}$.

Proof: The idea is to use the expression from Theorem 4.2.1 of the horizontal Dirac operators involving the right-regular representation R of the Carnot groups $\mathbb{G}_2$ and $\mathbb{G}_{2,\nu}$ on the spaces $L^2(\Sigma^H_\delta M_2)$ and $L^2(\Sigma^H_\delta M_{2,\nu})$. This means in the current situation that the operators $D^H_{M_2}$ and $D^H_{M_{2,\nu}}$, applied to σ and σ_ν, are given by

$$D^H_{M_2}\sigma = \sum_{j=1}^{d_1} c^H_{M_2}(X_{1,j}) \, \frac{d}{dt} R(\exp_{\mathbb{G}_2} tX_{1,j})\sigma\Big|_{t=0} \tag{4.71}$$

and

$$D^H_{M_{2,\nu}}\sigma_\nu = \sum_{j=1}^{d_1} c^H_{M_{2,\nu}}\left(\mathrm{pr}_\nu(X_{1,j})\right) \frac{d}{dt} R\left(\exp_{\mathbb{G}_{2,\nu}} t\mathrm{pr}_\nu(X_{1,j})\right)\sigma_\nu\Big|_{t=0}. \tag{4.72}$$

We want to show that for any σ_ν such that $D^H_{M_{2,\nu}}\sigma_\nu = 0$ we have $D^H_M\sigma = 0$, where

$$\sigma = \sigma_\nu \circ \psi_\nu.$$

Following the discussion preceding this lemma, we have $c^H_{M_2}(X_{1,j}) = c^H_{M_{2,\nu}}(\mathrm{pr}_\nu(X_{1,j}))$ as endomorphisms on the vector bundles of rank $2^{[d/2]}$. Hence, from the expressions (4.71) and (4.72) the desired statement will follow if we have

$$\left(R(\exp_{\mathbb{G}_2} tX_{1,j})\sigma\right)(x) = \left(R\left(\exp_{\mathbb{G}_{2,\nu}} t\mathrm{pr}_\nu(X_{1,j})\right)\sigma_\nu\right)(\psi_\nu(x)) \tag{4.73}$$

for any $x \in M_2$. Will check this via a small calculation.
Let $t \in \mathbb{R}$. Then for the left hand side of (4.73) we get for every $\sigma \in \Gamma^\infty(\Sigma^H_\delta M_2)$, using exponential coordinates of $\mathbb{G}_2$ and the Baker-Campbell-Hausdorff formula (see Equation (2.5) in Section 2.2),

$$\begin{aligned}
& R(\exp_{\mathbb{G}_2} tX_{1,j})\sigma\left(x^{(1)}, x^{(2)}\right) \\
= \; & \sigma\left(\left(x^{(1)}, x^{(2)}\right).\exp_{\mathbb{G}_2} tX_{1,j}\right) \\
= \; & \sigma\left(\exp_{\mathbb{G}_2}\left(\sum_{k=1}^{d_1} x_{1,k}X_{1,k} + \sum_{\mu=1}^{d_2} x_{2,\mu}X_{2,\mu}\right).\exp_{\mathbb{G}_2} tX_{1,j}\right) \\
= \; & \sigma\left(\exp_{\mathbb{G}_2}\left(\sum_{k=1}^{d_1} x_{1,k}X_{1,k} + \sum_{\mu=1}^{d_2} x_{2,\mu}X_{2,\mu} + tX_{1,j} + \frac{1}{2}\left[\sum_{k=1}^{d_1} x_{1,k}X_{1,k}, tX_{1,j}\right]\right)\right).
\end{aligned}$$

Now we can calculate the commutators occurring in the last line using for $1 \le \mu \le d_2$ the μ-Levi forms (see Definition 2.3.4) $L^{(\mu)}$ of $\mathbb{G}_2$, which gives us the identity

$$[X_{1,k}, X_{1,j}] = \sum_{\mu=1}^{d_2} L^{(\mu)}_{jk} X_{2,\mu}$$

for all $1 \le k \le d$. Plugging this into the above calculation we get

$$\begin{aligned}
& R(\exp_{\mathbb{G}_2} tX_{1,j})\sigma\left(x^{(1)}, x^{(2)}\right) \\
& = \sigma\left(\exp_{\mathbb{G}_2}\left(\sum_{k=1}^{d_1} x_{1,k}X_{1,k} + \sum_{\mu=1}^{d_2} x_{2,\mu}X_{2,\mu} + tX_{1,j} + \frac{1}{2}t\sum_{\mu=1}^{d_2}\sum_{k=1}^{d_2} x_{1,k}L^{(\mu)}_{kj}X_{2,\mu}\right)\right).
\end{aligned} \tag{4.74}$$

Now we set

$$\sigma = \sigma_\nu \circ \psi_\nu = \sigma_\nu \circ \left(\exp_{\mathbb{G}_{2,\nu}} \circ \mathrm{pr}_\nu \circ \exp_{\mathbb{G}_2}^{-1}\right)$$

in (4.74) and get from the definition (4.68) of pr_ν

$$R(\exp_{\mathbb{G}_2} tX_{1,j})\sigma\left(x^{(1)}, x^{(2)}\right)$$
$$= \sigma_\nu\left(\exp_{\mathbb{G}_{2,\nu}}\left(\sum_{k=1}^{d_1} x_{1,k}\mathrm{pr}_\nu(X_{1,k}) + x_{2,\nu}\mathrm{pr}_\nu(X_{2,\nu}) + t\mathrm{pr}_\nu(X_{1,j}) + \frac{1}{2}t\sum_{k=1}^{d_1} x_{1,k}L_{kj}^{(\nu)}\mathrm{pr}_\nu(X_{2,\nu})\right)\right) \tag{4.75}$$

since $\mathrm{pr}(X_{2,\mu}) = 0$ for all $\mu \neq 0$. Since pr_ν is a Lie algebra homomorphism, we have

$$[\mathrm{pr}_\nu(X_{1,j}), \mathrm{pr}_\nu(X_{1,k})] = \mathrm{pr}_\nu([X_{1,j}, X_{1,k}])$$

for all $1 \leq j, k \leq d_1$. Thus we see immediately (after using the Baker-Campbell-Hausdorff formula on $\mathbb{G}_{2,\nu}$ in the same way we did above on $\mathbb{G}_2$) that (4.75) is exactly the right hand side of (4.73), and therefore we have

$$D^H_{M_2}\sigma(x) = D^H_{M_{2,\nu}}\sigma_\nu(\pi_\nu(x)) = 0.$$

This shows the statement of the lemma. □

Now we can put the things together to prove that the horizontal pull-back Dirac operator we constructed has an infinite dimensional eigenspace on any Carnot group one can choose.

Theorem 4.4.2
Let $M = \Gamma\backslash\mathbb{G}$ be the compact nilmanifold of a Carnot group $\mathbb{G}$ of rank d and step R. Let D^H be the horizontal pull-back Dirac operator acting on the horizontal Clifford bundle $\Sigma^H_\delta M$ which is arising from a spin structure $\Sigma^{\mathbb{T}^d}_\delta$ of the torus $\mathbb{T}^d$.
Then the kernel of D^H is infinite-dimensional. This means in particular that D^H does not have a compact resolvent.

Proof: Let $\mathfrak{g} = \bigoplus_{S=1}^R V_S$ be the grading of the Lie algebra of $\mathbb{G}$. We consider the decomposition

$$L^2(\Sigma^H_\delta M) = \mathcal{H}_0 \oplus \bigoplus_{\tau\in\mathbb{Z}^{\dim V_R}\setminus\{0\}} \mathcal{H}_\tau$$

of $L^2(\Sigma^H_\delta M)$ from Theorem 4.2.3, where all the spaces $\mathcal{H}_\tau$ are invariant under D^H. By the second statement of Theorem 4.2.3, we have

$$\mathcal{H}_0 \cong L^2(\Sigma^H_\delta M_{R-1}),$$

where M_{R-1} is the compact nilmanifold of the Carnot group $\mathbb{G}_{R-1} \cong \mathbb{G}/Z_R(\mathbb{G})$ and the restriction of D^H to $\mathcal{H}_0$ can be identified with a horizontal pull-back Dirac operator $\tilde{D}^H$ acting on $\Sigma^H_\delta M_{R-1}$. But this means that we can apply the same decomposition to the space

$\mathcal{H}_0$, which contains a Hilbert space isomorphic to $L^2(\Sigma_\delta^H M_{R-2})$, with M_{R-2} the compact nilmanifold of the step $R-2$ Carnot group $\mathbb{G}_{R-2} \cong \mathbb{G}_{R-1}/Z_{R-1}(\mathbb{G}_{R-1})$, and so on.

Inductively, we find a Hilbert space $\tilde{\mathcal{H}} \cong L^2(\Sigma_\delta^H M_2)$, where M_2 is the compact nilmanifold of a Carnot group $\mathbb{G}_2$ of step 2, which is invariant under D^H and on which D^H can be identified with a horizontal pull-back Dirac operator $D^H_{M_2}$, acting on $\Sigma_\delta^H M_2$. But for this operator, we find an infinite dimensional kernel by Lemma 4.4.1: Since $\mathbb{G}$ is not abelian, we find a $\nu \in \{1, \dots, d_2\}$, where d_2 is the dimension of the space V_2 from the grading $\mathfrak{g}_2 = V_1 \oplus V_2$ of the Lie algebra of $\mathbb{G}_2$, such that

$$\mathbb{G}_{2,\nu} = \psi_\nu(\mathbb{G}_2) \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$$

for some $m \geq 1$. We can define a horizontal Clifford bundle $\Sigma_\delta^H M_{2,\nu}$ and a horizontal pull-back operator $D^H_{M_{2,\nu}}$ from the corresponding objects on M_2 like we did in the discussion preceding Lemma 4.4.1.

Now we know by Theorem 4.3.4 that $D^H_{M_{2,\nu}}$ has an infinite dimensional kernel, and that we can choose a basis $\{\tilde{\sigma}_1, \tilde{\sigma}_2, \dots\}$ of this kernel. Using Lemma 4.4.1, we can lift this basis to an orthonormal system of infinitely many independent sections of $L^2(\Sigma_\delta^H M_2)$, given by $\sigma_j := \tilde{\sigma}_j \circ \psi_\nu$, which all lie in the kernel of D^H. (Note that these sections are indeed linear independent since this is the case for the $\tilde{\sigma}_j$'s and ψ_ν is a submersion.) Hence we have shown that D^H has an infinite-dimensional kernel on $L^2(\Sigma_\delta^H M_2)$, and from the above argumentation this is also the case on $L^2(\Sigma_\delta^H M)$.

The statement that D^H cannot have a compact resolvent follows trivially from the fact that $\mathrm{Ker} D^H$ is infinite-dimensional. □

We have shown by Theorem 4.4.2 that the horizontal pull-back Dirac operator D^H does not furnish a spectral triple on arbitrary compact Carnot nilmanifolds. Theoretically it is possible to do spectral decompositions like in Section 4.3 for any given Carnot group and thus get statements about the asymptotic behavior of the non-degenerate eigenvalues: We have to know about the representation theory of $\mathbb{G}$.

Now there is an algorithm to determine all the irreducible unitary representation of a Carnot group $\mathbb{G}$ (up to equivalence) developed by Alexander Kirillov ([Kir62]), which is also referred to as the orbit method (see e.g. [CG89] or [Kir04]). But since this algorithm makes use of the concrete structure of $\mathbb{G}$ it is hard to get general results in our context. And even for a given Carnot group which is not isomorphic to $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ we expect the calculations to be very long and complicated.

But anyway we will show in the following chapters that the phenomenon of the degeneracy of the horizontal Dirac operator occurs in general, such that the description of the Carnot-Carathéodory geometry via spectral triples does not work as one would expect.

Chapter 5

Calculus on Heisenberg Manifolds

In the previous chapter we presented an explicit construction for horizontal Dirac operators on compact Carnot nilmanifolds and we saw that these operators do not have a compact resolvent. To put these observations into a greater generality, we want to adopt tools from pseudodifferential calculus. In the classical case, the Dirac operator D on a compact manifold is an elliptic operator of order 1, and therefore it follows from pseudodifferential theory that it admits a parametrix of order -1, which is compact because of the Sobolev embedding theorem. The existence of a parametrix leads to the possibility to construct complex powers within the calculus, which shows that the resolvent $(D^2+I)^{-1/2}$ of D is compact. Now we intend to present something analogous for operators on Carnot manifolds, respecting the grading of a graded nilpotent Lie algebra.

Indeed there is a pseudodifferential symbol calculus for Heisenberg manifolds: It has been developed simultaneously by Richard Beals and Peter Greiner (see [BG84]) and Michael Taylor (see [Tay84]) in the 1980s. In the last decade, some properties which are important for our work have been presented by Raphaël Ponge (see [Pon08]). We will see that in this calculus hypoellipticity takes the place of ellipticity, since hypoellipticity implies the existence of complex powers. In addition, we will see that on a compact manifold operators of negative order are compact, such that one can get a compact resolvent for a given operator of positive Heisenberg order. For hypoelliptic self-adjoint horizontal Laplacians on compact Heisenberg manifolds it is also known that their eigenvalues grow polynomial with a rate which gives back the graded dimension of the manifold. This can be seen as an analogy to the Weyl asymptotics in the elliptic case.

In this chapter we give an overview over the Heisenberg calculus developed by Richard Beals and Peter Greiner, explain the composition of symbols and explore the role of hypoellipticity. For details we refer to the books by Beals and Greiner ([BG84]) and by Raphaël Ponge ([Pon08]). Afterwards we present results concerning the asymptotic growth of the eigenvalues and the existence of complex powers of hypoelliptic operators, which can be derived by an expansion of the heat kernel of the operator similarly to the classical case. Here we refer to [BGS84] and also to [Pon08]. It is expected that there are analogous results for arbitrary Carnot manifolds, see [Pon08], but we did not find this generalization worked out in the literature and hence we restrict ourselves to Heisenberg manifolds.

5.1 The Heisenberg Calculus

Let M be a Heisenberg manifold of dimension $n = d+1$, which means by Section 2.3 that we have a grading $TM = HM \oplus VM$ of the tangent bundle such that HM is a bracket generating horizontal distribution of rank d and $VM = [HM, HM]$ is of rank 1. An orthonormal frame for HM shall be given by the vector fields $\{X_1, \ldots, X_d\}$, while $\{X_{d+1}\}$ shall span VM. As we know by Section 2.2, the graded (and therefore the Hausdorff) dimension of the metric space (M, d_{CC}), where d_{CC} is the Carnot-Carathéodory metric, is equal to $d+2$.

In Section 2.3 we have seen how one can identify the tangent space of M with a bundle $\mathfrak{g}M$ of graded nilpotent Lie algebras, and hence we also have this structure on the cotangent bundle T^*M which we denote by $\mathfrak{g}^*M$. At a point $x_0 \in M$, we have $\mathfrak{g}_{x_0}M \cong \mathbb{R}^n$ (as a vector space), and we have the dilations

$$\lambda.(\xi + \xi_{d+1}) = \lambda\xi + \lambda^2\xi_{d+1}. \tag{5.1}$$

for coordinates $(\xi, \xi_{d+1}) \in \mathbb{R}^n$ (with $\xi = (\xi_1, \ldots, \xi_d)$). We will further use the Koranyi gauge

$$\|\xi\|_{\mathbb{H}} = \left(\sum_{j=1}^{d} |\xi_j|^4 + |\xi_{d+1}|^2\right)^{\frac{1}{4}}, \tag{5.2}$$

see Definition 2.2.8.

First we will consider $M = U \subset \mathbb{R}^n$ to be an open subset of $\mathbb{R}^n$; the generalization to vector bundles and manifolds will be the content of a theorem we mention at a later point of this section. Let $\sigma_j(x,\xi) = \sigma(-iX_j)$ denote the (classical) symbol of the vector fields $-iX_j$, and set $\sigma(x,\xi) := (\sigma_1(x,\xi), \ldots, \sigma_n(x,\xi))$. Note that by Proposition 2.2.6 we can detect ξ from the symbol σ of the homogeneous operator. Using the Koranyi gauge $\|\cdot\|_{\mathbb{H}}$ and the notation

$$\langle\alpha\rangle = \sum_{j=1}^{d} \alpha_j + 2\alpha_{d+1} \tag{5.3}$$

for a multi-index $\alpha \in \mathbb{N}^{d+1}$ we take care of the homogeneity of the X_j considered as differential operators. We can define the following symbol classes on which the Heisenberg calculus will be based (see [BG84], (10.5)-(10.18), and [Pon08], Section 3.1.2).

Definition 5.1.1
Let $U \subset \mathbb{R}^n$ be an open subset. Then:

(i) For $m \in \mathbb{Z}$, we set

$$F_{\mathbb{H},m}(U) := \{f \in C^\infty(U \times \mathbb{R}^n \setminus \{0\}) : f(x, \lambda.\sigma) = \lambda^m f(x,\sigma)\ \forall \lambda > 0\}.$$

We will further denote the class of these homogeneous functions, which do not depend on x, by $F_{\mathbb{H},m}$.

(ii) *For $m \in \mathbb{Z}$, we set*

$$F_{\mathbb{H}}^m(U) := \left\{ f \in C^\infty(U \times \mathbb{R}^n) : f \sim \sum_{j=0}^{\infty} f_{m-j},\ f_k \in F_{\mathbb{H},k}(U) \right\},$$

where the asymptotic expansion $f \sim \sum f_{m-j}$ is meant in the sense that for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ and all $N > 0$, we have

$$\left| D_x^\alpha D_\xi^\beta \left(f(x,\sigma) - \sum_{j<N} f_{m-j}(x,\sigma) \right) \right| \leq C_{\alpha\beta N}(x) \|\sigma\|_{\mathbb{H}}^{m-N-\langle\beta\rangle}, \tag{5.4}$$

for a locally bounded function $C_{\alpha\beta N}$ on U.

We will further denote the class of these functions, which do not depend on x, by $F_{\mathbb{H}}^m$:

$$f \in F_{\mathbb{H}}^m \ \Leftrightarrow\ f \sim \sum_{j=0}^{\infty} f_{m-j},\ f_k \in F_{\mathbb{H},k}.$$

(iii) *For $m \in \mathbb{Z}$, we set*

$$S_{\mathbb{H},m}(U) := \{ q \in C^\infty(U \times \mathbb{R}^n \setminus \{0\}) : \exists f \in F_{\mathbb{H},m}(U) \text{ with } q(x,\xi) = f(x,\sigma(x,\xi)) \}.$$

(iv) *For $m \in \mathbb{Z}$, we set*

$$S_{\mathbb{H}}^m(U) := \{ q \in C^\infty(U \times \mathbb{R}^n) : \exists f \in F_{\mathbb{H}}^m(U) \text{ with } q(x,\xi) = f(x,\sigma(x,\xi)) \}.$$

For $f \sim \sum_{j=0}^{\infty} f_{m-j}$, the asymptotic expansion of $q(x,\xi) = f(x,\sigma(x,\xi))$ is given by

$$q \sim \sum_{j=0}^{\infty} q_{m-j} \quad \text{with } q_k(x,\xi) = f_k(x,\sigma(x,\xi)). \tag{5.5}$$

We call elements belonging to the class $S_{\mathbb{H}}^m(U)$ Heisenberg symbols *of order m.*

(v) *The symbol class*

$$S_{\mathbb{H}}^\infty(U) := \bigcup_{m \in \mathbb{Z}} S_{\mathbb{H}}^m(U)$$

induces the class of Heisenberg pseudodifferential operators *on U, which we denote by $\Psi_{\mathbb{H}}(U)$.*

In detail, for $q \in S^m(U)$, the corresponding operator $\mathrm{Op}(q) \in \Psi_{\mathbb{H}}^m(U)$ is given by

$$\mathrm{Op}(q)u(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi) đ\xi \tag{5.6}$$

for any function $u \in C_c^\infty(U)$. On the other hand, if an operator Q can be written in the form (5.6) for a function $q \in C^\infty(U \times \mathbb{R}^n)$, we call $\sigma_{\mathbb{H}}(Q) := q$ the Heisenberg symbol of Q.

(vi) The class

$$S_{\mathbb{H}}^{-\infty}(U) := \bigcap_{m\in\mathbb{Z}} S_{\mathbb{H}}^m(U)$$

is called the class of smoothing operators *in the Heisenberg calculus.*

◁

Before we go on with the theory, we will state an example for Heisenberg pseudodifferential operators we already know.

Example 5.1.2
Let $M = \mathbb{R}^n$ equipped with a frame $\{X_1, \dots, X_{d+1}\}$ such that $\{X_1, \dots, X_d\}$ forms a bracket generating horizontal distribution. Then any graded differential operator

$$D = p(X_1, \dots, X_d, X_{d+1})$$

of order $\mu \in \mathbb{N}$ with respect to this frame (see Definition 2.2.7) is a Heisenberg pseudodifferential operator of Heisenberg order μ. Its Heisenberg symbol is a polynomial $p(\sigma_1, \dots, \sigma_{d+1}) \in F_{\mathbb{H}}^{\mu}(\mathbb{R}^n)$ of (homogeneous) degree μ, and its asymptotic expansion is given by the homogeneous terms of this polynomial. In case the coefficients of D are constant, we have a Heisenberg symbol belonging to the class $F_{\mathbb{H}}^{\mu}$.
In particular, a horizontal Laplacian is a Heisenberg pseudodifferential operator of Heisenberg order 2. ◁

We do not know yet if the expression (5.6) from the last item of Definition 5.1.1 makes sense. But this will be the case, since a symbol of $S_{\mathbb{H}}^m(U)$ belongs to a Hörmander symbol class

$$S_{\rho,\delta}^m(U) = \left\{ q \in C^\infty(U \times \mathbb{R}^n) : \left| D_x^\alpha D_\xi^\beta q(x,\xi) \right| \leq C_{\alpha\beta}(x)\,(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|} \right\}, \tag{5.7}$$

where $C_{\alpha\beta}$ is again a locally bounded function on U. These symbol classes were established by Lars Hörmander in [Hoe66], where it was also shown that the corresponding operators can be extended to bounded operators on certain Sobolev spaces. For the following theorem, we refer to [BG84], Proposition (10.22).

Theorem 5.1.3
For every $m \in \mathbb{Z}$, we have

$$S_{\mathbb{H}}^m(U) \subset \begin{cases} S_{\frac{1}{2},\frac{1}{2}}^m(U) & \text{for } m \geq 0 \\ S_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}m}(U) & \text{for } m < 0 \end{cases}.$$

□

From this embedding, we see that the operator from equation (5.6) is well defined for every test function $u \in C_c^\infty(U)$. Moreover, we immediately get some regularity properties for Heisenberg pseudodifferential operators, which follow immediately from the corresponding regularity properties of operators belonging to the class $S^m_{1/2,\,1/2}(U)$.

Corollary 5.1.4
Let $m \in \mathbb{Z}$, $q \in S^m_{\mathbb{H}}(U)$. We can define an operator

$$Qu(x) := \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)đ\xi$$

which has the following properties:

(i) $Q : C_c^\infty(U) \to C^\infty(U)$ *is a continuous linear operator.*

(ii) For every $s \in \mathbb{R}$, Q can be extended to a bounded linear operator

$$Q : H^{s+m}(U) \to H^s(U),$$

where $H^{s+m}(U)$ and $H^s(U)$ denote the L^2-Sobolev spaces.

In particular, this means $Q \in \mathcal{B}(L^2(U))$ if $m \leq 0$. □

In [BG84], it is also shown that the class $S^m_{\mathbb{H}}(U)$ does not depend on the choice of the frame $\{X_1, \ldots, X_d, X_{d+1}\}$ of TM, as long as it respects the grading structure (see [BG84], Proposition (10.46)).

After defining our symbol classes, we show that they induce a meaningful calculus. This is in general not the case for symbols of the class $S_{1/2,\,1/2}(U)$, but it turns out that the composition of two Heisenberg pseudodifferential operators gives a $\Psi_{\mathbb{H}}DO$ again. We briefly sketch how this composition is defined, referring to [BG84], Chapters 12-14, and to [Pon08]. Section 3.1.3, for the details. Assume first that the symbols of the operators which shall be composed are given by $p_1 \in S_{\mathbb{H},m_1}(U)$ and $p_2 \in S_{\mathbb{H},m_2}(U)$, where $p_k(x,\xi) = f_k(x,\sigma(x,\xi))$ for homogeneous functions $f_k \in F_{\mathbb{H},m_k}$.

- We fix an $x \in U$ and choose coordinates on U which are centered at x. This provides us with symbols
$$f_k^{(x)}(\sigma) := f_k(x,\sigma)$$
with $f_k^{(x)} \in F_{\mathbb{H},m_1}$ for $k \in \{1,2\}$, defined on the dual $\mathfrak{g}_x^*U$ of the tangent graded Lie algebra $\mathfrak{g}_xU$ at x, which corresponds to a Carnot group $\mathbb{G}^x$ via the exponential mapping. Then $\mathbb{G}^x$ can be identified with the tangent Carnot group $\mathbb{G}_xU$ of U.

- After these identifications, we can use the Lie group composition . of the tangent Carnot group $\mathbb{G}_xM$ to define a convolution product
$$*^{(x)} : F_{\mathbb{H},m_1} \times F_{\mathbb{H},m_2} \to F_{\mathbb{H},m_1+m_2} \tag{5.8}$$

in the following way: For $f_k^{(x)}$, $k = 1, 2$, the operator associated to the symbol is simply the convolution operator

$$\mathrm{Op}\left(f_k^{(x)}\right) u(z) = \int_{\mathbb{G}_x M} \check{f}_k^{(x)}(y) u(z.y^{-1}) đy,$$

acting on the Carnot group $\mathbb{G}_x U$. The composition of two convolution operators of this type gives the bilinear mapping (5.8) (see [BG84], Proposition (12.14)).

- Finally, one finds that the product $*^{(x)}$ from (5.8) depends smoothly on x (see [BG84], Proposition (13.3)), which gives us a continuous bilinear product

$$\begin{aligned} * :& S_{\mathbb{H},m_1}(U) \times S_{\mathbb{H},m_2}(U) \to S_{\mathbb{H},m_1+m_2}(U) \\ &(p_1 * p_2)(x,\xi) = \left(f_1^{(x)} *^{(x)} f_2^{(x)}\right)(\sigma(x,\xi)). \end{aligned} \tag{5.9}$$

The above construction shows how homogeneous symbols can be composed. Now every Heisenberg symbol has an asymptotic expansion (5.5) into homogeneous symbols, and therefore one can find an asymptotic expansion for the composition of two arbitrary Heisenberg symbols, which is again a Heisenberg symbol and furnishes a $\Psi_{\mathbb{H}} DO$. The details for this expansion are formulated within the next theorem (see [BG84], Theorems (14.1) and (14.7), and [Pon08], Proposition 3.1.9).

Theorem 5.1.5
For $j = 1, 2$ let $P_j \in \Psi_{\mathbb{H}}^{m_j}(U)$ have the symbol

$$p_j \sim \sum_{k \geq 0} p_{j,m_j-k}$$

in the sense of (5.5) *and assume that one of these operators is properly supported. Then for the operator $P = P_1 P_2$ we have $P \in \Psi_{\mathbb{H}}^{m_1+m_2}(U)$, and its symbol p is given by*

$$p \sim \sum_{k \geq 0} p_{m_1+m_2-k}$$

in the sense of (5.5), *where*

$$p_{m_1+m_2-k}(x,\xi) = \sum_{k_1+k_2 \leq k} \sum_{\alpha,\beta,\gamma,\delta}^{(k-k_1-k_2)} h_{\alpha,\beta,\gamma,\delta}(x) \cdot \left(D_\xi^\delta p_{1,m_1-k_1}(x,\xi)\right) * \left(\xi^\gamma \partial_x^\alpha \partial_\xi^\beta p_{2,m_2-k_2}(x,\xi)\right). \tag{5.10}$$

In (5.10), $\sum_{\alpha,\beta,\gamma,\delta}^{(l)}$ *denotes the sum over all the indices such that*

$$|\alpha| + |\beta| \leq \langle\beta\rangle - \langle\gamma\rangle + \langle\delta\rangle = l \quad \text{and} \quad |\beta| = |\gamma|,$$

and the functions $h_{\alpha,\beta,\gamma,\delta}$ are polynomials in the derivatives of the coefficients of the vector fields $X_1, \ldots, X_d, X_{d+1}$. □

So far, we have developed the Heisenberg calculus on open subsets of $\mathbb{R}^n$. To define this calculus for operators acting on vector bundles over arbitrary Heisenberg manifolds, one has to show that the class of $\Psi_{\mathbb{H}}DO$s is invariant under the change of charts respecting the Heisenberg structure. In detail, we have the following theorem which shows us that we are able to extend the theory to the manifold case (see [BG84], Theorem (10.67), and [Pon08], Proposition 3.1.18).

Theorem 5.1.6
Let U_1 and U_2 be open subsets on $\mathbb{R}^{d+1}$ together with hyperplane bundles $HU_1 \subset TU_1$ and $HU_2 \subset TU_2$, and let $\phi : (U_1, HU_1) \to (U_2, HU_2)$ be a Heisenberg diffeomorphism, which means we have $D\phi HU_1 = HU_2$ for the differential of ϕ.

Then if $P_2 \in \Psi_{\mathbb{H}}^m$ is a Heisenberg pseudodifferential operator of order m on U_2, the pullback $P_1 := \phi^* P_2$ of this operator to U_1 is a Heisenberg pseudodifferential operator of order m on U_1. □

Because of this theorem, we can consider Heisenberg pseudodifferential operators on Heisenberg manifolds M, acting on vector bundles E from now on, and write $\Psi_{\mathbb{H}}(M, E)$ for this class of operators. In particular, we can derive the following consequence from Corollary 5.1.4 which states that $\Psi_{\mathbb{H}}DO$s of negative order defined on a compact manifold are compact. It can be used if we want to show that a certain Heisenberg pseudodifferential operator has a compact resolvent.

Corollary 5.1.7
For $m < 0$, let $Q \in \Psi^m(M, E)$ be a Heisenberg pseudodifferential operator acting on a vector bundle E over a Heisenberg manifold M. Then Q is a compact operator on the Hilbert space $L^2(M, E)$.

Proof: By Corollary 5.1.4 and Theorem 5.1.6, for every $s \in \mathbb{R}$ the operator

$$Q : H^{s+m}(M, E) \to H^s(M, E)$$

is bounded on the L^2-Sobolev spaces $H^{s+m}(M, E)$. We can now choose $s = -m$ and use the fact that the embedding $H^{-m}(M, E) \hookrightarrow H^0(M, E) = L^2(M, E)$ is compact by the Sobolev embedding theorem to derive that that operator

$$Q : L^2(M, E) \to H^{-m}(M, E) \hookrightarrow L^2(M, E)$$

is compact as the composition of a bounded and a compact operator. □

After having established a composition rule inside the class of $\psi_{\mathbb{H}}DO$s in Theorem 5.1.5, the next thing to examine is the existence and the regularity of parametrices inside this class. First of all let us recall the definition of a parametrix.

Definition 5.1.8
Let E be a vector bundle over a Heisenberg manifold M, and let $P : C_c^\infty(M,E) \to C^\infty(M,E)$ be a $\Psi_{\mathbb{H}}DO$. Then an operator $Q : C_c^\infty(M,E) \to C^\infty(M,E)$ with $Q \in \Psi_{\mathbb{H}}(M,E)$ is called a Heisenberg parametrix *or a* Heisenberg pseudodifferential inverse *of P, if we have*

$$QP = PQ = I \mod \Psi_{\mathbb{H}}^{-\infty}(M,E),$$

which means that the operators $PQ - I$ and $QP - I$ are smoothing. ◁

In the classical calculus for symbols of the type $S_{1,0}$, a parametrix for a pseudodifferential operator P exists if the operator is elliptic. We will see that a necessary condition for the existence of a parametrix in the Heisenberg calculus is the hypoellipticity of the operator. The following classical definition of hypoellipticity is due to Lars Hörmander and can also be found in [Pon08] and [BG84].

Definition 5.1.9
Let P be a $\Psi_{\mathbb{H}}DO$ of order $m \in \mathbb{Z}$, acting on a vector bundle E over a Heisenberg manifold M. Then P is called hypoelliptic, *if for any distribution $u \in \mathcal{D}'(M,E)$ we have*

$$Pu \in C^\infty(M,E) \quad \Rightarrow \quad u \in C^\infty(M,E).$$

In more detail, we call P hypoelliptic with the loss of k derivatives, *if we have for any $s \in \mathbb{R}$:*

$$Pu \in H^s(M,E) \quad \Rightarrow \quad u \in H^{s+k}(M,E).$$

◁

In the classical case, a pseudodifferential operator is elliptic if its principal symbol is invertible. This generalizes to the case of Heisenberg pseudodifferential operators in terms of hypoellipticity: The main result will be that a $\Psi_{\mathbb{H}}DO$ is hypoelliptic if the $\Psi_{\mathbb{H}}DO$ associated to the principal part in the asymptotic (Heisenberg) expansion of its symbol is invertible in the Heisenberg calculus. We first introduce the notion of the principal symbol and the model operator, as it is done in [BG84] and [Pon08].

Definition 5.1.10
Let P be a $\Psi_{\mathbb{H}}DO$ of order $m \in \mathbb{Z}$ with Heisenberg symbol $p \in S_{\mathbb{H}}^m(M,E)$, acting on a vector bundle E over a Heisenberg manifold M. For a point $a \in M$, we choose an open subset $U \ni a$ of M and local coordinates of U centered at a to consider the asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} \tilde{p}_{m-j} \quad \text{with } \tilde{p}_{m-j} \in S_{\mathbb{H},m}(U,E) \tag{5.11}$$

in the sense of Definition 5.1.1.

For $\sigma(x,\xi) = (\sigma_1(x,\xi),\dots,\sigma_{d+1}(x,\xi))$ *with* $\sigma_j(x,\xi) = \sigma(-iX_j)$ *the classical symbol of* $-iX_j$*, we find a* $p_m \in F^m_{\mathbb{H}}(U)$ *such that in (5.11)* $\tilde{p}_m(x,\xi) = p_m(x,\sigma(x,\xi))$*. Then the symbol*

$$p_m^{(a)}(\sigma) := p_m(a,\sigma(a,\xi)) \ \in F_{\mathbb{H}m}(U).$$

is called the (homogeneous) principal symbol *of* P *at* a*. The corresponding operator*

$$P^{(a)} := \operatorname{Op}\left(p_m^{(a)}\right) : C_c^\infty(\mathbb{G}_a M, E_a) \to C^\infty(\mathbb{G}_a M, E_a), \tag{5.12}$$

where $\mathbb{G}_a M$ *is the tangent Carnot group at* a*, is called the* (homogeneous) model operator *of* P *at* a*.* ◁

Remark: It is also possible to define a global principal symbol $\sigma_m(P) \in F_{\mathbb{H},m}(M,E)$ on M using the kernel representation of Heisenberg pseudodifferential operators, for which we refer to [Pon08], Theorem 3.2.2. ◁

Example 5.1.11
For a graded differential operator

$$D = p(X_1,\dots,X_{d+1})$$

from Example 5.1.2, the principal symbol of D at a point $a \in M$ is given by the leading homogeneous term of the polynomial p after freezing the coefficients of p in a. Hence the model operator of D is given by the homogeneous graded differential operator with constant coefficients belonging to this homogeneous term.

In particular, for the case of a horizontal Laplacian of the form

$$\Delta^{\text{hor}} = -\sum_{j=1}^d X_j^2 + B(x)X_{d+1} + \sum_{j=1}^d a_j(x)X_j + a(x),$$

the model operator of Δ^{hor} at $a \in M$ is the operator

$$\Delta^{\text{hor}}_{\text{mod}} = -\sum_{j=1}^d X_j^2 + B(a)X_{d+1}.$$

◁

One can show that the convolution of two principal symbols gives the principal symbol of the composition of the corresponding operators, and hence the composition of two model operators gives the model operator of the composition of the original operators. See for example [Pon08], Proposition 3.2.9.

In particular, the existence of a parametrix for a Heisenberg pseudodifferential operator implies the existence of a parametrix of its model operator at each point. We even have equivalence for these two statements, and both statements imply the hypoellipticity of each model operator, which is formulated in the next theorem (see [Pon08], Proposition 3.3.1, Theorem 3.3.18 and Proposition 3.3.20).

Theorem 5.1.12
Let E be a vector bundle over a Heisenberg manifold M, and let

$$P : C_c^\infty(M, E) \to C^\infty(M, E)$$

be a $\Psi_{\mathbb{H}}DO$ of order $m \in \mathbb{N}$. Then the following statements are equivalent.

(i) P has a Heisenberg parametrix Q with symbol $q \in S_{\mathbb{H}}^{-m}(M, E)$.

(ii) At each point $a \in M$, the model operator $P^{(a)}$ from (5.12) of P has a Heisenberg parametrix $Q^{(a)}$ with symbol $q^{(a)} \in F_{\mathbb{H},-m}$.

(iii) The global principal symbol $\sigma_m(P)$ of P is invertible with respect to the convolution product for homogeneous symbols.

If any of these conditions is fulfilled, P is hypoelliptic with loss of $\frac{m}{2}$ derivatives. □

We note that in [Pon08] it is only shown that hypoellipticity is implied by the conditions (i) - (iii) of Theorem 5.1.12. But in the case of a graded differential operator D with constant coefficients, the hypoellipticity of D is equivalent to the invertibility of D in the Heisenberg calculus. This can be seen by the so-called *Rockland condition*, which we will introduce in Chapter 6: It is shown in [Pon08], that this Rockland condition (in a more general version compared to the one we will introduce) for a Heisenberg pseudodifferential operator P is equivalent to the statements (i) - (iii) of Theorem 5.1.12 (see [Pon08], Theorem 3.3.18). But for graded differential operators with constant coefficients, the Rockland condition is equivalent to the hypoellipticity of D, as we will see in Section 6.1.

In case of a horizontal Laplacian of the form

$$\Delta^{\text{hor}} = -\sum_{j=1}^{d} X_j^2 + B(x)X_{d+1} + \sum_{j=1}^{d} a_j(x)X_j + a(x) \tag{5.13}$$

it was proved in [BG84] that the hypoellipticity of this operator is equivalent to its invertibility in the Heisenberg calculus. Since we will work in particular with horizontal Laplacians, we state this theorem here (see [BG84], Theorem (18.4)).

Theorem 5.1.13
For the horizontal Laplacian Δ^{hor} from (5.13), the following statements are equivalent:

(i) Δ^{hor} has a Heisenberg parametrix $Q \in \Psi_{\mathbb{H}}^{-2}(U)$

(ii) At each $a \in U$, the model operator of Δ^{hor} has a Heisenberg pseudodifferential inverse.

(iii) Δ^{hor} is hypoelliptic with loss of one derivative. □

We will see in the next section that from the point of view of constructing and analyzing spectral triples using the techniques of Heisenberg calculus indeed hypoellipticity is the central property. Right now, we close this section with a remark concerning a possible generalization to arbitrary Carnot manifolds.

Remark 5.1.14
Let M be an arbitrary Carnot manifold of step R with grading $TM \cong V_1M \oplus \ldots \oplus V_RM$ such that for any $1 \leq S \leq R$ $X_{S,1}, \ldots, X_{S,d_S}$ forms an orthonormal frame for V_SM (with $d_S = \text{Rank } V_SM$). Then we assume that we can generalize Definition 5.1.1 as follows: For the Carnot group $\mathbb{G}$ corresponding to M we consider the exponential coordinates $\xi \in \mathfrak{g} = \exp^{-1}\mathbb{G}$ with $\xi = (\xi^{(1)}, \ldots, \xi^{(R)})$, where $\xi^{(S)} \in \mathbb{R}^{d_S}$, the dilations

$$\lambda.\xi = \left(\lambda\xi^{(1)}, \lambda^2\xi^{(2)}, \ldots, \lambda^R\xi^{(R)}\right)$$

and the Koranyi gauge

$$\|x\|_{\mathbb{G}} := \left(\sum_{S=1}^{R}\sum_{j=1}^{d_S} |x_{S,j}|^{\frac{2R!}{S}}\right)^{\frac{1}{2R!}},$$

see Definition 2.2.8. In addition, for multi-indices $\alpha \in \mathbb{N}^{d_1+\ldots+d_R}$ we set

$$\langle\alpha\rangle_{\mathbb{G}} := \sum_{S=1}^{R} S \cdot \sum_{j=1}^{d_S} \alpha_{S,j}.$$

Then we can define symbol classes in analogy to Definition 5.1.1, and it seems natural that it is possible to generalize the further definitions and theorems of this section to the general Carnot case. The proofs should work more or less analogous to the presentation in [BG84] and [Pon08], but it would by very laborious to write everything down in detail. $\triangleleft$

5.2 Complex Powers and Eigenvalue Asymptotics

In this short section we present some results in Heisenberg calculus which open the door to constructing meaningful spectral triples from this calculus. As we will see, the crucial assumption in all these results is the hypoellipticity of the operator.

First of all, we provide a theorem from which it can be shown that the operator $(D^2+I)^{-1/2}$ is compact for a hypoelliptic self-adjoint operator D of Heisenberg order 1. We formulate a combinations of the Theorems 5.3.1 and 5.4.10 from [Pon08].

Theorem 5.2.1
Let M be a Heisenberg manifold. Suppose that P is a hypoelliptic self-adjoint Heisenberg pseudodifferential operator of order $\nu \in \mathbb{Z}$ which is bounded from below and which is

satisfying $\mathrm{Ker} P = \{0\}$. *Then, for any* $s \in \mathbb{C}$, *the operator* P^s *defined via functional calculus is a* $\Psi_{\mathbb{H}}DO$ *of order* νs. □

Remark: Note that we have not introduced $\Psi_{\mathbb{H}}DO$s of non-integer order properly in this thesis. For this, we refer to [Pon08]. Since in our case every operator appearing in this context will be of integer order, the theorems formulated in Section 5.1 are sufficient.

Note also that the formulation in [Pon08] of this theorem is more general in the sense that he does not assume that the kernel of P is only assumed to be finite dimensional, which is the case because of the hypoellipticity of P. In this general case, one can also construct complex powers using projections onto the orthonormal complement of the kernel. ◁

The next thing is to discover the asymptotic behavior of the eigenvalues of a self-adjoint Heisenberg pseudodifferential operator, from which one can detect the metric dimension of a spectral triple. In classical pseudodifferential calculus, one can consider the heat kernel of a positive self-adjoint elliptic operator P to get asymptotics for the growth of its eigenvalues. This is done by expanding the trace of the operator e^{-tP}. Now something similar works for hypoelliptic self-adjoint horizontal Laplacians which are bounded from below: Mostly, this is the content of the paper [BGS84] by Richard Beals, Peter Greiner and Nancy Stanton. Some further considerations have been carried out by Raphaël Ponge (see e.g. [Pon08]). Without going into the details, we just state the results here.

The following theorem shows how the trace expansion of the heat kernel of such a horizontal Laplacian looks like (see [BGS84], Theorem (5.6), or [Pon08], Proposition 6.1.1).

Theorem 5.2.2
Let Δ^{hor} *be a hypoelliptic and self-adjoint horizontal Laplacian which is bounded from below, acting on a vector bundle* E *over a Heisenberg manifold* M *of dimension* $d+1$ *(which means that the horizontal distribution of* M *has rank* d*). Then for* $t \to 0^+$ *we have the expansion*

$$\mathrm{Tr}\, e^{-t\Delta^{\mathrm{hor}}} \sim t^{-\frac{d+2}{2}} \sum_{j=0}^{\infty} t^{\frac{2j}{m}} A_j(\Delta^{\mathrm{hor}}) \tag{5.14}$$

with $A_j(\Delta^{\mathrm{hor}}) = \int_M \mathrm{tr}_E a_j(\Delta^{\mathrm{hor}})(x)dx$, *where* a_j *can be computed from the term of degree* $-2-2j$ *in the asymptotic expansion of the symbol of the parametrix of* Δ^{hor} *in local coordinates.* □

Now we denote by

$$\lambda_0(\Delta^{\mathrm{hor}}) \leq \lambda_1(\Delta^{\mathrm{hor}}) \leq \dots$$

the eigenvalues of Δ^{hor}, counted with multiplicity. A consequence of (5.14), in connection with Karamata‘s Tauberian Theorem, is that the following asymptotic behavior of these eigenvalues holds (see [Pon08], Proposition 6.1.2).

Theorem 5.2.3
Let Δ^{hor} be a hypoelliptic and self-adjoint horizontal Laplacian which is bounded from below, acting on a vector bundle E over a Heisenberg manifold M of dimension $d+1$. Then for $j \to \infty$ we have

$$\lambda_j(\Delta^{\mathrm{hor}}) \sim \left(\frac{j}{\nu_0\left(\Delta^{\mathrm{hor}}\right)} \right)^{\frac{2}{d+2}} \tag{5.15}$$

for the eigenvalues λ_j of Δ^{hor}, where $\nu_0(\Delta^{\mathrm{hor}})$ is a constant depending on the dimension d of the horizontal distribution and the term $A_0(\Delta^{\mathrm{hor}})$ in the heat trace expansion (5.14) *of Δ^{hor}.* □

Note that the above theorem just gives a qualitative statement about the growth of the eigenvalues, which suffices to detect the metric dimension of a spectral triple constructed from a first order hypoelliptic and self-adjoint operator on M. We will carry out this construction in Section 7.1. But first of all, we want to return to the horizontal Dirac operator and show that this theory cannot be applied to D^H, since D^H is not hypoelliptic.

Chapter 6

Hypoellipticity of Graded Differential Operators

As we saw in the last chapter, the condition of hypoellipticity allows us to define complex powers of a self-adjoint Heisenberg pseudodifferential operator within the Heisenberg calculus. Since operators of negative Heisenberg order on a compact Carnot manifold are compact, one can argue that a hypoelliptic graded differential operator has a compact resolvent. This means that if a horizontal Dirac operator D^H acting on a compact Carnot manifold M is hypoelliptic, condition (ii) for a spectral triple will be fulfilled on $(C(M), L^2(M), D^H)$. But in Chapter 4 we already constructed a horizontal Dirac operator on an arbitrary compact Carnot nilmanifold which does not have a compact resolvent.

The aim of this chapter is to show a generalization of the results of Chapter 4 in the setting of pseudodifferential calculus: We show that any horizontal Dirac operator on a Carnot manifold is not hypoelliptic. From this we will draw the conclusion that a horizontal Dirac operator does not have a compact resolvent, and hence does not furnish a spectral triple.

We start this chapter by reviewing some well-known hypoellipticity criteria, starting with Hörmanders *Sum-of-Squares Theorem* and leading to the *Rockland Condition*, which states an equivalence between hypoellipticity of a graded differential operator and the non-degeneracy in the irreducible representations of its associated Lie algebra. Then we will draw special attention to the situation where this Lie algebra is a Heisenberg algebra, since in this situation one has a good classification for the hypoellipticity of horizontal Laplacians. This will pay off, because in the second section we develop a criterion to exclude hypoellipticity of a graded differential operator by reducing the case to the co-dimension 1 case. Similar to the argument given in Chapter 4 for a specific example on compact Carnot nilmanifolds, we will make use of the submersions between graded nilpotent Lie algebras introduced in Section 2.4.

Finally, in Section 6.3 we prove that any horizontal Dirac operator is not hypoelliptic. This follows quickly from the previous work by considering its square.

6.1 Some Classical Theorems

The development of hypoellipticity criteria for certain differential operators was a great matter in the 1970s and 1980s, and there are some celebrated results. The origin of all these criteria is the famous *sum-of-squares theorem* by Lars Hörmander (see [Hoe67]).

Theorem 6.1.1
Let $X_0, X_1, \dots, X_d$ be homogeneous vector fields on $\mathbb{R}^n$ with real C^∞-coefficients on an open set $\Omega \subset \mathbb{R}^n$ and $c \in C^\infty(\Omega)$ real valued. Then the operator

$$P = \sum_{j=1}^{d} X_j^2 + X_0 + c$$

is hypoelliptic, if among the operators X_j and all their commutators there exist n which are linearly independent at any given point in Ω. □

The problem is that this theorem only works for vector fields with real coefficients, so it will not apply to our case of the square of a horizontal Dirac operator where the Clifford action causes complex coefficients. During the following years there were several generalizations of Hörmander's theorem, for example by Kohn ([Koh70] and [Koh71]) or Rothschild and Stein ([RS77]). Rothschild and Stein developed a close-to-complete characterization for the hypoellipticity of horizontal Laplacians of the form

$$\Delta_{RS}^{\text{hor}} = -\sum_{j=1}^{d} X_j^2 - \frac{i}{2} \sum_{j,k=1}^{d} b_{jk}[X_j, X_k], \tag{6.1}$$

where the X_j are homogeneous vector fields on $\mathbb{R}^n$ and $b = (b_{jk}) \in \text{Skew}_{d\times d}(\mathbb{R})$ is assumed to be a real skew-symmetric matrix. Then Rothschild and Stein proved the following theorem in a slightly more general version (see [RS77], Theorem 1' and Theorem 2):

Theorem 6.1.2
Consider the space

$$\mathcal{R} := \left\{ r = (r_{jk}) \in \text{Skew}_{d\times d}(\mathbb{R}) : \sum_{j,k=1}^{d} r_{jk}[X_j, X_k] = 0 \right\}, \tag{6.2}$$

and its orthonormal complement $\mathcal{R}^\perp$ with respect to the inner product $(s_1, s_2) = -\text{tr}(s_1 s_2)$ on $\text{Skew}_{d\times d}(\mathbb{R})$.

(i) If

$$\sup_{\rho \in \mathcal{R}^\perp,\ \|\rho\|_1 \le 1} |\text{tr}\,(b\rho)| < 1,$$

then Δ_{RS}^{hor} from (6.1) is hypoelliptic.

(ii) *Assume that the Lie algebra* $\mathfrak{g}$ *spanned by* $\{X_1, \ldots, X_d\}$ *is graded, which means* $\mathfrak{g} = \bigoplus_{S=1}^{R} V_R$ *such that* $[V_S, V_T] \subset V_{S+T}$ *if* $S + T \leq R$ *and* $[V_S, V_T] = 0$ *if* $S + T > R$ *(see also Section 2.2). Then if*

$$\sup_{\rho \in \mathcal{R}^\perp,\ \|\rho\|_1 \leq 1} |\mathrm{tr}\,(b\rho)| \geq 1$$

and if the algebra $\mathfrak{g}_2 := \mathfrak{g} \Big/ \bigoplus_{S=3}^{R} V_S$ *is not the Lie algebra of a Heisenberg group* $\mathbb{H}^{2m+1}$, $\Delta_{RS}^{\mathrm{hor}}$ *from* (6.1) *is not hypoelliptic.*

□

Remark: For $\exp \mathfrak{g}_2 \cong \mathbb{H}^{2m+1}$, the situation is more involved: It can be shown that there are situations where $\mathfrak{g}_2$ is the Lie algebra of a Heisenberg group, in which the operator $\Delta_{RS}^{\mathrm{hor}}$ is hypoelliptic, even though

$$\sup_{\rho \in \mathcal{R}^\perp,\ \|\rho\|_1 \leq 1} |\mathrm{tr}\,(b\rho)| = 1.$$

For a more detailed treatment of the co-dimension 1 case, we refer to Theorem 6.1.4 below. ◁

The idea for the proof of the second statement of this theorem is to reduce the situation to the case $R = 2$, and to describe the operator Δ^{hor} on $\mathfrak{g}_2$ via irreducible unitary representation of $\mathfrak{g}_2$. Doing this, it is possible to write down a function which is not C^∞ but belongs to the kernel of Δ^{hor}, which is a contradiction to the hypoellipticity of the operator.

Remember that we already used techniques from representation theory for the case of horizontal Dirac operators on compact Carnot nilmanifolds in Chapter 4, and it turned out that there is indeed a close connection between the hypoellipticity of graded differential operators and the representation theory of the underlying Carnot group. We will describe this connection now.

As we noted in Section 2.2, a graded differential operator is an operator of the form

$$D = p\,(X_1, \ldots, X_d, X_{2,1}, \ldots, X_{R,d_R}),$$

where p is a polynomial with matrix-valued C^∞ coefficients and where $\{X_{S,1}, \ldots, X_{S,d_S}\}$ is a frame for the vector space V_S appearing in the grading $\mathfrak{g} = V_1 \oplus \ldots \oplus V_R$ of a graded Lie algebra $\mathfrak{g}$. If the coefficients of D are constant, This suggests to consider D as an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, the exponential mapping is an isomorphism from $\mathfrak{g}$ onto its Lie group $\mathbb{G}$ which is a Carnot group. Now let π be an irreducible unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, consisting of L^2-functions. Since π gives rise to an irreducible, unitary representation of $\mathfrak{g}$ on $\mathcal{H}$ via

$$(d\pi(X)\varphi)(x) = \frac{d}{dt}\pi(\exp t_X)\varphi(x)\Big|_{t=0},$$

we obtain a representation $d\pi(D)$ of the graded differential operator D.

Let $\hat{\mathbb{G}}$ denote the unitary dual of $\mathbb{G}$, which is the space of all irreducible unitary representations of $\mathbb{G}$. Using the above concept, there is a representation theoretic criterion which characterizes the hypoellipticity of graded differential operators completely. We will now state this criterion, which is called the *Rockland condition* since it was developed by Charles Rockland [Roc78] for Heisenberg algebras. B. Helffer and J. Nourrigat extended the Rockland condition to the case of arbitrary graded nilpotent Lie algebras (see [HN79] or also [Rot83]).

Theorem 6.1.3
Let $\mathfrak{g}$ be a graded nilpotent Lie algebra, $\mathbb{G} = \exp(\mathfrak{g})$, and let $L \in \mathcal{U}(\mathfrak{g})$ be a graded differential operator which is homogeneous of degree m. Then L is hypoelliptic if and only if $\pi(L)$ is injective for all nontrivial $\pi \in \hat{\mathbb{G}}$. □

Although there is a way to determine the irreducible unitary representations of an arbitrary nilpotent Lie group (see e.g. [CG89], we also mentioned this at the end of Section 4.4), this is a quite difficult task for specific examples. Hence we follow the same approach as in Chapter 4: For the co-dimension 1 case, it is not too difficult to formulate hypoellipticity criteria deduced from the Rockland condition. Then in the next section, we will see how this case enables us to make more general statements.

For the rest of this section, we assume $n = d + 1$ and $\mathfrak{g} \cong V_1 \oplus V_2$ with $d = \dim V_1$. Let $\{X_1, \dots, X_{d+1}\}$ be a frame for $T\mathbb{R}^n$ such that $V_1 = \operatorname{span}\{X_1, \dots, X_d\}$ and $V_2 = \operatorname{span}\{X_{d+1}\}$ for the representation of this Lie algebra as vector fields on $\mathbb{R}^n$ from Proposition 2.2.6. We want to check horizontal Laplacians of the form

$$\Delta^{\text{hor}} := -\sum_{j=1}^{d} X_j^2 - iAX_{d+1} + O_H(1), \tag{6.3}$$

acting on a vector bundle E of rank $p \in \mathbb{N}$ over an open subset $\Omega \subset \mathbb{R}^{d+1}$ with $A \in \operatorname{Mat}_{p\times p}(\mathbb{C})$ for hypoellipticity. Here, the term $O_H(1)$ denotes a graded differential operator of order smaller or equal to 1 (which means it is a differential operator of order 1 only depending on the horizontal vector fields $X_1, \dots, X_d$). To formulate the criterion we recall the notion of the Levi form from Chapter 2.3 (see Definition 2.3.2), which is the bilinear form

$$\mathcal{L} : V_1 \times V_1 \to V_2, \quad (Y_1, Y_2) \mapsto [Y_1, Y_2] \mod V_1.$$

For $\mathcal{L}(X_j, X_k) = L_{ik}X_{d+1}$ with $L_{ik} \in \mathbb{R}$, we denote by $L = (L_{jk})$ the antisymmetric matrix describing $\mathcal{L}$.

The following result states that the hypoellipticity of Δ^{hor} from (6.3) only depends on how the eigenvalues of A behave in comparison with the eigenvalues of $\mathcal{L}$. It is well known and can be found at various places in the literature (see e.g. [Pon08] or [BG84]), but because of its importance for our future arguments we give a proof, orientated towards the one given in [Pon08].

Theorem 6.1.4
Let Δ^{hor} be the horizontal Laplacian given by (6.3) *with corresponding graded nilpotent Lie algebra $\mathfrak{g} = \operatorname{span}\{X_1, \dots, X_d\} \oplus \operatorname{span}\{X_{d+1}\}$, Carnot group $\mathbb{G} = \exp \mathfrak{g}$ and Levi form $\mathcal{L}$, which is described by the Levi matrix $L \in \operatorname{Skew}_{d\times d}(\mathbb{R})$ corresponding to this basis. The non-zero eigenvalues of $\mathcal{L}$ are denoted by $\pm i\lambda_1, \dots, \pm i\lambda_m$ (including multiplicity) with $\lambda_j > 0$ for all $j \in \{1, \dots, m\}$, where $2m \leq d$ is the rank of $\mathcal{L}$..*
Then the hypoellipticity of Δ^{hor} can be characterized as follows:

(i) If $\mathbb{G} \cong \mathbb{H}^{2m+1}$ with $\mathbb{H}^{2m+1}$ the $(2m+1)$-dimensional Heisenberg group (with $d = 2m$), then Δ^{hor} is hypoelliptic if and only if no eigenvalue of A belongs to the set

$$\Lambda := \left\{ \pm \left(\frac{1}{2} \|L\|_1 + 2 \sum_{1\leq j\leq m} \alpha_j |\lambda_j| \right) : \alpha_j \in \mathbb{N}^m \right\}. \tag{6.4}$$

(ii) If $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ with $2m < d$, then Δ^{hor} is hypoelliptic if and only if no eigenvalue of A belongs to the set

$$\Lambda := \left(-\infty, -\frac{1}{2} \|L\|_1 \right] \cup \left[\frac{1}{2} \|L\|_1, \infty \right). \tag{6.5}$$

In both cases, $\|L\|_1 = \operatorname{tr}(|L|) = 2\sum_{j=1}^m |\lambda_j|$ denotes the trace norm of L.

Proof: Since Δ^{hor} is a horizontal Laplacian, by Theorem 5.1.13 its hypoellipticity is equivalent to the hypoellipticity of its model operator, which is in this case the homogeneous horizontal Laplacian

$$\Delta^{\text{hor}}_{\text{mod}} = -\sum_{j=1}^{d} X_j^2 - iAX_{d+1}. \tag{6.6}$$

Hence we only have to check the operator (6.6) for hypoellipticity to prove the theorem, which can be done by the Rockland condition (see Theorem 6.1.3).
First of all, for every Carnot group $\mathbb{G}$ of co-dimension 1 we have $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$, where $2m = \operatorname{Rank} L \leq d$, by Proposition 2.3.3. By the same proposition, there exists an orthonormal basis transformation of $V_1 = \operatorname{span}\{X_1, \dots, X_d\} \cong \mathbb{R}^d$ such that after this transformation we have

$$L = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.7}$$

for the Levi matrix of $\mathcal{L}$, where D is a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_m > 0$. Since an orthonormal change of the frame of V_1 does not change the form of the horizontal Laplacian (6.6), as one can see after a small calculation, we can assume (6.7) to be the matrix of the Levi form $\mathcal{L}$ of $\mathbb{G}$. But this gives the commutator relations

$$[X_j, X_{j+m}] = \lambda_j X_{d+1} \quad \text{and} \quad [X_{j+m}, X_j] = -\lambda_j X_{d+1}$$

if $1 \leq j \leq m$, while all the other commutators are zero. For $\lambda_1 = \ldots = \lambda_m = 1$ these are exactly the commutator relations for the Lie algebra of $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$. Thus, the isomorphism $\phi : \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m} \to \mathbb{G}$ is coordinate-wise defined by

$$\phi(x_j) = \begin{cases} \sqrt{\lambda_j}\, x_j & \text{for } 1 \leq j \leq m \\ \sqrt{\lambda_{j-m}}\, x_j & \text{for } m+1 \leq j \leq 2m\ , \\ x_j & \text{for } j > 2m \end{cases} \tag{6.8}$$

where we work on exponential coordinates of the Carnot groups.

The next step is to show that the general case of $\Delta^{\text{hor}}_{\text{mod}}$ acting on a vector bundle E of rank p can be restricted to the scalar case: We can choose (point-wise) a basis of E such that in this basis the matrix A in (6.6) is given by

$$A = \begin{pmatrix} \mu_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mu_p \end{pmatrix},$$

where $\mu_1, \ldots, \mu_p$ are the eigenvalues of A. With respect to this basis the operator $\Delta^{\text{hor}}_{\text{mod}}$ takes the form

$$\Delta^{\text{hor}}_{\text{mod}} = \begin{pmatrix} \Delta^{\text{hor}}_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \Delta^{\text{hor}}_p \end{pmatrix}, \tag{6.9}$$

where for each j with $1 \leq j \leq p$ we have a horizontal Laplacian

$$\Delta^{\text{hor}}_j = -\sum_{j=1}^{d} X_j^2 - i\mu_j X_{d+1}$$

acting on scalar valued functions. Now it is obvious that the operator (6.9) fulfills the Rockland condition from Theorem 6.1.3 if and only if each of the scalar operators Δ^{hor}_j does (because otherwise the matrix would not be invertible which would be a contradiction to the injectivity of the corresponding irreducible representations). Therefore we can restrict our considerations to the scalar case, working with the eigenvalues $\mu_1, \ldots, \mu_p$ of A. From now on, we will assume A is scalar.

After the above simplifications, our task is to check the Rockland condition for the scalar case with Levi form of $\mathbb{G}$ given by (6.7). We start with the situation where $2m = d$ (which means $\mathbb{G} \cong \mathbb{H}^{2m+1}$)). For the Lie algebra of the Heisenberg group the representation theory is well known: Up to equivalence, the nontrivial irreducible representations of the operators $X_1, \ldots, X_{d+1}$, considered as basis vectors of the Heisenberg algebra $\mathfrak{h}_{2m+1}$ are of two types (see e.g. [Pon08], (3.3.5) - (3.3.7), or [Fol89]; compare also to the proof of Theorem 4.3.4):

(i) The infinite dimensional representations $\pi_t : \mathbb{H}^{2m+1} \to \mathcal{U}(L^2(\mathbb{R}^m))$ of $\mathbb{H}^{2m+1}$, parametrized by $t \in \mathbb{R} \setminus \{0\}$, give rise to the representations

$$\begin{aligned} d\pi_t(X_j)f(\xi) &= |t|\partial_{\xi_j} f(\xi) \quad \text{for } 1 \le j \le m \\ d\pi_t(X_{m+j})f(\xi) &= it\xi_j f(\xi) \quad \text{for } 1 \le j \le m \\ d\pi_t(X_{2m+1})f(\xi) &= it|t|f(\xi). \end{aligned}$$

(ii) The one dimensional representations $\pi_\zeta : \mathbb{H}^{2m+1} \to \mathbb{C}$ of $\mathbb{H}^{2m+1}$, parametrized by $\zeta \in \mathbb{R}^{2m} \setminus \{0\}$, give rise to the representations

$$\begin{aligned} d\pi_\zeta(X_j) &= i\zeta_j \quad \text{for } 1 \le j \le 2m \\ d\pi_\zeta(X_{2m+1}) &= 0. \end{aligned}$$

Under the Lie group isomorphism $\phi : \mathbb{H}^{2m+1} \to \mathbb{G}$ given by (6.8), the representations of the basis vectors of the Lie algebra $\mathfrak{g}$ we are interested in are given by the above representations applied to $\sqrt{\lambda_j} X_j$ and $\sqrt{\lambda_j} X_{m+j}$ for $1 \le j \le m$. Hence all the irreducible representation for Δ^{hor} are

$$d\pi_{t,\xi}\left(\Delta^{\text{hor}}_{\text{mod}}\right) = -\sum_{j=1}^{m} \lambda_j \left(|t|^2 \partial^2_{\xi_j} - t^2 \xi_j^2\right) + At|t| \quad \text{for } t \in \mathbb{R} \setminus \{0\} \tag{6.10}$$

and

$$d\pi_\zeta\left(\Delta^{\text{hor}}_{\text{mod}}\right) = -\sum_{j=1}^{2m} \zeta_j^2 \quad \text{for } \xi \in \mathbb{R}^{2m} \setminus \{0\}. \tag{6.11}$$

Obviously (6.11) is injective for any $\zeta \in \mathbb{R}^{2m} \setminus \{0\}$, so we only have to check the operators given by (6.10) for the Rockland condition. For these operators, their injectivity is equivalent to the injectivity of the operators

$$-\sum_{j=1}^{m} \lambda_j \left(\partial^2_{\xi_j} - \xi_j^2\right) \pm A. \tag{6.12}$$

It is known that the spectrum of the harmonic oscillator $\sum_{j=1}^{m} \lambda_j \left(\partial^2_{\xi_j} - \xi_j^2\right)$ is exactly the set $\sum_{j=1}^{m} \lambda_j (1 + 2\mathbb{N})$ (see e.g. [Pon08], Section 3.4, or [Fol89]; compare also to the proof of Theorem 4.3.4), and therefore the invertibility of (6.12) is equivalent to the condition

$$A \notin \left\{ \pm \sum_{j=1}^{m} \lambda_j (1 + 2\mathbb{N}) \right\} = \left\{ \pm \left(\frac{1}{2} \|L\|_1 + 2 \sum_{j=1}^{m} \alpha_j |\lambda_j| \right) : \alpha_j \in \mathbb{N}^m \right\}.$$

But this shows statement (i) of the theorem for a scalar $A \in \mathbb{C}$, and because of the above simplifications statement (i) is also proved for $A \in \text{Mat}_{p\times p}(\mathbb{C})$.

The second statement where $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ with $2m < d$ can be proved via the Rockland condition in a similar way: One can use that the irreducible representations of the abelian group $\mathbb{R}^{d-2m}$ are simply the trivial ones. Note that for the case $A \in \mathbb{R}$ the second statement can also be proved using Theorem 6.1.2, if we set

$$b_{j,k} = \begin{cases} \frac{|\lambda_j| A}{\sum_{j=1}^m |\lambda_j|} & \text{for } 1 \leq j \leq m, \ k = m+j \\ -\frac{|\lambda_k| A}{\sum_{k=1}^m |\lambda_k|} & \text{for } 1 \leq k \leq m, \ j = k+m \\ 0, & \text{otherwise} \end{cases}$$

in (6.1). This works for both directions of the equivalence we want to prove since in this situation $\mathbb{G}$ is not a Heisenberg group. By a more general version of the Theorem by Rothschild and Stein (see [RS77]), the case $A = B + iC \in \mathbb{C}$ is also covered. □

Remark: If we look at the proof of this theorem, we note that the strategy is very similar to the strategy of proving Theorem 4.3.4: In both cases, we make use of the fact that we can describe our operator using the irreducible representations of $\mathbb{G}$. While in Theorem 4.3.4 we deduced that the horizontal pull-back Dirac operator on a compact Carnot nilmanifold has infinite dimensional eigenspaces, we will deduce from this theorem that the square of any horizontal Dirac operator (and hence the operator D^H itself) on a Heisenberg manifold is not hypoelliptic in Section 6.3. ◁

Before we close this section, we remark that of course one can combine Theorem 6.1.4 with Theorem 6.1.2 to deduce a better characterization of hypoellipticity for horizontal Laplacians of the form (6.1), acting on scalar-valued functions. Since this is straightforward, we will not write it down here.

6.2 A Reduction Criterion for Non-hypoellipticity

In this section, we want to develop a criterion which we can use to show that any horizontal Dirac operator on a Carnot manifold is not hypoelliptic. Note that Theorem 6.1.2 is not very practical for this situation: First of all, it has to be extended to the case of vector bundles, and even after doing this is seems to be a quite complicated task to work with the spaces $\mathcal{R}$ from (6.2), which depends heavily on the concrete structure of the Carnot group, in a general setting. But on the other hand Theorem 6.1.4 can be applied easily to the square of such a horizontal Dirac operator, as long as we are in the case of Heisenberg manifolds.

Thus the idea we want to follow is to reduce the general problem of excluding hypoellipticity on an arbitrary Carnot manifold to the co-dimension 1 case. We have already seen that this approach works for the case of horizontal pull-back Dirac operators on compact Carnot nilmanifolds in Chapter 4: It was possible to lift the infinite dimensional eigenspaces of D^H to a higher dimensional nilmanifold. In Chapter 4 we used the submersions between the corresponding Carnot group introduced in Section 2.4, and for our current (and more general) situation this strategy will also work.

Like in Section 6.1, we consider a graded differential operator with constant coefficients

$$D = D(X_1, \ldots, X_n) \in \mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}^p$$

as an element of the universal enveloping algebra of a graded Lie algebra $\mathfrak{g} = \bigoplus_{S=1}^{R} V_S$ belonging to a Carnot group $\mathbb{G}$ of step R. The (vector-valued) functions D^H is acting on are supposed to be defined on a Carnot group $\mathbb{G}$, realized as a non-abelian group structure on $\mathbb{R}^n$. We remember the orthogonal projection

$$\mathrm{pr} : \mathfrak{g} \to \tilde{V}, \quad v \mapsto v \mod \tilde{V}^{\perp}, \tag{6.13}$$

from Section 2.4, where

$$\tilde{V} := \bigoplus_{S=1}^{M-1} V_S \oplus \tilde{V}_M$$

for a linear subspace $\tilde{V}_M \subset V_M$ for some $1 \leq M \leq R$. By Proposition 2.4.1, $\tilde{\mathfrak{g}} = \mathrm{pr}(\mathfrak{g})$ has the structure of a nilpotent graded Lie algebra which is induced by $\mathfrak{g}$. After applying the map pr to the elements of $\mathfrak{g}$, we get a new differential operator

$$\mathrm{pr}(D) := D\left(\mathrm{pr}(X_1), \ldots, \mathrm{pr}(X_n)\right) \in \mathcal{U}(\tilde{\mathfrak{g}}) \otimes \mathbb{C}^q$$

considered as an element of the universal enveloping algebra of $\tilde{\mathfrak{g}}$. Note that of course the operator $\mathrm{pr}(D)$ is supposed to act on the (compared to $\mathbb{G}$ lower dimensional) Carnot group $\tilde{G} = \exp(\tilde{\mathfrak{g}})$, realized on the Euclidean space $\mathbb{R}^{\dim \tilde{\mathfrak{g}}}$.

By Proposition 2.4.1 the projection pr gives rise to a submersive Lie group homomorphism

$$\psi := \exp_{\tilde{\mathbb{G}}} \circ \; \mathrm{pr} \; \circ \exp_{\mathbb{G}^{-1}} : \mathbb{G} \to \tilde{\mathbb{G}}$$

between the Carnot groups $\mathbb{G} = \exp \mathfrak{g}$ and $\tilde{\mathbb{G}} = \exp \tilde{\mathfrak{g}}$. The operator $\mathrm{pr}(D)$ is supposed to act on vector-valued functions living on the Carnot group $\tilde{\mathbb{G}}$. Since ψ is a submersion, a function or distribution on $\tilde{\mathbb{G}}$ can be extended to a function or distribution on $\mathbb{G}$ via pullback along ψ. This observation leads us to the idea to deduce that D is not hypoelliptic if $\mathrm{pr}(D)$ is not hypoelliptic for some projection of the type (6.13), as we will do via the following theorem.

Theorem 6.2.1
Let $\mathbb{G}$ be a Carnot group with grading $\mathfrak{g} = \bigoplus_{S=1}^{R} V_S$ of its Lie algebra, and let $D \in \mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}^p$ be a graded differential operator with constant coefficients.

Assume that for some $2 \leq M \leq R$ there is a linear space $\tilde{V}_M \subset V_M$ such that the corresponding orthonormal projection

$$\mathrm{pr} : \mathfrak{g} \to \tilde{\mathfrak{g}} := \tilde{V} = \bigoplus_{S=1}^{M-1} V_S \oplus \tilde{V}_M, \quad X \mapsto \tilde{X} = \mathrm{pr}(X)$$

defined by (6.13) *furnishes a graded differential operator*

$$\mathrm{pr}(D) = D(\tilde{X}_1, \ldots, \tilde{X}_n) \in \mathcal{U}(\tilde{\mathfrak{g}}) \otimes \mathbb{C}^q,$$

which is not hypoelliptic. Then D is not hypoelliptic.

Proof: We write down the argument for the case $p = 1$ where D is acting on complex-valued functions functions; the vector-valued case works analogously. Assume the operator $\tilde{D}$ constructed above is not hypoelliptic, which means that there exists a distribution $\tilde{\varphi} \in \mathcal{D}'(\tilde{\mathbb{G}})$ such that $\tilde{\varphi} \notin C^\infty(\tilde{\mathbb{G}})$, but $\tilde{D}\tilde{\varphi} \in C^\infty(\tilde{\mathbb{G}})$ for the Carnot group $\tilde{\mathbb{G}} = \exp(\tilde{\mathfrak{g}})$. We have to find a $\varphi \in \mathcal{D}'(\mathbb{G})$ which is not a C^∞-function, such that $D\varphi \in C^\infty(\mathbb{G})$.

We consider the submersive Lie group homomorphism

$$\psi := \exp_{\tilde{\mathbb{G}}} \circ \mathrm{pr} \ \circ \exp_{\mathbb{G}}^{-1} : \mathbb{G} \to \tilde{\mathbb{G}} \tag{6.14}$$

arising from pr, see (6.2). In addition we use the projection $\mathrm{pr}^\perp : \mathfrak{g} \to \mathrm{Ker}\,(\mathrm{pr})$, which is given by $\mathrm{pr}^\perp(v) = v \mod \tilde{V}$, of $\mathfrak{g}$ onto the kernel of pr. Then from Proposition 2.4.1 we get an isomorphism

$$\alpha : \mathbb{G} \xrightarrow{\sim} \tilde{\mathbb{G}} \times N, \quad x \mapsto (\tilde{x}, n) = (\psi(x), \nu(x)) \tag{6.15}$$

from ψ, where $N = \mathrm{Ker}(\psi)$ and

$$\nu := \exp_N \circ \mathrm{pr}^\perp \ \circ \exp_{\mathbb{G}}^{-1} : \mathbb{G} \to N$$

is the projection onto the kernel of ψ, arising from $\mathrm{pr}^\perp$. From now on, we will use the coordinates $(\tilde{x}, n)$ on $\mathbb{G} \cong \tilde{\mathbb{G}} \times N$ which are induced by the isomorphism α from (6.15).

If $x = \exp_{\mathbb{G}} X$ are exponential coordinates (corresponding to a vector field $X \in \mathfrak{g}$) on $\mathbb{G}$, then we have

$$\alpha\left(\exp_{\mathbb{G}} X\right) = \left(\exp_{\tilde{\mathbb{G}}} \mathrm{pr}(X), \exp_N \mathrm{pr}^\perp(X)\right), \tag{6.16}$$

as one sees immediately from the definition of α via pr. But from these exponential coordinates we see that for the differential of α we have

$$D\alpha(X) = \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right),$$

and hence we see how the application of a vector field $X \in \mathfrak{g}$ to a function $f \in C^\infty(\mathbb{G})$ carries over to the push-forward $\alpha_* f$ of f on $\tilde{\mathbb{G}} \times N$: We have

$$Xf(x) = \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) \alpha_* f(\tilde{x}, n), \tag{6.17}$$

where $(\tilde{x}, n) = \alpha(x)$. From now on, we will consider functions and distributions on $\mathbb{G}$ as functions and distributions on $\tilde{\mathbb{G}} \times N$, where the identification is given via the isomorphism α.

After these preparations, we are ready to prove the theorem. Let $\tilde{\varphi} \in \mathcal{D}'(\tilde{\mathbb{G}})$ be given such that $\tilde{\varphi} \notin C^\infty(\tilde{\mathbb{G}})$, but $\mathrm{pr}(D)\tilde{\varphi} \in C^\infty(\tilde{\mathbb{G}})$ for our graded differential operator D. For a test function $f \in C_c^\infty(\tilde{\mathbb{G}} \times N)$ we can define the push-forward $\psi_* f$ of f along ψ via

$$\psi_* f(\tilde{x}) := \int_N f(\tilde{x}, n) d\mu(n), \tag{6.18}$$

where $\mu(n)$ is the Haar measure on the nilpotent Lie group N. Note that the expression (6.18) is well-defined since N is a normal subgroup of $\mathbb{G}$ as the kernel of the Lie group homomorphism ψ, and that $\psi_* f$ is a C^∞-function with compact support on $\tilde{\mathbb{G}}$ since this is the case for f on $\tilde{\mathbb{G}} \times N$. But this means we get a distribution $\varphi \in \mathcal{D}'(\tilde{\mathbb{G}} \times N)$ via

$$\langle \varphi, f \rangle := \langle \tilde{\varphi}, \psi_* f \rangle. \tag{6.19}$$

Obviously, since we assumed $\tilde{\varphi} \notin C^\infty(\tilde{\mathbb{G}})$, we have $\varphi \notin C^\infty(\tilde{\mathbb{G}} \times N)$. We will show now that $D\varphi \in C^\infty(\mathbb{G})$, from which the theorem will be proved.

If we apply a vector field $X \in \mathfrak{g}$ to the distribution, which means by (6.17) to apply $(\mathrm{pr}(X), \mathrm{pr}^\perp(X)) \in \tilde{\mathfrak{g}} \times \mathfrak{n}$ to $\varphi \in \mathcal{D}'(\tilde{\mathbb{G}} \times N)$, we get from (6.19)

$$\begin{aligned} \left\langle \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) \varphi, f \right\rangle &= \left\langle \varphi, \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) f \right\rangle \\ &= \left\langle \tilde{\varphi}, \psi_* \left[\left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) f\right] \right\rangle \end{aligned} \tag{6.20}$$

for any test function $f \in C_c^\infty(\tilde{\mathbb{G}} \times N)$. Now for the push-forward of $(\mathrm{pr}(X), \mathrm{pr}^\perp(X))f$ to $\tilde{\mathbb{G}}$ via ψ we calculate, using (6.18),

$$\begin{aligned} \psi_* \left[\left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) f\right](\tilde{x}) &= \int_N \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) f(\tilde{x}, n) d\mu(n) \\ &= \int_N \frac{d}{dt} f\left(\tilde{x} \cdot_{\tilde{\mathbb{G}}} \exp_{\tilde{\mathbb{G}}} t\mathrm{pr}(X),\ n \cdot_N \exp_N t\mathrm{pr}^\perp(X)\right)\Big|_{t=0} d\mu(n) \\ &= \frac{d}{dt} \int_N f\left(\tilde{x} \cdot_{\tilde{\mathbb{G}}} \exp_{\tilde{\mathbb{G}}} t\mathrm{pr}(X),\ n \cdot_N \exp_N t\mathrm{pr}^\perp(X)\right) d\mu(n)\Big|_{t=0} \\ &= \frac{d}{dt} \int_N f\left(\tilde{x} \cdot_{\tilde{\mathbb{G}}} \exp_{\tilde{\mathbb{G}}} t\mathrm{pr}(X),\ n\right) d\mu(n)\Big|_{t=0} \\ &= \frac{d}{dt} \psi_* f\left(\tilde{x} \cdot_{\tilde{\mathbb{G}}} \exp_{\tilde{\mathbb{G}}} t\mathrm{pr}(X)\right)\Big|_{t=0} \\ &= \mathrm{pr}(X)\psi_* f(\tilde{x}). \end{aligned}$$

Here we are allowed to interchange differentiation and integration since f is a test function, and the forth equation is true since the Haar measure is translation invariant. The same calculation works if we apply another vector field Y to $X\varphi$, which provides

$$\psi_* \left[\left(\mathrm{pr}(Y), \mathrm{pr}^\perp(Y)\right) \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) f\right](\tilde{x}) = \mathrm{pr}(Y)\mathrm{pr}(X)\psi_* f(\tilde{x}), \tag{6.21}$$

and we can go on inductively for higher order applications of vector fields. Now we can apply these identities to (6.20) to find that

$$\left\langle \left(\mathrm{pr}(X), \mathrm{pr}^\perp(X)\right) \varphi, f \right\rangle = \langle \tilde{\varphi}, \mathrm{pr}(X)\psi_* f \rangle = \langle \mathrm{pr}(X)\tilde{\varphi}, \psi_* f \rangle,$$

and because of (6.21) the same identity is true if we apply further vector fields to $X\varphi$. But this means that for any graded differential operator D we get

$$\langle D\varphi, f \rangle = \langle \mathrm{pr}(D)\tilde{\varphi}, \psi_* f \rangle. \tag{6.22}$$

Now we assumed $\mathrm{pr}(D)\tilde{\varphi} \in C^\infty(\tilde{\mathbb{G}})$ to be a smooth function. But from this it follows immediately from (6.22) that $D\varphi$ is a smooth function on $\tilde{\mathbb{G}} \times N$: We have by this equation and by the definition of $\psi_* f$

$$\begin{aligned}\langle D\varphi, f\rangle &= \langle \mathrm{pr}(D)\tilde{\varphi}, \psi_* f\rangle \\ &= \int_{\tilde{\mathbb{G}}} \mathrm{pr}(D)\tilde{\varphi}(\tilde{x}) \int_N f(\tilde{x}, n)\ d\mu(n) d\mu(\tilde{x}) \\ &= \int_{\tilde{\mathbb{G}}\times N} \mathrm{pr}(D)\tilde{\varphi}\,(\tilde{x})\, f(\tilde{x}, n)\ d\mu(n) d\mu(\tilde{x}),\end{aligned}$$

where $d\mu(n)$ and $d\mu(\tilde{x})$ are the corresponding Haar measures on N and $\tilde{\mathbb{G}}$. But this shows that we must have

$$D\varphi(\tilde{x}, n) = \mathrm{pr}(D)\tilde{\varphi}\left(\psi(\tilde{x}, n)\right),$$

and since $\mathrm{pr}(D)\tilde{\varphi}$ is smooth this must also be the case for $D\varphi$.

Altogether we have found a distribution φ on $\mathbb{G} \cong \tilde{\mathbb{G}} \times N$, for which we have $D\varphi \in C^\infty(\mathbb{G})$, but $\varphi \notin C^\infty(\mathbb{G})$, whenever there is a projection pr of the type (6.13) such that $\mathrm{pr}(D)$ is not hypoelliptic. But this shows that the graded differential operator D is not hypoelliptic in this situation, such that the theorem is proved. □

Remark 6.2.2

We did the proof of Theorem 6.2.1 in the very general setting of distributions. If in a more specific sense the operator $\mathrm{pr}(D)$ is assumed to be hypoelliptic in the sense that there is a function $\tilde{\varphi}$ on $\tilde{\mathbb{G}}$, which lies in the domain of D but which is not C^∞, such that $\mathrm{pr}(D)\tilde{\varphi} \in C^\infty(\tilde{\mathbb{G}})$, then the hypoellipticity of D is neglected by the function $\varphi(x) = \tilde{\varphi}(\psi(x))$ on $\mathbb{G}$. The calculation is straight forward, making use of the isomorphism $\mathbb{G} \cong \tilde{\mathbb{G}} \times N$ like in the proof of the above theorem.

Via the same calculation one can show that $\mathrm{pr}(D)\tilde{\varphi} = 0$ implies $D\varphi = 0$. But from this, we immediately get Theorem 4.4.2, which states the degeneracy of the horizontal pull-back operator on any compact Carnot nilmanifold (if we use the degeneracy in the Heisenberg case). This is no surprise, since the techniques of lifting an operator from a low-dimensional Carnot manifold to a higher dimensional one work very similar in both cases. In this way, the results from this chapter are the more general ones, while we get some extra features (like the detection of the metric dimension) from the specific examples in Chapter 4 which we were not able to derive in the general case. ◁

In the next section, we want to use Theorem 6.2.1 to show that any horizontal Dirac operator constructed in Section 3.2 is not hypoelliptic. The idea is to use Theorem 6.2.1 after reducing the operator to an operator acting on a Carnot group of co-dimension 1, which will be assumed to be of the type $\mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$. For this operator, the hypoellipticity can be neglected by Theorem 6.1.4.

Like before, let $d_S := \dim V_S$ be the dimension of V_S and let $\{X_{S,1}, \ldots, X_{S,d_S}\}$ be an orthonormal basis of V_S for the grading $\mathfrak{g} = \bigoplus_{S=1}^{R} V_S$ of $\mathfrak{g}$. Then a projection of $\mathfrak{g}$ onto a graded nilpotent Lie algebra of co-dimension 1 can be constructed as follows: For any $\nu \in \{1, \ldots, d_2\}$, consider the 1-dimensional linear subspace $\tilde{V}_{2,\nu} := \operatorname{span}\{X_{2,\nu}\}$ of V_2. We then define $\tilde{V}_\nu := V_1 \oplus \tilde{V}_{2,\nu} \subset \mathfrak{g}$, such that our orthogonal projection (6.13) becomes

$$\operatorname{pr}_\nu : \mathfrak{g} \to \tilde{V}_\nu, \quad v \mapsto v \mod \left(\tilde{V}_\nu\right)^{\perp}. \tag{6.23}$$

We can apply Proposition 2.4.1 to pr_ν, which gives us graded nilpotent Lie algebras $\mathfrak{g}_{2,\nu} := \operatorname{pr}_\nu(\mathfrak{g})$ and $\mathfrak{n}_\nu := \operatorname{Ker}(\operatorname{pr}_\nu)$ with corresponding Carnot groups $\mathbb{G}_{2,\nu}$ and N_ν.

Now let

$$\Delta^{\text{hor}} = -\sum_{j=1}^{d_1} X_j^2 - i \sum_{j<k} A_{j,k}[X_j, X_k] + O_H(1) \quad \in \mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}^q \tag{6.24}$$

be a horizontal Laplacian acting on a vector bundle E of rank p, which is considered as an element of the universal enveloping algebra of $\mathfrak{g}$ tensored with $\mathbb{C}^p$. The term $O_H(1)$ denotes a graded differential operator of order smaller or equal to 1 (which is a first order differential operator depending only on the vector fields $X_1, \ldots, X_d$). The $A_{j,k}$ are $(p \times p)$-matrices with complex valued entries.

Using the Lie algebra homomorphism pr_ν from (6.23) we can define a horizontal Laplacian $\tilde{\Delta}_\nu^{\text{hor}} \in \mathcal{U}(\mathfrak{g}_{2,\nu}) \otimes \mathbb{C}^p$, acting on a Carnot group with a horizontal distribution $\{\tilde{X}_1, \ldots, \tilde{X}_d\}$ of co-dimension 1: For $\tilde{X}_j := \operatorname{pr}_\nu(X_j)$ we get

$$\tilde{\Delta}_\nu^{\text{hor}} := \operatorname{pr}(\Delta^{\text{hor}}) = -\sum_{j=1}^{d_1} \tilde{X}_j^2 - i \sum_{j<k} A_{j,k} \left([\tilde{X}_j, \tilde{X}_k]\right) + O_H(1) \quad \in \mathcal{U}(\mathfrak{g}_{2,\nu}) \otimes \mathbb{C}^p.$$

After calculating the commutators we have an $A_\nu \in \operatorname{Mat}_{p\times p}(\mathbb{C})$ such that (for $d = d_1$)

$$\tilde{\Delta}_\nu^{\text{hor}} = -\sum_{j=1}^{d} \tilde{X}_j^2 - i A_\nu \tilde{X}_{d+\nu} + O_H(1), \tag{6.25}$$

with $\tilde{X}_{d+\nu} = \operatorname{pr}_\nu(X_{2,\nu})$, which is a horizontal Laplacian for which its hypoellipticity can be determined by Theorem 6.1.4.

This argument enables us to formulate non-hypoellipticity criteria involving the ν-Levi form introduced in Section 2.3: In the above situation, for $\nu \in \{1, \ldots, d_2\}$ this is the bilinear form

$$\mathcal{L}_\nu : V_1 \times V_1 \to \operatorname{span}\{X_{2,\nu}\}, \quad (Y_1, Y_2) \mapsto [Y_1, Y_2] \mod (\operatorname{span}\{X_{2,\nu}\})^{\perp}. \tag{6.26}$$

If we fix a basis for V_1, $\mathcal{L}_\nu$ is described by an antisymmetric matrix $L_\nu = \left(L_{jk}^{(\nu)}\right)$, such that $\mathcal{L}_\nu(X_j, X_k) = L_{jk} X_{2,\nu}$. Now we can use Theorem 6.2.1 together with Theorem 6.1.4 to formulate the following criterion.

Corollary 6.2.3
Let $\mathbb{G}$ be a Carnot group with corresponding Lie algebra $\mathfrak{g} = \bigoplus_{S=1}^{R} V_S$. For any number $\nu \in \{1, \ldots, \dim V_2\}$, we consider the projection pr_ν from (6.23) of $\mathfrak{g}$ onto the Lie algebra $\mathfrak{g}_{2,\nu} = \mathrm{pr}_\nu(\mathfrak{g})$ together with its corresponding Carnot group $\mathbb{G}_{2,\nu}$. Let $\Delta^{\mathrm{hor}} \in \mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}^p$ be a horizontal Laplacian of the type (6.24), such that the horizontal Laplacian $\tilde{\Delta}_\nu^{\mathrm{hor}} = \mathrm{pr}_\nu(\Delta^{\mathrm{hor}})$ on $\mathbb{G}_{2,\nu}$ is given via

$$\tilde{\Delta}_\nu^{\mathrm{hor}} = -\sum_{j=1}^{d} \tilde{X}_j^2 - iA_\nu \tilde{X}_{d+\nu} + O_H(1),$$

like in (6.25), where $A_\nu \in \mathrm{Mat}_{p\times p}(\mathbb{C})$ and $O_H(1)$ denotes a graded differential operator of order smaller or equal to 1.

Assume there is a $\nu \in \{1, \ldots, \dim V_2\}$ such that there is an eigenvector of A_ν which is contained in the singular set Λ_ν defined as follows:

(i) If $\mathbb{G}_{2,\nu} \cong \mathbb{H}^{2m+1}$ is isomorphic to the $(2m+1)$-dimensional Heisenberg group (with $2m = d$), we define

$$\Lambda_\nu := \left\{ \pm \left(\frac{1}{2} \|L_\nu\|_1 + 2 \sum_{1\le j\le m} \alpha_j |\lambda_j| \right) : \alpha_j \in \mathbb{N}^m \right\},$$

where $\pm i\lambda_1, \ldots, \pm i\lambda_m$ denote the non-zero eigenvalues and $\|L_\nu\|_1 = 2\sum_{j=1}^{m} |\lambda_j|$ denotes the trace norm of the ν-Levi form L_ν.

(ii) If $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ with $2m < d$ is not isomorphic to a Heisenberg group, then we define

$$\Lambda_\nu := \left(-\infty, -\frac{1}{2}\|L_\nu\|_1\right] \cup \left[\frac{1}{2}\|L_\nu\|_1, \infty\right),$$

where $\|L_\nu\|_1$ denotes the trace norm of the ν-Levi form L_ν.

Then Δ^{hor} is not hypoelliptic.

Remark: Note that for an arbitrary $\nu \in \{1, \ldots \dim V_2\}$, the cases $\mathbb{G}_{2,\nu} \cong \mathbb{H}^{2m+1}$ and $\mathbb{G}_{2,\nu} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$, with $m \ge 1$, are indeed the only possible cases since $\mathbb{G}_{2,\nu}$ is a Carnot group of step 2 and horizontal co-dimension 1.

Proof: By Theorem 6.2.1, Δ^{hor} is not hypoelliptic if $\tilde{\Delta}_\nu^{\mathrm{hor}}$ is not hypoelliptic for some $\nu \in \{1, \ldots, \dim V_2\}$. But to the operator $\tilde{\Delta}_\nu^{\mathrm{hor}}$ we can apply Theorem 6.1.4, which states that the operator $\tilde{\Delta}_\nu^{\mathrm{hor}}$ is not hypoelliptic if and only if there is an eigenvalue of A_ν belonging to one of the singular sets Λ_ν from (i) and (ii), depending on how the structure of $\mathbb{G}_{2,\nu}$ concretely looks like. □

From Corollary 6.2.3, we can immediately derive a simple criterion which ensures us that a horizontal Laplacian is not hypoelliptic if one of the matrices from (6.25) has a certain eigenvalue. In detail we can use the matrices $L^{(\nu)}$ of the ν-Levi forms $\mathcal{L}_\nu$ from (6.26) according to the frame $\{X_1, \ldots, X_d\}$ of V_1, such that for all $j, k \in \{1, \ldots, d\}$ the commutators have the form

$$[X_j, X_k] = \sum_{\nu_1}^{d_2} L_{jk}^{(\nu)} X_{2,\nu}.$$

Thus the horizontal Laplacian (6.24) can be rewritten in the form

$$\begin{aligned} \Delta^{\text{hor}} &= -\sum_{j=1}^{d_1} X_j^2 - i \sum_{\nu=1}^{d_2} \sum_{j<k} A_{j,k} L_{jk}^{(\nu)} X_{2,\nu} + O_H(1) \\ &= -\sum_{j=1}^{d_1} X_j^2 - i \sum_{\nu=1}^{d_2} A_\nu X_{2,\nu} + O_H(1), \end{aligned} \tag{6.27}$$

such that the matrices A_ν are given by

$$A_\nu = \sum_{j<k} A_{j,k} L_{jk}^{(\nu)}. \tag{6.28}$$

But from these matrices one can check immediately that a given horizontal Laplacian is not hypoelliptic:

Corollary 6.2.4
If there is a $\nu \in \{1, \ldots, \dim V_2\}$ such that there is an eigenvalue μ of the $(p \times p)$-matrix A_ν from (6.27) with

$$\mu = \pm \frac{1}{2} \|L_\nu\|_1 = \pm \sum_{j=1}^{m} |\lambda_j|,$$

then the operator Δ^{hor} is not hypoelliptic. Here, for $j \in \{1, \ldots, m\}$ the numbers $\pm i\lambda_j$ are supposed to be the non-zero eigenvalues of the ν-Levi form $\mathcal{L}_\nu$ of $\mathbb{G}$.

Proof: Using the projection pr_ν for any $\nu \in \{1, \ldots, d_2\}$ from (6.23), we get from (6.27)

$$\mathrm{pr}_\nu \left(\Delta^{\text{hor}}\right) = -\sum_{j=1}^{d_1} \mathrm{pr}_\nu(X_j)^2 - i A_\nu \mathrm{pr}_\nu(X_{2,\nu}) + O_H(1).$$

Now we can apply Corollary 6.2.3 to this operator and see that it is not hypoelliptic if one of the eigenvalues of A_ν belongs to one the sets Λ_ν from Corollary 6.2.3. Since the numbers $\pm\frac{1}{2} \|L_\nu\|_1$ are included in both sets, the operator Δ^{hor} fails to be hypoelliptic if one of them is an eigenvalue of one of the matrices A_ν, no matter how the Carnot groups $\mathbb{G}_{2,\nu} = \psi_\nu(\mathbb{G})$ looks like. □

6.3 Non-hypoellipticity of D^H

Using the preparing work done in the last section we are finally ready to show that the horizontal Dirac operators we constructed in Chapter 3 cannot be hypoelliptic.

We have constructed our horizontal Dirac operators D^H more or less analogously to classical Dirac operators, acting on a horizontal Clifford bundle which is arising from the natural horizontal connection on a Carnot manifold M. We have seen that these operators detect the Carnot-Carathéodory metric and that they are therefore also a generalization of the classical case from the Connes metric point of view. If D^H would be hypoelliptic, it would follow that it has a compact resolvent by Heisenberg calculus (which is developed for the case of Heisenberg manifolds in Chapter 5), and hence it would give a spectral triple. But this is not the case: We will prove that D^H is not hypoelliptic, and therefore the machinery of graded pseudodifferential calculus does not work. We have already seen this in detail for a specific example in Chapter 4, and from the following theorems it will follow that these results fit into the general situation.

To prove the non-hypoellipticity of D^H, we will consider its square calculated locally in Proposition 3.2.8 from Chapter 3. This strategy is justified by the following simple observation.

Proposition 6.3.1
Let D be a (pseudo-)differential operator acting on a vector bundle E over a manifold M. Then D is hypoelliptic of and only if D^2 is hypoelliptic.

Proof: Assume D is hypoelliptic, which means for any distribution φ we have that $D\varphi \in C^\infty(M,E)$ implies $\varphi \in C^\infty(M,E)$. This gives the implication

$$D^2\varphi \in C^\infty(M,E) \;\Rightarrow\; D\varphi \in C^\infty(M,E) \;\Rightarrow\; \varphi \in C^\infty(M,E),$$

and hence D^2 is hypoelliptic.

On the other hand, let $D^2\varphi \in C^\infty(M,E)$ imply that $\varphi \in C^\infty(M,E)$. We have to show that D is hypoelliptic. But this follows immediately since for $D\varphi \in C^\infty(M,E)$, we also have $D^2\varphi \in C^\infty(M,E)$, which shows $\varphi \in C^\infty(M,E)$ by assumption. Altogether the proposition is proved. □

Since $(D^H)^2$ is a horizontal Laplacian, we can use Theorem 6.1.4 to decide about its hypoellipticity for the case it is acting on a Heisenberg manifold. For the case of a general Carnot manifold, we can use the criterion developed in Section 6.2 to get a corresponding horizontal Laplacian on a Heisenberg manifold to which we can apply Theorem 6.1.4. We have already fulfilled this reduction step in Section 6.2, such that we can simply use the Corollaries 6.2.3 and 6.2.4 to prove the following theorem.

Theorem 6.3.2
Let M be a Carnot manifold with horizontal distribution HM, equipped with a horizontal Clifford bundle S^H arising from the horizontal connection on M, and let D^H be any horizontal Dirac operator acting on S^HM. Then D^H is not hypoelliptic.

Proof: Assume the grading of TM is given by

$$TM \cong HM \oplus V_2M \oplus \ldots \oplus V_RM,$$

and that we have a Riemannian metric on M such that these sub-bundles are point-wise orthogonal to each other. Further we assume $\{X_1, \ldots, X_d\}$ (with $d = \text{Rank } HM$) to be an orthonormal frame for HM and $\{X_{2,1}, \ldots, X_{2,d_2}\}$ to be an orthonormal frame for V_2M (with $d_2 = \text{Rank } V_2M$). After fixing a $\nu_0 \in \{1, \ldots, d_2\}$ we can assume the frame of HM to have the additional property that the matrix $L^{(\nu_0)} \in \text{Skew}_{d\times d}(\mathbb{R})$ describing the ν_0-Levi form $\mathcal{L}_{\nu_0}$ with respect to this frame is given by

$$L^{(\nu_0)} = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.29}$$

where D is a diagonal matrix carrying the absolute values $\lambda_1, \ldots, \lambda_m$ of the non-zero eigenvalues of $\mathcal{L}_{\nu_0}$ on its diagonal. This can always be achieved by on orthonormal transformation of the horizontal frame because any ν-Levi matrix is skew symmetric. For the proof of this theorem, we work with the expression of D^H according to this horizontal frame $\{X_1, \ldots, X_d\}$, see e.g. Equation (3.9) from Theorem 3.2.5.

We show that $(D^H)^2$ is not hypoelliptic, then the non-hypoellipticity of D^H follows from Proposition 6.3.1. Note that it suffices to show the non-hypoellipticity locally in an environment of any point $x \in M$. By Proposition 3.2.8 we have locally

$$\left(D^H\right)^2 = -\sum_{j=1}^{d} X_j^2 + \sum_{j<k} c^H(X_j)c^H(X_k)\,[X_j, X_k] + O_H(1) \tag{6.30}$$

with horizontal Clifford action $c^H : HM \to \text{End}S^HM$ on the horizontal Clifford bundle S^HM. As before X_j is to be understood as a component wise directional derivative in a local chart and $O_H(1)$ denotes a graded differential operator of order smaller than or equal to 1.

For any $\nu \in \{1, \ldots, \text{Rank } V_2M\}$ let $L^{(\nu)}$ denote the matrix of the ν-Levi form corresponding to the frame $\{X_1, \ldots, X_d\}$ of V_1M. Hence for any pair $j, k \in \{1, \ldots, d\}$ we have

$$[X_j, X_k] = \sum_{\nu=1}^{d_2} L^{(\nu)}_{j,k} X_{2,\nu},$$

and plugging this into (6.30) we find that

$$\begin{aligned}
\left(D^H\right)^2 &= -\sum_{j=1}^{d} X_j^2 + \sum_{\nu=1}^{d_2}\sum_{j<k} c^H(X_j)c^H(X_k)L^{(\nu)}_{j,k}X_{2,\nu} + O_H(1) \\
&= -\sum_{j=1}^{d} X_j^2 - i\sum_{\nu=1}^{d_2}\sum_{j<k} ic^H(X_j)c^H(X_k)L^{(\nu)}_{j,k}X_{2,\nu} + O_H(1).
\end{aligned}$$

Now we can use Corollary 6.2.4 applied to the matrix

$$A_{\nu_0} = \sum_{j<k} ic^H(X_j)c^H(X_k)L^{(\nu_0)}_{j,k},$$

where $\nu_0 \in \{1, \ldots, d_2\}$ is the number we chose in the beginning of the proof such that the ν_0-Levi matrix $L^{(\nu_0)}$ has the form (6.29). But this means that we have

$$A_{\nu_0} = \sum_{j=1}^{m} ic^H(X_j)c^H(X_{j+m})\lambda_j, \tag{6.31}$$

where $\lambda_1, \ldots, \lambda_m$ denote the absolute values of the non-zero eigenvalues of $\mathcal{L}_\nu$.
According to Corollary 6.2.4, we have to show that there is an eigenvalue of the matrix A_ν from (6.31) with absolute value $\sum_{j=1}^m \lambda_j$. But this has already been done by Proposition 3.2.9 from Section 3.2: This proposition states that $i\sum_{j=1}^m \lambda_j$ and $-i\sum_{j=1}^m \lambda_j$ are eigenvalues of the matrix $\sum_{j=1}^m \lambda_j c(X_j)c(X_{m+j})$. Hence the Theorem is proved because of Corollary 6.2.4. □

Remark: The property of being hypoelliptic only depends on the leading term of the local expression (6.30) for the square of D^H. As we already noted in the remark after Proposition 3.2.8, this term does not change if we modify D^H by adding a section of the endomorphism bundle of S^HM, meaning an order zero term in the language of differential operators. Hence there is no chance of getting a hypoelliptic first order horizontal differential operator on S^HM via a modification of the connection on S^HM by an endomorphism. ◁

In Chapter 5, Theorem 5.2.1 we have seen that on a compact Heisenberg manifold M we need the condition of being hypoelliptic to get a resolvent for a Heisenberg pseudodifferential operator of positive order which is compact. On the other hand, it is clear that from the existence of a compact resolvent of a differential operator D of order m one can expect this operator to be hypoelliptic: For every $s \in \mathbb{R}$, D can be extended to a bounded operator

$$D : H^s(M) \to H^{s-m}(M)$$

between the L^2-Sobolev spaces $H^s(M)$ and $H^{s-m}(M)$. Hence we expect a (compact) resolvent of D to be a mapping from $H^{s-m}(M)$ to $H^s(M)$ for any $s \in \mathbb{R}$. Now assume D is not hypoelliptic, i.e. that there is an element $\varphi \notin C^\infty(M)$ such that $D\varphi \in C^\infty(M)$. But this means that the resolvent of D maps the C^∞-function $D\varphi$ (which is an element of $H^s(M)$ for every $s \in \mathbb{R}$) to a distribution which does not belong to $H^{s-m}(M)$ for one $s \in \mathbb{R}$. This is a contradiction.
From the above discussion, we can formulate the following corollary.

Corollary 6.3.3
Let M be a Carnot manifold with horizontal distribution HM, equipped with a horizontal Clifford bundle S^H arising from the horizontal connection on M, and let D^H be any horizontal Dirac operator acting on S^HM. Then D^H does not have a compact resolvent.
□

We emphasize once again that Theorem 6.3.2 is true for any example of a horizontal Dirac operator according to the horizontal Levi-Civita connection on HM one can imagine. It is true for the horizontal pull-back Dirac operators we discussed in Chapter 4, and it is also true for horizontal Dirac operators defined on classical Clifford or spinor bundles (by skipping the non-horizontal derivatives), see Proposition 3.2.7. In particular, it is true for the operator $d^H + (d^H)^*$, where d^H is the horizontal exterior derivative (see Example 3.2.4). Note that all these constructions are based on the horizontal connection on M, which is induced by the Levi-Civita connection.

This shows that the degeneracy we detected in Chapter 4 for the horizontal pull-back Dirac operator on compact Carnot nilmanifolds is not because of a bad choice for D^H. Rather it reflects a general phenomenon: The natural differential operator, which detects the Carnot-Carathéodory metric on a Carnot manifold via Connes' formula, does not give a spectral triple, and the classical construction of a spectral triple on a compact spin manifold cannot be transported to the Carnot case.

Chapter 7

Spectral Triples from Horizontal Laplacians

In the last chapter we saw that the canonical candidate for a spectral triple over a Carnot manifold detecting the horizontal geometry, the horizontal Dirac operator D^H, reproduces the Carnot-Carathéodory metric but is not hypoelliptic, and therefore does not have a compact resolvent in the Heisenberg calculus. But on the other hand the classical hypoellipticity criteria imply that there are a lot of horizontal Laplacians which are hypoelliptic, and in addition (at least in the Heisenberg case) give back the Hausdorff dimension of (M, d_{CC}) via their eigenvalue asymptotics.

Now this chapter is devoted to studying horizontal Laplacians and to discussing how they furnish the geometry in the sense of Connes. In the first section we show explicitly how one can construct a spectral triple from a positive hypoelliptic horizontal Laplacian using the Heisenberg calculus. We present some operators which are induced by a small perturbation of the square of a horizontal Dirac operator. Then in Section 7.2 we will show that any horizontal Laplacian detects the Carnot-Carathéodory metric of a Carnot manifold via a formula similar to Connes' one. We also discuss what this means for a first order operator whose square is a horizontal Laplacian.

The last section of this chapter is rather speculative: We intend to give some ideas how one can find estimates for the Connes metric of spectral triples arising from horizontal Laplacians towards the Carnot-Carathéodory metric (which is exactly detected by a horizontal Dirac operator D^H). Therefore we use some of the observations we made in Section 1.2 concerning the convergence of the metrics belonging to compact quantum order unit spaces. But sadly we have not been able to prove such an estimate, so there are a few open problems formulated in that section.

7.1 Spectral Triples via Heisenberg Calculus

We now point out how we can construct spectral triples on compact Heisenberg manifolds using the Heisenberg calculus we introduced in Chapter 5. The idea is to start with a hypoelliptic, positive horizontal Laplacian Δ^{hor} and to consider the operator $D_{\mathrm{hor}} = \sqrt{\Delta^{\mathrm{hor}}}$, defined via functional calculus.

To show that such a construction indeed furnishes a spectral triple, we check that the commutator of any Heisenberg pseudodifferential operator of order 1 with a smooth function f is bounded.

Proposition 7.1.1
Let M be a Heisenberg manifold and $P \in \Psi^1_{\mathbb{H}}(M, E)$ a Heisenberg pseudodifferential operator of order 1 acting on a vector bundle E over M. For any function $f \in C^\infty(M)$ we denote by M_f the operator of multiplication by f on the Hilbert space $L^2(M, E)$. Then the commutator $[P, M_f]$ is bounded.

Proof: We assume that E is the trivial line bundle over M, the general case works analogously. Our strategy is to show that the symbol of the commutator $[P, M_f]$ lies in $S^0_{\mathbb{H}}(M)$, and hence provides a Heisenberg pseudodifferential operator of order 0. Since $S^0_{\mathbb{H}}(M) \subset S^0_{1/2,1/2}$ for the Hörmander class $S^0_{1/2,1/2}$ by Theorem 5.1.3, this implies by classical pseudodifferential calculus that $[P, M_f]$ is bounded (see also Corollary 5.1.4).

First of all we note that the multiplication operator M_f is a Heisenberg pseudodifferential operator of order 0, and its symbol is exactly the function f by the definition of the symbol classes (see Definition 5.1.1). Let $p \in S^1_{\mathbb{H}}(M)$ denote the symbol of P. Then we can calculate the symbol of the commutator $[P, M_f]$ using Theorem 5.1.5: For general Heisenberg symbols q_1 of order m_1 and q_2 of order m_2, the asymptotic expansion of the symbol

$$q = q_1 \# q_2 \sim \sum_{k \geq 0} q_{m_1+m_2-k},$$

denoting the symbol of the composition of $\mathrm{Op}(q_1)\mathrm{Op}(q_2)$, is given by the terms

$$q_{m_1+m_2-k}(x,\xi) = \sum_{k_1+k_2 \leq k} \sum_{\alpha,\beta,\gamma,\delta}^{(k-k_1-k_2)} h_{\alpha,\beta,\gamma,\delta}(x) \cdot \left(D^\delta_\xi q_{1,m_1-k_1}(x,\xi)\right) * \left(\xi^\gamma \partial^\alpha_x \partial^\beta_\xi q_{2,m_2-k_2}(x,\xi)\right), \tag{7.1}$$

where $\sum^{(l)}_{\alpha,\beta,\gamma,\delta}$ denotes the sum over all the indices such that

$$|\alpha| + |\beta| \leq \langle\beta\rangle - \langle\gamma\rangle + \langle\delta\rangle = l \quad \text{and} \quad |\beta| = |\gamma|, \tag{7.2}$$

and the functions $h_{\alpha,\beta,\gamma,\delta}$ are polynomials in the derivatives of the coefficients of the vector fields $X_1, \ldots, X_d, X_{d+1}$ forming a graded frame for TM. The operation $*$ denotes the operation of point-wise convolution, varying smoothly over x (see Equations (5.8) and (5.9) from Section 5.1). See the discussion before Theorem 5.1.5 for a more detailed explanation.

To get the boundedness of the commutator we only have to show that the order 1 part of this asymptotic expansion vanishes, since all the lower order terms have maximal Heisenberg order 0 which gives rise to a bounded operator by Corollary 5.1.4. Since $m_1 + m_2 = 1$ in our situation, these are exactly the terms in (7.1) such that $k = 0$. But this means $k_1 = k_2 = 0$ and therefore also $\alpha, \beta, \gamma, \delta = 0$ by (7.2). Now the Heisenberg symbol of the commutator $[P, M_f] = Pf - fP$ is given by

$$\sigma_{\mathbb{H}}([P, M_f]) = p\#f - f\#p,$$

and for its leading (Heisenberg order 1) symbol we have because of the above argumentation

$$(p\#f - f\#p)_1 (x, \xi) = h_{0,0,0,0}(x) \cdot (p_1(x, \xi) * f(x) - f(x) * p_1(x, \xi)), \tag{7.3}$$

where p_1 denotes the leading term of the Heisenberg symbol of P. Note that the polynomial $h_{0,0,0,0}(x)$ only depends on the coefficients of the vector fields $X_1, \ldots, X_{d+1}$, but not on the symbols of the operators to be composed.

Hence all we have to show is that the convolution of p_1 and f commutes. But this is clear by the definition of the convolution: Whenever we fix an $x \in M$, the terms commute point-wise in x since in this case $f(x)$ is a constant. Now since $*$ varies smoothly in x, this shows that

$$p_1(x, \xi) * f(x) - f(x) * p_1(x, \xi) = 0,$$

and from (7.3) we get $(p\#f - f\#p)_1 = 0$.

We have seen that the leading term in the asymptotic expansion of the symbol belonging to the commutator $[P, M_f]$ has mostly Heisenberg order zero, and we can conclude that this commutator is bounded by Corollary 5.1.4. Therefore the proposition is proved. □

To check the compactness of the resolvent, we refer to Theorem 5.2.1, which states that for a hypoelliptic self-adjoint $\Psi_{\mathbb{H}}DO$ P of order ν the operator P^s is a $\Psi_{\mathbb{H}}DO$ of order νs. Then it is clear that $\Psi_{\mathbb{H}}DO$s of order one can provide spectral triples.

Theorem 7.1.2

Let M be a compact Heisenberg manifold, and let P be a hypoelliptic self-adjoint $\Psi_{\mathbb{H}}DO$ of order 1 acting on a vector bundle E over M, which is bounded from below.

Then the triple

$$(\mathcal{A}, \mathcal{H}, D) = \big(C(M), L^2(M, E), P\big)$$

is a spectral triple, where the representation $\pi : C(M) \to \mathcal{B}(L^2(M, E))$ is given by left multiplication of a function $f \in C(M)$.

Proof: We have to show that

(i) There is a dense sub-algebra $\mathcal{A}' \subset C(M)$ such that $[P, \pi(f)]$ is bounded for any $f \in \mathcal{A}'$.

(ii) The operator P has a compact resolvent.

Statement (i) is immediately clear from Proposition 7.1.1, since $C^\infty(M)$ is a dense subalgebra of $C(M)$. And statement (ii) follows immediately from Theorem 5.2.1: By this theorem, the operator $(P^2+I)^{-1/2}$ exists in the Heisenberg calculus and is a $\Psi_{\mathbb{H}}DO$ of order -1. Since M is compact, this is a compact operator by Corollary 5.1.7, and therefore P possesses a compact resolvent. □

From the above considerations we find that it is in general possible to construct spectral triples via the Heisenberg calculus. We will see that concrete examples can be comfortably derived from horizontal Laplacians, and by the results stated in Section 5.2 concerning the eigenvalue asymptotics we will also see that the metric dimension of these spectral triples provides the Hausdorff dimension of (M, d_{CC}).

Theorem 7.1.3
Let M be a compact Heisenberg manifold equipped with a horizontal distribution of rank d, and let Δ^{hor} be a hypoelliptic self-adjoint horizontal Laplacian which is bounded from below, acting on a vector bundle E over M. Then the triple

$$\left(C(M), L^2(M,E), \left(\Delta^{\text{hor}}\right)^{\frac{1}{2}}\right) \tag{7.4}$$

is a spectral triple of metric dimension $d+2$.

In particular, the metric dimension of this spectral triple coincides with the Hausdorff dimension of the Carnot manifold (M, d_{CC}).

Proof: Without loss of generality we assume Δ^{hor} to be invertible (since it is bounded from below, this can always be achieved by adding a constant). Thus, by Theorem 5.2.1, $\left(\Delta^{\text{hor}}\right)^{1/2}$ exists and is a $\Psi_{\mathbb{H}}DO$ of order one, which is hypoelliptic, self-adjoint and positive. Then the statement that (7.4) is a spectral triple follows from Theorem 7.1.2.

To get the additional statement about the metric dimension of this spectral triple, we can apply Theorem 5.2.3, which says that the eigenvalues of Δ^{hor} have the asymptotic behavior

$$\lambda_k\left(\Delta^{\text{hor}}\right) \sim \left(\frac{k}{\nu_0\left(\Delta^{\text{hor}}\right)}\right)^{\frac{2}{d+2}},$$

which gives us

$$\lambda_k\left(\left(\Delta^{\text{hor}}\right)^{\frac{1}{2}}\right) \sim \left(\frac{k}{\nu_0\left(\Delta^{\text{hor}}\right)}\right)^{\frac{1}{d+2}}.$$

This asymptotic behavior shows that $\left(\Delta^{\text{hor}}\right)^{-p/2}$ is trace class if and only if $p > d+2$. Thus the metric dimension of the spectral triple is $d+2$, which is also the Hausdorff dimension of (M, d_{CC}) by Theorem 2.1.6. □

From now on, we consider once again the orthonormal frame $\{X_1, \ldots, X_d, X_{d+1}\}$ of the tangent bundle of our Heisenberg manifold M, such that $\{X_1, \ldots, X_d\}$ span the horizontal distribution HM. We assume that the Levi form of M according to this frame is given by the matrix

$$L = \begin{pmatrix} 0 & D_m & 0 \\ -D_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with } D_m = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}, \ \lambda_1, \dots, \lambda_m > 0 \tag{7.5}$$

for an $m \leq d/2$. Note that such a Levi-form can always be achieved by an orthonormal change of the frame (see Proposition 2.3.3). But this means we have $\mathbb{G} \cong \mathbb{H}^{2m+1} \times \mathbb{R}^{d-2m}$ for the underlying Carnot group, and the commutator relation of the vector fields belonging to the frame of M become

$$[X_j, X_k] = \begin{cases} \lambda_j, & \text{for } 1 \leq j \leq m,\ k = m+j \\ -\lambda_k & \text{for } 1 \leq k \leq m,\ j = k+m\ . \\ 0 & \text{otherwise} \end{cases}$$

Now an obvious candidate for a horizontal Laplacian which provides a spectral triple is the sum-of-squares operator

$$\Delta^H = \nabla^*_{X_1} \nabla_{X_1} + \ldots + \nabla^*_{X_d} \nabla_{X_d}, \tag{7.6}$$

where ∇ is any connection acting on a vector bundle E over M. This operator is obviously self-adjoint, and after applying Hörmander's sum-of-squares theorem (see Theorem 6.1.1) in any local chart we see that is is hypoelliptic. It is also known that this operator is bounded from below, as it is the case for any self-adjoint horizontal Laplacian, whose principal symbol is invertible in the Heisenberg calculus. See [Pon08], Remark 5.2.10 for that.

As we have seen in the previous chapters, the horizontal Dirac operator D^H detecting the Carnot-Carathéodory metric on M is not hypoelliptic and therefore does not provide a spectral triple. To see why the hypoellipticity fails, we take a look at the proof of Theorem 6.3.2 adapted to this situation: The square of D^H is (locally) given by

$$\left(D^H\right)^2 = -\sum_{j=1}^{d} X_j^2 + \sum_{j=1}^{d} \lambda_j c^H(X_j) c^H(X_{m+j}) X_{d+1} + O_H(1), \tag{7.7}$$

and this operator fails to be hypoelliptic since $i \sum_{j=1}^m \lambda_j$ and $-i \sum_{j=1}^m \lambda_j$ are eigenvalues of the matrix $\sum_{j=1}^d \lambda_j c(X_j) c(X_{m+j})$. This is the content of Theorem 6.1.4, but the same theorem also states that the operator from (7.7) would be hypoelliptic if the absolute value of every eigenvalue of the matrix coefficient of X_{d+1} would be smaller than $\sum_{j=1}^m \lambda_j$. Since this is fulfilled for any other eigenvalue of $\sum_{j=1}^d \lambda_j c(X_j) c(X_{m+j})$ (see Proposition 3.2.9), one can disturb this operator by a small number $\theta > 0$ and conclude that the operator

$$\left(D^H_\theta\right)^2 := -\sum_{j=1}^{d} X_j^2 + (1-\theta) \sum_{j=1}^{d} \lambda_j c^H(X_j) c^H(X_{m+j}) X_{d+1} + O_H(1)$$

is hypoelliptic for any $0 < \theta \leq 1$.

We thus construct a spectral triple which shall be a small perturbation of the horizontal Dirac operator, using the sum-of-squares operator from (7.6). The connection we are using is the horizontal Clifford connection ∇^{S^H} from Section 3.2, acting on a Clifford bundle $S^H M$ over M on which the horizontal Dirac operator D^H is defined.

Corollary 7.1.4

Let M be a closed Heisenberg manifold, and let $S^H M$ be a horizontal Clifford bundle over M on which a horizontal Dirac operator D^H is acting. We further define the horizontal Laplacian

$$\Delta^H = \left(\nabla^{S^H}_{X_1}\right)^* \nabla^{S^H}_{X_1} + \ldots + \left(\nabla^{S^H}_{X_d}\right)^* \nabla^{S^H}_{X_d}.$$

Then for any $0 < \theta \leq 1$ the operator

$$D^H_\theta := \left((1-\theta)\left(D^H\right)^2 + \theta\Delta^H\right)^{\frac{1}{2}} \tag{7.8}$$

provides a spectral triple $(C(M), L^2(S^H M), D^H_\theta)$ of metric dimension $d+2$.

Proof: We have to show that for any $\theta \in (0,1]$ the operator

$$\Delta^H_\theta := (1-\theta)\left(D^H\right)^2 + \theta\Delta^H \ \in \Psi^2_{\mathbb{H}}(S^H M) \tag{7.9}$$

is positive (which also means that (7.8) is well-defined) and hypoelliptic. In this case we can apply Theorem 7.1.3 to get that D^H_θ provides a spectral triple of metric dimension $d+2$ and the corollary is proved.

We assume that $\Delta^H \geq 0$, which can always be achieved by adding a constant because it is bounded from below. Since the horizontal Dirac operator D^H is self-adjoint by Theorem 3.2.5, its square $\left(D^H\right)^2$ is positive, and therefore Δ^H_θ from (7.9) is positive as a convex combination of positive operators.

To show that Δ^H_θ is hypoelliptic, we note that

$$\Delta^H_\theta = \theta\Delta^H + (1-\theta)\left(D^H\right)^2 = \Delta^H + (1-\theta)\left(\left(D^H\right)^2 - \Delta^H\right). \tag{7.10}$$

We consider this operator in the local frame $\{X_1, \ldots, X_{d+1}\}$ introduced above, such that $HM = \operatorname{span}\{X_1, \ldots, X_d\}$ and the matrix of the Levi form corresponding to this basis is given by (7.5). But in these local coordinates, the argument we sketched in the discussion preceding this corollary applies: We have locally $\Delta^H = -\sum_{j=1}^d X_j^2$ and therefore

$$\left(D^H\right)^2 - \Delta^H = \sum_{j=1}^m c^H(X_j)c^H(X_{m+1})X_{d+1} + O_H(1)$$

by the local expression of $(D^H)^2$ from Proposition 3.2.8. Hence (7.10) has locally the form

$$\Delta^H_\theta = -\sum_{j=1}^d X_j^2 + (1-\theta)\sum_{j=1}^m c^H(X_j)c^H(X_{m+1})X_{d+1} + O_H(1), \tag{7.11}$$

where like in the previous chapters $O_H(1)$ denotes a graded differential operator of order smaller than or equal to 1. Now by Proposition 3.2.9, the eigenvalues of the matrix

$$(1-\theta)\sum_{j=1}^{m}\lambda_j c^H(X_j)c^H(X_{m+j})$$

are included in the interval

$$\left[-(1-\theta)i\sum_{j=1}^{d}\lambda_j, (1-\theta)i\sum_{j=1}^{d}\lambda_j\right] \subset \mathbb{R}i$$

on the imaginary line, and since $1 \geq \theta > 0$ the absolute value of any of these eigenvalues is smaller than $\sum_{j=1}^{d}\lambda_j = \frac{1}{2}\mathrm{tr}|L|$. But this means by Theorem 6.1.4 that Δ_θ^H is (locally) hypoelliptic.

The global hypoellipticity then follows by Theorem 5.1.13, and the theorem is proved. □

7.2 Detection of the Metric via Horizontal Laplacians

After we have constructed spectral triples from horizontal Laplacians, we ask ourselves whether one can detect the Carnot-Carathéodory metric from these spectral triples. While we postpone the discussion of estimates for the Connes metric formula of these operators to Section 7.3, we want to show now that there is a formula to detect the Carnot-Carathéodory metric directly from any horizontal Laplacian. In the case where we have an ordinary Laplacian $\Delta = -\sum_{j=1}^{n} X_j^2$, it is known that for $f \in C^\infty(M)$ we have

$$[[\Delta, f], f] = -2\,\|df\|^2\,,$$

see for example [BGV04], Proposition 2.3. Now an analogous result is true for horizontal Laplacians: The key observation is the following lemma.

Lemma 7.2.1
Let Δ^{hor} be a horizontal Laplacian, acting on a vector bundle E over a Carnot manifold M. Then we have for any function $f \in C^\infty(M)$

$$\frac{1}{2}\left[\left[\Delta^{\mathrm{hor}}, f\right], f\right]\sigma = -\left\|\mathrm{grad}^H f\right\|^2 \sigma \tag{7.12}$$

for $\sigma \in \Gamma^\infty(M,E)$. Here, $\|\cdot\|$ denotes the (point-wise) norm of a vector in E_xM induced by the Riemannian metric g on M.

Proof: Let $TM = V_1M \oplus \ldots \oplus V_RM$ be the grading of M, such that $\{X_1,\ldots,X_d\}$ is an orthonormal frame of V_1M and $\{X_{d+1},\ldots,x_{d+d_2}\}$ is an orthonormal frame of V_2M. We prove the statement locally, meaning that Δ^{hor} is given in the form

$$\Delta^{\mathrm{hor}} = -\sum_{j=1}^{d}\partial_{X_j}^2 + B(\partial_{X_1},\ldots,\partial_{X_{d+d_2}}) + b, \tag{7.13}$$

where ∂_{X_k} denotes the partial derivative along X_k in any chart, $B(\partial_{X_1}, \ldots, \partial_{X_{d+d_2}})$ is any differential operator of (classical) order 1 and b is a (matrix-valued) function.

We now plug the expression (7.13) term by term into the double commutator of (7.12) and use the linearity of the commutator. Since b commutes with any smooth function f, this is zero for the last summand. It is also zero for the second summand of (7.13), since for any first order differential operator $B = B(\partial_{X_1}, \ldots, \partial_{X_{d+d_2}})$ we have because of the Leibniz rule

$$\begin{aligned} [[B, f], f]\,\sigma &= [Bf - fB, f]\,\sigma \\ &= B(f^2\sigma) - 2f \cdot B(f\sigma) + f^2 \cdot B\sigma \\ &= 2(Bf) \cdot f\sigma + f^2 B\sigma - 2f \cdot (Bf) \cdot \sigma - 2f^2 \cdot B\sigma + f^2 \cdot B\sigma \\ &= 0 \end{aligned}$$

locally for any section $\sigma \in \Gamma^\infty(M, E)$.

Calculating the first term of (7.13), we find that for the double commutator applied to any $\partial^2_{X_j}$ we have

$$\begin{aligned} & \left[\left[\partial^2_{X_j}, f\right], f\right] \sigma \\ = \;& \partial^2_{X_j}(f^2\sigma) - 2f \cdot \partial^2_{X_j}(f\sigma) + f^2 \cdot \partial^2_{X_j}\sigma \\ = \;& \partial_{X_j}\left(2(\partial_{X_j} f) \cdot f\sigma + f^2 \cdot \partial_{X_j}\sigma\right) - 2f \cdot \partial_{X_j}\left((\partial_{X_j} f) \cdot \sigma + f \cdot \partial_{X_j}\sigma\right) + f^2 \cdot \partial^2_{X_j}\sigma \\ = \;& 2(\partial^2_{X_j} f) \cdot f\sigma + 2(\partial_{X_j} f)^2 \cdot \sigma + 4(\partial_{X_j} f) \cdot f \cdot \partial_{X_j}\sigma + f^2 \cdot \partial^2_{X_j}\sigma \\ & -2f \cdot (\partial^2_{X_j} f) \cdot \sigma - 4f \cdot (\partial_{X_j} f) \cdot (\partial_{X_j}\sigma) - 2f^2 \cdot \partial^2_{X_j}\sigma + f^2 \cdot \partial^2_{X_j}\sigma \\ = \;& 2(\partial_{X_j} f)^2 \cdot \sigma. \end{aligned}$$

Finally, we plug everything together into the double commutator expression (7.13), which shows us that

$$\begin{aligned} \frac{1}{2}\,[[\Delta, f], f]\,\sigma &= -\frac{1}{2}\sum_{j=1}^{d} \left[\left[\partial^2_{X_j}, f\right], f\right] \sigma \\ &= -\sum_{j=1}^{d} (\partial_{X_j} f)^2 \cdot \sigma \\ &= -\left\|\mathrm{grad}^H f\right\|^2 \sigma. \end{aligned}$$

Hence the statement of the lemma is proved. □

From the identity (7.12) we get an expression depending on a horizontal Laplacian instead of a horizontal Dirac operator which can detect the Carnot-Carathéodory metric, following the theory from Section 3.3. By Corollary 3.3.7, we have for the Carnot-Carathéodory metric

$$d_{CC}(x, y) = \sup\left\{|f(x) - f(y)| : f \in \mathrm{Lip}_{CC}(M), \operatorname*{ess\,sup}_{\xi \in M} \left\|\mathrm{grad}^H f(\xi)\right\| \leq 1\right\}. \tag{7.14}$$

Using this expression, we can prove the following theorem.

Theorem 7.2.2
Let Δ^{hor} be a horizontal Laplacian, acting on a vector bundle E over a closed Carnot manifold M. Then we have for any $x, y \in M$

$$d_{CC}(x,y) = \sup\left\{|f(x) - f(y)| : f \in C^\infty(M), \left\|\frac{1}{2}\left[\left[\Delta^{\text{hor}}, f\right], f\right]\right\| \leq 1\right\}, \tag{7.15}$$

where d_{CC} is the Carnot-Carathéodory metric on M.

Proof: By Lemma 7.2.1 we have

$$\left\|\frac{1}{2}\left[\left[\Delta^{\text{hor}}, f\right], f\right]\right\| = \sup_{x \in M} \left\|\text{grad}^H f(x)\right\|^2.$$

In addition, we have shown in Section 3.3 that

$$\left\|[D^H, f]\right\| = \sup_{x \in M} \left\|\text{grad}^H f(x)\right\|$$

for any smooth function $f \in C^\infty(M)$. Thus (7.14) provides us

$$d_{CC}(x,y) = \operatorname{ess\,sup}\left\{|f(x) - f(y)| : f \in C^\infty(M),\ \sup_{\xi \in M}\left\|\text{grad}^H f(\xi)\right\| \leq 1\right\}.$$

Since $\left\|\text{grad}^H f(x)\right\|^2 \leq 1$ if and only if $\left\|\text{grad}^H f(x)\right\| \leq 1$, the statement of the theorem follows from Lemma 7.2.1. □

Theorem 7.2.2 states that it is possible to detect the Carnot-Carathéodory metric by arbitrary positive horizontal Laplacians. In the case D^H is a horizontal Dirac operator constructed in Chapter 3, both D^H and $(D^H)^2$ detect the Carnot-Carathéodory metric via the corresponding formulas.

Now it is a natural question whether one can use the metric detection by a positive horizontal Laplacians to find estimates for the Connes metric given by its square root operator. For the rest of this section, we take the algebra $C^\infty(M)$ as the dense sub-algebra of $C(M)$ over which the Connes metric is defined, i.e. we consider the metric

$$d_D(x,y) = \sup\left\{|f(x) - f(y)| : f \in C^\infty(M), \|[D, f]\| \leq 1\right\} \tag{7.16}$$

for an operator D defining a spectral triple. Then we can note the following criterion.

Proposition 7.2.3
Let $\tilde{D}$ be a self-adjoint operator acting on a closed Carnot manifold M and defining a spectral triple such that $\tilde{D}^2 = \Delta^{\text{hor}}$ is a horizontal Laplacian. Let $d_{\tilde{D}}$ denote the Connes metric defined by $\tilde{D}$ via (7.16).

Then if the condition

$$\left[\tilde{D}, f\right]^2 = \frac{1}{2}\left[[\Delta^{\mathrm{hor}}, f], f\right] \tag{7.17}$$

is fulfilled, we have the estimate

$$d_{\tilde{D}}(x,y) \le d_{CC}(x,y)$$

for all points $x, y \in M$.

Proof: This follows immediately from the formulas for the metrics. By (7.17) we see that

$$\left\|\frac{1}{2}\left[[\Delta^{\mathrm{hor}}, f], f\right]\right\| = \left\|\left[\tilde{D}, f\right]^2\right\| \le \left\|\left[\tilde{D}, f\right]\right\|^2. \tag{7.18}$$

But this shows that $\|[\tilde{D}, f]\| \le 1$ implies $\|[[\Delta^{\mathrm{hor}}, f], f]\| \le 1$, which gives together with Theorem 7.2.2 the desired estimates for the corresponding metrics:

$$\begin{aligned} d_{\tilde{D}}(x,y) &= \sup\left\{|f(x) - f(y)| : \left\|\left[\tilde{D}, f\right]\right\| \le 1\right\} \\ &\le \sup\left\{|f(x) - f(y)| : \left\|\frac{1}{2}\left[[\Delta^{\mathrm{hor}}, f], f\right]\right\| \le 1\right\} \\ &= d_{CC}(x,y). \end{aligned}$$

□

Now in case D^H is a horizontal Dirac operator, condition (7.17) is fulfilled as we will see in a minute. In this situation we even have equality of the metrics, since estimate (7.18) in the proof of the proposition is an equality. The reason for this is that $[D^H, f]$ is exactly the horizontal Clifford action c^H of the horizontal gradient of f, and that at each point $x \in M$ $c^H : H_xM \to \mathrm{End}_{\mathbb{C}}(S^HM)$ is an isometry.

We now develop a condition equivalent to (7.17) from elementary commutator calculations.

Proposition 7.2.4
Let A, B *be linear operators on a Hilbert space. Then we have*

$$[A,B]^2 = \frac{1}{2}\left[\left[A^2, B\right] B\right]$$

if and only if

$$A\,[[A,B],B] + [[A,B],B]\,A = 0 \tag{7.19}$$

Proof: This is just a simple calculation involving commutator rules:

$$\begin{aligned} \left[\left[A^2, B\right] B\right] &= [A[A,B] + [A,B]A, B] \\ &= [A[A,B],B] + [[A,B]A, B] \\ &= A\,[[A,B],B] + [A,B][A,B] + [A,B][A,B] + [[A,B],B]\,A. \end{aligned}$$

From this equation, we see the equivalence of the two statements. □

From the aspect of spectral triples, we choose in Proposition 7.2.4 $A = D$ (the Dirac operator of the triple) and $B = f$ (the representation of the corresponding algebra on the Hilbert space, which is multiplication by functions in the commutative case). Hence it is obvious that (7.19) is fulfilled if D is a first-order differential operator, since in this case we have $[D, f] = 0$.
More generally, (7.19) is fulfilled if the operator D fulfills the order-one condition for a spectral triple by Alain Connes (see e.g. [Con96]), or [GVF01], Section 10.5), which states in the commutative situation that $[[D, f], g] = 0$ for all $f, g \in C^\infty(M)$. Therefore we can write down the following corollary.

Corollary 7.2.5
Let M be a closed Carnot manifold. If $(C(M), L^2(\Sigma M), \tilde{D})$ is a spectral triple such that $\tilde{D}^2 = \Delta^{\mathrm{hor}}$ is a horizontal Laplacian, which fulfills the order one condition, then we have for any points $x, y \in M$

$$d_{CC}(x, y) \geq \sup \left\{ |f(x) - f(y)| : f \in C^\infty(M), \left\| [\tilde{D}, f] \right\| \leq 1 \right\}.$$

Proof: This follows immediately by Proposition 7.2.4 and Proposition 7.2.3 together with the above discussion. □

But despite this corollary, we note that the condition (7.17) or, equivalently by Proposition 7.2.4,

$$\tilde{D} \left[[\tilde{D}, f], f \right] + \left[[\tilde{D}, f], f \right] \tilde{D} = 0$$

seem to be rather strong conditions for the case where $\tilde{D}$ is not a differential operator.

7.3 Approaches for Approximation of the Metric

Although the horizontal Dirac operator, which detects the Carnot-Carathéodory metric, does not furnish a spectral triple, we have seen in Section 7.1 that there are quite a lot of spectral triples arising from Heisenberg calculus which give at least the right dimension. In particular the spectral triples constructed in Corollary 7.1.4 only differ by a small parameter from the horizontal Dirac operator. But sadly we do not know how their Connes metric behaves with respect to the Carnot-Carathéodory metric.

Now in this final section we want to present a few ideas how one can approximate the Carnot-Carathéodory metric by a family of spectral triples. Therefore we use the observations made in Section 1.2 concerning arbitrary compact quantum metric spaces to develop criteria for this approximation. First of all, we give a reformulation of Proposition 1.2.5 and Proposition 1.2.6 for our setting.

Proposition 7.3.1
Let M be a closed Carnot manifold, where d_{CC} is the Carnot-Carathéodory metric on M. Let further D^H denote a horizontal Dirac operator on M, acting on a horizontal Clifford bundle $S^H M$.

(i) Let $\tilde{D}$ be an operator on $L^2(S^H M)$ such that $(C(M), L^2(S^H M), \tilde{D})$ is a spectral triple. If there exists a constant $0 < C < 1$ such that

$$\left\| [\tilde{D} - D^H, f] \right\| \leq C \left\| [D^H, f] \right\| \tag{7.20}$$

for any f belonging to a suitable sub-algebra of $C(M)$, then the Connes metric $d_{\tilde{D}}$ arising from $\tilde{D}$ is equivalent to d_{CC}.

(ii) Let for $0 < \theta \leq 1$ the family $\tilde{D}_\theta$ be a family of operators on $S^H M$ with the property that

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; 0 < \theta < \delta \; \Rightarrow \; \left\| [\tilde{D}_\theta - D^H, f] \right\| < \varepsilon \; \forall f \in \Sigma_0, \tag{7.21}$$

where $\Sigma_0 := \{f \in \mathcal{A}' : \mathrm{Lip}_{CC}(f) = 1\}$ for a suitable sub-algebra $\mathcal{A}'$ of $C(M)$. Then for every $\varepsilon > 0$ with $\varepsilon < 1$ there is a $\delta > 0$ such that

$$(1 - \varepsilon) \, d_{\tilde{D}_\theta}(x, y) \leq d_{CC}(x, y) \leq (1 + \varepsilon) \, d_{\tilde{D}_\theta}(x, y) \quad \forall x, y \in M \tag{7.22}$$

for every $0 < \theta < \delta$.

Proof: We set $L_0(f) := \|[D^H, f]\|$, which is a Lip-norm on $C(M)$ coinciding with the Carnot-Carathéodory-Lipschitz constant $\mathrm{Lip}_{CC}(f)$ of f by Corollary 3.3.8. Further $L_1 := \|[\tilde{D}, f]\|$ in (i) and $L_\theta := \|[\tilde{D}_\theta, f]\|$ in (ii) are also Lip-norms on $C(M)$, since they arise from spectral triples. But in this context (i) is just a reformulation of Proposition 1.2.6 and (ii) is just a reformulation of Proposition 1.2.5. □

This proposition implies that we would be able to prove good metric approximations by spectral triples for the Carnot-Carathéodory metric if the following (far more general) assumption is true.

Assumption 7.3.2
Let P be a first order Heisenberg pseudodifferential operator, acting on a Heisenberg manifold M, and let $f \in C^\infty(M)$ or (more generally) $f \in \mathrm{Lip}_{CC}(M)$. Then there exists a $C > 0$ such that

$$\|[P, f]\| \leq C \cdot \mathrm{Lip}_{CC}(f), \tag{7.23}$$

where $\mathrm{Lip}_{CC}(f)$ denotes the Carnot-Carathéodory-Lipschitz constant of f from Definition 3.3.2. ◁

This assumption is motivated by the fact that something analogous is true for the classical Lipschitz functions Lip(M) and for classical pseudodifferential operators of order 1 belonging to the class $S^1_{1,0}$. For this classical situation, the statement can be found in [Tay91], Section 3.6. But the proof presented in the book by Taylor is quite involved, such that it is far away from being trivial to transfer this proof to the Carnot-Carathéodory situation.

Unfortunately, we have not found a way to prove the estimates (7.20) or (7.21), and the proof of the above assumption (if possible) seems to be an even harder and much involved problem. But we assume that the Connes metrics arising from the spectral triples constructed in Corollary 7.1.4 are a good approximation of the Carnot-Carathéodory metric (note that we have in the formulation of the corollary $D^H_0 = |D^H|$ with D^H the horizontal Dirac operator).

As an open problem, we present an approximation of the degenerate spectral triple

$$(C(M), L^2(\Sigma^H_\delta M), D^H)$$

for the case where M is the compact nilmanifold of a Heisenberg group $\mathbb{H}^{2m+1}$, and D^H is the horizontal pull-back Dirac operator arising from a spin structure δ of the $2m$-dimensional torus constructed in Chapter 4. Let $\{X_1, \dots, X_d\}$ be an orthonormal frame of the horizontal distribution HM of M (with $d = 2m$), such that the horizontal pull-back Dirac operator is given by

$$D^H = \sum_{j=1}^{d} c^H(X_j)\nabla^{\Sigma^H}_{X_j},$$

see Section 4.1 for the explicit construction. We have seen in Section 4.3 that the only space where this horizontal Dirac operator degenerates is its infinite dimensional kernel, but otherwise it has all the properties we ask for (see Corollary 4.3.5 and Corollary 4.3.7).

The idea is now to fix the gap of degeneracy on the kernel of D^H by using the sum-of-squares operator

$$\Delta^H = \sum_{j=1}^{d} \left(\nabla^{\Sigma^H}_{X_j}\right)^* \nabla^{\Sigma^H}_{X_j}.$$

It is known that Δ^H is hypoelliptic and bounded from below (see also Section 7.1 for this). Now let P denote the orthogonal projection onto the kernel of D^H. Since the spectrum of D^H is discrete, this projection operator is a pseudodifferential operator of order 0, and it is also an order-zero operator in the Heisenberg calculus. Then for a small $\theta > 0$ we define an operator D^H_θ via

$$D^H_\theta := D^H + \theta \cdot P \left(\Delta^H\right)^{\frac{1}{2}} P. \tag{7.24}$$

By the above argumentation, this is a hypoelliptic Heisenberg pseudodifferential operator, and hence it does define a spectral triple. In addition it detects the metric dimension of (M, d_{CC}) via its eigenvalue asymptotics, since we have the right eigenvalues asymptotics on $(\ker D^H)^\perp$ by Corollary 4.3.7 and on $\ker D^H)$ because of the eigenvalue asymptotics of Δ^H (see Theorem 5.2.3).

Now if Assumption 7.3.2 is true, we have a constant $C > 0$ such that

$$\left\| [P \left(\Delta^H\right)^{\frac{1}{2}} P, f] \right\| \leq C \cdot \mathrm{Lip}_{CC}(f). \tag{7.25}$$

Note that in this situation it would be enough to have the estimate (7.25) on the kernel of D^H, which may be much easier to prove than the general case of (7.23). This would lead to the estimate

$$\left\| [D^H - D^H_\theta, f] \right\| \leq \left\| \theta [P \left(\Delta^H\right)^{\frac{1}{2}} P, f] \right\| \leq \theta C \cdot \mathrm{Lip}_{CC}(f),$$

because of the definition (7.24) of D^H_θ, and we could use Proposition 7.3.1 to show that $d_{D^H_\theta}(x, y) \to d_{CC}(x, y)$ for $\theta \to 0$ for all points $x, y \in M$.

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Stefan Hasselmann

Persönliche Daten

Geboren	am 12.04.1981 in Ibbenbüren, Deutschland
Nationalität	Deutsch

Ausbildung

12/2009 - 07/2013	Promotionsstudium der Mathematik an der Leibniz Universität Hannover
Titel	*Spectral Triples on Carnot Manifolds*
10/2009	Diplom in Mathematik an der Leibniz Universität Hannover
Titel	*Geometrische und topologische Untersuchungen der Sierpinski-Pyramide mit Methoden der Nichtkommutativen Geometrie*
10/2003 - 10/2009	Diplomstudiengang *Mathematik mit Studienrichtung Informatik* an der Leibniz Universität Hannover
06/2000	Abitur am Hannah Arendt Gymnasium Lengerich

Berufliche Tätigkeit

12/2009 - 07/2013	Wissenschaftlicher Mitarbeiter, Institut für Analysis, Leibniz Universität Hannover
12/2009 - 12/2012	Mitglied im *Graduiertenkolleg 1463 - Analysis, Geometry and String Theory* der DFG