Localization and Recurrence in Quantum Walks

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Abstract

In this thesis we study the recurrence and localization properties of discrete-time quantum walks. Quantum walks describe the discrete-time dynamics of a single particle on a lattice under the assumption of a finite propagation speed. Hence, the particle can only move a finite number of lattice site per time step. In order to characterize the propagation behaviour of such evolutions, we analyze the transition probabilities between different states of the system. Depending on whether we consider transitions back to the initial state or transitions between distant lattice sites, we obtain the following results on the recurrence and localization behaviour of discrete-time quantum walks.

Recurrence properties: We analyze the probability that a particle evolving according to a discrete-time unitary dynamics eventually returns to its initial state. In order to detect such a recurrence event, we perform after each unitary time step a von Neumann measurement that projects onto the initial state. In case of a positive measurement outcome, the evolution is stopped, otherwise, we continue with the next time step. We define an initial state to be recurrent with respect to the time evolution if the probability that the system is eventually detected in the initial state is equal to one. Using the theory of probability measures on the unit circle, we show that a state is recurrent if and only if its spectral measure with respect to the evolution operator does not contain an absolutely continuous component. We also find that the expectation value of the recurrence time is either infinite or an integer. A topological explanation of this quantization effect is provided by identifying the expected recurrence time with a winding number.

Localization properties: When analyzing the properties of quantum walks a common simplification is to assume translation invariance with respect to the underlying lattice. However, random fluctuations due to unavoidable experimental imperfections often violate this assumption in an actual implementation. Therefore, we go beyond this translation-invariant regime and study the effect of randomly chosen position dependent coin operations on the standard one-dimensional single-shift-and-coin quantum walk. Adapting techniques from the theory of random Schrödinger operators, we prove that in contrast to the ballistic propagation behaviour observed in the translation-invariant case such disordered quantum walks generically exhibit Anderson localization, that is, a complete breakdown of any transport in the system. In fact, we show for a wide class of coin distributions the strongest form of such localization behaviour, namely dynamical localization, which means that for all times the transition probability between two arbitrary lattice sites decays exponentially with their distance.

Keywords : quantum walks, dynamical localization, recurrence

Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit dem Wiederkehrverhalten und den Lokalisierungseigenschaften zeitdiskreter Quantenwalks. Quantenwalks beschreiben zeit- und ortsdiskrete Dynamiken eines einzelnen Teilchens unter der Annahme einer endlichen Ausbreitungsgeschwindigkeit. In dieser Arbeit werden die Propagationseigenschaften solcher Systeme mithilfe der Übergangswahrscheinlichkeiten zwischen den Systemzuständen analysiert. Je nachdem, ob man die Rückkehrwahrscheinlichkeiten zum Anfangszustand oder die Übergangswahrscheinlichkeiten zwischen beliebigen Gitterplätzen betrachtet, erhält man die folgenden Resultate über das Propagationsverhalten zeitdiskreter Quantenwalks:
Wiederkehrverhalten: Es wird die Wahrscheinlichkeit untersucht, mit der ein System unter einer zeitdiskreten unitären Zeitentwicklung in seinen Anfangszustand zurückkehrt. Um dieses Ereignis zu detektieren, wird in jedem Zeitschritt eine projektive Messung durchgeführt, die testet, ob sich das System wieder im Anfangszustand befindet. Bei einem positiven Messausgang wird die Dynamik angehalten andernfalls fortgesetzt. Ein Anfangszustand heißt rekurrent bezüglich der Zeitentwicklung falls die Wahrscheinlichkeit, dass dieser Messprozess das Teilchen irgendwann detektieren wird, eins ist. Es stellt sich heraus, dass das Rekurrenzverhalten eines Zustandes allein durch sein Spektralmaß charakterisiert ist. Es wird gezeigt, dass ein Zustand genau dann rekurrent ist, falls sein Spektralmaß keinen absolutstetigen Anteil enthält. Weiterhin wird bewiesen, dass die mittlere Rückkehrzeit rekurrenter Systeme entweder eine natürliche Zahl oder unendlich ist. Die topologische Erklärung dieses Quantisierungseffekts erfolgt über die Identifikation der mittleren Rückkehrzeit mit einer Windungszahl.
Lokalisierung: Eine häufige Annahme zur einfacheren theoretischen Behandlung von Quantenwalks ist Translationsinvarianz bezüglich des zugrunde liegenden Gitters. Allerdings können schon geringe experimentelle Ungenauigkeiten diese Annahme verletzen. Daher wird in dieser Arbeit diese Einschränkung fallen gelassen und der Einfluss zufällig gewählter Münzoperationen auf das dynamische Verhalten des eindimensionalen Standardquantenwalks untersucht. Unter der Verwendung von Techniken aus der Theorie zufälliger Schrödingeroperatoren wird bewiesen, dass im Gegensatz zu dem in translationsinvarianten Systemen beobachteten ballistischen Ausbreitungsverhalten solche ungeordneten Quantenwalks typischerweise Andersonlokalisierung, das heißt vollständige Abwesenheit jeglichen Transports, zeigen. Tatsächlich wird für eine große Klasse von Münzverteilungen sogar dynamische Lokalisierung gezeigt. Dies bedeutet, dass die Übergangswahrscheinlichkeiten zwischen beliebigen Gitterplätzen für alle Zeiten exponentiell mit ihrem Abstand abfallen.

Schlagwörter: Quantenwalks, dynamische Lokalisierung, Rekurrenz

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Nomenclature

All chapters

$(x_n)_n$	sequence $x_1, x_2, \ldots$
$\mathbb{C}$	complex numbers
χ_A	characteristic function of the set A
$K\mu$	Cauchy transform of a measure μ on $\mathbb{T}$
$\mathbb{D}$	open unit disc
$\ell^2(\mathbb{Z}^s, \mathcal{K})$	Hilbert space of square summable $\mathcal{K}$-valued functions on $\mathbb{Z}^s$
Γ_x	lattice translation
$\mathrm{GL}(\mathbb{C}, d)$	space of invertible matrices on $\mathbb{C}^d$
$\mathcal{H}, \mathcal{K}$	Hilbert spaces
$\Im(a)$	imaginary part of $a \in \mathbb{C}$
$\mathcal{L}^2([0,2\pi)^s, \mathcal{K})$	Hilbert space of square integrable functions on $[0,2\pi)^s$
$\mathcal{L}^2(\mu)$	Hilbert space of μ-square integrable functions
$\mathcal{L}^p(\mu)$	vector space of p-integrable functions with respect to the measure μ
$\mathbb{Q}$	rational numbers
$\phi, \psi, \ldots$	quantum states
$\mathbb{P}\mathbb{C}^d$	projective space
$\overline{x}$	element of projective space $\mathbb{P}\mathbb{C}^d$
$\Re(a)$	real part of $a \in \mathbb{C}$
$SL_{\mathbb{T}}(d)$	group of invertible matrices with determinant of modulus one
$\mathcal{SU}(k)$	special unitary group
$\mathbb{T}$	unit circle
$\mathbb{Z}$	natural numbers
T_ω	random (ergodic) operator

Chapter 3

$(\tau_\alpha)_{\alpha\in J}$	family of measure preserving transformations
N_ν	integral with invariant measure ν
$\delta(\overline{x},\overline{y})$	metric on projective space $\mathbb{PC}^d$
γ	Lyapunov exponent of a product of random matrices
$\mathcal{L}_\alpha$	Banach space of Hölder-continuous functions of order α
$m_\alpha(f)$	Hölder-coefficient of order α of the function f
R_μ	Markov operator
$\mu * \nu$	convolution of measures μ and ν
$\|\cdot\|_\alpha$	norm on $\mathcal{L}_\alpha$
$\|\cdot\|_\infty$	uniform norm
ν	invariant measure
g_{ω_n}	random matrix
$S_{n(\omega)}$	product of n random matrices
τ	measure preserving transformation
T_μ	convolution operator

Chapter 4

P_{tg}	projector onto target states
τ	expected return time
$\tilde{U}$	modified time evolution where the return to the initial state is checked (see (4.2))
$\widehat{\mu}(z)$	generating function of n-step return amplitudes
$\widehat{a}(z)$	generating function of first return amplitudes
$f(z)$	Schur function of a probability measure on $\mathbb{T}$
$G(z), \widetilde{G}(z)$	Neumann series of the operators zU and $z\widetilde{U}$
R	total return probability
s_n, s_∞	n-step and asymptotic survival probability
V_τ	variance of expected return time

Chapters 5 and 6

χ_A	spectral projection onto the set A
$\mathcal{N}$	integrated density of states
$\gamma(z)$	Lyapunov exponent for the spectral parameter z
$\mathfrak{I}$	subset of $\mathbb{T}$
ν	density of states
$\nu_\Theta(f)$	expectation of f with respect to the invariant measure ν_Θ associated to the spectral parameter Θ
P_2	subset of $\mathbb{C}^2$ left invariant by the transfer matrices
G_z	resolvent of disordered quantum walk W_ω
$\rho_{\omega,N}^{x,y}$	spectral measure between lattice sites x, y
τ_z	mapping from $\mathcal{U}(2)$ to the transfer matrices with respect to the spectral parameter z
W_ω	disordered quantum walk
$W_\omega(N)$	finite unitary restriction of $W_\omega(N)$
U_ω	random coin operator
$T_{x(z)}$	transfer matrix associated with lattice site x with spectral parameter z

1. Introduction

Motivation

The field of Quantum information theory encompasses areas from quantum physics, computer science, information theory and mathematics. The two major themes in this area of research are, on one hand, the application of information theoretic concepts to the foundations of quantum theory and, on the other, the idea of harnessing the properties of quantum mechanical systems in order to perform communication and computational tasks. Already in 1982, Richard Feynman noted that due to an exponential scaling of the number of system parameters with the particle number, classical computers are not capable of efficiently simulating large many-body states [Fey82]. To avoid this dilemma, Feynman proposed to use another quantum system for the simulation rather than a classical computer. More precisely, he suggested to realize a quantum system with very precise experimental control, a quantum simulator, such that the quantum state of interest, or even its dynamics, can be simulated by changing these system parameters in a controlled way.

This simulation idea motivates the study of discrete-time unitary evolutions, where the complex evolution induced by a possibly time dependent Hamiltonian is decomposed into simpler building blocks. One particular class of such systems are quantum cellular automata, which are defined as quantum lattice systems undergoing a discrete-time evolution that satisfies an additional locality constraint [WS04, Vog09]. More precisely, the propagation speed of information within the system has to be finite, *e.g.* a localized excitation at one lattice site can after one time step only influence a finite number of neighbouring cells. In a bottom-up approach one can also consider the corresponding single-particle dynamics. This means that a single particle with some internal degree of freedom moves on a lattice, such that the distance traveled per time step is finite. Such discrete-time single-particle systems are called quantum walks, since originally they arose in an attempt to develop a quantum mechanical analogue of classical random walks both in discrete [ADZ93] as well as in continuous time [FG98]. The prototypical example of such a quantum walk dynamics is given by a coined quantum walk that describes a spin-$1/2$ particle on the line with an evolution that consists of the alternation of two processes [ABNW01]: The first one is a shift operation that moves the particle one lattice site to the left or to the right, depending on its internal state; the second one, a so-called coin operation, consists of a unitary operator acting exclusively on the internal spin state.

This intuitive description of a quantum walk already suggests some relation to classical random walks, where the particle's next position is determined by a coin toss. Since classical random walks have found applications in such diverse areas as biology [BdLVC05, KS83], genetics [vdEST92, LS91], neuroscience [GM64, TYZ$^+$09], the development of stock market prices [Man91], epidemic spread [DG11] and the analysis of

social networks [SM11], one might hope that quantum walks also have some computational application. As it turns out, quantum walks are interesting both as a model of single particle dynamics as well as as a generalization of classical random walks. Treated as a computational tool, they outperform classical random walks [CFG02] in the context of search algorithms [SKW03, AKR05, MNRS11] as well as graph traversals [CCD$^+$03] and have even been shown to allow for universal quantum computation [Chi09, LCE$^+$10].

Most of these results rely on an enhanced spreading behaviour of quantum walks as compared to their classical counter parts [Kem05, MNRS09]. More precisely, if the evolution is translation-invariant and coherent, meaning that the evolution operator does not change in time, quantum walks are known to behave ballistically, that is, the particle spreading behaviour is linear in time. This is in contrast to the diffusive spreading observed in a classical random walk [MNRS09, ABNW01]. This behaviour can be understood by regarding a quantum walk as the translation-invariant discrete-time dynamics of a single particle. In particular, it is possible to derive a group velocity operator that governs the asymptotic ballistic transport of the particle in this translation-invariant scenario [ABNW01, AVWW11]. In addition, random fluctuations that respect the translation invariance of the evolution tend to destroy the coherence of the problem and asymptotically result in diffusive, that is, classical behaviour [AVWW11, Joy11a]. Aside from some specific examples [SK10, LS09] and perturbations on a finite number of lattice sites [Kon10], current literature focusses mostly on translation-invariant quantum walks [GJS04, AVWW11].

Quantum walks exhibit a rich variety of quantum effects such as Landau-Zener tunneling [RBH$^+$11], the Klein-paradox [Kur08], topological phases [KRBD10, Kit12], Bloch oscillations [RBH$^+$11] and the effects of electric fields [CRW$^+$13] as well as the formation of molecules for two interacting particles each performing a quantum walk [AAM$^+$12]. They therefore hold a great promise as a simulation tool. Another important advantage from the perspective of an actual quantum simulation is that quantum walks can be realized with current technology. In recent years, quantum walks have been implemented in such diverse physical systems as neutral atoms in optical lattices [KFC$^+$09, GAS$^+$13], trapped ions [ZKG$^+$10, SMS$^+$09], wave guide lattices [PLM$^+$10, SSV$^+$12] and light pulses in optical fibres [SCP$^+$10, SGR$^+$12] as well as single photons in free space [BFL$^+$10].

This thesis studies two aspects of the dynamical properties of quantum walks, one inspired from their connection to classical random walks and one stemming from the goal to simulate single-particle quantum dynamics. In both cases, the central quantities of interest are the transition probabilities between different states the system induced by the quantum walk dynamics.

The first part of this thesis concentrates on transitions back into the initial state, that is, the probability to find the system again in the state it was prepared in at the beginning. In the theory of classical random walks and Markov chains an initial state is called recurrent if the probability of eventually returning to the initial state is equal to one. Otherwise, the state is called transient. If the Markov chain is irreducible, all states are either recurrent or transient and in addition there is a renewal equation connecting the first return probability to the probability to return in exactly n steps [KT75]. Moreover, it was shown by Pólya that for classical random walks with nearest neighbour hopping,

there is a critical lattice dimension ($d = 3$), such that all states are recurrent for strictly smaller lattice dimensions and transient otherwise. This motivates the investigation of the recurrence behaviour of general discrete-time quantum evolutions.

In the second part of this thesis transition probabilities between arbitrary lattice sites are considered. As described above, most results known about quantum walks rely on the translation invariance of the system. However, inevitable imperfections in an actual experimental realization often violates this assumption. Therefore, one motivation in this thesis is to analyze a regime where translation invariance is broken. More precisely,the influence of random position dependent static fluctuations on the propagation properties of the system are studied. To this end, the previously described standard model of a one-dimensional coined quantum walk of a spin-1/2 particle is considered, where the coin operation at each lattice is chosen randomly according to some common probability measure on the unitary 2×2 matrices. This manifestly breaks the translation invariance.

Outline of the thesis

In **chapter 2** we introduce the basic mathematical and physical concepts. The covered topics include the spectral theory of unitary operators, the characterization of probability measures on the unit circle by analytic functions on the unit disc and the theory of quantum walks as well as an introduction to the theory of random ergodic operators and their localization properties.

In preparation of the proof of dynamical localization, we review in **chapter 3** results on products of random matrices. In comparison to the approach commonly found in the literature [BL85, CL90], we develop the theory for matrices with complex entries.

In **chapter 4** we investigate the recurrence behaviour of discrete-time unitary evolutions and characterize the probability that such systems return to their initial state.

The last two chapters deal with localization properties of disordered quantum walks. In **chapter 5** we define disordered quantum walks as random position dependent perturbations of the standard one-dimensional coined quantum walk of a spin-$\frac{1}{2}$ particle and derive their basic properties. Building on these results, we analyze in **chapter 6** under which assumptions transitions between different lattice sites are exponentially suppressed in their distance, which implies dynamical localization. We also provide explicit examples where these assumptions are met, including the certification of dynamical localization observed in an actual experiment by Schreiber et al. that was performed with photons in an optical fibre [SCP+11], thereby showing the practical applicability of the result.

We note that since chapter **chapter 4** can be read independently from the **chapters 5 and 6** all three chapters contain a section with conclusions and further research directions.

2. Basic concepts

This chapter contains a short introduction into the mathematical and physical background of quantum mechanics, spectral theory, quantum walks and random operators. In section 2.1 we make some introductory remarks on quantum mechanics and concepts in quantum information theory. In the context of this thesis, we are mostly dealing with unitary operators and therefore, some results on their spectral theory are provided in section 2.2. This naturally leads us to the study of probability measures on the unit circle which we discuss in section 2.3. Finally in sections 2.4 and 2.5 we introduce the theory of quantum walks and random ergodic operators.

2.1. Quantum information

The main purpose of this section is to introduce notation and since the results presented here are well known, we omit their proofs. Specific references will be pointed out at the level of the discussed results, however, for a general overview on mathematical methods in quantum mechanics and quantum information theory we mention the following references [RS80, Tes09, KBDW83, NC10].

The basic goal of a theory in physics is to model a physical system on such a level of complexity that once certain parameters, which identify the state of the system, are given, we can predict its behaviour, with respect to a defined set of questions we want to ask about the system. In order to achieve this we first have to fix the set of all allowed states of the system. In quantum mechanics this state space is given by a Hilbert space $\mathcal{H}$, the normalized elements ψ of which describe different possible pure states that can be realized. Asking questions to the system is formalized by the concept of measurements or observables, which are modeled by hermitian operators on the Hilbert space $\mathcal{H}$. We adopt the common notation of $\mathcal{B}(\mathcal{H})$ for all bounded operators on $\mathcal{H}$ with respect to the usual operator norm. The dynamics of an isolated quantum system that does not interact with any environment is given by some unitary operator U on $\mathcal{H}$.

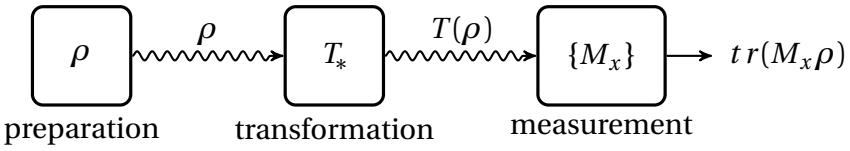


Figure 2.1.: Prepare and measure scheme

In quantum information it is common to describe quantum mechanical setups in the so-called operational approach, which is also appropriate for the discrete-time systems we are going to study. Figure 2.1 depicts the basic setup, where the system undergos

some transformation between the preparation procedure that fixes an initial state, and a final measurement.

If we want to allow for some uncertainty during the preparation or transformation in the sense that we have some probability distribution according to which either initial state or transformation is chosen, it is appropriate to describe the system within the framework of density operators. That is, the state of a system is described by a positive operator ρ on $\mathcal{H}$ with unit trace. Consider as an example the situation, where our preparation produces one of k possible states $\phi_l \in \mathcal{H}$ with probability p_l each. In this case, the corresponding density matrix ρ is given by

$$\rho = \sum_{l=1}^{k} p_l |\phi_l\rangle\langle\phi_l| \quad \text{with} \sum_l p_l = 1$$

where $|\phi\rangle\langle\phi|$ is the usual bra-ket-notation for the projector onto the subspace of $\mathcal{H}$ spanned by the vector $\phi \in \mathcal{H}$.

The next question we have to answer is which subset of possible transformations on the system should be allowed. It turns out that under the physically reasonable assumption that density operators are mapped to density operators, all allowed transformations take the form of completely positive trace preserving maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ for some target Hilbert space $\mathcal{K}$. We often call a completely positive trace preserving map a quantum channel. In order to study the action of a transformation there are two equivalent points of view the so-called Schrödinger picture, where the quantum channel T_* acts on the state ρ of the system and the Heisenberg picture, where the quantum channel T acts on the observables of the system. Both pictures are connected via the relation

$$\mathrm{tr}(T_*(\rho)A) = \mathrm{tr}(\rho\, T(A)) \ ,$$

where ρ is a density matrix and $A \in \mathcal{B}(\mathcal{K})$ is an observable.

Instead of a this axiomatic characterization, we can take the opposite approach and define a set of basic operations that should definitely be allowed. Three such basic operations are certainly unitary transformations and the addition and removal of auxiliary subsystems to and from the system. Stinespring's dilation theorem basically tells us that these three operations are enough to implement an arbitrary completely positive and trace preserving map [KBDW83, Pau02, Sti55].

Theorem 2.1.1 (Stinespring)**.** *Let T be a quantum channel with input Hilbert space $\mathcal{H}_1$ and output Hilbert space H_2. Then, there are a Hilbert space $\mathcal{K}$ called a dilation space for T and an isometry $V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathcal{K}$ such that for all $A \in \mathcal{B}(\mathcal{H}_1)$ and all density operators ρ on $\mathcal{H}_1$*

$$T(A) = V^*(A \otimes \mathbb{1})V \quad \textit{and} \quad T_*(\rho) = \mathrm{tr}_{\mathcal{K}}(V \rho\, V^*) \ .$$

In particular, this result implies that in order to study the possible dynamics on a fixed quantum system $\mathcal{H}$, we can restrict our attention to unitary operators if we include the environment into our description. The idea of such quantum channels and motivates the following definition.

Definition 2.1.2 (discrete unitary evolution)**.** *Let $\mathcal{H}$ be a separable Hilbert space. We say that a quantum system described by $\mathcal{H}$ undergoes a discrete unitary time evolution if the dynamics is given by the subsequent application of a unitary operator U acting on $\mathcal{H}$. That is, given an initial state $\phi_0 \in \mathcal{H}$, the state of the system after $t \in \mathbb{N}$ time steps is given by $\psi_t = U^t \phi_0$.*

Let us briefly compare this setting to the usual evolution in continuous time. In this case, the dynamics is given by the operator semigroup $e^{\mathbb{i}tH}$ generated by the self-adjoined Hamiltonian H. If H is time independent or periodic, with periodicity T we can turn this situation into a discrete-time one, by integrating $e^{\mathbb{i}tH}$ over one period $[0, T]$. In addition, this fixes the time scale of a single time step.

With this definition we have outlined the class of systems we are going to investigate. In particular, quantum walks are defined as a class of unitary discrete time evolutions on a lattice at least until we take into account (experimental) imperfections (see section 2.4). For this reason, the next section deals with some aspects of the spectral theory of unitary operators. We are in particular interested in the propagation properties of such discrete-time unitary systems. That is, the Hilbert space carries a notion of position and we want to make statements about the spreading behaviour of localized initial states.

2.2. Spectral theory of unitary operators

In this section we provide some basic results on the spectral theory of unitary operators. Again we omit the proofs in favour of a compact presentation and refer the reader to the references [Lax02, RS80, DS58, Tes09].

Although many of the results carry over to unbounded operators, we restrict our attention to bounded operators A on a separable Hilbert space $\mathcal{H}$ and we even assume that T is normal so it commutes with its adjoint. As usual we define the resolvent set $\rho(A)$ of an operator A as

$$\rho(A) := \{z \in \mathbb{C};\, \mathrm{G}_z(A) := (A - z\,\mathbb{1})^{-1} \in \mathcal{B}(\mathcal{H})\}\ ,$$

where $\mathrm{G}_z(A)$ is called the resolvent (operator) of A. As a mapping from $\mathbb{C}$ to $\mathcal{B}(\mathcal{H})$ the function $\mathrm{G}_z(A)$ is analytic on the set $\rho(A)$. The spectrum of A, denoted by $\sigma(A)$, is given via the expression $\sigma(A) = \mathbb{C} \setminus \rho(A)$, which is compact for $A \in \mathcal{B}(\mathcal{H})$. The operator A is called self-adjoint if $\sigma(A) \subset \mathbb{R}$ and A is called unitary if $\sigma(A) \subset \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle. Decompositions of the spectrum into various subsets are discussed at the end of this section.

Next we formulate the spectral theorem for bounded and in particular unitary operators. Since we are going to use the connection to complex Borel measures, we chose an approach via projection valued measures.

Definition 2.2.1 (projection valued measure)**.** *Let $\mathcal{H}$ be a Hilbert space and $(\Omega, \mathfrak{B})$ be a measurable space with Borel σ-algebra $\mathfrak{B}$. A mapping $E : \mathfrak{B} \mapsto \mathcal{B}(\mathcal{H})$ is called a projection valued measure or resolution of the identity, if it satisfies the relations*

(i) $E(\Omega) = \mathbb{1}$ *and* $E(\emptyset) = 0$

(ii) *For all* $S \subset \mathfrak{B}$ *:* $E(S) \in \mathcal{B}(\mathcal{H})$ *is an orthogonal projection*

(iii) *For all* $S_1, S_2 \in \mathfrak{B}$ *:* $E(S_1)E(S_2) = E(S_2)E(S_1) = E(S_2 \cap S_2)$

(iv) *If* $(S_n)_n \subset \mathfrak{B}$*, such that for* $m \neq n$ $S_n \cap S_m = \emptyset$*,*
then $E(\bigcup_n S_n)\phi = \sum_n E(S_n)\phi$ *for all* $\phi \in \mathcal{H}$*, i.e.* E *is strongly* σ*-additive.*

If E only satisfies $E(\Omega) < \mathbb{1}$, we call it a subresolution of the identity. For $\phi \in \mathcal{H}$ is the expression $\langle \phi \, , E(S)\phi \rangle$ a linear functional on $\mathcal{B}$. From properties (i) to (iv) we see that $\langle \phi \, , E(S)\phi \rangle$ constitutes a complex valued measure for $S \in \mathfrak{B}$ [Con90]. Invoking the Riesz-Markov representation theorem then tells us that $\langle \phi \, , E(S)\phi \rangle$ gives rise to a unique Borel measure on $\mathbb{C}$, which we denote by μ_ϕ. The spectral theorem exactly tells us, that every bounded operator can be uniquely represented by a spectral measure. In particular, we can choose Ω equal to $\mathbb{R}$ for self-adjoint and equal to the unit circle $\mathbb{T}$ for unitary operators [RS80, Tes09, Lax02]. A proof of the spectral theorem for normal bounded operators can be found in [Rud06].

Theorem 2.2.2 (spectral theorem)**.** *Let* T *be a bounded, normal or self-adjoint or unitary operator on a separable Hilbert space* $\mathcal{H}$*. There exists a unique projection valued measure* $E : \Sigma \subset \mathfrak{B} \mapsto \mathcal{B}(\mathcal{H})$ *with* Σ *being a compact subset of* $\mathbb{C}$*, in the bounded,* $\mathbb{R}$ *in the self-adjoint and* $\mathbb{T}$ *in the unitary case, such that for all* $\phi \in \mathcal{H}$

$$\langle \phi \, , T\phi \rangle = \int_\Sigma \lambda \, \langle \phi \, , E(\mathrm{d}\lambda)\phi \rangle =: \int_\Sigma \lambda \, \mu_\phi(\mathrm{d}\lambda) \, . \tag{2.1}$$

Conversely, for every projection valued measure E *there exists a unique bounded, normal or self-adjoint or unitary operator* $T \in \mathcal{B}(\mathcal{H})$ *satisfying* (2.1)*, depending on whether* Σ *is a compact subset of* $\mathbb{C}$*,* $\mathbb{R}$ *or* $\mathbb{T}$*.*

In order to translate results known for ergodic self-adjoined operators to unitary operators in section 2.5 we remark that for a unitary operator U there always exists a projection valued measure on the Borel sets of $\mathbb{R}$ instead of $\mathbb{T}$ [Wei80].

Lemma 2.2.3. *For every unitary operator* U *on a separable Hilbert space* $\mathcal{H}$ *there exists a resolution of the identity* E_U *on the Borel* σ*-algebra of* $\mathbb{R}$ *such that for all* $\phi, \psi \in \mathcal{H}$

$$\langle \phi \, , U\psi \rangle = \int_{\mathbb{R}} e^{\mathbb{i}t} \, \langle \phi \, , E_U(\mathrm{d}t)\psi \rangle \, .$$

In addition, E_U *satisfies* $E_U(t) = 0$ *for* $t < 0$ *and* $E_U(t) = \mathbb{1}$ *for* $t \geq 2\pi$*.*

Note that it suffices to define the spectral measures μ_ϕ, with respect to diagonal scalar products $\langle \phi \, , \cdot \phi \rangle$, because we can always compute expressions of the form $\mu_{\phi\psi}$ via the polarization identity

$$\mu_{\psi\phi} := \langle \psi \, , T\phi \rangle = \frac{1}{4}\left(\mu_{\phi+\psi} - \mu_{\psi-\phi} + \mathbb{i}\,(\mu_{\phi+\psi} - \mu_{\phi-\psi})\right) \, . \tag{2.2}$$

Given any spectral measure μ_ϕ, we can integrate any bounded Borel function f with respect to it. Since the measure μ_ϕ corresponds uniquely to an operator T we might

hope that this procedure allows us to define the operator $f(T)$. That this a well defined definition is content of the functional calculus, which establishes a one to one correspondence between bounded operators on $\mathcal{H}$ and the integration of bounded Borel functions with respect to spectral measures [RS80, Tes09, Lax02, Rud06].

Theorem 2.2.4 (functional calculus)**.** *Let T be a bounded normal operator on a separable Hilbert space $\mathcal{H}$ and denote by E_T its corresponding spectral measure. For every bounded Borel function f there is a unique operator, which we denote by $f(T)$, that satisfies*

$$\langle \phi \, , f(T)\phi \rangle = \int_{\mathbb{C}} f(\lambda) \, \langle \phi \, , E_T(\mathrm{d}\lambda)\phi \rangle := \int_{\mathbb{C}} f(\lambda) \, \mu_\phi(\mathrm{d}\lambda) \; .$$

We remark that a functional calculus can also be developed for unbounded self-adjoint operators. In this case the support of its spectral measure is still a closed but no longer compact subset of the real line.

The spectral theorem tells us that in order to analyze the properties of an operator T we can also study the properties of its corresponding spectral measure. This raises the question, how measure theory helps our understanding of a quantum system. First we state a structural result about the decomposition of the Hilbert space of the system. In the next section we study the relevance of this decomposition for dynamical questions.

Remember, that by the Lebesgue decomposition theorem we can divide any regular Borel measure on $\mathbb{R}$ into three mutually singular components [Rud87, Tes09, RS80]. More precisely we find

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc} \; .$$

Here, the pure point part μ_{pp} consists of a sum of Dirac measures,the absolutely continuous part μ_{ac} has a density with respect to the Lebesgue measure and the singular continuous part is supported on a null set with respect to the Lebesgue measure, but assigns zero weight to single points. Note that because of lemma 2.2.3 these considerations also apply to unitary operators.

Since all three measures in the Lebesgue decomposition are mutually singular, they are supported on disjoined sets $\Omega_{pp}, \Omega_{sc}, \Omega_{ac}$ which we can choose in such a way that Ω_{sc} has Lebesgue measure zero. Property (iii) of definition 2.2.1 implies that integrating a projection valued measure E over each of the sets Ω_{xx}, the three resulting projections $E(\Omega_{xx})$ will be mutually orthogonal and each provides a subresolution of the identity. Therefore, this measure theoretic decomposition of the spectrum induces a decomposition of the Hilbert space $\mathcal{H}$ into three orthogonal components

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{ac} \; ,$$

where the different summands are defined as

$$\begin{aligned}
\mathcal{H}_{pp} &:= \{\phi \in \mathcal{H} : \; \mu_\phi \text{ is pure point } \} \\
\mathcal{H}_{sc} &:= \{\phi \in \mathcal{H} : \; \mu_\phi \text{ is absolutely continuous w.r.t. the Lebesgue measure } \} \\
\mathcal{H}_{ac} &:= \{\phi \in \mathcal{H} : \; \mu_\phi \text{ is singular continuous w.r.t. the Lebesgue measure} \} \; .
\end{aligned}$$

In turn, this result implies a corresponding decomposition of the operator T

$$T = (TP_{pp}) \oplus (TP_{sc}) \oplus (TP_{ac}) \ ,$$

where P_{xx} denotes the projection onto the corresponding subspace $\mathcal{H}_{xx}$ [Tes09]. We call the three spectra corresponding to the three different components of T the pure point $\sigma_{pp}(T)$, the singular continuous $\sigma_{sc}(T)$ and the absolutely continuous spectrum $\sigma_{ac}(T)$ of T. In general $\sigma_{pp}(T) = \overline{\{\lambda : \ \lambda \text{ is eigenvalue of } T\}}$, so it also contains the limit points of the set of eigenvalues. With these definitions we obtain the decomposition

$$\sigma(T) = \sigma_{pp}(T) \cup \sigma_{sc}(T) \cup \sigma_{ac}(T) = \sigma_{pp}(T) \cup \sigma_{c}(T) = \sigma_{s}(T) \cup \sigma_{ac}(T) \ ,$$

where the singular spectrum $\sigma_s(T)$ and the continuous spectrum $\sigma_c(T)$ are defined in the obvious way. As a remark we note that there is an additional useful decomposition of the spectrum of an operator T, namely into the discrete spectrum $\sigma_{disc}(T)$ and the essential spectrum $\sigma_{ess}(T)$. In this case, $\sigma_{disc}(T)$ contains all isolated eigenvalues of T of finite multiplicity and $\sigma_{ess}(T)$ contains the continuous part of spectrum together with all limit points of the set of eigenvalues of T and all eigenvalues of infinite multiplicity. This decomposition of the spectrum can also be characterized in terms of the dimensionality of the projection valued measure E_T of the normal operator $T \in \mathcal{B}(\mathcal{H})$ [RS80].

Theorem 2.2.5. *Let T be a unitary or bounded self-adjoined operator on a separable Hilbert space $\mathcal{H}$. A point $\lambda \in \mathbb{C}$ lies in the essential spectrum of T if and only if the projector $E_T(\lambda-\varepsilon, \lambda+\varepsilon)$ is infinite dimensional for all $\varepsilon > 0$ and λ is an element of $\sigma_{disc}(T)$ if and only if $E_T(\lambda-\varepsilon, \lambda+\varepsilon)$ is finite dimensional for some $\varepsilon > 0$.*

In the next section we discuss, whether the different spectral components have an observable effect on the dynamics of a quantum system.

2.2.1. Dynamical significance of spectral components

In the last section we have described the spectral theorem for bounded self-adjoined and unitary operators. As we have seen, it is possible to decompose an operator with respect to the different measure theoretic components of its spectral measure. To begin with, this is just a mathematical result that does not reveal any information about the physics of the system under consideration. However, in this section we show that this decomposition is connected to its dynamical properties.

The main result of this section is the so-called RAGE theorem named after Ruelle, Amrein, Georgescu and Enss [Gol85, Rue69, AG73, Ens78]. It has been developed for the continuous time case, where the dynamics is generated by a Hamiltonian. A very good overview, also on additional results concerning upper and lower bounds on the minimal spreading speed, can be found in the excellent paper by Last [Las95].

However, in the following we consider again systems evolving in discrete time steps. Therefore, let us fix a separable Hilbert space $\mathcal{H}$ and some unitary operator U implementing a single time step. One dynamical question we could investigate for such systems is the time dependence of the transition amplitudes between two states $\phi, \psi \in \mathcal{H}$

$$\langle \psi \, , U^n \, \phi \rangle = \int \Theta^n \, \mu_{\phi\psi}(\mathrm{d}\Theta) = \int_{-\pi}^{\pi} e^{\mathrm{i}pt} \mu_{\phi\psi}(\mathrm{d}p) \ .$$

The second formulation of this integral shows that we consider the Fourier transform of the measure $\mu_{\phi\psi}$. Therefore, we know from the Riemann-Lebesgue theorem that the expression tends to zero for n to infinity, if μ_ϕ is absolutely continuous. This implies in particular that if K is a compact operator, $\phi \in \mathcal{H}$ and P_{ac} the projection onto $\mathcal{H}_{ac}$, then the time evolution satisfies [Tes09]

$$\lim_{n\to\infty} \|K\, U^n\, P_{ac}\, \phi\| = 0 \ . \tag{2.3}$$

We can also consider the Cesaro mean of the transition probability, that is

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{l=0}^{n} |\langle \psi\, , U^l\, \phi \rangle|^2 \ .$$

A unitary version of Wiener's theorem tells us that this expression tends to zero if the measure $\mu_{\phi\psi}$ has no pure point part (see theorem A.0.1). The dynamical interpretation of these abstract results is that if the spectral measure $\mu_{\phi\psi}$ is continuous the overlap between the two fixed states will go to zero in the mean. The RAGE theorem strengthens these results in the following way .

Theorem 2.2.6 (RAGE)**.** *Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$ and let $(G_n)_n$ be a sequence of compact operators converging strongly to the identity. The subspaces $\mathcal{H}_c = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$ and H_{pp} admit the following characterization in terms of the dynamical behaviour of vectors $\phi \in \mathcal{H}$*

$$\begin{aligned} \mathcal{H}_c &= \{\phi \in \mathcal{H};\ \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \|G_n U^t \psi\|^2 = 0\} \\ \mathcal{H}_{pp} &= \{\phi \in \mathcal{H};\ \lim_{n\to\infty} \sup_{t\geq 0} \|(\mathbb{1} - G_n) U^t \psi\|^2 = 0\} \ . \end{aligned} \tag{2.4}$$

If we choose as a sequence (G_n) projections onto increasing subspaces of $\mathcal{H}$ the RAGE theorem tells us that vectors corresponding to pure point spectral measures can be found up to small corrections inside some finite dimensional subspace. Vectors corresponding to the continuous part of the spectral measure on the other hand, tend to leave every finite subspace, at least in the Cesaro mean.

We can also turn this argument around and say that the RAGE theorem gives us a dynamical way to probe the nature of the spectral measure of an initial state. More precisely, this test distinguishes between the continuous and the pure point part of the spectrum. In chapter 4 we develop, in the context of recurrence properties of discrete-time systems, a criterion that provides the other distinction, that is between the singular and the absolutely continuous part of the spectrum.

As a final remark in this section let us note that the RAGE theorem gives a first hint on how to characterize localization properties of an operator. According to (2.4) is a state, that is initially only supported on a finite dimensional subspace basically contained in a possible larger finite dimensional subspace for all times if the vector lies in $\mathcal{H}_{pp}$. If we could therefore show that an operator only possesses point spectrum one could assume that any transport in the system should be suppressed. For obvious reasons this weak form of localization is called spectral localization and we will discuss its

properties in section 2.5. There we will also see that the mere existence of pure point spectrum is too weak to guarantee the absence of transport in a physical or operational sense.

2.3. Measures on the unit circle

We have seen in the last section that the spectral measure of an operator contains information about the dynamical properties of the time evolution it induces. Since we want to study unitary and discrete-time evolutions induced by some unitary operator U on a Hilbert space $\mathcal{H}$, the spectral measure we are going to encounter will be supported on the unit circle. Measures on the unit circle however are a well-studied mathematical topic in their own right.

Instead of giving a complete overview, we rather concentrate on the important results. The main idea we are going to exploit is that one can set up a one to one correspondence between probability measures on the unit circle and different classes of analytic functions on the open unit disc. The boundary behaviour of these functions then indicates, where the absolutely and the singular part of the measure are supported. As references for this section we recommend the books and papers by Simon [Sim05b, Sim05c, Sim05a, Sim10].

In the mathematical context the study of measures on the unit circle arises from the study of orthogonal polynomials on the unit circle. Let us denote by $\mathcal{L}^2(\mathbb{T},\mu)$ the space of square integrable functions on $\mathbb{T}$ with respect to a probability measure μ. One key question in this context is the following one: Given a probability measure μ on $\mathbb{T}$, perform a Gram-Schmidt-orthogonalization of the monomials $\{\Theta^n\}_{n\in\mathbb{N}}$. The resulting polynomials Φ_n then satisfy the relation

$$\langle \Phi_m , \Phi_n \rangle = \int \overline{\Phi}_m(\Theta)\, \Phi_n(\Theta)\ \mu(\mathrm{d}\Theta) = \delta_{m,n}\ .$$

Let us denote with ϕ_n the polynomial resulting from normalizing Φ_n with respect to μ in the $\mathcal{L}^1$ sense. Is it possible to infer directly from the measure μ, whether the set $\{\phi_n\}$ is an orthonormal basis of $\mathcal{L}^2(\mathbb{T},\mu)$?

This question can be positively answered, but before formulating the precise result, we have to introduce a bit of additional notation. In what follows let μ always be a probability measure on $\mathbb{T}$. We call a probability measure μ trivial, if it is only supported on finitely many points or in other words μ is given by a finite convex combination of point or Dirac measures. If the support is infinite we call μ non-trivial.

Definition 2.3.1. *Let $\mathbb{D} = \{z \in \mathbb{C};: |z| < 1\}$ be the open unit disc. An analytic function F on $\mathbb{D}$ is a Carathéodory function if and only if; $F(0) = 1$ and $\Re F(z) > 0$. A function f on $\mathbb{D}$ is a Schur function, if and only if $\sup_{z\in\mathbb{D}} |f(z)| \leq 1$. A Schur function f is called trivial if there exists $\{z_l \in \mathbb{D}\}$ such that f can be decomposed into a finite Blaschke product*

$$f(z) = e^{\mathrm{i}\theta} \prod_{l=1}^{m} \frac{z - z_l}{1 - \overline{z_l} z}\ . \tag{2.5}$$

In addition to these definitions we need the notion of an inner function on the unit disc, which is a bounded analytic function g on $\mathbb{D}$ such that $|g(z)| = 1$ for a.e. $z \in \mathbb{T}$. Rational inner functions are especially easy to represent [RR94].

Lemma 2.3.2. *A bounded analytic function f on $\mathbb{D}$ is inner and rational if and only if it can be decomposed into a finite Blaschke product.*

In general, a Schur function can always be decomposed into a product of two inner and a non-inner function as described in the following lemma, a proof of which can be found in [Sim05b].

Lemma 2.3.3. *Any inner Schur function f admits a decomposition into a possibly infinite Blaschke product B and a singular inner function f_{SI}, without zeroes on $\mathbb{D}$*

$$f(z) = B(z)\, f_{SI}(z) = e^{\mathrm{i}\theta} \prod_{l=1}^{m} \frac{z - z_l}{1 - \overline{z_l} z}\, f_{SI}(z) \ ,$$

with $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$ and $1 \leq m \leq \infty$. If $f(z)$ is a Schur, but not an inner function, an inner part f_I can be separated and the remaining outer part f_O has an integral representation with respect to $g \in \mathcal{L}^1$

$$f(z) = f_I(z)\, f_O(z) = f_I(z) e^{\frac{\mathrm{i}}{2\pi}\int_{\mathbb{T}} g(u)\frac{u+z}{u-z}\,\mathrm{d}u}$$

This result implies that it is not sufficient to restrict to infinite Blaschke products in order to exhaust the class of inner and particular of inner Schur functions. The next lemma due to Frostman however shows how to transform any inner function into a Blaschke product [Gar81].

Lemma 2.3.4. *If f is a non-constant inner function on $\mathbb{D}$, then, for almost all $\xi \in \mathbb{D}$ the function f_ξ defined as*

$$f_\xi(z) := \frac{\xi - f(z)}{1 - \overline{\xi}\, f(z)} \tag{2.6}$$

admits a representation as a Blaschke product.

Next we note that Carathéodory and Schur functions can actually be used interchangeably and identified with each other by a simple transformation. This change of representation will be very useful in our analysis of the recurrence properties of discrete-time unitary evolutions in chapter 4.

Lemma 2.3.5. *There is a one-to-one correspondence between Schur and Carathéodory functions. For every Carathéodory function F there exists a corresponding Schur function f and vice versa via the transformations*

$$F(z) = \frac{1 + z f(z)}{1 - z f(z)} \quad \textit{and} \quad f(z) = \frac{1}{z}\frac{1 - F(z)}{1 + F(z)} \ .$$

If f is trivial, the corresponding F is a rational function with all its poles lying on $\mathbb{T}$.

We are now going to connect these functions to probability measures on the unit circle. The result presented here can be considered as a special formulation of Heroglotz' representation theorem for Carathéodory functions [Sim05a, Sim05b]. Together with the preceding lemma this result also implies a connection between Schur functions and measures on the unit circle.

Lemma 2.3.6. *For all Carathéodory functions F, there exists a unique probability measure μ_F on $\mathbb{D}$ such that F agrees with the Cauchy-Hilbert transform of μ_F*

$$F(z) = \int \frac{\Theta + z}{\Theta - z}\, \mu_F(\mathrm{d}\Theta) \ . \tag{2.7}$$

Conversely, there is a unique Carathéodory function F_μ for every probability measure μ on $\mathbb{T}$, such that (2.7) *is satisfied.*

Before we turn to the results characterizing the measure μ in terms of its Schur and Carathéodory function we include a short remark on the connection of the former one to the Fourier coefficients of the measure μ. Starting from (2.7) and using the definition of the geometric series we find for $z \in \mathbb{D}$

$$F(z) = \int \frac{2\,\Theta}{\Theta - z}\, \mu(\mathrm{d}\Theta) - 1 = 1 + 2\sum_{n=1}^{\infty} \left(\int \overline{\Theta^n}\, \mu(\mathrm{d}\Theta) \right) z^n \ . \tag{2.8}$$

Let us denote by μ_n the moments of a probability measure μ and introduce its moment generating function, which is also called the Stieltjes transform of the measure

$$\widehat{\mu}(z) = \int \frac{1}{1 - \Theta z}\, \mu(\mathrm{d}\Theta) = \sum_{n=0}^{\infty} \int (\Theta z)^n\, \mu(\mathrm{d}\Theta) = \sum_{n=0}^{\infty} \mu_n z^n \ . \tag{2.9}$$

In addition, we define for an analytic function f the function $\overline{f}(z)$, which we construct from f by conjugating its Taylor coefficients

$$\overline{f}(z) := \overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{a}_n z^n \quad \text{if} \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \ .$$

This enables us to write the identity in (2.8) and the connection between the Schur function of a measure and its Stieltjes transform in the following compact form.

$$F(z) = 2\overline{\widehat{\mu}}(z) - 1 \quad \text{and} \quad f(z) = \frac{1}{z}\, \frac{\overline{\widehat{\mu}}(z) - 1}{\overline{\widehat{\mu}}(z)} \ . \tag{2.10}$$

We now turn to the connection between the boundary behaviour of these functions and properties of the corresponding probability measure μ. We collect the results we need for the characterization of the recurrence behaviour of discrete-time evolutions in the following lemma [Sim05b].

Lemma 2.3.7. *Let μ be a probability measure on the unit circle and write $\mu = \mu_{ac} + \mu_s = \omega(\theta)\frac{\mathrm{d}\theta}{2\pi} + \mu_s$ for its decomposition into its absolutely continuous and singular part. Let us denote by F, f and $\widehat{\mu}(z)$ its corresponding Carathéodory function, its Schur function and its Stieltjes transform. Then,*

(i) $\Re F(re^{\mathrm{i}\theta})\frac{\mathrm{d}\Theta}{2\pi}$ *converges weakly to* $\mu(\mathrm{d}\theta)$

(ii) $\lim_{r\to 1^-} F(re^{\mathrm{i}\theta})$ *exists for a.e.* $\theta \in [0,2\pi)$ *and* $\omega(\theta) = \Re F(e^{\mathrm{i}\Theta})$

(iii) μ_s *has a pointmass at* θ *iff* $\mu(\{\theta\}) = \lim_{r\to 1^-} \frac{1-r}{2} F(re^{\mathrm{i}\theta}) \neq 0$.

(iv) μ_{ac} *is essentially supported on the set* $\left\{\theta \,:\, |f(e^{\mathrm{i}\theta})| < 1\right\}$

We remark that if the Schur function f of μ is inner, *i.e.* $|f(e^{\mathrm{i}\theta})| = 1$ a.e. point (iv) of lemma 2.3.7 implies that μ has no absolutely continuous part. This concludes our overview on results of measures on the unit circle connected to the theory of orthogonal polynomials on the unit circle. In the next section we describe one additional way to characterize measures on the unit circle, the so-called Cauchy transform.

2.3.1. Cauchy transform

In the remainder of this section we introduce one additional theme in the characterization program of measures on the unit circle. This so-called Cauchy transform is closely related to the Stieltjes transform of a measure. The crucial result on the Cauchy transform is theorem 2.3.9, which characterizes the singular part of the measure in terms of the limiting behaviour of its Cauchy transform. For a reference providing a thorough introduction as well as a good overview over the existing literature we refer the reader to [CMR06].

Definition 2.3.8 (Cauchy transform)**.** *For a finite complex Borel measure μ on $\mathbb{T}$ we define for $z \in \mathbb{C}$ its Cauchy transform $K\mu$ as*

$$(K\mu)(z) := \int \frac{1}{1-\overline{\Theta}z}\mu(\mathrm{d}\Theta) \ .$$

Since we can express $K\mu$ in terms of the moments μ_n of μ via

$$(K\mu)(z) = \sum_{n=0}^{\infty} \mu_{-n} z^n \ ,$$

we know that $K\mu$ is an analytic function on $\mathbb{D}$. In the case of the Hilbert transform of a positive Borel measure μ a refined variant of Boole's theorem allows to certify the absence of a singular part of the measure, if the distribution function grows too linearly. For the Cauchy transform, Poltoratski proved that we can even recover the variation of the singular part of the measure, when we consider the distribution function of $K\mu$ [Pol96]. Let us denote by χ_A the characteristic function of a set A.

Theorem 2.3.9 (Poltoratski). *Let μ be a finite complex Borel measure on $\mathbb{T}$, μ_s its singular part, $\mathfrak{I} \subset \mathbb{T}$ an open set and denote by $|\mu|$ the total variation of μ. Then, all Borel functions f satisfy*

$$\int_{\mathfrak{I}} f(\Theta)\,|\mu_s|(\mathrm{d}\Theta) = \lim_{\kappa\to\infty} \pi\kappa \int_{\mathfrak{I}} f(\Theta)\,\chi_{\{\tilde{\Theta}\in\mathbb{T}\,:\,|(K\mu)(\tilde{\Theta})|>\kappa\}}(\Theta)\,\mathrm{d}\Theta \ .$$

In addition, the Cauchy transform projects a function onto their positive Fourier series. To see this, consider $f \in \mathcal{L}^2(\mathbb{T}, m)$, where m is the Lebesgue measure on $\mathbb{T}$. The product $f\,m$ is then also a measure on the unit circle and we can define the Cauchy transform of the function f as

$$(Kf)(z) := (Kf\,m)(\Theta) = \int m(\mathrm{d}\Theta)\,\frac{f(\Theta)}{1-\overline{\Theta}z} \ .$$

Expressing f in terms of its Fourier transform, we find that the Cauchy transform indeed projects a function to its positive Fourier coefficients [CMR06]

$$\begin{aligned}(Kf)(z) &= \sum_{n=-\infty}^{\infty} \int m(\mathrm{d}\Theta)\,\frac{f_n\Theta^n}{1-\overline{\Theta}z} = \sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty} f_n z^l \left(\int m(\mathrm{d}\Theta)\,\Theta^{n-l}\right) \\ &= \sum_{n=0}^{\infty} f_n z^n =: f_+(z) \ .\end{aligned}$$

This concludes our short overview on polynomials and measures on the unit circle and the Cauchy transform. As a last remark note that since we are integrating over the unit circle, we can rewrite the integrand of the Cauchy transform as $\Theta(\Theta-z)^{-1}$. If the measure μ corresponds to the spectral measure $\langle \phi\,, E(\cdot)\psi\,\rangle$ of some unitary operator U, then computing the Cauchy transform gives us exactly the matrix elements $\langle U^*\phi\,, (U-z)^{-1}\psi\,\rangle$ of the resolvent of U. This relation is used in chapter 5 to relate the transition probabilities of a disordered quantum walk to the distribution function of the Cauchy transform of its spectral measure.

2.4. Quantum walks

In this section we review the basic theory of quantum walks. We begin with some general remarks on the development and general framework, before we give more detailed results in the translation-invariant as well as in the non-translation-invariant case. Although we cite many of the results mentioned, we refer the reader to the following general review papers both of which contain a more complete literature overview [Kem03, VA12].

In essence, an ideal quantum walk describes a strictly local unitary discrete time evolution of a single particle with some internal degree of freedom on a lattice. In this context strict locality refers to the requirement that an initially localized particle stays confined to a finite region after a single time step. We reserve the term quantum walk to describe discrete dynamics. If we address quantum processes with a continuous time

evolution with tunneling probabilities to different lattice sites we use the term continuous time quantum walk.

The most prominent property of an ideal and translation-invariant quantum walk dynamics, in particular in comparison with its classical counterpart, a random walk, is certainly its ballistic spreading behaviour. Here ballistic spreading refers to the fact that the width of the position distribution of the particle increases linearly in time ($\sim t$), whereas in the classical setting a diffusive spreading ($\sim \sqrt{t}$) is observed.

Originally, the concept of quantum walk like dynamics arose from the effort to quantize classical random walks. A first operational proposal to realize such a quantum random walk was given in 1993 by Aharonov, Davidovich and Zagury [ADZ93]. Their scheme requires indeed a particle with internal degree of freedom, but instead of a coherent evolution they propose to measure and reprepare the particle in every time step. However, already in this proposal a spreading behaviour deviating from the classical one was observed.

Classical random walks have found countless applications in such diverse areas as biology [BdLVC05, KS83], genetics [vdEST92, LS91] , neuroscience [GM64, TYZ$^+$09], and the analysis of social networks [SM11]. Therefore, the observed speedup sparked an interest in the analyses of quantum walks. On the one hand, so-called coined quantum walks on one-dimensional lattices were analyzed, where the dynamics is composed of a transport step and a unitary transformation of the internal degree of freedom [ABNW01]. On the other hand, also the problem of a concise definition of quantum walks including the dynamics on more general graph structures has been investigated [AAKV01, CHKS09, CKS10, AD10].

One key motivation was to compare the performance of classical and quantum walks on such graphs. This led to the reformulation of performance measures known for classical random walks like hitting times [KT03a, KB06, CG11, MNRS09, AR05], mixing times [AAKV01, KT03a, MPAD08, MPA10] and standard deviation [MBSS02, LKBK10] within the quantum mechanical context. In addition, the behaviour of quantum walks in terms of absorbing lattice sites [ABNW01, BCG$^+$04] and the dependence on the initial state [TFMK03, dVRR10, YKI03] have been studied. Generically, a quadratic speed up with respect to classical random walks on the same graph structure was observed.

In addition to the performance measures inspired by the connection with classical random walks, the transport properties of quantum walks where also analyzed from a more quantum mechanically viewpoint by considering the time evolved position operator [ABNW01, GJS04]. It was shown that in the translation-invariant case the random variable Q_n corresponding to a position measurement after n time steps converges weakly in ballistic rather then diffusive scaling to some distribution, that is, Q_n/n instead of $Q_n/\sqrt{n}$ has a meaningful limit for large n [WKKK08, Kon05]. These results where then refined to include higher moments [Kon02] and finally the full generating function of Q_n [AVWW11]. The relevant mathematical techniques to show these results include Fourier methods [ABNW01, GJS04, AVWW11], path counting combinatorics [Kon05, Kon02, KNSS03], the theory of Jacobi polynomials [CIR03, CRT05] and differential geometrically considerations [BGPP09, BBBP11]. From the physical point of view it is interesting that in the translation-invariant case it is possible to define a group velocity operator, which on the one hand contains the full information about the asymptotic spreading behaviour of the walk and on the other hand can be computed

from the dispersion relation, which is given by the momentum dependent eigenvalues of the Fourier transformed walk operator, by derivation with respect to momentum [AVWW11, Ahl13].

A second line of research evolved around algorithmic applications. It was noted early on that there are instances of quantum walks where the spreading is faster then that of a classical random walk on the same lattice structure [Kem05]. This fact led to the development of search algorithms, which make use of quantum walks and which were shown to provide a speed up compared to the most efficient known classical search algorithms in one [SKW03, Amb07] and higher-dimensional arrays [Tul08, AKR05]. In the context of search algorithms also the quantization of general Markov chains was addressed [San08, MNRS11]. An overview about possible applications including element distinctness, matrix product verification and group commutativity can be found in [San08], see also [BŠ06, Amb07, MN07].

Parallel to the development of quantum algorithms for quantum walks, experimental proposals for actual implementations were put forward based on ions in ion-traps, neutral atoms in optical traps and NMR systems, as well as photons passing through several beam splitters [DRKB02, RLBL05]. In the meantime, quantum walks have been realized experimentally in a variety of physical systems, *e.g.* neutral atoms in optical lattices [KFC+09, GAS+13], trapped ions [ZKG+10, SMS+09], light pulses in optical fibres [SCP+10, SGR+12] as well as wave guide arrays [PLM+10].

These experimental realizations together with the question how the aforementioned search algorithms might perform under realistic conditions in turn raised the question how experimental imperfections might influence the system [KT03a, AR05, AAA10, CB13, EPS06, HS08]. In particular in the one-dimensional lattice case and for coined quantum walks several different error models, like broken links and decoherence of the internal state, have been investigated [KT02, LP11, RSA+05]. In conclusion, the generic effect of noise that preserves translation invariance but fluctuates in time is decoherence that causes a transition from a ballistic to a diffusive spreading behaviour [MK10, KT03b, BCA03b]. This means that the position distribution of the decoherent quantum walk agrees for small time scales with the ideal dynamics, but that the spreading behaviour of the particle for larger timescales is proportional to $\sqrt{t}$. These results have been extended to higher-dimensional quantum walks and correlated noise, where the time evolution is driven by some additional classical Markov process [AVWW11, BCA03a, Joy11a]. In the case of non-translation-invariant noise mostly the effect of single defects on some specific lattice site and its impact on the spectrum of the walk have been studied by the use of CMV-matrices [CGMV10, CGMV12, KS10] and Fourier methods [AAM+12].

In the course of this more physical approach to quantum walks as a discrete unitary time evolutions the above mentioned axiomatic definition as a space and discrete-time local evolution of a single particle with an internal degree of freedom was put forward [AVWW11, GNVW12]. Starting from such an axiomatic approach it would be interesting to find simple building blocks that allow for the construction of quantum walks and that exhaust this definition similar to the Stinespring dilation in the case of completely positive maps. One possible candidate of such a set of operations are concatenations of finite step size shift operations and local unitaries, however, as of today this result has only been proven in the one-dimensional case [Vog09].

A question related to this constructive approach is the following one. Assume we relax our conditions and no longer require translation invariance. Can we learn something about the general structure of a discrete-time unitary operation if we determine its local action? An affirmative answer was given in [GNVW12], where such a locally computable invariant called the index of a quantum walk is defined. More precisely only quantum walks of the same index, which in the coined quantum walk scenario in one dimension means that they basically share the same shift operation, can be combined into one quantum walk if one insists on unitarity and locality [Vog09, GNVW12]. In the translation-invariant case it is also possible to define an index in higher dimensions that is vector valued and measures the shift imbalance with respect to the corresponding lattice dimension [Vog09].

As mentioned in the beginning a quantum walk describes the dynamics of a single particle. From an experimental point of view it is therefore very suggestive to see what physical effects can be observed in the system. It turns out that they include such diverse effects as Landau-Zener tunneling [RBH$^+$11], the Klein-paradox [Kur08], topological phases [KRBD10, Kit12], Bloch oscillations [RBH$^+$11] and the effects of electric fields [CRW$^+$13]. Also the correlation and entanglement properties of quantum walks have been studied extensively [RFR12, ADF07, ASRD06, AGG12]. In addition, it was investigated how the addition of more particles changes the system. This has been done in the two and few particle regime for non-interacting as well as interacting systems. In the non-interacting case mostly the dependency on the symmetry and entanglement of the initial state has been studied [EB05, VAB09]. In the interacting case, it has been shown that the formation of bound states can be observed and that these molecules themselves evolve according to an effective quantum walk [AAM$^+$12]. There is one other way to connect quantum walks to many particle systems related to reversible quantum cellular automata [WS04]. Those are quantum systems with infinitely many particles that evolve according to a discrete time evolution that is reversible and local. The notion of locality is similar to the quantum walk case. Namely we demand that local observables of the system are mapped into local observables on a larger region, which is called the neighbourhood of the automaton. It can be shown that for any translation-invariant quantum walk there exists a reversible quantum cellular automaton such that the walk dynamics corresponds to its single particle sector [Vog09]. Note however, that this identification is not unique and one can find quantum cellular automata either minimizing the local cell dimension or the size of the neighbourhood.

This finishes our overview over the literature and results on quantum walks. In the next two subsection we give precise formulation of the results we build on later in this thesis. The first section deals with the theory of translation-invariant walks and contains a quick introduction to the Fourier method and the results on the asymptotic position distribution. The second part gives some definition in the non-translation-invariant case with a special emphases on the one-dimensional case which is the bases for the definition of disordered quantum walks in chapter 5.

2.4.1. Translation invariant quantum walks

We already characterized quantum walks as the strictly local and discrete-time dynamics of a single particle with an internal degree of freedom on a lattice. The Hilbert space

of the system is therefore given by the tensor product of the Hilbert space $\ell_2(\mathbb{Z}^s)$ of square-summable sequences on $\mathbb{Z}^s$ and some k dimensional Hilbert space $\mathcal{K}$ representing the internal state space. Sometimes it is also useful to identify $\ell_2(s)\otimes\mathcal{K}$ with the Hilbert space $\ell^2(\mathbb{Z}^s,\mathcal{K})$, the Hilbert space of $\mathcal{K}$-valued functions ψ on $\mathbb{Z}^s$ such that $\sum_{x\in\mathbb{Z}^s}\|\psi(x)\|^2_{\mathcal{K}}\leq\infty$.

Under the additional assumption of translation invariance, a formal definition of a general translation-invariant quantum walk on $\mathbb{Z}^s$ of a particle with a k dimensional internal degree of freedom in terms of quantum channels (see section 2.1) was given in [AVWW11]. On $\ell_2(\mathbb{Z}^s)$ we introduce the basis $\{\delta_x;\ x\in\mathbb{Z}^s\}$ of sequences that are equal to one at position $x\in\mathbb{Z}^s$ and zero everywhere else. Let us define for any operator $A\in\mathcal{B}(\ell_2(\mathbb{Z}^s)\otimes\mathcal{K})$ and any $x,y\in\mathbb{Z}^s$ the transition operators

$$A_{xy}:=\mathrm{tr}_{\ell_2(\mathbb{Z}^s)}\big((|\delta_x\rangle\langle\delta_y|\otimes\mathbb{1})A\big)=\langle\delta_y|A\,\delta_x\rangle\ . \tag{2.11}$$

Note that A_{xy} is an operator on $\mathcal{K}$ for all $x,y\in\mathbb{Z}^s$ and any operator $A\in\mathcal{B}(\ell_2(\mathbb{Z}^s)\otimes\mathcal{K})$ admits a decomposition in terms of $|\delta_x\rangle\langle\delta_y|\otimes A_{xy}$. In order to define what we mean by translation invariance let us introduce the lattice translation operation τ_x on $\mathcal{B}(\ell_2(\mathbb{Z}^s)\otimes\mathcal{K})$ by its action on operators of the form $|\delta_x\rangle\langle\delta_y|\otimes A$ with $A\in\mathcal{B}(\mathcal{K})$

$$\tau_z\big(|\delta_x\rangle\langle\delta_y|\otimes A\big)=\Gamma_z\big(|\delta_x\rangle\langle\delta_y|\otimes A\big)\Gamma_z^*=|\delta_{x+z}\rangle\langle\delta_{y+z}|\otimes A\ .$$

Here Γ_z denotes the unitary operator that implements the shift operation on $\ell^2(\mathbb{Z}^s,\mathbb{C}^d)$, *e.g.* $(\Gamma_z\phi)(x)=\phi(x-z)$. This allows us to characterize translation invariance of an operator by commutativity with arbitrary lattice translations. Note that we do not require unitarity at this point, hence this definition also covers decoherent quantum walks.

Definition 2.4.1. *A quantum channel $\mathcal{W}$ is called a translation-invariant quantum walk if $\mathcal{W}$ commutes with τ_z for all $z\in\mathbb{Z}^s$ and admits a finite neighbourhood scheme, i.e. there exists a finite subset $\mathcal{N}\subset\mathbb{Z}^s$ such that for all internal states $\phi,\psi\in\mathcal{K}$ and a basis $\{\alpha_i\}$ of $\mathcal{K}$*

$$\langle\delta_q\otimes\phi|\mathcal{W}(|\delta_x\rangle\langle\delta_y|\otimes|\alpha_i\rangle\langle\alpha_j|)\,\delta_r\otimes\psi\rangle=0\quad\textit{whenever } q-x\notin\mathcal{N}\textit{ or }y-r\notin\mathcal{N}\ .$$

This definition can naturally be extended to more general graph structures. Note that with this definition the concatenation of two quantum walks W_1 and W_2 is again a quantum walk, because its neighbourhood scheme is contained in the set theoretic sum of the two single neighbourhood schemes. We have discussed in section 2.1 that every quantum channel admits a representation in terms of an isometry into some higher-dimensional Hilbert space. An equivalent representation of a channel can be given in terms of Kraus operators satisfying [KBDW83]

$$\mathcal{W}(X)=\sum_l K_l X K_l^*\quad\text{with}\quad\sum_l K_l K_l^*=\mathbb{1}\ .$$

However, translation invariance of the quantum channel $\mathcal{W}$ itself does not imply that all the Kraus operators K_l have to commute with translations. If this is not the case the quantum walk is said to transfer momentum and examples of such walks can be found in [AVWW11, Ahl13]. In the following we want to exclude this possibility and only

study quantum walks where the Kraus operators themselves can be chosen translation-invariant. Except for a short paragraph on the effect of decoherence, we even restrict our attention to unitary quantum walks. In this case, there is only a single unitary and translation-invariant Kraus operator W, which means that the action of $\mathcal{W}$ is given by

$$\mathcal{W}(X) = W^* A W \quad \text{for all } A \in \mathcal{B}(\ell_2(\mathbb{Z}^s) \otimes \mathcal{K}) \ . \tag{2.12}$$

For further results on the structure of general translation-invariant quantum walks and their explicit Kraus decomposition we refer to [AVWW11, Ahl13].

Let us now turn to the discussion of unitary translation-invariant quantum walks. Since we are dealing with a translation-invariant system it seems very suggestive to consider the Fourier transform of the implementing unitary operator W from (2.12). The dual group to $\mathbb{Z}^s$ is isomorphic to the s dimensional torus $\mathbb{T}^s$. Hence, the Fourier transform $\mathcal{F}$ is a map from $\ell_2(\mathbb{Z}^s, \mathcal{K})$ to the Hilbert space $\mathcal{L}^2([0,2\pi)^s, \mathcal{K})$ of square integrable functions on the interval $[0,2\pi)^2$ with values in $\mathcal{K}$ [Dei05]. Explicitly, we define the Fourier transform and its inverse of vectors $\psi \in \ell_2(\mathbb{Z}^s, \mathcal{K})$ and $\phi \in \mathcal{L}^2([0,2\phi)^s, \mathcal{K})$

$$\begin{aligned} \psi(p) \equiv (\mathcal{F}\psi)(p) &:= \sum_{x \in \mathbb{Z}^s} e^{\mathbb{i} x \cdot p} \psi(x) \\ (\mathcal{F}^* \phi)(x) &:= \frac{1}{(2\pi)^s} \int_0^{2\pi} e^{-\mathbb{i} x \cdot p} \phi(p) \, \mathrm{d}p \ . \end{aligned} \tag{2.13}$$

The action of an operator $A \in \mathcal{B}(\mathcal{L}^2([0,2\pi)^s, \mathcal{K}))$ on a vector ψ is then given by

$$(A\psi)(p) = \int_0^{2\pi} A(p,p')\psi(p') \, \mathrm{d}p \ ,$$

and we can think of $A(p,p')$ as an integral kernel representing the operator A. If a translation-invariant quantum walk $\mathcal{W}$ is assumed to be unitary and possesses no momentum transfer, also the implementing unitary operator W commute with translations. This implies that the transition operators $W_{x,y}$ from (2.11) satisfy for all $x, y \in \mathbb{Z}^s$, $\phi, \psi \in \mathcal{K}$

$$\langle \psi \, , W_{xy} \, \phi \rangle = \langle \delta_y \otimes \psi \, , W \Gamma_x \, \delta_0 \otimes \phi \rangle = \langle \delta_{x-y} \otimes \psi \, , W \delta_0 \otimes \phi \rangle = \langle \psi \, , W_{x-y0} \phi \rangle \ .$$

Therefore, the transition operators W_{xy} depend only on $x - y$ and moreover, due to the locality condition, W_{xy} vanishes whenever $x - y \notin \mathcal{N}$. Expanding W with respect to the local transition operators from (2.11) we obtain for its Fourier transform

$$\begin{aligned} (\mathcal{F} W \mathcal{F}^*)(p,p') &= \sum_{x,y \in \mathbb{Z}} e^{\mathbb{i} p \cdot x} e^{-\mathbb{i} p' y} W_{x-y0} = \sum_{x,z \in \mathbb{Z}} e^{\mathbb{i}(p-p') \cdot x} e^{\mathbb{i} p' \cdot z} W_{z0} \\ &= \delta(p-p') \sum_{z \in \mathcal{N}} e^{\mathbb{i} p \cdot z} W_{z0} \ , \end{aligned}$$

where we used in the last step that the transition operators W_{z0} vanish if z is not an element of the neighbourhood $\mathcal{N}$. In other words, the Fourier transform of a translation-invariant walk operator W takes the form of a p dependent unitary matrix $W(p) \in \mathcal{B}(\mathcal{K})$

Table 2.1.: Building blocks of coined quantum walks. The set $\{\alpha_i\}$ denotes a basis of $\mathbb{C}^k$, e_l the l^{th} unit direction in $\mathbb{Z}^s$ and $V \in \mathcal{SU}(k)$ a $k \times k$ unitary matrix with determinant 1

Coin operator U	$U = \mathbb{1} \otimes V = \bigoplus_{x \in \mathbb{Z}^s} V$
Shift operator S_l	$S_{l,j}\lvert x, \alpha_i\rangle = \begin{cases} \lvert x, \alpha_i\rangle & i \neq j \\ \lvert x + e_l, \alpha_j\rangle & i = j \end{cases}$

the entries of which are Fourier polynomials in the variable $e^{\mathbb{i}p}$ of finite degree, *i.e.*

$$W(p) := (\mathcal{F} W \mathcal{F}^*)(p) = \sum_{z \in \mathcal{N}} e^{\mathbb{i}p \cdot z} W_{z0} \quad \text{and} \quad (W\psi)(p) = W(p)\psi(p) \quad \forall \psi \in \mathcal{L}^2([0, 2\pi)^s, \mathcal{K}) \ . \tag{2.14}$$

The vectors $z \in \mathcal{N}$ correspond exactly to the allowed transitions in the position space representation of the quantum walk.

As already mentioned in the introduction there is a simple construction scheme to generate unitary quantum walks by concatenating a finite number of local unitary operations acting exclusively on the internal degree of freedom and shift operations that translate the particle depending on its initial state. To be more precise, set $\mathcal{K} = \mathbb{C}^k$ and let $\mathcal{SU}(k)$ be the group of $k \times k$ dimensional unitary matrices with determinant one, then a translation-invariant local unitary operation U is defined by

$$U := \mathbb{1}_{\mathbb{Z}^s} \otimes V$$

for $V \in \mathcal{SU}(k)$. To define the state dependent shift operations we introduce the one step shift operators $S_{l,j}$, which move the particle to next lattice site in the l^{th} lattice direction, if its internal state is equal to some basis state α_j of an orthonormal basis $\{\alpha_i\}$ of $\mathbb{C}^k$ and acts as the identity otherwise. The action of the two operators is summarized in table 2.1. Note that with this definition $S_{l,j}$ as well as U are unitary operators on $\ell_2(\mathbb{Z}^s) \otimes \mathbb{C}^k$. Often, the local unitary operator is called coin operator in analogy to the coin toss of a classical random walk that determines the change of the walker's position.

The general idea is depicted in figure 2.2 in the one-dimensional situation. As noted above in the one-dimensional case every unitary translation-invariant quantum walk according to definition 2.4.1 can be constructed in such a way [Vog09]. With the notation from table 2.1 we can write down our definition of a coined quantum walk.

Definition 2.4.2 (coined quantum walks)**.** *A coined quantum walk W on $\ell_2(\mathbb{Z}^s) \otimes \mathbb{C}^k$ is a unitary quantum walk according to definition 2.4.1 that admits a decomposition into a finite product of shift and coin operations*

$$W = S_{l_n, j_n} U_n \cdots S_{l_1, j_1} U_1 \ .$$

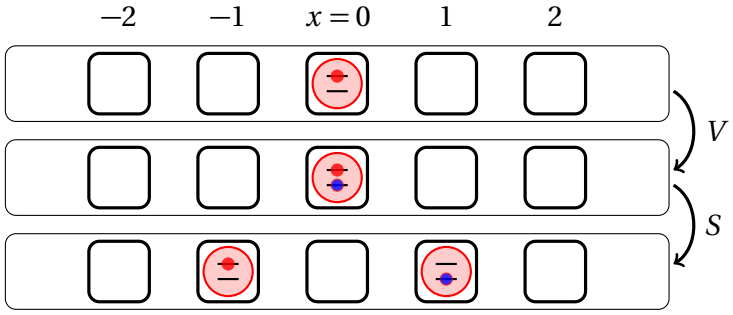


Figure 2.2.: Schematics of a quantum walk on the line with shift and coin evolution. Initially the particle is localized at the origin and the local coin operation V is applied generating a superposition of the internal degrees of freedom depending on which the shift S translates it.

Let us quickly compute the Fourier transform of a coined quantum walk. To this end we analyze how each of the two building blocks transforms separately. Let us begin with the action of the coin operation on an arbitrary $\phi \in \mathcal{H}$

$$(\mathcal{F}U\phi)(p) = (\mathcal{F}((\mathbb{1}\otimes V))\phi)(p) = \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}((\mathbb{1}\otimes V)\phi)(x) = \sum_{x^s\in\mathbb{Z}} e^{\mathbb{i}p\cdot x} V\phi(x) = V\phi(p)\ .$$

Therefore, the Fourier representation of $\mathbb{1}\otimes V$ is given by the p independent operator $V \in \mathcal{SU}(k)$. In the same manner we compute the Fourier representation of the shift operation $S_{l,j}$. Choosing again $\phi \in \mathcal{H}$ arbitrary, we obtain

$$\begin{aligned}
(\mathcal{F}S_{l,j}\phi)(p) &= \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}(S_{l,j}\phi)(x) \\
&= \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}\left(\langle\alpha_j,\phi(x-e_l)\rangle|\alpha_j\rangle + \phi(x) - \langle\alpha_j,\phi(x)\rangle|\alpha_j\rangle\right) \\
&= \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x} e^{\mathbb{i}\cdot p\cdot e_l}\langle\alpha_j,\phi(x)\rangle|\alpha_j\rangle + \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}\phi(x) - \langle\alpha_j,\phi(x)\rangle|\alpha_j\rangle \\
&= e^{\mathbb{i}p\cdot e_l}|\alpha_j\rangle\langle\alpha_j| + (\mathbb{1} - |\alpha_j\rangle\langle\alpha_j|)\psi(p) \\
&= \left(\mathbb{1} + \left(e^{\mathbb{i}p\cdot e_l} - 1\right)|\alpha_j\rangle\langle\alpha_j|\right)\psi(p)\ .
\end{aligned}$$

If we choose for $\{\alpha_j\}$ the standard basis in $\mathbb{C}^k$, the shift operator $S_{l,j}$ assumes the more transparent form of a diagonal matrix with all entries except for the j^{th} one are equal to one

$$S_{l,j} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^{\mathbb{i}p\cdot e_l} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}\ .$$

We summarize the preceding discussion in the following lemma.

Lemma 2.4.3. *Let W be a coined quantum walk given by a set of n shift operators S_{l_i,j_i} and coin operators $U_i = \mathbb{1} \otimes V_i$. Its Fourier transform is given by a p dependent multiplication operator of the form*

$$W(p) = S_{l_n,j_n}(p) \cdot V_n \dots S_{l_1,j_1} \cdot V_1 \quad \textit{with} \quad S_{l_r,j_r}(p) = \mathbb{1} + \left(e^{\mathbb{i}p \cdot e_{l_r}} - 1\right) |\alpha_{j_r}\rangle\langle\alpha_{j_r}| \ .$$

Naturally, a single coin operator as well as all shift operators $S_{l,j}$ are quantum walks themselves, but they comprise the rather boring examples of a particle that rests or moves with a constant speed, respectively. The canonical example of a quantum walk on a one-dimensional lattice, which has been investigated intensively [Kon02, GJS04, NV00] draws on the connection to classical random walks.

In this class of examples one considers a single particle on the line with a two dimensional internal state space, so its Hilbert space is given by $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$. The time evolution alternates between the application of one fixed translation-invariant coin operation induced by $V \in \mathcal{SU}(2)$ and two shift operations that shift one internal state to the left and the other one to the right. Therefore the walk operator is given by

$$W = S_{-1,0} \cdot S_{1,1} \cdot (\mathbb{1} \otimes V) = S \cdot (\mathbb{1} \otimes V) \quad \text{with} \quad S|x,\pm\rangle = |x \pm 1,\pm\rangle \ , \tag{2.15}$$

where $|\pm\rangle$ labels the two basis vectors of some chosen basis of $\mathbb{C}^2$. Employing the Fourier description of the shift and coin parts as discussed in lemma 2.4.3 the p-dependent multiplication operator $W(p)$ representing the quantum walk in momentum space takes the form

$$W(p) = \begin{pmatrix} e^{-\mathbb{i}p} & 0 \\ 0 & e^{\mathbb{i}p} \end{pmatrix} \cdot V \ ,$$

if we identify $|+\rangle$ with $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|-\rangle$ with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The scheme of unitary rotation of the internal space and then a shift operation depending on that internal state looks similar to a classical random walk, where in every time step a coin toss decides whether the particle moves to the left or to the right. However, as already pointed out in the introduction to this section the dynamical behaviour of this systems is remarkably different. A point which we will analyze now.

Asymptotic position distribution for unitary quantum walks

In this section we exclusively consider unitary translation-invariant quantum walks without momentum transfer. Therefore, the Fourier representation of the walk operator is given by a p dependent unitary matrix the entries of which are Fourier polynomials (see (2.14)).

As described before the spreading behaviour of these walks is remarkable different from the case of a classical random walk. In this regard the connection to random walks is even misleading and one should as we emphasised in our definition of a quantum walk, better think about the system as the discrete-time evolution of a single particle.

In order to analyze the dynamical properties of the system we have to specify the questions we want to ask to the system. One interesting question is to study the orbits $\{W^n \phi_0\}$ for some given initial state under the time evolution as given by the walk

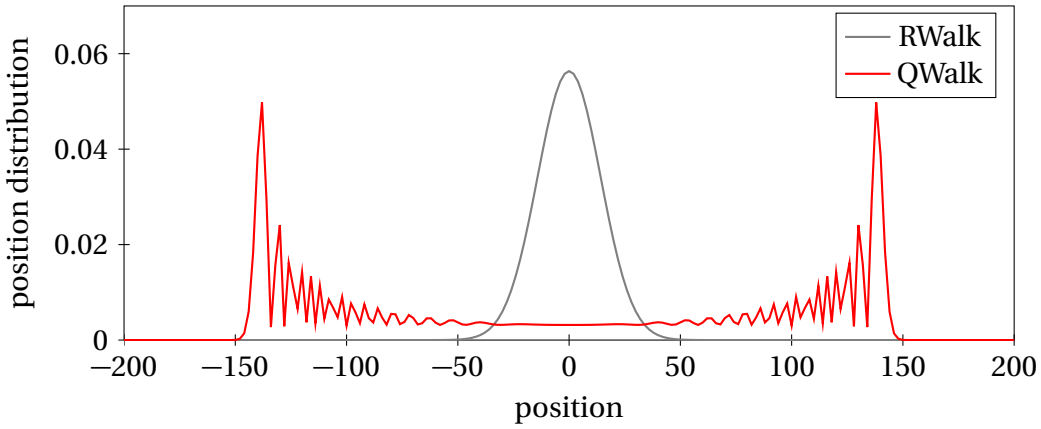


Figure 2.3.: Position distribution after 200 time steps of Hadamard and classical random walk of a particle starting at the origin and with internal state $\phi = \frac{1}{\sqrt{2}}\binom{1}{\mathrm{i}}$.

operator W up to some finite time N. In the translation-invariant case we can answer this question rather easily in the Fourier representation: We can diagonalize the matrix $W(p)$ for every p, expand the initial state ϕ_0 in the p-dependent eigenstates of $W(p)$ and take the eigenvalues $e^{\mathrm{i}\omega_l(p)}$ to the right power.

Figure 2.3 shows the position distribution of an initial state ϕ_0 localized at the origin after 200 time steps for the so-called Hadamard walk. This constitutes a coined quantum walk according to (2.15) where the coin operator is chosen to be

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} . \tag{2.16}$$

To compare the results with the classical case figure 2.3 also contains the position distribution of a random walk, where a fair coin toss decides whether the particle is shifted to the left or to the right. In the following we want to find a good figure of merit that enables us to distinguish between these two kinds of dynamical behaviour.

As it turns out a good concept is given by the asymptotic position distribution of the dynamics, which we briefly describe. For a more thorough introduction see [AVWW11, Ahl13, ACM+12]. The position observable Q of a particle moving on a s dimensional lattice constitutes a random variable with values in $\mathbb{Z}^s$. In order to define the asymptotic position distribution of a particle we want to find the minimal exponent $\alpha \in \mathbb{R}$ such that the distribution of the scaled and time evolved random variable $\frac{Q(t)}{t^\alpha}$ converges to a limiting distribution on the interval $[-1,1]$.

There are several equivalent ways to uniquely characterize the distribution of a random variable and the convergence of different distributions to each other. We could for example look at the probability measure or the cumulative distribution function. Another formulation which turns out to be the convenient choice in the case at hand is to use the characteristic function $C(\lambda)$, which is defined as an integral with respect to the cumulative distribution function F of a random variable Q in the Riemann-Stieltjes

sense [Chu01]

$$C_Q(\lambda) := \mathbb{E}\left(e^{\mathrm{i}\lambda Q}\right) = \int e^{\mathrm{i}\lambda Q}\mathrm{d}F_Q(q) \ .$$

Whenever the random variable Q has a probability density function f_Q then the characteristic function $C_Q(\lambda)$ is simply its Fourier transform. We note that a sequence of random variables Q_t converges weakly to a random variable Q if and only if the corresponding sequence of characteristic functions converges pointwise.

Given the characteristic function of a random variable Q we can obtain the distribution of Q by an inverse Fourier transform. However, this is only strictly true if $C_Q(\lambda)$ is an $\mathcal{L}^1$ function [Don69]. Otherwise, the inverse Fourier transform could be unbounded and we could evaluate it only with respect to suitable test functions. Indeed as we will see in a moment, this is what generically happens for unitary translation-invariant quantum walks. However, there is a way to circumvent this problem, because $C_Q(\lambda)$ itself is given as a Fourier transform [AVWW11, Ahl13].

Although we defined the asymptotic position distribution $\lim_t \frac{Q(t)}{t^\alpha}$ for an arbitrary $\alpha \in \mathbb{R}$ it turns out that the interesting distinction is between two cases. One is the case $\alpha = \frac{1}{2}$, which corresponds to a diffusive spreading of the particle's position distribution, which is the generic case for classical random walks. The other case, $\alpha = 1$, is called ballistic spreading, because the distribution scales linearly with time, which, as we will see, is the generic behaviour of unitary translation-invariant quantum walks.

As described before, our goal is to determine the minimal exponent α such that the characteristic function of the position measurement converges pointwise in λ

$$\begin{aligned} C_{Q,\alpha}(\lambda) &:= \lim_{t\to\infty} C_{Q(t)t^{-\alpha}}(\lambda) = \lim_{t\to\infty} \mathrm{tr}\left(\rho_0 e^{\mathrm{i}\lambda\frac{Q(t)}{t^\alpha}}\right) \\ &= \lim_{t\to\infty} \mathrm{tr}\left(\rho_0 W^{*t} e^{\mathrm{i}\lambda\frac{Q}{t^\alpha}} W^t\right) = \lim_{t\to\infty} \mathrm{tr}\left(\rho_0 \mathcal{W}^t\left(e^{\mathrm{i}\lambda\frac{Q}{t^\alpha}}\right)\right) \ . \end{aligned} \tag{2.17}$$

This pointwise convergence then in turn implies weak convergence of the probability measures corresponding to position measurements after a fixed number of time steps. First note that in the last expression the operator $e^{\mathrm{i}\lambda\frac{Q}{t^\alpha}}$ converges to the identity so the two unitary matrices would cancel. At the same time we conjugate this operator close to the identity with a high power of the unitary matrix W. To analyse this limit we consider a slightly different quantum channel

$$\mathcal{W}_\alpha(X) := W^* X e^{-\mathrm{i}\lambda\frac{Q}{t^\alpha}} W e^{\mathrm{i}\lambda\frac{Q}{t^\alpha}} \ . \tag{2.18}$$

Since two exponential factors cancel if we apply $\mathcal{W}_\alpha$ repeatedly to an observable X we get the following relation between $\mathcal{W}$ and $\mathcal{W}_\alpha$

$$\mathcal{W}^t\left(e^{\mathrm{i}\lambda\frac{Q}{t^\alpha}}\right) = \mathcal{W}_\alpha^t(\mathbb{1})\, e^{\mathrm{i}\lambda\frac{Q}{t^\alpha}}$$

and we can drop the exponential factor on the right-hand side for localized initial states ρ_0 in the limit of large t if we insert this relation into (2.17). The important thing to notice is that $\mathcal{W}_\alpha$ still commutes with lattice translations, because the two phase factors

contributed by the two exponential factors in (2.18) cancel each other. Therefore, $\mathcal{W}_\alpha$ just like $\mathcal{W}$ maps translation-invariant, that is p-dependent $k \times k$ matrices in momentum representation, to translation-invariant operators. Using the fact that conjugation by the operator $e^{\mathbb{i}\lambda Q}$ implements a shift in the momentum variable we find according to (2.18)

$$(W_\alpha(A))(p) = W^*(p)A(p)\Big(e^{-\mathbb{i}\lambda\frac{Q}{t^\alpha}} W e^{\mathbb{i}\lambda\frac{Q}{t^\alpha}}\Big)(p) = W^*(p)A(p)W(p+\tfrac{\lambda}{t^\alpha}) \ .$$

From the relation between $\mathcal{W}$ and $\mathcal{W}_\alpha$ we know that we are interested in the image of the identity operator. Note that in the limit, $\mathbb{1}$ is an eigenvector of the operator $\mathcal{W}_\alpha$ with corresponding eigenvalue 1. However, it is not the only eigenvector to this eigenvalue, because every operator that commutes with $W(p)$ also has this property. Since this certainly holds for the eigenprojections, the eigenspace of the eigenvalue 1 is at least k-dimensional. Note that the degeneracy might be even higher if $W(p)$ itself is degenerate for some p. As a unitary operator on $\mathbb{C}^k$, the matrix $W(p)$ admits a spectral decomposition

$$W(p) = \sum_l e^{\mathbb{i}\omega_l(p)} P_l(p) \tag{2.19}$$

with the dispersion relations or quasi-energies $\omega_l(p)$ and the eigenprojections $P_l(p)$. This decomposition is not unique if there are degeneracies. However, perturbation theory assures that in some neighbourhood around $\varepsilon = 0$ the eigenprojections $P_l(p)$ can be chosen such that the functions $P_l(p+\varepsilon\lambda)$ as well as $\omega_l(p+\varepsilon\lambda)$ are simultaneously analytic with respect to ε. From now on we fix such a choice for (2.19).

The analyticity of $W(p+\varepsilon\lambda)$ carries over to $\mathcal{W}_\alpha$ in the following way. First we decompose $\mathcal{W}_\alpha$ with respect to an operator basis of $\mathcal{B}(\mathbb{C}^k)$ and the Hilbert-Schmidt scalar product $\langle X, Y\rangle = \operatorname{tr} X^*Y$. Note that for any two families $\{P_l^i\}_{l=1}^k$ of k orthogonal projections on $\mathbb{C}^k$ the operators $R_{kl}(X) := P_k^1 X P_l^2$ on $\mathcal{B}(\mathbb{C}^k)$ are hermitian with respect to the Hilbert-Schmidt scalar product and themselves a family of orthogonal projections. Now choosing $P_k^1 = P_r(p)$ and $P_l^2 = P_l(p')$ we can decompose $\mathcal{W}_\alpha$ as

$$\mathcal{W}_\alpha = \sum_{k,l} e^{\mathbb{i}(\omega_l(p+\lambda t^{-\alpha})-\omega_l(p))} R_{kl} \ ,$$

which comprises a spectral decomposition in terms of eigenvalues and eigenvectors, which all depend analytically on $t^{-\alpha}$. Applying this operator t times to the identity we find

$$\mathcal{W}_\alpha^t(\mathbb{1}) = \sum_{k,l} e^{\mathbb{i}\, t(\omega_l(p+\lambda t^{-\alpha})-\omega_l(p))} P_k(p) P_l(p+t^{-\alpha}\lambda) \ ,$$

which is correct up to all orders. For large times analyticity implies that $P_l(p+t^{-\alpha}\lambda) \to P_l(p)$ and since the $P_l(p)$ are orthogonal projections only the diagonal terms $k = l$ remain. Furthermore, considering the limit in ballistic scaling, *i.e.* setting $\alpha = 1$, we see that the exponent in the prefactor converges to a derivative. So in total we find

$$\lim_{t\to\infty} \mathcal{W}_1^t(\mathbb{1}) = \sum_l \exp\left(\frac{\mathrm{d}\omega_l(p+\varepsilon\lambda)}{\mathrm{d}\varepsilon}\right)\Bigg|_{\varepsilon=0} P_l(p) \ ,$$

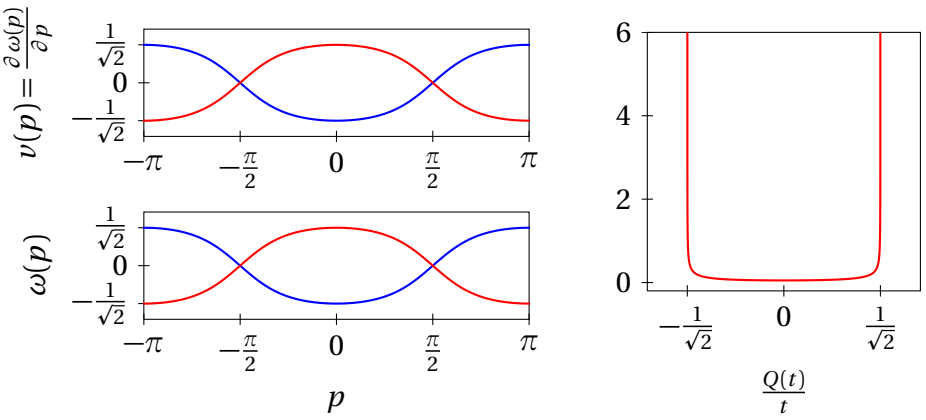


Figure 2.4.: Left panel: dispersion relation and group velocity of the Hadamard walk. Right panel: asymptotic position distribution of the Hadamard walk with initial state $(1,\mathbb{i})/\sqrt{2}$

where we substituted ε for t^{-1} in the last expression. At momenta p, where the dispersion relation of $W(p)$ is degenerate there might be no analytic choice of the projections P_l and eigenvalues ω_l [AVWW11]. However, one can show that the set of these points has Lebesgue measure zero [Ahl13]. We call points p regular if there exists an analytic choice of projections and dispersion relations around p, which in particular include the momenta where $W(p)$ is non-degenerate [AVWW11]. For the regular momenta we can define the group velocity operator V as a vector operator with the components

$$V_k := \sum_l \frac{\mathrm{d}\omega_l(p)}{\mathrm{d}p_k} P_l(p) \tag{2.20}$$

With this definition we can formulate the main technical result on unitary quantum walks.

Theorem 2.4.4. *Let $\mathcal{W}$ be a unitary translation-invariant quantum walk and assume that $W(p)$ is regular for almost all $p \in [0,2\pi)$. The time evolved position operator $Q(t) = W^{*t} Q W^t$ satisfies for all bounded continuous function $f : \mathbb{R}^s \to \mathbb{C}$*

$$\lim_{t\to\infty} \mathrm{tr}\left(\rho\, f(\tfrac{Q(t)}{t})\right) = \mathrm{tr}\left(\rho\, f(V)\right) \ ,$$

where V is the group velocity operator according to (2.20)*. Therefore, for any initial state ρ with finite support, the distribution of $\frac{Q(t)}{t}$ converges weakly to the distribution of V with respect to ρ.*

This result further strengthens our interpretation of a quantum walk as the discrete-time evolution of a single particle. As in the time continuous case of a particle in a periodic, that is translation-invariant, potential, the dispersion relation dictates the asymptotic velocity of the particle.

As an example we present the dispersion relation and momentum dependent group velocities of a one-dimensional coined quantum walk in figure 2.4. As one can see the

asymptotic position distribution agrees with the allowed group velocities. The extremal points of the group velocity with respect to p are responsible for the divergences in the asymptotic position distribution. Such caustic effects also occur in higher-dimensional quantum walks, see [AVWW11, Ahl13] for additional examples.

Of course, the results presented here rely heavily on the coherence of the whole process. The question arises, how we can deal with experimental imperfections. In the next section we give an overview on how to classify different regimes of noise and what is known about their influence on the dynamics.

Decoherent quantum walks

In an actual experiment one has to deal with unavoidable imperfections of the setup. In the case of quantum walks, fluctuations of the parameters of the system could for example affect the assumptions of translation invariance and unitarity. In the following we want to classify the fluctuations in the system with regard to two characteristics: On what time scale do the fluctuations occur and do the fluctuations break translation invariance?

Restricting our attention to these two possibilities leaves us with four different cases of temporal and spatial fluctuations. One of them, where neither temporal nor spatial fluctuations are present, corresponds to the ideal case. There could also be slow fluctuations that leave the walk operator unchanged during a single experimental run, but break translation invariance. This case will be analysed in chapters 5 and 6 of this thesis.

Within this subsection we focus our attention mainly on the other extreme, where translation invariance is preserved but the fluctuations change rapidly, such that the walk operator varies from time step to time step. For this kind of fluctuations that respects translation invariance usually the term decoherent or random time-dependent quantum walk is used [AVWW11, Ahl13, Joy11a]

One specific model of this kind of decoherence is the following one. In each time step one of a collection of L possible walk operators $\{W_l\}$ could be chosen independently with probabilities p_l. This scenario corresponds to a quantum channel where the different Kraus operators are given by these walk operators weighted with the square root of their occurrence probability. Hence, we obtain the following representation in the Heisenberg picture according to (2.12)

$$\mathcal{W}(X) = \sum_l p_l W_l^* X W_l \ .$$

More generally, instead of independent fluctuations we could have an additional classical Markov process choosing the walk operator applied in the next time step [AVWW11, Ahl13, HJ12].

Since this change of parameters in time destroys the coherence of the evolution, physical intuition lets us suspect that the quantum features should vanish over time. It turns out that this intuition can be made precise using similar Fourier techniques as in the unitary case. The idea is again to consider the perturbed operation $\mathcal{W}_\alpha$ from (2.18).

After subtracting a possible deterministic drift with velocity v_0 from the position distribution generically the right scaling is given by $\alpha = \frac{1}{2}$. Then, the random variable

Table 2.2.: Asymptotic behaviour of quantum walks under the influence of different noise models. The purely position dependent case is treated in chapters 5 and 6.

		position dependent fluctuations	
		No	Yes
temporal fluctuations	No	$\sim t$	~ 1
	Yes	$\sim \sqrt{t}$	$\sim \sqrt{t}$

$Q(t)/\sqrt{t}$ converges to a meaningful limit, which exactly agrees with the scaling of the position distribution of a classical random walk. In this sense we can observe a quantum to classical transition of the dynamics in the limit of large times. The transition can for example be observed in the variance of the position distribution that changes from a linear to a square root behaviour. For a precise statement of this result as well as the proofs and additional examples we refer again to [AVWW11, Ahl13, HJ12, Joy11a].

Diffusive behaviour can also be observed in the case of fast fluctuations that change the parameters locally and independently. More precisely, consider a situation, where in every time step at every lattice side the local coin operation is chosen according to some common probability distribution. Also in this case Fourier techniques can be applied to show the suppression of any ballistic evolution and to compute the diffusion constant. Again the basic technique involves an analysis of the perturbation of the eigenvalue 1 of the corresponding operator W_α. For more details we refer to [ACM$^+$12, Ahl13]. Table 2.4.1 summarizes the spreading behaviour within the four different regimes of fluctuations. As one can see decoherence in the sense of temporal fluctuations makes translation-invariant quantum walks slower and disordered quantum walks, as defined in chapter 5, faster, but in both cases the spreading behaviour of a classical random walk is obtained. In the next section we provide a condensed overview over non-translation-invariant results.

2.4.2. Non translation invariant quantum walks

Beyond the translation-invariant regime there are only few results on the general propagation behaviour of quantum walks apart from some numerical experiments [RSA$^+$05, WŁK$^+$04]. The main problem that arises here, is the lack of Fourier techniques that as discussed in the previous sections provide a powerful tool in the translation-invariant case. The existing results however deal almost exclusively with the one-dimensional case, which we briefly summarize here. In almost all cases the one-dimensional situation with the standard shift operation on a two level system from (2.15) is considered and only the local coin operations are changed. Nevertheless, there is no technical difficulty in extending this definition to higher dimensions. However, we will restrict our discussion here to non translation-invariant coined quantum walks. To this end we just generalize our coin definition by allowing the local unitary operator to vary from lattice site to lattice site.

Definition 2.4.5 (non-translation-invariant coined quantum walk)**.** *A non-translation-invariant coined quantum walk W on $\ell_2(\mathbb{Z}^s)\otimes\mathbb{C}^k$ is a unitary operator that admits a decomposition into a finite product of the form*

$$W = S_{l_n,j_n} U_n \cdots S_{l_1,j_1} U_1 \quad \textit{with} \quad U_i = \bigoplus_{x\in\mathbb{Z}^s} V_{i,x}$$

where the S_{l_i,j_i} are shift operations according to table 2.1 and $V_{i,x}\in\mathcal{U}(\mathbb{C}^k)$ for all i and all $x\in\mathbb{Z}^s$.

As already mentioned, similar to the translation-invariant case mostly the restricted class of the product of a single coin and a single shift operators is studied, *i.e.* the walk operator W is given by

$$W = S\cdot\bigoplus_{x\in\mathbb{Z}} V_x \quad \text{with} \quad V_x\in\mathcal{SU}(2)\ , \tag{2.21}$$

where S is the standard state dependent shift from (2.15) and often a specific dependence of V_x on x is assumed [LS09, SK10].

In contrast to such case studies, the theory of CMV matrices provides a general tool to analyze quantum walks of the form (2.21) [CGMV10, CGMV12]. The framework of CMV matrices developed by Cantero et al. introduces a normal form of unitary operators acting on Hilbert spaces [CMV03]. By the spectral theorem every unitary operator can be decomposed into a direct sum of multiplication operators U_μ each acting on $\mathcal{L}^2(\mu)$ for some probability measure μ on $\mathbb{T}$, *i.e.* the action of U_μ on $f\in\mathcal{L}^2(\mu)$ can we written as $f(z)\mapsto z f(z)$.

In order to construct the CMV matrix representation of the multiplication operator U_μ, we apply the Gram-Schmidt procedure to the Laurent polynomials $\{1,z,z^{-1},z^2,\dots\}$ with respect to the standard scalar product in $\mathcal{L}^2(\mu)$, which induces an orthonormal basis $\{\chi_k\}$ with respect to μ. Setting $\rho_k=\sqrt{1-|\alpha_k|^2}$, the matrix representation of U_μ with respect to the basis $\{\chi_k\}$ is given by [CMV03]

$$\begin{pmatrix}
\overline{\alpha}_0 & \overline{\alpha}_1\rho_0 & \rho_0\rho_1 & 0 & 0 & 0 & 0 & \cdots\\
\rho_0 & -\alpha_0\overline{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & 0 & 0 & \\
0 & -\rho_1\overline{\alpha}_2 & -\alpha_1\overline{\alpha}_2 & -\overline{\alpha}_3\rho_2 & \rho_2\rho_3 & 0 & 0 & \\
0 & -\rho_1\rho_2 & -\alpha_1\rho_2 & -\alpha_2\overline{\alpha}_3 & -\alpha_2\rho_3 & 0 & 0 & \\
0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\alpha_3\overline{\alpha}_4 & \rho_4\overline{\alpha}_5 & \rho_4\rho_5 & \\
0 & 0 & 0 & \rho_3\rho_4 & -\alpha_3\rho_4 & -\alpha_4\overline{\alpha}_5 & -\alpha_4\rho_5 & \\
\vdots & & & & & & & \ddots
\end{pmatrix},$$

where the sequence (α_k) satisfies $|\alpha_k|<1$ and corresponds to the Verblunsky coefficients of the measure μ. Although developed for the half infinite case, this framework has been adapted to the doubly infinite context of quantum walks and to accommodate the internal degree of freedom of the particle. This is basically done by choosing a suitable order of the basis states $\{\delta_x\otimes e_\alpha;\ x\in\mathbb{Z},\ \alpha\in\{0,1\}\}$ for the Gram-Schmidt algorithm, namely $\{\delta_0\otimes e_0,\delta_{-1}\otimes e_1,\delta_{-1}\otimes e_0,\delta_0\otimes e_1,\delta_1\otimes e_0,\delta_{-2}\otimes e_1,\delta_1\otimes e_1,\dots\}$. In the corresponding basis the coined quantum walk $U=S\cdot\bigotimes_{x\in\mathbb{Z}}V_x$ can be decomposed

into a product $\Lambda L \Lambda$, where Λ is a diagonal matrix consisting basically of the phase factors of the diagonal coin entries and L is the following matrix valued CMV or CGMV matrix [CGMV10]

$$\begin{pmatrix} \overline{\alpha}_0 & 0 & \rho_0^L & & & & & \\ \rho_0^R & 0 & -\alpha_0 & 0 & & & & \\ 0 & \alpha_2^* & 0 & 0 & \rho_2^L & & & \\ & \rho_2^R & 0 & 0 & -\alpha_2 & 0 & & \\ & & 0 & \alpha_4^* & 0 & 0 & \rho_4^L & \\ & & \rho_4^R & 0 & 0 & -\alpha_4 & 0 & \\ & & & \ddots & & & & \ddots \end{pmatrix},$$

where the α_k are given by 2×2 matrices, the entries of which are constructed from the coin operations U_k, $\rho_k^L = (\mathbb{1}_2 - \alpha_k^* \alpha_k)^{1/2}$ and $\rho_k^R = (\mathbb{1}_2 - \alpha_k \alpha_k^*)^{1/2}$. For additional details we refer to [CGMV10] and [CGMV12]. Since this approach allows to relate the evolution of an initial state to its spectral measure, also non-translation-invariant cases can be treated. For example, the asymptotic return probability for a localized initial state is characterized in terms of the spectral decomposition of the spectral measure [CGMV12]. In particular, it is shown that an absolutely continuous spectral measure corresponds to a vanishing asymptotic return probability, whereas a mass or Dirac point contained in the spectral measure implies a strictly positive asymptotic return probability.

The CGMV method also allows to study quantum walks on the half line, whereas higher-dimensional analogues are complicated due to the lack of a corresponding theory of orthogonal polynomials on higher-dimensional tori. For a walk on the half line these techniques were also used by Konno and Segawa to study the influence of different boundary conditions [KS10].

Before finishing our discussion on position dependent quantum walks we shortly mention one family of position dependent quantum walks studied by Shikano and Katsura [SK10, SK11]. There the local coin operation $V_x(\alpha)$ is given by a real rotation matrix

$$V_x(\alpha) = \begin{pmatrix} \cos(2\pi\alpha x) & -\sin(2\pi\alpha x) \\ \sin(2\pi\alpha x) & \cos(2\pi\alpha x) \end{pmatrix},$$

where $\alpha \in \mathbb{R}$ parameterizes the family of walk operators W_α. Rationality of α ensures that the coin operations $V_x(\alpha)$ are periodic with respect to the denominator of α, which in turn implies absolutely continuous spectrum and ballistic transport except for the case of total reflecting coins, where $V_x(\alpha)$ becomes purely off-diagonal (see [LS09] and section 5.3). In the irrational case Shikano and Katsura study the α-dependence of $\sigma(W_\alpha)$ numerically [SK10, SK11] and find a behaviour similar to the Hofstadter butterfly, which indicates self similarity [Hof76].

This concludes our overview over results in the non-translation-invariant case and we continue with a short introduction into the theory of random operators and their localization properties.

2.5. Random operators and localization

In this section we give an overview of localization results in disordered lattice systems in the continuous time case and discuss its appropriate translations to the discrete unitary case. Most of the results are borrowed from the excellent review articles [Kir08, MS87] and the textbooks [CL90, PF92, CFS87] on the subject of random Schrödinger operators.

One of the main questions in condensed matter physics, aside from the crystal structure of solids, is to explain the conductivity of different materials. If we neglect the movement of the nuclei in the solid and make a one-particle approximation, which means that we forget about any electron-electron interaction, the Hamiltonian of an electron moving under the influence of the background potential generated by the atoms is given by

$$H = H_0 + V = -\frac{\hbar^2}{2}\Delta + V \ .$$

Here Δ is the usual Laplace operator on $\mathcal{L}^2(\mathbb{R}^s)$ and V is the potential generated by the atoms forming the solid. If all atoms are identical and placed at the sites y of a perfect regular lattice, for example $\mathbb{Z}^s$, the potential V takes the form

$$V(x) = \sum_{y\in\mathbb{Z}^s} q f(x-y) \ ,$$

where q is some coupling constant and the function f describes the single site or single atom potential in the solid. It is easy to see that $V(x)$ is a periodic function with respect to any $y \in \mathbb{Z}^s$ and from the general theory of electrons in periodic lattices it is clear that the operator H has only absolutely continuous spectrum, which according to the RAGE theorem from section 2.2.1 indicates transport in the system.

However, in a real solid we will never find such a perfect crystal. In contrast, every sample contains random impurities and displaced atoms. Such deviations alter the potential that is experienced by the electrons. A random displacement can for example be modeled by a sequence of random vectors $d_y(\omega)$ that describe for every lattice site y how the atom deviates from the ideal position

$$V_\omega(x) = \sum_{y\in\mathbb{Z}^s} q f(x-y+d_y(\omega)) \ .$$

By increasing the magnitude of the $d_y(\omega)$ we can go from a regime of small disorder corresponding to a crystalline structure all the way to amorphic and glass-like materials, where the underlying lattice no longer dominates the distribution of the atoms. The case of a random impurity on the other hand, where on atom is substituted by the atom of another element, can be described by a random variation of the coupling constants $q_y(\omega)$ such that the random potential takes the form

$$V_\omega(x) = \sum_{y\in\mathbb{Z}^s} q_l(\omega) f(x-i) \ .$$

How does this kind of disorder influence the conductivity of the resulting material? Considering the classical Drude-theory that explains conductivity in terms of free path length of the electrons, one would expect that the conductivity decreases because more collisions take place. However, although such an increased number of collisions slows down the particle this model would not completely prevent diffusive transport.

In 1958 however, Anderson introduced a simple model of such random impurities, where a quantum mechanical treatment predicted a complete breakdown of the transport in a certain energy region [And58]. This effect which is now called Anderson localization has been extensively studied within the last five decades. Aside from numerical studies of the time evolution also for interacting particles [SK79, SBZ91] a large community within mathematical physics formalized those ideas and proved the absence of transport in various regimes [AM93, GK01, FFES85, FS83].

The most prominent example studied in this area is the so-called Anderson tight binding model. In this scenario the electrons only move on a s-dimensional lattice and it is assumed that the potential generated by a single atom affects the electron only at the corresponding lattice side. In this case, the Hilbert space of the system is given by the square summable sequences $\ell_2(\mathbb{Z}^d)$ and H_0 is the usual lattice Laplacian. The action of the Hamiltonian H_ω of the Anderson tight-binding model on some $\phi \in \ell_2(\mathbb{Z}^s)$ can then be written as

$$(H_\omega \phi)(x) = (H_0\phi)(x) + (V_\omega \phi)(x) = -\sum_{\langle x,y\rangle} (\phi(y) - \phi(x)) + q_x(\omega)\phi(x) \ , \qquad (2.22)$$

where the sum is taken over the nearest neighbours y of x and the set $\{q_x\}$ is a collection of independent and identically distributed random variables, each modelling the random potential seen by the electron at lattice site x. Before we look at known results about the Anderson model we define a bit more formally what a general random operator should be.

2.5.1. Random operators

Let us introduce the problem a bit more formally and since we are mainly concerned with the connection to quantum walks, we restrict our attention to the lattice case. However, before we can define random operators, first we have to recap some results from probability and ergodic theory. We assume that the reader is familiar with basic measure theory and refer to common textbooks in this area [Gra09, Kal10, Rud87]. Remember that a probability space $\mathfrak{X}$ is a triple $(\Omega, \mathcal{A}, \mu)$. Its constituents are given by a sample space Ω, which we assume to be a separable metric space, a σ-algebra $\mathcal{A}$ on Ω and a probability measure μ on $\mathcal{A}$. In turn, a random variable F, which we often denote by F_ω, is defined as a measurable function from a probability space $\mathfrak{X}$ into a measurable space $(\mathcal{Y}, \mathcal{B})$. As usual, a function F is measurable if its preimage satisfy $F^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{B}$.

To turn a separable Hilbert space $\mathcal{H}$ into a measurable space we choose the Borel σ-algebra $\mathcal{B}_\mathcal{H}$ generated by its norm topology. However, since we assume $\mathcal{H}$ to be separable, the identical σ-algebra is generated by the weak topology as well as the weak-$*$ topology [CL90]. In turn we can define the measurability of a random variable taking values in the space of bounded linear operators of a Hilbert space $\mathcal{H}$ with respect to

either of these generating sets. Note however that in the general case there is a difference between norm-measurable, weak-measurable and weak-$*$-measurable operators [AB07].

Definition 2.5.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathcal{H}$ be a separable Hilbert space. A function $T : \Omega \mapsto \mathcal{B}(\mathcal{H})$ is measurable and denoted by T_ω iff one of the following three equivalent conditions is satisfied*

(i) For all $\phi, \psi \in \mathcal{H}$ the mapping $\Omega \ni \omega \mapsto \langle \psi , T_\omega \phi \rangle \in \mathbb{C}$ is measurable.

(ii) For all $\phi \in \mathcal{H}$ the mapping $\Omega \ni \omega \mapsto T_\omega \phi \in \mathcal{H}$ is measurable.

(iii) $\Omega \ni \omega \mapsto T_\omega \in \mathcal{B}(\mathcal{H})$ is measurable with respect to the Borel σ-algebra on $\mathcal{B}(\mathcal{H})$ generated by the operator norm.

A proof of these equivalences can be found in [CL90]. In the case of Schrödinger operators it is of course not enough to consider bounded operators. The following result, also borrowed from [CL90], provides a criterion to extend measurability to unbounded self-adjoint operators. We omit the proof in favor of a compact presentation.

Proposition 2.5.2. *Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathcal{H}$ be a separable Hilbert space such that for all $\omega \in \Omega$, the function T_ω is a self-adjoint or a unitary operator on $\mathcal{H}$. Denote by E_ω the projection valued measure corresponding to T_ω. The function T_ω is measurable iff one of the following two equivalent conditions is satisfied*

(i) $\Omega \ni \omega \mapsto E_\omega(A) \in \mathcal{B}(\mathcal{H})$ is measurable for all $\begin{cases} A \in \mathcal{B}_{\mathbb{R}} & \text{if } \mathbb{T}_\omega \text{ self-adjoined} \\ A \in \mathcal{B}_{\mathbb{T}} & \text{if } \mathbb{T}_\omega \text{ unitary} \end{cases}$.

(ii) $\Omega \ni \omega \mapsto e^{\mathbb{i} t T_\omega} \in \mathcal{B}(\mathcal{H})$ is measurable for all $t \in \mathbb{R}$ (for T_ω self-adjoint)

We could define a random operator simply by requiring measurability according to definition 2.5.1, however since we will exclusively deal with ergodic operators we include that requirement in our definition. Therefore we have to introduce one additional concept before we can define random operators. To make our life easier we restrict our attention from now on to the Hilbert spaces $\ell_2(\mathbb{Z}^d) \otimes \mathbb{C}^k$, which we already introduced in section 2.4.1 in the context of translation-invariant quantum walks.

Recall that a function τ mapping a probability space $(\Omega, \mathcal{A}, \mu)$ into itself is called a measure preserving transformation iff $\mu(\tau^{-1} A) = \mu(A)$ for all $A \in \mathcal{A}$. Note that we have defined this notion on the preimage, because in general $\tau(A) \notin \mathcal{A}$. A set $A \in \mathcal{A}$ is invariant with respect to a family of measure preserving transformations $(\tau_\alpha)_{\alpha \in J}$ if $\tau_\alpha^{-1}(A) = A$ for all $\alpha \in J$, some index set. A family of measure preserving transformations $(\tau_\alpha)_{\alpha \in J}$ is called ergodic iff $\tau_\alpha^{-1}(A) = A$ implies $\mu(A) = 1$ or $\mu(A) = 0$. If furthermore $(\tau_\alpha)_{\alpha \in J}$ forms a semigroup, then the four tuple $(\Omega, \mathcal{B}, \mu, (\tau_\alpha)_{\alpha \in J})$ is called an ergodic random dynamical system.The following result is the basis for the proof of the deterministic nature of the spectrum of an ergodic operator [CFS87].

Theorem 2.5.3. *Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and $(\tau_\alpha)_{\alpha \in J}$ an ergodic family of measure preserving transformations. A real random variable $f : \Omega \mapsto \mathbb{R}$ that is invariant with respect to this family, i.e. it satisfies $f(\tau_\alpha \omega) = f(\omega)$ for all $\alpha \in I$, is constant μ almost everywhere.*

A family of random variables or stochastic process $(X_\alpha(\omega))_{\alpha\in J}$ on an ergodic random dynamical system, is called ergodic iff $X_\alpha(\tau_\beta\omega) = X_{\alpha-\beta}(\omega)$ for all $\beta \in J$. This motivates the following definition of a random (ergodic) operator [Kle08].

Definition 2.5.4 (random operator)**.** *Let $(\Omega,\mathcal{B},\mu)$ be a probability space, $(\tau_\alpha)_{\alpha\in J}$ a family of ergodic measure preserving transformations and $\mathcal{H}$ a separable Hilbert space. A mapping $T_\omega : \Omega \mapsto \mathcal{B}(\mathcal{H})$ is called a random (ergodic) operator, iff T_ω is measurable and there exists a family of unitary operators $(V_\alpha)_{\alpha\in J}$ on $\mathcal{H}$ such that T_ω satisfies for all $\alpha \in J$ the covariance condition*

$$T_{\tau_\alpha\omega} = V_\alpha T_\omega V_\alpha^* \ . \tag{2.23}$$

The important property of random ergodic operators is that their spectrum is deterministic. In the following we sketch a proof of this result, before we provide an example. We begin with a result on random projections [CFS87].

Lemma 2.5.5. *Let P_ω be a random projection valued operator on a separable Hilbert space $\mathcal{H}$, then the random variable $d_\omega = \dim P_\omega\mathcal{H}$ is constant almost surely.*

Proof. Since $d_\omega = \operatorname{tr} P_\omega = \sum_l \langle e_l\,, P_\omega\ e_l\,\rangle$, the measurability of d_ω follows from the measurability of P_ω. Using ergodicity of P_ω we find for all $\alpha \in J$

$$d_{\tau_\alpha\omega} = \operatorname{tr} P_{\tau_\alpha\omega} = \operatorname{tr} V_\alpha P_\omega V_\alpha^\star = \operatorname{tr} P_\omega = d_\omega \ .$$

So d_ω is invariant with respect to the family of ergodic transformations and therefore constant almost surely due to theorem 2.5.3. □

This result suffices in order to show that the spectrum and even the spectral components of a random operator are deterministic [MK82, CL90, PF92].

Theorem 2.5.6. *Let T_ω be a random unitary or random self-adjoined operator on a separable Hilbert space $\mathcal{H}$. There exists a set Ω_0 of full measure and a subset Σ of the interval $[0,2\pi)$ in the unitary, respectively of $\mathbb{R}$ in the self adjoined case such that $\sigma(T_\omega) = \Sigma$ for all $\omega \in \Omega_0$. In addition, the same result holds separately for each spectral component: $\Sigma_{xx} = \sigma_{xx}(T_\omega)$ for $xx \in \{pp, sc, ac\}$.*

Proof. Denote by E_ω the projection valued measure corresponding to T_ω. We know that $\lambda \in \sigma(T_\omega)$ iff

$$g_\omega(p,q) := \dim((E_\omega(q) - E_\omega(p))\mathcal{H}) > 0 \quad \text{for all } p,q \in \mathbb{Q} \text{ with } p < \lambda < q$$

Since E_ω is measurable by assumption, all functions $g_\omega(p,q)$ are measurable. From the relation

$$f(U T_\omega U^*) = U f(T_\omega) U^*$$

for any bounded Borel function f and unitary operator U it follows by setting f equal to the characteristic function χ_A that the projection valued measure $E_\omega(A)$ also satisfies $E_{\tau_x\omega} = V_\alpha E_\omega V_\alpha^*$.

This implies that for every pair (p,q) of rational numbers $E_\omega([p,q])$ is a random projection, so by lemma 2.5.5 there exists a set of full measure $\Omega_{p,q}$ such that $f_\omega(p,q)$ is equal to a constant $f_{p,q}$ almost surely. Since the rational numbers are countable, the intersection of all $\Omega_{p,q}$ with $p<q$ still has full measure. So we can set

$$\Sigma = \{\lambda\,;\, f_{p,q}>0\,, \forall p,q\in\mathbb{Q} \text{ with } p<\lambda<q\}\ .$$

The non-randomness of the single spectral components can be shown in a similar way. Note for example that $\lambda\in\sigma c(T_\omega)$ or $\lambda\in\sigma_{ac}(T_\omega)$ if the range of the projections $E_\omega P_\omega^{xx}$ is nonzero, where P^{xx} denotes the projection onto $\mathcal{H}_c$, respectively $\mathcal{H}_{ac}$. The only thing that has to be checked beforehand is that $E_\omega P_\omega^{xx}$ or more precisely P_ω^{xx} is measurable. However, this follows directly from the measurability of the spectral measures $\mu_{\phi,\omega} = \langle\phi\,, E_\omega\phi\,\rangle$ and its Lebesgue decomposition [MK82, PF92]. □

Under the additional assumption that the orbits $(V_\alpha\phi)_{\alpha\in J}$ of the family $(V_\alpha)_\alpha$ that implements the ergodicity condition in a certain sense span the full Hilbert space, we can also show the absence of discrete spectrum. To this end, we call a family of unitary operators $\{U_\alpha\}_{\alpha\in J}$ total [MK82] if the set

$$A_J = \{\phi\in\mathcal{H}\,;\, \langle U_\alpha^*\phi\,, U_\beta^*\phi\,\rangle = 0 \text{ for all } \alpha\neq\beta\} \tag{2.24}$$

is dense in $\mathcal{H}$. If $\mathcal{H}$ is separable and A_J is total then J has to be countable. Totality of the unitary operators corresponding to an ergodic family of measure preserving transformations implies the absence of discrete spectra [MK82, CFS87].

Lemma 2.5.7. *Let $\mathcal{H}$ be a separable Hilbert space and T_ω a unitary or self-adjoined random operator with respect to an ergodic random dynamical system $(\Omega,\mathcal{B},\mu,(\tau_l)_l)$. If the family of unitary operators V_l corresponding to τ_l is total, then $\sigma_{disc}(T_\omega)=\emptyset$ almost surely.*

Proof. Because the spectral projections satisfy the covariance condition (2.23) we know from the proof of theorem 2.5.6 and by lemma 2.5.5 that for all $q<p\in\mathbb{Q}$ the functions

$$g_\omega(p,q) = \dim(E_\omega(q)-E_\omega(q))\mathcal{H} = \operatorname{tr} E_\omega([p,q])$$

are equal to a constant $g_{p,q}$ on a set of full measure. By totality the sequences $\{U_l\phi\}$ are orthogonal for every $\phi\in A_J$. If we choose some normalized $\phi\in A_J$ as well as $n\in\mathbb{N}$ and use that $g_{p,q}$ is constant almost surely, we find

$$\begin{aligned}\operatorname{tr} E_\omega([p,q]) = \mathbb{E}\big(\operatorname{tr} E_\omega([p,q])\big) &\geq \sum_{l=1}^{n}\mathbb{E}\big(\langle U_l^*\phi\,, E_\omega([p,q])U_l^*\phi\,\rangle\big)\\ &= \sum_{l=1}^{n}\mathbb{E}\big(\langle\phi\,, E_{\tau_l\omega}([p,q])\phi\,\rangle\big) = n\,\mathbb{E}\big(\langle\phi\,, E_\omega\phi\,\rangle\big)\ ,\end{aligned}$$

where we used that τ_l is measure preserving in the last step. Since we have chosen n arbitrary $\operatorname{tr} E_\omega([p,q])$ is either zero or infinite for all $p<q\in\mathbb{Q}$ depending on whether $\mathbb{E}\big(\langle\phi\,, E_\omega\phi\,\rangle\big)$ is strictly positive. Therefore, with probability one there is no $\lambda\in\mathbb{R}$ respectively $\lambda\in[0,2\pi)$ such that $0<\dim E_\omega([p,q])<\infty$ for all $p<\lambda<q\in\mathbb{Q}$, which rules out discrete spectrum. □

In the remainder of this subsection we show that the Anderson Hamiltonian as introduced in (2.22) is indeed a random operator. The system Hilbert space is given by $\ell_2(\mathbb{Z}^d)$ and the only random part of the Anderson Hamiltonian H_ω is of course the potential V_ω. We assumed that V_ω acts as a pointwise multiplication operator by a real random variable $(V_\omega\phi)(x) = V_\omega(x)\phi(x)$ and that the random variables $V_\omega(x)$ at each lattice side $x \in \mathbb{Z}^d$ are drawn i.i.d according to some common probability measure μ_0. The probability space is then just given by the infinite product space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ equipped with the generic infinite product measure that we denote by μ_∞ and $\omega \in \Omega$ has the form of a vector in $\mathbb{Z}^d$. It is easy to see that the shift operation $(\tau_x\omega)_y = \omega_{y-x}$ is a measure preserving transformation if one considers cylinder sets, which will be done in section 3.1 more explicitly. Ergodicity follows from the fact that the τ_x are even mixing, that is

$$\lim_{\|x\|\to\infty} \mu(A \cap \tau_{-x}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{A} \ .$$

Mixing implies ergodicity, because if we consider an invariant set $A \in \mathcal{A}$ we find

$$\mu(A) = \mu(A \cap A) = \mu(A \cap \tau_{-x}(A)) \to \mu(A)^2 \ ,$$

which exactly means that $\mu(A)$ has to be equal to zero or one [Kre85] (see also section 3.1). Now it is easy to see that the unitary operator Γ_x that implements lattice translation in the sense that $(\Gamma_x\phi)(y) = \phi(y-x)$ satisfies the relation

$$V_{\tau_x\omega} = \Gamma_x V_\omega \Gamma_x \ ,$$

because by construction $V_{\tau_x\omega}(y) = V_\omega(y-x)$. Therefore V_ω is indeed a random operator once measurability is verified, which under some regularity conditions of the single site probability measure μ is for example done in [MK82, CFS87]. This in turn implies that the Anderson Hamiltonian

$$H_\omega = H_0 + V_\omega \tag{2.25}$$

is indeed an (ergodic) random operator. Therefore, by theorem 2.5.6 the spectrum of H_ω is deterministic and since the unitary operators Γ_x are obviously total H_ω also posses only essential spectrum.

Let us for the moment treat the random potential V_ω as a perturbation of the lattice Laplacian H_0. How does the spectrum of H_0 change due to this random potential? To answer this question let us first determine the spectrum of H_0.

To this end, we define the Fourier transform on this lattice system in the same manner as for the quantum walk on $\mathbb{Z}^d$ (see (2.13)). Since the lattice Laplacian H_0 is a translation-invariant operator we can use the Fourier transform to determine its spec-

trum. Computing $\mathcal{F}^* H_0 \mathcal{F}$ and denoting by e_l the standard basis vectors of $\mathbb{R}^s$ we find

$$\begin{aligned}
(\mathcal{F}H_0\phi)(p) &= \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}(H_0\phi)(x) = -\sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x} \sum_{\langle x,y\rangle} \phi(y)-\phi(x)\ . \\
&= 2s\sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}\phi(x) - \sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}\sum_{l=1}^{s} \phi(x+e_l)+\phi(x-e_l) \\
&= \sum_{l=1}^{s}\sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}\left((1-e^{\mathbb{i}p\cdot e_l})+(1-e^{-\mathbb{i}p\cdot e_l})\right)\phi(x) \\
&= 2\left(\sum_{l=1}^{s}(1-\cos(p_l))\right)\sum_{x\in\mathbb{Z}^s} e^{\mathbb{i}p\cdot x}\phi(x)\ .
\end{aligned}$$

So we have shown that the lattice Laplacian H_0 acts as a multiplication operator by the function $\sum_l(1-\cos p_l)$ in the momentum representation. Therefore, its spectrum $\sigma(H_0)$ is absolutely continuous and identical to $[0,4d]$, the range of this function. The continuous time version of the RAGE theorem (see 2.2.6) tells us that this implies transport of the electron. We claimed at the beginning of this section that the Anderson model exhibits localization effects due to the random impurities in the system. So at the very least the absolutely continuous spectrum of H_0 has to be suppressed by the random potential V_ω, but is this enough to exclude any transport in the system? This leads us to the question, what the precise mathematical definition of Anderson localization should be and which we discuss in the next subsection.

2.5.2. Anderson and dynamical localization

In the introduction to this section we claimed that transport breaks down if we consider a disordered system. What should be the mathematical formalization of this concept? There are at least two possible approaches to this problem, leading to two different definitions of localization. The first approach starts from the mathematical point of view, or more precisely, from the RAGE theorem and defines an unitary or self-adjoined operator to exhibit localization if its spectrum solely consists of point spectrum. In the second approach a physical or dynamical point of view is taken and a system is considered to be localized if the time dependent transition probabilities between two lattice sites are exponentially suppressed in their distance with time independent constants. We continue our discussion with the first approach.

In section 2.2.1 we have seen that as long as there is some continuous part in the spectrum unavoidably some transport of the particle takes place. This motivates the following definition [Kle08].

Definition 2.5.8 (spectral localization)**.** *A random self-adjoined or unitary operator T_ω is said to exhibit spectral localization in $\mathfrak{I}\subset\mathbb{R}$ respectively $\mathfrak{I}\subset\mathbb{T}$ iff $(\sigma(\mathbb{T}_\omega)\cap I)\subset\sigma_{pp}(\mathbb{T})$ almost surely. If furthermore, the eigenfunctions ϕ_ω of T_ω corresponding to the spectral interval $\mathfrak{I}$ decay exponentially μ almost surely, that is $\|\chi_x\phi\|$ decays exponentially with x, then T_ω is said to exhibits exponential or strong spectral localization with respect to $\mathfrak{I}$.*

To begin with this is just a mathematical definition for a notion of localization although it is motivated by the RAGE theorem. In addition, Simon [Sim90] showed that for a large class of random Schrödinger operators the absence of continuous spectrum implies at least for compactly supported initial states ϕ_0

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle}{t^2} = \lim_{t\to\infty} \frac{\|Q e^{\mathbb{i}Ht}\phi_0\|^2}{t^2} = 0 \ .$$

We recognize this expression from our discussion of the spreading behaviour of quantum walks in section 2.4. It readily implies the absence of ballistic transport in the system, since otherwise the expectation value of the position operator Q should grow linearly. On the other hand, we have seen that the absence of ballistic transport does not exclude diffusive spreading behaviour of the particle. In the case of Schrödinger operators the absence of continuous spectrum turns out to be even more inconclusive. Indeed, in [dRJLS95] del Rio et al. constructed a deterministic and exponentially localized Schrödinger operator, which nevertheless satisfies

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle}{t^{2-\delta}} = \infty$$

for any $\delta > 0$. Obviously this violates Anderson's idea of localization as the absence of transport. Therefore, a strengthened definition of localization has been introduced. In order to avoid cumbersome case distinctions we define

$$T^t := \begin{cases} T^t & \text{if T is unitary} \\ e^{\mathbb{i}T} & \text{if T is self-adjoined} \end{cases} .$$

Definition 2.5.9 (dynamical localization (weak form))**.** *Let T_ω be a random unitary or self-adjoined operator on $\ell_2(\mathbb{Z}^d)\otimes\mathbb{C}^k$ and $\mathfrak{I}$ an open subset of $\mathbb{T}$, respectively $\mathbb{R}$. We say that T_ω exhibits dynamical localization with respect to $\mathfrak{I}$ if the time evolution generated by T_ω satisfies*

$$\mathbb{E}\left(\sup_t \||Q|^p T_\omega^t \chi(\mathfrak{I})\phi_0\|^2\right) < \infty$$

for all $p > 0$ and all compactly supported $\phi_0 \in \ell_2(\mathbb{Z}^d)\otimes\mathbb{C}^k$.

As announced in the introduction to this section, we know discuss a more operational definition of localization. In this regard localization should imply a decoupling of the dynamics of far apart regions of the system similar to a zero velocity Lieb-Robinson bound [LR72]. We demand that if we prepare a wave packet localized around some lattice site x then a detector that we place at some distant lattice site y should never detect the particle. This basic idea corresponds to the following formal definition.

Definition 2.5.10 (dynamical localization (strong form))**.** *Let T_ω be a random unitary or self-adjoined operator on $\ell_2(\mathbb{Z}^d)\otimes\mathbb{C}^k$ and let $\mathfrak{I}$ be an open subset of $\mathbb{T}$ or $\mathbb{R}$. We say that T_ω exhibits dynamical localization (of the strong form) with respect to $\mathfrak{I}$ if there are positive constants C_1 and C_2 such that for all $x, y \in \mathbb{Z}^s$, $\phi, \psi \in \mathbb{C}^k$ and all $0 < \zeta < 1$*

$$\mathbb{E}\left(\sup_t |\langle \delta_y \otimes \psi\,, \chi(\mathfrak{I}) T_\omega^t \ \delta_x \otimes \phi\,\rangle|\right) \leq C_1 e^{-C_2\|x-y\|^\zeta} \tag{2.26}$$

We remark that some authors refer to definition 2.5.9 as dynamical localization and to definition 2.5.10 as strong dynamical localization. In the context of random unitary models however it is common to use the term dynamical localizations for systems satisfying definition 2.5.10. We will adopt this convention and say that systems, which only satisfy definition 2.5.9 exhibit dynamical localization of the weak form.

Let us briefly show that dynamical localization implies dynamical localization of the weak form. To this end, fix some $p > 0$ and some compactly supported $\phi_0 \in \ell_2(\mathbb{Z}^s) \otimes \mathbb{C}^k$ with $\phi(x) = 0$ for $\|x\| > D$. We obtain for the p^{th} moment of the position operator

$$\begin{aligned}
\| |Q|^p T_\omega^t \chi(\mathfrak{I}) \phi_0 \|^2 &= \sum_{\substack{x \in \mathbb{Z}^d \\ 1 \le i \le k}} \|x\|^{2p} \left| \langle \delta_x \otimes e_i \,,\, T_\omega^t \chi(\mathfrak{I}) \, \phi_0 \rangle \right|^2 \\
&= \sum_x \|x\|^{2p} \left| \sum_{\substack{\|y\| < D \\ 1 \le i,j \le k}} \langle \delta_x \otimes e_i \,,\, T_\omega^t \chi(\mathfrak{I}) \, \delta_y \otimes e_j \rangle \, \langle \delta_y \otimes e_j \,,\, \phi_0 \rangle \right|^2 \\
&\le \sum_x \|x\|^{2p} \sum_{\substack{\|y\| < D \\ 1 \le i,j \le k}} \left| \langle \delta_x \otimes e_i \,,\, T_\omega^t \chi(\mathfrak{I}) \, \delta_y \otimes e_j \rangle \right|^2 \|\phi_0\|^2 \ .
\end{aligned}$$

Taking the supremum with respect to t on both sides and then the expectation value we find for a strongly dynamical localized system

$$\begin{aligned}
\mathbb{E}\left(\sup_t \| |Q|^p T_\omega^t \chi(\mathfrak{I}) \phi_0 \|^2 \right) &\le \sum_x \|x\|^{2p} \sum_{\substack{\|y\| < D \\ 1 \le i,j \le k}} \mathbb{E}\left(\sup_t |\langle \delta_x \otimes e_i | \,,\, T_\omega^t \chi(\mathfrak{I}) \, \delta_y \otimes e_j \rangle|^2 \right) \\
&\le \sum_x \|x\|^{2p} \sum_{\|y\| < D} C_1 k^2 e^{-C_2 \|x-y\|^\zeta} \ ,
\end{aligned}$$

which is indeed finite for all $0 < \zeta < 1$.

Lemma 2.5.11. *Dynamical localization with respect to $\mathfrak{I} \subset \mathbb{T}$ implies spectral localization with respect to the same interval $\mathfrak{I}$ for a random unitary operator W_ω.*

Proof. We borrow an argument from [HJS09] for the unitary Anderson model. The idea is to show that with probability one the projection onto the random subspace $\mathcal{H}_c$ corresponding to the continuous part of the spectrum, only contains the null vector. For the definition of $\mathcal{H}_c$ we refer to section 2.2. Let us denote with P_c the projector onto $\mathcal{H}_c$ and define the projectors G_n onto balls of radius n around the origin via

$$G_n = \sum_{\|x\| < n} |\delta_x\rangle\langle\delta_x| \otimes \mathbb{1} \ .$$

Since the W_ω is unitary it leaves the norm invariant and we obtain for all $T, n \in \mathbb{N}$ and

all $\psi \in \mathcal{H}$

$$\begin{aligned}\|\mathrm{P}_c\psi\|^2 &= \frac{1}{1+T}\sum_{t=0}^{T}\left(\|(\mathbb{1}-G_n)W_\omega^t P_c\psi\|^2 + \|G_n W_\omega^t P_c\psi\|^2\right)\\ &= \frac{1}{1+T}\sum_{t=0}^{T}\left(\|(\mathbb{1}-G_n)W_\omega^t\psi\|^2 - \|(\mathbb{1}-G_n)W_\omega^t P_{pp}\psi\|^2 + \|G_n W_\omega^t P_c\psi\|^2\right)\ ,\end{aligned}$$

where we used the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{pp}$ from section 2.2 in the second step. By the RAGE theorem (2.2.6) the second and the third term in this expression tend to zero if we take the limit first with respect to T and then with respect to n, because the G_n constitute a sequence of operators with finite range that converges strongly to the identity.

To deal with the remaining sum we now consider more specifically the subspace $\chi(\mathfrak{I})\mathcal{H}$, where $\chi(\mathfrak{I})$ denotes the spectral projection onto the subspace corresponding to the interval $\mathfrak{I}$. Let us pick the projection of some arbitrary basis state $\delta_x \otimes e_i$ onto this subspace. Taking the expectation value and inserting the definition of G_n we obtain by Fatou's lemma and Fubini's theorem

$$\begin{aligned}\mathbb{E}\left(\|\mathrm{P}_c\chi(\mathfrak{I})(\delta_x\otimes e_i)\|^2\right) &\leq \liminf_{n\to\infty}\mathbb{E}\left(\lim_{T\to\infty}\frac{1}{1+T}\sum_{t=0}^{T}\|(\mathbb{1}-G_n)W_\omega^t\chi(\mathfrak{I})(\delta_x\otimes e_i)\|^2\right)\\ &\leq \liminf_{n\to\infty}\sum_{j,\|y\|>n}\mathbb{E}\left(\sup_{t\geq 0}|\langle\delta_y\otimes e_j\,, W_\omega^t\chi(\mathfrak{I})\,(\delta_x\otimes e_i)\rangle|^2\right)\\ &\leq \liminf_{n\to\infty}\sum_{j,\|y\|>n} C_1 e^{-C_2\|x-y\|^\xi}\ ,\end{aligned}$$

where we used in the second to last step the dynamical localization of W_ω with respect to the interval $\mathfrak{I}$. Employing the inverse triangle inequality and making n larger than $\|x\|$ such that we can forget about the absolute value in the difference of $\|x\|$ and $\|y\|$ we find

$$\begin{aligned}\mathbb{E}\left(\|\mathrm{P}_c\chi(\mathfrak{I})\delta_x\otimes e_i\|^2\right) &\leq \liminf_{n\to\infty}\sum_{j,\|y\|>n} C_1 e^{-C_2\left|\|x\|-\|y\|\right|^\xi}\\ &\leq C_1 e^{C_2\|x\|^\xi}\liminf_{n\to\infty}\sum_{j,\|y\|>n} e^{-C_2\|y\|^\xi}\ .\end{aligned}$$

The last term converges exponentially fast to zero, so the projection onto $\mathcal{H}_c$ does in the mean only contain the null vector. Positivity of the norm implies that this is true on a x dependent set Ω_x of measure one and since the basis $\{\delta_x\otimes e_i\}$ is countable the intersection of all Ω_x still has full measure, which completes this part of the argument. □

2.5.3. Results on localization

We conclude this section with a short overview over the literature and the basic techniques used to prove spectral and dynamical localization. We do not claim to exhaust

all relevant publications, but refer the reader again to one of the following review articles [Kir08, Kle08, Sto11] or textbooks [CL90, PF92, Sto01].

We already mentioned the seminal paper of Anderson in 1958 [And58], where he argued for the absence of transport in the presence of static disorder. From a physical point of view, supplemented by physical arguments, the intuitive picture raises the following expected localization behaviour of disordered systems [Kir08, Sto11].

First of all, the localization properties depend crucially on the dimensionality of the problem. In one-dimension it is expected that every disordered system exhibits at least exponential spectral localization that is dense pure point spectrum and a complete set of exponentially decaying eigenfunctions [Kir08]. So in the one-dimensional case any kind of disorder should cause reduced transport or even localization of the particles. The same behaviour is assumed to be true in two dimensions, however perturbations such as external magnetic fields could break the localization [Kir08].

The behaviour in three and higher dimensions is expected to be more complicated. The basic intuition is that the band structure corresponding to absolutely continuous spectrum that is present in the translation-invariant case is more stable, because, in a figure of speech, the particle has more space to manoeuver around the impurities. If disorder is added to the system an effect called band edge localization is expected to occur [Kir08]. This just means that point spectrum appears at the band edges of the dispersion relation of the unperturbed system. If the strength of the disorder is increased the localization regions should increase until at some point there is no absolutely continuous spectrum left. This amounts to a predicted phase transition from an conduction phase to an insulator phase, when the level of disorder is increased.

A first rigourous result on exclusive pure point spectrum of a continuous Schrödinger operator was given in 1977 by Gol'dshtein, Molchanov and Pastur [GMP77]. A proof for the original Anderson model was then obtained by Kunz and Souillard in 1980 [KS80]. Both papers treated exclusively the one-dimensional case.

In order to cope with higher-dimensional situations as well as to show dynamical localization essentially two different proof techniques have been developed, one called the multiscale analysis [FS83], the other one the fractional moment method [AM93]. The first of the afore mentioned techniques can also deal with singular distributions, whereas the later, if applicable, provides stronger bounds.

The basic building block in the fractional moment method is to show that the expectation value of fractional powers of the matrix elements of the resolvent decay exponentially in the distance between different lattice sites $x, y \in \mathbb{Z}^d$. More precisely, one has to find an $0 < s < 1$ such

$$\mathbb{E}\left(|\langle y\,, (T_\omega - z)^{-1}\, x\rangle|^s\right) \leq C_1 e^{-C_2\|x-y\|}$$

for some positive and finite constants C_i, which have to be uniform with respect to z and $x, y \in \mathbb{Z}^s$. Such bounds imply localization due to the following theorem, which we state for the self-adjoined case [Gra94, Sto11].

Theorem 2.5.12. *Let $\mathfrak{I} \subset \mathbb{R}$ be open and bounded and let H_ω be the Anderson Hamiltonian from* (2.25)*. If there exist constants $0 < s < 1$ and $0 < C_1, C_2 < \infty$ such that for all $x, y \in \mathbb{Z}^s$*

$$\mathbb{E}\left(|\langle y\,, (H_\omega - E + \mathrm{i}\varepsilon)^{-1}\, x\rangle|^s\right) \leq C_1 e^{-C_2\|x-y\|}$$

uniformly with respect to $E \in \mathfrak{I}$ and $\varepsilon > 0$, then H_ω exhibits dynamical localization with respect to the interval $\mathfrak{I}$.

To prove this theorem, the bound on the fractional moments is first translated into a bound on the second moments of the Green's function [Gra94], which in turn implies an exponential bound on the total variation of the spectral measure $\mu_{x,y}$ for any two lattice sites $x, y \in \mathbb{Z}^d$. Since the total variation upper bounds the transition probabilities in (2.26), dynamical localization follows [Gra94].

The second proof strategy called multiscale analysis can be considered as a probabilistic generalization of the well known mathematical induction. One abstract formulation of the concept is given in section 6.4 and its application to the proof of localization for disordered quantum walks can be found in section 6.5.

The basic idea of the method is to perform a probabilistic iteration procedure, which relies on two assumptions. One is an initial scale estimate that assures that at some fixed length scale n_0 the transition probability within an energy interval $\mathfrak{I}$ is smaller than one, so the particle is not transported with certainty. The second assumption is a so-called Wegner estimate that provides an upper bound on the probability that two independent disordered regions are conductive for the same energy. In the iteration step the distance between two lattice sites x and y is now divided into regions of size n_0. For each individual patch the probability to be transported is less than one, so intuitively one could expect exponential decay to traverse all of these patches. At the same time the probability of resonant transport for one fixed energy through several of these patches is controlled by the afore mentioned Wegner estimate. Iterating this procedure to larger and larger length scales then yields the desired exponential decay of the transition probabilities.

One advantage of the multiscale analysis is that it provides the means to deal with singular measures, which is not possible with fractional moments estimates. It was developed by Fröhlich and Spencer and coworkers to prove spectral localization for random Schrödinger operators [FS83, FFES85]. Damanik and coworkers showed that in this setting the applicability of the multiscale analysis is sufficient to obtain dynamical localization [DS01]. The method was subsequently generalized in terms of assumptions and applicable models [vDK89, GK01, DSS02].

3. Products of random matrices

In this chapter we provide some results on the asymptotic properties of products of random matrices. Although there exist good reviews on this topic [BL85, CL90, GR86, Gui08] their presentation is mostly centered around the case of matrices in $\mathrm{GL}(\mathbb{R}, d)$. However, in chapter 5 we need to consider complex transfer matrices in order to prove dynamical localization for disordered quantum walks.

The most influential papers in this area are the results by Fürstenberg, Fürstenberg and Kesten on the existence and positivity of the Lyapunov exponent, *e.g.*the asymptotic growth rate, of products of random matrices [Für63, FK60]. These results have been successively refined, which led to limit theorems and large deviation estimates for the Lyapunov exponent [GR86, Rue79, LP82].

However, since we have to deal with matrices in $\mathrm{GL}(\mathbb{C}, d)$ we include the results for reference, while some of the more lengthy arguments will be shifted to the appendix. The results on measure, probability and ergodic theory used can be found in one of the excellent textbooks [Gra09, Kal10, Kre85].

3.1. The upper Lyapunov exponent

The basic situation we are interested in is the following one. Assume we draw a sequence of independent and identically distributed (i.i.d) random matrices $(g_{\omega_i})_{i\in\mathbb{N}}$, $g_{\omega_i} \in \mathrm{GL}(\mathbb{C}, d)$ according to some common probability measure μ on $\mathrm{GL}(\mathbb{C}, d)$. We are interested in products

$$S_n(\omega) = g_{\omega_n} \cdot g_{\omega_{n-1}} \cdots g_{\omega_1} \tag{3.1}$$

and want study their asymptotic properties as n tends to infinity. In particular, we are interested in the asymptotic behaviour of the norm of such products of random matrices.

Let us now model this situation a bit more formally and in order to keep this chapter as self-contained as possible, we repeat some results on random dynamical systems that were also introduced in section 2.5. Consider a separable probability space $\mathfrak{X} = (\mathcal{X}, \mathcal{A}, \mu)$ consisting of a sample space $\mathcal{X}$ that is a separable metric space, a σ-algebra $\mathcal{A}$ and a probability measure μ. We can construct the product measure space given by the cartesian product $\Omega = \bigtimes_{i\in\mathbb{N}} \mathcal{X}_i$, the product σ-algebra $\bigotimes_{i\in\mathbb{N}} \mathcal{A}_i$, which is generated by the cylinder sets, and the corresponding unique infinite product measure which we denote by μ_∞ [Kal10]. An element $\omega \in \Omega$ can then be regarded as a sequence of elements of $\mathcal{X}$, *e.g.*$\omega = (\omega_i)_{i\in\mathbb{N}}$, $\omega_i \in \mathcal{X}_i$. On Ω we define the shift τ in the natural way by its action on some $\omega \in \Omega$

$$\tau(\omega_1, \omega_2, \omega_3, \dots) = (\omega_2, \omega_3, \dots) \ .$$

The preimage of an arbitrary cylinder set

$$A = \{\omega \in \Omega : \omega_i \in A_i,\ A_i \in \mathcal{B}_i, i \in L \subset \mathbb{N},\ |L| < \infty\}$$

under the shift τ is given by

$$\tau^{-1}A = \{\omega \in \Omega : \omega_{i+1} \in A_i,\ A_i \in \mathcal{B}_i, i \in L \subset \mathbb{N},\ |L| < \infty\}\ .$$

Hence, $\mu_\infty(A) = \mu_\infty(\tau^{-1}A)$ holds, so τ leaves μ_∞ invariant. In general we call a map τ from Ω to itself a μ preserving transformation if it satisfies $\mu_\infty(A) = \mu_\infty(\tau^{-1}A)$ for all $A \in \sigma$ [Arn10]. We recover the situation of a sequence of i.i.d random matrices if we pick a measurable map $f : \mathcal{X} \to \mathrm{GL}(\mathbb{C}, d)$ and define

$$g_{\omega_n} := f((\tau^n \omega)_1)\ .$$

These considerations imply that sequences of i.i.d random matrices are an instance of the more general concept of a random dynamical systems [Arn10]. Abstractly, a random dynamical system is a four tuple $(\Omega, \mathcal{A}, \tau_n, \mu)$ consisting of a probability space $(\Omega, \mathcal{A}, \mu)$ and a semigroup $\{\tau_n;\ n \in \mathbb{N}\}$ of μ preserving transformations.

Definition 3.1.1 (ergodic random dynamical system)**.** *A random variable f defined on a random dynamical system $(\Omega, \mathcal{A}, \tau_n, \mu)$ is called invariant iff $f(\omega) = f(\tau_n \omega)$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. If in addition, the only random variables that are invariant with respect to a random dynamical system are constant μ-a.e. we say that the random dynamical system is ergodic.*

Ergodicity is a very strong property since it allows in a sense to compute expectation values with respect to the semigroup τ_n by looking at long sample sequences. The example of the product measure space Ω of i.i.d sequences with the right shift is indeed ergodic. This follows from the fact that the corresponding random dynamical system is strong mixing, *i.e.*

$$\lim_{n\to\infty} \mu_\infty(B \cap \tau^{-n}A) = \mu_\infty(B)\mu_\infty(A)$$

holds for all measurable sets A, B, which implies ergodicity [Kre85]. Let us for any real valued function f define

$$f^+(a) := \sup(f(a), 0)$$

We summarize this discussion in the following definition.

Definition 3.1.2 ((regular) multiplicative process)**.** *Let $(\Omega, \mathcal{A}, \tau_n, \mu)$ be an ergodic random dynamical system and f a random variable that maps Ω into $\mathrm{GL}(\mathbb{C}, d)$. The random process $\{S_n,\ n \in \mathbb{N}\}$ defined via*

(i) $S_0(\omega) = \mathbb{1}$

(ii) $S_n(\omega) = f(\tau_n \omega) \cdot f(\tau_{n-1}\omega) \cdots f(\omega)$

is called a multiplicative process. the process is regular if

$$\int (\log^+(\|S_1(\omega)\|) + \log^+(\|S_1^{-1}(\omega)\|)\, \mu(d\omega) < \infty \ . \tag{3.2}$$

In particular, the sequence of products of i.i.d random matrices of length n from (3.1) is a regular multiplicative process if we set $g_{\omega_n} := (\tau_n \omega)$, which implies

$$S_n(\omega) = g_{\omega_n} \cdots g_{\omega_1} \ .$$

It is a regular multiplicative process if the logarithm of the matrix norm is integrable with respect to the probability measure μ. Before we continue, let us shortly comment on the different notions of convergence that exist for random variables [Kal10].

Definition 3.1.3 (convergence of random variables)**.** *For a probability space $(\Omega, \mathcal{A}, \mu\}$ let $(f_n)_n$ be a sequence of random variables. Taking expectations and probability with respect to μ, we say that the sequence converges to a random variable f*

(i) μ almost surely iff $\mathbb{P}\big(\{\omega \in \Omega\,;\, \lim_{n\to\infty} f_n(\omega) = f(\omega)\}\big) = 1$.

(ii) in probability iff for all $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\big(|f_n - f| > \varepsilon\big) = 0$.

(iii) weakly iff for all continuous bounded functions X: $\mathbb{E}\big(X(f_n)\big) \to \mathbb{E}\big(X(f)\big)$.

(iv) in the $\mathcal{L}^p(\mu)$-sense for $p > 0$ iff $\lim_{n\to\infty} \mathbb{E}\big(|f_n - f|\big) = 0$.

We are interested in the asymptotic growth behaviour of such regular multiplicative processes. One way to study this behaviour is to consider exponential growth rates, *i.e.* the random sequences $\frac{1}{n}\log\|S_n(\omega)\|$. From sub-multiplicativity of the operator norm it follows that

$$\log\|S_{n+m}(\omega)\| \le \log\|S_n(\tau_m\omega)\| + \log\|S_m(\omega)\| \tag{3.3}$$

and together with the regularity condition (3.2) this implies that $\log\|S_n(\omega)\|$ is integrable. In addition we see from (3.3) that

$$\mathbb{E}\big(\log\|S_n(\omega)\|\big) = \int \log\|S_n(\omega)\| \mu(d\omega)$$

is a subadditive sequence and it follows from Fekete's lemma that the limit

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\big(\log\|S_n(\omega)\|\big) = \inf_n \frac{1}{n}\mathbb{E}\big(\log\|S_n(\omega)\|\big) \tag{3.4}$$

exists in $\mathbb{R}\cup\{-\infty\}$. We collect this in the following definition.

Definition 3.1.4 (upper Lyapunov exponent)**.** *Let $\{S_n,\, n \in \mathbb{N}\}$ be a regular multiplicative process, then it's upper Lyapunov exponent γ defined via*

$$\gamma := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\big(\log\|S_n(\omega)\|\big) = \inf_{n\in\mathbb{N}} \frac{1}{n}\mathbb{E}\big(\log\|S_n(\omega)\|\big) \tag{3.5}$$

exists and $\gamma \in \mathbb{R}\cup\{-\infty\}$.

It is clear that the limit on the right-hand side of (3.5) is hard to calculate explicitly. However, in many applications it is already enough to conclude positivity, which implies exponential growth of the operator norm of the matrix product. Fürstenberg's theorem gives sufficient conditions for the positivity of γ in the i.i.d case. A first step in this direction is the following theorem by Fürstenberg and Kesten that shows that almost all realizations of a product of random matrices realize the upper Lyapunov exponent [FK60].

Theorem 3.1.5 (Fürstenberg & Kesten)**.** *Let $\{S_n,\, n\in\mathbb{N}\}$ be a regular multiplicative process on an ergodic random dynamical system $(\Omega,\mathcal{A},\tau_n,\mu)$, then μ-a.e. every product of matrices converges to the upper Lyapunov exponent, i.e.*

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log\|S_{n(\omega)}\| \ , \tag{3.6}$$

with probability one. If in addition γ is lower bounded the convergence in (3.6) *holds in the $\mathcal{L}^1(\mu)$ sense.*

Proof. From equation (3.3) it follows that the sequence $(\log\|S_{n(\omega)}\|)_n$ forms a subadditive stochastic process and together with the regularity condition (3.2) and the fact that τ_n preserves μ this implies that $\log\|S_{n(\omega)}\|$ is an $\mathcal{L}^1(\mu)$ function. Thus are all requirements of Kingman's subadditive ergodic theorem met, from which together with the ergodicity of the process all statements follow directly. For a variety of statements and proofs of the subadditive ergodic theorem see [Arn10, Kre85, Ste89]. □

As a side remark we note that the original proof by Fürstenberg and Kesten was given before Kingman's results and demanded considerable more effort to allow for the application of Birkhoff's theorem, see for example [FK60, Kin68, BL85].

Via the exterior algebra of $\mathbb{C}^d$ it is possible to define a full spectrum of Lyapunov exponents for a product of random matrices in $\mathrm{GL}(\mathbb{C},d)$. This allows for a description of the different possible growth rates of $\|S_{n(\omega)}v\|$ for different $v\in\mathbb{C}^d$. First we built up some notation for the exterior algebra of a vector space but we refer to the literature for more extensive reviews [Bha97, SR13].

To reduce notation set $V := \mathbb{C}^d$. For two vectors $x, y \in V$ we introduce their wedge product $x\wedge y$ which is multilinear and antisymmetric, *e.g.* $x\wedge y = -y\wedge x$, and therefore $x\wedge x = 0$. We denote by $\wedge^p V$ the vector space generated by the decomposable p-vectors that is vectors of the form $x_1\wedge\cdots\wedge x_p$. If $\{e_i\}$ is a orthonormal basis of $\mathbb{C}^d$ then the set

$$\{e_{i_1}\wedge\cdots\wedge e_{i_p}\,; 1\le i_1<i_2\cdots<i_p\le d\}$$

is an orthonormal bases of $\wedge^p V$ with respect to the scalar product

$$\langle x_1\wedge\cdots\wedge x_p\,, y_1\wedge\cdots\wedge y_p\rangle := \det(\langle x_i\,, y_j\rangle) \ ,$$

where $(\langle x_i\,, y_j\rangle)$ denotes the matrix of all possible scalar products between the p-tuples $(x_1,\ldots,x_p)$ and $(y_1,\ldots,y_p)$. In addition we define for $M\in\mathrm{GL}(\mathbb{C},d)$ the action of $\wedge^p M$ on $\wedge^p V$ via

$$\wedge^p M\ x_1\wedge\cdots\wedge x_p = Mx_1\wedge\cdots\wedge Mx_p \ .$$

Note, that this definition is compatible with matrix products, *e.g.* $\wedge^p AB = (\wedge^p A)(\wedge^p B)$. As a consequence if $\{m_i\}$ denotes the set of singular values of $M \in \mathrm{GL}(\mathbb{C}, d)$ in decreasing order, the operator norm of $\wedge^p M$ is given by the product of the first p singular values [Bha97]

$$\|\wedge^p M\| := \sup_{\|x\|=1} \|\wedge^p Mx\| = m_1 \cdots m_p, \tag{3.7}$$

which implies $\|\wedge^p A\| \leq \|A\|^p$ and as one might expect of the operator norm it satisfies $\|\wedge^p AB\| \leq \|\wedge^p A\|\|\wedge^p B\|$. These prerequisites allow for the definition of the Lyapunov spectrum of a product of random matrices.

Lemma 3.1.6 (Lyapunov spectrum)**.** *Let* $\{S_n,\ n \in \mathbb{N}\}$ *be a regular multiplicative process on the ergodic random dynamical system* $(\Omega, \mathcal{A}, \tau_n, \mu)$ *and define the sequence of Lyapunov exponents* γ_l *for* $l \in \{1, \ldots, p\}$ *recursively via*

$$\gamma_l(\omega) = \lim_{n\to\infty} \frac{1}{n} \log\|\wedge^l S_n(\omega)\| - \sum_{k=1}^{l-1} \gamma_k(\omega)\ . \tag{3.8}$$

Then, for μ*-a.e. sequence of random products* $S_n(\omega)$ *the Lyapunov spectrum* $\{\gamma_l\}$ *is constant. If in addition* γ_l *is lower bounded, the convergence in* (3.8) *holds in the* $\mathcal{L}^1(\mu)$ *sense.*

Proof. We note that for $l = 1$ the result is just given by the Fürstenberg and Kesten Lemma. Submultiplicativity of the operator norm together with the upper bound $\|\wedge^i M\| \leq \|M\|^i$ for $M \in \mathrm{GL}(\mathbb{C}, d)$ and the regularity condition on multiplicative processes ((3.2)) results in subadditivity as well as integrability of the sequence $(\log\|\wedge^i S_n(\omega)\|)_n$ and this in turn implies as in the case of the upper Lyapunov exponent that

$$\mathbb{E}\big(\log\|\wedge^p S_n(\omega)\|\big) = \int \log\|\wedge^p S_n(\omega)\|\, \mu \mathrm{d}\omega\ .$$

Therefore the limit of $\frac{1}{n}\log\|\wedge^i S_n(\omega)\|$ exists again by Fekete's lemma. Now the result follows as in the case of Fürstenberg and Kesten directly from Kingman's subadditive ergodic theorem. □

3.2. Positivity of the upper Lyapunov exponent

From now on we turn to the i.i.d case, that is we fix a probability measure μ on $\mathrm{GL}(\mathbb{C}, d)$ and consider the infinite product measure space. As discussed in the previous section this corresponds to an ergodic random dynamical system for which the upper Lyapunov γ exponent is well defined and constant almost surely with respect to the infinite product measure μ_∞. The arguments used in this section follow along the lines of [Gui06, CL90, BL85, GR86]. The necessary conditions on the probability measure μ under which we prove positivity of γ are given in terms of the smallest group that contains the support of μ.

Definition 3.2.1. *Let μ be a probability measure on* $\mathrm{GL}(\mathbb{C},d)$*. We denote by* (μ) *the semigroup and by* $\langle\mu\rangle$ *the group generated by* $\mathrm{supp}(\mu)\subset\mathrm{GL}(\mathbb{C},d)$*. In addition, we denote by* $SL_{\mathbb{T}}(d)$ *the subgroup of matrices in* $\mathrm{GL}(\mathbb{C},d)$ *with determinant of modulus one*

$$SL_{\mathbb{T}}(d) := \{M\in\mathrm{GL}(\mathbb{C},d)\,;\,|\det(M)|=1\}$$

$\mathrm{GL}(\mathbb{C},d)$ acts naturally on $\mathbb{C}^d$ via matrix multiplication. The key assumption that guarantees a positive Lyapunov exponent is a strengthened version of irreducibility.

Definition 3.2.2 (strong irreducibility)**.** *A subset* $\Gamma\subset\mathrm{GL}(\mathbb{C},d)$ *is strongly irreducible iff there is no nontrivial union of proper subspaces* $C_i\subsetneq\mathbb{C}^d$ *of* $\mathbb{C}^d$ *that is* Γ *invariant, i.e. satisfies*

$$g\cdot(\cup_i C_i)=\cup_i C_i\quad\forall g\in\Gamma\ .$$

A measure μ *is called strongly irreducible if its generated subgroup* $\langle\mu\rangle$ *or equivalently its generated semigroup* (μ) *are strongly irreducible.*

We have now all definitions at hand to state Fürstenberg's theorem on the positivity of γ.

Theorem 3.2.3 (Fürstenberg)**.** *Let* μ *be a probability measure on* $SL_{\mathbb{T}}(d)$ *such that* $\langle\mu\rangle$ *is strongly irreducible and non-compact and* $\log\|g_\omega\|$ *is* μ*-integrable. Then the upper Lyapunov exponent* γ *of the corresponding i.i.d product of random matrices* $S_{n(\omega)}$ *is strictly positive, constant with probability one and given by*

$$\gamma=\lim_{n\to\infty}\frac{1}{n}\log\|S_{n(\omega)}\|=\lim_{n\to\infty}\frac{1}{n}\log\|\prod_{l=1}^{n}g_{\omega_l}\|\ ,$$

almost surely, where the convergence is in the $\mathcal{L}^1$ *sense.*

Note that $g\in SL_{\mathbb{T}}(d)d$ satisfies $\|g\|\geq 1$, which via the Fürstenberg and Kesten theorem directly implies $\gamma\geq 0$ and convergence in the $\mathcal{L}^1$-sense. The proof we are going to present here relies on the following proposition, which is proven in the appendix B.

Proposition 3.2.4. *Let* μ *be a probability measure on* $SL_{\mathbb{T}}(d)$ *such that* $\langle\mu\rangle$ *is strongly irreducible and non-compact. Define the convolution operator* T_μ *on* $\mathcal{L}^2(\mathbb{C}^d)$ *as*

$$(T_\mu f)(x) := \int_{SL_{\mathbb{T}}(d)} f(g^{-1}x)\mu(dg).$$

Then r_μ *the spectral radius of* T_μ *satisfies* $r_\mu<1$.

Proof of Fürstenberg's theorem. As discussed in section 3.1, the scenario of a product of i.i.d random matrices is an instance of an ergodic multiplicative system for which theorem 3.1.5 provides all claims except positivity of γ if we can show the regularity conditions (3.2). This follows from the condition $|\det M|=1$ for $M\in SL_{\mathbb{T}}(d)$, because it gives an upper bound on the norm of the inverse via the largest singular value of M. More precisely we have

$$\|g^{-1}\|\leq\|g\|^{d-1}\ , \tag{3.9}$$

so as claimed, the integrability of $\log^+\|g_\omega\|$ and $\log^+\|g_\omega^{-1}\|$ follows from the integrability of $\log\|g_\omega\|$. In addition, $\|M\|\geq 1$ for $\|\det M\|=1$ implies that the Lyapunov exponent is bounded from below by 0. Hence, the convergence holds in the $\mathcal{L}^1(\mu)$ sense.

For positivity of γ the proof follows the strategy used in [GR86] for the real case. We define the function

$$q(x) := \min(1, \|x\|^\delta)$$

with $\delta > d/2$ chosen such that $g(x)\in\mathcal{L}^2(\mathbb{C}^d,\mu)$. In addition, we define the set $M=\{x\in\mathbb{C}^d\,;\,1\leq\|x\|\leq 2\}$. To find a lower bound for γ denote by χ_A the characteristic function of a set A and consider

$$\langle\chi_M\,,T_\mu^n q\rangle \geq \mathrm{vol}(M)\int\|g^{-1}x\|^{-\delta}\,\mu(\mathrm{d}g)\geq 2^{-\delta}\int\|g\|^{-\delta(d-1)}\mu(\mathrm{d}g)\ ,$$

where the volume of M is computed with respect to the d dimensional Lebesgue measure and we used (3.9) to bound the norm of the inverse of g. Using Jensen's inequality we find for any n

$$\frac{1}{n}\int\log\|gx\|\,\mu(\mathrm{d}g)\geq\frac{const}{n}+\frac{1}{\delta(d-1)}\log(\langle\chi_M\,,T_\mu^n q\rangle^{-1/n})\ .$$

Taking the limes inferior on the left and limes superior on the right-hand side implies

$$\gamma\geq\liminf_{n\to\infty}\frac{1}{n}\int\log\|gx\|\,\mu(\mathrm{d}g)\geq\frac{1}{\delta(d-1)}\limsup_{n\to\infty}\log(\langle\chi_M\,,T_\mu^n q\rangle^{-1/n})\ .$$

From the definition of the spectral radius and proposition 3.2.4, we obtain

$$\limsup_{n\to\infty}\langle\chi_M\,,T_\mu^n q\rangle^{1/n}\leq\limsup_{n\to\infty}\|T_\mu^n\|^{1/n}\|q\|^{1/n}\|\chi_M\|^{1/n}\leq r_\mu<1\ .$$

Hence, γ is indeed strictly larger than zero. □

We are now going to extend these results from the norm of the products or random matrices to the norm of vectors the product is applied to. For this purpose, we introduce the complex projective space $\mathbb{PC}^d$, which is the set of all one-dimensional complex subspaces of $\mathbb{C}^d$. Elements of $\mathrm{GL}(\mathbb{C},d)$ act naturally on $\overline{x}\in\mathbb{PC}^d$ and we write $M\overline{x}:=\overline{Mx}$ to reduce notation. We choose a metric on $\mathbb{PC}^d$ via the second exterior power of $\mathbb{C}^d$

$$\delta(\overline{x},\overline{y}) := \frac{\|x\wedge y\|}{\|x\|\|y\|} = \left(1-\left(\frac{|\langle x\,,y\rangle|}{\|x\|\|y\|}\right)^2\right)^{1/2}\ . \tag{3.10}$$

This identity can be easily checked from the definition of $\|x\wedge y\|$. Furthermore, the right-hand expression is nothing but the Hilbert Schmidt distance for pure quantum states. Hence, it is clear that this expression defines a metric on the projective Hilbert space $\mathbb{PC}^d\,\mathbb{C}^d$ [BŻ06].

Given two vectors $\overline{x}, \overline{y} \in \mathbb{P}\mathbb{C}^d$ we can always find two representatives $x, y \in \mathbb{C}^d$ such that $\Re \langle x, y \rangle = |\langle x, y \rangle|$, where $\Re x$ denotes the real part of x. Therefore, we obtain the relation

$$\frac{\delta(\overline{x},\overline{y})^2}{\|x-y\|^2} = \frac{1}{2}\frac{1-|\langle x, y \rangle|^2}{1-\Re \langle x, y \rangle} = \frac{1}{2}(1+|\langle x, y \rangle|) \ ,$$

which together with $0 \leq |\langle x, y \rangle| \leq 1$ implies the inequality

$$\frac{1}{\sqrt{2}}\|x-y\| \leq \delta(\overline{x},\overline{y}) \leq \|x-y\| \ . \tag{3.11}$$

For later use we also note the following lemma.

Lemma 3.2.5. *For $M \in \mathrm{GL}(\mathbb{C}, d)$, denote by m_1 and m_2 its two largest singular values, then*

$$\frac{\delta(M\overline{x}, M\overline{y})}{\delta(\overline{x},\overline{y})} \leq m_1 m_2 \frac{\|x\|\|y\|}{\|Mx\|\|My\|}$$

This is an immediate consequence from the definition of the metric δ in (3.10) and the fact that the operator norm of $\wedge^2 M$ is exactly given by the product of the two largest singular values (see (3.7)). Given two probability measures μ and ν on $\mathrm{GL}(\mathbb{C}, d)$ and $\mathbb{P}\mathbb{C}^d$ respectively, we define their pseudo convolution as the measure $\mu * \nu$ on $\mathbb{P}\mathbb{C}^d$ satisfying

$$\int f(\overline{x})\, \mu * \nu(\mathrm{d}\overline{x}) := \int f(M\overline{x})\, \mu(\mathrm{d}M)\, \nu(\mathrm{d}\overline{x}) \ ,$$

for all bounded Borel functions f on $\mathbb{P}\mathbb{C}^d$. A measure ν on $\mathbb{P}\mathbb{C}^d$ is called μ-invariant if ν satisfies

$$\int f(\overline{x})\ \nu(\mathrm{d}\overline{x}) = \int f(M\overline{x})\, \mu(\mathrm{d}M)\, \nu(\mathrm{d}\overline{x}) \ ,$$

e.g. if $\mu * \nu = \mu$. Given a probability measure μ on $\mathrm{GL}(\mathbb{C}, d)$, a μ-invariant measure ν allows for the construction of a dynamical system on $(\mathrm{GL}(\mathbb{C}, d)^{\mathbb{N}} \times \mathbb{P}\mathbb{C}^d, \mu_\infty \times \nu)$, if we define the shift operation $\tilde{\tau}$ via

$$\tilde{\tau}((g_{\omega_n})_n,\ \overline{x}) \to ((g_{\tau\omega_n})_n,\ g_{\omega_1}\overline{x}) = ((g_{\omega_{n+1}})_n,\ g_{\omega_1}\overline{x}) \tag{3.12}$$

for all sequences $(g_{\omega_n})_n \in \mathrm{GL}(\mathbb{C}, d)^{\mathbb{N}}$ and vectors $\overline{x} \in \mathbb{P}\mathbb{C}^d$ with τ as before. This definition ensures that $\{\tilde{\tau}_n\}$ is a semigroup and the μ-invariance of ν implies that $\tilde{\tau}$ is indeed a $\mu_\infty \times \nu$ invariant transformation. So the tuple

$$(\mathrm{GL}(\mathbb{C}, d)^{\mathbb{N}} \times \mathbb{P}\mathbb{C}^d, \mathcal{B}, \tilde{\tau}_n, \mu_\infty \times \nu) \tag{3.13}$$

forms a random dynamical system. We call a function $\sigma : \mathrm{GL}(\mathbb{C}, d) \times \mathbb{P}\mathbb{C}^d \to \mathbb{R}$ an additive cocycle if it satisfies the relation

$$\sigma(g_2 g_1, \overline{x}) = \sigma(g_2, g_1\overline{x}) + \sigma(g_1, \overline{x}) \ .$$

Since the natural action of $\mathrm{GL}(\mathbb{C},d)$ on $\mathbb{PC}^d$ is up to a phase factor given by $M\cdot\overline{x}=\frac{Mx}{\|Mx\|}$ for $\overline{x}\in\mathbb{PC}^d$ the function $\sigma(g,\overline{x}):=\log g\overline{x}$ is an additive cocycle on $\mathrm{GL}(\mathbb{C},d)\times\mathbb{PC}^d$, indeed

$$\log g_2g_1\overline{x}=\log\left(\frac{g_2g_1x}{\|g_2g_1x\|}\frac{\|g_1x\|}{\|g_1x\|}\right)=\log g_2\overline{g_1x}+\log g_1\overline{x}\ . \tag{3.14}$$

Note that this construction is by no means limited to the projective space $\mathbb{PC}^d$. In exactly the same manner we can choose any other compact separable space B on which $\mathrm{GL}(\mathbb{C},d)$ acts and define a pseudo convolution and invariant measures. Using (3.12) analogously to introduce a shift operation on the measure space $\mathrm{GL}(\mathbb{C},d)\times B$ also gives a random dynamical system and the idea of an additive cocycle makes sense as well. We need this generalization in order to prove the simplicity of the upper Lyapunov exponent. The compact separable spaces that we will consider there are in particular the normalized endomorphism on $\mathbb{C}^d$ and $\wedge^2\mathbb{C}^d$. Bearing this in mind, we state some of the following results for general compact $\mathrm{GL}(\mathbb{C},d)$ spaces.

Next we show that Birkhoff's pointwise ergodic theorem applies to additive cocycles.

Lemma 3.2.6. *Let μ be probability measure on $\mathrm{GL}(\mathbb{C},d)$, ν a μ-invariant measure on a $\mathrm{GL}(\mathbb{C},d)$-space B. If σ is an additive cocycle with either σ^+ or σ^- in $\mathcal{L}^1(\mu\times\nu)$ then for $\mu_\infty\times\nu$ almost all $((g_{\omega_n})_n,b)$ exists the limit*

$$\widehat{\sigma}((g_{\omega_n})_n,b)=\lim_{n\to\infty}\frac{1}{n}\sigma(S_{n(\omega)},b)\ . \tag{3.15}$$

In addition, the function $\widehat{\sigma}$ is equal a.e. to the conditional expectation of σ with respect to the invariant sigma field, which in particular implies

$$\int\widehat{\sigma}((g_{\omega_n})_n,b)\,\mu_\infty(\mathrm{d}\omega)\,\nu(\mathrm{d}b)=\int\sigma(g_{\omega_1},b)\,\mu(\mathrm{d}g_{\omega_1})\,\nu(\mathrm{d}b)\ .$$

With the stronger assumption $\sigma\in\mathcal{L}^1(\mu\times\nu)$ also $|\widehat{\sigma}|$ is integrable and the convergence in (3.15) *is in the $\mathcal{L}^1$ sense.*

Proof. We have seen that the tuple $(\mathrm{GL}(\mathbb{C},d)^{\mathbb{N}}\times B,\mathcal{B},\tilde{\tau}_n,\mu_\infty\times\nu)$ constitutes a random dynamical system. Now setting $\Theta((g_{\omega_n})_n,b)=\sigma(g_{\omega_1},b)$ we find

$$\frac{1}{n}\sigma((S_{n(\omega)})_n,b)=\frac{1}{n}\sum_{l=1}^{n}\Theta(\tilde{\tau}_l((g_{\omega_n})_n,b))\ .$$

The function Θ thus satisfies all assumptions of Birkhoff's pointwise ergodic theorem and all claims follow [Pet90, Thm 2.3 & Rm 2.4]. In particular we obtain 3.2.6 from

$$\int\widehat{\sigma}((g_{\omega_n})_n,b)\mu_\infty(\mathrm{d}\omega)\,\nu(\mathrm{d}b)=\int\Theta((g_{\omega_n})_n,b)\mu_\infty(\mathrm{d}\omega)\,\nu(\mathrm{d}b)=\int\sigma(g_{\omega_1},b)\mu(\mathrm{d}g_{\omega_1})\,\nu(\mathrm{d}b)$$

□

We also note the following corollary to the pointwise ergodic theorem showing that the limit in (3.15) is strictly larger then zero in most cases.

Corollary 3.2.7. *Let μ be a probability measure on $\mathrm{GL}(\mathbb{C},d)$, ν an μ-invariant measure on a compact separable $\mathrm{GL}(\mathbb{C},d)$-space B and σ a additive cocycle on $\mathrm{GL}(\mathbb{C},d)\times B$ with either σ^+ or σ^- in $\mathcal{L}^1(\mu\times\nu)$. If $\mu_\infty\times\nu$-a.s.*

$$\lim_{n\to\infty}\sigma(S_{n(\omega)},b)=\infty\ ,$$

then the limit from (3.15) *is strictly larger than zero $\mu_\infty\times\nu$-a.e..*

A proof can be found in [BL85, lem 2.3] or [CL90, thm IV.1.4]. Since $\mathrm{GL}(\mathbb{C},d)$ also acts on itself, it is natural to define the convolution $\mu*\nu$ of two probability measures μ and ν on $\mathrm{GL}(\mathbb{C},d)$ via

$$\int f(g)\,\mu*\nu(\mathrm{d}g)=\int f(gh)\,\mu(\mathrm{d}g)\,\nu(\mathrm{d}h) \tag{3.16}$$

for all bounded Borel functions on $\mathrm{GL}(\mathbb{C},d)$. Note that $\mu*\nu$ is again a probability measure on $\mathrm{GL}(\mathbb{C},d)$ and not on the product space. In particular, we denote by μ^n the n fold convolution of μ with itself. Next we show that for any probability measure μ on $\mathrm{GL}(\mathbb{C},d)$ there exists an invariant measure on a compact separable $\mathrm{GL}(\mathbb{C},d)$-space B.

Lemma 3.2.8. *Let μ be a probability measure on $\mathrm{GL}(\mathbb{C},d)$ and η be a probability measure on a compact $\mathrm{GL}(\mathbb{C},d)$ space B. Any limit point of the sequence $(\nu_n)_n$ with*

$$\nu_n=\frac{1}{n}\sum_{i=1}^{n}\mu^i*\eta$$

is invariant with respect to μ.

Proof. The space of probability measures of a compact metric space is compact and sequentially compact due to the Banach-Alaoglu and Prokhorov's theorem [Rud06, thm 3.17]. Since B is compact and separable by assumption, this implies that the sequence $(\nu_n)_n$ has a limit point and for every limit point ν we can find a weakly converging subsequence of $(\nu_n)_n$. Passing to such a subsequence and denoting it again by (ν_n) we find

$$\mu*\nu_n=\frac{1}{n}\sum_{i=1}\mu^i*\eta+\frac{1}{n}(\mu^{n+1}*\eta-\mu*\eta)=\nu_n+\frac{1}{n}(\mu^{n+1}*\eta-\mu*\eta)\ ,$$

which implies $\nu=\mu*\nu$ in the limit $n\to\infty$. □

In the special case $B=\mathbb{P}\mathbb{C}^d$ and for a strongly irreducible probability measure μ on $\mathrm{GL}(\mathbb{C},d)$ we can show that an invariant measure can contain no Dirac points.

Lemma 3.2.9. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C},d)$. Then, any invariant measure μ on $\mathbb{P}\mathbb{C}^d$ is proper, meaning that if V is a proper subspace of $\mathbb{C}^d$ then $\nu(\overline{V})=0$, where $\overline{V}=\{\overline{x}\in\mathbb{P}\mathbb{C}^d\ ;\ x\in V\setminus\{0\}\}$.*

The proof of this statement is given in the appendix for lemma B.2.3. In order to study the convergence in direction of the products $S_{n(\omega)} \cdot x$ as well as their growth rate, we need a concept that quantifies the asymptotic rank of the product $S_{n(\omega)}$.

Definition 3.2.10. *A subset $M \subset \mathrm{GL}(\mathbb{C}, d)$ is said to have index p if p is the smallest integer for which M contains a sequence (M_n) such that $\frac{M_n}{\|M_n\|}$ converges to a matrix with rank p. M is called contractive if $p = 1$.*

The following technical proposition concerns the limit behaviour of products of random matrices applied to a sequence of vectors. The proof is given in appendix B.

Proposition 3.2.11. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that (μ) has index p. Then, for any sequence $(x_n)_n \subset \mathbb{C}^d$ that converges to a nonzero vector we have almost surely*

$$\sup_{n \geq 1} \frac{\|S_{n(\omega)}\|}{\|S_{n(\omega)} x_n\|} \leq \infty \ .$$

If in addition, $m_{i(n)}$ denotes the i th largest singular value of the random product $S_{n(\omega)}$, then with probability one

$$\lim_{n \to \infty} \frac{m_{p+1(n)}}{\|S_{n(\omega)}\|} = 0$$

Strong irreducibility also ensures the almost sure convergence of $\log \|S_{n(\omega)} x\|$ to the upper Lyapunov exponent for all $x \in \mathbb{C}^d$ in the following sense:

Lemma 3.2.12. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that $\log^+ \|g_{\omega_1}\|$ is integrable. If $(x_n)_n \subset \mathbb{C}^d$ is a convergent sequence with a limit different from the zero vector then μ-almost surely*

$$\lim_{n \to \infty} \frac{1}{n} \log \|S_{n(\omega)} x_n\| = \gamma \tag{3.17}$$

and if ν is a μ-invariant measure on $\mathbb{P}\mathbb{C}^d$ the Lyapunov exponent admits the representation

$$\gamma = \int \log \frac{\|g\, x\|}{\|x\|} \, \mu(\mathrm{d}g)\, \nu(\mathrm{d}x) \ .$$

If in addition, $\log^+ \|g_{\omega_1}^{-1}\|$ is also integrable, the sequence $(\mathbb{E}\left(\frac{1}{n} \log \|S_{n(\omega)} x\|\right))_n$ converges uniformly to γ on $S^d = \{x \in \mathbb{C}^d \ ; \ \|x\| = 1\}$.

Proof. From proposition 3.2.11 we immediately get for any sequence (x_n) that does not converge to the zero vector, the following n independent relation between operator and vector norm

$$\frac{1}{n} \log \|S_{n(\omega)} x_n\| \leq \frac{1}{n} \log \|S_{n(\omega)}\| \leq \frac{1}{n} \left(\log C_\omega + \log \|S_{n(\omega)} x_n\| \right) \tag{3.18}$$

with some positive constant C_ω. The limit of the expression in the middle tends to the upper Lyapunov exponent γ by theorem 3.1.5 and since the additive term on the right-hand side becomes negligible in the limit this proofs (3.17).

From (3.14) it is clear that $\log\|g\overline{x}\|$ is an additive cocycle and by lemma 3.2.6 we have

$$\lim_{n\to\infty}\frac{1}{n}\int \log\|S_{n(\omega)}x\|\mu^n(\mathrm{d}S_{n(\omega)})\,\nu(\overline{x}) = \int \log\|g_{\omega_1}\overline{x}\|\mu(\mathrm{d}g_{\omega_1})\,\nu(\overline{x})\ .$$

From (3.18) we know that in the limit the $\overline{x}$ dependent integrant can be replaced by the operator norm of the product and this is equal to the upper Lyapunov exponent γ.

Since the unit sphere in $\mathbb{C}^d$ is compact we only have to show that for any converging sequence (x_n) in the unit sphere we have

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\big(\log\|S_{n(\omega)}x\|\big) = \gamma\ .$$

We know from (3.17) that this is true without the expectation value. The additional regularity condition on $\log\|S_1^{-1}(\omega)\|$ establishes uniform integrability of the sequence $(\frac{1}{n}\log\|S_{n(\omega)}x\|)_n$ from which the result follows. □

Lemma 3.2.12 provides a first analogue to the law of large numbers for the computation of the upper Lyapunov exponent. However, in contrast to the commuting case it is not enough to average the function $\log\|g\|$ with respect to the common probability measure μ. Instead we need an invariant measure on the projective space, which is in general is hard to compute. Equipped with these preliminaries we can show the simplicity of the upper Lyapunov exponent in the Lyapunov spectrum.

Lemma 3.2.13. *Let μ be a strongly irreducible probability measure on* $\mathrm{GL}(\mathbb{C},d)$ *such that* (μ) *is contractive and* $\log^+\|g_{\omega_1}\| \in \mathcal{L}^1(\mu)$*, then the upper two Lyapunov exponents satisfy* $\gamma_1 > \gamma_2$.

Proof. Let us denote by K_1 and K_2 the spaces of endomorphism on $\mathbb{C}^d$ or $\wedge^2\mathbb{C}^d$ with operator norm one, respectively. As described above, there is a natural way in which $\mathrm{GL}(\mathbb{C},d)$ acts on $\mathbb{P}\mathbb{C}^d$. In a similar way we can define an action of $\mathrm{GL}(\mathbb{C},d)$ on the spaces K_1 and K_2 if we define the operation $\cdot$ via

$$g\cdot A := \frac{gA}{\|gA\|} \quad \text{and} \quad (\wedge^2 g)\cdot B := \frac{(\wedge^2 g)\,B}{\|(\wedge^2 g)\,B\|}$$

for all $g\in\mathrm{GL}(\mathbb{C},d)$, $A\in K_1$ and $K\in K_2$. It is shown in the appendix, lemma B.3.1 that, as in the case of the projective space, there exist μ-invariant measures ν_i on K_i such that we rediscover the Fürstenberg and Kesten result in the sense that $\mu_\infty\times\nu_i$ a.e.

$$\lim_{n\to\infty}\frac{1}{n}\log\|S_{n(\omega)}A\| = \gamma_1 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\log\|\wedge^2 S_{n(\omega)}A\| = \gamma_1+\gamma_2\ . \tag{3.19}$$

Now consider the space $K_1\times K_2$. We can turn this space into a $\mathrm{GL}(\mathbb{C},d)$ space, if we define the action componentwise via

$$g\cdot(A,B) := (g\cdot A,(\wedge^2 g)\cdot B)\ .$$

By lemma 3.2.8 we also know that any limit point of the sequence $(\frac{1}{n}\sum_{i=1}^{n}\mu^i*(\nu_1\times\nu_2))_n$ is a μ-invariant measure on the product space. Let us pick one and denote it by ν. In addition let us define the function

$$\sigma(g,(A,B)) := \log\frac{\|gA\|^2}{\|(\wedge^2 g)B\|} \ .$$

By virtually the same arguments as in (3.14) it can be shown that σ constitutes an additive cocycle on $\mathrm{GL}(\mathbb{C},d)\times(K_1\times K_2)$. The restriction of the invariant measure ν to either K_1 or K_2 is equal to ν_1 or ν_2, respectively. Using (3.19) this implies that the limit of σ is $\mu_\infty\times\nu$-a.e. equal to

$$\lim_{n\to\infty}\frac{1}{n}\sigma(S_{n(\omega)},(A,B)) = \lim_{n\to\infty}\frac{2}{n}\log\|gA\| - \lim_{n\to\infty}\frac{1}{n}\log\|(\wedge^2 g)A\| = \gamma_1-\gamma_2 \ . \quad (3.20)$$

Bounding the term $\|(\wedge^2 g)B\|$ in the denominator of σ by its operator norm we find the lower bound

$$\sigma(S_{n(\omega)},(A,B)) \geq \log\frac{\|S_{n(\omega)}A\|}{\|(\wedge^2 S_{n(\omega)})\|} \ . \quad (3.21)$$

In addition lemma 3.2.11 implies that due to the contractivity of (μ)

$$\inf_n\frac{\|S_{n(\omega)}x\|}{\|S_{n(\omega)}\|}\neq 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{m_{2(n)}}{\|S_{n(\omega)}\|} = \lim_{n\to\infty}\frac{\|\wedge^2 S_{n(\omega)}\|}{\|S_{n(\omega)}\|^2} = 0 \ ,$$

where $m_{2(n)}$ is the second largest singular value of $S_{n(\omega)}$. Therefore for any non-zero matrix A the limit on the lower bound in (3.21) is infinite. This in turn implies by lemma 3.2.7 that a.s.

$$\lim_{n\to\infty}\frac{1}{n}\sigma(S_{n(\omega)},(A,B)) > 0$$

Using (3.20) we see that indeed $\gamma_1>\gamma_2$ as claimed. □

We remark that it is possible to generalize lemma 3.2.13 if (μ) has index p. Then the upper Lyapunov exponent is exactly p-fold degenerate and $\gamma_p>\gamma_{p+1}$ [GR85].

For the remainder of this chapter we often need additional regularity conditions on the common probability measure of the random matrices. To shorten the statement of this conditions, we define for any $g\in\mathrm{GL}(\mathbb{C},d)$ the function $\mathcal{F}(g)$ as

$$\mathcal{F}(g) := \sup(\log^+\|g\|,\log^+\|g^{-1}\|) \quad (3.22)$$

As an immediate consequence we find for the operator norm of a matrix.

Lemma 3.2.14. *Let $\mathcal{F}$ be defined as in* (3.22) *then we have for* $1\leq p\leq d$, *all* $g\in\mathrm{GL}(\mathbb{C},d)$ *and* $x\in\wedge^p C^d$ *normalized*

$$|\log\|\wedge^p g\|| \leq p\mathcal{F}(g) \quad \textit{and} \quad |\log\|\wedge^p gx\|| \leq p\mathcal{F}(g)$$

The lemma follows from a straight forward calculation and our comments about the connection between singular values and operator norm after (3.7).

3.3. Large deviation estimates

The goal of this section is to derive a large deviation principle for the growth rate of a product of random matrices applied to a normalized vector. After this result is established, we will see that it can be extended to the single matrix elements of the random products. We already know from (3.14) that the norm of a product of random matrices applied to a unit vector constitutes an additive cocycle on $\mathrm{GL}(\mathbb{C},d)\times\mathbb{PC}^d$ if we take the norm

$$\sigma(S_{n(\omega)},\overline{x}) = \log\|S_{n(\omega)}\overline{x}\| = \sum_{l=1}^{n}\log\left\|g_{\omega_l}\frac{S_{l-1(\omega)}x}{\|S_{l-1(\omega)}x\|}\right\| \ .$$

If we define the random variables $F_{\omega(l)} = S_{l(\omega)}\overline{x}$, we can consider σ as a Markov process on $\mathbb{PC}^d$ with the transition or Markov operator R_μ acting on the space of bounded measurable functions on $\mathbb{PC}^d$ via

$$(R_\mu f)(\overline{x}) := \int f(g\overline{x})\,\mu(\mathrm{d}g) \ . \tag{3.23}$$

The crucial step in proving the large deviation bounds is to define an analytic family of operators that contain the Markov operator R_μ and identify this family with the Fourier-Laplace transform on a suitable Banach space. It turns out that if we consider the space of Hölder-continuous functions, the Fourier-Laplace transform around zero gives rise to such a family of analytic operators to which perturbation theory can be applied. Under the assumption of contractivity and strong irreducibility on the measure μ, the spectrum of R_μ contains a simple eigenvalue of maximal modulus, which dominates the asymptotic concatenation of R_μ with itself. To this end, note that we can express the n^{th} application of R_μ to a function f via the n^{th} convolution of the probability measure μ with itself

$$(R_\mu^n f)(\overline{x}) = \int f(g\overline{x})\mu^n(\mathrm{d}g) \ . \tag{3.24}$$

This is obvious for the case $n=1$ and if we assume it is true for n we find for $n+1$

$$(R_\mu(R_\mu^n f))(\overline{x}) = \int f(g_1g_2\overline{x})\mu(\mathrm{d}g_1)\mu^n(\mathrm{d}g_2) = \int f(g\overline{x})\mu^{n+1}(\mathrm{d}g) \ ,$$

where we used the definition of μ^{n+1} in the last step (see (3.16)).

3.3.1. Space of Hölder continuous functions

After this preliminaries we proceed by defining the space of Hölder continuous function which turns out to be the suitable function space to make the Fourier Laplace transform of R_μ an analytic family of operators.

Definition 3.3.1. *Let $C(\mathbb{PC}^d)$ be the space of continuous functions on $\mathbb{PC}^d$ and define for all $f\in C(\mathbb{PC}^d)$ and $\alpha>0$ and the metric δ on $\mathbb{PC}^d$ as defined on* (3.10).

- *the uniform norm* $\|f\|_\infty = \sup_{\overline{x}\in\mathbb{PC}^d} |f(\overline{x})|$
- *the map* $m_\alpha(f) = \sup_{\overline{x}\neq\overline{y}\in\mathbb{PC}^d} \frac{|f(\overline{x})-f(\overline{y})|}{\delta(\overline{x},\overline{y})^\alpha}$

Then we define $\mathcal{L}_\alpha$ *as the subset of functions of* $C(\mathbb{PC}^d)$ *for which*

$$\|f\|_\alpha = \|f\|_\infty + m_\alpha(f)$$

is finite.

Lemma 3.3.2. $\mathcal{L}_\alpha$ *with the norm* $\|f\|_\alpha$ *is a Banach space.*

Proof. That $\|f\|_\alpha$ satisfies all properties of a norm is easily verified. Now assume we are given a Cauchy sequence $(f_n)_n \in \mathcal{L}_\alpha$. The space uniformly bounded functions $\mathcal{L}^\infty$ on $\mathbb{PC}^d$ is a Banach space with respect to the uniform norm, so the limit of $(f_n)_n$ exists in $\mathcal{L}^\infty$. Since $(f_n)_n$ is Cauchy, we also know that for every $\varepsilon > 0$ as well as every n, m large enough and $\overline{x} \neq \overline{y} \in \mathbb{PC}^d$

$$\frac{|f_n(\overline{x}) - f_n(\overline{y}) + f_m(\overline{y}) - f_m(\overline{x})|}{\delta(\overline{x},\overline{y})} < \varepsilon \ .$$

Taking the limit with respect to either n or m shows that the sequence $(f_n)_n$ converges also in $\|\ \|_\alpha$ to f. So we are left to prove that the limit f is in $\mathcal{L}_\alpha$. Since we already know that f_n converges to f uniformly this amounts to the question whether $m_\alpha(f)$ is finite. Choosing n large enough we find for all $\varepsilon > 0$ and $\overline{x}, \overline{y} \in \mathbb{PC}^d$

$$|f(\overline{x}) - f(\overline{y})| \leq |f(\overline{x}) - f_n(\overline{x})| + |f_n(\overline{x}) - f_n(\overline{y})| + |f_n(\overline{y}) - f(\overline{y})| \leq \|f_n\|_\alpha + 2\varepsilon \ .$$

We know that $(f_n)_n$ is Cauchy and therefore bounded, so taking ε small enough finishes the proof. □

For convenience we restate the definition of the function $\mathcal{F}$ that describes the regularity conditions we require for the probability measures μ, we consider.

$$\mathcal{F}(g) := \sup(\log^+ \|g\|, \log^+ \|g^{-1}\|)$$

Under an integrability condition on $\mathcal{F}$, we are now able to show the uniqueness of the invariant measure and the exponential convergence of the convolutions μ^n to it.

Lemma 3.3.3. *Let* μ *be a strongly irreducible probability measure on* $\mathrm{GL}(\mathbb{C}, d)$ *such that* (μ) *is contracting and let* ν *be a* μ*-invariant probability measure on* $\mathbb{PC}^d$*. Suppose that* μ *has an exponential moment, that is there exists* $\tau > 0$ *such that*

$$\mathbb{E}\left(e^{\tau\mathcal{F}}\right) = \int e^{\tau\mathcal{F}(g)}\mu(\mathrm{d}g) < \infty \ .$$

Then, there exists $\alpha_0 > 0$ *such that for all* $0 < \alpha \leq \alpha_0$ *the Markov operator* R_μ *and the operator* N_ν *defined as*

$$(N_\nu f)(\overline{x}) := \int f(\overline{y})\,\nu(\mathrm{d}\overline{y}) \tag{3.25}$$

are bounded operators on $\mathcal{L}_\alpha$ and in addition satisfy the relation

$$\lim_{n\to\infty} \|R_\mu - N_\nu\|^{\frac{1}{n}}_{\mathcal{B}(\mathcal{L}_\alpha)} < 1 \ ,$$

where $\|\cdot\|_{\mathcal{B}(\mathcal{L}_\alpha)}$ denotes the operator norm induced by $\|\cdot\|_\alpha$. This implies in particular that the invariant measure ν is unique.

For the proof we will need the following technical proposition the proof of which is given in the appendix.

Proposition 3.3.4. *Let μ be a strongly irreducible and contracting probability measure on $\mathrm{GL}(\mathbb{C}, d)$, such that for some $\tau > 0$ the function $\exp(\tau\mathcal{F}(g))$ is μ-integrable, then there is $\alpha_0 > 0$ such that for all $0 < \alpha \leq \alpha_0$*

$$\lim_{n\to\infty} \left(\sup_{\overline{x}\neq\overline{y}\in\mathbb{PC}^d} \int \frac{\delta(S_n(\omega)\overline{x}, S_n(\omega)\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha}\, \mu^n(\mathrm{d}g) \right)^{\frac{1}{n}} < 1 \ .$$

This implies in particular that there exit constants $0 < \rho < 1$ and $C > 0$ such that

$$\sup_{\overline{x}\neq\overline{y}\in\mathbb{PC}^d} \int \frac{\delta(S_n(\omega)\overline{x}, S_n(\omega)\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha}\, \mu^n(\mathrm{d}g) \leq C\rho^n$$

proof of lemma 3.3.3. We first note the following estimate that we use several times in the argument. Given a function $f \in \mathcal{L}_\alpha$ we have for all $\overline{x}, \overline{y} \in \mathbb{PC}^d$

$$|f(\overline{x}) - f(\overline{y})| = \frac{|f(\overline{x}) - f(\overline{y})|}{\delta(\overline{x},\overline{y})^\alpha}\, \delta(\overline{x},\overline{y})^\alpha \leq \|f\|_\alpha\, \delta(\overline{x},\overline{y})^\alpha \ . \tag{3.26}$$

We start with the boundedness of R_μ. By definition we have $f(\overline{x}) \leq \|f\|_\infty$ and therefore $\|R_\mu f\|_\infty \leq \|f\|_\infty$, so R_μ is a contraction of $\mathcal{L}_\alpha$. For the second summand in the α-norm we find

$$m_\alpha(R_\mu f) = \sup_{\overline{x}\neq\overline{y}\in\mathbb{PC}^d} \frac{|(R_\mu f)(\overline{x}) - (R_\mu f)(\overline{y})|}{\delta(\overline{x},\overline{y})^\alpha} \leq \int \frac{|f(g\overline{x}) - f(g\overline{y})|}{\delta(g\overline{x}, g\overline{y})^\alpha} \frac{\delta(g\overline{x}, g\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha} \mu(\mathrm{d}g) \ .$$

As we have seen in (3.26), the first factor under the integral on the right-hand side is for all g upper bounded by $\|f\|_\alpha$, which is finite by assumption. For the second factor we have by the definition of the metric δ and lemma 3.2.14 for all $\overline{x}, \overline{y} \in \mathbb{PC}^d$

$$\frac{\delta(g\overline{x}, g\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha} \leq \|\wedge^2 g\|^\alpha \frac{\|x\|^\alpha \|y\|^\alpha}{\|gx\|^\alpha \|gy\|^\alpha} \leq e^{4\alpha\mathcal{F}(g)}$$

and so by assumption the right-hand side is μ integrable for $4\alpha \leq \tau$. Since N_ν projects every function to a constant, its boundedness in the α norm is trivial. So we can turn

to the convergence of R_μ^n to N_ν. Again we first consider the first term in the α norm. For all $\overline{x} \in \mathbb{P}\mathbb{C}^d$ we find by (3.24) and the invariance of μ

$$|((R_\mu^n - N_\nu)f)(\overline{x})| = \left| \int f(g\overline{x})\, \mu^n(\mathrm{d}g) - \int f(\overline{y})\ \nu(\mathrm{d}\overline{y}) \right| \leq \int |f(g\overline{x}) - f(g\overline{y})|\, \mu^n(\mathrm{d}g)\mu(\mathrm{d}\overline{y}) \ .$$

We can use (3.26) for the integrand on the right-hand side, which leaves us with an integral over the δ distance between two vectors in the projective space to which the same product of n random matrices is applied. Since $\delta(\overline{x},\overline{y}) \leq 1$ this can be upper bounded via

$$\int \delta(g\overline{x}, g\overline{y})^\alpha \mu^n(\mathrm{d}g)\, \nu(\mathrm{d}\overline{v}) \leq \sup_{\overline{x}\neq\overline{y}} \int \frac{\delta(g\overline{x}, g\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha}\, \mu^n(\mathrm{d}g) + \nu(\{\overline{x}\})\, \delta(g\overline{x}, g\overline{x})^\alpha \ .$$

Since ν is μ-invariant and μ is a strongly irreducible probability measure, ν it is proper according to lemma 3.2.9. Therefore, the weight of $\overline{x}$ is zero, as is the distance of a vector to itself, which implies that the second term on the right-hand side vanishes. Therefore, by proposition 3.3.4, we have for all $0 < \alpha \leq \alpha_0$

$$\|(R_\mu^n - N_\nu)f\|_\infty \leq m_\alpha(f) C_\alpha \rho_\alpha^n \ ,$$

with $C_\alpha > 0$ and $0 < \rho_\alpha < 1$. Since N_ν projects a function to a constant value, the second term in the α norm gives

$$m_\alpha((R_\mu^n - N_\nu)f) = \sup_{\overline{x}\neq\overline{y}\in\mathbb{P}\mathbb{C}^d} \int \frac{|f(g\overline{x}) - f(g\overline{y})|}{\delta(\overline{x},\overline{y})^\alpha} \leq \|f\|_\alpha \sup_{\overline{x}\neq\overline{y}\in\mathbb{P}\mathbb{C}^d} \int \frac{\delta(g\overline{x}, g\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha}$$

and we arrive at exactly the same situation as for the uniform norm and can apply proposition 3.3.4. Since we have found a bound in terms of the α norm for all functions $f \in \mathcal{L}_\alpha$, this implies the following bound on the induced operator norm for $0 < \alpha \leq \alpha_0$

$$\|(R_\mu^n - N_\nu)\|_{\mathcal{B}(\mathcal{L}_\alpha)} \leq 2 C_\alpha \rho_\alpha^n \ . \tag{3.27}$$

Due to the μ-invariance of ν it is easy to verify the identities $N_\nu R_\mu = R_\mu N_\nu = N_\nu$ and $N_\nu^n = N_\nu$, which imply by induction $R_\mu^n - N_\nu = (R_\mu - N_\nu)^n$. So the sequence $\|(R_\mu - N_\nu)^n\|$ is submultiplicative and by Fekete's lemma $\|(R_\mu - N_\nu)^n\|^{\frac{1}{n}}$ converges to its infimum, which (3.27) is strictly smaller than one. □

In order to prove Hölder-continuity of a family of invariant measure μ_z in chapter 5 we require the following result on properties of R_μ and N_ν.

Lemma 3.3.5. *Let μ be a strongly irreducible and contracting probability measure on* $\mathrm{GL}(\mathbb{C}, d)$*, such that for some $\tau > 0$ the function* $\exp(\tau \mathcal{F}(g))$ *is μ-integrable. Setting $Q := R_\mu - N_\nu$ we have*

(i) *The set* $\{a \in \mathbb{C}\,;\ |a| > \max(1, \limsup_{n\in\mathbb{N}} \|Q^n\|_\alpha\}$ *is contained in the resolvent set of* R_μ *and its resolvent satisfies*

$$\mathrm{G}_a := \big(R_\mu - a\,\mathbb{1}\big)^{-1} = \frac{N_\nu}{1-a} - \sum_{n=0}^{\infty} \frac{Q^n}{a^{n+1}}$$

(ii) *There are positive constants* C_α *and* $\rho_\alpha < 1$ *such that for* $a \in \mathbb{C}$ *with* $|a| > 1$

$$\|\,\mathrm{G}_a\,\|_\alpha \le \frac{1}{|1-a|} + \frac{C_\alpha}{1-\rho_\alpha}$$

(iii) *Denoting by* $\mathbb{T}_\varepsilon$ *the circle of radius* $1+\varepsilon$ *around the origin in* $\mathbb{C}$ *for* $\varepsilon > 0$ *we find the Cauchy identity*

$$N_\nu + \mathbb{1} = \frac{1}{2\pi\mathrm{i}} \int_{T_\varepsilon} \mathrm{G}_a \ \mathrm{d}a$$

Proof. Note again, that due to the μ-invariance of ν we obtain the relations $N_\nu R_\mu = R_\mu N_\nu = N_\nu$ and $N_\nu^n = N_\nu$, which imply by induction $R_\mu^n - N_\nu = (R_\mu - N_\nu)^n = Q^n$. Inserting this into the geometric series representation for the resolvent we find

$$\mathrm{G}_a = -a \sum_{n=0}^{\infty} \big(a^{-1} R_\mu\big)^n = -a^{-1} \frac{N_\nu}{1-a^{-1}} - \sum_{n=0}^{\infty} a^{-(n+1)} Q^n \ . \tag{3.28}$$

Since N_ν and Q are bounded operators completeness of $\mathcal{B}(\mathcal{L}_\alpha)$ together with the triangle inequality for the operator norm implies that this expression converges to a bounded operator if $|a| > \max(1, \limsup_{n\in\mathbb{N}} \|Q^n\|_\alpha)$, which proves (i). In order to prove (ii), note that all assumptions of lemma 3.3.3 are satisfied, therefore we can find positive constants C_α and $\rho_\alpha < 1$ such that $|Q^n| < C_\alpha \rho_\alpha^n$. Inserting this into 3.28, taking the α-norm and using the triangle inequality we obtain

$$\|\,\mathrm{G}_a\,\|_\alpha \le \frac{1}{|1-a|} + \sum_{n=0}^{\infty} \|Q^n\|_\alpha = \frac{1}{|1-a|} + C_\alpha \sum_{n=0}^{\infty} \rho_\alpha^n$$

and evaluating the geometric series finishes the proof. Returning to (3.28) and evaluating the integral along an ε-circle around the origin leaves by the residual theorem only the first term and the 0th summand in the sum, which already gives (iii). □

3.3.2. Fourier Laplace transform

In addition to the Markov operator R_μ we define now formally for a given probability measure μ on $\mathrm{GL}(\mathbb{C}, d)$ the Fourier-Laplace transform $R_\mu(z)$ of a function $f \in \mathcal{L}_\alpha$

$$(R_\mu(z) f)(\overline{x}) := \int e^{z \log\|g\overline{x}\|} f(g\overline{x}) \mu(\mathrm{d}g) \ ,$$

where $z \in \mathbb{C}$ and $\overline{x} \in \mathbb{P}\mathbb{C}^d$. It is clear that $R_\mu(0) = R_\mu$. First we have to establish under which assumptions this definition gives rise to an analytic family of bounded operators on the Banach space $\mathcal{L}_\alpha$. Since the proof is rather lengthy and not very instructive, we hide it in the appendix and just state the result.

Lemma 3.3.6. *Let μ be a probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that for some $\tau > 0$ the function $\exp \tau \mathcal{F}(g)$ is integrable. Then for any $0 < \alpha < \frac{\tau}{2}$ there is a $\eta > 0$ such that $\{R_\mu(z), |z| < \eta\}$, constitutes an analytic family of bounded operators on $\mathcal{L}_\alpha$.*

With this result we are ready to derive a representation of $R_\mu(z)$ in terms of its largest eigenvalue. To this end we make use of the following result in perturbation theory. The proof follows more or less directly from general results on perturbation theory, which can be found *e.g.* in [DS58, VII.3]. It relies on the fact that by Cauchy's integral theorem the contour integral of the resolvent G of an operator A around an isolated part of the spectrum $O \subset \sigma(A)$ projects onto the eigenspace corresponding to O. Under the assumption that for an analytic family $R(z)$ of operators the spectrum of $R(0)$ contains an isolated eigenvalue of maximal modulus, it is shown that this is also true for the operators $T(z)$ with $|z|$ small enough. The complete proof can be found in [BL85, Kel06].

Theorem 3.3.7. *Let $U \subset \mathbb{C}$ be a neighbourhood of 0 and $\{R(z), z \in U\}$ an analytic family of bounded operators on a Banach space $\mathcal{B}$ such that for some rank one projector N*

$$\rho = \lim_{n\to\infty} \|R(0)^n - N\|^{\frac{1}{n}} < 1 \ ,$$

then there is $\eta > 0$ such that for $\zeta \in \mathbb{C}$ with $|\zeta| < \eta$

$$R(\zeta) = \lambda(\zeta)N(\zeta) + Q(\zeta) \ , \tag{3.29}$$

where

(i) $\lambda(\zeta)$ is the unique eigenvalue of maximal modulus of $R(\zeta)$ and $\lambda(0) = 1$.

(ii) $N(\zeta)$ is a rank one projection with $N(\zeta)Q(\zeta) = Q(\zeta)N(\zeta) = 0$ and $N(0) = N$.

(iii) All three functions λ, N and Q are analytic in ζ.

(iv) $|\lambda(\zeta)| \geq \frac{2+\rho}{3}$.

(v) For each $p \in \mathbb{N}$ there is a $c > 0$ such that for all $n \in \mathbb{N}$: $\|\frac{\mathrm{d}^p}{\mathrm{d}\zeta^p} Q^n(\zeta)\| \leq c(\frac{1+2\rho}{3})^n$.

Theorem 3.3.7 together with lemma 3.3.3 and lemma 3.3.6 implies for the Fouler-Laplace transform $R_\mu(z)$.

Lemma 3.3.8. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that (μ) is contracting and $\exp(\tau \mathcal{F}(g))$ is integrable for some $\tau > 0$, then*

$$(R_\mu(z)^n f)(\overline{x}) = \int e^{z \log \|g\overline{x}\|} f(g\overline{x})\, \mu^n(\mathrm{d}g) \ . \tag{3.30}$$

In addition, there exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$ and $\eta > 0$ the family of operators $\{R_\mu(z)\,;\, |z| \leq \eta\}$ satisfy the assumptions of theorem 3.3.7.

If $\lambda(z)$ denotes the eigenvalue of maximal modulus from (3.29), then $\lambda'(0)$ is equal to the upper Lyapunov exponent γ.

Proof. From lemma 3.3.3 and lemma 3.3.6 we know that the family $\{R_\mu(z)\}$ satisfies the assumptions of theorem 3.3.7. Since (3.30) is by definition satisfied for $n = 1$ we continue by induction. For the induction step from n to $n+1$ we obtain

$$\begin{aligned}(R_\mu(R_\mu(z)^n f))(\overline{x}) &= \int \exp(z\frac{\log\|g_1 x\|}{\|x\|})(R_\mu(z)^n f)(g_1\overline{x})\mu(\mathrm{d}g_1)\\ &= \int \exp(z\frac{\log\|g_1 x\|}{\|x\|} + z\frac{\log\|g_2 g_1 x\|}{\|g_1 x\|} f(g_2 g_1\overline{x}))\mu(\mathrm{d}g_1)\mu^n(\mathrm{d}g_2)\\ &= \int \exp(z\frac{\|g x\|}{\|x\|} f(g\overline{x})\mu^{n+1}(\mathrm{d}g) = (R_\mu^{n+1} f)(\overline{x}) \ .\end{aligned}$$

Here we used the cocycle property of $\log\|g\overline{x}\|$ from (3.14) and the definition of the convolution measure μ^n from (3.16).

Let $1_{\mathcal{L}_\alpha} \in \mathcal{L}_\alpha$ be the function that is identical to one, *i.e.* $1_{\mathcal{L}_\alpha}\,\overline{x} = 1$ for all $\overline{x} \in \mathbb{PC}^d$. Now consider the Fourier-Laplace transformation of this function, which gives

$$(R_\mu(z)^n\, 1_{\mathcal{L}_\alpha})(\overline{x}) = \int \exp(z\log\|g\overline{x}\|)\,\mu^n(\mathrm{d}g) \ . \tag{3.31}$$

Next we choose $z = t \in \mathbb{R}$ and compute the derivatives of both sides of the last equation with respect to t at $t = 0$. For the right-hand side this implies by dominated convergence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \exp(t\log\|g\overline{x}\|)\,\mu^n(\mathrm{d}g)\right)\Big|_{t=0} = \int \log\|g\overline{x}\|\,\mu^n(\mathrm{d}g) = \mathbb{E}\big(\log(S_{n(\omega)}\overline{x})\big) \ ,$$

which converges by lemma 3.2.12 for n to infinity uniformly to γ if we divide by n. Using the decomposition of $R_\mu(z)$ provided by theorem 3.3.7, the left-hand of (3.31) evaluates for t small enough to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(R_\mu(t)^n\, 1_{\mathcal{L}_\alpha}\big)(\overline{x})\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\lambda(t)^n N(t) 1_{\mathcal{L}_\alpha}(\overline{x}) + Q^n(t) 1_{\mathcal{L}_\alpha}(\overline{x})\big)\Big|_{t=0}\\ &= \big(n\lambda(t)^{n-1}\lambda'(t)N(t)1_{\mathcal{L}_\alpha}(\overline{x}) + \lambda^n(t)N'(t)1_{\mathcal{L}_\alpha}(\overline{x}) + (Q^n)'(t)1_{\mathcal{L}_\alpha}(\overline{x})\big)\Big|_{t=0}\\ &= n\lambda'(0) + N'(0)1_{\mathcal{L}_\alpha}(\overline{x}) + (Q^n)'(0)1_{\mathcal{L}_\alpha}(\overline{x}) \ ,\end{aligned}$$

where we used the properties of N, Q and $1_{\mathcal{L}_\alpha}$ from theorem 3.3.7. If we divide the equation by n and take the limit, the second term in the last line vanishes since N is analytic and hence its first derivative is bounded. The norm of the third term is bounded by point (v) of theorem 3.3.7 and can therefore also be neglected. Therefore the only term surviving in this scaling is $\lambda'(0)$, which must then indeed be equal to γ. □

In order to shorten the notation, we define for any probability measure μ and any measurable set A and function f

$$\mathbb{P}(A) := \int \chi_A(g)\,\mu(\mathrm{d}g) \quad \text{and} \quad \mathbb{E}\big(f\big) := \int f(g)\,\mu(\mathrm{d}g) \ .$$

Now we gathered all the information to prove the desired large deviation result on the growth rate of the random products $\|S_{n(\omega)}\overline{x}\|$. The proof follows the general arguments for large deviation bounds [BL85, Kel06]. Note that we only proof a lower bound on the rate function or equivalently an upper bound on the decay of the probability. These results can be strengthened to a complete large deviation principle with an upper and lower bound on the rate function [BL85].

Theorem 3.3.9. *Let μ be a strongly irreducible and contracting probability measure on* $\mathrm{GL}(\mathbb{C},d)$ *such that* $\exp(\tau\mathcal{F}(g))$ *is integrable for some $\tau>0$. Then for every $\varepsilon>0$ there are $C_\varepsilon>0$ and $n_0\in\mathbb{N}$ such that for all $n\leq n_0$ and $x\in C^d$ with $\|x\|=1$ we have*

$$\mathbb{P}\left(|\log\|S_{n(\omega)}x\|-n\gamma|\geq n\varepsilon\right)\leq e^{-c_\varepsilon n}\ . \tag{3.32}$$

Proof. Starting from the left-hand side of (3.32) we make a distinction of cases for the absolute value inside the probability. Following the usual procedure for large deviation estimates we can exponentiate both sides of the inequalities and find for positive t the two upper bounds

$$\begin{aligned}\mathbb{P}\left(\log\|S_{n(\omega)}\overline{x}\|\geq n(\gamma+\varepsilon)\right)&\leq e^{-t\,n(\gamma+\varepsilon)}\mathbb{E}\left(e^{t\log\|S_{n(\omega)}\overline{x}\|}\right)\\ \mathbb{P}\left(-\log\|S_{n(\omega)}\overline{x}\|\geq -n(\gamma-\varepsilon)\right)&\leq e^{t\,n(\gamma-\varepsilon)}\mathbb{E}\left(e^{-t\log\|S_{n(\omega)}\overline{x}\|}\right)\ .\end{aligned} \tag{3.33}$$

In either case we can express the expectation value on the right-hand in terms of the nth power of $R_\mu(t)$ applied to $1_{\mathcal{L}_\alpha}$. Using (3.29) this implies that for t small enough we find

$$\begin{aligned}\mathbb{E}\left(e^{\pm t\log\|S_{n(\omega)}\overline{x}\|}\right)&=\lambda^n(\pm t)\,(N(\pm t)1_{\mathcal{L}_\alpha})(\overline{x})+(Q^n(\pm)1_{\mathcal{L}_\alpha})(\overline{x})\\ &=\lambda^n(\pm t)\left((N(\pm t)1_{\mathcal{L}_\alpha})(\overline{x})+\frac{(Q^n(\pm t)1_{\mathcal{L}_\alpha})(\overline{x})}{\lambda(\pm t)^n}\right)\ .\end{aligned} \tag{3.34}$$

Since the operator norm of Q^n decays faster than $|\lambda|^n$ and the expectation value on the left-hand side is positive, this is also true for $\lambda(\pm t)$ and in particular $\lambda(\pm t)\in\mathbb{R}$. From theorem 3.3.7 we even know that for some $c>0$, $\frac{\|Q(pmt)\|_\alpha}{|\lambda(\pm)|}\leq c$ In addition, $N(\pm t)1_{\mathcal{L}_\alpha})(\overline{x})$ is bounded for $|t|<\eta$, with η some positive constant. Therefore, after setting $C=\sup_{|t|\leq\eta}\|(N(\pm t)1_{\mathcal{L}_\alpha})(\overline{x})\|_\alpha+c$, taking the logarithm in (3.34) and dividing both sides by n we obtain

$$\frac{1}{n}\log\mathbb{E}\left(e^{\pm t\log\|S_{n(\omega)}\overline{x}\|}\right)\leq\log\lambda(\pm t)+\frac{1}{n}\log(C)\ .$$

The second summand on the right-hand side is bounded and goes to zero in the limit. Combining both expressions from (3.33) and inserting these estimates gives

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\pm\log\|S_{n(\omega)}\overline{x}\|\geq\pm n(\gamma\pm\varepsilon)\right)=-(t(\varepsilon\pm\gamma)-\log\lambda(\pm t))=:-c_\pm(t)\ . \tag{3.35}$$

From lemma 3.3.8 we know that $\lambda(0)=1$ and $\lambda'(0)=\gamma$, so computing the value of $c(t)$ and its first derivative at $t=0$ gives $c_\pm(0)=0$ and $c'_\pm(0)=\varepsilon>0$. Together with the

analyticity of $\lambda(t)$ this implies the existence of an interval $(0, t_0)$ on which $c_\pm(t)$ is strictly positive. Taking the supremum of $c_\pm(t)$ with respect to this interval provides a positive decay rate $\widetilde{c}(\varepsilon)$ for every $\varepsilon > 0$ in the limit of large n in (3.35).

The claimed bound follows now by the following standard argument. Fixing $\varepsilon > 0$ and $\delta > 0$ we can find an n_0 such that

$$e^{-n(\widetilde{c}(\varepsilon)+\delta)} \leq \mathbb{P}\left(|\log\|S_n(\omega)x\| - n\gamma| \geq n\varepsilon\right) \leq e^{-n(\widetilde{c}(\varepsilon)-\delta)}$$

for all $n \geq n_0$, which finishes the proof. □

As a last step before concluding this chapter we need to extend this large deviation result to the matrix elements of the random product $S_{n(\omega)}$. However, in order to keep the chapter in reasonable limits, most of the argument is shifted to the appendix. The proof is similar to the arguments in [DSS02] for the two dimensional real case.

Lemma 3.3.10. *Let μ be a strongly irreducible and contracting probability measure on* $\mathrm{GL}(\mathbb{C}, d)$ *with an exponential moment τ. Then there is an ε_0 such that for every $\varepsilon_0 > \varepsilon > 0$ there exits $\sigma > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and normalized $x, y \in \mathbb{C}^d$*

$$\mathbb{P}\left(|\langle y, S_{n(\omega)}x\rangle| \leq e^{(\gamma-\varepsilon)n}\right) \leq e^{-\sigma n} \ .$$

We will obtain this bound as a consequence of the following lemma.

Lemma 3.3.11. *Let μ be a strongly irreducible and contracting probability measure on* $\mathrm{GL}(\mathbb{C}, d)$ *with an exponential moment τ. Then there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is a $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $\overline{x} \in \mathbb{P}\mathbb{C}^d$*

$$\sup_{\overline{x}\in\mathbb{P}\mathbb{C}^d} \mathbb{P}\left(\frac{|\langle \overline{y}, S_{n(\omega)}\overline{x}\rangle|}{\|S_{n(\omega)}\overline{x}\|} < e^{-\varepsilon n}\right) < e^{-\delta n}$$

proof of lemma 3.3.10. The probability of the event we are interested in can be upper bounded in the following way

$$\begin{aligned}\mathbb{P}\left(|\langle \overline{y}, S_{n(\omega)}\overline{x}\rangle| \leq e^{(\gamma-\varepsilon)n}\right) &= \mathbb{P}\left(\left(|\langle \overline{y}, S_{n(\omega)}\overline{x}\rangle| \leq e^{\frac{-\varepsilon n}{2}}\|S_{n(\omega)}\overline{x}\|\right) \cap \left(\|S_{n(\omega)}\overline{x}\| \leq e^{(\gamma-\frac{\varepsilon}{2})n}\right)\right) \\ &\leq \mathbb{P}\left(|\langle \overline{y}, S_{n(\omega)}\overline{x}\rangle| \leq e^{\frac{-\varepsilon n}{2}}\|S_{n(\omega)}\overline{x}\|\right) + \mathbb{P}\left(\|S_{n(\omega)}\overline{x}\| \leq e^{(\gamma-\frac{\varepsilon}{2})n}\right) \ .\end{aligned}$$

According to theorem 3.3.9 and lemma 3.3.11 both summands on the right-hand side can be upper bounded by an exponentially decreasing function for n large enough, which concludes the argument. □

We now proceed with the proof of the lemma, which, as already mentioned uses ideas from the proof of [BL85, Prop. VI.2.2] and from a similar argument in [DSS02] for the two dimensional real case.

proof of lemma 3.3.11. Our argument uses a bound similar to the Chebyshev's inequality. A standard way to derive such relations is to identify the probability of an event E with the integral over the respective characteristic function $\chi_E(x)$ and then to choose

another function Γ, the integral of which can be controlled and which dominates $\chi_E(x)$. In the case at hand the event of interest for fixed $\overline{y}, \overline{x} \in \mathbb{PC}^d$ is

$$E := \{g \in \mathrm{GL}(\mathbb{C}, d)\,;\, \frac{|\langle \overline{y}\,, g\overline{x}\rangle|}{\|g\overline{x}\|} < e^{-\varepsilon n}\}\ .$$

A possible choice for a function, that simply interpolates the jump from one to zero of the characteristic function is given by

$$\Gamma_n(x) := h_n(|\langle y\,, \frac{x}{\|x\|}\rangle|) \quad \text{with} \quad h_n(t) := \begin{cases} 1 & 0 \le t \le e^{-\varepsilon n} \\ 2 - t\,e^{-\varepsilon n} & e^{-\varepsilon n} \le t \le 2e^{-\varepsilon n} \\ 0 & 2e^{-\varepsilon n} \le t \end{cases}\ . \tag{3.36}$$

This leads to the following upper bound for the probability of the scalar product

$$\begin{aligned} \mathbb{P}\left(\frac{|\langle \overline{y}\,, S_{n(\omega)}\overline{x}\rangle|}{\|S_{n(\omega)}\overline{x}\|} < e^{-\varepsilon n}\right) &= \int \chi_E(g\overline{x})\mu^n(\mathrm{d}g) \le \int \Gamma_n(g\overline{x})\,\mu^n(\mathrm{d}g) \\ &\le \left|\int \Gamma_n(g\overline{x})\,\mu^n(\mathrm{d}g) - \int \Gamma_n(\overline{y})\,\nu(\mathrm{d}\overline{y})\right| + \left|\int \Gamma_n(\overline{y})\,\nu(\mathrm{d}\overline{y})\right| \\ &= |(R_\mu^n\Gamma_n)(\overline{x}) - (N_\nu\Gamma_n)(\overline{x})| + |(N_\nu\Gamma_n)(\overline{x})|\ , \end{aligned} \tag{3.37}$$

where we used the definitions of R_μ and N_ν in the last step (see (3.24) and (3.25)). From (3.27) in the proof of lemma 3.3.3 we know that the first term in the last expression satisfies for α in some interval $(0, \alpha_0)$

$$|(R_\mu^n\Gamma_n)(\overline{x}) - (N_\nu\Gamma_n)(\overline{x})| \le \|(R_\mu^n - N_\nu)\Gamma_n)\|_\alpha \le C_\alpha \rho_\alpha^n \|\Gamma_n\|_\alpha\ , \tag{3.38}$$

with $0 < \rho_\alpha < 1$ and $C_\alpha > 0$. So in order to obtain an exponential bound, we only have to show that $\Gamma_n \in \mathcal{L}_\alpha$ for α small enough. Since it is clear from the definition that $\|\Gamma_n\|_\infty \le 1$, we only have to worry about the term $m_\alpha(\Gamma_n)$. Using the mean value theorem for $h_n(t)$ and choosing suitable representatives $x_i \in \mathbb{C}^d$ for $\overline{x}_i \in \mathbb{PC}^d$ with $\|x_i\| = 1$ we get

$$|\Gamma_n(\overline{x}_1) - \Gamma_n(\overline{x}_2)| \le \Big||\langle y\,, x_1\rangle| - |\langle y\,, x_2\rangle|\Big|\, e^{\varepsilon n} \le \|x_1 - x_2\|\, e^{\varepsilon n} \le \sqrt{2}\delta(\overline{x}, \overline{y}) e^{\varepsilon n}\ ,$$

where we used the identity (3.11) in the last step. Therefore for $0 < \alpha < 1$ the α-norm of Γ_n can be upper bounded by

$$\|\Gamma_n\|_\alpha = \|\Gamma_n\|_\infty + m_\alpha(\Gamma_n) \le 1 + \sqrt{2}e^{\varepsilon n}\ . \tag{3.39}$$

To bound the second summand in (3.37) we can use the regularity of the invariant measure ν. For fixed $\overline{y} \in \mathbb{PC}^d$ let us define the set $\mathcal{B} := \{\overline{x} \in \mathbb{PC}^d\ ;\ |\langle \overline{x}\,, \overline{y}\rangle| \le 2e^{-\varepsilon n}\}$. The projection $(N_\nu\Gamma_n)(\overline{x})$ of Γ_n onto the constant functions then satisfies

$$\int \Gamma_n(\overline{y})\ \nu(\mathrm{d}\overline{y}) \le \nu(\mathcal{B}) = \int_{\mathcal{B}} \frac{|\langle \overline{x}\,, \overline{y}\rangle|^\beta}{|\langle \overline{x}\,, \overline{y}\rangle|^\beta}\ \nu(\mathrm{d}\overline{x}) \le 2^\beta\, e^{-\varepsilon n\beta} \int \frac{1}{|\langle \overline{x}\,, \overline{y}\rangle|^\beta}\ \nu(\mathrm{d}\overline{x})\ . \tag{3.40}$$

The integral in the last expression can be shown to be finite for all $\overline{y} \in \mathbb{PC}^d$ if β is small enough, which is done in lemma B.5.1 in the appendix. Inserting (3.39), (3.38) and

(3.40) into (3.37), we find that for some $0 < \alpha < \alpha_0$ and n large enough there are $C_\alpha, K_\beta > 0$ and $0 < \rho_\alpha < 1$ such that

$$\mathbb{P}\left(\frac{|\langle \overline{y}, S_n(\omega)\overline{x}\rangle|}{\|S_n(\omega)\overline{x}\|} < e^{-\varepsilon n}\right) \leq C_\alpha(1+\sqrt{2}e^{\varepsilon n})\rho_\alpha^n + 2^\beta K_\beta e^{-\varepsilon n\beta} \ . \tag{3.41}$$

For $0 < \varepsilon < |\log\rho_\alpha| = \varepsilon_0$ we can therefore find a positive $\delta \leq \min(|\log\rho_\alpha + \varepsilon|, \varepsilon\beta)$ and a $n_0 \in \mathbb{N}$, such that the right-hand side of (3.41) is upper bounded by $e^{-\delta n}$ for all $n \geq n_0$. □

In a similar manner, we can show the almost sure convergence of the matrix elements of a product of transfer matrices to the Lyapunov exponent.

Lemma 3.3.12. *Let μ be a strongly irreducible and contracting probability measure on* $\mathrm{GL}(\mathbb{C}, d)$ *with exponential moment τ then almost surely for all nonzero $x, y \in \mathbb{C}^d$*

$$\lim_{n\to\infty} \frac{1}{n} \log|\langle y, S_n(\omega)x\rangle| = \gamma \ .$$

Proof. We show that for all $t > 0$ almost surely

$$\lim_{n\to\infty} \frac{1}{n^t} \log \frac{|\langle y, S_x(\omega), |\rangle}{\|S_x(\omega)\|} = 0 \ . \tag{3.42}$$

The result can then be obtain from lemma 3.2.12, which ensures the almost sure convergence of $\frac{1}{n}\|S_n(\omega)x\|$ to the Lyapunov exponent. More precisely, we prove that almost surely for $r > 0$ large enough

$$\lim_{n\to\infty} \frac{1}{n^r} \frac{\|S_x(\omega)\|}{|\langle y, S_x(\omega), |\rangle} = 0 \ ,$$

which implies (3.42). We use the Borel-Cantelli-lemma in order to show that the complementary event has zero probability. Hence, we have to prove that the expression

$$\sum_{i=1}^{\infty} \mathbb{P}\left(\left|\langle y, \frac{S_x(\omega)}{\|S_x(\omega)\|}\rangle\right| < \varepsilon n^{-r}\right)$$

is finite. As in the proof of lemma 3.3.11, we want to derive a Chebyshev like bound for these probabilities. To this end note that we can consider this probabilities as a expectation value of the characteristic functions χ_E with respect to the event

$$E := \left\{ g \in \mathrm{GL}(\mathbb{C}, d) \, ; \, \left|\langle y, \frac{S_x(\omega)}{\|S_x(\omega)\|}\rangle\right| < \varepsilon n^{-r} \right\} \ .$$

In the same manner as in (3.36) we can define the function $\Gamma_n(s)$, with $e^{-\varepsilon n}$ substituted

by εn^{-r}. Following all the steps after (3.36), we find according to (3.37) the upper bound

$$\sum_{i=1}^{\infty}\mathbb{P}\left(\left|\langle y\,,\frac{S_x(\omega)}{\|S_x(\omega)\|}\rangle\right|<\varepsilon n^{-r}\right) \tag{3.43}$$
$$\leq\sum_{n=1}^{\infty}\left|(R_\mu^n\Gamma_n)(\overline{x})-(N_\nu\Gamma_n)(\overline{x})\right|+\sum_{n=1}^{\infty}|(N_\nu\Gamma_n)(\overline{x})|$$
$$\leq\sum_{n=1}^{\infty}\|R_\mu^n-N_\nu\|_{\mathcal{B}(\mathcal{L}_\alpha)}\|\Gamma_n\|_\alpha+\sum_{n=1}^{\infty}|(N_\nu\Gamma_n)(\overline{x})|\ .$$

By an argument similar to the one leading to (3.39) and due to (3.27) we can bound every summand in the first series by

$$\|R_\mu^n-N_\nu\|_{\mathcal{B}(\mathcal{L}_\alpha)}\|\Gamma_n\|_\alpha\leq(1+2n^r\varepsilon^{-1})C\rho_\alpha^n\ ,$$

with C finite and $0<\rho_\alpha<1$ showing that the sequence can be upper bounded by a finite constant. Repeating the argument leading to the bound in (3.40) we obtain for the summands of the second series

$$|(N_\nu\Gamma_n)(\overline{x})|\leq(2\varepsilon)^\beta n^{-r\beta}\int\frac{1}{|\langle\overline{x}\,,\overline{y}\rangle|}\ \nu(\mathrm{d}\overline{x})\ .$$

The integral is finite for β small enough by lemma B.5.1 from the appendix. Therefore, the bound in (3.43) is finite if we choose r large enough, such that $\sum_{n=1}^{\infty}n^{-r\beta}$ is finite, which finishes the proof. □

4. Recurrence properties of discrete unitary evolutions

There are several ways to analyze the dynamical properties of a discrete unitary evolution. In section 2.4 we considered the asymptotic position distribution in order to show the ballistic spreading behaviour of translation-invariant quantum walks. In this chapter we study the transition probabilities between different states of the system. Again, our goal is to derive asymptotic statements, that is, we want to determine whether a certain state is attained by the system eventually, in the limit of large times. In particular, we study transitions back into the initial state, *i.e.*, the recurrence properties of a system.

The results presented in this chapter have been obtained together with Alberto Grünbaum, Luis Velázquez and Reinhard F. Werner and have been published in [GVWW13].

Since quantum mechanics fundamentally prevents us from knowing the particle's exact position without an actual measurement, we have to find notions of transition and recurrence that take into account the inevitable disturbance due to a measurement process. For the recurrence scenario there are at least two possible ways to define time dependent return probabilities. As it turns out however, one is conceptually and operationally more convenient than the other, even though it comes at the cost of a perturbation of the free evolution.

This chapter is organized as follows. First, we introduce a notion of transition amplitudes between an initial and a collection of target states that explicitly takes into account a measurement of the time evolved state in every time step. Subsequently, we give a brief account on related work on transition problems and recurrence properties. In section 4.2 we return to our definition of a transition amplitudes, but restrict to the case to self-transitions, *i.e.* the recurrence properties of an initial state. We derive a complete characterization of the recurrence properties in terms of the spectral measure of the initial state with respect to the unitary operator that implements the time evolution. Equipped with this characterization, we compare this notion of recurrence with other definitions that were put forward in the context of quantum walks.

We begin by specifying the scenario we are going to investigate in the remainder of this chapter. We consider a quantum system described by a separable Hilbert space $\mathcal{H}$ with a discrete unitary evolution according to definition 2.1.2. Therefore, there is a unitary operator U acting on $\mathcal{H}$ that implements the dynamics.

Starting with some pure initial state ϕ we can study the orbit of time evolved states $\{U^n\phi\ ,\ n\in\mathbb{N}\}$. We assume that in addition to the initial state ϕ, we are given a target state ψ and we wish to determine whether the system realizes this state eventually. Although we are mostly concerned with a single target state, we define the problem more general and allow for a finite set of k mutually orthonormal target states $\{\psi_l\}$.

Denote by P_{tg} the projector onto the target subspace

$$P_{tg} := \sum_{l=1}^{k} |\psi_l\rangle\langle\psi_l| \ . \tag{4.1}$$

A basic fact of quantum theory is that in order to decide the question, whether one of the target states has been realized by the system, we have to perform a measurement. Therefore, we consider a modified dynamics such that the evolution from time step n to time step $n+1$ consists of three parts:

1. The application of U to the current state of the system ϕ_n.
2. A projective measurement consisting of the orthogonal projectors $\{P_{tg}, \mathbb{1} - P_{tg}\}$ that checks, whether the system happens to be in one of the target states.
3. If we observe one of the target states, the evolution is stopped, because the transition to one of the target states already occurred. Otherwise, the next time step is executed.

These operations give rise to a modified evolution operator $\tilde{U}$, defined by

$$\tilde{U} := (\mathbb{1} - P_{tg})U \ . \tag{4.2}$$

According to this definition, ϕ_n is equal to the renormalized state of the system conditioned on the event that we did not observe a transition to one of the target states in the first n time steps. The probability to find the particle in the $n+1$ step in the target subspace is given by the probabilities to observe one of the target states, *i.e.* $\sum_l |\langle \psi_l , U\phi_n \rangle|^2$. Otherwise, the system will be, up to normalization, in the state $(\mathbb{1} - P_{tg})U\phi_n$. We can express ϕ_n in terms of the modified evolution operator $\tilde{U}$ and obtain

$$\phi_n = \frac{\tilde{U}^n \phi}{\|\tilde{U}^n \phi\|} \ .$$

The square of the renormalization factor of this expression corresponds to the probability not to find the system in one of the target states ψ_l within the first n steps. Hence, it can be interpreted as the survival probability s_n of the initial state ϕ up to step n

$$s_n := \|\tilde{U}^n \phi\|^2 \ .$$

We can now compute the loss of normalization per time step and identify this quantity as the n^{th} step transition probability q_n from the initial state to the target subspace. Evaluating the difference between s_n and s_{n+1} we obtain

$$\begin{aligned} q_{n+1} = s_n - s_{n+1} &= \|\tilde{U}^n \phi\|^2 - \|\tilde{U}^{n+1} \phi\|^2 \\ &= \mathrm{tr}\big(\tilde{U}^n |\phi\rangle\langle\phi| \tilde{U}^{*n} (\mathbb{1} - \tilde{U}^* \tilde{U})\big) \\ &= \mathrm{tr}\big(\tilde{U}^n |\phi\rangle\langle\phi| \tilde{U}^{*n} (\mathbb{1} - U^* (\mathbb{1} - P_{tg}) U)\big) \\ &= \langle \phi \, , \tilde{U}^{*n} U^* P_{tg} U \tilde{U}^n \, \phi \rangle \ . \end{aligned} \tag{4.3}$$

Using (4.1) and the mutual orthonormality of the target states we obtain the relation

$$q_{n+1} = \sum_{l=1}^{k} |\langle \psi_l , U\tilde{U}^n \phi \rangle|^2 =: \sum_{l=1}^{k} |a_{n(l)}|^2 \tag{4.4}$$

where we defined $a_{n(l)}$ as the n^{th} step transition amplitude to the state ψ_l. These transition amplitudes will be the key quantities in our theory. Starting from the identity $s_1 + q_1 = 1$ and repeatedly using (4.3) we obtain the relation

$$\sum_{m=1}^{n} \sum_{l=1}^{k} |a_{m(l)}|^2 + \|\tilde{U}^n \phi\|^2 = 1 \ ,$$

which just reflects the fact that due to the absence of other loss processes the particle either survives or is detected in one of the target states. We can therefore express the asymptotic or total transition or arrival probability T of the system in terms of the survival as well as in terms of the transition probabilities

$$T = \sum_{n=1}^{\infty} \sum_{l=1}^{k} |a_{n(k)}|^2 = 1 - \lim_{n\to\infty} \|\tilde{U}^n \phi\|^2 = 1 - s_\infty \ , \tag{4.5}$$

where s_∞ denotes the asymptotic survival probability. If s_∞ converges to zero, we know that a transition to one of the target states will occur eventually. The structure of this equation suggests to study the generating functions of the sequences $(a_{n(l)})_n$ of the transition amplitudes

$$\widehat{a}_l(z) := \sum_{n=1}^{\infty} a_{n(l)} z^n = \sum_{n=1}^{\infty} \langle \psi_l , U\tilde{U}^{n-1}\phi \rangle z^n = z \langle \psi_l , U \sum_{n=1}^{\infty} z^n \tilde{U}^{n-1} \phi \rangle \ , \tag{4.6}$$

where we used (4.4) in the second step. Before we further evaluate this equation via a Krein's formula approach that is especially successful in the context of a single target state, we give an overview over related results on modified time evolutions and recurrence properties.

4.1. Related work

Concepts like first arrival probabilities, escape probabilities and return probabilities are well established and very successful in the context of classical random walks and helped to analyze their dynamical properties. Due to the connection between quantum and random walks it is natural to define quantum analogues of these quantities. In the following we present some relevant approaches. First we review some results on quantum walks with absorbing boundary conditions, which represent a special case of the scheme of alternating unitary evolution and measurement presented above. Next we consider the recurrence properties of classical Markov chains in more detail before we discuss an alternative notion of recurrence for quantum walks in the last subsection.

4.1.1. Quantum walks with absorbing boundary conditions

The concept of an intermediate measurement in addition to a unitary time evolution is already present in the first definition of a quantum random walk in [AAKV01]. In addition, schemes alternating between a unitary time step and a projective measurement have been studied early on to introduce absorbing boundary conditions [ABNW01, BCG+04].

Aside from numerical studies of the influence of absorbing lattice sites in one and two dimensions [GAM09, GASM11], there are also analytic results. Ambainis et al. as well as Bach et al. both used such measurements as a means to introduce absorbing boundary conditions in a one-dimensional coined quantum walk according to definition 2.4.2 [ABNW01, BCG+04]. In this case, the Hilbert space is $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ and in the language of the previous section the target states ψ_l always live on a single lattice site x, *i.e.* is of the form $\psi_l = \delta_x \otimes \xi$ for some $\xi \in \mathbb{C}^2$. In addition, the target states ψ_l always come in pairs such that both internal states are tested for, which implies that the projector onto the target subspace P_{tg} can be decomposed in terms of lattice site projections

$$P_{tg} = \sum_{x_l} |\delta_{x_l}\rangle\langle\delta_{x_l}| \otimes \mathbb{1} \ .$$

Due to this choice of measurement, the walking particle cannot bypass the lattice site x without being detected and is therefore absorbed with certainty if it enters the lattice site x. This is different from our approach in the last section, where we allow the target state to be supported on several lattice sites or only on a subspace of the internal space. We also do not restrict our definition to quantum walks.

Both [ABNW01] as well as [BCG+04] consider one and two absorbing boundaries and compute the escape probabilities, that is, the arrival probabilities at the left or the right boundary if the particle starts in between. In both cases results are obtained by path counting combinatorics in combination with a generating function approach on the level of the transition amplitudes similar to (4.6). Since the measurement effectively changed the geometry of the quantum walk to a one sided infinite or finite lattice respectively, Bach et al. could show that it suffices to consider the generating functions for the absorbing boundary at lattice site 1 if the particle starts at the origin [BCG+04].

$$\begin{aligned}\widehat{a}_1(z) &:= \sum_{n=1}^{\infty} \langle \delta_1 \otimes |0\rangle, \left(U\tilde{U}^{n-1}\right) \delta_0 \otimes |0\rangle\rangle z^n \\ \widehat{a}_2(z) &:= \sum_{n=1}^{\infty} \langle \delta_1 \otimes |0\rangle, \left(U\tilde{U}^{n-1}\right) \delta_0 \otimes |1\rangle\rangle z^n \ ,\end{aligned}$$

where $\tilde{U} = (\mathbb{1} - |\delta_1\rangle\langle\delta_1| \otimes \mathbb{1})U$ and U denotes the unitary operator implementing the considered coined walk (see (2.15)). The generating functions corresponding to an absorbing boundary at positions different from 1 can be decomposed into a product of simpler generating functions. If, for example, the absorbing lattice site is located at position x and we are interested in the behaviour of the initial state $\phi = \delta_0 \otimes |0\rangle$ we have to consider the generating function $\widehat{a}_1(z)(\widehat{a}_2(z))^{x-1}$ [BCG+04].

In [ABNW01] and [BCG+04] results are obtained for an absorbing lattice site right next to the initial state and in the limiting case for an infinitely distant absorbing boundary. For the Hadamard walk for example, the absorbtion probability in the first case is $\frac{2}{\pi}$ and in the second case it is given by

$$\frac{1}{2}|\alpha|^2 + \frac{4-\pi}{2\pi}|\beta|^2 - 2\frac{\pi-2}{2\pi}\Re\overline{\alpha}\beta$$

for the initial state $\phi = \delta_0 \otimes \big(\alpha|0\rangle + \beta|1\rangle\big)$ and there is also an explicit formula in the general case [BCG+04]. In addition, Bach et al. obtain some results on quantum walks on the d-dimensional lattice if the absorbing boundary is $d-1$ dimensional [BCG+04].

For the Hadamard walk with two absorbing boundary conditions, which effectively means that we are dealing with a quantum walk on the finite line with N lattice sites, Bach and Borisov derived an explicit formula for the absorbtion probability at the origin [BB09]. More precisely, the probability that the walking particle is absorbed at $x = 0$ if it started at lattice site $0 < x < N$ is given by

$$\frac{\sqrt{2}\big((2+\sqrt{2})^{N-x} - (2-\sqrt{2})^{N-x}\big)\big((2+\sqrt{2})^{x-1} + (2-\sqrt{2})^{x-1}\big)}{4(2+\sqrt{2})^{N-1} + (2-\sqrt{2})^{N-1}} .$$

In the same explicit measurement setting, absorption probabilities of quantum walks on the finite and infinite one-dimensional lattice were investigated by Konno and coworkers by path counting methods [KNSS03]. In this paper explicit expressions for the absorption probability at zero are obtained if the particle starts at lattice site $x = 1$ in an arbitrary internal state. After this short overview on results of quantum walks with absorbing boundary conditions, we continue with some remarks on the recurrence properties of classical random walks.

4.1.2. Recurrence in classical random walks

We do not attempt to give a full account on the theory of classical Markov chains, but rather compose a brief introduction into their recurrence properties mostly to compare them to the quantum case. For a thorough introduction and a good review of the theory of classical Markov chains, we refer to the textbooks [Kel79, KT75]. Moreover, since we are interested in a comparison with discrete quantum evolutions, we limit our discussion to time homogeneous or stationary Markov chains. These are processes, where the transition probabilities do not change with time. In the remainder of this section we therefore synonymously use the terms Markov chain and stationary Markov chain.

Markov processes describe systems with short lived temporal correlations. In fact, the memory is reset in every time step. This implies that the new state of the system depends exclusively on the current state and the transition rule is given by a fixed probability distribution due to our assumption of stationarity.

We define a classical Markov chain $\{X(n)\}$ operating in discrete time steps $n \in \mathbb{N}$ on a countable state space X as a stochastic process, which is completely characterized by its transition matrix (P_{xy}) describing the transition probabilities from all states x to all states y [KT75]. Its matrix elements P_{xy} satisfy the conditions $P_{xy} \geq 0$ and $\sum_y P_{xy} = 1$ for all $x \in X$. The Markov property ensures that the probability of a transition from

x to y along the exact sequence $(x_1, x_2, \ldots, x_{n-1})$ is determined by the product of the corresponding matrix elements of (P_{xy})

$$P_{x,x_1} P_{x_1,x_2} \cdots P_{x_{n-1},y} \ .$$

If we are only interested in the overall probability to change from state x to state y in exactly n steps, we have to sum over all possible paths of length n, which gives

$$\sum_{x_1,\ldots,x_{n-1}} P_{x,x_1} P_{x_1,x_2} \cdots P_{x_{n-1},y} \ . \tag{4.7}$$

This expression is equal to the xy matrix element of the n^{th} power of the transition matrix (P_{xy}). In order to address questions about recurrence in this classical process we fix some specific state $0 \in X$ and consider only transitions from 0 to 0, that is, the 00 matrix element of the matrix (P^n_{xy}). This matrix element corresponds exactly to the probability to return in the n^{th} time step p_n, which can be expressed by (4.7) as

$$p_n := (P_{xy})^n_{00} = \sum_{x_1,\ldots,x_{n-1}} P_{0,x_1} P_{x_1,x_2} \cdots P_{x_{n-1},0} \ . \tag{4.8}$$

This quantity does not take into account whether or not the system already returned to the state 0 in an earlier time step. On the other hand it is quite easy to write down the definition of the first return probabilities q_n by restricting the sequences $(x_1, \ldots, x_{n-1})$ that are summed over in (4.7)

$$q_n := \sum_{\substack{x_1,\ldots,x_{n-1} \\ \forall_l : x_l \neq 0}} P_{0,x_1} P_{x_1,x_2} \cdots P_{x_{n-1},0} \ .$$

Returning to our expression for p_n, we can decompose the sum in (4.7) with respect to the largest index $k < n$ for which x_k is equal to zero. This means that up to the index k we do not care whether the system returned to the state 0, so this part of the product $P_{0,x_1} P_{x_1,x_2} \cdots P_{x_{n-1},0}$ is described by p_k. In between $x_k = 0$ and $x_n = 0$ however, the definition of k forces the system to stay away from the state 0, *i.e.* $x_i \neq 0$ for all $k < i < n$. Therefore this part of the product is equal to q_{n-k}. In total, this amounts for $n \geq 1$ to the identity

$$p_n = \sum_{k=0}^{n} p_k q_{n-k} \ , \tag{4.9}$$

if we choose $p_0 = 1$ and $q_0 = 0$ as initial values of the two sequences [KT75]. After defining the corresponding generating functions of the two sequences

$$\widehat{p}(z) := \sum p_n z^n \quad \text{and} \quad \widehat{q}(z) := \sum q_n z^n \ , \tag{4.10}$$

we can insert the identity (4.9) into $\widehat{p}(z)$ for $n > 0$, which together with $p_0 = 1$ implies the relation

$$\widehat{p}(z) = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} p_k q_{n-k} \right) z^n = 1 + \widehat{p}(z)\widehat{q}(z) \ . \tag{4.11}$$

Hence, we have found the well renewal equation that combines the first return probabilities with the probabilities to return in exactly n steps.

The natural definition of a recurrent Markov chain is to demand that the system returns with certainty to the initial state, *i.e.* that the asymptotic or total return probability R^C

$$R^C := \sum_{n=1}^{\infty} q_n$$

of the classical process is equal to one. The quantity R^C is also known as the Pólya number of the random walk [Pól21]. With the preceding identity from (4.11) we can formulate the following characterization of recurrent Markov chains [KT75].

Lemma 4.1.1. *A Markov chain is recurrent if and only if the series of its return probabilities p_n diverges.*

Proof. The total return probability R^C of a classical random walk can be computed as the sum of the first return probabilities. Therefore, using (4.11) we obtain

$$R^C = \sum_{n=0}^{\infty} q_n = \widehat{q}(1) = 1 - \frac{1}{\widehat{p}(1)} = 1 - \frac{1}{\sum_n p_n} \ .$$

So indeed $R^C = 1$ if and only if the series $\sum_n p_n$ diverges. □

Before we turn to recurrence results in the context of quantum walks, we discuss how the spectral measure of the transition matrix (P_{xy}) with respect to the initial state 0 is connected to the recurrence behaviour.

In order to establish a connection with the quantum case we represent Markov processes as self-adjoined or symmetric operators on Hilbert spaces. It turns out that this is possible for the class of so-called reversible Markov chains [Kel79]. This class contains all Markov processes, which are invariant with respect to time reversion. That is, they produce the same statistics if the process is run backwards. The formal definition of a reversible Markov chain can be given in terms of the random variables $X(n)$ composing the Markov chain. We demand that for any collection of times $n_1, \ldots, n_k$ and all $l \in \mathbb{N}$ the random variables $(X(n_1), \ldots X(n_k))$ and $(X(l-n_1), \ldots, X(l-n_k))$ have the same distribution. Note that this definition readily implies stationarity for the Markov chain $\{X(n)\}$ [Kel79].

For our purpose the important consequence of reversibility of a Markov process is the detailed balance equation [Kel79]

Theorem 4.1.2 (detailed balance equation)**.** *Let $\{X(n)\}$ be a stationary Markov chain in discrete time on a separable space X and (P_{xy}) its transition matrix. Then, $\{X(n)\}$ is reversible if and only if there exists a normalized vector π with positive components satisfying the detailed balance condition*

$$\pi_x P_{xy} = P_{yx} \pi_y \quad \forall x, y \in X \ .$$

If π exists it is an invariant state of the Markov chain, so $\pi = (P_{xy}) \cdot \pi$.

Proof. We begin the proof by considering a reversible chain. Firstly, stationarity implies that $\mathbb{P}(X(n)=x)$, the probability of the process being in state x at time n, is independent of n. Set $\pi_x = \mathbb{P}(X(n)=x)$. Due to reversibility the two distributions of the random variables $(X(n), X(n+1))$ and $(X(n+1), X(n))$ are identical, which indeed implies

$$\begin{aligned} &\mathbb{P}\big(X(n)=x \text{ and } X(n)=y\big)=\mathbb{P}\big(X(n)=y \text{ and } X(n)=x\big) \\ \Longleftrightarrow\ &\mathbb{P}(X(n)=x)\, P_{xy}=P_{yx}\,\mathbb{P}\big(X(n)=y\big) \\ \Longleftrightarrow\ &\pi_x\, P_{xy}=P_{yx}\,\pi_y\ . \end{aligned}$$

Summing the last expression with respect to y we also see that π is an invariant state.

If we now assume that π satisfies the detailed balance equation we can infer from the previous discussion that π is an eigenvector of (P_{xy}) with corresponding eigenvalue 1. Therefore, for any collection of states $x_1,\dots,x_l \in X$ and any $n,m \in \mathbb{N}$ we find by successively applying the detailed balance condition

$$\begin{aligned} &\mathbb{P}(X(n)=x_1 \text{ and } \dots \text{ and } X(n+l)=x_l)= \\ &\qquad \pi_{x_1}P_{x_1x_2}\cdots P_{x_{l-1}x_l}=P_{x_2x_1}\pi_{x_2}\cdots P_{x_{l-1}x_l}=\cdots=P_{x_2x_1}\cdots P_{x_lx_{l-1}}\pi_{x_l} \\ &\qquad\qquad =\mathbb{P}(X(m)=x_l \text{ and } \dots \text{ and } X(m+l)=x_1)\ . \end{aligned}$$

Due to this relation, we can infer that the random variables $(X(n),\dots,X(n+l))$ and $(X(k-n),\dots,X(k-n-l))$ have the same distribution by choosing $m=k-n-l$ for any $l \in \mathbb{N}$, which in turn implies reversibility. □

Given a reversible Markov process on some separable state space X with invariant state π, we can introduce an inner product between functions on X by

$$(f,g)_\pi = \sum_{x\in X} \overline{f_x}\pi_x g_x\ .$$

The transition matrix (P_{xy}) acts on a function f via matrix multiplication. In fact, the operator (P_{xy}) is self-adjoined on this function space, which results from

$$(Pf,g)_\pi = \sum_{x\in X} \overline{(Pf)_x}\pi_x g_x \sum_{x,y\in X} \overline{f_y}P_{xy}\pi_x g_x = \sum_{y\in X} \overline{f_y}\pi_y (Pg)_y = (f,Pg)_\pi\ .$$

So we can use the theory of self-adjoined operators in order to analyze the spectral properties of the transition matrix. In particular, for any probability vector $\psi \in X$ we can consider the spectral measures $\mu_\psi(\mathrm{d}\lambda) = \sum_{x,y\in X}\psi_x\psi_y E_{xy}(\mathrm{d}\lambda)$ induced by the projector valued measure E corresponding to (P_{xy}). Note that because the transition matrix is self-adjoined but can have negative eigenvalues those spectral measures are supported on the interval $[-1,1]$.

We can now express the overall probability p_n to return to an initial distribution ϕ after n time steps in terms of the moments of the spectral measure

$$\langle \phi\,, (P_{xy})^n \phi\,\rangle = \int_{-1}^{1} \lambda^n\, \mu_\phi(\mathrm{d}\lambda)\ .$$

By choosing ϕ equal to the vector $\delta_{x,0}$ that gives weight one to the initial state $0 \in X$ we get the overall return probability in the n^{th} step p_n from (4.8). Therefore, the corresponding generating function is given by an expression similar to the Stieltjes transform of a probability measure on the unit circle (see (2.9))

$$\widehat{p}(z) = \sum_n p_n z^n = \int_{-1}^{1} \lambda^n z^n \, \mu_0(\mathrm{d}\lambda) = \int_{-1}^{1} \frac{\mu_0(\mathrm{d}\lambda)}{1-\lambda z} \ .$$

Our recurrence criterium from lemma 4.1.1 therefore implies that the state 0 is recurrent if and only if

$$\lim_{z \to 1^-} \int_{-1}^{1} \frac{\mu_0(\mathrm{d}\lambda)}{1-\lambda z} = \infty \ .$$

Provided that the state 0 is recurrent, we might also investigate the expected first return time $\tau^C := \sum_n n q_n$. Expressing τ^C in terms of the generating function $\widehat{q}(z)$ from (4.10) we obtain

$$\tau^C = \sum_n n q_n = \left. \frac{\mathrm{d}\widehat{q}(z)}{\mathrm{d}z} \right|_{z=1} = \lim_{z \to 1^-} \frac{\widehat{q}(1) - \widehat{q}(z)}{1-z} = \lim_{z \to 1^-} \left((1-z)\widehat{p}(z) \right)^{-1} \ . \tag{4.12}$$

The limit in the last expression can be evaluated by dominated convergence and since the function $\frac{1-z}{1-\lambda z}$ converges to zero for $z \to 1^-$ for all $\lambda \neq 1$ it follows that

$$\tau^C = \lim_{z \to 1^-} \int_{-1}^{1} \frac{1-z}{1-\lambda z} \, \mu_0(\mathrm{d}\lambda) = \mu_0(\{1\}) \ .$$

In other words, the expected return time of a reversible Markov process is given by the weight that is assigned by the spectral measure μ_0 to the eigenvalue 1.

4.1.3. Recurrence without disturbance

An approach to a definition of recurrence properties that avoids any disturbance of the system was introduced in [ŠJK08]. It translates the concept of recurrence in the context of classical Markov chains to the scenario of quantum walks. As explained in the previous subsection, a classical Markov chain is recurrent if and only if the first return probabilities sum up to one. By lemma 4.1.1 this is equivalent to the divergence of the sum of the probabilities p_n to return in exactly n steps to the initial state. In close analogy to this classical case it was suggested in [ŠJK08] to consider the series p_n of probabilities to return in the n^{th} step

$$p_n = |\langle \phi \, , U^n \phi \rangle|^2 \tag{4.13}$$

and to consider a pair (U, ϕ) consisting of a unitary operator U and an initial state ϕ recurrent if and only if the series $\sum_n p_n$ diverges. We call this property SJK-recurrence.

It is clear that by the very nature of quantum theory the connection to the first return probabilities will be lost, since we simply cannot condition on specific properties of a

path without a measurement, which would disturb the system. As a way out it is noted in [ŠKJ08] that the divergence of $\sum_n p_n$ is equivalent to the condition

$$R^{SJK} = 1 - \prod_n (1 - p_n) = 1 \ .$$

There is also an operational meaning of this quantity [ŠKJ08]: "Take a system and measure the position of the walker after one time step at the origin, then discard the system. Take a second, identically prepared system and let it evolve for two time steps, measure at the origin, then discard the system. Continue similarly for arbitrarily long evolution time. The probability that the walker is found at the origin in a single series of such measurement records is the Pólya number [i.e., R^{SJK}]."

This definition of R^{SJK} ensures that in analogy to the classical case the sum of the p_n determines whether a process is recurrent. On the other hand the simple operational meaning in terms of paths is lost. Instead to evaluate this criterion we have to prepare and measure a large ensemble of systems and in particular do not check whether the particle already returned to the initial state some time prior to the measurement.

In order to establish comparability between this and our notions of recurrence, we finish this section with a characterization of SJK-transience and recurrence in terms of the spectral measure of the initial state ϕ and the unitary operator U.

Lemma 4.1.3. *A pair (U, ϕ) is SJK-transient if and only if the spectral measure μ is absolutely continuous with a square integrable density. Otherwise the pair is SJK-recurrent.*

Proof. Unitarity of U implies that its spectrum is a subset of $\mathbb{T}$ and therefore we can express the Fourier coefficients of its spectral measure μ_n for $n \geq 0$ as

$$\mu_n = \int \mu(d\,u)\, u^n = \langle \phi \,, U^n \phi \rangle \ .$$

Using the identity $\mu_{-n} = \overline{\mu_n}$ we see that the pair (U, ϕ) is not SJK-recurrent and therefore SJK-transient if and only if the Fourier coefficients of μ are square summable

$$\sum_n p_n = \frac{1}{2}(1 + \sum_{n=-\infty}^{\infty} |\mu_n|^2) \ .$$

This summability conditions implies via the Riesz-Fischer and the Plancherel theorem that μ is absolutely continuous with respect to the Lebesgue measure m with a square summable density, which finishes the proof. □

In a similar spirit the asymptotic return probability

$$P_{\text{asy}} := \limsup_n |\langle \phi \,, U^n \, \phi \rangle|^2$$

of one-dimensional quantum walks has been studied by several authors [KS10, KŁS13]. In this context the rather unfortunate term localization has been used for a quantum walk with non vanishing asymptotic return probability. Note however that this kind

of localization is not related to the notions of Anderson or dynamical localization as discussed in section 2.5.2.

With the CGMV approach discussed in section 2.4.2 Cantero et al. were able to attribute a non vanishing P_{asy} to the existence of a point mass in the spectral measure of the initial state and the case $P_{\text{asy}} = 0$ to an absolutely continuous spectral measure, provided that the existence of a singular continuous part can be excluded ([CGMV10, CGMV12], see also section 5.2).

This concludes our overview on related results on the recurrence properties of quantum walks.

4.2. Recurrence by absorption

In this section we continue our study of transition amplitudes and probabilities in the context of self-transitions. This means that we limit our analysis to a single target state, which furthermore is identical to the initial state ϕ.

It turns out that this restricted case has enough structure to obtain a complete characterization of recurrent and transient initial states of a unitary operator in terms of their spectral measure. In fact, we obtain a dynamical distinction between the absolutely continuous and the singular part of the spectral measure, which supplements the distinction obtained by the RAGE theorem 2.2.6, which distinguishes the continuous and the point spectrum.

Given an initial state $\phi \in \mathcal{H}$ and the unitary operator U the modified time evolution operator $\widetilde{U}$ defined in (4.2) takes now the simple form

$$\widetilde{U} = (\mathbb{1} - |\phi\rangle\langle\phi|)U \ .$$

In addition, we obtain for the first return probabilities, which correspond to the first transition probabilities from (4.3),

$$q_n = s_n - s_{n+1} = \|\tilde{U}^n\phi\|^2 - \|\tilde{U}^{n+1}\phi\|^2 = |\langle\phi\,, U\tilde{U}^n\phi\rangle|^2 =: |a_n|^2 \ . \tag{4.14}$$

Hence, in contrast to (4.4) we only have to deal with a single sequence of first return amplitudes a_n. The total return probability R can be defined analogously to the total transition probability T in (4.5) as

$$R := \sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} |a_n|^2 \ . \tag{4.15}$$

This discussion motivates the following definition of recurrent and transient behaviour of an initial state ϕ with respect to a unitary operator U.

Definition 4.2.1 (recurrence)**.** *Let $\mathcal{H}$ be a separable Hilbert space. A pair (U,ϕ), with $\phi \in \mathcal{H}$ and U a unitary operator acting on $\mathcal{H}$, is called recurrent if and only if the total return probability R is equal to one. Otherwise it is called transient, which by* (4.15) *implies that the asymptotic survival probability s_∞ is strictly positive.*

In order to decide the recurrence behavior of a pair (U,ϕ) we have to study the total return probability R of the system. Similar to (4.6) we construct the generating function corresponding to the first return probabilities a_n

$$\widehat{a}(z)=\sum_{n=1}^{\infty}a_n z^n = z\ \langle\phi\,,\left(U\sum_{n=0}^{\infty}z^n\tilde{U}^n\right)\phi\,\rangle\ . \tag{4.16}$$

This leads us to the study of the generating function of the operator-sequence $(\tilde{U}^n)_n$ or equivalently to the study of the Neumann series of the operator $z\tilde{U}$:

$$\tilde{G}(z)=\sum_{n=0}^{\infty}\tilde{U}^n z^n=(\mathbb{1}-z\tilde{U})^{-1}\ \text{, for } |z|<1. \tag{4.17}$$

The operator $\tilde{G}(z)$ is, up to a multiplication by $(-z^{-1})$, equal to the resolvent of $\tilde{U}$, so it comes at no surprise that we can derive a Krein's formula expressing the Neumann series of $z\tilde{U}$ as a function of the Neumann series of the operator zU. The second resolvent formula therefore implies the relation [RS80]

$$G(z)-\tilde{G}(z)=z\tilde{G}(z)(U-\tilde{U})G(z)\ .$$

The difference operator $\tilde{U}-U$ in (4.17) is equal to the rank one operator $|\phi\rangle\langle\phi|U$ and therefore we obtain

$$G(z)\phi=\tilde{G}(z)\big(\mathbb{1}+z|\phi\rangle\langle\phi|UG(z)\big)\phi=\tilde{G}(z)\phi\left(1+z\langle\phi|UG(z)|\phi\rangle\right)\ . \tag{4.18}$$

The scalar product containing the unperturbed operator $G(z)$ on the right-hand side of this equation has a simple expression in terms of the Stieltjes transform $\widehat{\mu}(z)$ (see (2.9))

$$z\langle\phi|UG(z)|\phi\rangle=-1+\int\mu(du)\left(\frac{1-zu}{1-zu}+\frac{zu}{1-zu}\right)=\widehat{\mu}(z)-1\ .$$

Solving (4.18) for $\tilde{G}(z)\phi$ and taking the scalar product with $U^*\phi$ implies the relation

$$z\,\langle\phi\,,U\tilde{G}(z)\phi\,\rangle=\frac{\widehat{\mu}(z)-1}{\widehat{\mu}(z)}\ .$$

Returning to (4.16) and using the connection between Stieltjes transform and Schur function (see relation (2.10)) we find

$$\widehat{a}(z)=z\,\langle\phi\,,U\tilde{G}(z)\phi\,\rangle=\frac{\widehat{\mu}(z)-1}{\widehat{\mu}(z)}=1-\widehat{\mu}(z)^{-1}=z\overline{f}(z)\ . \tag{4.19}$$

Hence, the generating function of the first return amplitudes is basically the Schur function of the spectral measure μ. This means in particular that we obtain a dynamical interpretation of the Taylor coefficients of f as the first return amplitudes a_n. In addition, (4.19) reminds us of the renewal equation for classical Markov chains. In order to compare classical and quantum scenario, table 4.1 lists the key quantities and their relations in both cases. It is interesting to note that the structure of the equations

Table 4.1.: Comparison between classical and quantum recurrence

	classical	quantum
return in n steps	probability p_n	amplitude μ_n
generating function	$\widehat{p}$, see (4.10)	$\widehat{\mu}(z)$, see (2.9)
first return in the n^{th} step	probability q_n	amplitude a_n
generating function	$\widehat{q}$, see (4.10)	$\widehat{a} = z\overline{f}$, see (4.16)
renewal equation	$\widehat{q} = 1 - \frac{1}{\widehat{p}}$	$\widehat{a} = 1 - \frac{1}{\widehat{\mu}(z)}$

and the dependencies between the quantities from the classical situation are reproduced in the quantum case, but rather on the level of the amplitudes than on the level of the probabilities. In both cases the renewal equation between first return and overall return in the n^{th} step takes the same form. Compare this with Feynman's famous explanation of quantum mechanical principles: "In quantum theory add amplitudes and square at the end to get probabilities" [Fey66]. This is indeed what happens here, the renewal equation is valid for the amplitudes rather than the probabilities in the quantum case. However, this metaphor breaks down, when we consider Feynman's definition of a monitored process in the context of the double slit experiment, where one first squares the amplitudes and then adds them up. From this perspective our approach would be the un monitored one, which does not at all comply with our discussion in section 4.1.3.

4.2.1. Spectral characterization of recurrence

Given a pair (U, ϕ) of unitary operator and initial state we would like to decide, whether the system is recurrent or transient. In this subsection we formulate a straightforward spectral criterion deciding this question, which can also be expressed in terms of the Schur function of the spectral measure.

Theorem 4.2.2. *Let $\mathcal{H}$ be a separable Hilbert space. Given a unitary operator U on $\mathcal{H}$ and an initial state $\phi \in \mathcal{H}$, let μ be the spectral measure of ϕ with respect to U then the following are equivalent*

1. *The pair (U, ϕ) is recurrent.*
2. *The Schur function f of μ is inner.*
3. *μ has no absolutely continuous part.*

Before we prove the theorem let us compare its content with respect to other results about the dynamical implications of the spectral measure of a state ϕ as discussed in

section 2.2.1. We have shown that an initial state can only be recurrent if its spectral measure contains no absolutely continuous part. In this respect, we have found a converse to (2.3) stating that the absolutely continuous part of an initial state eventually leaves any finite subspace.

The RAGE theorem (see theorem 2.2.6) provides in comparison a distinction between the continuous and pure point part of the spectrum. Note that if we combine the RAGE theorem and theorem 4.2.2 we are able to differentiate between all three spectral components in terms of their implied recurrence and localization properties.

In summary, theorem 4.2.2 complements the classical results on the dynamical significance of the spectral measure in the sense that it gives a dynamical characterization of states with purely singular and purely absolutely continuous spectral measure in terms of their recurrence behaviour.

Let us also remark that in the classical case the concept of recurrence is especially useful, because if the considered Markov chain is irreducible, either all or none of the states is recurrent. In this context, irreducibility means that independently of the starting point, all other states of the system are accessible for the evolution [KT75]. The quantum analogue of the set of all accessible points is the subspace spanned by the orbit $\{U^n\phi\}$ of an initial state under the time evolution. Due to the spectral theorem it is clear that a general unitary operator U cannot be irreducible in the classical sense, because the spectral decomposition of U already gives orthogonal subspaces that cannot be reached from each other. However, since we know from section 2.2 that the different spectral components of U correspond to orthogonal subspaces $\mathcal{H}_{xx}$ of $\mathcal{H}$ and which are U invariant, we obtain the following corollary of theorem 4.2.2.

Corollary 4.2.3. *If (U,ϕ) is recurrent then (U,ψ) is recurrent for all $\psi \in \operatorname{span}\{U^n\phi\}$.*

Note that the inverse for transient states is only true if the spectral measure μ was purely absolutely continuous.

Proof of theorem 4.2.2. The equivalence of the last two points follows from the theory of Schur functions of measures on the unit circle (see lemma 2.3.7 and subsequent remark) and are only included to have two different criteria that can be checked for a given system. We will now show the equivalence between 1 and 2.

According to our definition of recurrence in 4.2.1 we have to study the total return probability $R=\sum_n |a_n|^2$. From (4.19) we know that $\widehat{a}(z)$ is closely related to the Schur function $f(z)$ of the spectral measure μ. Hence, we can convert the requirement $R=1$ for recurrence into a condition on the Schur function: Fix $r<1$, then we can consider the series

$$\widehat{a}(re^{\mathbb{i}\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{\mathbb{i}\theta n} = re^{\mathbb{i}\theta}\overline{f}(re^{\mathbb{i}\theta}) \tag{4.20}$$

as a Fourier series of $\overline{f}(r\cdot)$. By the Plancherel theorem it is clear that the norms of the respective functions have to coincide which amounts to the relation

$$\sum_{n=1}^{\infty} r^{2n}|a_n|^2 = \|f\|^2 = \frac{r^2}{2\pi}\int_{-\pi}^{\pi} d\theta |f(re^{\mathbb{i}\theta})|^2 \ . \tag{4.21}$$

To determine R we have to study the limit of this expression for $r \to 1^-$. Since f is a Schur function, we know that $|f(x)|$ is bounded from above by one for all x in the unit disc. Therefore, if (4.21) is equal to the maximum 1 in the limit $r \to 1^-$ we must have $|f(re^{\mathbb{i}\theta})| \to 1$ for almost all θ, but by definition 2.3.1 this exactly means that f has to be an inner function. □

In comparison with our characterization of SJK-recurrence (see 4.1.3) theorem 4.2.2 also shows that there is a difference between these two notions of recurrence. For example in the SJK case any point mass contained in the spectral measure suffices to make the system recurrent whereas in our definition the absence of absolutely continuous spectrum is required which also fits very well with the result from the Riemann-Lebesgue-lemma. We illustrate this difference in the following explicit example.

4.2.2. A simple example

The example we study in this section appears slightly artificial, but certainly clarifies the distinction between our definition of recurrence in terms of an active measurement step and the SJK-definition from section 4.1.3. As our toy model we consider the Hilbert space $\mathcal{H}$ of the doubly infinite chain $\ell_2(\mathbb{Z})$ with one additional dynamically decoupled lattice site, which we call $\star$. Therefore, $\mathcal{H}$ is given by the direct sum $\mathbb{C} \oplus \ell_2(\mathbb{Z})$.

The time evolution U we want to study acts trivially on the first summand of this direct sum and is given by the right shift on the second factor, which is unitary on the square summable sequences. In the standard basis $\{\delta_\star\} \cup \{\delta_x \,;\, x \in \mathbb{Z}\}$ the action of U is given by

$$U\delta_\star = \delta_\star \quad \text{and} \quad U\delta_x = \delta_{x+1} \text{ for } x \in \mathbb{Z}.$$

Fixing the initial state $\phi = \alpha\delta_\star + \beta\delta_0$, with $|\alpha|^2 + |\beta|^2 = 1$, we find for the n^{th} step return probabilities p_n from (4.13)

$$p_n = |\langle \phi \,, U^n \phi \rangle|^2 = \left|\delta_{0,n} + |\alpha|^2(1-\delta_{0,n})\right|^2 \ . \tag{4.22}$$

Next we show by induction that for this example $\tilde{U}^n\phi$ can be expressed in terms of α and β as

$$\tilde{U}^n\phi = \beta\delta_n + \alpha|\beta|^{2n}\delta_\star - |\alpha|^2\beta \sum_{k=0}^{n-1} |\beta|^{2(n-1-k)}\delta_k \ . \tag{4.23}$$

To this end, we first note that the projection of the additional lattice site $|\star\rangle$ onto the initial state ϕ turns out to be

$$|\phi\rangle\langle\phi|\delta_\star = \Big(|\alpha|^2|\delta_\star\rangle\langle\delta_\star| + \overline{\alpha}\beta|\delta_0\rangle\langle\delta_\star| + \alpha\overline{\beta}|\delta_\star\rangle\langle\delta_0|\Big)|\star\rangle = |\alpha|^2\delta_\star + \overline{\alpha}\beta\delta_0 \ .$$

Therefore, $\tilde{U}\phi$ is given by

$$\tilde{U}\phi = (\mathbb{1} - |\phi\rangle\langle\phi|)U\phi = \beta\delta_1 + \alpha|\beta|^2\delta_\star - |\alpha|^2\beta\delta_0 \ ,$$

which establishes the base case. Now the induction step from n to $n+1$ can be carried out by applying $\tilde{U}$ to $\tilde{U}^n\phi$, using (4.23) and shifting indices. Using the normalization condition $|[|\alpha]^2 + |[|\beta]^2 = 1$, we can evaluate the expression for the survival probability s_n and its limit s_∞ to be equal to

$$s_\infty = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \|\tilde{U}^n\phi\|^2 = \lim_{n\to\infty} \frac{2|\beta|^2 + |\alpha||\beta|^{4n}}{1+|\beta|^2} = \frac{2|\beta|^2}{1+|\beta|^2} \ .$$

As one might have expected, the asymptotic survival probability is nonzero whenever the initial state ϕ has some overlap β with the state δ_0, which is orthogonal to the invariant state $\delta_\star$. This component of ϕ is then shifted to infinity by U. Therefore, all states with $\beta \neq 0$ are transient and only the initial state $\phi = \delta_\star$ is recurrent according to definition 4.2.1.

In contrast we find from (4.22) that for $n > 0$ the return probabilities p_n are all equal to $|\alpha|^4$. Therefore, $\sum_n p_n$ diverges, whenever α is nonzero, and by section 4.1.3 this implies that all pairs (U, ϕ) are SJK-recurrent, with the only exception of the initial state $\phi = \delta_0$.

After we have established the difference between our recurrence definition and the SJK notion from [ŠJK08] we turn our attention to expected return times in the next section.

4.3. Expected recurrence time

Let us assume that the pair (U, ϕ) is recurrent, so the total return probability R is equal to one. This in turn implies that the first return probabilities $|a_n|^2$ according to (4.14) constitute a probability distribution on the integers. What are the properties of this probability distribution? A very natural quantity to analyze is the expected return time of the initial state ϕ given by

$$\tau = \sum_{n=1}^{\infty} |a_n|^2 n \ . \tag{4.24}$$

It turns out that τ is either an integer or infinite. This is rather surprising given the fact that the single first return amplitudes a_n depend continuously on the spectral measure μ, so one could suspect that a small change in μ might also imply only a small change in τ. This intuition fails because the expected return time τ is computed as an asymptotic expression and small alterations in the a_n nevertheless add up eventually. As it turns out, we can give a topological explanation of this quantization effect by identifying $\tau - 1$ with the winding number of the phase of the Schur function of the probability measure around the unit circle.

As an example, we consider the probability measure $\mu_2(x) = 1/2(\delta_{\mathbb{i}x} + \delta_{1x})$ consisting of two equally weighted point masses at 1 and $\mathbb{i}$. The left panel of figure 4.1 shows that the corresponding Schur function has a single zero inside the unit circle and the winding number of the phase of the Schur function around the unit circle is equal to 1, which implies $\tau = 2$. The right panel of figure 4.1 depicts the effect of a small additional point mass. The perturbed measure is of the form $\mu_\varepsilon(x) = (1-\varepsilon)\mu_2(x) + \varepsilon\delta_{-1x}$ and

there appears an additional zero in the Schur function near the support of this additional point mass. In addition, the winding number of the phase of the Schur function increases to 2, which means that the expected return time τ is equal to 3.

The precise relation between the number of point masses in the probability measure and the expected return time τ is contained in the following theorem.

Theorem 4.3.1. *Let the pair (U,ϕ) be recurrent with spectral measure μ, Schur function f and expected return time $\tau \in \mathbb{R}\cup\infty$. Then the following is equivalent*

(1) $\tau < \infty$.

(2) f is a rational function.

(3) μ is equal to the sum of $n < \infty$ distinct point measures with non-zero weights.

Moreover, $\tau = n$, and the polynomial degree of numerator and denominator of the Schur function is $n-1$.

Proof of theorem 4.3.1. We will take the route $3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3$ in order to establish the result.
$(3 \Rightarrow 2)$ Since the probability measure μ consists of a finite number of point measures at u_i with weights w_i its Stieltjes function takes the form of a sum of n simple summands, which can be expressed as a rational function in the following way

$$\widehat{\mu}(z) = \sum_k^n \frac{w_k}{1-u_k z} = \frac{\sum_k^n w_k \prod_{l=1,l\neq k}^n (1-u_l z)}{\prod_{k=1}^n (1-u_k z)} = \frac{P(z)}{Q(z)} \ ,$$

where the two polynomials $P(z)$ and $Q(z)$ are of degree $n-1$ and n, respectively. Using the relation between Stieltjes-transform and Schur function (see (2.10)) this implies

$$f(z) = \frac{z^{-1}(\overline{P}(z)-\overline{Q}(z))}{\overline{P}(z)} \ . \tag{4.25}$$

The numerator of this expression is also a polynomial of degree $n-1$, since $P(0)/Q(0) = \sum_k w_k = 1$ and therefore the difference of $P(z)$ and $Q(z)$ has no constant term. This implies that f is indeed a rational function. Note that if numerator and denominator had a common root, so would $P(z)$ and $Q(z)$. But this would be a contradiction to $\widehat{\mu}(z)$ having n distinct poles on the unit circle, because μ consists of exactly n point measures. So the degree of both numerator and denominator in (4.25) is $n-1$ as claimed in the theorem.
$(2 \Rightarrow 1)$ Since in addition to (2) we assume the pair (U,ϕ) to be recurrent, we know by theorem 4.2.2 that the Schur function f will be rational and inner. According to lemma 2.3.3 we can therefore decompose f as a finite Blaschke product of the form

$$f(z) = \beta \prod_{k=1}^{n-1} B_k(z) = \beta \prod_{k=1}^{n-1} \frac{\alpha_k - z}{1-\overline{\alpha}_k z} \ , \tag{4.26}$$

with $|\alpha_k| < 1$ and $|\beta| = 1$. Note that we have chosen the upper limit of the product to be consistent with the degree of $P(z)$ and $Q(z)$ in the previous part of the proof. This

Figure 4.1.: Influence of an additional point mass with small weight on the winding number of the phase of the Schur function. The red spheres at the bottom of both images indicate the positions of the point masses and the radii of the spheres signify their weights. The surface inside the framed box shows the absolute value of the Schur function and the circle atop the box represents the value of phase of the Schur function on the unit circle. Left panel: The corresponding probability measure μ consists of two point masses of equal weight $1/2$ at the positions 1 and $\mathbb{i}$ on the unit circle. The unique zero of the Schur function lies on the line connecting both points. Right panel: A small point mass of weight $.02$ is added to μ at the point (-1) on the unit circle. The additional zero in the Schur function appears near this additional point mass and therefore close to the unit circle, as does the additional winding in the phase of the Schur function, which increases the expected return time τ from 1 to 2.

in turn implies by (4.20) that $|\widehat{a}(e^{\mathbb{i}\theta})| = 1$ for all $\theta \in \mathbb{T}$ holds for the generating function of the first arrival amplitudes. So by defining the periodic function g as

$$g(\theta) := \widehat{a}(e^{\mathbb{i}\theta}) = \sum_{n=1}^{\infty} a_n e^{\mathbb{i}\theta n} \tag{4.27}$$

we know that g winds around the origin an integer number $\omega(g)$ of times as θ is varied from 0 to 2π. As a function with its range within the unit circle we can express $g(\theta)$ locally as $e^{\mathbb{i}\eta(\theta)}$ for some function η. Thus, in order to compute $\omega(g)$ we have to integrate the angular velocity

$$\frac{\partial \eta(\theta)}{\partial \theta} = -\mathbb{i} g(\phi)^{-1} \frac{\partial g(\phi)}{\partial \theta} = \overline{g(\theta)} \left(-\mathbb{i} \frac{\partial g(\theta)}{\partial \theta} \right)$$

and divide by 2π:

$$\omega(g) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ \overline{g(\theta)} \left(-\mathbb{i} \frac{\partial g(\theta)}{\partial \theta} \right) = \langle g \, , -\mathbb{i} \frac{\partial g}{\partial \theta} \rangle \ . \tag{4.28}$$

Using the Plancherel identity $\langle \hat{u} \, , \hat{v} \rangle = \langle u \, , v \rangle$ and noting that the Fourier coefficients of $\frac{\partial g}{\partial \theta}$ can be determined from (4.27) to be $\mathbb{i} a_n n$ we find

$$\omega(g) = \sum_{n=1}^{\infty} |a_n|^2 n = \tau \ ,$$

so indeed τ is an integer.

To compute the exact value of τ we make use of the Blaschke product representation of $g(\theta) = z\overline{f}(z)$ (see (4.26)). Since $\overline{B}_k(z) = B_k(z)^{-1}$ for $|z| = 1$ we can use the argument principle in (4.28) to compute τ. By definition $|\alpha_k| < 1$ in every Blaschke factor, which implies that the function $g(z)$ has no poles inside the unit circle, but exactly n zeroes, one from each of the factors in the Blaschke product and one additional zero contributed by the multiplication by z. Hence, in total we proved $\tau = n$.

(1 $\Rightarrow$ 3) Actually we show the contrapositive, *i.e.* $\neg 3 \Rightarrow \neg 1$ or, more precisely, that if μ is not composed of a finite number of point masses then the expected return time τ is infinite. Note that if the measure μ is not given as a sum of point measures its corresponding Schur function f does not admit a decomposition as a finite Blaschke product.

Since we require the pair (U, ϕ) to be recurrent, f is inner and therefore admits according to lemma 2.3.3 a decomposition into a possible infinite Blaschke product and a singular inner function f_{SI}. Assume for the moment that $f_{SI} = 1$, which implies

$$f(z) = e^{\mathbb{i}\theta} \prod_{l=1}^{\infty} \frac{z - a_l}{1 - \overline{a_l} z} \ .$$

Of course, we can pull out $k-1$ factors in this product in order to decompose $f = f_1 f_2$ into a finite Blaschke product f_1 and an infinite product f_2. From our discussion in the

previous paragraph it is clear that $\tau_{f_1} = k$. Assume for the moment that the expected return time satisfies $\tau_{f_1 f_2} \geq \tau_{f_i}$ for any pair of inner Schur functions f_1 and f_2. Then our preceding discussion implies that $\tau_f = \tau_{f_1 f_2} \geq \tau_{f_1} = k$ for any $k \in \mathbb{N}$. Therefore τ_f has to be infinite.

In order to show the inequality we make use of the identity

$$\lim_{r\to 1^-} \frac{1-r^{2n}}{1-r^2} = n$$

and the connection between the total return probability R and the $\mathcal{L}^2$-norm of the Schur function from (4.21). Inserting these two relations into the definition of τ_f we find

$$\begin{aligned} \tau_f &= \sum_{n=0}^{\infty} n\,|a_n|^2 = \lim_{r\to 1^-} \sum_{n=0}^{\infty} \frac{1-r^{2n}}{1-r^2}\,|a_n|^2 \\ &= \lim_{r\to 1^-} \int_{-\pi}^{\pi} \frac{1-r^2|f(re^{\mathbb{i}t})|}{1-r^2}\frac{\mathrm{d}t}{2\pi} = 1 + \lim_{r\to 1^-} \frac{r^2}{2\pi(1-r^2)} \int_{-\pi}^{\pi} 1-|f(re^{\mathbb{i}t})|^2\,\mathrm{d}t \end{aligned} \tag{4.29}$$

where we used in the second step that recurrence implies $\sum_n |a_n| = 1$. Since an inner function f by definition satisfies $|f(z)| \leq 1$ we obtain the bound $|f_1(z)f_2(z)| \leq |f_i(z)|$ for any pair of inner functions f_1 and f_2. Inserting this bound into (4.29) implies $\tau_{f_1 f_2} \geq \tau_{f_1}$ as claimed.

In the case $f_{SI} \neq 1$ we can employ lemma 2.3.4 to transform f into a Blaschke product f_ξ for some $\xi \in \mathbb{D}$. Using (2.6) we can express f in terms of f_ξ and obtain for the absolute value of f the relation

$$\begin{aligned} 1-|f(z)|^2 = 1 - \frac{|\xi - f_\xi(z)|^2}{|1-\overline{\xi} f_\xi(z)|^2} &= \frac{1-|\xi|^2}{|1-\overline{\xi} f_\xi(z)|^2}(1-|f_\xi(z)|^2) \\ &\geq \frac{1-|\xi|^2}{(1+|\overline{\xi} f_\xi|)^2}(1-|f_\xi(z)|^2) \geq \frac{1-|\xi|}{1+|\xi|}(1-|f_\xi(z)|^2)\ . \end{aligned}$$

Inserting this bound into the expression of the expected return time τ_f from (4.29) we find

$$\tau_f \geq 1 + \frac{1-|\xi|}{1+|\xi|} \lim_{r\to 1^-} \frac{r^2}{2\pi(1-r^2)} \int_{-\pi}^{\pi} 1-|f_\xi(re^{\mathbb{i}t})|^2\,\mathrm{d}t \geq \frac{1-|\xi|}{1+|\xi|}\,\tau_{f_\xi}\ .$$

If f_ξ is an infinite Blaschke product our previous argument ensures $\tau_{f_\xi} = \infty$, which by the last inequality then also holds for τ_f. However, if f_ξ would be a finite Blaschke product, then by the relation between f and f_ξ the function f would be rational and therefore also a finite Blaschke product, which we excluded by assumption. So indeed $\tau_f = \infty$, which finishes the proof. □

In the remainder of this section we define an expected return time for the SJK-case and compare it to lemma 4.3.1.

4.3.1. Expected return time in the SJK-case

As discussed in section 4.1.3 the definition of SJK-recurrence via the quantum mechanical probabilities

$$p_n = |\langle \phi \, , U^n \, \phi \rangle|^2 \tag{4.30}$$

to return exactly in the n^{th} time step explicitly avoids the notion of first return probabilities. Therefore, the canonical definition of an expected return time via (4.24) is not available in this case. We investigate two other seemingly natural definitions of an expected return time in this section and show that one of them would lead to negative first return probabilities.

The approach for the SJK-criterion has been to take the quantum mechanical probabilities p_n and to apply to them the classical Pólya criterion of the diverging series $\sum_n p_n$. Therefore, it seems to be natural to draw again on the classical relations, namely the renewal equation from (4.11), in order to define an expected return time. Moreover, in our excursion to the theory of the recurrence properties of classical Markov chains in section 4.1.2 we derived an expression for the expected return time that only depended on the n^{th}-step return probabilities p_n (see (4.12)). Therefore, one possible candidate for τ^{Polya} is

$$\tau^{\text{Polya}} := \sum_n n\, q_n = \lim_{z \to 1^-} \left((1-z)\widehat{p}(z) \right)^{-1} \; . \tag{4.31}$$

Let us restrict our attention to the case where the spectral measure of ϕ with respect to U is given by a finite sum of point measure with respective masses m_k. In that case, the scalar products $\langle \phi \, , U^n \phi \rangle$ take the simple form of a linear combination of phase factors $\sum_k m_k \exp(\mathbb{i}\theta_k)$ and we can further evaluate (4.31)

$$\begin{aligned} \tau^{\text{Polya}} &= \lim_{z \to 1^-} \left(\sum_{n=0}^{\infty} \sum_{k,l} (1-z) r_k \, r_l \, e^{\mathbb{i} n(\theta_k - \theta_l)} z^n \right)^{-1} \\ &= \lim_{z \to 1^-} \left(\sum_{k,l} r_k \, r_l \, \frac{1-z}{1 - e^{\mathbb{i}(\theta_k - \theta_l)} z} \right)^{-1} = \frac{1}{\sum_k m_k^2} \; . \end{aligned} \tag{4.32}$$

Let us stress that in contrast to the Markov case the quantity τ^{Polya} has no operational interpretation in the quantum case. Due to the renewal equation (4.11) it is also possible to define corresponding first return probabilities q_n via the relation $\widehat{q}(z) = 1 - (\widehat{p}(z))^{-1}$. However, as we will see in the example below, these first return probabilities q_n need not to be positive numbers.

In our second approach to define an expected return time in the SJK-case we consider a Markov process on the integers such that the state n evolves into state 0 with probability p_n and into the state $n+1$ with probability $n-1$. Starting from the state 1 we obtain the following expression for the probabilities q_n of the events, first return to

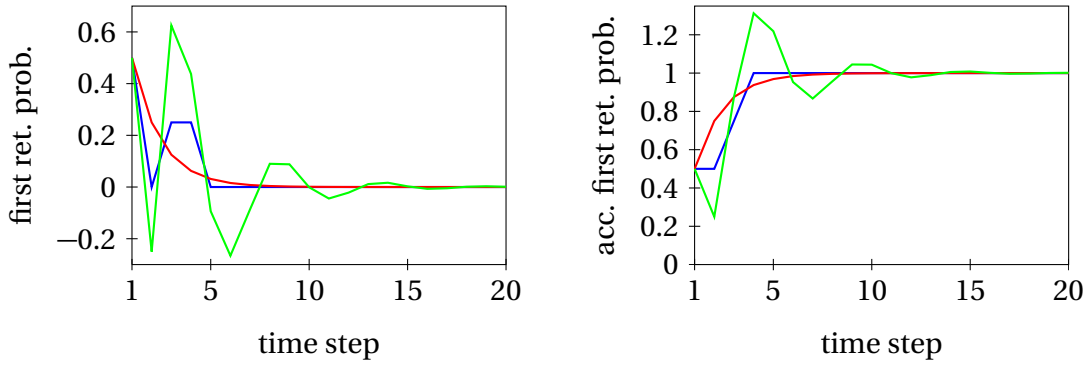


Figure 4.2.: The three sets of first return probabilities $|a_n|^2$ (red), q_n^{SJK} (blue) and q_n (green) for the pair $(U_{\pi/4},(1,0))$. Left panel: First return probabilities for the first 20 time steps. The sequence of the q_n (green curve) can drop below zero. Right panel: Accumulated first return probabilities. There is no inequality between the three quantities valid for all time steps.

0 in step n, of this Markov process

$$\begin{aligned} q_0^{\mathrm{SJK}} &= 0 \\ q_1^{\mathrm{SJK}} &= p_1 \\ q_n^{\mathrm{SJK}} &= p_n \prod_{l=1}^{n-1}(1-p_l) \quad n>1 \end{aligned} \tag{4.33}$$

with the corresponding expected return time $\tau^{\mathrm{SJK}} = \sum_n n q^{\mathrm{SJK}}$. This leaves us with three different definitions of *first return probabilities* and their corresponding expected return times for any pair (U,ϕ): the modulus square of the first return amplitudes $|a_n|^2$ according to 4.3 and the two quantities q_n^{SJK} and q_n derived from the sequence $(p_n)_n$. Next we analyse the behaviour of those quantities in a specific example.

A simple example

In order to illustrate the difference between the three sets of first return probabilities we derived in this section, we study a simple toy model in $\mathbb{C}^2$. To this end, we choose the unitary operator U equal to the real rotation matrix

$$U_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and some initial vector $\phi = (a,b) \in \mathbb{C}^2$. By explicitly inverting the 2×2 matrix $(\mathbb{1} - zU_\theta)$ we can compute the Stieltjes transform $\widehat{\mu}$ that is given by

$$\widehat{\mu} = \langle \phi\,, (\mathbb{1} - zU_\theta)^{-1}\ \phi \rangle = \frac{1 - z(\cos\theta + 2\mathrm{i}\,\Im(a\overline{b})\sin\theta)}{1 + z^2 - 2z\cos\phi}\ .$$

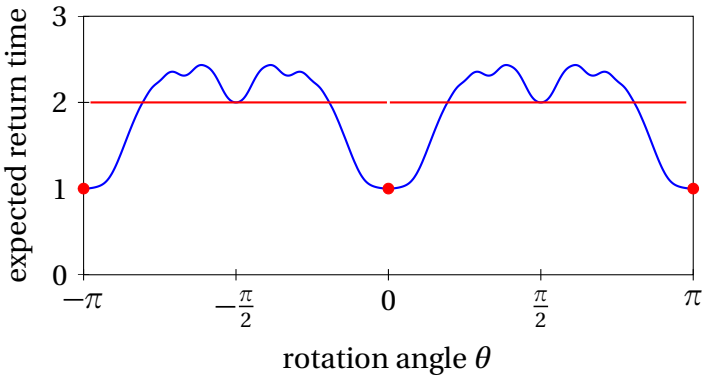


Figure 4.3.: Expected return times $\tau = \tau^{\text{Polya}}$ (red) and τ^{SJK} (blue, first 400 summands) depending on the rotation angle θ.

According to the identity (2.10) we can in turn obtain from this expression the Schur function f corresponding to this dynamics, which evaluates to

$$f(z) = \frac{1}{z}\frac{\overline{\widehat{\mu}}(z)-1}{\overline{\widehat{\mu}}(z)} = \frac{z-\cos\theta-2\mathrm{i}\Im(a\overline{b})\sin\theta}{z(\cos\theta-2\mathrm{i}\Im(a\overline{b})\sin\theta)-1} \ .$$

From Schur function and Stieltjes transform we can now compute the three sets of first return probabilities $|a_n|^2$, q_n^{SJK} and q_n. Figure 4.2 depicts all three quantities for the particular case of $\theta = \pi/4$ and for the initial state $\phi = (1,0)$. We see immediately that the derived first return probabilities q_n can drop below zero, which makes the corresponding expected return time a questionable definition.

Furthermore, for this specific example we find $U^2\phi = (0,1)$, $\mu_2 = \langle \phi\,, U^2\phi\,\rangle = 0$. In addition, the sequence of return probabilities p_n is given by $(1,1/2,0,1/2,\dots)$. However, this sequence cannot represent the return probabilities of a classical Markov process, because if there is some nonzero probability for the system to remain in the initial state, this has to be true in any time step, meaning that $p_2 = 0$ is impossible. In quantum information theory $U_{\pi/4}$ is called a square root of the NOT-gate [Bra94], because twice applied to a computational basis state it flips its logical value, an operation impossible to implement with two identical classical processes.

In addition, we can infer from figure 4.2 that there is no general inequality between the three different first return probabilities. Indeed, which of the three definitions assigns a higher probability to the event *arrival before some time t* does depend on the considered time t.

To finish our analysis of this example we also compute the three expected return times τ, τ^{Polya} and τ^{SJK}. Note that the eigenvectors of U are independent of the rotation angle θ and therefore their overlap with the initial state $\phi = (1,0)$ is also fixed and nonzero in both cases. By theorem 4.3.1 this implies that the expected recurrence time τ is equal to the number of distinct eigenvalues of U, which is 2 for all θ except for the extreme cases $\theta \in \{0, \pm\pi\}$. In these two remaining instances U has a single degenerate eigenvalue, which gives $\tau = 1$.

In order to determine τ^{Polya}, we note that both eigenvectors of U have overlap $1/2$ with the vector ϕ. This overlap also equals the weight of the two point masses in the spectral measure of ϕ with respect to U. Therefore, by (4.32) we obtain for all $\tau^{\text{Polya}} = 1$ for $\theta \in \{0, \pm\pi\}$ and $\tau^{\text{Polya}} = (1/4 + 1/4)^{-1} = 2$ for all other θ. Hence, $\tau^{\text{Polya}} = \tau$ in this example.

Before computing τ^{SJK} we have to determine the first return probabilities q_n^{SJK} via (4.33). Using the relation $p_n = \cos^2(n\theta)$ we find

$$\tau^{\text{SJK}} = 1 + \sum_{n\geq 1} \prod_{k=1}^{n} \sin^2(k\theta).$$

The dependence of the rotation angle on the three expected return times τ, τ^{Polya} and τ^{SJK} is depicted in figure 4.3. We note that there is again no general inequality between the three quantities. Instead, by choosing different rotation angles, we can achieve $\tau < \tau^{\text{SJK}}$ as well as $\tau^{\text{SJK}} < \tau$. Varying the initial state ϕ in such a way that it has no longer an equal overlap with both eigenvectors, we can also realize $\tau^{\text{Polya}} < \tau^{\text{SJK}}$.

It follows from lemma 4.1.3 that any point mass in the spectral measure makes a pair (U, ϕ) SJK-recurrent. In the remainder of this section we strengthen this result and show that the existence of a point mass already implies a finite expected return time τ^{SJK}.

Lemma 4.3.2. *Let μ be the spectral measure corresponding to a pair (U, ϕ) of unitary operator U and initial state ϕ. If μ contains a point mass then τ^{SJK} is finite.*

Proof. Given the sequence of arrival probabilities $p_n = |\langle \phi, U^n \phi \rangle|^2$ we can equivalently consider the corresponding survival probabilities $s_n = \prod_{n=1}^{n}(1 - p_k)$. Together with the first return probabilities the survival probabilities satisfy the relation $q_n + s_n = s_{n-1}$, which by summation implies

$$\sum_{n=1}^{m} n q_n = \sum_{n=0}^{m-1} (n+1) s_n - \sum_{n=1}^{m} n s_n = \sum_{n=0}^{m-1} s_n - m s_m \leq \sum_{n=0}^{m-1} s_n \ .$$

Employing this inequality together with the relation $e^{\alpha} \geq 1 + \alpha$ valid for all $\alpha \in \mathbb{R}$ and the definition of the sequence p_n according to (4.30) we find for the expected return time τ^{SJK}

$$\tau^{\text{SJK}} = \sum_{n=1}^{\infty} n q_n \leq 1 + \sum_{n=1}^{\infty} s_n = 1 + \sum_{n=1}^{\infty} \prod_{l=1}^{n} (1 - p_n) \leq 1 + \sum_{n=1}^{\infty} e^{-\sum_{l=1}^{n} p_n} \ . \tag{4.34}$$

Since the p_n are given by $|\mu_n|^2$, *i.e.* the moments of the spectral measure μ, the unitary version of Wieners theorem (A.0.1) tells us that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} p_l = \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} |\mu_n|^2 = \mu_{pp}(\mathbb{T}) \ .$$

By assumption μ does contain at least one point mass with weight $m_1 > 0$. Hence, we can find some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the finite sums are strictly bounded away

from zero, *i.e.* $\sum_{l=1}^{n} p_l \geq M > 0$. Inserting this into (4.34) we obtain the bound

$$\tau^{\mathrm{SJK}} \leq 1 + C_{n_0} + \sum_{n=n_0}^{\infty} e^{-Mn} \ ,$$

with some constant C_{n_0} depending on our choice of M. Summability of $\exp(-Mn)$ for $M > 0$ finishes the proof.

□

4.4. Variance of finite expected recurrence times

In this section we continue our study of the expected return time τ of a discrete time unitary evolution and analyze its variance. We restrict our attention to recurrent systems with a finite expected return time. According to theorem 4.3.1 this means that the spectral measure μ of ϕ consists of $n \in \mathbb{N}$ mass points at positions $u_i \in \mathbb{T}$ with corresponding weights m_i. In this case the Schur function can be expressed as a finite Blaschke product of $n-1$ factors (see (2.5)) and the Stieltjes transform is given by

$$\widehat{\mu}(z) = \sum_{k=1}^{n} \frac{m_k}{1 - u_k z} \ .$$

We have seen in section 4.3 that the expected return time jumps if we add or subtract a mass point from the measure μ independently of its corresponding weight. As we will show in this section this is accompanied by a divergent variance. Our arguments will be based on the following lemma that expresses the variance of τ in terms of the zeroes α_k of the Schur function.

Lemma 4.4.1. *Let $f(z) = \sum_k \overline{a}_k z^{k-1}$ be a rational Schur function and $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{D}$ its zeroes. The variance of the expected return time satisfies*

$$V_\tau = \sum_k k^2 |a_k|^2 - \left(\sum_k k |a_k|^2 \right)^2 = 2 \sum_{l,k} \frac{\alpha_l \overline{\alpha}_k}{1 - \alpha_l \overline{\alpha}_k} \ . \tag{4.35}$$

Before providing the proof of this lemma we discuss how it allows us to understand the behaviour of V_τ. In a first step we want to know, when the variance becomes minimal. To this end we rewrite (4.35) using the geometric series to obtain

$$V_\tau = \sum_{l,k} \sum_{n=0}^{\infty} \alpha_l^{n+1} \overline{\alpha}_k^{n+1} = \sum_{l=1}^{\infty} \left| \sum_{k=1}^{n-1} \alpha_k^l \right|^2 \ . \tag{4.36}$$

To minimize this expression we have to guarantee that the power sums $p_l = \sum_k \alpha_k^l$ are identically zero for all l. The elementary symmetric polynomials $e_l(\alpha_1, \ldots, \alpha_{n-1})$ satisfy for $l \geq 1$ the relation

$$e_l = \frac{1}{l} \sum_{k=1}^{l} (-1)^{k-1} e_{l-k} p_k \ ,$$

which implies that e_l has to vanish, too (see [Mac95]). If we now consider the decomposition

$$\prod_{k=1}^{n-1}(z-\alpha_k) = z^{n-1} - e_1 z^{n-2} \ldots (-1)^{n-1} e_{n-1} = z^{n-1} \ ,$$

we find that the Schur function $f(z)$ has to be of the form βz^{n-1} with $|\beta| = 1$, which represents a spectral measure with n equal point masses at the n^{th} root of $\overline{\beta}$. Such a measure corresponds to a clock unitary, which cyclically maps the basis states of one specific basis into each other. If we start in one of those basis states, it takes exactly n applications of the operator to return to the initial state, which clearly has zero variance. In addition, for a fixed number of mass points, a small variance also implies small α_k.

In the other extreme, we can obtain a divergent variance if one of the zeroes α_l of the Schur function approaches the unit circle. In that case the modulus of the corresponding $\alpha_{l'}$ tends to one, which implies that the term $l = k = l'$ in (4.35) diverges. Since the other terms stay finite this implies a diverging variance V_τ.

One way for this to happen is that the number of point masses n in the measure and equivalently the number $n-1$ of zeroes in the Schur function changes. There are two limiting processes to realize such a behaviour. We can either merge two different point masses to one, which corresponds to a limit $u_{k_1} \to u_{k_2}$, or we can decrease the weight of one point mass, which corresponds to a limit $m_k \to 0$. Both scenarios generate a sequence of probability measures each with n mass points that weakly converge to a limit measure μ with only $n-1$ mass points. For the corresponding Schur function this weak convergence implies uniform convergence on every disc around the origin of radius $(1-\varepsilon)$.

However, uniform convergence also implies that the number of zeroes of the function converges inside every such disc, therefore, the zero has to approach the unit circle and the variance diverges. If in addition, the sequence of measures only changes inside a specific region, as it is the case for the above described merging and weight-decrease processes, we can repeat the uniform convergence argument for other parts of the unit circle. This in turn implies that the convergence of this zero to the unit circle has to happen in the vicinity of the two merging mass points or the mass point with decreasing weight, respectively. This behaviour can also be observed in figure 4.1, where the additional zero of the Schur function indeed appears close to the added point mass.

Note however that the connection between the concentration of zeroes of the Schur function around the origin and low variance of the expected return time only holds for a fixed number of point masses, because any estimate will explicitly depend on the number of zeroes. An extreme case is given by the following example, where the variance is minimal although all zeroes of the Schur function move towards the unit circle. Consider the Schur function with zeroes $\alpha_k = \lambda e^{2\pi k/(n-1)}$ with $k = 1, 2 \ldots, n-1$. Inserting this into (4.36) we see that the sum over the phase factors $e^{2\pi k/(n-1)}$ to the power l inside the absolute value vanishes if l is not a multiple of $(n-1)$, in which case $\sum_r e^{2\pi r} = (n-1)$. Therefore, we obtain for the variance V_τ by adding and subtracting a

Figure 4.4.: Behaviour of the Schur function and its phase on the unit circle depending on the relative position of four equally weighted point masses. The upper circle shows the winding of the phase of the Schur function around the unit circle. In the middle, the absolute value of the Schur function is depicted and the coloured spheres indicate the position of the point masses on the unit circle. In the right and the left panel, the proximity of two respectively three point masses implies that one respectively two zeroes of the Schur function lie near the unit circle, which corresponds to a high variance in the expected return time. In the middle, where the mass points are more evenly distributed, also all the zeroes of the Schur function are nearer to the origin.

one in order to complete the geometric series in the second step

$$V_\tau = \sum_{l=1}^{\infty} \left| \sum_{k=1}^{n-1} \lambda e^{2\pi k l/(n-1)} \right|^2 = (n-1)^2 \sum_{r=1}^{\infty} |\lambda|^{2r(n-1)} = \frac{(n-1)^2 |\lambda|^{2(n-1)}}{1-|\lambda|^{2(n-1)}} \ .$$

Choosing $\log|\lambda|^2 = -3(n-1)^{-1}\log(n-1)$ we see that the modulus of λ converges to one for $n \to \infty$ and $V_\tau \sim (n-1)^{-1} \to 0$.

Proof of lemma 4.4.1. We start with the connection between the Schur function and the first return amplitudes, but to simplify notation we redefine the sequence a_k in such a way that we can ignore the complex conjugation and start counting from 0, such that a_k now represents the k^{th} Taylor coefficient of f. In addition, we note that $\tau = \sum_k k|a_k|^2 = n-1$, which agrees with the number of zeroes of f. Employing once more use of the Plancharel identity we find for the expectation value of k^2

$$\sum_k |k a_k|^2 = \int \frac{\mathrm{d}\theta}{2\pi} \left| \frac{\mathrm{d} f(e^{\mathbb{i}\theta})}{\mathrm{d}\theta} \right|^2 = \int \frac{\mathrm{d}\theta}{2\pi} \left| \frac{f'(e^{\mathbb{i}\theta})}{f(e^{\mathbb{i}\theta})} \right|^2 \ , \tag{4.37}$$

where we used that f is an inner function and therefore $|f| = 1$ holds on the unit circle. Starting from the decomposition of f into a finite Blaschke product we find for the quotient of f' and f

$$\frac{f'(z)}{f(z)} = \sum_l \left(\frac{z-\alpha_l}{1-\overline{\alpha}_l z} \right)' \left(\frac{z-\alpha_l}{1-\overline{\alpha}_l z} \right)^{-1} = \sum_l \left(\frac{-1}{\alpha_l - z} + \frac{\overline{\alpha}_l}{1-\overline{\alpha}_l z} \right) .$$

Inserting this expression into (4.37) and changing from an integration with respect to the phase to an integration around the unit circle we obtain

$$\sum_k k^2 |\alpha_k|^2 = \sum_{k,l} \oint \frac{\mathrm{d}z}{2\pi} \left(\frac{1}{z-\alpha_l} + \frac{\overline{\alpha}_l}{1-\overline{\alpha}_l z} \right) \left(\frac{1}{1-\overline{\alpha}_k z} + \frac{\alpha_k}{z-\alpha_k} \right) .$$

The value of this integral can be determined by residual calculus. The denominators of the form $1-\overline{\alpha}_k z$ have no roots inside the unit disc, because $|\alpha_k| < 1$. Hence, the product of two such denominators does not contribute to the integral. In addition, the product of the first summand in the first and the second summand in the second factor does also vanish, because for $k = l$ the pole $(z-\alpha_l)^2$ is of second order and for $k \neq l$ the contributions for the two poles at α_l and α_k cancel each other. This leaves us with two terms in the integral, which then evaluates to

$$\sum_k k^2 |\alpha_k|^2 = \sum_{l,k} \left(\frac{1}{1-\overline{\alpha}_l \alpha_k} + \frac{\alpha_l \overline{\alpha}_k}{1-\overline{\alpha}_k \alpha_l} \right) = \sum_{l,k} \frac{1+\overline{\alpha}_l \alpha_k}{1-\overline{\alpha}_l \alpha_k} \ . \tag{4.38}$$

To obtain the variance V_τ we have to subtract the squared expected return time, which according to theorem 4.3.1 is equal to $(n-1)^2$. The sum in (4.38) also runs over $(n-1)^2$ terms and therefore we subtract 1 from each summand which gives

$$\frac{1+\overline{\alpha}_l \alpha_k}{1-\overline{\alpha}_l \alpha_k} - 1 = \frac{2\overline{\alpha}_l \alpha_k}{1-\overline{\alpha}_l \alpha_k}$$

as claimed. □

4.5. Recurrence properties of quantum walks

As an example of a discrete-time unitary evolution with an absolutely continuous spectral measure, we analyse the return probability of one-dimensional translation-invariant quantum walks as introduced in section 2.4.

Fixing some lattice dimension s and some internal state space $\mathcal{K}$ of the particle with $\dim\mathcal{K} = k$, the system Hilbert space is given by $\ell_2(\mathbb{Z}^s)\otimes\mathbb{C}^k$. Since we are dealing with a translation-invariant system we can equivalently represent the walk operator W in moment space, where it acts as a p-dependent multiplication operator $W(p)$ (see section 2.4.1). Keeping in mind that the spectrum of W typically consists of absolutely continuous spectrum we know that the expected return time for any initial state will be infinite. Hence, we have to compute the series of first return amplitudes a_n or at least the total return probability

$$R = \sum_n |a_n|^2 = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} |f(e^{\mathrm{i}\theta})|^2 = \|f\|^2 \ .$$

In order to determine the Schur function f for a given initial state ϕ we can in a first step compute the Stieltjes transform of the corresponding spectral measure. In momentum representation the Stieltjes transform is given by

$$\widehat{\mu}(z) = \langle \phi \,, (\mathbb{1} - z\,W)^{-1}\ \phi \,\rangle = \int \mathrm{d}p\ \langle \phi(p), \big(\mathbb{1} - z\,W(p)\big)^{-1}\ \phi(p)\rangle \ . \tag{4.39}$$

If we limit our attention to an initial state localized at some lattice site x_0, its Fourier transform has the simple form $\phi(p) = e^{\mathrm{i}x_0 p}\psi$, with $\psi \in \mathbb{C}^d$. Since the p-dependent phases cancel in (4.39) we can evaluate $\widehat{\mu}(z) = \langle \psi \,, M(z)\,\psi \,\rangle$ by computing the Stieltjes operator

$$M(z) := \int \mathrm{d}p\ \big(\mathbb{1} - z\,W(p)\big)^{-1} \ . \tag{4.40}$$

In the case of a one-dimensional coined quantum walk with a $\mathcal{SU}(2)$ coin operation the walk operator takes the form (see section 2.4.1)

$$W(p) = \begin{pmatrix} e^{\mathrm{i}p} & 0 \\ 0 & e^{-\mathrm{i}p} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \ .$$

Via a change of variables to $u = e^{\mathrm{i}p}$, the integral in (4.40) can be evaluated by the Residue theorem. Aside from zero there are two other possibilities for poles in the expression at $u_{1,2} = (z^2+1 \pm \sqrt{(z^2+1)^2 - 4z^2|a|^2})/2az$. However, since $u_1\,u_2 \in \mathbb{T}$, at most one of them lies inside the unit circle. Evaluating the integral we therefore obtain for the Stieltjes-operator

$$M(z) = \frac{1}{2g(z)} \begin{pmatrix} 1 - z^2 + g(z) & \frac{b}{a}(1 + z^2 - g(z)) \\ \frac{-\overline{b}}{\overline{a}}(1 + z^2 - g(z)) & 1 - z^2 + g(z) \end{pmatrix} \quad \text{with } g(z) := \sqrt{(1+z^2)^2 - 4|a|^2\,z^2} \ .$$

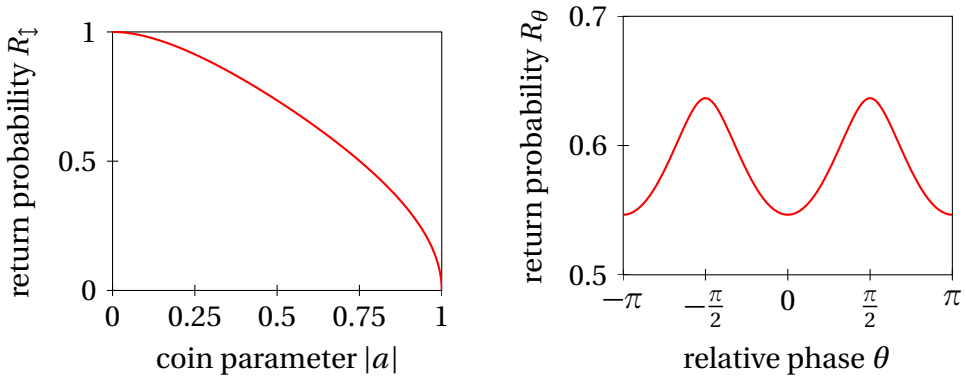


Figure 4.5.: Left panel: Return probability $R_\updownarrow$ of a one-dimensional quantum walk with initial state $|\uparrow\rangle$ or $|\downarrow\rangle$ at the origin and its dependence on the coin parameter $|a|$. For a coin matrix with real entries $R_\updownarrow$ is identical to the return probability R_α of the internal state $(\cos(\alpha), \sin(\alpha))$. Right panel: Influence of the relative phase θ between the two internal states on the return probability for the Hadamard walk ($a = b = 1/\sqrt{2}$).

Since the diagonal elements of $M(z)$ are identical, we find that for any localized initial state at some lattice site x, the internal states $|\uparrow\rangle = (1,0)$ and $|\downarrow\rangle = (0,1)$ show the same recurrence behaviour. In both cases we can derive the corresponding Schur function from (2.10), which leads to

$$f_\updownarrow = \frac{(1-z^2-g(z))^2}{4|a|^2 z^3} \ ,$$

where we have to pick the branch in the square root converging to 1 for z going to zero, so the real part should in particular be positive. This implies that in order to compute the total return probability R we have to evaluate the expression

$$R_\updownarrow = \|f_\updownarrow\|^2 = \int_{-\pi}^{\pi} \mathrm{d}p \, \frac{|h(p)^2|^2}{|a|^4} \ ,$$

where the function h is given by

$$h(p) = \mathrm{i}\sin p - \begin{cases} \mathrm{sign}(\cos p)\sqrt{\cos^2 p - |a|^2} & |\cos p| \geq |a| \\ -\mathrm{i}\,\mathrm{sign}(\sin p)\sqrt{|a|^2 - \cos^2 p} & |a| > |\cos p| \end{cases} \ .$$

Evaluating the integral we find for the total return probability the expression

$$R_\updownarrow = \frac{2}{\pi|b|^4}\left((1+2|a|^2)|a||b| + (1-4|a|^2)\arcsin|b|\right) \ ,$$

where due to unitarity of the coin operation $|b|^2 = 1-|a|^2$. The behavior of $R_\updownarrow$ with respect to $|a|$ is depicted in figure 4.5. As one can see a change in $|a|$ interpolates between none and full revival, where the extreme points correspond to the coin operation being the identity or a perfect reflection operation, respectively.

In general, the return probability does depend on the choice of the initial state. This is shown for the Hadamard walk in figure 4.5, where the recurrence behaviour for initial states of the form $(1, e^{\mathrm{i}\theta})/\sqrt{2}$ is depicted as a function of the relative phase θ. Note that without a relative phase between the two internal states there would be no observable difference in the return probabilities for the Hadamard walk. In fact, this is true for any coin matrix with only real valued entries. In that case, we can forget about the complex conjugation in M_{21} of the Stieltjes-operator $M(Z)$ and compute the Schur function, which turns out to be independent of the relative weight of the two internal levels, that is, f_α is independent of α for any vector of the form $(\cos\alpha, \sin\alpha)$

$$f_\alpha = \frac{1-z^2-g(z)}{z(1-z^2+g(z))} = \frac{(1-z^2-g(z))^2}{4a^2z^3} = f_\updownarrow \ .$$

Hence, the overall return probability is in this case again identical to $R_\updownarrow$.

This concludes our investigation of the return probability of one-dimensional translation-invariant quantum walks. As we have seen, we can achieve any return probability by choosing the right coin operation if we start with the particle at the origin with its initial state either up or down. Restricting to coins with real entries, we can also allow for superpositions of the internal state as long as we do not introduce a relative phase between the two levels.

4.6. Conclusion and outlook

In this chapter we have studied the recurrence properties of discrete-time unitary evolutions given by some unitary operator U on a separable Hilbert space. Since quantum mechanics necessitates us to perform a measurement in order to determine properties of the system, we introduced an additional monitoring step in the time evolution in order to check whether the system returned to its initial state. The time evolution is continued until the measurement process detects the particle in the initial state.

We could identify the loss of normalization in the n^{th} time step with n^{th}-step first return probability of the process. An initial state ϕ is called recurrent if is eventually detected the initial state, otherwise the state is called transient. We have shown that the recurrence properties of a pair (U, ϕ) of initial state ϕ and unitary operator U are completely determined by the spectral measure of ϕ with respect to U. More precisely, the state ϕ is recurrent with respect to U if and only if its spectral measure does not contain an absolutely continuous component. The key observation in order to obtain this result has been to realize that the generating function of the first return amplitudes is basically given by the Schur function f corresponding to the spectral measure. The condition of certain return was then equivalent to f being an inner function. An interesting point to note is that this relation also gives a dynamical interpretation to the Taylor coefficients of the Schur function in terms of the return amplitudes.

The characterization of recurrent states in terms of the absence of an absolutely continuous component in their spectral measure complements known results about the dynamical significance of the spectral types, in particular the RAGE theorem. Whereas the latter gives a dynamical distinction between the continuous and the pure point part of the spectrum of an operator, our recurrence result allows for a dynamical distinction

between the absolutely continuous and the singular part of the spectrum. Therefore, combining both results allows us to separate all three spectral components of an operator by dynamical properties. However, it should be noted that this is not an operational test that could be decided in an actual experiment. Since we are dealing with asymptotic concepts like the probability to return eventually, a single run of the experiment could require an infinite waiting time. However, from this perspective, it is always impossible to distinguish between the different spectral types, because a finite run of a discrete unitary evolution can always be implemented on a finite dimensional Hilbert space, which implies pure point spectrum for the unitary matrix representing U.

In addition, we studied the expected return time of an initial state in the recurrent case. We found that the expected return time is either infinite or an integer. More precisely, the expected return time is equal to the number of mass points that are contained in the spectral measure. The topological explanation of this quantization effect is the identification of the expected return time with the winding number of the phase of the Schur function of this spectral measure around the unit circle.

As an explicit example, we considered the expected return probability of translation-invariant one-dimensional coined quantum walks. Due to purely the absolutely continuous nature of their spectrum we could immediately conclude that all initial states are transient. Therefore, we studied the total return probability of these processes. In the case of localized initial states with either in spin up or down, we derived a closed formula for the total return probability in terms of the coin parameters of the walk.

We conclude this chapter with an overview of some open questions for further research.

- **Dimensionality:** Consider a classical random walk on a d-dimensional lattice. Pólya proved that all simple random walks, that is, random walks with nearest neighbour hopping, are recurrent for $d = 1$ and $d = 2$ and transient for $d > 2$ [Pól21]. In contrast, we have seen in section 4.5 that even the one-dimensional translation-invariant quantum walks are transient. In this sense, the critical dimension in the quantum case is $d = 1$ if we consider the coined quantum walks with nearest neighbour shifts as the quantum analogue of simple random walks. However, it still seems plausible that the overall return probability to the initial state should decrease with the lattice dimension, because intuitively it becomes easier to avoid a specific lattice site or low dimensional subspace. In the context of quantum walks there is in addition to the lattice dimension a second parameter, namely the dimension of the internal degree of freedom. Although it is not a priori clear which family of quantum walks with increasing internal dimension would be a good candidate to consider, the influence on the recurrence behaviour would be interesting to investigate.

- **Site recurrence:** In this chapter we exclusively considered the return of the system to the exact initial state. In the context of quantum walks another natural question would be whether the particle returns to the initially occupied lattice site regardless of its internal state. If we allow for arbitrary return subspaces this corresponds to the general transition problem with the additional condition that the initial state is contained in P_{tg}. However, there might be a way to evaluate the corresponding higher-dimensional Krein's formulas for specific examples, or to

formulate sufficient conditions for a state and unitary operator to be either site-recurrent or site-transient.

- **Characterization of transition behaviour:** Closely related to the question of higher-dimensional return subspaces is the question of general transition or arrival amplitudes with respect to arbitrary target subspaces. In order to obtain an analogue to (4.19) we would have to deal with a Krein's formula for perturbations with rank larger than one. If we restrict again to the scenario of a single target state ψ we can use the same arguments that led in the recurrence scenario to (4.19) and obtain the following expression for the generating function of the transition amplitudes from (4.6)

$$\widehat{a}_1(z) = \langle \psi \, , U \widetilde{G}(z) \, \phi \rangle = \frac{\langle \psi \, , U G(z) \, \phi \rangle}{1 + z \, \langle \psi \, , U G(z) \, \psi \rangle} \ .$$

  Hence, the direct connection to a single Schur function is lost, since we have to evaluate two different matrix elements of the operator $UG(z)$. Moreover, we have to consider the spectral measure between the initial and the target state, which does not need to be a probability measure. Therefore, the theory of probability measures on the unit circle and in particular their correspondence to Schur functions cannot be used directly. This could be circumvented by the polarization identity (see (2.2)) at the expense of having to deal with a sum over four different scalar products. In addition, similar problems arise if we allow for more than one target state.

- **Singular measures:** The examples that we considered so far encompass spectral measures consisting of a finite number of point masses and translation-invariant quantum walks, which generically only have absolutely continuous spectrum. It would therefore be interesting to also study examples with a singular continuous spectrum. This includes the quantum walk based on the Riesz measure as introduced in [GV12], as well as quantum walks with quasi-periodic coin operators, *e.g.* electric quantum walks [CRW+13]. Since the latter can possess a purely singular continuous spectrum they seem of particular interest: on the one hand, they exhibit a hierarchy of time scales, depending on the continuous fraction expansion of the electric field, for which almost perfect revivals can be observed; on the other, we know from lemma 4.3.1 that their expected return time is nevertheless infinite.

- **OPUC results:** The main observation that allowed us to classify recurrent states in terms of the spectral measure has been the identification of the generating function of the first return amplitudes with the Schur function of the spectral measure. However, we can also turn this relation around and say that we found a dynamical interpretation of the Taylor coefficients of the Schur function. This opens up the possibility to rephrase questions from the theory of orthogonal polynomials and probability measures on the unit circle in terms of the dynamical behaviour of discrete-time unitary quantum systems, which might provide new insights and proof ideas.

- **Quantization of expectation values:** The fact that in the recurrent case the expected return time is either quantized or infinite has been rather surprising to us and it was very interesting that τ could be related to the winding number of the phase of the Schur function around the unit circle. However, a similar effect has been observed in continuous time quantum walks, where in addition to the unitary time evolution the particle can be kicked out of the lattice at every second lattice site with some constant rate [RL09]. In this case the expectation value of the distance that the particle moves in the lattice before this loss process occurs is quantized and can also be considered as a winding number [RL09]. It would be very interesting, if there is a common principle or mechanism in the sense that the expectation values of such monitored observables generically show such a quantized behaviour, which can be attributed to a winding number.

5. Disordered quantum walks

In this chapter we introduce a class of one-dimensional quantum walks that strongly violates translation-invariance. More precisely, we allow the local coin operation at every lattice site to be randomly chosen according to some common probability measure on the unitary 2×2 matrices. This choice of local coins is then kept constant during the time evolution and we are again interested in the propagation properties of the system. Our goal in this chapter is to lay the groundwork for the proof of dynamical localization that is given in chapter 6 for a large class of probability distributions on the unitary 2×2 matrices.

The basic model is defined in section 5.1 and put into the context of the unitary Anderson model's and results on quantum walks in section 5.2. After an introductory example that exhibits dynamical localization is discussed in section 5.3, we return to the general case in section 5.4. There, we define unitary restrictions of the disordered quantum walks to finite sublattices and introduce the density of states. In section 5.5 we connect transition probabilities between two lattice sites with the resolvent of the walk operator, before we develop a general transfer matrix approach to position dependent coined quantum walks in section 5.6. The properties of these transfer matrices and their connection tho the resolvent of the walk operator are analyzed in section 5.7. In particular, we provide an initial scale estimate for the growth rate of products of transfer matrices and prove the Hölder continuity of the corresponding Lyapunov exponent with respect to the spectral parameter.

The results presented here have been obtained in collaboration with Andre Ahlbrecht and Volkher Scholz and have been published in [ASW11].

5.1. Model definition

The model of a one-dimensional coined quantum walk as described in section 2.4 is of course an idealization. In an actual experimental setup some of the underlying assumptions will not be completely satisfied but only be an approximation to a more complex situation.

As described in section 2.4 there are different ways in which such imperfections can affect the system. We identified four extreme cases corresponding to the respective time scales of the fluctuations and their impact on the translation invariance of the system. In this chapter we are going to investigate a regime where the translation invariance of the model is broken, but the temporal fluctuations of the parameters are very slow. This corresponds to a regime of frozen or quenched disorder, where the unitary operator implementing the evolution is assumed to be constant during each run of the experiment, but may vary randomly between different executions.

This is similar to the situation in the Anderson model discussed in section 2.5, where

the Hamiltonian is a random (ergodic) operator with respect to the underlying lattice, but also constant in time. In this case, the perturbation is induced by a position dependent potential and we correspondingly model the disorder for coined quantum walks by a position dependent random distortion of the coin operations.

Here, we concentrate on the standard one-dimensional coined quantum walk of a particle with a two-dimensional internal degree of freedom according to (2.15). This means that we assume that the shift operation is identical to the one in the ideal translation-invariant coined quantum walk, but that the local coin operation U_x at each lattice site $x \in \mathbb{Z}$ is now given by a random variable, which we denote by U_{ω_x}. Hence, the local coin operation U_{ω_x} is described by a mapping

$$U_{\omega_x} : \Omega_x \to \mathcal{U}(2)$$

from some measure spaces Ω_x to the space of unitary 2×2 matrices $\mathcal{U}(2)$. We call the probability measure μ_x corresponding to U_{ω_x} the local coin distribution and require in addition that the coins are independent and identically distributed with respect to the lattice sites. Hence, $\mu_x = \mu_c$ holds for all $x \in \mathbb{Z}$ and some probability measure μ_c on $\mathcal{U}(2)$.

For any finite number of lattice sites the joint distribution of the local coin operators $U_{\omega_{x_1}}, U_{\omega_{x_2}}, \ldots$ is then defined by product measure of the single site distribution μ_c and since this is true for any finite collection of lattice sites there exists a unique probability measure μ_∞ on the infinite product probability space (see section 3.1 and [Kal10])

$$\Omega = \times_{x \in \mathbb{Z}} \Omega_x \, .$$

From now on expectation values and probabilities will be taken with respect to this joint probability measure μ_∞ or finite restrictions of it, if not explicitly stated otherwise. The random coin operation U_ω as an operator on $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ can now be described as a random variable on Ω given by the direct sum

$$U_\omega = U_{\ldots,\omega_{x-1},\omega_x,\omega_{x+1},\ldots} = \bigoplus_{x \in \mathbb{Z}} U_{\omega_x} \quad \text{with } U_{\omega_x} = \begin{pmatrix} a_{\omega_x} & b_{\omega_x} \\ c_{\omega_x} & d_{\omega_x} \end{pmatrix} . \tag{5.1}$$

In the language of section 2.4.2 we can equivalently consider each realization U_ω as a position dependent coin according to definition 2.4.5. Similarly, every realization

$$W_\omega := U_\omega \cdot S$$

of such a disordered quantum walk operator represents a non-translation-invariant coined quantum walk. For later reference, we summarize the preceding discussion in the following definition.

Definition 5.1.1 (one-dimensional disordered quantum walk)**.** *Let $\left(U_{\omega_x}\right)_{x \in \mathbb{Z}}$, be a family of random variables with values in $\mathcal{U}(2)$, distributed according to some common probability measure μ_c. Then, a one-dimensional disordered quantum walk with single site distribution μ_c is defined as the random operator*

$$W_\omega = U_\omega \cdot S = \bigoplus_{x \in \mathbb{Z}} U_{\omega_x} \cdot S \ ,$$

where S denotes the standard nearest neighbour shift according to (2.15) *(see section 2.4.1).*

Let us remark that in the next two chapters we also use the term disordered quantum walk to denote a single realization W_ω of such a one-dimensional disordered quantum walk if there is no likelihood of confusion. Before we put the results presented here in a general context with respect to quantum walks and random unitary models, let us briefly comment on the measurability of the random operators W_ω and their ergodicity by proving the following proposition.

Proposition 5.1.2. *A disordered quantum walk W_ω is a random (ergodic) operator with almost sure constant spectral components $\sigma_{xx}(W_\omega)$, with $xx \in \{ac, sc, pp\}$. Moreover, $\sigma_{disc} = \varnothing$ with probability one.*

Proof. First let us check that W is weakly measurable and therefore, because we are dealing with a separable Hilbert space, measurable according to definition 2.5.1. Since the step size per time step is limited to one, due to our choice of the shift operation S, we have

$$\langle \delta_y \otimes \psi \, , W_\omega \, \delta_x \otimes \phi \rangle = 0 \quad \text{for } |x-y| > 1 \text{ or } x = y \ .$$

This already implies weak measurability for $|x-y| > 1$ and $x = y$, because constant functions are measurable. Hence, we are left with the cases $y = x \pm 1$ which lead to the expressions

$$\langle \delta_{x\pm1} \otimes \psi \, , W_\omega \, \delta_x \otimes \phi \rangle = \langle \delta_{x\pm1} \otimes \psi \, , U_{\omega_{x\pm1}} \, e_\pm \rangle \, \langle e_\pm \, , \phi \rangle \ .$$

By assumption, U_{ω_x} is a random variable on $\mathcal{U}(2)$ and therefore $\langle \psi \, , U_{\omega_x} \, \phi \rangle$ has to be measurable for any $\phi, \psi \in \mathbb{C}^2$. So by linearity $\langle \alpha \, , W_\omega \, \beta \rangle$ is measurable for any $\alpha, \beta \in \ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ and so is W_ω.

In accordance with section 2.5 the transformations $(\tau_y \omega)(x) = \omega(x-y)$ with $x \in \mathbb{Z}$ are an ergodic family of measure preserving transformations due the our i.i.d assumption for the single coin operations U_{ω_x}. Considering the lattice translations Γ_y acting as $\Gamma_y |\delta_x \otimes e_\pm\rangle = |\delta_{x+y} \otimes e_\pm\rangle$ we find

$$\Gamma_y \, W_\omega \Gamma_y^* = W_{\tau_y \omega} \ . \tag{5.2}$$

Since the operators Γ_y are unitary for all $y \in \mathbb{Z}$, we know that W_ω is a random (ergodic) operator, which by theorem 2.5.6 implies almost sure constancy of the spectral components $\sigma_{xx}(W_\omega)$. In order to exclude discrete spectrum we use lemma 2.5.7, so we have to show totality of our representation Γ_y of the measure preserving transformation τ_y. For any vector of the form $\delta_x \otimes \phi$, with $\phi \in \mathbb{C}^2$ we have

$$\langle \Gamma_y^* \, \delta_x \otimes \phi \, , \Gamma_z^* \delta_x \otimes \phi \rangle = \langle \delta_{x-y} \otimes \phi \, , \delta_{x-z} \otimes \phi \rangle = \delta_{z,y} \ .$$

Therefore, every vector of this form is an element of the set $A_\mathbb{Z}$ (see (2.24)), which makes this set total. Hence by lemma 2.5.7, the discrete part of the spectrum of W_ω is empty almost surely. □

Before we continue our analysis of disordered quantum walks in section 5.3, we introduce and discuss some related models and results.

5.2. Related work

As illustrated in chapter 2, the question of localization effects in quantum systems dates back to the late 50s and P.W. Anderson's paper [And58]. Since we have already given an overview on the Hamiltonian and time continuous case in section 2.5, we limit our discussion of related results here to random unitary models and quantum walks.

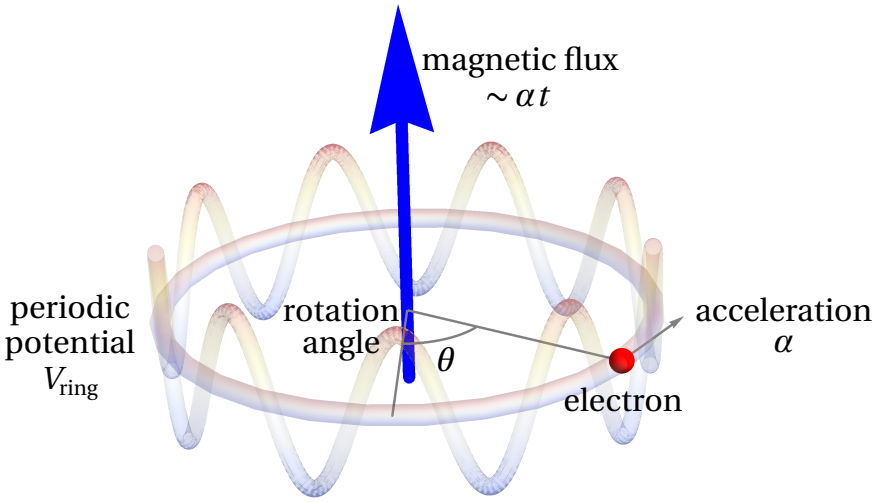


Figure 5.1.: Schematics of the physical setting that motivates the unitary Anderson model, illustration according to [Joy11b]. An electron moves in a periodic potential generated by a one-dimensional ring that is in addition threaded by a time dependent magnetic flux $\sim \alpha t$.

5.2.1. Unitary Anderson model

The term unitary Anderson model encompasses a class of random unitary operators on $\ell_2(\mathbb{N})$ or $\ell_2(\mathbb{Z}^s)$. The one dimensional version is connected to the physical model of an electron moving on a metallic ring, which is threaded by a time-dependent magnetic flux [Joy11b, Joy04, HJS09]. However, it can also be formulated in terms of non translation-invariant coined quantum walks.

The typical experimental scheme considered in this context, is depicted in figure 5.1. An electron moves according to a periodic potential V_{ring}, which is induced by a one-dimensional ring. In addition, the electron feels a magnetic flux inside the ring that is perpendicular to the ring and which increases linearly in time. One question of interest in this setting is whether the assumption of random impurities in the metallic ring is enough to show that the energy of the electron remains finite for all times.

After setting all physical constants like $\hbar$, the radius of the ring, the electron mass and its charge, etc. to one, the Hamiltonian for the angular variable θ takes the form

[BB88, Joy11b, HJS06]

$$H_{circ}(t) = \left(-\mathbb{i}\frac{\partial}{\partial_\theta} - \alpha t\right)^2 + V_{\text{ring}}(\theta) \ . \tag{5.3}$$

Analyzing this model in the adiabatic limit, which corresponds to a small α, allows no transitions between the different instantaneous energy eigenfunctions $\{e_l(t)\}$ of $H_{circ}(t)$, where t or rather αt now plays the role of a quasi-momentum for the periodic Schrödinger operator $H_{circ}(t)$ [Joy11b]. Since all eigenfunctions $e_l(t)$ are periodic with some period T we see that the eigenfunctions $e_l(0)$ at time 0 and $e_l(T)$ after one time step of length T can only differ by a phase factor.

Hence, transitions between the different energy bands, which we denote by $E_l(t)$, are only possible if the parameter α is large enough compared to the energy gap in order to allow for Landau-Zener tunneling between the bands. However, for all $E_l(t)$ with $l \neq 0$ this only happens twice during one period T once with the band $E_{l+1}(t)$ and once with $E_{l-1}(t)$ (see figure. 5.1). The idea of the unitary Anderson model is to describe the behaviour at those transition points, where the gap becomes small, by a scattering process between the states E_l and E_{l+1} which is governed by a scattering matrix [Joy11b]

$$S_l = e^{\mathbb{i}\theta}\begin{pmatrix} r_l e^{-\mathbb{i}\alpha_l} & \mathbb{i}\, t_l e^{\mathbb{i}\gamma_l} \\ \mathbb{i}\, t_l e^{-\mathbb{i}\gamma_l} & r_l e^{\mathbb{i}\alpha_l} \end{pmatrix} ,$$

with $\alpha_l, \gamma_l, \theta_l \in [-\pi, \pi)$ and reflection and transition coefficients $r_l, t_l \in [0,1]$ satisfying $r_l^2 + t_l^2 = 1$. Taking the period T as one time step, a scattering process from the lower, even energy levels to the higher odd energy levels, $E_{2l}(t)$ to $E_{2l+1}(t)$ is followed by a possible transition between the lower odd energy bands $E_{2l+1}(t)$ and the higher even energy bands $E_{2l}(t)$, see also figure 5.2. Associating with each standard basis vector of $\ell_2(\mathbb{N})$ one of the energy levels $E_l(t)$, the unitary operator $W_{\text{UA},\omega}$ describing this single time step is given by the product of these two scattering events

$$W_{\text{UA},\omega} := \left(\bigoplus_{l\in\mathbb{N}} S_{2l+1}\right) \cdot \left(1 \oplus \left(\bigoplus_{l\in\mathbb{N}} S_{2l}\right)\right) , \tag{5.4}$$

where the additional summand 1 in the second factor ensures that the scattering matrices S_k couple the right energy levels. Boundedness of the energy of the electron for all times is in this context equivalent to the question, whether the operator $U_{UA,\omega}$ can transport an initial state $\phi \in \ell_2(\mathbb{N})$ supported on a subspace corresponding to low energy bands $E_l(t)$ to infinity. This would in particular be the case, if dynamical localization of $U_{UA,\omega}$ could be shown as an effect of random impurities, because in that case transitions between different energy levels $E_l(t)$ and $E_k(t)$ would be exponentially suppressed in their distance $|k-l|$ for all times.

In principle we could vary all parameters in the scattering matrices S_k randomly, in order to describe the influence of random impurities within the ring. However, in most cases, the reflection and transition coefficients r_l and t_l are chosen translationally-invariant and fixed and only the phases α_l, γ_l and θ_l are drawn $i.i.d$ according to some probability measures on the interval $[-\pi, \pi)$.

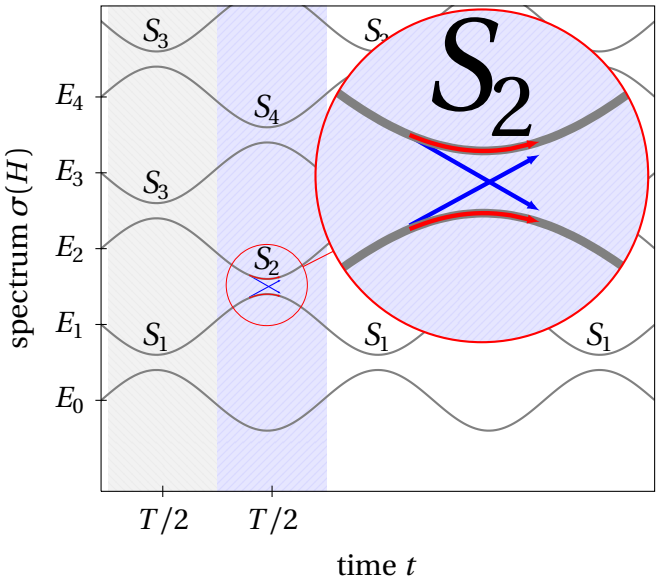


Figure 5.2.: Schematics of the dispersion relation of the Hamiltonian $\mathcal{H}_{circ}$ from (5.3), illustration according to [Joy11b]. Over one period T, Landau-Zener tunneling is only possible if the gaps are small. This happens twice per period for a fixed band $E_l(t)$, once with the band E_{l+1} and once with the band E_{l-1}. The transition and reflection amplitudes describing these inter-band transition processes are collected in the scattering matrices S_l.

In the unitary Anderson model, the system is studied on $\ell_2(\mathbb{Z})$ instead of $\ell_2(\mathbb{N})$ to avoid boundary terms [Joy11b]. In addition, the phases α_l, γ_l and θ_l in the scattering matrices are all set to zero and it is postulated that the randomness only enters via a diagonal matrix D_ω of random $i.i.d$ phase factors [HJS06, Joy11b]. In the case of the uniform distribution this can be shown to be equivalent to an independent choice of all parameters α_l, γ_l and θ_l from the uniform distribution [HJS06]. The unitary operator $W_{\mathrm{UA},\omega}$ on $\ell_2(\mathbb{Z})$ corresponding to the one from (5.4) for the half infinite case then has the matrix representation [HJS09]

$$W_{\mathrm{UA},\omega} := D_\omega \cdot S_0 = D_\omega \cdot \begin{pmatrix} \ddots & rt & -t^2 & & & \\ & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -tr & r^2 & -rt & \\ & & & rt & r^2 & \\ & & & -t^2 & -tr & \ddots \end{pmatrix} \tag{5.5}$$

Using the equivalence between $\ell_2(\mathbb{Z})$ and $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ we can identify W_{UA} also with a coined quantum walk of the form

$$W_{\mathrm{UA},\omega} = D_\omega \, \widetilde{S}^* \, U_{r,t} \, \widetilde{S} \, U_{r,t} \ , \quad \text{with } U_{r,t} = \mathbb{1} \otimes \begin{pmatrix} r & -t \\ t & r \end{pmatrix} . \tag{5.6}$$

Here $\widetilde{S}$ is the shift operation that only moves the upper internal state to the right, *i.e.* it acts on the standard basis as

$$\widetilde{S}\,(\delta_x \otimes e_\pm) = \delta_{x+(1\pm1)/2} \otimes e_\pm \ .$$

The unitary Anderson model has been extensively studied [Joy04, BHJ03]. It has been shown that $W_{\mathrm{UA},\omega}$ is a random ergodic operator with almost surely constant spectrum [BHJ03]. Hamza and coworkers were able to prove dynamical localization via a fractional moments analysis (see 2.5.3) for all $r, t \in (0,1)$ if the probability measure of the random phases in D_ω has an absolutely continuous component with respect to the Lebesgue measure on $[-\pi,\pi)$ [HJS06].

The unitary Anderson model on $\ell_2(\mathbb{Z})$ also admits a natural generalization to s-dimensional lattices if one takes the s-fold tensor product of the operator S_0 from (5.5) and generalizes the diagonal random phase matrix to a random multiplication operator such that $D_\omega \delta_x = e^{\phi_x}\delta_x$ for all $x \in \mathbb{Z}^s$ [Joy05, Joy04]. Dynamical localization has been proven for all lattice dimensions in the large disorder regime, which corresponds to a small value of the parameter t, which brings the deterministic operator S_0 in (5.5) close to a diagonal operator [Joy05, HJS09, Joy05]. In addition, dynamical localization at the band edges of the almost sure spectrum of the unitary operator $W_{\mathrm{UA},\omega}$ has been established [HJS09].

If we change one of the two coin operations in (5.6) to another translationally invariant coin operation it is possible to make a connection between the unitary Anderson model to a disordered coined quantum walk model, which we discuss in the next subsection.

5.2.2. Quantum walks

In this section we discuss some results on random operators related specifically to models of randomness in quantum walks. Since we are only dealing with discrete-time quantum walks we do not comment on numerical results that have been obtained in the time continuous case [KLMW07].

Starting from the unitary Anderson model as given in (5.6) we can obtain a standard one-dimensional coined quantum walk of the form $S \cdot C$ by changing the left of the two operator $U_{r,t}$ in (5.6) into another coin operation. Indeed if we define the translation-invariant coin operators

$$Y = \bigoplus_{x\in\mathbb{Z}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad X = \bigoplus_{x\in\mathbb{Z}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and use the relation $X^2 = \mathbb{1}$ we obtain

$$D_\omega \cdot \widetilde{S}^* \cdot Y \cdot \widetilde{S} \cdot X^2 \cdot U_{r,t} = D_\omega \cdot S \cdot X \cdot U_{r,t} = D_\omega\, S \bigoplus_{x\in\mathbb{Z}} \begin{pmatrix} t & r \\ -r & t \end{pmatrix} ,$$

where S denotes the standard state dependent shift operation according to (2.15). By a simple parameter change $r \mapsto -r$, this matches the disordered quantum walk model that was studied in [JM10]. For fixed parameters $r, t \in (0,1)$ and a diagonal operator D_ω

where all phases are drawn i.i.d with respect to some probability measure absolutely continuous with respect to the Lebesgue measure on the interval $[-\pi,\pi)$ dynamical localization has been shown by a fractional moments argument [JM10]. In this thesis, we generalize this result on the one hand by allowing for a random unitary coin instead of random phase factors and on the other hand by allowing singular local coin distributions μ_c, which were not covered in [JM10]. In particular, we obtain dynamical localization for a distribution of only two different local coin matrices (see section 6.6).

For higher lattice dimensions a prove of dynamical localization is known in a perturbative regime [Joy12]. Joye showed that starting from a static situation, a disordered quantum walk on a d-dimensional lattice with random relative phases stays localized if the perturbed operator is close to the original static walk operator in operator norm.

We continue our overview with a short report on one specific disorder model that was introduced by Konno et al. and that does not localize for a specific initial state [Kon10]. The local coin operator is in this case given by

$$U_\omega = \bigoplus_{x\in\mathbb{Z}} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\mathbb{i}\omega_x} & 1 \\ 1 & e^{-\mathbb{i}\omega_x} \end{pmatrix},$$

where ω_x is some random or non-random position dependent parameter. For a particle localized at the origin with internal state $(1,\mathbb{i})/\sqrt{2}$ ballistic behaviour is shown in [Kon10]. More specifically, the final spreading behaviour depends in this case exclusively on the parameter ω_0 and the probability that the particle moves with an asymptotic group velocity inside some interval $[a,b]$ is given by

$$\lim_{n\to\infty} \mathbb{P}\left(a \le \frac{Q(t)}{t} \le b\right) = \int_a^b \frac{1-\sin\omega_0}{\pi(1-x^2)\sqrt{1-2x^2}}\, \chi_{[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]}(x)\,\mathrm{d}x \ ,$$

where χ_A denotes the characteristic function of the set A. On way to see this independence is verify that the transfer matrices that we define in definition 5.6.1 do in this model not depend on the random phases. This implies that the Lyapunov exponent and hence the spectrum of the operator are also non random.

Before we return to our study of general disordered quantum walks, we want to comment on one other notion of localization that has been introduced in the context of translation-invariant quantum walks [IK05, IKK04, ŠKJ08]. In the view of section 2.4.1, where ballistic transport was shown for translation-invariant quantum walks, it is not clear how localization could occur in such systems. However, if we consider the momentum space representation of a translation-invariant walk operator W it can happen that one of the quasi-energy bands is independent of momentum, *i.e.*, $\omega_l(p)$ is constant. In this case, theorem 2.4.4 tells us that the asymptotic group velocity of the corresponding eigenstate is zero. Localization in the context of such quantum walks is defined as the property that some localized initial state ψ_0

$$\lim_{n\to\infty} \mathrm{tr}(P_\Lambda\, W^n |\psi_0\rangle\langle\psi_0| W^{*n}) \neq 0 \ , \tag{5.7}$$

where P_Λ denotes the projector onto some finite region $\Lambda \subset \mathbb{Z}^s$. In other words, with some finite probability the particle can asymptotically be found inside the region Λ.

In contrast to the notion of localization introduced by Anderson, which we discussed in section 2.5, no statement about the overall propagation behaviour of the particle is made. It could even be the case that the part of the wave function that does not localize behaves ballistically. This is similar to the different recurrence definitions discussed in chapter 4, where a single point mass contained in the spectral measure makes a particle recurrent in the SJK-sense, but only the absence of an absolutely continuous component implies recurrence in the monitored scenario. In view of chapter 4, it would be more appropriate to speak of a non-vanishing survival probability of the particle inside the region Λ.

A well known example where the simultaneous occurrence non-vanishing survival probability together with ballistic transport can be observed is given by the Grover walk in two lattice dimensions. Its walk operator W_{Grover} is given in quasi-momentum representation by the quasi-momentum dependent matrix

$$W_{\text{Grover}}(p_1,p_2)=\begin{pmatrix} e^{\mathbb{i}p_1} & 0 & 0 & 0 \\ 0 & e^{-\mathbb{i}p_1} & 0 & 0 \\ 0 & 0 & e^{\mathbb{i}p_2} & 0 \\ 0 & 0 & 0 & e^{-\mathbb{i}p_2} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} . \tag{5.8}$$

The unitary matrix W_{Grover} has two constant eigenvalues 1 and -1 and the corresponding eigenvectors can be shown to be strictly localized around the origin and therefore also according to the criterion in (5.7). The band structure as well as the separation between a localized and a spreading part of a localized initial state are depicted in figure 5.3. Moreover, it is possible to construct a coined quantum walk that is restricted to the non-localized subspace orthogonal to the eigenspaces of the two constant eigenvalues ± 1. The corresponding walk operator only acts on a two dimensional internal space and is given by

$$W_{red} = S_x \cdot H \cdot S_y H \ ,$$

where S_x and S_y denote the standard state dependent shifts in x or y direction respectively and H is the usual Hadamard coin from (2.16) [FGMB11, DFMGB11].

In a more general approach, the scenario of an arbitrary single defect or finite number of defects, *i.e.* positions where the translation invariance of the coin is broken, has been studied in the CGMV approach to quantum walks ([CGMV12, CGMV10], see also section 2.4.2). In particular, localized and transported states are characterized in terms of their spectral measure and their asymptotic survival probability at the origin is determined (see also section 2.4.2). For a walk on the half line these techniques were used by Konno and Segawa to study the effect of different boundary conditions, which also led to a non-zero asymptotic survival probability [KS10]. This concludes our discussion about related results and we continue our analysis of general disordered quantum walks.

5.3. A simple case

In order to continue our analysis, we consider a specific class of disordered quantum walks that exhibits dynamical localization. The reason is twofold: firstly, the proof is

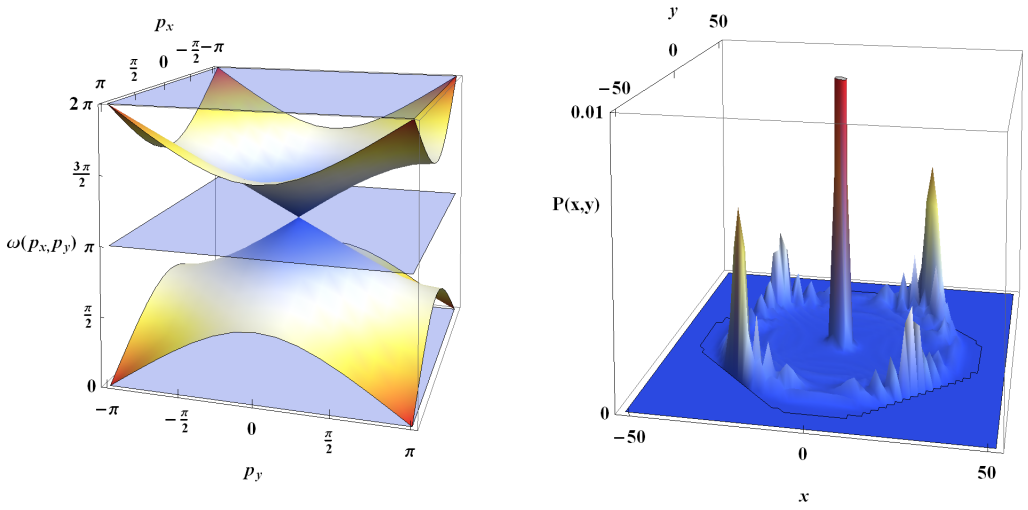


Figure 5.3.: Left panel: Band structure of the Grover walk from (5.8). There are two constant bands, in the dispersion relation at $\omega(p_x,p_y)=0,\pi$ (the band at 2π is identical to the one at 0 due to periodicity). In addition, the Grover walk possess two conical singularity. Right panel: Position distribution of the initial state $\delta_0\otimes(1,1,1,1)/2$ after 50 time steps showing the apparent separation between the spreading and immobile part. The later is responsible for the described non-vanishing asymptotic return probability.

shorter and more transparent than the general case presented in the remainder of this chapter and secondly, it will be convenient to exclude the class of walks discussed here in order to simplify the argument in the general case.

The specific disorder model we want to study is the occurrence of total reflections. More precisely, we consider a coin distribution that assigns a non-zero probability to such total reflections. It was already noted in [LS09] that a local coin of the form

$$U_{ref}=\begin{pmatrix}0 & e^{\mathrm{i}\phi}\\ e^{\mathrm{i}\theta} & 0\end{pmatrix} \tag{5.9}$$

at position x_0 effectively decouples the evolution of the lattice sites left of x_0 from the ones to the right of x_0, see also figure 5.4. This is due to the fact that a particle arriving from the left or the right at lattice site x_0 is in a definite internal state, *i.e.*, spin-up or spin-down. In the next time step U_{ref} is applied to the internal state at lattice site x_0, which, up to an unimportant phase factor, results in a flip of the internal state, *e.g.* the spin up state is mapped to the spin down state and vice versa. The following shift operation therefore transports the particle that arrived from x_0+1 in the previous step back to x_0+1. Hence, the operator U_{ref} acts as a total reflection of the incidenting particles with some additional phase shift.

Intuitively it seems very plausible that if such reflections occur with some finite probability a particle moving through the lattice will at some point inadvertently encounter

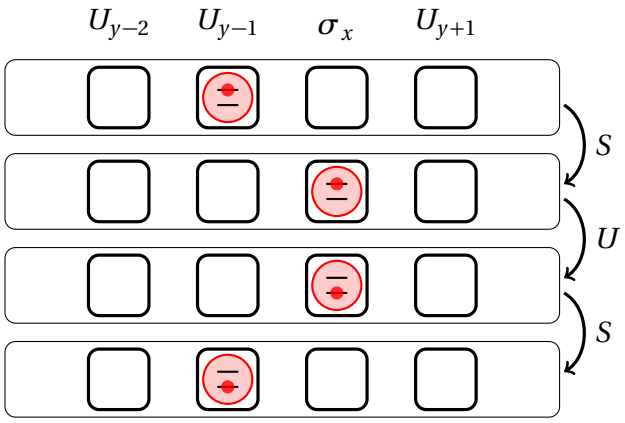


Figure 5.4.: Reflective action of the Pauli matrix σ_x at lattice site y.

such a barrier. Hence, we would suspect that this model exhibits localization. To make this idea precise we define the set of unitary 2×2 matrices with non-vanishing diagonal

$$\mathcal{U}_{nd} := \{U \in \mathcal{U}(2) \,|\, U_{11} \neq 0 \text{ and } U_{22} \neq 0\} \ .$$

The complement of $\mathcal{U}_{nd}$ in $\mathcal{U}(2)$ which we denote by $\mathcal{U}_{nd}^c$ contains exactly the reflections U_{ref} from (5.9). With these preliminaries we can formulate a localization result for one-dimensional disordered quantum walks that have a finite probability for such reflections.

Lemma 5.3.1. *Let W_ω be a one-dimensional disordered quantum walk such that the coin distribution μ_c satisfies $\mu_c(\mathcal{U}_{nd}) = p_{nd} < 1$. Then W_ω exhibits dynamical localization in the sense of definition 2.5.9 and the transition probabilities between two arbitrary lattice sites $x, y \in \mathbb{Z}$ satisfy*

$$\sup_{t\in\mathbb{N}} \mathbb{E}\big(|\langle \delta_y \otimes \psi \,, W_\omega^t \ \delta_x \otimes \phi \rangle|\big) \leq p_{nd}^{|x-y|-1}$$

Proof. As already discussed in the beginning of this section, the occurrence of a local coin $U_{ref} \in \mathcal{U}_{nd}^c$ at some point x_0 decouples the dynamics of the lattice sites left of x_0 from the ones right to it, which implies

$$|\langle \delta_y \otimes \psi \,, W_\omega^t \ \delta_x \otimes \phi \rangle| = 0 \quad \forall\, t \in \mathbb{N} \ ,$$

if there is a reflective coin U_{ref} in between the lattice sites x and y. For each individual lattice site the probability to be transmissive is assumed to be p_{nd}, so that the probability for no reflection at all in between x and y is $p_{nd}^{|x-y|-1}$. Due to the Cauchy-Schwarz-inequality and the unitarity of W_ω we have the trivial bound

$$|\langle \delta_y \otimes \psi \,, W_\omega^t \ \delta_x \otimes \phi \rangle| \leq 1 \ .$$

Hence, we obtain the claimed bound for the expectation value of the transition probability, because

$$\mathbb{E}\big(|\langle \delta_y \otimes \psi \,, W_\omega^t \ \delta_x \otimes \phi \rangle|\big) = p_{nd}^{|x-y|-1} |\langle \delta_y \otimes \psi \,, W_\omega^t \ \delta_x \otimes \phi \rangle| + 0 \cdot (1 - p_{nd}^{|x-y|-1}).$$

□

This result tells us that we can focus our attention on the case $\mu_c(\mathcal{U}_{nd}) = 1$ to prove localization and for later reference we define:

Definition 5.3.2 (regular disordered quantum walk)**.** *A disordered quantum walk is called regular if and only if its local coin distribution satisfies* $\mu_c(\mathcal{U}_{nd}) = 1$.

In addition, our discussion shows that a particle starting in between two reflective coins $U_1, U_2 \in \mathcal{U}_{nd}^c$ will be trapped in this region for all times. In the next section we use this idea to define finite restrictions of the walk operator by introducing reflective boundary conditions.

5.4. Finite restrictions

In this section we introduce a family of unitary operators on finite dimensional subspaces of $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ that approximate the disordered quantum walk and define the density of states. As explained in section 2.4, a defining property of a quantum walk is strict locality. Note that the local coin operations U_ω cause no transport of the particle at all. In addition, the shift operation S moves the particle exactly one lattice site to the left or to the right in a single time step, so the maximal propagation speed is one. This also means that W_ω as well as all its powers W_ω^t, with $t \in \mathbb{N}$ finite, are band matrices of finite width. Therefore, the evolution of an initially localized wave function up to some finite time t is completely characterized by a restriction of W_ω to some finite dimensional subspace of $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$.

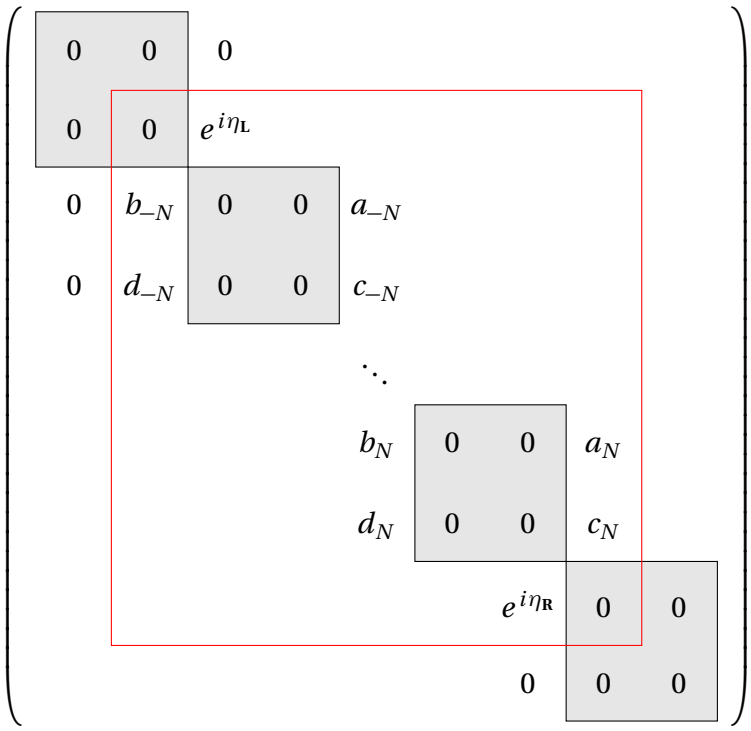
In particular, this implies a time dependent but finite upper bound on the region where an initially localized particle can be found with non-zero probability after t time steps. Hence, the transition probability between two lattice sites x and y in t time steps can be obtained by evaluating the scalar product with a finite restriction of the walk operator $\chi_{N_t} W_\omega \chi_{N_t}$, where χ_{N_t} denotes the projector onto the lattice sites $-N_t$ to N_t, if we choose N_t large enough

$$\langle \delta_y \otimes \psi \,, W_\omega^t \;\; \delta_x \otimes \phi \rangle = \langle \delta_y \otimes \psi \,, \left(\chi_{N_t} W_\omega \chi_{N_t}\right)^t \; \delta_x \otimes \phi \rangle \;\; .$$

In order to connect these transition probabilities to the resolvent of W_ω it is convenient to define an explicit sequence of restrictions $W_\omega(N)$ of W_ω to the lattice sites $-N$ to N in such a way that each $W_\omega(N)$ is a unitary operator on $\mathbb{C}^{4(N+1)}$. We do this by introducing reflective boundary conditions. As discussed in the preceding section this can be accomplished by changing the local coin operations U_l at the lattice sites $-(N+1)$ and $N+1$ from a random unitary coin to a coin with vanishing diagonal, *i.e.* an element of the set $\mathcal{U}_{nd}^c$. In particular, we choose multiples of the Pauli matrix σ_x for the reflections

$$U_{-(N+1)} = e^{\phi_{\mathbf{L}}} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad U_{N+1} = e^{\phi_{\mathbf{R}}} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrix representation of W_ω with $U_{-(N+1)}$ and U_{N+1} substituted by $e^{\phi_{\mathbf{R/L}}} \sigma_x$ is depicted in figure 5.5. From (5.10) it is then evident that we obtain a unitary submatrix if we include half of the internal state spaces at the lattice sites $-(N+1)$ and $N+1$, since all rows are then normalized and mutually orthogonal. This discussion motivates the following definition.



(5.10)

Figure 5.5.: Matrix representation of a disordered quantum walk. The red line indicates a possible unitary restriction, with boundary phases $e^{i\phi_{L/R}}$. The four-tuple (a_x, b_x, c_x, d_x) corresponds to the random coin matrix at lattice site x.

Definition 5.4.1 (unitary restrictions)**.** *For $N \in \mathbb{N}$ let $\widehat{W}_\omega(N)$ be equal to the disordered walk operator W_ω with reflective boundary conditions according to* (5.10) *at lattice sites $\pm(N+1)$. Then we define a unitary finite restriction of W_ω to the lattice sites $-N$ to N as the unitary matrix $W_\omega(N) : \mathbb{C}^{4(N+1)} \to \mathbb{C}^{4(N+1)}$ with matrix elements*

$$(W_\omega(N))_{k,l} = (\widehat{W})_{k,l} \ , \quad -(2N+1) \leq k,l \leq 2(N+1) \ ,$$

With this definition we can express the transition probabilities we have to bound in order to prove dynamical localization as the limit of these finite restrictions, *i.e.*

$$\sup_t |\langle \delta_y \otimes \psi \, , \chi(I) W_\omega^t \ \delta_x \otimes \phi \rangle| = \sup_t \lim_{N\to\infty} |\langle \delta_y \otimes \psi \, , \chi(I) W_\omega(N)^t \ \delta_x \otimes \phi \rangle| \ , \quad (5.11)$$

where $\chi(I)$ denotes the spectral projector of W_ω onto some closed interval I of the unit circle $\mathbb{T}$ and $\psi, \phi \in \mathbb{C}^2$ are arbitrary internal states of the particle. Next we introduce the density of states.

5.4.1. Density of states

In condensed matter physics, the density of states measures how many energy eigenstates are available inside a certain energy region in the system. It turns out that this

quantity or more precisely the integrated density of states can be connected to the Lyapunov exponent of the system via a Thouless formula (see section 6.1). This connection allows us to use regularity properties of the Lyapunov exponent with respect to a quasi-energy interval $\mathcal{I} \subset \mathbb{T}$ to obtain Wegner estimate (see section 6.3). This means that the probability that two independent realizations of a disordered quantum walk both have an eigenvalue inside a small quasi-energy interval is exponentially small.

The definitions and lemmas presented in this section are for the most part a translation of corresponding results obtained by Joye for the unitary Anderson model [Joy04] to the context of disordered quantum walks. The first observation is that due to the i.i.d nature of the coin operator, disordered quantum walks show a self-averaging behaviour.

Lemma 5.4.2. *For all continuous $f : \mathbb{T} \to \mathbb{C}$ we have with probability one*

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{4(N+1)} \operatorname{tr}(\chi_N f(W_\omega)) &= \frac{1}{2}\mathbb{E}\big(\operatorname{tr}\big(|\delta_0\rangle\langle\delta_0| f(W_\omega)\big)\big) \\ &= \frac{1}{2}\mathbb{E}\big(\langle \delta_0 \otimes e_1, f(W_\omega)\delta_0 \otimes e_1\rangle + \langle \delta_0 \otimes e_2, f(W_\omega)\delta_0 \otimes e_2\rangle\big) \ ,\end{aligned}$$

where χ_N denotes the projector onto the support of the unitary restriction $W_\omega(N)$. In terms of the projectors $P_{\delta_l \otimes e_\alpha}$ onto localized states and the boundary projector P_{bd}, χ_N takes the form

$$\chi_N := \sum_{l=-N}^{N} \sum_{\alpha} P_{\delta_l \otimes e_\alpha} + P_{\delta_{-(N+1)} \otimes e_2} + P_{\delta_{N+1} \otimes e_1} = \sum_{l=-N}^{N} \sum_{\alpha} P_{\delta_l \otimes e_\alpha} + P_{bd} \ .$$

Note that due to the additional boundary terms P_{bd}, the operator $\chi_N W_\omega \chi_N$ acts on exactly the same subspace as the unitary restriction $W_\omega(N)$ from definition 5.4.1.

Proof. Let us fix some continuous function f. Evaluating the trace on the left-hand side with respect to the canonical basis $\{\delta_i \otimes e_j\}$ of $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ we find

$$\begin{aligned}\frac{1}{4(N+1)} \operatorname{tr}(\chi_N f(W_\omega)) = \\ \frac{1}{4(N+1)} \Big(\sum_{l,j} \langle \delta_l \otimes e_j, f(W_\omega)\delta_l \otimes e_j\rangle + \operatorname{tr}(P_{bd} f(W_\omega))\Big) \ .\end{aligned} \tag{5.12}$$

As a continuous function on a compact interval f is bounded and the boundary projector P_{bd} has rank two. This implies that the second term on the right-hand side of (5.12) vanishes in the limit of large N. We have seen in (5.2) that W_ω is a random ergodic operator with respect to the lattice translation Γ_x. This implies in particular that the random variables

$$\begin{aligned}X_l := X(\tau^l(\omega)) &= \sum_j \langle \delta_0 \otimes e_j, f(W_{\tau_l(\omega)})\delta_0 \otimes e_j\rangle = \sum_j \langle \delta_0 \otimes e_j, \Gamma_l^* f(W_\omega)\Gamma_l\, \delta_0 \otimes e_j\rangle \\ &= \sum_j \langle \delta_l \otimes e_j, f(W_\omega)\delta_l \otimes e_j\rangle\end{aligned}$$

constitute an ergodic random process. Since in addition

$$\langle \delta_0 \otimes e_j \, , f(W_{\tau_l(\omega)}) \delta_0 \otimes e_j \rangle \leq \max_{\Theta \in \mathbb{T}} |f(\Theta)| \ ,$$

the expectation value of X_l for all l is finite. Therefore, Birkhoff's theorem applies and

$$\sum_{l=0}^{\infty} X(\tau^l(\omega)) = \mathbb{E}\left(\sum_j \langle \delta_l \otimes e_j \, , f(W_\omega) \delta_l \otimes e_j \rangle \right)$$

for each fixed $f \in C(\mathbb{T})$ on a set Ω_f of full measure. Now taking some dense countable subset of $C_0 \subset C(\mathbb{T})$, the intersection $\bigcap_{f \in C_0} \Omega_f$ as a countable intersection of sets of full measure still has full measure, which concludes the proof. □

Using the Riesz-Markov representation theorem, we now define the density of states in terms of this expectation value [Rud87]:

Definition 5.4.3 (density of states)**.** *The density of states of a disordered quantum walk W_ω is defined as the probability measure ϑ on the unit circle $\mathbb{T}$ satisfying for all $f \in C(\mathbb{T})$*

$$\int_{\mathbb{T}} f(\theta) \, \vartheta(d\theta) = \frac{1}{2} \mathbb{E}\big(\langle \delta_0 \otimes e_1 \, , f(W_\omega) \delta_0 \otimes e_1 \rangle + \langle \delta_0 \otimes e_2 \, , f(W_\omega) \delta_0 \otimes e_2 \rangle \big) \ .$$

In the same manner we can use the theorem of Riesz and Markov to define probability measures ϑ_N on $\mathbb{T}$ for the unitary restrictions $W_\omega(N)$ via

$$\int_{\mathbb{T}} f(\theta) \, \vartheta_N(d\theta) = \frac{1}{4(N+1)} \operatorname{tr}(f(W_\omega(N))) \ . \tag{5.13}$$

In section 6.1, we connect these probability measures to the Lyapunov exponent of the transfer matrices in order to obtain the Hölder continuity of the integrated density of states. To this end, we now establish an asymptotic trace formula between the unitary restrictions $W_\omega(N)$ and the non-unitary restrictions $\chi_N W_\omega \chi_N$ of the disordered walk operator W_ω. The specific formulation is given in the following lemma, which is also inspired by [Joy04].

Lemma 5.4.4. *For all continuous $f : \mathbb{T} \to \mathbb{C}$, the unitary restrictions $W_\omega(N)$ and the non-unitary restrictions $\chi_N W_\omega \chi_N$ of a disordered quantum walk W_ω satisfy*

$$\lim_{N \to \infty} \frac{1}{4(N+1)} \big(\operatorname{tr}(f(W_\omega(N))) - \operatorname{tr}(\chi_N f(W_\omega) \chi_N \big) = 0 \ . \tag{5.14}$$

Proof. By rescaling with the maximum of the function on the unit circle, we can assume without loss of generality that $\sup_{\Theta \in \mathbb{T}} |f(\Theta)| \leq 1$. As a continuous periodic function we can approximate $f(z)$ uniformly on $\mathbb{T}$ to an arbitrary precision ε by a trigonometric polynomial $p_\varepsilon(z)$ of finite degree $\kappa(\varepsilon)$. Remember that according to definition 5.4.1 the operator $\widehat{W}_\omega(N)$ is equal to the disordered walk operator W_ω except at the two lattice sites $\pm(N+1)$ and that it satisfies the relation $(W_\omega(N))^n = \chi_N (\widehat{W}_\omega)^n \chi_N$ for $n \in \mathbb{N}$. Adding

and subtracting the operators $p_\varepsilon(\widehat{W}_\omega(N))$ and $p_\varepsilon(W_\omega)$ in (5.14) and using the fact that $p_\varepsilon(z)$ approximates $f(z)$ uniformly on $\mathbb{T}$, we obtain

$$\begin{aligned}
&|\operatorname{tr}(f(W_\omega(N))-\chi_N f(W_\omega)\chi_N)| \\
&\quad = |\operatorname{tr}(\chi_N(p_\varepsilon(\widehat{W}_\omega)-p_\varepsilon(W_\omega)\chi_N))+\operatorname{tr}((f-p_\varepsilon)(W_\omega(N))-\chi_N(f-p_\varepsilon)(W_\omega)\chi_N)| \\
&\quad \leq |\operatorname{tr}(\chi_N(p_\varepsilon(\widehat{W}_\omega)-p_\varepsilon(W_\omega)\chi_N)|+8(N+1)\varepsilon \ . \qquad (5.15)
\end{aligned}$$

Therefore, we reduced the problem to the difference of two identical polynomials of W_ω and $\widehat{W}_\omega$, which we can evaluate monomial-wise. We use the following expansion formula for the difference of powers of two operators, which can be proved by induction

$$\chi_N(W_\omega^k-\widehat{W}_\omega^k)\chi_N=\sum_{i=0}^{k-1}\chi_N W_\omega^i(W_\omega-\widehat{W}_\omega)\widehat{W}_\omega^{k-(i+1)}\chi_N \ . \qquad (5.16)$$

Since both W_ω and $\widehat{W}_\omega$ are unitary, the relation is also valid for negative k, because we can substitute adjoins for inverses. Computing the trace of (5.16), we find for the single summands on the right-hand side

$$|\operatorname{tr}(W_\omega^i(W_\omega-\widehat{W}_\omega)\widehat{W}_\omega^{k-(i+1)}\chi_N)| \leq \|(W_\omega-\widehat{W}_\omega)\widehat{W}_\omega^{k-(i+1)}\chi_N\|_1 \ . \qquad (5.17)$$

Since $\widehat{W}_\omega(N)\chi_N$ is bounded and the difference of W and $\widehat{W}$ is of finite rank, because they only differ in the coin operators at the lattice sites $\pm(N+1)$, we apply the following inequality valid for A a trace class and B a bounded operator [Sim05d]

$$\|AB\|_1 \leq \|A\|_1\|B\|_{Op}$$

to the right-hand side of (5.17) in order to obtain the bound

$$|\operatorname{tr}(W_\omega^i(W_\omega-\widehat{W}_\omega)\widehat{W}_\omega^{k-(i+1)}\chi_N)| \leq \|W_\omega-\widehat{W}_\omega\|_1$$

Substituting everything back into (5.15) gives

$$\frac{1}{4(N+1)}\left|\operatorname{tr}(f(W_\omega(N))-\chi_N f(W_\omega)\chi_N)\right| \leq \frac{2\kappa(\varepsilon)\|(W_\omega-\widehat{W}_\omega)\|_1}{4(N+1)}+2\varepsilon \ ,$$

where $\kappa(\varepsilon)$ denotes the finite degree of $p_\varepsilon(z)$. Choosing ε small and using the fact that the difference of the two operator W_ω and $\widehat{W}_\omega$ is of rank at most four, which implies a bounded trace norm for their difference independent of N, finishes the proof. □

5.5. Dynamical localization and resolvent

In order to prove dynamical localization according to definition 2.5.9, we have to control the expectation value of transition probabilities between lattice sites x and y uniformly with respect to their distance within some spectral interval $I \subset \mathbb{T}$. Our next goal is to connect these transition probabilities to matrix elements of the resolvent of the walk operator.

Proposition 5.5.1. *Let W_ω be a disordered walk operator and $I \subset \mathbb{T}$ an open arc of the unit circle. Then we have that the transition probability from $\delta_x \otimes \phi$ to $\delta_y \otimes \psi$ for $x \neq y$ satisfies for all $t \in \mathbb{N}$*

$$\sup_t |\langle \delta_y \otimes \psi \,, \chi(I) W_\omega^t \;\delta_x \otimes \phi \rangle| \leq$$
$$4 \max_{i,j} \lim_{N\to\infty} \lim_{\kappa\to\infty} m(\{\theta \in I : |\langle \delta_y \otimes e_i \,, (W_\omega(N) - \theta)^{-1} \;\delta_x \otimes e_j \rangle| > \kappa\}) \,. \quad (5.18)$$

where $m(A)$ denotes the Lebesgue measure of a set A and e_1 and e_2 label the elements of some orthonormal basis of $\mathbb{C}^2$.

Proof. By decomposing the internal states ϕ, ψ in some orthonormal basis $\{e_1, e_2\}$ and applying the triangle inequality, it is clear that it suffices to show (5.18) for arbitrary combinations of basis states e_i, e_j and then taking the maximum. Suppressing for the moment the additional dependence on e_i and e_j in favor of readability, we denote by $\rho_{\omega,N}^{x,y}$ the spectral measure of $W_\omega(N)$ associated to the vectors $\delta_x \otimes e_i$ and $\delta_y \otimes e_j$. We obtain from (5.11) that

$$\sup_t |\langle \delta_y \otimes e_j \,, \chi(I) W_\omega^t \;\delta_x \otimes e_i \rangle| = \sup_t \lim_{N\to\infty} \left| \int_{\mathbb{T}} \rho_{\omega,N}^{x,y}(d\theta) \chi(I) \theta^t \right| \,, \quad (5.19)$$

where in a slight abuse of notation the symbol $\chi(A)$ also denotes the characteristic function of a set on the right-hand side. Taking the polar decomposition of the measure $\rho_{\omega,N}^{x,y}$ and using $|\theta| = 1$ for $\theta \in \mathbb{T}$, we can upper bound the right-hand side of (5.19) by the total variation of $\rho_{\omega,N}^{x,y}$

$$\sup_t \lim_{N\to\infty} \left| \int_I \rho_{\omega,N}^{x,y}(d\theta) \theta^t \right| \leq \lim_{N\to\infty} |\rho_{\omega,N}^{x,y}|(I) \,. \quad (5.20)$$

Let us fix for the moment some $N \in \mathbb{N}$. Since by construction $W_\omega(N)$ is a unitary operator acting on a finite dimensional Hilbert space, $\rho_{\omega,N}^{x,y}$ is a purely singular measure on the unit circle, *i.e.* it is only supported on isolated points. By lemma 2.3.9 from section 2.3.1 this implies that we can express the right-hand side of (5.20) as the limiting Lebesgue measure of a sequence of sets in the following way:

$$|\rho_{\omega,N}^{x,y}|(I) = |\rho_{\omega,N}^{x,y}|_s(I) = \lim_{\kappa\to\infty} \pi\kappa \cdot m(\{\theta \in I : |K\rho_{\omega,N}^{x,y}| > \kappa\}) \,, \quad (5.21)$$

where $K\rho$ denotes the Cauchy transform of a measure ρ on the unit circle, see section 2.3.1. We can express the Cauchy transform of the spectral measure $\rho_{\omega,N}^{x,y}$ of two different lattice sites $x, y \in \mathbb{Z}$ with respect to the unitary restriction $W_\omega(N)$ of a disordered quantum walk W_ω in terms of a matrix element of the resolvent via

$$\begin{aligned} K\rho_{\omega,N}^{x,y}(z) &= \int_{\mathbb{T}} \rho_{\omega,N}^{x,y}(d\theta) \frac{\theta}{\theta - z} \\ &= \langle \delta_y \otimes e_j \,, W_\omega(N)(W_\omega(N) - z)^{-1} \delta_x \otimes e_i \rangle \\ &= \langle \delta_y \otimes e_j \,, \mathbb{1} + z \cdot (W_\omega(N) - z)^{-1} \delta_x \otimes e_i \rangle \\ &= z \cdot \langle \delta_y \otimes e_j \,, (W_\omega(N) - z)^{-1} \delta_x \otimes e_i \rangle \,. \end{aligned}$$

Inserting this into (5.21) we obtain

$$|\rho^{x,y}_{\omega,N}|(I) = \lim_{\kappa\to\infty} \pi\kappa \cdot m(\{\theta \in I : \left|\langle \delta_y \otimes e_j, (W_\omega(N) - \theta)^{-1}\delta_x \otimes e_i\rangle\right| > \kappa\}) \ ,$$

which completes the proof. □

5.6. Resolvent formula and transfer matrices

Transfer matrix techniques translate the analysis of the spectrum and the eigenvectors of an operator to the study of products of finite dimensional matrices. This approach has been used very successfully to analyze spectral properties of operators acting on one-dimensional lattices in the area of random Schrödinger operators, random unitary models and transport properties of quasi-crystals, see section 2.5. In fact, one main obstacle to tackle higher-dimensional problems is exactly the lack of a transfer matrix approach in dimensions larger than one.

In this section we develop a transfer matrix approach for disordered quantum walks, in order to express the resolvent of W_ω in terms of suitably defined products of 2×2 matrices. We start with the infinite case on the whole lattice and discuss afterwards how to deal with finite restrictions and boundary conditions.

We want to study the matrix elements of the resolvent

$$\mathrm{G}_z := (W_\omega - z)^{-1} \ , \quad z \notin \sigma(W_\omega) \ ,$$

with respect to the standard basis $\{\delta_x \otimes e_i\}$. In order to simplify the discussion, we make use of the fact that $\ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$ is isomorphic to $\ell_2(\mathbb{Z})$ via the map

$$\begin{array}{ll} \delta_x \otimes e_1 \to l_{2x} & \\ \delta_x \otimes e_2 \to l_{2x+1} & \forall\, x \in \mathbb{Z} \ , \end{array}$$

where we now use $\{l_i\}$ to denote the standard basis of $\ell_2(\mathbb{Z})$. Since the resolvent is defined as the inverse of the operator $(W_\omega - z)$ we know that the r^{th} column g^r of G_z satisfies the relation

$$(W_\omega - z)g^r = l_r \ . \tag{5.22}$$

This implies in particular that except for the r^{th} component all entries of the vector $(W_\omega - z)g^r$ have to vanish. We show how this identity enables us to construct the vector g^r from some local data only, namely the two adjacent components g^r_{2x-1} and g^r_{2x}. To this end we say that a lattice site x lies not on the diagonal of the row g^r of the resolvent of G_z if

$$2x \neq r \quad \text{and} \quad 2x+1 \neq r \ .$$

Let us consider the matrix representation of a disordered walk operator $(W_\omega - z)$

$$\begin{pmatrix}
\ddots & & & & & & & \\
b_{x-1} & -z & 0 & a_{x-1} & & & & \\
d_{x-1} & 0 & -z & c_{x-1} & & & & \\
 & & b_x & -z & 0 & a_x & & \\
 & & d_x & 0 & -z & c_x & & \\
 & & & & b_{x+1} & -z & 0 & a_{x+1} \\
 & & & & d_{x+1} & 0 & -z & c_{x+1} \\
 & & & & & & & \ddots
\end{pmatrix}$$

If we choose x not on the diagonal of g^r, *i.e.*, such that $r \neq 2x$ or $r \neq 2x+1$, respectively, this expression together with (5.22) implies the following relations between the components of g^r

$$\begin{aligned} g^r_{2x-1} b_x - z g^r_{2x} + a_x g^r_{2x+2} &= 0 \\ g^r_{2x-1} d_x - z g^r_{2x+1} + c_x g^r_{2x+2} &= 0 \ . \end{aligned} \tag{5.23}$$

Using the fact that the 4-tuple (a_x, b_x, c_x, d_x) consists of the components of the unitary 2×2 matrix U_x that constitutes the local coin operation at lattice site x, we can derive the following relation between the two vectors (g^r_{2x-1}, g^r_{2x}) and (g^r_{2x+1}, g^r_{2x+2})

$$\begin{pmatrix} g^r_{2x+1} \\ g^r_{2x+2} \end{pmatrix} = \frac{1}{a_x} \begin{pmatrix} \frac{\det(U_x)}{z} & c_x \\ -b_x & z \end{pmatrix} \begin{pmatrix} g^r_{2x-1} \\ g^r_{2x} \end{pmatrix} , \tag{5.24}$$

which we take as the definition of our transfer matrices.

Definition 5.6.1 (transfer matrix)**.** *Let W_ω be a disordered quantum walk. The transfer matrices $T_x(z)$ of W_ω are defined for all lattice sites $x \in \mathbb{Z}$ with $a_x \neq 0$ as*

$$T_x(z) = \frac{1}{a_x} \begin{pmatrix} \frac{\det(U_x)}{z} & c_x \\ -b_x & z \end{pmatrix} , \tag{5.25}$$

where the quantities a_x, b_x and c_x are given by the components of the local coin operation U_x at lattice site x. Due to unitarity of U_x the determinant of a transfer matrix is given by

$$\det T_x(z) = \frac{d_x}{a_x} \ , \tag{5.26}$$

which in particular implies that it is independent of z and $|\det T_x(z)| = 1$.

The determinant of a transfer matrix can be evaluated to

$$\det T_{x(z)} = \frac{1}{a_x^2}(\det(U_x) + b_x c_x) \ .$$

Now, (5.26) follows if we insert the definition of U_x (see (5.1)). By unitarity of U_x it is also clear that $a_x^{-1} \cdot d_x \in \mathbb{T}$. Therefore, the modulus of the determinant is indeed equal to one.

As we have seen in section 5.5, the restriction to the case $a_x \neq 0$ is a technical one due to the transfer matrix approach, because coins with vanishing diagonal automatically lead to dynamical localization (see section 5.3). Therefore, we can assume without loss of generality that $\alpha_x \neq 0$ holds for all lattice sites x.

For x not on the diagonal of g^r, we study the doubly infinite sequence $(\Phi_y^{r,x})_{y\in\mathbb{Z}}$ obtained by iteratively applying transfer matrices or their inverses to the components (g_{2x-1}^r, g_{2x}^r) of g^r

$$\Phi_y^{r,x} = \begin{pmatrix} \phi_{2y-1}^{r,x} \\ \phi_{2y}^{r,x} \end{pmatrix} := \begin{cases} T_y(z)^{-1} \dots T_{x-1}(z)^{-1} \cdot \begin{pmatrix} g_{2x-1}^r \\ g_{2x}^r \end{pmatrix} & \text{, if } y < x \\ \begin{pmatrix} g_{2x-1}^r \\ g_{2x}^r \end{pmatrix} & \text{, if } y = x \\ T_{y-1}(z) \dots T_x(z) \cdot \begin{pmatrix} g_{2x-1}^r \\ g_{2x}^r \end{pmatrix} & \text{, if } y > x \end{cases} . \tag{5.27}$$

Note that in order to simplify notation, we often drop the subscript $y \in \mathbb{Z}$ when referring to the doubly infinite sequence $(\phi_y^{r,x})$. From (5.24) it is immediately clear that by construction the sequences $(\Phi_y^{r,x})$ solve the equation

$$(W_\omega - z)\Phi = 0 \ , \tag{5.28}$$

which implies two things. Fist, the sequence cannot agree everywhere with g^r since the latter has to respect 5.22; second, since z was chosen to lie outside the spectrum of W_ω, it cannot be square summable. These two facts raise two questions: Which elements of the sequence $(\phi_y^{r,x})$ and g^r are equal and what can we say about its growth behaviour?

Let us begin with the first question. From the definition of the sequence $(\Phi_y^{r,x})$ it is clear that at least the element $\Phi_x^{r,x}$ in (5.27) is equal to the components (g_{2x-1}^r, g_{2x}^r) of g^r. Due to (5.24) this relation remains true for the other elements $\Phi_y^{r,x}$ as long as y lies not on the diagonal of g^r. Therefore, starting from any two components (g_{2x-1}^r, g_{2x}^r) of g^r, we can construct any other component as long as we remain on the same side of the diagonal as x. We summarize this reasoning in the following proposition.

Proposition 5.6.2. *Let g^r be the r^{th} row of the resolvent G_z of a realization of a regular disordered quantum walk W_ω. Then, if x and y are on the same side of the diagonal of g^r, meaning that either $2x-1 < r$ and $2y-1 < r$ or $2x > r+1$ and $2y > r+1$ holds, the sequences $(\Phi_y^{r,x})_{y\in\mathbb{Z}}$ and g^r agree*

$$\Phi_y^{r,x} = \begin{pmatrix} \phi_{2y-1}^{r,x} \\ \phi_{2y}^{r,x} \end{pmatrix} = \begin{pmatrix} g_{2y-1}^r \\ g_{2y}^r \end{pmatrix} .$$

Since in particular all x on the same side of the diagonal give rise to the same sequence, we omit its dependence on x and write $(\Phi_y^{r,L})$ and $(\Phi_y^{r,R})$ for all sequences with $x < 2r-1$ (L) or $x > 2r$ (R), respectively.

We have seen in our discussion below (5.28) that the sequences $(\Phi_y^{r,L})$ and $(\Phi_y^{r,R})$ cannot be square summable, because otherwise z would be a point in the spectrum of W_ω. However, we now also know that the sequences have the same components as the resolvent if we stay on the same site of the diagonal of g^r. However, by definition G_z is a bounded operator and this implies that the rows g^r must be square summable, because otherwise the image of the corresponding basis vector $l_r \in \ell_2(\mathbb{Z})$ should lie outside of $\ell_2(\mathbb{Z})$. Therefore, both sequences $(\Phi_y^{r,L})$ and $(\Phi_y^{r,R})$ have to be square summable on the side where they agree with g^r. We introduce the terms left and right summability for this property.

Definition 5.6.3 (left/right summability)**.** *A doubly infinite sequence $(\phi_y)_{y\in\mathbb{Z}}$ is called left square summable if*

$$\sum_{y<k} |\phi_y|^2 < \infty \quad \forall k \in \mathbb{Z} \ .$$

The sequence is called right square summable if

$$\sum_{y>k} |\phi_y|^2 < \infty \quad \forall k \in \mathbb{Z} \ .$$

Next we want to discuss the dependence of the sequences $(\Phi_y^{r,L})$ and $(\Phi_y^{r,R})$ on the row index of the resolvent. To this end, we prove the following proposition.

Proposition 5.6.4. *Let W_ω be a regular disordered quantum walk, then up to scalar multiplication, there is a unique right square summable sequence $(\phi_{+(n)})$ and a unique left square summable sequence $(\phi_{-(n)})$ both solving*

$$(W_\omega - z)\Phi = 0 \ . \tag{5.29}$$

Proof. Let us assume that there are two linearly independent left square summable sequences $(\phi^1(n))$ and $(\phi^2(n))$ both solving (5.29). Due to (5.24), for all $y < x \in \mathbb{Z}$ we find the relation

$$T_{x-1}(z)\dots T_y(z)\cdot\begin{pmatrix} \phi^{1}(2y-1) & \phi^{2}(2y-1) \\ \phi^{1}(2y) & \phi^{2}(2y) \end{pmatrix} = \begin{pmatrix} \phi^{1}(2x-1) & \phi^{2}(2x-1) \\ \phi^{1}(2x) & \phi^{2}(2x) \end{pmatrix} \ .$$

Taking the determinant on both sides of this expression and using the fact $|\det T_x(z)| = 1$ for all x (see (5.26)), we find that the modulus of the determinant of the matrix formed by the two solutions is constant for all $x \in \mathbb{Z}$, *i.e.*,

$$\left|\det\begin{pmatrix} \phi^{1}(2x-1) & \phi^{2}(2x-1) \\ \phi^{1}(2x) & \phi^{2}(2x) \end{pmatrix}\right| = \text{const} \ .$$

Left square summability implies that both solutions have to be null sequences. Therefore, the determinant has to vanish in the limit of large x, which contradicts the linear independence of $(\phi^1(n))$ and $(\phi^2(n))$. The corresponding argument applies to the right square summable solution. □

Since the sequences $(\Phi_y^{r,L})$ and $(\Phi_y^{r,R})$ are left or right square summable solutions of (5.29),respectively, we know from proposition 5.6.4 that are up to an r-dependent factor the only solutions of (5.29).

Let us fix one particular choice of a pair of right and left square summable solutions $(\phi_+(n))$ and $(\phi_-(n))$. In the next step, we show how to construct the complete resolvent from this data. The idea is to choose $(\phi_+(n))$ below the diagonal of every g^r and $(\phi_-(n))$ above of the diagonal and scale both sequences by an r-dependent factor in such a way that the overall sequence solves (5.22). In other words, for $r=2x$ or $r=2x+1$, we express the columns of the resolvent g^r as the following linear combination

$$g^r = \quad \alpha_r \begin{pmatrix} \vdots \\ \phi_{-(2x-1)} \\ \phi_{-(2x)} \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \quad \beta_r \begin{pmatrix} \vdots \\ 0 \\ 0 \\ \phi_{+(2x+1)} \\ \phi_{+(2x+2)} \\ \vdots \end{pmatrix}$$

More formally, this corresponds to the following ansatz for the matrix elements of the resolvent, where we have to distinguish the cases of odd or even m:

$$\mathrm{G}_z(n,m)=g^m(n)=\begin{cases} \alpha_m\,\phi_{-(n)} & \text{if } (m \text{ even and } n\le m) \text{ or } (m \text{ odd and } n<m) \\ \beta_m\,\phi_{+(n)} & \text{if } (m \text{ even and } n>m) \text{ or } (m \text{ odd and } n\ge m) \end{cases} \tag{5.30}$$

We want to choose the coefficients α_m and β_m in such a way that our ansatz solves (5.22). In order to treat the cases m odd and m even simultaneously, we define y_m as the lattice site corresponding to m, *e.g.* y_m satisfies either $2y_m = m$ or $2y_m+1=m$. Then (5.22) implies that

$$\begin{aligned} \alpha_m(b_{y_m}\phi_{-(2y_m-1)}-z\phi_{-(2y_m)})+\beta_m a_{y_m}\phi_{+(2y_m+2)}&=\delta_{m,2y_m} \\ \alpha_m d_{y_m}\phi_{-(2y_m-1)}-\beta_m(z\phi_{-(2y_m+1)})-c_{y_m}\phi_{+(2y_m+2)})&=1-\delta_{m,2y_m+1} \end{aligned}$$

holds for the coefficients α_m and β_m. Since both sequences $(\phi_\pm(n))$ are solutions of (5.29), we can simplify in both equations the terms inside the parentheses by making use of (5.23), which gives

$$\begin{pmatrix} -a_y\phi_{-(2y+2)} & a_y\phi_{+(2y+2)} \\ d_y\phi_{-(2y-1)} & -d_y\phi_{+(2y-1)} \end{pmatrix}\begin{pmatrix}\alpha_m\\ \beta_m\end{pmatrix}=\begin{pmatrix}\delta_{m,2y_m}\\ 1-\delta_{m,2y_m}\end{pmatrix}.$$

We can now solve for α_m and β_m and by using Cramer's rule, we find

m even	m odd
$\alpha_m=\alpha_{2y_m}=\dfrac{\phi_{+(2y_m-1)}}{a_{y_m}\det A_{y_m}}$	$\alpha_m=\alpha_{2y_m+1}=\dfrac{\phi_{+(2y_m+2)}}{a_{y_m}\det A_{y_m}}$
$\beta_m=\beta_{2y_m}=\dfrac{\phi_{-(2y_m-1)}}{a_{y_m}\det A_{y_m}}$	$\beta_m=\beta_{2y_m+1}=\dfrac{\phi_{-(2y_m+2)}}{a_{y_m}\det A_{y_m}}$

where we defined the matrix

$$A_y := \begin{pmatrix} \phi_{-(2y-1)} & \phi_{+(2y-1)} \\ \phi_{-(2y+2)} & \phi_{+(2y+2)} \end{pmatrix} .$$

However, this definition does not respect the application of transfer matrices, because $A_{y+1} \neq T_y(z) \cdot A_y$. This is due to the fact that we combined the components $\phi_{\pm(2y-1)}$ and $\phi_{\pm(2y+2)}$. We can compensate for this by eliminating $\phi_{\pm(2y+2)}$ in favour of $\phi_{\pm(2y)}$. The upper equation of (5.23) gives the relation

$$\begin{pmatrix} \phi_{(2y-1)} \\ \phi_{(2y)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b_y}{z} & \frac{a_y}{z} \end{pmatrix} \begin{pmatrix} \phi_{(2y-1)} \\ \phi_{(2y+2)} \end{pmatrix} .$$

Therefore, by defining the matrices

$$B_y := \begin{pmatrix} \phi_{-(2y-1)} & \phi_{+(2y-1)} \\ \phi_{-(2y)} & \phi_{+(2y)} \end{pmatrix} , \tag{5.31}$$

we find the following relation between the determinants of A_y and B_y

$$\det B_y = \frac{a_y}{z} \det A_y .$$

In addition, the matrices B_y satisfy the relation $B_{y+1} = T_y(z) B_y$ and due to the identity $|\det T_y(z)| = 1$ for all $y \in \mathbb{Z}$, we can also infer that

$$|\det B_y| = |\det B_x| \quad \forall x, y \in \mathbb{Z} . \tag{5.32}$$

We can now express the coefficients α_m and β_m in terms of B_{y_m} such that

m even	m odd
$\alpha_m = \alpha_{2y_m} = \dfrac{\phi_{+(2y_m-1)}}{z \det B_{y_m}}$	$\alpha_m = \alpha_{2y_m+1} = \dfrac{d_{y_m} \phi_{+(2y_m+2)}}{a_{y_m} z \det B_{y_m}}$
$\beta_m = \beta_{2y_m} = \dfrac{\phi_{-(2y_m-1)}}{z \det B_{y_m}}$	$\beta_m = \beta_{2y_m+1} = \dfrac{d_{y_m} \phi_{-(2y_m+2)}}{a_{y_m} z \det B_{y_m}}$

By inserting these solutions into our ansatz (5.30), we obtain an expression for the matrix elements of the resolvent. If we are only interested in the absolute value of the matrix elements, we are free to evaluate the determinant in the denominator at an arbitrary lattice site $k \in \mathbb{Z}$ (see (5.32)). The next lemma summarizes the discussion of this section and contains the final formula for the matrix elements of the resolvent.

Lemma 5.6.5. *Let W_ω be a regular disordered quantum walk and $(\phi_{+(n)})$ and $(\phi_{-(n)})$ a right- or left square summable solution of* (5.29)*, respectively. Then the absolute value*

of the matrix elements of the resolvent $\mathrm{G}_{z\,(n,m)}$ *for* $z \in \mathbb{C}\setminus\{0\}$ *are given by the expression*

$$|\mathrm{G}_{z\,(n,m)}| = \frac{1}{|z\det B_k|}\begin{cases} |\phi_{+(m-1)}\,\phi_{-(n)}| & \text{if } m \text{ even and } n \le m \\ |\phi_{-(m-1)}\,\phi_{+(n)}| & \text{if } m \text{ even and } n > m \\ |\phi_{+(m+1)}\,\phi_{-(n)}| & \text{if } m \text{ odd and } n < m \\ |\phi_{-(m+1)}\,\phi_{+(n)}| & \text{if } m \text{ odd and } n \ge m \end{cases},$$

where $k \in \mathbb{Z}$ *is arbitrary,* B_k *is defined in* (5.31) *and* y_m *is determined form the equation* $2y_m = m$ *if* m *is even and from the equation* $2y_m + 1 = m$ *if* m *is odd.*

5.7. Properties of transfer matrices

In this section we investigate the properties of the transfer matrices as introduced in definition 5.6.1. In particular, this leads to a refined formulation of lemma 5.6.5 for the finite restriction $W_\omega(N)$ of a disordered walk operator W_ω encompassing the reflective boundary conditions (see section 5.5). In addition, we prove some continuity results for products of transfer matrices with respect to the spectral parameter z.

In section 5.3 we defined the subset $\mathcal{U}_{nd} \subset \mathcal{U}(2)$ as the set of all unitary 2×2 matrices with non-vanishing diagonal. We will now consider the transfer matrices $T_{x(z)}$ from definition 5.6.1 as a family of maps τ_z indexed by $z \in \mathbb{C}\setminus\{0\}$ from $\mathcal{U}_{nd}$ into $\mathrm{GL}(\mathbb{C},2)$

$$\begin{aligned} \tau_z : \mathcal{U}_{nd} &\mapsto \mathrm{GL}(\mathbb{C},2) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \frac{1}{a}\begin{pmatrix} \frac{ad-bc}{z} & c \\ -b & z \end{pmatrix}. \end{aligned} \tag{5.33}$$

We have to show that the function τ_z is well defined and in particular that its image of $\mathcal{U}_{nd}$ is contained in $\mathrm{GL}(\mathbb{C},2)$. Obviously, we have to demand that $a \neq 0$, but this is exactly the definition of the set $\mathcal{U}_{nd}$. In addition, we see from (5.33) that for $U \in \mathcal{U}_{nd}$

$$\det(\tau_z(U)) = \frac{d}{a} \neq 0\ .$$

Hence, the image of $\mathcal{U}_{nd}$ under τ_z is indeed a subset of $\mathrm{GL}(\mathbb{C},2)$. Next we analyze the operator norm and symmetries of the mapping τ_z.

Lemma 5.7.1. *Let* $\tau_z : \mathcal{U}_{nd} \mapsto \mathrm{GL}(\mathbb{C},2)$ *be the family of maps according to* (5.33)*. Then* τ_z *is injective on* $\mathcal{U}_{nd}$ *for* $z \in \mathbb{C}\setminus\{0\}$ *and the image of* $\mathcal{U}_{nd}$ *under* τ_z *leaves the plane*

$$P_2 = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 : |x_1| = |x_2|\}$$

invariant, that is, $\tau_z(U)P_2 \subset P_2$ for all $U \in \mathcal{U}_{nd}$. If $z \in \mathbb{T}$ then the operator norm of $\tau_z(U)$ for $U \in \mathcal{U}_{nd}$ is independent of z and given by

$$\|\tau_z(U)\|^2 = \frac{1+|U_{22}|}{1-|U_{22}|} \geq 1 \tag{5.34}$$

Proof. We can write down the inverse of τ_z for an arbitrary matrix in $\mathrm{GL}(\mathbb{C},2)$, which is given by

$$\tau_z^{-1} : \begin{pmatrix} q & u \\ v & w \end{pmatrix} \to \frac{z}{w} \begin{pmatrix} 1 & -v \\ u & q\,w - v\,u \end{pmatrix} ,$$

provided $w \neq 0$. Since we assumed that $z \neq 0$ this is satisfied for all $M \in \tau_z(\mathcal{U}_{nd})$ and because this implies $\tau_z^{-1}(\tau_z(U)) = U$ for all $U \in \mathcal{U}_{nd}$, τ_z is indeed injective on $\mathcal{U}_{nd}$.

To show the invariance of P_2, we consider an arbitrary element $M \in \mathrm{GL}(\mathbb{C},2)$ that leaves P_2 invariant. Up to an irrelevant overall phase factor, a generic vector k in P_2 has the form $(1,c)$ with $c \in \mathbb{T}$. Therefore, in order for $M \cdot k$ to be an element of P_2 its components have to have the same absolute value, which implies

$$|M_{11} + c\,M_{12}| = |M_{21} + c\,M_{22}| \quad \forall c \in \mathbb{T}$$

for the matrix elements of M. A family of solutions for this expression is given by the three conditions

$$\begin{aligned} |M_{11}| &= |M_{22}| \\ |M_{12}| &= |M_{21}| \\ \arg(M_{11}) + \arg(M_{22}) &= \arg(M_{21}) + \arg(M_{12}) \ . \end{aligned}$$

Using the definition of τ_z and unitarity of $U \in \mathcal{U}_{nd}$ it is straightforward to check that $\tau_z(U)$ satisfies these relations. Assuming $z \in \mathbb{T}$, we can compute the singular values $\alpha_\pm$ of $\tau_z(U)$, which are given by

$$\alpha_\pm = \frac{1 \pm |U_{22}|}{|U_{11}|} \ . \tag{5.35}$$

The positive sign corresponds to the larger value and therefore to the operator norm of $\tau_z(U)$. Noting that unitarity of U implies $|U_{11}|^2 + |U_{22}|^2 = 1$ completes the proof. □

This invariance result allows for a simplification of lemma 5.6.5 for the finite unitary restrictions $W_\omega(N)$. Since we deal in that case with a finite dimensional operator, summability is satisfied for every solution of

$$(W_\omega(N) - z)\phi = 0 \ . \tag{5.36}$$

Instead, we have to ensure that the boundary conditions at the reflecting sites are satisfied (see figure 5.5). Of course, if we were to take solutions of this equation that satisfy both boundary conditions, z would be an eigenvalue. Therefore, let us consider the

half infinite operators $W_\omega^\pm(N)$, which are constructed by introducing a reflective coin exclusively at lattice site $N+1$ $(+)$ or $-(N+1)$ $(-)$, respectively. A vector $\phi_\pm$ satisfying

$$(W_\omega^\pm(N)-z)\phi = 0$$

is then called right- or left-compatible with $W_\omega(N)$, respectively. Such vectors also solve (5.36) except for one of the boundary conditions. These two conditions then replace left or right summability and we can construct the resolvent of $W_\omega(N)$ in the same manner as in the doubly-infinite case with an ansatz of the form (5.30). This leads again to lemma 5.6.5 for a pair of left- and right compatible solutions. Due to the invariance of P_2 under the action of the transfer matrices, we can simplify this result in the following way.

Lemma 5.7.2. *Let W_ω be a regular disordered quantum walk and denote by G_z^N the resolvent of its unitary restriction $W_\omega(N)$ to the lattice sites $-N$ to N. We have that for all $z \in \mathbb{T}\setminus\sigma(W_\omega(N))$, all $N\in\mathbb{N}$, all $x,y\in[-N,N]\subset\mathbb{Z}$ and all $i,j\in\{0,1\}$, there exist normalized vectors $\Phi_{+(N,x,y,z,i,j)}, \Phi_{-(N,x,y,z,i,j)} \in P_2$ such that the absolute values of the matrix elements $\mathrm{G}_z^N{}_{(2x-i,2y-j)}$ of G_z^N are given by*

$$|\mathrm{G}_z^N{}_{(2x-i,2y-j)}| = \frac{1}{2}\begin{cases} |\langle \Phi_+, T_{y-1}(z)\cdots T_x(z)\Phi_-\rangle|^{-1} & \text{if } x<y \\ |\langle \Phi_+, T_{x-1}(z)\cdots T_y(z)\Phi_-\rangle|^{-1} & \text{if } x>y \end{cases} .$$

The dependence of $\Phi_\pm$ on N, x, y and z is non trivial.

Proof. Fix $N\in\mathbb{N}$ and $z\in\mathbb{T}\setminus\sigma(W_\omega(N))$ and let $\phi_\pm^N$ be right- respectively left compatible solutions of $W_\omega(N)$. From (5.10) we deduce that due to their compatibility with the reflective boundary conditions the two sequences satisfy

$$\begin{aligned} z\phi_-^N{}_{(-2N-1)} &= e^{\mathrm{i}\eta^L}\phi_-^N{}_{(-2N)} \\ z\phi_+^N{}_{(2(N+1))} &= e^{\mathrm{i}\eta^R}\phi_+^N{}_{(2N+1)} . \end{aligned} \tag{5.37}$$

Due to $z\in\mathbb{T}$ this implies that

$$|\phi_-^N{}_{(-2N-1)}| = |\phi_-^N{}_{(-2N)}| \qquad \text{and} \qquad |\phi_+^N{}_{(2(N+1))}| = |\phi_+^N{}_{(2N+1)}| .$$

Hence, the vectors

$$\begin{pmatrix} \phi_-^N{}_{(-2N-1)} \\ \phi_-^N{}_{(-2N)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_+^N{}_{(2N+1)} \\ \phi_+^N{}_{(2(N+1))} \end{pmatrix}$$

are elements of P_2. Since we know from (5.27) that we can construct other vectors from these two initial vectors by the application of transfer matrices, lemma 5.7.1 tells us that $|\phi_\pm^N{}_{(2k-1)}| = |\phi_\pm^N{}_{(2k)}|$ holds for any $-N\le k\le N+1$ and therefore, that any other vector satisfies

$$\begin{pmatrix} \phi_\pm^N{}_{(2k-1)} \\ \phi_\pm^N{}_{(2k)} \end{pmatrix} \in P_2 .$$

Starting with the case $x < y$ and dropping for the moment the explicit dependence on N, lemma 5.6.5 gives

$$|\mathrm{G}_z^N{}_{(2x-i,2y-j)}| = \left| \frac{\phi_+(2k)\phi_-(2k-1) - \phi_+(2k-1)\phi_-(2k)}{\phi_+(2y-1+j)\phi_-(2x-i)} \right|^{-1} \tag{5.38}$$

for all $-N < k < N$. Choosing $k = 2y$ and expressing the determinant in the denominator as a scalar product of two vectors, we can rewrite the right-hand side of this equation as

$$\left| \left\langle \frac{1}{|\phi_+(2y-1+j)|} \begin{pmatrix} -\overline{\phi_+(2y-1)} \\ \overline{\phi_+(2y)} \end{pmatrix}, \frac{1}{|\phi_-(2x-i)|} \begin{pmatrix} \phi_-(2y-1) \\ \phi_-(2y) \end{pmatrix} \right\rangle \right| . \tag{5.39}$$

By our preceding discussion it is clear that both vectors in this scalar product are elements of P_2. Using (5.27), we can express a vector at lattice site y in terms of the vector at lattice site $x \leq y$ via a product of transfer matrices

$$\begin{pmatrix} \phi_-(2y-1) \\ \phi_-(2y) \end{pmatrix} = T_{y-1}(z) \cdots T_x(z) \frac{1}{|\phi_-(2x-i)|} \begin{pmatrix} \phi_-(2x-1) \\ \phi_-(2x) \end{pmatrix} .$$

Inserting this into (5.39) and defining

$$\Phi_+ = \frac{1}{\sqrt{2}|\phi_+(2y-1+j)|} \begin{pmatrix} -\overline{\phi_+(2y-1)} \\ \overline{\phi_+(2y)} \end{pmatrix} \quad \text{and} \quad \Phi_- = \frac{1}{\sqrt{2}|\phi_-(2x-i)|} \begin{pmatrix} \phi_-(2x-1) \\ \phi_-(2x) \end{pmatrix}$$

then gives the desired result with normalized vectors $\Phi_\pm \in P_2$.

In the case $x > y$ lemma 5.6.5 results in an expression for the matrix elements of the resolvent identical to (5.38) except that the arguments of ϕ_+ and ϕ_- are exchanged in the denominator. Therefore, we can apply the same argument as before except that we evaluate the determinant at the lattice site $k = x$ and the result follows by setting

$$\Phi_+ = \frac{1}{\sqrt{2}|\phi_+(2x-i)|} \begin{pmatrix} -\overline{\phi_+(2x-1)} \\ \overline{\phi_+(2x)} \end{pmatrix} \quad \text{and} \quad \Phi_- = \frac{1}{\sqrt{2}|\phi_-(2y-1+j)|} \begin{pmatrix} \phi_-(2y-1) \\ \phi_-(2y) \end{pmatrix} .$$

□

Since we need the result in the proof of the Wegner bound in section 6.3, we also note the following converse to lemma 5.7.2.

Corollary 5.7.3. *Given $\Phi_\pm \in P_2$ normalized and a set $\{T_x(z)\}$ of L transfer matrices, then there exists for any $z \in \mathbb{T}$ a unitary finite restriction $W_\omega(L)$ of a regular disordered quantum walk W_ω to the lattice sites 0 to $L+1$ such that its resolvent G_z^N satisfies*

$$|\mathrm{G}_z^N{}_{(0,L+1)}| = \frac{1}{2} |\langle \Phi_+ , T_L(z) \cdots T_1(z) \Phi_- \rangle|^{-1} .$$

Proof. Fix some $z \in \mathbb{T}$. Since the mapping τ_z from $\mathcal{U}(2)$ to the transfer matrices is invertible (see lemma 5.7.1), we can find a unique sequence of local coin operations corresponding to $\{T_x(z)\}$. We can now take any regular disordered quantum walk W_ω,

whose local coin operations on the lattice sites 1 to L coincide with this sequence and introduce reflective boundary coins at the lattice sites 0 and $L+1$ according to (5.10). We choose the boundary phases η_L and η_R as

$$\begin{aligned}\eta_L &= \arg(z) + \arg(\Phi_{-(1)}) - \arg(\Phi_{-(2)}) \\ \eta_R &= -(\pi + \arg(z) - \arg(\Phi_{+(1)}) + \arg(\Phi_{+(2)}))\ ,\end{aligned}$$

which ensures that Φ_- satisfies the left and $-\sigma_x\overline{\Phi_+}$ satisfies the right boundary equation, *i.e.*, (5.37). Therefore, the application of the transfer matrices $\{T_x(z)\}$ or their inverses to Φ_- or $-\sigma_x\overline{\Phi_+}$ generates a sequence that is left or right compatible with the finite unitary restriction of the walk operator W_ω to the lattice sites 1 to L. Now starting from these sequences and W_ω, we can use lemma 5.7.2 and find that the vectors $\Phi_\pm$ correspond exactly to the two vectors in P_2 given in this lemma. □

Next we establish a decomposition relation for the matrix elements of a product of transfer matrices that allows us to subdivide such a matrix element into a product of matrix elements of shorter products of transfer matrices.

Lemma 5.7.4. *Let $\Phi_\pm \in P_2$ be normalized and $\{T_x(z)\}$ be a set of L transfer matrices, then there exists for any $z \in \mathbb{T}$ and any $1 \le l \le L$ a normalized vector $\widetilde{\Psi} \in P_2$ such that*

$$|\langle \Phi_+ , T_L(z)\cdots T_1(z)\Phi_-\rangle| = |\langle \Phi_+ , T_L(z)\cdots T_l(z)\Psi\rangle|\,|\langle \Psi , T_{l+1}(z)\cdots T_1(z)\,\Phi_-\rangle|\ . \tag{5.40}$$

Proof. The plane P_2 is invariant under the action of the transfer matrices. Hence, the normalized vector $\widetilde{\Psi}$ defined as

$$\Psi := \frac{T_{l+1}(z)\cdots T_1(z)\,\Phi_-}{\|T_{l+1}(z)\cdots T_1(z)\,\Phi_-\|}$$

lies in P_2 and we obtain for the left-hand side of (5.40) the relation

$$|\langle \Phi_+ , T_L(z)\cdots T_1(z)\,\Phi_-\rangle| = |\langle \Phi_+ , T_L(z)\cdots T_l(z)\,\Psi\rangle|\,\|T_{l+1}(z)\cdots T_1(z)\,\Phi_-\|\ .$$

This finishes the proof, because any vector $\Psi \in \mathbb{C}^2$ satisfies the relation

$$\|\Psi\| = \frac{\langle \Psi , \Psi\rangle}{\|\Psi\|} = \langle \frac{\Psi}{\|\Psi\|} , \Psi\rangle\ .$$

□

5.7.1. Continuity properties of products of transfer matrices

In this section we connect our results on transfer matrices with the general theory of products of random matrices as developed in chapter 3. In particular, we rephrase the various regularity and integrability conditions for the case of matrices with determinant of modulus one. First, we define the main regularity condition we are going to use.

Definition 5.7.5. *A probability measure μ on* $\mathrm{GL}(\mathbb{C}, d)$ *is ζ-integrable for $\zeta > 0$ if*

$$\int \|g\|^{\zeta} \, \mu(\mathrm{d}g) < \infty \ .$$

This kind of regularity is convenient to assume for the local coin distributions due to the following proposition.

Proposition 5.7.6. *Let μ be a ζ-integrable probability measure on $SL_{\mathbb{T}}(2)$. We have that* $\log\|g\|$ *as well as* $\log\|g^{-1}\|$ *are integrable with respect to μ and ζ is an exponential moment for μ.*

Proof. Since $g \in SL_{\mathbb{T}}(2)$ implies $|\det g| = 1$ the two singular values of g are reciprocal to one another and we only have to consider the case $\|g\|$. In addition the determinant condition tells us that $\|g\| \geq 1$ and therefore $\log\|g\| \leq \|g\|$, which proofs integrability as well as the fact that ζ is an exponential moment for μ. □

From the proof of lemma 5.7.1 we know that the singular values of a transfer matrix are upper bounded by $\frac{2}{|U_{11}|}$. This implies the following proposition.

Proposition 5.7.7. *Let μ be a probability measure on $\mathcal{U}(2)$, with $\mu(\mathcal{U}_{nd}) = 1$ and denote by U_{11} the upper left matrix element of $U \in \mathcal{U}(2)$. The measure on the transfer matrices induced by τ is ζ-integrable, if $\frac{1}{U_{11}}$ is ζ-integrable with respect to μ. This means in particular that ζ-integrability of μ can be verified independently of the parameter z.*

These two propositions allow us directly to rephrase the main result from chapter 3 for the case of matrices in $SL_{\mathbb{T}}(2)$. Recall that a set $M \subset \mathrm{GL}(2, \mathbb{C})$ is contractive if and only if we can find a sequence $(M_n) \subset M$ such that $\|M_n\|^{-1} M_n$ converges to a rank one matrix. Since for $M \in SL_{\mathbb{T}}(2)$ the singular values are reciprocal to each other, this is equivalent to the sequence $(\|M_n\|)_n$ being unbounded. Using the fact that a compact semigroup is a group [HR94] we see that non-compactness implies contractiveness for $SL_{\mathbb{T}}(2)$ matrices [CL90]. Hence, denoting by $\mathbb{P}(A)$ the probability of the event A with respect to μ_∞, we can reformulate lemma 3.3.10 for $SL_{\mathbb{T}}(2)$ matrices.

Lemma 5.7.8. *Let μ be a strongly-irreducible, non-compact and ζ-integrable probability measure on $SL_{\mathbb{T}}(2)$ and let $S_n(\omega)$ be a product of i.i.d random matrices drawn according to μ. There exist $\gamma, \varepsilon_0 > 0$ such that for every $\varepsilon_0 > \varepsilon > 0$ there exits $\sigma > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and normalized $x, y \in \mathbb{C}^2$*

$$\mathbb{P}\left(|\langle y \, , S_n(\omega) x \rangle| \leq e^{(\gamma-\varepsilon)n}\right) \leq e^{-\sigma n} \ .$$

If the transfer matrices of a regular disordered quantum walk satisfy this assumptions, this lemma provides us with the desired almost sure exponential decay of the matrix elements of the resolvent. However, the result or more precisely the resulting constants γ and σ depend on the point z on the unit circle we consider.

To overcome this limitation we conclude this section with some continuity results for products of random matrices if the parameter z is varied on the unit circle. Although these results depend explicitly on the length of the product, they allow for the formulation of an initial scale estimate needed in the proof of dynamical localization that is

carried out in the next chapter. We begin with a continuity result under one additional assumption. Since the only parameter that enters in all of the arguments is the length of the product rather than the exact lattice sites between which it is evaluated, we set $x = 1$ and $y - 2 = n$ for the remainder of this section.

Proposition 5.7.9. *Let $n \in \mathbb{N}$, $\Theta, \Theta' \in \mathbb{T}$ and assume that all preimages $U_i \in \mathcal{U}(2)$ of the transfer matrices T_i satisfy $|(U_i)_{11}| > \kappa > 0$. Then there exists $C_n > 0$ such that for all normalized $v \in \mathbb{C}^2$ we have*

$$\mathbb{P}\left(\left|\|T_{n(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} v\| - \|T_{n(\Theta')} \cdot \ldots \cdot T_{1(\Theta')} v\|\right| > \eta\right) \leq \frac{1}{\eta} C_n |\Theta - \Theta'| \ , \tag{5.41}$$

where the probability is taken with respect to the product measure μ^n.

Proof. Applying Markov's inequality and the reverse triangle inequality to the left-hand side of (5.41) and using the definition of the operator norm implies that we have to show

$$\mathbb{E}\left(\|T_{n(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} - T_{n(\Theta')} \cdot \ldots \cdot T_{1(\Theta')}\|\right) \leq C_n |\Theta - \Theta'| \ . \tag{5.42}$$

The difference between two products of k operators can be written as

$$A_k \cdots A_1 - B_k \cdots B_1 = (A_k - B_k) B_{k-1} \cdots B_1 + A_k (A_{k-1} \cdots A_1 - B_{k-1} \cdots B_1) \ .$$

Applying this decomposition iteratively to the two products of transfer matrices results in the following telescope sum like scheme

$$\begin{aligned} T_{n(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} - T_{n(\Theta')} \cdot \ldots \cdot T_{1(\Theta')} = &(T_{n(\Theta)} - T_{n(\Theta')}) T_{n-1(\Theta')} \cdot \ldots \cdot T_{1(\Theta')} \\ &+ T_{n(\Theta)} (T_{n-1(\Theta)} - T_{n-1(\Theta')}) T_{n-1(\Theta')} \cdot \ldots \cdot T_{1(\Theta')} \\ &+ \cdots \\ &+ T_{n(\Theta)} \cdot \ldots \cdot T_{2(\Theta)} (T_{1(\Theta)} - T_{1(\Theta')}) \ . \end{aligned}$$

Inserting this expression into (5.42), using that the operator norm is sub-multiplicative and the fact that the quantities $\|T_{i(z)}\|$ are i.i.d random variables we obtain the upper bound

$$\mathbb{E}\left(\|T_{n(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} - T_{n(\Theta')} \cdot \ldots \cdot T_{1(\Theta')}\|\right) \leq n \mathbb{E}\left(\|T_{1(\Theta)}\|\right)^{n-1} \mathbb{E}\left(\|T_{1(\Theta)} - T_{1(\Theta')}\|\right)$$

We know from lemma 5.7.1 that for all $\theta \in \mathbb{T}$ the singular values of the transfer matrices are less or equal than $2|U_{11}|^{-1}$. Therefore, using our assumptions, the first expectation value on the right-hand side can be upper bounded by $(2\kappa)^{-n+1}$. The difference in the last factor takes the simple form of a diagonal matrix

$$T_{1(\Theta)} - T_{1(\Theta')} = \frac{1}{U_{11}} \begin{pmatrix} \det(U)(\Theta^{-1} - \Theta'^{-1}) & 0 \\ 0 & \Theta - \Theta' \end{pmatrix} \tag{5.43}$$

of which the singular values can be computed to be both equal to $|U_{11}|^{-1}|\Theta - \Theta'|$. Therefore, using again our assumption on U_{11}, the lemma follows if we set $C_n = \frac{2^{n-1} n}{\kappa^n}$. □

In addition to the preceding norm estimate, we need the following corollary that translates the bound to the single matrix elements of a product of transfer matrices.

Corollary 5.7.10. *Let $n \in \mathbb{N}$, $\Theta, \Theta' \in \mathbb{T}$ and assume that all preimages $U_i \in \mathcal{U}(2)$ of the transfer matrices T_i satisfy $|(U_i)_{11}| > \kappa > 0$. Then there exists $C_n > 0$ such that for all normalized $v_1, v_2 \in \mathbb{C}^2$ we have*

$$\mathbb{P}\left(\left||\langle v_2, T_{n(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} v_1\rangle| - |\langle v_2, T_{n(\Theta')} \cdot \ldots \cdot T_{1(\Theta')} v_1\rangle|\right| > \eta\right) \leq \frac{1}{\eta} C_n |\Theta - \Theta'| \ , \quad (5.44)$$

where the probability is taken with respect to the product measure μ^n.

Proof. Applying again Markov's inequality and the reverse triangle inequality we see that the left-hand side of (5.44) is upper bounded by

$$\frac{1}{\eta} \mathbb{E}(|\langle v_2, T_{n(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} - T_{n(\Theta')} \cdot \ldots \cdot T_{1(\Theta')} v_1\rangle|) \ .$$

Using the Cauchy-Schwarz inequality on the scalar product and normalization of v_2 we arrive again at (5.42) and the whole argument presented in the proof of lemma 5.7.9 carries over with the same bound C_n as before. □

We can now state an initial scale estimate for the growth rate of the transfer matrices in an interval around a given point Θ on the unit circle $\mathbb{T}$.

Lemma 5.7.11 (initial scale estimate)**.** *Let $n \in \mathbb{N}$, μ be a probability measure on $\mathcal{U}_{nd}$ and assume that for $\Theta_0 \in \mathbb{T}$ the image measure of μ on $\tau_{\Theta_0}(\mathcal{U}_{nd})$ is strongly irreducible, non-compact and ζ-integrable. Then there are $\gamma_0, \sigma_0 > 0$ and $n_0 \geq n$ such that for some open arc $I(\Theta_0, \delta)$ around Θ_0 of arc length 2δ we have for all normalized vectors $v_1, v_2 \in \mathbb{C}^2$*

$$\mathbb{P}\left(\exists \Theta \in I(\Theta_0, \delta) \, ; \, |\langle v_2, T_{n_0(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} v_1\rangle| \leq e^{\gamma_0 n_0}\right) \leq e^{-\sigma_0 n_0} \ .$$

In particular, the constants γ_0 and σ_0 are independent of the vectors v_1 and v_2.

Proof. Starting with an arbitrary $\Theta \in \mathbb{T}$ and $m \in \mathbb{N}$, we define the events

$$E_1 := \{\left||\langle v_2, T_{m(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} v_1\rangle| - |\langle v_2, T_{m(\Theta_0)} \cdot \ldots \cdot T_{1(\Theta_0)} v_1\rangle|\right| < \eta\}$$
$$E_2 := \{|\langle v_2, T_{m(\Theta_0)} \cdot \ldots \cdot T_{1(\Theta_0)} v_1\rangle| \geq e^{(\gamma-\varepsilon)m}\} \ .$$

Denoting by $\complement E$ the complementary event of an event E, we can lower bound the probability that the matrix element of a product of n transfer matrices is large by

$$\begin{aligned}\mathbb{P}\left(|\langle v_2, T_{m(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} v_1\rangle| > e^{(\gamma-\varepsilon)m} - \eta\right) &\geq \mathbb{P}(E_1 \cap E_2) \geq 1 - \mathbb{P}(\complement E_1) - \mathbb{P}(\complement E_2) \\ &\geq 1 - \mathbb{P}\left(||\langle v_2, T_{m(\Theta)} \cdot \ldots \cdot T_{1(\Theta)} v_1\rangle| - |\langle v_2, T_{m(\Theta_0)} \cdot \ldots \cdot T_{1(\Theta_0)} v_1\rangle|| \leq \eta\right) \\ &\quad - \mathbb{P}\left(|\langle v_2, T_{n_0(\Theta_0)} \cdot \ldots \cdot T_{1(\Theta_0)} v_1\rangle| \leq e^{(\gamma-\varepsilon)m}\right) \ . \quad (5.45)\end{aligned}$$

We now fix ε and γ according to lemma 5.7.8 such that in addition $\gamma - 2\varepsilon > 0$ holds. The second probability in the last line of this expression can then be controlled by an exponential rate σ_ε if we make m larger than some N_ε. If $n > N_\varepsilon$ set $n_0 = n$ otherwise we choose $n_0 = N_\varepsilon$. Since $\mu(\mathcal{U}_{nd}) = 1$ we can find for any $\xi > 0$ a $\kappa > 0$ such that

$$\mathbb{P}(\exists l \in \{1, \ldots, n_0\} : |U_{11}| < \kappa) \leq \xi \ .$$

Thus, all assumptions of corollary 5.7.10 are met and we can apply it to the second term in (5.45). Inserting both bounds into the inequality we obtain

$$\mathbb{P}\left(|\langle v_2, T_{n_0(\Theta)}\cdot\ldots\cdot T_1(\Theta)v_1\rangle| > e^{(\gamma-\varepsilon)n_0}-\eta\right) \geq 1-e^{-\sigma_\varepsilon n_0}-\frac{C_{n_0}}{\eta}|\Theta-\Theta_0|-\xi \ .$$

Note that C_{n_0} does depend on the value of ξ. To prove the desired bound we first set $\gamma_0=\gamma-2\varepsilon$, which is positive by construction, and choose $\eta < e^{(\gamma-\varepsilon)n_0}(1-e^{-\varepsilon n_0})$. Next we pick constants ξ and σ_0 that ensure $e^{-\sigma_0 n_0} > e^{-\sigma_\varepsilon n_0}+\xi$. This fixes the value of the constant C_{n_0} which enables us to choose an allowed length δ by making $|\Theta-\Theta_0|$ small enough. □

To conclude this section we note one uniform regularity property of the expectation value of a product of transfer matrices with respect to normalized vectors in $\mathbb{C}^2$. This result is needed for the Wegner estimate we derive in section 6.3. It is an adaption of a similar result in [CKM87a]. The proof uses a standard chaining argument and is included in appendix C.1 for completeness.

Proposition 5.7.12. *Let μ be probability measure on $\mathcal{U}_{nd}$ and assume that for all $\Theta \in I \subset \mathbb{T}$ the image measure of μ on $\tau_\Theta(\mathcal{U}_{nd})$ is strongly irreducible, non-compact and ζ-integrable. Then there are positive constants α and δ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $\Theta \in I$*

$$\sup_{\|\phi\|=1} \mathbb{E}\left(\|T_{n(\Theta)}\cdot\ldots\cdot T_1(\Theta)\phi\|^\delta\right) \leq e^{-\alpha n} \ .$$

5.7.2. Hölder continuity of the Lyapunov exponent

With the initial scale estimate from the last section we have shown regularity on the unit circle for the norm growth for finite products of transfer matrices. The goal in this section is to establish regularity conditions for the Lyapunov exponent γ. To be more precise, since the transfer matrices $T(\Theta)$ depend on the spectral parameter Θ, so does in general the Lyapunov exponent $\gamma(\Theta)$. The goal of this section is to show that with respect to Θ, the Lyapunov exponent is Hölder continuous.

Our standing assumption is again the ζ-integrability of the measures μ_Θ according to definition 5.7.5 for all Θ. Since the norm of the transfer matrices $T(\Theta)$ allows for an Θ independent bound, as discussed in proposition 5.7.7, we can check this condition independently of Θ. In addition, we can transfer the Θ dependence of the image measure μ_Θ on $\mathrm{GL}(\mathbb{C},2)$ to an integration over the transfer matrices, thereby integrating over $\mathcal{U}(2)$. Therefore, we assume that there exists a Θ-independent measure μ and a function τ_Θ such that for all μ_Θ we have

$$\int f(g)\mu_\Theta(\mathrm{d}g) = \int f(\tau_\Theta(U))\mu(\mathrm{d}U) =: \mathbb{E}\left(f(g_\Theta)\right) \ . \tag{5.46}$$

In addition, the exact form of the transfer matrices does not enter our argument. Instead, we only need the following properties.

Proposition 5.7.13. *Let μ be a ζ-integrable probability measure on $\mathcal{U}_{nd}$. The transfer matrices $T_l(\Theta)$ satisfy for all $\Theta,\Theta' \in \mathbb{T}$, $l \in \mathbb{Z}$ and some $C > 0$ the relation*

$$\mathbb{E}\left(\log \|T_l(\theta) T_l^{-1}(\theta')\|\right) \leq C|\Theta - \Theta'|^{\frac{\zeta}{2}} . \tag{5.47}$$

Denoting by $[x, y]$ for $x, y \in \mathbb{C}^2$ the matrix with columns x and y, we have for all $\Theta,\Theta' \in \mathbb{T}$, $l \in \mathbb{Z}$ and $\overline{x} \in \mathbb{P}\mathbb{C}^2$ the relation

$$|\det[T_l(\Theta)\overline{x},\ T_l(\Theta')\overline{x}]| \leq \|T_l(\Theta)\|^2 |\Theta - \Theta'| . \tag{5.48}$$

Proof. To show the first relation, note that any pair of matrices A, B satisfies $\|AB^{-1}\| \leq \|(A-B)\|\|B^{-1}\| + 1$. Inserting this into (5.47) we find

$$\mathbb{E}\left(\log \|T_l(\theta) T_l^{-1}(\theta')\|\right) \leq \frac{2}{\zeta} \mathbb{E}\left(\log((\|T_l(\theta) - T_l^{-1}(\theta')\|\|T_l^{-1}(\theta')\| + 1)^{\frac{\zeta}{2}})\right) .$$

From (5.43) we know that the difference of two transfer matrices depending on different spectral parameters Θ and Θ' is given by a simple diagonal matrix. Furthermore, employing (5.34) implies that the operator norm of this difference is upper bounded by $\|T_l(\Theta)\||\Theta - \Theta'|$ for any $\Theta \in \mathbb{T}$. Inserting this into (5.47) and using the inequalities $\log(1+x) \leq x$ as well as $(1+x)^s \leq 1 + x^s$ for positive x and $0 < s < 1$ we find

$$\mathbb{E}\left(\log \|T_l(\theta) T_l^{-1}(\theta')\|\right) \leq \frac{2}{\zeta} \mathbb{E}\left(\|T_l(\theta')\|^{\frac{\zeta}{2}} \|T_l^{-1}(\theta')\|^{\frac{\zeta}{2}}\right) |\Theta - \Theta'|^{\frac{\zeta}{2}} .$$

The fact that the determinant of a transfer matrix has modulus one implies for the operator norm of a transfer matrix and its inverse the relation $\|T_l(\Theta)\| = \|T_\Theta^{-1}(z)\|$ and therefore the prefactor in the last expression is finite by ζ-integrability of μ_Θ. In order to show the second part, we use the relation $|\det[x, y]| = |\det[Ax, Ay]|$ for $A \in SL_{\mathbb{T}}(2)$. Choosing $T_l^{-1}(\Theta')$ and $\overline{x} = (1, \exp(\mathrm{i}\phi))$, we obtain by explicitly evaluating the determinant in the second step

$$|\det[T_l(\Theta')\overline{x},\ T_l(\Theta)\overline{x}]| = |\det[\overline{x},\ T_l^{-1}(\Theta') T_l(\Theta)\overline{x}]| = 2\,\frac{1 + |U_{22}|}{|U_{11}|^2}\,|\Theta - \Theta'| .$$

The bound now follows from (5.34). □

Since we do not need the exact form of the transfer matrices we return to the notation of chapter 3 and again denote by g_ω a single and by $S_{n(\omega)}$ the product of n random matrices. First we translate some continuity results that are valid in the Hamiltonian case to our setting of complex transfer matrices [BL85, CL90].

Proposition 5.7.14. *Let μ_Θ, $\Theta \in \mathbb{T}$, be a family of strongly irreducible, non-compact and ζ-integrable probability measures on* $\mathrm{GL}(\mathbb{C}, 2)$*, which satisfy* (5.47) *and let us define the functions $\Phi_\Theta(\overline{x})$ for $\overline{x} \in \mathbb{P}\mathbb{C}^2$ as*

$$\Phi_\Theta(\overline{x}) := \int \log \frac{\|g\,x\|}{\|x\|} \mu_\Theta(\mathrm{d}g) .$$

If we denote by $\gamma(\Theta)$ Lyapunov exponent and by ν_Θ the unique invariant measure on $\mathbb{P}\mathbb{C}^2$ corresponding to μ_Θ then we have:

(i) *The function* $\Phi_\Theta(\overline{x})$ *is continuous on* $\mathbb{PC}^2 \times \mathbb{T}$ *and Hölder continuous with order* α *on* $\mathbb{T}$ *for all* $0 < \alpha \leq \frac{\zeta}{2}$.

(ii) $\gamma(\Theta) = \nu_\Theta(\Phi_\Theta) = \int \Phi_\Theta(\overline{x})\, \nu_\Theta(\mathrm{d}\overline{x})$.

(iii) $\gamma(\Theta)$ *is continuous with respect to* Θ *and the convergence*

$$\gamma(\Theta) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log \|S_{n(\omega,\Theta)}\overline{x}\|\right)$$

is uniform in Θ *and* $\overline{x}$.

Proof. Part (ii) follows directly from the definition of Φ_Θ and lemma 3.2.12, which also provides the uniform convergence of the Lyapunov exponent in (iii) with respect to $\overline{x}$. Since $\log\|g\overline{x}\| \leq \log\|g\|$, the continuity of $\Phi_\Theta(\overline{x})$ for fixed Θ follows from dominated convergence and the integrability of $\log\|g_{\omega,\Theta}\|$. To complete the proof of (i) note that with C independent of $\overline{x}$ we have

$$|\Phi_\Theta(\overline{x}) - \Phi_{\Theta'}(\overline{x})| \leq \mathbb{E}\left(\log \frac{\|g_{\omega,\Theta}\overline{x}\|}{\|g^{-1}_{\omega,\Theta'}\overline{x}\|}\right) \leq \mathbb{E}\left(\log\|g_{\omega,\Theta} g^{-1}_{\omega,\Theta'}\|\right) \leq C|\Theta - \Theta'|^{\frac{\zeta}{2}} \ ,$$

where we used (5.47) in the last step.

Uniqueness of the invariant measure μ_Θ for every Θ, which holds according to lemma 3.3.3, already implies that μ_Θ has to be weakly continuous in Θ. Now assume that the sequence (Θ_n) converges to Θ. The representation of the Lyapunov exponent according to (ii) together with the fact that Φ_Θ is Lipschitz implies

$$\lim_{n\to\infty} \gamma(\Theta_n) = \lim_{n\to\infty} \nu_{\Theta_n}(\Phi_{\Theta_n}) = \lim_{n\to\infty} \nu_{\Theta_n}(\Phi_\Theta) + \nu_{\Theta_n}(\Phi_{\Theta_n} - \Phi_\Theta) = \nu_\Theta(\Phi_\Theta) = \gamma(\Theta) \ .$$

Hence, $\gamma(\Theta)$ is indeed continuous with respect to $\Theta \in \mathbb{T}$. Let us now define the probability measures $\nu_{n,\Theta,\overline{x}} = \frac{1}{n}\sum_{k=0}^{n} \mu_\Theta^k * \delta_x$. For two sequences $(\Theta_n)_n \subset \mathbb{T}$ and $(\overline{x}_n)_n \subset \mathbb{PC}^2$ converging to $\Theta \in \mathbb{T}$ or $\overline{x} \in \mathbb{PC}^2$ respectively, the sequence $(\nu_{n,\Theta_n,\overline{x}_n})_n$ converges weakly to the unique invariant measure ν_Θ, because any limit point of $(\nu_{n,\Theta_n,\overline{x}_n})_n$ is ν_Θ invariant according to lemma 3.2.8. Integrating the function Φ_Θ with respect to these measures, we obtain

$$\begin{aligned}
\nu_{n,\Theta,\overline{x}}(\Phi_\Theta) &= \frac{1}{n}\sum_{k=0}^{n-1} \int \Phi_\Theta(\overline{y})\, (\mu^n * \delta_{\overline{x}})(\mathrm{d}\overline{y}) = \frac{1}{n}\int \sum_{k=0}^{n-1} \log \frac{\|g_1\, g_2 x\|}{\|g_2 x\|}\, \mu_\Theta(\mathrm{d}g_1)\, \mu_\Theta^k(\mathrm{d}g_2) \\
&= \frac{1}{n}\int \log \frac{\|gx\|}{\|x\|} \mu_\Theta^n(\mathrm{d}g) = \frac{1}{n}\mathbb{E}\left(\log\|S_{n(\omega,\Theta)}\overline{x}\|\right) =: h_n(\Theta,\overline{x}) \ .
\end{aligned}$$

The functions h_n are continuous on the space $\mathbb{PC}^2 \times \mathbb{T}$ by a similar argument as for Φ_Θ. Now, since $\mathbb{PC}^2 \times \mathbb{T}$ is a compact space it suffices to show convergence of $h_n(\Theta_n, \overline{x}_n)$ to the Lyapunov exponent $\gamma(\Theta)$ in order to prove uniform convergence. We find

$$\begin{aligned}
\lim_{n\to\infty} h_n(\Theta_n, \overline{x}_n) &= \lim_{n\to\infty} \nu_{n,\Theta_n,\overline{x}_n}(\Phi_{\Theta_n}) \\
&= \lim_{n\to\infty} \nu_{n,\Theta_n,\overline{x}_n}(\Phi_\Theta) + \nu_{n,\Theta_n,\overline{x}_n}(\Phi_{\Theta_n} - \Phi_\Theta) \\
&= \nu_\Theta(\Phi_\Theta) = \gamma(\Theta) \ ,
\end{aligned}$$

which finishes the proof. □

We now use proposition 5.7.14 together with some general results on products of random matrices from chapter 3 in order to show the Hölder continuity of the Lyapunov exponent $\gamma(\Theta)$ with respect to $\Theta \in \mathbb{T}$.

Lemma 5.7.15. *Let μ_Θ, $\Theta \in \mathcal{I} \subset \mathbb{T}$, be a family of strongly irreducible, non-compact and ζ-integrable probability measures on* $\mathrm{GL}(\mathbb{C},2)$*, which all satisfy the assumptions of proposition 5.7.13 and* (5.46)*, then the Lyapunov exponent $\gamma(\Theta)$ is Hölder continuous for all $0<\alpha<\frac{\zeta}{8}$, that is there exists a strictly positive constant C_α which satisfies*

$$|\gamma(\Theta)-\gamma(\Theta')| \leq C_\alpha |\Theta-\Theta'|^\alpha \ .$$

Proof. The proof relies on the following decomposition of the Lyapunov exponent

$$\gamma(\Theta)-\gamma(\Theta') = (\nu_\Theta - \nu_{\Theta'})(\Phi_{\Theta'}) + \nu_\Theta(\Phi_\Theta - \Phi_{\Theta'}) \ . \tag{5.49}$$

The second term is Hölder continuous for $0<\alpha\leq\frac{\zeta}{2}$ due to proposition 5.7.14 and so we only have to check the Hölder continuity of the first summand. In section 3.3.1 we introduced the space of Hölder-continuous functions $\mathcal{L}_\alpha$ and from proposition 5.7.14 we know that $\Phi_\Theta \in \mathcal{L}_\alpha$ for $0<\alpha<\zeta/2$. Therefore, we can identify ν_Θ with the operator N_{ν_Θ} according to (3.25) that projects $f \in \mathcal{L}_\alpha$ to a constant functions via

$$N_{\nu_\Theta}(f) = \nu_\Theta(f) = \int f(\overline{y})\ \nu_\Theta(\mathrm{d}\overline{y}) \ .$$

By lemma 3.3.5 we can express N_{ν_Θ} as a Cauchy integral over the resolvent $\mathrm{G}_{a,\Theta}$ of the Markov operator R_{μ_Θ} (see (3.23)). Using the second resolvent identity we obtain for arbitrary $f \in \mathcal{L}_\alpha$

$$|N_{\nu_\Theta}(f)-N_{\nu_{\Theta'}}(f)| \leq \|N_{\nu_\Theta}(f)-N_{\nu_{\Theta'}}(f)\|_\alpha \leq \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}_\varepsilon} \|\,\mathrm{G}_{a,\Theta}(R_{\Theta'}-R_\Theta)\mathrm{G}_{a,\Theta'}(f)\|_\alpha \,\mathrm{d}\ell \tag{5.50}$$

Since part (ii) of lemma 3.3.5 provides a bound on $\sup_{\Theta\in\mathcal{I}}\|\,\mathrm{G}_{a,\Theta}\|_\alpha$ we are done if we can show Hölder-continuity of the expression

$$\|(R_{\Theta'}-R_\Theta)(f)\|_\alpha = \|(R_{\Theta'}-R_\Theta)(f)\|_\infty + m_\alpha((R_{\Theta'}-R_\Theta)(f)) \ , \tag{5.51}$$

which we do for both terms separately. Starting with the uniform norm and using the definition of $m_\alpha(f)$, we find for $f \in \mathcal{L}_\alpha$

$$\begin{aligned}\|(R_{\Theta'}-R_\Theta)(f)\|_\infty &\leq \sup_{x\in\mathbb{P}\mathbb{C}^2} \mathbb{E}\left(|f(g_{\Theta'}\overline{x})-f(g_\Theta\overline{x})|\frac{|\delta(g'_\Theta\overline{x},g_\Theta\overline{x})|^\alpha}{|\delta(g'_\Theta\overline{x},g_\Theta\overline{x})|^\alpha}\right)\\ &\leq m_\alpha(f)\sup_{\overline{x}\in\mathbb{P}\mathbb{C}^2}\mathbb{E}\left(|\delta(g'_\Theta\overline{x},g_\Theta\overline{x})|^\alpha\right) \ .\end{aligned}$$

Using the identity $\delta(\overline{x},\overline{y}) = |\det[x,y]|\|x\|^{-1}\|y\|^{-1}$ for $\overline{x},\overline{y}\in\mathbb{P}\mathbb{C}^2$, the bound $\|g\overline{x}\| \geq \|g\|^{-1}$ for $g \in SL_\mathbb{T}(2)$, and (5.48) this already implies

$$\|(R_{\Theta'}-R_\Theta)(f)\|_\infty \leq m_\alpha(f)\mathbb{E}\left(\|g_\Theta\|^{3\alpha}\|g_{\Theta'}\|^\alpha\right)|\Theta-\Theta'|^\alpha \ .$$

Due to ζ-integrability the expectation value is finite if we choose $\alpha < \zeta/4$, which means that we can find a finite constant A_α such that $\|(R_{\Theta'} - R_\Theta)(f)\|_\infty \leq A_\alpha |\Theta - \Theta'|^\alpha$. With the same arguments we also can bound the expression

$$|(R_\Theta f)(\overline{x}) - (R_\Theta f)(\overline{y})| \leq m_\alpha(f)\mathbb{E}\left(|\delta(g_\Theta \overline{x}, g_\Theta \overline{y})|^\alpha\right) \leq m_\alpha(f)\mathbb{E}\left(\|g_\Theta\|^{2\alpha}\right)\delta(\overline{x},\overline{y})^\alpha \ .$$

The prefactor is smaller than some finite constant B_α due to ζ-integrability of μ_Θ if we choose $\alpha < \zeta/2$. Therefore, let us now pick $\beta < \zeta/4$ and consider the second term in (5.51). Given the definition of $m_\beta(f)$ in (3.3.1) and these two bounds we consider for $\overline{x}, \overline{y} \in \mathbb{PC}^2$ the expression

$$|(R_{\Theta'} - R_\Theta)(f)(\overline{x}) - (R_{\Theta'} - R_\Theta)(f)(\overline{y})| \leq m_\beta(f) C_\beta \min\left(|\Theta - \Theta'|^\beta, \delta(\overline{x},\overline{y})^\beta\right) \ ,$$

where we set $C_\beta = \min(B_\beta, A_\beta)$. Now for $\beta/2$ we obtain

$$\begin{aligned} m_{\frac{\beta}{2}}(R_{\Theta'} - R_\Theta)(f) &= \sup_{\overline{x}\neq\overline{y}} \frac{|(R_{\Theta'} - R_\Theta)(f)(\overline{x}) - (R_{\Theta'} - R_\Theta)(f)(\overline{y})|}{\delta(\overline{x},\overline{y})^{\frac{\beta}{2}}} \\ &\leq 2 m_\beta(f) C_\beta \sup_{\overline{x}\neq\overline{y}} \min\left(|\Theta - \Theta'|^\beta \, \delta(\overline{x},\overline{y})^{\frac{-\beta}{2}}, \delta(\overline{x},\overline{y})^{\frac{\beta}{2}}\right) \ . \end{aligned}$$

Realizing that all positive numbers r, t satisfy the relation $\min(\frac{r^2}{t}, t) < r$, this formula implies the desired bound for all $\alpha < \zeta/8$

$$m_\alpha((R_{\Theta'} - R_\Theta)(f)) \leq 2 C_{2\alpha} 2 m_{2\alpha} f |\Theta - \Theta'|^\alpha \ .$$

Since we know from proposition 5.7.14 that $\Phi_\Theta \in \mathcal{L}_\alpha$ for $\alpha < \frac{\zeta}{2}$, we can insert this bound into (5.50), which via (5.49) then implies the Hölder-continuity for the Lyapunov exponent $\gamma(\Theta)$. □

5.8. Conclusion

In this chapter we have introduced a class of non-translation-invariant quantum walks where the local coin operations are given by random unitary matrices drawn according to some common probability measure μ on $\mathcal{U}(2)$. Such disordered quantum walks model a situation where the timescales on which fluctuations occur are long in comparison to a single run of an experiment. As such, disordered quantum walks can be related to other unitary models with random perturbations, *e.g.* the unitary Anderson model.

We showed that the system exhibits dynamical localization, *i.e.* has exponential decaying transition probabilities between distant lattice sites, if the coin distribution assigns a finite probability to coins implementing total internal reflections. Using such reflecting coins as boundary conditions we defined finite unitary restrictions of the infinite evolution operator. The finite propagation speed in the model allowed us to substitute the full evolution operator by a finite restriction in the computation of the transition probabilities between two lattice sites up to some finite time.

Next, we developed a transfer matrix approach for general one-dimensional quantum walks consisting of the standard shift and some position dependent unitary coin operation. Via the Cauchy transform of the spectral measure this allowed us to express the matrix elements of the resolvent of such a walk operator in terms of its transfer matrices.

Using the theory of products of random matrices developed in chapter 3, we have studied the properties of these transfer matrices. The two main results we established are the initial scale estimate in lemma 5.7.11 and the Hölder continuity of the Lyapunov exponent in lemma 5.7.15.

The first of these results provided us with a minimal length scale on which with high probability the absolute value of the matrix elements of the resolvent decay exponentially inside an interval of quasi-energies. This length scale constitutes the starting hypothesis of the multiscale analysis conducted in the next chapter to prove dynamical localization. The Hölder continuity of the Lyapunov exponent on the other hand is required to establish Hölder continuity of the integrated density of states. This regularity condition enabled us to show that with high probability the eigenvalues of an disordered walk operator do not cluster and that it is very unlikely that two independent realizations of a disordered quantum walk have a common eigenvalue. Such estimates are called Wegner bounds and they are the second prerequisite for the application of the multiscale analysis. In order to allow for a clearer presentation of open questions and further research directions, we postpone this discussion to the end of chapter 6.

6. Proof of dynamical localization

In this chapter we continue our analysis of disordered quantum walks. Our main goal is to prove dynamical localization for a wide class of local coin distributions. However, before we can finish this task in section 6.5, we have to lay some groundwork in the first four sections. The final result about the exponential decay of the transition probability between distant lattice sites relies on a so called multiscale analysis that iteratively establishes the decay on increasing length scales once we can guarantee some decay for a fixed distance.

The abstract form of this iteration procedure is proven in section 6.4, but in order to apply it to the setting of disordered quantum walks, we have to certify two assumptions. The first one, the initial scale estimate that provides some starting distance for which the transition probabilities decay with some positive probability, has already been shown in lemma 5.7.11 in the last chapter. The second ingredient, called a Wegner or decoupling estimate, bounds the probability that two independent restrictions of the quantum walk operator to different regions of the lattice share a common eigenvalue.

The preparatory work for the proof of the Wegner bound in section 6.3 is done in the first two sections of this chapter. It relies on the Hölder continuity of the integrated density of states, which we prove in section 6.2. For this purpose, we use the connection between the Lyapunov exponent and the density of states, via the Thouless formula that is provided in section 6.1.

After the proof of the main result in section 6.5, we consider some explicit examples that satisfy the assumptions of the technical main theorem 6.5.1. In particular, we show that any coin distribution with an absolutely continuous component with respect to the Haar measure exhibits dynamical localization.

6.1. Thouless formula

Our goal in this section is to express the Lyapunov exponent $\gamma(z)$ of a disordered walk operator W_ω in terms of the density of states ϑ as defined in section 5.4.1. This relation, called a Thouless formula after [Tho72], is central to our proof of the Hölder continuity of the integrated density of states in section 6.2. The precise connection between Lyapunov exponent and density of states is formulated in the following lemma.

Lemma 6.1.1 (Thouless formula)**.** *Let W_ω be the realization of a regular disordered quantum walk and ϑ its density of states. Then with probability one the Lyapunov exponent corresponding to the product of its transfer matrices $T_{l(z)}$ satisfies*

$$\gamma(z) = 2\int \vartheta(d\lambda)\log|z-e^{\mathrm{i}\lambda}| + \mathbb{E}\big(\log|a|\big) - \log|z|$$

For the proof of this lemma we need the following proposition about the eigenvalues of the finite unitary restrictions $W_\omega(N)$ of a disordered walk operator W_ω.

Proposition 6.1.2. *Let W_ω be a realization of a regular disordered quantum walk and $W_\omega(N)$ its unitary finite restriction to the sites $-N$ to N according to definition 5.4.1. Then z is an eigenvalue of $W_\omega(N)$ if and only if it is a root of the polynomial*

$$p(z) = z^{2N+1} \langle \phi_{\mathbf{R}}^{\perp}, \mathrm{T}_{-N}(z) \cdots \mathrm{T}_{N}(z) \phi_{\mathbf{L}} \rangle \ ,$$

where $\phi_{\mathbf{R}}^{\perp} = (-1, z e^{\mathrm{i}\eta_R})/\sqrt{2}$ and $\phi_{\mathbf{L}} = (1, z e^{-\mathrm{i}\eta_L})/\sqrt{2}$ and η_R and η_L are the right respectively left boundary phases of the finite restriction $W_\omega(N)$.

Proof. It is clear that $z \in \mathbb{T}$ is an eigenvalue of the finite matrix $W_\omega(N)$ if and only if z satisfies the eigenvalue equation $(W_\omega(N) - z)\phi = 0$ for some vector ϕ. As in the infinite case this implies the relation

$$\begin{pmatrix} \phi_{(2l+1)} \\ \phi_{(2l+2)} \end{pmatrix} = T_l(z) \begin{pmatrix} \phi_{(2l-1)} \\ \phi_{(2l)} \end{pmatrix} \tag{6.1}$$

between the vector components of ϕ, where $T_l(z)$ is a transfer matrix according to (5.25). In addition, it follows from definition 5.4.1 that at the left and right boundary of $W_\omega(N)$, the vector ϕ has to respect the boundary conditions

$$z\phi_{(-(2N+1))} = e^{\mathrm{i}\eta_L} \phi_{(-2N)} \quad \text{and} \quad z\phi_{(2N+1)} = e^{\mathrm{i}\eta_R} \phi_{(2N)} \ .$$

Therefore, the two most right respectively left components of ϕ have to be equal to a multiple of the vector $\phi_{\mathbf{L}} = (1, z e^{\mathrm{i}\eta_L})/\sqrt{2}$ on the left and to the vector $\phi_{\mathbf{R}} = (1, z e^{\mathrm{i}\eta_R})/\sqrt{2}$ on the right boundary. By (6.1) we can express $\phi_{\mathbf{R}}$ as the product of the transfer matrices $T_{-N}(z) \cdots T_N(z)$ applied to $\phi_{\mathbf{L}}$ and by noting that the vector $\phi_{\mathbf{R}}^{\perp}$ is orthogonal to $\phi_{\mathbf{R}}$ we arrive at the desired relation

$$\langle \phi_{\mathbf{R}}^{\perp}, T_{-N}(z) \cdots T_N(z) \phi_{\mathbf{L}} \rangle = \langle \phi_{\mathbf{R}}^{\perp}, \phi_{\mathbf{R}} \rangle = 0$$

To see that $p(z)$ is a polynomial consider the definition of the transfer matrices $T_l(z)$

$$T_l(z) = \frac{1}{a_l} \begin{pmatrix} \frac{\det U_l}{z} & c_l \\ -b_l & z \end{pmatrix} .$$

The matrices T_l depend only on the powers of z and z^{-1} and this is of course also true for their product. The highest power of z^{-1} of such a product is accumulated in the first diagonal element of the product and since every transfer matrix contributes exactly one factor, the highest power of z^{-1} is equal to $2N+1$. Hence, multiplying the scalar product by z^{2N+1} makes it indeed a polynomial with respect to the variable z. To determine the order of $p(z)$ note that the second diagonal element of the product of transfer matrices acquires the highest power of z namely one per factor $T_l(z)$. Note that there is an additional contribution of two by the two vectors $\phi_{\mathbf{L}}$ and $\phi_{\mathbf{R}}$. If we take also into account the z^{2N+1} prefactor, we see that the order of $p(z)$ is $4(N+1)$, which finishes the only if part. □

Proof of Thouless formula. Since $p(z)$ is a polynomial of order $4(N+1)$ that vanishes exactly on the $4(N+1)$ eigenvalues of the finite restriction $W_\omega(N)$, it is proportional to the characteristic polynomial of $W_\omega(N)$. More precisely, we will show the identity

$$p(z) = e^{-\mathrm{i}(\eta_R+\eta_L)} \prod_{k=-N}^{N} a_k^{-1} \prod_{l=1}^{4(N+1)} (z - e^{\theta_l}) \,, \tag{6.2}$$

where $\{\theta_l\}$ denote the eigenvalues of $W_\omega(N)$. From the definition of the $T_l(z)$ we see that the $z^{4(N+1)}$ contribution to $p(z)$ comes from the second diagonal element of the transfer matrix product. Every factor T_l contributes only a prefactor a_l^{-1} and the vectors $\phi_{\mathbf{L}}$ and $\phi_{\mathbf{R}}$ account for the additional phase factor $e^{-\mathrm{i}(\eta_R+\eta_L)}$, which proves the relation in (6.2).

Using the definition of $p(z)$ together with (6.2), taking the logarithm of the absolute value on both sides and dividing by $4(N+1)$ gives for $|z| \neq 1$ the relation

$$\begin{aligned} \frac{2N+1}{4(N+1)} \ln|z| + \frac{\ln|\langle \phi_{\mathbf{R}}^{\perp}, \mathrm{T}_{-N}(z)\cdots\mathrm{T}_{-N}(z)\phi_{\mathbf{L}}\rangle|}{4(N+1)} \\ = \frac{1}{4(N+1)} + \sum_{k=-N}^{N} \frac{\ln|a_k|}{4(N+1)} + \sum_{l=1}^{4(N+1)} \frac{\ln|z-e^{\theta_l}|}{4(N+1)} \,. \end{aligned} \tag{6.3}$$

Let us now analyze the behaviour of the different terms in the limit $N \to \infty$. The first terms of both sides are easy and tend to $\frac{1}{2}\ln|z|$ respectively 0. Since all the requirements of lemma 3.3.12 are met, the second term on the left-hand side will converge to the Lyapunov exponent $\frac{1}{2}\gamma(z)$ with probability one. Since the disordered walk is regular, we know from our discussion in section 5.7 that the transfer matrices $T_l(z)$ are non singular and $0 < |a| < 1$ so by the law of large numbers the second term on the right-hand side will converge to the expectation value $\frac{1}{2}\mathbb{E}(\ln a)$ in the limit.

This leaves us with the last term on the right-hand side of (6.3). Using functional calculus and the definition of the density of states (see (5.13)) we can express this term by the following integral

$$\sum_{l=1}^{4(N+1)} \frac{\ln|z-e^{\theta_l}|}{4(N+1)} = \frac{1}{4(N+1)} \operatorname{tr} \ln|z - W_\omega(N)| = \int \vartheta_N(d\lambda) \ln|z - e^{\mathrm{i}\lambda}| \,.$$

This expression converges to an integral over the density of states θ of the unrestricted disordered quantum walk W_ω for $N \to \infty$. Inserting everything into 6.3, we obtain the Thouless formula for all complex z with $|z| \neq 1$

$$\gamma(z) = 2\int \vartheta(d\lambda) \log|z - e^{\mathrm{i}\lambda}| + \mathbb{E}\big(\log|a|\big) - \log|z| \,. \tag{6.4}$$

We can extend the equality to $z \in \mathbb{T}$ by adapting an argument due to Craig and Simon from the Hamiltonian case [CS83]. Recall that a subharmonic function is a function from $\mathbb{C}$ into $\mathbb{R}$ that is upper semicontinuous and submean. The later property meaning that the value at any point $z \in \mathbb{C}$ is upper bounded by the average of the function

taken with respect to any circle around z [Gar81]. In our context we need that two subharmonic functions that agree except for a set of Lebesgue measure zero, must be equal everywhere on $\mathbb{C}$ [CL90]. Therefore, the proof is finished, if we can show that both sites of (6.4) constitute subharmonic functions.

For the Lyapunov exponent this follows because a product of n transfer matrices $T_{n(z)} \cdot \ldots \cdot T_1(z)$ is a holomorphic function of z and therefore

$$\gamma_n(z) = \frac{1}{n} \mathbb{E}\left(\log \| T_{n(z)} \cdot \ldots \cdot T_{1(z)} \|\right)$$

is subharmonic (see (3.4) and [CS83]). In turn $\gamma(z)$ as the pointwise limit of a sequence of locally lower bounded functions is also a subharmonic function[CL90].

In general, $\log|f(z)|$ is subharmonic for any analytic function f [Gar81], so we only have to discuss the integral part on the right-hand side of (6.4). Since $\log|z - e^{\mathrm{i}\lambda}|$ is subharmonic for every λ, Fubini's theorem implies

$$\int_{-\pi}^{\pi} \mathrm{d}\eta \int \nu(\mathrm{d}\lambda) \log|z + r e^{\mathrm{i}\eta} - e^{\mathrm{i}\lambda}| \geq \int \nu(\mathrm{d}\lambda) \log|z - e^{\mathrm{i}\lambda}| =: g(z)$$

therefore $g(z)$ is submean and it is also easy to see that $g(z)$ is lower semicontinuous [CS83], which finishes the proof. □

In the next section we use the Thouless formula in order to show the Hölder continuity of the integrated density of states.

6.2. Integrated density of states

The Thouless formula connects the Lyapunov exponent of the transfer matrices to the density of states ϑ of the disordered quantum walk W_ω. This connection allows us to transfer the Hölder continuity of the Lyapunov exponent to the integrated density of states, which we define for $\Theta \in \mathbb{T}$ as

$$\mathcal{N}(\Theta) := \int_0^{\arg(\Theta)} \nu(\mathrm{d}\lambda) \ .$$

We show the following proposition.

Proposition 6.2.1. *Let W_ω be a regular disordered quantum walk such that for some interval $\mathfrak{I} \subset \mathbb{T}$ the image measure of its local coin distribution μ on $\tau_\Theta(\mathcal{U}_{nd})$ is strongly irreducible, non-compact and ζ-integrable for all $\Theta \in \mathfrak{I}$. Then, the integrated density of states $\mathcal{N}(z)$ is Hölder continuous on $\mathcal{I}$, i.e. there exists a finite constant C together with some $\alpha > 0$ such that*

$$|\mathcal{N}(\Theta) - \mathcal{N}(\Theta')| \leq C|\Theta - \Theta'|^\alpha \quad \forall\, \Theta, \Theta' \in \mathcal{I} \ .$$

Proof. First, we show that the integrated density of states is continuous following an argument by Craig and Simon for log-Hölder continuity [CS83, GT94]. Let us choose

$\Theta_1, \Theta_2 \in \mathbb{T}$ with $|\Theta_2 - \Theta_1| < 4/5$ and assume without loss of generality that $\arg\Theta_1 < \arg\Theta_2$. From the Thouless formula (lemma 6.1.1), we obtain

$$\begin{aligned} 0 \leq \gamma(\Theta_1) &= \frac{1}{2}\mathbb{E}\big(\log|a|\big) + \int \vartheta(\mathrm{d}\lambda)\, \log|\Theta_1 - e^{\mathbb{i}\lambda}| \\ &= \frac{1}{2}\mathbb{E}\big(\log|a|\big) + \int_{\Theta_1}^{\Theta_2} \vartheta(\mathrm{d}\lambda) \log|\Theta_1 - e^{\mathbb{i}\lambda}| + \int\limits_{\substack{|\Theta_1 - \exp \mathbb{i}\lambda| < 1 \\ \lambda < \arg\Theta_1 \cup \lambda > \arg\Theta_2}} \vartheta(\mathrm{d}\lambda) \log|\Theta_1 - e^{\mathbb{i}\lambda}| \\ &\qquad + \int_{\Theta_1}^{\Theta_2} \vartheta(\mathrm{d}\lambda) \log|\Theta_1 - e^{\mathbb{i}\lambda}| + \int\limits_{1 \leq |\Theta_1 - \exp \mathbb{i}\lambda|} \vartheta(\mathrm{d}\lambda) \log|\Theta_1 - e^{\mathbb{i}\lambda}| \ . \end{aligned}$$

We see that the second integral in the last expression is negative and therefore, we find the relation

$$\begin{aligned} 0 \leq -\log|\Theta_1 - \Theta_2| &(\mathcal{N}(\Theta_2) - \mathcal{N}(\Theta_1)) \\ &\leq \frac{1}{2}\mathbb{E}\big(\log|a|\big) + \int\limits_{1 \leq |\Theta_1 - \exp \mathbb{i}\lambda|} \vartheta(\mathrm{d}\lambda) \log|\Theta_1 - e^{\mathbb{i}\lambda}| \leq \frac{1}{2}\mathbb{E}\big(\log|a|\big) + \log 2 \ , \end{aligned}$$

which implies continuity and even log-Hölder continuity for the integrated density of states. We now strengthen this regularity property to Hölder continuity on $\mathcal{I}$. To this end, let us define for $a \in \mathbb{T}$ and $\phi \in [0, \pi]$ the truncated density of states

$$\widehat{\mathcal{N}}_{a,\phi}(z) := \mathcal{N}(z) \chi_{\{b \in \mathbb{C};\, \arg(b\overline{a}) < \phi\}}(z)$$

and study its Cauchy transform for $z \in \mathbb{D}$ with $|\arg(z \cdot \overline{a})| \leq \phi/2$, for which $\mathcal{N}(z) = \widehat{\mathcal{N}}_{a,\phi}(z)$ holds. We have just shown that the density of states is continuous for z inside the unit disc and therefore we can integrate by parts in order to obtain

$$\begin{aligned} (K\widehat{\mathcal{N}}_{a,\phi})(z) &= \int_{\mathbb{T}} m(\mathrm{d}\Theta)\, \frac{\widehat{\mathcal{N}}_{a,\phi}(\Theta)}{1 - \overline{\Theta} z} \\ &= \mathcal{N}(a e^{\mathbb{i}\phi}) \frac{\log(1 - a e^{\mathbb{i}\phi})}{2\pi\mathbb{i}} - \mathcal{N}(a e^{-\mathbb{i}\phi}) \frac{\log(1 - a e^{-\mathbb{i}\phi})}{2\pi\mathbb{i}} \\ &\quad + \int_{\mathbb{T}} \vartheta(\mathrm{d}\Theta)\, \log(1 - \Theta\overline{z}) - \int_{|\arg\Theta\overline{a}| > \phi} \vartheta(\mathrm{d}\Theta)\, \log(1 - \Theta\overline{z}) \ . \end{aligned} \tag{6.5}$$

For the complex logarithm in this expression $\log(r e^{\mathbb{i}\phi}) = \log r + \mathbb{i}\theta$, we choose $\theta \in [-\pi, \pi)$ if $\arg a \leq \phi$ and $\theta \in [0, 2\phi)$ if $\arg a > \phi$. As a continuous and bounded function on $\mathbb{T}$ we know by Carlson's theorem that the Fourier series of $\mathcal{N}$ converges pointwise almost everywhere on $\mathbb{T}$ to $\mathcal{N}(z)$. The Cauchy transform truncates the negative powers of a Fourier series (see 2.3.1). Since $\mathcal{N}$ is a real function, its Fourier coefficients satisfy $f_n = \overline{f_{-n}}$ and we obtain the relation $\mathcal{N}(z) = (K\mathcal{N})(z) + \overline{(K\mathcal{N})(z)}$.

Hence, we can establish Hölder continuity of the integrated density of states in (6.5) if the real part of every summand is Hölder continuous for $|z|$ going to one. This is

certainly true for the first two and the last term, so we only have to consider the third term in (6.5). Rewriting the expression in terms of its real and imaginary part we obtain

$$\int_{\mathbb{T}} \vartheta(\mathrm{d}\Theta)\,\log(1-\Theta\overline{z}) = \int_{\mathbb{T}} \vartheta(\mathrm{d}\Theta)\,\log|1-\Theta\overline{z}| + \mathbb{i}\int_{\mathbb{T}} \vartheta(\mathrm{d}\Theta)\,\arg(1-\Theta\overline{z}) - \mathbb{i}\lambda_0 \ ,$$

where λ_0 is a real constant depending on the branch of the logarithm we take. Hence, the real part consists only of the first integral, which can be related via the Thouless formula (6.1.1) to the Lyapunov exponent $\gamma(z)$. Since $\gamma(z)$ is Hölder continuous on $\mathcal{I}$ by lemma 5.7.15 this implies Hölder continuity for the integrated density of states $\mathcal{N}$. □

Before we turn to the proof of the Wegner bound in the next section we derive an immediate consequence of the Hölder continuity of the density of states. Roughly speaking this property implies that the probability to find an eigenvalue corresponding to a localized eigenfunction within a small arc of the unit circle is generically small and therefore the eigenvalues cannot cluster arbitrarily close together. More precisely we show the following lemma the proof of which is an adaption from the self-adjoint case as considered in [CKM87a, DSS02].

Lemma 6.2.2. *Let W_ω be a regular disordered quantum walk such that for some interval $\mathcal{I} \subset \mathbb{T}$ the image measure of its local coin distribution μ on $\tau_\Theta(\mathcal{U}_{nd})$ is strongly irreducible, non-compact and ζ-integrable for all $\Theta \in \mathcal{I}$. Then, there are positive constants η and C such that for all $\theta_0 \in \mathcal{I}$ and all $0 < \varepsilon < 1$ we have*

$$\begin{aligned}\mathbb{P}\Big(&\exists \theta \in I(\theta_0,\varepsilon) \text{ and } \exists \phi \in \mathbb{C}^{4(N+1)} \text{ with } \|\phi\| = 1 \text{ such that}\\ &(W_\omega(N)-\theta)\phi = 0 \text{ and } |\phi(-2N-1)|^2 + |\phi(2N+2)|^2 \le \varepsilon^2\Big) \le 4(N+2)C\varepsilon^\eta \ ,\end{aligned}$$

where $W_\omega(N)$ denotes again the unitary restriction of W_ω to the $2N+3$ lattice sites centered around the origin.

Proof. Let us denote by $A_N(0)$ the event,

$$\begin{aligned}A_N(0) := \{&\exists \theta \in I(\theta_0,\varepsilon) \text{ and } \phi \in \mathbb{C}^{4(N+1)} \text{ with } \|\phi\| = 1 \text{ such that}\\ &(W_\omega(N)-\theta)\phi = 0 \text{ and } |\phi(-2N-1)|^2 + |\phi(2N+2)|^2 \le \varepsilon^2\} \ ,\end{aligned}$$

where we indicate by 0 that we consider the restriction $W_\omega(N)$ of W_ω around the origin and hide its implicit dependence on θ_0 and ε. We want to bound the probability of the event $A_N(0)$. Instead of the event $A_N(0)$ we might also consider the event $A_N(k)$ where we consider the restriction of W_ω around another lattice site k. If we choose the sequence $2k(N+2)$ with $k \in \mathbb{Z}$ the unitary restrictions $W_\omega(2k(N+2),N)$ do not overlap. Since the local coin distributions μ are independent and identically distributed so are the events $A_N(2k(N+2))$ and we might consider them as independent realizations of the same random variable. Hence, we can express the probability of the event $A_N(0)$ as the limit of the relative frequencies of the these events

$$\mathbb{P}(A_N(0)) = \lim_{L\to\infty} \frac{1}{2L+1} |\{k \in \{-L,\dots,L\} : A_N(2k(N+1)) \text{ occurs}\}| \ , \tag{6.6}$$

where $|M|$ counts the number of elements in the set M.

By a unitary variation of the Kato-Temple inequality from the self-adjoint case [ST85], which we prove appendix C.2, the number of occurrences of the event A_N is a lower bound for the number of eigenvalues of the unitary restriction $W_\omega(0,2L(N+2)+N+1)$ of the walk operator. Now, fix some $L\in\mathbb{N}$, let $\{k_l\}\subset\{-L,\dots,L\}$ be the set of k for which the event $A_N(k)$ occurs and denote by θ_{k_l} and Φ_{k_l} the corresponding eigenvalues and eigenvectors of $W(k_l(2N+2),N)$.

By setting their vector components equal to zero outside their domain of definition we can extend the vectors Φ_{k_l} to $\mathbb{C}^{4(N+2)(2L+1)}$. Due to the disjointness of their support and the nearest neighbour shift structure of the walk operator W_ω we have the relation

$$\langle\Phi_{k_l},\Phi_{k_t}\rangle=\langle\Phi_{k_l},W_\omega(0,2L(N+2)+N+1)\Phi_{k_t}\rangle=0\ \ \forall l\neq t\ .$$

In addition we obtain from the norm estimates in the definition of the events $A_N(k)$ and the structure of the W_ω the bound

$$\|(W_\omega(0,2L(N+2)+N+1)-\theta_{k_l})\Phi_{k_l}\|\leq 2\varepsilon\ .$$

Therefore, all requirements of corollary C.2.2 are met, so we know that the number of eigenvalues of the matrix $W_\omega(0,2L(N+2)+N+1)$ is lower bounded by the number of occurrences of the event A_N. Expressing the number of eigenvalues within an interval by the integrated density of states we conclude from (6.6)

$$\begin{aligned}
&\mathbb{P}(A_N(0))\\
&\leq 4(N+2)\lim_{L\to\infty}\frac{|\{\theta\in I(\theta_0,4\varepsilon):\theta\text{ eigenvalue of }W(0,2L(N+2)+N+1)\}|}{4(N+2)(2L+1)}\\
&=4(N+2)|\mathcal{N}(\theta_0+4\varepsilon)-\mathcal{N}(\theta_0-4\varepsilon)|\\
&=4(N+2)K8^\beta\varepsilon^\beta\ ,
\end{aligned}$$

where we used the Hölder continuity of $\mathcal{N}(z)$ on $\mathcal{I}$ (see lemma 6.2.1) in the last step. Setting C equal to $K8^\rho$ and $\eta=\beta$ finishes the proof. □

6.3. Wegner bound

One of the prerequisites to conduct a multiscale analysis is an estimate about the clustering of eigenvalues of different realizations of the considered random operators. Such estimates go under the name Wegner bounds going back to a paper by Franz Wegner [Weg81]. In the case of disordered quantum walks we can closely follow the strategies used in the self-adjoint case (see for example [DSS02, CKM87a] which carry over with only slight adjustments.

We begin with an estimate on the probability that the matrix elements of the transfer matrices of a disordered quantum walk decay exponentially.

Proposition 6.3.1 (Wegner estimate)**.** *Let W_ω be a regular disordered quantum walk such that for some interval $\mathfrak{I}\subset\mathbb{T}$ the image measure of its local coin distribution μ on $\tau_\Theta(\mathcal{U}_{nd})$ is strongly irreducible, non-compact and ζ-integrable for all $\Theta\in\mathfrak{I}$. Then for all*

$0 < \beta < 1$ *and for all* $\sigma > 0$ *there exists* $N_0(\beta,\sigma) \in \mathbb{N}$ *and* $\alpha(\beta,\sigma) > 0$ *such that for all* $N \geq N_0$, $\theta_0 \in \mathfrak{I}$ *and normalized vectors* $\Phi_{\mathbf{L},\mathbf{R}} \in P_2$

$$\mathbb{P}\left(\exists \theta \in I(\theta_0, e^{-\sigma(2N+1)^\beta}) : |\langle \Phi_{\mathbf{R}}, T_{-N}(\theta)\cdot \ldots \cdot T_N(\theta)\Phi_{\mathbf{L}}\rangle| < e^{-\sigma(2N+1)^\beta}\right) \leq e^{-\alpha N^\beta} \ .$$

Proof. Let us fix some $\varepsilon > 0$ and consider the case that the event

$$E_0 := \{\exists \theta \in I(\theta_0,\epsilon) : |\langle \Phi_{\mathbf{R}}, T_{-N}(\theta)\cdot \ldots \cdot T_N(\theta)\Phi_{\mathbf{L}}\rangle| < \epsilon\}$$

occurs. According to corollary 5.7.3, we can identify the sequence of transfer matrices $(T_i(\theta))_{i=-N}^N$ together with the vectors $\Phi_{\mathbf{L}}$ and $\Phi_{\mathbf{R}}$ as the matrix element of the resolvent G_z^N of the unitary restriction $\widetilde{W}_\omega(N)$ of some disordered quantum walk $\widetilde{W}_\omega$. In addition, we know that for any normal operator the inverse operator norm of the resolvent upper bounds the proximity to the spectrum [Wei80]

$$\|(A-z)^{-1}\|^{-1} \geq d(z,\sigma(A)) = \inf_{\lambda\in\sigma(A)} |\lambda - z| \ .$$

Therefore, the relation between the resolvent of a finite restriction and the product of transfer matrices implies $\|(\widetilde{W}_\omega(N)-\theta)^{-1}\|^{-1} \leq \epsilon$ for some $\theta \in I(\theta_0,\epsilon)$. Hence, there has to be an eigenvalue $\widetilde{\theta}$ of $\widetilde{W}_\omega(N)$ with $\widetilde{\theta} \in I(\theta_0, 2\epsilon)$. Let $\tilde{\phi}$ be its corresponding normalized eigenvector.

We would like to apply lemma 6.2.2 so we have to ensure that $|\widetilde{\phi}_{(-2N-1)}|$ and $|\widetilde{\phi}_{(2N+2)}|$ are both smaller than ε. This can be achieved in the following way. Note that due to our choice of boundary conditions for $W_\omega(N)$ the vectors

$$\Phi^{\mathbf{L}}_{\widetilde{\theta}} := \begin{pmatrix} \widetilde{\phi}_{(-2N-1)} \\ \widetilde{\phi}_{(-2N)} \end{pmatrix} \quad \text{and} \quad \Phi^{\mathbf{R}}_{\widetilde{\theta}} := \begin{pmatrix} \widetilde{\phi}_{(2N+1)} \\ \widetilde{\phi}_{(2N+2)} \end{pmatrix}$$

both lie in the plane P_2. Let us denote by $\widehat{\Phi}^{\mathbf{L},\mathbf{R}}_{\widetilde{\theta}}$ normalized vectors in P_2 proportional to $\Phi^{\mathbf{L},\mathbf{R}}_{\widetilde{\theta}}$. Because $\widetilde{\phi}$ as an eigenvector solves the eigenvalue equation

$$(\widetilde{W}_\omega(N) - \widetilde{\Theta})\widetilde{\phi} = 0 \ ,$$

we can express its components in terms of the transfer matrices in two equivalent ways either starting from the left or the right boundary via

$$\begin{pmatrix} \widetilde{\phi}_{(2l-1)} \\ \widetilde{\phi}_{(2l)} \end{pmatrix} = \frac{1}{C_{\text{norm}}} T_l(\widetilde{\Theta})\cdot \ldots \cdot T_{-N}(\widetilde{\Theta})\,\widehat{\Phi}^{\mathbf{L}}_{\widetilde{\theta}} = \frac{1}{C_{\text{norm}}} T_l^{-1}(\widetilde{\Theta})\cdot \ldots \cdot T_N^{-1}(\widetilde{\Theta})\,\widehat{\Phi}^{\mathbf{R}}_{\widetilde{\theta}} \ .$$

The normalization constant C_{norm} in turn admits the two representations

$$C^2_{\text{norm}} = \sum_{i=-N}^{N} \| T_i(\widetilde{\Theta})\cdot \ldots \cdot T_{-N}(\widetilde{\Theta})\,\widehat{\Phi}^{\mathbf{L}}_{\widetilde{\theta}}\|^2 = \sum_{i=-N}^{N} \| T_i^{-1}(\widetilde{\Theta})\cdot \ldots \cdot T_{-N}^{-1}(\widetilde{\Theta})\,\widehat{\Phi}^{\mathbf{R}}_{\widetilde{\theta}}\|^2$$

and therefore each summand on the middle- and right-hand side is a lower bound for C_{norm}. Bringing everything together we find the following upper bounds on the most left- respectively right vector components of the normalized eigenvector $\widetilde{\phi}$

$$|\widetilde{\phi}_{(2N-1)}| = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} \widetilde{\phi}_{(-2N-1)} \\ \widetilde{\phi}_{(-2N)} \end{pmatrix} \right\| \leq \frac{\|\widehat{\Phi}^{\mathbf{L}}_{\widetilde{\theta}}\|}{C_{\text{norm}}} \leq \frac{1}{\sqrt{2}} \| T_i(\widetilde{\Theta}) \cdot \ldots \cdot T_{-N}(\widetilde{\Theta}) \, \widehat{\Phi}^{\mathbf{L}}_{\widetilde{\theta}} \|^{-1}$$
$$|\widetilde{\phi}_{(2N+2)}| = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} \widetilde{\phi}_{(2N+1)} \\ \widetilde{\phi}_{(2N+2)} \end{pmatrix} \right\| \leq \frac{\|\widehat{\Phi}^{\mathbf{L}}_{\widetilde{\theta}}\|}{C_{\text{norm}}} \leq \frac{1}{\sqrt{2}} \| T_i^{-1}(\widetilde{\Theta}) \cdot \ldots \cdot T_N^{-1}(\widetilde{\Theta}) \, \widehat{\Phi}^{\mathbf{R}}_{\widetilde{\theta}} \|^{-1} \ ,$$

which are valid for all $-N \leq i \leq N$. So in order to show an upper bound on the vector components it suffices to show a lower bound on the respective norms on the right-hand sides of the equations.

Let us now fix two positive constants τ and β and define the integer $n_N = \lfloor \tau N^\beta \rfloor + 1$. The preceding discussion motivates the definition of the following two events

$$\begin{aligned} E_0 &:= \{ \exists \theta \in I(\theta_0, \epsilon) : |\langle \Phi_{\mathbf{R}}, T_{-N}(\theta) \cdot \ldots \cdot T_N(\theta) \Phi_{\mathbf{L}} \rangle| < \epsilon \} \\ A_{k,N}(\theta) &:= \{ \| T_{-N+n_N}(\theta) \cdot \ldots \cdot T_{-N}(\theta) \Phi^{\mathbf{L}}_{\widetilde{\theta}} \| > e^{kN^\beta} \} \\ B_{k,N}(\theta) &:= \{ \| T^{-1}_{N-n_N}(\theta) \cdot \ldots \cdot T^{-1}_{N}(\theta) \Phi^{\mathbf{R}}_{\widetilde{\theta}} \| > e^{kN^\beta} \} \ . \end{aligned}$$

For two arbitrary events E_1 and E_2 one can easily verify the inequality

$$\mathbb{P}(E_1) = \mathbb{P}(E_1 \cap E_2) + \mathbb{P}\left(E_1 \cap E_2^c\right) \leq \mathbb{P}(E_1 \cap E_2) + \mathbb{P}\left(E_2^c\right) \ . \tag{6.7}$$

For the sake of clarity let us also introduce the quantities $\kappa = \frac{\tau\alpha}{4\delta}$ and $\epsilon = e^{-\sigma(2N+1)^\beta}$, where α and δ are positive constants to be chosen later. Using (6.7) we can upper bound the probability of the lemma by a sum of five terms that we will control individually

$$\begin{aligned} &\mathbb{P}(\exists \theta \in I(\theta_0, \epsilon) : |\langle \Phi_{\mathbf{R}}, T_{-N}(\theta) \cdot \ldots \cdot T_N(\theta) \Phi_{\mathbf{L}} \rangle| < \epsilon) \\ &\quad \leq \mathbb{P}\left(E_0 \cap \bigcap_{\theta \in I(\theta_0, 3\epsilon)} \left(A_{\kappa,N}(\theta) \cap B_{\kappa,N}(\theta) \right) \right) && (I) \\ &\quad + \mathbb{P}\left(E_0 \cap A_{2\kappa,N}(\theta_0) \cap B_{2\kappa,N}(\theta_0) \cap \bigcup_{\theta \in I(\theta_0, 3\epsilon)} (A_{\kappa,N}(\theta))^c \right) && (II) \\ &\quad + \mathbb{P}\left(E_0 \cap A_{2\kappa,N}(\theta_0) \cap B_{2\kappa,N}(\theta_0) \cap \bigcup_{\theta \in I(\theta_0, 3\epsilon)} (B_{\kappa,N}(\theta))^c \right) && (III) \\ &\quad + \mathbb{P}\left(E_0 \cap (A_{2\kappa,N}(\theta_0))^c \right) + \mathbb{P}\left(E_0 \cap (B_{2\kappa,N}(\theta_0))^c \right) && (IV) \,\&\, (V) \ . \end{aligned}$$

We discussed the term (I) already in great detail and analyzed the occurrence of an eigenvalue in the interval $I(\theta_0, 3\epsilon)$. As we have seen, the expression can be directly

upper bounded by employing lemma 6.2.2. Therefore, we find for (I) the upper bound

$$\begin{aligned}(I) \leq \mathbb{P}\Big(&\exists \Theta \in I(\theta_0, 3\epsilon) \text{ and } \phi \in \mathbb{C}^{4(N+1)},\ \|\phi\| = 1,\ \text{such that} \\ &(W_\omega(N) - \Theta)\phi = 0 \text{ and } |\phi(1)|^2 + |4(N+1)|^2 < e^{\kappa N^\beta}\Big) \\ &\leq 4(N+2)C(\max(3\varepsilon, e^{-\kappa N^\beta}))^\rho \ .\end{aligned} \tag{6.8}$$

Let us continue with the terms (IV) and (V). Since in both cases we have to deal with a product of n_N transfer matrices applied to a vector Φ the same arguments will be valid for both cases and we only consider (IV) for the moment. To get rid of the event E_0 in the probability, we take the supremum of the probability with respect to all possible vectors Φ instead of just considering $\Phi^{\mathbf{L}}_{\tilde{\theta}}$:

$$\mathbb{P}\left(E_0 \cap \|T_{-N+n_N}(\theta)\cdot \ldots \cdot T_{-N}(\theta)\Phi^{\mathbf{L}}_{\tilde{\theta}}\| > e^{\kappa N^\beta}\right) \leq \sup_{\|\Phi\|=1} \mathbb{P}\left(\|T_{-N+n_N}(\theta)\cdot \ldots \cdot T_{-N}(\theta)\Phi\| > e^{\kappa N^\beta}\right) \ .$$

Raising the terms inside the probabilities on both sides of the inequality to the negative power of δ, applying Markov's inequality and inserting the definition of κ, we find

$$\begin{aligned}(IV) &\leq \sup_{\|\phi\|=1} \mathbb{P}\left(\|T_{-N+n_N}(\theta)\cdot \ldots \cdot T_{-N}(\theta)\Phi\|^{-\delta} \geq e^{-\frac{\tau\alpha}{2}N^\beta}\right) \\ &\leq \sup_{\|\phi\|=1} \mathbb{E}\left(\|T_{-N+n_N}(\theta)\cdot \ldots \cdot T_{-N}(\theta)\Phi\|^{-\delta}\right) e^{\frac{\tau\alpha}{2}N^\beta} \ .\end{aligned}$$

If we now choose α and δ according to proposition 5.7.12 we can bound the expectation value by $e^{-\alpha n_N}$ which amounts to the following bound on (IV) and (V)

$$(IV) + (V) \leq 2e^{-\frac{\alpha}{2}N^\beta} \ . \tag{6.9}$$

for N large enough. This leaves us to deal with the terms (II) and (III). We first turn to (II). In the case that the event in (II) occurs, there is a $\Theta \in I(\theta_0, 3\epsilon)$ such that the event $(A_{k,N}(\Theta))^C$ together with the event $A_{2k,N}(\Theta_0)$ occurs. This directly implies the following lower bound on the difference of the products of transfer matrices applied to the initial vector $\Phi^{\mathbf{L}}_{\tilde{\theta}}$

$$\begin{aligned}&\big|\|T_{-N+n_N}(\Theta_0)\cdot \ldots \cdot T_{-N}(\Theta_0)\Phi^{\mathbf{L}}_{\tilde{\theta}}\| - \|T_{-N+n_N}(\Theta)\cdot \ldots \cdot T_{-N}(\Theta)\Phi^{\mathbf{L}}_{\tilde{\theta}}\|\big| \\ &\geq e^{2\kappa N^\beta} - e^{\kappa N^\beta} \geq \frac{1}{2}e^{2\kappa N^\beta} \ ,\end{aligned} \tag{6.10}$$

where in the second step N, depending on β and κ has to be large enough. As before we remove the dependence on the event E_0 and the vector $\Phi^{\mathbf{L}}_{\tilde{\theta}}$ by taking the supremum of the probability with respect to $\Phi \in \mathbb{C}^2$. We remark, that this relation remains true, if we raise both sides to the power of some $0 < \eta < 1$. To continue, we note that for any two $\Theta, \Theta' \in \mathbb{T}$ the difference of the respective transfer matrices takes the simple form of a diagonal matrix

$$T_x(\Theta) - T_x(\Theta') = \frac{1}{a_x}\begin{pmatrix} \det(U_x)(\overline{\Theta} - \overline{\Theta}') & 0 \\ 0 & \Theta - \Theta' \end{pmatrix} \ . \tag{6.11}$$

Since both singular values of this diagonal matrix are equal $|a_x|^{-1}|\Theta_0-\Theta|$, its operator norm is also given by this value. As we have seen, the probability of the event corresponding to (6.10) being true, upper bounds (II) and applying Markov's inequality as well as the inverse triangle inequality, we find for all $0<\eta\leq 1$

$$(II)\leq 2^{\eta}e^{-2\eta\kappa N^{\beta}}\sup_{\|\phi\|=1}\mathbb{E}\left(\|(T_{-N+n_N}(\Theta_0)\cdot\ldots\cdot T_{-N}(\Theta_0)-T_{-N+n_N}(\Theta)\cdot\ldots\cdot T_{-N}(\Theta))\Phi\|^{\eta}\right)\ .$$

To control the expectation value on the right-hand side, we can proceed as in the proof of proposition 5.7.9 by decomposing the difference of the two operator products iteratively. Using the inequality $(|a|+|b|)^{\eta}\leq|a|^{\eta}+|b|^{\eta}$ and the independence of the transfer matrices, this results in the following upper bound on (II)

$$\begin{aligned}(II)\leq 2^{\eta}e^{-2\eta\kappa N^{\beta}}\mathbb{E}\Bigg(&\sum_{k=1}^{n_N+1}\|T_{-N+n_N}(\Theta_0)\cdot\ldots\cdot T_{-N+n_N-k+1}(\Theta_0)\|^{\eta}\cdot\\&\|T_{-N+n_N-k}(\Theta_0)-T_{-N+n_N-k}(\Theta)\|^{\eta}\cdot\|T_{-N+k-1}(\Theta_0)\cdot\ldots\cdot T_{-N}(\Theta_0)\|^{\eta}\Big)\\&\leq 2^{\eta}e^{-2\eta\kappa N^{\beta}}(n_N+1)\max\Big(\mathbb{E}\left(\|T_l(\Theta_0)\|^{\eta}\right)^{n_N+1},\mathbb{E}\left(\|T_l(\Theta)\|^{\eta}\right)^{n_N+1}\Big)(3\varepsilon)^{\eta}\ .\end{aligned}$$

In the second step (6.11) was used, together with the fact that $\mathbb{E}\left(|a_x|^{-\eta}\right)\leq\mathbb{E}\left(\|T(\Theta)\|^{\eta}\right)$ (see (5.35)). By assumption the single site distribution μ is ζ-integrable for some $\zeta>0$. Therefore, we can choose $\eta>0$ such that $\mathbb{E}\left(\|T(\Theta)\|^{\eta}\right)$ is finite. Let us denote that constant by C_{η}. Inserting the definition of all abbreviations we find the upper bound

$$(II)\leq 6^{\eta}(\tau N^{\beta}+1)C_{\eta}^{\lfloor\tau N^{\beta}\rfloor+1}e^{-\sigma\eta(2N+1)^{\beta}}e^{-2\frac{\eta\tau\alpha}{4\delta}N^{\beta}}\ .$$

The corresponding argument holds for the term (III) and we see that by choosing $\tau>0$ small we can find $\alpha_2>0$ such that starting from some N_0 we have for all $N\geq N_0$

$$(II)+(III)\leq 2e^{\alpha_2 N^{\beta}}\ . \tag{6.12}$$

The result now follows by combining the upper bounds in (6.8), (6.9) and (6.12). □

The formulation of the Wegner bound we are actually going to use as a building block in the multiscale analysis is the uniform version given in the following lemma. The intuitive idea is to bound the probability that two independent realizations of a disordered quantum walk share a common eigenvalue. More precisely, instead of the probability to find an eigenvalue within a small interval of the unit circle, the probability to find two close eigenvalues for two independent realizations of a disordered quantum walk is controlled.

Corollary 6.3.2 (uniform Wegner estimate)**.** *Let μ be a probability measure on the set $\mathcal{U}_{nd}$ such that for some interval $\mathfrak{I}\subset\mathbb{T}$ the image measure of μ on $\tau_{\Theta}(\mathcal{U}_{nd})$ is strongly irreducible, non-compact and ζ-integrable for all $\Theta\in\mathfrak{I}$. If $(T_{l(z)})_{l=-N}^{N}$ and $(\widetilde{T}_{l(z)})_{l=-N}^{N}$ are two sequences of transfer matrices, drawn independently and $\Phi^{\mathbf{L/R}},\widetilde{\Phi}^{\mathbf{L/R}}\in P_2$ then for all $0<\beta<1$ and $\sigma>0$ there exist $\alpha>0$ and $N_0\in\mathbb{N}$ such that for all $N\geq N_0$*

$$\begin{aligned}\mathbb{P}\Big(\exists\Theta\in\mathfrak{I}:|\langle\Phi^{\mathbf{R}},T_{N(\Theta)}\cdot\ldots\cdot T_{-N(\Theta)}\Phi^{\mathbf{L}}\rangle|<e^{-\sigma(2N+1)^{\beta}}\ \text{and}\\|\langle\widetilde{\Phi}^{\mathbf{R}},\widetilde{T}_{N(\Theta)}\cdot\ldots\cdot\widetilde{T}_{-N(\Theta)}\widetilde{\Phi}^{\mathbf{L}}\rangle|<e^{-\sigma(2N+1)^{\beta}}\Big)\leq e^{-\alpha(2N+1)^{\beta}}\ .\end{aligned}\tag{6.13}$$

Proof. The proof consists of a straight forward application of the union bound. We know by corollary 5.7.3 that both scalar products in (6.13) can be identified with the matrix element of the resolvent of some unitary restrictions $W_\omega(N)$ and $\widetilde{W}_\omega(N)$. From our argument to bound the probability of the term (I) in the proof of the Wegner bound we know that if the event

$$|\langle \Phi^{\mathbf{R}}, T_N(z)\cdot\ldots\cdot T_{-N}(z)\Phi^{\mathbf{L}}\rangle| < e^{-\sigma(2N+1)^\beta}$$

occurs for some $\Theta_0 \in \mathfrak{I}$, an eigenvalue of $W_\omega(N)$ has to lie the interval $I(\theta_0, e^{-\sigma(2N+1)^\beta})$. As a finite dimensional matrix acting on $\mathbb{C}^{4(N+1)}$, $W_\omega(N)$ can have at most $4(N+1)$ eigenvalues. Since by assumption the two sequences of transfer matrices and therefore $W_\omega(N)$ and $\widetilde{W}_\omega(N)$ are independent we can apply the union bound and find $\Theta_0 \in \mathfrak{I}$ for N large enough and suitable $\alpha' > 0$

$$\begin{aligned}(6.13) &\leq 4(N+1)\mathbb{P}\left(\exists \Theta \in I(\theta_0, e^{-\sigma(2N+1)^\beta}) : |\langle \widetilde{\Phi}^{\mathbf{R}}, \widetilde{T}_N(\Theta)\cdot\ldots\cdot\widetilde{T}_{-N}(\Theta)\widetilde{\Phi}^{\mathbf{L}}\rangle| < e^{-\sigma(2N+1)^\beta}\right)\\ &\leq 4(N+1)e^{-\alpha N^\beta}\\ &\leq e^{-\alpha'(2N+1)^\beta}\ ,\end{aligned}$$

where we used the Wegner estimate (prop 6.3.1) in the second step to determine α. □

6.4. Multiscale analysis

The original proof technique of a multiscale analysis was developed by Fröhlich and Spencer in the context of Anderson localization in Hamiltonian systems [FS83], which has been substantially refined and simplified [GK01, DS01, KSS98, CKM87b]. For the general idea of how a multiscale analysis can be employed to show dynamical localization in the Hamiltonian case we refer to section 2.5.3.

In the abstract formulation presented in this chapter, the method can be regarded as a probabilistic generalization of the well known mathematical induction to products of random variables. In an inductive proof one starts with the base case for some fixed value n_0 of a parameter n and then shows that if the claim is true for n it also holds for $n+1$. In a multiscale analysis the initial scale estimate for some n_0 corresponding to the base case is true only with some probability. The multiscale induction step then establishes the claim for all n_0^k with high probability provided an additional assumption about the occurrence of exceptional bad instances.

The presented lemma is an adaption of techniques used in [GK01, Kle08] to the setting of disordered quantum walks. It is tailored to the initial scale estimate and Wegner-bound we could obtain in that scenario. As in [GK01] we allow the number of bad instances to scale with length scale instead of being fixed in advance. In addition, we require the length scales to grow in exact multiples of the former length scale. The abstract formulation in terms of products of random variables should make the result also applicable in other circumstances. The lemma presented here constitutes the main technical ingredient for the proof of dynamical localization given in section 6.5.

Lemma 6.4.1 (MSA Induction step)**.** *Let $\{X_\theta\}_{\theta\in\mathbb{T}}$ be a family of positive, real valued random variables that satisfies*

A1 initial scale estimate: *For all $n\in\mathbb{N}$ there exists $n_0\geq n$ and positive constants $\gamma_0(n_0)$, $\sigma(n_0)$ and a subset $I(n_0)\subset\mathbb{T}$ such that*

$$\mathbb{P}\left(\textit{for all } \theta\in I(n_0)\,:\, X_\theta(\omega) > e^{\gamma_0 n_0}\right) > 1-e^{-\sigma n_0}\,.$$

A2 decoupling estimate: *Define $X_\theta^m(\underline{\omega})$ to be the m - fold product of i.i.d realizations of the random variable X_θ:*

$$X_\theta^m(\underline{\omega}) := \prod_{i=1}^m X_\theta(\omega_i)\,.$$

Then there exists $0<c<1$ such that for all $0<\delta<1$ and $m>n_0$ the estimate

$$\mathbb{P}\left(\exists\,\theta\in I(n_0)\,:\, X_\theta^m(\underline{\omega}) < e^{-c\gamma_0 m^\delta} \text{ and } X_\theta^m(\underline{\tilde{\omega}}) < e^{-c\gamma_0 m^\delta}\right) < 3m\,e^{-\sigma m^\delta} \tag{6.14}$$

holds for two i.i.d products $X_\theta^m(\underline{\omega})$ and $X_\theta^m(\underline{\tilde{\omega}})$.

Then, for all $0<\xi<1$ there exist $1<\alpha<\xi^{-1}$ and $n_0\in\mathbb{N}$ such that for all length scales $n_k=\max_{L\in 2n_{k-1}\mathbb{N}}(L\leq n_{k-1}^\alpha)$, $k\in\mathbb{N}$, pairs of independent products of length $\frac{n_k}{2}$ satisfy

$$\mathbb{P}\left(\exists\,\theta\in I(n_0)\,:\, X_\theta^{\frac{n_k}{2}}(\underline{\omega}) < e^{\gamma_k\frac{n_k}{2}} \text{ and } X_\theta^{\frac{n_k}{2}}(\underline{\tilde{\omega}}) < e^{\gamma_k\frac{n_k}{2}}\right) \leq e^{-\sigma\left(\frac{n_k}{2}\right)^\xi} \tag{6.15}$$

for some decreasing sequence (γ_k) with $\gamma_0\geq\gamma_k\geq c\gamma_0$.

In addition, there exists for all $1<\alpha<2$ and $0<\xi<\alpha^{-1}$ a $n_0\in\mathbb{N}$ such that for all length scales n_k, $k\in\mathbb{N}$ (6.15) holds.

Proof. The proof is carried out via an induction. Assuming (6.15) for some initial $k\in\mathbb{N}$, we use assumption **A2** to establish that it also holds for $k+1$, with an adjusted growth rate γ_{k+1}, which can nevertheless be lower bounded by $c\ \gamma_0$. This requires however that, depending on the choice of α and ξ, the initial scale n_0 has to be large enough. The result then follows by applying the first induction step to a suitable initial scale n_0, which can be constructed using assumption **A1**.

It is convenient to divide the proof of the lemma into a deterministic and a probabilistic part. In the deterministic part the new growth rate γ_{k+1} is determined conditioned on two hypotheses about the value of factors X_θ in the product $X_\theta^{n_k/2}$. In the probabilistic part we bound the probability that these hypotheses are violated or more precisely we bound the probability that the new growth rate is smaller then our estimate γ_{k+1}.

Deterministic part

First we deal with the deterministic part and assume that (6.15) holds for some $n_k\in\mathbb{N}$. Now consider for $1<\alpha$ the new length scale $n_{k+1}\in 2n_k\mathbb{N}$ given by

$$n_{k+1} = \max_{L\in 2n_k\mathbb{N}}(L\leq n_k^\alpha) =: 2R_{k+1}n_k\ , \tag{6.16}$$

where $R_{k+1} \in \mathbb{N}$ is defined by the left-hand side and we demand $n_0^\alpha > 4n_0$. Note that this choice also implies the following chain of inequalities for n_{k+1}

$$n_0^{\alpha^{k+1}} \geq n_k^\alpha \geq n_{k+1} \geq n_k^\alpha - 2n_k \geq n_k^\alpha(1-2n_0^{-(\alpha-1)}) \ . \tag{6.17}$$

To ensure positivity of this lower bound, we have to require that $n_0 > 2^{\frac{1}{\alpha-1}}$. We can express the product $X_\theta^{\frac{n_{k+1}}{2}}$ of length $\frac{n_{k+1}}{2}$ via a product of products $X_\theta^{\frac{n_k}{2}}$ of length $\frac{n_k}{2}$ in the following way

$$X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega}) = \prod_{i=1}^{\frac{n_{k+1}}{2}} X_\theta(\omega_i) = \prod_{i=1}^{2R_{k+1}} X_\theta^{\frac{n^k}{2}}(\underline{\omega}_i) \ . \tag{6.18}$$

Let us denote for any real number x by $\lfloor x \rfloor$ the largest natural number that is smaller than x. For the remainder of the deterministic estimate we work under the following two hypotheses, the failure of which will be covered in the probabilistic part.

H1 Only $2\lfloor\left(\frac{n_k}{2}\right)^\beta + 1\rfloor - 1$ of the $2R_{k+1}$ factors in the product of the $X_\theta^{\frac{n_k}{2}}(\underline{\omega})$ grow with a rate smaller than γ_k, where $\alpha > 1+\beta$ and n_0 has to be large enough such that $2R_{k+1} > 2\lfloor\left(\frac{n_k}{2}\right)^\beta + 1\rfloor - 1$, which we will check in the probabilistic part of the proof.

H2 None of the $2\lfloor\left(\frac{n_k}{2}\right)^\beta + \frac{1}{2}\rfloor - 1$ bad instances from hypotheses **H1** possesses a growth rate smaller than $e^{-c\gamma_0(\frac{n_k}{2})^\delta}$, for some $0 < \delta < 1$ to be specified later.

Note, that this choice of parameters ensures $2\lfloor\left(\frac{n_k}{2}\right)^\beta + 1\rfloor - 1 \leq 2\left(\frac{n_k}{2}\right)^\beta + 1 \leq 3\left(\frac{n_k}{2}\right)^\beta$. Next we compute a lower bound γ_{k+1} on the actual growth-rate of the product $X_\theta^{\frac{n_{k+1}}{2}}$. Taking the logarithm on both sides in (6.18) we find

$$\begin{aligned} \ln(X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})) &= \sum_{i=1}^{2R_{k+1}} \ln(X_\theta^{\frac{n_k}{2}}(\omega_i)) \\ &\geq \left(2R_{k+1} - 3\left(\frac{n_k}{2}\right)^\beta\right)\frac{n_k\gamma_k}{2} - 3c\gamma_0 2^{-\delta} n_k^\delta \left(\frac{n_k}{2}\right)^\beta \ , \end{aligned} \tag{6.19}$$

where we used the hypotheses **H1** and **H2** in the second step to obtain the inequality. In order to lower bound the new growth rate we define γ_{k+1} by the right-hand side of (6.19) divided by $\frac{n_{k+1}}{2}$. This leads to the recursion relation

$$\begin{aligned} \gamma_{k+1} &= (2R_{k+1} - 2^{-\beta}3n_k^\beta)\frac{\gamma_k}{2R_{k+1}} - \frac{2^{-\delta-\beta}3c\gamma_0}{R_{k+1}} n_k^\beta n_k^{\delta-1} \\ &\geq (1 - 2^{-\beta}3\frac{n_k^\beta}{R_{k+1}})\gamma_k - \frac{2^{-\delta}3c\gamma_0}{R_{k+1}} n_k^\beta \\ &\geq \gamma_k - (1 + 2^{-\delta-\beta}3c)\gamma_0 \frac{n_k^\beta}{R_{k+1}} \ , \end{aligned} \tag{6.20}$$

because $\delta < 1$ and $\gamma_k \leq \gamma_0$. To complete the deterministic part of the multiscale analysis we have to show that the new grow rate γ_{k+1} is indeed lower bounded by $c\gamma_0$ and upper bounded by γ_0. The upper bound follows easily from the last equation considering

$$\gamma_k - \gamma_{k+1} = \frac{n_k^\beta}{R_{k+1}}(\gamma_k + 2^{-\delta-\beta}3c\gamma_0 n_k^{\delta-1}) > 0 \ .$$

Therefore, the γ_k constitute a decreasing sequence. To verify the lower bound we have to ensure

$$\gamma_0 - \gamma_{k+1} \leq (1-c)\gamma_0 \ .$$

For the proof of this lower bound we need to employ an inductive argument that takes into account all length scales down to n_0. From (6.20) we find by an telescope sum expansion

$$\gamma_0 - \gamma_{k+1} = \sum_{l=0}^{k}(\gamma_l - \gamma_{l+1}) \leq (1+2^{-\delta-\beta}3c)\gamma_0 \sum_{l=0}^{k}\frac{n_k^\beta}{R_{k+1}} \ . \tag{6.21}$$

In order to proceed we need a lower bound on R_{k+1}. Considering the definition of R_{k+1} from (6.16) and the lower bound in (6.17) we obtain

$$R_{k+1} \geq (1-2n_0^{-(\alpha-1)})\frac{n_k^{\alpha-1}}{2} \geq (1-2n_0^{-(\alpha-1)})\frac{n_0^{(\alpha-1)\alpha^k}}{2}\prod_{r=1}^{k}\left(1-2n_{k-r}^{-(\alpha-1)}\right)^{(\alpha-1)\alpha^{r-1}} \ .$$

Since $n_k > n_0$ and $\alpha > 1$ we can lower bound each factor in the product by the term $(1-2n_0^{-(\alpha-1)})$. To evaluate this product of equal factors we just sum up the exponents which constitute a finite geometric series. In total this amounts to the lower bound

$$R_{k+1} \geq (1-2n_0^{-(\alpha-1)})\frac{n_0^{(\alpha-1)\alpha^k}}{2}\left(1-2n_0^{-(\alpha-1)}\right)^{\alpha^k} \ .$$

Inserting this bound into (6.21) and using $0 < \beta < 1$ as well as $0 < \delta < 1$ we obtain

$$\gamma_0 - \gamma_k \leq \frac{(1+3c)\gamma_0}{2(1-2n_0^{-(\alpha-1)})}\sum_{l=0}^{k}\left[n_0^{\alpha-1-\beta}(1-2n_0^{-(\alpha-1)})\right]^{-\alpha^l} \ . \tag{6.22}$$

Remember that we have to show $\gamma_{k+1} \geq c\gamma_0$, which is true provided the right-hand side of (6.22) is upper bounded by $(1-c)\gamma_0$. Hence, in total we have to guarantee

$$(1-2n_0^{-(\alpha-1)})^{-1}\sum_{l=0}^{k}\left[n_0^{\alpha-1-\beta}(1-2n_0^{-(\alpha-1)})\right]^{-\alpha^l} \leq \frac{2(1-c)}{(1+3c)} \ . \tag{6.23}$$

Next we show that the sum on the left-hand side is finite if we choose n_0 large enough. First note that we can make the term inside the square brackets strictly larger than one,

provided that we choose $n_0 > \exp(\frac{2}{\alpha-1-\beta})$. Doing this and using $\alpha^k \geq k(\alpha-1)+1$ the finite sum is dominated by a convergent geometric series times a prefactor

$$\sum_{l=0}^{k}\Big[n_0^{\alpha-1-\beta}(1-2n_0^{-(\alpha-1)})\Big]^{-\alpha^l} \leq n_0^{-(\alpha-1-\beta)}(1-2n_0^{-(\alpha-1)})^{-1}\sum_{l=0}^{\infty}\Big[n_0^{\alpha-1-\beta}(1-2n_0^{-(\alpha-1)})\Big]^{-l(\alpha-1)} . \quad (6.24)$$

Evaluating the series and inserting the result into (6.23) implies the following condition on the initial length scale n_0

$$\frac{1+3c}{2(1-c)} \leq \Big(1-2n_0^{-(\alpha-1)}\Big)^2\left(1-\Big(n_0^{\alpha-1-\beta}(1-2n_0^{-(\alpha-1)})\Big)^{-(\alpha-1)}\right)n_0^{\alpha-1-\beta} .$$

Since $\alpha > 1+\beta$ by assumption, the first two factors on the left-hand side tend to one and the third factor grows for large n_0. Therefore, by making n_0 large enough we can always satisfy this condition. Employing the already known lower bound $n_0 > \exp\Big(\frac{2}{\alpha-1-\beta}\Big)$ that we need in order to ensure the convergence of the geometric series in (6.24), we find the more explicit condition

$$n_0 > \left(\frac{2(1-2)}{1+3c}\big(C-C^\alpha\big)\right)^{-\frac{1}{\alpha-1-\beta}} \quad \text{with} \quad C := 1-2e^{-\frac{2(\alpha-1)}{\alpha-1-\beta}} .$$

This completes the deterministic estimate since the new growth rate γ_{k+1} now indeed satisfies the bound $\gamma_0 \geq \gamma_{k+1} \geq c\gamma_0$.

Probabilistic part

For the probabilistic estimate, we return to (6.15) and note, that we have to bound the probability that either one of the hypotheses **H1** or **H2** is violated for two independent products $X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})$ and $X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})$. Denoting by $\complement\mathbf{Hi}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\Big]$ the event that hypotheses **Hi** fails for the product $X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})$ we obtain the estimate

$$\begin{aligned}
&\mathbb{P}\Big(\exists\theta\in I(n_0) : \text{Either } \mathbf{H1} \text{ or } \mathbf{H2} \text{ is violated for both } X_\theta^{\frac{n_{k+1}}{1}}(\underline{\omega}) \text{ and } X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})\Big)\\
&= 2\,\mathbb{P}\Big(\exists\,\theta\in I(n_0) : \complement\mathbf{H1}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\Big] \text{ and } \complement\mathbf{H2}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})\Big]\Big)\\
&+ \mathbb{P}\Big(\exists\,\theta\in I(n_0) : \complement\mathbf{H1}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\Big] \text{ and } \complement\mathbf{H1}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})\Big]\Big)\\
&+ \mathbb{P}\Big(\exists\,\theta\in I(n_0) : \complement\mathbf{H2}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\Big] \text{ and } \complement\mathbf{H2}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})\Big]\Big)\\
&\leq 3\,\mathbb{P}\Big(\exists\,\theta\in I(n_0) : \complement\mathbf{H1}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\Big]\Big)\\
&+ \mathbb{P}\Big(\exists\,\theta\in I(n_0) : \complement\mathbf{H2}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\Big] \text{ and } \complement\mathbf{H2}\Big[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})\Big]\Big) .
\end{aligned}$$

Let us begin with the term in the last line of this expression. If hypothesis **H2** fails for a product $X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})$ at least one of the $2\lfloor\left(\frac{n_k}{2}\right)+1\rfloor-1$ instances has to grow at a rate smaller then $-c\gamma_0(\frac{n_k}{2})^\delta$. The probability that this happens for two independent products of length $\frac{n_k}{2}$ however, is governed by the decoupling assumption **A2** that gives us via (6.14) the following estimate

$$\mathbb{P}\left(\exists\,\theta\in I(n_0):\mathtt{CH2}\left[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\omega})\right]\text{ and }\mathtt{CH2}\left[X_\theta^{\frac{n_{k+1}}{2}}(\underline{\tilde{\omega}})\right]\right)\leq 3\,\frac{n_k}{2}\,e^{-\sigma\left(\frac{n_k}{2}\right)^\delta}\ . \tag{6.25}$$

Now we deal with the first term in the probabilistic estimate. If **H1** is violated then at least $2\lfloor\left(\frac{n_k}{2}\right)^\beta+1\rfloor$ factors of the product $X^{\frac{n_{k+1}}{2}}(\theta,\omega)$ grow with a rate smaller than γ_k. Equivalently, we can say that at least $\lfloor\left(\frac{n_k}{2}\right)^\beta+1\rfloor$ pairs of independent products grow with a rate smaller than γ_k. Hence, by invoking the induction hypotheses from (6.15) we find

$$\begin{aligned}&3\,\mathbb{P}\left(\exists\,\theta\in I(n_0):\mathtt{CH1}\left[X^{\frac{n_{k+1}}{2}}(\theta,\omega)\right]\right)\\&\leq 3\mathbb{P}\left(\exists\,\theta\in I(n_0):X^{\frac{n_k}{2}}(\theta,\underline{\omega})<e^{\gamma_k\frac{n_k}{2}}\text{ and }X^{\frac{n_k}{2}}<e^{\gamma_k\frac{n_k}{2}}(\theta,\underline{\tilde{\omega}})\right)^{\lfloor\left(\frac{n_k}{2}\right)^\beta+1\rfloor}\leq 3\,e^{-\sigma\left(\frac{n_k}{2}\right)^{\xi+\beta}}\ ,\end{aligned} \tag{6.26}$$

since $\lfloor x+1\rfloor\geq x$ for all positive $x\in\mathbb{R}$. Combining the estimates (6.25) and (6.26) and using the fact that $n_{k+1}<n_k^\alpha$ yields for large enough n_0

$$\begin{aligned}&\mathbb{P}\left(\exists\theta\in I(n_0):\text{Hypotheses }\mathbf{H1}\text{ or }\mathbf{H2}\text{ violated for }X^{\frac{n_{k+1}}{2}}(\theta,\omega)\text{ and }X^{\frac{n_{k+1}}{2}}(\theta,\tilde{\omega})\right)\\&\qquad\leq 3\,e^{-\sigma\left(\frac{n_k}{2}\right)^{\xi+\beta}}+3\,\frac{n_k}{2}\,e^{-\sigma\left(\frac{n_k}{2}\right)^\delta}\leq e^{-\sigma\left(\frac{n_{k+1}}{2}\right)^\xi}\end{aligned}$$

provided that we can choose $\xi+\beta>\alpha\cdot\xi$ and $1>\delta>\alpha\cdot\xi$. Given $0<\xi<1$ and we can always find η satisfying $0<\eta^2<\xi<\eta<1$. Setting $\alpha=\frac{\eta}{\xi}$, first note, that $\alpha\xi<1$ so we can find $\alpha\xi<\delta<1$ for the worst case estimate in $\mathbf{H_2}$. With this choice it also follows that $0<\xi(\eta-\xi)<\eta-\xi$, which amounts to $\eta-\xi<\frac{\eta}{\xi}-1$. Pick $\eta-\xi<\beta<\frac{\eta}{\xi}-1$. The first inequality implies $\alpha<1+\beta\xi^{-1}$ and the second one $\beta+1<\alpha$ as claimed. The only thing we have to check whether β satisfies the assumptions of hypothesis **H1**. This means that we have to guarantee that

$$2R_{k+1}\geq 2\left(\frac{n_k}{2}\right)^2+1\ .$$

Multiplying both sides by n_k and using the last inequality in (6.17) this is equivalent to the condition

$$2R_{k+1}n_k=n_{k+1}\geq n_k^\alpha-2n_k>\frac{2}{2^\beta}n_k^{1+\beta}+n_k\ .$$

Hence, we have to guarantee the relation

$$n_k^{\beta+1}\left(n_k^{\alpha-\beta-1}-\frac{2}{2^\beta}-3n_k^{-\beta}\right)\geq n_k^{\beta+1}\left(n_k^{\alpha-\beta-1}-\frac{5}{2^\beta}\right)>0\ ,$$

but since $\alpha > 1+\beta$ we can always achieve this by making n_0 large enough.

Conversely, if we fix $1 < \alpha < 2$, we see that the induction step can be carried out if $\xi < \beta < \alpha - 1$ and $\xi < \alpha^{-1}$, because then $\xi + \beta > 2\xi > \alpha\xi$. So if we pick $\xi < \alpha - 1$ and $\xi < \alpha^{-1}$, we can find $0 < \beta, \delta < 1$ accordingly, which finishes the proof. □

6.5. Final assembly

Finally we have developed all necessary tools to prove our main result on dynamical localization of one-dimensional disordered quantum walks. The basic argument is to facilitate the initial scale estimate from chapter 5 and the Wegner estimate from section 6.3 as the prerequisites of a multiscale analysis that ensures the almost sure decay of the resolvent of the disordered quantum walk operator. Together with the connection between the transition probabilities and the resolvent given by the Cauchy transform this is sufficient to guarantee dynamical localization.

Theorem 6.5.1. *Let W_ω be a regular disordered quantum walk, such that for some $\mathfrak{I} \subset \mathbb{T}$ the image measure of its coin distribution μ on $\tau_\Theta(\mathcal{U}_{nd})$ is strongly-irreducible, non-compact and ζ-integrable for all $\Theta \in \mathfrak{I}$. Then, for all $0 < \xi < 1$ there exist $C_1, C_2 > 0$ such that for all $x, y \in \mathbb{Z}$ and $\phi, \psi \in \mathbb{C}^2$*

$$\mathbb{E}\left(\sup_t |\langle \delta_y \otimes \psi, \chi(\mathfrak{I}) W_\omega^t \ \delta_x \otimes \phi \rangle| \right) \leq C_1 e^{-C_2 \|x-y\|^\xi} \ . \tag{6.27}$$

Proof. We show that if the assumptions of the lemma are satisfied for some fixed $\Theta_0 \in \mathfrak{I}$, there is some open arc I_δ around Θ_0 on which the claim holds. The result for the full interval $\mathfrak{I}$ then follows by compactness of $\mathbb{T}$.

By proposition 5.5.1 the expectation value on the left-hand side of (6.27) can be upper bounded by the total variation of the spectral measure, which in turn is upper bounded by the Lebesgue measure of a limiting set of matrix elements of the resolvent of W_ω in the following way

$$\begin{aligned} \mathbb{E}\left(\sup_t |\langle \delta_y \otimes \psi, \chi(\mathfrak{I}) W_\omega^t \ \delta_x \otimes \phi, | \rangle \right) &\leq \mathbb{E}\left(\lim_{N\to\infty} |\rho_{\omega,N}^{x,y}|(I) \right) \\ &\leq 4 \max_{i,j} \mathbb{E}\left(\lim_{N\to\infty} \lim_{\kappa\to\infty} m(\{\theta \in I : |\mathrm{G}_\Theta(2x-i, 2y-j)| > \kappa\}) \right) \ . \end{aligned} \tag{6.28}$$

Here G_Θ denotes again the resolvent of W_ω and we remark that by dominated convergence we can pull the limit over N out of the expectation value. From lemma 5.7.2 we know that any question we ask about the absolute value of a matrix element of the resolvent can be phrased in terms of a product of transfer matrices.

Before we proceed let us fix $\xi^2 < \eta\xi$ and define $\alpha = \frac{\eta}{\xi}$. In addition, we denote by $n_0 \in \mathbb{N}$ an initial length scale the size of which will be determined later and define the length scales n_k recursively via

$$n_{k+1} = \max_{K \in 2n_k\mathbb{N}} K =: 2R_{k+1} n_k \ . \tag{6.29}$$

Denote by k_0 the natural number such that $n_{k_0} - 2 \leq |x-y| \leq n_{k_0+1}$. Without loss of generality assume that $x > y$. Then we know from lemma 5.7.2 that there exists $\Phi_\pm \in P_2$ such that

$$|\,\mathrm{G}_\Theta(2x-i,2y-j)| = \frac{1}{2}|\,\langle \Phi_+ , T_{x-1}(\Theta_0)\cdot \ldots \cdot T_y(\Theta_0)\Phi_-\rangle|^{-1} \ .$$

Since $x-y$ is assumed to be larger than n_{k_0}, we can according to lemma 5.7.4 divide this product of transfer matrices into two products, one of length $\frac{n_{k_0}}{2}$, the other of length $|x-y|-\frac{n_{k_0}}{2}$ either starting from the x or y, if we choose appropriate normalized vectors $\widetilde{\Psi}_\pm, \widehat{\Psi}_\pm \in P_2$

$$\begin{aligned} &|\,\langle \Phi_+ , T_y(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| \\ &= |\,\langle \Phi_+ , T_y(\Theta_0)\cdot \ldots \cdot T_{x+\frac{n_{k_0}}{2}+1}(\Theta_0)\widetilde{\Psi}_-\rangle|\,|\,\langle \widetilde{\Psi}_+ , T_{x-\frac{n_{k_0}}{2}}(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| \\ &= |\,\langle \Phi_+ , T_y(\Theta_0)\cdot \ldots \cdot T_{y-\frac{n_{k_0}}{2}}(\Theta_0)\widehat{\Psi}_-\rangle|\,|\,\langle \widehat{\Psi}_+ , T_{y-\frac{n_{k_0}}{2}+1}(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| \ . \end{aligned} \tag{6.30}$$

Note, that due to the lower bound on the distance between x and y, the product of transfer matrices of length $\frac{n_{k_0}}{2}$ starting at x will be independent from the product of transfer matrices of length $\frac{n_{k_0}}{2}$ starting from y. Our definition of the length scales n_k from (6.29) also allows us to decompose a product of transfer matrices of length $\frac{n_k}{2}$ into $2R_k$ products of length $\frac{n_{k-1}}{2}$. Suppressing for the moment the dependency of the transfer matrices on Θ_0, this means that

$$|\,\langle \Phi_+ , T_{\frac{n_k}{2}}\cdot \ldots \cdot T_1\Phi_-\rangle| = \left(\prod_{l=1}^{2R_k} |\,\langle \Psi_{l,+} , T_{\frac{n_{k-1}}{2}}\cdot \ldots \cdot T_1\Psi_{l+1,-}\rangle|\right) \tag{6.31}$$

for some collection of vectors $\Psi_{l,\pm} \in P_2$, with $\Psi_{1,+} = \Phi_+$ and $\Psi_{R_k,-} = \Phi_-$. Our goal is to make the multiscale analysis from lemma 6.4.1 applicable to the two independent products in (6.30). Therefore, we have to verify the two prerequisites $\mathbf{A_1}$ and $\mathbf{A_2}$. Due to our choice of assumptions the initial scale estimate in lemma 5.7.11 as well as the uniform Wegner estimate from corollary 6.3.2 are valid if the products of transfer matrices is long enough.

We now have to find a consistent set of parameters, where both results can be used. Remember that we already fixed η with $0 < \xi^2 < \eta < \xi < 1$ and $\alpha = \frac{\eta}{\xi}$, which also implies $\alpha < \eta^{-1}$. To satisfy the initial scale estimate $\mathbf{A_1}$ we now fix some $n \in \mathbb{N}$ and use lemma 5.7.11 to obtain n_{IS}, γ_{IS}, σ_{IS} and δ_{IS} such that

$$\mathbb{P}\left(\exists \Theta \in I(\Theta_0, \delta_{IS})\,;\, |\,\langle \Phi_1 , T_{n_{IS}}(\Theta)\cdot \ldots \cdot T_1(\Theta)\Phi_2\rangle| \leq e^{\gamma_{IS} n_{IS}}\right) \leq e^{-\sigma_{IS} n_{IS}} \ ,$$

for all normalized $\Phi_i \in P_2$. With this choice assumption $\mathbf{A_1}$ of lemma 6.4.1 is indeed satisfied.

Next we pick $0 < c < 1$, $\alpha\eta < \delta < 1$, and feed the parameters η, c, $I(\Theta_0, \delta_{IS})$, σ_{IS} and δ into the uniform Wegner estimate, which outputs σ_W and n_W such that for all $n > n_W$ and $\Phi_i \in P_2$

$$\begin{aligned} \mathbb{P}\Big(\exists I(\Theta_0, \delta_{IS}) : |\,\langle \Phi_1 , T_n(\Theta)\cdot \ldots \cdot T_1(\Theta)\Phi_2\rangle| < e^{-\gamma_{IS} c\, n^\delta} \text{ and} \\ |\,\langle \Phi_3 , T_n(\Theta)\cdot \ldots \cdot T_1(\Theta)\Phi_4\rangle| < e^{-\gamma_{IS} c\, n^\delta}\Big) \leq e^{-\sigma_W n^\delta} \ . \end{aligned}$$

If $n_W \leq n_{IS}$ we can set our initial scale estimate n_0 equal to n_{IS} and are done, because now we can perform a multiscale analysis on the length scales n_k as defined in (6.29) by decomposing the product of length $\frac{n_{k+1}}{2}$ into $2R_{k+1}$ products of length $\frac{n_k}{2}$ according to (6.31).

If on the other hand $n_W > n_{IS}$, we reiterate the whole procedure by feeding n_W into the initial scale estimate. Let us furthermore define the event $E_{\Theta,\delta,n}$ as

$$E_{\Theta,\delta,n} = \{\exists I(\Theta,\delta) : |\langle \widetilde{\Phi}_+, T_{x-\frac{n_{k_0}}{2}}(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| < e^{\gamma \frac{n_k}{2}} \\ \text{and } |\langle \Phi_+, T_y(\Theta_0)\cdot \ldots \cdot T_{y-\frac{n_{k_0}}{2}}(\Theta_0)\widehat{\Phi}_-\rangle| < e^{\gamma \frac{n_k}{2}}\}$$

Setting $\sigma = \min(\sigma_W, \sigma_{IS})$ and $\gamma = c\gamma_{IS}$ and inserting all constants into the multiscale analysis implies that for all $x, y \in \mathbb{N}$ with $n_k \leq |x-y|-2 \leq n_{k+1}$ the transfer matrices satisfy

$$\mathbb{P}\left(E_{\Theta_0,\delta_{IS},\frac{n_k}{2}}\right) \leq e^{-\sigma\left(\frac{n_k}{2}\right)^\eta} < e^{-\frac{\sigma}{2^{\alpha\xi}} n_{k+1}^\xi} \tag{6.32}$$

where we used $\alpha\xi = \eta$ and $n_k^\alpha > n_{k+1}$. Returning to (6.28) we can split the computation of the expectation value with respect to the event $E_{\Theta_0,\delta_{IS},\frac{n_k}{2}}$. If $E_{\Theta_0,\delta_{IS},\frac{n_k}{2}}$ does not occur than at least one of the transfer matrix products of length $\frac{n_k}{2}$ in (6.30) grows exponentially with a rate of at least γ_k. Without loss of generality, assume that the product starting from x is growing. In this case, we find after redefining κ in the third step

$$\begin{aligned}
|\rho_{\omega,N}^{x,y}|(I(\Theta_0,\delta_{IS})) &= \lim_{\kappa\to\infty} \pi\kappa m(\{\theta \in I(\Theta_0,\delta_{IS}) : \\
&\quad |\langle \Phi_+, T_y(\Theta_0)\cdot \ldots \cdot T_{x+\frac{n_{k_0}}{2}+1}(\Theta_0)\widetilde{\Phi}_-\rangle||\langle \widetilde{\Phi}_+, T_{x-\frac{n_{k_0}}{2}}(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| < \frac{1}{2\kappa}\}) \\
&\leq \lim_{\kappa\to\infty} \pi\kappa m(\{\theta \in I(\Theta_0,\delta_{IS}) : e^{\gamma_k \frac{n_k}{2}}|\langle \widetilde{\Phi}_+, T_{x-\frac{n_{k_0}}{2}}(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| < \frac{1}{2\kappa}\}) \\
&= e^{-\gamma_k \frac{n_k}{2}} \lim_{\kappa\to\infty} \pi\kappa m(\{\theta \in I(\Theta_0,\delta_{IS}) : |\langle \widetilde{\Phi}_+, T_{x-\frac{n_{k_0}}{2}}(\Theta_0)\cdot \ldots \cdot T_x(\Theta_0)\Phi_-\rangle| < \frac{1}{2\kappa}\}) \ .
\end{aligned}$$

By corollary 5.7.3 we can interpret the remaining transfer matrix product in the last equation again as the matrix element of a resolvent of a disordered quantum walk. Therefore, the limit measure corresponds again to the variation of a spectral measure between two lattice sites $\widetilde{x}, \widetilde{y} \in \mathbb{Z}$ of distance $|x-y|-1-\frac{n_k}{2}$. Since such a spectral measure is upper bounded by one and $\gamma_k > c\gamma_0$ we obtain by the same arguments as in (6.32)

$$|\rho_{\omega,N}^{x,y}|(I(\Theta_0,\delta_{IS})) \leq e^{-\gamma_0 c\frac{n_k}{2}}\ |\rho_{\omega,N}^{\widetilde{x},\widetilde{y}}|(I(\Theta_0,\delta_{IS})) \leq e^{-\gamma_0 c\frac{n_k}{2}} \leq e^{-\gamma_0 c\frac{n_k^\eta}{2}} \leq e^{-\gamma_0 c\frac{n_{k+1}^\xi}{2}} \tag{6.33}$$

As already indicated we now combine the two estimates (6.32) and (6.33) in our estimate of the desired expectation value by splitting the computation of this expectation value with respect to the event $E_{\Theta_0,\delta_{IS},\frac{n_k}{2}}$. This amounts to the following bound for all

$x, y \in \mathbb{Z}$ with $n_k \leq |x-y|-2 \leq n_{k+1}$

$$\begin{aligned}\mathbb{E}\left(|\rho^{x,y}_{\omega,N}|(I(\Theta_0,\delta_{IS}))\right) &= \int_{E_{\Theta_0,\delta_{IS},\frac{n_k}{2}}} |\rho^{x,y}_{\omega,N}|(I(\Theta_0,\delta_{IS}))\,\mathbb{P}(\mathrm{d}\omega) + \int_{\complement E_{\Theta_0,\delta_{IS},\frac{n_k}{2}}} |\rho^{x,y}_{\omega,N}|(I(\Theta_0,\delta_{IS}))\,\mathbb{P}(\mathrm{d}\omega)\\ &\leq e^{-\frac{\gamma_0 c}{2} n_{k+1}^{\xi}} + e^{-\frac{\sigma}{2^{\alpha\xi}} n_{k+1}^{\xi}} \leq e^{-\frac{\gamma_0 c}{2}(|x-y|-2)^{\xi}} + e^{-\frac{\sigma}{2^{\alpha\xi}}(|x-y|-2)^{\xi}}\\ &\leq e^{\frac{2^{\xi}\gamma_0 c}{2}} e^{-\frac{\gamma_0 c}{2}|x-y|^{\xi}} + e^{\frac{2\sigma}{2^{\alpha\xi}}} e^{-\frac{\sigma}{2^{\alpha\xi}}|x-y|^{\xi}} \leq C_1 e^{-C_2|x-y|^{\xi}}\ ,\end{aligned}$$

with $C_1 = e^{\frac{2^{\xi}\gamma_0 c}{2}} + e^{\frac{2\sigma}{2^{\alpha\xi}}}$ and $C_2 = \min(\frac{\sigma}{2^{\alpha\xi}}, \frac{\gamma_0 c}{2})$. In order to extend the result also to lattice sites with a distance smaller than n_0, we might have to increase C_1 to obtain the trivial bound of 1 for the transition probability in those cases, namely we have to ensure that $1 \leq C_1 \exp(-C_2 n_0^{\xi})$, which finishes the proof. □

This concludes the proof of our main theorem, which establishes dynamical localization for large classes of disordered quantum walks. Note that we even proved a slightly stronger result than dynamical localization. Since we found an upper bound on the total variation of the spectral measure corresponding to two different lattice sites x and y we do not only control the powers of the disordered walk operator, but also the modulus of the matrix elements

$$|\langle \delta_y \otimes \psi\,, \chi(\mathfrak{I}) f\left(W_\omega^t\right) \delta_x \otimes \phi\,\rangle|$$

for any continuous function f on the unit circle with $\|f\|_\infty \leq 1$.

Theorem 6.5.1 also provides us with a direct criterion that allows us to check for a given distribution on $\mathcal{U}(2)$, whether it exhibits dynamical localization. We only have to consider the subgroup generated by the corresponding transfer matrices within the group $SL_{\mathbb{T}}(2)$. As a remark let us mention that lemma 2.5.11 together with our main theorem implies also spectral localization for those disordered quantum walks.

Corollary 6.5.2. *Let W_ω be a regular disordered quantum walk, such that for some $\mathfrak{I} \subset \mathbb{T}$ the image measure of its coin distribution μ on $\tau_\Theta(\mathcal{U}_{nd})$ is strongly-irreducible, non-compact and ζ-integrable for all $\Theta \in \mathfrak{I}$.*

Proof. The exclusively pure point nature of the spectrum follows from the dynamical localization of W_ω due to theorem 6.5.1 together with lemma 2.5.11. □

Although, theorem 6.5.1 provides a technical criterion in terms of the transfer matrices generated by the local coin distribution, it does not directly provide examples of local coin distributions satisfying these assumptions. To this end, we discuss a collection of different examples in the next section.

6.6. Examples of disordered quantum walks

Even though the technical result form theorem 6.5.1 allows us in principle to check whether a certain local coin distribution exhibits dynamical localization, we still have

to show that the criterion it can be evaluated for a given distribution and even more importantly that there are interesting distributions that satisfy these assumptions. Therefore, we study in this section a collection of explicit disorder models which correspond to a specific choice of local coin distributions and analyze whether they satisfy the requirements of theorem 6.5.1. Aside from measures absolutely continuous with respect to the Haar measure, we also give an example of a discrete measure exhibiting dynamical localization.

In order to apply theorem 6.5.1 we have to verify properties of the group $\langle\mu\rangle$ and the subgroup (μ) generated by corresponding transfer matrices. Remember that we denote by $\langle\mu_z\rangle$ the group and by (μ_z) the semi-group generated by the transfer matrices contained in the support of μ_z on $SL_{\mathbb{T}}(2)$.

We first discuss two preparatory results concerning the parametrization of the image of $\mathcal{U}_{nd}$ under the transfer matrix mapping τ_z defined in (5.33) and the strong irreducibility of $\mathrm{GL}(2,\mathbb{C})$ matrices.

Proposition 6.6.1. *The mapping $\tau_z : \mathcal{U}(2) \mapsto SL_{\mathbb{T}}(2)$ is injective on $\mathcal{U}_{nd}$ for all $z \in \mathbb{C}\setminus\{0\}$ and the image $\tau_z(\mathcal{U}_{nd})$ is given by the set*

$$\left\{ T(r,\alpha,\beta,\eta) := \begin{pmatrix} \sqrt{1+r^2}\,e^{\mathbb{i}\alpha}|z|^{-1} & r\,e^{\mathbb{i}\beta} \\ r\,e^{\mathbb{i}\eta} & \sqrt{1+r^2}\,e^{\mathbb{i}(\beta+\eta-\alpha)}|z| \end{pmatrix} ;\ r\in\mathbb{R}_+,\ \alpha,\beta,\eta\in[-\pi,\pi) \right\}$$

Proof. Injectivity has been already shown in lemma 5.7.1. Writing down the image of a general unitary matrix $U \in \mathcal{U}(2)$ we obtain

$$\tau_z\left(\begin{pmatrix} a & b \\ -\overline{b}\,e^{\mathbb{i}\phi} & \overline{a}\,e^{\mathbb{i}\phi} \end{pmatrix}\right) = \frac{1}{a}\begin{pmatrix} e^{\mathbb{i}\phi} & -\overline{b}\,e^{\mathbb{i}\phi} \\ -\overline{b} & z \end{pmatrix},$$

with $|a|^2+|b|^2=1$ and $|a|\neq 0$. Setting $r=\frac{|a|}{|b|}$ we get $r\geq 0$ and $|a|^2=(1+r^2)^{-1}$. The phases α, β and η now depend on the phases of a and b as well as of ϕ and can therefore each assume any value in $[-\pi,\pi)$. □

The following characterization for strong irreducibility is proven in [BL85] for the real case. Here we consider the case of $SL_{\mathbb{T}}(2)$ matrices.

Proposition 6.6.2. *Let μ be a measure on $SL_{\mathbb{T}}(2)$ such that its generated subgroup $\langle\mu\rangle$ is non-compact. Then μ is strongly irreducible if for any $\overline{x}\in\mathbb{P}\mathbb{C}^2$ the set $\{M\overline{x}\ ;\ M\in\langle\mu\rangle\}$ contains more than two elements.*

Proof. Since the only proper subspaces of $\mathbb{C}^2$ are one-dimensional, it suffices to show that the existence of k vectors $\overline{x}_i\in\mathbb{P}\mathbb{C}^2$ such that $M\bigcup_l x_l \subset \bigcup_l x_l$, implies $k\leq 2$. The matrices $M\in\langle\mu\rangle$ permute the set $\{\overline{x}_1,\dots\overline{x}_k\}$ according to some permutation π_M and the map $M\mapsto\pi_M$ constitutes a group homomorphism. The Kernel K of this homomorphism, that is, the elements $M\in\langle\mu\rangle$ such that $M\overline{x}_i=\overline{x}_i$ for all i is a normal subgroup of $\langle\mu\rangle$ and we can construct the quotient group $\langle\mu\rangle\setminus K$, which is isomorphic to a subgroup of the permutation group S_k and hence finite. By the group action of $SL_{\mathbb{T}}(2)$ on $\mathbb{P}\mathbb{C}^2$ the action of the matrices $M\in K$ on the corresponding vectors $x_i\in\mathbb{C}^2$ is given by $Mx_i=\lambda_i x_i$ with $\lambda_i\in\mathbb{C}$. Under the assumption $k>3$ we can find for any three distinct

vectors $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$, nonzero complex coefficients α, β such that $x_3 = \alpha x_1 + \beta x_2$ with x_1, x_2 linearly independent. Acting with an element $M \in K$ on both sides we obtain

$$M x_3 = \lambda_3 x_3 = \alpha\lambda_3 x_1 + \beta\lambda_3 x_2 = \alpha\lambda_1 x_1 + \beta\lambda_2 x_2 = M(\alpha x_1 + \alpha x_2) \ .$$

Therefore, $\lambda_1 = \lambda_2 = \lambda_3$ and $M \in K$ has to be a multiple of the identity. Furthermore, since we know that $|\det M| = 1$, we have $M = e^{i\theta}\mathbb{1}$, with $\theta \in \mathbb{R}$. But then K and $\langle\mu\rangle \setminus K$ would both be compact, which contradicts the non-compactness of $\langle\mu\rangle$. Therefore, k has to be smaller than three. □

Measures with a density

The first class of examples we want to consider consists of local coin distributions with an absolutely continuous component. Since it suffices to verify non-compactness and strong irreducibility on a subgroup generated by the transfer matrices this also implies that all coin distributions with an absolutely continuous part exhibit dynamical localization. We begin with the following corollary to theorem 6.5.1.

Corollary 6.6.3. *Let μ be a ζ-integrable probability measure on $\mathcal{U}(2)$ with $\mu(\mathcal{U}_{nd}) = 1$. If the support of μ has a non-empty interior in the standard topology of $\mathcal{U}(2)$ then the corresponding disordered quantum walk exhibits dynamical localization for all $z \in \mathbb{T}$.*

Proof. Given ζ-integrability we have to check non-compactness and strong irreducibility of the subgroup generated by the support of μ in order to satisfy the assumptions of theorem 6.5.1. We begin with the non-compactness. Since we assumed $\mathrm{supp}(\mu)$ to have a non-empty interior we can find an open set $O \subset \mathrm{supp}(\mu)$ and because τ_z as well as its inverse are continuous $\tau_z(O)$ is open for all $z \in \mathbb{C} \setminus \{0\}$. Therefore, we can find $r \neq t$ such that $T(r,\alpha,\beta,\eta)$ and $T(t,\alpha,\beta,\eta)$ are both in $\tau_z(O)$ for some $\alpha, \beta, \eta \in [-\pi, \pi)$ according to proposition 6.6.1. We are interested in the group generated by the support μ_z of the image measure of μ on the transfer matrices. Therefore, we can invert one of the matrices and study the expression

$$T(r,\alpha,\beta,\eta)\cdot T(r',\alpha,\beta,\eta)^{-1} = \begin{pmatrix} A & e^{\alpha-\beta)}\frac{B}{|z|} \\ e^{-(\alpha-\beta)}B|z| & A \end{pmatrix} \ ,$$

with

$$A = \sqrt{(1+r^2)(1+t^2)} - r\,t \quad \text{and} \quad B = r\sqrt{1+t^2} - t\sqrt{1+r^2} \ .$$

This matrix has two eigenvalues equal to $\lambda_\pm = A \pm B$. Due to the fact that $|\det T| = 1$ for any transfer matrix, the product $\lambda_+\lambda_-$ has also modulus one. If they do not lie both on the unit circle, one of them has modulus strictly larger than 1, in which case we would have found an unbounded sequence in the group generated by $\mathrm{supp}\,\mu_z$, which implies non-compactness. So we have to exclude the case $|\lambda_+| = |\lambda_-| = 1$. Since A is strictly positive, we find depending on whether B is positive and $A \geq |B|$ or vice versa

$$||\lambda_+| - |\lambda_-|| = |\pm(A+B) \pm (A-B)| = 2\min(A, |B|) \ .$$

From $A > 0$ and the fact that $B = 0$ implies $r = t$ we indeed find $|\lambda_+| \neq |\lambda_-|$.

Next we show strong irreducibility. Consider again the open set $\tau_z(O) \subset SL_{\mathbb{T}}(2)$. If we choose $M \in \tau_z(O)$, the set $\tau_z(O)\cdot M^{-1}$ is an open neighbourhood of the identity matrix. Therefore, we can find $\varepsilon > 0$ such that

$$\{\mathbb{1} + \delta U \,;\, 0 \leq \delta < \varepsilon, U \in \mathcal{U}(2)\} \subset \tau_z(O)\cdot M^{-1} \ .$$

This implies in particular, that for every normalized $v \in \mathbb{C}^2$ the ε-Ball $\mathcal{B}_\varepsilon(v)$ around v is contained in the set $\tau_z(O)\cdot M^{-1}\cdot v$, which therefore has a non-empty interior. This means that none of the sets $\tau_z(O)\cdot M^{-1}\cdot v$ can be contained within a finite union of subspaces of $\mathbb{C}^2$, because those do have empty interior. This already rules out the existence of an invariant union of subspaces and therefore the group generated by the support of μ is strongly irreducible. □

As a first explicit example with open interior we study a local coin distribution μ that has an absolutely continuous part with respect to the Haar measure. In that case it is clear that the local coin distribution has an open interior, so corollary 6.6.3 applies and we only have to check integrability. Returning to proposition 5.7.7 we see that we only have to guarantee ζ-integrability of $|U_{11}|^{-1}$ for $U_\omega \in \mathrm{supp}(\mu)$. Decomposing μ into its singular and absolutely continuous components μ_{ac} and μ_s we obtain

$$\mathbb{E}\left(|U_{11}|^{-\zeta}\right) = \mathbb{E}_{\mu_s}\left(|U_{11}|^{-\zeta}\right) + \mathbb{E}_{\mu_{ac}}\left(|U_{11}|^{-\zeta}\right) \ .$$

Since we are integrating a positive function, the expectation value with respect to the absolutely continuous part can be controlled by the expectation value with respect to the full Haar measure. Hence, denoting by $\mathbb{E}_{\mu_H}$ the expectation value with respect to the Haar measure on $\mathcal{U}(2)$, we obtain [SHH12]

$$\mathbb{E}_{\mu_{ac}}\left(|U_{11}|^{-\zeta}\right) \leq \mathbb{E}_{\mu_H}\left(|U_{11}|^{-\zeta}\right) = \int_0^\pi \frac{1}{|\sin\theta|^\zeta}\,\mathrm{d}\theta \ .$$

This expression is finite for all $0 \leq \zeta < 1$. If in addition, the measure μ_s consists of finitely many point masses lying in $\mathcal{U}_{nd}$ also the second expectation value is finite for any positive and finite ζ.

The dynamics of an initially localized particle with internal state $\phi = \frac{1}{\sqrt{2}}(1,-1)$ starting at the origin is depicted in figure 6.1. The exponential decay of the transition probabilities is clearly visible. In addition, we can also see a very fast saturation in the variance of the process to a constant value.

As a second example we consider the following situation. Assume that in an experimental setup the experimentalists try to implement one specific target coin C_{target} but at each lattice site due to random noise effects they only manage to realize a coin close to it in the following sense. The target coin is multiplied with a unitary operator close to the identity of the form

$$U_{\omega_x} = C_{target}\cdot e^{\mathbb{i} v_{\omega_x}\cdot\overline{\sigma}} \tag{6.34}$$

where $\overline{\sigma}$ is the vector of Pauli matrices and $v_\omega \in \mathbb{R}^3$ a random vector of norm less or equal to ε.

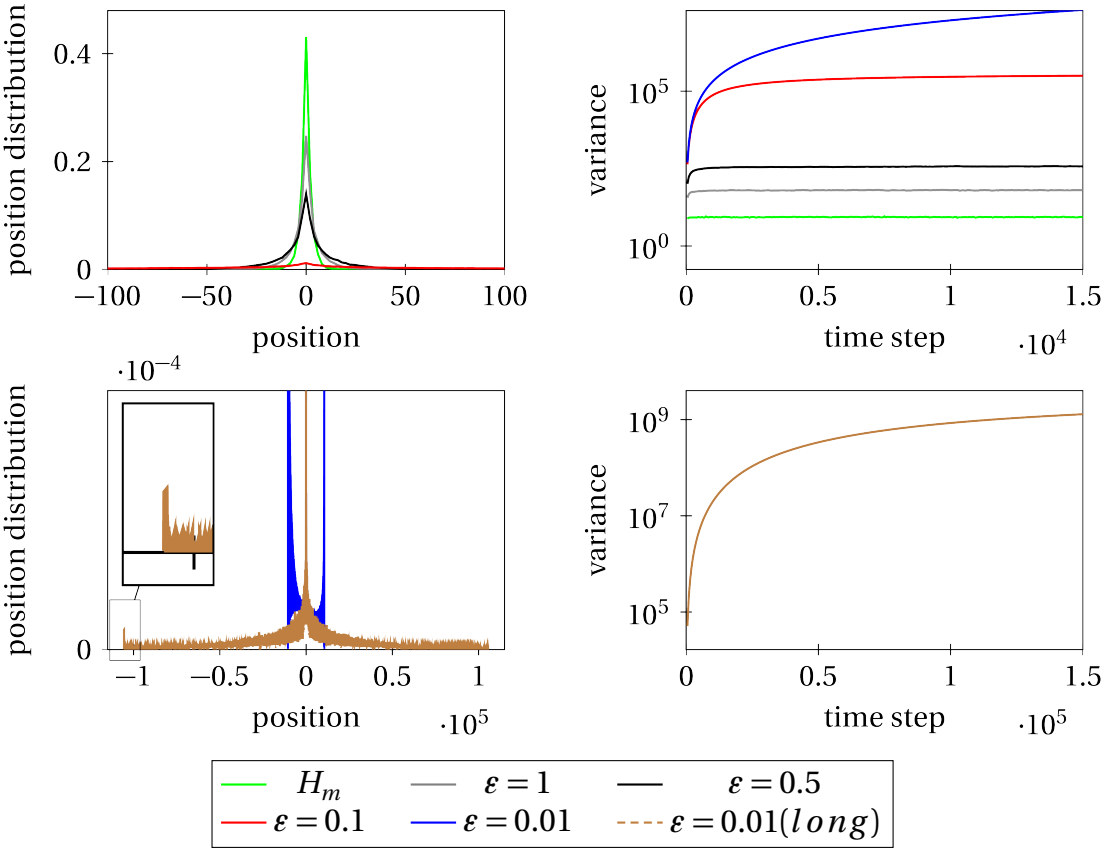


Figure 6.1.: Simulation of a disordered quantum walk with different local coin operations: Depicted are the full Haar measure on $\mathcal{U}(2)$ and the example from (6.34) for different values of ε. The initial state is equal to $\delta_0 \otimes \frac{1}{\sqrt{2}}(1,-1)$. Upper left panel: Position distribution of the particle after $1.5 \cdot 10^4$ time steps averaged over 500 samples. The localization length decreases with the increase of the disorder parameter. Lower left panel: Comparison of the position distribution in the case $\phi = 0.01$ after $1.5 \cdot 10^4$ time steps (500 samples) and after $1.5 \cdot 10^5$ time steps (10 samples). A small component of the initial state still shows ballistical spreading behaviour. Upper right panel: The variance of the time evolved states state is depicted. Except for $\varepsilon = 0.01$ all variances seem to saturate rather quickly. Lower right panel: For $\varepsilon = 0.01$ even $1.5 \cdot 10^5$ time steps are not sufficient for the variance to saturate completely.

By choosing ν_ω uniformly from the $\mathcal{B}_\varepsilon(0) \subset \mathbb{R}^3$ we obtain a local coin distribution μ with nonempty interior. In figure 6.1 the mean position distribution and the time dependence of mean variance with respect to 500 realization is depicted for the initial state $\phi = 1/\sqrt{2}(1,-1)$ after $1.5 \cdot 10^4$ time steps for different values of ε. The target coin C_{target} is chosen as the Hadamard coin according to (2.16). From the position distribution as well as from the variance it is clear that the spreading behaviour is neither

ballistic nor diffusive. However, we also infer that the localization length depends on the strength of the disorder. For large values of ε the decay is completely exponential. For the small value $\varepsilon = 0$ we find that even after $1.5 \cdot 10^5$ there remains some small ballistic tail that is not yet localized.

An experimental example

In this section we analyse an experimental implementation of a disordered quantum walk in an optical fibre performed by Schreiber et al. and show that dynamical localization can indeed be certified via theorem 6.5.1 [SCP+11]. The basic experimental idea is to use the horizontal and vertical polarization of a photon in an optical fibre network as the internal state of freedom and to encode the position information into the arrival times of the photons. A coin operation then corresponds to rotation of the polarization of the photon and the shift operation is implemented by sending the photon through a delay line depending on its polarization [SCP+11].

The most important property of this setup in our context is the possibility to change the coin operation from time-bin to time-bin which exactly implements a position dependent coin operation. Overall it is possible to implement the following family of walk operators [SCP+11]

$$W_{exp} = S \cdot \left(\bigoplus_{x \in \mathbb{Z}} \begin{pmatrix} e^{\mathbb{i}\phi_{\omega_x}} & 0 \\ 0 & e^{\mathbb{i}\eta_{\omega_x}} \end{pmatrix} \right) \cdot \left(\mathbb{1}_{\mathbb{Z}} \otimes \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \right) , \tag{6.35}$$

where S denotes the standard one-dimensional shift operator according to (2.15). Note that the rotation angle given by θ is the same for all $x \in \mathbb{Z}$ but that the phases ϕ_{ω_x} and η_{ω_x} may vary from lattice site to lattice site implementing a static disorder pattern. In [SCP+11] these phases are drawn uniformly from some interval $I \subset [-\pi, \pi)$ and θ is set to $\frac{\pi}{8}$. We will now show that the support of the corresponding probability

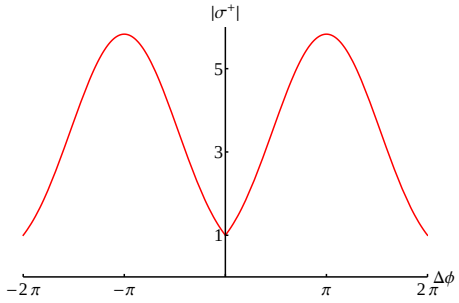


Figure 6.2.: Modulus of the eigenvalue σ^+.

measure on the transfer matrices satisfies the assumptions of theorem 6.5.1. For the non-compactness we compute for two arbitrary phases $\phi_1, \phi_2 \in I$ the product

$$\begin{aligned} M_{\phi 1,\phi 2} &:= \tau_z(U(\phi_1,\phi_2)) \cdot \left(\tau_z(U(\phi_2,\phi_1))\right)^{-1} \\ &= \begin{pmatrix} e^{\mathbb{i}(\Delta\phi)}(2-e^{\mathbb{i}\Delta\phi}) & \sqrt{2}z^{-1}e^{\mathbb{i}\phi_1}(1-e^{-\mathbb{i}\Delta\phi}) \\ \sqrt{2}ze^{-\mathbb{i}\phi_1}(1-e^{-\mathbb{i}\Delta\phi}) & -1+2e^{-\mathbb{i}\Delta\phi} \end{pmatrix} , \end{aligned}$$

where $\Delta\phi = \phi_1 - \phi_2$. The eigenvalues $\sigma_\pm$ of this matrix depend exclusively on $\Delta\phi$ and can be computed to be

$$\sigma_\pm = \frac{1}{2} e^{-2\mathbb{i}\Delta\phi} \left(-1 + 4e^{\mathbb{i}\Delta\phi} - e^{2\mathbb{i}\Delta\phi} \pm \sqrt{2}(1 - e^{\mathbb{i}\Delta\phi}) \sqrt{e^{\mathbb{i}\Delta\phi}(\cos\Delta\phi - 3)} \right) .$$

According to figure 6.2 the σ_+ solution is always larger than 1 if ϕ_1 and ϕ_2 are different and therefore the generated group is non-compact. For the proof of the strong irreducibility we provide the following result.

Proposition 6.6.4. *Let $M \in \mathrm{GL}(\mathbb{C}, 2)$ such that $v \in \mathbb{C}^2$ is not an eigenvector of M. Then, either the vectors $v, M \cdot v$ and $M^2 \cdot v$ are mutually linearly independent or $M^2 = \alpha \mathbb{1}$ for some $\alpha \in \mathbb{C}$.*

Proof. By assumption v and $M \cdot v$ are linearly independent and if we would find $\lambda \in \mathbb{C}$ such that $M^2 \cdot v = \lambda M \cdot v$ then v would be an eigenvector for M. Therefore we only have to check whether $M^2 \cdot v$ and v are linearly independent. Assuming the contrary we could find $\lambda \in \mathbb{C}$ such that $M^2 \cdot v = \lambda v$. Multiplying both sides with M we see that both v and $M \cdot v$ are eigenvectors of M^2 for the eigenvalue λ. Since v and $M \cdot v$ are linearly independent by assumption, this already implies that $M^2 = \lambda \mathbb{1}$. □

In order to show strong irreducibility for our experimental example via proposition 6.6.2 we have to guarantee that $\langle \mu \rangle \cdot \overline{v}$ as a subset of $\mathbb{P}\mathbb{C}^2$ contains three distinct elements for every $\overline{v}$, which is equivalent to the condition that $\langle \mu \rangle \cdot v$ contains three pairwise linearly independent vectors for any normalized $v \in \mathbb{C}^2$. By explicitly computing the square of M_{ϕ_1,ϕ_2} we can directly infer that this matrix does not square to a multiple of the identity unless $\Delta\phi = 0$. Therefore, the vectors $\{v, M_{\phi_1,\phi_2} \cdot v, M^2_{\phi_1,\phi_2} \cdot v\}$ are mutually linearly independent as required as long as v is not an eigenvector of M_{ϕ_1,ϕ_2}.

The corresponding argument goes through for the matrix $N_{\phi_1,\phi_2} := \left(\tau_z(U(\phi_1, \phi_2))\right)^{-1} \cdot \tau_z(U(\phi_2, \phi_1))$, so the only cases that are left are the ones, where M_{ϕ_1,ϕ_2} and N_{ϕ_1,ϕ_2} share an eigenvector. This condition can be solved with respect to the spectral parameter z, which results in eight points on the unit circle, where strong irreducibility could be violated. Abbreviating $\Delta\phi = \phi_1 - \phi_2$ those points are given by

$$z^2 = e^{2\mathbb{i}(\phi 1 + \phi 2)} \quad \text{and} \quad z^2 = \frac{1}{4} e^{\mathbb{i}\phi_2} \left((1 - e^{\mathbb{i}\Delta\phi})^2 \pm \mathbb{i}\sqrt{2(3 - \cos\Delta\phi)}(1 + e^{\mathbb{i}\Delta\phi}) e^{\mathbb{i}\frac{\Delta\phi}{2}} \right) .$$

Since those points depend continuously on our choice of ϕ_1 and ϕ_2, we can just pick a third phase ϕ_3 in order to guarantee strong irreducibility for these cases. Then the whole argument can be repeated for the matrices M_{ϕ_1,ϕ_3} and M_{ϕ_2,ϕ_3} which in turn establishes dynamical localization also for these remaining points z.

Hence, drawing the phases uniformly from an interval $I \subset [-\pi, \pi)$ is certainly sufficient to show dynamical localization for the whole unit circle. Moreover, we have seen that even a finite number of different phases suffices to guarantee this effect for all quasi-energies, regardless of the actual local coin distribution μ. However, the exact form of μ determines the Lyapunov exponent γ and therefore the actual decay rate that governs the dynamical localization.

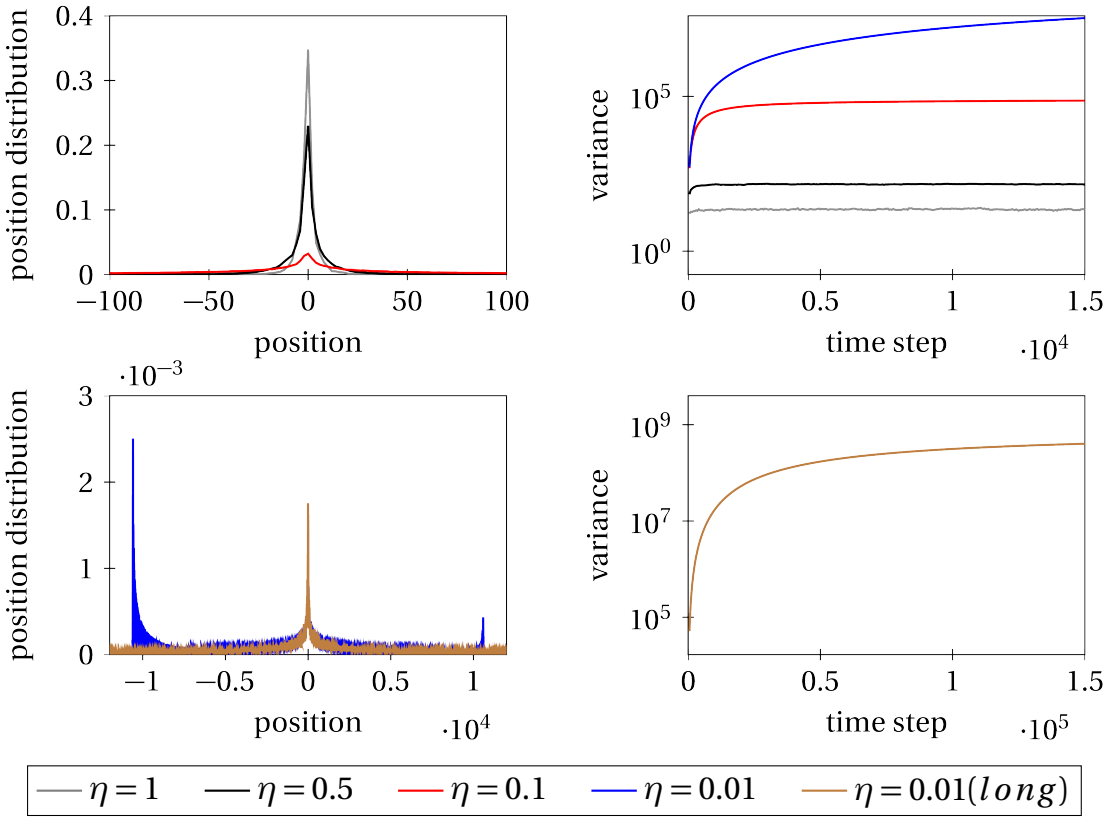


Figure 6.3.: Simulation of the experimental disorder model according to (6.35) with $\theta = \frac{\pi}{8}$ and for different intervals $\eta \cdot [-\pi, \pi]$. The initial state is equal to $\delta_0 \otimes \frac{1}{\sqrt{2}}(1,-1)$. Upper left panel: Position distribution of the particle after $1.5 \cdot 10^4$ time steps averaged over 500 samples. The localization length decreases with the increase of the disorder parameter. Lower left panel: Comparison of the position distribution in the case $\phi = 0.01$ after $1.5 \cdot 10^4$ time steps (500 samples) and after $1.5 \cdot 10^5$ time steps (10 samples). Even though the transport seems to be ballistic in the beginning, we see that eventually the disorder in the system suppresses further spreading of the particle. Upper right panel: Variance of the position distribution for the first $1.5 \cdot 10^4$ time steps (500 samples). Higher disorder implies faster saturation and lower variance. Lower right panel: Variance of the position distribution for the parameter $\phi = 0.01$ for the first $1.5 \cdot 10^5$ time steps. In comparison with the upper right panel saturation of the variance takes place eventually.

Figure 6.3 again depicts a simulated time evolution of the initial state $\frac{1}{\sqrt{2}}(1,-1)$ and $\theta = \frac{\pi}{8}$ for differently sized intervals $I \subset [-\pi, \pi)$ centered around 0. Although localization can be observed for all parameter regimes within the first $1.5 \cdot 10^5$ time steps, we can see that the actual localization length depends on the level of disorder. A larger interval from which the random phases are chosen from implies a shorter localization length, see figure 6.3.

Discrete measures

As a final example, we show that dynamical localization at least for almost all energies can be found disordered quantum walk even if we alternate only between two different coin operations. This discreetness and the consideration of a local coin distribution on general unitary 2×2 matrices also brings us beyond the scenario considered in [JM10]. The model we want to consider here is that the local coins are chosen either to be the Hadamard coin or some other, but fixed $M \in \mathcal{U}(2)$, so the coin distribution μ is only supported on the set

$$H := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, M := \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \right\} \quad \text{with} \quad |a|^2+|b|^2 = 1\,;\, |b|>0 \,\text{and}\, |\Im(b)| > |a| \ .$$

Since according to theorem 6.5.1 we only have to check properties of group $\langle \mu \rangle$ that is generated by transfer matrices of H and M, the actual weight $p \in (0,1)$ and $(1-p)$ we assign to these matrices are irrelevant, although they determine the Lyapunov exponent. The transfer matrices of H and M are given by

$$\tau_z(H) = \begin{pmatrix} -\sqrt{2}z^{-1} & 1 \\ -1 & \sqrt{2}z \end{pmatrix} \quad \text{and} \quad \tau_z(M) = \frac{1}{a} \begin{pmatrix} z^{-1} & -\overline{b} \\ -b & z \end{pmatrix} \ .$$

Similar to the preceding example for random phases we consider the group element

$$M := \tau_z(H) \cdot (\tau_z(M))^{-1} = \frac{1}{\overline{a}} \begin{pmatrix} -\sqrt{2}+b & (1-\sqrt{2}\overline{b})z^{-1} \\ (-1+\sqrt{2}b)z & \sqrt{2}-\overline{b} \end{pmatrix} \ . \tag{6.36}$$

In order to show non-compactness we again compute the modulus of the eigenvalues $\lambda_\pm$ of this matrix and obtain the z independent expression

$$|\lambda_\pm| = \frac{\left| \mathrm{i}\Im(b) \pm \sqrt{|a|^2 - |\Im(b)|^2} \right|}{|a|} \ .$$

Therefore, under the assumption $|\Im(b)| > |a|$ we have indeed one eigenvalue strictly larger than one, which already implies non-compactness of $\langle \mu \rangle$. Next we consider the question of strong irreducibility. We want to use proposition 6.6.2 again. Hence, we have to check, whether $\langle \mu_z \rangle v$ contains three mutually linearly independent vectors for all $z \in \mathbb{T}$. Squaring the matrix M from (6.36), we find the following conditions to obtain a diagonal matrix

$$2\mathrm{i}(-1+\sqrt{2}b)z\Im(b) = 0 \quad \text{and} \quad 2\mathrm{i}(-1+\sqrt{2}\overline{b})z\Im(b) = 0 \ ,$$

which cannot be met, since we assumed $\Im(b) \neq 0$. This implies by proposition 6.6.4 that the vectors $v, M\cdot v$ and $M^2 \cdot v$ are mutually linearly independent and therefore the condition for strong irreducibility is met for $v \in \mathbb{C}^2$ that are not eigenvectors of M. Similar to the example in the previous section, we can repeat the whole argument for the element $N := (\tau_z(H))^{-1} \cdot \tau_z(H)$, thereby showing strong irreducibility for all $z \in \mathbb{T}$ for which N and M do not share an eigenvector. This can only happen for those z, for

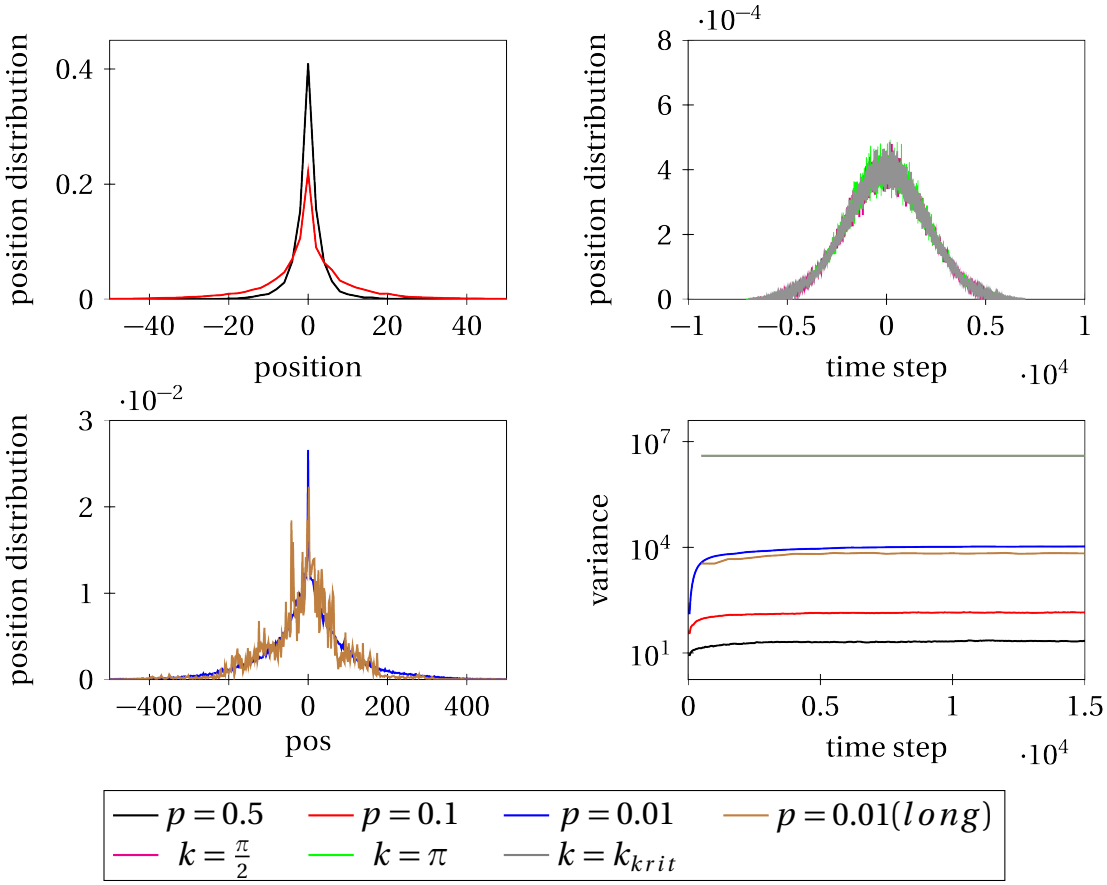


Figure 6.4.: Simulation of a local coin distribution supported on two matrices. At every lattice site the probability for the Hadamard coin is p and $(1-p)$ for the unitary matrix with coin parameters $a = \frac{1}{3}$ and $b = \frac{\mathrm{i}2\sqrt{2}}{3}$. The initial state is equal to $\delta_0 \otimes \frac{1}{\sqrt{2}}(1,-1)$. Upper left panel: Position distribution after $1.5 \cdot 10^4$ time steps averaged over 500 trails. The exponential decay as well as the dependence of the localization length on the Bernoulli parameter p is clearly visible. Lower left panel: Comparison of the position distribution in the case $p = 0.01$ after $1.5 \cdot 10^4$ time steps (500 samples) and after $1.5 \cdot 10^5$ time steps (10 samples). The localization length seems to be constant. Upper right panel: Gaussian initial states peaked around different pseudo-momenta k after $1.5 \cdot 10^5$ (100 samples). The critical value of the pseudo-momentum corresponding to a solution of (6.37) is $k_{\mathrm{crit}} = (\pi - \arctan(\frac{24}{7}))/2$. There is no observable difference between the different k values. Lower right panel: Time dependence of the variance of the position distribution. The width of the final distribution increases with the imbalance between the two possible coins. In the case $p = 0.01$ it also takes longer until the variance is saturated. The uppermost line corresponds to all three Gaussian wave packages depicted in the upper right panel.

which one of the following four conditions corresponding to the possible choices of signs, is satisfied

$$z^2 = \frac{\sqrt{2}\,b+1}{\sqrt{2}\,b-1}\,\frac{\Re(b)-\sqrt{2}\pm\mathbb{i}\sqrt{|\Im(b)|^2-|a|^2}}{\Re(b)+\sqrt{2}\pm\mathbb{i}\sqrt{|\Im(b)|^2-|a|^2}}\ . \tag{6.37}$$

Note that due to the assumption $|\Im(b)|>|a|>0$, b cannot be a real number and therefore $\sqrt{2}b\pm 1\neq 0$ hold. So by this argument there are eight points $\Theta_k\in\mathbb{T}$, where the strong irreducibility might not holds. Since the intuition from the RAGE theorem is that only continuous measures corresponds to transport in the system, one might be tempted to conclude that this finite number of exceptional energies is not enough to cause a break down of dynamical localization. However, as mentioned in section 2.5 point spectrum alone is not strong enough to imply dynamical localization. Therefore it is possible that the localization length, corresponding to the Lyapunov exponent of the system diverges as we approach one of those exceptional points. Such a behaviour was for example numerically investigated by Obuse and Kawakami for a disordered quantum walk with an additional reflecting boundary imposed at the origin [OK11].

Of course, we could extend the support of the local coin distribution by a finite number of additional coins to exclude this situation, assigning a very small probability to their occurrence and obtain dynamical localization for the whole unit circle. However, it would be quite interesting to observe such a mobility edge in the system, since the basic intuition about one-dimensional systems in the Hamiltonian case is that any amount of disorder should result in localization for all energies [Kir08].

We chose the parameters of the second coin M such that $a\in\mathbb{R}$ and $b\in\mathbb{C}\setminus\mathbb{R}$, with $a=\frac{1}{3}$ and $b=\mathbb{i}\frac{2\sqrt{2}}{3}$. Then one of the critical points on the unit circle is given by $\Theta_{crit}=\pm\mathbb{i}e^{-\mathbb{i}\arctan(\frac{24}{7})/2}$. In the left panel of figure 6.4 we again see that for a generic localized initial state dynamical localization can be observed. In addition, we study the dynamics of a gaussian wave packet peaked around Θ_{crit}. If the localization length would diverge at these points, one would expect that some broadening of the localization area should be observable for such a wave packet. However, taking 100 samples of such a walk with a highly peaked wave package around Θ_{crit} as an initial state does not indicate any different behaviour as can be observed for the localized state at the origin.

Nonzero index walk

In our last example we go beyond the applicability of the disordered quantum walk model discussed so far and study numerically a quantum walk with nonzero index (see section 2.4). We keep the random coin operation, but the shift operation now shifts one of the internal states two instead of one lattice site

$$W_\omega=\left(\bigoplus_{x\in\mathbb{Z}}U_{\omega_x}\right)\cdot S_{\text{alt}}\quad\text{, with}\quad S_{\text{alt}}|x,\pm\rangle=\begin{cases}|x+2,+\rangle & \text{for }+\\ |x-1,-\rangle & \text{for }-\end{cases}\ . \tag{6.38}$$

The results of a numerical simulation of the position distribution and the variance are shown in figure 6.5. The altered shift operation S_{alt} is combined with the random coin distributions of all four examples we considered so far in this section. In each case the

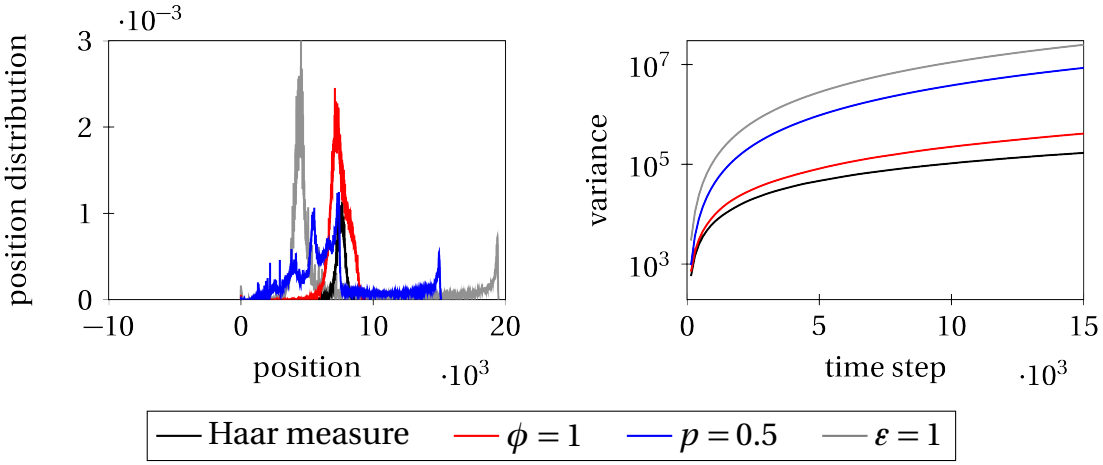


Figure 6.5.: Position distribution and variance of the disordered quantum walk with non-zero index according to (6.38). The local coin distributions agree with the examples studied in the zero index case: The full Haar measure on $\mathcal{U}(2)$, the random rotation by a unitary matrix close to the identity (see (6.34)), the experimental example with random phases (see (6.35)) and the discrete example with two different matrices (see 6.36). The initial state is equal to $\delta_0 \otimes \frac{1}{\sqrt{2}}(1,-1)$. Left panel: Mean position distribution of the four different coin distributions over 200 samples for $1.5 \cdot 10^4$ time steps. A drift of all particles to the right can be observed. Right panel: Time dependence of the variance for the different models.

parameter is chosen such that strongest localization behaviour could be observed in the zero index case.

In comparison with the numerical results for the other examples figure 6.5 seems to indicate that localization can no longer be observed in this system. The variance of the process is still increasing and there seems to be at least a drift term that moves the particle constantly to the right in all four cases. This concludes our overview over explicit examples of disordered quantum walks exhibiting dynamical localization.

6.7. Conclusion and outlook

In this chapter we have completed the proof of dynamical localization of disordered quantum walks for a wide class of local coin distributions. Our main theorem extends previous results by Joye and Merkli, who established dynamical localization for disordered quantum walks with a translation-invariant coin multiplied with random position dependent phases [JM10]. In comparison with the fractional moment method employed by Joye and Merkli, our approach via a multiscale analyses also allowed us to apply our result to singular coin distributions.

The main ingredients of the proof are the Thouless formula and the multiscale analysis lemma. The Thouless formula enabled us to transfer the Hölder continuity of the Lyapunov exponent to the integrated density of states, which in turn allowed us to es-

tablish a uniform Wegner bound for the system. The uniform Wegner estimate allowed us to control the occurrence of resonant energies that are at the same time eigenvalues of different realizations of a finite restriction of a disordered quantum walk.

The multiscale analysis lemma, as a generalized probabilistic version of the induction principle, gave us the means to iteratively show exponential decay on larger scales once the decay was established at one specific scale. To this end the new length scale was split into boxes of the size of the previous length scale and their respective decay properties were used to show decay at the level of the new length scale.

For the purpose of proving dynamical localization we began with the initial scale estimate from chapter 5. The multiscale principle then allowed us to iterate this bound over increasing length scales, which provided an exponential decay of matrix elements of the resolvent of the disordered walk operator. In this context, the uniform Wegner bound enabled us to exclude resonant energies, that is energies at which the matrix elements do not decay exponentially over many length scales. All those building blocks were then combined in theorem 6.5.1 to establish dynamical localization for disordered quantum walks.

In the final part of this chapter, we constructed explicit examples, where the assumptions of theorem 6.5.1 have been met. In particular this included, measures on the group of unitary 2×2 matrices with an absolutely continuous component with respect to the Haar measure and an example where the measure is only supported on a discrete number of different coin operations. In addition, the situation studied in an actual experimental realization performed by Schreiber et al. in the context of optical fibres [SCP+11] has been analyzed and we have been able to certify dynamical localization within their experimental setup.

The generic occurrence of dynamical localization has a positive and negative aspect with respect to the usefulness of quantum walks in quantum computation. In the context of single-particle simulations it seems promising that, as in the continuous-time case, any amount of disorder leads to dynamical localization. Hence, quantum walks reproduce this behaviour faithfully. However, all algorithms that use quantum walks as a building block achieve their speed-up due to the faster spreading behaviour of the quantum walk as compared to classical random walks. From this perspective, localization certainly is an undesired effect that should be avoided.

We conclude this chapter with some remarks on possible further research directions and open questions.

- **Random shift operation:** Instead of a random coin operation, one could also think of noise models that affect the shift operation of the quantum walk. For example instead of a perfect shift operation there could be a random position dependent reflection probability p_x such that shift operator S_ω acts as

$$\begin{aligned} S_\omega \delta_x \otimes + &= \sqrt{p_{\omega_x}}\,\delta_{x+1}\otimes + - \sqrt{1-p_{\omega_x}}\,\delta_x \otimes - \\ S_\omega \delta_x \otimes - &= \sqrt{p_{\omega_{x-1}}}\,\delta_{x-1}\otimes - + \sqrt{1-p_{\omega_{x-1}}}\,\delta_x \otimes + \ . \end{aligned}$$

  With this definition it is easy to check that S_ω is a unitary operator, so $W_\omega = S_\omega \cdot (\mathbb{1}\otimes U)$ defines another kind of disordered quantum walk. One can now follow the steps of chapter 5 and in a similar manner define transfer matrices, which are

given by [Cho12]

$$T_x(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} z^{-1} & z^{-1} + \sqrt{2(1-p_{\omega,x}} \\ z^{-1} - \sqrt{2(1-p_{\omega,x})} & z^{-1}(1-2z^2) \end{pmatrix} .$$

Note that as in the case of coin disorder, the single transfer matrix only depends on the random parameter on one single lattice site. Assuming that the reflection probabilities p_{ω_x} are independent and identically distributed, the full theory of products of random matrices, as discussed in chapter 3, is applicable. However, a simple identity between products of transfer matrices and matrix elements of the resolvent, as in corollary 5.7.3 is not so easy to obtain. In particular, it is no longer true that set P_2 is left invariant. In this respect it would be interesting to generalize the results of this thesis to prove dynamical localization for these random reflections and to also look at the case of combined shift and coin disorder.

- **General one-dimensional QW:** In the previous paragraph as well as in the case of the disordered quantum walks studied in this thesis, we dealt with a specific model of disorder, but also with the specific model of a standard shift and coin quantum walk. It would be interesting to see what can be said about localization on the level of general one-dimensional quantum walks, that is for random unitary band matrices with strictly finite width. In the light of the last example in section 6.6 it would also be interesting to make the connection to the index of a quantum walk and whether a nonzero index generically prevents localization. An additional question that arises for quantum walks is how the dimensionality of the internal state space influences the localization behaviour. As seen in chapter 3 the theory of products of random matrices also works for higher-dimensional transfer matrices. However, at the moment the connection between transfer matrices and the resolvent relies on the specific form and the two-dimensionality of the problem.

- **Quasi-periodic QWs:** The transfer matrix approach developed in chapter 5 is not restricted to a randomly chosen coin operation. In particular, one could study the propagation behaviour if the local coin operations are chosen according to some quasi periodic process. A model of particular interest in this context is the choice $U_x := e^{\mathrm{i}2\pi\theta x} U$, with some fixed $U \in \mathcal{SU}(2)$ and $\theta \in \mathbb{R}$.

  This position dependent phase factor implements the action of an electric field θ on the particle, see [CRW+13, GAS+13]. If the field θ is given by a rational number, the system is again periodic and asymptotically the particle will move ballistically. However, if the field θ is irrational, the underlying dynamical system becomes ergodic again, which hints at a positive Lyapunov exponent for the product of transfer matrices. The positivity in turn excludes an absolutely continuous component in the spectrum.

  In addition, one should investigate, under which conditions on the field it is possible to also exclude singular continuous spectrum. From a comparison with the Hamiltonian case [BG00] it seems promising to impose a diophantine condition on the field θ. Hence, there should be irrational values of the electric field such that the system exhibits spectral and possibly even dynamical localization.

- **Higher lattice dimensions:** Localization of quantum walks in higher lattice dimensions has up to now only been investigated perturbatively [Joy12]. In this case, one starts from a static situation and then shows that localization survives if one considers a small enough perturbation of the operator. However, those results do not cover all interesting cases. For example, they are not applicable to two dimensional quantum walks of the form $U_{\omega_1} \cdot S_y \cdot U_{\omega_2} \cdot S_x$, where shifts in the x and y direction are alternated with two random coin operations.

  The obvious problems one faces in higher dimensions is that the main tool employed in the proof of dynamical localization, namely the transfer matrices, is no longer available. A way out would be to directly prove a uniform Wegner bound and an initial scale estimate on the level of the resolvent, using similar techniques as in the Hamiltonian case, *e.g.* the geometric resolvent inequality. In this regard it seems more difficult to show the initial scale estimate, which in our case was based solely on the properties of transfer matrices. In addition, since the number of time steps that can be simulated decreases in higher-dimensional models, the numerical results as of today are rather inconclusive, neither showing complete localization nor ballistic behaviour of the particle.

  The main question in higher dimensions is whether one can engineer a situation where the spectrum contains a pure point part as well as an absolutely continuous component. The occurrence of these mobility edges is also still an open problem in the Hamiltonian case. The numerical inconclusiveness mentioned above could be a hint of such a behaviour since a localized initial state is essentially supported on all energies, but it could also mean that the number of simulated time steps should be increased.

- **More particles QWs:** Another further research direction is to study the localization properties within a few- or even many-body scenario. As long as we consider a non-interacting situation, results directly carry over from the one particle case. It is already known that in the two particle case with on-site interactions the formation of dynamically stable molecules is possible and that these molecules admit again an effective description as a quantum walk [AAM$^+$12].

  It would be interesting to see if this binding phenomenon can also occur in disordered systems and whether the resulting compounds are still dynamically stable. In addition, it is not at all clear if dynamical localization survives in such an interacting system or whether the particles can overcome the localization in a bound state.

  In the next step one could also allow for an infinite number of particles, which would bring the system to the regime of quantum cellular automata. Note that for such infinite systems only the observables contained in the quasi-local algebra are well defined. Therefore, the concept of dynamical localization would have to be reformulated in terms of local observables. This approach might also allow the study of the dynamics of collective excitations as they arise, for example in fractional quantum Hall systems [Lau83].

A. A unitary version of the RAGE theorem

In this chapter we provide a proof of the unitary and discrete-time version of the RAGE theorem [Gol85] described in section 2.2.1. The whole section has been taken from the appendix of the paper [ASW11] of the author of this thesis.

The proof strategy follows along the lines of a proof for the Hamiltonian case, where the time evolution is induced by a self-adjoint operator [Tes09, Kir08]. The proof is almost identical to the one which appeared in [Ens83] which was also reproduced in [HJS09]. We nevertheless included it for completeness.

The goal of this section is therefore to connect the spectral properties of a unitary operator W on a separable Hilbert space $\mathcal{H}$ with the dynamical behavior of vectors of this Hilbert space under the time evolution induced by W. In order to do so, we are interested in the properties of the spectral measure $\hat{\rho}^{x,y}(t)$ of the time evolution of induced by W

$$\hat{\rho}^{x,y}(t) := \langle \delta_y \otimes \phi \, , W^t \delta_x \otimes \psi \rangle = \int_{\mathbb{T}} \theta^t \rho^{x,y}(d\theta).$$

We begin by proving the following version of Wiener's theorem:

Theorem A.0.1. *Let μ be a complex Borel measure on the unit circle $\mathbb{T}$ and define*

$$\hat{\mu}(t) = \int_{\mathbb{T}} \theta^t \mu(d\theta)$$

then the time average of $\hat{\mu}(t)$ has the limit

$$\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} |\hat{\mu}(t)|^2 = \sum_{\lambda\in\mathbb{T}} |\mu(\{\lambda\})|^2 \ .$$

Proof. Starting from the definition of $\hat{\mu}(t)$, using linearity of the integral and the that $\theta \in \mathbb{T}$ we get

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} |\hat{\mu}(t)|^2 = \\ &= \lim_{T\to\infty} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{T+1} \sum_{t=0}^{T} e^{\mathrm{i}(x-y)t} \right) \mu(d(e^x)) \mu^\star(d(e^y)). \end{aligned}$$

Since the geometric series within the parenthesis is bounded by T and converges pointwise to the indicator function $\chi_{\{0\}}(x)$, we can interchange the limit with the integration

by dominated convergence and arrive at

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\chi_{\{0\}}(x-y)\mu(d(e^x))\mu^\star(d(e^y)) =$$
$$=\int_{-\pi}^{\pi}\mu(\{y\})\mu^\star(dy)=\sum_{x\in[-\pi,\pi)}|\mu(\{e^x\})|^2\,.$$

□

For a given unitary W we can decompose the Hilbert space into three orthogonal subspaces $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$ and $\mathcal{H}_{pp}$ each containing the vectors ϕ for which the spectral measure ρ_ϕ is absolutely continuous, singular continuous or pure point, respectively and these subspaces are left invariant by W. In addition define $\mathcal{H}_c=\mathcal{H}_{ac}\oplus\mathcal{H}_{sc}$. A connection between this decomposition and the time evolution of the quantum walk is given by the following theorem:

Theorem A.0.2. *Let W be a unitary operator and G a compact operator then for $\psi_c\in\mathcal{H}_c$*

$$\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}\|GW^t\psi_c\|^2=0$$

holds.

Proof. Choose a vector $\psi_c\in\mathcal{H}_c$. Since W leaves $\mathcal{H}_c$ invariant we can infer from the relation $\langle\phi\,,W\psi\rangle=\langle P_{\mathcal{H}_c}\phi\,,W\psi\rangle$ that if ρ_ψ is continuous so is $\rho_{\phi,\psi}$. For the compact operator G there is a sequence of finite rank operators $G_n=\sum_{k=0}^{n}\alpha_k\langle\phi_k\,,.\rangle\eta$ converging to it. By the triangle inequality we only have to check the single rank one summands so we end up with

$$\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}|\langle\phi\,,W^t\psi_c\rangle|^2=\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}|\hat{\rho}_{\phi,\psi}(t)|^2=0\,,$$

which follows from Wiener's theorem. □

Equipped with this result, we can prove the discrete time version of the RAGE theorem:

Theorem A.0.3 (RAGE)**.** *Let W be a unitary operator and G_n a sequence of compact operators converging strongly to the identity. Then we have for $\mathcal{H}_c$ and H_{pp}*

$$\mathcal{H}_c=\{\psi\in\mathcal{H};\,\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}\|G_nW^t\psi\|^2=0\}$$
$$\mathcal{H}_{pp}=\{\psi\in\mathcal{H};\,\lim_{n\to\infty}\sup_{t\geq0}\|(\mathbb{1}-G_n)W^t\psi\|^2=0\}\,.$$

Proof. We start with the continuous case, which by theorem A.0.2 holds for every $\psi \in \mathcal{H}_x$. Decomposing now an arbitrary $\phi \in \mathcal{H}$ into $\psi_c \in \mathcal{H}_c$ and $\psi_p \in \mathcal{H}_{pp}$ we can infer that we have to find a lower bound on

$$\lim_{n\to\infty} \lim_{T\to\infty} \sum_{t=0}^{T} \|G_n W^t \Psi_p\|^2 + \|G_n W^t \Psi_c\|^2 - 2|\langle G_n W^t \Psi_p, G_n W^t \Psi_c \rangle| \ .$$

Since the norm of the G_n is uniformly bounded, because they converge to the identity strongly, and by theorem A.0.2 the last two summands tend to zero. So we have to show that $\|G_n W^t \psi_p\|$ is bounded away from zero for n large enough. Instead we prove that

$$\lim_{n\to\infty} \sup_{t\geq 0} \|(\mathbb{1} - G_n) W^t \psi_p\| = 0 \ .$$

Being an element of $\mathcal{H}_{pp}$ ψ_p can be decomposed into eigenvectors ψ_k of the unitary operator W. Inserting this decomposition for ψ_p we can upper bound the norm by

$$\lim_{n\to\infty} \sup_{t\geq 0} \sum_{k=1}^{N} |\alpha_k e^{-\mathrm{i}\lambda_k}| \|(\mathbb{1} - G_n)\psi_k\| + \|\mathbb{1} - G_n\| \sum_{k=N+1}^{\infty} |\alpha_k e^{-\mathrm{i}\lambda_k}| \|\psi_k\| \ .$$

The first sum goes to zero by strong convergence of the G_n and the second goes to zero if we make N large enough and using the fact that a strong convergent sequence of operators is bounded. At the same time this proves the second claim of the RAGE theorem for $\psi \in \mathcal{H}_{pp}$.

If we now again decompose an arbitrary vector $\psi \in \mathcal{H}$ in its components in $\mathcal{H}_c$ and $\mathcal{H}_{pp}$ as in the first case we are left to prove that $\sup_{t\geq 0} \|(\mathbb{1} - G_n) W^t \psi_c\|$ stays strictly larger than zero for all n. Assuming the contrary we find

$$\begin{aligned} \|W^t \psi_c\| &\leq \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \|(\mathbb{1} - G_n) W^t \psi_c\| + \|G_n W^t \psi_c\| \\ &\leq \sup_t \|(\mathbb{1} - G_n) W^t \psi_c\| \xrightarrow{n\to\infty} 0 \, , \end{aligned}$$

by strong convergence of the $\{G_n\}$. This contradiction then concludes the proof of the RAGE theorem. □

B. Supplement to the theory of products of random matrices

In this appendix we provide some of the proofs omitted in chapter 3. The order follows their occurrence in the main text.

B.1. Positivity of the Lyapunov exponent

We begin with the proof of proposition 3.2.4 for which we need the following result on invariant measures on the projective spaces. The real case can be found in [Gui08, Gui06]

Lemma B.1.1. *Let G be a non-compact and strongly irreducible subgroup of $SL_{\mathbb{T}}(d)$, then there is no G-invariant probability measure on $\mathbb{PC}^d$.*

Proof. Assume that the probability measure ν on $\mathbb{PC}^d$ is G invariant and let us choose an unbounded sequence $(g_n)_n \subset G$. Setting $u_n := \frac{g_n}{\|g_n\|}$, we find that $\det(u_n) = \frac{1}{\|g_n\|^d}$ converges to zero. In addition, the sequence $(u_n)_n$ is bounded so we can pass to a converging subsequence with a limit u, such that $\|u\| = 1$ and $\det(u) = 0$.

Denoting by $X \subset \mathbb{PC}^d$ the projective subspace corresponding to the kernel of u and by Y one corresponding to the image of u, we can decompose the measure ν into to measures ν_1, ν_2 supported on X and $\mathbb{PC}^d \setminus X$ such that $\nu = \nu_1 + \nu_2$. Due to G-invariance of ν, this implies the relation

$$\nu = \lim_{n\to\infty} g_n \cdot \nu = \lim_{n\to\infty} g_n \cdot \nu_1 + g_n \cdot \nu_2 = \lim_{n\to\infty} g_n \cdot \nu_1 + u \cdot \nu_2 \ ,$$

where the symbol $\cdot$ signifies that we consider the action of G on $\mathbb{PC}^d$. Note that $\mathbb{PC}^d$ is compact, hence we can again pass to a subsequence such that $g_n \cdot \nu_1$ converges to a measure $\nu_{1,lim}$ supported on the subspace $X_{lim} := \lim_{n\to\infty} g_n \cdot X$. This implies that the measure ν is supported on the union of X_{lim} and Y. Let Γ be the set of subsets of $\mathbb{PC}^d$, consisting of finite unions of projective subspaces, on which ν is supported. This set contains a least element $L = \bigcap_{M\in\Gamma} M$ and by construction $\nu(L) = 1$. Since $g \cdot \nu = \nu$ by assumption, we also have $g \cdot L = L$, which contradicts strong irreducibility. □

We now turn to proposition 3.2.4.

Proposition B.1.2. *Let μ be a probability measure on $SL_{\mathbb{T}}(d)$ such that $\langle\mu\rangle$ is strongly irreducible and non-compact. Denote by T_g for $g \in SL_{\mathbb{T}}(d)$ the operator $(T_g f)(x) = f(g^{-1})$ and define the convolution operator T_μ on $\mathcal{L}^2(\mathbb{C}^d)$*

$$(T_\mu f)(x) := \mathbb{E}\big((T_g f)(x)\big) = \int_{SL_{\mathbb{T}}(d)} f(g^{-1}x)\mu(\mathrm{d}g) \ ,$$

then ρ_μ, the spectral radius of T_μ satisfies $\rho_\mu < 1$.

Proof. Assume that $\rho_\mu = 1$ and let $z \in \mathbb{T}$ be an element of $\sigma(T_\mu)$. Then either z is an element of the point spectrum of $T\mu$ and we can find a sequence $(f_n) \subset \mathcal{L}^2$ of normalized functions f_n such that

$$\|(T_\mu f_n) - z f_N\| \to 0 \tag{B.1}$$

or the image of the operator $T_\mu - z\mathbb{1}$ is not dense in $\mathcal{L}^2$. However, the latter just means z lies in point spectrum of the adjoined operator T_μ^* [RS80], which corresponds to a convolution operator T_{μ^*} for the measure $\mu^*(\mathrm{d}g) := \mu(\mathrm{d}g^{-1})$. Since μ and μ^* generate the same groups, μ is strongly irreducible and non-compact if and only if μ^*. Hence, we can without loss of generality assume that (B.1) holds for some normalized sequence $(f_n)_n$. By the inverse triangle inequality we obtain

$$\lim_{n\to\infty} \|(T_\mu |f_n|) - |f_n|\| \leq \lim_{n\to\infty} \|(T_\mu f_n) - f_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|T_\mu |f_n|\| = 1 \ ,$$

which in turn implies

$$\lim_{n\to\infty} \langle |f_n|, (T_\mu |f_n|)\rangle = \lim_{n\to\infty} \int \langle |f_n|, (T_g |f_n|)\rangle \, \mu(\mathrm{d}g) = 1 \ .$$

Since $\langle |f_n|, (T_g|f|)\rangle \leq 1$ there exists a set $\Omega_0 \subset \mathrm{supp}(\mu)$ of full measure and a subsequence $(f_{n'})_n$ of $(f_n)_n$ such that for all $g \in \Omega_0$ the expression $\langle |f_{n'}|, (T_g |f_{n'}|)\rangle$ converges to 1. Hölder's inequality gives

$$\| T_g |f_n|^2 - |f_n|^2 \|_1 \leq 2 \| T_g |f_n| - |f_n| \| \ ,$$

so the two functions on the left-hand side also converge in 1-norm for all $g \in \Omega_0$. If we consider the probability measure $|f_n|^2 \nu$ on $\mathbb{C}^d$ and its projection on $\mathbb{P}\mathbb{C}^d$ we see that the expression $g \cdot |f_n|^2 \nu - |f_n|^2 \nu$ converges to zero for all $g \in \Omega_0$ in variation norm. Since $\mathbb{P}\mathbb{C}^d$ is compact we can find a measure η such that a subsequence of $|f_n|^2 \nu$ converges weakly to it. But this means that $g \cdot \eta = \eta$ for $g \in \Omega_0$ and because $\mu(\Omega_0) = 1$, η would be an μ invariant measure on $\mathbb{P}\mathbb{C}^d$, which is impossible due to lemma B.1.1. □

B.2. Contraction properties

In this chapter we show the technical propositions needed for the proof of proposition 3.2.11. The main ingredient is the lemma B.2.1 on additive cocycles, a prove of which we can be found in [BL85, lemma II.2.1]. Since the lemma holds for right products of elements of a semigroup we introduce the following notation for this section. If $\{g_n, n \geq 1\}$ is a sequence of matrices, we define their left product S_n and their right product M_n as

$$S_n := g_n \cdot g_{n-1} \cdots g_1 \quad \text{and} \quad M_n := g_1 \cdot g_2 \cdots g_n \ .$$

Lemma B.2.1. *Let G be a topological semigroup acting on a second countable locally compact space B. Consider a sequence $\{X_n, n \geq 1\}$ of independent random elements of G with common distribution μ defined on $(\Omega, \mathcal{A}, \mathbb{P})$. If ν is an μ-invariant distribution on B then for almost all ω there exists a probability measure ν_ω on B such that the sequence*

$$\{X_1(\omega)X_1(\omega)\dots X_n(\omega)g\,\nu\,,\ n \geq 1\}$$

converges weakly to ν_ω as $n \to \infty$ for almost all $g \in G$ with respect to $\lambda = \sum_{n=0}^{\infty} 2^{-(n+1)}\,\nu^n$. Moreover for each bounden Borel function f on B

$$\int f(x)\ \nu(\mathrm{d}x) = \mathbb{E}\left(\int f(x)\ \nu_\omega(\mathrm{d}x)\right)\,.$$

The following proposition shows that we can trade left products of matrices for right products, if we allow for a complex conjugation. More precisely, we show that strong irreducibility and the index are conserved if we consider the measure μ^* generated by μ via conjugate transposition.

Proposition B.2.2. *Let μ be a probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that (μ) has index p and is strongly irreducible. Denote by μ^* the measure induced by μ via $\mu^*(A) = \mu(\{M^*\,;\ M \in A\})$ for all $A \subset \mathrm{GL}(\mathbb{C}, d)$. The semigroup (μ^*) generated by the measure μ^* has the same index and is also strongly irreducible.*

Proof. It is clear that $(\mu^*) = \{M^*\,;\ M \in (\mu)\}$ and since the rank of a matrix is invariant under the conjugate transpose both semigroups have the same index. Now assume that (μ^*) is not strongly irreducible. Hence, there exists a set of subspaces $V_i \subset \mathbb{C}^d$, whose union is (μ^*) invariant. Since $g \in \mathrm{GL}(\mathbb{C}, d)$ has full rank, $\dim V_i = \dim V_j$ and therefore its action on the set $\{V_i\}$ is a permutation of subspaces of equal dimension. So the invariance property holds for every dimension separately and we can choose to study only the subsets V_i with maximal dimension. For every g we have $g\,V_i = V_{j(g,i)}$, which implies $g^* V_{j(i,g)}^{\perp} = V_i^{\perp}$, so g^* permutes the orthogonal complements of V_i. But this exactly means that the union of the orthogonal complements of the single V_i is invariant under the action of all g^*. □

In particular, strong irreducibility implies that the invariant measure is regular on the projective space in the following way.

Lemma B.2.3. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C}, d)$ then any invariant measure μ on $\mathbb{P}\mathbb{C}^d$ is proper, meaning that if V is a proper subspace of $\mathbb{C}^d$ then $\nu(\overline{V}) = 0$, where $\overline{V} = \{\overline{x} \in \mathbb{P}\mathbb{C}^d\,;\ x \in V \setminus \{0\}\}$.*

Proof. The proof follows along the lines of [BL85] for the real case. Assume ν is invariant with respect to μ, and let l_0 be the minimal dimension such that there exists a subspace $V \subset \mathbb{C}^d$ with $\nu(\overline{V}) > 0$. Set

$$r = \sup\{\,\nu(\overline{V})\,;\, V \subset \mathbb{C}^d \text{ subspaces with } \dim V = l_0\} > 0$$

and define B as the set of all subspaces of $\mathbb{PC}^d$ with weight r. Then B is nonempty and any two elements $V_1, V_2 \in B$ are either identical or $\nu(V_1 \cup V_2) = 0$, because $\dim(V_1 \cup V_2) < l_0$ in that case. Therefore the measure of the union of distinct elements of B is

$$\nu(\overline{V}_1 \cup \cdots \cup \overline{V}_n) = r\,n$$

so B has to be finite. By definition we have for any $V \in B$

$$r = \nu(\overline{V}) = \int \chi_{\overline{V}}(M\overline{x})\ \nu(\mathrm{d}\overline{x})\,\mu(\mathrm{d}M) = \int \nu(\{M^{-1}\overline{V}\})\,\mu(\mathrm{d}M)\ .$$

On the other hand $\nu(M^{-1}\overline{V}) \leq r$ by the definition of r so for μ almost all $M \in GLd$ we find $\mu(M^{-1}\overline{V}) = r$. Let Γ be the union over all $V \in B$ then $g\Gamma \subset \Gamma$ for all g supp μ. But this would also imply $(\mu)\Gamma \subset \Gamma$ and if $l_0 < d$ then Γ is a union of proper subspaces which contradicts strong irreducibility. □

Next we turn to the proof of proposition 3.2.11. We obtain it as a corollary to the following theorem that shows that asymptotically the rank of a right product of random matrices agrees with its index (see thm III.3.1 of [BL85] for the real case).

Theorem B.2.4. *Let μ be a strongly irreducible probability measure on* $\mathrm{GL}(\mathbb{C}, d)$ *with index p. For almost all ω there exists a p dimensional subspace $V(\omega) \subset \mathbb{C}^d$ such that any limit point of $M(\omega) := \{\frac{M_n(\omega)}{\|M_n(\omega)\|}\ ;\ n \geq 1\}$ is a rank p matrix with range $V(\omega)$ and for any non-zero $x \in \mathbb{C}^d$*

$$\mathbb{P}\left(x \text{ lies in the orthogonal complement of } V(\omega)\right) = 0\ .$$

If furthermore μ is contracting, there exists a unique μ-invariant probability measure ν on $\mathbb{PC}^d$ and $M_{n(\omega)}\nu$ converges weakly to a point measure $\delta_{Z(\omega)}$, and $Z(\omega)$ is distributed according to the invariant measure ν.

Proof. By lemma B.2.1 we can find for an μ-invariant distribution ν on $\mathbb{PC}^d$ and for almost all ω a probability measure ν_ω such that the sequence $M_{n(\omega)}g\,\nu$ converges weakly to ν_ω for almost all $g \in \mathrm{GL}(\mathbb{C}, d)$ with respect to the measure $\lambda := \sum_{n=0}^{\infty} 2^{-(n+1)}\mu^n$. We will now show that the subspace

$$V(\omega) := \mathrm{span}(\{x \in \mathbb{C}^d\ ;\ \overline{x} \in \mathrm{supp}(\nu_\omega)\})$$

is p dimensional and equal to the range of each limit point of M_ω.

If $A(\omega)$ is a limit point of $M(\omega)$ with $A(\omega)x \neq 0$ we can find a subsequence $M_{n(\omega)}\overline{x}$ that converges to $A(\omega)\overline{x}$ on $\mathbb{PC}^d$. The matrix $A(\omega)$ is not the zero matrix and therefore its kernel is a proper subspace of $\mathbb{PC}^d$. Hence, as a subspace of $\mathbb{PC}^d$ the kernel of $A(\omega)$ has zero weight with respect to the measure ν due to lemma 3.2.9, because μ is strongly irreducible and ν as a μ-invariant measure therefore proper. In addition, for λ-almost all g we have $A(\omega)g\,\nu = \lim_{n\to\infty} M_{n(\omega)}g\,\nu = \nu_\omega$. Since the set of matrices g, for which this is convergence holds, is certainly closed and because the support of λ contains the semigroup (μ), the convergence holds for all $g \in (\mu)$.

Now take any sequence $(g_n)_n \subset (\mu)$ such that $\frac{g_n}{\|g_n\|}$ converges to a rank p matrix $h \in (\mu)$. Note that there exists at least one such sequence because we assumed that μ and therefore also (μ) have index p. Since (μ) is a semigroup $g\,g_n \in (\mu)$, which implies $A(\omega)g\,g_n\,\nu = \nu_\omega$.

Next we show that there is a $g \in (\mu)$ such $A(\omega)g\,g_n\,\nu$ converges weakly to $A(\omega)g\,h\,\nu$, which just means that the set of vectors $\overline{x}$ for which $A(\omega)g\,h\,x = 0$ has zero weight with respect to ν. Since ν is proper, the contrary would imply that

$$K := \{g\,\mathrm{Im}(h)\,;\; g \in (\mu)\} \subset \mathrm{Kern}(A(\omega))\ .$$

However, since $A(\omega)$ is of rank at least p, K is a proper subspace, which would then be also invariant with respect to $g \in (\mu)$; this contradicts strong irreducibility. Therefore we can find $g \in (\mu)$ such that $A(\omega)g\,h\,\nu = \nu_\omega$, which means that $\dim V(\omega) \leq p$, because $V(\omega) \subset \mathrm{Im}(A(\omega)g\,h)$. Since $A(\omega)$ has at least rank p and $A(\omega)\nu = \nu_\omega$, we even have $\dim V(\omega) = p$ and $V(\omega) = \mathrm{range}(A(\omega))$

Denote for $x \in \mathbb{C}^d$ by $H(x) \subset \mathbb{P}\mathbb{C}^d$ the projective image of the orthogonal complement of x. Since ν_ω also satisfies $\nu = \int \nu_\omega \mu(d\omega)$ by lemma B.2.1, we obtain

$$\begin{aligned}\mathbb{P}\left(x \text{ lies in the orthogonal complement of } V(\omega)\right) &= \mathbb{P}\left(\mathrm{supp}\,\nu_\omega \subset H(x)\right)\\ &= \mathbb{P}\left(\nu_\omega(H(x)) = 1\right) = \nu(H(x))\ ,\end{aligned}$$

which is zero, because ν is proper. If (μ) is contracting, $p = 1$ and therefore $V(\omega)$ is one-dimensional and therefore corresponds to a single point $Z(\omega) \in \mathbb{P}\mathbb{C}^d$, which is in addition distributed according to ν. □

Proposition B.2.5. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that (μ) has index p. Then, for any sequence $(x_n)_n \subset \mathbb{C}^d$ that converges to a nonzero vector we have almost surely for the left products of random matrices $S_{n(\omega)}$*

$$\sup_{n\geq 1} \frac{\|S_{n(\omega)}\|}{\|S_{n(\omega)}x_n\|} \leq \infty\ .$$

If in addition, $m_{i(n)}$ denotes the ith largest singular value of the random product $S_{n(\omega)}$, then almost surely

$$\lim_{n\to\infty} \frac{m_{p+1(n)}}{\|S_{n(\omega)}\|} = 0$$

Proof. Let us start with the second part. By complex conjugation we can turn the left product $S_{n(\omega)}$ into a right product $S^*_{n(\omega)}$ to which by proposition B.2.2 then theorem B.2.1 applies. Therefore, we know almost surely the range of each limit point $M(\omega)$ of $\{\frac{S^*_n(\omega)}{\|S^*_n(\omega)\|}; n \in \mathbb{N}\}$ is given by a p dimensional subspace $V(\omega)$. Since $\|S_{n(\omega)}\|$ is given by the largest singular value $m_{1(n)}$, the polar decomposition of an element of this set can be written as

$$\frac{S^*_n(\omega)}{\|S^*_n(\omega)\|} = K_n \begin{pmatrix} 1 & 0 & \dots & \\ 0 & \frac{m_2(n)}{m_1(n)} & \ddots & \\ 0 & 0 & \frac{m_3(n)}{m_1(n)} & \ddots \\ 0 & 0 & 0 & \ddots \end{pmatrix} U_n\ .$$

Since with probability one any limit point $M_n(\omega)$ is a p dimensional matrix, exactly p singular values of $M_n(\omega)$ are non-zero, which is then also true for the left products $S_{n(\omega)}$. In order to show the first part of the proposition we use the polar decomposition of $K_n \cdot A_n \cdot U_n$ of $S_{n(\omega)}$ and obtain

$$\frac{\|S_{n(\omega)}x_n\|^2}{\|S_{n(\omega)}\|^2} = \frac{\|A_n U_n x_n\|^2}{\|A_n\|^2} = \sum_{i=1}^{d} \frac{m_{i(n)}}{m_{1(n)}} |\langle x_n, U_n^* e_i \rangle|^2 \geq \sum_{i=1}^{p} \frac{m_{i(n)}}{m_{1(n)}} |\langle x_n, U_n^* e_i \rangle|^2 \ .$$

Denoting by x the non-zero limit of $(x_n)_n$ and writing $X_{V(\omega)}$ for the orthogonal projection of x onto $V(\omega)$ we obtain

$$\liminf_{n\to\infty} \frac{\|S_{n(\omega)}x_n\|^2}{\|S_{n(\omega)}\|^2} \geq \left(\inf_n \left(\frac{m_{p(n)}}{m_{1(n)}} \right) \|X_{V(\omega)}\| \right) \ .$$

The first factor on the right-hand side is nonzero, because $S_{n(\omega)}$ is at least a rank p matrix and the second factor vanishes if x is orthogonal to $V(\omega)$, but by theorem B.2.4 this happens with probability 0. □

B.3. Theorem of Fürstenberg and Kesten for normalized endomorphisms

In this section we prove an addition to the Fürstenberg and Kesten theorem. To this end, we define K_1 and K_2 as the spaces of endomorphism on $\mathbb{C}^d$ or $\wedge^2\mathbb{C}^d$ with operator norm one. As described in section 3.1 there is a natural way in which $\mathrm{GL}(\mathbb{C}, d)$ acts on $\mathbb{P}\mathbb{C}^d$. In a similar way we can make K_1 and K_2 into a $\mathrm{GL}(\mathbb{C}, d)$ space, if we define the operation $\cdot$ in each case

$$g \cdot A := \frac{gA}{\|gA\|} \quad \text{respectively} \quad (\wedge^2 g) \cdot B := \frac{(\wedge^2 g)\, B}{\|(\wedge^2 g)\, B\|}$$

for all $g \in \mathrm{GL}(\mathbb{C}, d)$, $A \in K_1$ and $K \in K_2$.

Lemma B.3.1. *Let μ be a probability measure on $\mathrm{GL}(\mathbb{C}, d)$ with $\log^+ \|g_{\omega_1}\| \in \mathcal{L}^1(\mu)$ then there exist μ-invariant measures ν_1 and ν_2 on K_1 and K_2 such that $\mu \times \nu_i$-almost surely for $A \in K_1$ and $B \in K_2$*

$$\lim_{n\to\infty} \frac{1}{n} \log \|S_{n(\omega)} \cdot A\| = \gamma_1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \|(\wedge^2 S_{n(\omega)})B\| = \gamma_1 + \gamma_2 \ .$$

Proof. We treat both cases in parallel as much as this is possible, which means in particular that if we write $\sigma_i(g, A)$ we take $A \in K_i$. First, fix $k \in \mathbb{N}$ then we can find for any integer n two integers q and $0 \leq s < k$ such that $n = mq + s$ and we decompose a product of random matrices as

$$\frac{1}{n} \log \| \wedge^i S_{n(\omega)} \| \leq \frac{1}{n} \sum_{l=qm}^{n} \log \| \wedge^i g_{\omega_l} \| + \frac{1}{n} \sum_{l=0}^{p-1} \log \| \wedge^i (g_{\omega_{(l+1)m}} \cdots g_{\omega_{lm+1}}) \| \ .$$

From the law of large numbers, we directly find the two relations

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\wedge^i S_n(\omega)\| \le \frac{1}{m}\mathbb{E}\left(\log\|\wedge^i S_m(\omega)\|\right) \tag{B.2}$$
$$\limsup_{n\to\infty}\frac{1}{n}\log\|\wedge^i S_n(\omega)\| \le \lim_{m\to\infty}\frac{1}{m}\mathbb{E}\left(\log\|\wedge^i S_m(\omega)\|\right) = \gamma_1+\delta_{i,2}\gamma_2 \ .$$

Given two invariant measures ν_i, we can define a random dynamical system according to (3.13), with the $\mu\times\nu$ invariant shift transformation $\tilde{\tau}$ defined by

$$\tilde{\tau}((g_{\omega_n})_n,\, A)\to((g_{\tau\omega_n})_n,\, g_{\omega_1}A)=((g_{\omega_{n+1}})_n,\, g_{\omega_1}A) \ .$$

With these definitions we find that the functions σ_1 and σ_2 given by

$$\sigma_1(g,A):=\log\|gA\| \quad \text{and} \quad \sigma_2(g,B):=\log\|(\wedge^2 g)B\|$$

become additive cocycles. Due to the relations $\sigma_i^+(g,A)\le\log^+\|g\|^i$, and the assumption $\mathbb{E}\left(\log^+\|g\|\right)<\infty$, we know that both cocycles are μ-integrable. Hence, the integral $\int\sigma_i^+(g,A)\mu(\mathrm{d}g)$ results in a bounded continuous function on K_i. In addition, both $\sigma_i(g,A)$ satisfy the assumptions of lemma 3.2.6. Therefore, we know that the sequences $\frac{1}{n}\sigma_i(S_{n(\omega)},A)$ converges $\mu\times\nu$ almost surely, to a $\mathcal{L}^1(\mu\times\nu)$ random variable $\widehat{\sigma}_i((g_{\omega_n})_n,A)$.

From lemma 3.2.8 we know that any limit point of the sequence $\nu_n=\frac{1}{n}\sum_{k=1}^n\mu^n*\delta_{\mathrm{Id}}$, where δ_{Id} is the point measure at the identity map in K_i, is invariant with respect to μ. We choose one such limit point ν and obtain

$$\begin{aligned}\int\sigma_i(g,A)\,\mu(\mathrm{d}g)\,\nu_n(\mathrm{d}A) &= \frac{1}{n}\sum_{n=1}^{\infty}\int\sigma_i(g,A)\mu(\mathrm{d}g)(\mu*\delta_{\mathrm{Id}})(\mathrm{d}A)\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\sigma_i(g_{\omega_{n+1}},S_{n(\omega)}\cdot\mathrm{Id})\right)\\ &= \frac{1}{n}\mathbb{E}\left(\sigma_i(S_{n(\omega)},\mathrm{Id})\right)\\ &= \frac{1}{n}\mathbb{E}\left(\log\|\wedge^i S_{n(\omega)}\|\right) \ .\end{aligned}$$

Hence, according to lemma 3.1.6, we find that for any convergent subsequence of ν_{n_l}

$$\lim_{l\to\infty}\int\sigma_i(g,A)\mu(g)\,\nu_{n_l}(\mathrm{d}A)=\lim_{l\to\infty}\frac{1}{n}\mathbb{E}\left(\log\|\wedge^i S_{n+1(\omega)}\|\right)=\gamma_1+\delta_{i2}\gamma_2 \ .$$

By Fatou's lemma, we also have

$$\limsup_{l\to\infty}\int\sigma_i(g,A)\mu(g)\,\nu_{n_l}(\mathrm{d}A)\le\int\sigma_i(g,A)\mu(g)\,\nu(\mathrm{d}A) \ ,$$

which with lemma 3.2.6 implies for the expectation value of the limiting random variable $\widehat{\sigma}_i((g_{\omega_n})_n,A)$

$$\mathbb{E}\left(\widehat{\sigma}_i((g_{\omega_n})_n,A)\right)=\int\sigma_i(g,A)\mu(g)\,\nu(\mathrm{d}A)\ge\gamma_1+\delta_{i2}\gamma_2 \ .$$

In addition, we have due to (B.2) that almost everywhere

$$\widehat{\sigma}_i((g_{\omega_n})_n, A) \leq \liminf_{n\to} \frac{1}{n} \log \| \wedge^i S_n(\omega)\| \leq \gamma_1 + \delta_{i,2}\gamma_2 \ .$$

Thus we found that on the one hand, the expectation value of $\widehat{\sigma}_i((g_{\omega_n})_n, A)$ is larger or equal than $\gamma_1 + \delta_{i,2}\gamma_2$, but on the other $\widehat{\sigma}_i((g_{\omega_n})_n, A)$ is almost surely smaller or equal to it. Hence, $\widehat{\sigma}_i((g_{\omega_n})_n, A)$ has to be $\mu_\infty \times \nu$ almost everywhere equal to $\gamma_1 + \delta_{i,2}\gamma_2$. □

B.4. The space of Hölder continuous functions

In this section we prove some of the more technical results about the Banach space of Hölder continuous functions. We require the following general result on semigroups and cocycles, which is proven in [BL85, lemma V.2.3].

Lemma B.4.1. *Let T be a topological semigroup acting on a set B and σ be an additive cocycle on $T \times B$ and let μ be a probability measure on T such that*

(i) for $r(g) = \sup_{x\in B} |\sigma(g,x)|$ there exists $\tau > 0$ such that $\mathbb{E}\left(e^{\tau r(g)}\right) < \infty$

(ii) for some positive integer p, $\sup_{x\in B} \int \sigma(g,x)\mu^n(\mathrm{d}g) < \infty$.

Then, there exists $\alpha > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\sup_{x\in B} \int e^{\alpha\sigma(g,x)}\, \mu^n(\mathrm{d}g)\right) < 1 \ .$$

The large deviation estimates, we are going to prove in the next section, rely on the following result by Le Page [LP82].

Proposition B.4.2. *Let μ be a strongly irreducible probability measure on $\mathrm{GL}(\mathbb{C}, d)$ such that (μ) is contractive and $\mathcal{F} \in \mathcal{L}^1(\mu)$ then*

(i) For any $\overline{x}, \overline{y} \in \mathbb{P}\mathbb{C}^d$ a.s.

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\delta(S_n(\omega)\overline{x}, S_n(\omega)\overline{y})}{\delta(\overline{x},\overline{y})} < 0$$

(ii) If we take the expectation value with respect to μ^n

$$\limsup_{n\to\infty} \sup_{\overline{x},\overline{y}} \frac{1}{n} \mathbb{E}\left(\log \frac{\delta(S_n(\omega)\overline{x}, S_n(\omega)\overline{y})}{\delta(\overline{x},\overline{y})}\right) < 0$$

Proof. By the definition of the metric δ in (3.10) we have

$$\frac{\delta(S_n(\omega)\overline{x}, S_n(\omega)\overline{y})}{\delta(\overline{x},\overline{y})} \leq \frac{\|S_n(\omega)x \wedge S_n(\omega)y\|\, \|x\|\|y\|}{\|x\wedge y\|\, \|S_n(\omega)x\|\, \|S_n(\omega)y\|} \leq \|\wedge^2 S_n\| \frac{\|x\|\|y\|}{\|S_n(\omega)x\|\|S_n(\omega)y\|} \ .$$

Lemma 3.1.6 about the Lyapunov spectrum tells us that the almost surely the limit of $\frac{1}{n}\log\|\wedge^2 S_{n(\omega)}\|$ is the sum of the upper two Lyapunov exponents γ_1 and γ_2. In addition the almost sure limit of the sequence $\frac{1}{n}\log\|S_{n(\omega)}x\|$ is γ_1 by (3.17). This amounts to

$$\lim_{n\to\infty}\frac{1}{n}\frac{\delta(S_{n(\omega)}\overline{x},S_{n(\omega)}\overline{y})}{\delta(\overline{x},\overline{y})}\leq\gamma_1+\gamma_2-2\gamma_1=\gamma_2-\gamma_1\ ,$$

which is strictly negative by lemma 3.2.13. Using the same upper bound for the metric δ as before we find in the second case

$$\limsup_{n\to\infty}\sup_{\overline{x},\overline{y}}\frac{1}{n}\mathbb{E}\left(\frac{\delta(S_{n(\omega)}\overline{x},S_{n(\omega)}\overline{y})}{\delta(\overline{x},\overline{y})}\right)$$
$$\leq\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\log\|\wedge^2 S_n\|\right)-2\lim_{n\to\infty}\sup_x\mathbb{E}\left(\frac{1}{n}\log\frac{\|S_{n(\omega)}x\|}{\|x\|}\right)\ .$$

The first term converges again to the sum of the first two Lyapunov exponents. Due to the integrability condition lemma 3.2.12 implies that the expectation value in the second term converges uniformly to twice the upper Lyapunov exponent, which finishes the proof if we use lemma 3.2.13 again. □

Proposition B.4.3. *Let μ be a strongly irreducible and contracting probability measure on $\mathrm{GL}(\mathbb{C},d)$, such that for some $\tau>0$ the function $\exp(\tau\mathcal{F}(g))$ is μ-integrable, then there is $\alpha_0>0$ such that for all $0<\alpha\leq\alpha_0$*

$$\lim_{n\to\infty}\left(\sup_{\overline{x}\neq\overline{y}\in\mathbb{P}\mathbb{C}^d}\int\frac{\delta(S_{n(\omega)}\overline{x},S_{n(\omega)}\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha}\,\mu^n(\mathrm{d}g)\right)^{\frac{1}{n}}<1\ . \tag{B.3}$$

This implies in particular that there exit constants $0<\rho<1$ and $C>0$ such that

$$\sup_{\overline{x}\neq\overline{y}\in\mathbb{P}\mathbb{C}^d}\int\frac{\delta(S_{n(\omega)}\overline{x},S_{n(\omega)}\overline{y})^\alpha}{\delta(\overline{x},\overline{y})^\alpha}\,\mu^n(\mathrm{d}g)\leq C\rho^n$$

Proof. Looking at the content of lemma B.4.1, we see that we are finished once we can identify a cocycle that satisfies the assumptions of the lemma. We can make the set $B=\{(\overline{x},\overline{y})\,;\overline{x},\overline{y}\mathbb{P}\mathbb{C}^d\ \overline{x}\neq\mathbb{P}\mathbb{C}^d\,y\}$ into a $\mathrm{GL}(\mathbb{C},d)$ space by defining its action on B via $M(\overline{x},\overline{y}):=(M\overline{x},M\overline{y})$. With this definition becomes the mapping σ given by

$$\sigma(M,(\overline{x},\overline{y})):=\log\frac{\delta(M\overline{x},M\overline{y})}{\delta(\overline{x},\overline{y})}$$

a cocycle on $\mathrm{GL}(\mathbb{C},d)\times B$. Inserting the definition of the metric δ from (3.10) and with the use of lemma 3.2.14 we find

$$\sup_{\overline{x}\neq\overline{y}}\sigma(M,(\overline{x},\overline{y}))=\sup_{\overline{x}\neq\overline{y}}\log\left(\frac{\|Mx\wedge My\|}{\|x\wedge y\|}\frac{\|Mx\|\|My\|}{\|x\|\|y\|}\right)\leq 4\mathcal{F}(M)\ .$$

Since we assumed that μ has an exponential moment τ, this implies assumption (i) of lemma B.4.1. To obtain assumption (ii) we note that due to part (ii) of lemma B.4.2 there exists an $n>0$ such that

$$\sup_{\overline{x}\neq\overline{y}}\int\sigma(g,(\overline{x},\overline{y}))\mu^n(\mathrm{d}g)=\sup_{\overline{x}\neq\overline{y}}\mathbb{E}\left(\log\left(\frac{\delta(S_n(\omega)\overline{x},S_n(\omega)\overline{y})}{\delta(\overline{x},\overline{y})}\right)\right)<0\ .$$

Thus, all assumptions of lemma B.4.1 are met, which implies (B.3). This means, that we have a sequence $(a_n)_n$ such that $(a_n^{\frac{1}{n}})_n$ converges to some limit smaller α than 1 picking ε such that $\alpha+\varepsilon<1$ we can find n_0 such that for all $n\geq n_0$ the relation $a_n^{\frac{1}{n}}<\alpha+\varepsilon<1$ holds, which proves the second part of the proposition. □

Lemma B.4.4. *Let μ be a probability measure on $\mathrm{GL}(\mathbb{C},d)$ such that for some $\tau>0$ the function $\exp\tau\mathcal{F}(g)$ is integrable. Then for any $0<\alpha<\frac{\tau}{2}$ there is a $\eta>0$ such that $\{R_\mu(z),|z|<\eta\}$, constitutes an analytic family of bounded operators on $\mathcal{L}_\alpha$.*

In the proof of the lemma we rely on the following two estimates:

Lemma B.4.5. *Let $z\in\mathbb{C}$ and $0<\alpha\leq 1$, then there exist finite constants $C_1,C_2>0$ such that for all $M\in\mathrm{GL}(\mathbb{C},d)$*

(i) $$\sup_{\overline{x},\overline{y}}\frac{|e^{z\log\|Mx\|}-e^{z\log\|My\|}|}{\delta(\overline{x},\overline{y})^\alpha}\leq C_1e^{((1+\alpha)|\Re z|+2\alpha)\mathcal{F}(M)}$$

(ii) $$\sup_{\overline{x},\overline{y}}\frac{|\log\|Mx\|-\log\|My\||}{\delta(\overline{x},\overline{y})^\alpha}\leq C_2\mathcal{F}(M)e^{2\alpha\mathcal{F}(M)}\quad,$$

Proof. We begin with point (i). Assuming without loss of generality $\|Mx\|\leq\|My\|$ and using the mean value theorem for the function x^z we find

$$|e^{z\log\|Mx\|}-e^{z\log\|My\|}|\leq|z|\left(\sup_{\|Mx\|\leq c\leq\|My\|}c^{\Re z-1}\right)|\|Mx\|-\|My\||\ .$$

Depending on whether the real part $\Re z$ is positive or negative, the supremum over c is upper bounded by $\|Mx\|^{-1}\|My\|^{\Re z}$ or $\|Mx\|^{\Re z-1}$, which, using the relation $|Mx|\geq\|M^{-1}\|$ for $\|x\|=1$ vectors, is in either case upper bounded by $e^{(1+|\Re z|)\mathcal{F}(M)}$. Bringing everything together, we find

$$\frac{|e^{z\log\|Mx\|}-e^{z\log\|My\|}|}{\delta(\overline{x},\overline{y})}\leq\left(|z|\sup_{\overline{x},\overline{y}}\frac{\|\overline{x}-\overline{y}\|}{\delta(\overline{x},\overline{y})}\right)e^{(2+|\Re z|)\mathcal{F}}\leq 2\sqrt{2}|z|e^{(2+|\Re z|)\mathcal{F}}=:A\ ,$$

where we used the relation $\|x-y\|\leq\sqrt{2}\delta(\overline{x},\overline{y})$ from (3.11) in the last step. In addition, we have the trivial bound

$$|e^{z\log\|Mx\|}-e^{z\log\|My\|}|\leq 2\sup_{\|x\|=1}|e^{z\log\|Mx\|}|=:B\ .$$

Therefore, for $A\delta(\overline{x},\overline{y}) \leq 1$, we obtain

$$|e^{z\log\|Mx\|} - e^{z\log\|My\|}| \leq A\delta(\overline{x},\overline{y}) \leq A^{\alpha}\delta(\overline{x},\overline{y})^{\alpha} \ .$$

For the case $A\delta(\overline{x},\overline{y}) \geq 1$ we find

$$|e^{z\log\|Mx\|} - e^{z\log\|My\|}| \leq B \leq BA^{\alpha}\delta(\overline{x},\overline{y})^{\alpha} \ ,$$

which in total gives the bound of the theorem. The second bound is proven analogously after noting that by using the mean value theorem again

$$|\log\|Mx\| - \log\|My\|| \leq \left(\sup_{\|Mx\|\leq c\leq\|My\|} c^{-1}\right)\|M\|\|x-y\| \leq e^{2\mathcal{F}(M)}\|x-y\|$$

for $\|Mx\| \leq \|My\|$ and x normalized. □

Proof of lemma 3.3.6. We first show that $R_{\mu}(z)$ for $|z|$ small enough is a bounded operator $\mathcal{L}_{\alpha}$. First note that for $f \in \mathcal{L}_{\alpha}$ and x normalized, we find

$$|(R_{\mu}(z)f)(\overline{x})| = |\int e^{z\log\|gx\|}\, f(g\overline{x})\,\mu(\mathrm{d}g)| \leq \|f\|_{\infty}\int e^{|\Re z|\mathcal{F}(g)}\,\mu(\mathrm{d}g)$$

The integral can be bounded by some constant A_1 as long as $\tau > |\Re z|$. For the second term of $\|\cdot\|_{\alpha}$ we find with the help of lemma B.4.5 the upper bound

$$\begin{aligned}\frac{|(R_{\mu}(z)f)(\overline{x}) - (R_{\mu}(z)f)(\overline{y})|}{\delta(\overline{x},\overline{y})^{\alpha}} &\leq \int \left|e^{z\log\|gx\|}\right| \frac{|f(g\overline{x}) - f(g\overline{y})|}{\delta(\overline{x},\overline{y})^{\alpha}}\,\mu(\mathrm{d}g) \\ &\quad + \int \left|f(g\overline{y})\right| \frac{|e^{z\log\|gx\|} - e^{z\log\|gy\|}|}{\delta(\overline{x},\overline{y})^{\alpha}}\,\mu(\mathrm{d}g) \\ &\leq m_{\alpha}(f)\,\mathbb{E}\left(e^{|\Re z|\mathcal{F}(g)}\right) + C_1\|f\|_{\infty}\mathbb{E}\left(e^{((1+\alpha)|\Re z|+2\alpha)\mathcal{F}(g)}\right) \\ \leq A_2(m_{\alpha}(f) + \|f\|_{\infty}) \ ,\end{aligned} \tag{B.4}$$

as long as $(1+\alpha)|\Re z| + 2\alpha \leq \tau$, which implies that for $|\Re z| \leq \frac{\tau-2\alpha}{1+\alpha}$ the operator $R_{\mu}(z)$ is indeed bounded. To prove analyticity we have to show that the operator R_{μ} can be represented as a power series of bounded operators. To this end we define

$$(R_{\mu,n}(z)f)(\overline{x}) := \int \log^n\|gx\|\, f(g\overline{x})\mu(\mathrm{d}g) \ .$$

By a telescope sum construction, we can use part (ii) of lemma B.4.5 in order to obtain the bound

$$\begin{aligned}\frac{|\log^n\|gx\| - \log^n\|gy\|||}{\delta(\overline{x},\overline{y})^{\alpha}} &\leq \frac{|\log\|gx\| - \log\|gy\|||}{\delta(\overline{x},\overline{y})^{\alpha}} \sum_{l=0}^{n-1} |\log\|gx\||^{l}\,|\log\|gy\||^{n-1-l} \\ &\leq C_2(\mathcal{F}(g))^{n+1}e^{2\alpha\mathcal{F}(g)} \ .\end{aligned}$$

□

This allows us to bound $m_\alpha(()R_{\mu,n}(z)f)$ in a similar way as in (B.4)

$$\begin{aligned}\frac{|(R_{\mu,n}(z)f)(\overline{x})-(R_{\mu,n}(z)f)(\overline{y})|}{\delta(\overline{x},\overline{y})^\alpha} &\leq \int \left|\log^n \|g\,x\|\right| \frac{|f(g\overline{x})-f(g\overline{y})|}{\delta(\overline{x},\overline{y})^\alpha}\,\mu(\mathrm{d}g) \\ &\quad + \int \left|f(g\overline{y})\right| \frac{|\log^n \|g\,x\| - \log^n \|g\,y\||}{\delta(\overline{x},\overline{y})^\alpha}\,\mu(\mathrm{d}g) \\ &\leq m_\alpha(f)\int \left|\log^n \|g\,x\|\right|\,\mu(\mathrm{d}g) + \|f\|_\infty C_2 \mathbb{E}\left((\mathcal{F}(g))^{n+1} e^{2\alpha\mathcal{F}(g)}\right)\,.\end{aligned}$$

In addition, we have for the uniform norm

$$\|R_{\mu,n}(z)f\|_\infty \leq \|f\|_\infty \sup_{\overline{x}} \int |\log^n \|g\,x\||\,\mu(\mathrm{d}g) \leq \|f\|_\infty \int (\mathcal{F}(g))^n \mu(\mathrm{d}g)\,.$$

Bringing everything together, we find that the operator norm of $R_{\mu,n}(z)$ is bounded by

$$\|R_{\mu,n}(z)\|_{\mathcal{B}(\mathcal{L}_\alpha)} \leq n\,C_2 \mathbb{E}\left((\mathcal{F}(g))^{n+1} e^{2\alpha\mathcal{F}(g)}\right) + 2\mathbb{E}\left((\mathcal{F}(g))^n\right)\,,$$

which implies the following bound on the corresponding power series of $R_\mu(z)$

$$\|R_\mu(z)\|_{\mathcal{B}(\mathcal{L}_\alpha)} \leq \sum_n \frac{|z|^n}{n!}\|R_{\mu,n}(z)\|_{\mathcal{B}(\mathcal{L}_\alpha)} \leq C_2 \mathbb{E}\left(|z|\mathcal{F}(g) e^{(2\alpha+|z|)\mathcal{F}(g)}\right) + 2\mathbb{E}\left(e^{|z|\mathcal{F}(g)}\right)\,,$$

which is finite for $|z|+2\alpha \leq \tau$. Therefore $\{R_\mu(z)\,;\,|z|<\zeta\}$ is a family of analytic operators on $\mathcal{L}_\alpha$ for $\alpha < \frac{\tau}{2}$ and $\zeta < \tau - 2\alpha$.

B.5. Regularity properties of the invariant measure

In this section we derive the following regularity property of the invariant measure that is used in the proof of lemma 3.3.11. We first state the proposition, but we need some additional results on the asymptotic behaviour of the Iwasawa-decomposition of a product of random matrices in order to prove it.

Proposition B.5.1. *Let μ be a strongly irreducible and contracting probability measure on $\mathrm{GL}(\mathbb{C},d)$ with an exponential moment $\tau > 0$ and denote by ν its unique invariant measure. Then there exists $\alpha > 0$ such that*

$$\sup_{\|y\|=1} \int \frac{\|x\|^\alpha}{|\langle x\,,\,y\rangle|^\alpha}\,\nu(\mathrm{d}\overline{x}) < \infty\,.$$

First, we introduce the Iwasawa-decomposition of a matrix.

Lemma B.5.2 (Iwasawa-decomposition)**.** *Let $M \in \mathrm{GL}(\mathbb{C},d)$ then there exists a unique pair of a unitary matrix $U(M)$ and a lower triangular matrix $s(M)$ with positive diagonal, such that*

$$M = s(M)\cdot U(M)\,.$$

The form of the decomposition as well as its uniqueness can be obtained by a Gram-Schmidt iteration on the vectors $M^*e_1,\dots,M^*e_n$ [AM07].

Lemma B.5.3. *Let μ be a non-compact and strongly-irreducible probability distribution on $\mathrm{GL}(\mathbb{C},d)$ and denote by $M_{n(\omega)}=s(M_{n(\omega)})U(M_{n(\omega)})$ the Iwasawa-decomposition of the right product $M_{n(\omega)}$, then almost surely*

$$\lim_{n\to\infty}\frac{s(M_{n(\omega)})e_1}{\langle s(M_{)(\omega)}n\,e_1\,,e_1\rangle}=\frac{Z}{\langle Z\,,e_1\rangle}\ ,$$

where $Z\in\mathbb{C}^d$ is chosen arbitrarily from the subspace $\lambda\overline{Z}$, $\lambda\in\mathbb{C}$ and $\overline{Z}\in\mathbb{P}\mathbb{C}^d$ is distributed according to the invariant measure ν on $\mathbb{P}\mathbb{C}^d$.

Proof. Since $s(M_{n(\omega)})$ is a lower triangular matrix, we obtain the relation $s^*(M_{n(\omega)})e_1=\langle s(M_{n(\omega)})e_1\,,e_1\rangle e_1$, which in turn implies $|\langle s(M_{n(\omega)})e_1\,,e_1\rangle|=\|s^*(M_{n(\omega)})e_1\|=\|M^*_{n(\omega)}e_1\|$ and

$$\frac{M_{n(\omega)}M^*_{n(\omega)}e_1}{\|M^*_{n(\omega)}e_1\|^2}=\frac{s(M_{n(\omega)})e_1}{\langle s(M_{n(\omega)})e_1\,,e_1\rangle}\ .$$

Let us consider the singular value decomposition $M_{n(\omega)}=K(n)A(n)V(n)$ with K, V unitary and A being a diagonal matrix composed of the singular values $m_{i(n)}$ in decreasing order. Since the μ is contracting proposition 3.2.11 and theorem B.2.4 imply that almost surely $\lim m_1^{-1}{}_{(n)}m_{l(n)}=0$ for all $l\geq 2$ and that $U^*(n)\overline{e}_1$ converges almost surely to a vector $\overline{Z}\in\mathbb{P}\mathbb{C}^d$. From the singular value decomposition we also obtain for an arbitrary $x\in\mathbb{C}^d$ arbitrary $\langle M_{n(\omega)}M^*_{n(\omega)_1}\,,x\rangle=\lim m_1{}_{(n)}{}^2\,\langle K(n)e_1\,,e_1\rangle\,\langle e_1\,,K_n\,y\rangle+o(m_1^{-1}{}_{(n)}m_{l(n)})$, which already implies

$$\lim_{n\to\infty}\frac{\langle s(M_{n(\omega)})e_1\,,x\rangle}{\langle s(M_{n(\omega)})e_1\,,e_1\rangle}=\lim_{n\to\infty}\frac{\langle K^*(n)e_1\,,x\rangle}{\langle K^*(n)e_1\,,e_1\rangle}=\frac{\langle Z\,,y\rangle}{\langle z\,,e_1\rangle}\ .$$

□

Proof of proposition B.5.1. We first set $y=e_1$ in equation B.5.1. Since the vector $\overline{Z}$ in lemma B.5.3 is distributed according to the invariant measure, we find

$$\int\frac{\|Z\|^\alpha}{|\langle Z\,,e_1\rangle|^\alpha}\,\nu(\mathrm{d}\overline{Z})=\lim_{n\to\infty}\mathbb{E}\left(\frac{\|s(M_{n(\omega)})e_1\|^\alpha}{|\langle s(M_{n(\omega)})e_1\,,e_1\rangle|^\alpha}\right)\ . \tag{B.5}$$

Therefore, we have to show that the expectation value on the right-hand side is finite. To simplify the argument let us introduce the following abbreviations

$$\Gamma(M):=\frac{s(M)}{\langle s(M)e_1\,,e_1\rangle}\quad\text{and}\quad\eta(M):=\Gamma(M)e_1-e_1\ . \tag{B.6}$$

So, we are interested in the quantity $\|\Gamma(M_{n(\omega)})e_1\|$. Let us consider the Iwasawa decomposition of a product AB of two matrices A and B. Depending, whether we decompose the whole product at once or first the matrix A and then the rest, we find

$$AB=s(A)s\,(U(A)B)\,U(U(A)B)=s(AB)U(AB)\ .$$

Uniqueness of the decomposition implies the relation $s(AB) = s(A)s\,(U(A)B)$ and since $s(m)$ is triangular, we even find $\langle S(AB)e_1\,,e_1\rangle = \langle S(A)e_1\,,e_1\rangle\,\langle s(U(A)B)e_1\,,e_1\rangle$. Inserting this into the definition of $\eta(M)$ from (B.6) we find

$$\eta(AB) = \eta(A) + \Gamma(M)\eta(U(A)B)\ .$$

Iterating this equation for a right product $M_{n(\omega)}$ of matrices, we obtain

$$\|\Gamma(M_{n(\omega)})e_1\| = \|e_1 + \eta(M_{n(\omega)})\| \leq \|\Gamma(g_{\omega_1})e_1\| + \sum_{k=1}^{\infty}\|\Gamma(M_{k(\omega)})\eta(U(M_{k(\omega)})g_{\omega_{k+1}})\|\ .$$

By construction, $\eta(M)$ is orthogonal to e_1 and therefore we can upper bound the summands in this equation by the following two relations using the function $\mathcal{F}$ defined in (3.22)

$$\|\eta(M)\| \leq e^{2\mathcal{F}(M)} \quad \text{and} \quad \sup_{y\perp e_1}\left(\frac{\|s(M)y\|}{\|y\|}\right) \leq \frac{\|\Lambda^2 M^*\|}{\|M^* e_1\|}\ .$$

Inserting those bounds, using the Hölder-inequality and the $i.i.d$ assumption for the random matrices $S_{k(\omega)}$, we obtain in (B.5) the upper bound

$$\begin{aligned}\int \frac{\|Z\|^\alpha}{|\langle Z\,,e_1\rangle|^\alpha}\ \nu(\mathrm{d}\overline{Z}) &\leq \sum_{k=0}^{\infty}\left(\mathbb{E}\left(\frac{\|\Lambda^2 M^*_{k(\omega)}\|^{2\alpha}}{\|M^*_{k(\omega)}e_1\|^{4\alpha}}\right)\mathbb{E}\left(e^{\,4\alpha\mathcal{F}(g_{\omega_k})}\right)\right)^{\frac{1}{2}} \\ &= \mathbb{E}\left(e^{\,4\alpha\mathcal{F}(g_{\omega_1})}\right)^{\frac{1}{2}}\sum_{k=0}^{\infty}\mathbb{E}\left(\frac{\|\Lambda^2 M^*_{k(\omega)}\|^{2\alpha}}{\|M^*_{k(\omega)}e_1\|^{4\alpha}}\right)^{\frac{1}{2}}\ .\end{aligned} \tag{B.7}$$

The first expectation value on the right-hand side is finite for $4\alpha < \tau$, because we assumed that ν has an exponential moment. In order to bound the series, we want to employ lemma B.4.1 in order to show that each summand satisfies an upper bound of the form $C\beta^n$ with C positive and $0 < \rho < 1$ in which case the corresponding geometric series is finite. To this end consider the following cocycle on $\mathrm{GL}(\mathbb{C},d)\times\left(\mathbb{P}\mathbb{C}^d \times \mathbb{P}\Lambda^2\mathbb{C}^d\right)$

$$\sigma(M,(\overline{x},\overline{y})) := \frac{\|\Lambda^2 M\,y\|\|x\|^2}{\|y\|\|M\,x\|^2}\ ,$$

where $M\,\mathrm{GL}(\mathbb{C},d)$, $x\,\mathbb{C}^d$ and $y \in \Lambda^2\mathbb{C}^d$. Using lemma 3.2.14, we obtain

$$\sup_{\overline{x},\overline{y}}|\sigma(M,(\overline{x},\overline{y}))| \leq \sup_{\overline{x},\overline{y}}|\log\frac{\|M\,x\|}{\|x\|}| + |\log\frac{\|\Lambda^2 M\,y\|}{\|y\|}| \leq 3\mathcal{F}(M)\ .$$

Therefore, $e^{\frac{\tau}{3}\sup_{\overline{x},\overline{y}}|\sigma(M,(\overline{x},\overline{y}))|}$. is integrable, because μ possess an exponential moment τ. In addition, we find for a right product $M_{n(\omega)}$ the relation

$$\sup_{\overline{x},\overline{y}}\mathbb{E}\left(\sigma(M^*_{n(\omega)},(\overline{x},\overline{y}))\right) \leq \mathbb{E}\left(\log\|\Lambda^2 M^*_{n(\omega)}\|\right) - 2\inf\mathbb{E}\left(\log\frac{\|M^*_{n(\omega)}x\|}{\|x\|}\right)\ .$$

According to lemma 3.2.12 the last term becomes close to $n\gamma_1$ and the first term converges to $n(\gamma_1+\gamma_2)$ since $\gamma_1>\gamma_2$ by lemma 3.2.13, we can find n large enough, such that the expression becomes negative. Thus are all assumptions of lemma B.4.1 met, which implies that we can indeed find for $\alpha>0$ small enough some $C>0$ and $0<\rho<1$ such that

$$\mathbb{E}\left(\frac{\|\Lambda^2 M_k^*(\omega)\|^{2\alpha}}{\|M_k^*(\omega)e_1\|^{4\alpha}}\right)\leq C\rho^k \ .$$

After evaluating the corresponding geometric series in (B.7) we therefore obtain the finite upper bound

$$\int \frac{\|Z\|^\alpha}{|\langle Z\,,e_1\rangle|^\alpha}\ \nu(\mathrm{d}\overline{Z})\leq \frac{C^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}\mathbb{E}\left(e^{4\alpha\mathcal{F}(g_{\omega_1})}\right)^{\frac{1}{2}} \ .$$

To extend this result from e_1 to general vectors normalized vectors y, we choose some unitary matrix K such that $Ke_1=y$. The transformation $g_{\omega_k}\mapsto K^*g_{\omega_k}K$ on the level of the random matrices, induces a transformation $M_k(\omega)\mapsto K^*M_k(\omega)K$ on the level of the products and Z has to be replaced by K^*Z. All the bounds are still valid in this situation and we obtain

$$\int \frac{\|Z\|^\alpha}{|\langle Z\,,k\rangle|^\alpha}\ \nu(\mathrm{d}\overline{Z})=\int \frac{\|K^*Z\|^\alpha}{|\langle K^*Z\,,e_1\rangle|^\alpha}\ \nu(\mathrm{d}\overline{Z})\leq \frac{C^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}\mathbb{E}\left(e^{4\alpha\mathcal{F}(K^*g_{\omega_1}K)}\right)^{\frac{1}{2}} \ .$$

This finishes the proof, because $\mathcal{F}$ is unchanged under unitary conjugations of its argument. □

C. Supplement to the Wegner estimate

In this appendix we provide proofs on two general results needed in the proof of the Wegner estimate in section 6.3.

C.1. Proof of proposition 5.7.12

From the first order Taylor expansion of $\exp(x)$ and the mean value form of the remainder one can derive the inequality

$$\exp(x) \leq 1 + x + x^2 \exp(|\log x|) \ ,$$

which implies for $x = \log(a^{-\delta})$

$$a^{-\delta} \leq 1 - \delta \log x + \delta^2 |\log x|^2 e^{\delta |\log x|} \ .$$

In the following we are going to use a slightly strengthened version of this inequality, which is given by

$$a^{-\delta} \leq 1 - \delta \log x + \delta^2 |\log(x^2)| e^{\delta |\log x|} \ . \tag{C.1}$$

Note that neither of the right-hand sides can be bounded by the other one. The second version can be obtained via a case distinction. One case is given when the right-hand side of the last equation is lower bounded by the right-hand side of the first inequality. In the remaining case one can lower bound the right-hand side by the expression $-\delta + e^{\delta |\log x|}$, which can be shown to satisfy the relation.

Let us now fix some normalized $\phi \in \mathbb{C}^2$ and $\Theta \in I$. Inserting the logarithm of the argument of the expectation value of interest into (C.1) we find

$$\begin{aligned} \|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\|^\delta \leq 1 - \delta \log \|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\| \\ + 2\delta^2 |\log \|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\|| e^{\delta |\log \|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\||} \ . \end{aligned} \tag{C.2}$$

Since the norm of transfer matrices is lower bounded by one and equal to the norm of its inverse, the norm of their product satisfies

$$|\log \|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\|| \leq \sum_{i=1}^{n} \log \|T_i(\Theta)\| \ .$$

Inserting this into (C.2), taking the expectation value and once using Hölder's inequality as well as the independence of different transfer matrices yields

$$\begin{aligned} \mathbb{E}\left(\|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\|^\delta \right) \leq 1 - \delta \mathbb{E}\left(\log \|T_n(\Theta) \cdot \ldots \cdot T_1(\Theta)\phi\| \right) \\ + 2\delta^2 n \mathbb{E}\left((\log \|T_1(\Theta)\|)^2 \right)^{1/2} \mathbb{E}\left(\|T_1(\Theta)\|^2 \right)^{n/2} \ . \end{aligned}$$

The integrability condition on μ ensures that the expectation values in the third summand are finite if we choose $2\delta \leq \zeta$.

By proposition 5.7.14 we know that the Lyapunov exponent $\gamma(\Theta)$ is continuous with respect to Θ and that the sequence $n^{-1}\mathbb{E}\left(\log\|T_{n(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi\|\right)$ converges to the $\gamma(\Theta)$ uniformly in the arguments ϕ and γ. Set $\gamma = \inf_{\Theta\in I}\gamma(\Theta)$. Then we can find $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\mathbb{E}\left(\|T_{n(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi\|^{\delta}\right) \leq 1-\delta n(\frac{1}{2}\gamma - \delta C_1 C_2^n) \leq 1-\varepsilon$$

for some $\varepsilon > 0$, where the constants C_1 and C_2 are given by the supremum over the interval I. This gives us the starting point for the chaining argument. We can decompose a given $n \in \mathbb{N}$ as $n = kn_0 + r$ with $0 \leq n_0 - 1$. Using the inequality $\|AB\|^{-}1 \leq \|A^{-1}\|$ we get

$$\begin{aligned}\mathbb{E}\left(\|T_{n(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi\|^{\delta}\right) \leq \mathbb{E}\left(\|T_{1(\Theta)}\|^{\delta}\right)^{n_0}\cdot \\ \mathbb{E}\left(\|T_{n(\Theta)}\cdot\ldots\cdot T_{n_0+1(\Theta)}\frac{T_{n_0(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi}{\|T_{n_0(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi\|^{-\delta}}\|\|T_{n_0(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi\|^{-\delta}\right) \\ \leq \widetilde{C}(1-\varepsilon)\sup_{\|\phi\|=1}\mathbb{E}\left(\|T_{n(\Theta)}\cdot\ldots\cdot T_{n_0+1(\Theta)}\phi\|\right)\ ,\end{aligned}$$

where the constant $\widetilde{C}$ is given by the supremum of the expectation value of $\|T_{1(\Theta)}\|^{\zeta/2}$ over all $\Theta \in I$ to the power of n_0. Iterating this decomposition gives the desired estimate

$$\mathbb{E}\left(\|T_{n(\Theta)}\cdot\ldots\cdot T_{1(\Theta)}\phi\|^{\delta}\right) \leq C(1-\varepsilon)^n \leq e^{-\alpha n}\ ,$$

for some $\alpha > 0$ and n larger then some n_{min}.

C.2. A variant of Temple's inequality

This version of temple's inequality is used in the proof of lemma 6.2.2 in section 6.2. The self-adjoint case appeared in [ST85], but we include the short proof for completeness.

Lemma C.2.1. *Let U by a unitary matrix, $\{\Phi_l\}$ a set of k orthonormal vectors and $\theta_0 \in \mathbb{T}$ such that*

1. $\|(U-\theta_0)\phi_l\| \leq \varepsilon$
2. $\langle\phi_l, U\phi_{l'}, =\rangle 0, \forall l \neq l'$

Then U has at least k eigenvalues within the interval $I(\theta_0, \varepsilon)$.

Proof. Define $V = \mathrm{span}(\Phi_l)$. Then using 1. and 2. we have for every $\psi \in V$

$$\|(U-\theta_0)\psi\|^2 = \sum_l \|(U-\theta_0)\Phi_l\|^2 |\langle\Phi_l, \psi\rangle|^2 \leq \varepsilon^2\|\psi\|\ .$$

Now assuming U has only $k-1$ eigenvalues within the interval $I(\theta_0, \varepsilon)$ we could find $\psi \in V$ which is orthogonal to the combined eigenspace of the interval $I(\theta_0, \varepsilon)$. Therefore

we can decompose ψ into a sum of eigenvectors $\widehat{\Phi}_l$ with corresponding eigenvalues outside $I(\theta_0, \varepsilon)$, which yields the desired contradiction

$$\|(U-\Theta_0)\psi\|^2 = \sum_l \|(U-\theta_0)\widehat{\Phi}_l\|^2 |\langle \widehat{\Phi}_l, \psi \rangle|^2 > \varepsilon^2 \|\psi\| \ .$$

□

Corollary C.2.2. *Let U by a unitary matrix, $\{\Phi_l\}$ a set of k orthonormal vectors and $\theta_l \in I(\theta_0, \varepsilon)$ for some $\theta_0 \in \mathbb{T}$ such that*

1. $\|(U-\theta_l)\phi_l\| \leq \varepsilon$
2. $\langle \phi_l, U\phi_{l'}, = \rangle 0, \forall l \neq l'$

Then U has at least k eigenvalues within the interval $I(\theta_0, 2\varepsilon)$.

Proof. Because $\|(U-\Theta_0)\phi_l\| \leq \|(U-\Theta_l)\phi_l\| + \|(\Theta_l-\Theta_0)\phi_l\| \leq 2\varepsilon$ holds, all assumptions of lemma C.2.1 are fulfilled with 2ε. □

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