Geometry and Analysis on Black Hole Spacetimes

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Abstract

Black holes are among the most fascinating predictions of Einstein's theory of general relativity. They are expected to be the final state of gravitational collapse. To substantiate this physically reasonable scenario, a proof of non-linear stability of the Kerr solution is needed.

This thesis explores aspects of Maxwell and linearized gravity equations as a model-problem on vacuum spacetimes of Petrov type D. These include the Kerr solution describing a rotating black hole. A combination of exterior and spinor calculus with the Geroch-Held-Penrose (GHP) formalism provides a powerful tool for the analysis.

Decoupled wave-like equations for all tetrad components for the linearized curvature are presented. The fields admit time-independent, finite energy solutions. These non-radiating modes for the linearized gravitational field are investigated, using the concept of spin-lowering with a Killing spinor. Obstructions for the existence of a quasi-local angular momentum charge in terms of curvature are extracted.

Symmetry operators for the Maxwell equations and linearized gravity are discussed. For this reason the Lie derivative of GHP-weighted fields along isometries is introduced. In the discussion of second order operators, the Carter operator is generalized to spin-s fields. Also an anti-linear symmetry operator for the Fackerell-Ipser equation is derived using the method of adjoint operators.

Keywords: General Relativity, black holes, stability

Zusammenfassung

Schwarze Löcher gehören zu den faszinierendsten Vorhersagen der Allgemeinen Relativitätstheorie. Erwartungsgemäß wird das Endstadium eines gravitativen Kollapses durch ein solches Objekt beschrieben. Mit einem Beweis der nichtlinearen Stabilität der Kerr-Lösung kann dieses Szenarium weiter bekräftigt werden.

In dieser Arbeit werden Aspekte der Maxwell Gleichungen und der lineariserten Gravitation als Modellproblem auf Vakuum-Raumzeiten vom Petrov-Typ D untersucht. Diese Klasse beinhaltet die Kerr Lösung eines rotierenden schwarzen Loches. Für die weitere Analyse wird eine Kombination des Differentialformen- und Spinorkalküls mit dem Geroch-Held-Penrose (GHP) Formalismus entwickelt.

Entkoppelte Wellengleichungen für alle Tetradenkomponenten der linearisierten Krümmung werden abgeleitet. Die Felder lassen zeitunabhängige Lösungen endlicher Energie zu. Diese nichtstrahlenden Moden für das linearisierte Gravitationsfeld werden mithilfe von Killing-Spinoren untersucht. Einschränkungen für die Existenz einer quasilokalen Drehimpulsladung aus Krümmungsgrößen werden angegeben.

Im Anschluss werden Symmetrieoperatoren für die Maxwell-Gleichungen sowie die Feldgleichungen der linearisierten Gravitation betrachtet. Aus diesem Grund wird die Lie-Ableitung von GHP-gewichteten Feldern entlang von Isometrien eingeführt. In der Diskussion von Operatoren zweiter Ordnung wird der Carter-Operator auf spin-s Felder verallgemeinert. Weiterhin wird mit der Methode der adjungierten Operatoren ein antilinearer Symmetrieoperator für die Fackerell-Ipser Gleichung hergeleitet.

Schlagworte: Allgemeine Relativitätstheorie, schwarze Löcher, Stabilität

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Contents

1. Introduction

General Relativity is a geometric description of gravity interacting with matter. This dynamical characterization of space and time by Albert Einstein in 1915 has increased our understanding of nature tremendously. The theory, condensed into the at first sight not too complicated looking equation

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab}\,, \tag{1.1}$$

relating the curvature on the left hand side to properties of matter fields on the right hand side, is extraordinary rich of physical effects and mathematical structure. One of its most fascinating predictions is the existence of black holes as asymptotic states of gravitational collapse. For half a century, these extreme scenarios were viewed as mathematical curiosities and only over the years, with deeper physical insights these objects became accepted. Nowadays, there is pretty plausible evidence of a super massive black hole in the center of our galaxy, see e.g. [58]. To quote Chandrasekhar, [29, p. 1]:

> "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well."

The solutions he refers to are the family of Kerr spacetimes with the two parameters for mass M and angular momentum Ma. It is expected to be the end state of gravitational collapse. To substantiate this physically reasonable scenario, a proof of non-linear stability of the Kerr solution is needed. This is a very complicated mathematical problem. An important result in this direction is the proof of non-linear stability of Minkowski spacetime by Christodoulou and Klainerman, [31]. Roughly speaking and heavily oversimplifying, the proof is based on energy estimates adapted to the symmetries of the background. A generalization of these methods towards a proof of non-linear stability of Kerr spacetime creates many difficult problems. A constructive procedure is to split the problem into a series of model problems of increasing complexity. This program enables one to understand the obstructions step by step. There are basically two complexity-increasing "directions". The first is the spacetime under consideration. Examples are the non-rotating ($a = 0$) Schwarzschild solution or the slowly rotating ($a << M$) Kerr solution as intermediate cases. The second direction is to look at various field equations on such backgrounds. Here, it is natural to start with the scalar wave equation and generalize e.g. to higher spin fields. A crucial step in the above mentioned stability proof is the analysis of linear field equations, see [30].

We now briefly review some achievements in this program and refer to chapter 6 for further information. On a Schwarzschild background, decay estimates for the wave equation were proven in [22], [39]. Maxwell equations in that background were analyzed in

[21] and solutions to (1.1) approaching the Schwarzschild spacetime asymptotically were investigated in [67]. The scalar wave equation $\nabla_a \nabla^a \phi = 0$ can be viewed as a model problem for the linearization of the vacuum Einstein equations. Great progress has been made on this equation on a Kerr background in the last years. Boundedness and decay for $a << M$ were proven in [9], see also [52, 40, 107], and very recently results for the Maxwell equations on that background were obtained in [10]. The first results for the wave equation in the whole range $a < M$ are proven in [41].

In order to generalize concepts from Minkowski spacetime to a curved background, a good understanding of the geometry is inevitable. Important examples are the role of symmetries and, strongly related, the existence of a non-trivial Killing tensor on the Kerr background. In the Minkowski limit it becomes a symmetrized product of Killing vectors and so does not contain any new information compared to the isometries. Many miraculous relations of fields on Kerr spacetime are contained in the monograph *The Mathematical Theory of Black Holes*, [29]. In a review of Chandrasekhar's work, Penrose remarks in [89, p. 228]:

> "In view of the many intriguing relationships with different features of integral systems that Chandra's work has thrown up, I feel sure that there is a good deal that is deep, yet to be learned, from a study of the insights that he gained from his work in general relativity.
>
> The same can also be said of a study of his analysis of the separation of gravitational perturbations and of other systems of equations in stationary black hole backgrounds. There is yet much mystery to be unravelled. Some of this has already been achieved in the work of Carter (1968), Walker and Penrose (1970), Carter and McLenaghan (1979), Kamran and McLenaghan (1984) and many others, whereby separation can be related to the existence of a Killing Tensor, Killing spinor and Killing-Yano tensor. There are relations to twistor theory here also, and it is my guess that a further study of Chandra's work from this direction may well throw some profound light on these issues."

The results for Maxwell fields and linearized gravity on Minkowski spacetime in [30] can be understood quite naturally in the null tetrad formalism of Newman and Penrose [87]. There, the Maxwell field strength F_{ab} is encoded in the three complex scalars $\phi_i, i = 0, 1, 2$ and the linearized Weyl curvature $\dot{C}_{abcd}$ is described by the five complex scalars $\dot{\Psi}_i, i = 0, 1, 2, 3, 4$. All these scalars satisfy decoupled wave-like equations and a remarkable fact is that this also holds for a Kerr background. For the "extreme components" $\phi_0, \phi_2, \dot{\Psi}_0, \dot{\Psi}_4$, these are known as Teukolsky equations, [110]. They have the great advantage of being gauge invariant and separable. The vector field method (and so the energy estimates), which are briefly reviewed in section 6.1, have not yet been generalized to handle such complex wave-like equations. On the other hand, gravitational perturbations on a Schwarzschild background can be described by two scalar wave equations with potential, known as Regge-Wheeler [97] and Zerilli [123] equations. Motivated by the intriguing relation to the wave equation for the middle curvature component $\dot{\Psi}_2$, it seems natural to ask for a generalization. Such an equation is presented in (3.39c). Assuming a suitable gauge condition, it is of the form

$$(\nabla^a \nabla_a + c_s \Psi_2)\Phi_s = 0\,,$$

with $s = 2$, $c_2 = 8$, and $\Phi_2 = \Psi_2^{-2/3}\dot{\Psi}_2$. This equation also includes the Fackerell-Ipser equation [50] for the middle Maxwell scalar with $s = 1$, $c_1 = 2$, $\Phi_1 = \Psi_2^{-1/3}\phi_1$ and the scalar wave equation for the case $s = 0$, $c_0 = 0$, $\phi_0 = \phi$. It turned out that the Fackerell-Ipser equation plays an important role in deriving energy estimates for Maxwell fields on Schwarzschild, [21], and Kerr spacetime, [10]. The parameter s is an example of the above mentioned complexity-increasing directions. For $s = 1, 2$ the equation allows for non-trivial, stationary finite energy solutions. These additional solutions are obstructions to decay and its characterization will be discussed in this thesis.

The separability properties of the scalar wave equation on Kerr spacetime can be characterized by symmetry operators which are the Lie derivatives along the two isometries and the Carter operator. These structures are also very important for the vector field method, see [9]. For this reason, symmetry operators for the more general Maxwell and linearized gravity equations are analyzed. They admit symmetry operators of two different types and connections to the above mentioned miraculous relations are extracted. This fits quite nicely into Penrose's suggestion in the above quote and it turns out that some relations occur quite naturally if one uses geometry as a guiding principle. This means in particular that results are coordinate independent.

With the geometric and analytic tools discussed in this thesis, we try to give a glimpse on the upcoming interaction of two fields of research, which only in the last decade began to merge and to benefit from each other. This is on the one hand the theory of gravitational perturbations and on the other hand the analytic tools based on energy estimates.

The thesis is structured as follows. In chapter 2, basics about null tetrad formalisms and spinors are reviewed. Killing spinors, the Lie derivative along isometries and Petrov type D spacetimes are discussed afterwards and in particular properties of the Kerr solutions are reviewed. The spinorial form of the Maxwell equations and Bianchi identities are derived in chapter 3. Then, decoupled equations for linearized gravity are presented and the gauge dependence is reviewed. Chapter 4 is an account on the non-radiating modes of the above mentioned fields. A conservation law for linearized mass is derived and charges for Kerr spacetime are discussed. From the results for Minkowski space, obstructions for an angular momentum charge in terms of linearized curvature are derived. Symmetry operators are introduced in chapter 5. The connection to separability of the Teukolsky equation is clarified and an anti-linear symmetry operator for the Fackerell-Ipser equation is deduced from the method of adjoint operators. The vector fields method is reviewed in chapter 6. Finally, chapter 7 contains a summary and proposals for further research. Parts of the results are already published in [2], [3].

2. Mathematical structures

The theory of General Relativity describes gravity as a geometric effect. In mathematical terms, spacetime is a four-dimensional manifold $\mathcal{N}$ and the notion of distance is described by a symmetric tensor field g_{ab} of Lorentzian signature, the metric. In this chapter we introduce basic notions of Lorentzian geometry. If not stated differently, we will assume the vacuum Einstein equations to hold. Those can be expressed as the condition of vanishing Ricci curvature,

$$R_{ab} = 0\,. \tag{2.1}$$

An important structure in Lorentzian geometry is the light cone spanned by null vectors V^a, $V^aV^bg_{ab} = 0$. A formulation tied to this is the null tetrad formalism of Newman and Penrose [87] (hereafter NP) and its advancement, the Geroch-Held-Penrose [57] (hereafter GHP) formalism. In addition, we will assume the existence of a spinor structure, so that the NP formalism can be traced back to spinors as the fundamental quantities, see [90, 91] for a thorough description.

Exterior calculus on the space of 2-forms (sometimes called bivector formalism, [77, 19, 26, 71]) provides an elegant framework for linearized gravity. Following [71, 37], the equations of structure are derived and applied in chapter 4 and section 5.2. Relations to the GHP formalism are shown.

Properties of Killing spinors and the Petrov classification are reviewed in sections 2.3 and 2.4, respectively. Finally, some properties of Black holes and in particular the Kerr solution are discussed in section 2.5.

Parts of sections 2.2, 2.3 and 2.4 are adapted from my publications [2], [3] with Lars Andersson.

2.1. Preliminaries and notation

We only consider real spacetimes $\mathcal{N}$ with metric g_{ab} of Lorentzian signature $(+,-,-,-)$. Furthermore, we assume the existence of a spinor structure, so that a spinor dyad can be introduced. On globally hyperbolic spacetimes this structure is inherent as shown in [56]. The unique torsion-free and metric Levi-Civita connection will be used throughout the thesis and we denote its covariant derivative by ∇. For convenience, we will sometimes write $X^a\nabla_aT = \nabla_XT$ and $\nabla_aT = T_{;a}$. The "abstract index notation" is used, in which indices merely characterize a quantity instead of denoting components. The notation $2x_{[a}y_{b]} = x_ay_b - y_ax_b$ for anti-symmetrization and $2x_{(a}y_{b)} = x_ay_b + y_ax_b$ for symmetrization is used. Geometric units are used in which the speed of light c and the gravitational constant G are set to 1.

In tensor notation (in the tangent bundle) we use lowercase latin indices $a, b, c, \dots$ with values $0, 1, 2, 3$. For spinors, we use uppercase latin indices $A, B, C, \dots A', B', C', \dots$.

with values $0,1$. Uppercase Latin indices starting at $I,J,K,\dots$ will be used for bivectors and take values in the range $0,1,2$.

2.2. Some differential geometry

The 2-spinor calculus, developed in [90], is a powerful tool to describe 4-dimensional geometries. Following [118, Chapter 13], we introduce a complex 2-dimensional vector space V and its dual $V^* = \{f : V \to \mathbb{C}, f \text{ linear}\}$. There is also the space of anti-linear maps $\bar{V}^*$ and its dual $\bar{V}$. For $v \in V, f \in V^*$, the natural anti-isomorphism $j : V \to \bar{V}$ given by $j(v)(f) = \overline{f(v)}$ defines complex conjugation of vectors. By *spinors* we understand elements in the tensor algebra over the vector spaces $V, V^*, \bar{V}, \bar{V}^*$. Conventionally, the notation $\phi^A \in V$, $\phi_A \in V^*$, $\psi^{A'} \in \bar{V}$ and $\psi_{A'} \in \bar{V}^*$ for their elements is used with the natural generalization to tensor products. The space of anti-symmetric 2-spinors is 1-dimensional and we define ϵ_{AB} to be a representative (fixed up to a complex constant). It is an isomorphism from V^* to V and together with the element ϵ^{AB}, fixed by $\epsilon_{AC}\epsilon^{BC} = \delta_A^B$, used to "raise and lower indices",

$$\epsilon_{AB}\phi^A = \phi_B\,, \qquad \epsilon_{AB}\phi^B = -\phi_A\,, \qquad \epsilon^{AB}\phi_B = \phi^A\,, \qquad \epsilon^{AB}\phi_A = -\phi^B\,.$$

Care has to be taken of the sign change depending on index positions. We will find the mnemonic

$$\textbf{L}\text{eft} \leftrightarrow \textbf{L}\text{owering}\,, \qquad \textbf{R}\text{ight} \leftrightarrow \textbf{R}\text{aising}\,, \tag{2.2}$$

convenient, which prevents us from inserting minus signs.

We start by introducing a spinor dyad o_A, ι_A for V^* and the "spinor metric"

$$\epsilon_{AB} = o_A\iota_B - \iota_A o_B\,, \qquad \epsilon^{AB} = o^A\iota^B - \iota^A o^B\,, \tag{2.3}$$

so that $o_A\iota^A = 1$. It is preserved under dyad transformations

$$L\begin{pmatrix} o_A \\ \iota_A \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} o_A \\ \iota_A \end{pmatrix}, \qquad \det L = \alpha\delta - \beta\gamma = 1\,, \tag{2.4}$$

i.e. $L \in \mathrm{SL}(2,\mathbb{C})$. The spinors $\phi^{A'A}$ comprise a complex 4-dimensional vector space and the subspace of spinors $\phi^{A'A} = \bar{\phi}^{AA'}$ will be identified with the tangent space of $\mathcal{N}$. Another convenient basis for the complexified tangent space is given by

$$l^a = o^A\bar{o}^{A'}\,, \qquad m^a = o^A\bar{\iota}^{A'}\,, \qquad \bar{m}^a = \iota^A\bar{o}^{A'}\,, \qquad n^a = \iota^A\bar{\iota}^{A'}\,. \tag{2.5}$$

This null tetrad,

$$l^a l_a = 0\,, \qquad n^a n_a = 0\,, \qquad m^a m_a = 0\,, \qquad \bar{m}^a\bar{m}_a = 0\,, \tag{2.6}$$

with its real null vectors l^a and n^a adapted to the light-cone is the basis of the NP formalism. Furthermore $l^a n_a = -m^a\bar{m}_a = 1$ and all other inner products being zero. The metric can be expanded into

$$g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}\,. \tag{2.7}$$

Of particular importance in physics are 2-forms $\eta_{ab} = \eta_{[ab]}$, which occur as field strengths in Cartan's equations of structure, see (2.20). Given the spinor dyad above, a natural choice[1] for the 6-dimensional space of 2-forms is

$$Z^0_{ab} = 2\bar{m}_{[a}n_{b]} = \iota_A\iota_B\bar{\epsilon}_{A'B'} \tag{2.8a}$$
$$Z^1_{ab} = 2n_{[a}l_{b]} - 2\bar{m}_{[a}m_{b]} = -2o_{(A}\iota_{B)}\bar{\epsilon}_{A'B'} \tag{2.8b}$$
$$Z^2_{ab} = 2l_{[a}m_{b]} = o_Ao_B\bar{\epsilon}_{A'B'}\,, \tag{2.8c}$$

together with the complex conjugated forms $\bar{Z}^0, \bar{Z}^1, \bar{Z}^2$. The metric g_{ab} induces a triad metric G_{IJ} and its inverse G^{IJ}, given by

$$G^{IJ} = Z^I \cdot Z^J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad G_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here, $\cdot$ is the induced inner product on 2-forms, $Z^I \cdot Z^J = \frac{1}{2}Z^I{}_{ab}Z^{Jab}$. Triad indices are raised and lowered with this metric,

$$Z_0 = Z^2, \qquad Z_1 = -\tfrac{1}{2}Z^1, \qquad Z_2 = Z^0.$$

More generally, we have

Proposition 2.2.1. *The bivectors* (2.8) *satisfy the equations*

$$Z^J{}_a{}^cZ^K{}_{bc} = \frac{1}{2}G^{JK}g_{ab} + \epsilon^{JKL}Z_{Lab}\,, \tag{2.9a}$$
$$Z^J{}_{[a}{}^c\bar{Z}^K{}_{b]c} = 0\,, \tag{2.9b}$$
$$Z^{Jab}\bar{Z}^K{}_{ab} = 0\,, \tag{2.9c}$$

with ϵ^{JKL} the totally antisymmetric symbol fixed by $\epsilon^{012} = 1$.

The action of the 6-dimensional group of Lorentz transformations can be decomposed into three abelian subgroups ($a, b, \lambda \in \mathbb{C}$),

I null rotations around $l^c : n^c \to n^c + a\bar{m}^c + \bar{a}m^c + a\bar{a}l^c\,, m^c \to m^c + al^c\,,$ (2.10a)

II null rotations around $n^c : l^c \to l^c + b\bar{m}^c + \bar{b}m^c + b\bar{b}n^c\,, m^c \to m^c + bn^c\,,$ (2.10b)

III nl-boosts and $\bar{m}m$-rotations : $l^c \to \lambda\bar{\lambda}l^c,\ n^c \to (\lambda\bar{\lambda})^{-1}n^c,\ m^c \to \lambda\bar{\lambda}^{-1}m^c\,,$ (2.10c)

leaving the tetrad normalization invariant. Restricting to transformations of the third class[2], parametrized by a complex scalar λ, we define the notion of weighted fields.

Definition 2.2.2. *A field T_α, with some (multi)-index α, transforming under* (2.10c) *according to*

$$T_\alpha \to (\lambda\bar{\lambda})^b(\lambda\bar{\lambda}^{-1})^sT_\alpha = \lambda^p\bar{\lambda}^qT_\alpha\,, \tag{2.11}$$

is of spin-weight s and boost-weight b. Here, s and b can be arbitrary integers and we also defined the weights $p = b + s$ and $q = b - s$.

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[1] We use the convention of [51], which differs from [71, 49] by a factor of 2 in the middle component and the numbering.

[2] This is of particular interest for geometries having distinguished null directions, see section 2.4 below.

The (p,q) weights encode the same information as s and b but are sometimes more convenient to work with (e.g. components of spin-s fields are $q=0$ weighted).

Example 2.2.3. *The NP tetrad* (2.5) *is by definition of* (p,q) *weight*

$$l^a:(1,1)\,,\qquad n^a:(-1,-1)\,,\qquad m^a:(1,-1)\,,\qquad \bar{m}^a:(-1,1)\,. \tag{2.12}$$

The representation of class III on the spinor dyad yields $o_A\to\lambda o_A$ *and* $\iota_A\to\lambda^{-1}\iota_A$*, so they are of* (p,q) *weight*

$$o_A:(1,0)\,,\qquad \iota_A:(-1,0)\,,\qquad \bar{o}_A:(0,1)\,,\qquad \bar{\iota}_A:(0,-1)\,. \tag{2.13}$$

We also find the weights of the 2-forms (2.8) *to be*

$$Z^0:(-2,0)\,,\qquad Z^1:(0,0)\,,\qquad Z^2:(2,0)\,. \tag{2.14}$$

We note that tensor fields are a priori unweighted and that NP tetrad vectors (or any other fields from the example above) can be understood as maps from the cotangent space into the space (complex line-bundle) of weighted fields. Also partial projection is possible, e.g. for an unweighted field H_{ab} we have $H_{ab}l^b:(1,1)$. The spinor dyad (o_A,ι_A) makes also half integer spin and boost weights possible.

Remark 2.2.4. *In GHP formalism, discrete transformations of the tetrad,*

$$':l^a\leftrightarrow n^a\,,m^a\leftrightarrow\bar{m}^a\,, \tag{2.15a}$$

$$^-:m^a\leftrightarrow\bar{m}^a\,, \tag{2.15b}$$

$$*:l^a\to m^a\,,n^a\to-\bar{m}^a\,,m^a\to-l^a\,,\bar{m}^a\to n^a\,, \tag{2.15c}$$

denoted prime[3]*, bar and star, respectively, are introduced. The number of equations can be effectively reduced by using these operations. We note, that* $''=id$ *for tetrad projections and find for the 2-forms* (2.8)*,*

$$Z^{0\prime}=-Z^2\,,\qquad Z^{1\prime}=-Z^1\,,\qquad Z^{2\prime}=-Z^0\,. \tag{2.16}$$

For later use it is convenient to define connection 1-forms σ_{Ia} or equivalently the NP spin coefficients $\kappa,\kappa',\sigma,\sigma',\tau,\tau',\rho,\rho',\epsilon,\epsilon',\beta,\beta'$ by

$$\sigma_{0a}=m^b\nabla_a l_b \qquad = \tau l_a+\kappa n_a-\rho m_a-\sigma\bar{m}_a\,, \tag{2.17a}$$

$$\sigma_{1a}=\frac{1}{2}\left(n^b\nabla_a l_b-\bar{m}^b\nabla_a m_b\right)=-\epsilon' l_a+\epsilon n_a+\beta' m_a-\beta\bar{m}_a\,, \tag{2.17b}$$

$$\sigma_{2a}=-\bar{m}^b\nabla_a n_b \qquad = -\kappa' l_a-\tau' n_a+\sigma' m_a+\rho'\bar{m}_a\,. \tag{2.17c}$$

The middle component σ_{1a} is not properly weighted and transforms inhomogeneously via

$$\begin{aligned}\sigma_{1a}\to&\frac{1}{2}\Big(\frac{1}{\lambda\bar{\lambda}}n^b\nabla_a(\lambda\bar{\lambda}l_b)+\frac{\lambda}{\bar{\lambda}}m^b\nabla_a(\frac{\bar{\lambda}}{\lambda}\bar{m}_b)\Big)\\ =&\,\sigma_{1a}+\frac{1}{2}(\nabla_a\ln(\lambda\bar{\lambda})-\nabla_a\ln\frac{\bar{\lambda}}{\lambda})\\ =&\,\sigma_{1a}+\frac{\nabla_a\lambda}{\lambda}\,,\end{aligned}$$

[3]The symbol $'$ will never be used for a derivative in this thesis, but always denote the GHP prime or a spinor index. This should be clear from context.

under (2.10c). It defines the GHP connection, [57],

$$\Theta_a T_\alpha = (\nabla_a - p\sigma_{1a} - q\bar{\sigma}_{1a}) T_\alpha \,, \tag{2.18}$$

on the line-bundle of weighted fields, because on a (p, q) weighted field T_α we find

$$\begin{aligned}
\Theta_a T_\alpha \to & \nabla_a(\lambda^p \bar{\lambda}^q T_\alpha) - \left[p\sigma_{1a} + p\frac{\nabla_a \lambda}{\lambda} + q\bar{\sigma}_{1a} + \frac{\nabla_a \bar{\lambda}}{\bar{\lambda}} \right] \lambda^p \bar{\lambda}^q T_\alpha \\
&= \lambda^p \bar{\lambda}^q \Theta_a T_\alpha + T_\alpha \left[\nabla_a(\lambda^p \bar{\lambda}^q) - p\lambda^p \bar{\lambda}^q \frac{\nabla_a \lambda}{\lambda} - q\lambda^p \bar{\lambda}^q \frac{\nabla_a \bar{\lambda}}{\bar{\lambda}} \right] \\
&= \lambda^p \bar{\lambda}^q \Theta_a T_\alpha \,,
\end{aligned}$$

so Θ_a is of proper weight $(0, 0)$. Since the middle connection form σ_1 is absorbed into the connection, we will sometimes find it convenient to write $\sigma_0 = \Gamma$ and $\sigma_2 = -\Gamma'$. Derivatives of the spinor dyad can now be written in the compact form

$$\Theta_a o^A = -\Gamma_a \iota^A \,, \qquad \Theta_a \iota^A = -\Gamma'_a o^A \,. \tag{2.19}$$

Next, we will rederive the equations of structure in bivector formalism following [71], see also [77, 19, 26]. This provides an intermediate description between the tensorial and component formulation. Making use of Cartan's equations of structure for tetrad 1-forms[4]

$$\mathrm{d}e^{\underline{a}} = -\omega^{\underline{a}}{}_{\underline{b}} \wedge e^{\underline{b}} \,, \qquad \Omega^{\underline{a}}{}_{\underline{b}} = \mathrm{d}\omega^{\underline{a}}{}_{\underline{b}} + \omega^{\underline{a}}{}_{\underline{c}} \wedge \omega^{\underline{c}}{}_{\underline{b}} \,, \tag{2.20}$$

Bianchi identities

$$\Omega^{\underline{a}}{}_{\underline{b}} \wedge e^{\underline{b}} = 0 \,, \qquad \mathrm{d}\Omega^{\underline{a}}{}_{\underline{b}} = \Omega^{\underline{a}}{}_{\underline{c}} \wedge \omega^{\underline{c}}{}_{\underline{b}} - \omega^{\underline{a}}{}_{\underline{c}} \wedge \Omega^{\underline{c}}{}_{\underline{b}} \,, \tag{2.21}$$

the relation to the connection 1-forms σ_J in (2.17) and the definition of curvature 2-forms Σ_J,

$$\omega_{\underline{ab}}\, e^{\underline{a}} \wedge e^{\underline{b}} = -2\sigma_J Z^J - 2\bar{\sigma}_J \bar{Z}^J \,, \qquad \Omega_{\underline{ab}} e^{\underline{a}} \wedge e^{\underline{b}} = -2\Sigma_J Z^J - 2\bar{\Sigma}_J \bar{Z}^J \,, \tag{2.22}$$

we find the following result.

Proposition 2.2.5. *The bivector equations of structure are*

$$\mathrm{d}Z^J = -2\epsilon^{JKL} \sigma_K \wedge Z_L \,, \qquad \Sigma_J = \mathrm{d}\sigma_J + \frac{1}{2}\epsilon_{JKL}\sigma^K \wedge \sigma^L \,, \tag{2.23}$$

while the Bianchi identities read

$$\Sigma_{[J} \wedge Z_{K]} = 0 \,, \qquad d\Sigma_J = -\epsilon_{JKL}\Sigma^K \wedge \sigma^L \,. \tag{2.24}$$

Here $\wedge$ is the usual wedge product of 1-forms σ^J and 2-forms Z^J, Σ^J.

[4] For a given tetrad $e^{\underline{a}}{}_a$, connection and curvature are defined by $\omega^{\underline{a}}{}_{\underline{b}a} = e^{\underline{a}}{}_b \nabla_a e_{\underline{b}}{}^b$ and $\Omega^{\underline{a}}{}_{\underline{b}ab} = 2e^{\underline{a}}{}_c \nabla_{[a}\nabla_{b]} e_{\underline{b}}{}^c$, respectively. The underlined indices number the tetrad components.

Proof. Expanding the bivectors $Z^J = \frac{1}{2} Z^J_{\underline{ab}} e^{\underline{a}} \wedge e^{\underline{b}}$, we find

$$\begin{aligned}
\mathrm{d}Z^J &= \frac{1}{2} Z^J_{\underline{ab}} \left(\mathrm{d}e^{\underline{a}} \wedge e^{\underline{b}} - e^{\underline{a}} \wedge \mathrm{d}e^{\underline{b}} \right) = Z^J_{\underline{ab}} \mathrm{d}e^{\underline{a}} \wedge e^{\underline{b}} \\
&= -Z^J_{\underline{ab}} \omega^{\underline{a}}{}_{\underline{c}} e^{\underline{c}} \wedge e^{\underline{b}} \\
&= Z^J_{\underline{ab}} \left(\sigma_K Z^{K\underline{a}}{}_{\underline{c}} + \bar{\sigma}_K \bar{Z}^{K\underline{a}}{}_{\underline{c}} \right) \wedge e^{\underline{c}} \wedge e^{\underline{b}} \\
&= \epsilon^{JKL} Z_{L\underline{bc}} \sigma_K \wedge e^{\underline{c}} \wedge e^{\underline{b}} \\
&= -2\epsilon^{JKL} \sigma_K \wedge Z_L \,,
\end{aligned}$$

where proposition 2.2.1 has been used in the third step. For the second equation of structure, we plug (2.22) into (2.20),

$$-\Sigma_J Z^J_{ab} - \bar{\Sigma}_J \bar{Z}^J_{ab} = -\mathrm{d}\sigma_J Z^J_{ab} - \mathrm{d}\bar{\sigma}_J \bar{Z}^J_{ab} + \left(\sigma_J Z^J_{ac} + \bar{\sigma}_J \bar{Z}^J_{ac} \right) \wedge \left(\sigma_K Z^{Kc}{}_b + \bar{\sigma}_K \bar{Z}^{Kc}{}_b \right) .$$

Using proposition 2.2.1, the self-dual part reads

$$\Sigma_J Z^J_{ab} = \mathrm{d}\sigma_J Z^J_{ab} + \epsilon^{KLJ} Z_{Jab} \sigma_K \wedge \sigma_L \,.$$

Changing index positions on ϵ^{KLJ} and using $\det G_{JK} = \frac{1}{2}$ gives the second equation of structure. For the first Bianchi identity, look at

$$\begin{aligned}
0 &= \frac{1}{2} \mathrm{d}^2 Z^J \\
&= -\epsilon^{JKL} \left(\mathrm{d}\sigma_K \wedge Z_L - \sigma_K \wedge \mathrm{d}Z_L \right) \\
&= -\epsilon^{JKL} \left(\Sigma_K \wedge Z_L - \frac{1}{2} \epsilon_{KNM} \sigma^N \wedge \sigma^M \wedge Z_L + \sigma_K \wedge \epsilon_{LNM} \sigma^N \wedge Z^M \right) \\
&= -\epsilon^{JKL} \Sigma_K \wedge Z_L \underbrace{+\sigma^L \wedge \sigma^J \wedge Z_L - \sigma^J \wedge \sigma^L \wedge Z_L - 2\sigma_L \wedge \sigma^J \wedge Z^L}_{=0} + \underbrace{\sigma_K \wedge \sigma^K}_{=0} \wedge Z^J \,,
\end{aligned}$$

where the identity $\epsilon^{IJK} \epsilon_{INM} = \delta^J_N \delta^K_M - \delta^J_M \delta^K_N$ has been used. Finally, the second Bianchi identity is given by

$$\begin{aligned}
\mathrm{d}\Sigma_J &= -\epsilon_{JKL} \mathrm{d}\sigma^K \wedge \sigma^L \\
&= -\epsilon_{JKL} (\Sigma^K - \epsilon^{KMN} \sigma_M \wedge \sigma_N) \wedge \sigma^L \\
&= -\epsilon_{JKL} \Sigma^K \wedge \sigma^L + \underbrace{\sigma_L \wedge \sigma_J \wedge \sigma^L}_{=0} - \sigma_J \wedge \underbrace{\sigma_L \wedge \sigma^L}_{=0} \,.
\end{aligned}$$

□

Remark 2.2.6. *Instead of using Cartan equations* (2.20) *for the tetrad and translating to 2-forms, one could have used directly the bivector connection form*

$$\omega_{IJa} := \epsilon_{IJK} \sigma^K_a = Z^{bc}_{[J} \nabla_a Z_{I]bc} \,. \tag{2.25}$$

Let us collect some equations which will turn out to be useful for later calculations. The components of the first equations of structure in (2.23) read

$$\mathrm{d}^\Theta Z^0 = \Gamma' \wedge Z^1 \qquad \Leftrightarrow \qquad \mathrm{d}Z^0 = -2\sigma_1 \wedge Z^0 - \sigma_2 \wedge Z^1, \tag{2.26a}$$

$$\mathrm{d}^\Theta Z^1 = 2\Gamma \wedge Z^0 + 2\Gamma' \wedge Z^2 \qquad \Leftrightarrow \qquad \mathrm{d}Z^1 = 2\sigma_0 \wedge Z^0 - 2\sigma_2 \wedge Z^2, \tag{2.26b}$$

$$\mathrm{d}^\Theta Z^2 = \Gamma \wedge Z^1 \qquad \Leftrightarrow \qquad \mathrm{d}Z^2 = 2\sigma_1 \wedge Z^2 + \sigma_0 \wedge Z^1. \tag{2.26c}$$

Here we introduced a covariant exterior derivative $\mathrm{d}^\Theta = \mathrm{d} - p\sigma_1 \wedge -q\bar{\sigma}_1\wedge$, cf. (2.18), to present a properly weighted form of the equations. Note that the middle component can be simplified to $\mathrm{d}Z^1 = -h \wedge Z^1$ with the 1-form

$$h_a = 2(\rho' l_a + \rho n_a - \tau' m_a - \tau \bar{m}_a)\,. \tag{2.27}$$

This fact and a relation between type D curvature Ψ_2 and h, see (2.60) below, will be crucial in the derivation of a conservation law in section 4.3.2. The components of the second equation of structure in (2.23) read

$$\Sigma_0 = \mathrm{d}^\Theta \Gamma \quad \Leftrightarrow \quad \Sigma_0 = \mathrm{d}\sigma_0 - 2\sigma_1 \wedge \sigma_0\,, \tag{2.28a}$$
$$\Sigma_1 = \mathrm{d}\sigma_1 - \Gamma \wedge \Gamma' \quad \Leftrightarrow \quad \Sigma_1 = \mathrm{d}\sigma_1 + \sigma_0 \wedge \sigma_2\,, \tag{2.28b}$$
$$\Sigma_2 = -\mathrm{d}^\Theta \Gamma' \quad \Leftrightarrow \quad \Sigma_2 = \mathrm{d}\sigma_2 + 2\sigma_1 \wedge \sigma_2\,. \tag{2.28c}$$

The middle equation is of weight $(0,0)$ because $\mathrm{d}^2 \ln \lambda = 0$.

From the point of view of weighted line-bundles, it does not belong to the second equations of structure. It is rather the curvature of the bundle connection σ_{1a}. In GHP formalism this is sometimes phrased as "part of the Ricci identities are encoded in the commutators", see (2.38). The Weyl scalars are

$$\Psi_0 = \Psi_{ABCD}\, o^A o^B o^C o^D = -C_{abcd} l^a m^b l^c m^d \; = -C \cdot (Z_0, Z_0)\,, \tag{2.29a}$$
$$\Psi_1 = \Psi_{ABCD}\, o^A o^B o^C \iota^D = -C_{abcd} l^a n^b l^c m^d \; = -C \cdot (Z_0, Z_1)\,, \tag{2.29b}$$
$$\Psi_2 = \Psi_{ABCD}\, o^A o^B \iota^C \iota^D = -C_{abcd} l^a m^b \bar{m}^c n^d \; = -C \cdot (Z_0, Z_2) = C \cdot (Z_1, Z_1)\,, \tag{2.29c}$$
$$\Psi_3 = \Psi_{ABCD}\, o^A \iota^B \iota^C \iota^D = -C_{abcd} l^a n^b \bar{m}^c n^d \; = -C \cdot (Z_2, Z_1)\,, \tag{2.29d}$$
$$\Psi_4 = \Psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d = -C \cdot (Z_2, Z_2)\,. \tag{2.29e}$$

They contain the full information of the Weyl curvature $C_{abcd} = \Omega_{\underline{ab}ab} e^{\underline{a}}{}_c e^{\underline{b}}{}_d$ (in vacuum) and we used the notation $C \cdot (Z, Z) = \frac{1}{4} C_{abcd} Z^{ab} Z^{cd}$. The curvature spinor Ψ_{ABCD} is introduced in (3.21). The GHP prime (2.15a) yields $\Psi_4' = \Psi_0$, $\Psi_3' = \Psi_1$, $\Psi_2' = \Psi_2$ and the curvature 2-forms Σ_J can now be expanded into

$$\Sigma_I = C_{IJ} Z^J\,, \qquad C_{IJ} = \begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix}, \tag{2.30}$$

with the Weyl 2-bivector C_{IJ}. Finally, the Bianchi identities in (2.24) can be decomposed into

$$\mathrm{d}^\Theta \Sigma_0 = -2\Gamma \wedge \Sigma_1 \quad \Leftrightarrow \quad \mathrm{d}\Sigma_0 = 2\sigma_1 \wedge \Sigma_0 - 2\sigma_0 \wedge \Sigma_1, \tag{2.31a}$$
$$\mathrm{d}^\Theta \Sigma_1 = -\Gamma' \wedge \Sigma_0 - \Gamma \wedge \Sigma_2 \quad \Leftrightarrow \quad \mathrm{d}\Sigma_1 = \sigma_0' \wedge \Sigma_0 - \sigma_1 \wedge \Sigma_2, \tag{2.31b}$$
$$\mathrm{d}^\Theta \Sigma_2 = -2\Gamma' \wedge \Sigma_1 \quad \Leftrightarrow \quad \mathrm{d}\Sigma_2 = -2\sigma_1 \wedge \Sigma_2 + 2\sigma_2 \wedge \Sigma_1. \tag{2.31c}$$

The GHP derivatives are defined to be the projections of the covariant derivative (2.18),

$$\text{þ} = l^a \Theta_a\,, \qquad \text{þ}' = n^a \Theta_a\,, \qquad \eth = m^a \Theta_a\,, \qquad \eth' = \bar{m}^a \Theta_a\,, \tag{2.32}$$

and so $\Theta = l^a \text{þ}' + n^a\, \text{þ} - m^a \eth' - \bar{m}^a\, \eth$. The divergence of a vector field X^a expands into

$$\Theta_a X^a = (\text{þ} - \rho - \bar{\rho}) X^n + (\text{þ}' - \rho' - \bar{\rho}') X^l - (\eth - \bar{\tau}' - \tau) X^{\bar{m}} - (\eth' - \tau' - \bar{\tau}) X^m\,. \tag{2.33}$$

For NP based calculations it turns out to be very useful to have explicit expressions for the weighted derivatives of the tetrad,

$$\begin{aligned}\Theta_a l_b &= -(\overline{\tau} l_a + \overline{\kappa} n_a - \overline{\sigma} m_a - \overline{\rho m}_a) m_b - (\tau l_a + \kappa n_a - \rho m_a - \sigma \overline{m}_a)\overline{m}_b \\ &= -\overline{\sigma}_{0a} m_b - \sigma_{0a}\overline{m}_b \,, \qquad (2.34a)\\ \Theta_a n_b &= -(\overline{\kappa}' l_a + \overline{\tau}' n_a - \overline{\rho}' m_a - \overline{\sigma}'\overline{m}_a)\overline{m}_b - (\kappa' l_a + \tau' n_a - \sigma' m_a - \rho'\overline{m}_a) m_b \\ &= \overline{\sigma}_{2a}\overline{m}_b + \sigma_{2a} m_b \,, \qquad (2.34b)\\ \Theta_a m_b &= -(\overline{\kappa}' l_a + \overline{\tau}' n_a - \overline{\rho}' m_a - \overline{\sigma}'\overline{m}_a) l_b - (\tau l_a + \kappa n_a - \rho m_a - \sigma\overline{m}_a) n_b \\ &= \overline{\sigma}_{2a} l_b - \sigma_{0a} n_b \,, \qquad (2.34c)\\ \Theta_a \overline{m}_b &= -(\kappa' l_a + \tau' n_a - \sigma' m_a - \rho'\overline{m}_a) l_b - (\overline{\tau} l_a + \overline{\kappa} n_a - \overline{\sigma} m_a - \overline{\rho m}_a) n_b \\ &= \sigma_{2a} l_b - \overline{\sigma}_{0a} n_b \,, \qquad (2.34d)\end{aligned}$$

following from (2.17). So the GHP derivatives of the tetrad are,

$$\begin{aligned}
þ\, l_a &= -\overline{\kappa} m_a - \kappa \overline{m}_a \,, & þ\, m_a &= -\overline{\tau}' l_a - \kappa n_a \,, \qquad (2.35a)\\
þ' l_a &= -\overline{\tau} m_a - \tau \overline{m}_a \,, & þ' m_a &= -\overline{\kappa}' l_a - \tau n_a \,, \qquad (2.35b)\\
ð\, l_a &= -\overline{\rho} m_a - \sigma \overline{m}_a \,, & ð\, m_a &= -\overline{\sigma}' l_a - \sigma n_a \,, \qquad (2.35c)\\
ð' l_a &= -\overline{\sigma} m_a - \rho \overline{m}_a \,, & ð' m_a &= -\overline{\rho}' l_a - \rho n_a \,, \qquad (2.35d)
\end{aligned}$$

$$\begin{aligned}
þ\, n_a &= -\tau' m_a - \overline{\tau}' \overline{m}_a \,, & þ\, \overline{m}_a &= -\tau' l_a - \overline{\kappa} n_a \,, \qquad (2.35e)\\
þ' n_a &= -\kappa' m_a - \overline{\kappa}' \overline{m}_a \,, & þ' \overline{m}_a &= -\kappa' l_a - \overline{\tau} n_a \,, \qquad (2.35f)\\
ð\, n_a &= -\rho' m_a - \overline{\sigma}' \overline{m}_a \,, & ð\, \overline{m}_a &= -\rho' l_a - \overline{\rho} n_a \,, \qquad (2.35g)\\
ð' n_a &= -\sigma' m_a - \overline{\rho}' \overline{m}_a \,, & ð' \overline{m}_a &= -\sigma' l_a - \overline{\sigma} n_a \,, \qquad (2.35h)
\end{aligned}$$

and we also find the weighted derivatives of the 2-forms (2.8) to be

$$\begin{aligned}\Theta_a Z^0_{bc} &= \Gamma'_a Z^1_{bc} \,, \qquad (2.36a)\\ \Theta_a Z^1_{bc} &= 2\Gamma_a Z^0_{bc} + 2\Gamma'_a Z^2_{bc} \,, \qquad (2.36b)\\ \Theta_a Z^2_{bc} &= \Gamma_a Z^1_{bc} \,. \qquad (2.36c)\end{aligned}$$

The GHP equations are found to be the tetrad components of (2.28) and (2.31). For convenience, we give the list of scalar equations (the full list contains the GHP primed (2.15a) and complex conjugated, (2.15b), versions as well). The GHP form of the second equations of structure (2.28) are the vacuum Ricci identities,

$$\begin{aligned}(þ - \rho)\rho &= (ð' - \tau')\kappa + \sigma\overline{\sigma} - \overline{\kappa}\tau \,, \qquad (2.37a)\\ (ð - \tau)\tau &= (þ' - \rho')\sigma - \overline{\sigma}'\rho + \kappa\overline{\kappa}' \,, \qquad (2.37b)\\ (ð - \tau)\rho &= ð'\sigma - \overline{\rho}\tau + (\overline{\rho}' - \rho')\kappa - \Psi_1 \,, \qquad (2.37c)\\ (þ - \rho)\tau &= þ'\kappa - \overline{\tau}'\rho + (\overline{\tau} - \tau')\sigma + \Psi_1 \,, \qquad (2.37d)\\ (þ' - \overline{\rho}')\rho &= (ð' - \overline{\tau}')\tau + \sigma\sigma' - \kappa\kappa' - \Psi_2 \,, \qquad (2.37e)\\ (þ - \rho - \overline{\rho})\sigma &= (ð - \tau - \overline{\tau}')\kappa + \Psi_0 \,, \qquad (2.37f)\end{aligned}$$

and GHP commutators,

$$\begin{aligned}[þ, þ'] &= (\bar{\tau} - \tau')\, ð + (\tau - \bar{\tau}')ð' - p(\kappa\kappa' - \tau\tau' + \Psi_2) - q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2) \,, \qquad (2.38a)\\ [þ, ð] &= \bar{\rho}\, ð + \sigma ð' - \bar{\tau}'\, þ - \kappa þ' - p(\rho'\kappa - \tau'\sigma + \Psi_1) - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}') \,, \qquad (2.38b)\\ [ð, ð'] &= (\bar{\rho}' - \rho')\, þ + (\rho - \bar{\rho})þ' + p(\rho\rho' - \sigma\sigma' + \Psi_2) - q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\Psi}_2) \,. \qquad (2.38c)\end{aligned}$$

The Bianchi identities (2.31) expand into

$$(\text{þ} - 4\rho)\Psi_1 = (\eth' - \tau')\Psi_0 - 3\kappa\Psi_2\,, \tag{2.39a}$$
$$(\text{þ} - 3\rho)\Psi_2 = (\eth' - 2\tau')\Psi_1 + \sigma'\Psi_0 - 2\kappa\Psi_3\,, \tag{2.39b}$$
$$(\text{þ} - 2\rho)\Psi_3 = (\eth' - 3\tau')\Psi_2 + 2\sigma'\Psi_1 - \kappa\Psi_4\,, \tag{2.39c}$$
$$(\text{þ} - \rho)\Psi_4 = (\eth' - 4\tau')\Psi_3 + 3\sigma'\Psi_2\,. \tag{2.39d}$$

2.3. Killing spinors and conformal Killing-Yano tensors

An interesting mathematical structure is the valence (n, m) Killing spinor equation,

$$\nabla^{(A'}{}_{(A}\kappa^{A'_1 \dots A'_m)}{}_{A_1 \dots A_n)} = 0\,, \tag{2.40}$$

for a spinor $\kappa^{A'_1 \dots A'_m}_{A_1 \dots A_n}$, symmetric in its m primed and n unprimed indices. It unifies certain well known equations, e.g. for

- $n = 1, m = 0$, it is known as twistor equation

$$\nabla_{A'(A}\kappa_{B)} = 0\,. \tag{2.41}$$

- $n = 1 = m$, it reduces to the conformal Killing vector equation,

$$\nabla_{(a}\kappa_{b)} - \tfrac{1}{4} g_{ab}\nabla^c\kappa_c = 0\,, \tag{2.42}$$

  for a complex vector $\kappa^a = \kappa^{A'A}$, see [91, p. 82].

- $n = 2, m = 0$, it is the conformal Killing-Yano equation,

$$Y_{a(b;c)} = g_{bc}\xi_a - g_{a(b}\xi_{c)}, \text{ with } \xi_a = \tfrac{1}{3}\nabla^b Y_{ab}\,, \tag{2.43}$$

  for a complex 2-form $Y_{ab} = \kappa_{AB}\epsilon_{A'B'}$, [91, p. 77]. This is the case of most interest in this thesis[5].

- $n = 2 = m$ is the conformal Killing tensor equation for a complex, trace-free symmetric tensor $K_{ab} = \kappa_{A'AB'B}$, [91, p. 106],

$$K_{(ab;c)} = g_{(ab}K_{c)}, \text{ with } K_c = \tfrac{1}{6}(2K^a{}_{c;a} + K^a{}_{a;c})\,. \tag{2.44}$$

Note that symmetrized products of valence (n, m) and (k, l) solutions to (2.40), are solutions of valence $(n + k, m + l)$. A solution which can not be represented as a product is called *irreducible*. Moreover on a vacuum background, the fields

$$\psi^{A'}{}_{A_2 \dots A_n} = \nabla^{A'A_1}\kappa_{A_1 \dots A_n}\,, \qquad \chi^{A'_2 \dots A'_m}{}_A = \nabla_{A'_1 A}\kappa^{A'_1 \dots A'_m}\,, \tag{2.45}$$

[5] The more general conformal Killing-Yano equation for n-forms does not contain any new information in four dimensions, because the dual of a solution is again a solution of rank $4 - n$, see [55, p. 427]. In particular conformal Killing-Yano 3-forms always correspond to conformal Killing vectors (2.42).

for $n, m = 2, 3, \dots$ are solutions of valence $(n-1, 1)$ and $(1, m-1)$, respectively. This follows from the commutator relations (2.6) in [7] together with the footnote on p. 14. For $n = 2$, the first equation expresses the fact that the divergence of a valence $(2, 0)$ Killing spinor is a conformal Killing vector. Another interesting connection along the same lines is that the anti-symmetrized derivative of a Killing vector is a spin-1 field, sometimes called Killing potential, see section 4.2.3. We shift further discussion of the deep relationship between all these quantities to chapter 7 and focus now on the Killing spinor equation (2.40). It is heavily restricted in curved spacetime due to an integrability condition[6]. For convenience, we restrict to valence $(n, 0)$ Killing spinors in vacuum. Contracting a second derivative $\nabla^{A'}{}_B$ on (2.40) and symmetrizing gives[7]

$$\begin{aligned} 0 &= \nabla^{A'}{}_{(B}\nabla_{|A'|A}\kappa_{A_1\dots A_n)} \\ &= -\,\Box_{(AB}\,\kappa_{A_1\dots A_n)} \\ &= \Psi_{(ABA_1}{}^{C}\kappa_{CA_2\dots A_n)} + \dots + \Psi_{(ABA_n}{}^{C}\kappa_{A_1\dots A_{n-1}C)} \\ &= n\Psi_{(ABA_1}{}^{C}\kappa_{CA_2\dots A_n)}\,. \end{aligned} \tag{2.46}$$

For Killing spinors of valence $(1, 0)$ this yields $0 = \Psi_{ABCD}\kappa^D$, so κ^D has to be a four-fold principal spinor. To find any non trivial solution, the spacetime has to be of Petrov type N or O. We will now stick to the valence $(2, 0)$ case,

$$\nabla_{A'(A}\kappa_{BC)} = 0\,, \tag{2.47}$$

because it has an irreducible solution in physically interesting spacetimes and will be of frequent use in subsequent sections. The above integrability condition yields

$$0 = \Psi_{(ABC}{}^{D}\kappa_{DE)}\,. \tag{2.48}$$

The existence of a non-trivial solution κ_{AB} to (2.47) restricts the spacetime to be of Petrov type D, N or O. On the other hand, on a spactimes of Petrov type D, (2.48) restricts the space of Killing spinors considerably, see section 2.4. A divergence of (2.47) yields

$$\Box\kappa_{AB} = \Psi_{ABCD}\kappa^{CD}\,, \tag{2.49}$$

as follows from the footnote on p. 14. The divergence of the Killing spinor itself,

$$\xi_{AA'} = \frac{1}{3}\nabla^b Y_{ab} = -\frac{1}{3}\nabla_{A'}{}^{B}\kappa_{AB}\,, \tag{2.50}$$

introduced in (2.43) as an abbreviation, is not only a conformal Killing vector as follows from (2.45), but a proper Killing vector because $\nabla^{A'A}\xi_{A'A} = 3\,\Box^{AB}\,\kappa_{AB} = 0$. In case it vanishes, Y_{ab} in (2.43) solves the Killing-Yano equation

$$Y_{a(b;c)} = 0\,. \tag{2.51}$$

[6]This is similar to the restrictions on spin-s fields due to the Buchdahl constraint, see section 3.1.

[7] We use the notation $2\nabla_{[a}\nabla_{b]} = \epsilon_{A'B'}\,\Box_{AB} + \epsilon_{AB}\Box_{A'B'}$ with $\Box_{AB} = \epsilon^{A'B'}\nabla_{[AA'}\nabla_{BB']} = \nabla_{A'(A}\nabla_{B)}{}^{A'}$, following [91, Section 4.9]. In vacuum, this derivation acts via $\Box_{AB}\phi_C{}^D{}_{C'}{}^{D'} = -\Psi_{ABC}{}^E\phi_E{}^D{}_{C'}{}^{D'} + \Psi_{AB}{}^D{}_E\phi_C{}^E{}_{C'}{}^{D'}$. Note also the decomposition $\nabla_{A'A}\nabla^{A'}{}_B = \Box_{AB} - \frac{1}{2}\epsilon_{AB}\Box$.

Symmetrized products $X_{c(a}Y_{b)}{}^{c} = K_{ab}$ of conformal Killing-Yano tensors X_{ab}, Y_{ab} are conformal Killing tensors (2.44) as already mentioned above. If X_{ab} and Y_{ab} have vanishing divergence (2.50), then K_{ab} is a solution to the Killing tensor equation

$$\nabla_{(a}K_{bc)} = 0 . \tag{2.52}$$

The metric g_{ab} is a solution, but it is pure trace and therefore not constructable from a Killing spinor. More general the conformal metric $f g_{ab}$ is a conformal Killing tensor for any function f. The well known applications of Killing tensors in constructing constants of motion for geodesics and symmetry operators for scalar waves due to Carter [27] will be reviewed in chapter 5.

The projection of (2.47) into a spinor dyad yields for the components $\kappa_{AB} = \kappa_0 \iota_A \iota_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_2 o_A o_B$ the following set of eight scalar equations:

$$þ\,\kappa_0 = -2\kappa\kappa_1\,, \quad ð\,\kappa_0 = -2\sigma\kappa_1\,, \quad þ'\kappa_2 = -2\kappa'\kappa_1\,, \quad ð'\kappa_2 = -2\sigma'\kappa_1\,, \tag{2.53a}$$

$$\begin{aligned}(ð' + 2\tau')\kappa_0 + 2(þ + \rho)\kappa_1 = -2\kappa\kappa_2\,, \quad (þ' + 2\rho')\kappa_0 + 2(ð + \tau)\kappa_1 = -2\sigma\kappa_0\,,\\ (ð + 2\tau)\kappa_2 + 2(þ' + \rho')\kappa_1 = -2\kappa'\kappa_0\,, \quad (þ + 2\rho)\kappa_2 + 2(ð' + \tau')\kappa_1 = -2\sigma'\kappa_2\,.\end{aligned} \tag{2.53b}$$

Thus, the three (sets of) equations, (2.43), (2.47) and (2.53) characterize the same geometric object. We will find the component form (2.53) most convenient to calculate exact solutions in sections 2.4 and 4.2.1. The tensorial form (2.43) can also be studied in other dimensions. It can be shown that the dual of a conformal Killing-Yano tensor is again a conformal Killing-Yano tensor. This is less obvious in the spinorial formulation, if we e.g. want to show that the dual of a conformal Killing-Yano 3-form is a conformal Killing vector.

Remark 2.3.1 (Twistor equation in components)**.** *Defining the components of a twistor by* $\lambda^0 = \lambda_A \iota^A, \lambda^1 = -\lambda_A o^A$*, the twistor equation* (2.41) *has components*

$$\begin{aligned} þ\,\lambda^1 = \kappa\lambda^0\,, \quad & þ'\lambda^0 = \kappa'\lambda^1\,, \quad (ð' + \tau')\lambda^1 = (þ + \rho)\lambda^0\,,\\ ð\,\lambda^1 = \sigma\lambda^0\,, \quad & ð'\lambda^0 = \sigma'\lambda^1\,, \quad (þ' + \rho')\lambda^1 = (ð + \tau)\lambda^0\,,\end{aligned} \tag{2.54}$$

cf. [90, eq. (4.12.46)]. If the spin frame is aligned with a 2-sphere such that $m^a = o^a \iota^{A'}$ *and* $\bar{m}^a = o^{A'}\iota^A$ *are tangential, the restricted equations are*

$$ð\,\lambda^1 = \sigma\lambda^0\,, \qquad ð'\lambda^0 = \sigma'\lambda^1\,. \tag{2.55}$$

Penrose uses these 2-surface twistors in [91, Section 9.9] to construct charges in general spacetimes. Note that the Kinnersley and Carter tetrad (2.107) *cannot be used to study these equations on a Kerr background since* $(m, \bar{m})$ *are not tangential to the* (θ, φ) *coordinate spheres. A Lorentz transformation of class I and II with parameter* $a = b = -\frac{\mathrm{i}a\sqrt{\Delta}}{2(r^2+a^2)}$ *makes them tangential.*

2.4. Spacetimes of Petrov type D

So far, we discussed general 4-dimensional Lorentzian vacuum spacetimes. But for certain structures, e.g. Killing spinors, to exist we have to focus on a subclass restricted by the curvature tensor. In vacuum, the curvature tensor equals the Weyl tensor and it can be classified by a procedure due to Petrov. This can be thought of as a classification of the eigenvectors for the symmetric 2-tensor (2.30) over bivector space. These eigenvectors are named *principal null directions* (hereafter PND) and the multiplicity of the vectors defines the classes. Many equivalent approaches can be found in the literature, e.g. [29] for a tensorial version, [118] using spinors or [60, Appendix B] with a bivector approach. We will not repeat the arguments here, but only give the result. The Weyl tensor can be classified according to the following Petrov types:

- **I** The four PND are linearly independent
- **II** Two PND coincide
- **D** Two pairs of coinciding PND
- **III** Three PND coincide
- **N** All four PND coincide
- **O** Flat space, PND not defined

The advantage of the NP formulation is that the null tetrad can be aligned with the PND (wich is then called principal tetrad) so that field equations simplify considerably. For the vacuum case, we cite the Goldberg-Sachs theorem, see e.g. [29][8],

Theorem 2.4.1 (Goldberg-Sachs)**.** *If the Riemann tensor is of type II and a null basis is chosen that l^a is the repeated null direction and $\Psi_0 = \Psi_1 = 0$, then $\kappa = \sigma = 0$; and, conversely, if $\kappa = \sigma = 0$, then $\Psi_0 = \Psi_1 = 0$ and the Riemann tensor is of type II.*

The vanishing connection coefficients $\kappa = 0$ and $\sigma = 0$ imply that l^a is geodesic and shear-free, as seen from (2.17). It should be noted that the algebraic classification holds pointwise and general solutions will have different types in different parts of spacetime. An interesting approach to "detect" whether some spacetime settles down to a Kerr black hole by introducing a Killing spinor candidate is given in [13]. This candidate is tightly related to a Killing spinor of valence two, which restricts spacetime to Petrov type D. We will focus on this class in the remainder of this thesis.

The vacuum field equations in the algebraically special case of Petrov type D have been integrated explicitly and classified by Kinnersley in [81]. We also want to mention the coordinate independent integration method by Held, [65]. An explicit type D line element solving the Einstein-Maxwell equations with cosmological constant is known, from which all type D line elements of this type can be derived by certain limiting procedures, see [104, Section 19.1.2], [42]. The family of type D spacetimes contains the Kerr and Schwarzschild solutions, but also solutions with more complicated topology and asymptotic behavior, such as the NUT- or C-metrics, and solutions whose orbits of the isometry group are null. In the following, we again restrict to the vacuum case.

A Newman-Penrose tetrad with the two real null vectors l^a, n^a aligned along the two repeated principal null directions of a Weyl tensor of Petrov type D is called a principal tetrad. In a spacetime of Petrov type D in a principal tetrad, we have

$$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4 \,, \qquad \kappa = \kappa' = 0 = \sigma = \sigma' \,, \tag{2.56}$$

[8] A more general version can be found in [91, p. 195].

due to the Goldberg-Sachs theorem 2.4.1. Only the middle curvature component Ψ_2 is non-vanishing and for convenience, we introduce a new variable

$$\zeta \propto \Psi_2^{-1/3} . \tag{2.57}$$

The constant of proportionality is chosen such that ζ is non-vanishing in the Minkowski limit, see section 4.2.2. The Ricci-identities (2.37) simplify to

$$\text{þ}\rho = \rho^2 , \tag{2.58a}$$
$$\eth\tau = \tau^2 , \tag{2.58b}$$
$$\eth\rho = (\rho - \bar\rho)\tau , \tag{2.58c}$$
$$\text{þ}\tau = (\tau - \bar\tau')\rho , \tag{2.58d}$$
$$\text{þ}'\rho = \eth'\tau + \rho\bar\rho' - \tau\bar\tau - \Psi_2 , \tag{2.58e}$$

and the Bianchi identities (2.39) reduce in this case to

$$(\text{þ} - 3\rho)\Psi_2 = 0 , \quad (\eth - 3\tau)\Psi_2 = 0 , \quad (\text{þ}' - 3\rho')\Psi_2 = 0 , \quad (\eth' - 3\tau')\Psi_2 = 0 . \tag{2.59}$$

With the 1-form (2.27) this can be expressed in the more compact form

$$2\Theta_a\Psi_2 = 3h_a\Psi_2 . \tag{2.60}$$

Applying further derivatives and commutators to the above equations yields for a general vacuum type D background the additional identities

$$\rho\bar\rho' = \rho'\bar\rho , \quad \tau\bar\tau = \tau'\bar\tau' , \quad \text{þ}'\rho = \text{þ}\rho' , \quad \text{þ}\tau' = \eth'\rho = -\bar\rho\tau' + 2\tau'\rho - \rho\bar\tau , \tag{2.61}$$

as shown in [47].

We also note the simplified first equations of structure (2.26),

$$\mathrm{d}^\Theta Z^0 = -\tfrac{1}{2}h \wedge Z^0 , \quad \mathrm{d}^\Theta Z^1 = -h \wedge Z^1 , \quad \mathrm{d}^\Theta Z^2 = -\tfrac{1}{2}h \wedge Z^2 , \tag{2.62}$$

with h the 1-form corresponding to (2.27).

We now come to the Killing spinor equation (2.47). It has been shown in [120] that a spacetime of Petrov type D admits a Killing spinor of valence two. This follows from the integrability condition (2.48), which becomes

$$\begin{aligned} 0 &= 2\Psi_2\, o_{(A}o_B\iota_C\iota_D\left(\kappa_0\iota^D\iota_{E)} - \kappa_1 o^D\iota_{E)} - \kappa_1\iota^D o_{E)} + \kappa_2 o^D o_{E)}\right) \\ &= -\Psi_2\kappa_0 o_{(A}\iota_B\iota_C\iota_{E)} + \Psi_2\kappa_2\iota_{(A}o_B o_C o_{E)} , \end{aligned} \tag{2.63}$$

so $\kappa_0 = 0 = \kappa_2$. Hence, the components (2.53) of the Killing spinor equation simplify to

$$(\text{þ} + \rho)\kappa_1 = 0 , \quad (\eth + \tau)\kappa_1 = 0 , \quad (\text{þ}' + \rho')\kappa_1 = 0 , \quad (\eth' + \tau')\kappa_1 = 0 , \tag{2.64}$$

and comparison with the Bianchi identities (2.59) shows that $\kappa_1 = \zeta$, with ζ given in (2.57), is a solution. It follows from this argument that it is in fact the only non-trivial solution of the Killing spinor equation in Petrov type D. So explicit coordinate calculations, as done in [59], are not necessary.

Remark 2.4.2 (Rescaling)**.** *In component calculations, a multiplication by some power of Ψ_2 is often used to absorb lower order terms, e.g. in the Teukolsky equation. We prefer to use the Killing spinor coefficient, which is defined so that it does not vanish in the Minkowski limit. Therefore it is convenient to have a full table of derivatives for later calculations:*

$$\begin{aligned} &þ\zeta = -\rho\zeta\,, && þ'\zeta = -\rho'\zeta\,, && ð\zeta = -\tau\zeta\,, && ð'\zeta = -\tau'\zeta\,, \\ &þ\bar{\zeta} = -\bar{\rho}\bar{\zeta}\,, && þ'\bar{\zeta} = -\bar{\rho}'\bar{\zeta}\,, && ð\bar{\zeta} = -\bar{\tau}'\bar{\zeta}\,, && ð'\bar{\zeta} = -\bar{\tau}\bar{\zeta}\,. \end{aligned} \tag{2.65}$$

Summarized, the general solution to the Killing spinor equation (2.47) on a vacuum spacetime of Petrov type D in a principal tetrad with ζ given by (2.57) reads

$$\kappa_{AB} = -2\zeta o_{(A}\iota_{B)}\,. \tag{2.66}$$

Therefore, the 2-form

$$Y_{ab} = \kappa_{AB}\bar{\epsilon}_{A'B'} = \zeta Z^1_{ab}\,, \tag{2.67}$$

is a complex conformal Killing-Yano tensor. We will look at the real and imaginary parts in more detail below. Let us now calculate the complex Killing vector (2.50),

$$\begin{aligned} \xi_{AA'} &= -\frac{1}{3}\nabla_{A'}{}^{B}\kappa_{AB} = \frac{1}{3}\nabla^b\left(\zeta Z^1_{ab}\right) \\ &= \frac{1}{3}\zeta\left(-\frac{1}{2}h^b Z^1_{ab} + 2\Gamma^b Z^0_{ab} + 2\Gamma'^b Z^2_{ab}\right) \\ &= \zeta\left(\rho' l_a - \rho n_a - \tau' m_a + \tau\bar{m}_a\right)\,, \end{aligned} \tag{2.68}$$

using (2.57), (2.60) and (2.36). It either spans the 2-dimensional space of isometries of type D, or a Killing tensor exists from which the space can be constructed, see below. Note that, because $\zeta' = -\zeta$, ξ_a is GHP prime invariant as we expect for a (formalism independent) isometry. We find the squared norm in coordinate independent form,

$$\xi^a\xi_a = 2\zeta^2(\tau\tau' - \rho\rho')\,. \tag{2.69}$$

We might ask ourself, if it is possible to construct valence two Killing spinors from Killing vectors. Given a Killing vector ξ^a, one reasonable candidate is the Komar form $\omega_{AB} = \nabla_{A'(A}\xi^{A'}_{B)}$, see section 4.2.3. But we find

$$\nabla^{A'}{}_{(A}\omega_{BC)} = 2\Psi_{ABCD}\xi^{DA'}\,, \tag{2.70}$$

which has no solutions on Petrov type D backgrounds.

2.4.1. The Kerr-NUT class

The Kerr-NUT subclass of Petrov type D spacetimes in which real and imaginary parts of the Killing vector (2.68) are linearly dependent will be investigated in this section. This excludes only class IIIB (e.g. Ehlers and Kundts C-metric) in the Kinnersley classification [81] of Petrov type D spacetimes and is sometimes called *non-accelerating*, see [34]. In the Kerr-NUT class, the Hamilton Jacobi eq. is separable due to the existence of a Killing tensor. It can be characterized by the condition[9]

$$
\begin{aligned}
0 = 2\mathrm{iIm}\,\xi^a &= \zeta\left(\rho' l^a - \rho n^a - \tau' m^a + \tau \bar{m}^a\right) - \bar{\zeta}\left(\bar{\rho}' l^a - \bar{\rho} n^a - \bar{\tau}' \bar{m}^a + \bar{\tau} m^a\right) \\
&= -n^a(\zeta\rho - \bar{\zeta}\bar{\rho}) + l^a(\zeta\rho' - \bar{\zeta}\bar{\rho}') - m^a(\zeta\tau' + \bar{\zeta}\bar{\tau}) + \bar{m}^a(\zeta\tau + \bar{\zeta}\bar{\tau}')\,. \qquad (2.71)
\end{aligned}
$$

To vanish, all coefficients must be zero,

$$
\frac{\rho}{\bar{\rho}} = \frac{\rho'}{\bar{\rho}'} = -\frac{\tau'}{\bar{\tau}} = -\frac{\tau}{\bar{\tau}'} = \frac{\bar{\zeta}}{\zeta}\,. \qquad (2.72)
$$

Due to these equations, we can add to the above list (2.58) of Ricci identities

$$
\eth'\rho = 2\tau'\rho\,, \qquad \text{þ}'\tau = 2\tau\rho'\,, \qquad \text{þ}\rho = \rho\rho' + \tau(\tau' - \bar{\tau}) - \tfrac{1}{2}\Psi_2 - \frac{\bar{\zeta}}{2\zeta}\bar{\Psi}_2\,, \qquad (2.73)
$$

which can be derived from the commutators relations (2.38) applied to spin coefficients.

Remark 2.4.3. *A neat way to derive the last identity is to expand* $0 = \mathcal{L}_\xi \rho$ *with the Lie derivative on weighted scalars introduced below in* (2.88).

An investigation of Komar forms in section 4.2.3 leads to the purely algebraic identities

$$
\begin{aligned}
\epsilon\rho' + \epsilon'\rho - \tau'\beta - \tau\beta' &= -\tfrac{1}{2}\Psi_2\,, \qquad (2.74a)\\
2I^2(\epsilon\rho' + \epsilon'\rho) + 2R^2(\tau'\beta + \tau\beta') &= \tfrac{1}{4}(\zeta^2 + \bar{\zeta}^2)\Psi_2 - \tfrac{1}{2}\bar{\zeta}^2\bar{\Psi}_2 \\
&\quad + \rho\rho'\zeta(\bar{\zeta} - \zeta) + \tau\tau'\zeta(\bar{\zeta} + \zeta)\,, \qquad (2.74b)
\end{aligned}
$$

if a tetrad invariant under $\mathcal{L}_\xi$ and $\mathcal{L}_\eta$ is chosen, see remark 2.4.4 on p. 21. We used the abbreviations $R = \mathrm{Re}\,\zeta$ and $I = \mathrm{Im}\,\zeta$. The straight forward but tedious proof is shifted to appendix A.3.

The conformal Killing-Yano tensor (2.67) can be decomposed into a real 2-form f_{ab} and its dual, $Y_{ab} = f_{ab} + \mathrm{i} * f_{ab}$. Since real and imaginary parts decouple, f_{ab} and $*f_{ab}$ are conformal Killing-Yano tensors,

$$
f_{ab} = \frac{1}{2}(Y_{ab} + \bar{Y}_{ab}) = 2R\, n_{[a}l_{b]} - 2\mathrm{i}\, I\, \bar{m}_{[a}m_{b]}\,, \qquad (2.75a)
$$

$$
*f_{ab} = \frac{1}{2\mathrm{i}}(Y_{ab} - \bar{Y}_{ab}) = 2I\, n_{[a}l_{b]} + 2\mathrm{i}\, R\, \bar{m}_{[a}m_{b]}\,. \qquad (2.75b)
$$

On Kerr-NUT, $*f_{ab}$ is a proper Killing-Yano tensor, because of (2.71). We can now build the conformal Killing tensors

$$
K_{ab} = f_{ac}f^c{}_b = 2\left[R^2 n_{(a}l_{b)} + I^2 \bar{m}_{(a}m_{b)}\right], \qquad (2.76a)
$$

$$
\overset{*}{K}_{ab} = f_{ac} * f^c{}_b = 2RI\left[n_{(a}l_{b)} - \bar{m}_{(a}m_{b)}\right], \qquad (2.76b)
$$

$$
\overset{**}{K}_{ab} = *f_{ac} * f^c{}_b = 2\left[I^2 n_{(a}l_{b)} + R^2 \bar{m}_{(a}m_{b)}\right]. \qquad (2.76c)
$$

[9]Instead of adding a complex phase $e^{\mathrm{i}c}$ for full generality, see [36], we choose a complex prefactor in the Killing spinor solution (2.66).

Their traces are given by $2(R^2 - I^2)$, $4RI$ and $2(I^2 - R^2)$, respectively. The last one is a proper Killing tensor. Of particularly simple form are the complex conformal Killing tensors directly derived from Y_{ab},

$$Y_{ac}Y^c{}_b = \frac{1}{4}\zeta^2 g_{ab}\,, \tag{2.77a}$$

$$\bar{Y}_{ac}\bar{Y}^c{}_b = \frac{1}{4}\bar{\zeta}^2 g_{ab}\,, \tag{2.77b}$$

$$\bar{Y}_{ac}Y^c{}_b = \frac{1}{2}\zeta\bar{\zeta}(n_{(a}l_{b)} + \bar{m}_{(a}m_{b)})\,, \tag{2.77c}$$

with traces ζ^2, $\bar{\zeta}^2$ and 0, respectively. This set is of course equivalent to (2.76). Because fg_{ab} for an arbitrary function f is a conformal Killing tensor, we find from (2.76) the real conformal Killing tensors

$$\zeta\bar{\zeta}l_{(a}n_{b)}\,, \qquad \zeta\bar{\zeta}\bar{m}_{(a}m_{b)}\,, \tag{2.78}$$

by using $f = \zeta\bar{\zeta}$. The second Killing vector can be constructed via

$$\eta^a = \overset{**}{K}{}^{ab}\xi_b = \zeta\left[I^2(\rho' l^a - \rho n^a) - R^2(\tau\bar{m}^a - \tau' m^a)\right]\,, \tag{2.79}$$

with the Killing tensor (2.76c). On a Kerr background, ξ^a and η^a are linearly independent and span the space of isometries, see [69]. In the special case of a Schwarzschild background, η^a vanishes, see (2.115) and [36] for details.

2.4.2. Lie derivative of weighted fields

Let us have a look at the Lie derivative of spinor fields, following Penrose-Rindler, [91, p. 101]. For a general spinor field $\phi_{AA'}^{BB'}$ and a conformal Killing vector ω^a it is of the form

$$\mathcal{L}_\omega\phi_{AA'}^{BB'} = \omega^a\nabla_a\phi_{AA'}^{BB'} + h_A{}^C\phi_{CA'}^{BB'} + \bar{h}_{A'}{}^{C'}\phi_{AC'}^{BB'} - h_C{}^B\phi_{AA'}^{CB'} - \bar{h}_{C'}{}^{B'}\phi_{AA'}^{BC'}\,, \tag{2.80}$$

with $h_A{}^B = \frac{1}{2}\nabla_{AA'}\omega^{BA'} - \frac{1}{8}\epsilon_A{}^B\nabla_{CC'}\omega^{CC'}$, cf. [91, eq. (6.6.11)]. For spinors of higher valence an additional term for each index must be added in the usual way. We are mostly interested in the case of proper Killing fields ω^a, for which we find

$$h_{AB} = \tfrac{1}{2}\nabla_{A'(A}\omega_{B)}^{A'}\,. \tag{2.81}$$

This is the anti-self-dual part of the exterior derivative of the Killing 1-form ω_a, as will be discussed in section 4.2.3. With an expansion $h_{AB} = h_0\iota_A\iota_B - 2h_1 o_{(A}\iota_{B)} + h_2 o_A o_B$, the Lie derivative (2.80) of the dyad itself reads

$$\mathcal{L}_\omega o_A = \omega^a\nabla_a o_A - h_1 o_A + h_0\iota_A\,, \qquad \mathcal{L}_\omega\iota_A = \omega^a\nabla_a\iota_A + h_1\iota_A - h_2 o_A\,, \tag{2.82}$$

and the covariant derivatives can be rewritten into algebraic form by using (2.19),

$$\omega^a\nabla_a o_A = \omega^a\sigma_{1a}o_A - \omega^a\sigma_{2a}\iota_A\,, \qquad \omega^a\nabla_a\iota_A = -\omega^a\sigma_{1a}\iota_A + \omega^a\sigma_{2a}o_A\,. \tag{2.83}$$

So far the equations hold for any Killing vector ω in any NP tetrad. In the remainder, we will write ${}_\omega h_{AB}$ for (2.81) to distinguish the fields for different Killing vectors.

Remark 2.4.4. *By definition, we have* $\mathcal{L}_\omega g_{ab} = 0$*, but this does not imply that the Lie derivative of a tetrad or spinor dyad vanish. However, for a set* $\{{}_1\omega, \ldots {}_n\omega\}$ *of* $n \leq 4$ *commuting Killing vectors, there exist tetrads with vanishing Lie derivatives along all* ${}_i\omega, 1 \leq i \leq n$*, see [35]. If we restrict to the Kerr-NUT class*[10] *and choose such a tetrad which is invariant under* ξ *and* η *given in* (2.68) *and* (2.79)*, this indeed implies*

$$\mathcal{L}_\xi o_A = 0\,, \qquad \mathcal{L}_\xi \iota_A = 0\,, \qquad \mathcal{L}_\eta o_A = 0\,, \qquad \mathcal{L}_\eta \iota_A = 0\,, \tag{2.84}$$

for the normalized dyad (2.5)*. This condition holds for the Kinnersley and Carter tetrads* (2.103) *and* (2.107)*.*

Restricting to the Kerr-NUT class, we show in section 4.2.3 that (2.81) for the Killing vector (2.50) becomes

$${}_\xi h_{AB} = \tfrac{1}{4}\Psi_{ABCD}\kappa^{CD} = \zeta\Psi_2 o_{(A}\iota_{B)}\,, \qquad \Leftrightarrow \qquad {}_\xi h_1 = -\tfrac{1}{2}\zeta\Psi_2\,. \tag{2.85}$$

Thus (2.82) simplifies for this particular Killing vector to

$$\mathcal{L}_\xi o_A = \xi^a\sigma_{1a}o_A + \tfrac{1}{2}\zeta\Psi_2 o_A\,, \qquad \mathcal{L}_\xi \iota_A = -\xi^a\sigma_{1a}\iota_A - \tfrac{1}{2}\zeta\Psi_2\iota_A\,, \tag{2.86}$$

by using (2.83) together with $\xi^a\sigma_{0a} = 0 = \xi^a\sigma_{2a}$.

As an example, the Lie derivative of a symmetric 2-spinor $\phi_{AB} = \phi_0\iota_A\iota_B - 2\phi_1 o_{(A}\iota_{B)} + \phi_2 o_A o_B$ reads

$$\mathcal{L}_\xi\phi_{AB} = (\xi^a\Theta_a\phi_0 - \zeta\Psi_2\phi_0)\iota_A\iota_B + (\xi^a\nabla_a\phi_1)o_{(A}\iota_{B)} + (\xi^a\Theta_a\phi_2 + \zeta\Psi_2\phi_2)o_A o_B\,, \tag{2.87}$$

and the generalization to arbitrary valence follows from repeated application of (2.86) and Leibniz rule. Hence, the Lie derivative of a $\{p, q\}$ weighted scalar ϕ along ξ^a reads,

$$\begin{aligned}\mathcal{L}_\xi\phi &= \left[-\zeta\left(\rho\text{þ}' - \rho'\,\text{þ} - \tau\eth' + \tau'\,\eth\right) - \frac{p}{2}\zeta\Psi_2 - \frac{q}{2}\bar\zeta\bar\Psi_2\right]\phi \\ &= \left[\xi^a\Theta_a - \frac{p}{2}\zeta\Psi_2 - \frac{q}{2}\bar\zeta\bar\Psi_2\right]\phi \qquad (2.88\text{a}) \\ &\stackrel{*}{=} \xi^a\nabla_a\phi\,. \qquad (2.88\text{b})\end{aligned}$$

The last equality holds under the condition of remark 2.4.4. To summarize, if a dyad is chosen, such that (2.84) holds, then the Lie derivative of weighted scalars along ξ simplifies to a directional derivative. This in turn implies the interesting identity $\xi^a\sigma_{1a} = -\frac{1}{2}\zeta\Psi_2$ relating the curvature to products of connection coefficients in a purely algebraic way. An explicit form is given in (2.74a). We also note that (2.88a) applied to NP scalars gives non-trivial identities, see remark 2.4.3 on p. 19. The calculation of ${}_\eta h_{AB}$ for the second Killing vector is rather lengthy and shiftet to section A.3 on p. 95. It is found that ${}_\eta h_0 = \eta^a\sigma_{0a}$ and ${}_\eta h_2 = \eta^a\sigma_{2a}$ are satisfied identically. This leads to the Lie derivative

$$\mathcal{L}_\eta\phi = \left[\eta^a\Theta_a + p\,{}_\eta h_1 + q\,{}_\eta\bar h_1\right]\phi \stackrel{*}{=} \eta^a\nabla_a\phi\,, \tag{2.89}$$

[10] One needs to check, whether real and imaginary parts of ξ commute, to generalize to all Petrov type D spacetimes.

with

$$ {}_{\eta}h_1 = \tfrac{1}{8}\zeta(\zeta^2 + \bar{\zeta}^2)\Psi_2 - \tfrac{1}{4}\zeta\bar{\zeta}^2\bar{\Psi}_2 + \tfrac{1}{2}\rho\rho'\zeta^2(\bar{\zeta} - \zeta) + \tfrac{1}{2}\tau\tau'\zeta^2(\bar{\zeta} + \zeta)\,. \tag{2.90} $$

Under condition (2.84) we find for the second Killing vector the identity ${}_{\eta}h_1 = \eta^a\sigma_{1a}$, which is expanded in (2.74b) and yields the * equality in (2.89). The Lie derivatives (2.88a) and (2.89) are the symmetry operators $\mathcal{V}$ and $\mathcal{P}$ found in [94, p. 37].

The identities implied by condition (2.84) can be summarized by

$$ {}_{\omega}h_I = \omega^a\sigma_{Ia}\,, \tag{2.91} $$

for all Killing vectors ω^a. This will turn out to be useful to analyze Komar integrals in section 4.2.3.

2.5. Black holes and the Kerr spacetime

The no-hair theorem states that a black hole equilibrium state is completely described by only three parameters, the mass M, the angular momentum Ma and the charge q. It can be described by an explicit solution of the Einstein-Maxwell equation, known as the Kerr-Newman metric. For vanishing charge q it reduces to the Kerr metric and the Schwarzschild metric is contained in the limit of vanishing angular momentum Ma. More generally, there are uniqueness and singularity theorems due to Hawking and Penrose, which describe black hole formation as a generic process.

Assuming spherical symmetry, the Schwarzschild solution can be calculated rather straightforwardly from the vacuum field equations (2.1). A derivation of the Kerr metric from an axially symmetric line element is much more involved, see [29] for a full account. Most commonly, the metric is written in Boyer-Lindquist coordinates (t, r, θ, φ),

$$ g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4Mra\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 - \frac{\Pi\sin^2\theta}{\Sigma}\mathrm{d}\varphi^2\,, \tag{2.92} $$

where

$$ \Delta = r^2 - 2Mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta, \qquad \Pi = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \tag{2.93} $$

This representation of the metric does have the advantage that mass M and angular momentum Ma can be read off directly for large r . We will only consider the situation $a < M$, where proper horizons exist (they disappear for $a \geq M$ and cosmic censorship is violated). The case $a = M$ is called extreme and $a > M$ over-extreme. At present, no "inner solution", e.g. rotating star, is known which connects to the Kerr solution in the vacuum region. The two roots of $\Delta = 0$,

$$ r_+ = M + \sqrt{M^2 - a^2}\,, \qquad r_- = M - \sqrt{M^2 - a^2}\,, \tag{2.94} $$

describe the *outer horizon* and *inner horizon*, respectively. The ring singularity is parametrized by the circle $r = 0$, $\theta = \pi/2$ and equivalent to the condition $\Sigma = 0$. It is not a coordinate artifact, because the invariant $R_{abcd}R^{abcd}$ diverges. The condition $g_{tt} = 0$ holds for

$$ r_o = M + \sqrt{M^2 - a^2\cos^2\theta}\,, \qquad r_i = M - \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{2.95} $$

The radius r_o is the outer boundary of the ergo-region. Particles and fields can extract energy from the black hole inside this region due to the Penrose process and super-radiance, respectively.

Of particular interest in the analysis of dispersive estimates, see section 6.1, are null geodesics with orbits of constant r. In the special case of Schwarzschild spacetime, this is only possible at $r = 3M$ and called photon-sphere. These orbits are unstable to perturbations in the radial direction and a scalar field disperses away from this sphere. On a Kerr background the $r =$ const. null geodesics (photon orbits) fill a whole region characterized by the inequality [63],

$$(4r\Delta - \Sigma\partial_r\Delta)^2 \le 16a^2r^2\Delta\sin^2\theta\,. \tag{2.96}$$

The minimal and maximal radii,

$$r_{ph\pm} = 2M\left[1 + \cos\left(\tfrac{2}{3}\arccos\left(\pm\tfrac{|a|}{M}\right)\right)\right], \tag{2.97}$$

are possible only for null geodesics in the equatorial plane. Here the minus sign holds for direct orbits and the plus for retrograde orbits. Null geodesics outside the equatorial plane can have stationary orbits between these radii and so the set (2.96) is not a 2-dimensional sphere, as it is on Schwarzschild, see [29, p. 330] and [108] for a more detailed discussion of the bounded null-geodesics.

The inverse metric to (2.92) reads

$$g^{ab}\partial_a\partial_b = \frac{\Pi}{\Sigma\Delta}\partial_t^2 + \frac{4Mra}{\Sigma\Delta}\partial_t\partial_\varphi - \frac{\Delta}{\Sigma}\partial_r^2 - \frac{1}{\Sigma}\partial_\theta^2 - \frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta}\partial_\varphi^2\,. \tag{2.98}$$

The wave operator on a scalar field can in general be expanded into

$$\Box_g u = \nabla^a\nabla_a u = \nabla^a\partial_a u = \partial^a\partial_a u - g^{ab}\Gamma_{ba}{}^c\partial_c u = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b u\right), \tag{2.99}$$

where the last equality follows from $\partial_a\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{bc}\partial_a g_{bc}$. In Boyer-Lindquist coordinates (t, r, θ, ϕ), this yields

$$\begin{aligned}\Sigma\,\Box\,u = &\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]u_{tt} + \frac{4aMr}{\Delta}u_{t\phi}\\ &-\left[\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right]u_{\phi\phi} - \partial_r(\Delta u_r) - \frac{1}{\sin\theta}\partial_\theta(\sin\theta u_\theta)\,.\end{aligned} \tag{2.100}$$

Specializing to Schwarzschild ($a = 0$) yields $\Sigma = r^2$ and $\Delta = r^2 - 2Mr$. With $f = (1 - 2M/r)$ the wave operator reduces to

$$\begin{aligned}\Box u &= f^{-1}u_{tt} - \frac{1}{r^2}\partial_r(fr^2u_r) - \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta u_\theta) - \frac{1}{r^2\sin^2\theta}u_{\phi\phi}\\ &= \frac{1}{fr}\left[\partial_t^2 - \partial_{r_*}^2 - \frac{f}{r^2}\Delta_{S^2} + f\frac{2M}{r^3}\right](ru)\,.\end{aligned} \tag{2.101}$$

Further specialization to Minkowski ($M = 0$) gives $f = 1$ and the wave operator in spherical coordinates,

$$\begin{aligned}\Box u &= u_{tt} - \frac{1}{r^2}\partial_r(r^2 u_r) - \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta u_\theta) - \frac{1}{r^2\sin^2\theta}u_{\phi\phi} \\ &= u_{tt} - \frac{1}{r^2}\partial_r(r^2 u_r) - \frac{1}{r^2}\Delta_{S^2}u \\ &= u_{tt} - \Delta_{\mathbb{R}^3}u\,. \end{aligned} \tag{2.102}$$

2.5.1. Kinnersley and Carter tetrad

The mostly used Newman-Penrose tetrad for the Kerr solution is due to Kinnersley. With $p := r - \mathrm{i}a\cos\theta$ it is given by

$$l^b = \frac{1}{\Delta}\Big[r^2 + a^2, \Delta, 0, a\Big], \tag{2.103a}$$

$$n^b = \frac{1}{2\Sigma}\Big[r^2 + a^2, -\Delta, 0, a\Big], \tag{2.103b}$$

$$m^b = \frac{1}{\sqrt{2}\bar{p}}\Big[\mathrm{i}a\sin\theta, 0, 1, \frac{\mathrm{i}}{\sin\theta}\Big], \tag{2.103c}$$

$$\bar{m}^b = \frac{1}{\sqrt{2}p}\Big[-\mathrm{i}a\sin\theta, 0, 1, -\frac{\mathrm{i}}{\sin\theta}\Big]. \tag{2.103d}$$

For exterior calculations, it is convenient to also have explicit expressions for their duals,

$$l_b = \Big[1, -\frac{\Sigma}{\Delta}, 0, -a\sin^2\theta\Big], \tag{2.104a}$$

$$n_b = \frac{\Delta}{2\Sigma}\Big[1, \frac{\Sigma}{\Delta}, 0, -a\sin^2\theta\Big], \tag{2.104b}$$

$$m_b = \frac{1}{\sqrt{2}\bar{p}}\Big[\mathrm{i}a\sin\theta, 0, -\Sigma, -\mathrm{i}(r^2 + a^2)\sin\theta\Big], \tag{2.104c}$$

$$\bar{m}_b = \frac{1}{\sqrt{2}p}\Big[-\mathrm{i}a\sin\theta, 0, -\Sigma, \mathrm{i}(r^2 + a^2)\sin\theta\Big]. \tag{2.104d}$$

The spin coefficients in the Kinnersley tetrad read

$$\rho = -\frac{1}{p}, \qquad \rho' = \frac{\Delta}{2p\bar{p}^2}, \qquad \tau = -\frac{\mathrm{i}a\sin\theta}{\sqrt{2}\Sigma}, \qquad \tau' = -\frac{\mathrm{i}a\sin\theta}{\sqrt{2}p^2}, \tag{2.105a}$$

$$\epsilon' = \rho' - \frac{r - M}{2\Sigma}, \qquad \beta = \frac{\cot\theta}{2\sqrt{2}\bar{p}}, \qquad \beta' = \bar{\beta} + \tau'. \tag{2.105b}$$

It has in addition to $\kappa = \kappa' = \sigma = \sigma' = 0$ also $\epsilon = 0$. The identity $\beta' = \tau' + \bar{\beta}$ has no invariant meaning. Because of the special form $\rho = -\zeta^{-1}$, this spin coefficient is sometimes used for rescaling. This does not have invariant meaning and is very unnatural from the GHP weighted point of view.

To simplify the expressions as much as possible, a tetrad transformation of class III,

$$l^b \to A^{-1} l^b\,, \qquad n^b \to A n^b\,, \qquad m^b \to e^{\mathrm{i}\Theta} m^b\,, \qquad \bar{m}^b \to e^{-\mathrm{i}\Theta} \bar{m}^b\,, \tag{2.106}$$

with $A = \sqrt{2\Sigma/\Delta}$ and $e^{\mathrm{i}\Theta} = \sqrt{\bar{p}/p}$ transforms it into the symmetric Carter tetrad, [124],

$$l^b = \frac{1}{\sqrt{2\Sigma}}\left[\frac{r^2+a^2}{\sqrt{\Delta}}, \sqrt{\Delta}, 0, \frac{a}{\sqrt{\Delta}}\right], \tag{2.107a}$$

$$n^b = \frac{1}{\sqrt{2\Sigma}}\left[\frac{r^2+a^2}{\sqrt{\Delta}}, -\sqrt{\Delta}, 0, \frac{a}{\sqrt{\Delta}}\right], \tag{2.107b}$$

$$m^b = \frac{1}{\sqrt{2\Sigma}}\left[\mathrm{i}a\sin\theta, 0, 1, \frac{\mathrm{i}}{\sin\theta}\right], \tag{2.107c}$$

$$\bar{m}^b = \frac{1}{\sqrt{2\Sigma}}\left[-\mathrm{i}a\sin\theta, 0, 1, -\frac{\mathrm{i}}{\sin\theta}\right], \tag{2.107d}$$

$$l_b = \frac{1}{\sqrt{2\Sigma}}\left[\sqrt{\Delta}, -\frac{\Sigma}{\sqrt{\Delta}}, 0, -a\sqrt{\Delta}\sin^2\theta\right], \tag{2.108a}$$

$$n_b = \frac{1}{\sqrt{2\Sigma}}\left[\sqrt{\Delta}, \frac{\Sigma}{\sqrt{\Delta}}, 0, -a\sqrt{\Delta}\sin^2\theta\right], \tag{2.108b}$$

$$m_b = \frac{1}{\sqrt{2\Sigma}}\left[\mathrm{i}a\sin\theta, 0, -\Sigma, -\mathrm{i}(r^2+a^2)\sin\theta\right], \tag{2.108c}$$

$$\bar{m}_b = \frac{1}{\sqrt{2\Sigma}}\left[-\mathrm{i}a\sin\theta, 0, -\Sigma, \mathrm{i}(r^2+a^2)\sin\theta\right]. \tag{2.108d}$$

The spin coefficients in this tetrad read

$$\beta = \beta' = \frac{\tau}{2} + \frac{\cot\theta}{2\sqrt{2\Sigma}}\,, \qquad \tau = \tau' = -\frac{ia\sin\theta}{\sqrt{2\Sigma}p}\,, \tag{2.109a}$$

$$\epsilon = -\epsilon' = \frac{\rho}{2} + \frac{r-M}{2\sqrt{2\Sigma}\sqrt{\Delta}}\,, \qquad \rho = -\rho' = -\frac{\sqrt{\Delta}}{\sqrt{2\Sigma}p}\,. \tag{2.109b}$$

The components of the Weyl tensor in both tetrads are

$$\Psi_2 = -\frac{M}{p^3}\,, \qquad \Psi_{i\neq 2} = 0\,. \tag{2.110}$$

The coefficient ζ of the type D Killing spinor (2.57) is determined up to a constant, which we fix by setting $M\zeta^{-3} = -\Psi_2$, or $\zeta = p$. The bivectors and connection forms in a

Carter tetrad are given by

$$Z^1_{ab} = \begin{pmatrix} 0 & -1 & -\mathrm{i}a\sin\theta & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ \mathrm{i}a\sin\theta & 0 & 0 & -\mathrm{i}(r^2+a^2)\sin\theta \\ 0 & a\sin^2\theta & \mathrm{i}(r^2+a^2)\sin\theta & 0 \end{pmatrix}, \tag{2.111a}$$

$$Z^0_{ab} = \frac{1}{2\sqrt{\Delta}} \begin{pmatrix} 0 & -\mathrm{i}a\sin\theta & \Delta & -\mathrm{i}\Delta\sin\theta \\ \mathrm{i}a\sin\theta & 0 & \Sigma & -\mathrm{i}(r^2+a^2)\sin\theta \\ -\Delta & -\Sigma & 0 & a\Delta\sin^2\theta \\ \mathrm{i}\Delta\sin\theta & \mathrm{i}(r^2+a^2)\sin\theta & -a\Delta\sin^2\theta & 0 \end{pmatrix}, \tag{2.111b}$$

$$Z^2_{ab} = \frac{1}{2\sqrt{\Delta}} \begin{pmatrix} 0 & \mathrm{i}a\sin\theta & -\Delta & -\mathrm{i}\Delta\sin\theta \\ -\mathrm{i}a\sin\theta & 0 & \Sigma & \mathrm{i}(r^2+a^2)\sin\theta \\ \Delta & -\Sigma & 0 & -a\Delta\sin^2\theta \\ \mathrm{i}\Delta\sin\theta & -\mathrm{i}(r^2+a^2)\sin\theta & a\Delta\sin^2\theta & 0 \end{pmatrix}, \tag{2.111c}$$

$$\sigma_{0a} = \left(0, \frac{\mathrm{i}a\sin\theta}{2p\sqrt{\Delta}}, -\frac{\sqrt{\Delta}}{2p}, -\frac{\mathrm{i}\sqrt{\Delta}\sin\theta}{2p}\right), \tag{2.112a}$$

$$\sigma_{1a} = \left(\frac{M}{2p^2}, 0, 0, -\frac{Ma\sin^2\theta}{2p^2} - \frac{a+\mathrm{i}r\cos\theta}{2p}\right), \tag{2.112b}$$

$$\sigma_{2a} = \left(0, \frac{\mathrm{i}a\sin\theta}{2p\sqrt{\Delta}}, -\frac{\sqrt{\Delta}}{2p}, \frac{\mathrm{i}\sqrt{\Delta}\sin\theta}{2p}\right). \tag{2.112c}$$

The real Killing-Yano tensors (2.75) become

$$f_{ab} = 2r\, n_{[a}l_{b]} + 2\mathrm{i}\, a\cos\theta\, \bar{m}_{[a}m_{b]}\,, \tag{2.113a}$$

$$\ast f_{ab} = 2a\cos\theta\, n_{[a}l_{b]} - 2\mathrm{i}\, r\, \bar{m}_{[a}m_{b]}\,. \tag{2.113b}$$

While $\ast f_{ab}$ is a Killing-Yano tensor, f_{ab} is a conformal Killing-Yano tensor and does have the time-like Killing field as divergence,

$$\xi^a \partial_a = \partial_t\,. \tag{2.114}$$

The second Killing vector (2.79) in coordinates becomes

$$\eta^b \partial_b = a^2 \partial_t + a\partial_\varphi\,, \tag{2.115}$$

so the axial Killing vector is given by

$$\Xi = \frac{1}{a}\eta - a\xi = \partial_\varphi\,. \tag{2.116}$$

In addition to the continuous symmetries due to the Killing vectors and tensors, there is also a discrete symmetry

$$P : t \to -t\,,\ \varphi \to -\varphi\,. \tag{2.117}$$

The conformal Killing tensors (2.76) have the simple form

$$K_{ab} = 2\big[r^2 n_{(a}l_{b)} + a^2\cos^2\theta\bar{m}_{(a}m_{b)}\big]\,, \tag{2.118a}$$

$$\overset{*}{K}_{ab} = -2ra\cos\theta\big[n_{(a}l_{b)} - \bar{m}_{(a}m_{b)}\big]\,, \tag{2.118b}$$

$$\overset{**}{K}_{ab} = 2\big[a^2\cos^2\theta n_{(a}l_{b)} + r^2\bar{m}_{(a}m_{b)}\big]\,. \tag{2.118c}$$

Their traces are $2(r^2 - a^2\cos^2\theta), -4ra\cos\theta$ and $-2(r^2 - a^2\cos^2\theta)$. Note that $\overset{**}{K}_{ab}$ is the well known Killing tensor on Kerr spacetime and that $\overset{*}{K}_{ab}= -ra\cos\theta g_{ab}$. It also follows that $\overset{**}{K}$ is reducible in the case $a = 0$.

We want to point out another potentially interesting set of coordinates introduced by Doran in [46], see also [100] for Kerr-Newman. It generalizes the Painlevé-Gullstrand coordinates of Schwarzschild spacetime. The "time coordinate" t is the local proper time of specific freely falling observers, $g^{tt} = 1$, and the metric is regular at the horizon.

2.5.2. Geometric choice of coordinates

The Boyer-Lindquist and Doran coordinates do have particular advantages, but one might ask whether the geometry itself singles out preferred coordinates. This is indeed the case as shown in [55, Appendix D] (they use the opposite signature). We will briefly review this from the point of view taken in this thesis. Viewing the Killing spinor equation (2.40) as fundamental, we derived the valence two solution in (2.66). Its divergence is the Killing vector (2.68), which can be made real for the Kerr-Nut class. In that case the imaginary part of the Killing spinor yields a Killing-Yano tensor (2.75b), which itself squares to the Killing tensor (2.76c). In case, the spacetime admits less than four isometries, a second Killing vector can be constructed according to (2.79). One can check $\mathcal{L}_\xi\eta = 0$ and the first two coordinates x^0, x^4 are defined to have these Killing vectors as its flow,

$$\xi = \partial_0\,, \qquad\qquad \eta = \partial_3\,. \tag{2.119}$$

The remaining coordinate freedom is fixed by analyzing the eigenvalue problem for the Killing tensor,

$$\overset{**}{K}{}^a{}_b n^b = \lambda n^a\,. \tag{2.120}$$

From the representation (2.76c) the eigenvectors $Al^a, Bn^a, Cm^a, D\bar{m}^a$ with free functions A, B, C, D follow immediately. The first two have the common eigenvalue I^2 and the second two have the common eigenvalue $-R^2$.[11] The two eigenspaces are orthogonal, because $l^a m_a = 0 = l^a\bar{m}_a$ and $n^a m_a = 0 = n^a\bar{m}_a$. To construct coordinates, we need four mutually commuting vector fields. With the ansatz

$$\big[Al^a + Bn^a, Cm^a + D\bar{m}^a\big] = 0\,, \tag{2.121}$$

[11] In [55, p. 420], the Killing-Yano tensor (2.75b) is called non-degenerate, if the associated Killing tensor (2.76c) does have two different eigenvalues. For Schwarzschild spacetime it is degenerate, because of $I^2 = 0$ and geometric coordinates can be constructed purely from Killing vectors.

we find an element in each eigenspace, such that their commutator vanishes,

$$e_1^a = \zeta^2\bar{\zeta}(\rho' l^a + \rho n^a)\,, \qquad e_2^a = \mathrm{i}\zeta^2\bar{\zeta}(\tau' m^a + \tau\bar{m}^a)\,, \qquad [e_1, e_2] = 0\,. \tag{2.122}$$

These vectors are real, because of (2.72). Also $\mathcal{L}_\xi e_i = 0 = \mathcal{L}_\eta e_i$, $i = 1, 2$ holds. More generally one can show that the Schouten-Nijenhuis bracket $[X, K]^{ab} = X^c\nabla_c K^{ab} - K^{ac}\nabla_c X^b$ of the Killing vectors with the Killing tensor vanishes, see [44]. We find the norms

$$\begin{aligned} \xi^a\xi_a &= -2\zeta^2(\rho\rho' - \tau\tau')\,, & \eta^a\eta_a &= -2\zeta^2(I^4\rho\rho' - R^4\tau\tau')\,, \\ e_1^a e_{1a} &= 2\zeta^4\bar{\zeta}^2\rho\rho'\,, & e_2^a e_{2a} &= 2\zeta^4\bar{\zeta}^2\tau\tau'\,, \\ \xi^a\eta_a &= -2\zeta^2(I^2\rho\rho' + R^2\tau\tau')\,, & e_i^a\xi_a &= 0 = e_i^a\eta_a = e_1^a e_{2a}\,. \end{aligned}$$

Now, we introduce coordinates x^i with ($\partial_{x^i} = \partial_i$),

$$\xi = \partial_0\,, \qquad e_1 = \partial_1\,, \qquad e_2 = \partial_2\,, \qquad \eta = \partial_4\,, \tag{2.123}$$

and we read off the metric

$$\begin{aligned} g_{ab}\mathrm{d}x^a\mathrm{d}x^b =& -2\zeta^2(\rho\rho' - \tau\tau')\mathrm{d}x_0^2 - 2\zeta^2(I^4\rho\rho' - R^4\tau\tau')\mathrm{d}x_4^2 \\ &+ 2\zeta^4\bar{\zeta}^2\rho\rho'\mathrm{d}x_1^2 + 2\zeta^4\bar{\zeta}^2\tau\tau'\mathrm{d}x_2^2 - 2\zeta^2(I^2\rho\rho' + R^2\tau\tau')\mathrm{d}x_0\mathrm{d}x_4\,. \end{aligned} \tag{2.124}$$

It is expressed purely in terms of the Killing spinor, since $\text{þ}\ln\zeta = \rho$ etc.. Comparison to the Kerr spacetime in the usual Boyer-Lindquist coordinates shows

$$\partial_0 = \partial_t\,, \qquad \partial_1 = \Delta\partial_r\,, \qquad \partial_2 = a\sin\theta\partial_\theta\,, \qquad \partial_4 = a^2\partial_t + a\partial_\varphi\,. \tag{2.125}$$

We note that the coordinate x^1 shifts the horizon to $-\infty$ like the tortoise coordinates. The Schwarzschild limit $a \to 0$ is singular in these coordinates. Separability properties are not affected by coordinate changes $\partial_{x^i} \to f(x^i)\partial_{x^i}$ for any fixed i. The choice in [55] leads to coordinates (τ, r, y, ψ) in which the metric takes the particularly nice form

$$\mathrm{d}s^2 = \frac{1}{r^2 + y^2}\left[-\Delta_r(\mathrm{d}\tau + y^2\mathrm{d}\psi)^2 + \Delta_y(\mathrm{d}\tau - r^2\mathrm{d}\psi)^2\right] + (r^2 + y^2)\left[\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}y^2}{\Delta_y}\right].$$

For the Kerr spacetime the functions $\Delta_r(r)$ and $\Delta_y(y)$ read

$$\Delta_r = r^2 - 2Mr + a^2\,, \qquad \Delta_y = a^2 - y^2\,,$$

but they can also be solved for in the whole Kerr-NUT class. The scalar wave operator takes the symmetric form,

$$(r^2 + y^2)\Box = \partial_r\Delta_r\partial_r - \frac{(r^2\partial_\tau + \partial_\psi)^2}{\Delta_r} + \partial_y\Delta_y\partial_y + \frac{(y^2\partial_\tau - \partial_\psi)^2}{\Delta_y}\,.$$

The overall factor is the density $\Sigma = r^2 + y^2 = \sqrt{-\det g}$.

For further research, it would be interesting to investigate possible extensions of this result to Petrov type D spacetimes. This might give a simplified approach to the Kinnersley classification [81] and to Held's coordinate free integration method [65].

3. Spin-s fields and linearized gravity

In the last chapter we investigated mathematical descriptions of vacuum spacetimes and their inherent structure. The next step is to describe fields on such backgrounds. For that reason, the spin-s equation is introduced in the next section and its integrability conditions are derived. We will not deal with fields coupled to the curvature[12]. In section 3.2, we will translate the Maxwell equations into spinorial form and show the equivalence of the Bianchi identities to the spin-2 field equations. Finally, field equations for linearized gravity and its gauge freedom are discussed and compared to the spin-2 equations in section 3.4.

The results of section 3.4 are partly based on the publication [2].

3.1. The spin-s field equation

The formulation in terms of spinors yields in many cases much simpler equations then the corresponding tensorial version. An example is the Killing spinor equation discussed in section 2.3, which unifies several tensorial equations related to conservation laws. More examples can be found in the books [90] and [91] of Penrose and Rindler. Another one is the spin-s equation,

$$\nabla^{A'A_1}\phi_{A_1A_2\ldots A_{2s}} = 0\,, \tag{3.1}$$

for a symmetric spinor $\phi_{A_1A_2\ldots A_{2s}} = \phi_{(A_1A_2\ldots A_{2s})}$ with s a positive half-integer.

It can be shown to be equivalent to the neutrino equation for $s = 1/2$, the Maxwell equation for $s = 1$, the Rarita-Schwinger equation for $s = 3/2$ and the Bianchi identities for $s = 2$. Before we proceed with the explicit derivation for $s = 1, 2$ in the next section, we derive the integrability conditions of the spin-s equation. Applying another derivative to (3.1) and using the footnote on p. 14 yields,

$$\begin{aligned} 0 &= \nabla_{A'B}\nabla^{A'A_1}\phi_{A_1A_2\ldots A_{2s}} \\ &= -\tfrac{1}{2}\,\Box\,\phi_{BA_2\ldots A_{2s}} + \Box_B{}^{A_1}\phi_{A_1A_2\ldots A_{2s}} \\ &= -\tfrac{1}{2}\,\Box\,\phi_{BA_2\ldots A_{2s}} - \Psi_B{}^{A_1}{}_{A_2}{}^{C}\phi_{A_1C\ldots A_{2s}} - \cdots - \Psi_B{}^{A_1}{}_{A_{2s}}{}^{C}\phi_{A_1A_2\ldots C}\,. \end{aligned} \tag{3.2}$$

So the totally symmetric part becomes the wave equation,

$$\Box\phi_{A_1\ldots A_{2s}} - 2(2s-1)\Psi_{(A_1A_2}{}^{BC}\phi_{A_3\ldots A_{2s})BC} = 0\,, \tag{3.3}$$

[12] This test field approximation neglects the quadratic occurrence of the field in the energy-momentum tensor on the right hand side of the Einstein equations.

while the only non trivial trace part is the contraction with ϵ^{BA_2} resulting in

$$\begin{aligned} 0 &= \Box^{A_1A_2}\phi_{A_1A_2\dots A_{2s}} \\ &= -\Psi^{A_1A_2}{}_{A_3}{}^{B}\phi_{A_1A_2B\dots A_{2s}} - \dots - \Psi^{A_1A_2}{}_{A_{2s}}{}^{B}\phi_{A_1A_2\dots A_{2s-1}B} \\ &= -(2s-2)\Psi^{A_1A_2B}{}_{(A_3}\phi_{A_4\dots A_{2s})A_1A_2B}\,. \end{aligned} \tag{3.4}$$

This last purely algebraic identity is known as the *Buchdahl constraint*, [25]. Note that this construction is non-trivial only for $s \geq 3/2$. So only fields of spin greater than one are constrained in this way. For the spin-2 (e.g. gravity) case, we have explicitly

$$0 = \Psi^{ABC}{}_{(D}\phi_{E)ABC}\,, \tag{3.5}$$

which results on a type-D background in a principal frame ($\Psi_{ABCD} = 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)}$) in $0 = \phi_3 o_{(D}o_{E)} - \phi_1\iota_{(D}\iota_{E)}$, so $\phi_1 = 0 = \phi_3$. Attempts have been made to reformulate the spin-s equation into a constraint free form by non-minimal coupling to curvature, see e.g. [70].

The wave equation with a curvature potential (3.3) for $s = 2$ and $\phi = \Psi$ is called Penrose wave equation. The projection into a NP-tetrad for $s = 1, 2$ can be found in [20]. To linear order this leads to the Teukolsky equations for the extreme components, see section 3.4, and the form is also convenient to extract perturbation equations of higher order.

3.2. Maxwell equations and Bianchi identities

Let us first have a look at the source-free Maxwell equations

$$\nabla_{[a}F_{bc]} = 0\,, \qquad \nabla^a F_{ab} = 0\,, \tag{3.6}$$

for a real 2-form F_{ab}. Following [90, eq. (3.3.31)], we define the alternating tensor

$$e_{abcd} = \mathrm{i}\epsilon_{AC}\epsilon_{BD}\bar{\epsilon}_{A'D'}\bar{\epsilon}_{B'C'} - \mathrm{i}\epsilon_{AD}\epsilon_{BC}\bar{\epsilon}_{A'C'}\bar{\epsilon}_{B'D'}\,, \tag{3.7}$$

and the Hodge dual on 2-forms,

$$*F_{ab} = \tfrac{1}{2}e_{abcd}F^{cd}\,. \tag{3.8}$$

Now, the first equation in (3.6) can be rewritten in divergence form,

$$0 = \tfrac{1}{2}e_{abcd}\nabla^b F^{cd} = \nabla^b *F_{ab}\,. \tag{3.9}$$

Alternatively, the second equation yields,

$$0 = \nabla^a e_{abcd} * F^{cd} = e_{abcd}\nabla^{[a} *F^{cd]}\,, \tag{3.10}$$

because of $**F = -F$. So with the complex linear combination $\mathcal{F}_{ab} = \frac{1}{2}(F_{ab} + \mathrm{i} * F_{ab})$, the Maxwell equations do have the two alternative representations,

$$\mathrm{d}\mathcal{F} = 0 \qquad \Leftrightarrow \qquad \nabla^a\mathcal{F}_{ab} = 0\,. \tag{3.11}$$

Since any real 2-form can be expanded into

$$F_{ab} = \phi_{AB}\bar{\epsilon}_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}\,, \tag{3.12}$$

with a symmetric spinor $\phi_{AB} = \phi_{(AB)}$ and the Hodge dual reads,

$$\star F_{ab} = -\mathrm{i}\phi_{AB}\bar{\epsilon}_{A'B'} + \mathrm{i}\bar{\phi}_{A'B'}\epsilon_{AB}\,, \tag{3.13}$$

we find the spinorial version of (3.6) for $\mathcal{F}_{ab} = \phi_{AB}\bar{\epsilon}_{A'B'}$ to be

$$0 = \nabla^{A'A}\phi_{AB}\bar{\epsilon}_{A'B'} = \nabla_{B'}{}^{A}\phi_{AB}\,, \tag{3.14}$$

so the source-free Maxwell equations (3.6) are equivalent to

$$\nabla_{A'}{}^{A}\phi_{AB} = 0\,, \tag{3.15}$$

which is (3.1) for $s = 1$. The wave equation (3.3) in this case yields

$$\Box\phi_{AB} = 2\Psi_{ABCD}\phi^{CD}\,. \tag{3.16}$$

A valence-2 spinor can be decomposed into $\phi_{AB} = \phi_2 o_A o_B - 2\phi_1 o_{(A}\iota_{B)} + \phi_0 \iota_A \iota_B$ and the six real degrees of freedom of F_{ab} are encoded in 3 complex scalars

$$\phi_0 = \phi_{AB}o^A o^B = F_{ab}l^a m^b = F\cdot Z_0\,, \tag{3.17a}$$

$$\phi_1 = \phi_{AB}\iota^A o^B = \tfrac{1}{2}F_{ab}(l^a n^b - m^a\bar{m}^b) = F\cdot Z_1\,, \tag{3.17b}$$

$$\phi_2 = \phi_{AB}\iota^A\iota^B = F_{ab}\bar{m}^a n^b = F\cdot Z_2\,. \tag{3.17c}$$

So the real 2-form does have the bivector representation

$$F = \phi_0 Z^0 + \phi_1 Z^1 + \phi_2 Z^2 + \overline{\phi}_0\overline{Z}^0 + \overline{\phi}_1\overline{Z}^1 + \overline{\phi}_2\overline{Z}^2\,, \tag{3.18}$$

or in index notation $\phi_I = F\cdot Z_I$ and $F = \phi_I Z^I + \overline{\phi}_I\overline{Z}^I$. The components of (3.15) yield the vacuum Maxwell equations in GHP form,

$$(\text{þ} - 2\rho)\phi_1 = (\eth' - \tau')\phi_0 - \kappa\phi_2\,, \tag{3.19a}$$

$$(\eth - 2\tau)\phi_1 = (\text{þ}' - \rho')\phi_0 - \sigma\phi_2\,, \tag{3.19b}$$

$$(\text{þ}' - 2\rho')\phi_1 = (\eth - \tau)\phi_2 - \kappa'\phi_0\,, \tag{3.19c}$$

$$(\eth' - 2\tau')\phi_1 = (\text{þ} - \rho)\phi_2 - \sigma'\phi_0\,. \tag{3.19d}$$

The Bianchi identity for the Weyl or conformal curvature $C_{abcd} = C_{[ab]cd} = C_{ab[cd]} = C_{cdab}$ reads

$$\nabla_{[a}C_{bc]de} = 0\,. \tag{3.20}$$

Defining the left dual, ${}^*C_{abcd}$, and right dual, C^*_{abcd}, by taking the dual with respect to the first and last index pair, respectively, one finds ${}^*C^*_{abcd} = -C_{abcd}$. So analogous to Maxwell's equations, we can rewrite (3.20) into divergence form $\nabla^{a*}C_{abcd} = 0$. On the other hand, contracting indices in (3.20) (there is only one non-trivial possibility) yields $\nabla^a C_{abcd} = 0$.

The Weyl tensor does have the spinor representation

$$-C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}\,, \tag{3.21}$$

where Ψ_{ABCD} is a completely symmetric 4-spinor. The 10 degrees of freedom of the Weyl tensor are given by 5 complex scalars $\Psi_0, \dots \Psi_4$ given in (2.29). The vacuum Bianchi identity in the form $\nabla^a(C_{abcd} + \mathrm{i}^*C_{abcd}) = 0$ then translates into the spin-2 equation

$$\nabla^{A'A}\Psi_{ABCD} = 0\,. \tag{3.22}$$

3.3. Spin lowering

Suppose $\phi_{A\ldots D}$ is a solution of the spin-s field equation (3.1) and $\kappa_{C\ldots D}$ solves the valence $(n,0)$ Killing spinor equation (2.40). The concept of *spin lowering* is based on the observation [91, p. 75] that

$$\nabla^{A'A}\left[\phi_{A\ldots BC\ldots D}\kappa^{C\ldots D}\right] = \kappa^{C\ldots D}\nabla^{A'A}\phi_{A\ldots BC\ldots D} + \phi_{A\ldots BC\ldots D}\nabla^{A'A}\kappa^{C\ldots D} = 0\,. \tag{3.23}$$

So the spin lowered field

$$\psi_{A\ldots B} = \phi_{A\ldots BC\ldots D}\kappa^{C\ldots D}\,, \tag{3.24}$$

solves the spin-$(s-\frac{n}{2})$ equation. The argument holds for arbitrary backgrounds, but the fields involved are in general heavily restricted due to the integrability conditions (3.4) and (2.46). Of particular interest is the construction of Maxwell fields (spin-1) from spin-2 fields and Killing spinors of valence two, because their charges can be calculated by integration, see chapter 4. For later reference, we present the component form of this case. For symmetric valence two and four spinors,

$$\kappa_{AB} = \kappa_0\iota_A\iota_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_2 o_A o_B\,, \tag{3.25}$$

$$\begin{aligned}\Psi_{ABCD} &= \Psi_0\iota_A\iota_B\iota_C\iota_D - 4\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)}\\ &\quad - 4\Psi_3 o_{(A}o_B o_C\iota_{D)} + \Psi_4 o_A o_B o_C o_D\,,\end{aligned} \tag{3.26}$$

the contraction over two indices results in the symmetric 2-spinor

$$\begin{aligned}\phi_{AB} &= \Psi_{ABCD}\kappa^{CD}\\ &= \left[\Psi_0\kappa_2 - 2\Psi_1\kappa_1 + \Psi_2\kappa_0\right]\iota_A\iota_B\\ &\quad - 2\left[\Psi_1\kappa_2 - 2\Psi_2\kappa_1 + \Psi_3\kappa_0\right]o_{(A}\iota_{B)}\\ &\quad + \left[\Psi_2\kappa_2 - 2\Psi_3\kappa_1 + \Psi_4\kappa_0\right]o_A o_B\,.\end{aligned} \tag{3.27}$$

3.4. Linearized gravity

The field equations (1.1) of General Relativity are a set of ten non-linear, coupled second order partial differential equations. A vast field of research is the study of exact solutions to these equations [104]. But mostly, symmetry assumptions have to be made to simplify the equations into manageable form. Another approach is to linearize the field equations in some parameter ϵ around a given exact solution (background) and analyze the resulting linear system. This can be used to investigate effects like gravitational radiation, but of course only for “small field strength“ so that higher order effects can be neglected. For a general solution G_{ab} to the field equations and a given background metric g_{ab}, we define the linearized metric

$$h_{ab} = \lim_{\epsilon\to 0,\epsilon\neq 0}\frac{G_{ab}(\epsilon) - g_{ab}}{\epsilon}\,. \tag{3.28}$$

The linearized field equations then take the form

$$-\tfrac{1}{2}\Box\, h_{ab} - \tfrac{1}{2}\nabla_a\nabla_b h^c{}_c + \nabla^c\nabla_{(a}h_{b)c} + \tfrac{1}{2}g_{ab}\left(\Box h^c{}_c - \nabla^c\nabla^d h_{cd}\right) = 8\pi\dot{T}_{ab}\,. \tag{3.29}$$

This is still a set of ten coupled partial differential equations, but it is linear. An additional difficulty is the gauge dependence of h_{ab} due to infinitesimal coordinate changes, see section 3.5. On a flat background with $g_{ab} = \eta_{ab}$ the Minkowski metric, the trace reversed metric $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h^c{}_c$ solves the reduced wave equation

$$\Box \bar{h}_{ab} = -16\pi \dot{T}_{ab}\,, \tag{3.30}$$

after imposing the gauge condition $\nabla^a \bar{h}_{ab} = 0$. This choice of gauge-source-function is compatible with the radiation gauge, which can be imposed on the initial slice and then propagates. There are two remaining degrees of freedom which characterize the linearized gravitational field, see e.g. [118, Section 4.4] for more details.

On a spherically symmetric background with non-vanishing curvature, the extraction of the true degrees of freedom is much more involved. Due to the symmetry, everything can be decomposed into tensor spherical harmonics and a natural decomposition into odd and even parity[13] is inherent. For example the metric can be expanded into ($f = 1-2Mr^{-1}, s = \sin\theta, c = \cos\theta$),

$$h^-_{ab} := \begin{pmatrix} 0 & 0 & -h_0 s^{-1}\partial_\varphi & h_0 s\partial_\theta \\ * & 0 & -h_1 s^{-1}\partial_\varphi & h_1 s\partial_\theta \\ * & * & h_2(s^{-1}\partial^2_{\theta\varphi} - cs^{-2}\partial_\varphi) & \frac{1}{2}h_2(s^{-1}\partial^2_\varphi + c\partial_\theta - s\partial^2_\theta) \\ * & * & * & -h_2(s\partial^2_{\theta\varphi} - c\partial_\varphi) \end{pmatrix} Y_{lm}, \tag{3.31}$$

with 3 free functions h_0, h_1, h_2 depending on t, r for the odd parity and

$$h^+_{ab} := \begin{pmatrix} fH_0 & H_1 & h_0\partial_\theta & h_0\partial_\varphi \\ * & f^{-1}H_2 & h_1\partial_\theta & h_1\partial_\varphi \\ * & * & r^2K + r^2G\partial^2_\theta & r^2G(\partial^2_{\theta\varphi} - cs^{-1}\partial_\varphi) \\ * & * & * & r^2s^2K + r^2G(\partial^2_\varphi + sc\partial_\theta) \end{pmatrix} Y_{lm}, \tag{3.32}$$

with 7 free functions $H_0, H_1, H_2, h_0, h_1, K, G$ depending on t, r for the even parity part. The tensor spherical harmonics are generated by the differential operators acting on the scalar spherical harmonics Y_{lm}. After imposing a suitable gauge condition in [97], Regge and Wheeler were able to extract a wave equation with curvature potential governing the odd parity sector. More then ten years later a similar equation for the even parity sector has been found by Zerilli, [123]. Moncrief then showed in [84] the gauge invariant significance of the equations using a Hamiltonian formulation and a derivation without spherical harmonic decomposition can be found in the article [74] of Jezierski. Using tortoise coordinates $\partial_{r_*} = f\partial_r$, which transform the horizon at $r = 2M$ to $r_* = -\infty$, the equations take the form,

$$\left[\partial^2_t - \partial^2_{r_*} + V^\pm\right] Q^\pm = 0\,. \tag{3.33}$$

Here, we denote the even parity by + and the odd parity by −. The variables and potentials can be written in the form

$$Q^- = \frac{f}{2r}\left[2h_1 + (\partial_r - 2r^{-1})h_2\right], \qquad V^- = f\left[l(l+1)r^{-2} - 6Mr^{-3}\right], \tag{3.34a}$$

$$Q^+ = \frac{q_1}{2\Lambda}, \qquad V^+ = V^- - 2\partial^2_{r_*}\ln\Lambda\,, \tag{3.34b}$$

[13] This corresponds to the sign change for transformations $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$ on the 2-sphere.

with $\Lambda = (l-1)(l+2)/2 + 3Mr^{-1}$ and q_1 related to the even parity metric (3.32) by

$$
\begin{aligned}
q_1 &= 4f^2 r k_2 + l(l+1) r k_1 \,, && (3.35a)\\
k_1 &= K + \tfrac{f}{r}\left[r^2\partial_r G - 2h_1\right] , && (3.35b)\\
2fk_2 &= H_2 - r\partial_r k_1 + (1 - r\partial_r \ln\sqrt{f}) - (f\partial_r + Mr^{-2})(r^2\partial_r G - 2h_1) \,. && (3.35c)
\end{aligned}
$$

The Regge-Wheeler and Zerilli equations (3.33) are most common to study effects of the linearized gravitational field around a Schwarzschild black hole. Applications in the stability analysis will be discussed in chapter 6.

This approach heavily depends on the spherical symmetry of the background and cannot be directly carried over to Kerr spacetime in a natural way. However, Teukolsky derived in [109] complex wave equations for the linearized and gauge-invariant curvature components $\dot{\Psi}_0$ and $\dot{\Psi}_4$. We use a dot to distinguish linearized quantities from the background fields. The derivation is based on the Bianchi identities (2.39) in linearized form and so has a different origin than the Regge-Wheeler approach. The implications will be discussed below, but first we present the equations. With a modification of the GHP connection (2.18)[14,15] introduced in [2],

$$
D_a = \Theta_a - pB_a - q\bar{B}_a \,, \qquad\qquad B_a = \rho n_a - \tau \bar{m}_a \,, \tag{3.36}
$$

the weighted wave operator reads

$$
\begin{aligned}
\boxplus_{p,q} &= D_a D^a \\
&= 2(\text{þ} - p\rho - (q+1)\bar{\rho})(\text{þ}' - \rho') - 2(\eth - p\tau - \bar{\tau}')(\eth' - q\bar{\tau} - \tau') \\
&\quad + (3p-2)\Psi_2 - q(2(\eth\,\bar{\tau}) + 2\bar{\rho}\rho' - \bar{\Psi}_2) \,.
\end{aligned} \tag{3.37}
$$

The last term can be simplified further if restricted to the Kerr-NUT family. However, only the case $q = 0$ is of interest in this chapter and we write $\boxplus_{p,0} = \boxplus_p$. We also not that for unweighted fields, it reduces to the usual wave operator, $\boxplus_{0,0} = \Box$.

Remark 3.4.1. *Note that B_a in* (3.36) *is not GHP prime invariant. With the 1-form h_a defined in* (2.27) *we find $h_a = 2(B_a + B'_a)$ and it follows*

$$
(D_a\phi)' = D'_a\phi' = \zeta^{-2} D_a(\zeta^2\phi') \,. \tag{3.38}
$$

It is also interesting that the time like isometry (2.68) *can be written $\xi_a = \zeta(B'_a - B_a)$.*

On a vacuum background of Petrov type D, a set of decoupled wave equations for the linearized curvature scalars can be derived. The full calculation will not be repeated here. We only mention that one can either apply another derivative[16] to the linearized Bianchi identities (2.39) or project the linearized wave equation (3.3) into a tetrad to derive the

[14] Adding a $(0,0)$ weighted field B_a to σ_{1a} yields again a connection on the weighted line bundle.
[15] Notation differs from [2] by a sign in B_a.
[16] This approach can be formalized by the *decoupling operators* $\mathcal{S}$, further discussed in section 5.2.

equations

$$\left[\boxplus_4 - 16\Psi_2\right]\dot{\Psi}_0 = 0\,, \tag{3.39a}$$

$$\left[\boxplus_2 - 4\Psi_2\right]\left(\zeta\dot{\Psi}_1\right) = -6\zeta\Psi_2[(\text{þ}' + 2\rho' - \bar{\rho}')\dot{\kappa} - (\eth' + 2\tau' - \bar{\tau})\dot{\sigma} + 2\dot{\Psi}_1]\,, \tag{3.39b}$$

$$\left[\,\square + 8\Psi_2\right]\left(\zeta^2\dot{\Psi}_2\right) = -3\dot{\square}\left(\zeta^2\Psi_2\right)\,, \tag{3.39c}$$

$$\left[\boxplus_{-2} - 4\Psi_2\right]\left(\zeta^3\dot{\Psi}_3\right) = -6\zeta^3\Psi_2[(\text{þ} + 2\rho - \bar{\rho})\dot{\kappa}' - (\eth + 2\tau - \bar{\tau}')\dot{\sigma}' + 2\dot{\Psi}_3]\,, \tag{3.39d}$$

$$\left[\boxplus_{-4} - 16\Psi_2\right]\left(\zeta^4\dot{\Psi}_4\right) = 0\,. \tag{3.39e}$$

Remark 3.4.2. *Sometimes the spin coefficient ρ is used in the literature for rescaling the fields in the Teukolsky equation. This only works in the Kinnersley tetrad, since in that case the relation $\rho \sim \zeta^{-1}$ holds. More often, a fractional power of Ψ_2 is used for rescaling. This works for all principal tetrads, because of* (2.57)*, but it degenerates in the Minkowski limit. The Killing spinor does have a well defined Minkowski limit and also gives the correct interpretation to the conservation law* (4.18) *discussed below.*

Details about the derivation of (3.39) can be found in [2]. The spin-2 Teukolsky equations are (3.39a) and (3.39e). The other three equations are in general gauge dependent as we discuss in the next section. However, on Schwarzschild $\Psi_2 = -Mr^{-3}$ and $\zeta = r$ are real and therefore

$$[\square + 8\Psi_2]\left(\zeta^2 \mathrm{Im}\,\dot{\Psi}_2\right) = 0\,, \tag{3.40}$$

follows for the (gauge invariant) imaginary part of (3.39c). After a spherical harmonic decomposition and an expansion of the wave operator (2.101), this is for $l > 1$ the Regge-Wheeler equation (3.33). This was already noticed by Price in [95, Section 4A] and an expansion of $\dot{\Psi}_2$ in terms of linearized metric shows

$$-r^3\partial_t \mathrm{Im}\,\dot{\Psi}_2 = \frac{(l+2)!}{4(l-2)!}Q^-\,, \tag{3.41}$$

as was also calculated in [2]. It follows that t-independent as well as $l = 0, 1$ solutions of (3.40) cannot be described by Q^-. This suggests to use $\mathrm{Im}\,\dot{\Psi}_2$ as a representative for odd parity perturbations of Schwarzschild spacetime. Moreover, we showed for even parity perturbations

$$r^3 \mathrm{Re}\,\dot{\Psi}_2 + \frac{3M}{4}\left[2K + l(l+1)G\right] = 4\Lambda Q^+\,, \tag{3.42}$$

so a "slight" modification[17] of $\mathrm{Re}\,\dot{\Psi}_2$ makes it gauge invariant and, up to functions in case of spherical harmonic decomposition, equivalent to the gauge invariant Zerilli variable Q^+. However, the even parity part is more complicated, because it is non-trivial already in the background. On a Kerr background, Ψ_2 is complex and therefore real and imaginary parts of (3.39c) are gauge dependent.

[17] The correction term is the spherical trace of the linearized metric (3.32). We note the similarity to the metric term in $\dot{\Psi}_2$ in the expansion (3.51c).

Remark 3.4.3. *The decomposition into odd and even parity on a spherically symmetric background corresponds to imaginary and real parts of unweighted scalars. For weighted scalars the decomposition is most easily done by "despinning" first, see [95]. On a Kerr background the discrete transformation $(t,\varphi) \to -(t,\varphi)$ would be a natural generalization, see also [94, Section 3.1]. The operators for "despinning" are presented in appendix A.2.*

Remark 3.4.4. *In a vacuum type D background the Maxwell components* (3.17) *fulfill the equations*

$$\left[\boxplus_2 - 4\Psi_2\right]\phi_0 = 0\,, \tag{3.43a}$$

$$\left[\,\square + 2\Psi_2\right]\left(\zeta\phi_1\right) = 0\,, \tag{3.43b}$$

$$\left[\boxplus_{-2} - 4\Psi_2\right]\left(\zeta^2\phi_2\right) = 0\,, \tag{3.43c}$$

The middle equation is known as Fackerell-Ipser equation, [50], and the other two are the spin-1 Teukolsky equations.

The equations (3.43a) and (3.39a) together with their versions for spin $0, 1/2, 3/2$ can be unified into the Teukolsky master equation (5.53). It is the most common equation to analyze linearized gravitational perturbations of the Kerr spacetime, because of its gauge invariance and separability. It is the starting point of Chandrasekhar's integration program [29] and also used in the proof of mode stability by Whiting in [121]. On the other hand the operator does have a complex connection and a long range potential. This makes it difficult to analyze with the methods reviewed in chapter 6. We will take this, together with the fact that (3.39c) and (3.43b) are scalar wave equations, as a motivation to further investigate the middle equations in this thesis.

Before we discuss the gauge transformations in the next section, let us briefly review the situation for coupled electromagnetic and gravitational perturbations of a Kerr-Newman background. A single decoupled scalar wave equation for this case is not known. Following the approach for the vacuum case and using the gauge-invariant variables

$$\dot\Psi_0\,, \qquad \dot\chi_0 = 3\Psi_2\dot\phi_0 - 2\phi_1\dot\Psi_1\,, \qquad \dot\chi_2 = 3\Psi_2\dot\phi_2 - 2\phi_1\dot\Psi_3\,, \qquad \dot\Psi_4\,, \tag{3.44}$$

Lee [82] ended with equations of the form (neglecting lower order terms)

$$\left[\text{þ}\,\text{þ}' - f(\Psi_2,\phi_1)\,\eth\,\eth'\right]\dot\Psi_0 = g(\phi_1,\Psi_2)\,\text{þ}\,\eth\,\dot\chi_0\,, \tag{3.45}$$

$$\left[\text{þ}'\,\text{þ} - \frac{1}{f(\Psi_2,\phi_1)}\eth'\,\eth\,\right]\dot\chi_0 = h(\Psi_2,\phi_1)\text{þ}'\eth\dot\Psi_0\,, \tag{3.46}$$

and the GHP primed versions for $\dot\chi_2$ and $\dot\Psi_4$. The equations do not separate, see also [29, p. 582]. In the special case of vanishing angular momentum (Reissner-Nordström), the equations decouple *after* a separation of variables is performed, see [116]. The right hand side of (3.45) vanishes in the case of vanishing background electromagnetic field and the equations reduce to the usual Teukolsky equations for spin $s = 1, 2$. Another approach to Kerr-Newman perturbations by Fackerell and Crossmann, [37], is based on the middle three linearized curvature components and the Maxwell scalars. However, their equations also do not decouple, even though a gauge condition for a decoupled gravitational part is derived, see also [15].

Finally, we mention a recent result of [88], where it is used that the equations decouple to first order in a and give an approximation for slowly rotating Kerr backgrounds.

3.5. Gauge freedom

For linearized gravity in a tetrad based approach, there are gauge degrees of freedom corresponding to infinitesimal changes of the tetrad (tetrad gauge) and of the coordinates (coordinate gauge). In this section, we will review these gauge transformations and collect equations which are needed in section 4.3.2. For further details and tabulated results, we refer to [105],[23].

The tetrad gauge freedom is the freedom of infinitesimal Lorentz transformations (2.10) of the linearized NP tetrad. With a, b complex and A, ϑ real functions they read,

$$\delta \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}_B = \begin{pmatrix} A & 0 & \bar{b} & b \\ 0 & -A & \bar{a} & a \\ a & b & \mathrm{i}\vartheta & 0 \\ \bar{a} & \bar{b} & 0 & -\mathrm{i}\vartheta \end{pmatrix} \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix} . \tag{3.47}$$

Here the subscript B denotes linearized quantities, δ stands for an infinitesimal tetrad transformation and the matrix entries itself are linear in the perturbation parameter. The effect on any NP scalar can now be calculated. As an example, we derive the tetrad gauge freedom of the curvature scalars.

With four real functions N_1, N_2, L_1, L_2 and six complex functions $L_3, N_3, M_i, i = 1, .., 4$, the linearized tetrad can be expanded in terms of the background tetrad via

$$\begin{pmatrix} l^a \\ n^a \\ m^a \\ \overline{m}^a \end{pmatrix}_B = \begin{pmatrix} L_1 & L_2 & L_3 & \overline{L}_3 \\ N_1 & N_2 & N_3 & \overline{N}_3 \\ M_1 & M_2 & M_3 & M_4 \\ \overline{M}_1 & \overline{M}_2 & \overline{M}_4 & \overline{M}_3 \end{pmatrix} \begin{pmatrix} l^a \\ n^a \\ m^a \\ \overline{m}^a \end{pmatrix} . \tag{3.48}$$

Here, we use a subscript B instead of a dot for the linearized tetrad. Note that the matrix entries are by definition linearized quantities and we suppress an overdot to avoid clutter. There are 16 degrees of freedom at a point, ten correspond to metric perturbations and six are infinitesimal Lorentz transformations (tetrad gauge). The linearized tetrad 1-forms have the representation

$$\begin{pmatrix} l_a \\ n_a \\ m_a \\ \overline{m}_a \end{pmatrix}_B = \begin{pmatrix} -N_2 & -L_2 & \overline{M}_2 & M_2 \\ -N_1 & -L_1 & \overline{M}_1 & M_1 \\ \overline{N}_3 & \overline{L}_3 & -\overline{M}_3 & -M_4 \\ N_3 & L_3 & -\overline{M}_4 & -M_3 \end{pmatrix} \begin{pmatrix} l_a \\ n_a \\ m_a \\ \overline{m}_a \end{pmatrix} . \tag{3.49}$$

For the bivectors (2.8), it follows

$$\dot{Z}^0 = -(L_1 + M_3)Z^0 + \tfrac{1}{2}(\overline{M}_1 + N_3)Z^1 - \overline{M}_4\overline{Z}^0 - \tfrac{1}{2}(\overline{M}_1 - N_3)\overline{Z}^1 + N_1\overline{Z}^2 , \tag{3.50a}$$

$$\begin{aligned} \dot{Z}^1 = &-(M_2 + \overline{L}_3)Z^0 - \tfrac{1}{2}(L_1 + N_2 + M_3 + \overline{M}_3)Z^1 - (\overline{M}_1 + N_3)Z^2 \\ &+ (L_3 - \overline{M}_2)\overline{Z}^0 - \tfrac{1}{2}(L_1 + N_2 - M_3 - \overline{M}_3)\overline{Z}^1 + (\overline{N}_3 - M_1)\overline{Z}^2 , \end{aligned} \tag{3.50b}$$

$$\dot{Z}^2 = -\tfrac{1}{2}(M_2 + \overline{L}_3)Z^1 - (N_2 + \overline{M}_3)Z^2 + L_2\overline{Z}^0 + \tfrac{1}{2}(M_2 - \overline{L}_3)\overline{Z}^1 - M_4\overline{Z}^2 . \tag{3.50c}$$

Linearization of the tetrad representation of the metric, $g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$, yields

$$h_{ln} = -L_1 - N_2 , \qquad h_{m\bar{m}} = \overline{M}_3 + M_3 , \qquad h_{n\bar{m}} = N_3 - \overline{M}_1 , \qquad h_{lm} = \overline{L}_3 - M_2 ,$$

and therefore $\mathrm{tr}_g h = -2(L_1 + N_2 + M_3 + \overline{M}_3)$. Linearization of the curvature scalars (2.29) shows

$$\dot\Psi_0 = -\dot C\cdot(Z_0, Z_0)\,, \tag{3.51a}$$
$$\dot\Psi_1 = -\dot C\cdot(Z_0, Z_1) + \tfrac{3}{2}\Psi_2(\overline{L}_3 + M_2)\,, \tag{3.51b}$$
$$\dot\Psi_2 = -\dot C\cdot(Z_1, Z_1) - \tfrac{1}{2}\Psi_2 \mathrm{tr}_g h\,, \tag{3.51c}$$
$$\dot\Psi_3 = -\dot C\cdot(Z_2, Z_1) + \tfrac{3}{2}\Psi_2(N_3 + \overline{M}_1)\,, \tag{3.51d}$$
$$\dot\Psi_4 = -\dot C\cdot(Z_2, Z_2)\,, \tag{3.51e}$$

which follows from e.g. $\dot\Psi_1 = -\dot C\cdot(Z_0, Z_1) - C\cdot(\dot Z_0, Z_1) - C\cdot(Z_0, \dot Z_1)$ and (3.50). The linearized curvature $\dot C_{abcd}$ is of course independent of any tetrad and therefore only the additional terms transform non-trivial under (3.47). It follows that $\dot\Psi_0, \dot\Psi_4$ and $\dot\Psi_2$ are tetrad gauge-invariant. The remaining components transform via

$$\dot\Psi_1 \to \dot\Psi_1 + 3b\Psi_2\,, \qquad \dot\Psi_3 \to \dot\Psi_3 + 3\bar a\Psi_2\,. \tag{3.52}$$

From (3.47), we find for (3.50b)

$$\delta\dot Z^1 = -2bZ^0 - 2\bar a Z^2\,. \tag{3.53}$$

Under infinitesimal coordinate transformations $x^a \to x^a + \xi^a$, a tensor field $\dot T$ transforms with the Lie derivative according to

$$\dot T \to \dot T + \delta\dot T\,, \qquad \delta T = -\mathcal{L}_\xi T\,. \tag{3.54}$$

The gauge term can be expressed in terms of covariant derivatives,

$$\begin{aligned}\mathcal{L}_\xi T^{b_1\ldots b_m}_{a_1\ldots a_n} = \xi^c\nabla_c T^{b_1\ldots b_m}_{a_1\ldots a_n} &+ (\nabla_{a_1}\xi^c)T^{b_1\ldots b_m}_{c a_2\ldots a_n} + \ldots + (\nabla_{a_n}\xi^c)T^{b_1\ldots b_m}_{a_1\ldots a_{n-1}c}\\ &- (\nabla_c\xi^{b_1})T^{c b_2\ldots b_m}_{a_1\ldots a_n} - \ldots - (\nabla_c\xi^{b_m})T^{b_1\ldots b_{m-1}c}_{a_1\ldots a_n}\,,\end{aligned}$$

and for the linearized metric this implies

$$h_{ab} \to h_{ab} - \nabla_{(a}\xi_{b)}\,. \tag{3.55}$$

The linearized curvature around a Minkowski spacetime is gauge invariant, because it vanishes in the background. Moreover $\dot\Psi_0, \dot\Psi_1, \dot\Psi_3, \dot\Psi_4$ are coordinate gauge invariant on Petrov type D spacetimes in a principal tetrad, because of (2.56). Also $\mathrm{Im}\,\Psi_2$, the variable in the Regge-Wheeler equation (3.40), is gauge-invariant on a Schwarzschild background, because $\delta\dot\Psi_2 = -\mathcal{L}_\xi\Psi_2$ and Ψ_2 is real in that background.

For the middle bivector component $Z^1 = n\wedge l - \bar m\wedge m$ and with gauge vector $\xi = Xl + Yn + Zm + \bar Z\bar m$ we find

$$\begin{aligned}\mathcal{L}_\xi Z^1_{ab} &= \xi^c\nabla_c Z^1_{ab} + (\nabla_a\xi^c)Z^1_{cb} + (\nabla_b\xi^c)Z^1_{ac}\\ &= 2\nabla_{[a}(\xi^c Z^1_{|c|b]}) + \xi^c(\nabla_c Z^1_{ab} - \nabla_a Z^1_{cb} + \nabla_b Z^1_{ca})\\ &= (\mathrm{d}\hat\xi)_{ab} + 3\xi^c(\mathrm{d}Z^1)_{cab}\,,\end{aligned}$$

where we put $\hat\xi = Xl - Yn + Zm - \bar Z\bar m$.

A general 1-form ξ_a has spherical harmonic decomposition

$$\xi_a^- = (0, 0, -\Lambda s^{-1}\partial_\varphi, \Lambda s\partial_\theta)Y_{lm}\,, \qquad \xi_a^+ = (N_0, N_1, N\partial_\theta, N\partial_\varphi)Y_{lm}\,,$$

with functions Λ, N_0, N_1, N depending on t, r. The metric transforms according to (3.55), for which we find

$$\delta h_0 = \partial_t\Lambda\,, \qquad \delta h_1 = (\partial_r - 2/r)\Lambda\,, \qquad \delta h_2 = -2\Lambda$$

for the odd parity perturbations (3.31) and

$$\begin{aligned} &\delta v H_0 = 2\partial_t N_0 - 2Mr^{-3}(r-2M)N_1\,, && \delta H_1 = (\partial_r + \tfrac{2M}{r(r-2M)})N_0 + \partial_t N_1\,, \\ &\delta v^{-1} H_2 = 2(\partial_r + \tfrac{M}{r(r-2M)})N_1\,, && \delta h_0 = N_0 + \partial_t N\,, \\ &\delta h_1 = N_1 + (\partial_r - 2/r)N\,, && \delta r^2 K = 2(r-2M)N_1\,, \qquad \delta r^2 G = 2N\,, \end{aligned}$$

for the even parity perturbations (3.32). The Regge-Wheelers gauge, [97], is given by

- $\Lambda = \frac{1}{2}h_2$ for odd parity. It gauges $h_2 \to 0$.
- $N = -\frac{1}{2}r^2 G$, $N_0 = -h_0 - \partial_t N$, $N_1 = -h_1 - (\partial_r - 2/r)N$ for even parity. It gauges $h_0, h_1, G \to 0$.

A neat formulation of this gauge conditions independent of a decomposition into spherical harmonics is derived in [94, p. 57]. It is of the form

$$h_{mm} = 0 = h_{\bar{m}\bar{m}}\,, \qquad \eth' h_{lm} + \eth\, h_{l\bar{m}} = 0 = \eth\, h_{n\bar{m}} + \eth' h_{nm}\,. \tag{3.56}$$

A different and very flexible point of view on the gauge freedom is the notion of gauge-source-functions devised in [54]. Here one extracts a reduced system of evolution equations, which is equivalent to the actual system for a particular choice of gauge-source-function. Consider an equation of the form $F = \mathrm{d}A$ for a 1-form A, e.g. Maxwell equations or each of (2.20). In equations of this form, the gauge freedom $\delta A = \mathrm{d}f$ for a function f is inherent. Instead of fixing this function directly to fix the gauge, e.g. the Coulomb gauge $A^0 = 0$ for Maxwell's equations, one can add an equation of the form

$$\nabla_a A^a = \Box f = G\,, \tag{3.57}$$

with the gauge-source-function G depending on A, see [53, Section 3.2]. For Einstein equations in first order form, given by the Cartan equations, the coordinate and tetrad gauge-source-functions G^a and $G^{\underline{bc}}$ take the form

$$\Box x^a = G^a\,, \qquad \nabla^a e^{\underline{ba}} \nabla_a e_{\underline{b}}{}^{\underline{b}} = G^{\underline{bc}}\,, \tag{3.58}$$

for coordinates x^a. For the linearized Ricci tensor, the coordinate gauge-source-function leads to

$$\delta R_{ab} = -\frac{1}{2}\Box\, h_{ab} - R_a{}^c{}_b{}^d h_{cd} + \nabla_{(a} G_{b)}\,. \tag{3.59}$$

The harmonic coordinates gauge $G^a = 0$ leads to the vacuum Einstein equations in wave form, $\Box h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} = 0$. Since G^a is of the form $G^a = \nabla_b h^{ba} - \frac{1}{2}\nabla^a h^b{}_b$, we find for the linearized scalar wave operator

$$\dot{\Box} = -h^{ab}\nabla_a\nabla_b - G^a\nabla_a\,. \tag{3.60}$$

So, one can choose the gauge-source-function such that

$$\dot{\Box}(\zeta^2\Psi_2) = 0\,. \tag{3.61}$$

This makes (3.39c) into the generalized Regge-Wheeler equation

$$(\Box + 8\Psi_2)(\zeta^2\dot{\Psi}_2) = 0\,. \tag{3.62}$$

4. Non-radiating modes and conserved charges

In contrast to the scalar wave equation, the spin-s field equations with $s > 0$ on a Kerr background admit non-trivial, time-independent, finite energy solutions. We will call these solutions non-radiating modes. There is a close relation between gauge-invariant, non-radiating modes and conserved quasi-local charge integrals. For the Maxwell field, there is a two parameter family of non-radiating, Coulomb type solutions which carry the two conserved electric and magnetic charges. In fact, a Maxwell field on the Kerr exterior will disperse exactly when it has vanishing charges, see [21] for the Schwarzschild case and [10] for a slowly rotating Kerr background.

For linearized gravity, however, there are both non-radiating modes corresponding to gauge-invariant conserved charges and "pure gauge" non-radiating modes. Thus conditions ensuring that a solution of linearized gravity will disperse must be a combination of charge-vanishing and gauge conditions. For the Kerr background, the non-radiating modes include perturbations within the Kerr family, i.e. infinitesimal changes of mass and axial rotation. We denote the parameters for these deformations $\dot{M}, \dot{a}$. Since M, a are gauge-invariant quantities, it is not possible to eliminate these modes by imposing a gauge condition. A canonical analysis along the lines of [72], see below, yields conserved charges corresponding to the Killing fields $\partial_t, \partial_\phi$, which in turn correspond to the gauge-invariant deformations $\dot{M}, \dot{a}$ mentioned above.

The infinitesimal boosts, translations and (non-axial) rotations of the black hole yield further non-radiating modes which are, however, "pure gauge" in the sense that they are generated by infinitesimal coordinate changes. If one imposes suitable regularity conditions on the perturbations which exclude e.g. those which turn on the NUT charge, a 10-dimensional space of non-radiating modes remains. This is spanned by the 2-dimensional space of non-gauge modes which carry the $\dot{M}, \dot{a}$ charges, together with the "pure gauge" non-radiating modes, and corresponds in a natural way to the Lie algebra of the Poincaré group. It can be seen from this discussion that a combination of charge vanishing conditions and gauge conditions allows one to eliminate all non-radiating solutions of linearized gravity.

We also note that the constraint equations implied by the Maxwell and linearized gravity equations are underdetermined elliptic systems, and therefore admit solutions of compact support, see [43] and references therein. In particular, one may find solutions of the constraint equations with arbitrarily rapid fall-off at infinity. The corresponding solutions of the Maxwell equations have vanishing charges. For the case of linearized gravity, the charges corresponding to $\dot{M}, \dot{a}$ vanish for solutions of the field equations with rapid fall-off at infinity. For such solutions, all non-radiating modes may therefore be eliminated by imposing suitable gauge conditions.

There are many candidates for a quasi-local mass expression in the literature including, to mention just a few, those put forward by Penrose, Brown and York, and Wang and Yau. See the review of Szabados [106] for background and references. But in order to make effective use of such charge vanishing conditions, it is necessary to have simple expressions for the charge integrals in terms of the field strengths. Therefore the goal is to find quasi-local mass and angular momentum for linearized gravity on Kerr spacetime in terms of the curvature. Here the approach of spin lowering due to Penrose, reviewed in section 3.3, seemed to be most promising, see also [62].

We start by discussing the relation between non-radiating modes and charges for the case of a Maxwell field in section 4.1. The spin-2 equation on Minkowski and Petrov type D spacetimes is discussed in section 4.2, where the results are also compared to the Komar integrals. In section 4.3, non-radiating modes for linearized gravity on Petrov type D backgrounds are derived, in particular an expression for the conserved charge corresponding to the linearized mass in terms of linearized curvature quantities on the Kerr background is presented in section 4.3.2. Obstructions for angular momentum charges are discussed in section 4.3.3. Finally, the lowest modes for the scalar wave equation with curvature potential, as they occur for the middle components for spin one and two, are derived in section 4.4.

Sections 4.2.1, 4.2.2 and 4.3.2 are based on the publication [3].

4.1. Maxwell equations

Let F_{ab} be a solution of the source-free Maxwell equations $\mathrm{d}F = 0 = \mathrm{d} * F$. The charge integral

$$q_E + \mathrm{i}q_B = \frac{1}{4\pi}\int_S *F + \mathrm{i}F\,, \tag{4.1}$$

depends only on the homology class of the closed 2-surface S. Here real and imaginary parts correspond to electric and magnetic charges, respectively. The Kerr exterior, being diffeomorphic to $\mathbb{R}^4$ with a solid cylinder removed, contains topologically non-trivial 2-spheres, and hence the source-free Maxwell equations on the Kerr exterior admits solutions with non-vanishing charges. In view of the fact that the charges are conserved, it is natural that there is a time-independent solution which "carries" the charge. It is most conveniently expressed in the equivalent representation of Newman-Penrose, see section (3.2), in which it takes the explicit form

$$\phi_1 = \frac{q_E + \mathrm{i}q_B}{\zeta^2}\,, \qquad \phi_0 = 0 = \phi_2\,, \tag{4.2}$$

where ζ is the coefficient of the type D Killing spinor (2.66) and known as the Coulomb solution.

In order to prove boundedness and decay for the Maxwell field, it is necessary to eliminate the non-radiating modes by imposing the charge vanishing condition

$$\int_S *F + \mathrm{i}F = 0\,, \tag{4.3}$$

see section 6.1 and [10, Section 3]. Written in terms of the Newman-Penrose scalars ϕ_I, $I = 0, 1, 2$, the charge vanishing condition (4.3) in the Carter tetrad (2.107) takes the form

$$\int_{S^2(t,r)} 2(r^2 + a^2)\phi_1 + \mathrm{i}a\sqrt{\Delta}\sin\theta(\phi_0 - \phi_2)\mathrm{d}\mu = 0\,, \tag{4.4}$$

as follows from (3.18) and (2.111). Here $S^2(t,r)$ is a sphere of constant t, r in Boyer-Lindquist coordinates and $\mathrm{d}\mu = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$.

4.2. Spin-2 equations

The situation is much more involved for spin-2. The heuristic picture is that the $l = 0$ and $l = 1$ modes are non-radiating and their charges correspond to mass and angular momentum, respectively. But an "$l = 0, 1$ mode" of a spin-2 field needs to be defined appropriately, spherical harmonics are tight to spherically symmetric situations and gauge dependence on curved backgrounds affects the discussion.

In addition, the choice of variables is non-trivial in the case of linearized gravity, since the equations for the linearized curvature components admit solutions[18] which do not correspond to solutions of the linearized Einstein equations. This will be explained in more detail in the next section, where the concept of Penrose charges is applied on Minkowski space.

The basic idea of Penrose charges is to use solutions of the Killing spinor equation (2.47) to spin-lower a spin-2 field into a spin-1 field, see section 3.3. The resulting field can be integrated in a natural way over any 2-sphere, as mentioned already for the Maxwell field strength in (4.1). The resulting charges only depend on the homology class of the 2-sphere and are therefore called *quasi-local*.

In the next section we calculate all valence-2 Killing spinors on Minkowski spacetime in spherical coordinates. This gives a clear correspondence of the charges (generated by some source) to the lowest modes of the spin-2 field. In section 4.2.2, the remaining Killing spinor (2.66) on type D backgrounds is used to derive a quasi-local mass in terms of the curvature. The absence of an angular momentum charge in this context is discussed and compared to Komar integrals in section 4.2.3.

4.2.1. Non-radiating modes on Minkowski spacetime

Minkowski spacetime is topologically $\mathbb{R}^4$ and all spheres can be contracted to a point. It follows, that quasi-local charges for spin-s fields vanish identically. However, removing a ball (or just a point) gives a good model to motivate the approach.

The Killing spinor equation (2.47) and its tensorial counterpart, the conformal Killing-Yano equation (2.43), on Minkowski space have been widely discussed in the literature [91, 76, 66] and the explicit solution in Cartesian coordinates is well known. Denoting them by $x^{A'A}$, it is of the form

$$\kappa^{AB} = U^{AB} + 2x^{A'(A}V_{A'}^{B)} + x^{A'A}x^{B'B}W_{A'B'}\,. \tag{4.5}$$

[18] The space of solutions depends on regularity and fall-off conditions. For example a magnetic monopole $q_B \neq 0$ in Maxwell theory is excluded, if the existence of a regular vector potential is assumed. The field strength remains regular, even with magnetic charge, see (4.2).

Table 4.1.: Poincaré isometries and corresponding charges

label	isometry	charge	#
$\mathcal{T}_t$	time translation	mass	1
$\mathcal{T}_i$	spatial translations	linear momenta	3
$\mathcal{L}_{ij}$	rotations	angular momenta	3
$\mathcal{L}_{ti}$	boosts	center of mass	3

Here $U^{AB}, W_{A'B'}$ are constant, symmetric spinors and $V_{A'}^{B}$ is a constant complex vector, so the space of real solution is of dimension $2 \cdot 6 + 8 = 20$. Each solution, contracted into a spin-2 field, e.g. the linearized Weyl tensor on Minkowski spacetime, and integrated over a 2-sphere results in a quasi-local charge.

We follow [91, p. 99], where 10 of these charges are related to a source for linearized gravity in the following sense. Given a divergence free, symmetric energy momentum tensor T_{ab}, one has for each Killing field ξ^b the divergence free current $j_a = T_{ab}\xi^b$. Using linearized Einstein equations,

$$\dot{G}_{ab} = \dot{R}_{acb}{}^{c} - \frac{1}{2} g_{ab} \dot{R}_{cd}{}^{cd} = -8\pi \dot{T}_{ab} \,, \tag{4.6}$$

and the conformal Killing-Yano equation (2.43), it is shown that

$$3 \int_{\partial\Sigma} \dot{R}_{abcd} \, {*Y}^{cd} \mathrm{d}x^a \wedge \mathrm{d}x^b = 16\pi \int_{\Sigma} e_{abc}{}^{d} \dot{T}_{df} \xi^f \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \,. \tag{4.7}$$

Here Σ denotes a 3 dimensional hyper-surface with boundary $\partial\Sigma$ and e_{abcd} is the Levi-Civita tensor. Outside the support of $\dot{T}_{ab}$, the left hand side is the charge integral described above, while the right hand side gives the more familiar form of a conserved 3-form corresponding to a linearized source and a Killing vector $\xi^a = \frac{1}{3} Y^{ab}{}_{;b}$. Note that it is the dual conformal Killing-Yano tensor on the left hand side, which gives the charge associated to the isometry ξ^a. In Cartesian coordinates $x^a = (t, x, y, z)$ the Poincaré isometries read

$$\mathcal{T}_a = \frac{\partial}{\partial x^a} \,, \qquad \mathcal{L}_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} \,, \tag{4.8}$$

and the relation to the charges is listed in table 4.1. The angular momentum around the z-axis is found in the component $\mathcal{L}_{xy} = \partial_\phi$. Explicit expressions for linearized sources generating these charges can be found in [73, eq. (27)].

The ten remaining charges cannot be generated this way, since the corresponding conformal Killing-Yano tensors have vanishing divergence (they are Killing-Yano tensors). Their occurrence is much more subtle, see [73] and the discussion below. One of these charges corresponds to the NUT parameter[19], and the remaining nine are three dual linear momenta and six *ofam*[20] charges. In the expression (4.5) for a general Killing spinor, they correspond to U^{AB} and the imaginary part of $V_{A'}^{B}$.

[19] The parameter is sometimes called dual mass, because a duality rotation from Schwarzschild to NUT spacetime exists, see the appendix in [96].

[20] This name was introduced in [75] and means *obstructions for angular momentum*. Non-vanishing ofam charges imply very slow fall-off of the spin-2 field.

Table 4.2.: Solutions to the Killing spinor equation on Minkowski spacetime in spherical coordinates. Real and imaginary parts of their divergences (2.50) are given in the last two columns.

label	components $\kappa_0/\sqrt{2}$	κ_1	$\kappa_2/\sqrt{2}$	combination	divergence Re	Im
				Ω^0_{-1}	0	0
Ω^0_m	${}_1Y_{1m}$	${}_0Y_{1m}$	${}_{-1}Y_{1m}$	Ω^0_0	0	0
				Ω^0_1	0	0
				Ω^1	$\mathcal{T}_t$	0
Ω^1	0	$r\,{}_0Y_{00}$	0	$\Omega^1_1 - \Omega^1_{-1}$	$\mathcal{T}_x$	0
Ω^1_m	$(t-r)\,{}_1Y_{1m}$	$t\,{}_0Y_{1m}$	$(t+r)\,{}_{-1}Y_{1m}$	$\mathrm{i}\Omega^1_1 + \mathrm{i}\Omega^1_{-1}$	$\mathcal{T}_y$	0
				Ω^1_0	$\mathcal{T}_z$	0
				$\Omega^2_1 - \Omega^2_{-1}$	$\mathcal{L}_{tx}$	$\mathcal{L}_{yz}$
Ω^2_m	$(t-r)^2\,{}_1Y_{1m}$	$(t^2-r^2)\,{}_0Y_{1m}$	$(t+r)^2\,{}_{-1}Y_{1m}$	$\mathrm{i}\Omega^2_1 + \mathrm{i}\Omega^2_{-1}$	$\mathcal{L}_{ty}$	$\mathcal{L}_{xz}$
				Ω^2_0	$\mathcal{L}_{tz}$	$\mathcal{L}_{xy}$

To understand the charges as projections into $l = 0$ and $l = 1$ modes, we rederive the complete set of solutions to the Killing spinor equation (2.47) in spherical coordinates, using spin-weighted spherical harmonics, see section A.1 and [90, Section 4.15]. A null tetrad for Minkowski spacetime in spherical coordinates (t, r, θ, ϕ) (the $M, a \to 0$ limit of the symmetric Carter tetrad (2.107)) is given by

$$l^a = \frac{1}{\sqrt{2}}\Big[1, 1, 0, 0\Big], \qquad n^a = \frac{1}{\sqrt{2}}\Big[1, -1, 0, 0\Big], \qquad m^a = \frac{1}{\sqrt{2}r}\Big[0, 0, 1, \frac{\mathrm{i}}{\sin\theta}\Big],$$

and the non-vanishing spin coefficients are

$$\rho = -\frac{1}{\sqrt{2}r} = -\rho', \qquad \beta = \frac{\cot\theta}{2\sqrt{2}r} = \beta'.$$

For a Killing spinor $\kappa_{AB} = \kappa_0 \iota_A \iota_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_2 o_A o_B$, the subset (2.53a) of the Killing spinor equation becomes

$$(\partial_t + \partial_r)\,\kappa_0 = 0, \qquad \left(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\varphi - \cot\theta\right)\kappa_0 = 0,$$

$$(\partial_t - \partial_r)\,\kappa_2 = 0, \qquad \left(\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\varphi - \cot\theta\right)\kappa_2 = 0,$$

so $\kappa_0 = f_0(t-r)\,{}_1Y_{1m}$ and $\kappa_2 = f_1(t+r)\,{}_{-1}Y_{1m}$, where the functions f_i depend on advanced and retarded coordinates only and ${}_sY_{lm}$ are the spin-weighted spherical harmonics, see (A.10) for the explicit form of the first few harmonics. Finally (2.53b) can be solved for κ_1, which is only possible for particular functions f_i. The result is given in table 4.2.

In that table, Ω^1 represents one complex solution of the Killing spinor equation, while $\Omega^i_m, i = 0, 1, 2$ represent 3 complex solutions each ($m = 0, \pm 1$), so they sum up to 20 real solutions. Furthermore, Ω^1 is the only $l = 0$ solution and the corresponding charges are mass and dual mass. The remaining solutions Ω^i_m are all $l = 1$ and a rough power counting shows that a spin-2 field has to fall-off $\sim r^{-2-i}$ to have a non-vanishing charge

Table 4.3.: Solutions to the Killing spinor equation on Minkowski spacetime in spheroidal coordinates.

components		
$\kappa_0/\sqrt{2}$	κ_1	$\kappa_2/\sqrt{2}$
$\frac{\sqrt{\Delta}}{\zeta}\ {}_1Y_{1m}$	$\frac{r\cos\theta-\mathrm{i}a}{\zeta}$	$\frac{\sqrt{\Delta}}{\zeta}\ {}_{-1}Y_{1m}$
0	ψ	0
$\frac{\sqrt{\Delta}}{\zeta}u_-\ {}_1Y_{1m}$	$\frac{r\cos\theta-\mathrm{i}a}{\zeta}t$	$\frac{\sqrt{\Delta}}{\zeta}u_+\ {}_{-1}Y_{1m}$
$\frac{\sqrt{\Delta}}{\zeta}u_-^2\ {}_1Y_{1m}$	$\frac{r\cos\theta-\mathrm{i}a}{\zeta}u_+u_-$	$\frac{\sqrt{\Delta}}{\zeta}u_+^2\ {}_{-1}Y_{1m}$

from Ω^i_m. For this reason, the charges of Ω^0_m are called *ofam*. We remark the following correspondence to the solutions (4.5) in Cartesian coordinates

$$\Omega^0_m \leftrightarrow U^{AB}, \qquad \Omega^1, \Omega^1_m \leftrightarrow V^A_{A'}, \qquad \Omega^2_m \leftrightarrow W_{A'B'}.$$

In the limit $M \to 0, a \neq 0$, the Kerr metric becomes the flat Minkowski metric in spheroidal coordinates. In these coordinates, the full set of solutions to the Killing spinor equation is given in table 4.3. Here we use the abbreviation $\Delta = r^2 + a^2$ and the null coordinates

$$u_+ = t + r - \mathrm{i}a\cos\theta\,, \qquad u_- = t - r + \mathrm{i}a\cos\theta\,. \tag{4.9}$$

Remark 4.2.1 (Twistors on Minkowski spacetime)**.** *The twistor equation* (2.41) *can be solved explicitly along the same lines. Using the component form* (2.54)*, we find the complete set of solutions,*

$$\lambda_{Am} = {}_{-\frac{1}{2}}Y_{\frac{1}{2}m}o_A + {}_{\frac{1}{2}}Y_{\frac{1}{2}m}\iota_A\,, \qquad \chi_{Am} = (t+r)\,{}_{-\frac{1}{2}}Y_{\frac{1}{2}m}o_A + (t-r)\,{}_{\frac{1}{2}}Y_{\frac{1}{2}m}\iota_A\,, \tag{4.10}$$

with ${}_sY_{lm}$ spin-weighted spherical harmonics, given for $l = \frac{1}{2}$ in (A.9)*. Since $m = \pm\frac{1}{2}$ in this case, the space is complex 4-dimensional. All solutions of the Killing spinor equation, listed in table 4.2, can be written as symmetric products of these twistors. It should be noted, that also all conformal Killing vectors can be written as products of twistors and their complex conjugated versions. On a background of Petrov type D a solution to the twistor equation cannot exist, because of the integrability condition* (2.46)*.*

We also remark that, locally, a metric can be reconstructed from the linearized curvature via,

$$h_{ab} = 2x^c x^d \int_0^1 \dot{R}_{acbd}(\lambda x)\lambda(1-\lambda)\mathrm{d}\lambda + 2\xi_{(a,b)}\,, \tag{4.11}$$

with x^a Cartesian coordinates. This was derived originally in [99][21]. Globally, this is not true anymore for non-singular metrics. This follows from (4.7), which holds under the assumption of an existing, non-singular metric. In that case ten charges vanish identically because the divergence ξ^a vanishes, see also [90, p. 364],[91, Section 6.5] for further discussion. For that reason, we introduce the name *singular charges* and discuss the

[21] A metric reconstruction on a curved background is much more involved, see [122] for details.

effects on the angular momentum charges in section 4.3.3. It would be nice to characterize such a vanishing condition for the singular charges analogous to magnetic monopoles in the Maxwell theory by showing that the spin-lowered curvature is exact. We tried the following ansatz. Suppose κ_{AB} is one of the Killing spinors with vanishing divergence

$$\nabla_{A'}{}^{A}\kappa_{AB} = 0, \tag{4.12}$$

which means that it is one-to-one to two Killing-Yano tensors and that these 2-forms are closed (it is in fact simply the spin-1 equation). It leads to vanishing charges, if a metric is present. The analogy to electromagnetism is plainest, if we switch to bivector formalism. Spin lowering the linearized curvature $\dot{\Sigma}_I Z_J$ then corresponds to the closed 2-form

$$F = \kappa^I \dot{\Sigma}_I .$$

Here κ^I are the components of the Killing spinor in a bivector triad (2.8) and $\dot{\Sigma}_I$ are the linearized bivector curvature 2-forms. Using the second equations of structure (2.23) (this means, assuming that a connection as potential exists) in linearized form, we find

$$\begin{aligned} F &= \kappa^I \left(\mathrm{d}\dot{\sigma}_I + \epsilon_{IJK}\dot{\sigma}^J \wedge \sigma^K \right) \\ &= \mathrm{d}\left(\kappa^I \dot{\sigma}_I\right) - \left(\mathrm{d}\kappa_I + \kappa^J \epsilon_{JIK}\sigma^K\right) \wedge \dot{\sigma}^I . \end{aligned} \tag{4.13}$$

The first term integrates to zero (assuming, that $\kappa^I \dot{\sigma}_I$ is non-singular) and the second term looks similar to the component form of (4.12),

$$\begin{aligned} 0 &= \mathrm{d}(\kappa_I Z^I) \\ &= \mathrm{d}\kappa_I \wedge Z^I - 2\kappa_I \epsilon^{IJK}\sigma_J \wedge Z_K \\ &= \left(\mathrm{d}\kappa_I + \kappa^J \epsilon_{JIK}\sigma^K\right) \wedge Z^I . \end{aligned} \tag{4.14}$$

However, we cannot conclude the disappearance of the second term in (4.13), because of the summation over I. It remains to be seen, whether this can be made into an argument. It might be possible to use (2.25), which introduces a "second order potential" analogous to a metric.

4.2.2. Conserved charges for type D spacetimes

Next, we consider a curved vacuum background and the curvature itself as the spin-2 field (because of the Bianchi identity (3.22)). For spin lowering (3.27) to apply, a valence-2 Killing spinor is needed and this restricts the spacetime to be at least of Petrov type D (type N or O is also possible), see (2.48). Using the type D Killing spinor (2.66), we consider the complex 2-form

$$\mathcal{M}_{ab} = -\tfrac{1}{2}\Psi_{ABCD}\kappa^{CD}\bar{\epsilon}_{A'B'} , \tag{4.15}$$

satisfying the Maxwell equations $d\mathcal{M} = 0$. It may be represented as

$$\mathcal{M}_{ab} = \zeta \Psi_2 Z^1_{ab} , \tag{4.16}$$

if a principal tetrad (2.56) is chosen. For an explicit calculation of the charge, we restrict to the Kerr solution. The coordinate form of (4.16) (the ingredients to are given in section 2.5) yields,

$$\frac{1}{4\pi \mathrm{i}}\int_S \mathcal{M} = -\frac{1}{4\pi \mathrm{i}}\int_S \frac{M}{(r-\mathrm{i}a\cos\theta)^2}(-\mathrm{i})(r^2+a^2)\sin\theta \mathrm{d}\theta \mathrm{d}\varphi = M\,. \tag{4.17}$$

Here M is the ADM mass [12] of the Kerr spacetime[22], see also [76] for a purely tensorial version of the argument. The relation between the mass and charge for the spin-lowered curvature $\mathcal{M}$ is natural in view of the fact that the divergence (2.68) is given by $\xi^a = (\partial_t)^a$ and the discussion of the preceding section.

We note that the charge (4.17) is in general complex. The imaginary part corresponds to the NUT parameter, which is the gravitational analog of a magnetic charge. Details are not discussed in this thesis, see [96] for the construction of charge integrals in NUT spacetime.

An expression very similar to (4.16) has been derived already in 1961 by Jordan, Ehlers and Sachs [77]. By investigating the Bianchi identities in bivector form, they found

$$\mathrm{d}\left(\Psi_2^{2/3} Z^1\right) = 0\,. \tag{4.18}$$

Because of (2.57), this is in fact equivalent to the exterior derivative of (4.16). However, integrating (4.18) results in $CM^{2/3}$, with some constant C and only the Killing spinor construction explains the discrepancy in the exponent of the mass parameter. We will repeat the derivation of (4.18) from the Bianchi identities here, because this formulation can be generalized most easily to linearized gravity as we will see in section 4.3.2. On a type D background, the curvature and connection forms simplify to

$$\Sigma_0 = \Psi_2 Z^2\,, \qquad \Sigma_1 = \Psi_2 Z^1\,, \qquad \Sigma_2 = \Psi_2 Z^0\,, \qquad \Gamma = \tau l - \rho m\,, \tag{4.19}$$

so the middle Bianchi identity (2.31b) becomes

$$\begin{aligned} 2\mathrm{d}\Sigma_1 &= 2\Psi_2\left[(\rho'\bar{m} - \tau' n)\wedge l \wedge m + (\rho m - \tau l)\wedge \bar{m}\wedge n\right] \\ &= 2\Psi_2(\rho' l + \rho n - \tau' m - \tau \bar{m})\wedge Z^1 \\ &= h \wedge \Sigma_1\,, \end{aligned}$$

where $h = 2(\rho' l + \rho n - \tau' m - \tau\bar{m})$ is used. Because of (2.60), one obtains

$$\mathrm{d}(\Psi_2 Z^1) = \mathrm{d}\Sigma_1 = \frac{1}{2} h \wedge \Sigma_1 = \frac{1}{3}\mathrm{d}\Psi_2 \wedge Z^1\,,$$

which yields the *Jordan-Ehlers-Sachs conservation law* (4.18). An interesting generalization of this result to vacuum spacetimes of Petrov type II is given in [71].

Now, one might ask whether a quasi-local angular momentum charge can be constructed by spin lowering the curvature. This of course cannot work out directly, because a corresponding Killing spinor does not exist, see (2.63). However, we believe that the approach needs to be modified to also cover this situation. This belief stems on the one hand from the well known quasi-local angular momentum charge due to Komar, discussed

[22] Equivalently, the mass parameter in the Boyer-Lindquist form of the Kerr line element.

in the next section, and on the other hand from the structure of the non-radiating modes for a Minkowski background, as further discussed in section 4.3.3.

Let us first try to follow the derivation of (4.18) also for the other curvature components Σ_0 and Σ_2 to derive further closed 2-forms. We have from (2.62) together with (2.60) the weighted "conservation laws"

$$\mathrm{d}^\Theta(\Psi_2^{1/3} Z^0) = 0\,, \qquad \mathrm{d}^\Theta(\Psi_2^{1/3} Z^2) = 0\,. \tag{4.20}$$

To make the first equation (the second one is its GHP prime (2.15a)) into a proper closed 2-form, a multiplication by a function f of opposite weight,

$$\mathrm{d}(f\Psi_2^{1/3} Z^0) = \Psi_2^{1/3} Z^0 \wedge \mathrm{d}^\Theta f + f\mathrm{d}^\Theta(\Psi_2^{1/3} Z^0) \stackrel{!}{=} 0\,, \tag{4.21}$$

is needed. For the $(2,0)$ weighted scalar f, one has to solve $Z^0 \wedge \mathrm{d}^\Theta f = 0$, which is equivalent to $\text{þ}' f = 0 = \eth' f$. An expansion in the Carter tetrad (2.107) yields the (t,φ)-independent solution

$$f = A \frac{\zeta}{\sqrt{\Delta}\sin\theta}\,, \tag{4.22}$$

for some constant A. The corresponding charge integral for $A = 2$ results in,

$$\frac{1}{2\pi}\int_{S^2} f\zeta^2 \Psi_2 Z^0 = Ma\,. \tag{4.23}$$

However, the solution (4.22) is singular at the poles and the horizon. Also the choice of just one component for an angular momentum integral looks unnatural. So we allow all three components to be non-trivial, which results in the requirement that the spin lowered curvature, (3.27),

$$\phi_{AB} = \Psi_2 \kappa_0 \iota_A \iota_B + 4\Psi_2 \kappa_1 o_{(A}\iota_{B)} + \Psi_2 \kappa_2 o_A o_B\,,$$

is a solution to the vacuum Maxwell equations[23]. The integrand in (4.23) corresponds to the special case $\kappa_0 = f\zeta^2\,, \kappa_1 = 0 = \kappa_2$. We know already other (stationary, axi-symmetric) solutions, namely Komar forms. These will be discussed in detail in the next section.

Before we proceed, we note two further interesting references. Another generalization of the conformal Killing-Yano equation is discussed in [16]. A further interesting approach due to Bergqvist and Ludvigsen [17] introduces a modified connection for the Kerr solution, which ensures solutions to the twistor equation to exist. The corresponding Nester-Witten form yields an angular momentum charge.

4.2.3. Komar integrals for Kerr spacetime

On a Kerr background in Boyer-Lindquist coordinates (2.92) with Killing vectors $\xi = \partial_t$ and $\Xi = \partial_\varphi$, the mass M and angular momentum Ma can be calculated via the Komar integrals

$$M = -\frac{1}{8\pi}\int_S *\mathrm{d}\xi\,, \qquad Ma = \frac{1}{16\pi}\int_S *\mathrm{d}\Xi\,, \tag{4.24}$$

[23] This is equivalent to the condition $\Psi_{ABCD}\nabla^{A'A}\kappa^{BC} = 0$. Since $\Psi_{ABCD} = 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)}$, the components $o_A o_B o_C \nabla^{A'A}\kappa^{BC}$ and $\iota_A\iota_B\iota_C\nabla^{A'A}\kappa^{BC}$ are not assumed to vanish anymore. In components, this equation reduces to the subset (2.53b) of the Killing spinor equation, so it weakens the conditions to allow for more solutions.

with S a 2-sphere as in the above discussion. The integrands itself are solutions to the vacuum Maxwell equations. In [114] they were interpreted as stationary, axisymmetric test electromagnetic fields without magnetic charge. The first one, generated by ξ, vanishes asymptotically, while the second one, generated by Ξ, asymptotically approaches a uniform magnetic field.

Note that we now have two seemingly different ways to calculate the mass quasi-locally, namely (4.24) in terms of connection coefficients and (4.17) in terms of curvature. However, the following calculation shows that not only the integrals coincide but also the integrands itself,

$$-2{}_{\xi}h_{ab} = \mathcal{M}_{ab}\,, \tag{4.25}$$

with ${}_{\xi}h_{ab}$ defined in (4.27). Let us first have a look at the form of the integrands in (4.24). We start with an arbitrary real 1-form ω. The exterior derivative can be expanded into

$$\tfrac{1}{2}(\mathrm{d}\omega)_{ab} = \nabla_{[AA'}\omega_{BB']} = {}_{\omega}h_{AB}\bar{\epsilon}_{A'B'} + {}_{\omega}\bar{h}_{A'B'}\epsilon_{AB}\,, \tag{4.26}$$

with a symmetric 2-spinor ${}_{\omega}h_{AB}$. It is convenient to work with the complex 2-form

$${}_{\omega}h_{ab} = {}_{\omega}h_{AB}\bar{\epsilon}_{A'B'} = \tfrac{1}{4}\left(\mathrm{d}\omega + \mathrm{i} * \mathrm{d}\omega\right)_{ab}\,, \tag{4.27}$$

instead. We note that its imaginary part is of the same form as the integrand in the Komar integral (4.24) and that its real part integrates to zero. Therefore one can use (4.27) as a complex Komar form. Contracting (4.26) with $\frac{1}{2}\bar{\epsilon}^{A'B'}$ yields

$${}_{\omega}h_{AB} = \tfrac{1}{4}(\nabla_{AA'}\omega_{B}{}^{A'} - \nabla_{B}{}^{A'}\omega_{AA'}) = \tfrac{1}{2}\nabla_{A'(A}\omega_{B)}{}^{A'}\,. \tag{4.28}$$

This proves the assertion made after (2.81). The first Killing vector ξ^a is expressed as a divergence of the type D Killing spinor in (2.68). We find with (2.49) and the footnote on p. 14 that

$$\begin{aligned}
{}_{\xi}h_{AB} &= -\tfrac{1}{12}\big[\nabla_{AA'}\nabla^{CA'}\kappa_{BC} + \nabla_{BA'}\nabla^{CA'}\kappa_{AC}\big]\\
&= -\tfrac{1}{12}\big[-\tfrac{1}{2}\epsilon_{A}{}^{C}\,\Box\,\kappa_{BC} + \Box_{A}{}^{C}\kappa_{BC} - \tfrac{1}{2}\epsilon_{B}{}^{C}\,\Box\,\kappa_{AC} + \Box_{B}{}^{C}\kappa_{AC}\big]\\
&= -\tfrac{1}{12}\big[-\Psi_{ABCD}\kappa^{CD} - \Psi_{A}{}^{C}{}_{B}{}^{D}\kappa_{DC} - \Psi_{B}{}^{C}{}_{A}{}^{D}\kappa_{DC}\big]\\
&= \tfrac{1}{4}\Psi_{ABCD}\kappa^{CD}\,.
\end{aligned} \tag{4.29}$$

The result (4.25) follows now from (4.15) and holds for the whole Kerr-NUT class. Can we do a similar analysis for the second isometry? This is not obvious, because a Killing spinor potential, as needed in the above calculation, does not exist for Ξ. However, using the "almost potential" form (2.79) for η we proof in section A.3 the following result,

$${}_{\eta}h_0 = -\zeta^2\bar{\zeta}\tau\rho\,, \tag{4.30a}$$
$${}_{\eta}h_1 = \tfrac{1}{8}\zeta(\zeta^2 + \bar{\zeta}^2)\Psi_2 - \tfrac{1}{4}\zeta\bar{\zeta}^2\bar{\Psi}_2 + \tfrac{1}{2}\rho\rho'\zeta^2(\bar{\zeta} - \zeta) + \tfrac{1}{2}\tau\tau'\zeta^2(\bar{\zeta} + \zeta)\,, \tag{4.30b}$$
$${}_{\eta}h_2 = -\zeta^2\bar{\zeta}\tau'\rho'\,. \tag{4.30c}$$

On a Kerr background, $\eta = a^2\xi + a\Xi$ as discussed in (2.116). We now have a representation of the Komar integral for Ξ, which is partly in terms of curvature and partly in terms of products of connection coefficients. An expansion in coordinates shows that these parts contribute equally to the integral. Since we do not have a geometric characterization of this splitting, we cannot express this Komar integral purely in terms of curvature.

Let us finally give an explicit form of the integrand in a Carter tetrad

$$ {}_{\xi}h_1 = \frac{M}{2\zeta^2}, \quad {}_{\Xi}h_0 = -\frac{\mathrm{i}\sin\theta\sqrt{\Delta}}{2\zeta} = -{}_{\Xi}h_2, \quad {}_{\Xi}h_1 = -\frac{Ma\sin^2\theta}{2\zeta^2} - \frac{a+\mathrm{i}r\cos\theta}{2\zeta}. \tag{4.31} $$

This is most simply extracted from (2.112) together with (2.91). Here, the coefficient $\zeta = r - \mathrm{i}a\cos\theta$ is used. The formal similarity with the solution of the Killing spinor equation on Minkowski spacetime in spheriodal coordinates, given in table 4.3, is remarkable.

4.3. Linearized gravity

So far we constructed charges for spin-2 fields by Penrose's concept of spin lowering. This included linearized gravity on a flat background. However, the theory differs from the spin-2 theory on a curved background, as discussed in chapter 3, and it is not clear whether the concept can be generalized to this situation.

On the other hand, quasi-local charge integrals for linearized gravity are known e.g. from the Hamiltonian approach where a charge for each isometry can be calculated, see section 4.3.4. A translation into linearized curvature is intricate and we only succeeded for an angular momentum charge on a Schwarzschild background. So it would be interesting to derive quasi-local charges directly from the linearized Bianchi identities.

We will start by linearizing the Kerr solution in its parameters M, a around itself to construct prototypes for charge integrals, based on linearized curvature, in the next section. In section 4.3.2 we show that Penrose's concept generalizes and present a suitable formulation for linearized mass on type D backgrounds in terms of curvature. For the angular momentum charge we explain the obstructions to derive a curvature conservation law and possible ways to proceed in section 4.3.3. In section 4.3.4, charges from a Hamiltonian approach are discussed. Finally the lowest modes for the wave equation with curvature potential, as they occur in Maxwell and linearized gravity theory, are investigated.

4.3.1. Non-radiating modes on a Kerr background

We have explicit representations of the gauge-invariant non-radiating modes by linearizing the Kerr solution around itself. This does not directly lead to gauge-invariant charge integrals, but it is a helpful test case and gives some hints to their construction.

Let us take a Kerr solution with mass M and angular momentum MA and linearize it (tetrad, spin coefficients and Weyl scalars) around a solution with mass m and angular momentum ma. With perturbation parameter ϵ, we write

$$ M = m + \epsilon\dot{m}, \qquad A = a + \epsilon\dot{a}. \tag{4.32} $$

Everything can be calculated explicitly and a choice of gauge is inherent.

We start with the linearized mass charge. With $M = m+\epsilon\dot{m}+O(\epsilon^2)$, $A = a$, we calculate the deviation from (m,a)-Carter tetrad to first order in ϵ,

$$\frac{d}{d\epsilon}l^a(M,a)\Big|_{\epsilon=0} = \dot{l}^b = \frac{\dot{m}r}{\sqrt{2\Sigma\Delta}}\left[\frac{(r^2+a^2)}{\Delta}, -1, 0, \frac{a}{\Delta}\right], \tag{4.33a}$$

$$\dot{n}^b = \frac{\dot{m}r}{\sqrt{2\Sigma\Delta}}\left[\frac{(r^2+a^2)}{\Delta}, 1, 0, \frac{a}{\Delta}\right], \tag{4.33b}$$

$$\dot{m}^b = 0\,. \tag{4.33c}$$

Note that the spherical $m, \bar{m}$ part is not affected to first order in this canonical gauge. To calculate the linearized spin coefficients, we can either use the perturbed tetrad and linearize again, or linearize the already known (M,A)-spin coefficients by hand. The result is

$$\dot{\epsilon} = -\dot{\epsilon}' = \frac{\dot{m}}{2\sqrt{2\Delta\Sigma}}\left(\frac{r}{p} - 1 + \frac{(r-m)r}{\Delta}\right), \qquad \dot{\beta} = \dot{\beta}' = 0, \tag{4.34a}$$

$$\dot{\rho} = -\dot{\rho}' = \frac{\dot{m}r}{\sqrt{2\Sigma\Delta}p}, \qquad \dot{\tau} = \dot{\tau}' = 0\,. \tag{4.34b}$$

The linearized Weyl scalars are given by

$$\dot{\Psi}_2 = -\frac{\dot{m}}{(r - ia\cos\theta)^3} = \Psi_2\frac{\dot{m}}{m}, \qquad \dot{\Psi}_{i\neq 2} = 0\,. \tag{4.35}$$

So with the type D Killing spinor coefficient $\zeta = r - \mathrm{i}a\cos\theta$ we find

$$-\frac{1}{4\pi}\int_{S^2}(r^2+a^2)\zeta\dot{\Psi}_2\,\mathrm{d}\mu = \dot{m}\,, \tag{4.36}$$

where $d\mu = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. The integrand can be interpreted as the linearization of the Jordan-Ehlers-Sachs conservation law (4.18), because the middle bivector (2.111) has component $Z^1_{\theta\varphi} \sim (r^2+a^2)\sin\theta$. However, this combination does in general not fulfill the requirement of gauge-invariance. We will derive the correct generalization in section 4.3.2.

Next, we look at the linearized angular momentum mode. With $A = a + \epsilon\dot{a} + O(\epsilon^2)$, $M = m$, we calculate the deviation from (m,a)-Carter tetrad to first order in ϵ:

$$\frac{d}{d\epsilon}l^a(m,A)\Big|_{\epsilon=0} = \dot{l}^b = -\frac{a\dot{a}\cos\theta}{\Sigma}l^b + \frac{a\dot{a}}{\sqrt{2\Sigma\Delta}}\left[2 - \frac{(r^2+a^2)}{\Delta}, 1, 0, \frac{1}{a} - \frac{1}{\Delta}\right], \tag{4.37a}$$

$$\dot{n}^b = -\frac{a\dot{a}\cos\theta}{\Sigma}n^b + \frac{a\dot{a}}{\sqrt{2\Sigma\Delta}}\left[2 - \frac{(r^2+a^2)}{\Delta}, -1, 0, \frac{1}{a} - \frac{1}{\Delta}\right], \tag{4.37b}$$

$$\dot{m}^b = -\frac{a\dot{a}\cos\theta}{\Sigma}m^b + \frac{1}{\sqrt{2\Sigma}}\left[\mathrm{i}\dot{a}\sin\theta, 0, 0, 0\right]. \tag{4.37c}$$

The linearized tetrad is more involved in this case and the spin coefficients look quite complicated. We have for example

$$\dot{\rho} = -\dot{\rho}' = \frac{\dot{a}}{\sqrt{2\Sigma}p}\left(-\frac{a}{\sqrt{\Delta}} + \frac{a\sqrt{\Delta}\cos\theta}{\Sigma} - \frac{\mathrm{i}\sqrt{\Delta}\cos\theta}{p}\right). \tag{4.38}$$

The linearized Weyl scalars still have a simple structure,

$$\dot{\Psi}_2 = -\frac{3mi\dot{a}\cos\theta}{(r - ia\cos\theta)^4} = \Psi_2 \frac{3\mathrm{i}\dot{a}\cos\theta}{r - \mathrm{i}a\cos\theta}, \qquad \dot{\Psi}_{i\neq 2} = 0\,. \tag{4.39}$$

Here we find for a Schwarzschild background

$$-\frac{1}{4\pi}\int_{S^2} r^2 \cos\theta\, \zeta^2 \dot{\Psi}_2 \,\mathrm{d}\mu = \mathrm{i}m\dot{a}\,. \tag{4.40}$$

The $\cos\theta$ is interpreted as $l = 1$ spherical harmonic ${}_0Y_{10}$ and we indeed find this conservation law for the imaginray part of $\dot{\Psi}_2$ from a Hamiltonian analysis in section 4.3.4. The problems in deriving the non-radiating mode directly from the Bianchi identities are discussed in section 4.3.3.

4.3.2. Mass

One can of course linearize the 2-form (4.15), which would provide a charge for perturbations within the class of type D spacetimes. But more generally a closed 2-form for arbitrary linear perturbations around a type D background[24] can be derived.

In this section we shall show that the natural linearization of the spin-lowered Weyl tensor $\mathcal{M}$ is the 2-form

$$\dot{\mathcal{M}} = \zeta\dot{\Psi}_1 Z^0 + \zeta\dot{\Psi}_2 Z^1 + \zeta\dot{\Psi}_3 Z^2 + \tfrac{3}{2}\zeta\Psi_2 \dot{Z}^1.$$

As will be demonstrated below, $\dot{\mathcal{M}}$ is closed and hence the integral

$$\int_S \dot{\mathcal{M}} \tag{4.41}$$

defines a conserved charge. A charge vanishing condition for the linearized mass, analogous to the one discussed above for the charges of the Maxwell field, may be introduced by requiring that this integral vanishes. The coordinate form of this charge vanishing condition is

$$\int_{S^2(t,r)} \big(2(r^2 + a^2)\dot{\widehat{\Psi}}_2 + \mathrm{i}a\sqrt{\Delta}\sin\theta \dot{\Psi}_{diff}\big)(r - \mathrm{i}a\cos\theta)\mathrm{d}\mu = 0\,, \tag{4.42}$$

which should be compared to the corresponding condition for the Maxwell case, cf. (4.4). Here, $\dot{\widehat{\Psi}}_2$ and $\dot{\Psi}_{diff}$ are suitable combinations of the linearized curvature scalars $\dot{\Psi}_1, \dot{\Psi}_2, \dot{\Psi}_3$ and linearized tetrad, see (4.52).

The closed 2-form, with $\dot{\mathcal{M}}$ in the form (4.45), was first calculated by Fackerell. Together with Crossmann he used it to derive field equations for perturbations of Kerr-Newman spacetime, see [49]. However, the corresponding charge was not investigated. In this section we give a short and simple proof of the identity (4.43), from which Fackerell's conservation law for the vacuum case can be deduced. We also calculate the explicit gauge transformation behavior of $\dot{\mathcal{M}}$, from which gauge-invariance of linearized mass follows. The interpretation of (4.46) as the linearized ADM mass $\dot{M}$, and also its relation to Penrose's idea of spin lowering are given in [3] for the first time.

[24] One can expect that such a structure for perturbations of algebraically special solutions exists also for other signatures and a result in that direction is given in remark 4.3.3 on p. 56. A classification of the Weyl tensor for arbitrary signature is given in [14].

Lemma 4.3.1. *A series expansion of the middle Bianchi identity* (2.31b) *around a vacuum spacetime of Petrov type D yields*

$$(\mathrm{d} - \tfrac{1}{2}h\wedge)\Sigma_1 = O(\epsilon^2)\,, \tag{4.43}$$

with $h = 2(\rho' l + \rho n - \tau' m - \tau\bar{m})$.

Proof. We expand the the right hand side of the middle Bianchi identity (2.31b),

$$\mathrm{d}\Sigma_1 = -\Gamma' \wedge \Sigma_0 - \Gamma \wedge \Sigma_2\,, \tag{4.44}$$

in some parameter ϵ, using the explicit form of the curvature forms (2.30),

$$\begin{aligned}
\Sigma_0 &= \Psi_0 Z^0 + \Psi_1 Z^1 + \Psi_2 Z^2\,,\\
\Sigma_1 &= \Psi_1 Z^0 + \Psi_2 Z^1 + \Psi_3 Z^2\,,\\
\Sigma_2 &= \Psi_2 Z^0 + \Psi_3 Z^1 + \Psi_4 Z^2\,,
\end{aligned}$$

and the 2-forms (2.8), which in exterior notation read

$$Z^0 = \bar{m}\wedge n\,, \qquad Z^1 = n\wedge l - \bar{m}\wedge m\,, \qquad Z^2 = l\wedge m\,.$$

Due to the type-D conditions (2.56) the first term becomes

$$\begin{aligned}
\Gamma'\wedge\Sigma_0 &= (\tau' n + \kappa l - \rho'\bar{m} - \sigma m)\wedge(\Psi_0 Z^0 + \Psi_1 Z^1 + \Psi_2 Z^2)\\
&= \Psi_0(\kappa l\wedge\bar{m}\wedge n - \sigma m\wedge\bar{m}\wedge n)\\
&\quad - \Psi_1(\rho'\bar{m}\wedge n\wedge l + \sigma m\wedge n\wedge l + \tau' n\wedge\bar{m}\wedge m + \kappa l\wedge\bar{m}\wedge m)\\
&\quad + \Psi_2(\tau' n\wedge l\wedge m - \rho'\bar{m}\wedge l\wedge m)\\
&= -\Psi_1(\rho'\bar{m}\wedge n\wedge l + \tau' n\wedge\bar{m}\wedge m) + \Psi_2(\tau' n\wedge l\wedge m - \rho'\bar{m}\wedge l\wedge m) + O(\epsilon^2)\\
&= \Psi_1(-\rho' l + \tau' m)\wedge Z^0 + \Psi_2(\tau' m - \rho' l)\wedge Z^1 + O(\epsilon^2)\\
&= (\tau' m - \rho' l)\wedge\Sigma_1 + O(\epsilon^2).
\end{aligned}$$

The last equality holds, because $\Psi_3(\tau' m - \rho' l)\wedge Z^2 = 0$. A calculation along the same lines (or using the GHP prime operation (2.15a)) yields

$$\Gamma\wedge\Sigma_2 = (-\tau\bar{m} + \rho n)\wedge\Sigma_1 + O(\epsilon^2)\,,$$

and therefore

$$\mathrm{d}\Sigma_1 = \tfrac{1}{2}h\wedge\Sigma_1 + O(\epsilon^2)\,.$$

□

This enables us to present a conserved quasi-local charge in

Theorem 4.3.2. *For linearized gravity on a vacuum type D background in a principal tetrad, there exists a closed 2-form*

$$\dot{\mathcal{M}} = \zeta\dot{\Psi}_1 Z^0 + \zeta\dot{\Psi}_2 Z^1 + \zeta\dot{\Psi}_3 Z^2 + \tfrac{3}{2}\zeta\Psi_2\dot{Z}^1\,, \tag{4.45}$$

which can be used to calculate the "linearized mass". Here ζ *is the coefficient of the Killing spinor* (2.66).

The 2-form $\dot{\mathcal{M}}$ *is tetrad gauge-invariant and changes only with an exact term* $\chi = \mathrm{d}f$*, under coordinate gauge transformations. Hence, the integral*

$$\frac{1}{4\pi\mathrm{i}}\int_{S^2}\dot{\mathcal{M}}\,, \tag{4.46}$$

is conserved and gauge-invariant.

Proof. For linearized gravity, making use of (4.43) and $3h\Psi_2 = 2\mathrm{d}\Psi_2$, we find the identity

$$\begin{aligned} 0 &= \zeta(\mathrm{d} - \tfrac{1}{2}h\wedge)\dot{\Sigma}_1 - \tfrac{1}{2}\zeta\dot{h}\wedge\Sigma_1 \\ &= \mathrm{d}(\zeta\dot{\Psi}_1 Z^0 + \zeta\dot{\Psi}_2 Z^1 + \zeta\dot{\Psi}_3 Z^2 + \zeta\Psi_2\dot{Z}^1) - \tfrac{1}{2}\zeta\Psi_2\dot{h}\wedge Z^1 \\ &= \mathrm{d}(\zeta\dot{\Psi}_1 Z^0 + \zeta\dot{\Psi}_2 Z^1 + \zeta\dot{\Psi}_3 Z^2 + \tfrac{3}{2}\zeta\Psi_2\dot{Z}^1)\,, \end{aligned} \tag{4.47}$$

were the linearized version of $\mathrm{d}Z^1 = -h\wedge Z^1$ is used in the last step. Note, that also

$$0 = \mathrm{d}(\zeta\dot{\Psi}_1 Z^0 + \zeta\dot{\Psi}_2 Z^1 + \zeta\dot{\Psi}_3 Z^2) - \tfrac{3}{2}\zeta\Psi_2\dot{h}\wedge Z^1 \tag{4.48}$$

holds, which looks similar to Maxwell equations with a source.

Now consider the coordinate gauge transformations, (3.54) and use Cartan's identity $\mathcal{L}_\xi\omega = \mathrm{d}(\xi\lrcorner\omega) + \xi\lrcorner\mathrm{d}\omega$, which holds for arbitrary forms ω. It follows for $\dot{\mathcal{M}}$,

$$\begin{aligned} \delta\dot{\mathcal{M}} &= -\zeta\xi(\Psi_2)Z^1 - \tfrac{3}{2}\zeta\Psi_2[\mathrm{d}(\xi\lrcorner Z^1) + \xi\lrcorner\mathrm{d}Z^1] \\ &= -\tfrac{3}{2}\zeta\Psi_2(\mathrm{d} + h\wedge)(\xi\lrcorner Z^1) \\ &= -\tfrac{3}{2}\mathrm{d}[\zeta\Psi_2(\xi\lrcorner Z^1)]\,, \end{aligned} \tag{4.49}$$

where $\xi\lrcorner h = \frac{2}{3}\Psi_2^{-1}\xi(\Psi_2)$ and $\xi\lrcorner(h\wedge Z^1) = (\xi\lrcorner h)Z^1 - h\wedge(\xi\lrcorner Z^1)$ was used. The 2-form (4.49) is exact and hence integrates to zero.

The tetrad gauge dependence of the curvature scalars and $\dot{Z}^1$ is calculated in (3.52) and (3.53). It follows that the second term in (4.45) is tetrad gauge invariant, because Ψ_2 is. The non-trivial transformations of $\dot{\Psi}_1, \dot{\Psi}_3$ and $\dot{Z}^1$ exactly cancel each other. This shows the tetrad gauge-invariance of $\dot{\mathcal{M}}$ and therefore gauge-invariance of (4.46). □

Equation (4.43) is to zeroth order the Jordan-Ehlers-Sachs conservation law (4.18) and to first order Fackerell's conservation law, $\mathrm{d}\dot{\mathcal{M}} = 0$. In the Minkowski limit, $M, a \to 0$, it reduces to the $l = 0$ Penrose charge with Killing spinor Ω^1, see table 4.2 on p. 45.

Finally, to express the charge integral in a form similar to the Maxwell case (4.4), we need the $\theta\phi$ components of the bivectors (2.111),

$$Z^1_{\theta\phi} = -\mathrm{i}(r^2 + a^2)\sin\theta\,, \qquad Z^0_{\theta\phi} = -Z^2_{\theta\phi} = \frac{a\sqrt{\Delta}}{2}\sin^2\theta\,. \tag{4.50}$$

The charge integral becomes

$$2\mathrm{i}\int_{S^2(t,r)}\dot{\mathcal{M}} = \int_{S^2(t,r)}\Big(2(r^2 + a^2)\dot{\widehat{\Psi}}_2 + \mathrm{i}a\sqrt{\Delta}\sin\theta\Psi_{diff}\Big)(r - \mathrm{i}a\cos\theta)\mathrm{d}\mu\,, \tag{4.51}$$

with $\mathrm{d}\mu = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ and

$$\hat{\dot{\Psi}}_2 = \dot{\Psi}_2 - \Psi_2(M_3 + \bar{M}_3)\,, \tag{4.52a}$$

$$\Psi_{diff} = \dot{\Psi}_1 - \dot{\Psi}_3 - 3\Psi_2\big[\mathrm{Re}\,(M_2 - M_1) - \mathrm{i}\,\mathrm{Im}\,(L_3 + N_3)\big]\,. \tag{4.52b}$$

Here, M_i, L_3, N_3 are the coefficients of $\dot{Z}^1$, see (3.50b). The test-case (4.36) is compatible with this result, because $\dot{\Psi}_1 = \dot{\Psi}_3 = \dot{Z}^1 = 0$ holds in that (induced) gauge.

Remark 4.3.3 (Riemannian signature)**.** *A Newman-Penrose formalism for 4-dimensional Manifolds of Riemannian signature can be found in [4]. For linearized fields, assuming $\Psi_0 = \Psi_1 = 0 = \kappa = \sigma$ in the background, we find the conservation law*

$$d\big[\zeta\dot{\theta}_0{}^0 - \zeta\Psi_2\dot{L}_0{}^0\big] = 0\,, \tag{4.53}$$

where $\zeta \sim \Psi_2^{-1/3}$, θ is a curvature form and L is one component of the self-dual bivector triad, see [4]. In components, this reads

$$d\Big[\zeta\overline{\dot{\Psi}}_1 L_0{}^1 + \zeta\dot{\Psi}_1 L_1{}^0 - 2\zeta\dot{\Psi}_2 L_0{}^0 - 3\zeta\Psi_2\dot{L}_0{}^0\Big] = 0\,. \tag{4.54}$$

It seems reasonable, that a similar conservation law for the anti-self-dual part exists.

4.3.3. Angular momentum

After we learned that spin-lowering also works in the derivation of a linearized mass charge, we might look for a similar structure for an angular momentum charge. The obstruction to calculate such a charge for the Kerr solution itself due to the integrability condition (2.48) was already discussed in section 4.2.2. A possible conclusion is that this approach simply does not work. On the other hand we know that such quasi-local charge integrals do exist from e.g. the canonical or Hamiltonian approach, see section 4.3.4. A possible reason for this discrepancy is, that linearized gravity, formulated via the linearized spin-2 equations, admits more solutions than the theory formulated in terms of a metric[25]. These additional spurious solutions, carrying the singular charges of the linearized spin-2 equations, discussed on p. 46, prevent the direct derivation of an angular momentum charge in terms of linearized curvature.

To better understand this idea, we again review the charges on Minkowski spacetime. Integrating the spin-lowered spin-2 field (3.27) and using the fact that only the middle bivector has non-vanishing θ, φ component in spherical coordinates, (2.111), we find the simplified integral

$$C^n_m = -\mathrm{i}r^2 \int_S (\dot{\Psi}_1\kappa_2 - 2\dot{\Psi}_2\kappa_1 + \dot{\Psi}_3\kappa_0)\mathrm{d}\mu\,, \tag{4.55}$$

for a charge C^n_m corresponding to a Killing spinor solution κ_{AB}, labeled by Ω^n_m in table 4.2. Here $\mathrm{d}\mu = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ is the volume form of the unit sphere. More explicitly, for the quadratic Killing spinor Ω^2_0, singling out the angular momentum of interest, we have

$$C^2_0 = -\mathrm{i}r^2 \int_S \big[\sqrt{2}(t+r)^2\dot{\Psi}_1\,{}_{-1}Y_{10} - 2(t^2 - r^2)\dot{\Psi}_2\,{}_0Y_{10} + \sqrt{2}(t-r)^2\dot{\Psi}_3\,{}_1Y_{10}\big]\mathrm{d}\mu\,. \tag{4.56}$$

[25]More precisely, this depends on regularity and fall-off conditions.

The t-dependence can be expanded out and rewritten in the following two steps. Firstly, the charge for Ω_0^0 reads

$$C_0^0 = -\mathrm{i}r^2 \int_S \left[\sqrt{2}\dot{\Psi}_{1\,{-1}}Y_{10} - 2\dot{\Psi}_{2\,0}Y_{10} + \sqrt{2}\dot{\Psi}_{3\,1}Y_{10}\right]\mathrm{d}\mu\,,$$

so the charge linear in t, r takes the form

$$\begin{aligned} C_0^1 &= -\mathrm{i}r^2 \int_S \left[\sqrt{2}(t+r)\dot{\Psi}_{1\,{-1}}Y_{10} - 2t\dot{\Psi}_{2\,0}Y_{10} + \sqrt{2}(t-r)\dot{\Psi}_{3\,1}Y_{10}\right]\mathrm{d}\mu \\ &= tC_0^0 - \mathrm{i}\sqrt{2}r^3 \int_S \left[\dot{\Psi}_{1\,{-1}}Y_{10} - \dot{\Psi}_{3\,1}Y_{10}\right]\mathrm{d}\mu \\ &= tC_0^0 + D_0^1\,, \end{aligned}$$

where D_0^1 is defined as the second term in the second line. Note that in the case of vanishing charges C_0^0, the part D_0^1 with time independent coefficients is conserved. Secondly, the same holds for (4.56), namely

$$\begin{aligned} C_0^2 &= t^2C_0^0 + 2tD_0^1 - \mathrm{i}r^4 \int_S \left[\sqrt{2}\dot{\Psi}_{1\,{-1}}Y_{10} + 2\dot{\Psi}_{2\,0}Y_{10} + \sqrt{2}\dot{\Psi}_{3\,1}Y_{10}\right]\mathrm{d}\mu \\ &= t^2C_0^0 + 2tD_0^1 + E_0^2\,, \end{aligned}$$

with E_0^2 defined to be the third term on the right hand side. Note that it is t-dependent in general and can be rewritten

$$E_0^2 = r^2C_0^0 - 4\mathrm{i} \int_S r^4\dot{\Psi}_{2\,0}Y_{10}\mathrm{d}\mu\,,$$

so in case of vanishing charges C_0^0, C_0^1, we have the conserved (and therefore t-independent) quantity

$$C_0^2 = -2\mathrm{i} \int_S r^4\dot{\Psi}_{2\,0}Y_{10}\mathrm{d}\mu\,. \tag{4.57}$$

However, the condition $C_0^0 = 0 = C_0^1$ is not satisfied a priori. If the existence of a linearized metric with the above curvature is assumed, the singular charges vanish identically, $C_0^0 = 0 = \mathrm{Re}\, C_0^1$, and it follows that the linearized angular momentum charge is given by the real part of (4.57). The situation is similar to magnetic monopole solutions in Maxwell theory. If one assumes that a global vector potential A exists, $F = \mathrm{d}A$, then the magnetic charge has to vanish identically because of (4.1). We will derive a charge of the form (4.57) in section 4.3.4 from a Hamiltonian analysis.

Spatial Killing spinors and instant charges

We have seen in the last section that only the t-independent part of the angular momentum Killing spinor contributes to the charge (4.57), if $C_0^0 = 0 = C_0^1$. So one might try to restrict to a spatial slice, calculate quasi-local charges there, eliminate the ones with too slow fall-off and with these conditions prove conservation in the direction of evolution.

On a Schwarzschild background in a Carter tetrad (2.107), we have the time-like vector field $l^a + n^a = \frac{1}{\sqrt{2(1-2M/r)}}(\partial_t)^a$. The Killing spinor equation (2.47) can be decomposed into 3 evolution equations and 5 constraints. We refer to the set of constraint equations,

$$\eth\,\kappa_0 = 0\,, \tag{4.58a}$$
$$\eth'\kappa_2 = 0\,, \tag{4.58b}$$
$$(\text{þ} - \text{þ}' + 2\rho)\kappa_2 + 2(\eth' + \tau')\kappa_1 = 0\,, \tag{4.58c}$$
$$(\text{þ}' - \text{þ} + 2\rho')\kappa_0 + 2(\eth + \tau)\kappa_1 = 0\,, \tag{4.58d}$$
$$(\eth' + 2\tau')\kappa_0 - (\eth + 2\tau)\kappa_2 + 2(\text{þ} - \text{þ}' - \rho' + \rho)\kappa_1 = 0\,, \tag{4.58e}$$

as spatial Killing spinor equation. For a 2-spinor κ_{AB} with the ansatz

$$\kappa_0 = f(t,r)\,{}_1Y_{10}\,, \qquad \kappa_1 = g(t,r)\,{}_0Y_{10}\,, \qquad \kappa_2 = h(t,r)\,{}_{-1}Y_{10}, \tag{4.59}$$

for an $l = 1$ mode, we look for solutions to (4.58) and find,

$$f(t,r) = \left[-r^2\sqrt{f} + r(r-M)\right]A(t) \quad - \left[r^2\sqrt{f} + r(r-M)\right]B(t) \quad + rC(t)\,, \tag{4.60a}$$
$$g(t,r) = \left[-r^2\sqrt{f} + r(r-M)\right]A(t) \quad + \left[r^2\sqrt{f} + r(r-M)\right]B(t)\,, \tag{4.60b}$$
$$h(t,r) = \left[-r^2\sqrt{f} + r(r-M)\right]A(t) \quad - \left[r^2\sqrt{f} + r(r-M)\right]B(t) \quad - rC(t)\,, \tag{4.60c}$$

where $f = 1 - 2M/r$. Note the limit

$$-r^2\sqrt{f} + r(r-M) = M^2/2 + O(M^3)\,, \tag{4.61}$$
$$r^2\sqrt{f} + r(r-M) = 2r^2 + O(M)\,, \tag{4.62}$$

so we recover all solutions in the Minkowski limit $M \to 0$. The first solution (A) gives the constant coefficient Killing spinor Ω^0_m, the second one gives the quadratic Ω^2_m and the third one (C) is the Killing spinor Ω^1_m, linear in t, r. The application of this to a spatial version of spin lowering has not yet been worked out.

Bianchi identities on Schwarzschild spacetime

On a Schwarzschild background $\operatorname{Im}\dot{\Psi}_2$ is gauge-invariant and describes the even parity perturbations, see section 3.4. Rewriting the Bianchi identities for this variable, one should find other gauge-invariant variables. For a null derivative of $\operatorname{Im}\dot{\Psi}_2$ we find from the linearized (2.39b) and its complex conjugated version

$$(\text{þ} - 3\rho)(\dot{\Psi}_2 - \dot{\overline{\Psi}}_2) = \eth'\dot{\chi}_1 - \eth\dot{\overline{\chi}}_1\,, \tag{4.63}$$

with $\dot{\chi}_1 = \dot{\Psi}_1 - 3\Psi_2 M_2$. The coefficient M_2 defined in (3.48) arises from an expansion of $(\dot{\text{þ}} - 3\dot{\rho})\Psi_2$. The right hand side is in fact gauge-invariant because a coordinate gauge transformation $\delta M_2 = \eth\xi^l$ gives the vanishing commutator $[\eth', \eth]\xi^l = 0$, as follows from (2.38) using the realness of ρ, ρ' and Ψ_2. For the right hand side of (4.63) we find from the Bianchi identity (2.39a) after commuting derivatives,

$$(\text{þ} - 5\rho)(\eth'\dot{\chi}_1 - \eth\dot{\overline{\chi}}_1) = \eth'\eth'\dot{\Psi}_0 - \eth\eth\dot{\overline{\Psi}}_0\,. \tag{4.64}$$

Projecting this equation into the $l = 1$ mode and using properties (A.4) of ð for partial integration, we find

$$(\text{þ} - 5\rho) \int_S \left(\eth' \dot{\chi}_1 - \eth \dot{\bar{\chi}}_1\right) {}_0Y_{10} \mathrm{d}\mu = 0\,, \tag{4.65}$$

with $\mathrm{d}\mu = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$. Inserting (4.63) and doing the same for the GHP primed equations yields

$$(\text{þ} - 5\rho)(\text{þ} - 3\rho) \int_S \left(\dot{\Psi}_2 - \dot{\bar{\Psi}}_2\right) {}_0Y_{10} \mathrm{d}\mu = 0\,, \tag{4.66a}$$

$$(\text{þ}' - 5\rho')(\text{þ}' - 3\rho') \int_S \left(\dot{\Psi}_2 - \dot{\bar{\Psi}}_2\right) {}_0Y_{10} \mathrm{d}\mu = 0\,. \tag{4.66b}$$

So both second null derivatives of the $l = 1$ mode of $\operatorname{Im} \dot{\Psi}_2$ vanish. However, this is not enough to conclude, that the integral is conserved.

4.3.4. Hamiltonian and canonical charges

There are several approaches to derive quasi-local charge integrals for linearized gravity in the metric formulation. They automatically deliver a charge for each isometry of the background. Let us review some of them.

Superpotential One way is to expand the linearized vacuum Einstein equations in a parameter ϵ, writing everything linear in ϵ on the left hand side and the leftover as an energy momentum tensor on the right hand side. Due to the Bianchi identities, this tensor is divergence free and, after introducing a superpotential, leads to a divergence free 2-form

$$F_{ab} = \xi_c H^c{}_{[b;a]} + \xi_{c;[b} H_{a]}{}^c + \xi_{[b} Z_{a]}\,. \tag{4.67}$$

Here $H_{ab} = h_{ab} - \frac{1}{2} g_{ab} h$, $Z_a = \nabla^b H_{ab}$ and ξ^a is a Killing vector, see [1] for more details. This is also discussed in [61] in terms of so called Taub numbers.

Noether charge The canonical analysis following [72] shows the following. Given an asymptotically flat vacuum spacetime (N, g_{ab}), a solution of the linearized Einstein equations h_{ab} (satisfying suitable asymptotic conditions) and a Killing field ξ^a, the variation of the Hamiltonian current is an exact form, which yields the relation

$$\dot{\mathcal{P}}_{\xi;\infty} = \int_S \dot{\mathbf{Q}}[\xi] - \xi \cdot \mathbf{\Theta}\,. \tag{4.68}$$

Here, $\mathcal{P}_{\xi;\infty}$ is the Hamiltonian charge at infinity, generating the action of ξ, $\mathbf{Q}[\xi]$ is the Noether charge 2-form for ξ, and $\mathbf{\Theta}$ is the symplectic current three-form, defined with respect to the variation h_{ab}. We use a $\dot{}$ to denote variations along h_{ab}, thus $\dot{\mathcal{P}}_{\xi;\infty}$ and $\dot{\mathbf{Q}}[\xi]$ denote the variation of the Hamiltonian and the Noether 2-form, respectively. The integral on the right hand side of (4.68) is evaluated over an arbitrary sphere, which generates the second homology class.

For the case of $\xi = \partial_t$, and considering solutions of the linearized Einstein equations on the Kerr background it follows that

$$\dot{M} = \dot{\mathcal{P}}_{\partial_t;\infty}\,,$$

gives the linearized ADM mass. The same is true for the charge integral (4.46) and thus we have the relation

$$\int_S \dot{\mathbf{Q}}[\partial_t] - \partial_t \cdot \mathbf{\Theta} = \frac{1}{4\pi i}\int_S \dot{\mathcal{M}}\,, \tag{4.69}$$

for any surface S in the Kerr exterior. We remark that the left hand side of (4.69) can be evaluated in terms of the metric perturbation using the expressions for $\mathbf{Q}$ and $\mathbf{\Theta}$ given in [72, Section V]. On the other hand, the right hand side has been calculated in terms of linearized curvature. It would be of interest to have a direct derivation of the resulting identity.

In addition to the conserved charge corresponding to $\dot{M}$, equation (4.68) with $\xi = \partial_\phi$, the angular Killing field, gives a conserved charge integral for linearized angular momentum $\dot{a}$. If ∂_ϕ is tangent to S, then the term $\partial_\phi \cdot \mathbf{\Theta}$ does not contribute in (4.68).

Hamiltonian charge In the ADM formulation of General Relativity, the field equations split into constraints and evolution equations with respect to some time-like vector field. Denoting the metric and momentum on the slice by (g_{ij}, π_{ij}), the hamiltonian and momentum constraints read

$$\mathcal{H} = \mu_g(\pi^{ij}\pi_{ij} - \tfrac{1}{2}(\pi^i{}_i)^2) - \mu_g R\,, \qquad \mathcal{H}^i = -2\pi^{ij}{}_{|j}\,. \tag{4.70}$$

The linearized form leads to the quasi-local charge integrals,

$$\mathcal{C}_\xi = \int_S \big[\mu_g(-Ch^{mn}{}_{|n} + Ch^{|m} - C^{|m}h + C_{|n}h^{mn}) - 2X^i\dot{\pi}^m{}_i + X^m\pi^{lj}h_{lj}\big]n_m\,.$$

Here, $(h_{ij}, \dot{\pi}_{ij})$ are the linearized metric and momentum on a spatial slice and $\xi = (C, X^i)$ is a Killing field of the background spacetime.

Now, we relate this charge for $\xi = \partial_\varphi$ to the linearized curvature on a Schwarzschild background. With $f = 1 - \frac{2m}{r}$, the metric reads

$${}^4g = f\mathrm{d}t^2 - f^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\varphi^2)\,. \tag{4.71}$$

The charge expression simplifies to

$$\mathcal{C}_{\partial_\varphi} = -2\int_S \dot{\pi}^r{}_\varphi\,. \tag{4.72}$$

This angular momentum charge is also given in [74, Section 3]. Introducing a potential 2-form for ∂_φ on the sphere, it can be rewritten into

$$\int_S \dot{\pi}^r{}_\varphi = \int_S \dot{\pi}^r{}_A(r^2\varepsilon^{AB}\cos\theta)_{||B} = -\int_S \dot{\pi}^r{}_{A||B}r^2\varepsilon^{AB}\cos\theta\,. \tag{4.73}$$

Here $||$ denotes the covariant derivative on the 2-sphere S with metric $\eta = r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$ has $\det(\eta) = r^2\sin\theta$ and connection $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,\ \Gamma^\varphi_{\varphi\theta} = \cot\theta$. For the first equality $\varepsilon^{\varphi\theta} = -(r^2\sin\theta)^{-1}$ is used and the second step is partial integration.

Let $\varepsilon_{abef} = \mu_4\epsilon_{abef}$ be the volume element, where $\epsilon_{abef} = 0, \pm 1$ and $\mu_4 = \sqrt{\det {}^4g}$. The left dual of the Weyl tensor ${}^*C_{abcd}$ was introduced on p. 31. With a timelike unit vector T^a, normal to some spacelike hypersurface Σ, define the magnetic part of the Weyl tensor by

$$B_{ij} = {}^*C_{iajb}T^aT^b\,. \tag{4.74}$$

The induced volume element on Σ is $\varepsilon_{ijk} = \varepsilon_{aijk}T^a$. Using the Gauss equation for the second fundamental form k_{ij},

$$D_i k_{jm} - D_j k_{im} = \varepsilon_{ij}{}^l B_{lm}\,, \tag{4.75}$$

and contract with $\varepsilon^{ij}{}_n$ which gives

$$2\varepsilon^{ij}{}_n D_i k_{jm} = B_{nm}\,. \tag{4.76}$$

With $\pi^{ij} = \mu_3(kg^{ij} - k^{ij})$ and inverse $k^{ij} = -\mu_3^{-1}\pi^{ij} + (3\mu_3)^{-1}\pi g^{ij}$ we find

$$B_{nm} = 2\varepsilon^{ij}{}_n \mu_3^{-1}\left(-\pi_{jm|i} + \frac{1}{3}\pi_{|i} g_{jm}\right). \tag{4.77}$$

We choose $T = f^{-1/2}\partial_t$ and $n = f^{1/2}\partial_r$ as unit normal to the $t, r =$ const. spheres. In this slicing $B_{mn} = 0 = k_{mn}$ holds in the background and therefore linearization is simple. We find for the rr component of the magnetic part of the Weyl tensor

$$n_k n_l \dot{B}^{kl} = f^{-1}\dot{B}^{rr} = \mu_3^{-1}\varepsilon^{ijl} n_k n_l \dot{\pi}^k{}_{i|j}\,. \tag{4.78}$$

The second term vanishes, because $g_{r\theta} = 0 = g_{r\varphi}$. We find

$$\begin{aligned}
\tfrac{1}{2}\dot{B}^{rr} &= \mu_3^{-1}\varepsilon^{ijr}\dot{\pi}^r{}_{i|j} \\
&= \mu_3^{-1}\varepsilon^{ijr}(\dot{\pi}^r{}_{i,j} + \Gamma^r_{jl}\dot{\pi}^l{}_i \underbrace{-\Gamma^l_{ji}\dot{\pi}^r{}_l}_{\cdot\varepsilon^{ijr}=0} - \Gamma^l_{jl}\dot{\pi}^r{}_i) \\
&= \mu_3^{-1}v^{1/2}\varepsilon^{AB}(\dot{\pi}^r{}_{A,B} + \Gamma^r_{Bl}\dot{\pi}^l{}_A - \Gamma^l_{Bl}\dot{\pi}^r{}_A) \\
&= \mu_3^{-1}v^{1/2}\varepsilon^{AB}(\dot{\pi}^r{}_{A,B} + \underbrace{\Gamma^r_{BC}}_{\propto {}^2g_{BC}}\dot{\pi}^C{}_A + \underbrace{\Gamma^r_{Br}}_{=0}\dot{\pi}^r{}_A - \Gamma^C_{BC}\dot{\pi}^r{}_A) \\
&= \mu_3^{-1}v^{1/2}\varepsilon^{AB}(\dot{\pi}^r{}_{A,B} \underbrace{-\Gamma^C_{BA}\dot{\pi}^r{}_C}_{\cdot\varepsilon^{AB}=0} - \Gamma^C_{BC}\dot{\pi}^r{}_A) \\
&= \mu_3^{-1}v^{1/2}\varepsilon^{AB}\dot{\pi}^r{}_{A||B} \\
&= \frac{v}{\mu_2}\varepsilon^{AB}\dot{\pi}^r{}_{A||B}\,.
\end{aligned}$$

We used that $\mu_2^2 = \det {}^2g = f \det {}^3g = f\mu_3^2 = r^2 \sin\theta$. On the other hand,

$$\dot{B}^{rr} = -\frac{2f}{\mu_4}\dot{C}_{tr\theta\varphi} = -f \operatorname{Im}\dot{\Psi}_2\,. \tag{4.79}$$

The charge integral (4.72) then reads

$$\mathcal{C}_{\partial_\varphi} = -2\int_{S^2}\dot{\pi}^r{}_\varphi = -2\int_{S^2}\dot{B}^{rr} r^2\cos\theta\frac{\mu_2}{v}\mathrm{d}\theta\mathrm{d}\varphi = 2\int_{S^2} r^4 \operatorname{Im}\dot{\Psi}_2\cos\theta\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,. \tag{4.80}$$

So the angular momentum around the $\theta = 0, \pi$ axis on a Schwarzschild background can be calculated by projecting $\operatorname{Im}\dot{\Psi}_2$ into the $l = 1$ mode. Comparison with the charge (4.57) on Minkowski spacetime shows that the singular charges are excluded by construction. This is not the case for the linearized Bianchi identities alone and therefore, we were not able to extract the charge (4.80) from (4.66).

4.3.5. Gauge modes

We reviewed in section 4.2.1 that linearized gravity in terms of a spin-2 field admits a priori 20 real charges. Ten singular charges vanish identically, if the existence of a metric (with its curvature the spin-2 field) is assumed. The remaining ten charges can be associated with Poincaré symmetries. But what happens to this structure on a black hole background? Let us review the Schwarzschild case. The charges for mass and angular momentum were calculated quasi-locally in terms of the $l = 0$ and $l = 1$ modes of $\dot{\Psi}_2$, see (4.41) and (4.80), respectively. Because of the spherical symmetry, this extends to a set of four charges corresponding to the four remaining isometries. The angular momentum charge was derived from a Hamiltonian analysis, so the assumption of a metric potential is implicit in (4.80). Comparison to the Minkowski charge (4.57) suggests the absence of singular charges. Because only four of the ten Poincaré symmetries remain valid, six non-radiating modes can be eliminated by a gauge transformation. We call those *gauge modes* to contrast them from the modes leading to conserved charges.

On metric level, the $l = 0$ and $l = 1$ modes were analyzed in the original papers [97] and [123]. For an alternative formulation see also [74]. In the first two references, the perturbed metric is expanded in tensor spherical harmonics as represented in (3.31) and (3.32).

l=0 For the lowest mode $Y_{00} = C = \text{const.}$ we find

$$h^{\text{odd}}_{ab} \equiv 0\,, \qquad h^{\text{even}}_{ab} = C\begin{pmatrix} fH_0 & H_1 & 0 & 0 \\ H_1 & f^{-1}H_2 & 0 & 0 \\ 0 & 0 & r^2K & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta K \end{pmatrix}.$$

There is no $l = 0$ odd parity perturbation in accordance with the vanishing singular charges (here the NUT parameter). For the even parity part, Zerilli chose a gauge $K = H_1 = 0$ in which one finds $H_2 = \frac{c}{r-2M}$, $H_0 = H_2 + g(t)$ and $g(t)$ can be gauged away. This is linearized mass, which we expressed in (4.46) without any gauge ambiguities.

Calculating the wave operator for this $l = 0$ metric to first order leads to

$$2r^4\dot{\Box}r^{-1} = r(r-2M)\partial_r(H_0 + H_2 - 2K) - 4MH_2 - 2r^2\partial_t H_1\,. \tag{4.81}$$

This is non-vanishing in the Zerilli gauge and therefore not compatible with the gauge condition (3.61).

l=1 For this mode, we have 3 spherical harmonics, $Y_{10} = C\cos\theta$, $Y_{1\pm1} = \mp D\sin\theta e^{\pm i\varphi}$, with some constants C, D. We write $s = \sin\theta, c = \cos\theta$ and find for odd parity,

$${}^0h^{\text{odd}}_{ab} = C\begin{pmatrix} 0 & 0 & 0 & -h_0s^2 \\ 0 & 0 & 0 & -h_1s^2 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \qquad {}^{\pm}h^{\text{odd}}_{ab} = De^{\pm i\varphi}\begin{pmatrix} 0 & 0 & ih_0 & \mp h_0sc \\ 0 & 0 & ih_1 & \mp h_1sc \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}.$$

It describes the angular momentum charges. For all 3 cases, $m = 0, \pm 1$, we find $\dot{\Box}r^{-1} = 0$, so this mode is by construction compatible with the gauge condition (3.61). For even parity, we find

$$
{}^{0}h^{\text{even}}_{ab} = C \begin{pmatrix} fH_0 c & H_1 c & -h_0 s & 0 \\ * & f^{-1}H_2 c & -h_1 s & 0 \\ * & * & r^2 c(K-G) & 0 \\ 0 & 0 & 0 & r^2 s^2 c(K-G) \end{pmatrix},
$$

$$
{}^{\pm}h^{\text{even}}_{ab} = \mp D e^{\pm i\varphi} \begin{pmatrix} fH_0 s & H_1 s & h_0 c & \pm i h_0 s \\ * & f^{-1}H_2 s & h_1 c & \pm i h_1 s \\ * & * & r^2 s(K-G) & 0 \\ * & * & 0 & r^2 s^3 (K-G) \end{pmatrix}.
$$

Zerilli chose a gauge in which $K = 0$ and he then used the linearized field equations $\dot{R}_{ab} = 0$ to show that the whole $l = 1$ even parity perturbation can be removed by the remaining gauge freedom. The explicit form of the gauge vector is singular for $m \to 0$. This is consistent with the picture of conserved charges on Minkowski spacetime. Furthermore, the gauge transformation was interpreted for large r as a "transformation to the center-of-momentum system". The linearized wave operator in this case reads

$$
2r^5 \dot{\Box} r^{-1} = r^2(r-2M)\partial_r(H_0 + H_2 - 2K + 2G) + 4MrH_2 - 4r(r-2M)h_1 - 2r^3\partial_t H_1 .
$$

This is compatible with he gauge condition (3.61), if we restrict the Zerilli gauge on the initial data.

A generalization of "gauge modes" to a Kerr background is still lacking. An alternative would be to consider only data with sufficiently fast fall-off.

4.4. Lowest modes for the wave equation with potential

We presented in section 3.4 wave equations with potential for the middle components of a Maxwell field and linearized gravity (in a particular gauge), see (3.43b) and (3.39c), respectively. Do those equations alone contain non-radiating modes without referring to the whole tensor equation? To analyze this, we look at

$$
(\Box + n\Psi_2)\,\phi = 0\,, \tag{4.82}
$$

with the background curvature potential Ψ_2. It contains the following three cases:

	scalar field	Maxwell field	linearized gravity
n	0	2	8
ϕ	φ	$\zeta\phi_1$	$\zeta^2\dot{\Psi}_2$

Assuming a decomposition into spherical harmonics on a Schwarzschild background, (4.82) simplifies to

$$
\left[\partial_t^2 - \partial_{r_*}^2 + f\frac{l(l+1)}{r^2} - f\frac{(n-2)M}{r^3}\right](r\phi) = 0\,. \tag{4.83}
$$

Here the coordinate expression (2.101) and tortoise coordinates $\partial_{r*} = f\partial_r$ with $f = 1 - 2M/r$ have been used. For $l = 0$, we have the following time independent solutions:[26]

	$l = 0$
$\varphi(n = 0)$	$A + B\ln(-f)$
$\phi_1(n = 2)$	$A/r^2 - B\left(r + 2M\ln(2M - r)\right)/r^2$
$\dot{\Psi}_2(n = 8)$	$\frac{A(3M-r)}{r^4} + B\frac{8M(3M-r)\ln(2M-r)+3Mr-r^2+27M^2}{r^4}$

Here A and B are complex constants. The first solution for $n = 2$ is the Coulomb mode (4.2) and the first solution for $n = 8$ contributes to the linearized mass mode (4.45). All the other solutions (thought of in terms of ϕ) have slower fall off. We note the difference between this mass mode and (4.35), which is given by a gauge transformation (3.54) with $\xi^r = -\dot{M}$.

For $l = 2$, we find:

	$l = 1$
$\varphi(n = 0)$	$A(M - r) + B\left(2M + (r - M)\ln(-f)\right)$
$\phi_1(n = 2)$	$A + B\left(\ln(f) + 2M/r + 2M^2/r^2\right)$
$\dot{\Psi}_2(n = 8)$	$\frac{A}{r^4} + B\frac{12M^2r+3Mr^2+r^3+24M^3(2M-r)}{r^4}$

Here, the first solution for $n = 8$ is the angular momentum mode (4.39) and all the other solutions have slower fall-off. The equivalence of the angular momentum modes is not a coincidence, but follows from the gauge-invariance of odd parity perturbations ($\mathrm{Im}\,\dot{\Psi}_2$) on a Schwarzschild background.

Using the wave operator on a Kerr background, (2.100), we find (4.82) in the form

$$(\partial_r\Delta\partial_r + \frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - \Sigma V)\phi(r,\theta) = 0\,, \tag{4.84}$$

with

$$\Delta = r^2 - 2Mr + a^2\,, \qquad V = -nM\zeta^{-3}\,, \qquad \zeta = r - \mathrm{i}a\cos\theta\,, \qquad \Sigma = \zeta\bar{\zeta}\,. \tag{4.85}$$

This equation is not known to be separable for $n \neq 0$ and therefore difficult to analyze. A coordinate independent approach shows

$$(\Box + n\Psi_2)\,\phi = 2\zeta\left[(\text{þ}' - \rho' - \bar{\rho}')(\text{þ} - 2\rho) - (\eth' - \tau' - \bar{\tau})(\eth - 2\tau) + (n/2 - 1)\Psi_2\right](\zeta^{-1}\phi)\,.$$

For $n = 0$, the solution $\phi = A =$ const. follows from (2.58e) and for the Maxwell case $n = 2$, the solution $\phi = \zeta\phi_1 = A\zeta^{-1}$ follows from (2.65). For linearized gravity, $n = 8$, we make the ansatz $\phi = \zeta^2\dot{\Psi}_2 = \zeta^{-1} - c\zeta^{-2}$ for some constant c and find

$$\begin{aligned}(\Box + 8\Psi_2)\,\phi &= 2\zeta\left[3\Psi_2\zeta^{-2} - c(\text{þ}' - \rho' - \bar{\rho}')(\rho\zeta^{-3}) + c(\eth' - \tau' - \bar{\tau})(\tau\zeta^{-3}) - 3c\Psi_2\zeta^{-3}\right]\\ &= 2\zeta\left[3\Psi_2\zeta^{-2} - 2c\zeta^{-3}(\rho\rho' - \tau\tau' + \Psi_2)\right]\,.\end{aligned}$$

[26]During the calculation we found that the time independent equation can be solved for arbitrary n in terms of hyper-geometric functions, but only $n = 0, 2, 8, 18, 32, 50, \ldots$ could be expressed elementary.

Here, we used (2.65) and (2.58e) in the second step. Expansion in a Carter tetrad in Boyer-Lindquist coordinates yields

$$\begin{aligned}\rho\rho' - \tau\tau' + \Psi_2 &= -\frac{\Delta - a^2 \sin^2\theta + 2M\bar{\zeta}}{2\bar{\zeta}\zeta^3} \\ &= -\frac{1}{2\zeta^2} - \frac{M\mathrm{i}a\cos\theta}{\bar{\zeta}\zeta^3}\,. \end{aligned} \tag{4.86}$$

Because $\Psi_2 = -M\zeta^{-3}$, the potential term cancels, iff $c = 3M$ and in that case we are left with

$$\begin{aligned}(\Box + 8\Psi_2)\,\phi &= 2\zeta\left[3\Psi_2\zeta^{-2} + c(\text{þ}' - \rho' - \bar{\rho}')(\rho\zeta^{-3}) - c(\eth' - \tau' - \bar{\tau})(\tau\zeta^{-3}) + 3c\Psi_2\zeta^{-3}\right] \\ &= -12M^2\frac{\mathrm{i}a\cos\theta}{\bar{\zeta}\zeta^5}\,.\end{aligned}$$

This term vanishes in the Schwarzschild spacetime and gives the result discussed above. For a Kerr solution, we were not able to derive solutions for $n = 8$.

The choice $c = 3M$ has a geometric interpretation. The massless geodesic equation admits bounded solutions at the *photon sphere* $r = 3M$. On Kerr spacetime the set of bounded solution is not a sphere anymore, but an open photon region, see (2.96). Whether the stationary solutions of (4.82) for $n = 8$ on a Kerr background can be expressed in closed form is not clear at the moment and needs further research.

5. Symmetries

In the context of General Relativity, important applications of symmetries are exact solutions, integrability of the geodesic equation or separability of field equations. In this chapter, we investigate symmetry properties of the spin-s field equations (3.1). This is of interest not only for the construction of symmetry operators, which are essential ingredients for the vector field method of section 6.1, but also to rederive well known results from a geometric point of view. For motivation, we review constants of motion for geodesics in

Example 5.0.1. *Let $K_{a_1\dots a_n}$ be a solution to the Killing tensor equation $\nabla_{(a}K_{a_1\dots a_n)} = 0$. Given the four-momentum p^a of a geodesic, $p^b\nabla_b p^a = 0$, the scalar*

$$C = p^{a_1}\dots p^{a_n}K_{a_1\dots a_n}\,, \tag{5.1}$$

is conserved along the geodesic, since

$$p^b\nabla_b C = nK_{a_1\dots a_n}p^{a_1}\dots p^{a_{n-1}}p^b\nabla_b p^{a_n} + p^{a_1}\dots p^{a_n}p^b\nabla_b K_{a_1\dots a_n} = 0\,. \tag{5.2}$$

In the class of vacuum Kerr-NUT spacetimes, there exist two isometries ξ^a, Ξ^a and besides the metric g_{ab} a second irreducible rank two Killing tensor $\overset{**}{K}_{ab}$ given in (2.76c)*, so the geodesic equation is completely integrable with four constants of motion. The fourth conserved quantity,*

$$C = \overset{**}{K}_{ab}\, p^a p^b = \mathrm{Im}\,(\zeta)^2 p^n p^l + \mathrm{Re}\,(\zeta)^2 p^m p^{\bar{m}}\,, \tag{5.3}$$

is the Carter constant. On a Schwarzschild background, because of $\mathrm{Im}\,\zeta = 0$ *and therefore $C = \frac{1}{2}(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2)$, it reduces to the squared total angular momentum. The interpretation on Kerr spacetime is more subtle. It can be related to the velocity of a null geodesic in θ-direction, passing through the equatorial plane, see [108]. For null geodesics, $0 = p^a p_a = p^n p^l - p^m p^{\bar{m}}$, the integrability holds for all Petrov type D spacetimes, because a conformal Killing tensor is sufficient to build constants of motion in that case. The Carter constant simplifies to*

$$C_{null} = \zeta\bar{\zeta}p^m p^{\bar{m}}\,, \tag{5.4}$$

in accordance with (2.78).

The existence of an irreducible Killing tensor (other than the metric) is sometimes called a *hidden symmetry*, because it does not correspond to an isometry of the background.

For a field theory, the role of constants of motion is played by symmetry operators, which are defined in the next section. Results for the Maxwell equations on vacuum type D backgrounds are reviewed.

In section 5.2 we discuss a constructive method to find such operators using Wald's approach [115] based on adjoint operators. The method is applied to the spin-1 equations and compared to previous results. Then components of covariant spin-1 symmetry operators are investigated for the Fackerell-Ipser and the Teukolsky equation in sections 5.3 and 5.4, respectively. In particular, a second order anti-symmetry operator for the Fackerell-Ipser equation is given in (5.49) and a second order symmetry operator for the spin-s equation is proposed in section 5.4.1.

5.1. Symmetry operators

Given a field equation $Of = 0$ for some field f, a symmetry operator S is a differential operator which maps solutions to solutions, $OSf = 0$[27]. In particular, operators which commute with O are symmetry operators. Let us first have a look at the scalar wave equation

$$\Box\phi = \nabla^a \nabla_a \phi = 0\,. \tag{5.5}$$

With ϕ a solution and ξ^a an isometry, the Lie derivative $\mathcal{L}_\xi \phi$ is also a solution because

$$[\Box, \mathcal{L}_\xi] = 0\,. \tag{5.6}$$

Compared to example 5.0.1, these first order symmetry operators correspond to the constants of motion which are linear in p^a. The field equation (5.5) itself corresponds to the norm of the four momentum. In [27], Carter calculated the most general second order operator which commutes with (5.5). In particular, he found in the vacuum case that, with K_{ab} a Killing tensor, the *Carter operator*

$$\mathcal{Q} = \nabla^a K_{ab} \nabla^b \tag{5.7}$$

commutes with the scalar wave operator. So with the Killing tensors mentioned in example 5.0.1, four mutually commuting operators exist and a full separation of variables of (5.5) in Kerr-NUT spacetime is possible, see also section 5.4.1.

For spin s greater than zero, another interesting structure occurs. Besides a map

$$\phi_{A_1 \dots A_{2s}} \mapsto \omega_{A_1 \dots A_{2s}}\,, \tag{5.8}$$

from solutions to solutions, a helicity flipping map

$$\phi_{A_1 \dots A_{2s}} \mapsto \bar{\chi}_{A'_1 \dots A'_{2s}}\,, \tag{5.9}$$

from solutions to solutions of the complex conjugated equation exists under certain conditions. In [78], Kalnins, McLenaghan and Williams refer to these maps as symmetry operators of first and second type, respectively. In many cases the symmetry operator of first type can be related to a separation of variables. The spin-1/2 (massless Dirac or Weyl) equation admits a symmetry operator built from a Killing-Yano tensor, see [28].

[27] Regularity and fall-off conditions of the operators will not be discussed in this section. We also exclude discrete operators corresponding to (2.117)

Also the spin-1 equation admits such a symmetry operator of first type, if a valence-2 Killing spinor does exist. It is first published in [79] for a Kerr background under the condition that the spin-1 field is generated by a Debye potential. The separation constants can be interpreted as eigenvalues of the symmetry operator. A recent exploration of the conditions for existence and explicit forms of spin $0, 1/2$ and 1 symmetry operators of second order can be found in [7].

Symmetry operators of second type have been derived several times from different points of view in the literature. For example in the Kerr-NUT class, the extremal helicity components of such an operator can be shown to decouple and are known as *Teukolsky-Starobinski identities*. After a separation of variables is performed, the "eigenvalues" of this symmetry operator correspond to the Starobinski constants, see [79]. The Debye potential approach, developed in [33], also leads to this symmetry operator[28]. Another powerful method to calculate such operators was invented by Wald in [115] and is discussed in the next section.

The spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ Teukolsky equations are also known to be separable in the Kerr-NUT class. However, a geometric formulation of this fact is lacking, even though the $s = \frac{1}{2}, 1$ cases are investigated in some detail as mentioned above. An obstruction to continue this program to higher spin, beside the complexity of the equations, are algebraic constraints (3.4) and deviations of the linearized gravity equations from the spin-2 equations.

For a Kerr background, two symmetry operators (one of each type) were given explicitly in [79]. Very recently, conditions for the existence of symmetry operators and simplified expressions for the Kerr-NUT case in terms of potentials were derived in [7]. The symmetry operators of first and second type were expressed in the form

$$\chi_{AB} = \nabla_{(B}{}^{A'} A_{A)A'} \,, \tag{5.10a}$$

$$\omega_{A'B'} = \nabla^{B}{}_{(A'} B_{|B|B')} \,, \tag{5.10b}$$

with potentials

$$\Theta_{AB} = -2\kappa_{(A}{}^{C}\phi_{B)C} \,, \tag{5.11a}$$

$$A_{AA'} = \bar{\kappa}_{A'}{}^{B'}\nabla_{BB'}\Theta_{A}{}^{B} - \tfrac{1}{3}\Theta_{A}{}^{B}\nabla_{BB'}\bar{\kappa}_{A'}{}^{B'} \,, \tag{5.11b}$$

$$B_{AA'} = \kappa_{A}{}^{B}\nabla_{CA'}\Theta_{B}{}^{C} + \tfrac{1}{3}\Theta_{A}{}^{B}\nabla_{CA'}\kappa_{B}{}^{C} \,. \tag{5.11c}$$

Here κ_{AB} is the Killing spinor and the resulting fields solve the spin-1 equation and its complex conjugated version,

$$\nabla^{A}{}_{A'}\chi_{AB} = 0 \,, \qquad \nabla_{A}{}^{A'}\omega_{A'B'} = 0 \,. \tag{5.12}$$

The components of (5.10) will be analyzed in more detail in the next sections.

5.2. The method of adjoint operators

To decouple the field equations for the $\phi_i, i = 0, 1, 2$ components of a Maxwell test field or the $\dot{\Psi}_i, i = 0, ..., 4$ components for linearized gravity, a first order operator acting on

[28] More precisely, a Debye potential has additional gauge freedom and in the commonly chosen gauge it gives the symmetry operator. Otherwise the Debye potential equation differs from the original field equation.

the Maxwell or linearized Bianchi equations is needed, see section (3.4). This fact has been used in a systematic way by Wald in [115], to connect various results about Hertz potentials [33], symmetry operators [103] and Teukolsky-Starobinski identities [29, p. 436]. In this section, we will review this *method of adjoint operators* using bivector formalism. The symmetry operator (5.10b) of second type for spin-1 is rederived and a conserved current for the Teukolsky equation is presented. In particular, a symmetry operator of second type for the Fackerell-Ipser equation is extracted in the next section.

We now restrict to the Maxwell case and comment on the linearized gravity case in section 5.2.1. Let A be a real vector potential for a source-free Maxwell field $F = \mathrm{d}A$. While for a regular potential, $\mathrm{d}F = 0$ is identically satisfied, $\mathrm{d} * F = 0$ yields the wave equation

$$\mathcal{E}(A)_b = \nabla^c\nabla_c A_b - \nabla^c\nabla_b A_c = \nabla^c F_{cb} = 0\,. \tag{5.13}$$

To decouple these equations, *decoupling operators*

$$\mathcal{S}_I(j) = Z_I^{ab}\Theta_a(\zeta^2 j_b) \tag{5.14}$$

are introduced, which yield the operators $\mathcal{O}_I$ of Teukolsky ($I = 0, 2$) and Fackerell-Ipser ($I = 1$) acting on the (rescaled) Maxwell scalars ϕ_I[29], according to (5.16). Here the Z_I are the index-lowered self-dual bivectors (2.8). The scalars ϕ_I are components of the field strength and can be written in terms of the potential as

$$\phi_I = \mathcal{T}_I(A) = Z_I^{ab}\Theta_a A_b\,. \tag{5.15}$$

The resulting identity

$$\mathcal{S}_I\mathcal{E} = \mathcal{O}_I\mathcal{T}_I \qquad \text{(no sum)}\,, \tag{5.16}$$

is a set of three decoupled (in terms of ϕ_I) third order partial differential equations for the vector potential A. It should be noted that one has to use commutation relations to decouple the field equations into $\mathcal{O}_I\mathcal{T}_I$, so the identity (5.16) is true only after using the GHP commutator

$$(\text{þ} - a\rho - \bar{\rho})(\eth - a\tau) - (\eth - a\tau - \bar{\tau}')(\text{þ} - a\rho) = 0\,, \tag{5.17}$$

for any constant a. The (p, q) weights (2.11) of the operators $\mathcal{E}$ and $\mathcal{O}$ are zero, while the projection operators $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$ (and the decoupling operators $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$) have weights $(2, 0), (0, 0), (-2, 0)$, respectively.

The formal adjoint $\mathcal{L}^\dagger$ of an operator $\mathcal{L}$ is defined by

$$\psi(\mathcal{L}\phi) - (\mathcal{L}^\dagger\psi)\phi = \nabla_a j^a\,, \tag{5.18}$$

from which

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger \tag{5.19}$$

[29] One should note that the decoupling operators are not unique. For example, the middle component can be decoupled using $q(j) = (n_a l_b - \bar{m}_a m_b)\zeta^{-2}\Theta^a(\zeta^2 j^b)$ or its GHP primed version q'. Because $Z_{1ab} = l_{[a}n_{b]} - m_{[a}\bar{m}_{b]}$, one finds $\mathcal{S}_1 = q - q'$.

follows. In coordinates, with α a multi-index, we find for an operator of order m,

$$\mathcal{L} = \Sigma_{|\alpha|\leq m} a^\alpha \partial_\alpha \,, \tag{5.20}$$

the formal adjoint (acting on a function ϕ)

$$\mathcal{L}^\dagger \phi = \Sigma_{|\alpha|\leq m} (-1)^{|\alpha|} \partial_\alpha (a^\alpha \phi) \,. \tag{5.21}$$

In particular, the adjoint of the operator identity (5.16) reads

$$\mathcal{E}^\dagger \mathcal{S}_I^\dagger = \mathcal{T}_I^\dagger \mathcal{O}_I^\dagger \qquad \text{(no sum)} \,. \tag{5.22}$$

So for a self-adjoint operator $\mathcal{E}^\dagger = \mathcal{E}$ (e.g. (5.13) of the wave operator for a linearized metric), the results of [115] can be summarized in the following

Theorem 5.2.1.

1. *For a field ψ solving $O_I^\dagger \psi = 0$, $B = \mathcal{S}_I^\dagger \psi$ is a (complex) solution to Maxwell's equation, $\mathcal{E}(B) = 0$.*

2. *Applying the operator $\mathcal{S}_J$ to* (5.22) *and using* (5.16) *gives*

$$\mathcal{S}_J \mathcal{E} \mathcal{S}_I^\dagger = \mathcal{O}_J \mathcal{T}_J \mathcal{S}_I^\dagger = \mathcal{S}_J \mathcal{T}_I^\dagger \mathcal{O}_I^\dagger \,, \tag{5.23}$$

so with ψ as above, $\chi = \mathcal{T}_J \mathcal{S}_I^\dagger \psi$ solves $\mathcal{O}_J \chi = 0$.

Remark 5.2.2. *In the above definition of adjoint, no complex conjugation is involved and the current j^a in* (5.18) *is in general complex.*

From (2.34) we find the covariant divergence of the Newman-Penrose tetrad,

$$\begin{aligned} \nabla_a l^a &= \epsilon + \bar{\epsilon} - \rho - \bar{\rho} \,, & \nabla_a n^a &= \epsilon' + \bar{\epsilon}' - \rho' - \bar{\rho}' \,, \\ \nabla_a m^a &= \beta + \bar{\beta}' - \tau - \bar{\tau}' \,, & \nabla_a \bar{m}^a &= \beta' + \bar{\beta} - \tau' - \bar{\tau} \,, \end{aligned} \tag{5.24}$$

and from (5.21) follows a weighted version for the adjoint GHP operators,

$$\begin{aligned} þ^\dagger &= -þ + \rho + \bar{\rho} \,, & þ'^\dagger &= -þ' + \rho' + \bar{\rho}' \,, \\ ð^\dagger &= -ð + \tau + \bar{\tau}' \,, & ð'^\dagger &= -ð' + \tau' + \bar{\tau} \,, \end{aligned} \tag{5.25}$$

which is equivalent to the skew-adjointness property of the GHP connection,

$$\Theta_a^\dagger = -\Theta_a \,. \tag{5.26}$$

The generalized wave operator (3.37) is not self-adjoint because the additional connection B_a is not GHP-prime invariant. However, with the choice of decoupling operators (5.14) made above, we find the following form of the operators $\mathcal{O}_I$,

$$\begin{aligned}
\zeta^{-2}\mathcal{O}_0\phi_0 &= \left[(þ - 2\rho - \bar{\rho})(þ' - \rho') - (ð - 2\tau - \bar{\tau}')(ð' - \tau')\right]\phi_0 \\
&= \left[(þ' - \bar{\rho}')(þ - 3\rho) - (ð' - \bar{\tau})(ð - 3\tau) - 6\Psi_2\right]\phi_0 \\
&= \frac{1}{2}\left[\boxplus_2 - 4\Psi_2\right]\phi_0\,, \qquad (5.27a)\\
\zeta^{-2}\mathcal{O}_1\phi_1 &= \left[(þ - \rho - \bar{\rho})(þ' - 2\rho') - (ð - \tau - \bar{\tau}')(ð' - 2\tau')\right]\phi_1 \\
&= \left[(þ' - \rho' - \bar{\rho}')(þ - 2\rho) - (ð' - \tau' - \bar{\tau})(ð - 2\tau)\right]\phi_1 \\
&= \frac{1}{2}\zeta^{-1}\left[\Box + 2\Psi_2\right](\zeta\phi_1)\,, \qquad (5.27b)\\
\zeta^{-2}\mathcal{O}_2\phi_2 &= \left[(þ' - 2\rho' - \bar{\rho}')(þ - \rho) - (ð' - 2\tau' - \bar{\tau})(ð - \tau)\right]\phi_2 \\
&= \left[(þ - \bar{\rho})(þ' - 3\rho') - (ð - \bar{\tau}')(ð' - 3\tau') - 6\Psi_2\right]\phi_2 \\
&= \frac{1}{2}\zeta^{-2}\left[\boxplus_{-2} - 4\Psi_2\right](\zeta^2\phi_2)\,. \qquad (5.27c)
\end{aligned}$$

The operators differ from the Teukolsky and Fackerell-Ipser wave operators (3.43) only by a prefactor $\zeta^{\frac{p+2}{2}}$, but have the following nice property.

Proposition 5.2.3. *The adjoints of the wave-like operators* $\mathcal{O}_I$, $I = 0, 1, 2$ *are*

$$\mathcal{O}_0^\dagger = \mathcal{O}_2\,, \qquad \mathcal{O}_1^\dagger = \mathcal{O}_1\,, \qquad \mathcal{O}_2^\dagger = \mathcal{O}_0\,. \qquad (5.28)$$

Proof. To calculate the adjoint and re-express it in terms of $\mathcal{O}$, we use (5.19), the adjoint GHP operators (5.25) and the Killing spinor (2.66) to rescale with ζ according to (2.65),

$$\begin{aligned}
\mathcal{O}_0^\dagger &= \left[\zeta^2(þ - 2\rho - \bar{\rho})(þ' - \rho') - \zeta^2(ð - 2\tau - \bar{\tau}')(ð' - \tau')\right]^\dagger \\
&= (þ' - \rho')^\dagger(þ - 2\rho - \bar{\rho})^\dagger\zeta^2 - (ð' - \tau')^\dagger(ð - 2\tau - \bar{\tau}')^\dagger\zeta^2 \\
&= (þ' - \bar{\rho}')(þ + \rho)\zeta^2 - (ð' - \bar{\tau})(ð + \tau)\zeta^2 \\
&= \zeta^2\left[(þ' - 2\rho' - \bar{\rho}')(þ - \rho) - (ð' - 2\tau' - \bar{\tau})(ð - \tau)\right] \\
&= \mathcal{O}_2\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_1^\dagger &= \left[\zeta^2(þ - \rho - \bar{\rho})(þ' - 2\rho') - \zeta^2(ð - \tau - \bar{\tau}')(ð' - 2\tau')\right]^\dagger \\
&= (þ' - 2\rho')^\dagger(þ - \rho - \bar{\rho})^\dagger\zeta^2 - (ð' - 2\tau')^\dagger(ð - \tau - \bar{\tau}')^\dagger\zeta^2 \\
&= (þ' + \rho' - \bar{\rho}')\,þ\,\zeta^2 - (ð' + \tau' - \bar{\tau})\,ð\,\zeta^2 \\
&= \zeta^2\left[(þ' - \rho' - \bar{\rho}')(þ - 2\rho) - (ð' - \tau' - \bar{\tau})(ð - 2\tau)\right] \\
&= \mathcal{O}_1\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_2^\dagger &= \left[\zeta^2(þ' - 2\rho' - \bar{\rho}')(þ - \rho) - \zeta^2(ð' - 2\tau' - \bar{\tau})(ð - \tau)\right]^\dagger \\
&= (þ - \rho)^\dagger(þ' - 2\rho' - \bar{\rho}')^\dagger\zeta^2 - (ð - \tau)^\dagger(ð' - 2\tau' - \bar{\tau})^\dagger\zeta^2 \\
&= (þ - \bar{\rho})(þ' + \rho')\zeta^2 - (ð - \bar{\tau}')(ð' + \tau')\zeta^2 \\
&= \zeta^2\left[(þ - 2\rho - \bar{\rho})(þ' - \rho') - (ð - 2\tau - \bar{\tau}')(ð' - \tau')\right] \\
&= \mathcal{O}_2\,,
\end{aligned}$$

□

Because to the self-adjointness of $\mathcal{O}_1$, the map $\mathcal{T}_1\mathcal{S}_1^\dagger\phi_1$ is a symmetry operator. Details will be discussed in section 5.3. We also note the simple relation between the decoupling and projection operators, (5.14), (5.15),

$$\mathcal{S}_I(j^b) = \mathcal{T}_I(\zeta^2 j^b)\,, \qquad\qquad \mathcal{S}_{Ib}^\dagger = \zeta^2\mathcal{T}_{Ib}^\dagger\,. \tag{5.29}$$

Furthermore, the Maxwell equation (5.13) can be rewritten in the following form,

$$\mathcal{E}(A)_b = \Theta^c F_{cb} = \Theta^c(Z^I_{cb}\phi_I) = -\zeta^{-2}\mathcal{S}_b^{I\dagger}\mathcal{T}_I(A) = -\mathcal{T}_b^{I\dagger}\mathcal{T}_I(A)\,. \tag{5.30}$$

Here, the real field strength F is replaced by its self-dual version since the Maxwell equation $\mathrm{d}F = 0$ is satisfied identically for $F = \mathrm{d}A$. It follows the simple representation

$$\mathcal{O}_I = -\mathcal{S}_I\zeta^{-2}\mathcal{S}^{I\dagger} = -\mathcal{T}_I\mathcal{S}^{I\dagger} = -\mathcal{T}_I\zeta^2\mathcal{T}^{I\dagger} \qquad \text{(no sum)}\,, \tag{5.31}$$

for the Teukolsky and Fackerell-Ipser operators.

The current j^a in (5.18) has many useful applications, see e.g. [112]. Another application of conserved currents constructed from adjoint operators is the classification of all local conserved currents of the Maxwell equations on Minkowski space in [6]. For the Teukolsky equation, we also find

Proposition 5.2.4. *Let ϕ_0 and ϕ_2 be solutions of the Teukolsky equations $\mathcal{O}_0\phi_0 = 0 = \mathcal{O}_2\phi_2$. Then the current*

$$j^a = \tilde{\phi}_2 D^a\phi_0 - D^a(\tilde{\phi}_2)\phi_0\,, \tag{5.32}$$

with $\tilde{\phi}_2 = \zeta^2\phi_2$, is conserved.

Proof. This is a direct consequence of (5.18) and (5.28). Using (5.27a) and the representation (3.37), it follows from Leibniz rule,

$$\begin{aligned}
2\phi_2\mathcal{O}_0\phi_0 &= \zeta^2\phi_2[D^aD_a - 4\Psi_2]\phi_0 \\
&= \nabla^a[\zeta^2\phi_2 D_a\phi_0] - D^a(\zeta^2\phi_2)D_a\phi_0 - 4\zeta^2\Psi_2\phi_2\phi_0 \\
&= \nabla^a[\zeta^2\phi_2 D_a\phi_0 - D_a(\zeta^2\phi_2)\phi_0] + [D^aD_a - 4\Psi_2](\zeta^2\phi_2)\phi_0 \\
&= \nabla_a j^a + 2\phi_0\mathcal{O}_2\phi_2.
\end{aligned} \tag{5.33}$$

□

Now, we make the first point of theorem 5.2.1 more explicit in

Proposition 5.2.5. *Suppose ψ_I are solutions to $\mathcal{O}_I\psi_I = 0$. Then the vector fields*

$$B_{0b} := \mathcal{S}_{0b}^\dagger(\psi_2) = -\zeta^2\Theta^a(Z_{0ab}\psi_2) = -\zeta Z_{0ab}\Theta^a(\zeta\psi_2)\,, \tag{5.34a}$$

$$B_{1b} := \mathcal{S}_{1b}^\dagger(\psi_1) = -\zeta^2\Theta^a(Z_{1ab}\psi_1) = -Z_{1ab}\Theta^a(\zeta^2\psi_1)\,, \tag{5.34b}$$

$$B_{2b} := \mathcal{S}_{2b}^\dagger(\psi_0) = -\zeta^2\Theta^a(Z_{2ab}\psi_0) = -\zeta Z_{2ab}\Theta^a(\zeta\psi_0)\,, \tag{5.34c}$$

are (complex) solutions to the Maxwell equations (5.13)*. For a solution $F = \psi_I Z^I$ of the Maxwell equations, all three vector fields B_{Ib} generate the same Maxwell field strength, $\hat{F} = \mathrm{d}B_I$, $I = 0, 1, 2$.*

Proof. It follows directly from the method of adjoint operators that $\mathcal{E}(\mathcal{S}_I^\dagger\psi) = 0$ if $\mathcal{O}_I^\dagger\psi = 0$. With (5.28), $\mathcal{E}(B_{Ib}) = 0$ follows. Next, we show that the three resulting vector potentials B_{Ib} generate the same field strength. This is mentioned already in [103], but not proved. On type D backgrounds, apart from (2.60) we also have

$$\Theta^a Z_{0ab} = -\frac{1}{2}h^a Z_{0ab}\,, \qquad \Theta^a Z_{1ab} = -h^a Z_{1ab}\,, \qquad \Theta^a Z_{2ab} = -\frac{1}{2}h^a Z_{2ab}\,. \tag{5.35}$$

On the complex field strength $\mathcal{F} = \phi_I Z^I$, the equations $\mathrm{d}\mathcal{F} = 0$ and $\delta\mathcal{F} = 0$ are equivalent. Expanding the latter in a bivector triad gives

$$\begin{aligned}(\delta\mathcal{F})_b &= \Theta^a \mathcal{F}_{ab} \qquad (5.36)\\ &= \Theta^a(\psi_0 Z^0_{ab} + \psi_1 Z^1_{ab} + \psi_2 Z^2_{ab})\\ &= Z^0_{ab}\Theta^a\psi_0 + Z^1_{ab}\Theta^a\psi_1 + Z^2_{ab}\Theta^a\psi_2 + \psi_0\Theta^a Z_{2ab} - 2\psi_1\Theta^a Z_{1ab} + \psi_2\Theta^a Z_{0ab}\\ &= Z^0_{ab}\zeta^{-1}\Theta^a(\zeta\psi_0) + Z^1_{ab}\zeta^{-2}\Theta^a(\zeta^2\psi_1) + Z^2_{ab}\zeta^{-1}\Theta^a(\zeta\psi_2).\end{aligned}$$

In the last step, (5.35) was used. Next, define

$$q_{ab} = n_a l_b - \bar{m}_a m_b\,, \qquad q'_{ab} = l_a n_b - m_a\bar{m}_b\,. \tag{5.37}$$

Then $Z^1_{ab} = q_{ab} - q'_{ab}$ and $g_{ab} = q_{ab} + q'_{ab}$. Since

$$\Theta^a(q_{ab}\phi) = q_{ab}\Theta^a\phi + (B_b - B'_b)\phi\,, \tag{5.38}$$

we find a partial decoupling of (5.36) into

$$0 = \Theta^a(\psi_0 Z^0_{ab} - \psi_1 q'_{ab}) + (B_b + B'_b)\phi_1\,, \tag{5.39a}$$
$$0 = \Theta^a(\psi_2 Z^2_{ab} + \psi_1 q_{ab}) - (B_b + B'_b)\phi_1\,. \tag{5.39b}$$

Here, the first equation only involves ψ_0, ψ_1, while the second involves ψ_1, ψ_2. From this form of the Maxwell equations we find

$$\begin{aligned}B_{0b} &= -\zeta^2\Theta^a(Z^2_{ab}\psi_2)\\ &= -\zeta^2\left[\Theta^a(Z^0_{ab}\psi_0) - \Theta_b\psi_1 + h_b\psi_1\right]\\ &= -\zeta^2\Theta^a(Z^0_{ab}\psi_0) + \Theta_b(\zeta^2\psi_1)\\ &= B_{2b} + \text{gradient}\,. \end{aligned} \tag{5.40}$$

Using this and

$$B_{0b} - \tfrac{1}{2}B_{1b} + B_{2b} = 0, \tag{5.41}$$

which is a consequence of (5.36), the equivalence up to a gradient follows. □

It should be noted that the vector potentials B_{0b} and B_{2b} are by construction in radiation gauge, $B_{0b}l^b = 0 = B_{2b}n^b$. For linearized gravity, the analogous potentials solving the Teukolsky equation also generate metrics in radiation gauge, see e.g. [32]. This is not the case for the middle potential B_{1b}. However, none of the potentials is in Lorenz gauge, since $\nabla^b B_{Ib} = h^b B_{Ib}$. This in turn follows from the divergence-freeness of $\zeta^{-2}B_{Ib}$ as seen from (5.34)[30] and the fact that any 2-form Z_{ab} in vacuum satisfies $\nabla^a\nabla^b Z_{ab} = 0$. The potentials B_{Ib} will be compared to the symmetry operator (5.10b) of second type in section 5.4.2.

[30] The potentials B_{Ib} are of GHP weight $\{0,0\}$ and therefore we can use $\Theta_a = \nabla_a$.

5.2.1. Linearized gravity

For linearized gravity, Wald [115] used the self-adjoint operator

$$\mathcal{E}(h)_{ab} = -\Box\, h_{ab} - \nabla_a \nabla_b h^c{}_c + 2\nabla^c \nabla_{(a} h_{b)c} + g_{ab}(\Box h^c{}_c - \nabla^c \nabla^d h_{cd})\,. \tag{5.42}$$

He then continued the program for the linearized curvature components $\dot{\Psi}_0, \dot{\Psi}_4$, which solve the decoupled Teukolsky equations (3.39a) and (3.39e), respectively. The projection operators

$$\mathcal{T} : h_{ab} \to \dot{\Psi}_i\,, \tag{5.43}$$

mapping from linearized metric to curvature, are of second order in this case. Also the decoupling operators $\mathcal{S}$ are of second order so that the analog to the simple relation (5.13), saying that $\mathcal{E}$ corresponds to the linearized Bianchi identities, does not hold. However, Wald used (B.11), (B.12) of [32] as projection operators for the extreme components and reinterpreted equation (6.13) of that reference as the map $\mathcal{S}^\dagger$. So applying 2. of theorem 5.2.1 together with the fact that the operators in (3.39a) and (3.39e) are adjoint to each other[31] allowed him to write the maps

$$\dot{\Psi}_0 = D^4 \bar{\Psi}\,, \qquad \dot{\Psi}_4 = L^4 \bar{\Psi} + T\Psi\,, \tag{5.44}$$

with D, L, T first order differential operators, see [83] for details, and Ψ a solution of the Teukolsky equation (3.39e) for $\dot{\Psi}_4$. A new feature, compared to spin-1, is the term $T\Psi$, which can be shown to reduce to a multiple of ∂_t on Kerr spacetime in Boyer-Lindquist coordinates, see [83] and also [119] for the Schwarzschild case. In terms of symmetry operators, this map mixes first and second type.

Whether the method of adjoint operators can be applied to all curvature components is not obvious. The form of the operators might heavily depend on gauge conditions for the components $\dot{\Psi}_1, \dot{\Psi}_2, \dot{\Psi}_3$. For a discussion of the gauge issues in the method of adjoint operators, we refer to [116]. Chandrasekhar [29] even chose a gauge in which these components vanish identically. Nevertheless, we know from (4.45) that gauge-invariant information is contained once the right linear combination with lower order terms is used. We conclude this section with a form of the linearized Bianchi identities which most closely resembles the structure of Maxwell's equations,

$$\left(\mathrm{d}^\Theta - h\wedge\right)\Sigma_0 = (\kappa m - \sigma l) \wedge \Sigma_2 + O(\epsilon^2)\,, \tag{5.45a}$$

$$\left(\mathrm{d}^\Theta - \frac{1}{2}h\wedge\right)\Sigma_1 = O(\epsilon^2)\,, \tag{5.45b}$$

$$\left(\mathrm{d}^\Theta - h\wedge\right)\Sigma_2 = (\kappa'\bar{m} - \sigma' n) \wedge \Sigma_0 + O(\epsilon^2)\,. \tag{5.45c}$$

On a flat background, the RHS of all equations is $O(\epsilon^2)$, since the curvature Σ_I and the connection coefficients $\kappa, \kappa', \sigma, \sigma'$ vanish. The middle equation was used in section 4.3.2 to calculate the linearized mass.

It might also be worth thinking about this approach in terms of the linearized connection (potentially the bivector valued 1-forms σ_{ia} introduced in (2.17)) instead of the metric. Whether the Bianchi identities can in this case be reinterpreted as a self-adjoint operator on σ_{Ia} is not known to the author.

[31] More precisely, this is true after a rescaling by some power of ζ, analogous to (5.27a).

5.3. Fackerell-Ipser equation

The components $\phi_J = F_{ab} Z_J^{ab}$ of $F = \mathrm{d}B_I$ given in (5.34) are again solutions to $\mathcal{O}_J$. In particular $\mathcal{T}_I \mathcal{S}^{I\dagger}$ (no sum) maps solutions of $\mathcal{O}_I$ into solutions of $\mathcal{O}_I$. In [115], only the $I = 0, 2$ components were considered. The symmetry operator for the Fackerell-Ipser equation is the $I = 1$ component. We found $\mathcal{O}_1^\dagger = \mathcal{O}_1$, from which it follows that $\mathcal{T}_1 \mathcal{S}^{1\dagger}$ maps solutions of the Fackerell-Ipser equation into solutions.

Wald also pointed out that for real Maxwell fields one has to take the real part of (5.34). However, it follows from (5.31) that $\mathcal{S}^{I\dagger}(\psi_I)$ is always in the kernel of $\mathcal{T}_I$, so it is only the complex conjugate vector potential $\overline{B}_I$ which contributes to the new solution $\phi_I = \mathcal{T}_I \overline{\mathcal{S}^{I\dagger}(\psi_I)}$.

The decoupling operator (5.14) for the middle component reads

$$\begin{aligned}\mathcal{S}_1(j) = -\frac{1}{2}\zeta^2\big[&(\text{þ}' - \rho' - \bar{\rho}')(j_b l^b) - (\text{þ} - \rho - \bar{\rho})(j_b n^b) \\ &- (\eth' - \tau' - \bar{\tau})(j_b m^b) + (\eth - \tau - \bar{\tau}')(j_b \bar{m}^b)\big],\end{aligned} \tag{5.46}$$

and its adjoint (as can be read off from (5.34)) is

$$\mathcal{S}^\dagger_{1a}(\psi_1) = \frac{1}{2}\big[l_a \text{þ}' - n_a\, \text{þ} - m_a \eth' + \bar{m}_a\, \eth\,\big](\zeta^2 \psi_1)\,. \tag{5.47}$$

We then find

$$\begin{aligned}\mathcal{T}_1 \overline{\mathcal{S}_1^\dagger \psi_1} &= -Z_1^{ab}\Theta_a\left(\overline{Z_{1bc}}\Theta^c\left(\overline{\zeta^2\psi_1}\right)\right) \\ &= -\frac{1}{2}\left[(\text{þ}' + \rho' - \bar{\rho}')\,\text{þ} + (\eth + \tau - \bar{\tau}')\eth'\right](\overline{\zeta^2\psi_1})\,,\end{aligned} \tag{5.48}$$

which can be checked to be equivalent to the middle component of the symmetry operator of second type, (5.10b). Since it decouples from the other components, we have

Proposition 5.3.1. *Let ψ_1 be a solution of the Fackerell-Ipser equation, $\mathcal{O}_1\psi_1 = 0$. The anti-linear operator $\mathcal{Q}$, defined by*

$$\mathcal{Q}\psi_1 = -\big[(\rho' - \bar{\rho}')\,\text{þ} + \eth\,\eth'\big](\overline{\zeta^2\psi_1})\,, \tag{5.49}$$

maps into the space of solutions of the Fackerell-Ipser equation,

$$\mathcal{O}_1 \mathcal{Q}\psi_1 = 0\,. \tag{5.50}$$

Proof. Since ψ_1 solves the Fackerell-Ipser equation, we add $0 = \frac{1}{2}\overline{\mathcal{O}_1\psi_1}$ to (5.48),

$$\begin{aligned}\mathcal{Q}\psi_1 &= \mathcal{T}_1\overline{\mathcal{S}_1^\dagger\psi_1} + \frac{1}{2}\overline{\mathcal{O}_1\psi_1} \\ &= -\frac{1}{2}\Bigg[(\text{þ}' + \rho' - \bar{\rho}')\,\text{þ} + (\eth + \tau - \bar{\tau}')\eth' - (\text{þ}' - \rho' + \bar{\rho}')\,\text{þ} + (\eth + \bar{\tau} - \tau)\eth'\Bigg](\overline{\zeta^2\psi_1}) \\ &= -\big[(\rho' - \bar{\rho}')\,\text{þ} + \eth\,\eth'\big](\overline{\zeta^2\psi_1}).\end{aligned}$$

□

The Coulomb solution (4.2) is given by $\psi_1 \propto \zeta^{-2}$ and is therefore in the kernel of $\mathcal{Q}$. On a Schwarzschild background in a Carter tetrad in Boyer-Lindquist coordinates, ρ' and ζ are real. Since the Fackerell-Ipser operator $\mathcal{O}_1$ is also real in that case, $\mathcal{Q}$ reduces to the spherical Laplacian and a complex conjugation.

The middle component of the symmetry operator (5.10a) of first type reads

$$\chi_1 = 2\bar{\zeta}\left[-2\rho'\tau'\zeta\phi_0 - 2\rho\tau\zeta\phi_2 + (\eth' - \bar{\tau})(\eth - \tau)(\zeta\phi_1) + (\eth - \bar{\tau}')(\eth' - \tau')(\zeta\phi_1)\right] . \quad (5.51)$$

Here, we note the coupling to ϕ_0 and ϕ_2 in the first two terms, which have the same θ-dependence as the Coulomb mode (4.4). The Coulomb solution (4.2) is again in the kernel. On a Schwarzschild background, $\tau = 0 = \tau'$ and the couplings vanish. The symmetry operator reduces to the spherical Laplacian and leads to a separation of variables. In [79], a symmetry operator of first type for the middle component is derived for the restricted solution space generated by Debye potentials (this excludes at least the Coulomb mode).

5.4. Teukolsky master equation

In this section, we investigate the extreme components of the symmetry operators (5.10). For both types, these components decouple. The operator of first type leads us to a generalized Carter operator, as we will see in the next section. The operator of second type yields the Teukolsky-Starobinski identities and will be discussed further in section 5.4.2.

5.4.1. Generalized Carter operator and separation of variables

Due to the GHP prime operation (2.15a) it is sufficient to look at the χ_0 component of (5.10a). An expansion in GHP formalism yields

$$\begin{aligned}\chi_0 = \Big[&-\frac{1}{2}(\zeta + \bar{\zeta})\Psi_2 + \rho\rho'\zeta - \bar{\zeta}(\text{þ}\rho') - \frac{1}{2}\tau'(\zeta + \bar{\zeta})\,\eth + \frac{3}{2}\tau(\zeta - \bar{\zeta})\eth' \\ &+ \bar{\zeta}(\eth + \tau)\eth' + \frac{1}{2}\rho'(\zeta - \bar{\zeta})\,\text{þ} - \frac{1}{2}\rho(\zeta - \bar{\zeta})\text{þ}'\Big](\zeta\phi_0) \\ = {} & \zeta\bar{\zeta}(\eth - \tau - \bar{\tau}')(\eth' - 2\tau')\phi_0 + \frac{1}{2}(\zeta - \bar{\zeta})\mathcal{L}_\xi\phi_0 . \end{aligned} \quad (5.52)$$

Here, we used (2.58e) and the notion of a Lie derivative of a weighted scalar, given in (2.88a). This discussion is restricted to the Kerr-NUT class (2.72), where a Killing spinor (2.66) with real divergence (2.68) exists. On a Kerr background in Boyer-Lindquist coordinates $\mathcal{L}_\xi\phi = \partial_t\phi$ holds. Since both scalars, ϕ_0 and χ_0 are solutions of the $s = 1$ Teukolsky equation, (5.52) defines a second order symmetry operator. Comparison of (5.52) with (3.37) leads to a generalization to other spin weights, as we will now show. Writing the vacuum spin $s = 0, 1, 2$ Teukolsky equation in the form

$$T_s\phi = \left[(\text{þ} - 2s\rho - \bar{\rho})(\text{þ}' - \rho') - (\eth - 2s\tau - \bar{\tau}')(\eth' - \tau') - (2s^2 - 3s + 1)\Psi_2\right]\phi = 0 , \quad (5.53)$$

we state

Theorem 5.4.1 (Spin-s Carter operator)**.** *In the vacuum Kerr-NUT class, the operators*

$$\mathcal{R}_s = \zeta\bar{\zeta}(\text{þ} - \rho - \bar{\rho})(\text{þ}' - 2s\rho') + \frac{2s-1}{2}(\zeta + \bar{\zeta})\mathcal{L}_\xi \,, \tag{5.54a}$$

$$\mathcal{S}_s = \zeta\bar{\zeta}(\eth - \tau - \bar{\tau}')(\eth' - 2s\tau') + \frac{2s-1}{2}(\zeta - \bar{\zeta})\mathcal{L}_\xi \,, \tag{5.54b}$$

are symmetry operators for the spin-s Teukolsky equation $T_s\phi = 0$.

Corollary 5.4.2. *The Teukolsky operator* (5.53) *can be cast into the simple form*

$$\zeta\bar{\zeta}T_s = \mathcal{R}_s - \mathcal{S}_s \,, \tag{5.55}$$

with the symmetry operators $\mathcal{R}_s$ *and* $\mathcal{S}_s$ *given in* (5.54).

Proof. A calculation shows

$$\begin{aligned}
\frac{1}{\zeta\bar{\zeta}}(\mathcal{R}_s - \mathcal{S}_s) &= (\text{þ} - \rho - \bar{\rho})(\text{þ}' - 2s\rho') - (\eth - \tau - \bar{\tau}')(\eth' - 2s\tau') + \frac{2s-1}{\zeta}\mathcal{L}_\xi \\
&= (\text{þ} - \rho - \bar{\rho})(\text{þ}' - \rho') - (\eth - \tau - \bar{\tau}')(\eth' - \tau') \\
&\quad - (2s-1)\big[(\text{þ} - \rho - \bar{\rho})\rho' - (\eth - \tau - \bar{\tau}')\tau' + (\rho\text{þ}' - \rho'\text{þ} - \tau\eth' + \tau'\eth + s\Psi_2)\big] \\
&= (\text{þ} - \rho - \bar{\rho})(\text{þ}' - \rho') - (\eth - \tau - \bar{\tau}')(\eth' - \tau') \\
&\quad - (2s-1)\big[-\rho\rho' + \tau\tau' + \rho\text{þ}' - \tau\eth' + (s-1)\Psi_2\big] \\
&= (\text{þ} - 2s\rho - \bar{\rho})(\text{þ}' - \rho') - (\eth - 2s\tau - \bar{\tau}')(\eth' - \tau') - (2s-1)(s-1)\Psi_2 \,.
\end{aligned}$$

Here, (2.88a) with $q = 0$ is used in the second step and the GHP primed version of the Ricci identity (2.58e) is used in the third step. □

Proof of theorem 5.4.1. (incomplete). In the $s = 0$ case for the scalar wave equation, we find

$$\mathcal{R}_0 = \zeta\bar{\zeta}\Theta^a l_{(a}n_{b)}\Theta^b, \qquad \mathcal{S}_0 = \zeta\bar{\zeta}\Theta^a m_{(a}\bar{m}_{b)}\Theta^b \,, \tag{5.56}$$

as follows from (2.34). On solutions ϕ, we have $\mathcal{R}_0\phi = \mathcal{S}_0\phi$. On the other hand, the Carter operator (5.7) with $I = \operatorname{Im}\zeta, R = \operatorname{Re}\zeta$ on solutions can be rewritten

$$\begin{aligned}
\mathcal{Q}\phi &= \Theta^a I^2 l_{(a}n_{b)}\Theta^b\phi + \Theta^a R^2 m_{(a}\bar{m}_{b)}\Theta^b\phi \\
&= I^2\Theta^a l_{(a}n_{b)}\Theta^b\phi + R^2\Theta^a m_{(a}\bar{m}_{b)}\Theta^b\phi \\
&= [I^2 + R^2]\Theta^a l_{(a}n_{b)}\Theta^b\phi \\
&= \mathcal{R}_0\phi \,,
\end{aligned} \tag{5.57}$$

because of (2.72). The spin-1 operator was identified to be the 0-component (5.52) of the covariant symmetry operator (5.10a) at the beginning of this section.

For $s = 2$ linearized gravity on a Kerr background, the explicit form of the operators in a Kinnersley tetrad is given below, see (5.58). The commutation property $[\mathcal{R}_s, \mathcal{S}_s] = 0$ in coordinates follows immediately. It seems reasonable that these properties also hold in the Kerr-NUT class, but this has not been checked yet. □

A covariant second order symmetry operator of first type for linearized gravity is not known in the literature and the author does not expect such an operator to exist, because of the gauge dependence of the middle curvature components. From this point of view, the coordinate independent symmetry operators $\mathcal{R}_s$ and $\mathcal{S}_s$ are the most geometric characterizations of separability one can hope for.

It would be interesting to check a commutation property directly in GHP formalism, which might be possible (in finite time) with *GHPtools* developed by L. Price, see [94].

The half integer spin case is not investigated in this thesis, but it is plausible that the same splitting (5.55) holds, see [28] for the spin-$\frac{1}{2}$ case.

Using the Kinnersley tetrad (2.103), we find the coordinate form of the operators,

$$\begin{aligned} 2\mathcal{R}_s =& -\Delta^{-\mathrm{s}}\partial_r\Delta^{\mathrm{s}+1}\partial_r + \frac{[(r^2+a^2)\partial_t + a\partial_\varphi]^2}{\Delta} \\ &- \frac{2\mathrm{s}(r-M)}{\Delta}[(r^2+a^2)\partial_t + a\partial_\varphi] + 4\mathrm{s}r\partial_t - 2\mathrm{s}\,, \end{aligned} \tag{5.58a}$$

$$2\mathcal{S}_s = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{\left[a\sin^2\theta\partial_t + \partial_\varphi + \mathrm{is}\cos\theta\right]^2}{\sin^2\theta} - 4\mathrm{si}a\cos\theta\partial_t - \mathrm{s}\,. \tag{5.58b}$$

A simplification can be made for $\mathcal{R}_s$ after rescaling $\phi \to \Delta^{-\mathrm{s}/2}\phi$, which gives[32]

$$2\tilde{\mathcal{R}}_s = -\partial_r\Delta\partial_r + \frac{[(r^2+a^2)\partial_t + a\partial_\varphi - \mathrm{s}(r-M)]^2}{\Delta} + 4\mathrm{s}r\partial_t - \mathrm{s}. \tag{5.59}$$

The Teukolsky equation written in this form was used in [121] to prove mode stability of Kerr spacetime. Note that in other tetrads (and coordinates), the commutation property is less obvious, because r,θ-dependent transformations change also the variables (e.g. in the Carter tetrad).

Remark 5.4.3. *To avoid clutter in the notation, we wrote both operators $\mathcal{S}_s, \mathcal{R}_s$ with spin weight s dependence. It does not matter for explicit calculations, because $s = p/2 = b$ for spin-s field components, but the more natural choice would be to write*

$$0 = [\mathcal{R}_b - \mathcal{S}_s]\left(R_b(t,r,\varphi)S_s(t,\theta,\varphi)\right) \tag{5.60}$$

for the separated ansatz. The operator $\mathcal{R}_b$ would then be unaffected by complex conjugation (2.15b)*, which changes $p \leftrightarrow q, s \to -s, b \to b$. We also note that the Sachs star operation* (2.15c) *maps $\mathcal{R}_s \leftrightarrow \mathcal{S}_s$.*

Remark 5.4.4. *In [18] a symmetry operator for the scalar wave equation with only first order time derivatives is derived. Let us briefly review and generalize the result. Define $\rho = (r^2+a^2)^2\Delta^{-1}$ and $\sigma = a^2\sin^2\theta$ for the coefficients of the second time derivatives in* (5.58)*. It follows that $\mathcal{O}_s := A(\rho^{-1}\mathcal{R}_s - \sigma^{-1}\mathcal{S}_s)$ has only first order time derivatives for any function A. With $\overline{\Sigma} = \rho - \sigma$ we find for the choice $A = \rho\sigma/\overline{\Sigma}$,*

$$\begin{aligned} \overline{\Sigma}\left[\mathcal{O}_s, \tfrac{\Sigma}{\overline{\Sigma}}T_s\right] &= \overline{\Sigma}\left[\tfrac{1}{\overline{\Sigma}}(\sigma\mathcal{R}_s - \rho\mathcal{S}_s), \tfrac{1}{\overline{\Sigma}}(\mathcal{R}_s - \mathcal{S}_s)\right] \\ &= \left\{\sigma(\mathcal{R}_s\tfrac{1}{\overline{\Sigma}}) - \rho(\mathcal{S}_s\tfrac{1}{\overline{\Sigma}}) - \mathcal{R}_s(\tfrac{\sigma}{\overline{\Sigma}}) + \mathcal{S}_s(\tfrac{\sigma}{\overline{\Sigma}})\right\}\mathcal{R}_s \\ &\quad + \left\{-\sigma(\mathcal{R}_s\tfrac{1}{\overline{\Sigma}}) + \rho(\mathcal{S}_s\tfrac{1}{\overline{\Sigma}}) + \mathcal{R}_s(\tfrac{\rho}{\overline{\Sigma}}) - \mathcal{S}_s(\tfrac{\rho}{\overline{\Sigma}})\right\}\mathcal{S}_s \\ &= \left\{-\overline{\Sigma}(\mathcal{S}_s\tfrac{1}{\overline{\Sigma}}) + \tfrac{1}{\overline{\Sigma}}(\mathcal{S}_s\sigma)\right\}\mathcal{R}_s + \left\{\overline{\Sigma}(\mathcal{R}_s\tfrac{1}{\overline{\Sigma}}) + \tfrac{1}{\overline{\Sigma}}(\mathcal{R}_s\rho)\right\}\mathcal{S}_s \\ &= 0\,, \end{aligned}$$

[32] This is part of the transformation to the Carter tetrad, see (2.107).

because $\mathcal{R}_s\sigma = 0 = \mathcal{S}_s\rho$, $\mathcal{R}_s\frac{1}{\overline{\Sigma}} = -\frac{\mathcal{R}_s\rho}{\overline{\Sigma}^2}$, *by Leibniz rule* $\mathcal{S}_s\frac{1}{\overline{\Sigma}} = \frac{\mathcal{S}_s\sigma}{\overline{\Sigma}^2}$ *and* $[\mathcal{R}_s, \mathcal{S}_s] = 0$. *The result of [18] is the special case* $s = 0$ *for the function spaces with* $e^{\mathrm{i}m\varphi}$ *dependence of the azimutal coordinate* φ.

5.4.2. Teukolsky-Starobinski identities

In the last sections, we investigated the middle components of both types of symmetry operators and the extreme component of the symmetry operator of first type. The remaining components of the second type operator are well known in the literature and will be discussed in this section.

Expansion of the potential $B_{AA'}$ given in (5.11c) and comparison with the potentials B_{Ib} given in (5.34) yields the simple relation

$$B_{AA'} = B_{0a} + B_{2a}\,, \tag{5.61}$$

which shows the equivalence of the symmetry operators (5.10b) and $\mathcal{T}_J\overline{\mathcal{S}_I^\dagger}$ given in theorem 5.2.1. We learned in proposition 5.2.5 that each scalar ϕ_I, solving $\mathcal{O}_I\phi_I = 0$, can be used to generate a whole Maxwell field and that the resulting fields are equivalent. This is also known as the Debye potential approach, see [33]. Expanding the operators $\mathcal{T}_I\overline{\mathcal{S}^{J\dagger}}$, we find

$$\begin{aligned}
\omega_0 &= -(\eth - \bar\tau')(\eth + \bar\tau')(\overline{\zeta^2\phi_0}) &&= -\eth\,\eth(\overline{\zeta^2\phi_0}) && (5.62a)\\
&= [(\eth - \bar\tau')\,\text{þ} + (\text{þ} - \bar\rho)\,\eth](\overline{\zeta^2\phi_1}) &&= 2(\eth - \bar\tau')\,\text{þ}(\overline{\zeta^2\phi_1}) && (5.62b)\\
&= -(\text{þ} - \bar\rho)(\text{þ} + \bar\rho)(\overline{\zeta^2\phi_2}) &&= -\text{þ}\,\text{þ}(\overline{\zeta^2\phi_2})\,, && (5.62c)\\
\omega_1 &= [-\text{þ}'(\eth + \bar\tau') - (\tau - \bar\tau')(\text{þ}' + \bar\rho')](\overline{\zeta^2\phi_0}) &&= [-\text{þ}'\,\eth - \tau\text{þ}'](\overline{\zeta^2\phi_0}) && (5.62d)\\
&= [(\text{þ}' + \rho' - \bar\rho')\,\text{þ} + (\eth + \tau - \bar\tau')\eth'](\overline{\zeta^2\phi_1}) && && (5.62e)\\
&= [-\text{þ}(\eth' + \bar\tau) - (\tau' - \bar\tau)(\text{þ} + \bar\rho)](\overline{\zeta^2\phi_2}) &&= [-\text{þ}\eth' - \tau'\text{þ}](\overline{\zeta^2\phi_2})\,, && (5.62f)\\
\omega_2 &= -(\text{þ}' - \bar\rho')(\text{þ}' + \bar\rho')(\overline{\zeta^2\phi_0}) &&= -\text{þ}'\text{þ}'(\overline{\zeta^2\phi_0}) && (5.62g)\\
&= [(\text{þ}' - \bar\rho')\eth' + (\eth' - \bar\tau)\text{þ}'](\overline{\zeta^2\phi_1}) &&= 2(\eth' - \bar\tau)\text{þ}'(\overline{\zeta^2\phi_1}) && (5.62h)\\
&= -(\eth' - \bar\tau)(\eth' + \bar\tau)(\overline{\zeta^2\phi_2}) &&= -\eth'\eth'(\overline{\zeta^2\phi_2})\,. && (5.62i)
\end{aligned}$$

After a separation of variables $\phi_I = R_I(r)S_I(\theta)e^{-\mathrm{i}\omega t}e^{\mathrm{i}m\varphi}$, $I = 0, 2$ in Boyer-Lindquist coordinates, e.g. (5.62a) leads to the Teukolsky-Starobinski identity

$$S_0(\theta)e^{-\mathrm{i}\omega t}e^{\mathrm{i}m\varphi} = -\eth\,\eth\left[S_2(\theta)e^{\mathrm{i}\omega t}e^{-\mathrm{i}m\varphi}\right], \tag{5.63}$$

relating S_0 and S_2 via a second order differential operator. Another Teukolsky-Starobinski identity relating R_0 and R_2 follows from (5.62c). Such identities exist for all spins s and the operators are of order $2s$. In particular the Teukolsky-Starobinski identities for linearized gravity are of order four and thus difficult to analyze covariantly.

The Sasaki-Nakamura transformation, see [101], maps the separated radial Teukolsky equation for spin-2 into a "Regge-Wheeler like" equation with short range potential. It would be interesting to see how it relates to (5.62a) in the spin-1 case, see [68], and whether this transformation can be generalized to the full Teukolsky variable.

Summarizing this chapter, we have the following picture. For the spin-1 equation in the vacuum Kerr-NUT class a non-trivial second order symmetry operator of each type does exist. Non-trivial here means that it is not the product of Lie derivatives along isometries. The symmetry operator of first kind leads (together with the isometries) to a separation of variables for the Teukolsky equations and generalizes the spin-0 Carter operator. The middle component does not decouple, which is consistent with the non-separability of the Fackerell-Ipser equation. The symmetry operator of second kind does decouple for all three components. It leads to the (anti-linear) symmetry operator (5.49) for the middle component and to the Teukolsky-Starobinski identities for the extreme components. In section 5.4.1 we were able to generalize the extreme components of the symmetry operator of first type to spin-2. We do not expect this result to generalize to a covariant operator for linearized gravity, because the linearized Bianchi identities couple to lower order terms. It is only the Teukolsky system, defined in terms of the extreme components, which is mostly unaffected.

6. Stability analysis

This chapter is intended to review some results about black hole perturbations. In section 6.1 we review the vector field method, present basic examples and explain how the results of the preceding chapters may be integrated into this scheme.

Let us first review the situation for a Schwarzschild background. As described in section 3.4, the linearized metric can be characterized by the two functions $Q^{\pm}$. With the separated ansatz $Q^{\pm}(t,r) = e^{-\mathrm{i}\omega t}R^{\pm}(r)$, the Regge-Wheeler and Zerilli equations (3.33) become second order ordinary differential equations. Stability in the sense of non-existence of exponential growing modes (with negative imaginary part in ω) was shown e.g. by Moncrief in [84]. In [117], Wald proofed the stronger result that $Q^{\pm}(t,r)$ is pointwise bounded in terms of initial data. The proof relies on a conserved energy for the field.

This shows some ambiguity in the usage of the word "stability". However, the ultimate notion, considering the full Einstein equations, is non-linear stability. From the physical point of view, a small disturbance of some equilibrium state should "decay". More precisely, we quote from a recent talk[33] presented by Dafermos:

Conjecture 1 (Stability of Kerr)**.** *Let $(\Sigma, \bar{g}, K)$ be a vacuum initial data set sufficiently close to the initial data on a Cauchy hypersurface Σ in the Kerr solution $(\mathcal{M}, g_{M_i,a_i})$ for some parameters $0 \leq |a_i| < M_i$. Then the maximal Cauchy development $(\mathcal{M}, g)$ possesses a complete null infinity $\mathcal{I}^+$ such that the metric restricted to $J^-(\mathcal{I}^+)$ approaches a Kerr solution $(\mathcal{M}, g_{M_f,a_f})$ in a uniform way with quantitative decay rates, where M_f, a_f are near M_i, a_i respectively.*

Here M_i, a_i and M_f, a_f are the initial and final parameters of the Kerr solutions, respectively. Furthermore, Σ is a Cauchy hypersurface, $\bar{g}$ the induced metric, K the second fundamental form and $J^-(\mathcal{I}^+)$ the causal past of future null infinity.

An important analytical tool based on energy estimates was developed in the monumental proof of non-linear stability of Minkowski spacetime by Christodoulou and Klainerman in [31]. Several groups are working towards a solution of the Kerr stability problem by generalizing and extending this tool. It is nevertheless essential to understand linear fields first.

Linear gravitational perturbations of Kerr spacetime are most commonly described by the Teukolsky equations (3.39a), (3.39e) for the linearized, gauge invariant curvature scalars $\dot{\Psi}_0$ and $\dot{\Psi}_4$. In fact it is enough to control one of the scalars, as shown by Wald in [113]. Due to the separability property (5.55), the ansatz

$$\phi_s = e^{-\mathrm{i}\omega t} e^{\mathrm{i}m\varphi} R_s(r) S_s(\theta) \tag{6.1}$$

is usually made. The second order radial ordinary differential equation for $R_s(r)$ has eigenvalues depending on ω and m. In [93], Teukolsky analyzed the stability numerically up to the $l = 4$ spheroidal mode and concluded "good evidence of stability".

[33]Notes from the Kerr conference, 4th – 5th July 2013 at the Albert Einstein Institute, Golm.

A rigorous result is the proof of mode stability for the Kerr black hole by Whiting, [121]. In this reference, the Teukolsky equation (5.53) for $s = 2$ with the separated ansatz (6.1) is analyzed. It is noted that no conserved energy exists, because of the ergo-region, and that the radial differential equation cannot be related to a scalar wave equation as in the Schwarzschild case. After a differential transformation of $S_s(\theta)$ and an integral transformation of $R_s(r)$ is applied, a conserved energy for the new field is found, from which the existence of exponentially growing modes of the original variable can be excluded. The method was recently extended in [102] to show instabilities of solutions to the Klein-Gordon equation on Kerr spacetime.

6.1. The vector field method

In this section, we review some basics about energy estimates. First the idea of using conformal isometries on Minkowski spacetime to control solutions of spin-s equations is presented in section 6.1.1 and afterwards the achievements and obstructions on curved spacetimes are reviewed in section 6.1.2. The first part is based on [30], [5] and [98].

Classically, asymptotic behavior of solutions of differential equations were analyzed from its fundamental solution. The vector field method is able to extract information about boundedness and decay without referring to explicit solutions. It enables the analyses of late time behavior in terms of the initial data. We need the following setup. The globally hyperbolic spacetime is topologically $\mathbb{R} \times \Sigma$ and admits a time-function t, parameterizing the family of spatial hypersurfaces $\Sigma_t = \Sigma(t)$. Let u be a real solution to a well-posed initial value problem (6.2a) with given data $u_{t=0} = u_0, \partial_t u_{t=0} = \dot{u}_0$ on Σ_0. Suppose further that it admits a symmetric, trace-free and divergence-free energy-momentum tensor (6.2b), which satisfies the dominant energy condition $T_{ab}Y^aZ^b \geq 0$ for any time-like, future directed vector fields Y^a, Z^a. We define the momentum[34] with respect to a vector field X^a by (6.2c) and its energy at time t by (6.2d). Here, $\mathrm{d}\mu^a = n^a \mathrm{d}^3\mu$ with $\mathrm{d}^3\mu$ the induced volume element on Σ_t and n^a the future directed unit normal, see [118, Appendix B.2]. Fundamental properties for the following discussion are positivity and monotonicity of such energies.

Evolution equation	$\mathcal{E}u = 0\,,$	(6.2a)
Energy-momentum tensor of u	$T_{ab} = T[u]_{ab}\,,$	(6.2b)
Momentum of u in X^a direction	$P_a = T_{ab}X^b\,,$	(6.2c)
Energy	$E(t) = \int_{\Sigma_t} P_a \mathrm{d}\mu^a\,,$	(6.2d)
Divergence	$\nabla^a P_a = T_{ab}\pi^{ab}\,,$	(6.2e)
Deformation tensor of X^a	$\pi^{ab} = \nabla^{(a}X^{b)}\,.$	(6.2f)

Integrating (6.2e) over a region $[t_0, t_1] \times \Sigma$ yields by the Gauss theorem $E(t_1) - E(t_0)$ on the left hand side, if sufficient fall-off guarantees vanishing integrals in the asymptotic regions. The bulk integral on the right hand side is the energy-momentum tensor contracted

[34] In this simplified notation we suppress the dependence of the energy momentum tensor on u and the momentum on X^b and u etc.. When several different solutions to (6.2a) and different vector fields X^a are discussed below, the additional dependencies will be added.

into the deformation tensor (6.2f). This contraction vanishes if X^a is conformal Killing (2.42), so the energy (6.2d) is conserved in this case. Other vector fields X^a of crucial importance are dispersive (or Morawetz) vector fields discussed below. In that case the bulk term does not vanish, but the goal is to control it via other (conserved) energies.

Let us have a look at the important case of the scalar wave equation $\Box u = 0$ with energy-momentum tensor

$$T_{ab} = \nabla_a u \nabla_b u - \tfrac{1}{2} g_{ab} \nabla^c u \nabla_c u \,. \tag{6.3}$$

It satisfies the above requirements except for trace-freeness. This means in particular that pure-trace deformation tensors (6.2f) lead to non-vanishing divergences (6.2e). However, if the momentum is modified according to

$$\hat{P}_a = P_a + qu\nabla_a u - \tfrac{1}{2} u^2 \nabla_a q \,, \tag{6.4}$$

for some function q, we find

$$\begin{aligned} \nabla^a \hat{P}_a &= T_{ab}\pi^{ab} + q\nabla^a u \nabla_a u - \tfrac{1}{2} u^2 \Box q \\ &= \left(\tfrac{1}{4}\pi - \tfrac{1}{2}\pi + q\right) \nabla^a u \nabla_a u + \pi^{ab}_{\text{tf}} \nabla_a u \nabla_b u - \tfrac{1}{2} u^2 \Box q \,, \end{aligned} \tag{6.5}$$

with $\pi = \pi^{ab} g_{ab}$ and the decomposition into trace and trace-free part, $\pi^{ab} = \frac{1}{4} g^{ab}\pi + \pi^{ab}_{\text{tf}}$. So if X^a is conformal Killing, $\pi^{ab} = \frac{\pi}{4} g^{ab}$ is pure-trace and with $q = \frac{\pi}{4}$ the momentum is divergence free ($\Box q = 0$ follows from the conformal Killing equation (2.42)). The alternative choice $q = 0$ leads to $-\frac{\pi}{4}|\nabla u|^2$ in the bulk, which is a multiple of the usual energy. Examples will be discussed in section 6.1.1 below.

Remark 6.1.1 (Method of multipliers)**.** *An alternative way of constructing momenta* (6.2c) *is the method of multipliers. For a vector field X^a, introduce the multiplier $Xu = X^a \nabla_a u = \mathcal{L}_X u$ such that $(Xu)(\mathcal{E}u) = \nabla^a \tilde{P}_a + p$. It should be noted that multipliers can also be understood in terms of adjoint operators as discussed in section 5.2. Since for solutions u, v of $\mathcal{E}u = 0$ and $\mathcal{E}^\dagger v = 0$, we have the conserved momentum*

$$0 = v\mathcal{E}u - u\mathcal{E}^\dagger v = \nabla_a \tilde{P}^a \,, \tag{6.6}$$

the multiplier $Xu = v$ yields the above result with $p = u\mathcal{E}^\dagger(Xu)$. For self-adjoint operators $\mathcal{E}^\dagger = \mathcal{E}$ and symmetry operators X, p vanishes.

For the (self-adjoint) scalar wave operator $\Box$, the momentum reads

$$\tilde{P}^a = (Xu)\nabla^a u - u\nabla^a (Xu) \,. \tag{6.7}$$

Because the Lie derivative along conformal Killing vectors is a symmetry operator in the sense of section 5.1, $p = 0$ and the momentum is conserved. So it has similar properties to (6.4)*.*

If symmetry operators $\mathcal{S}_i$ for (6.2a) exist, we have $\mathcal{E}(\mathcal{S}_i u) = 0$ and the higher order energies

$$E[\mathcal{S}_i u](t) \,, \tag{6.8}$$

can be constructed from momenta (6.2c) for the energy-momentum tensor $T[\mathcal{S}_i u]_{ab}$. In case of the scalar wave equation, examples of symmetry operators are Lie derivatives along isometries. This additional information (energy-strengthening property) can in some cases be used to control the field pointwise.

6.1.1. Linear fields on Minkowski spacetime

On Minkowski spacetime, we use global cartesian coordinates x^a and a slicing with respect to the time function $x^0 = t$. $x^i, i = 1,2,3$ are the coordinates of the spatial hypersurface and we set $r = \sqrt{x^i x^i}$. The space of conformal Killing vectors on Minkowski spacetime in 3+1 dimensions is 15-dimensional. It consists of four translations and six Lorentz transformations,

$$\mathcal{T}_\alpha = \frac{\partial}{\partial x^\alpha}\,, \qquad \mathcal{L}_{\alpha\beta} = x_\alpha \frac{\partial}{\partial x^\beta} - x_\beta \frac{\partial}{\partial x^\alpha}\,, \tag{6.9}$$

which form the 10-dimensional space of Poincaré isometries, a scaling vector field and four conformal accelerations given by

$$\mathcal{S} = x^\alpha \mathcal{T}_\alpha\,, \qquad \mathcal{K}_\alpha = -2x_\alpha \mathcal{S} + x^\beta x_\beta \mathcal{T}_\alpha\,. \tag{6.10}$$

In this set, the only time-like, future-directed vectors are $\mathcal{T}_0$ and $\mathcal{K}_0$ with the squared norms 1 and $(t+r)^2(t-r)^2$, respectively. We find in the null tetrad

$$l^a = \tfrac{1}{\sqrt{2}}(\partial_t + \partial_r) = \sqrt{2}\partial_{u_+}, \quad n^a = \tfrac{1}{\sqrt{2}}(\partial_t - \partial_r) = \sqrt{2}\partial_{u_-}, \quad m^a = \tfrac{1}{\sqrt{2}r}(\partial_\theta + \tfrac{\mathrm{i}}{\sin\theta}\partial_\varphi), \tag{6.11}$$

that $\mathcal{T}_0 = \partial_{u_+} + \partial_{u_-}$ and $\mathcal{K}_0 = u_+^2\partial_{u_+} + u_-^2\partial_{u_-}$.

The method outlined above is applied to the spin-$s = 0,1,2$ equations in [30]. We discussed the modified momentum (6.4) for the scalar wave equation $\Box u = 0$, which can be made divergence free for conformal Killing vectors X^a. With the vector fields $X = \mathcal{T}_0$ and $X = \mathcal{K}_0$, the conserved and positive energy and conformal energy of the scalar wave equation are

$$E_{\mathcal{T}_0}(t) = \tfrac{1}{2}\int_{\Sigma_t} |\partial_t u|^2 + |\partial_i u|^2 \mathrm{d}^3x = \tfrac{1}{2}\int_{\Sigma_t} |\nabla u|^2\,\mathrm{d}^3x\,, \tag{6.12a}$$

$$E_{\mathcal{K}_0}(t) = \int_{\Sigma_t} (r^2+t^2)\,|\nabla u|^2 + 4rt(\partial_r u)(\partial_t u) + 4tu\partial_t u - 2u^2\,\mathrm{d}^3x\,. \tag{6.12b}$$

Here, we added the vector field as a subscript to the energy (6.2d). The form of the conformal energy follows from (6.4) and $\mathcal{K}_0 = (r^2+t^2)\partial_t + 2rt\partial_r$ in spherical coordinates. It leads to the weak decay result

$$\left(\int_{r\le 1+t/2} |\nabla u|^2 \mathrm{d}^3x\right)^{1/2} = O(t^{-1})\,, \tag{6.13}$$

inside the light-cone, as shown in [85]. The result can be improved by using other vector fields, e.g. the Morawetz vector field $X = r\partial_r$, see [85, 86] for applications in obstacle problems and higher order energies (6.8). For more general equations, vector fields different from the conformal isometries or curved backgrounds, conserved energies (6.2d) cannot be constructed a priori. The task is to estimate the bulk in terms of initial data, which leads to energy inequalities.

The next step is to control the field pointwise. A powerful tool for this is the Klainerman-Sobolev inequality,

$$(1+t+r)^2(1+|r-t|)|u(t,x)|^2 \le C\sum_{k\le 2} \|Z^k u(t,x)\|^2_{L^2_x}\,. \tag{6.14}$$

Here $Z \in \{\mathcal{T}_\alpha, \mathcal{L}_{\alpha\beta}, \mathcal{S}\}$ are symmetry operators and $u \in C^\infty([0,\infty) \times \mathbb{R}^3)$ with suitably fast fall-off for $r \to \infty$. The Sobolev inequality can be understood as the special case with $Z \in \{\mathcal{T}_\alpha\}$. This gives pointwise control over u in terms of higher order energies (6.8). If the right hand side can be controlled in terms of energies on the initial slice, this means in particular that the field is controlled pointwise.

For a spin-1 field ϕ_{AB}, an energy-momentum tensor is given by

$$\begin{aligned} T_{ab} &= \phi_{AB}\bar{\phi}_{A'B'} \\ &= |\phi_0|^2 n_a n_b + 2|\phi_1|^2 (l_{(a}n_{b)} + m_{(a}\bar{m}_{b)}) + |\phi_2|^2 l_a l_b + \phi_2\bar{\phi}_0 m_a m_b + \phi_0\bar{\phi}_2 \bar{m}_a\bar{m}_b \\ &\quad - 2\phi_1\bar{\phi}_0 m_{(a}n_{b)} - 2\phi_1\bar{\phi}_2 l_{(a}\bar{m}_{b)} - 2\bar{\phi}_1\phi_0 \bar{m}_{(a}n_{b)} - 2\bar{\phi}_1\phi_2 l_{(a}m_{b)} \,. \end{aligned} \tag{6.15}$$

It is symmetric, trace-free and divergence-free by construction. It also satisfies the dominant energy condition, as follows from $\phi_{AB}\phi^B{}_C = (\phi_0\phi_2 - \phi_1^2)\epsilon_{AC}$. In the null tetrad (6.11) we find the components

$$2T(\mathcal{T}_0, u_+^n\partial_{u_+} + u_-^n\partial_{u_-}) = u_+^n|\phi_0|^2 + (u_+^n + u_-^n)|\phi_1|^2 + u_-^n|\phi_2|^2 \,.$$

These are the integrands for the $\mathcal{T}_0$-energy ($n = 0$) and $\mathcal{K}_0$-energy ($n = 2$) and decay rates for the components can be extracted as for the scalar wave equation above.

For the spin-2 field Ψ_{ABCD}, the Bel-Robinson tensor

$$T_{abcd} = \Psi_{ABCD}\bar{\Psi}_{A'B'C'D'} \,, \tag{6.16}$$

is the appropriate generalization of an energy-momentum tensor. It is totally symmetric, trace-free on all index pairs and divergence free by construction. Also a dominant energy condition of the form $T_{abcd}W^aX^bY^cZ^d \geq 0$ for time-like vectors W^a, X^b, Y^c, Z^d can be derived. This implies that the energy densities

$$T(\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0)\,, \quad T(\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{K}_0)\,, \quad T(\mathcal{T}_0,\mathcal{T}_0,\mathcal{K}_0,\mathcal{K}_0)\,, \quad T(\mathcal{T}_0,\mathcal{K}_0,\mathcal{K}_0,\mathcal{K}_0)\,,$$

lead to positive, conserved energies and decay rates for the components can be calculated. We also want to point out that Maxwell-like energy-momentum tensors for the spin-lowered curvature (3.27) with the full set of Minkowski Killing spinors can be constructed. These include in particular the above energy densities. The detailed relations to the Bel-Robinson energies are investigated in [75].

An alternative approach to prove decay rates of spin-s fields on Minkowski spacetime based on Hertz potentials can be found in [8]. It is based on decay results for the scalar wave equation and uses the Hertz potential map to deduce decay results for higher spin fields.

6.1.2. Challenges on Kerr spacetime

Several aspects of the Kerr background prevent the direct application of the vector field method as introduced in the last section. Let us first review the additional difficulties for the scalar wave equation along the lines of [9] and afterwards add a list of additional difficulties for higher spin fields.

The first immediate obstruction is a lack of symmetries in the background. Only the isometries ∂_t and ∂_φ corresponding to the stationarity and axial symmetry are left. In [9] the hidden symmetry of the Kerr background, characterized by the Killing tensor (2.76c), is added to the list and used to prove boundedness and decay for the scalar wave equation for small a. A similar result based on Fourier methods can be found in [38]. Recently the results were proven for the whole range $a < M$ in [41].

The Killing vector ∂_t changes sign at the boundary of the ergo-region (2.95) outside the horizon. So the usual energy, built from T_{ab} contracted into ∂_t and integrated, is not positive definite anymore. However, a vector field $T = \partial_t + \chi\omega_H\partial_\varphi$ with $\omega_H = a/(r_+^2 + a^2)$, a sufficiently small and $\chi(r)$ a compactly supported smooth function is time-like everywhere and Killing outside the set $\{\partial_r\chi \neq 0\}$. The non-vanishing bulk in that region can be controlled by a Morawetz estimate. The absence of a positive, conserved energy is strongly related to the effect of super-radiance.

The third obstruction is *trapping*. On a Schwarzschild background, null geodesics have stationary orbits at the photon-sphere $r_{ph} = 3M$. For fields this corresponds to arbitrarily slow dispersing wave packets, which are "trapped". The Morawetz vector field has to change sign at this photon sphere, to always point into the direction of dispersion. The trapping on Kerr spacetime is more complicated. There is an open set of $r =$ const. null geodesics described in (2.96). In the equatorial plane, there are exactly two orbits, one direct orbit at r_{ph+} and one retrograde orbit at r_{ph-} given by (2.97). The Carter constant is zero in both cases and positive inside this interval with a maximum at $r = 3M$. A non-zero Carter constant implies non-vanishing momentum orthogonal to the equatorial plane. We note that even though the $r =$ const. null geodesics are radially unstable, there is a (r,θ)-dependent direction, which points to another member of this family. It would be interesting to investigate a spatial orthogonal direction of maximal dispersion for a generalized Morawetz vector field.

The next step is to investigate more complicated fields. As mentioned in the introduction, recent results for the Maxwell field can be found in [10]. There, it was essential to control the middle component of the field-strength via an energy for the Fackerell-Ipser equation (3.43b). This, together with a charge-vanishing condition (4.4), results in a proof of boundedness and decay of solutions to the Maxwell equations on slowly rotating Kerr backgrounds. A major problem is the complex potential in the Fackerell-Ipser equation (3.43b) in connection with trapping. A uniform bound on the energy is proved in [11] using a Fourier transformation in t. The idea of controlling the field energy via an energy for the middle component is generalized from an earlier result [21] on the Schwarzschild background.

The linearized Bel-Robinson tensor is not conserved anymore on a curved background, because of linearized connection terms coupling to the background curvature. The construction of conserved energies in this setup is quite subtle and still under investigation.

7. Conclusions

7.1. Summary

In this thesis, we developed certain aspects of Maxwell and linearized gravitational fields on vacuum spacetimes of Petrov type D. The main motivation was a generalization of the vector field method towards a proof of Kerr stability. Properties of the scalar wave equation on this background are by now quite well understood. This motivated us to treat higher spin fields starting with the scalar wave equation (with potential) for their middle components. These are the Fackerell-Ipser equation (3.43b) for spin-1 and the generalized Regge-Wheeler equation (3.39c) for linearized gravity presented in chapter 3. The gauge freedom of the tetrad based formulation is discussed and the concept of gauge-source-function for the linearized gravitational field is used. The choice of gauge (3.61) for the generalized Regge-Wheeler equation $(\square + 8\Psi_2)(\zeta^2\dot{\Psi}_2) = 0$ is natural in the sense that the gauge-source-function does not depend on the linearized curvature and that the $l = 0, 1$ modes are non-radiating.

These non-radiating modes, which we analyzed in chapter 4, are obstructions to decay. Non-radiating modes on Minkowski spacetime were reviewed and the spin-2 field on type D backgrounds was discussed. Working in terms of linearized curvature, we derived the linearized mass charge (4.41) corresponding to the time translation isometry for Petrov type D backgrounds by using Penrose's idea of spin-lowering with a Killing spinor. A quasi-local angular momentum charge in terms of linearized curvature has not yet been found and obstructions for the construction were discussed. It seems that an assumption about the existence of a regular potential (metric) has to be used implicitly. Assuming the existence of such a potential on a Schwarzschild background, a partial integration on the sphere yields the charge (4.80) in terms of linearized curvature. We also analyzed the Komar integrals for Kerr spacetime as an alternative to spin-lowering, because a Killing spinor for the axial isometry does not exist. This led to the quite complicated integrand (4.30), which has not yet been generalized to the linearized setting. Another additional difficulty, compared to the Minkwoski case, is the gauge dependence of $\dot{\Psi}_2$. For a Schwarzschild background, gauge conditions are known which eliminate the gauge dependent non-radiating modes [123, 74]. A generalization to Kerr spacetime needs further work.

To generate higher order energies (6.8), a good understanding of the symmetry operators for the equation of interest is important. The two isometries of Kerr spacetime yield first order symmetry operators. That this also works for weighted scalars by taking the Lie derivative followed from (2.88b). We reviewed in chapter 5 the complete set of second order symmetry operators for the spin-1 equations on a vacuum type D spacetime. The component forms of the operators (5.10) of first and second type are related to certain well known aspects of the equations. So do the extreme components of the operator of first kind lead to separation of variables. The middle component does not decouple and the Fackerell-Ipser equation is not separable (as far as we know). The zeroth order coupling looks quite similar to the integrand of the Coulomb charge integral (4.4). For Schwarzschild spacetime, the coupling vanishes and the operator reduces to the spherical Laplacian. The operator of second kind decouples in all components and is equivalent to the results of the method of adjoint operators. It leads to the Teukolsky-Starobinski identities for the extreme components (after a separation of variables is performed) and the anti-linear symmetry operator (5.49) for the middle component. This operator for the middle equation has not yet been analyzed in detail. In the Schwarzschild case, it reduces to the spherical Laplacian (so there is a certain degeneracy of the symmetry operators in spherical symmetry) and a complex conjugation. We also point out that there are no further (non-trivial) symmetry operators of first and second order. In particular, we showed that a linear combination of the above results leads to the result of Beyer [18] (and its higher spin generalization).

For linearized gravity the situation seems to be similar. We generalized the Carter operator in GHP form for the extreme components in (5.54). It is responsible for the separation in Boyer-Lindquist r, θ variables. The existence of a covariant second order symmetry operator behind this result seems rather unlikely to us, because of the gauge ambiguities of the other components. The analog of the spin-1 second type symmetry operator are the fourth order Teukolsky-Starobinski identities for $\dot{\Psi}_0$ and $\dot{\Psi}_4$. An analysis of the other components in terms of the adjoint operator method of section 5.2 has not yet been done.

To summarize, the main results of the thesis are:

1. An analysis of the Lie derivative of weighted fields along isometries on vacuum Kerr-Nut spacetimes, which yield
   - algebraic identities (2.74) between curvature and connection.
   - simplifications (2.88b), (2.89) of the Lie derivative on weighted scalars.
   - a reformulation of the Komar integrals (4.29), (4.30) (partly) in terms of curvature.
2. The formulation (4.46) of linearized mass on type D backgrounds.
3. An anti-linear symmetry operator (5.49) for the Fackerell-Ipser equation.
4. A coordinate independent second order symmetry operator (5.54) for the Teukolsky equations, responsible for separation of variables.

Certain minor results are:

1. A construction of geometric coordinates (2.124) directly from the Killing spinor in section 2.5.2.
2. A generalization of the GHP-formalism to tensorial fields, in particular bivectors in section 2.2.
3. Identification of obstructions to a quasi-local angular momentum charge in terms of linearized curvature in section 4.3.3.

4. The second order symmetry operator of [18] is rederived in a simplified way and generalized to spin-s fields in section 5.4.1.

5. Wald's method of adjoint operators has been cast into a more systematic component form in section 5.2, which might have interesting applications in the linearized gravity case.

7.2. Outlook

There are still various problems that need to be solved for a stability proof along the lines outlined in this thesis. To derive a Morawetz estimate for the generalized Regge-Wheeler equation along the lines of [10, Section 6.3] a characterization of the linearized angular momentum charge and a gauge condition ensuring the vanishing of gauge modes (initially) is needed. Because the linearized mass charge (4.41) already couples to the linearized tetrad, it seems that the whole system of linearized tetrad, connection and curvature has to be treated simultaneously. The linearization of the system given in [54] including the gauge-source-functions is currently investigated.

During the course of this thesis further (independent) ideas emerged, which might also be worth pursuing. The reader is warned that some of them are quite speculative.

The first point is about Killing spinors. Many of the exceptional features of Petrov type D spacetimes can be traced back to the irreducible $(2, 0)$ Killing spinor. This was noted many times in the literature from different points of view and is also reflected in this thesis. We remarked on p. 15 that conformal Killing vectors, conformal Killing-Yano tensors and trace-free conformal Killing tensors can be handled in a uniform way. The metric itself is a Killing tensor and so is a conformal metric a conformal Killing tensor. To include also this case into the Killing spinor equation, the symmetry assumption on the spinor has to be dropped. It then immediately follows that any $\kappa_{AB} = f\epsilon_{AB}$ is a Killing spinor and so the conformal metric is included as the product $\kappa_{AB}\bar{\kappa}_{A'B'}$. **(1)** It would be interesting to further analyze this generalization. If we use the algebra of Killing spinors to describe symmetries, the unnatural splitting into conformal Killing vectors and "hidden symmetries" vanishes. It connects the 20-dimensional space of conformal Killing-Yano tensors and the 15-dimensional space of conformal Killing vectors on Minkowski spacetime via products of twistors as we show in section A.4.1 below. The twistors as basic ingredients explain the relation between the numbers 20 and 15 – this was asked for on p. 1 of [75]. Because of this structure, one can build conformal Killing tensors of arbitrarily high rank, but they will certainly be reducible. We conjecture the existence of a *highest irreducible order* in section A.4.2, which states that nothing new exists beyond a certain valence. This would in particular say that higher rank Killing tensors on type D spacetimes are reducible. **(2)** Can the idea of a proof given below made rigorous? There is more structure to be explored. The fact that the divergence of a conformal Killing-Yano tensor is a Killing vector is included in (2.45) for $n = 2$. **(3)** Does a similar result exist for mixed valence? **(4)** Can one also understand the construction of the second isometry (2.79) on Kerr spacetime in this way? Since symmetrized products of Killing spinors are again Killing spinors, they form a symmetric tensor-algebra. **(5)** It would be very interesting to check if it forms a Lie-algebra with a spinorial version of the Schouten-Nijenhuis bracket. Partial results are that this is true for conformal Killing vectors and conformal Killing tensors, see e.g. [45]. There is also a counterexample for pure Killing-Yano tensors in [80], but this subspace might be too restrictive.

Below, we list further ideas, which are not directly related to the original problem, but emerged along the way.

- The geometric background of GHP formalism described in [48], [64] might be worth a look. We understand σ_{1a} as a connection on weighted line bundles and $\mathrm{d}\sigma_1$ as its curvature. Can the modified covariant derivative (3.36) be used to construct a Lagrangian for the Teukolsky equations, once one interprets the geometry in the right way?

- The *Chandrasekhar-Sasaki-Nakamura transformation* [101] makes the long range potential of the radial Teukolsky equation into short range. Is this related to the component (5.62d) of the Teukolsky-Starobinski identities for the separated form of the Teukolsky variable? Does it lead to another kind of "generalized Regge-Wheeler equation" for the linearized gravity case?

- Does the geometric coordinates approach of section 2.5.2 generalize to all type D space times? In general, the second Killing vector comes from the imaginary part of the Killing spinor divergence. The canonical coordinates introduced in [55, p. 424] might then generalize to the Plebanski-Demianski form [92]. This is in particular interesting concerning the structure of duality transformations. More concretely, where does the "c-metric charge" occur in table 4.2 on p. 45 and is it in any reasonable sense dual to the Kerr parameter a?

- Because the conformal Killing vectors of Minkwoski spacetime can be traced back to products of twistors, it would be natural to reformulate the results of [30] from that point of view. This would also incorporate the hidden symmetry naturally in generalizations to type D spacetimes. Together with the spin-s equation, it naturally includes half-integer spin fields, e.g. the massless Dirac equation and it may even lead to a twistorial form of the Klainermann-Sobolev inequality.

- The orbiting null geodesics on Kerr spacetime form a 1-parameter family as follows from [108, eq. (11b)]. This means in particular that there is a r, θ dependent direction of stable perturbations of these null geodesics. Can the Morawetz vector field $f(r_*)\partial_{r_*}$ be modified to be orthogonal to the 1-parameter family of orbiting null geodesics on Kerr spacetime?

A. Appendix

A.1. Spin-weighted spherical harmonics

The spin weighted spherical harmonics are the eigenfunctions of the Laplace operator on the complex line bundle of spin-weighted fields over the 2-sphere S^2. Introducing coordinates (θ, φ), we read off the weighted derivatives along the sphere from (2.107) for $a = 0$,

$$ð = \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\varphi - s\cot\theta\right), \qquad ð' = \frac{1}{\sqrt{2}r}\left(\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\varphi + s\cot\theta\right). \tag{A.1}$$

The first operator is of spin-weight 1 and the second one of weight -1 so they raise and lower the spin weight of a field. The weighted Laplace operator reads

$$\Delta_s = ð\,ð' + ð'\,ð = \frac{1}{r^2\sin\theta}\partial_\theta\sin\theta\partial_\theta - \frac{(\mathrm{i}\partial_\varphi + s\cos\theta)^2}{r^2\sin^2\theta}. \tag{A.2}$$

Here, we note that the zeroth order term cancels out, because $[ð, ð'] = -sr^{-2}$, cf. (2.38). Integer s harmonics can be calculated from the usual spherical harmonics $Y_{lm} = {}_0Y_{lm}$ as follows,

$${}_sY_{lm} = \begin{cases} \sqrt{\frac{(l-s)!}{(l+s)!}}(-\sqrt{2}r\,ð)^s Y_{lm}, & 0 \le s \le l, \\ \sqrt{\frac{(l+s)!}{(l-s)!}}(\sqrt{2}rð')^s Y_{lm}, & -l \le s \le 0. \end{cases} \tag{A.3}$$

Applying (A.1) yields

$$ð\;{}_sY_{lm} = -\frac{1}{\sqrt{2}r}\sqrt{(l-s)(l+s+1)}\,{}_{s+1}Y_{lm}, \tag{A.4a}$$

$$ð'\;{}_sY_{lm} = \frac{1}{\sqrt{2}r}\sqrt{(l+s)(l-s+1)}\,{}_{s-1}Y_{lm}, \tag{A.4b}$$

and in particular

$$ð\;{}_sY_{lm} = 0, \quad \text{if } s = l \text{ or } s = -(l+1), \tag{A.5a}$$

$$ð'\;{}_sY_{lm} = 0, \quad \text{if } s = -l \text{ or } s = l+1. \tag{A.5b}$$

Remark A.1.1. *The eigenfunctions of* (5.58b)*, after a Fourier transformation in t and φ, are the spin-weighted spheroidal harmonics. But the 2-surface spanned by m^a and $\bar{m}^a$ is not closed and such a nice geometric interpretation as above cannot be given in this case, see [24].*

Integration by parts yields

$$\int_{S^2} (\eth f) g \mathrm{d}\mu = -\int_{S^2} f(\eth g)\mathrm{d}\mu = -\int_{S^2} f\overline{(\eth' \bar{g})}\mathrm{d}\mu\,, \tag{A.6a}$$

$$\int_{S^2} (\eth' h) k \mathrm{d}\mu = -\int_{S^2} h(\eth' k)\mathrm{d}\mu = -\int_{S^2} h\overline{(\eth \bar{g})}\mathrm{d}\mu\,, \tag{A.6b}$$

where $\mathrm{d}\mu = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ and f, g, h and k are of spin weights $s-1, -s, s$ and $-(s-1)$ respectively. From properties of the spherical harmonics together with (A.3) we find

$$_{-s}Y_{lm} = (-1)^{m+s}\, {}_s\bar{Y}_{l-m}\,, \tag{A.7}$$

which is used to calculate the normalization Normalize such that

$$\int_{S^2} {}_sY_{lm}\, {}_s\bar{Y}_{l'm'}\mathrm{d}\mu = \delta_{ll'}\delta_{mm'}\,. \tag{A.8}$$

For $l = 0$ there is only ${}_0Y_{00} = 1/\sqrt{4\pi}$. The more general case of half-integer spin can be extracted from [90, eq. (4.15.95)]. We give the explicit form of the $s = \frac{1}{2}$ harmonics for further use in (4.10),

$$\begin{aligned} {}_{-\frac{1}{2}}Y_{\frac{1}{2}-\frac{1}{2}} &= \frac{\mathrm{i}}{\sqrt{4\pi}}\sqrt{(1-\cos\theta)}e^{-\mathrm{i}\varphi/2}\,, & {}_{-\frac{1}{2}}Y_{\frac{1}{2}\frac{1}{2}} &= \frac{\mathrm{i}}{\sqrt{4\pi}}\sqrt{(1+\cos\theta)}e^{\mathrm{i}\varphi/2}\,, \\ {}_{\frac{1}{2}}Y_{\frac{1}{2}-\frac{1}{2}} &= \frac{\mathrm{i}}{\sqrt{4\pi}}\sqrt{(1+\cos\theta)}e^{-\mathrm{i}\varphi/2}\,, & {}_{\frac{1}{2}}Y_{\frac{1}{2}\frac{1}{2}} &= \frac{\mathrm{i}}{\sqrt{4\pi}}\sqrt{(1-\cos\theta)}e^{\mathrm{i}\varphi/2}\,. \end{aligned} \tag{A.9}$$

Also the $l = 1$ harmonics will be useful for the solutions of the Killing spinor equation in Minkowski spacetime on p. 45. With $C = \sqrt{3/(4\pi)}$ they read

$$\begin{aligned} {}_{-1}Y_{1-1} &= -\frac{C}{2}(1-\cos\theta)e^{-\mathrm{i}\varphi}\,, & {}_{-1}Y_{10} &= -\frac{C}{\sqrt{2}}\sin\theta\,, & {}_{-1}Y_{11} &= -\frac{C}{2}(1+\cos\theta)e^{\mathrm{i}\varphi}\,, \\ {}_{0}Y_{1-1} &= \frac{C}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\varphi}\,, & {}_{0}Y_{10} &= C\cos\theta\,, & {}_{0}Y_{11} &= -\frac{C}{\sqrt{2}}\sin\theta e^{\mathrm{i}\varphi}\,, \\ {}_{1}Y_{1-1} &= -\frac{C}{2}(1+\cos\theta)e^{-\mathrm{i}\varphi}\,, & {}_{1}Y_{10} &= \frac{C}{\sqrt{2}}\sin\theta\,, & {}_{1}Y_{11} &= -\frac{C}{2}(1-\cos\theta)e^{\mathrm{i}\varphi}\,. \end{aligned} \tag{A.10}$$

A.2. Despinning

In the classical work [95], Richard Price introduced the notion of *despun quantities*. This refers to applying as many $\eth, \eth'$ operators to weighted fields on a Schwarzschild background as needed for the resulting field to be of spin-weight zero. We only note here that a certain natural generalization for the Kerr spacetime exists, due to the form of the spin coefficients in the Carter tetrad (2.107). For a $\{p, q\}$-weighted field η we find,

$$\left(\eth + \frac{p}{2}\tau - \frac{q}{2}\bar{\tau}'\right)\eta = \frac{1}{\sqrt{2\Sigma}}\left(\mathrm{i}a\sin\theta\partial_t + \partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\varphi - s\cot\theta\right)\eta\,, \tag{A.11a}$$

$$\left(\eth' - \frac{p}{2}\tau' + \frac{q}{2}\bar{\tau}\right)\eta = \frac{1}{\sqrt{2\Sigma}}\left(-\mathrm{i}a\sin\theta\partial_t + \partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\varphi + s\cot\theta\right)\eta\,. \tag{A.11b}$$

This might be of interest for the generalized Regge-Wheeler gauge as proposed in [94, Section 3.2.1]. We also note the similarity to the conformally weighted GHP operators in [90, p. 360]. Because of the formal symmetry characterized by the Sachs star operation (2.15c), the operators

$$\left(\text{þ} + \frac{p}{2}\rho + \frac{q}{2}\bar{\rho}\right)\eta = \frac{1}{\sqrt{2\Sigma}}\left(\frac{r^2 + a^2}{\sqrt{\Delta}}\partial_t + \sqrt{\Delta}\partial_r + \frac{a}{\sqrt{\Delta}}\partial_\varphi - b\frac{r - M}{\sqrt{\Delta}}\right)\eta\,, \tag{A.12a}$$

$$\left(\text{þ}' - \frac{p}{2}\rho' - \frac{q}{2}\bar{\rho}'\right)\eta = \frac{1}{\sqrt{2\Sigma}}\left(\frac{r^2 + a^2}{\sqrt{\Delta}}\partial_t - \sqrt{\Delta}\partial_r + \frac{a}{\sqrt{\Delta}}\partial_\varphi - b\frac{r - M}{\sqrt{\Delta}}\right)\eta\,, \tag{A.12b}$$

follow. Here b denotes the boost weight.

A.3. Algebraic identities on Kerr-NUT

In this section, we prove the algebraic identities (2.74). Define the 1-forms

$$L = \zeta\left(\rho' l - \rho n\right)\,, \qquad\qquad M = \zeta\left(\tau\bar{m} - \tau' m\right)\,, \tag{A.13}$$

with $l, n, m, \bar{m}$ being the 1-forms corresponding to the NP tetrad (2.5). In the Kerr-NUT class, they satisfy $\bar{L} = L$ and $\bar{M} = M$, because of (2.72). The 1-forms corresponding to the isometries (2.68) and (2.79) take the simple form

$$\xi = L + M\,, \qquad\qquad \eta = I^2 L - R^2 M\,. \tag{A.14}$$

Here $R = \frac{1}{2}(\zeta + \bar{\zeta})$, $I = \frac{1}{2\mathrm{i}}(\zeta - \bar{\zeta})$ are real and imaginary parts of the Killing spinor coefficient in (2.66), respectively.

Proposition A.3.1. *The exterior derivatives of* (A.13) *read,*

$$\mathrm{d}L = \zeta\left[l \wedge n\left(2\tau(\bar{\tau} - \tau') + \Psi_2 + \tfrac{\bar{\zeta}}{\zeta}\bar{\Psi}_2\right) + m \wedge \bar{m}2\rho'(\rho - \bar{\rho})\right]\,, \tag{A.15a}$$

$$\mathrm{d}M = \zeta\left[l \wedge n 2\tau(\tau' - \bar{\tau}) + m \wedge \bar{m}\left(2\rho'(\bar{\rho} - \rho) - \Psi_2 + \tfrac{\bar{\zeta}}{\zeta}\bar{\Psi}_2\right)\right]\,. \tag{A.15b}$$

Proof. Since the forms (A.13) are unweighted we can use Leibniz rule and the weighted exterior derivative introduced below (2.62), which yields

$$\mathrm{d}L = \mathrm{d}\zeta \wedge (\rho' l - \rho n) + \zeta\left(\mathrm{d}^\Theta\rho' \wedge l + \rho'\mathrm{d}^\Theta l - \mathrm{d}^\Theta\rho \wedge n - \rho\mathrm{d}^\Theta n\right)\,. \tag{A.16}$$

We find $\mathrm{d}\zeta = -\frac{1}{2}h\zeta$ with h given in (2.27), as follows from (2.60). The weighted exterior derivative of the connection coefficients,

$$\begin{aligned}\mathrm{d}^\Theta\rho &= (l\text{þ}' + n\,\text{þ} - m\eth' - \bar{m}\,\eth)\,\rho \\ &= l\left(\rho\rho' + \tau(\tau' - \bar{\tau}) - \tfrac{1}{2}\Psi_2 - \tfrac{\bar{\zeta}}{2\zeta}\bar{\Psi}_2\right) + n\rho^2 - m2\tau'\rho - \bar{m}(\rho - \bar{\rho})\tau\,,\end{aligned}$$

follows from (2.58) and (2.73). The weighted exterior derivative of the tetrad follows from anti-symmetrizing (2.34) and yields after using (2.56),

$$\mathrm{d}^\Theta l = -\bar{\tau} l \wedge m + \bar{\rho}\bar{m} \wedge m - \tau l \wedge \bar{m} + \rho m \wedge \bar{m}\,.$$

For $\mathrm{d}^\Theta n$ and $\mathrm{d}^\Theta\rho'$, one can do the same calculation or alternatively use the GHP prime (2.15a). Adding terms up according to (A.16) yields (A.15a). The derivation of (A.15b) follows along the same lines or alternatively by applying the GHP star (2.15c) to (A.15a). However, the GHP star is subtle in this case, because $(\bar{\zeta})^* = -\bar{\zeta}$. □

The relation between the complex Komar form for ξ and the curvature we calculated covariantly in (4.29) can also be deduced from (A.15). For completeness, we formulate this in

Corollary A.3.2. *The exterior derivative of the Killing spinor divergence reads*

$$\mathrm{d}\xi = -\zeta\Psi_2 Z^1 - \bar{\zeta}\bar{\Psi}_2\bar{Z}^1 . \tag{A.17}$$

Proof. In exterior notation, the middle component of (2.8) reads $Z^1 = n \wedge l - \bar{m} \wedge m$. We find for given functions f, g the decomposition

$$fZ^1 + g\bar{Z}^1 = (f+g) n \wedge l - (f-g)\bar{m} \wedge m . \tag{A.18}$$

Adding the equations (A.15) yields

$$\begin{aligned} \mathrm{d}\xi &= \mathrm{d}(L+M) \\ &= l \wedge n \left(\zeta\Psi_2 + \bar{\zeta}\bar{\Psi}_2\right) + m \wedge \bar{m}\left(-\zeta\Psi_2 + \bar{\zeta}\bar{\Psi}_2\right) \\ &= -\zeta\Psi_2 Z^1 - \bar{\zeta}\bar{\Psi}_2\bar{Z}^1 , \end{aligned}$$

by setting $f = -\zeta\Psi_2$ and $g = -\bar{\zeta}\bar{\Psi}_2$. □

Because the complex Komar form (4.27) is naturally expressed in terms of connection coefficients and we have for tetrads which are invariant under $\mathcal{L}_\xi$ the identity (2.91), (A.17) yields

$${}_\xi h_1 = \zeta(\epsilon\rho' + \epsilon'\rho - \tau'\beta - \tau\beta') = -\tfrac{1}{2}\zeta\Psi_2 .$$

This proves (2.74a). For the second Killing vector in (A.14), the Komar form is calculated in

Proposition A.3.3. *The exterior derivative of the second Killing vector reads*

$$\mathrm{d}\eta = AZ^0 + BZ^1 + CZ^2 + \bar{A}\bar{Z}^0 + \bar{B}\bar{Z}^1 + \bar{C}\bar{Z}^2 , \tag{A.19}$$

with the coefficients A, B, C given by

$$A = -2\zeta^2\bar{\zeta}\tau\rho , \tag{A.20a}$$

$$B = \tfrac{1}{4}\zeta(\zeta^2 + \bar{\zeta}^2)\Psi_2 - \tfrac{1}{2}\zeta\bar{\zeta}^2\bar{\Psi}_2 + \rho\rho'\zeta^2(\bar{\zeta} - \zeta) + \tau\tau'\zeta^2(\bar{\zeta} + \zeta) , \tag{A.20b}$$

$$C = -2\zeta^2\bar{\zeta}\tau'\rho' . \tag{A.20c}$$

Proof. The exterior derivative decomposes into

$$\mathrm{d}\eta = \mathrm{d}I^2 \wedge L + I^2\mathrm{d}L - \mathrm{d}R^2 \wedge M - R^2\mathrm{d}M . \tag{A.21}$$

We find $\mathrm{d}I^2 = -\zeta(\zeta - \bar{\zeta})(\tau' m + \tau\bar{m})$ and $\mathrm{d}R^2 = -\zeta(\zeta + \bar{\zeta})(\rho' l + \rho n)$. Using this together with (A.15) and reordering terms yields the result. □

We can now directly relate this result to the coefficients of the Komar form. Because of the numerical factor in (4.27), we find

$${}_\eta h_0 = \tfrac{1}{2}A , \qquad {}_\eta h_1 = \tfrac{1}{2}B , \qquad {}_\eta h_2 = \tfrac{1}{2}C . \tag{A.22}$$

The extreme components give nothing new as can be checked by expanding (2.91), but the middle component yields the second algebraic identity for the connection coefficients,

$$\begin{aligned} 2I^2(\epsilon\rho' + \epsilon'\rho) + 2R^2(\tau'\beta + \tau\beta') = {}& \tfrac{1}{4}(\zeta^2 + \bar{\zeta}^2)\Psi_2 - \tfrac{1}{2}\bar{\zeta}^2\bar{\Psi}_2 \\ &+ \rho\rho'\zeta(\bar{\zeta} - \zeta) + \tau\tau'\zeta(\bar{\zeta} + \zeta) . \end{aligned}$$

This proves (2.74b).

A.4. Notes on Killing spinors

In this section, we collect some properties of the Killing spinor equation. In particular we show that all conformal Killing vectors and all Killing 2-spinors on Minkowski spacetime are reducible and can be constructed from solutions of the twistor equation (2.41). Section A.4.2 deals with more speculative ideas about Killing spinors.

A.4.1. Killing spinors from twistors on Minkowski spacetime

Introducing affine coordinates x^a with respect to some chosen origin on Minkowski spacetime, the general solution of the twistor equation (2.41) is of the form

$$\kappa^A = C^A - \mathrm{i}x^{AA'}\bar{D}_{A'}\,, \tag{A.23}$$

with constant spinors C^A and $\bar{D}_{A'}$. Following [91, eq. (6.1.10)], solutions of $\nabla^{A'(A}\kappa^{B)} = 0$ do have the property $\nabla^A_{A'}\nabla^B_{B'}\kappa^C = 0$ (it is skew in BC, skew in AC, symmetric in AB and hence vanishes). So $\nabla^B_{B'}\kappa^C$ must be constant and proportional to ϵ^{BC}, e.g. $\nabla^B_{B'}\kappa^C = -\mathrm{i}\epsilon^{BC}\bar{D}_{B'}$. An integration yields (A.23). It has four complex degrees of freedom and therefore the space of solutions is complex four dimensional. We now show that the complete real 15 dimensional space of conformal Killing vectors is reducible and can be expressed in terms of products of the form

$$\kappa^A\bar{\kappa}^{A'} = C^A\bar{C}^{A'} - \mathrm{i}\left(x^{AB'}\bar{D}_{B'}\bar{C}^{A'} - x^{BA'}D_BC^A\right) + x^{AB'}x^{BA'}\bar{D}_{B'}D_B\,. \tag{A.24}$$

The first term yields the four translations (6.9)(a). With an abuse of (index) notation, this reads $C^A\bar{C}^{A'} = \mathcal{T}^{AA'}$ for some choice of constant C_A. With an irreducible decomposition of the last term,

$$\begin{aligned} x^{AB'}x^{BA'} &= x^{B'(A}x^{B)A'} - \tfrac{1}{2}\epsilon^{AB}x^{B'C}x_C{}^{A'} \\ &= x^{B'(A}x^{B)A'} - \tfrac{1}{2}\epsilon^{AB}\left(x^{C(B'}x_C{}^{A')} + \tfrac{1}{2}\bar{\epsilon}^{A'B'}x^{CC'}x_{CC'}\right) \\ &= \tfrac{1}{2}x^{AB'}x^{BA'} + \tfrac{1}{2}x^{BB'}x^{AA'} - \tfrac{1}{4}\epsilon^{AB}\bar{\epsilon}^{A'B'}x^{CC'}x_{CC'}\,, \end{aligned}$$

we can rewrite it into

$$x^{AB'}x^{BA'}\bar{D}_{B'}D_B = -\tfrac{1}{2}\left(-2x^{BB'}x^{AA'} + \epsilon^{AB}\bar{\epsilon}^{A'B'}x^{CC'}x_{CC'}\right)\bar{D}_{B'}D_B\,. \tag{A.25}$$

So it yields the four conformal accelerations (6.10)(b) for some choice of constant D_A. For the middle term in (A.24) we define the constant spinor $E_{AB} = C_AD_B$. It has the irreducible decomposition $E_{AB} = S_{AB} + \frac{1}{2}\epsilon_{AB}S$ with $S_{AB} = E_{(AB)}, S = E_A{}^A$ and we find for the middle term (up to the prefactor $-\mathrm{i}$ to make it real),

$$\begin{aligned} x^{AB'}\bar{D}_{B'}\bar{C}^{A'} - x^{BA'}D_BC^A &= -x_{BB'}\epsilon^{AB}\bar{E}^{A'B'} + x_{BB'}\bar{\epsilon}^{A'B'}E^{AB} \\ &= x_{BB'}\left(S^{AB}\bar{\epsilon}^{A'B'} - \bar{S}^{A'B'}\epsilon^{AB}\right) + \tfrac{1}{2}x^{AA'}(S - \bar{S})\,. \end{aligned}$$

The first term corresponds via $S^{ab} = -\mathrm{i}(S^{AB}\bar{\epsilon}^{A'B'} - \bar{S}^{A'B'}\epsilon^{AB})$ to a real, anti-symmetric two tensor and describes the Lorentz transformations (6.9)(b). The second term is the scaling transformation (6.10)(a). We note, that only the imaginary part of S contributes.

This is the origin of the discrepancy between the naively expected dimension 16 for (A.24) and the dimension 15 for conformal Killing vectors.

The general solution of the Killing 2-spinor equation given in (4.5) is also reducible and can be decomposed into $\kappa^A\kappa^B$ with the constants being related via

$$U^{AB} = C^A C^B\,, \qquad V_{A'}^{B} = -i\bar{D}_{A'}C^B\,, \qquad W_{A'B'} = \bar{D}_{A'}\bar{D}_{B'}\,. \tag{A.26}$$

Here the expected real dimension 20 for symmetric products $\kappa^A\kappa^B$ coincides with the dimension of the solution space of the Killing 2-spinor equation (2.47). This result gives a (representation theoretic) relation between the space of conformal Killing-Yano tensors and the conformal group in dimension four based on twistors. This was asked for in [75]. The formulation used in this thesis is not able to analyze such relations in other dimensions.

A.4.2. Further relations

The results on Minkowski spacetime suggest to use the solutions of the twistor equation as the fundamental characterizations of symmetries. We have seen in the previous section, that all valence $(1,1)$ and $(2,0)$ solutions are reducible. A natural question is, whether all higher valence Killing spinors are reducible. More general, one might ask if there is an upper bound, also for Petrov type D spacetimes. For a conjecture, we need the following definitions.

Definition A.4.1 (Killing spinor representative)**.** *With a Killing spinor of valence (n,m) and $n \geq m$, the complex conjugated spinor is also Killing and of valence (m,n). In this case, we call the order (n,m) Killing spinor a representative for both solutions.*

Definition A.4.2. *For a given spacetime, the **highest irreducible order** (HIO) is defined as the pair (n,m) of integers, such that:*

1. *Irreducible representatives of valence (n,k), with $k \leq m$ and of valence (l,m), with $l \leq n$ to the Killing spinor equation do exist.*
2. *All solutions of valence (k,l), with $k > n$ or $l > m$ are reducible.*

Conjecture 2. *On a 4 dimensional vacuum spacetime of Lorentzian signature, the HIO of Petrov type*

- *O is $(1,0)$.*
- *D is $(2,1)$.*

For the conformally flat case, a representational theoretic argument might generalize the results of section A.4.1 to general valence. For Petrov type D, we found a partial result in [111], where it is proven that all valence $(2n,0)$ Killing spinors with integer $n > 1$ are reducible and that valence $(2n{+}1)$ Killing spinors do not exist. The calculation is based on the integrability condition (2.46). A decomposition of the Killing spinor equation (or any other first order equation) in a principal dyad shows that only neighboring components are coupled, see e.g. (2.39) or (2.53). It is reasonable that the integrability condition restricts the extreme components in such a way that the remaining components can be associated to a Killing spinor of lower valence. The general integrability condition is quite complicated, see [7, Section 2.3], and has not been analyzed in components, as far as the author is aware.

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- Steffen Aksteiner and Lars Andersson. Charges for linearized gravity. *Classical and Quantum Gravity*, 30(15):155016, August 2013. arXiv:1301.2674