Classical Solutions for Stabilized Periodic Hele-Shaw Flows with a Free Surface

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation
von

M.Sc. Michael Wenzel
geboren am 23. April 1985 in Celle

2013

Referent: Prof. Dr. Joachim Escher
Korreferent: Prof. Dr. Gerhard Starke
Tag der Promotion: 12. November 2013

Abstract. The flow of incompressible Stokesian fluids obeying Darcy's law in vertical Hele-Shaw cells that have a stabilizing source at the bottom is studied. A proper analytic framework for the free surface of the fluid and a velocity potential in a periodic geometry is developed leading to a free boundary problem for the potential and the free surface. Two different types of problems are addressed.

As a first result the existence of a unique classical solution for the flow of a non-Newtonian ferrofluid under the influence of gravity, surface tension, and magnetic forces is proved for small initial data. Moreover, the influence of the parameters in the model and of the external source on the stability of flat solutions is studied and global existence for solutions sufficiently close to a flat solution as well as exponential stability or instability of flat solutions is proved.

The well-posedness of the flow of a Newtonian fluid is proved for large initial data in the sense that unbounded sets of initial conditions for which a unique classical solution exists are characterized. This second result is developed for flows driven either by gravity only or by gravity and surface tension effects.

The aforementioned results are proved with methods from the theory of analytic semigroups and abstract parabolic equations of fully nonlinear and of quasilinear type. Moreover, maximal regularity arguments and results from the theory of Fourier multipliers are used.

In a third part a numerical scheme is developed that solves the flow problem for Newtonian fluids. A spline interpolation is chosen for the representation of the free surface and the finite element method is used to approximate the solution of the potential problem. Though an explicit Euler scheme is used for the evolution in time, accurate simulation results are obtained in the sense that the stability results derived from the abstract theory are reproduced numerically.

Keywords: Darcy's law; free surface flow; parabolic evolution equation; maximal regularity; fully nonlinear equation; quasilinear equation; stability analysis; finite element method

AMS Subject Classification: 35J57; 35K90; 35Q35; 65M60; 76B07

Zusammenfassung. Der Fluss von inkompressiblen Stokesschen Flüssigkeiten in vertikalen Hele-Shaw Zellen wird betrachtet. Die Flüssigkeit habe ein Geschwindigkeitspotential, das das Darcysche Gesetz erfülle, und die Hele-Shaw Zelle habe eine externe Quelle an ihrer unteren Randkomponente. Der obere Rand der Flüssigkeit ist eine freie Oberfläche, die zusammen mit dem Potential als freies Randwertproblem in periodischer Geometrie formuliert wird. Zwei Arten dieses Problems werden behandelt.

Zunächst wird die Existenz einer eindeutigen klassischen Lösung des Problems für nicht-Newtonsche Ferrofluide für kleine Anfangswerte gezeigt. Der Fluss unterliegt der Gravitations- und einer magnetischen Kraft sowie dem Einfluss durch Oberflächenspannung, deren Effekte zusammen mit dem Einfluss der externen Quelle auf die Stabilität flacher Lösungen untersucht wird. Die Existenz von globalen Lösungen nahe flacher Lösungen sowie die exponentielle Stabilität oder die Instabilität flacher Lösungen wird bewiesen.

Als zweites Hauptresultat wird die Wohlgestelltheit des Flusses von Newtonschen Flüssigkeiten für große Anfangswerte bewiesen. Unbeschränkte Mengen von zulässigen Anfangsbedingungen werden für Flüsse, die entweder nur Gravitation oder Gravitation und Oberflächenspannung unterliegen, beschrieben.

Die obigen Resultate werden mit Methoden der Theorie analytischer Halbgruppen und abstrakter parabolischer Gleichungen voll nichtlinearen oder quasilinearen Typs bewiesen. Zusätzlich werden Argumente aus der Theorie der maximalen Regularität und der Fourier Multiplikatoren benutzt.

Ein dritter Teil beschäftigt sich mit einem numerischen Verfahren um das freie Randwertproblem für Newtonsche Flüssigkeiten zu lösen. Hierbei wird für die Interpolation der freien Oberfläche ein Spline genommen und das Potentialproblem wird mit der Methode der finiten Elemente gelöst. Obwohl ein explizites Euler Verfahren für die Zeitdiskretisierung benutzt wird, ergeben sich präzise Simulationen. Insbesondere werden die oben genannten abstrakten Stabilitätsergebnisse numerisch verifiziert.

Schlagworte: Darcy Gesetz; Fluss mit freier Oberfläche; parabolische Evolutionsgleichung; maximale Regularität; voll nichtlineare Gleichung; quasilineare Gleichung; Stabilitätsanalyse; Methode der finiten Elemente

Preface

The field of research of applied mathematics combining real world problems with abstract mathematical knowledge forms an intriguing branch of science. A little contribution to this vast field is the thesis at hand constituting an advancement in the understanding of both the behavior of fluids under certain conditions and the mathematical theory of free boundary problems.

Hele-Shaw flows and also other types of free boundary problems have permanently been an object of research in mathematics for over a century and many results on their solvability and qualitative behavior are available. In particular, the approach of equivalently reformulating a free boundary problem, which in general depends on more than one function, as an evolution equation for the part of the boundary that is not fixed only has proved to be very fruitful for the analysis of this kind of problems. Besides the question of the solvability it is the statements on the qualitative behavior of these problems that show the particular strength of this approach: the possibility to obtain a feedback from the mathematical analysis to the real world application, which is also one aspect of this thesis.

In the introductory part of the thesis the groundwork is laid to advance properly to the main results concerning the classical solvability of Hele-Shaw free boundary problems. Both the modeling aspects—providing a sound understanding of basic concepts in fluid dynamics and the proper formulation of equations governing Hele-Shaw flows—and the most important theorems from the mathematical theory for attacking free boundary problems with the approach described in the preceding paragraph are presented.

The equations governing the motion of a fluid in a Hele-Shaw cell—Darcy's law—are widely accepted and numerous derivations of the classical Darcy law are available. However, I chose to include a detailed survey of the derivation of this law because for the generalized Darcy law as it is assumed to hold for a large class of fluids I could not find a rigorous derivation from first principles in the literature that discloses its validity in detail.

The expository part on parabolic evolution equations and analytic semigroups has the same intention as the part on the modeling of Hele-Shaw flows: The reader is given a concise summary on the most important results from this field of mathematics to quickly acquire the abstract knowledge needed for the ensuing main parts of the thesis. Also here I decided to concentrate on a certain topic: the interpolation property of the function spaces in which the free boundary problems are formulated. This result is widely used in the literature and it is of utmost importance for the derivation of the main results also in this thesis. However, a *proof* of it does not seem to exist in the literature, yet.

The results in the thesis at hand were developed during the last two and a half years at Leibniz Universität Hannover, where I had the pleasure to be a member of the International Research Training Group 'Virtual Materials and Structures and Their Validation' granted by the Deutsche Forschungsgemeinschaft. I am thankful for the financial and educational support I had in this period of time. In particular, I would like to express my gratitude to my advisor Joachim Escher, who had always interest in my work and who was ready to have a discussion if need be. But also the communication in the Research Training Group and at the Institute of Applied Mathematics—specifically, the exchange with the research group of Gerhard Starke, whose readiness to get involved as a referee I do appreciate—was a benefit; for this thesis and personally.

The part of this thesis that does not have an abstract mathematical character—the numerical simulation of Hele-Shaw flows—came into being during a stay at Ecole Normale Supérieure de Cachan in France. Though several approaches have already been made to solve free boundary problems, I chose to use this stay abroad to develop my own approach being suitable for the simulation of Hele-Shaw free boundary problems as they are studied in this thesis. The aim of this was the validation and the visualization of the abstract results in the qualitative study of solutions of the considered problem as mentioned above. I am particularly grateful to Pierre Gosselet from Laboratoire de Mécanique et Technologie for being his guest in Cachan in this context.

One last gratitude appertains to my wife Daniela; for her love and steady support during the last years, which were sometimes a strain.

Hannover, July 2013 Michael Wenzel

Contents

Chapter I

Introduction and Outline

In mathematical fluid dynamics free boundary problems constitute a class of problems that has gained a lot of research interest over the last decades. In these problems a state variable describing some property of a fluid (e.g., pressure) complies to a certain set of equations in an a priori unknown domain where the fluid is located. This means that determining the domain of the flow problem is itself a part of the flow problem. A major difficulty in treating these free boundary problems stems from this circumstance.

Let us begin with agreeing on some basic conventions that are crucial for the formulation of free boundary problems. First, a *continuum* approach is always understood. This means that no phenomena on the microstructure of the respective materials are considered. Only the macroscopic behavior is of interest in such a way as all state variables are continuous functions over their respective domains being open subsets of the Euclidean space.

Second, the boundaries of the domains are *sharp*. This means that every domain occupied by a specific fluid is saturated so that there are no zones of phase transition where a mixing of fluids occurs. As a consequence, the fluid domains can be ideally separated from each other and their respective boundaries can be unambiguously identified. Thus, finally, determining the domain of a flow problem means determining the unknown parts of these boundaries of the different domains.

In general, the unknown boundaries are understood to be *moving*, i.e., they continuously change their shape and their location. In contrast, there are problems whose domain is unknown but *stationary*. This means that for such a free boundary problem the domain has to be determined *once* and for all. However, free boundary problems having a moving boundary exhibit a time-dependent domain, i.e., at *every* time the domain of the problem has to be determined anew. They are often referred to as *moving* boundary problems whereas problems with stationary boundaries are widely called *free* boundary problems. However, in this thesis these two cases are not distinguished and usually the term 'free boundary problem' is employed even if the boundaries are not stationary.

There are many examples of free boundary problems appearing in a wide range of applications, e.g., civil and chemical engineering, see [10], [18], and [22]. One outstanding example is the flow of a fluid in a porous medium, e.g., ground water in the soil, seepage in a dam, oil in a sediment layer, or a solute in a packed bed. Thin films in the context of lubrication processes and heat flux in melting processes are further examples. But also in medical science free boundary problems can be of interest in helping to understand phenomena like the growth of tumors, which can interestingly be modeled as fluids, see, e.g., [24] and [25].

Besides classifying free boundary problems according to the field of research they come from, a classification of the respective phase boundaries is reasonable, too. Here, any two phases having different physical properties regarding their tendency to mix with one another can be separated by a sharp interface. For instance, a separation due to different states of aggregation of a fluid, e.g., a liquid-gaseous boundary, can be considered. But also two liquid fluids can be separated by a sharp interface, e.g., if one fluid is hydrophilic and the other one is hydrophobic or even if they are different aqueous solutions. Furthermore, if the phases do not consist of 'real' fluids but, e.g., of a tumor in an organ, the phase boundary is between healthy tissue and the tumor.

I.1 The Hele-Shaw Model

The first investigations leading to free boundary problems are certainly the works by J. Stefan and H. S. Hele-Shaw in the late 19th century. The Austrian physicist J. Stefan investigated the formation and melting of polar ice, see [69], [70], and also [56]. Here, a liquid-solid boundary between ice and water is considered and changes in state of aggregation due to changes in temperature lead to a moving interface between these two phases. Depending on the assumed temperature of the ice two different types of this problem can be studied. If the temperature of the ice is assumed to be uniformly 0 °C, the ice phase is considered as an empty phase where no motion is present. The remaining liquid phase constitutes a *one-phase* free boundary problem. However, if a non-uniform temperature distribution in the ice phase is assumed, both phases are relevant leading to the *two-phase* Stefan problem. The flow of a fluid in a so-called Hele-Shaw cell conceived by and named after British scientist H. S. Hele-Shaw also leads to a (one-phase) free boundary problem, which can formally be considered as a quasi-static Stefan problem. In such a cell, which consists of two parallel plates at a narrow distance to each other, the Navier-Stokes equations for the motion of a fluid can be reduced to an elliptic equation for one state variable defining—together with suitable boundary conditions—a boundary value problem on the a priori unknown domain of the fluid.

Hele-Shaw cells can be arranged in two different ways essentially. The plates can be aligned horizontally or vertically. In the first case the cell is parallel to the ground floor and a fluid blob between the plates of the cell can develop intricate shapes under certain conditions, which is known as fingering or Saffman-Taylor instability. In a vertical cell the fluid behaves as in a basin filling it to the bottom.

In this work the focus lies on the vertical orientation of a Hele-Shaw cell. The free surface of a fluid in such a cell is considered to be the graph of a continuous single-valued function and the motion of the free surface is reduced to a single equation for this function only. A first existence result for the free boundary problem governing the flow of a fluid in such a cell on an unbounded domain was given by J. Duchon and R. Robert in [21]. Taking into account surface tension effects they prove the existence of a local solution, which can be extended to a global one under certain restrictions. However, no statement on the uniqueness of solutions is given there. Another approach was given by H. Kawarada and H. Koshigoe in [49], where they prove local existence of a solution in Sobolev spaces. Also there no results on the uniqueness of solutions are given. Moreover, their results exhibit a loss of regularity of the solution. It was not until the publication of [34] by J. Escher and G. Simonett that the question of uniqueness of solutions was answered by formulating the problem as an abstract Cauchy problem and by applying maximal regularity arguments and results from the theory of analytic semigroups and Fourier multipliers to this Cauchy problem. This approach proved to be well suited for many free boundary problems. For instance, it was applied by J. Escher and G. Prokert in [33] to prove the local well-posedness of the same set of equations in a periodic geometry for small initial data. Moreover, a first result on the stability of solutions can be found there.

For the flow of a fluid in a horizontal (multi-dimensional) Hele-Shaw cell J. Escher and G. Simonett and, independently, X. Chen, J. Hong, and F. Yi were the first to show existence and uniqueness of a classical solution, cf. [37], [36] and [16], respectively. A few years before, Reissig showed the existence of analytic solutions to a Hele-Shaw problem, see [63]. However, this was shown for a planar, i.e., two-dimensional, flow problem only.

While all results mentioned above are shown only for Newtonian fluids, there are more recent results for a larger class of fluids. For Newtonian fluids a *linear* boundary value problem has to be solved at every time, i.e., for a known fixed domain where the fluid is located. However, the extension to *non*-Newtonian (or *generalized* Newtonian) fluids leads to a *nonlinear* boundary value problem. A governing equation for this setting was proposed by L. Kondic, P. Palffy-Muhoray, and M. J. Shelley in [50]. Using this approach, J. Escher and B.-V. Matioc were able to show existence and uniqueness of a classical solution for non-Newtonian Hele-Shaw flows in vertical Hele-Shaw cells, see [29], [30], [31], and [32]. Together with A.-V. Matioc they extended these results to non-Newtonian fluids in horizontal Hele-Shaw cells, see [28].

I.2 Outline of the Thesis

The aim of the thesis at hand is the development of the theory of classical solutions of a free boundary problem for the flow of a Stokesian fluid in a vertical Hele-Shaw cell. The focus lies on the structure of a special boundary condition and its influence on the solvability of the corresponding free boundary problem. Qualitative investigations—in particular, stability results of special solutions—are of interest, too.

In Chapter II a thorough derivation of the equations of motion in a Hele-Shaw cell leading to the free boundary problem of interest is given. Fundamental relations from continuum mechanics are derived from first principles and they are used to go (almost) all the way to the Navier-Stokes equations. However, as these equations are not needed in the first place, the main goal is the derivation of the constitutive equation governing the motion of a fluid under the special restrictions in the Hele-Shaw geometry: Darcy's law. This is done in a rather detailed manner—in particular, with precise computations in dimensionless variables involving power series for a (small) parameter.

Serving as a preparation for the mathematical analysis in the main part of this thesis Chapter III is devoted to the presentation of results from the theory of abstract parabolic equations and analytic semigroups as they are needed in the following chapters. Moreover, the basic function spaces in which the free boundary problems are posed are introduced. Several properties of these spaces are proved; in particular, an important interpolation property with the continuous interpolation method.

The results by J. Escher and B.-V. Matioc from [29], [30], and [31] and in particular the joint work with A.-V. Matioc in [27] on ferrofluids are the origin of the present thesis. In Chapter IV existence and uniqueness of a classical solution for a Hele-Shaw free boundary problem modeling the flow of a non-Newtonian ferrofluid in a vertical Hele-Shaw cell is proved. Moreover, conditions on the parameters in the model are derived under which flat surfaces are exponentially stable or unstable. An exposition of parts of this chapter and also bifurcation results for nontrivial stationary solutions can be found in [38].

In Chapter V the point of perspective changes from an abstract analytical to a numerical one. The free boundary problem for which existence and uniqueness of a classical solution is shown in the preceding chapter is considered in its Newtonian version, and a numerical scheme visualizing the motion of the free surface of a fluid for a given initial condition is developed. Furthermore, the chapter has the aim to reproduce the stability results from Chapter IV in a visualized manner, which makes it possible to study the behavior of the solution in the stable and unstable regime. The simulations are implemented in Matlab.

Numerical simulations raise the question what initial conditions are admissible so that the Hele-Shaw problem under consideration has a unique classical solution. Chapter VI being the final part of the thesis at hand extends known results on the existence and uniqueness of classical solutions of Hele-Shaw flows in such a way as (in some way) maximal sets of initial conditions are described for which unique classical solutions exist.

Chapter II

Modeling Aspects of Hele-Shaw Flows

A Hele-Shaw cell is a container consisting of two parallel plates located at a small distance to each other. At the end of the 19th century, as a part of his probably most famous experiments concerning the stream line flow of a fluid past an object visualized in this manner, H. S. Hele-Shaw discovered a sudden and drastic change in behavior of the flow of a fluid in such a cell from a turbulent to a stable velocity-independent stream line flow when he placed the plates at a sufficiently small distance to each other in his experiments. These results were published in the award-winning paper [45]. A presentation of the experiments and findings concerning Hele-Shaw's discovery and a further description of his life and research can be found in the *Obituary Notices of Fellows of the Royal Society* from his year of death 1941.

Suppose that a typical length scale of the plates in a Hele-Shaw cell in, say, x- and y-direction is L and that the distance between them (in z-direction) is b, see Figure II.1. Then the relation between these two values—the aspect ratio—is expressed by $\varepsilon := b/L$. If $\varepsilon \ll 1$ is sufficiently small, the flow in such a cell behaves as in Hele-Shaw's experiment and becomes uniform in z-direction. It turns out that in this case the distance of the two plates can be considered as

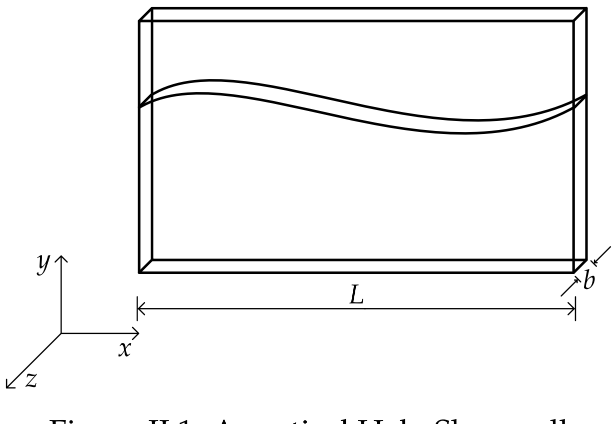


Figure II.1: A vertical Hele-Shaw cell

infinitesimal, and the z-coordinate, where the flow is uniform, can be neglected. This, in turn, means that the flow of a fluid trapped between the plates of a Hele-Shaw cell can be considered as two-dimensional. In the following these aspects are investigated in more detail.

II.1 The Equations of Motion

We begin the presentation of the equations of motion of a fluid in a Hele-Shaw cell by describing the behavior of a fluid as a continuum and by exploiting first principles from fluid mechanics. The concept of a control volume, the physical interpretation of integral identities, and mathematical results on mutually transforming volume and surface integrals are the main ingredients. At the end the continuum approach allows the derivation of partial differential equations (PDEs) governing the motion of a fluid, of which the Navier-Stokes equations are the most prominent ones. For their derivation we closely follow the books [43] and [67]. Several assumptions on the flow behavior lead to simplifications of these equations, which eventually admit the statement of a free boundary problem for Hele-Shaw flows.

Studying a fluid on the very microstructure, i.e., on a molecular level, might appear to be the most accurate approach to describing the behavior of a fluid. However, there are two severe drawbacks: On the one hand, relativistic uncertainties in location and momentum of a molecule prohibit the availability of precise values for a quantity making it impossible to perform reasonable computations. On the other hand, the mere amount of molecules in a fluid portion visible to the naked eye exceeds probably every existing storage and computation power or, at least, causes huge computational costs that are in no reasonable relation to the fluid portion under consideration. Fortunately, the continuum approach to describing the motion of a fluid remedies these defects.

II.1.1 The Continuum Hypothesis and Particle Paths

Following [61, p. 2] we introduce the notion of a fluid particle. Let Ω be the domain in Euclidean space covered by a fluid. As already mentioned in the introduction, every property of a fluid is given by a continuous function $f\colon \Omega \to \mathbb{R}$. This means that at every point $\xi \in \Omega$ the respective quantity can be evaluated as $f(\xi)$. To disclose the substantial difference to the molecular approach, where ξ would be identified with a *molecule* at this position, we define a *particle* as the amount of fluid in a sufficiently small (infinitesimal) volume V_ξ around the chosen point ξ. In V_ξ any property of the fluid is *assumed* to be uniform and we identify the *point* ξ with a corresponding *particle* V_ξ. In the literature on fluid mechanics this is referred to as the *continuum hypothesis*.

It has to be emphasized that a particle V_ξ must be *saturated* in the sense that in V_ξ *enough* molecules have to be present to give a reasonable (average) value for some quantity in this particle. This is related to a minimal density of the fluid. The density of a liquid fluid (nearly) always exceeds this minimal density.

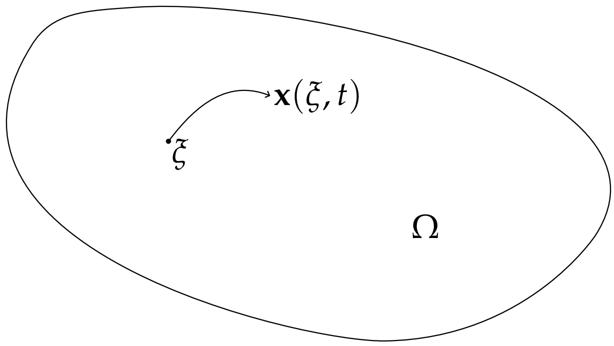


Figure II.2: A fluid continuum with a particle path

However, it may happen that the density of a gas goes below this limit, e.g., air in the atmosphere at the border to space.

A second case where the continuum approach is not valid is when effects appearing on smaller scales than V_ξ have to be taken into account. Then one has to work on this finer (probably molecular) scale. An example for this is the three-dimensional study of thin films in lubrication processes whose heights are that minuscule that working on a finer scale is indispensable.

With the notion of a fluid particle we can introduce the notion of a particle path. Let Ω be open and let $\xi \in \Omega$ be a particle. Its movement in the fluid can be parametrized by a curve $\mathbf{x}(\xi, \cdot)\colon (-T, T) \to \Omega$ for some $T > 0$ with the convention that $\mathbf{x}(\xi, 0) = \xi$, see Figure II.2. The mapping $\mathbf{x}(\xi, \cdot)$ is assumed to be continuously differentiable and its derivative is the velocity $\mathbf{v}$ of the particle. This means that

$$\frac{d}{dt}\mathbf{x}(\xi, t) = \mathbf{v}\big(\mathbf{x}(\xi, t), t\big). \tag{II.1}$$

This is an ordinary differential equation (ODE) and the function $\mathbf{x}$ is the solution of the initial value problem

$$\frac{d}{dt}\mathbf{x}(\xi, t) = \mathbf{v}\big(\mathbf{x}(\xi, t), t\big), \quad \mathbf{x}(\xi, 0) = \xi. \tag{II.2}$$

This initial value problem is posed in *Lagrangian* coordinates, i.e., the dependent variable is the (moving) particle ξ. It is convenient to choose other coordinates where a spatial point is the dependent variable. If we assume the mapping $\mathbf{x}$ to be invertible in its first component, this transformation is readily achieved turning problem (II.2) into

$$\frac{d}{dt}\xi(\mathbf{x}, t) = \mathbf{v}\big(\xi(\mathbf{x}, t), t\big), \quad \xi(\mathbf{x}(0), 0) = x. \tag{II.3}$$

Problem (II.3) having the spatial point $\mathbf{x}$ as dependent variable is said to be in *Eulerian* coordinates. Note that in contrast to problem (II.2) the total time derivative in problem (II.3) can be computed using the chain rule since the dependent variable $\mathbf{x}$ now depends on t. Thus differentiating with respect to t and using (II.1) gives

$$\frac{d}{dt}\xi = \frac{\partial}{\partial t}\xi + \frac{d}{dt}\mathbf{x} \cdot \nabla\xi = \frac{\partial}{\partial t}\xi + \mathbf{v} \cdot \nabla\xi. \tag{II.4}$$

The operator $D/Dt := \partial/\partial t + \mathbf{v} \cdot \nabla$ in this equation is sometimes called *material derivative*.

Consider an open subset $V_0 \subset \Omega$ of (infinitely many) fluid particles. In analogy to a particle path the sets

$$V(t) := \{\mathbf{x}(\xi, t) \colon \xi \in V_0\} \quad \text{for all} \quad 0 \le t < T \tag{II.5}$$

can be defined. The set V_0 is referred to as a *control volume* and the sets $V(t)$ represent the fluid portion at time $t > 0$ occupied by the particles that initially form V_0. Their respective boundary is denoted by $S(t) := \partial V(t)$. Observe that $V(0) = V_0$ and—for the sake of completeness—put $S_0 := S(0)$. With this notion of a control volume the equations of motion of a fluid are derived in the next subsection.

II.1.2 Conservation Laws

Let us begin with characterizing the fluids studied in the following in more detail by formulating some postulates that are assumed to hold. First, all processes are assumed to be isothermal so that effects caused by a nonhomogeneous temperature distribution are neglected. Second, external body moments are assumed to be absent so that the stress tensor (see (II.10)) of a fluid is symmetric, cf. [67, p. 183] and [6, Section 5.13]. Finally, all considered fluids are assumed to be of *Stokesian* type according to the following definition from [6, p. 107]:

(1) The stress tensor is a continuous function of the deformation tensor (see (II.11)) and the local pressure distribution only.

(2) The fluid is homogeneous, i.e., the stress tensor is independent of the spatial variables.

(3) The fluid is isotropic, i.e., there is no directional preference.

(4) Without deformation the stress tensor consists of the hydrostatic pressure only.

The author of [6] points out that 'a large class' of real fluids satisfies these conditions. In fact, all fluids in upcoming examples do so.

For performing integral manipulations properly in the sequel the following theorem from integration theory known as Reynolds' Transport Theorem, see [43, Eq. (1.3)], is indispensable.

Theorem II.1. *Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be open. Let $V_0 \subset \Omega$ be open, too. For a sufficiently smooth function $f \colon \Omega \to \mathbb{R}$ it holds that*

$$\frac{d}{dt} \int_{V(t)} f(x,t)\, dx = \int_{V(t)} \Big(\frac{\partial}{\partial t} f(x,t) + \operatorname{div}(f\mathbf{v})(x,t) \Big)\, dx,$$

where $V(t)$ for $0 < t < T$ is defined in (II.5) *and* $\mathbf{v}$ *is the velocity from* (II.1).

This theorem is widely used in continuum mechanics. Here it serves as a tool for deriving the equations governing conservation of mass and conservation of momentum in a fluid. Note that $\mathbf{v}$ is assumed to be a column vector.

To deduce an equation for conservation of mass let ρ denote the density of a fluid being a function of space and time. Reynolds' Transport Theorem yields

$$\frac{d}{dt}\int_{V(t)} \rho(x,t)\,dx = \int_{V(t)} \Big(\frac{\partial}{\partial t}\rho(x,t) + \operatorname{div}(\rho\mathbf{v})(x,t)\Big)\,dx.$$

Since the expression on the left hand side is the total mass of the fluid portion $V(t)$, the assumption of conservation of mass means

$$\frac{d}{dt}\int_{V(t)} \rho(x,t)\,dx = \int_{V(t)} \Big(\frac{\partial}{\partial t}\rho(x,t) + \operatorname{div}(\rho\mathbf{v})(x,t)\Big)\,dx = 0.$$

The integrand in the right expression has to vanish since $V(t)$ and t are arbitrary. Hence, the differential equation for conservation of mass is

$$\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{v}) = 0. \tag{II.6}$$

If the fluid is assumed to be incompressible and homogeneous[1], the density ρ is a constant and thus equation (II.6) simplifies to

$$\operatorname{div}\mathbf{v} = 0. \tag{II.7}$$

This equation is called *continuity equation*. Note that, of course, equations (II.6) and (II.7) are supposed to hold for $x \in \Omega$ and $0 < t < T$.

From Newton's second law, which states that the change of momentum of a body equals the total of forces exerted on that body, an equation for the conservation of momentum in a fluid can be derived. The momentum of a fluid portion $V(t)$, cf. (II.5), is given by

$$\int_{V(t)} \rho\mathbf{v}\,dx,$$

where again $\mathbf{v}$ is the velocity from (II.1) and ρ is the fluid's density. The forces on $V(t)$ are divided into body (or volume) and surface (or contact) forces. Body forces, like gravity, act directly on the volume $V(t)$ whereas surface forces, e.g., pressure, act on $V(t)$ only through its boundary $S(t)$.

Let $\mathbf{f}$ denote the vector for the body force and $\mathbf{t_n}$ the one for the surface force. The subscript at the latter indicates the dependence on the outer unit normal vector $\mathbf{n}$ at $S(t)$. Indeed, $\mathbf{t_n}$ depends linearly on $\mathbf{n}$ as a principle from continuum mechanics due to Cauchy says, see [43, p. 4]. Therefore, the surface force vector can be written as $\mathbf{t_n} = \Sigma\mathbf{n}$, where $\Sigma \in \mathbb{R}^{3\times 3}$ is the (symmetric) *stress tensor*, see also [64, pp. 25–29]. Then Gauß' Theorem implies that the surface force acting on $S(t)$ can be written as a volume integral over $V(t)$, i.e.,

$$\int_{S(t)} t_{\mathbf{n}}\,ds = \int_{S(t)} \Sigma\mathbf{n}\,ds = \int_{V(t)} \operatorname{div}\Sigma\,dx.$$

[1] See [67, Section 10.10.2] for a discussion of the difference between an incompressible *fluid* and an incompressible *flow*.

Note that the divergence acts on Σ line by line. Now Newton's second law reads

$$\frac{d}{dt}\int_{V(t)} \rho\mathbf{v}\,dx = \int_{V(t)} (\mathbf{f} + \operatorname{div}\Sigma)\,dx,$$

and Reynolds' Transport Theorem gives

$$\int_{V(t)} \Big(\frac{\partial}{\partial t}(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\cdot\mathbf{v}^\top)\Big)\,dx = \int_{V(t)} (\mathbf{f} + \operatorname{div}\Sigma)\,dx. \tag{II.8}$$

Again, the choice of $V(t)$ and t is arbitrary. Hence, the integral signs in (II.8) can be omitted and the conservation of momentum can be written as the PDE

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\cdot\mathbf{v}^\top) = \mathbf{f} + \operatorname{div}\Sigma.$$

As an identity from vector calculus and the continuity equation (II.7) show, this is equivalent to

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \rho\mathbf{v}\operatorname{div}\mathbf{v} = \frac{\partial}{\partial t}(\rho\mathbf{v}) + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} + \operatorname{div}\Sigma,$$

where the row vector ∇ is the gradient in Cartesian coordinates. The assumption of incompressibility for the fluid, i.e., constant density, simplifies this equation to

$$\rho\frac{\partial}{\partial t}\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} + \operatorname{div}\Sigma. \tag{II.9}$$

This *momentum equation* can also be written with the material derivative D/Dt, cf. (II.4), i.e.,

$$\rho\frac{D}{Dt}\mathbf{v} = \mathbf{f} + \operatorname{div}\Sigma.$$

Now the procedure for deriving the Navier-Stokes equations for Newtonian fluids would start with assuming a law for the stress tensor where the viscosity of the fluid is assumed to be a constant. In this case it can be justified that

$$\Sigma = -p\mathrm{id} + \mathcal{L}\mathbf{S}, \tag{II.10}$$

where p is the pressure distribution in the fluid and $\mathcal{L}$ is a linear map of the deformation (or rate-of-strain) tensor

$$\mathbf{S}(\mathbf{v}) := \frac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\big), \tag{II.11}$$

where the gradient is defined component-by-component, cf. [43, Eq. (1.9)]. Then it holds that

$$\operatorname{div}\Sigma = -\nabla p + \operatorname{div}\mathcal{L}\mathbf{S}.$$

It is convenient to merge the pressure gradient in the divergence of the stress tensor Σ with the body force vector $\mathbf{f}$. This defines a *velocity potential* u by means of $\nabla u := \nabla p - \mathbf{f}$. The total of body and surface forces is then given by $\operatorname{div}\mathcal{L}\mathbf{S} - \nabla u$.

The viscosity of a Newtonian fluid appears as a scalar in the linear map $\mathcal{L}$. However, as it is the aim to model also non-Newtonian fluids, which have a nonconstant viscosity, a modified law for the stress tensor has to be used allowing for a nonconstant viscosity.

II.2 Darcy's Law

For the flow of a fluid in a Hele-Shaw cell it is possible to merge the continuity equation (II.7) and the momentum equation (II.9) for the flow to one equation for one state variable only. Here, a gap-averaging method in z-direction being orthogonal to the lateral plates of the cell is the main tool resulting in Darcy's law.

Darcy's law is an empirical law from hydrology. In the mid 19th century French engineer H. Darcy worked (among other things) on a satisfactory water supply system for the city of Dijon. Besides he was also interested in scientific questions concerning the early stages of fluid dynamics and hydraulics.[2] In his report *Les fontaines publiques de la ville de Dijon* from 1856, where also the water supply system for the city of Dijon is presented, Darcy describes an experiment of the flow of water through sand in a cylindrical column where water is supplied at the top and the discharge is measured at the bottom. He recognized a linear relation of the flow rate of the water to the piezometric head as well as to the inverse of the thickness of the sand layer. This relation can be reformulated as a linear relation of the hydraulic gradient acting on the fluid to its velocity. An excellent account on Darcy's law in more detail can be found in [58, Chapter II].

Darcy's law in its classical linear form is applicable only for Newtonian fluids having constant viscosity, e.g., water as in Darcy's experiments. However, the attempt of generalizing Darcy's law to non-Newtonian fluids naturally involves a viscosity *function* $\mu\colon [0,\infty) \to (0,\infty)$ deduced from experimental observations.

There are different classes of non-Newtonian fluids exhibiting (to a certain extent) the same change in viscosity. For every class the viscosity function μ has a certain form involving several parameters whose choice reflects the behavior of the respective fluid under shear observed in experiments. Two examples are the classes of Oldroyd-B and powerlaw fluids, see, e.g, [61, Chapter 14] and also [29]. Fluids of these types are shear-thinning, i.e., they become less viscous under shear. A prominent example is blood flowing fast in a (narrow) blood vessel but becoming rather viscous when dropping out of a wound. Shear-thickening fluids exhibit the opposite behavior. Exerting a shear stress on such a fluid leads to an increase in viscosity. For instance, hitting on a shear-thickening mixture of water and cornstarch makes it become almost rubber-like.

II.2.1 The Dimensionless Equations

Darcy's law can be derived from the momentum equation (II.9) for both Newtonian and non-Newtonian fluids. To do this, dimensionless variables and a more precise notation are necessary.

Let us write the velocity field component-by-component $\mathbf{v} := \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^\top$. The standard Cartesian coordinates are denoted by x, y, z and, as above, t represents time. The introductory paragraphs of this chapter motivate the following dimensionless scaling of variables for deriving a dimensionless form of the momen-

[2]For a descriptive account on Darcy's life and achievements see the historical article [41] and also [12, p. 308].

tum equation. The new spatial variables are $\bar{x} := \frac{1}{L}x$, $\bar{y} := \frac{1}{L}y$, and $\bar{z} := \frac{1}{b}z = \frac{1}{\varepsilon L}z$. Likewise, the velocity is scaled to

$$\bar{\mathbf{v}} := \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix} := \begin{pmatrix} \frac{1}{\varepsilon U_0} v_1 \\ \frac{1}{\varepsilon U_0} v_2 \\ \frac{1}{\varepsilon^2 U_0} v_3 \end{pmatrix},$$

where U_0 is a characteristic (scalar) velocity. Moreover, put $\bar{t} := \frac{\varepsilon U_0}{L} t$ for time. Finally, the dimensionless pressure is given by $\bar{p} := \frac{\varepsilon}{P_0} p$ for a characteristic pressure P_0 and the dimensionless viscosity is scaled by $\hat{\mu}_0 := \frac{P_0 L}{U_0}$. Such a scaling is performed in [39] for a special class of non-Newtonian fluids obeying a Johnson–Segalman–Oldroyd model.

As the stress tensor Σ can be decomposed into the pressure p and an extra deformation dependent stress tensor, the viscosity appears in the latter. It may thus be assumed that the stress tensor is given by

$$\Sigma = -p\mathrm{id} + 2\hat{\mu}(2\,\mathrm{tr}\,\mathbf{S}^2)\mathbf{S},$$

cf. also (II.10), where $\hat{\mu}\colon [0,\infty) \to (0,\infty)$ is a viscosity function and again $\mathbf{S}$ (see (II.11)) is the deformation tensor, cf. [8, p. 55] and [50, Eq. (2)]. Moreover, $\mathrm{tr}\,\mathbf{A} := \sum_{ii} a_{ii}$ denotes the trace operator for a tensor $\mathbf{A} := (a_{ij})$. Plugging in Σ into the momentum equation (II.9) yields

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla u + 2\,\mathrm{div}\,\hat{\mu}(2\,\mathrm{tr}\,\mathbf{S}^2)\mathbf{S}, \tag{II.12}$$

where u is a velocity potential given by $\nabla u := \nabla p - \mathbf{f}$ with the vector for the body force $\mathbf{f}$. Observe that u is scaled in the same way as p.

To compute $\mathbf{S}$ in dimensionless form note that

$$\nabla \mathbf{v} = \begin{pmatrix} \nabla v_1 \\ \nabla v_2 \\ \nabla v_3 \end{pmatrix} = \frac{U_0}{L} \begin{pmatrix} \varepsilon\partial_{\bar{x}}\bar{v}_1 & \varepsilon\partial_{\bar{y}}\bar{v}_1 & \partial_{\bar{z}}\bar{v}_1 \\ \varepsilon\partial_{\bar{x}}\bar{v}_2 & \varepsilon\partial_{\bar{y}}\bar{v}_2 & \partial_{\bar{z}}\bar{v}_2 \\ \varepsilon^2\partial_{\bar{x}}\bar{v}_3 & \varepsilon^2\partial_{\bar{y}}\bar{v}_3 & \varepsilon\partial_{\bar{z}}\bar{v}_3 \end{pmatrix}.$$

Then $\mathbf{S} = \frac{U_0}{L}\bar{\mathbf{S}}$ with the dimensionless deformation tensor

$$\bar{\mathbf{S}} := \frac{1}{2} \begin{pmatrix} 2\varepsilon\partial_{\bar{x}}\bar{v}_1 & \varepsilon(\partial_{\bar{y}}\bar{v}_1 + \partial_{\bar{x}}\bar{v}_2) & \partial_{\bar{z}}\bar{v}_1 + \varepsilon^2\partial_{\bar{x}}\bar{v}_3 \\ \varepsilon(\partial_{\bar{y}}\bar{v}_1 + \partial_{\bar{x}}\bar{v}_2) & 2\varepsilon\partial_{\bar{y}}\bar{v}_2 & \partial_{\bar{z}}\bar{v}_2 + \varepsilon^2\partial_{\bar{y}}\bar{v}_3 \\ \partial_{\bar{z}}\bar{v}_1 + \varepsilon^2\partial_{\bar{x}}\bar{v}_3 & \partial_{\bar{z}}\bar{v}_2 + \varepsilon^2\partial_{\bar{y}}\bar{v}_3 & 2\varepsilon\partial_{\bar{z}}\bar{v}_3 \end{pmatrix}.$$

Obviously, $\mathrm{tr}\,\mathbf{S}^2 = \frac{U_0^2}{L^2}\,\mathrm{tr}\,\bar{\mathbf{S}}^2$ and

$$\mathrm{tr}\,\bar{\mathbf{S}}^{\mathbf{2}} = \frac{1}{2}(\partial_{\bar{z}}\bar{v}_1)^2 + \frac{1}{2}(\partial_{\bar{z}}\bar{v}_2)^2 + \mathcal{O}(\varepsilon). \tag{II.13}$$

Now the right hand side of (II.12) becomes

$$-\frac{P_0}{\varepsilon L}\begin{pmatrix} \partial_{\bar{x}}\bar{u} \\ \partial_{\bar{y}}\bar{u} \\ \varepsilon^{-1}\partial_{\bar{z}}\bar{u} \end{pmatrix} + \frac{\hat{\mu}_0 U_0}{\varepsilon L^2}\begin{pmatrix} \partial_{\bar{z}}[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\bar{v}_1] + \mathcal{O}(\varepsilon) \\ \partial_{\bar{z}}[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\bar{v}_2] + \mathcal{O}(\varepsilon) \\ \varepsilon\big(\partial_{\bar{x}}[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\bar{v}_1] + \partial_{\bar{y}}[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\bar{v}_2]\big) + \mathcal{O}(\varepsilon^2) \end{pmatrix},$$

where $\mu := \frac{1}{\hat{\mu}_0}\hat{\mu}\left(\frac{U_0^2}{L^2}\,\cdot\,\right)$ is a dimensionless viscosity and $\bar{u}$ is the scaled velocity potential, and the left hand side of (II.12) is

$$\begin{aligned}\rho\big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) &= \frac{\varepsilon^2 U_0^2 \rho}{L}\begin{pmatrix}\partial_{\bar{t}}\bar{v}_1\\ \partial_{\bar{t}}\bar{v}_2\\ \varepsilon\partial_{\bar{t}}\bar{v}_3\end{pmatrix} + \varepsilon U_0 \rho\begin{pmatrix}\bar{v}_1\nabla\\ \bar{v}_2\nabla\\ \varepsilon\bar{v}_3\nabla\end{pmatrix}\mathbf{v}\\ &= \frac{\varepsilon^2 U_0^2\rho}{L}\begin{pmatrix}\partial_{\bar{t}}\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix} + \left[\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix}\cdot\bar{\nabla}_2\right]\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix} + \varepsilon^{-1}\begin{pmatrix}\bar{v}_1\partial_{\bar{z}}\bar{v}_3\\ \bar{v}_2\partial_{\bar{z}}\bar{v}_3\end{pmatrix}\\ \varepsilon(\partial_{\bar{t}}\bar{v}_3 + \bar{v}_3\partial_{\bar{x}}\bar{v}_1 + \bar{v}_3\partial_{\bar{y}}\bar{v}_2 + \varepsilon^{-1}\bar{v}_3\partial_{\bar{z}}\bar{v}_3)\end{pmatrix},\end{aligned}$$

where $\bar{\nabla}_2 := \begin{pmatrix}\partial_{\bar{x}} & \partial_{\bar{y}}\end{pmatrix}$ is the two-dimensional gradient in the dimensionless variables.

Recall that the characteristic pressure is given by $P_0 = \frac{\hat{\mu}_0 U_0}{L}$ and put $Re := \varepsilon^3\frac{\rho U_0 L}{\hat{\mu}_0}$. Then the first two components of (II.12) are given in dimensionless form by

$$Re\frac{\bar{D}}{D\bar{t}}\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix} + \frac{Re}{\varepsilon}\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix}\partial_{\bar{z}}\bar{v}_3 = -\bar{\nabla}_2\bar{u} + \partial_{\bar{z}}\left[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix}\right] + \mathcal{O}(\varepsilon),$$

where $\frac{\bar{D}}{D\bar{t}} := \partial_{\bar{t}} + \begin{pmatrix}\bar{v}_1 & \bar{v}_2\end{pmatrix}^\top\cdot\bar{\nabla}_2$ is the dimensionless material derivative. The third component is

$$\begin{aligned}&\varepsilon^2 Re(\partial_{\bar{t}}\bar{v}_3 + \bar{v}_3\partial_{\bar{x}}\bar{v}_1 + \bar{v}_3\partial_{\bar{y}}\bar{v}_2 + \varepsilon^{-1}\bar{v}_3\partial_{\bar{z}}\bar{v}_3)\\ &= \partial_{\bar{z}}\bar{u} + \varepsilon^2\big(\partial_{\bar{x}}[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\bar{v}_1] + \partial_{\bar{y}}[\mu(2\,\mathrm{tr}\,\bar{\mathbf{S}}^2)\partial_{\bar{z}}\bar{v}_2]\big) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

As $\varepsilon \ll 1$ is small the *Reynolds number Re* nearly vanishes. Thus neglecting all $\mathcal{O}(\varepsilon)$-terms (see also (II.13)) yields the equations

$$-\bar{\nabla}_2\bar{u} + \partial_{\bar{z}}\left[\mu\left(\left|\partial_{\bar{z}}\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix}\right|^2\right)\partial_{\bar{z}}\begin{pmatrix}\bar{v}_1\\ \bar{v}_2\end{pmatrix}\right] = 0, \quad \partial_{\bar{z}}\bar{u} = 0$$

governing the motion of a fluid in the small gap limit $\varepsilon \to 0$ in a Hele-Shaw cell. For better readability all bars over the variables are omitted and $\mathbf{v_2} := \begin{pmatrix}\bar{v}_1 & \bar{v}_2\end{pmatrix}^\top$ denotes the lateral velocity. In this way the equations above turn into the reduced Stokes equations

$$-\nabla_2 u + \partial_z[\mu(|\partial_z\mathbf{v_2}|^2)\partial_z\mathbf{v_2}] = 0, \quad \partial_z u = 0. \tag{II.14}$$

Starting from these equations, the authors of [50] derive a modified Darcy law for non-Newtonian fluids in a Hele-Shaw cell. The main steps are as follows.

Equations (II.14) integrate to

$$z\nabla_2 u = \mu(|\partial_z\mathbf{v_2}|^2)\partial_z\mathbf{v_2}. \tag{II.15}$$

Squaring this equation gives

$$z^2|\nabla_2 u|^2 = h(|\partial_z\mathbf{v_2}|^2), \tag{II.16}$$

where $h(r) := r\mu(r)^2$ for $r \geq 0$. Under the assumption that

$$h'(r) = \mu(r)^2 + 2r\mu'(r)\mu(r) > 0 \quad \text{for} \quad r \geq 0 \tag{II.17}$$

equation (II.16) is uniquely invertible, i.e., it is equivalent to

$$|\partial_z \mathbf{v_2}|^2 = h^{-1}(z^2|\nabla_2 u|^2).$$

In (II.15) this gives

$$z\nabla_2 u = \tilde{\mu}(z^2|\nabla_2 u|^2)\partial_z \mathbf{v_2}, \tag{II.18}$$

where $\tilde{\mu} := \mu \circ h^{-1}$. Under the assumption of no slip on the lateral plates, i.e., $\mathbf{v}(\,\cdot\,,\,\cdot\,,\pm b/2) = 0$, integrating this equation gives the two-dimensional velocity field explicitly, i.e.,

$$\mathbf{v_2} = \int_{-b/2}^{z} \frac{\zeta\nabla_2 u}{\tilde{\mu}(\zeta^2|\nabla_2 u|^2)}\, d\zeta.$$

Finally, gap averaging $\mathbf{v_2}$ gives

$$\begin{aligned}
\bar{\mathbf{v}}_\mathbf{2} &:= \frac{1}{b}\int_{-b/2}^{b/2} \mathbf{v_2}\, dz = \frac{1}{b}\int_{-b/2}^{b/2}\int_{\zeta}^{b/2} \frac{z\nabla_2 u}{\tilde{\mu}(z^2|\nabla_2 u|^2)}\, dz d\zeta \\
&= \frac{1}{b}\int_{-b/2}^{b/2} \frac{z(b/2 - z)\nabla_2 u}{\tilde{\mu}(z^2|\nabla_2 u|^2)}\, dz \\
&= -\frac{1}{b}\int_{-b/2}^{b/2} \frac{z^2}{\tilde{\mu}(z^2|\nabla_2 u|^2)}\, dz \nabla_2 u = -\frac{1}{\bar{\mu}(|\nabla_2 u|^2)}\nabla_2 u,
\end{aligned}$$

where

$$\frac{1}{\bar{\mu}(r)} := \frac{1}{b}\int_{-b/2}^{b/2} \frac{s^2}{\tilde{\mu}(rs^2)}\, ds \quad \text{for} \quad r \geq 0. \tag{II.19}$$

In the gap averaged two-dimensional regime all state variables do not depend on z anymore and it is reasonable to consider the gradient as the lateral gradient in x- and y-direction. Thus, without the subscript at the gradient the gap averaged velocity is

$$\bar{\mathbf{v}}_\mathbf{2} = -\frac{1}{\bar{\mu}(|\nabla u|^2)}\nabla u. \tag{II.20}$$

This is the modified Darcy law for non-Newtonian fluids. Note that this nonlinear relation becomes linear for a Newtonian fluid. This is due to the fact that the function $1/\bar{\mu}$ is constant if the viscosity μ is constant.

Plugging in $\bar{\mathbf{v}}_\mathbf{2}$ from (II.20) into the continuity equation (II.7) gives

$$\operatorname{div} \bar{\mathbf{v}}_\mathbf{2} = -\operatorname{div} \frac{1}{\bar{\mu}(|\nabla u|^2)}\nabla u = 0, \tag{II.21}$$

where the divergence—like the gradient—has to be understood in two space dimensions now. This equation can be considered as a PDE for the function u. In combination with suitable boundary conditions it is used to formulate a boundary value problem for u in Section II.3.

It is not evident that (II.21) holds for every non-Newtonian fluid. In fact, the crucial step in the derivation of (II.21)—the inversion of the function $h(r) =$

$r\mu(r)^2$, $r \geq 0$—is not always feasible. If the viscosity μ is assumed to be positive, the invertibility condition (II.17) is satisfied if and only if

$$0 < \mu(r) \qquad \text{for all} \quad r \geq 0, \tag{II.22}$$
$$0 < \mu(r) + 2r\mu'(r) \quad \text{for all} \quad r \geq 0. \tag{II.23}$$

These inequalities are standard assumptions on admissible viscosity functions for non-Newtonian fluids, see also [14]. For a shear-thickening fluid, i.e., its viscosity increases with increasing shear, condition (II.23) is always satisfied. But for a shear-thinning fluid condition (II.23) need not be satisfied. There might be a range for the variable r, where $\mu(r) + 2r\mu'(r) < 0$, cf. also [50]. For a shear-thinning fluid the viscosity μ has to be carefully examined to determine whether or not condition (II.23) holds.

II.2.2 The Potential Function

The function u is introduced as a scalar function representing the pressure p and the body forces $\mathbf{f}$ of a fluid. The constitutive relation is $\nabla u = \nabla p - \mathbf{f}$. The basis for constructing such a potential function lies in knowing the considered body forces. However, to determine these forces in the context of Hele-Shaw flows it has to be agreed on the geometry of the problem first.

If a Hele-Shaw cell is aligned horizontally, i.e., it is situated parallel to the ground floor, a centrifugal body force can be exerted on a fluid between the plates by rotating the cell. However, in a vertical (immobile) Hele-Shaw cell a natural body force is gravity. In this work this vertical geometry is solely studied and, by convention, x is the horizontal and y is the vertical coordinate. The vector for the gravitational body force is then given by $\mathbf{f}_g := \begin{pmatrix} 0 & -g\rho \end{pmatrix}^\top$, where $g \approx 9.81\,\mathrm{m\,s^{-2}}$ is the (Earth's) gravitational acceleration.

Another possible body force is a magnetic one. As gravitation can be considered as a force field, a magnetic field can induce a body force on a fluid, too. However, the fluid under consideration does not necessarily react to magnetic forces. In fact, no natural fluid does. In contrast, ferromagnetic liquids do.

A ferromagnetic liquid (or ferrofluid) reacts to a magnetic field in a macroscopic way: In the presence of a magnetic field it becomes highly polarized and, retaining its flowability, it aligns itself with such a field—often by developing a mesh of peaks along the lines of this field. This macroscopic phenomenon relies on the microstructure of a ferromagnetic liquid. After a thorough grinding process of a ferromagnetic solid minuscule particles of it having the size of about 10^2 Å are suspended in a carrier fluid, e.g., some acid, a hydrocarbon or simply water. To dissolve this powder properly, every particle has to adhere to a coating of a dispersant to prevent agglomeration. In such a dilute solution the particles behave due to Brownian motion as long as no magnetic field is present. For more details see the technical reports [65] and [47].

Ferromagnetic liquids do not occur in nature. Their fabrication and hence their scientific investigation dates back only a few decades, see Patent 3,215,572, U.S. Patent Office, 1965, by S. S. Papell and, e.g., the early research articles [59] and [17]. However, ferrofluids have already gained important fields of application, such as in loudspeakers and in hard disk drives, cf. [9].

To determine a magnetic body force $\mathbf{f}_m$ exerted on a ferrofluid in a vertical Hele-Shaw cell, the source of the magnetic field has to be identified. Consider a wire placed above the cell at height $y = 1$, say. It carries a current of intensity ι_w. The wire is considered to be very (infinitely) long such that only the magnetic field induced by the portion of the wire above the cell has effects on the ferrofluid and not the magnetic field induced by the rest of the wire forming a closed circuit far away from the cell. Under this assumption the magnetic field is radial around the wire piercing perpendicularly through the fluid in the cell. If the wire is assumed to be infinitely thin, Ampère's circuital law states that the intensity of the magnetic field H is proportional to the current's intensity ι_w and inversely proportional to the distance from the wire, i.e., cf. [64, Eq. (3.40)], see also [57],

$$H = \frac{\iota_w}{2\pi|1-y|}, \quad y \neq 1. \tag{II.24}$$

Then the magnetic body force is given by, see [64, Eq. (4.33)],

$$\mathbf{f}_m = -\nabla(p_s + p_m) + \mu_0 M \nabla H,$$

where $\mu_0 := 4\pi \times 10^{-7}\,\mathrm{m\,kg\,s^{-2}\,A^{-2}}$ is the permeability of free space, M is the magnetization of the ferrofluid, and p_s and p_m are the magnetostrictive and the fluid-magnetic pressure, respectively, see [64, Eq. (4.36)] for a precise definition. In [64, p. 111] it is argued that for dilute ferrofluids dipole interaction is negligible and, as a consequence, both p_s and p_m vanish. The remaining Kelvin force density $\mu_0 M \nabla H$ can be further simplified with the linear relation $M = \chi H$ between the magnetization and the intensity of the magnetic field given by the material dependent magnetic susceptibility χ. The magnetic body force is then given by

$$\mathbf{f}_m = \mu_0 \chi H \nabla H = \frac{\mu_0 \chi}{2} \nabla H^2 = \frac{\mu_0 \chi}{8\pi^2} \nabla \left[y \mapsto \frac{\iota_w^2}{(1-y)^2} \right], \quad y \neq 1,$$

where use of (II.24) is made, cf. also [57, Eq. (2)].

The combination of gravitational and magnetic forces in the dimensionless velocity potential u for the flow of a ferrofluid in a vertical Hele-Shaw cell gives

$$\nabla u = \nabla p - (\bar{\mathbf{f}}_g + \bar{\mathbf{f}}_m),$$

where $\bar{\mathbf{f}}_g$ and $\bar{\mathbf{f}}_m$ are the dimensionless forces due to gravity and magnetization, respectively, and p is the dimensionless pressure. Recall that its scaling is given by the characteristic pressure P_0. Therefore, with

$$\bar{\mathbf{f}}_g := \frac{1}{P_0} \mathbf{f}_g \quad \text{and} \quad \bar{\mathbf{f}}_m := \frac{\mu_0 \chi \iota_0^2}{8\pi^2 P_0} \nabla \left[y \mapsto \frac{\iota^2}{(1-y)^2} \right], \quad y \neq 1,$$

where $\iota := \iota_w / \iota_0$ is a dimensionless intensity of the current in the wire, the function u can easily be written down explicitly as

$$u(x,y) = p(x,y) + \frac{g\rho}{P_0} y - \frac{\mu_0 \chi \iota_0^2}{8\pi^2 P_0} \frac{\iota^2}{(1-y)^2}, \quad y \neq 1.$$

For the sake of simplicity let us choose the quantities specific for the ferrofluid under consideration as $\rho := \frac{P_0}{g}$ and $\chi := \frac{8\pi^2 P_0}{\mu_0 \iota_0^2}$. The potential function thus reads

$$u(x,y) = p(x,y) + y - \frac{\iota^2}{(1-y)^2}, \quad y \neq 1. \tag{II.25}$$

II.3 The Free Boundary Problem

A boundary value problem for the potential function u is readily derived from (II.21) and (II.25). To begin with, let $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle. Functions over $\mathbb{S}$ are naturally identified with 2π-periodic functions. For a given function f over $\mathbb{S}$ let $\Omega(f) := \{(x,y) \in \mathbb{S} \times \mathbb{R} : 0 < y < f(x)\}$ denote the domain of a fluid in a vertical Hele-Shaw cell having the extent $\mathbb{S} \times [0,1]$ with bottom $\Gamma_0 := \mathbb{S} \times \{0\} \equiv \mathbb{S}$. The free surface of the fluid is given by $\Gamma(f) := \{(x, f(x)) : x \in \mathbb{S}\}$ and the cell is filled with air at atmospheric pressure p_a above $\Gamma(f)$.

The differential equation for the function u in $\Omega(f)$ is (II.21). With the definition of the differential operator $\mathcal{Q} := -\operatorname{div} \frac{1}{\bar{\mu}(|\nabla \cdot|^2)} \nabla$ this is simply

$$\mathcal{Q}u = 0 \quad \text{in} \quad \Omega(f). \tag{II.26}$$

To write down a boundary value problem for the function u in the domain $\Omega(f)$, boundary conditions have to be imposed on the free surface $\Gamma(f)$ and on the bottom of the cell Γ_0. Under suitable assumptions on the (dimensionless) pressure p at the free surface the *dynamic* boundary condition on $\Gamma(f)$ is given by (II.25) at $y = f(x)$ for $x \in \mathbb{S}$. Here essentially two cases have to be distinguished: prescribed pressure or surface tension effects. In the first case the pressure is given by the atmospheric pressure p_a of the surrounding air. In the second case the pressure exhibits a jump across the free surface from the fluid phase to the adjacent air phase, i.e., the pressure of the fluid at the free surface is assumed to be different from the pressure of the air right above. This is due to different types of molecules on the two phases separated by the free surface, see, e.g., [67, Section 2.8]. In the force balance at the free surface $\Gamma(f)$ this pressure jump is compensated by surface tension effects. More precisely, the *Laplace–Young equation*

$$p - p_a = \sigma \operatorname{div} \nu \tag{II.27}$$

holds, where $\nu := n/\sqrt{1+f_x^2}$ with $n := \begin{pmatrix} -f_x & 1 \end{pmatrix}$ is the unit outward normal at $\Gamma(f)$ and the surface tension coefficient σ is a positive constant, cf. [43, Section 1.1.2 & p. 15]. Note that x-subscripts stand for partial differentiation with respect to the spatial variable x. The pressure jump in (II.27) can be expressed as the local curvature of the free surface, i.e.,

$$p - p_a = -\sigma \partial_x \left[\frac{f_x}{\sqrt{1+f_x^2}} \right] = -\sigma \frac{f_{xx}}{(1+f_x^2)^{3/2}} =: -\sigma \kappa_f.$$

If the air pressure is normalized to zero, i.e., $p_a = 0$, the dynamic boundary condition on the free surface is

$$u = -\sigma \kappa_f - \frac{\iota^2}{(1-f)^2} + f \quad \text{on} \quad \Gamma(f), \tag{II.28}$$

where $\sigma > 0$ if surface tension effects are considered and $\sigma = 0$ otherwise. Likewise, $\iota > 0$ means that a current of intensity ι flows through the wire above the cell whereas $\iota = 0$ means that the current is switched off.

At the lower boundary component Γ_0 an external source is modeled by the source term $\hat{b}(f)$ depending on the free surface f, see also [23]. The term $\hat{b}$ has to be considered as a function from $\mathbb{R}$ to $\mathbb{R}$ and the function $\hat{b}(f)$ at some point $x \in \mathbb{S}$ is defined by $b(f(x))$. This expression can be used to prescribe the pressure on the bottom of the cell or to define a flux through the bottom of the cell. In the first case one sets $p(\,\cdot\,,0) = \hat{b}(f)$ and (II.25) says that this is equivalent to $u(\,\cdot\,,0) = \hat{b}(f) - \iota^2$. In the second case the flux through Γ_0 is given by the normal velocity

$$\bar{\mathbf{v}}_\mathbf{2} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{\bar{\mu}(|\nabla u|^2)} \partial_y u = \hat{b}(f),$$

see (II.20). Therefore, if

$$\tilde{b}(f) := \hat{b}(f) - (1-\delta)\iota^2, \tag{II.29}$$

the boundary condition on Γ_0 can be written as

$$(1-\delta)u + \frac{\delta}{\bar{\mu}(|\nabla u|^2)} \partial_y u = \tilde{b}(f) \quad \text{on} \quad \Gamma_0, \tag{II.30}$$

where the parameter $\delta \in \{0,1\}$ determines whether the pressure ($\delta = 0$) or the flux ($\delta = 1$) at the bottom of the cell is prescribed. A simple choice for $\hat{b}$ could be the identity, i.e., $\hat{b}(f) = f$, modeling (for $\delta = 1$) a flux through the bottom of the cell that equals the height of the fluid head. However, any other choice that is reasonable in a certain experimental setup can also be treated.

The boundary value problem consisting of the three equations (II.26), (II.28), and (II.30) is now extended by a *kinematic* boundary condition at the free surface. If it is assumed that particles at the free surface stay there forever, the (total) time derivative of the constitutive relation for the free surface $0 = -y + f(t,x)$ is

$$0 = \partial_t[-y + f(t,x)] + \big(x'(t) \quad y'(t)\big) \cdot \nabla\big(-y + f(t,x)\big), \quad (x,y) \in \Gamma(f), \quad t > 0.$$

Observe that $\big(x'(t) \quad y'(t)\big) = \bar{\mathbf{v}}_\mathbf{2}$, cf. (II.1) and (II.20). It follows that

$$0 = \partial_t f + \bar{\mathbf{v}}_\mathbf{2} \cdot \big(f_x \quad -1\big) \quad \text{on} \quad \Gamma(f).$$

Hence, by means of (II.20) this is

$$\partial_t f = -\frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla u|^2)} \partial_\nu u \quad \text{on} \quad \Gamma(f), \tag{II.31}$$

where $\partial_\nu u := \nabla u \cdot \nu$.

In sum, equations (II.26), (II.28), (II.30), and (II.31) constitute the free boundary problem

$$\begin{aligned} \mathcal{Q}u &= 0 && \text{in} \quad \Omega(f), \\ (1-\delta)u + \frac{\delta}{\bar{\mu}(|\nabla u|^2)}\partial_y u &= \tilde{b}(f) && \text{on} \quad \Gamma_0, \\ u &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on} \quad \Gamma(f), \\ \partial_t f &= -\frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla u|^2)}\partial_\nu u && \text{on} \quad \Gamma(f), \\ f(0) &= f_0, && \text{on} \quad \mathbb{S} \end{aligned} \tag{II.32}$$

where f_0 is some initial condition.

Observe that depending on the choice of $\tilde{b}$ the amount of fluid in the cell is not necessarily constant. In fact, in case $\delta = 1$ the total mass in the cell can be directly controlled by the choice of $\tilde{b}$: As the density of the fluid is assumed to be constant and as its volume is given by $\int_{\mathbb{S}} f(x)\,dx$, the change of mass in the cell is given by

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} f\,dx &= \int_{\mathbb{S}} \partial_t f\,dx \\ &= -\int_{\Gamma(f)} \frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla u|^2)} \nabla u \cdot \nu\,ds \\ &= -\int_{\Omega(f)} \mathcal{Q}u\,d(x,y) - \int_{\Gamma_0} \frac{1}{\bar{\mu}(|\nabla u|^2)}\partial_y u\,ds \\ &= -\int_{\mathbb{S}} \tilde{b}(f)\,dx. \end{aligned}$$

This means that if $\tilde{b}(f)$ has negative integral mean, additional fluid is injected through the bottom of the cell whereas a positive integral mean of $\tilde{b}(f)$ leads to a decrease of fluid volume in the cell. Only if $\tilde{b}(f)$ has exactly integral mean zero, the total mass in the cell is conserved. In particular, this is the case when the no-flux condition $\partial_y u = 0$ is employed on the lower boundary component, see also [27, Remark 2.1].

Chapter III

Parabolic Evolution Equations and Analytic Semigroups

The kinematic boundary condition at the free surface in a Hele-Shaw free boundary problem is an evolution equation for the function f defining the free surface. The corresponding evolution operator in this equation contains both the function f and the solution of the boundary value problem for the potential function u. In turn, due to the dependence of the domain of this boundary value problem on the function f, its solution u also depends on f. From this point of view, given an initial condition f_0 for the free surface, the dynamics of the Hele-Shaw flow under consideration is solely given by an abstract Cauchy problem of the form

$$\partial_t f = \Phi(f), \quad f(0) = f_0, \tag{III.1}$$

provided the boundary value problem for the function u is uniquely solvable.

It turns out that the boundary value problem for u in $\Omega(f)$ can be solved with the theory of elliptic PDEs of second order. The subsequent study of the Cauchy problem (III.1) then strongly relies on results from the theory of abstract parabolic equations and analytic semigroups, where the main focus lies on the proof that the linearization of the evolution operator Φ generates an analytic semigroup in suitable function spaces. The choice of these function spaces has two important constraints: They have to comply with the elliptic theory and they have to be interpolation spaces.

III.1 Little Hölder Spaces and Interpolation Theory

Let us begin with introducing some basic function spaces. For a bounded and open set $U \subset \mathbb{R}^n$, $n \geq 1$, with a C^2-boundary and an integer $k \geq 0$ let $BC^k(U)$ denote the Banach space of all k times differentiable functions from U to $\mathbb{R}$ having bounded and continuous derivatives up to order k with norm

$$\|u\|_{k,U} := \max_{|a| \leq k} \sup_{x \in U} |\partial^a u(x)| \quad \text{for} \quad u \in BC^k(U).$$

Here the multiindex notation $a := (a_{n_1}, \ldots, a_{n_k})$ with $\partial^a u := \partial^{\tau(a)}$ for an arbitrary permutation τ of a with repetitions is understood. We put $BC(U) := BC^0(U)$.

The space $BUC^k(U)$ consisting of all bounded and uniformly continuous functions from U to $\mathbb{R}$ up to order k is a closed linear subspace of $BC^k(U)$ having the same norm. Again, we set $BUC(U) := BUC^0(U)$.

For a function $u \in BUC(U)$ and $\alpha \in (0,1)$ the α-seminorm is given by

$$[u]_{\alpha,U} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Given $k \in \mathbb{N}$ and $\alpha \in (0,1)$, the Hölder spaces

$$BUC^{k+\alpha}(U) := \{u \in BUC^k(U) \colon \max_{|a|=k} [\partial^a u]_{\alpha,U} < \infty\}$$

equipped with the norm $\|\cdot\|_{k+\alpha,U} := \|\cdot\|_{k,U} + \max_{|a|=k}[\partial^a \cdot]_{\alpha,U}$ are Banach spaces.

Given $s > 0$, we define the class of *little* Hölder continuous functions of order s as the closure of the smooth functions over U, i.e.,

$$BUC^\infty(U) := \bigcap_{k=0}^{\infty} BUC^k(U),$$

in the Hölder spaces $BUC^s(U)$ denoted by $buc^s(U)$ and endowed with the same norm as the spaces $BUC^s(U)$.

For functions over the unit circle $\mathbb{S}$ we define the space $C^k := C^k(\mathbb{S})$ as the Banach space of continuous functions of order $k \in \mathbb{N}$ with norm

$$\|u\|_k := \|u\|_{k,\mathbb{S}} := \max_{a\le k} \sup_{x\in\mathbb{S}} |\partial^a u(x)| \quad \text{for} \quad u \in C^k.$$

For a continuous function u over the unit circle the α-seminorm, $\alpha \in (0,1)$, is defined by

$$[u]_\alpha := [u]_{\alpha,\mathbb{S}} := \sup_{\substack{x,y\in \mathbb{S}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Equipped with the norm $\|\cdot\|_{k+\alpha} := \|\cdot\|_{k+\alpha,\mathbb{S}} := \|\cdot\|_{k,\mathbb{S}} + \max_{a=k}[\partial^a \cdot]_{\alpha,\mathbb{S}}$, the space $C^{k+\alpha} := C^{k+\alpha}(\mathbb{S})$ of all k times continuously differentiable functions over $\mathbb{S}$ whose k-th derivative has finite α-seminorm is a Banach space.

Given $s > 0$, the *little* Hölder spaces over the unit circle $h^s := h^s(\mathbb{S})$ are defined as the closure of the smooth functions

$$C^\infty := C^\infty(\mathbb{S}) := \bigcap_{k=0}^{\infty} C^k(\mathbb{S})$$

in the Hölder spaces C^s equipped with the same norm as the spaces C^s.

We formulate our results in little Hölder spaces because they combine properties indispensable for the application of both the elliptic and the parabolic theory. In the next proposition we collect some fundamental properties about these spaces. A proof in particular of the interpolation property (vi), though widely used in the literature, is not known to the author. The subsequent presentation of basic interpolation theory has the special aim of proving this property.

Proposition III.1. *Let $s, t \in \mathbb{R} \setminus \mathbb{Z}$ be given such that $0 < s < t$ and put $s := k + \alpha$ for $k \in \mathbb{Z}$ and $\alpha \in (0,1)$. The following properties hold.*

(i) $f \in h^s$ *if and only if* $f \in C^s$ *and*

$$\lim_{\delta \to 0} \sup_{\substack{x,y \in \mathbb{S} \\ 0<|x-y|\leq\delta}} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^\alpha} = 0.$$

(ii) h^s *is a (true) closed linear subspace of* C^s.

(iii) $h^s = \overline{C^{t_0}}^{\|\cdot\|_s}$ *for arbitrary* $t_0 > s$.

(iv) h^t *is densely injected in* h^s.

(v) h^t *is compactly imbedded in* h^s.

(vi) Let $\theta \in (0,1)$ be given such that $\theta t + (1-\theta)s \notin \mathbb{Z}$ and let $(\,\cdot\,,\,\cdot\,)_\theta$ denote the continuous interpolation functor introduced in [19]. Then it holds that

$$(h^s, h^t)_\theta = h^{\theta t + (1-\theta)s}.$$

Proof of (i)–(v). A proof for the intrinsic characterization (i) of the spaces h^s, $s > 0$, and an example of a function in $C^s \setminus h^s$ can be found in [55, Lemma 2.2.3 & p. 31].

To prove that h^s is closed in C^s pick $f \in \overline{h^s}^{\|\cdot\|_s}$ and choose a sequence $(f_n)_n \subset h^s$ such that $\|f_n - f\|_s \to 0$. Then, given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\|f_N - f\|_s, \sup_{\substack{x,y \in \mathbb{S} \\ 0<|x-y|\leq\delta}} \frac{|\partial^k f_N(x) - \partial^k f_N(y)|}{|x-y|^\alpha} < \varepsilon/2,$$

see (i). Since

$$\begin{aligned} &\sup_{\substack{x,y \in \mathbb{S} \\ 0<|x-y|\leq\delta}} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^\alpha} \\ &\leq \sup_{x,y \in \mathbb{S}} \frac{|[\partial^k f - \partial^k f_N](x) - [\partial^k f - \partial^k f_N](y)|}{|x-y|^\alpha} + \sup_{\substack{x,y \in \mathbb{S} \\ 0<|x-y|\leq\delta}} \frac{|\partial^k f_N(x) - \partial^k f_N(y)|}{|x-y|^\alpha} \\ &< \varepsilon, \end{aligned}$$

the assertion follows from (i).

To prove assertion (iii) let $f \in h^s$ be given and let $t_0 > s$. There exist smooth functions $(f_n)_n \subset C^\infty$ such that $\|f_n - f\|_s \to 0$. Since $C^\infty \subset C^{t_0}$ the inclusion $h^s \subset \overline{C^{t_0}}^{\|\cdot\|_s}$ follows. To show the other inclusion we start with showing $C^{t_0} \subset h^s$. Indeed, let $g \in C^{t_0}$ be given. Then it holds that $\partial^k g$ is Hölder continuous. More precisely, $\partial^k g$ is a member of C^α and also of C^β for $\beta \in (\alpha, t_0 - k)$. From this and (i) we infer that $g \in h^s$ thanks to

$$\lim_{\delta \to 0} \sup_{\substack{x,y \in S \\ 0<|x-y|\le\delta}} \frac{|\partial^k g(x) - \partial^k g(y)|}{|x-y|^\alpha} \le \lim_{\delta\to 0} [\partial^k g]_\beta \delta^{\beta-\alpha} = 0.$$

From (ii) we now deduce that $\overline{C^{t_0}}^{\|\cdot\|_s} \subset \overline{h^s}^{\|\cdot\|_s} = h^s$.

Let $s, t \in \mathbb{R} \setminus \mathbb{Z}$ be given such that $0 < s < t$. The dense inclusion $h^t \overset{d}{\hookrightarrow} h^s$ follows directly from the definition of the little Hölder spaces and the compact inclusion is due to the fact that C^t is compactly imbedded in C^s. □

To prove assertion (vi) we need some preparation. To begin with we provide some basic material about interpolation theory including the introduction of the real and the continuous interpolation method. We closely follow [2, Section I.2] and [53, Chapter 1], see also [72, Section 1.2] for a very general approach.

Let X be a locally convex space. A pair of Banach spaces (E_0, E_1) is said to be an *interpolation couple* if both E_0 and E_1 are continuously imbedded in X. Then $E_0 \cap E_1$ and $E_0 + E_1$ are well defined and Banach spaces. A space E with $E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1$ is called *intermediate space* with respect to (E_0, E_1). Observe that, of course, E_0, E_1, $E_0 \cap E_1$, and $E_0 + E_1$ are all intermediate spaces with respect to (E_0, E_1).

To make things simpler we only consider the case where $E_1 \hookrightarrow E_0$. Then it obviously holds that $E_0 \cap E_1 = E_1$ and $E_0 + E_1 = E_0$. In this case E is an intermediate space if and only if $E_1 \hookrightarrow E \hookrightarrow E_0$ and we call such an intermediate space *interpolation space* with respect to (E_0, E_1) if for every linear operator $T \in \mathcal{L}(E_0)$ with $T|E_1 \in \mathcal{L}(E_1)$ it holds that $T|E \in \mathcal{L}(E)$. Again, E_0 and E_1 are trivially interpolation spaces with respect to (E_0, E_1).

In the sequel we define concrete nontrivial interpolation spaces for the special case of a densely injected interpolation couple (E_0, E_1), i.e., $E_1 \overset{d}{\hookrightarrow} E_0$. We distinguish between the *real* and the *continuous* interpolation method.

The real interpolation method for constructing (true) interpolation spaces between E_1 and E_0 can be characterized in several equivalent ways. In the following we delineate the *K-method*, see [53, Section 1.2] and [11, Section 3.1].

Given $x \in E_0$ and $t > 0$, we define the *K-functional*

$$K(t,x) := \inf_{\substack{x=a+b \\ a\in E_0, b\in E_1}} \big(\|x\|_0 + t\|b\|_1\big),$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ are the respective norms in E_0 and E_1. For $\theta \in (0,1)$ and $q \in [1,\infty]$ we define the spaces

$$(E_0, E_1)_{\theta,q} := \{x \in E_0 \colon [t \mapsto t^{-\theta-1/q} K(t,x)] \in L^q(0,\infty)\}, \qquad \text{(III.2)}$$

where we declare $1/\infty := 0$. Endowed with the norm

$$\|\cdot\|_{(E_0,E_1)_{\theta,q}} := \|\cdot\|_{\theta,q} := \|[t \mapsto t^{-\theta-1/q} K(t,\cdot)]\|_{L^q(0,\infty)},$$

the spaces $(E_0, E_1)_{\theta,q}$ are Banach spaces. Furthermore, for $\theta \in (0,1)$ we put

$$(E_0, E_1)_\theta := \{x \in E_0 \colon \lim_{t\to 0} t^{-\theta} K(t,x) = 0\}. \tag{III.3}$$

Equipped with the norm $\|\cdot\|_{\theta,\infty}$ also this space is a Banach space.

Observe that the spaces $(E_0, E_1)_{\theta,q}$ and $(E_0, E_1)_\theta$ with $\theta \in (0,1)$ and $q \in [1,\infty]$ are nontrivial in the sense that they are different from E_0, E_1, and $\{0\}$. Moreover, the following inclusions hold, cf. [53, Proposition 1.2.3].

Proposition III.2. *Let $\theta, \theta_0, \theta_1 \in (0,1)$ and $p_1, p_2 \in [1,\infty]$ be given such that $\theta_0 < \theta < \theta_1$ and $p_1 \leq p_2$. Then it holds that*

$$\begin{aligned} E_1 &\hookrightarrow (E_0,E_1)_{\theta_1,\infty} \hookrightarrow (E_0,E_1)_{\theta,p_1} \hookrightarrow (E_0,E_1)_{\theta,p_2} \\ &\hookrightarrow (E_0,E_1)_\theta \hookrightarrow (E_0,E_1)_{\theta,\infty} \hookrightarrow (E_0,E_1)_{\theta_0,1} \hookrightarrow E_0. \end{aligned}$$

For the densely injected Banach spaces $E_1 \overset{d}{\hookrightarrow} E_0$ we define the continuous interpolation functor as

$$(E_0, E_1)^0_{\theta,\infty} := \overline{E_1}^{\|\cdot\|_{\theta,\infty}}, \tag{III.4}$$

where $\theta \in (0,1)$, cf. [2, Section I.2.5]. It is a consequence of Proposition 1.2.12 in [53] that $(E_0,E_1)^0_{\theta,\infty} = (E_0,E_1)_\theta$. Moreover, it can be shown (see [20]) that these spaces coincide with the continuous interpolation spaces introduced by G. da Prato and P. Grisvard in [19], where they are also denoted with the symbol $(E_0, E_1)_\theta$. We adopt this notation in the following.

Let us conclude with the following central result from interpolation theory, see [11, Section 3.5] and [53, Section 1.2.3].

Theorem III.3 (Reiteration Theorem)**.** *Let $0 \leq \theta_0 < \theta_1 \leq 1$ and $\theta \in (0,1)$. Let E_0, E_1, F_0, and F_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$ and $(E_0,E_1)_{\theta_j,1} \hookrightarrow F_j \hookrightarrow (E_0,E_1)_{\theta_j,\infty}$ for $j \in \{0,1\}$. Then it holds that*

$$(F_0,F_1)_{\theta,q} = (E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1,q} \quad \text{for all} \quad q \in [1,\infty].$$

As a last step in the preparation of the proof of Proposition III.1 (vi) we introduce the periodic Besov spaces $B^s_{pq} := B^s_{pq}(\mathbb{S})$, where $s \in \mathbb{R}$ and $p,q \in (0,\infty]$. For a precise definition we refer to [66, Section 3.5.1]. Here we only give the following properties of these spaces, cf. Theorem 1 and Remark 2 in Section 3.6.1 and the Theorem in Section 3.5.4 in [66].

Proposition III.4. *(i) Let $\theta \in (0,1)$, $p,q,q_0,q_1 \in (0,\infty]$, $s_0,s_1 \in \mathbb{R}$, and $s := (1-\theta)s_0 + \theta s_1$, where $s_0 \neq s_1$. Then it holds that $(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,q} = B^s_{pq}$.*

(ii) Given $k \in \mathbb{N}$, it holds that $B^k_{\infty 1} \hookrightarrow C^k \hookrightarrow B^k_{\infty\infty}$.

(iii) If $t > 0$ is not an integer then it holds that $B^t_{\infty\infty} = C^t$.

We remark that $B^t_{\infty\infty}$ coincides with a Zygmund space if t is an integer, see [66, Section 3.5.4].

Proof of Proposition III.1 (vi). Let $s_0 < s_1$ with $s_1 \in \mathbb{N}$ and $\theta \in (0,1)$ be given and put $s_\theta := (1-\theta)s_0 + \theta s_1$. From Theorem III.4 (i) we infer that

$$B^{s_\theta}_{\infty\infty} = (B^{s_0}_{\infty\infty}, B^{s_1}_{\infty\infty})_{\theta,\infty} = (B^{s_0}_{\infty\infty}, B^{s_1}_{\infty 1})_{\theta,\infty}.$$

Owing to Theorem III.4 (ii) we get $B^{s_\theta}_{\infty\infty} = (B^{s_0}_{\infty\infty}, C^{s_1})_{\theta,\infty}$. From this relation, the definition of the continuous interpolation spaces (see (III.4)), Propositions III.4 (iii) and III.1 (iii), and the fact that $s_\theta < s_1$ we conclude that

$$(B^{s_0}_{\infty\infty}, C^{s_1})_\theta = \overline{C^{s_1}}^{\|\cdot\|_{B^{s_\theta}_{\infty\infty}}} = h^{s_\theta} = \overline{C^\infty}^{\|\cdot\|_{B^{s_\theta}_{\infty\infty}}}. \qquad \text{(III.5)}$$

The smooth functions can be identified with the space $B^\infty_\infty := \bigcap_{t>0} B^t_{\infty q}$. Indeed, since for every $t > 0$ there is a $k \in \mathbb{N}$ such that $k \le t < k+1$, we have $B^{k+1}_{\infty q} \hookrightarrow B^t_{\infty q} \hookrightarrow B^k_{\infty q}$ and hence $B^\infty_\infty = \bigcap_{k\in\mathbb{N}} B^k_{\infty q}$. From Theorem III.4 (ii) and the fact that B^∞_∞ does not depend on q we finally infer that

$$B^\infty_\infty = \bigcap_{k\in\mathbb{N}} C^k = C^\infty \qquad \text{(III.6)}$$

as claimed.

Given $t \in \mathbb{R}$, we define the little Besov spaces $b^t_{\infty\infty}$ as the closure of the space B^∞_∞ in $B^t_{\infty\infty}$. Then we see from (III.5) and (III.6) that

$$(B^{s_0}_{\infty\infty}, C^{s_1})_\theta = b^{s_\theta}_{\infty\infty}. \qquad \text{(III.7)}$$

The little Besov spaces are stable under continuous interpolation as the following application of Theorem III.3 shows.

Pick $t_0 \in \mathbb{R}$ and $t_1 \in \mathbb{N}$ such that $t_0 < s_0 < s_1 < t_1$ and choose θ_j in such a way that $s_j = (1-\theta_j)t_0 + \theta_j t_1$ for $j \in \{0,1\}$. To shorten our notation let $E_0 := B^{t_0}_{\infty\infty}$, $E_1 := C^{t_1}$, $F_0 := (E_0,E_1)_{\theta_0}$, and $F_1 := (E_0,E_1)_{\theta_1}$. With this notation we have, cf. (III.7), $(b^{s_0}_{\infty\infty}, b^{s_1}_{\infty\infty})_\theta = (F_0,F_1)_\theta$. Now Proposition III.2 shows that $(E_0,E_1)_{\theta_j,1} \hookrightarrow F_j \hookrightarrow (E_0,E_1)_{\theta_j,\infty}$, $j \in \{0,1\}$, and Theorem III.3 yields $(F_0,F_1)_{\theta,q} = (E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1,q}$ for $q \in [1,\infty]$. To apply this result for the continuous interpolation method, note that by definition $(F_0,F_1)_\theta = \overline{F_1}^{\|\cdot\|_{(E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1,\infty}}}$, cf. (III.4), where we already use this reiteration result. Observe that

$$F_1 \overset{d}{\hookrightarrow} (F_0,F_1)_{\theta,1} = (E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1,1} \overset{d}{\hookrightarrow} (E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1},$$

cf. [2, Chapter I, Eq. (2.5.2)]. This implies

$$\begin{aligned}(F_0,F_1)_\theta &= \overline{(E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1}}^{\|\cdot\|_{(E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1,\infty}}} \\ &= \overline{E_1}^{\|\cdot\|_{(E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1,\infty}}} = (E_0,E_1)_{(1-\theta)\theta_0+\theta\theta_1}\end{aligned}$$

and thus the assertion because

$$\begin{aligned}s_\theta &= (1-\theta)s_0 + \theta s_1 = (1-\theta)\big((1-\theta_0)t_0 + \theta_0 t_1\big) + \theta\big((1-\theta_1)t_0 + \theta_1 t_1\big) \\ &= \big(1-(1-\theta)\theta_0 - \theta\theta_1\big)t_0 + \big((1-\theta)\theta_0 + \theta\theta_1\big)t_1.\end{aligned}$$

Eventually, we obtain from Theorem III.4 (iii), (III.6), and the definition of the little Besov spaces that

$$h^s = \overline{C^\infty}^{\|\cdot\|_s} = \overline{C^\infty}^{\|\cdot\|_{B^s_{\infty\infty}}} = \overline{B^\infty_\infty}^{\|\cdot\|_{B^s_{\infty\infty}}} = b^s_{\infty\infty},$$

where $s > 0$ is not an integer. The assertion now follows from the stability of the little Besov spaces under continuous interpolation. □

We remark that the preceding proof loosely relies on some ideas from unpublished material of H. Amann.

Let us close this section with two important inequalities for norms in continuous interpolation spaces, cf. [2, Section I.2.2].

Proposition III.5. *Let E_0, E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$ and fix $\theta \in (0,1)$. Put $E_\theta := (E_0, E_1)_\theta$.*

(i) The following convexity property holds: There is a positive constant c_θ depending on θ such that

$$\|x\|_{E_\theta} \leq c_\theta \|x\|_0^{1-\theta} \|x\|_1^\theta \quad \textit{for all} \quad x \in E_1.$$

(ii) For all $\varepsilon > 0$ there is a positive constant c depending on θ and ε such that

$$\|x\|_{E_\theta} \leq \varepsilon \|x\|_1 + c\|x\|_0 \quad \textit{for all} \quad x \in E_1.$$

Proof. The first inequality is a consequence of Corollary 1.2.7 in [53] for the case $p = \infty$ since $\|\cdot\|_{E_\theta} = \|\cdot\|_{\theta,\infty}$.

From the convexity property (i) and Young's inequality we obtain

$$\begin{aligned}\|x\|_{E_\theta} &\leq c_\theta \Big(\frac{\varepsilon}{c_\theta \theta}\|x\|_1\Big)^\theta \bigg(\Big(\frac{\varepsilon}{c_\theta\theta}\Big)^{\theta/(\theta-1)} \|x\|_0\bigg)^{1-\theta} \\ &\leq \varepsilon\|x\|_1 + c_\theta(1-\theta)\Big(\frac{\varepsilon}{c_\theta\theta}\Big)^{\theta/(\theta-1)}\|x\|_0 = \varepsilon\|x\|_1 + c\|x\|_0,\end{aligned}$$

where $c = c(\theta,\varepsilon) := (1-\theta)\theta^{\theta/(1-\theta)} c_\theta^{1/(1-\theta)} \varepsilon^{\theta/(\theta-1)}$. □

III.2 Existence Theorems for Cauchy Problems in Continuous Interpolation Spaces

Let X be a Banach space. A semigroup T is said to be *analytic* on X if the mapping $[t \mapsto T(t)]\colon (0,\infty) \to \mathcal{L}(X)$ is analytic. It is called *strongly continuous* on X if $[t \mapsto T(t)x]\colon [0,\infty) \to X$ is continuous for all $x \in X$.

Consider semigroups of the form $T(t) := \exp(-At)$, $t \geq 0$, i.e., there is a linear operator, the (negative) *generator*, $A\colon D(A) \to X$ with domain $D(A) \subset X$ that 'generates' the semigroup T in the above way. For semigroups in this form a complete characterization of analytic ones is available. In [40, Theorems 4.2.1

& 4.2.2] it is shown that $-A$ generates the[1] analytic semigroup $\big(\exp(-At)\big)_{t\geq 0}$ if and only if it is *sectorial*, i.e., the resolvent set $\rho(-A)$ contains a sector $S_{\theta,\omega} := \{\lambda \in \mathbb{C} \colon \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$, where $\omega \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$, and the resolvent operator $R(\lambda, -A) := (\lambda + A)^{-1}$ satisfies the estimate $\|R(\lambda, -A)\|_{\mathcal{L}(X)} \leq M|\lambda - \omega|^{-1}$ for some $M > 0$ and all $\lambda \in S_{\theta,\omega}$, see also [53, Definition 2.0.1], where this equivalence is used to *define* analytic semigroups in the above form. It is important to note that the sectoriality condition can be equivalently reformulated in a (complex) half plane, cf. [53, Proposition 2.1.11] and [40, Lemma 4.2.3].

Proposition III.6. *Let $A \colon D(A) \subset X \to X$ be a linear operator. Then $-A$ is sectorial if and only if there exist constants $\omega \in \mathbb{R}$ and $M > 0$ such that*

$$\rho(-A) \supset \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda \geq \omega\}, \tag{III.8}$$
$$|\lambda| \|R(\lambda, -A)\|_{\mathcal{L}(X)} \leq M, \quad \operatorname{Re}\lambda \geq \omega. \tag{III.9}$$

Observe that if $-A$ is sectorial, the semigroup $\big(\exp(-At)\big)_{t\geq 0}$ is strongly continuous if and only if $[t \mapsto \exp(-At)x] \colon [0, \infty) \to X$ is continuous in 0 for all $x \in X$. In [53, Proposition 2.1.4] it is shown that $\lim_{t\to 0} \exp(-At)x = x$ if and only if $x \in \overline{D(A)}$, i.e., $\big(\exp(-At)\big)_{t\geq 0}$ is strongly continuous if and only if $D(A)$ is dense in X.

III.2.1 The Class of Generators of Strongly Continuous Analytic Semigroups in Continuous Interpolation Spaces

Let X be a Banach space and let $A \colon D(A) \to X$ be a linear mapping with domain $D(A) \subset X$. If A is closed, $D(A)$ endowed with the graph norm $\|\cdot\|_{D(A)} := \|\cdot\|_A := \|A\cdot\|_X + \|\cdot\|_X$ is a Banach space. Let E_0, E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$ and $A \in \mathcal{L}(E_1, E_0)$. In this case $D(A) = E_1$ (with equivalent norms) and $X = E_0$. The following result follows from the introductory remarks of this section.

Theorem III.7. *Let E_0, E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$. The semigroup $\big(\exp(-At)\big)_{t\geq 0}$ is strongly continuous and analytic if and only if the linear operator $-A \in \mathcal{L}(E_1, E_0)$ is sectorial.*

Observe that the operator A has to be considered as an (unbounded) operator in E_0 with dense domain $D(A) = E_1 \subset E_0$. Then, for fixed $t \geq 0$, $\exp(-At) \in \mathcal{L}(E_0)$.

The set of all negative generators of strongly continuous analytic semigroups in E_0 with domain $E_1 \overset{d}{\hookrightarrow} E_0$ is denoted by $\mathcal{H}(E_1, E_0)$. It turns out that this class is

[1] In fact, it can be shown that two sectorial operators $-A \colon D(A) \subset X \to X$ and $-B \colon D(B) \subset X \to X$ with $\exp(-At) = \exp(-Bt)$ for all $t > 0$ coincide and $D(A) = D(B)$, cf. [53, Corollary 2.1.8].

a union of subclasses $\mathcal{H}(E_1, E_0, \kappa, \omega)$ consisting of all $A \in \mathcal{L}(E_1, E_0)$ with

$$\omega + A \in \mathcal{L}\text{is}(E_1, E_0), \tag{III.10}$$

$$\kappa^{-1} \le \frac{\|(\lambda + A)x\|_0}{|\lambda|\|x\|_0 + \|x\|_1} \le \kappa \quad \text{for all} \quad x \in E_1 \setminus \{0\} \quad \text{and} \quad \operatorname{Re}\lambda \ge \omega, \tag{III.11}$$

where $\kappa \ge 1$ and $\omega > 0$ are given constants. In fact, in [2, Chapter I, Theorem 1.2.2] it is shown that

$$\mathcal{H}(E_1, E_0) = \bigcup_{\substack{\kappa \ge 1 \\ \omega > 0}} \mathcal{H}(E_1, E_0, \kappa, \omega).$$

This means that one only has to verify $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$, i.e., (III.10) and (III.11), for *some* $\kappa \ge 1$ and $\omega > 0$ to show that $-A$ generates a strongly continuous analytic semigroup. Moreover, note that (III.11) can be expressed in several equivalent ways, see Remark 1.2.1 (a) and the proof of Theorem 1.2.2 in [2, Chapter I].

Proposition III.8. *Let $\kappa \ge 1$ and $\omega > 0$ be given. Then the following two statements as well as* (III.9) *and* (III.11) *are equivalent.*

$$|\lambda|\|x\|_0 \le \kappa\|(\lambda + A)x\|_0 \quad \textit{for all} \quad x \in E_1 \quad \textit{and} \quad \operatorname{Re}\lambda \ge \omega, \tag{III.12}$$

$$\|x\|_1 \le \kappa\|(\lambda + A)x\|_0 \quad \textit{for all} \quad x \in E_1 \quad \textit{and} \quad \operatorname{Re}\lambda \ge \omega. \tag{III.13}$$

The following perturbation result holds, cf. [2, Chapter I, Theorem 1.3.1].

Theorem III.9. *Let E_0 be a Banach space and let E_1 be a Banach space that is dense in E_0. Then the following holds.*

(i) *$\mathcal{H}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$.*

(ii) *Let $A \in \mathcal{H}(E_1, E_0)$ and let $\varepsilon > 0$ and $C \ge 0$ be given. If $B \in \mathcal{L}(E_1, E_0)$ satisfies $\|Bx\|_0 \le \varepsilon\|x\|_1 + C\|x\|_0$ for $x \in E_1$, then $A + B \in \mathcal{H}(E_1, E_0)$.*

III.2.2 Fully Nonlinear Equations

Fully nonlinear Cauchy problems of the form (III.1) can be conveniently solved in interpolation spaces. With maximal regularity arguments the existence of a unique solution $f \in C([0,T], E_1) \cap C^1([0,T], E_0)$ is readily established for continuous interpolation spaces E_0 and E_1, where the latter is densely injected in E_0.

Let X be a Banach space and let $A\colon D(A) \to X$ be a linear operator with dense domain $D(A) \subset X$. Given $\alpha \in (0,1)$ and $p \in [1, \infty]$, consider the intermediate spaces with respect to $D(A)$ and X

$$D_A(\alpha, p) := \{x \in X \colon [t \mapsto v(t) := \|t^{1-\alpha-1/p} A \exp(At)x\|_X] \in L^p(0,1)\}$$

endowed with the norm $\|x\|_{D_A(\alpha,p)} := \|x\|_X + \|v\|_{L^p(0,1)}$, $x \in D_A(\alpha, p)$, and

$$D_A(\alpha) := \{x \in D_A(\alpha, \infty) \colon \lim_{t \to 0} t^{1-\alpha} A \exp(At)x = 0\}$$

endowed, of course, with the norm $\|\cdot\|_{D_A(\alpha,\infty)}$, cf. [53, Section 2.2.1]. Again, we put $1/\infty := 0$. In [53, Proposition 2.2.2] it is shown that these intermediate spaces are even interpolation spaces. More precisely, it holds that $D_A(\alpha, p) = \big(X, D(A)\big)_{\alpha,p}$, see (III.2), and $D_A(\alpha) = \big(X, D(A)\big)_{\alpha}$, see (III.3). We also introduce the spaces

$$\begin{aligned} D_A(\alpha+1, p) &:= \{x \in D(A)\colon Ax \in D_A(\alpha, p)\}, \\ D_A(\alpha+1) &:= \{x \in D(A)\colon Ax \in D_A(\alpha)\} \end{aligned}$$

equipped with the usual graph norm. It is important to note that the operators

$$\begin{aligned} A_{\alpha,p}\colon D_A(\alpha+1,p) \to D_A(\alpha,p), &\quad x \mapsto Ax, \\ A_\alpha\colon D_A(\alpha+1) \to D_A(\alpha), &\quad x \mapsto Ax \end{aligned}$$

are sectorial if A is sectorial, cf. [53, Proposition 2.2.7]. These operators are called the *parts* of A in $D_A(\alpha, p)$ and $D_A(\alpha)$ or the *maximal* $D_A(\alpha, p)$- and $D_A(\alpha)$-*realizations* of A, respectively.

A result on the existence and uniqueness of classical solutions for problems of type (III.1) can now be formulated, cf. [53, Theorem 8.4.1].

Theorem III.10. *Let $E_1 \overset{d}{\hookrightarrow} E_0 \hookrightarrow X$ be Banach spaces and let $\alpha \in (0,1)$. Let the operator $\Phi\colon \mathcal{O} \to E_0$ and its Fréchet derivative $\partial\Phi\colon \mathcal{O} \to E_0$ be continuous on an open subset $\mathcal{O} \subset E_1$. Moreover, for every $x \in \mathcal{O}$ let $\partial\Phi(x)\colon E_1 \to E_0$ be the part in E_0 of a sectorial operator $A\colon D(A) \subset X \to X$ such that $D_A(\alpha) = E_0$ and $D_A(\alpha+1) = E_1$. Then there exists $T > 0$ and problem* (III.1) *has a unique solution in the space $C([0,T], E_1) \cap C^1([0,T], E_0)$ for every $f_0 \in \mathcal{O}$.*

This result can be equivalently reformulated in an elegant way with the class of maximal regularity introduced in [19] and generalized in [4]. To do so we introduce a new class of function spaces. Given $\theta \in (0,1)$, put

$$V(\theta) := \{u\colon (0,1] \to E_0\colon \|u_\theta\|_1, \|v_\theta\|_0 < \infty\}, \tag{III.14}$$

where $u_\theta(t) := t^\theta u(t)$ and $v_\theta(t) := t^\theta u'(t)$ for $t \in (0,1]$. Equipped with the norm $\|u\|_{V(\theta)} := \sup_{0<t\le 1}\big(\|u_\theta\|_1 + \|v_\theta\|_0\big)$ this space is a Banach space. Moreover, the space

$$V_0(\theta) := \{u \in V(\theta)\colon \lim_{t\to 0} \|u\|_{V(\theta)} = 0\} \tag{III.15}$$

is a closed subspace of $V(\theta)$.

Observe that the upper bound of the interval in the definition above is arbitrary in the sense that equivalent norms are obtained when changing this value and, given $u \in V(\theta)$, the derivative u' has to be understood in the sense of distributions. Moreover, from the mean value theorem we infer for $0 < s < t \le 1$ that

$$\begin{aligned} \|u(t) - u(s)\|_0 &\le \int_s^t \tau^{-\theta} \|\tau^\theta u'(\tau)\|_0 \, d\tau \\ &\le \sup_{0<\tau\le 1} \|\tau^\theta u'(\tau)\|_0 \, \frac{\tau^{1-\theta}}{1-\theta}\bigg|_s^t \le \frac{\|u\|_{V(\theta)}}{1-\theta} |t-s|^{1-\theta}. \end{aligned}$$

Thus, u is Hölder continuous on $(0, 1]$ and a meaning can be assigned to the value of u at $t = 0$, cf [19, p. 334].

In [53, Proposition 1.2.10] it is shown that the real and continuous interpolation spaces coincide with spaces of *traces* of functions belonging to (III.14) and (III.15), respectively. More precisely, it holds that

$$(E_0, E_1)_{\theta,\infty} = \{u(0) \colon u \in V(1-\theta)\},$$

where

$$\|x\|_{\theta,V} := \inf\{\|u\|_{V(1-\theta)} \colon x = u(0), \quad u \in V(1-\theta)\}, \quad x \in (E_0, E_1)_{\theta,\infty},$$

is an equivalent norm on $(E_0, E_1)_{\theta,\infty}$, and

$$(E_0, E_1)_\theta = \{u(0) \colon u \in V_0(1-\theta)\}, \tag{III.16}$$

see also the *espaces de traces* in [11, Section 3.12]. Note that in [19, Definition 2.2] and [4, Eq. (2.1)] equality (III.16) is used to *define* the continuous interpolation spaces, see (III.3). Moreover, relation (III.16) (and, of course, also (III.4)) allows us to use results for continuous interpolation spaces independent of their definition.

Let us now introduce the classes of maximal regularity $\mathcal{M}_\theta$, $\theta \in (0, 1]$, as it is done in [4]. Given $A \in \mathcal{H}(E_1, E_0)$, $F \in C((0, 1], E_0)$, and $f_0 \in (E_0, E_1)_\theta$, where we put $(E_0, E_1)_1 := E_1$, consider the linear Cauchy problem

$$\partial_t f = Af + F, \quad f(0) = f_0. \tag{III.17}$$

The generator A is supposed to belong to the class $\mathcal{M}_\theta(E_1, E_0)$, $\theta \in (0, 1]$, if (III.17) has a unique solution $x \in V(1-\theta)$ for any data $F \in C((0, 1], E_0)$ and $f_0 \in (E_0, E_1)_\theta$. In [68, Theorem 2.3] the following criterion for $A \in \mathcal{M}_\theta(E_1, E_0)$ in continuous interpolation spaces is given.

Proposition III.11. *Let X_0 and X_1 be Banach spaces such that $X_1 \overset{d}{\hookrightarrow} X_0$ and suppose that $A \in \mathcal{H}(X_1, X_0)$. Fix $\sigma \in (0, 1)$ and let A_σ be the part of A in $E_0 := (X_0, X_1)_\sigma$ with domain E_1. Then $A_\sigma \in \mathcal{M}_\theta(E_1, E_0)$ for all $\theta \in (0, 1]$.*

In concrete applications verifying the hypotheses of Proposition III.11 is often the main part in establishing an existence result similar to Theorem III.10 as the following result from [4, Theorem 2.7] shows.

Theorem III.12. *If the Fréchet derivative $\partial\Phi(x)$ of the evolution operator in* (III.1) *belongs to the class $\mathcal{M}_1(E_1, E_0)$ for all x in an open set $\mathcal{O} \subset E_1$, then there is $T > 0$ and the Cauchy problem* (III.1) *has a unique solution $f \in C([0, T], E_1) \cap C^1([0, T], E_0)$ for all $f_0 \in \mathcal{O}$.*

Let us conclude with two properties that hold for a solution of (III.1) whose existence and uniqueness is verified, see [4, Theorem 2.9] and [53, p. 311].

Theorem III.13. *(i) Let the hypotheses of Theorem III.12 hold. Moreover, let $\Phi \in C^k(\mathcal{O}, E_0)$ for[2] $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. Then the solution of problem* (III.1) *generates a local C^k-semiflow on $\mathcal{O}$.*

(ii) Let the solution f of (III.1) *be uniformly continuous on $[0, T)$. Then it is either a global solution, i.e., $T = \infty$, or $\lim_{t\to T} f(t) \in \partial\mathcal{O}$.*

[2] $k = \omega$ indicates that Φ is (real) analytic.

III.2.3 Quasilinear Equations

We consider initial value problems of the form

$$\partial_t f = A(f)f + F(f), \quad f(0) = f_0, \tag{III.18}$$

where, given f in some suitable space, the operator $A(f)$ is a linear operator in $\mathcal{L}(E_1, E_0)$ for the Banach spaces E_0 and E_1 with $E_1 \overset{d}{\hookrightarrow} E_0$ and F is nonlinear. Clearly, problem (III.18) is of the form (III.1). Thus, we already know that it has a unique solution in the space $C([0,T], E_1) \cap C^1([0,T], E_0)$ for some $T > 0$. But in continuous interpolation spaces it is possible to avoid calculating the 'whole' linearization of the evolution operator under certain assumptions on A and F, cf. Theorem III.12. In fact, it turns out that it suffices to prove that the linear operator $A(f)$ generates a strongly continuous analytic semigroup. The following result makes this precise, cf. Theorem 2.11 and Theorem 2.12 in [4].

Theorem III.14. *Let $\theta, \sigma \in (0,1)$ be given such that $\theta < \sigma$ and put $E_\theta := (E_0, E_1)_\theta$ and $E_\sigma := (E_0, E_1)_\sigma$. Let $\mathcal{O} \subset E_\theta$ be an open set and let $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. Suppose that $A \in C^k\big(\mathcal{O}, \mathcal{L}(E_1, E_0)\big)$ and that $A(f) \in \mathcal{M}_\sigma(E_1, E_0)$ for all $f \in \mathcal{O} \cap E_\sigma$. Finally, suppose that $F \in C^k(\mathcal{O}, E_0)$. Then there is $T > 0$ and the initial value problem* (III.18) *has a unique solution $f \in C([0,T], E_1) \cap C^1([0,T], E_0)$ for any initial data $f_0 \in \mathcal{O} \cap E_\sigma$. Furthermore, this solution generates a local C^k semiflow on $\mathcal{O} \cap E_\sigma$.*

Chapter IV

Classical Solutions for Stabilized non-Newtonian Hele-Shaw Flows of Ferrofluids

We shortly recall the Hele-Shaw free boundary problem from Chapter II. Given a positive function $f \in C^4$ with $\|f\|_\infty < 1$, the domain of a fluid in a vertical Hele-Shaw cell is defined by $\Omega(f) := \{(x,y) \in \mathbb{S} \times \mathbb{R} : 0 < y < f(x)\}$ with bottom $\Gamma_0 := \mathbb{S} \times \{0\} \equiv \mathbb{S}$ and confining free surface $\Gamma(f) := \{(x, f(x)) : x \in \mathbb{S}\}$ at the top. Then the motion of a non-Newtonian ferrofluid in a Hele-Shaw cell is governed by the set of equations

$$\begin{aligned}
\mathcal{Q}u &= 0 && \text{in} \quad \Omega(f), \\
(1-\delta)u + \frac{\delta}{\bar{\mu}(|\nabla u|^2)}\partial_y u &= b(f) && \text{on} \quad \Gamma_0, \\
u &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on} \quad \Gamma(f), \\
\partial_t f &= -\frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla u|^2)}\partial_\nu u && \text{on} \quad \Gamma(f) \\
f(0) &= f_0, && \text{on} \quad \mathbb{S}
\end{aligned} \tag{IV.1}$$

where $\sigma \geq 0$ is the surface tension parameter, $\iota \geq 0$ is the intensity of a current in a straight wire at $y = 1$, and $b := \tilde{b}$ is a source term at the bottom of the cell, cf. Section II.3. We assume that $b \in C^\omega(\mathbb{R}, \mathbb{R})$. Given $f \in C^4$, recall that the *function* $b(f)$ is defined by $b(f)(x) := b(f(x))$ for $x \in \mathbb{S}$. The parameter $\delta \in \{0, 1\}$ is fixed from the very beginning determining whether a Dirichlet ($\delta = 0$) or a Neumann ($\delta = 1$) boundary condition is imposed at the bottom.

Given a viscosity function $\mu \in C^\omega([0,\infty),(0,\infty))$, the differential operator in problem (IV.1) is given by

$$\mathcal{Q} := -\operatorname{div} \frac{1}{\bar{\mu}(|\nabla \cdot|^2)}\nabla,$$

where the function $1/\bar{\mu} \colon [0,\infty) \to (0,\infty)$ is defined in II.19 by means of the function μ. We assume that there are positive constants c, C such that

$$c < \mu(r) < C \qquad \text{for all} \quad r \geq 0, \tag{IV.2}$$

$$c < \mu(r) + 2r\mu'(r) < C \quad \text{for all} \quad r \geq 0. \tag{IV.3}$$

Note that these conditions are slightly more restrictive than conditions (II.22), (II.23). But they are general enough to include many classes of non-Newtonian fluids, see, e.g., [29, p. 720]. Furthermore, the function $1/\bar{\mu}$ is of class C^ω on $(0, \infty)$.

Fix $\alpha \in (0,1)$ and let

$$\mathcal{V} := h^{4+\alpha} \cap \{f \in C^4 \colon 0 < f(x) < 1, x \in \mathbb{S}\}.$$

For some $T > 0$ we seek classical solutions (u, f) with

$$f \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha}),$$
$$u \in buc^{2+\alpha}(\Omega(f(t))), \quad 0 \leq t \leq T,$$

that fulfill (IV.1) pointwise. Note that the set $\{f \in C^4 \colon 0 < f(x) < 1, x \in \mathbb{S}\}$ can be considered as the set of *admissible* free surfaces in the sense that they neither touch the bottom of the cell leading to a separated fluid blob nor are in contact with the wire at $y = 1$, which is not included in our model.

Theorem IV.1. *Assume that conditions* (IV.2),(IV.3) *hold and let* $c \in (0,1)$. *There is an open neighborhood* $\mathcal{O}$ *of* c *in* $h^{4+\alpha}$ *such that for all* $f_0 \in \mathcal{O}$ *problem* (IV.1) *has a unique maximal classical solution in* $\mathcal{O}$.

Theorem IV.1 is the first main result in this chapter. Note that the statement is a local one in the sense that the *size* of $\mathcal{O}$ is not known. We only know that an initial condition has to be 'near' a flat solution (in the $(4+\alpha)$-norm). But for a concrete initial condition we cannot determine if it is admissible or not. We remedy this drawback in Chapter VI, where we characterize the set $\mathcal{O}$ explicitly. However, this only works for Newtonian fluids.

As a first step in the proof of Theorem IV.1 problem (IV.1) is transformed to a problem on a fixed domain and the elliptic problem for the transformed potential function consisting of the first three equations is solved. The solution, which depends on f, is plugged into the fourth equation yielding a parabolic evolution equation on $\mathbb{S}$, whose linearization is studied as a multiplication operator.

IV.1 The Transformation

For the transformation of problem (IV.1) on a fixed domain we use the following diffeomorphism. Put $\Omega := \mathbb{S} \times (0,1)$. Given $f \in \mathcal{V}$, the map

$$\phi_f(x,y) := \big(x, 1 - y/f(x)\big) \quad \text{for} \quad (x,y) \in \Omega(f)$$

is a $C^{4+\alpha}$-diffeomorphism between $\Omega(f)$ and Ω. Using ϕ_f we introduce the push forward and the pull back operators

$$\phi_*^f : BUC^{2+\alpha}(\Omega(f)) \to BUC^{2+\alpha}(\Omega), \quad u \mapsto u \circ \phi_f^{-1},$$
$$\phi_f^* : BUC^{2+\alpha}(\Omega) \to BUC^{2+\alpha}(\Omega(f)), \quad v \mapsto v \circ \phi_f.$$

Obviously, $\phi_f^* = (\phi_*^f)^{-1}$. Furthermore, these operators can be restricted to $buc^{2+\alpha}$, cf. Lemma IV.4. Given $f \in \mathcal{V}$, system (IV.1) is transformed into the system

$$\begin{aligned} \mathcal{A}(f,v) &= 0 && \text{in} \quad \Omega, \\ \mathcal{C}(f,v) &= b(f) && \text{on} \quad \Gamma_1, \\ v &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on} \quad \Gamma_0, \\ \partial_t f &= \mathcal{B}(f,v) && \text{on} \quad \Gamma_0, \\ f(0) &= f_0 && \text{on} \quad \mathbb{S} \end{aligned} \tag{IV.4}$$

where $\Gamma_1 := \mathbb{S} \times \{1\}$. The transformed operators in system (IV.4) are

$$\begin{aligned} \mathcal{A}(f,\,\cdot\,) &:= \phi_*^f \mathcal{Q} \phi_f^* : buc^{2+\alpha}(\Omega) \to buc^{\alpha}(\Omega), \\ \mathcal{B}(f,\,\cdot\,) &:= \phi_*^f \Big[-\frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla(\phi_f^*\,\cdot\,)|^2)} \partial_\nu(\phi_f^*\,\cdot\,)|\Gamma(f) \Big] \\ &= -\gamma_0 \phi_*^f \frac{\nabla(\phi_f^*\,\cdot\,)}{\bar{\mu}(|\nabla(\phi_f^*\,\cdot\,)|^2)} \cdot n : buc^{2+\alpha}(\Omega) \to h^{1+\alpha}, \\ \mathcal{C}(f,\,\cdot\,) &:= (1-\delta)\phi_*^f \gamma_0 \phi_f^* + \delta \phi_*^f \gamma_0 \frac{1}{\bar{\mu}(|\nabla(\phi_f^*\,\cdot\,)|^2)} \partial_2(\phi_f^*\,\cdot\,) : buc^{2+\alpha}(\Omega) \to h^{2-\delta+\alpha}, \end{aligned}$$

where $n := \begin{pmatrix} -f_x & 1 \end{pmatrix}$, $\nu := n/\|n\|$, and γ_0 is the trace operator with respect to Γ_0. To get a representation for the transformed operator $\mathcal{A}(f,\,\cdot\,)$ in the new coordinates, we use the fact that the quasilinear operator $\mathcal{Q}$ has the representation

$$u \mapsto \mathcal{Q}u = \sum_{i,j=1}^{2} a_{ij}(\nabla u)\partial_{ij}u \quad \text{for} \quad u \in C^2(\Omega(f)),$$

where

$$a_{ij}(p) = \frac{\delta_{ij}}{\bar{\mu}(|p|^2)} + 2\Big(\frac{1}{\bar{\mu}}\Big)'(|p|^2)p_i p_j \quad \text{for} \quad p \in \mathbb{R}^2, \quad 1 \le i,j \le 2, \tag{IV.5}$$

cf. also [29]. With this representation the coefficients of the transformed operator

$$v \mapsto \mathcal{A}(f,v) = \sum_{i,j=1}^{2} b_{ij}(f,\nabla v)\partial_{ij}v + b_2(f,\nabla v)\partial_2 v \quad \text{for} \quad v \in C^2(\Omega)$$

can be computed as

$$\begin{aligned} b_{11}(f,\nabla v) &= a_{11}(D_f v), \\ b_{12}(f,\nabla v) &= b_{21}(f,\nabla v) = \frac{\pi f_x}{f} a_{11}(D_f v) - \frac{1}{f} a_{12}(D_f v), \\ b_{22}(f,\nabla v) &= \frac{\pi^2 {f_x}^2}{f^2} a_{11}(D_f v) - 2\frac{\pi f_x}{f^2} a_{12}(D_f v) + \frac{1}{f^2} a_{22}(D_f v), \\ b_2(f,\nabla v) &= \pi \frac{f_{xx} f - 2f_x^2}{f^2} a_{11}(D_f v) + 2\frac{f_x}{f^2} a_{12}(D_f v), \end{aligned}$$

where we use the notation $D_f v := \big(\partial_1 v + \frac{\pi f_x}{f}\partial_2 v, -\frac{1}{f}\partial_2 v\big)$ and $\pi(x,y) := 1-y$ for $(x,y) \in \Omega$. The representation for the boundary operators in the new coordinates are

$$\mathcal{B}(f,v) = \frac{f_x\gamma_0\partial_1 v + \frac{1+f_x^2}{f}\gamma_0\partial_2 v}{\bar{\mu}\big((\gamma_0\partial_1 v + \frac{f_x}{f}\gamma_0\partial_2 v)^2 + \frac{1}{f^2}(\gamma_0\partial_2 v)^2\big)},$$
$$\mathcal{C}(f,v) = (1-\delta)\gamma_1 v - \frac{\delta}{f\bar{\mu}\big((\gamma_1\partial_1 v)^2 + \frac{1}{f^2}(\gamma_1\partial_2 v)^2\big)}\gamma_1\partial_2 v,$$

where γ_1 is the trace operator with respect to Γ_1. The following result follows from elementary calculations.

Lemma IV.2. *Let $f \in \mathcal{V}$ and $v \in buc^{2+\alpha}(\Omega)$. Then $\mathcal{B} \in C^\omega(\mathcal{V} \times buc^{2+\alpha}, h^{1+\alpha})$ and*

$$\begin{aligned}\partial_f\mathcal{B}(f,v)h = {}& \frac{1}{\bar{\mu}\big((\gamma_0\partial_1 v + \frac{f_x}{f}\gamma_0\partial_2 v)^2 + \frac{1}{f^2}(\gamma_0\partial_2 v)^2\big)}\Big(h_x\gamma_0\partial_1 v \\ & + \Big(-\frac{1+f_x^2}{f^2}h + \frac{2f_x}{f}h_x\Big)\gamma_0\partial_2 v\Big) \\ & + 2\Big(\frac{1}{\bar{\mu}}\Big)'\big((\gamma_0\partial_1 v + \tfrac{f_x}{f}\gamma_0\partial_2 v)^2 + \tfrac{1}{f^2}(\gamma_0\partial_2 v)^2\big) \\ & \times \Big(f_x\gamma_0\partial_1 v + \frac{1+f_x^2}{f}\gamma_0\partial_2 v\Big)\Big(\Big(-\frac{f_x}{f^2}h + \frac{1}{f}h_x\Big)\gamma_0\partial_1 v\gamma_0\partial_2 v \\ & + \Big(\frac{f_x}{f^2}h_x - \frac{1+f_x^2}{f^3}h\Big)(\gamma_0\partial_2 v)^2\Big)\end{aligned}$$

for $h \in h^{4+\alpha}$. Furthermore, it holds that

$$\begin{aligned}\partial_v\mathcal{B}(f,v)u = {}& \frac{1}{\bar{\mu}\big((\gamma_0\partial_1 v + \frac{f_x}{f}\gamma_0\partial_2 v)^2 + \frac{1}{f^2}(\gamma_0\partial_2 v)^2\big)}\Big(f_x\gamma_0\partial_1 u \\ & + \frac{1+f_x^2}{f}\gamma_0\partial_2 u\Big) + 2\Big(\frac{1}{\bar{\mu}}\Big)'\big((\gamma_0\partial_1 v + \tfrac{f_x}{f}\gamma_0\partial_2 v)^2 + \tfrac{1}{f^2}(\gamma_0\partial_2 v)^2\big) \\ & \times \Big(f_x\gamma_0\partial_1 v + \frac{1+f_x^2}{f}\gamma_0\partial_2 v\Big)\Big(\gamma_0\partial_1 v\gamma_0\partial_1 u \\ & + \frac{f_x}{f}(\gamma_0\partial_1 u\gamma_0\partial_2 v + \gamma_0\partial_1 v\gamma_0\partial_2 u) + \frac{1+f_x^2}{f^2}\gamma_0\partial_2 v\gamma_0\partial_2 u\Big]\end{aligned}$$

for $u \in buc^{2+\alpha}(\Omega)$.

Lemma IV.3. *Let $f \in \mathcal{V}$ be given and suppose that conditions* (IV.2) *and* (IV.3) *hold. Then the differential operator $\mathcal{Q}$ is uniformly elliptic in $\Omega(f)$ and the differential operator $\mathcal{A}(f,\cdot)$ is uniformly elliptic in Ω.*

Proof. In [31, pp. 122–123] it is shown that there are positive constants $\tilde{c}$ and $\tilde{C}$ such that

$$\tilde{c} < \frac{1}{\bar{\mu}(r)} < \tilde{C} \qquad \text{for all} \quad r \geq 0, \tag{IV.6}$$

$$\tilde{c} < \frac{1}{\bar{\mu}(r)} + 2r\Big(\frac{1}{\bar{\mu}}\Big)'(r) < \tilde{C} \quad \text{for all} \quad r \geq 0, \tag{IV.7}$$

provided conditions (IV.2) and (IV.3) hold. Given $p \in \mathbb{R}^2$, the eigenvalues of the matrix $\big(a_{ij}(p)\big)_{1\leq i,j\leq 2}$ are given by

$$\frac{1}{\bar{\mu}(|p|^2)}, \quad \frac{1}{\bar{\mu}(|p|^2)} + 2|p|^2\Big(\frac{1}{\bar{\mu}}\Big)'(|p|^2).$$

From this and (IV.6), (IV.7) the uniform ellipticity of $\mathcal{Q}$ follows.

Note that, given $f \in \mathcal{V}$ and $p \in \mathbb{R}^2$, the following decomposition of the coefficient matrix of the principal part of the operator $\mathcal{A}(f, \cdot\,)$ holds:

$$\big(b_{ij}(f,p)\big)_{1\leq i,j\leq 2} = T(f,p)\Big(a_{ij}\big(p_1 + \tfrac{(1-p_2)f_x(p_1)}{f(p_1)}p_2, -\tfrac{1}{f(p_1)}p_2\big)\Big)_{1\leq i,j\leq 2} T(f,p)^\top,$$

where

$$T(f,p) := \begin{pmatrix} 1 & 0 \\ \frac{(1-p_2)f_x(p_1)}{f(p_1)} & -\frac{1}{f(p_1)} \end{pmatrix}.$$

Now the uniform ellipticity of $\mathcal{A}(f, \cdot\,)$ follows from the uniform ellipticity of $\mathcal{Q}$ and Observation 7.1.6 in [46]. □

We call a pair (f,u) a classical solution of problem (IV.4) if it fulfills the equations pointwise and if there is a $T > 0$ such that

$$f \in C([0,T],\mathcal{V}) \cap C^1([0,T],h^{1+\alpha}),$$
$$u \in buc^{2+\alpha}(\Omega).$$

Then problem (IV.1) and problem (IV.4) are equivalent in the following sense.

Lemma IV.4. *If* (u,f) *is a classical solution of* (IV.1) *then* $(\phi_f^* u, f)$ *is a classical solution of* (IV.4) *and if* (v,f) *is a classical solution of* (IV.4) *then* $(\phi_*^f v, f)$ *is a classical solution of* (IV.1). *In short, given* $f \in \mathcal{V}$, $\phi_f^* : buc^{2+\alpha}(\Omega(f)) \to buc^{2+\alpha}(\Omega)$ *is an isomorphism with* $(\phi_f^*)^{-1} = \phi_*^f$.

Proof. It is clear that if (u,f) is a classical solution of (IV.1) then $(\phi_f^* u, f)$ is a solution of (IV.4) and if (v,f) is a classical solution of (IV.4) then $(\phi_*^f v, f)$ is a solution of (IV.1). It is left to check the respective regularity of the transformed functions, i.e., $\phi_f^*\big(buc^{2+\alpha}(\Omega(f))\big) = buc^{2+\alpha}(\Omega)$. We show only $\phi_f^*\big(buc^{2+\alpha}(\Omega(f))\big) \subset buc^{2+\alpha}(\Omega)$ for $h \in h^{4+\alpha}$. The other inclusion is similar.

For $u \in buc^{2+\alpha}(\Omega(f))$ we have to show that each neighborhood of $v := \phi_f^* u$ contains a smooth function. We construct such a function by choosing a sequence

$(u_n)_n \subset BUC^\infty(\Omega(f))$ such that $u_n \to u$ in $BUC^{2+\alpha}(\Omega(f))$ and another sequence of positive functions $(f_n)_n \subset C^\infty$ with $\|f_n\|_\infty < 1$ for all n such that $f_n \to f$ in $C^{4+\alpha}$. Set $v_{n,m} := \phi^*_{f_m} u_n \in BUC^\infty(\Omega)$. For arbitrary $\varepsilon > 0$ we seek $n, m \in \mathbb{N}$ such that $\|v_{n,m} - v\|_{2+\alpha,\Omega} < \varepsilon$.

First, we extend $v_{n,m} - v = \phi^*_{f_m} u_n - \phi^*_f u_n + \phi^*_f u_n - \phi^*_f u$. We know that ϕ_f is a $C^{4+\alpha}$-diffeomorphism. Then $\|\phi^*_f u_n - \phi^*_f u\|_{2+\alpha,\Omega}$ can be made small due to

$$\begin{aligned} \|\phi^*_f u_n - \phi^*_f u\|_{0,\Omega} &= \|u_n - u\|_{0,\Omega(f)}, \\ \|\partial(\phi^*_f u_n - \phi^*_f u)\|_{0,\Omega} &\le C\|\partial(u_n - u)\|_{0,\Omega(f)}, \\ \|\partial^2(\phi^*_f u_n - \phi^*_f u)\|_{0,\Omega} &\le C\big(\|\partial^2(u_n - u)\|_{0,\Omega(f)} + \|\partial(u_n - u)\|_{0,\Omega(f)}\big), \end{aligned}$$

where C is independent of u and u_n, and the Hölder estimate

$$\begin{aligned} &\frac{|\partial^2\phi^*_f u_n(x,y) - \partial^2\phi^*_f u(x,y) - \partial^2\phi^*_f u_n(\tilde{x},\tilde{y}) + \partial^2\phi^*_f u(\tilde{x},\tilde{y})|}{|(x,y) - (\tilde{x},\tilde{y})|^\alpha} \\ &\le C\big([\partial^2(u_n - u)]_{\alpha,\Omega(f)} + [\partial(u_n - u)]_{\alpha,\Omega(f)}\big)\frac{|\phi_f(x,y) - \phi_f(\tilde{x},\tilde{y})|^\alpha}{|(x,y) - (\tilde{x},\tilde{y})|^\alpha} \\ &\le C\tilde{C}\|\partial^2(u_n - u)\|_{\alpha,\Omega(f)} \end{aligned}$$

since there is a positive constant $\tilde{C}$ such that

$$\begin{aligned} \frac{|\phi_f(x,y) - \phi_f(\tilde{x},\tilde{y})|}{|(x,y) - (\tilde{x},\tilde{y})|} &\le 1 + \frac{|y/f(x) - \tilde{y}/f(\tilde{x})|}{|(x,y) - (\tilde{x},\tilde{y})|} \\ &\le 1 + 1/\inf_S f =: \tilde{C} \end{aligned}$$

if we assume $x \neq \tilde{x}$ and $y \neq \tilde{y}$. Otherwise, obvious modifications can be applied in both inequalities. Hence, $\|\phi^*_f u_n - \phi^*_f u\|_{2+\alpha,\Omega} < \varepsilon/2$ for $n \in \mathbb{N}$ large enough.

Now, with n fixed, u_n is smooth and the mean value theorem implies the existence of a positive constant C independent of u_n such that

$$\begin{aligned} \|\phi^*_{f_m} u_n - \phi^*_f u_n\|_{2+\alpha,\Omega} &= \Big\| \int_0^1 \partial u_n(t\phi_f + (1-t)\phi_{f_m})\,dt|\phi_{f_m} - \phi_f| \Big\|_{2+\alpha,\Omega} \\ &\le \sup_{t\in[0,1]} \|\partial u_n(t\phi_f + (1-t)\phi_{f_m})\|_{2+\alpha,\Omega} \\ &\quad \times |\big(0, y(1/f_m(x) - 1/f(x))\big)| \\ &\le C\|1/f_m - 1/f\|_\infty. \end{aligned}$$

Since $\|1/f_m - 1/f\|_\infty < \varepsilon/(2C)$ for $m \in \mathbb{N}$ sufficiently large, the proof is complete. □

The equivalence of system (IV.1) and system (IV.4) permits the statement of the following existence and uniqueness result.

Theorem IV.5. *Let $f \in \mathcal{V}$ be given. Then there exists a unique solution $\mathcal{T}(f) \in buc^{2+\alpha}(\Omega)$ of the problem*

$$\begin{aligned} \mathcal{A}(f,v) &= 0 && \text{in} \quad \Omega, \\ \mathcal{C}(f,v) &= b(f) && \text{on} \quad \Gamma_1, \\ v &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on} \quad \Gamma_0. \end{aligned} \tag{IV.8}$$

Moreover, $\mathcal{T} \in C^\omega(\mathcal{V}, buc^{2+\alpha}(\Omega))$.

Proof. Let $f \in \mathcal{V}$ be fixed for the entire proof.

Assume that $\delta = 0$ and consider the Dirichlet problem

$$\begin{aligned} \mathcal{Q}u &= 0 && \text{in} \quad \Omega(f), \\ u &= b(f) && \text{on} \quad \Gamma_0, \\ u &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on} \quad \Gamma(f). \end{aligned}$$

Since the form of the quasilinear operator $\mathcal{Q}$ exactly meets [42, Eq. (11.8)] and its ellipticity is uniform in $\Omega(f)$, it follows from Theorem 11.5 in [42], the remark following the proof of this result (see [42, p. 285]), and the remark after [42, Theorem 14.1] that the above Dirichlet problem has a solution in $BUC^{2+\alpha}(\Omega(f))$. Its uniqueness is obtained by the comparison principle.

Let $\delta = 1$. Now we have to solve an elliptic problem with two boundary portions. We still have the Dirichlet boundary portion $\Gamma(f)$. But now a differential boundary operator on Γ_0 comes into play.

Observe that the operator $\mathcal{Q}$ is in divergence form, i.e., $\mathcal{Q} = \operatorname{div} \mathbf{A}$, where $\mathbf{A}(\nabla u) = -\frac{1}{\bar{\mu}(|\nabla u|^2)}\nabla u$ for $u \in BUC^2(\Omega(f))$. Also the boundary operator on Γ_0 can be expressed in terms of the operator $\mathbf{A}$ since

$$\gamma_0 \frac{1}{\bar{\mu}(|\nabla u|^2)}\partial_y u = \gamma_0 \frac{1}{\bar{\mu}(|\nabla u|^2)}\nabla u \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} =: -\gamma_0 \mathbf{A}(\nabla u) \cdot \nu_0,$$

where ν_0 is the inner normal at Γ_0. Thanks to this structure of the differential operators in $\Omega(f)$ and on Γ_0, we can apply the results on quasilinear equations in divergence form due to O. Ladyzhenskaya and N. Ural'tseva, cf. [51, Section 10.2]. The results from [51] were improved by G. Lieberman in [52], where the solvability of uniformly elliptic equations in divergence form is reduced to a priori estimates for solutions in the space $BUC^{1+\beta}(\Omega(f))$ for some $\beta \in (0,1)$ and the solvability of an associated linear boundary value problem.

Fix $\beta \in (0,1)$. The fact that the boundary value problem

$$\begin{aligned} \operatorname{div} \mathbf{A}(\nabla u) &= 0 && \text{in} \quad \Omega(f), \\ \mathbf{A}(\nabla u) \cdot \nu_0 &= -b(f) && \text{on} \quad \Gamma_0, \\ u &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on} \quad \Gamma(f) \end{aligned} \tag{IV.9}$$

contains a Dirichlet boundary component and arguments similar to Lemma 2.1 from [51] yield an a priori bound for the supremum of solutions u in $\Omega(f)$. Note that the operator $\mathbf{A}$ is independent of the spatial variable and depends on u only through ∇u. Moreover, inequalities (IV.6) and (IV.7) imply the existence of a positive constant $\tilde{C}$ such that

$$|\mathbf{A}(p)| + (1+|p|)|\mathbf{A}'(p)| \leq \tilde{C}(1+|p|) \quad \text{for} \quad p \in \mathbb{R}^2.$$

These facts, the structure and the uniform ellipticity of the operator $\operatorname{div}\mathbf{A}$ and Lemmas 6 and 7 from [52] imply an a priori bound for solutions of (IV.9) in the space $BUC^{1+\beta}(\Omega(f))$.

Suppose now that $\beta \in (0,\alpha)$ and define the map

$$P := (\mathcal{Q}, N, \gamma)\colon BUC^{2+\beta}(\Omega(f)) \to BUC^{\beta}(\Omega(f)) \times C^{1+\beta}(\Gamma_0) \times C^{2+\beta}(\Gamma(f)),$$

where $Nu := \gamma_0 \mathbf{A}(\nabla u)\cdot \nu_0 + b(f)$ and $\gamma u := \gamma_f u + \sigma\kappa_f + \frac{\iota^2}{(1-f)^2} - f$ for $u \in BUC^{2+\beta}(\Omega(f))$ and γ_f denotes the trace operator with respect to $\Gamma(f)$. Fix $u \in BUC^{2+\beta}(\Omega(f))$. Then the Fréchet derivatives of the components of P in u are given by

$$\begin{aligned}
\partial\mathcal{Q}\psi &= \sum_{i,j=1}^{2} a_{ij}(\nabla u)\partial_{ij}\psi + \sum_{i,j=1}^{2} \partial_{ij}u\nabla a_{ij}(\nabla u)\cdot\nabla\psi,\\
\partial N\psi &= \gamma_0\nabla(\mathbf{A}(\nabla u)\cdot\nu_0)\cdot\nabla\psi,\\
\partial\gamma\psi &= \gamma_f\psi
\end{aligned}$$

for $\psi \in BUC^{2+\beta}(\Omega(f))$. We have to prove that P is onto. More precisely, given $u \in BUC^{2+\beta}(\Omega(f))$, consider the problem $\partial P\psi + Pu = 0$ in $BUC^{2+\beta}(\Omega(f))$. This is equivalent to the boundary value problem

$$\begin{aligned}
\sum_{i,j=1}^{2} a_{ij}(\nabla u)\partial_{ij}\psi + \sum_{i,j=1}^{2} \partial_{ij}u\nabla a_{ij}(\nabla u)\cdot\nabla\psi + \sum_{i,j=1}^{2} a_{ij}(\nabla u)\partial_{ij}u &= 0 \quad \text{in} \quad \Omega(f),\\
\nabla(\mathbf{A}(\nabla u)\cdot\nu_0)\cdot\nabla\psi + b(f) + \gamma_0\mathbf{A}(\nabla u)\cdot\nu_0 &= 0 \quad \text{on} \quad \Gamma_0\\
\psi + \sigma\kappa_f + \frac{\iota^2}{(1-f)^2} - f + \gamma_f u &= 0 \quad \text{on} \quad \Gamma(f).
\end{aligned}$$

But the solvability of this problem is guaranteed by the linear theory, see, e.g., Theorems 6.14 and 6.31 in [42]. Hence, we may infer from Theorem 1′ in [52] and the comparison principle that there exists a unique solution of problem (IV.9) in the space $BUC^{2+\beta}(\Omega(f))$. It follows from the Remark after Theorem 2 in [52] that this solution is also in $BUC^{2+\alpha}(\Omega(f))$.

In the following we show that $\mathcal{T}$ is analytic. We start by associating linear operators to the differential operators in (IV.8). Let

$$\begin{aligned}
\mathcal{S} &: \mathcal{V} \times BUC^{2+\alpha}(\Omega) \to \mathcal{L}(BUC^{2+\alpha}(\Omega), BUC^{\alpha}(\Omega)),\\
\mathcal{N} &: \mathcal{V} \times BUC^{2+\alpha}(\Omega) \to \mathcal{L}(BUC^{2+\alpha}(\Omega), h^{2-\delta+\alpha})
\end{aligned}$$

be defined by

$$\mathcal{S}(f,v) := \sum_{i,j=1}^{2} b_{ij}(f,\nabla v)\partial_{ij} + b_2(f,\nabla v)\partial_2,$$

$$\mathcal{N}(f,v) := (1-\delta)\gamma_1 - \frac{\delta}{f\bar{\mu}\big((\gamma_1\partial_1 v)^2 + \frac{1}{f^2}(\gamma_1\partial_2 v)^2\big)}\gamma_1\partial_2,$$

respectively. As $\mathcal{A}(f,\cdot)$ is uniformly elliptic in Ω, so is also $\mathcal{S}(f,v)$, where $v \in BUC^{2+\alpha}(\Omega)$. Moreover, $\mathcal{S}$ and $\mathcal{N}$ depend analytically on its variables. Now we define the operator

$$\mathcal{I} : \mathcal{V} \times BUC^{2+\alpha}(\Omega) \to BUC^{2+\alpha}(\Omega))$$

via

$$\mathcal{I}(f,v) := \big(\mathcal{S}(f,v), \mathcal{N}(f,v), \gamma_0\big)^{-1}\big(0, b(f), -\sigma\kappa_f - \tfrac{\iota^2}{(1-f)^2} + f\big).$$

The operator $\mathcal{I}$ is also analytic because the second and the third argument on the right hand side depend analytically on f and the map mapping a bijective linear operator on its inverse is analytic. Observe that the operators $\mathcal{S}$ and $\mathcal{N}$ and also their derivatives have extensions to $\mathcal{V} \times BUC^{1+\alpha}(\Omega)$. Since the imbedding $BUC^{2+\alpha}(\Omega) \hookrightarrow BUC^{1+\alpha}(\Omega)$ is compact, the linear operator $\mathrm{id}_{BUC^{2+\alpha}(\Omega)} - \partial_v\mathcal{I}(f,v)$ is bijective if and only if it is injective. Now, let $\mathcal{T}(f)$ be the unique solution of (IV.8) in $BUC^{2+\alpha}(\Omega)$, i.e., it is a fixed point of $\mathcal{I}(f,\cdot)$. Given $w \in BUC^{2+\alpha}(\Omega)$, the function $\partial_v\mathcal{I}(f,\mathcal{T}(f))w$ is the unique solution of the linear problem

$$\begin{aligned}
\mathcal{S}(f,\mathcal{T}(f))z &= -\sum_{i,j=1}^{2} \partial_{ij}\mathcal{I}(f,v)\partial_v b_{ij}(f,\nabla v) - \partial_2\mathcal{I}(f,v)\partial_v b_2(f,\nabla v) && \text{in} \quad \Omega,\\
\mathcal{N}(f,\mathcal{T}(f))z &= \frac{2\delta}{f}\Big(\gamma_1\partial_1 v\gamma_1\partial_1 w + \frac{1}{f^2}\gamma_1\partial_2 v\gamma_1\partial_2 w\Big)\gamma_1\partial_2\mathcal{I}(f,\mathcal{T}(f)) && \text{on} \quad \Gamma_1,\\
&\quad \times \Big(\frac{1}{\bar{\mu}}\Big)'\big((\gamma_1\partial_1 v)^2 + \tfrac{1}{f^2}(\gamma_1\partial_2 v)^2\big) &&\\
z &= 0 && \text{on} \quad \Gamma_0.
\end{aligned}$$

Take $w \in \ker\big(\mathrm{id}_{BUC^{2+\alpha}(\Omega)} - \partial_v\mathcal{I}(f,\mathcal{T}(f))\big)$, i.e., $w = \partial_v\mathcal{I}(f,\mathcal{T}(f))w$. Thus substituting z by w in the above system implies that $w = 0$, i.e., $\mathrm{id}_{BUC^{2+\alpha}(\Omega)} - \partial_v\mathcal{I}(f,\mathcal{T}(f))$ is bijective. Hence, $\partial_v\mathcal{I}(f,\mathcal{T}(f))$ is bijective, too. From this fact, the analyticity of $\mathcal{I}$ and the implicit function theorem we finally get the analyticity of $\mathcal{T}$ in a neighborhood of f.

An immediate consequence of the analyticity of $\mathcal{T}$ is the fact that $\mathcal{T}(f) \in buc^{2+\alpha}(\Omega)$ for given $f \in \mathcal{V}$: Pick positive functions $(f_n)_n \subset C^\infty$ with $\|f_n\|_\infty < 1$ for all $n \in \mathbb{N}$ and $f_n \to f$ in $C^{4+\alpha}$. Set $v_n := \mathcal{T}(f_n) \in BUC^\infty(\Omega)$ and $v := \mathcal{T}(f)$. Due to the loss of regularity in the boundary condition we find $v_n \to v$ in $BUC^{2+\alpha}(\Omega)$. Hence $\mathcal{T}(f) \in buc^{2+\alpha}(\Omega)$. □

Let us conclude this section with the following elementary differentiability result for the operator $\mathcal{A}$.

Lemma IV.6. *It holds that $\mathcal{A}(\,\cdot\,,\mathcal{T}(\,\cdot\,)) \in C^\omega(\mathcal{V}, buc^\alpha)$. Given $f \in \mathcal{V}$, we have*

$$\begin{aligned}\partial[\mathcal{A}(f,\mathcal{T}(f))]h &= \partial\mathcal{A}(f,\mathcal{T}(f))[h, \partial\mathcal{T}(f)h] \\ &= \sum_{i,j=1}^{2} \big(\partial_{ij}\mathcal{T}(f)\partial b_{ij}h + b_{ij}\partial_{ij}\partial\mathcal{T}(f)h\big) \\ &\quad + \partial_2\mathcal{T}(f)\partial b_2 h + b_2\partial_2\partial\mathcal{T}(f)h,\end{aligned}$$

where

$$\begin{aligned}\partial b_{11}h &= \partial a_{11}h, \\ \partial b_{12}h &= \partial b_{12}h = \pi\Big(-\frac{f_x h}{f^2} + \frac{h_x}{f}\Big)a_{11} + \frac{\pi f_x}{f}\partial a_{11}h \\ &\qquad\qquad + \frac{h}{f^2}a_{12} - \frac{1}{f}\partial a_{12}h, \\ \partial b_{22}h &= 2\pi^2\Big(\frac{f_x h_x}{f^2} - \frac{f_x^2 h}{f^3}\Big)a_{11} + \pi^2\frac{f_x^2}{f^2}\partial a_{11}h \\ &\quad - 2\pi\Big(\frac{h_x}{f^2} - \frac{2f_x h}{f^3}\Big)a_{12} - 2\pi\frac{f_x}{f^2}\partial a_{12}h \\ &\quad - 2\frac{h}{f^3}a_{22} + \frac{1}{f^2}\partial a_{22}h, \\ \partial b_2 h &= \pi\Big(\frac{4f_x^2 h}{f^3} - \frac{4f_x h_x + f_{xx}h}{f^2} + \frac{h_{xx}}{f}\Big)a_{11} \\ &\quad + \pi\Big(-\frac{2f_x^2}{f^2} + \frac{f_{xx}}{f}\Big)\partial a_{11}h + \Big(-\frac{4f_x h}{f^3} + \frac{2h_x}{f^2}\Big)a_{12} \\ &\quad + \frac{2f_x}{f^2}\partial a_{12}h,\end{aligned}$$

and $h \in h^{4+\alpha}$.

IV.2 The Evolution Equation

By defining the operator $\Phi := \mathcal{B}(\,\cdot\,,\mathcal{T}(\,\cdot\,)) : \mathcal{V} \to h^{1+\alpha}$ we can reduce problem (IV.4) to the abstract Cauchy problem

$$\partial_t f = \Phi(f), \quad f(0) = f_0. \tag{IV.10}$$

A function having the regularity $C([0,T],\mathcal{V}) \cap C^1([0,T], h^{1+\alpha})$ for some $T > 0$ that solves (IV.10) pointwise is called a classical solution of problem (IV.10).

Lemma IV.7. *Problem* (IV.10) *and system* (IV.4) *are equivalent in the sense that if f is a classical solution of* (IV.10) *then $(f, \mathcal{T}(f))$ is a classical solution of* (IV.4) *and if (f, v) is a classical solution of* (IV.4) *then $v = \mathcal{T}(f)$ and f is a classical solution of* (IV.10).

Proof. This follows immediately from the regularities of the respective functions and Theorem IV.5. □

For a given $c \in (0,1)$ the function $f = c$ is a classical solution of (IV.10). In this section we characterize the linearization of the evolution operator Φ from (IV.10) about c being a preparation for the following section where we prove that this linearization $\partial\Phi(c)$ generates an analytic semigroup in $h^{1+\alpha}$.

Let $c \in (0,1)$ be fixed. For the linearization of Φ about c we obtain

$$\partial\Phi(c)h = \partial\mathcal{B}(c,\mathcal{T}(c))[h,\partial\mathcal{T}(c)h], \quad h \in h^{4+\alpha}. \tag{IV.11}$$

In order to characterize this linearization in more detail, let us start with the following result.

Lemma IV.8. *Let the function $\bar{h}\colon [0,\infty) \to [0,\infty)$ be given by $\bar{h}(r) := \frac{1}{\bar{\mu}(r)^2}r$ and let* (IV.2)*,* (IV.3) *hold. Then $\bar{h}$ is uniquely invertible.*

Proof. It holds that $\bar{h}'(r) = \frac{1}{\bar{\mu}(r)^2} + 2r\frac{1}{\bar{\mu}(r)}\big(\frac{1}{\bar{\mu}}\big)'(r)$ and hence, $\bar{h}' > 0$ if and only if

$$\frac{1}{\bar{\mu}(r)}, \quad \frac{1}{\bar{\mu}(r)} + 2r\Big(\frac{1}{\bar{\mu}}\Big)'(r) > 0, \quad r \geq 0.$$

In [31, pp. 122–123] it is shown that assumptions (IV.2), (IV.3) are equivalent to the existence of two positive constants $\tilde{c}, \tilde{C}$ such that

$$\begin{aligned} &\tilde{c} < \frac{1}{\bar{\mu}(r)} < \tilde{C} && \text{for all} \quad r \geq 0,\\ &\tilde{c} < \frac{1}{\bar{\mu}(r)} + 2r\Big(\frac{1}{\bar{\mu}}\Big)'(r) < \tilde{C} && \text{for all} \quad r \geq 0. \end{aligned}$$

The assertion follows. □

Lemma IV.9. *Let $\beta_c := (1-\delta)b(c) - \delta\,\mathrm{sgn}(b(c))c\sqrt{\bar{h}^{-1}(b(c)^2)}$. Moreover, set $\theta_c := (1-\delta)\big(\frac{\iota^2}{(1-c)^2} - c\big) + \beta_c$ and*

$$\Gamma_{\bar{\mu},c} := \frac{1}{\bar{\mu}\big((\frac{\theta_c}{c})^2\big)} + 2\frac{\theta_c^2}{c^2}\Big(\frac{1}{\bar{\mu}}\Big)'\big((\tfrac{\theta_c}{c})^2\big).$$

It holds that

$$\partial\Phi(c)h = \frac{\Gamma_{\bar{\mu},c}}{c}\Big(\gamma_0\partial_2\partial\mathcal{T}(c)h - \frac{\theta_c}{c}h\Big), \tag{IV.12}$$

where $h \in h^{4+\alpha}$.

Proof. In case $\delta = 0$ the function v with

$$v(x,y) = (1-y)\Big(-\frac{\iota^2}{(1-c)^2} + c\Big) + b(c)y, \quad (x,y) \in \Omega,$$

is the unique solution of the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}(c,v) &= 0 && \text{in} \quad \Omega, \\ v &= b(c) && \text{on} \quad \Gamma_1, \\ v &= -\frac{\iota^2}{(1-c)^2} + c && \text{on} \quad \Gamma_0. \end{aligned}$$

Let us assume that in case $\delta = 1$ the unique solution of the problem

$$\begin{aligned} \mathcal{A}(c,v) &= 0 && \text{in} \quad \Omega, \\ -\frac{1}{c\bar{\mu}\big((\partial_1 v)^2 + \frac{1}{c^2}(\partial_2 v)^2\big)}\partial_2 v &= b(c) && \text{on} \quad \Gamma_1, \\ v &= -\frac{\iota^2}{(1-c)^2} + c && \text{on} \quad \Gamma_0 \end{aligned}$$

does not depend on x, too. Then this system is equivalent to

$$\begin{aligned} \partial_{22} v &= 0 && \text{in} \quad \Omega, \\ -\frac{1}{c\bar{\mu}\big(\frac{1}{c^2}(\partial_2 v)^2\big)}\partial_2 v &= b(c) && \text{on} \quad \Gamma_1, \\ v &= -\frac{\iota^2}{(1-c)^2} + c && \text{on} \quad \Gamma_0. \end{aligned}$$

Squaring the second equation in this system gives $\frac{1}{\bar{\mu}\left((\frac{1}{c}\partial_2 v)^2\right)^2}\left(\frac{1}{c}\partial_2 v\right)^2 = b(c)^2$ on Γ_1. Thanks to Lemma IV.8 this can be equivalently reformulated by $\partial_2 v = \pm c\sqrt{\bar{h}^{-1}(b(c)^2)}$. Since c and $1/\bar{\mu}$ are both positive, $\partial_2 v$ and $b(c)$ have different signs. Hence, we consider $\partial_2 v = -\operatorname{sgn}(b(c))c\sqrt{\bar{h}^{-1}(b(c)^2)}$ in the following.

Obviously, the unique solution of the problem

$$\begin{aligned} \partial_{22} v &= 0 && \text{in} \quad \Omega, \\ \partial_2 v &= -\operatorname{sgn}(b(c))c\sqrt{\bar{h}^{-1}(b(c)^2)} && \text{on} \quad \Gamma_1, \\ v &= -\frac{\iota^2}{(1-c)^2} + c && \text{on} \quad \Gamma_0 \end{aligned}$$

is given by

$$v(x,y) = -\frac{\iota^2}{(1-c)^2} + c - \operatorname{sgn}(b(c))c\sqrt{\bar{h}^{-1}(b(c)^2)}y.$$

The definition of β_c and both results on $\delta = 0$ and $\delta = 1$ imply that the unique solution of the problem

$$\begin{aligned} \mathcal{A}(c,v) &= 0 && \text{in} \quad \Omega, \\ \mathcal{C}(c,v) &= b(c) && \text{on} \quad \Gamma_1, \\ v &= -\frac{\iota^2}{(1-c)^2} + c && \text{on} \quad \Gamma_0 \end{aligned}$$

is given by

$$\mathcal{T}(c)(x,y) = \big(1-(1-\delta)y\big)\Big(-\frac{\iota^2}{(1-c)^2}+c\Big)+\beta_c y. \tag{IV.13}$$

Now the assertion follows from Lemma IV.2, the facts that $\partial_1\mathcal{T}(c) = 0$ and $\partial_2\mathcal{T}(c) = \theta_c$, and the formulas (IV.11) and

$$\partial\mathcal{B}(f,v)[h,u] = \partial_f\mathcal{B}(f,v)h + \partial_v\mathcal{B}(f,v)u,$$

where $f \in \mathcal{V}$, $h \in h^{4+\alpha}$, and $u, v \in buc^{2+\alpha}(\Omega)$. □

Given $f \in \mathcal{V}$ and $h \in h^{4+\alpha}$, the function $\partial\mathcal{T}(f)h$ solves the problem

$$\begin{aligned}
\partial\mathcal{A}(f,\mathcal{T}(f))[h,w] &= 0 && \text{in } \Omega,\\
\partial\mathcal{C}(f,\mathcal{T}(f))[h,w] &= b'(f)h && \text{on } \Gamma_1,\\
w &= -\frac{\sigma h_{xx}}{(1+f_x^2)^{3/2}} + \frac{3\sigma f_x f_{xx} h_x}{(1+f_x^2)^{5/2}} + \Big(1-2\frac{\iota^2}{(1-f)^3}\Big)h && \text{on } \Gamma_0.
\end{aligned}$$

We compute

$$\begin{aligned}
&\partial\mathcal{C}(f,\mathcal{T}(f))[h,\partial\mathcal{T}(f)h]\\
&= (1-\delta)\gamma_1\partial\mathcal{T}(f)h - \delta\Bigg[\frac{f\gamma_1\partial_2\partial\mathcal{T}(f)h - h\partial_2\mathcal{T}(f)}{f^2\bar\mu\big((\gamma_1\partial_1\mathcal{T}(f))^2+\frac{1}{f^2}(\gamma_1\partial_2\mathcal{T}(f))^2\big)}\\
&\quad + 2\frac{\gamma_1\partial_2\mathcal{T}(f)}{f}\Big(\gamma_1\partial_1\mathcal{T}(f)\gamma_1\partial_1\partial\mathcal{T}(f)h - \frac{h}{f^3}(\gamma_1\partial_2\mathcal{T}(f))^2\\
&\quad + \frac{1}{f^2}\gamma_1\partial_2\mathcal{T}(f)\gamma_1\partial_2\partial\mathcal{T}(f)h\Big)\Big(\frac{1}{\bar\mu}\Big)'\big((\gamma_1\partial_1\mathcal{T}(f))^2+\tfrac{1}{f^2}(\gamma_1\partial_2\mathcal{T}(f))^2\big)\Bigg].
\end{aligned}$$

For $f = c$ we know from the proof of Lemma IV.9 that the solution $\mathcal{T}(c)$ of the elliptic boundary value problem consisting of the first three equations in (IV.4) is given by (IV.13). Hence, $\partial_1\mathcal{T}(c) = 0$ and $\partial_2\mathcal{T}(c) = \theta_c$ and an elementary calculation shows that

$$\partial\mathcal{C}(c,\mathcal{T}(c))[h,\partial\mathcal{T}(c)h] = (1-\delta)\gamma_1\partial\mathcal{T}(c)h - \delta\frac{\Gamma_{\bar\mu,c}}{c}\Big(\gamma_1\partial_2\partial\mathcal{T}(c) - \frac{\theta_c}{c}\Big)h. \tag{IV.14}$$

Moreover, $a_{ij} = a_{ij}(D_c\mathcal{T}(c)) = a_{ij}\big(0,-\frac{\theta_c}{c}\big)$ for $1 \le i,j \le 2$, see (IV.5). This implies that

$$a_{11} = \frac{1}{\bar\mu\big((\frac{\theta_c}{c})^2\big)}, \quad a_{12} = 0, \quad \text{and} \quad a_{22} = \Gamma_{\bar\mu,c}, \tag{IV.15}$$

and the coefficients of the elliptic operator $\mathcal{A}(c,\,\cdot\,)$ are given by

$$b_{11} = \frac{1}{\bar\mu\big((\frac{\theta_c}{c})^2\big)}, \quad b_{12} = 0, \quad b_{22} = \frac{\Gamma_{\bar\mu,c}}{c^2}, \quad \text{and} \quad b_2 = 0.$$

Thus, equation (IV.14), Lemma IV.6, and the fact that $\partial_{ij}\mathcal{T}(c) = 0$ for $1 \le i, j \le 2$ imply that $\partial\mathcal{T}(c)h$ solves the problem

$$\begin{aligned} c^2\partial_{11}w + K_{\bar\mu,c}\partial_{22}w &= -h_{xx}\pi c\theta_c && \text{in} \quad \Omega, \\ (1-\delta)w + \delta\partial_2 w &= \beta_c' h && \text{on} \quad \Gamma_1, \\ w &= -\sigma h_{xx} + \Big(1 - 2\frac{\iota^2}{(1-c)^3}\Big)h && \text{on} \quad \Gamma_0, \end{aligned} \tag{IV.16}$$

where we put $K_{\bar\mu,c} := \bar\mu\big((\frac{\theta_c}{c})^2\big)\Gamma_{\bar\mu,c}$ and $\beta_c' := \big(-\frac{c}{\Gamma_{\bar\mu,c}}\big)^{\delta} b'(c) + \delta\frac{\theta_c}{c}$. Recall that $\pi(x,y) = 1-y$ for $(x,y) \in \Omega$.

Lemma IV.10. *Suppose that* (IV.2) *and* (IV.3) *hold. Then there are positive constants $\tilde C$ and $\tilde c$ such that $\tilde c \le \Gamma_{\bar\mu,c} \le \tilde C$ and $\tilde c \le K_{\bar\mu,c} \le \tilde C$.*

Proof. This follows immediately from the proof of Lemma IV.3. □

To solve problem (IV.16) we introduce the Fourier expansions

$$h(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \quad \text{and} \quad w(x,y) = \sum_{k\in\mathbb{Z}} C_k(y)e^{ikx} \quad \text{for} \quad (x,y) \in \Omega.$$

Note that we have convergence in C^3 of $\sum_{k\in\mathbb{Z}} c_k e^{ikx}$ whereas w is expanded only formally for the time being. From (IV.16) we obtain the conditions

$$\begin{aligned} K_{\bar\mu,c}C_k''(y) - c^2k^2C_k(y) &= c\theta_c k^2(1-y)c_k \quad \text{for} \quad 0 < y < 1, \\ C_k^{(\delta)}(1) &= \beta_c' c_k, \\ C_k(0) &= \Big(\sigma k^2 + 1 - 2\frac{\iota^2}{(1-c)^3}\Big)c_k \end{aligned} \tag{IV.17}$$

on the Fourier coefficients of h and w. For $k = 0$ we have

$$\begin{aligned} C_0''(y) &= 0 \quad \text{for} \quad 0 < y < 1, \\ C_0^{(\delta)}(1) &= \beta_c' c_0, \\ C_0(0) &= \Big(1 - 2\frac{\iota^2}{(1-c)^3}\Big)c_0 \end{aligned} \tag{IV.18}$$

with the solution $C_0 = c_0 d_0$, where

$$d_0(y) = \big(1 - (1-\delta)y\big)\Big(1 - 2\frac{\iota^2}{(1-c)^3}\Big) + \beta_c' y, \quad 0 \le y \le 1. \tag{IV.19}$$

For $k \ne 0$ let us assume that the Fourier coefficients of w can be represented as $C_k(y) = c_k d_k(y)$ for $0 \le y \le 1$. Then standard ODE techniques yield

$$\begin{aligned} d_k(y) = &\Big(\frac{e^{-cRky}}{1-(-1)^\delta e^{-2cRk}} + \frac{(-1)^\delta e^{cRky}}{1-(-1)^\delta e^{2cRk}}\Big)(\sigma k^2 + G_c) \\ &+ \frac{(-1)^\delta e^{cRk}}{1-(-1)^\delta e^{2cRk}}\big(e^{-cRky} - e^{cRky}\big)\frac{\beta_c' - \delta\theta_c/c}{(cRk)^\delta} - \frac{\theta_c}{c}(1-y), \end{aligned} \tag{IV.20}$$

where we put $G_c := 1 - 2\frac{\iota^2}{(1-c)^3} + \frac{\theta_c}{c}$ and $R := R_{\bar{\mu},c} := 1/\sqrt{K_{\bar{\mu},c}}$. Note that $R \in \mathbb{R}$, cf. Lemma IV.10.

Since we obtain the C_k's only formally by differentiating w and investigating the resulting systems (IV.17) and (IV.18), it is left to check that $w(x,y) = \sum_{k\in\mathbb{Z}} C_k(y)\mathrm{e}^{\mathrm{i}kx}$ really solves system (IV.16). First we infer from $h \in h^{4+\alpha}$ that there is $L > 0$ such that $|c_k| \le L/k^4$ for all $k \in \mathbb{Z}$. Together with the uniform boundedness of the $d_k(y)$'s in y this yields the uniform convergence

$$\sum_{|k|<N} C_k(y)\mathrm{e}^{\mathrm{i}kx} \to w(x,y) \quad \text{for} \quad N \to \infty.$$

From this we infer that $w \in BUC(\Omega)$. Now choose a sequence $(h_n)_n \subset C^\infty$ with $h_n \to h$ in $C^{4+\alpha}$. Let $\bar{w}$ be the solution of (IV.16) and let $\bar{w}_n$ be the solution of (IV.16) if we replace h by h_n. Expanding $h_n(x) = \sum_{k\in\mathbb{Z}} c_{n,k}\mathrm{e}^{\mathrm{i}kx}$ we observe that for all $l \in \mathbb{N}$ there is $L_{n,l} > 0$ such that $|c_{n,k}| \le L_{n,l}/|k|^l$ for all $k \in \mathbb{Z}$ since the h_n's are smooth. Hence, the Fourier series

$$\bar{w}_n(x,y) = \sum_{k\in\mathbb{Z}} d_k(y)c_{n,k}\mathrm{e}^{\mathrm{i}kx}, \quad (x,y) \in \Omega,$$

are smooth and converge to $\bar{w}$ in $BUC^{2+\alpha}(\Omega)$. For every $0 \le y \le 1$ we estimate

$$\|w(\,\cdot\,,y) - \bar{w}_n(\,\cdot\,,y)\|^2_{L^2(\mathbb{S})} \le M^2\|h - h_n\|^2_{L^2(\mathbb{S})},$$

where $M := \sup_{k\in\mathbb{Z}} \sup_{0\le y\le 1} d_k(y)$. From this and the facts that $\|h_n - h\|_\infty \to 0$ and $\|\bar{w}_n - \bar{w}\|_{\infty,\Omega} \to 0$ we conclude that

$$w(\,\cdot\,,y) = \bar{w}(\,\cdot\,,y) \quad \text{in} \quad L^2(\mathbb{S}).$$

Finally, the continuity of w and $\bar{w}$ yields $w = \bar{w}$.

Theorem IV.11. *We have $\Phi \in C^\omega(\mathcal{V}, h^{1+\alpha})$ and the linearization $\partial\Phi(c)$ is given by*

$$\partial\Phi(c)\Big[\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}kx}\Big] = \sum_{k\in\mathbb{Z}} \lambda_k c_k \mathrm{e}^{\mathrm{i}kx}, \tag{IV.21}$$

where (for $k \ne 0$)

$$\lambda_k = -\Gamma_{\bar{\mu},c}\left(Rk(\sigma k^2 + G_c) - (Rk)^{1-\delta}\Big(\frac{\beta_c'}{c^\delta} - \delta\frac{\theta_c}{c^2}\Big)\frac{\coth(cRk)^\delta}{\cosh(cRk)}\right)\coth(cRk)^{(-1)^\delta}$$

and

$$\lambda_0 = \Gamma_{\bar{\mu},c}\frac{\beta_c' - (1-\delta)G_c - \delta\theta_c/c}{c}.$$

Proof. From formula (IV.12) we easily get

$$\partial\Phi(c)\Big[\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}kx}\Big] = \frac{\Gamma_{\bar{\mu},c}}{c}\sum_{k\in\mathbb{Z}}\Big(d_k'(0) - \frac{\theta_c}{c}\Big)c_k\mathrm{e}^{\mathrm{i}kx}.$$

Now the representation for $\partial\Phi(c)$ follows immediately from (IV.19) and (IV.20). Finally, the analyticity of $\mathcal{B}$ and $\mathcal{T}$ imply the analyticity of Φ. □

The form of the λ_k's, the definition of R, and Lemma IV.10 imply the following result.

Corollary IV.12. *It holds that* $\lambda_k = \lambda_{-k}$ *for all* $k \in \mathbb{Z}$. *Moreover,* $\lambda_k \to -\infty$ *for* $|k| \to \infty$.

IV.3 Proof of Theorem IV.1

In this section we essentially prove the following generation result with methods from Chapter III.

Theorem IV.13. *Fix* $c \in (0,1)$. *The linearization* $\partial\Phi(c)$ *generates a strongly continuous analytic semigroup in* $h^{1+\alpha}$, *i.e.,* $-\partial\Phi(c) \in \mathcal{H}(h^{4+\alpha}, h^{1+\alpha})$.

To prove this result we need some preparation. We make use of the following equivalent characterization for $-\partial\Phi(c) \in \mathcal{H}(h^{4+\alpha}, h^{1+\alpha})$, cf. Section III.2.1:

$$\lambda - \partial\Phi(c) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{1+\alpha}), \tag{IV.22}$$

$$|\lambda| \|R(\lambda, \partial\Phi(c))\|_{\mathcal{L}(h^{1+\alpha})} \leq \chi \tag{IV.23}$$

for the densely injected Banach spaces $h^{4+\alpha} \overset{d}{\hookrightarrow} h^{1+\alpha}$, some $\chi, \omega > 0$, and all $\mathrm{Re}\,\lambda \geq \omega$. Recall that $R(\lambda, \partial\Phi(c)) := (\lambda - \partial\Phi(c))^{-1}$ is the resolvent operator of $\partial\Phi(c)$.

Thanks to Corollary IV.12 we can choose $\omega > \sup_{k\in\mathbb{Z}} \mathrm{Re}\,\lambda_k$ such that $|\lambda - \lambda_k|$ is bounded away from 0 for all $\mathrm{Re}\,\lambda \geq \omega$ and we consider $\partial\Phi(c)$ as an operator between Sobolev spaces

$$H^r := H^r(\mathbb{S}) := \{f \in L^2(\mathbb{S}) : \sum_{k\in\mathbb{Z}} (1+k^2)^r |\hat{f}(k)|^2 < \infty\}, \quad r > 0,$$

with scalar product $\langle f, g\rangle = \sum_{k\in\mathbb{Z}} (1+k^2)^r \hat{f}(k)\overline{\hat{g}(k)}$, where $\hat{f}(k)$ denotes the k-th Fourier coefficient of f, and norm $\|\cdot\|_{H^r} := \sqrt{\langle\cdot,\cdot\rangle}$. Our first result is the following

Lemma IV.14. $\lambda - \partial\Phi(c) \in \mathcal{L}\mathrm{is}(H^{r+3}, H^r)$ *for some* $r > 0$ *and all* $\mathrm{Re}\,\lambda \geq \omega$.

Proof. It is $\lambda_k/k^3 = \mathcal{O}(1)$ for $|k| \to \infty$. In particular, there is an $M > 0$ such that $|\lambda_k| \leq M(1+k^2)^{3/2}$ for all $k \in \mathbb{Z}$. As a consequence, for $\mathrm{Re}\,\lambda \geq \omega$ and some function $[x \mapsto h(x) := \sum_{k\in\mathbb{Z}} \hat{h}(k)\mathrm{e}^{\mathrm{i}kx}] \in H^{r+3}$ we have

$$\|\partial\Phi(c)h\|_{H^r} \leq M\|h\|_{H^{r+3}}.$$

Thus, $\lambda - \partial\Phi(c)$ is well-defined and the fact that $\partial\Phi(c)$ is injective is a direct consequence of the choice of ω. It is surjective (and the inverse mapping $R(\lambda, \partial\Phi(c))$ is well-defined) due to

$$\|R(\lambda, \partial\Phi(c))f\|^2_{H^{r+3}} = \underbrace{\sum_{k\in\mathbb{Z}} (1+k^2)^r |\hat{f}(k)|^2}_{=\|f\|^2_{H^r}} \underbrace{\frac{(1+k^2)^3}{|\lambda - \lambda_k|^2}}_{=\mathcal{O}(1) \text{ for } |k|\to\infty} \qquad \text{for} \quad f \in H^r.$$

This completes the proof. ☐

We also need the following lemmas.

Lemma IV.15. $H^{m+s} \overset{d}{\hookrightarrow} h^{m+\alpha}$ *for all* $m \in \mathbb{N}$, $\alpha \in [0,1]$ *and* $s > 3/2$.

Proof. See [31, Proposition 3.1]. □

Lemma IV.16. *Assume that* $R(\lambda, \partial\Phi(c)) \in \mathcal{L}(C^{1+\alpha}, C^{4+\alpha})$ *for some* $\operatorname{Re}\lambda \geq \omega$. *Then* $\lambda - \partial\Phi(c) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{1+\alpha})$.

Proof. Let $f \in h^{1+\alpha}$. By Lemma IV.15 there is a sequence $(f_n)_n \subset H^r$ for some $r > 5/2$ with $f_n \to f$ in $C^{1+\alpha}$. Hence, there is $C > 0$ such that

$$\|R(\lambda, \partial\Phi(c))f_n - R(\lambda, \partial\Phi(c))f\|_{4+\alpha} \leq C\|f_n - f\|_{1+\alpha}$$

due to the linearity and the continuity of $R(\lambda, \partial\Phi(c))$. Consequently, applying again Lemma IV.15, we find $R(\lambda, \partial\Phi(c))f \in h^{4+\alpha}$ since $R(\lambda, \partial\Phi(c))f_n \in H^{r+3}$ for all $n \in \mathbb{N}$. This implies $R(\lambda, \partial\Phi(c)) \in \mathcal{L}(h^{1+\alpha}, h^{4+\alpha})$. Since we already know that $\lambda - \partial\Phi(c) \in \mathcal{L}(h^{4+\alpha}, h^{1+\alpha})$, the proof is complete. □

From [31, Theorem 3.4] (cf. also [5]) we need the following result, which gives conditions for a multiplication operator to act between Besov spaces $B^r_{\infty\infty}$ over the unit circle, see Theorem III.4 and the definition prior to that.

Theorem IV.17. *Let* $r, s > 0$ *be given and let* $(M_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ *be a sequence satisfying the following conditions:*

(i) $\sup_{k\in\mathbb{Z}} |k^{r-s}||M_k| < \infty$,

(ii) $\sup_{k\in\mathbb{Z}} |k^{r-s+1}||M_{k+1} - M_k| < \infty$,

(iii) $\sup_{k\in\mathbb{Z}} |k^{r-s+2}||M_{k+2} - 2M_{k+1} + M_k| < \infty$.

Then the mapping

$$\sum_{k\in\mathbb{Z}} \hat{h}(k)\mathrm{e}^{\mathrm{i}kx} \mapsto \sum_{k\in\mathbb{Z}} M_k\hat{h}(k)\mathrm{e}^{\mathrm{i}kx}$$

belongs to $\mathcal{L}(B^s_{\infty\infty}, B^r_{\infty\infty})$.

In the following lemma Theorem IV.17 is used to prove relation (IV.22).

Lemma IV.18. $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \omega\} \subset \rho(\partial\Phi(c))$.

Proof. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \omega$. By Lemma IV.16 it suffices to show that $R(\lambda, \partial\Phi(c)) \in \mathcal{L}(C^{1+\alpha}, C^{4+\alpha})$. By means of the identification of Besov spaces $B^r_{\infty\infty}$ with spaces of Hölder continuous functions C^r provided $r \notin \mathbb{N}$, cf. Theorem III.4, we show that $R(\lambda, \partial\Phi(c)) \in \mathcal{L}(B^{1+\alpha}_{\infty\infty}, B^{4+\alpha}_{\infty\infty})$ invoking the conditions from Theorem IV.17 with $r = 4 + \alpha$ and $s = 1 + \alpha$, where now $M_k := 1/(\lambda - \lambda_k)$. Condition (i) holds since $\lambda_k/k^3 = \mathcal{O}(1)$ for $|k| \to \infty$. Condition (ii) holds because of

$$\begin{aligned} k^4|M_{k+1} - M_k| &= \Big|\frac{k^4(\lambda_{k+1} - \lambda_k)}{(\lambda - \lambda_k)(\lambda - \lambda_{k+1})}\Big| \\ &= \Big|\frac{k^3}{\lambda - \lambda_k}\Big|\Big|\frac{k^3}{\lambda - \lambda_{k+1}}\Big|\Big|\frac{\lambda_{k+1} - \lambda_k}{k^2}\Big| = \mathcal{O}(1) \quad \text{for} \quad |k| \to \infty. \end{aligned}$$

Since

$$\begin{aligned}&|k^5||M_{k+2}-2M_{k+1}+M_k|\\=&\Big|\frac{k^5}{(\lambda-\lambda_{k+2})(\lambda-\lambda_{k+1})(\lambda-\lambda_k)}\Big||(\lambda-\lambda_{k+1})(\lambda-\lambda_k)\\&-2(\lambda-\lambda_{k+2})(\lambda-\lambda_k)+(\lambda-\lambda_{k+2})(\lambda-\lambda_{k+1})|\\\le&\Big|\frac{k^3}{\lambda-\lambda_{k+2}}\Big|\Big|\frac{k^3}{\lambda-\lambda_{k+1}}\Big|\Big|\frac{k^3}{\lambda-\lambda_k}\Big|\\&\times\Big(\Big|\frac{\lambda(\lambda_{k+2}-2\lambda_{k+1}+\lambda_k)}{k^4}\Big|\\&+\Big|\frac{\lambda_k(\lambda_{k+1}-\lambda_{k+2})+\lambda_{k+2}(\lambda_{k+1}-\lambda_k)}{k^4}\Big|\Big)=\mathcal{O}(1)\end{aligned}$$

for $|k|\to\infty$, condition (iii) holds, too. The proof is complete. □

Lemma IV.19. *Suppose* $R(\lambda,\partial\Phi(c))\in\mathcal{L}(C^{1+\alpha})$ *for some* $\operatorname{Re}\lambda\ge\omega$. *Then we have* $R(\lambda,\partial\Phi(c))\in\mathcal{L}(h^{1+\alpha})$.

Proof. Let $f\in h^{1+\alpha}$. Thanks to Lemma IV.15 there is a sequence $(f_n)_n\subset H^r$ for some $r>5/2$ such that $f_n\to f$ in $C^{1+\alpha}$. Similarly to the proof of Lemma IV.16 there is a $C>0$ such that

$$\|\underbrace{R(\lambda,\partial\Phi(c))f_n}_{\in H^{r+3}}-R(\lambda,\partial\Phi(c))f\|_{1+\alpha}\le C\|f_n-f\|_{1+\alpha}.$$

Hence, $R(\lambda,\partial\Phi(c))f\in\overline{H^{r+3}}^{\|\cdot\|_{1+\alpha}}=h^{1+\alpha}$, cf. Lemma IV.15, and we are done. □

Lemma IV.20. *There exist positive constants* ω,χ *such that*

$$|\lambda|\|R(\lambda,\partial\Phi(c))\|_{\mathcal{L}(h^{1+\alpha})}\le\chi\quad\textit{for all}\quad\operatorname{Re}\lambda\ge\omega.$$

Proof. It is sufficient to show that the conditions from Theorem IV.17 for $r=s=1+\alpha$ and all $\operatorname{Re}\lambda\ge\omega$ are fulfilled by $M_k^\lambda:=|\lambda|/(\lambda-\lambda_k)$. Obviously, condition (i) holds. Since

$$|k||M_{k+1}^\lambda-M_k^\lambda|=\underbrace{\Big|\frac{\lambda}{\lambda-\lambda_{k+1}}\Big|}_{\to0}\underbrace{\Big|\frac{k^3}{\lambda-\lambda_k}\Big|}_{=\mathcal{O}(1)}\underbrace{\frac{|\lambda_{k+1}-\lambda_k|}{k^2}}_{=\mathcal{O}(1)}\quad\text{for}\quad|k|\to\infty,$$

condition (ii) holds. From

$$\begin{aligned}&|k^2||M_{k+2}^\lambda-2M_{k+1}^\lambda+M_k^\lambda|\\\le&\underbrace{\Big|\frac{\lambda}{\lambda-\lambda_{k+2}}\Big|}_{\to0}\underbrace{\Big|\frac{k^3}{\lambda-\lambda_{k+1}}\Big|}_{=\mathcal{O}(1)}\underbrace{\Big|\frac{k^3}{\lambda-\lambda_k}\Big|}_{=\mathcal{O}(1)}\Big(\underbrace{\Big|\frac{\lambda(\lambda_{k+2}-2\lambda_{k+1}+\lambda_k)}{k^4}\Big|}_{\to0}\quad\text{for}\quad|k|\to\infty\\&+\underbrace{\Big|\frac{\lambda_k(\lambda_{k+1}-\lambda_{k+2})+\lambda_{k+2}(\lambda_{k+1}-\lambda_k)}{k^4}\Big|}_{=\mathcal{O}(1)}\Big)\end{aligned}$$

we finally deduce condition (iii). Now Lemma IV.19 completes the proof. □

Lemmas IV.18 and IV.20 and the results from Section III.2.1 now imply Theorem IV.13.

Proof of Theorem IV.1. Let $\beta \in (0, \alpha)$. From Theorem IV.13 we know that $-\partial\Phi(c) \in \mathcal{H}(h^{4+\alpha}, h^{1+\alpha})$. Along the lines of the proof of Theorem IV.13 we can prove that $-\partial\Phi(c) \in \mathcal{H}(h^{4+\gamma}, h^{1+\gamma})$ for all $\gamma \in [\beta, \alpha]$. From Theorem III.9 we know that $\mathcal{H}(h^{4+\gamma}, h^{1+\gamma})$ is open in $\mathcal{L}(h^{4+\gamma}, h^{1+\gamma})$. Hence, there is an open neighborhood $\mathcal{O}_\beta$ of c in $h^{4+\beta}$ such that $-\partial\Phi(f) \in \mathcal{H}(h^{4+\beta}, h^{1+\beta})$ for all $f \in \mathcal{O}_\beta$. From this and the fact that $(h^{1+\beta}, h^{4+\beta})_{(\alpha-\beta)/3} = h^{1+\alpha}$, see Theorem III.1, we infer that $-\partial\Phi(f) \in \mathcal{M}_1(h^{4+\alpha}, h^{1+\alpha})$ for all $f \in \mathcal{O} := h^{1+\alpha} \cap \mathcal{O}_\beta$, cf. Proposition III.11. Now the statement follows from Theorem III.12. □

IV.4 Stability Analysis

In this section we investigate the stability of stationary solutions of system (IV.1). The pair (f, u) is a stationary solution of (IV.1) if and only if it fulfills the set of equations

$$\begin{aligned} \mathcal{Q}u &= 0 && \text{in } \Omega(f), \\ (1-\delta)u + \frac{\delta}{\bar{\mu}(|\nabla u|^2)}\partial_y u &= b(f) && \text{on } \Gamma_0, \\ u &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on } \Gamma(f), \\ \partial_\nu u &= 0 && \text{on } \Gamma(f), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \mathcal{A}(f,v) &= 0 && \text{in } \Omega, \\ \mathcal{C}(f,v) &= b(f) && \text{on } \Gamma_1, \\ v &= -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f && \text{on } \Gamma_0, \\ \mathcal{B}(f,v) &= 0 && \text{on } \Gamma_0. \end{aligned} \tag{IV.24}$$

Lemma IV.21. *Given $c \in (0,1)$, there is a unique function $v \in buc^{2+\alpha}(\Omega)$ such that the pair (c, v) is a stationary solution of system* (IV.24).

Proof. For $f = c$ system (IV.24) turns into

$$\begin{aligned} \mathcal{A}(c,v) &= 0 && \text{in } \Omega, \\ \mathcal{C}(c,v) &= b(c) && \text{on } \Gamma_1, \\ v &= -\frac{\iota^2}{(1-c)^2} + c && \text{on } \Gamma_0, \\ \mathcal{B}(c,v) &= 0 && \text{on } \Gamma_0. \end{aligned} \tag{IV.25}$$

In the proof of Lemma IV.9 we derive a representation for the unique solution $\mathcal{T}(c) \in buc^{2+\alpha}(\Omega)$ of the system consisting of the first three equations of (IV.25), cf. (IV.13). This solution complies to the fourth equation in system (IV.25) if and only if $\gamma_0 \partial_2 \mathcal{T}(c) = 0$, which is equivalent to $\beta_c = (1-\delta)(-\iota^2/(1-c)^2 + c)$. From the definition of β_c, see again Lemma IV.9, we see that $-\iota^2/(1-c)^2 + c = \beta_c = b(c)$ if $\delta = 0$ and $0 = \beta_c = -\operatorname{sgn}(b(c))c\sqrt{\bar{h}^{-1}(b(c)^2)}$ if $\delta = 1$. Hence, in both cases we obtain from (IV.13) that the function v with $v(x,y) = -\iota^2/(1-c)^2 + c$, $(x,y) \in \Omega$, is the unique solution of (IV.25) being a stationary solution of (IV.24). $\square$

Given $c \in (0,1)$, it is crucial to investigate the spectrum $\sigma(\partial\Phi(c))$ of the linearization about $f = c$ of the evolution operator Φ from (IV.10).

Lemma IV.22. *$\sigma(\partial\Phi(c)) = \{\lambda_k : k \in \mathbb{Z}\}$, where λ_k, $k \in \mathbb{Z}$, is taken from Theorem IV.11.*

Proof. First observe that the eigenvalues of the linear operator $\partial\Phi(c)$ are exactly the λ_k's from Theorem IV.11 and that $(\lambda_k)_k$ has no finite accumulation point. Furthermore, the resolvent set of $\partial\Phi(c)$ is not empty, cf. Lemma IV.18. Hence, for some $\lambda \in \rho(\partial\Phi(c))$ we can decompose $R(\lambda, -\partial\Phi(c)) \in \mathcal{L}(h^{1+\alpha})$ such that $R(\lambda, -\partial\Phi(c)) = i \circ \tilde{R}(\lambda, \partial\Phi(c))$, where $i : h^{4+\alpha} \to h^{1+\alpha}$ is a compact imbedding and $\tilde{R}(\lambda, -\partial\Phi(c)) \in \mathcal{L}(h^{1+\alpha}, h^{4+\alpha})$. Thus, $R(\lambda, -\partial\Phi(c))$ is compact and we can apply Theorem 6.29 from [48, Chapter III], which implies that the spectrum of an operator with compact resolvent consists only of isolated eigenvalues with finite multiplicities, i.e., $\sigma(\partial\Phi(c)) \subset \{\lambda_k : k \in \mathbb{Z}\}$. Because of $\{\lambda_k : k \in \mathbb{Z}\} \subset \sigma(\partial\Phi(c))$ equality holds. $\square$

Let $\delta = 0$. Note that $\lambda_0 < 0$ if and only if $\beta_c' < G_c$. Now let $k \neq 1$. We have

$$\lambda_k = -\Gamma_{\bar{\mu},c} R\Big(\sigma k^2 + G_c - \beta_c' \frac{1}{\cosh(cRk)}\Big) \coth(cRk)k.$$

Since $\Gamma_{\bar{\mu},c} R \coth(cRk)k$ is positive for all $k \in \mathbb{Z} \setminus \{0\}$, we have to require $\sigma k^2 + G_c - \beta_c'/\cosh(cRk) > 0$ for all $|k| \geq 1$. Thanks to the symmetry of the λ_k's (see Corollary IV.12) we only consider $k \geq 1$. If $\beta_c' \geq 0$, $\lambda_k < 0$ for all $k \geq 1$ is equivalent to $\cosh(cR)(\sigma + G_c) > \beta_c'$, which is fulfilled if $\beta_c' < G_c$. For $\beta_c' < 0$ we have to proceed differently. We have to ensure that $\mathrm{I} := \mathrm{I}(\beta_c') := \inf_{k\geq 1}\big(\sigma k^2 - \beta_c'/\cosh(cRk)\big) > -G_c$. However, the value of I cannot be computed explicitly for all b. To sum up, $\lambda_k < 0$ for all $k \in \mathbb{Z}$ if and only if $\beta_c' < G_c$ and additionally $G_c > -\mathrm{I}$ if $\beta_c' < 0$, i.e.,

$$\begin{aligned} &0 \leq \beta_c' < 1 - \frac{2\iota^2}{(1-c)^3} \quad \text{or} \\ &0 > \beta_c', \quad \max\{\beta_c', -\mathrm{I}\} < 1 - \frac{2\iota^2}{(1-c)^3}. \end{aligned} \tag{IV.26}$$

Note that $\theta_c = \partial_2\mathcal{T}(c) = 0$ for a steady-state solution $\mathcal{T}(c)$, see the proof of Lemma IV.21. On the contrary there is a $k_0 \in \mathbb{Z}$ such that $\lambda_{k_0} > 0$ if this condition is violated, i.e.,

$$\begin{aligned} &0 \leq \beta_c', \quad \beta_c' > 1 - \frac{2\iota^2}{(1-c)^3} \quad \text{or} \\ &0 > \beta_c', \quad \max\{\beta_c', -\mathrm{I}\} > 1 - \frac{2\iota^2}{(1-c)^3}. \end{aligned} \tag{IV.27}$$

Let $\delta = 1$. In this case $\lambda_0 < 0$ if and only if $\beta_c' < 0$. Again, Corollary IV.12 implies that $\lambda_k < 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ if and only if $\lambda_k < 0$ for all $k \geq 1$, which is equivalent to $cRk(\sigma k^2 + G_c)\sinh(cRk) > \beta_c'$ for all $k \geq 1$. If $\beta_c' \leq 0$ this holds independently of the choice of b if and only if $\sigma k^2 + G_c > 0$ for all $k \geq 1$, which is equivalent to $\sigma + G_c > 0$. Thus, $\lambda_k < 0$ for all $k \in \mathbb{Z}$ and all b with $\beta_c' < 0$ if and only if

$$\sigma + 1 - \frac{2\iota^2}{(1-c)^3} > 0. \tag{IV.28}$$

However, if $\beta_c' > 0$ it holds that $\lambda_0 > 0$. Moreover, if

$$\sigma + 1 - \frac{2\iota^2}{(1-c)^3} < 0, \tag{IV.29}$$

there is a b with $\beta_c' < 0$ and $cR(\sigma + G_c)\sinh(cR) < \beta_c'$, i.e., $\lambda_1 > 0$.

Theorem 9.1.2 (principle of linearized stability) and Theorem 9.1.3 from [53] and Lemmas IV.21 and IV.22 now imply the following main result of this section.

Theorem IV.23. *Fix $c \in (0,1)$ and let $\omega_0 := -\sup \sigma(\partial\Phi(c))$.*

(i) Let $\delta = 0$. If (IV.26) *holds then the flat solution $f = c$ of* (IV.10) *is exponentially stable. More precisely, given $\omega \in (0, \omega_0)$, there exist positive constants M and ε such that for all $f_0 \in h^{4+\alpha}$ with $\|f_0 - c\|_{4+\alpha} \leq \varepsilon$ the solution of* (IV.10) *corresponding to f_0 exists in the large and*

$$\|f(t) - c\|_{4+\alpha} + \|f'(t)\|_{1+\alpha} \leq M e^{-\omega t}\|f_0 - c\|_{4+\alpha} \quad \text{for all} \quad t \geq 0.$$

If (IV.27) *holds then the flat solution is unstable.*

(ii) Let $\delta = 1$. If $\beta_c' < 0$ and if (IV.28) *holds then the flat solution $f = c$ of* (IV.10) *is exponentially stable exhibiting a decay estimate as in (i). If $\beta_c' > 0$ then the flat solution is unstable. If* (IV.29) *holds then there exists b with $\beta_c' < 0$ such that the flat solution is unstable.*

Let $\delta = 0$. Given $c \in (0,1)$, notice that if $\beta_c' \geq 0$ there is only the stability restriction

$$\beta_c' < 1 - \frac{2\iota^2}{(1-c)^3}.$$

A necessary condition for this inequality to hold is the sharp upper bound 1 for β_c'. Moreover, the current's intensity ι and the height of the flat surface c have to be related to each other in a special way: The closer the free surface is to the wire carrying the current (at $y = 1$) the less intensity is allowed for the current, i.e., there is a critical value

$$\iota_*^0 := \iota_*^0(c) := \sqrt{\frac{(1-c)^3}{2}(1 - \beta_c')}$$

for the current's intensity such that the flat solution $f = c$ is stable only for $\iota < \iota_*^0$.

Also in case $\delta = 1$ such a bound for the current's intensity can be derived. Define

$$\iota_*^1 := \iota_*^1(c) := \sqrt{\frac{(1-c)^3}{2}(\sigma+1)}.$$

Condition (IV.28) is fulfilled if and only if $\iota < \iota_*^1$. Note that ι_*^1 is exactly the stability threshold from [27, Theorem 2.2] if we could choose $c = 0$, which is admissible in [27] but is unphysical in our setting. Observe additionally that in [27] the gravitation and the fluid's density are not normalized to 1.

Note that in both cases $\delta \in \{0,1\}$ we can make the critical threshold ι_*^δ for the current's intensity arbitrarily large (at least theoretically) by choosing β_c' or σ appropriately.

Suppose that $\delta = 1$ and $\beta_c' = 0$ and that (IV.28) holds. Then $\lambda_0 = 0$ and $\lambda_k < 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. In this critical case of stability, where $\sigma\big(\partial\Phi(c)\big) \cap \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda > 0\} = \emptyset$ but $\sigma\big(\partial\Phi(c)\big) \cap i\mathbb{R} \neq \emptyset$, the application of the principle of linearized stability fails. However, we would like to include this case in our stability analysis. In the following we construct a *center manifold* $\mathcal{M}^c$ for (IV.10) rendering this possible, cf. [53, Section 9.2.1].

Let P_0 be the projection of $h^{1+\alpha}$ to the spectral set $\{0\}$ in $h^{4+\alpha}$, i.e., $P_0(h^{1+\alpha}) = \{f \in h^{4+\alpha} \colon f \text{ is constant}\}$. Note that $P_0(h^{1+\alpha})$ is one-dimensional and that $\{1\}$ is a basis.

Given $c \in (0,1)$, the Cauchy problem (IV.10) can be written as

$$\partial_t f = \partial\Psi(0)f + G(f), \quad f(0) = f_0,$$

where $\Psi(f) := \Phi(c+f)$ with $\partial\Psi(0) \in \mathcal{H}(h^{4+\alpha}, h^{1+\alpha})$, cf. Theorem IV.13, and $G \in C^\omega(\mathcal{V}, h^{1+\alpha})$ fulfills $G(0) = \partial G(0) = 0$, see Theorem IV.11. This, in turn, can be equivalently decomposed as

$$\begin{aligned} \partial_t g &= \partial\Psi(0)g + P_0 G(g+h), \quad g(0) = g_0, \\ \partial_t h &= \partial\Psi(0)h + (\mathrm{id} - P_0)G(g+h), \quad h(0) = h_0, \end{aligned}$$

where $g := P_0 f$ and $h := (\mathrm{id} - P_0)f$ and $g_0 := P_0 f_0$ and $h_0 := (\mathrm{id} - P_0)f_0$. Note that for notational reasons we do not distinguish between $\partial\Psi(0) \in \mathcal{L}(h^{4+\alpha}, h^{1+\alpha})$ and its parts in $P_0(h^{1+\alpha})$ and $(\mathrm{id} - P_0)(h^{4+\alpha})$, respectively. However, observe that they also generate strongly continuous analytic semigroups with their respective domain.

We may choose a cut-off function $\rho \in C^\infty(P_0(h^{1+\alpha}), \mathbb{R})$ with $0 \le \rho \le 1$ and the property that $\rho(x) = 1$ if $\|x\|_{1+\alpha} \le 1/2$ and that $\rho(x) = 0$ if $\|x\|_{1+\alpha} \ge 1$. Observe that the stability behavior of problem (IV.10) is the same as the one of the system

$$\begin{aligned} \partial_t g &= \partial\Psi(0)g + \phi_r(g,h), \quad g(0) = g_0 \in P_0(h^{1+\alpha}), \\ \partial_t h &= \partial\Psi(0)h + \psi_r(g,h), \quad h(0) = h_0 \in (\mathrm{id} - P_0)(h^{1+\alpha}), \end{aligned} \tag{IV.30}$$

where $\phi_r(g,h) := P_0 G(\rho(g/r)g + h)$ and $\psi_r(g,h) := (\mathrm{id} - P_0)G(\rho(g/r)g + h)$ and $r > 0$. Even more, the two problems are equivalent if $\|g\|_{1+\alpha} \le r/2$.

Theorem IV.24. *There is $r_0 > 0$ such that for all $r \in (0, r_0]$ there exists a positive constant C such that if $\|g_0\|_{1+\alpha} + \|h_0\|_{4+\alpha} \le C$ system* (IV.30) *has a unique global solution $g \in C^1([0,\infty), h^{1+\alpha})$, $h \in C((0,\infty), h^{4+\alpha}) \cap C([0,\infty), h^{1+\alpha})$.*

Proof. This follows immediately from [53, Proposition 9.2.1]. □

Given $z_0 \in P_0(h^{1+\alpha})$ and $r > 0$, consider the problem

$$\partial_t z = \partial\Psi(0)z + \phi_r(z, \gamma(z)), \quad z(0) = z_0, \tag{IV.31}$$

in $P_0(h^{1+\alpha})$, where $\gamma\colon P_0(h^{1+\alpha}) \to (\mathrm{id} - P_0)(h^{4+\alpha})$ is a Lipschitz continuous function. From Theorem 9.2.2 in [53] and the regularity of G we infer that there is $r_0 > 0$ such that for all $r \in (0, r_0]$ such a function γ exists being of class C^ω and the graph of γ is invariant under the flow of system (IV.30).

Theorem IV.25. *Let $\delta = 1$ and suppose that $\beta_c' = 0$ and* (IV.28) *hold. Furthermore, put $\omega_0 := -\sup\{\lambda_k\colon k \neq 0\}$, i.e., ω_0 is the smallest modulus of the nonzero eigenvalues of $\partial\Phi(c)$. Then problem* (IV.10) *has a uniquely determined center manifold $\mathcal{M}^c$ consisting of all constant functions near* 0*. Moreover, for every $\omega \in (0, \omega_0)$ there are positive constants M, r_0, and ε such that for all $c + f_0 \in h^{4+\alpha}$ with $\|f_0\|_{4+\alpha} < \varepsilon$ the solution of* (IV.10) *exists in the large and there is a unique flat stationary solution of* (IV.10) *denoted by $c + z_0$ such that*

$$\|P_0 f - z_0\|_{1+\alpha} + \|(\mathrm{id} - P_0)f - \gamma(z_0)\|_{4+\alpha} \leq M\mathrm{e}^{-\omega t}\|(\mathrm{id} - P_0)f_0 + \gamma(P_0 f_0)\|_{1+\alpha},$$

where $t \geq 0$ and $\gamma \in C^\omega(P_0(h^{1+\alpha}), (\mathrm{id} - P_0)(h^{4+\alpha}))$ whose graph is invariant under the flow of (IV.30) *for $r \in (0, r_0]$.*

Proof. Fix $r_0 > 0$ and pick $r \in (0, r_0]$. Theorem 9.2.2 from [53] implies the existence of a function $\gamma \in C^\omega(P_0(h^{1+\alpha}), (\mathrm{id} - P_0)(h^{4+\alpha}))$ such that $g = P_0 g + \gamma(P_0 g)$ if $\|P_0 g\|_{1+\alpha} \leq r/2$ and $\|(\mathrm{id} - P_0)g\|_{4+\alpha} \leq r$. Theorem IV.24 yields that g is a unique global solution of (IV.30).

Choose a constant function f with $\|f\|_\infty \leq r/2$. Then f is a unique global solution of (IV.30) and $P_0 f = \varepsilon_f$ for some $\varepsilon_f \in \mathbb{R}$. The map $i\colon \mathbb{R} \to h^{1+\alpha}$ with $i(\varepsilon_f) = \varepsilon_f$ trivially parametrizes the function $P_0 f$. Moreover $\gamma(P_0 f) = 0$. Obviously, $i(-\varepsilon, \varepsilon)$ is an open neighborhood of 0 in $h^{4+\alpha}$ and hence, $\mathcal{O} := P_0 i(-\varepsilon, \varepsilon)$ is an open neighborhood of 0 in $P_0(h^{1+\alpha})$. Put $\mathcal{M}^c := \{(g, \gamma(g))\colon g \in \mathcal{O}\}$. Note that $\mathcal{M}^c$ is invariant under the flow of (IV.30), see again [53, Theorem 9.2.2], and that

$$\begin{aligned}\mathcal{M}^c &= \{(P_0 i(x), \gamma(P_0 i(x)))\colon x \in (-\varepsilon, \varepsilon)\} \\ &= \{(x, \gamma(x))\colon x \in (-\varepsilon, \varepsilon)\} = \{f \in h^{1+\alpha}\colon f \text{ is constant and } \|f\|_\infty < \varepsilon\},\end{aligned}$$

which means that $\mathcal{M}^c$ consists of all constant functions in a neighborhood of 0 in $P_0(h^{1+\alpha})$.

The decay estimate follows from the fact that constant functions are stationary solutions of (IV.10), see Lemma IV.21, and Proposition 9.2.4 in [53] since $\mathcal{M}^c$ consists of constant functions only. □

Chapter V

A Numerical Scheme for Stabilized Newtonian Hele-Shaw Flows

The coupling of two functions in the equations governing free surface flows, e.g., the flow of a fluid in a Hele-Shaw cell, involves severe difficulties in the development of suitable numerical schemes while the corresponding decoupled problems can be very easy to treat numerically. In periodic Hele-Shaw flows these single problems are the solution of an elliptic equation in a periodic domain and an initial value problem with periodic initial data. For both problems a wide range of different solution strategies is available. However, for the coupled problem this is not the case.

V.1 The Numerical Scheme

Let $[0,1] \times [0,1]$ be a Hele-Shaw cell with bottom $\Gamma_0 := [0,1] \times \{0\}$. The free surface $\Gamma(f)$ of a fluid in the cell is given by the graph of a continuous single-valued periodic function f from $[0,1]$ to $[0,1]$ and the fluid covers the domain $\Omega(f) := \{(x,y) \in [0,1] \times [0,1] \colon 0 < y < f(x)\}$. Consider the set of equations

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega(f), \\ (1-\delta)u + \delta\partial_y u &= b && \text{on } \Gamma_0, \\ u &= H(f;\lambda) && \text{on } \Gamma(f), \\ \partial_t f &= -f_x u_x + u_y && \text{on } \Gamma(f), \\ f(0) &= f_0, && \text{on } [0,1] \end{aligned} \tag{V.1}$$

where $b \in \mathbb{R}$, H is a function of f (and probably also of the derivatives of f) and a parameter (or family of parameters) λ, and f_0 is an initial condition, and observe its similarity to the one from the preceding chapter.

The road map to solving problem (V.1) is as follows.

(i) For a given function f (in the beginning $f = f_0$) generate a mesh to discretize the domain $\Omega(f)$.

(ii) Solve the elliptic boundary value problem consisting of the first three equations in (V.1) for the potential u with the Galerkin finite element method.

(iii) Retrieve the gradient of the finite element solution for u at the free surface.

(iv) Evolve the free surface according to the fourth equation in (V.1) and go back to step (i).

Though this procedure might look simple, there are several difficulties to overcome. While the initial free surface f_0 might be known as a function, i.e., $f_0(x)$ is known for every $0 \leq x \leq 1$, this is not the case in all the following time steps. As a consequence, the free surface has to be interpolated in a reasonable manner. A second problem is the discretization process for the domain $\Omega(f)$: Since the domain is not known a priori at some future time step, a flexible meshing procedure has to be conceived. Moreover, the values of the gradient on the boundary have to be determined carefully.

Several approaches have been proposed to solve free boundary problems of various types, see, e.g, Chapters 4, 5, and 8 in [18] for a broad overview. One first approach to a problem similar to (V.1) was given by A. Amar in [3], where a finite difference approach is made. R. Bonnerot and P. Jamet describe a numerical scheme for the one-phase Stefan problem, see [13]. Their approach resembles ours in a way that the finite element method is used and that it is the aim to make coincide discretization points of the free surface with element nodes. Three different finite element approaches were proposed by H. Rasmussen and D. Salhani, see [62], with restrictions on the geometry of the free surface such as symmetry requirements and prescribed slopes. They use special ansatz functions for the finite element solution that comply with the geometrical restrictions. N. Asaithambi, cf. [7], solves the problem of discretizing the fluid's domain by choosing a mesh that can be transformed to a mesh consisting of identical rectangles after the transformation of the fluid's domain to a rectangular domain. The problem is solved in the new coordinates with a finite difference scheme. Though this procedure yields accurate results, due to the coordinate transformation rather involved techniques have to be used for solving a (simple) Laplace equation. F. Mashayek and N. Ashgriz present a discretization method of the fluid's domain that is similar to ours, cf. [54]. For employing the finite element method they discretize the domain in a mesh of squares and they decompose the squares intersected by the free surface into smaller elements. However, their approach to moving the free surface in time relies on the so called volume-of-fluid method, which is not applied here since it does not track the free surface.

V.1.1 The Potential Problem

For a given suitable function f the potential problem consisting of the first three equations in (V.1) can be solved with the finite element method in the domain $\Omega := \Omega(f)$, whose upper boundary $\Gamma := \Gamma(f)$ is the free surface. Given $k \in \mathbb{N}$, let $H^k(\Omega)$ denote the standard Sobolev spaces on Ω and let $H_0^k(\Omega)$ be the subspace of functions in $H^k(\Omega)$ that vanish on $\partial\Omega$. Furthermore, define the spaces

$$V_0 = \{v \in H^1(\Omega) : v|\Gamma = H(f;\lambda), \quad v|\Gamma_0 = b\},$$
$$V_1 = \{v \in H^1(\Omega) : v|\Gamma = H(f;\lambda)\}.$$

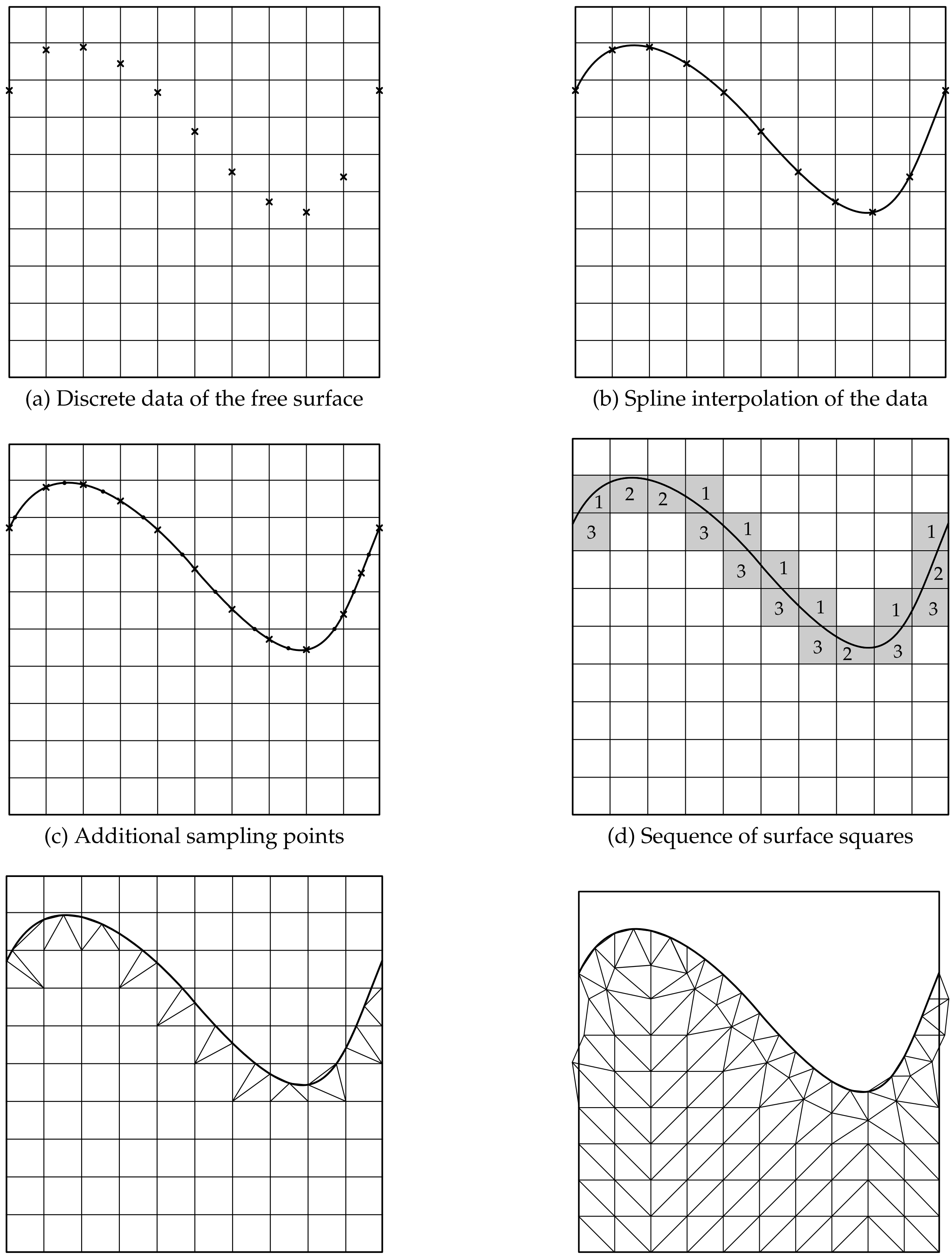
Proposition V.1. *Choose* $\delta \in \{0,1\}$. *A function* $u \in V_\delta$ *is a solution of the problem consisting of the first three equations in* (V.1) *if and only if it satisfies*

$$\int_\Omega \nabla u \nabla v \, d(x,y) - \delta b \int_{\Gamma_0} v \, dx = 0 \quad \textit{for all} \quad v \in H_0^1(\Omega).$$

In the subspace of V_δ consisting of continuous and piecewise linear functions an approximate solution to this problem can be constructed. The first step in this construction is the generation of a suitable decomposition of the domain Ω.

As a starting point for the discretization of Ω the Hele-Shaw cell $[0,1] \times [0,1]$ is subdivided into a mesh of $n \in \mathbb{N}$ squares of length $h := 1/n$. For a free surface function f of (almost) any shape there are uniquely determined pairwise different and adjacent squares containing the free surface Γ, see Figure V.1d. The squares below these 'surface squares' are naturally part of the discretization of Ω and they are later split in two triangles each. To deal with the surface squares observe that there are three types of them distinguished by the rough shape of the area below the free surface: Surface squares of type 1 have a surface segment connecting either the left and the bottom boundary or the bottom and the right boundary. Alternatively, the surface can cut the square in two 'halfs' by connecting either the left and the right boundary or the bottom and the top boundary (type 2). The third type has a surface segment from the left to the top boundary or from the top to the right boundary. The area of the surface squares below the free surface, which belongs to the domain Ω, is decomposed into triangles due to the type of the respective square. In order to do this, however, a proper discretization of the free surface is needed.

In general, the free surface is only known at certain (discrete) points, see Figure V.1a. These are the points $x_j = jh$ for $j = 0, \ldots, n-1$ and x_n is identified with x_0. Together with the values $f(x_j)$ for $j = 0, \ldots, n-1$ the free surface is determined by the pairs $(x_j, f(x_j))$ for $j = 0, \ldots, n-1$. Because of the intended triangulation of the surface squares mentioned above the free surface has to be known at additional points. Thus, an interpolation of the known values is needed. Here one has a choice. A reasonable (and well working) choice is a periodic cubic spline s_3 such that $s_3(x_j) = f(x_j)$ for $j = 0, \ldots, n-1$, see Figure V.1b. Now the additional points can be obtained by inserting the pairs $(x_{j,k}, s_3(x_{j,k}))$ for $j = 0, \ldots, n-1$ and $k = 1, \ldots, N_j$, where N_j is the number of additional sampling points of the free surface between x_j and x_{j+1}, see Figure V.1c. The N_j's are determined by the number of horizontals in the rectangular grid cut by the free surface between x_j and x_{j+1}. If this is zero, $N_j = 1$ and $x_{j,1} = x_j + h/2$. If only one horizontal in the grid is cut, then again $N_j = 1$ but $x_{j,1}$ is chosen such that $s_3(x_{j,1}) = \lceil f(x_j) \rceil_h$ if $f(x_{j+1}) > f(x_j)$, and $s_3(x_{j,1}) = \lfloor f(x_j) \rfloor_h$ if $f(x_{j+1}) < f(x_j)$, where $\lfloor \cdot \rfloor_h$ means rounding to the largest lower integer multiple of h and $\lceil \cdot \rceil_h$ rounding to the lowest larger integer multiple of h. If more than one horizontal in the grid is cut by the free surface between x_j and x_{j+1}, every intersection with such a horizontal is taken as a sampling point in a similar manner. Furthermore, the intersections of the free surface with the horizontals precisely between the horizontals of the grid between $\lceil f(x_j) \rceil_h$ and $\lfloor f(x_{j+1}) \rfloor_h$ are taken as sampling points. These points are of special importance because they are moved according



(a) Discrete data of the free surface

(b) Spline interpolation of the data

(c) Additional sampling points

(d) Sequence of surface squares

(e) Triangulation of surface squares

(f) Final triangulation

Figure V.1: Triangulation of Ω

to the evolution equation together with the given data points $(x_j, f(x_j))$. In this way every other sampling point is eventually moved. These points are denoted by $p_j := p_j(f)$ for $j = 1, \dots, \big(n + \sum_{k=0}^{n-1} N_k\big)/2$. Together with the other points on the free surface they form a polygonal line that represents the free surface.

The only problem with the discretization process for the free surface occurs when the interpolation function s_3 has a local minimum between values x_j and x_{j+1} such that $s_3(x_{j,1})$ for $x_{j,1} = x_j + h/2$ lies below $\lfloor f(x_j) \rfloor_h$ in type 2 surface squares. In this case the triangulation process breaks down and another value has to be chosen for $x_{j,1}$ such that $s_3(x_{j,1}) > \lfloor f(x_j) \rfloor_h$. A natural choice is the intersection of the interpolating spline with a diagonal of the surface square.

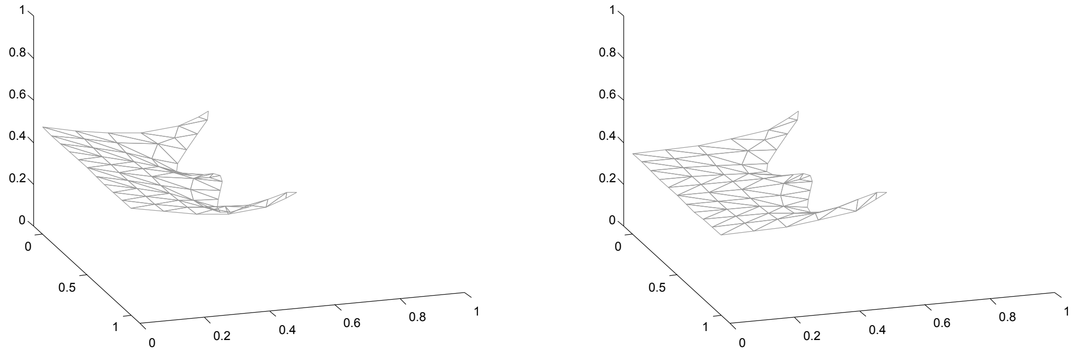
The sampling points of the free surface serve as element nodes for the triangulation of the area below the free surface in the surface squares, see Figure V.1e. In type 1 squares the area below the free surface is triangulated just by one triangle. In squares of the other types three triangles are used to subdivide the area below the free surface. If the type is 2, one side of the triangle in mid-position coincides with one side of the adjacent inner square from the rectangular grid, whereas the other two triangles each have one side in common with the polygonal line representing the free surface. If the type is 3, there is only the triangle in mid-position having one side in common with this polygonal line. The others have two sides in common with the rectangular grid.

All squares below the surface squares—being part of the domain—are cut in two triangles each. The orientation of this 'dissection' is determined by the rough shape of the free surface above. If it increases the squares below are cut from the bottom right to the upper left corner and if it decreases this is done from the bottom left to the upper right corner.

Note that the triangulation obtained so far is of low quality. The occurrence of tiny or highly distorted triangles is inevitable. To correct these undesirable effects, vertices of the surface squares and auxiliary nodes on the free surface different from the p_j's are moved, see Figure V.1f. In type 1 squares the inner node of the only triangle is moved away from the free surface whereas the inner node of the triangle in mid-position in type 3 squares is moved toward the free surface. In type 2 squares the nodes that do not lie on the free surface are moved away from it. Finally, if the distance of the surface node between p_j and p_{j+1} to one of these points is too low it is moved away from it. In this way the following quality of the triangulation can be achieved.

Proposition V.2. *Let $n \in \mathbb{N}$ and put $h := 1/n$. Let f be a periodic single-valued function from $[0,1]$ to $[0,1]$ with no oscillations of the size of h. Let $\mathcal{T}_h$ be a triangulation of the domain $\Omega = \Omega(f)$ as described above. Then there is $\rho > 0$ such that for every triangle $T \in \mathcal{T}_h$ there exist two concentric circular discs $D_1, D_2 \subset \mathbb{R}^2$ having the diameter ρ_1 and ρ_2, respectively, such that $D_1 \subset T \subset D_2$ and $\rho_2/\rho_1 \le \rho$. Moreover, each triangle in $\mathcal{T}_h$ has at most one side in common with the polygonal line representing the free surface.*

The interpolation procedure for the free surface and the discretization method for the fluid's domain described above are highly flexible in a way that the shape of the free surface is (nearly) arbitrary; the only restriction concerns too oscillatory



(a) Essential boundary condition $u =$ const. on Γ_0

(b) Natural boundary condition $\partial_y u = 0$ on Γ_0

Figure V.2: Finite element solutions of (V.3)

surfaces yielding incorrect triangulations. Moreover, one always keeps control over the surface nodes. These two aspects are of utmost importance for a proper implementation of the solution algorithm for the whole free boundary problem.

We are about to use the triangulation $\mathcal{T}_h$ described above to solve the potential problem on Ω. In doing so we obviously commit a variational crime since the union of all triangles from $\mathcal{T}_h$ does not coincide with Ω. We have to take this into account when estimating the error of the finite element solution.

In the following we delineate the construction of an approximate solution of the potential problem on Ω. Due to Proposition V.1 we seek a function $u \in V_\delta$ such that

$$\int_\Omega \nabla u \nabla v \, d(x,y) = \delta b \int_{\Gamma_0} v \, dx \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{V.2}$$

Let $\mathcal{T}_h$ be a triangulation of Ω as described above and let V_h be the space consisting of the continuous functions on $\Omega_h := \bigcup_{T \in \mathcal{T}_h} \overline{T}$ that are linear on each $T \in \mathcal{T}_h$ and that comply with the essential boundary conditions from the definition of the space V_δ at element nodes on $\partial\Omega$. Observe that $V_h \not\subset V_\delta$. Finding an approximate solution of (V.2) means finding a function $u_h \in V_h$ such that

$$\int_{\Omega_h} \nabla u_h \nabla v_h \, d(x,y) = \delta b \int_{\Gamma_0} v_h \, dx \quad \text{for all} \quad v_h \in H_0^1(\Omega_h). \tag{V.3}$$

Using linear triangular elements, we readily construct a solution of (V.3) with the finite element method, see Figures V.2a, V.2b, whose error is linear in h. More precisely, the error estimate

$$\|u - u_h\|_{H^1(\Omega)} \le Ch\|u\|_{H^2(\Omega)},$$

where h is sufficiently small and C is a positive constant depending on ρ, is a consequence of Theorem 10.2.36 in [15] and Proposition V.2.

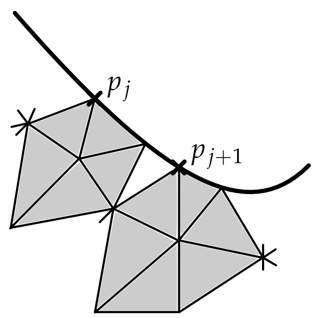


Figure V.3: Two boundary patches

V.1.2 The Evolution of the Free Surface

The fourth equation in problem (V.1) is an evolution equation for the function f that defines the free surface. To solve this equation for some initial value f_0 the gradient ∇u of the solution of the potential problem on $\Omega(f)$ is needed at the points p_j, where the free surface is moved. However, a naive differentiation of the approximate solution obtained in the preceding section leads to contradictory values for its gradient at the points p_j (in fact, at all element nodes) since it is constant on every triangle $T \in \mathcal{T}_h$, i.e., it is not continuous across element boundaries. A method avoiding this problem is performing a patch recovery due to Zienkiewicz and Zhu, i.e., finding a local representation of the gradient of the approximate solution in the same finite element space as the approximate solution, see [74]. Then one obtains a continuous *recovered* gradient in a neighborhood of some point p_j that can be properly evaluated. Moreover, this recovered gradient has the same accuracy as the solution u itself. The first step in obtaining the gradient at some point p_j is to find a *patch* of elements for p_j. Such a patch is a complete set of triangles in $\mathcal{T}_h$ surrounding an internal node in the mesh adjacent to p_j, see Figure V.3. In this patch the gradient ∇u can be locally represented as a member of V_h and evaluated at p_j.

The second part that has to be known for the evolution of the free surface is its outward normal vector. This is completely characterized by the derivative of the function f. Being considered as a cubic spline, the function f can be analytically differentiated by differentiating the interpolating spline.

To move the free surface forward in time a forward Euler scheme is used. After the evaluation of ∇u and f_x at every p_j at time t the free surface at the next time step $t + \Delta t$ is given by the points

$$\tilde{p}_j = p_j + \Delta t\big(-f_x(p_j)u_x(p_j) + u_y(p_j)\big). \tag{V.4}$$

Unfortunately, the use of a forward Euler scheme implies a severe constraint on the time step, i.e., $\Delta t = \mathcal{O}(h^2)$ for a small spatial discretization h, and an implicit or semi-implicit scheme would be preferable. However, the implementation of such a scheme seems to be impossible since ∇u at the next time step would be involved. It is noteworthy that other applications of explict time stepping schemes to similar problems can be found in the literature, see [39].

Since the global error in equation (V.4) is of order $\mathcal{O}(\Delta t)$, which is the same as $\mathcal{O}(h^2)$, and the error in ∇u is of order $\mathcal{O}(h)$, the total error in the coupled scheme is also of order $\mathcal{O}(h)$ for small h.

V.2 Numerical Experiments

In this section we present some simulations of periodic Newtonian Hele-Shaw flows involving the dynamic boundary condition at the free surface

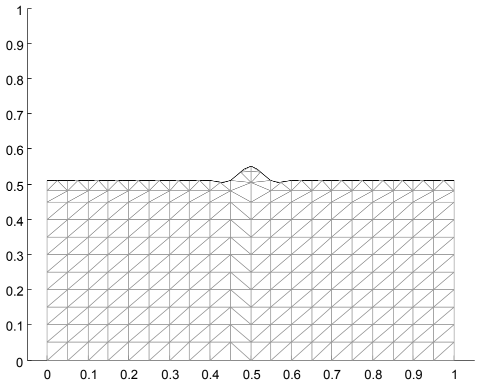
$$H(f;\iota,\sigma) := -\sigma\kappa_f - \frac{\iota^2}{(1-f)^2} + f \quad \text{for} \quad \sigma,\iota \geq 0$$

being exactly the one from Chapter IV. With this choice the free boundary problem (V.1) models the flow of a Newtonian (ferro-)fluid under the influence of gravity and surface tension effects (if $\sigma \neq 0$) in a magnetic field induced by the current of intensity ι in a straight wire above the cell at $y = 1$, see Chapter II. In case $\iota \neq 0$ problem (V.1) might appear to be rather unphysical since virtually every ferrofluid has to be considered as a non-Newtonian fluid. However, we disregard this fact because we are interested only in the numerical validation of the stability results from Section IV.4, which equally work for Newtonian fluids.

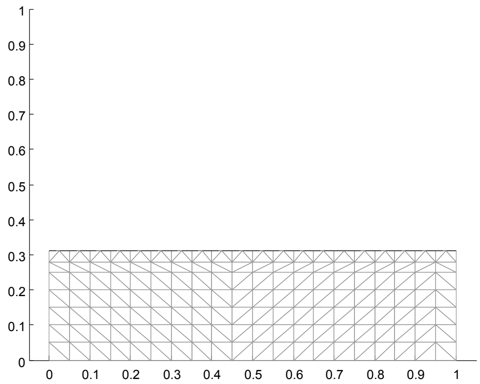
We can employ two different boundary conditions on the lower boundary component Γ_0: prescribed pressure for $\delta = 0$ or mass flux for $\delta = 1$. When studying the stability of some flat solution $c \in (0,1)$ we disclose in the proof of Lemma IV.21 the 'right' value for this boundary condition, which is $-\iota^2/(1-c)^2 + c$ if $\delta = 0$ and 0 if $\delta = 1$. In the following this choice is always understood and indicated only by the respective value of δ unless stated otherwise.

Numerical simulations confirm the necessity of the above choice for the lower boundary value. If a constant boundary value is not chosen in this way, mass is not conserved and an initial condition considered as a perturbation of some flat solution $c \in (0,1)$ (see Figure V.4a) does not converge to c in the stable regime. In fact, in case $\delta = 0$ there is a $\tilde{c} \neq c$ such that the free surface converges to $\tilde{c}$, see Figure V.4b. But in case $\delta = 1$ even this is not true since mass increases or decreases all the time. As a consequence, the free surface either approaches 0 or increases without bounds, which—in both cases— becomes unphysical and terminates the simulation; see Figure V.4c for an overflowing cell and Figure V.4d for a nearly empty one. Observe that at the end of Chapter II we even *prove* such a behavior in case $\delta = 1$.

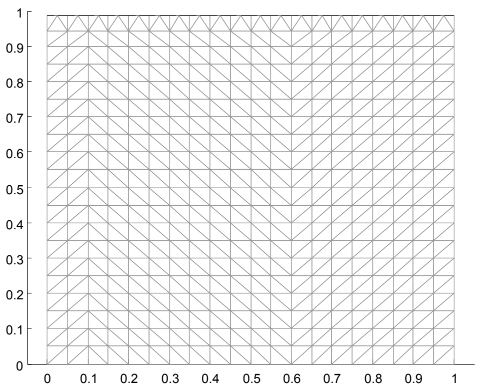
The simulations presented here are implemented in Matlab. They are performed on an Intel Xeon X5450 CPU with 3 GHz speed and 6 MB cache. For the spatial discretization $n = 20$ is chosen which is sufficient for our purposes. Indeed, a higher resolution does not entail a noticeable improvement in the simulations; it only renders them slower. In fact, with the exception of Figure V.6f all simulations (with $n = 20$) do not take more than one minute; the majority only few seconds. Note that the variable T in the figures stands for the number of computed time steps.



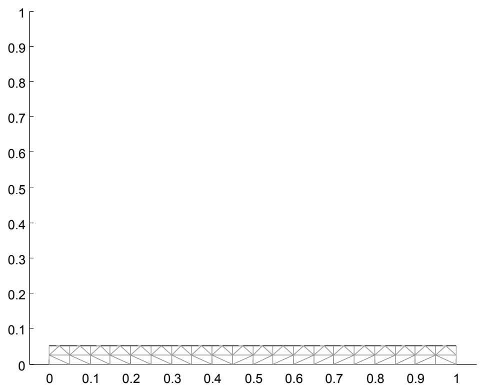
(a) Perturbation of $c = 0.51$



(b) Solution at $T = 2000$ near the limiting flat solution $\tilde{c} = 0.31 \neq c$ with pressure $-\iota^2/(1-\tilde{c})^2 + \tilde{c}$ at Γ_0, $\delta = 0$



(c) Increasing free surface at $T = 947$ with injection rate 0.01 at Γ_0, $\delta = 1$



(d) Decreasing free surface at $T = 923$ with suction rate 0.01 at Γ_0, $\delta = 1$

Figure V.4: Flows not conserving mass

V.2.1 The Strongly Stable Regime

Let us recall the stability constraints from Section IV.4. If only the simple boundary condition $b = \text{const.}$ (in agreement with the convention mentioned above) is imposed at Γ_0, the stability thresholds for the parameter ι are

$$\iota_*^\delta = \iota_*^\delta(c) = \sqrt{\frac{(1-c)^3}{2}(\delta\sigma + 1)} \quad \text{for} \quad \delta \in \{0,1\}. \tag{V.5}$$

For $\iota < \iota_*^\delta$ a flat solution $c \in (0,1)$ is exponentially stable; for $\iota > \iota_*^\delta$, unstable, see Theorem IV.23.

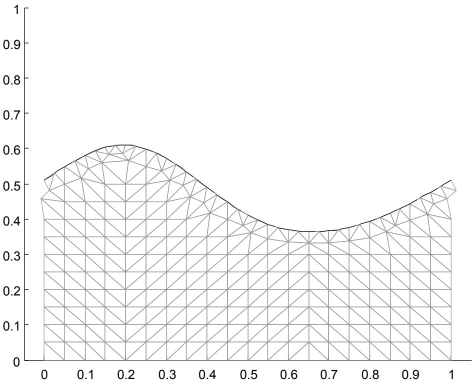
We first examine the flow governed by system (V.1) for $\sigma = \iota = 0$. In this case we obtain the classical gravity driven flow, which has been studied extensively in the past. The simulations show that the free surface flattens out rather quickly converging to a steady state, which is a flat solution, see Figure V.5c. We observe this behavior for virtually *any* initial surface configuration, see, e.g., Figure V.5b for an initial configuration with a steep gradient.

Consider the case $\sigma = 0$ and $0 < \iota \ll \iota_*^\delta$. Heuristically, in a weak magnetic field the gravity force prevails the small counteracting magnetic force. In accordance with this the simulated flows exhibit a similar behavior as in the case without magnetic forces: For a large class of initial configurations, e.g., Figure V.5a, the convergence of the free surface to a flat solution can still be observed, see Figure V.5e, albeit slower. However, by increasing ι we decrease the freedom in choosing the initial configuration showing a stable behavior; see Figure V.5f for a free surface that does not converge to a flat solution although—as seen in Figure V.5d—the gravity flow for the same initial condition is stable.

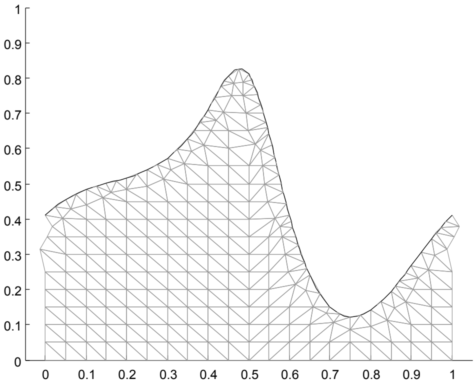
V.2.2 The Threshold to Instability

When the values of ι approach ι_*^δ, the initial configurations behaving in a stable manner become more and more 'flat'. This means that distinctive shapes in the initial configuration—like strong oscillations or steep gradients—are not allowed anymore. For such non-flat initial configurations the magnetic forces are strong enough to boost the 'bumps' in the initial configuration. This eventually leads to an unstable evolution of the free surface, see again Figure V.5f. However, the results from Section IV.4, of course, still hold, i.e., for *every* value $\iota \in (0, \iota_*^\delta)$ the corresponding flat solution is stable. This can be visualized according to the following rule: The closer ι is to ι_*^δ, the closer the initial configuration has to be to the corresponding flat solution. Indeed, in our simulations we are able to approach the stability threshold ι_*^δ up to one per mill, see Figures V.6a–V.6f. Most probably an even higher precision could be achieved. But the simulations take rather long times since the movement of a free surface very close to a flat solution is very slow.

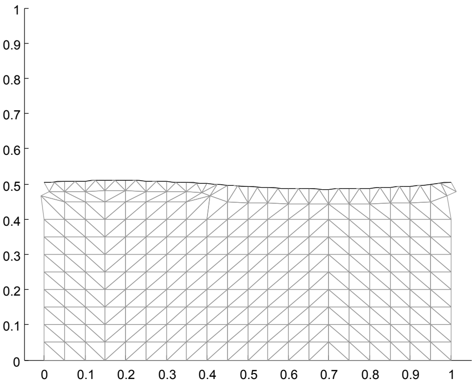
So far we have only considered the case $\sigma = 0$. Since σ does not appear in the stability threshold in case $\delta = 0$ and—in case $\delta = 1$—the choice $\sigma = 0$ does not render the threshold useless, it is not unreasonable to choose $\sigma = 0$ to study the stability of flat solutions. However, in case $\delta = 1$ it is interesting to study the effects of the choice of $\sigma \neq 0$ on the stability behavior of flat solutions.



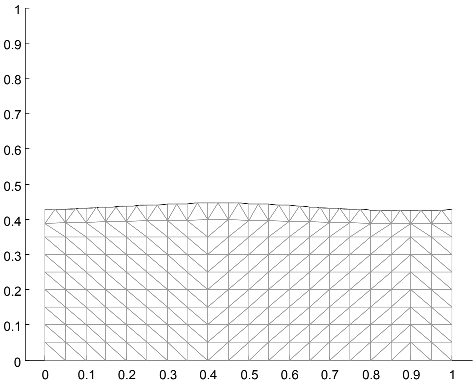
(a) Initial configuration



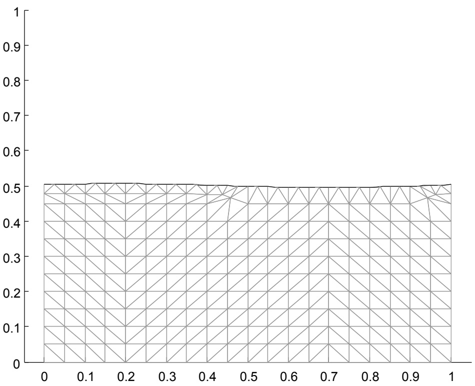
(b) Initial configuration with a steep gradient



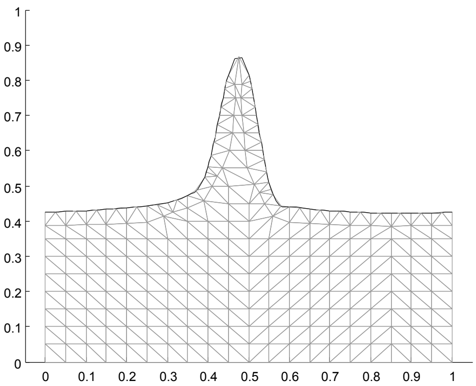
(c) Solution of (a) at $T = 300$ with $\iota = 0$, $\delta = 0$



(d) Solution of (b) at $T = 300$ with $\iota = 0$, $\delta = 1$

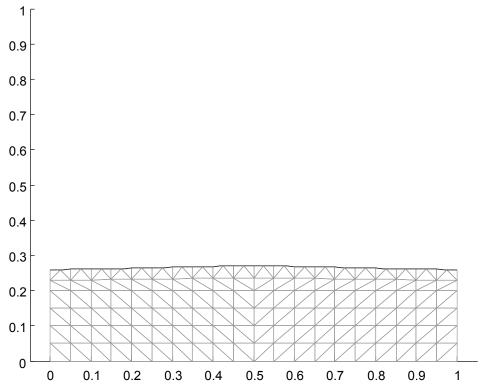


(e) Solution of (a) at $T = 500$ with $\iota = 0.45 \times \iota_*^\delta$, $\delta = 0$

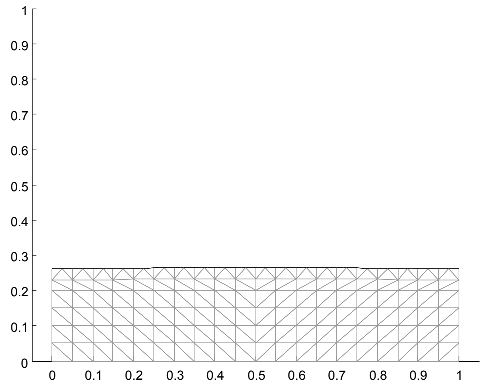


(f) Solution of (b) at $T = 663$ with $\iota = 0.45 \times \iota_*^\delta$, $\delta = 1$

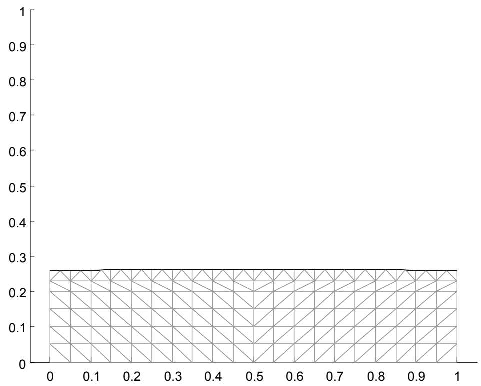
Figure V.5: Flows with $\sigma = 0$ and small ι



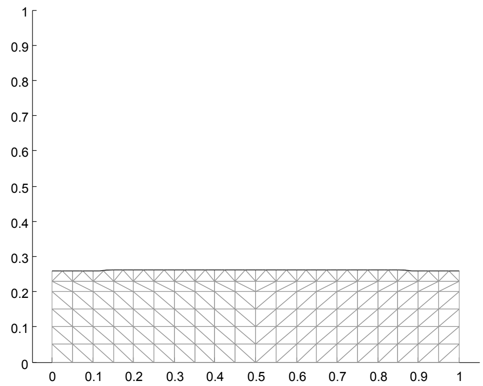
(a) Initial configuration



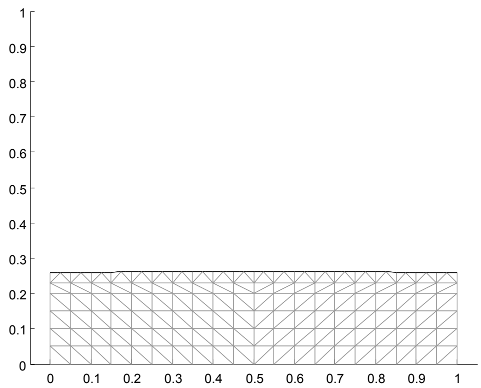
(b) Solution of (a) at $T = 500$ with $\iota = 0.9 \times \iota_*^\delta$, $\delta = 0$



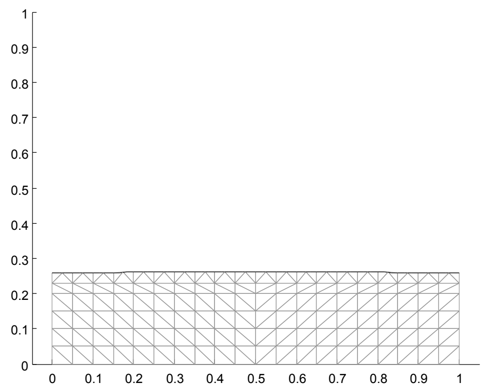
(c) Initial configuration



(d) Solution of (c) at $T = 5000$ with $\iota = 0.99 \times \iota_*^\delta$, $\delta = 0$

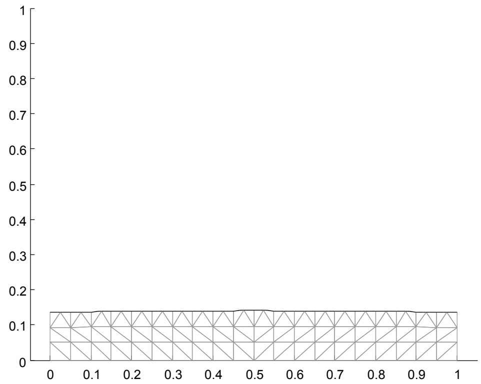


(e) Initial configuration

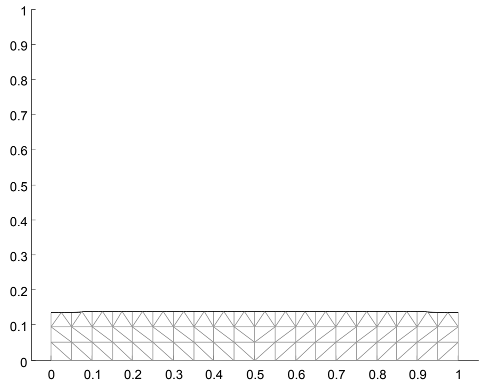


(f) Solution of (e) at $T = 10000$ with $\iota = 0.999 \times \iota_*^\delta$, $\delta = 0$

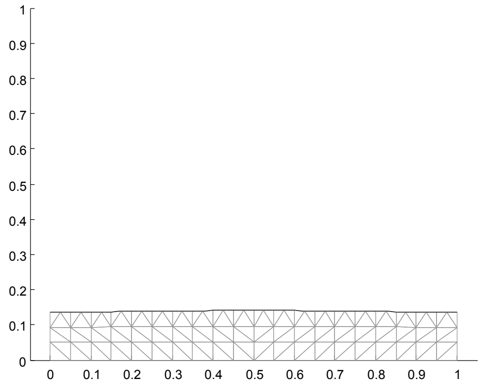
Figure V.6: Flows with $\sigma = 0$ and ι near ι_*^δ



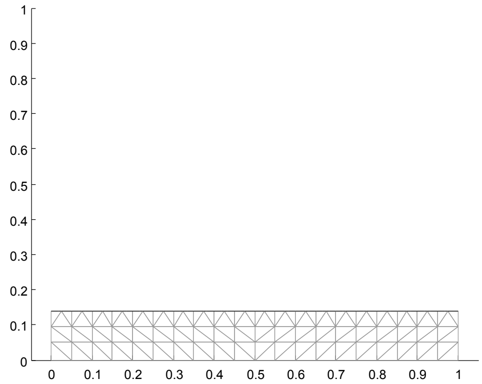
(a) Initial configuration near the flat surface $c = 0.137$.



(b) Solution at $T = 5000$ with $\sigma = 0$ and $\iota = 0.995 \times \iota_*^\delta$, $\delta = 1$



(c) Solution at $T = 5000$ with $\sigma = 0$ and $\iota = 0.995 \times \sqrt{\frac{(1-c)^3}{2} \times 1.015}$, $\delta = 1$



(d) Solution at $T = 100$ with $\sigma = 0.015$ and $\iota = 0.995 \times \iota_*^\delta$, $\delta = 1$

Figure V.7: Flows with $0 \leq \sigma \ll 1$ and ι near ι_*^δ

It is a delicate issue to handle surface tension effects in our numerical experiments because it is a source for possible numerical instabilities if the curvature of the free surface is not represented properly. Thus, we have to be careful with the choice of σ, i.e., the value has to be chosen sufficiently small.

Let $c = 0.137$. Suppose that $\sigma = 0.015$ and let $\iota = 0.995 \times \iota_*^1 \approx 0.568$. Note that for this choice of parameters it holds that $\iota > \iota_*^0 \approx 0.567$, i.e., in case $\sigma = 0$ the flat solution c would be unstable since ι exceeds the stability threshold, see Figure V.7c. However, the simulation (Figure V.7d) shows a stable behavior about c. Hence, the choice of σ does have the expected effect in the simulations according to the stability analysis from Section IV.4, cf. also Figure V.7b, where the evolution in the stable regime for $\sigma = 0$ is depicted. Moreover, it can be observed that the convergence to a flat solution is much faster if $\sigma \neq 0$ compared to the case without surface tension effects. This can be expected due to the structure of the eigenvalues of the linearized evolution operator governing the movement of the free surface, cf. Theorem IV.11, since—for increasing modes—they decay linearly for $\sigma = 0$ but to the power of 3 if $\sigma > 0$.

Chapter VI

Classical Solutions for Stabilized Newtonian Hele-Shaw Flows for Large Initial Data

In Chapter IV we prove a well-posedness result for the laminar flow of a non-Newtonian fluid in a vertical Hele-Shaw cell. We prove the existence of an open subset of the phase space such that the Hele-Shaw free boundary problem with some initial condition in this set has a unique classical solution. However, this is a local result in the sense that it is not clear how 'large' the set of admissible initial conditions is. The aim of this chapter is to prove local existence of a unique classical solution of a Hele-Shaw free boundary problem for a set of initial conditions that can be described explicitly. However, to accomplish such a result we reduce our model from Chapter IV and restrict our considerations to Newtonian fluids only. In this way a well developed linear theory for boundary value problems and Fourier multiplier operators in little Hölder spaces can be applied. The key results of this theory were developed by J. Escher and G. Simonett in [34]. They examine the well-posedness of a free boundary problem that is similar to the one below. The major difference is the geometry on which it is studied. Whereas we consider a periodic geometry, an unbounded geometry is studied there.

As in Chapter IV we turn again our attention to a special boundary condition on the fixed part of the boundary in the free boundary problem. In imposing such a boundary condition we can again control either the pressure or the flux at the fixed boundary component. We remark that for an unbounded geometry such a boundary condition was studied in [23] by J. Escher and Z. Feng. However, they are mainly interested in stability results for flat solutions and only the flux at the fixed boundary component can be controlled.

Let us start with briefly introducing the Hele-Shaw free boundary problem of interest. Given a positive function $\tilde{f} \in C^{2\,\mathrm{sgn}(\sigma)+2}$, we consider a Newtonian fluid in a periodic vertical Hele-Shaw cell having the free surface $\Gamma(\tilde{f}) := \{(x, \tilde{f}(x)) : x \in \mathbb{S}\}$. The fluid below the free surface covers the area

$$\Omega(\tilde{f}) := \{(x, y) \in \mathbb{S} \times \mathbb{R} : 0 < y < \tilde{f}(x)\}$$

having the fixed bottom part $\Gamma_0 := \mathbb{S} \times \{0\} \equiv \mathbb{S}$. Then the velocity potential u of the fluid and the free surface $\tilde{f}$ are governed by the set of equations

$$\begin{aligned} \Delta u &= 0 && \text{in} \quad \Omega(\tilde{f}), \\ (1-\delta)u + \delta\partial_2 u &= \tilde{b}(\tilde{f}) && \text{on} \quad \Gamma_0, \\ u &= -\sigma\kappa_{\tilde{f}} + \tilde{f} && \text{on} \quad \Gamma(\tilde{f}), \\ \partial_t \tilde{f} &= -\sqrt{1+\tilde{f}_x^2}\,\partial_\nu u && \text{on} \quad \Gamma(\tilde{f}), \\ \tilde{f}(0) &= \tilde{f}_0 && \text{on} \quad \mathbb{S}, \end{aligned} \tag{VI.1}$$

where $\nu := n/\|n\|$ is the unit outward normal at $\Gamma(\tilde{f})$ with $n := \begin{pmatrix} -\tilde{f}_x & 1 \end{pmatrix}$. The parameters $\sigma \geq 0$ and $\delta \in \{0, 1\}$ are fixed from the very beginning. The first is the surface tension parameter and the latter determines whether a Dirichlet ($\delta = 0$) or a Neumann ($\delta = 1$) boundary condition on Γ_0 is considered. In case $\delta = 0$ the pressure is prescribed at the lower boundary component whereas in case $\delta = 1$ a rate of injection/suction is given by $\tilde{b} \in C^\omega(\mathbb{R}, \mathbb{R})$. Given $\tilde{f} \in C^{2\,\mathrm{sgn}(\sigma)+2}$ and $x \in \mathbb{S}$, recall that $\tilde{b}(\tilde{f})(x) := \tilde{b}(\tilde{f}(x))$. Finally, observe that the function $\bar{\mu}$ from (II.30) is a constant for Newtonian fluids and we may choose without loss of generality $\bar{\mu} = 1$.

Let $\mathrm{Ad}_y := \{f \in C^{2\,\mathrm{sgn}(\sigma)+2} \colon \min_{x\in\mathbb{S}} f(x) > y\}$. Obviously, if $\tilde{f} \in \mathrm{Ad}_0$, system (VI.1) is meaningful in the sense that $\Omega(\tilde{f})$ is connected. Given $s > 0$, let us define $h^s_+(\mathbb{S}) := h^s(\mathbb{S}) \cap \mathrm{Ad}_0$. Note that $h^s_+(\mathbb{S})$ is the cone of strictly positive elements in $h^s(\mathbb{S})$. Let us again drop the unit circle from our notation and define $h^s_+ := h^s_+(\mathbb{S})$ for $s > 0$.

VI.1 Classical Solutions for Newtonian Hele-Shaw Flows Without Surface Tension Effects

Throughout this section we set $\sigma = 0$. Then we call a pair $(\tilde{f}, u)$ a classical solution of problem (VI.1) if there is a $T > 0$ such that

$$\begin{aligned} \tilde{f} &\in C([0,T], h^{2+\alpha}_+) \cap C^1([0,T], h^{1+\alpha}), \\ u(\,\cdot\,, t) &\in buc^{2+\alpha}(\Omega(\tilde{f}(t,\,\cdot\,))), \quad 0 \leq t \leq T, \end{aligned}$$

and $\tilde{f}, u$ fulfill (VI.1) pointwise. Given $\tilde{f} \in h^{2+\alpha}_+$, put

$$\lambda_{\tilde{f}} := \frac{\tilde{f}^2}{(1+\tilde{f}^2+\tilde{f}_x^2)(1+\tilde{f}_x^2)}$$

and let $u_{\tilde{f}}$ be the solution of

$$\Delta u = 0 \quad \text{in} \quad \Omega(\tilde{f}), \quad (1-\delta)u + \delta\partial_2 u = \tilde{b}(\tilde{f}) \quad \text{on} \quad \Gamma_0, \quad u = \tilde{f} \quad \text{on} \quad \Gamma(\tilde{f}).$$

Finally, defining the set

$$\mathcal{W} := \{\tilde{f} \in h^{2+\alpha}_+ \colon \partial_2 u_{\tilde{f}}(x, \tilde{f}(x)) < \lambda_{\tilde{f}}(x), x \in \mathbb{S}\}, \tag{VI.2}$$

we can formulate our main result.

Theorem VI.1. *Given* $c > 0$*, suppose that* $(-1)^{\delta}\tilde{b}(c) > (-1)^{\delta}c^{1+\delta}/(1+c^2)$*. For every* $\tilde{f}_0 \in \mathcal{W}$ *there exists* $T > 0$ *and a unique maximal classical solution* $(\tilde{f}, u)$ *of system* (VI.1) *on* $[0, T)$*. Furthermore,* $\tilde{f}$ *generates a local analytic semiflow on* $\mathcal{W}$ *and if* $\tilde{f}$ *is uniformly continuous in time it holds that*

$$\lim_{t\to T} \tilde{f}(t, \cdot\,) \in \partial\mathcal{W} \quad \textit{or} \quad T = +\infty.$$

VI.1.1 The Transformation

In this section we transform system (VI.1) on a fixed reference domain where we perform its further analysis. This transformation introduces nonlinear terms in the differential operators in the following way. Let $\alpha \in (0,1)$ and $c > 0$ be fixed in the following and let $\tilde{f} \in h_+^{2+\alpha}$ be given. Put

$$\tilde{\phi}_{\tilde{f}}(x,y) := \big(x, 1 - y/\tilde{f}(x)\big), \quad (x,y) \in \Omega(\tilde{f}).$$

The function $\tilde{\phi}_{\tilde{f}}$ is a $C^{2+\alpha}$-diffeomorphism between $\Omega(\tilde{f})$ and $\Omega := \mathbb{S} \times (0,1)$. Put $f := \tilde{f} - c$ and observe that

$$f \in h^{2+\alpha} \cap \mathrm{Ad}_{-c} =: \mathcal{V}_{2+\alpha} =: \mathcal{V}.$$

Put further $\phi_f := \tilde{\phi}_{f+c}$. This function induces the push forward and pull back operators

$$\begin{aligned} \phi_*^f u &:= u \circ \phi_f^{-1}, \quad u \in BUC(\Omega(\tilde{f})), \\ \phi_f^* v &:= v \circ \phi_f, \quad v \in BUC(\Omega), \end{aligned}$$

and the transformed operators

$$\begin{aligned} \mathcal{A}(f) &:= -\phi_*^f \Delta \phi_f^*, \\ \mathcal{B}(f) &:= \phi_*^f \Big(-\sqrt{1+\tilde{f}_x^2}\,\partial_\nu(\phi_f^* \cdot\,)|\Gamma(\tilde{f})\Big) = -\gamma_0 \phi_*^f \big(\nabla(\phi_f^* \cdot\,) \cdot n\big), \\ \tilde{\mathcal{C}}(f) &:= (1-\delta)\phi_*^f \gamma_0 \phi_f^* + \delta \phi_*^f \gamma_0 \partial_2(\phi_f^* \cdot\,), \end{aligned}$$

where γ_0 denotes the trace operator with respect to Γ_0. These operators are given by

$$\mathcal{A}(f) = -\sum_{j,k=1}^{2} a_{jk}(f)\partial_{jk} + a_2(f)\partial_2, \quad \mathcal{B}(f) = \sum_{j=1}^{2} b_j(f)\gamma_0\partial_j,$$

where

$$a_{11}(f) := 1, \quad a_{12}(f) := \frac{\pi f_x}{\tilde{f}}, \quad a_{22}(f) := \frac{1+\pi^2 {f_x}^2}{\tilde{f}^2}, \quad a_2(f) := \frac{\pi}{\tilde{f}}\Big(\frac{2{f_x}^2}{\tilde{f}} - f_{xx}\Big),$$

$$b_1(f) := f_x, \quad b_2(f) := \frac{1+{f_x}^2}{\tilde{f}},$$

and $\pi(x,y) := 1 - y$ for $(x,y) \in \overline{\Omega}$ (see also [34, Lemma 2.2]), and

$$\tilde{\mathcal{C}}(f) = (1-\delta)\gamma_1 - \frac{\delta}{\tilde{f}}\gamma_1\partial_2,$$

where γ_1 is the trace operator with respect to $\Gamma_1 := \mathbb{S} \times \{1\}$.

Lemma VI.2. *Given $f \in \mathcal{V}$, let $\tilde{f} := f + c$ and define $\underline{\alpha}(f) := \left(1 + \tilde{f}^2 + \pi^2 f_x^2\right)^{-1}$. The operator $\mathcal{A}(f)$ is elliptic in the sense that*

$$\sum_{j,k=1}^{2} a_{jk}(f)\xi_j\xi_k \geq \underline{\alpha}(f)|\xi|^2, \quad \xi \in \mathbb{R}^2.$$

Proof. See Lemma 2.2 in [34]. □

The next result follows from elementary calculations.

Lemma VI.3. *The operators $\mathcal{A}$ and $\mathcal{B}$ depend analytically on its variable, i.e.,*

$$\mathcal{A} \in C^{\omega}\big(\mathcal{V}, \mathcal{L}(buc^{2+\alpha}(\Omega), buc^{\alpha}(\Omega))\big),$$
$$\mathcal{B} \in C^{\omega}\big(\mathcal{V}, \mathcal{L}(buc^{2+\alpha}(\Omega), h^{1+\alpha})\big).$$

Given $f \in \mathcal{V}$, their respective linearizations are given by

$$\begin{aligned}
\partial\mathcal{A}(f)[h,v] &= \frac{\pi}{\tilde{f}^2}(f_x h - \tilde{f} h_x)\partial_{12} v \\
&\quad - \frac{2}{\tilde{f}^3}\big(\pi^2 \tilde{f} f_x h_x - (1 + \pi^2 f_x^2)h\big)\partial_{22} v \\
&\quad + \frac{\pi}{\tilde{f}}\Big(\frac{f_{xx}}{\tilde{f}}h - \frac{4 f_x^2}{\tilde{f}^2}h + \frac{4 f_x}{\tilde{f}}h_x - h_{xx}\Big)\partial_2 v, \\
\partial\mathcal{B}(f)[h,v] &= h_x\gamma_0\partial_1 v + \frac{1}{\tilde{f}^2}\big(2\tilde{f} f_x h_x - (1 + f_x^2)h\big)\gamma_0\partial_2 v,
\end{aligned}$$

where $\tilde{f} := f + c$, $h \in h^{2+\alpha}$, and $v \in buc^{2+\alpha}(\Omega)$.

Given $f \in \mathcal{V}$, note that in case $\delta = 1$ the problem

$$\mathcal{A}(f)v = 0 \quad \text{in} \quad \Omega, \quad \tilde{\mathcal{C}}(f)v = \tilde{b}(f+c) \quad \text{on} \quad \Gamma_1, \quad v = f \quad \text{on} \quad \Gamma_0$$

is obviously equivalent to the problem

$$\mathcal{A}(f)v = 0 \quad \text{in} \quad \Omega, \quad \partial_2 v = -(f+c)\tilde{b}(f+c) \quad \text{on} \quad \Gamma_1, \quad v = f \quad \text{on} \quad \Gamma_0.$$

With this in mind let us define the modified boundary operator $\mathcal{C} := (1-\delta)\gamma_1 + \delta\gamma_1\partial_2$, which is independent of f. Now we can formulate an equivalent version of problem (VI.1) in the new coordinates. Given $f_0 \in \mathcal{V}$, consider the system

$$\begin{aligned}
\mathcal{A}(f)v &= 0 && \text{in} \quad \Omega, \\
v &= f && \text{on} \quad \Gamma_0, \\
\mathcal{C}v &= b(f) && \text{on} \quad \Gamma_1, \\
\partial_t f &= \mathcal{B}(f)v && \text{on} \quad \Gamma_0, \\
f(0) &= f_0 && \text{on} \quad \mathbb{S},
\end{aligned} \tag{VI.3}$$

where $b(f) := (-f-c)^{\delta}\tilde{b}(f+c) - (1-\delta)c$. We call a pair (f, v) a classical solution of problem (VI.3) if there is $T > 0$ such that

$$f \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha}),$$
$$v \in buc^{2+\alpha}(\Omega)$$

and f, v fulfill (VI.3) pointwise. Observe that system (VI.1) and system (VI.3) are equivalent in the following sense.

Lemma VI.4. *If $(\tilde{f}, u)$ is a classical solution of problem* (VI.1)*, $(\tilde{f} - c, \phi_*^{\tilde{f}-c} u - c)$ is a classical solution of problem* (VI.3) *and if (f, v) is a classical solution of problem* (VI.3)*, $(f + c, \phi_f^* v + c)$ is a classical solution of problem* (VI.1).

Given $f \in \mathcal{V}$, define $\mathcal{R}(f) := \big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big)^{-1}$ and its natural decomposition

$$\mathcal{S}(f) := \mathcal{R}(f)|buc^{\alpha}(\Omega) \times \{0\} \times \{0\},$$
$$\mathcal{T}(f) := \mathcal{R}(f)|\{0\} \times h^{2+\alpha} \times \{0\},$$
$$\mathcal{U}(f) := \mathcal{R}(f)|\{0\} \times \{0\} \times h^{2-\delta+\alpha}.$$

Theorem VI.5. *Given $f \in \mathcal{V}$ and $\mu > 0$ it holds that*

$$\big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big) \in \mathcal{L}\mathrm{is}\big(buc^{2+\alpha}(\Omega), buc^{\alpha}(\Omega) \times h^{2+\alpha} \times h^{2-\delta+\alpha}\big),$$
$$\big(\mathcal{A}(f), \mu\gamma_0 - \mathcal{B}(f), \mathcal{C}\big) \in \mathcal{L}\mathrm{is}\big(buc^{2+\alpha}(\Omega), buc^{\alpha}(\Omega) \times h^{1+\alpha} \times h^{2-\delta+\alpha}\big).$$

There is $C > 0$ independent of u such that

$$\|u\|_{2+\alpha,\Omega} \leq C\big(\|\mathcal{A}(f)u\|_{\alpha,\Omega} + \|\gamma_0 u\|_{2+\alpha} + \|\mathcal{C}u\|_{2-\delta+\alpha}\big)$$

for all $u \in buc^{2+\alpha}(\Omega)$.

Proof. We prove only the first isomorphism result. The other one is similar. Given $f \in \mathcal{V}$, it can be shown that $\big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big)$ is a regular elliptic system in Ω in the sense of [71, Section 4.1.2, Definition 4]. Moreover, the only smooth solution of the problem

$$\mathcal{A}(0)u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma_0, \quad \mathcal{C}u = 0 \quad \text{on} \quad \Gamma_1$$

is the trivial one. Then if $F \in BUC^{\infty}(\Omega)$ and $g, h \in C^{\infty}$, the boundary value problem

$$\mathcal{A}(0)u = F \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \Gamma_0, \quad \mathcal{C}u = h \quad \text{on} \quad \Gamma_1$$

has a unique solution $u \in BUC^{\infty}(\Omega)$. This means

$$\big(\mathcal{A}(0), \gamma_0, \mathcal{C}\big) \in \mathcal{L}\mathrm{is}\big(BUC^{\infty}(\Omega), BUC^{\infty}(\Omega) \times C^{\infty} \times C^{\infty}\big).$$

Let $u \in buc^{2+\alpha}(\Omega)$. There is a sequence $(u_n)_n \subset BUC^{\infty}(\Omega)$ with $u_n \to u$ in $BUC^{2+\alpha}(\Omega)$ as $n \to \infty$. The identification of $BUC^{\gamma+2+s}(\Omega)$ with the Besov spaces

$B^{\gamma+2+s}_{\infty\infty}(\Omega)$, where $\gamma \in (-1,0)$ and $s \in \mathbb{Z}_{\geq 0}$, and Corollary 4.3.2 in [71] ensure the existence of two positive constants C_1, C_2 such that

$$C_1\|u\|_{2+\alpha,\Omega} \leq \|\mathcal{A}(0)u\|_{\alpha,\Omega} + \|\gamma_0 u\|_{2+\alpha} + \|\mathcal{C}u\|_{2-\delta+\alpha} \leq C_2\|u\|_{2+\alpha,\Omega}$$

for $u \in BUC^{2+\alpha}(\Omega)$. This estimate implies the convergence

$$\big(\mathcal{A}(0), \gamma_0, \mathcal{C}\big)u_n \to \big(\mathcal{A}(0), \gamma_0, \mathcal{C}\big)u \quad \text{in} \quad BUC^{\alpha}(\Omega) \times C^{2+\alpha} \times C^{2-\delta+\alpha},$$

which means $\big(\mathcal{A}(0), \gamma_0, \mathcal{C}\big)u \in buc^{\alpha}(\Omega) \times h^{2+\alpha} \times h^{2-\delta+\alpha}$. Analogously one shows that $u = \big(\mathcal{A}(0), \gamma_0, \mathcal{C}\big)^{-1}(F, g, h) \in buc^{2+\alpha}(\Omega)$ provided $F \in buc^{\alpha}(\Omega)$, $g \in h^{2+\alpha}$, $h \in h^{2-\delta+\alpha}$. Consequently, it holds that

$$\big(\mathcal{A}(0), \gamma_0, \mathcal{C}\big) \in \mathcal{L}\mathrm{is}\big(buc^{2+\alpha}(\Omega), buc^{\alpha}(\Omega) \times h^{2+\alpha} \times h^{2-\delta+\alpha}\big). \tag{VI.4}$$

Next we prove the a priori estimate. From Theorem 4.3.4 in [71] we infer that, given $f \in \mathcal{V}$, $F \in BUC^{\alpha}$, $g \in C^{2+\alpha}$, and $h \in C^{2-\delta+\alpha}$, the boundary value problem

$$\mathcal{A}(f)u = F \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \Gamma_0, \quad \mathcal{C}u = h \quad \text{on} \quad \Gamma_1$$

has a unique solution in $BUC^{2+\alpha}$. Now choose a compact subset K of $\mathcal{V}$ and note that $\big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big)$ fulfills the hypotheses of Theorem 7.3 in [1] for all $f \in K$. This result together with the uniqueness of solutions (cf. Remark 2 following Theorem 7.3 in [1]) implies the existence of a positive constant C depending on K but independent of u such that

$$\|u\|_{2+\alpha,\Omega} \leq C\big(\|F\|_{\alpha,\Omega} + \|g\|_{2+\alpha} + \|h\|_{2-\delta+\alpha}\big).$$

Since K is chosen arbitrarily, the assertion follows.

Now, this a priori estimate, the isomorphism result (VI.4), and an application of the method of continuity from [42, Theorem 5.2] similar to the proof of Theorem 3.5 in [34] implies the isomorphism result. □

VI.1.2 The Evolution Equation and Fourier Operators

In this section we reduce problem (VI.3) to an abstract Cauchy problem on $h^{1+\alpha}$. To do this observe that, given $f \in \mathcal{V}$, the function $\mathcal{T}(f)f + \mathcal{U}(f)b(f)$ is the unique solution in $buc^{2+\alpha}(\Omega)$ of the problem

$$\mathcal{A}(f)u = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \Gamma_0, \quad \mathcal{C}u = b(f) \quad \text{on} \quad \Gamma_1.$$

Put $\Phi(f) := \mathcal{B}(f)(\mathcal{T}(f)f + \mathcal{U}(f)b(f))$. Then problem (VI.3) turns into the abstract Cauchy problem

$$\partial_t f = \Phi(f), \quad f(0) = f_0, \tag{VI.5}$$

on $\mathbb{S}$. We seek functions $f \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha})$ with $T > 0$ that fulfill (VI.5) pointwise and we call these functions classical solutions of (VI.5). Thanks to Theorem VI.5 the following result holds.

Lemma VI.6. *Problem* (VI.5) *and system* (VI.3) *are equivalent in the sense that if* f *is a classical solution of* (VI.5)*, then* $\big(f, \mathcal{T}(f)f + \mathcal{U}(f)b(f)\big)$ *is a classical solution of* (VI.3) *and if* (f, v) *is a classical solution of* (VI.3)*, it holds that* $v = \mathcal{T}(f)f + \mathcal{U}(f)b(f)$ *and* f *is a classical solution of* (VI.5)*.*

Given $f \in \mathcal{V}$, let us introduce the abbreviations $v_f := \mathcal{T}(f)f + \mathcal{U}(f)b(f)$ and $\mathcal{P}(f) := \partial\mathcal{A}(f)[\,\cdot\,, v_f]$.

Lemma VI.7. *Given* $f \in \mathcal{V}$*, it holds that* $\Phi \in C^\omega(\mathcal{V}, h^{1+\alpha})$ *and its linearization is given by*

$$\partial\Phi(f)h = \mathcal{B}(f)(\mathcal{T}(f) + \mathcal{U}(f)b'(f))h + \partial\mathcal{B}(f)[h, v_f] - \mathcal{B}(f)\mathcal{S}(f)\mathcal{P}(f)h,$$

where $h \in h^{2+\alpha}$*.*

Proof. Note that b is supposed to be analytic and that $\partial b(f)h = b'(f)h$ for $h \in h^{2+\alpha}$. Then the assertion follows from Lemma VI.3 and a slight modification of Lemma 2.6 in [23]. □

Let $f \in \mathcal{V}$, $x_0 \in \mathbb{S}$, and $\mu_0 > 0$ be fixed in the following. The cornerstone of the proof of Theorem VI.1 is the proof of the fact that, given $\hat{f} + c \in \mathcal{W}$, the linearization $-\partial\Phi(\hat{f})$ is the negative generator of an analytic semigroup in $h^{1+\alpha}$. To proof this fact we begin with proving another generation result. We associate a pseudo differential operator A_π to the linear operator $-\partial\Phi(f)$ that can be considered as its principal part with coefficients fixed in f and x_0 and we show that this operator generates an analytic semigroup in a (suitable) space over $\mathbb{R}$.

To achieve our aims, we need some basic concepts from the calculus of pseudo differential operators, see, e.g., [60, Chapter 3] and also [44] for an exhaustive treatment. We work in function spaces over $\mathbb{R}$ and we frequently use the Fourier transform. In contrast to the convention in [60] we define the Fourier transform in $\mathbb{R}^n$ with $n \in \mathbb{N}$ for a function g in some (suitable) space of distributions as

$$\mathcal{F}_n g(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} g(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

where—for the sake of simplicity—we put $\mathcal{F} := \mathcal{F}_1$. Moreover, in the context of elliptic boundary value problems on unbounded domains solution operators of *Poisson* and *singular Green* type appear in a natural way. A detailed survey of these types of operators can also be found in [44].

Given $s > 0$ and $n \in \mathbb{N}$, define $h^s(\mathbb{R}^n)$ as the closure of the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ in $BUC^s(\mathbb{R}^n)$. Furthermore, let H be a half space in $\mathbb{R}^2$ and put $\mathcal{S}(H) := r_H\mathcal{S}(\mathbb{R}^2)$, where r_H is the restriction map from $\mathbb{R}^2$ to H. Given $s > 0$, the space $h^s(H)$ is defined as the closure of $\mathcal{S}(H)$ in $BUC^s(H)$. To shorten the notation we put $h^s_{\mathbb{R}^n} := h^s(\mathbb{R}^n)$, $n \in \mathbb{N}$, and $h^s_H := h^s(H)$.

Note that the spaces $buc^s(X)$, $s > 0$, are only defined on bounded domains $X \subset \mathbb{R}^2$ whereas in this section we work on unbounded domains. Moreover, observe that for functions in $h^s(\mathbb{R})$ and $h^s(H)$ we aim to include a decay estimate at infinity. Observe further that when working in these function spaces we completely ignore the fact that we study problem (VI.1) in a periodic geometry.

It is the aim of this section to prove that $A_\pi \in \mathcal{H}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$. Then we obtain a generation result for $-\partial\Phi(f)$ in the 'classical' little Hölder spaces on the unit circle by essentially estimating the difference of these two operators. Here one has to be careful with the correct use of the norms since the operators act on different function spaces. For this the crucial step is providing a suitable localization of the functions in use. For instance, for a test function $\theta \in \mathcal{D}([\pi/2, 3\pi/2])$ and a function $f \in h^s$, $s > 0$, the function θf can be considered as a function in $h^s_{\mathbb{R}}$. In this manner all *cut-off functions* are considered as multiplication operators on little Hölder spaces over the unit circle.

Let us begin by fixing the coefficients of the differential operators $\mathcal{A}$ and $\mathcal{B}$ in f and x_0. We may associate to these operators the differential operators with constant coefficients

$$\mathcal{A}_\pi := -\partial_1^2 - 2a_{12}\partial_{12} - a_{22}\partial_2^2 \quad \text{and} \quad \mathcal{B}_\pi := b_1\gamma_0\partial_1 + b_2\gamma_0\partial_2,$$

where

$$a_{j2} := a_{j2}(f, x_0) := a_{j2}(f)(x_0, 0), \quad b_j := b_j(f, x_0) := b_j(f)(x_0), \quad 1 \le j \le 2,$$

which can be interpreted as their respective principal parts with coefficients fixed in f and x_0. Furthermore, for $\eta, \theta \in \mathbb{R}$ let $p(\eta, \theta) := \eta^2 + 2a_{12}\eta\theta + a_{22}\theta^2$. Clearly, p is the symbol of the operator $\mathcal{A}_\pi$.

While it is not difficult to fix the coefficients of the differential operators $\mathcal{A}$ and $\mathcal{B}$, it is not immediately clear how to freeze the coordinates of the operators $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{U}$ in a reasonable way. It turns out that it is useful to interpret these operators as operators on function spaces in a half space in $\mathbb{R}^2$. Basic ideas of elliptic boundary value problems of second order in half spaces and of homogeneous elliptic symbols are developed in [34, Appendices A, B]. We present the main results of this in the following.

Put $\mathbb{H} := \mathbb{R} \times (0, \infty)$. Given $\eta \in \mathbb{R}$, let $q_\eta(z) := \mu_0^2 + \eta^2 + 2\mathrm{i}a_{12}\eta z - a_{22}z^2$ for $z \in \mathbb{C}$. This quadratic polynomial has the roots

$$\frac{1}{a_{22}}\Big(\mathrm{i}a_{12}\eta \pm \sqrt{a_{22}(\mu_0^2 + \eta^2) - (a_{12}\eta)^2}\Big).$$

If we assume that there is a positive constant α_0 such that

$$p(\eta, \theta) \ge \alpha_0 |(\eta, \theta)|^2 \quad \text{for all} \quad \eta, \theta \in \mathbb{R}, \tag{VI.6}$$

it holds that q_η has exactly one root with positive real part since

$$a_{22} - a_{12}^2 = p(a_{12}, -1) \ge \alpha_0(1 + a_{12}^2) \ge \alpha_0. \tag{VI.7}$$

Let $\lambda(\eta, \mu_0)$ be the root of q_η with positive real part and let $a(\eta)$ and $d(\eta, \mu_0)$ denote its imaginary and real part, respectively.

If we choose $\alpha_0 := \big(1 + (c + f(x_0))^2 + f_x(x_0)^2\big)^{-1}$ we infer from Lemma VI.2 that condition (VI.6) is satisfied and we infer from [34, Lemma B.2] that the boundary value problem

$$(\mu_0^2 + \mathcal{A}_\pi)u = 0 \quad \text{in} \quad \mathbb{H}, \quad u(\,\cdot\,, 0) = g \quad \text{on} \quad \partial\mathbb{H} \equiv \mathbb{R},$$

where $g \in h^{2+\alpha}_{\mathbb{R}}$, has a unique solution in $h^{2+\alpha}_{\mathbb{H}}$ given by $u = \mathcal{T}_\pi g$ with

$$(\mathcal{T}_\pi g)(x,y) := [\mathcal{F}^{-1} \mathrm{e}^{-\lambda(\,\cdot\,,\mu_0)y} \mathcal{F} g](x), \quad (x,y) \in \mathbb{H}.$$

Moreover, $\mathcal{T}_\pi \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{2+\alpha}_{\mathbb{H}})$. This operator is an example of a Poisson operator according to [44, Eq. (1.2.29)].

Let $\mathcal{E}$ denote the reflection operator from $\mathbb{H}$ to $\mathbb{R}^2$. Moreover, given $F \in \mathcal{S}(\mathbb{H})$, put

$$k_F(\eta) := -\frac{1}{a_{22}} \int_0^\infty \mathrm{e}^{-d(\eta,\mu_0)y} \cos(a(\eta)y) \mathcal{F}F(\eta,y)dy, \quad \eta \in \mathbb{R},$$

where $\mathcal{F}F$ has to be understood as the Fourier transform of the function $F(\,\cdot\,,y)$ for fixed $y \in (0,\infty)$. Given $F \in \mathcal{S}(\mathbb{H})$, Lemmas B.8 and B.9 in [34] say that the operators

$$\mathcal{S}_{\pi,1} F := r_{\mathbb{H}} \mathcal{F}_2^{-1} (\mu_0^2 + p)^{-1} \mathcal{F}_2 \mathcal{E} F,$$
$$(\mathcal{S}_{\pi,2} F)(x,y) := [\mathcal{F}^{-1} (\mathrm{e}^{-\lambda(\,\cdot\,,\mu_0)y} / d(\,\cdot\,,\mu_0)) k_F](x), \quad (x,y) \in \mathbb{H},$$

fulfill $\mathcal{S}_{\pi,i} \in \mathcal{L}(h^{\alpha}_{\mathbb{H}}, h^{2+\alpha}_{\mathbb{H}})$, $i = 1,2$, and $\mathcal{S}_{\pi,2} = -\mathcal{T}_\pi \gamma_0 \mathcal{S}_{\pi,1}$ being a singular Green operator, cf. [44, p. 30]. Put $\mathcal{S}_\pi := \mathcal{S}_{\pi,1} + \mathcal{S}_{\pi,2}$. Then [34, Corollary B.11] implies that the boundary value problem

$$(\mu_0^2 + \mathcal{A}_\pi) u = F \quad \text{in} \quad \mathbb{H}, \quad u(\,\cdot\,,0) = 0 \quad \text{on} \quad \mathbb{R},$$

where $F \in h^{\alpha}_{\mathbb{H}}$, has a unique solution in $h^{2+\alpha}_{\mathbb{H}}$ given by $u := \mathcal{S}_\pi F$.

In the sequel it is our aim to find similar results for elliptic boundary value problems in the shifted half space $\tilde{\mathbb{H}} := \mathbb{R} \times (-\infty, 1)$. Let

$$\tilde{q}_\eta(z) := \mu_0^2 + \eta^2 - 2\mathrm{i} a_{12} \eta z - a_{22} z^2 \quad \text{for} \quad \eta \in \mathbb{R}, \quad z \in \mathbb{C}.$$

If we again assume that condition (VI.6) holds, we conclude that this quadratic polynomial has the unique root with positive real part denoted by $\tilde{\lambda}(\eta,\mu_0)$ with real and imaginary part $\tilde{d}(\eta,\mu_0) := d(\eta,\mu_0)$ and $\tilde{a}(\eta) := -a(\eta)$, respectively. Given $h \in h^{2-\delta+\alpha}$, we define the (Poisson) operator $\mathcal{U}_\pi$ by

$$(\mathcal{U}_\pi h)(x,y) := \big[\mathcal{F}^{-1} \mathrm{e}^{-\tilde{\lambda}(\,\cdot\,,\mu_0)(1-y)} / \tilde{\lambda}(\,\cdot\,,\mu_0)^\delta \mathcal{F} h\big](x), \quad (x,y) \in \tilde{\mathbb{H}}.$$

Proposition VI.8. *Suppose that $h \in h^{2-\delta+\alpha}$. Then $\mathcal{U}_\pi \in \mathcal{L}(h^{2-\delta+\alpha}_{\mathbb{R}}, h^{2+\alpha}_{\tilde{\mathbb{H}}})$ and the unique solution of the boundary value problem*

$$(\mu_0^2 + \mathcal{A}_\pi) u = 0 \quad \textit{in} \quad \tilde{\mathbb{H}}, \quad \mathcal{C} u = h \quad \textit{on} \quad \mathbb{R}$$

in $h^{2+\alpha}_{\tilde{\mathbb{H}}}$ is given by $u = \mathcal{U}_\pi h$.

Proof. The first assertion can be proved in the same way as Lemma B.2 in [34]. It remains to show that $\mathcal{U}_\pi h$ is the unique solution of the boundary value problem. For $(x,y) \in \tilde{\mathbb{H}}$ it holds that

$$(\mu_0^2 + \mathcal{A}_\pi)(\mathcal{U}_\pi h)(x,y) = \mathcal{F}^{-1}\big[\eta \mapsto \tilde{q}_\eta(\tilde{\lambda}(\eta,\mu)) \mathrm{e}^{-\tilde{\lambda}(\eta,\mu_0)(1-y)} / \tilde{\lambda}(\eta,\mu_0)^\delta\big] \mathcal{F} h(x).$$

Since $\tilde{q}_\eta(\tilde{\lambda}(\eta,\mu)) = 0$ it follows that $(\mu_0^2 + \mathcal{A}_\pi)u = 0$ in $\tilde{\mathbb{H}}$. Furthermore, it holds that

$$\mathcal{C}\mathcal{U}_\pi h = \gamma_1 \mathcal{F}^{-1} \mathrm{e}^{-\tilde{\lambda}(\,\cdot\,,\mu_0)(1-y)} \mathcal{F} h = h.$$

In case $\delta = 0$ the uniqueness of this solution is obtained with the same argument as in the proof of [34, Lemma B.2]. In case $\delta = 1$ the uniqueness of the solution of the Neumann problem is guaranteed by Theorem 6.31 in [42]—a result on the classical solvability of oblique derivative problems. It has to be mentioned that this result cannot be applied directly since only the derivative is prescribed on the boundary. However—as is pointed out after the proof of this result in [42, p. 130]—the statement still holds in this case since uniqueness of solutions can still be proved with a Fredholm alternative argument. □

Given $u \in BUC(\tilde{\mathbb{H}})$, let us define the operator

$$\tilde{\mathcal{E}} u(x,y) := \begin{cases} u(x, 1-y), & (x,y) \in \overline{\tilde{\mathbb{H}}} \\ u(x, 1+y), & (x,-y) \in \mathbb{H} \end{cases}.$$

It holds that $\tilde{\mathcal{E}} \in \mathcal{L}(h^\alpha_{\tilde{\mathbb{H}}}, h^\alpha_{\mathbb{R}^2})$. For a function $F \in \mathcal{S}(\tilde{\mathbb{H}})$ its Fourier transform now has to be understood as the Fourier transform of the function $F(\,\cdot\,,y)$ for fixed $y \in (-\infty, 1)$. Moreover, let $\mathcal{R}_{\pi,1}$ be the restriction of the linear operator $(\mu_0^2 + \mathcal{A}_\pi)^{-1}$ to $\tilde{\mathbb{H}}$, i.e., $\mathcal{R}_{\pi,1} := r_{\tilde{\mathbb{H}}} \mathcal{F}_2^{-1} (\mu_0^2 + p)^{-1} \mathcal{F}_2 \tilde{\mathcal{E}}$, see also [44, p. 23].

Lemma VI.9. *It holds that $\mathcal{R}_{\pi,1} \in \mathcal{L}(h^\alpha_{\tilde{\mathbb{H}}}, h^{2+\alpha}_{\tilde{\mathbb{H}}})$ and that $(\mu_0^2 + \mathcal{A}_\pi)\mathcal{R}_{\pi,1} = \mathrm{id}_{h^\alpha_{\tilde{\mathbb{H}}}}$.*

Proof. The proof is the same as the one of Lemma B.9 in [34] in the shifted half space $\tilde{\mathbb{H}}$. □

Given $F \in \mathcal{S}(\tilde{\mathbb{H}})$, put

$$\begin{aligned} k_{F,1}(\eta) &:= (-1)^{\delta+1} \int_{-\infty}^0 \mathrm{e}^{-\lambda(\eta,\mu_0)(1-y)} \mathcal{F}F(\eta, 1+y)\, dy, \\ k_{F,2}(\eta) &:= (-1)^{\delta+1} \int_{-1}^0 \mathrm{e}^{-\lambda(\eta,\mu_0)(1+y)} \mathcal{F}F(\eta, 1+y)\, dy, \\ k_{F,3}(\eta) &:= -\int_{-\infty}^{-1} \mathrm{e}^{\tilde{\lambda}(\eta,\mu_0)(1+y)} \mathcal{F}F(\eta, 1+y)\, dy, \quad \eta \in \mathbb{R}, \end{aligned}$$

and set

$$(\mathcal{R}_{\pi,2}F)(x,y) := \mathcal{F}^{-1}\left[\frac{\mathrm{e}^{-\tilde{\lambda}(\,\cdot\,,\mu_0)(1-y)}}{2a_{22} d(\,\cdot\,,\mu_0)} \left(\left(\frac{\lambda(\,\cdot\,,\mu_0)}{\tilde{\lambda}(\,\cdot\,,\mu_0)} \right)^\delta (k_{F,1} + k_{F,2}) + k_{F,3} \right) \right](x)$$

for $(x,y) \in \tilde{\mathbb{H}}$. For the next lemma we shortly introduce the class of slowly increasing functions

$$\mathcal{O}_M := \{ u \in C^\infty(\mathbb{R}) \colon \forall k \in \mathbb{N} \exists m_k \in \mathbb{N} \exists c_k > 0 \colon |\partial^k u(x)| \le c_k (1+x^2)^{m_k}, x \in \mathbb{R} \}.$$

Lemma VI.10. *The linear (singular Green) operator $\mathcal{R}_{\pi,2}$ is a well-defined operator on $\mathcal{S}(\tilde{\mathbb{H}})$ and can be extended to an operator in $\mathcal{L}(h^\alpha_{\tilde{\mathbb{H}}}, h^{2+\alpha}_{\tilde{\mathbb{H}}})$ having the decomposition $\mathcal{R}_{\pi,2} = -\mathcal{U}_\pi \mathcal{C} \mathcal{R}_{\pi,1}$.*

Proof. Observe that there is a positive constant α_* such that

$$\operatorname{Re}\tilde{\lambda}(\eta,\mu_0) = \operatorname{Re}\lambda(\eta,\mu_0) = d(\eta,\mu_0) \geq \alpha_*\sqrt{\mu_0^2+\eta^2} \geq \alpha_*|\eta|, \quad \eta \in \mathbb{R}. \qquad \text{(VI.8)}$$

Furthermore, given $F \in \mathcal{S}(\tilde{\mathbb{H}})$, it holds that $\mathcal{F}F(\,\cdot\,,1+y) \in \mathcal{S}(\mathbb{R})$ for all $y < 0$. Then we see that $k_{F,i} \in \mathcal{S}(\mathbb{R})$ for all $i \in \{1,2,3\}$ and that $\lambda(\,\cdot\,,\mu_0)/\tilde{\lambda}(\,\cdot\,,\mu_0)$ and $\mathrm{e}^{-\tilde{\lambda}(\,\cdot\,,\mu_0)(1-y)}/d(\,\cdot\,,\mu_0)$ for all $y \in (-\infty,1)$ belong to $\mathcal{O}_M$. Now, formula (4.1.8) in [2, Chapter III], which says that the product of a rapidly decreasing and a slowly increasing function is again rapidly decreasing, implies the first assertion.

Choose $F \in \mathcal{S}(\tilde{\mathbb{H}})$ and observe that

$$\begin{aligned}(\mathcal{CR}_{\pi,1}F)(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}(\mathrm{i}\theta)^\delta \mathrm{e}^{\mathrm{i}(x\eta+\theta)}(\mu_0^2+p(\eta,\theta))^{-1}\mathcal{F}_2\tilde{\mathcal{E}}F(\eta,\theta)\,d\eta\,d\theta \\ &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\Big[\int_{\mathbb{R}}(\mathrm{i}\theta)^\delta \mathrm{e}^{\mathrm{i}\theta}(\mu_0^2+p(\,\cdot\,,\theta))^{-1}\mathcal{F}_2\tilde{\mathcal{E}}F(\,\cdot\,,\theta)\,d\theta\Big](x), \quad x \in \mathbb{R}.\end{aligned}$$

Given $\eta,\theta \in \mathbb{R}$, we compute

$$\begin{aligned}\mathcal{F}_2\tilde{\mathcal{E}}F(\eta,\theta) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}x\eta}\mathrm{e}^{-\mathrm{i}y\theta}\tilde{\mathcal{E}}F(x,y)\,dx\,dy \\ &= \frac{1}{\sqrt{2\pi}}\Big(\int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}y\theta}\mathcal{F}F(\eta,1+y)\,dy + \int_0^\infty \mathrm{e}^{-\mathrm{i}y\theta}\mathcal{F}F(\eta,1-y)\,dy\Big) \\ &= \frac{1}{\sqrt{2\pi}}\Big(\int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}y\theta}\mathcal{F}F(\eta,1+y)\,dy - \int_0^{-\infty} \mathrm{e}^{\mathrm{i}y\theta}\mathcal{F}F(\eta,1+y)\,dy\Big) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 (\mathrm{e}^{-\mathrm{i}y\theta}+\mathrm{e}^{\mathrm{i}y\theta})\mathcal{F}F(\eta,1+y)\,dy.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathcal{CR}_{\pi,1}F \\ &= \frac{1}{2\pi}\mathcal{F}^{-1}\Big[\int_{-\infty}^0\int_{\mathbb{R}}(\mu_0^2+p(\,\cdot\,,\theta))^{-1}(\mathrm{i}\theta)^\delta\big(\mathrm{e}^{\mathrm{i}(1-y)\theta}+\mathrm{e}^{\mathrm{i}(1+y)\theta}\big)\,d\theta\,\mathcal{F}F(\,\cdot\,,1+y)\,dy\Big].\end{aligned}$$

A short calculation shows that $\mu_0^2 + p(\eta,\theta) = a_{22}(d(\eta,\mu_0)^2 + (\theta + a(\eta))^2)$. Hence, making the change of variables $\tau := \theta + a(\eta)$ yields for $y \in (-\infty,0)$

$$\begin{aligned}&\int_{\mathbb{R}}(\mu_0^2+p(\eta,\theta))^{-1}\mathrm{e}^{\mathrm{i}(1-y)\theta}\,d\theta \\ &= \frac{1}{a_{22}d(\eta,\mu_0)}\int_{\mathbb{R}}\frac{d(\eta,\mu_0)}{d(\eta,\mu_0)^2+\tau^2}\mathrm{e}^{\mathrm{i}(\tau-a(\eta))(1-y)}\,d\tau \\ &= \frac{\mathrm{e}^{-\mathrm{i}a(\eta)(1-y)}}{a_{22}d(\eta,\mu_0)}\pi\mathrm{e}^{-d(\eta,\mu_0)|1-y|} = \frac{\pi}{a_{22}d(\eta,\mu_0)}\mathrm{e}^{-\lambda(\eta,\mu_0)(1-y)}.\end{aligned}$$

Analogously we obtain

$$\int_{\mathbb{R}}(\mu_0^2+p(\eta,\theta))^{-1}\mathrm{e}^{\mathrm{i}(1+y)\theta}\,d\theta = \frac{\mathrm{e}^{-\mathrm{i}a(\eta)(1+y)}}{a_{22}d(\eta,\mu_0)}\pi\mathrm{e}^{-d(\eta,\mu_0)|1+y|}.$$

However, we now distinguish between the cases $y \in [-1,0)$ and $y \in (-\infty,-1)$. To be precise we have

$$\int_{\mathbb{R}} (\mu_0^2 + p(\eta,\theta))^{-1} \mathrm{e}^{\mathrm{i}(1+y)\theta}\, d\theta = \frac{\pi}{a_{22} d(\eta,\mu_0)} \begin{cases} \mathrm{e}^{-\lambda(\eta,\mu_0)(1+y)}, & y \in [-1,0) \\ \mathrm{e}^{\tilde{\lambda}(\eta,\mu_0)(1+y)}, & y \in (-\infty,-1). \end{cases}$$

Next,

$$\begin{aligned}
&\int_{\mathbb{R}} (\mu_0^2 + p(\eta,\theta))^{-1} \mathrm{i}\theta \mathrm{e}^{\mathrm{i}(1-y)\theta}\, d\theta \\
&= -\frac{1}{a_{22}} \int_{\mathbb{R}} \frac{-\mathrm{i}\theta}{d(\eta,\mu_0)^2 + \tau^2} \mathrm{e}^{\mathrm{i}(\tau - a(\eta))(1-y)}\, d\tau \\
&= -\frac{\mathrm{e}^{-\mathrm{i}a(\eta)(1-y)}}{a_{22}} \int_{\mathbb{R}} \Big(\frac{-\mathrm{i}\tau}{d(\eta,\mu_0)^2 + \tau^2} \mathrm{e}^{\mathrm{i}\tau(1-y)} + \frac{\mathrm{i}a(\eta)}{d(\eta,\mu_0)} \frac{d(\eta,\mu_0)}{d(\eta,\mu_0)^2 + \tau^2} \mathrm{e}^{\mathrm{i}\tau(1-y)} \Big)\, d\tau \\
&= -\frac{\mathrm{e}^{-\mathrm{i}a(\eta)(1-y)}}{a_{22}} \Big(\pi \mathrm{e}^{-d(\eta,\mu_0)|1-y|} \operatorname{sgn}(1-y) + \frac{\mathrm{i}a(\eta)}{d(\eta,\mu_0)} \pi \mathrm{e}^{-d(\eta,\mu_0)|1-y|} \Big) \\
&= -\frac{\pi \mathrm{e}^{-\mathrm{i}a(\eta)(1-y)} \mathrm{e}^{-d(\eta,\mu_0)(1-y)}}{a_{22} d(\eta,\mu_0)} (d(\eta,\mu_0) + \mathrm{i}a(\eta)) \\
&= -\frac{\pi}{a_{22} d(\eta,\mu_0)} \lambda(\eta,\mu_0) \mathrm{e}^{-\lambda(\eta,\mu_0)(1-y)},
\end{aligned}$$

where again $y \in (-\infty, 0)$. In the same way we finally get

$$\begin{aligned}
&\int_{\mathbb{R}} (\mu_0^2 + p(\eta,\theta))^{-1} \mathrm{i}\theta \mathrm{e}^{\mathrm{i}(1+y)\theta}\, d\theta \\
&= -\frac{\pi \mathrm{e}^{-\mathrm{i}a(\eta)(1+y)} \mathrm{e}^{-d(\eta,\mu_0)|1+y|}}{a_{22} d(\eta,\mu_0)} \big(d(\eta,\mu_0) \operatorname{sgn}(1+y) + \mathrm{i}a(\eta) \big) \\
&= \frac{\pi}{a_{22} d(\eta,\mu_0)} \begin{cases} -\lambda(\eta,\mu_0) \mathrm{e}^{-\lambda(\eta,\mu_0)(1+y)}, & y \in [-1,0) \\ \tilde{\lambda}(\eta,\mu_0) \mathrm{e}^{\tilde{\lambda}(\eta,\mu_0)(1+y)}, & y \in (-\infty,-1). \end{cases}
\end{aligned}$$

These four computations show that

$$\begin{aligned}
&\int_{-\infty}^{0} \int_{\mathbb{R}} (\mu_0^2 + p(\eta,\theta))^{-1} (\mathrm{i}\theta)^{\delta} \big(\mathrm{e}^{\mathrm{i}(1-y)\theta} + \mathrm{e}^{\mathrm{i}(1+y)\theta} \big)\, d\theta\, \mathcal{F}F(\eta, 1+y)\, dy \\
&= \frac{\pi}{a_{22} d(\eta,\mu_0)} \Big(\int_{-\infty}^{0} (-\lambda(\eta,\mu_0))^{\delta} \mathrm{e}^{-\lambda(\eta,\mu_0)(1-y)} \mathcal{F}F(\eta,1+y)\, dy \\
&\qquad + \int_{-\infty}^{-1} \tilde{\lambda}(\eta,\mu_0)^{\delta} \mathrm{e}^{\tilde{\lambda}(\eta,\mu_0)(1+y)} \mathcal{F}F(\eta,1+y)\, dy \\
&\qquad + \int_{-1}^{0} (-\lambda(\eta,\mu_0))^{\delta} \mathrm{e}^{-\lambda(\eta,\mu_0)(1+y)} \mathcal{F}F(\eta,1+y)\, dy \Big).
\end{aligned}$$

Consequently,

$$\mathcal{C}\mathcal{R}_{\pi,1} F = -\mathcal{F}^{-1} \Big[\frac{1}{2a_{22} d(\,\cdot\,,\mu_0)} \Big(\Big(\frac{\lambda(\,\cdot\,,\mu_0)}{\tilde{\lambda}(\,\cdot\,,\mu_0)} \Big)^{\delta} (k_{F,1} + k_{F,2}) + k_{F,3} \Big) \Big].$$

The application of $\mathcal{U}_\pi$ to both sides of this equation and the definition of the operator $\mathcal{R}_{\pi,2}$ yield the third assertion.

The second assertion now follows from Proposition VI.8, Lemma VI.9, and the facts that $\mathcal{C} \in \mathcal{L}(h^{2+\alpha}_{\tilde{\mathbb{H}}}, h^{2-\delta+\alpha}_{\mathbb{R}})$ and $\mathcal{S}(\tilde{\mathbb{H}}) \overset{d}{\hookrightarrow} h^{\alpha}_{\tilde{\mathbb{H}}}$. This completes the proof. □

For the formulation of the following second main result of this paragraph put $\mathcal{R}_\pi := \mathcal{R}_{\pi,1} + \mathcal{R}_{\pi,2}$.

Proposition VI.11. *It holds that $\mathcal{R}_\pi \in \mathcal{L}(h^{\alpha}_{\tilde{\mathbb{H}}}, h^{2+\alpha}_{\tilde{\mathbb{H}}})$. Furthermore, given $F \in h^{\alpha}_{\tilde{\mathbb{H}}}$, the boundary value problem*

$$(\mu_0^2 + \mathcal{A}_\pi)u = F \quad \text{in} \quad \tilde{\mathbb{H}}, \quad \mathcal{C}u = 0 \quad \text{on} \quad \mathbb{R}$$

has a unique solution in $h^{2+\alpha}_{\tilde{\mathbb{H}}}$. It is given by $u = \mathcal{R}_\pi F$.

Proof. The first assertion follows immediately from Lemmas VI.9 and VI.10. Another consequence of Lemma VI.9 is $(\mu_0^2 + \mathcal{A}_\pi)\mathcal{R}_{\pi,1} = \mathrm{id}_{h^{\alpha}_{\tilde{\mathbb{H}}}}$. Moreover, from Lemma VI.8 we infer that $(\mu_0^2 + \mathcal{A}_\pi)\mathcal{U}_\pi = 0$ and that $\mathcal{C}\mathcal{U}_\pi = \mathrm{id}_{h^{2-\delta+\alpha}_{\mathbb{R}}}$. Hence, given $F \in h^{\alpha}_{\tilde{\mathbb{H}}}$, it follows that

$$\begin{aligned}(\mu_0^2 + \mathcal{A}_\pi)\mathcal{R}_\pi F &= (\mu_0^2 + \mathcal{A}_\pi)\mathcal{R}_{\pi,1}F - (\mu_0^2 + \mathcal{A}_\pi)\mathcal{U}_\pi\mathcal{C}\mathcal{R}_{\pi,1}F = F,\\ \mathcal{C}\mathcal{R}_\pi F &= \mathcal{C}\mathcal{R}_{\pi,1}F - \mathcal{C}\mathcal{U}_\pi\mathcal{C}\mathcal{R}_{\pi,1}F = 0,\end{aligned}$$

which implies that $u = \mathcal{R}_\pi F$ is a solution of the boundary value problem. Its uniqueness can again be proved as in the proof of Proposition VI.8. □

Given $\alpha_* > 0$ and $k \in \mathbb{N}$, let us introduce the class of smooth elliptic symbols being homogeneous of degree k:

$$\begin{aligned}\mathcal{E}ll\mathcal{S}^\infty_k(\alpha_*) := \{a \in C^\infty(\mathbb{R} \times (0,\infty), \mathbb{C})\colon\ & a(\omega\eta, \omega\mu) = \omega^k a(\eta,\mu), \quad \omega > 0,\\ & \text{all derivatives of } a \text{ are bounded on } |(\eta,\mu)| = 1,\\ & \operatorname{Re} a(\eta,\mu) \geq \alpha_* |(\eta,\mu)|^k, \quad (\eta,\mu) \in \mathbb{R} \times (0,\infty)\}.\end{aligned} \tag{VI.9}$$

It turns out (cf. [34, Appendix A]) that for elliptic symbols of this type the corresponding Fourier operators are linear operators between (little) Hölder spaces. Even more, they generate analytic semigroups on these spaces.

Theorem VI.12. *Suppose that $\alpha_*, \mu_* > 0$ and that $a \in \mathcal{E}ll\mathcal{S}^\infty_k(\alpha_*)$ for some $k \in \mathbb{N}$. Then for $l \in \mathbb{Z}$ and $\alpha \in (0,1)$ it holds that*

$$\mathcal{F}^{-1}a(\,\cdot\,,\mu)\mathcal{F} \in \mathcal{H}(h^{l+k+\alpha}_{\mathbb{R}}, h^{l+\alpha}_{\mathbb{R}}) \quad \text{for all} \quad \mu \geq \mu_*.$$

Moreover, these semigroups can be written down explicitly. To do this we need more information about Fourier multiplier operators on intermediate spaces between $\mathcal{S}(\mathbb{R})$ and its dual space $\mathcal{S}(\mathbb{R})'$—the space of tempered distributions on $\mathbb{R}$. For such an intermediate space E define the set M_E of all Fourier multipliers for E as the set of all tempered distributions $a \in \mathcal{S}(\mathbb{R})'$ such that the linear operator $\mathcal{F}^{-1}a\mathcal{F}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})'$ has an extension operator in $\mathcal{L}(E)$. Its norm is denoted by $\|a\|_{M_E} := \|\mathcal{F}^{-1}a\mathcal{F}\|_{\mathcal{L}(E)}$. Because the space M_E is rarely known for concrete cases, we have to content ourselves with a subspace of M_E that is, however, large enough to satisfy our needs. Define

$$\mathcal{M} := \{a \in W^1_\infty(\mathbb{R})\colon \sup_{\eta\in\mathbb{R}} \sqrt{1+\eta^2}|\partial a(\eta)| < \infty\}$$

with norm

$$\|a\|_{\mathcal{M}} := \max\{\|a\|_{\infty,\mathbb{R}}, \|[\eta \mapsto \sqrt{1+\eta^2}\partial a(\eta)]\|_{\infty,\mathbb{R}}\}, \quad a \in \mathcal{M}.$$

In [34, Appendix A] it is shown that for $k \in \mathbb{Z}$, $\alpha \in (0,1)$ it holds that $\mathcal{M} \hookrightarrow M_{h^{k+\alpha}_{\mathbb{R}}}$ and that this fact implies the following result.

Theorem VI.13. *Suppose that* $\alpha_*, \mu_* > 0$ *and that* $a \in \mathcal{E}ll\mathcal{S}^\infty_1(\alpha_*)$. *Then*

$$\mathrm{e}^{-a(\,\cdot\,,\mu)t} \in \mathcal{M} \quad \text{and} \quad \mathrm{e}^{-t\mathcal{F}^{-1}a(\,\cdot\,,\mu)\mathcal{F}} = \mathcal{F}^{-1}\mathrm{e}^{-a(\,\cdot\,,\mu)t}\mathcal{F}$$

for $t \geq 0$ *and* $\mu \geq \mu_*$.

Put $b_\pi := b_\pi(f, x_0) := b'(f)(x_0)$ and define the symbols

$$\begin{aligned} a_{1,1}(\eta,\mu_0) &:= \mathrm{i}b_1\eta - b_2\lambda(\eta,\mu_0),\\ a_{1,2}(\eta,\mu_0) &:= b_\pi(\mathrm{i}b_1\eta + b_2\tilde\lambda(\eta,\mu_0))\mathrm{e}^{-\tilde\lambda(\eta,\mu_0))}/\tilde\lambda(\eta,\mu_0)^\delta, \end{aligned}$$

where $\eta \in \mathbb{R}$. Put $A_{\pi,1,i} := \mathcal{F}^{-1}a_{1,i}(\,\cdot\,,\mu_0)\mathcal{F}$ for $i \in \{1,2\}$. Lemma 5.1 in [34] shows that $-A_{\pi,1,1} \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$. Moreover, $a_{1,2}(\,\cdot\,,\mu_0) \in \mathcal{M}$ as the following computations show, where we use (VI.8):

$$\begin{aligned} |a_{1,2}(\eta,\mu_0)|^2 &= \frac{b_\pi^2\mathrm{e}^{-2\tilde d(\eta,\mu_0)}}{|\tilde\lambda(\eta,\mu_0)|^{2\delta}}\big((b_1\eta + b_2\tilde a(\eta))^2 + b_2^2\tilde d(\eta,\mu_0)^2\big)\\ &\leq \frac{b_\pi^2}{|\tilde\lambda(\eta,\mu_0)|^{2\delta}}\big((b_1\eta + b_2\tilde a(\eta))^2 + b_2^2\tilde d(\eta,\mu_0)^2\big)\mathrm{e}^{-2\alpha_*|\eta|},\\ |\partial_\eta a_{1,2}(\eta,\mu_0)|^2 &= b_\pi^2\Big|(\mathrm{i}b_1 + b_2\partial_\eta\tilde\lambda(\eta,\mu_0))\frac{\mathrm{e}^{-\tilde\lambda(\eta,\mu_0)}}{\tilde\lambda(\eta,\mu_0)^\delta} - \frac{(\mathrm{i}b_1\eta + b_2\tilde\lambda(\eta,\mu_0))}{|\tilde\lambda(\eta,\mu_0)|^{2\delta}}\\ &\qquad \times \big(\partial_\eta\tilde\lambda(\eta,\mu_0)\mathrm{e}^{-\tilde\lambda(\eta,\mu_0)}\tilde\lambda(\eta,\mu_0)^\delta + \delta\partial_\eta\tilde\lambda(\eta,\mu_0)\mathrm{e}^{-\tilde\lambda(\eta,\mu_0)}\big)\Big|^2\\ &\leq b_\pi^2\Big|\frac{1}{\tilde\lambda(\eta,\mu_0)^\delta}(\mathrm{i}b_1 + b_2\partial_\eta\tilde\lambda(\eta,\mu_0))\\ &\qquad - \frac{(\mathrm{i}b_1\eta + b_2\tilde\lambda(\eta,\mu_0))(\tilde\lambda(\eta,\mu_0)^\delta + \delta)}{|\tilde\lambda(\eta,\mu_0)|^{2\delta}}\partial_\eta\tilde\lambda(\eta,\mu_0)\Big|^2\mathrm{e}^{-2\alpha_*|\eta|}. \end{aligned}$$

Consequently, it holds that

$$\sup_{\eta\in\mathbb{R}}|a_{1,2}(\eta,\mu_0)| < \infty \quad \text{and} \quad \sup_{\eta\in\mathbb{R}}\sqrt{1+\eta^2}|\partial_\eta a_{1,2}(\eta,\mu_0)| < \infty,$$

i.e., $a_{1,2}(\,\cdot\,,\mu_0) \in \mathcal{M}$ as claimed. Finally, we infer from $\mathcal{M} \hookrightarrow h^{1+\alpha}_{\mathbb{R}}$ that there is an extension operator of $-A_{\pi,1,2} = \mathcal{F}^{-1}[-a_{1,2}(\,\cdot\,,\mu_0)]\mathcal{F}\colon \mathcal{S} \to \mathcal{S}'$ in $\mathcal{L}(h^{1+\alpha}_{\mathbb{R}})$ denoted with the same symbol. To conclude this paragraph note that $A_{\pi,1,1} + A_{\pi,1,2} = \mathcal{B}_\pi(\mathcal{T}_\pi + \mathcal{U}_\pi b_\pi)$ and that this operator can be considered as the principal part of the operator $\mathcal{B}(f)\big(\mathcal{T}(f) + \mathcal{U}(f)b'(f)\big)$ with coefficients fixed in f and x_0.

Recall that $v_f = \mathcal{T}(f)f + \mathcal{U}(f)b(f)$ and set

$$c_\pi := \partial_1 v_f(x_0,0) + \frac{2f_x(x_0)}{c+f(x_0)}\partial_2 v_f(x_0,0). \tag{VI.10}$$

Moreover, put $a_2(\eta) := \mathrm{i}c_\pi\eta$, $\eta \in \mathbb{R}$, and $A_{\pi,2} := c_\pi\partial$. Then $A_{\pi,2} = \mathcal{F}^{-1}a_2\mathcal{F} \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$ is the Fourier operator associated to $\partial\mathcal{B}(f)[\,\cdot\,,v_f]$ that may be considered as its principal part with coefficients fixed in f and x_0, cf. Lemma VI.3.

Finally, let $A_{\pi,3}$ be the Fourier operator associated $-\mathcal{B}(f)\mathcal{S}(f)\mathcal{P}(f)$ that can be considered as its principal part with coefficients fixed in f and x_0. To determine its symbol let

$$w_f(x,y) := -\frac{1-y}{c+f(x)}\partial_2 v_f(x,y), \quad (x,y) \in \Omega, \tag{VI.11}$$

and put $w_\pi := w_f(x_0,0)$, see Lemma VI.3. Moreover, for $h \in h^{2+\alpha}_{\mathbb{R}}$ let

$$\mathcal{P}_\pi h := \mathcal{P}_\pi(f,x_0)h := [y \mapsto w_\pi h_{xx}\mathrm{e}^{-y}], \quad y > 0.$$

We define the symbol

$$a_{3,1}(\eta,\mu) := \frac{\mathrm{i}b_2}{\sqrt{2\pi}}\eta^2\int_{\mathbb{R}}\frac{\theta w_\pi[\theta \mapsto \mathcal{F}\mathrm{e}^{-|y|}]}{\mu^2 + p(\eta,\theta)}d\theta, \quad \eta \in \mathbb{R}, \quad \mu > 0,$$

and the operator $A_{\pi,3,1} := -b_2\gamma_0\partial_2\mathcal{S}_{\pi,1}\mathcal{P}_\pi$. Let $\beta \in (0,\alpha)$. Then it can be shown that $A_{\pi,3,1} = \mathcal{F}^{-1}a_{3,1}(\,\cdot\,,\mu_0)\mathcal{F} \in \mathcal{L}(h^{2+\gamma}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$ for $\gamma \in [\beta,\alpha]$, see [34, Lemma 5.3]. Introducing the operator $A_{\pi,3,2} := -b_2\gamma_0\partial_2\mathcal{S}_{\pi,2}\mathcal{P}_\pi$ with symbol

$$a_{3,2}(\eta,\mu_0) := \frac{b_2 w_\pi}{a_{22}}\eta^2\frac{\lambda(\eta,\mu_0)}{d(\eta,\mu_0)}\frac{d(\eta,\mu_0)+1}{|\lambda(\eta,\mu_0)+1|^2}, \quad \eta \in \mathbb{R},$$

we deduce from [34, Corollary 5.5] that $A_{\pi,3} := A_{\pi,3,1} + A_{\pi,3,2} = -\mathcal{B}_\pi\mathcal{S}_\pi\mathcal{P}_\pi \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$ is the Fourier operator we look for having the symbol $a_{3,1}(\,\cdot\,,\mu_0) + a_{3,2}(\,\cdot\,,\mu_0)$.

For $t \in [0,1]$, $\eta \in \mathbb{R}$, and $\mu > 0$ put

$$a_{\pi,t}(\eta,\mu) := -(b_1 + tc_\pi)\mathrm{i}\eta + b_2\lambda(\eta,\mu)\Big(1 - t\frac{w_\pi}{a_{22}}\eta^2\frac{\mu_0^2 d(\eta,\mu) + \mu_0\mu}{d(\eta,\mu)|\mu_0\lambda(\eta,\mu)+\mu|^2}\Big).$$

Given $t \in [0,1]$, $a_{\pi,t}(\,\cdot\,,\mu_0)$ is obviously the symbol of the operator $-A_{\pi,1,1} - t(A_{\pi,2} + A_{\pi,3,2})$.

Theorem VI.14. *Suppose that $w_\pi < \alpha_0/a_{22}$. Then the operator*

$$-A_\pi := A_{\pi,1,1} + A_{\pi,1,2} + A_{\pi,2} + A_{\pi,3,1} + A_{\pi,3,2}$$

generates an analytic semigroup in $h^{1+\alpha}_{\mathbb{R}}$, i.e., $A_\pi \in \mathcal{H}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$.

Proof. As in the proof of Lemma 5.6 in [34] we can prove the existence of a positive constant α_π such that $a_{\pi,t} \in \mathcal{E}ll\mathcal{S}_1^\infty(\alpha_\pi)$ for all $t \in [0,1]$ provided $w_\pi < \alpha_0/a_{22}$. Then Theorem VI.12 implies that

$$-A_{\pi,1,1} - A_{\pi,2} - A_{\pi,3,2} \in \mathcal{H}(h_{\mathbb{R}}^{2+\alpha}, h_{\mathbb{R}}^{1+\alpha}).$$

The operators $-A_{\pi,1,2}$ and $-A_{\pi,3,1}$ are considered as perturbations of this operator. We know that $-A_{\pi,1,2} \in \mathcal{L}(h_{\mathbb{R}}^{1+\alpha})$. Thus there is a positive constant C such that

$$\| - A_{\pi,1,2}h\|_{1+\alpha,\mathbb{R}} \leq C\|h\|_{1+\alpha,\mathbb{R}} \quad \text{for all} \quad h \in h_{\mathbb{R}}^{2+\alpha}. \tag{VI.12}$$

From $-A_{\pi,3,1} \in \mathcal{L}(h_{\mathbb{R}}^{2+\beta}, h_{\mathbb{R}}^{1+\alpha})$, $\beta \in (0,\alpha)$, and the interpolation property $h_{\mathbb{R}}^{2+\beta} = (h_{\mathbb{R}}^{1+\alpha}, h_{\mathbb{R}}^{2+\alpha})_{1+\beta-\alpha}$ for the continuous interpolation method we infer

$$\| - A_{\pi,3,1}h\|_{1+\alpha,\mathbb{R}} \leq C\|h\|_{2+\beta,\mathbb{R}} \leq \varepsilon\|h\|_{2+\alpha,\mathbb{R}} + \tilde{C}\|h\|_{1+\alpha,\mathbb{R}} \quad \text{for all} \quad h \in h_{\mathbb{R}}^{2+\alpha}$$

with arbitrary $\varepsilon > 0$ and positive constants C and $\tilde{C} = \tilde{C}(\varepsilon)$. Now Theorem III.9 (ii) implies the assertion. □

Note that the hypothesis in Theorem VI.14 is independent of μ_0. The set $\mathcal{W}$ from Theorem VI.1 is defined in such a way that, given $c + f \in \mathcal{W}$, this hypothesis is satisfied for all $x \in \mathbb{S}$.

Let $\tilde{f} := f + c \in h_+^{2+\alpha}$. Recall $\alpha_0 = \big(1 + \tilde{f}(x_0)^2 + f_x(x_0)^2\big)^{-1}$ and $a_{22} = a_{22}(x_0, 0) = (1 + f_x(x_0)^2)/\tilde{f}(x_0)^2$ so that

$$\lambda_{\tilde{f}}(x_0) = \frac{\tilde{f}(x_0)^2}{(1 + \tilde{f}(x_0)^2 + \tilde{f}_x(x_0)^2)(1 + \tilde{f}_x(x_0)^2)} = \frac{\alpha_0}{a_{22}}.$$

Recall $v_f = \mathcal{T}(f)f + \mathcal{U}(f)b(f)$ and consider the set

$$\{\tilde{f} \in h_+^{2+\alpha} \colon \min_{x \in \mathbb{S}} \big(\partial_2 v_f(x,0)/\tilde{f}(x) + \lambda_{\tilde{f}}(x)\big) > 0\}.$$

Observe that—according to the coordinate transformation induced by the diffeomorphism ϕ_f—this set coincides with the set $\mathcal{W}$ from (VI.2). Now, given $\tilde{f} := f + c \in \mathcal{W}$, we have $-\partial_2 v_f(x,0)/\tilde{f}(x) < \lambda_{\tilde{f}}(x)$ for all $x \in \mathbb{S}$. Recall $w_f(x,y) = -(1-y)\partial_2 v_f(x,y)/\tilde{f}(x)$, $(x,y) \in \Omega$, and $w_\pi = w_f(x_0,0)$, $x_0 \in \mathbb{S}$. Hence, $w_\pi < \alpha_0/a_{22}$ holds for all $x \in \mathbb{S}$ provided that $\tilde{f} = f + c \in \mathcal{W}$. Setting $\beta_c := b(0) = (-c)^\delta \tilde{b}(c) - (1-\delta)c$ we have the following result.

Lemma VI.15. *Suppose that $c > 0$ and that $\beta_c > -c^3/(1+c^2)$. Then $\mathcal{W}$ is an open neighborhood of c with unbounded diameter in $h^{2+\alpha}$.*

Here, the diameter of a subset $X \subset h^{2+\alpha}$ is defined as

$$\operatorname{diam}_{2+\alpha} X := \sup_{g,h \in X} \|g - h\|_{2+\alpha}.$$

Proof. Obviously, $0 < c \in h_+^{2+\alpha}$ and $\lambda_c = c^2/(1+c^2)$. Moreover, the function $v_0 = \mathcal{U}(0)\beta_c$ is the unique solution of the boundary value problem

$$\mathcal{A}(0)v = 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \Gamma_0, \quad \mathcal{C}v = \beta_c \quad \text{on} \quad \Gamma_1.$$

It follows that $v_0(x,y) = \beta_c y$, $(x,y) \in \Omega$, and consequently $\partial_2 v_0(x,0) = \beta_c$, $x \in \mathbb{S}$. Hence, $c \in \mathcal{W}$ provided $\beta_c > -c^3/(1+c^2)$. Now the assertion follows along the lines of the proof of Lemma 5.10 in [34]. □

VI.1.3 Proof of Theorem VI.1

In this section we carry over the generation property of the operator A_π to the whole linearization $-\partial\Phi(f)$, $f + c \in \mathcal{W}$, of the evolution operator. To do this we first introduce some notation. We define the operators

$$\begin{aligned} \partial\Phi_t(f) &:= \mathcal{B}(f)\mathcal{T}(f) + t\big(\partial\Phi(f) - \mathcal{B}(f)\mathcal{T}(f)\big), \\ A_{\pi,t} &:= \mathcal{F}^{-1}\big[a_{\pi,t}(\,\cdot\,,\mu_0) - t\big(a_{\pi,1,2}(\,\cdot\,,\mu_0) + a_{\pi,3,1}(\,\cdot\,,\mu_0)\big)\big]\mathcal{F}, \end{aligned}$$

where $t \in [0,1]$, $\mu_0 > 0$, and $f \in \mathcal{V}$. Moreover, given $\rho > 0$, we need the notion of a ρ-localization sequence $\{(U_j,\theta_j) \colon 1 \le j \le m_\rho\}$ of $\mathbb{S}$. Here $\{U_j \colon 1 \le j \le m_\rho\}$ is an open covering of the strip $\mathbb{S} \times (-\rho/2, \rho/2)$ and for all $1 \le j \le m_\rho$ it holds that $\operatorname{diam}(U_j) \le \rho$ and $U_j \cap \mathbb{R} \ne \emptyset$. Furthermore, $\theta_j \in \mathcal{D}(U_j)$, $1 \le j \le m_\rho$, are smooth test functions on U_j and $\{(U_j,\theta_j) \colon 1 \le j \le m_\rho\}$ is a partition of unity. For each $1 \le j \le m_\rho$ we fix $(x_j, 0) \in U_j$. The fundamental tool in this section is the following result.

Lemma VI.16. *Let $\kappa > 0$, $\beta \in (0,\alpha)$ and suppose that $K \subset \mathcal{W}$ is compact. Then there exists $\rho \in (0,1]$, a ρ-localization sequence $\{(U_j,\theta_j) \colon 1 \le j \le m_\rho\}$ for $\mathbb{S}$, and a positive constant C depending on K, κ, and ρ such that*

$$\|(A_{\pi,t}(f,x_j)\theta_j + \theta_j\partial\Phi_t(f))h\|_{1+\alpha,\mathbb{R}} \le \kappa\|\theta_j h\|_{2+\alpha,\mathbb{R}} + C\|h\|_{2+\beta}$$

for all $h \in h^{2+\alpha}$, $1 \le j \le m_\rho$, $t \in [0,1]$, and $f \in K$.

To prove Lemma VI.16 we split the operator $A_{\pi,t}(f,x_j)\theta_j + \theta_j\partial\Phi_t(f)$ into several expressions each of which we study independently. Let us begin with characterizing the cut-off functions that we frequently use in the sequel. Given $\rho \in (0,1]$ and a ρ-localization sequence $\{(U_j,\theta_j) \colon 1 \le j \le m_\rho\}$ for $\mathbb{S}$, we define for each $1 \le j \le m_\rho$ functions

$$\chi_j, \psi_j \in \mathcal{D}(U_j) \quad \text{such that} \quad \chi_j | \operatorname{supp}\theta_j = 1 \quad \text{and} \quad \psi_j | \operatorname{supp}\chi_j = 1. \tag{VI.13}$$

These functions can be chosen in such a way that there is a $C > 0$ independent of ρ such that

$$\|\chi_j\|_{0,U_j} + \|\psi_j\|_{0,U_j} + \rho^\alpha\big([\chi_j]_{\alpha,U_j} + [\psi_j]_{\alpha,U_j}\big) \le C. \tag{VI.14}$$

Furthermore, there is a constant $Z = Z(\rho)$ with

$$\|\chi_j\|_{2+\alpha,U_j} + \|\psi_j\|_{2+\alpha,U_j} + \|\theta_j\|_{2+\alpha,U_j} \le Z. \tag{VI.15}$$

By restriction a function $\theta \in C^\infty(\mathbb{R}^2)$ with $\operatorname{diam}\operatorname{supp}\theta < 2\pi$ induces a pointwise multiplier on the spaces h^s and $buc^s(\Omega)$, $s > 0$. In turn, for $1 \le j \le m_\rho$ a function $\theta \in \mathcal{D}(U_j)$ can be identified with the function $\theta_* \in C^\infty(\mathbb{R}^2)$, where $\theta_*|U_j = \theta$ and $\theta_* = 0$ otherwise. With this convention we can define the commutator $[A, \theta] := A\theta - \theta A$, where A belongs to $\mathcal{L}(h^s, buc^t(\Omega))$, $\mathcal{L}(h^s, h^t)$, $\mathcal{L}(buc^s(\Omega), buc^t(\Omega))$, or $\mathcal{L}(buc^s(\Omega), h^t)$ for $s, t > 0$.

In the sequel—for the sake of notational simplicity—we drop the dependence of the operators on $f \in \mathcal{V}$, $x_j \in \mathbb{S}$, $1 \le j \le m_\rho$, and $t \in [0, 1]$ from our notation.

Lemma VI.17. *Let $\theta \in C^\infty(\mathbb{R}^2)$ with* $\operatorname{diam}\operatorname{supp}\theta < 2\pi$ *and $\|\theta\|_{2+\alpha,\mathbb{R}^2} \le Z$. The commutators*

$$
\begin{aligned}
&[\theta, \mathcal{A}_\pi] \in \mathcal{L}(h^{2+\alpha}_{\mathbb{H}}, h^{\alpha}_{\mathbb{H}}), \quad [\theta, \mathcal{B}_\pi] \in \mathcal{L}(h^{2+\alpha}_{\mathbb{H}}, h^{1+\alpha}_{\mathbb{R}}),\\
&[\theta, \mathcal{P}_\pi] \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{\alpha}_{\mathbb{H}}), \quad [\theta, A_{\pi,2}] \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})
\end{aligned}
$$

are extendable to $\mathcal{L}(h^{1+\alpha}_{\mathbb{H}}, h^{\alpha}_{\mathbb{H}})$, $\mathcal{L}(h^{1+\alpha}_{\mathbb{H}}, h^{1+\alpha}_{\mathbb{R}})$, $\mathcal{L}(h^{1+\alpha}_{\mathbb{R}}, h^{\alpha}_{\mathbb{H}})$, and $\mathcal{L}(h^{1+\alpha}_{\mathbb{R}})$, respectively.

Proof. Performing all the differentiations using Leibniz' rule, we see that all terms of highest order drop out of the expressions. Since $h^{2+\alpha}_{\mathbb{H}} \hookrightarrow h^{1+\alpha}_{\mathbb{H}}$ and $h^{2+\alpha}_{\mathbb{R}} \hookrightarrow h^{1+\alpha}_{\mathbb{R}}$, the assertions follow. □

Lemma VI.18. *Let $K \subset \mathcal{V}$ be compact and let θ be as in the preceding lemma. Moreover, let $f \in K$ and $\beta \in (0, \alpha)$ be given. The commutators*

$$
[\theta, \mathcal{A}] \in \mathcal{L}(buc^{2+\alpha}(\Omega), buc^{\alpha}(\Omega)), \quad [\theta, \mathcal{B}] \in \mathcal{L}(buc^{2+\alpha}(\Omega), h^{1+\alpha})
$$

are extendable to operators in $\mathcal{L}(buc^{2+\beta}(\Omega), buc^{\alpha}(\Omega))$ and $\mathcal{L}(buc^{2+\beta}(\Omega), h^{1+\alpha})$, respectively. Furthermore, the commutators

$$
[\theta, \mathcal{B}\mathcal{T}], [\theta, \mathcal{B}\mathcal{S}\mathcal{P}], [\theta, \partial\mathcal{B}[\,\cdot\,, v]] \in \mathcal{L}(h^{2+\alpha}, h^{1+\alpha})
$$

are extendable to operators in $\mathcal{L}(h^{2+\beta}, h^{1+\alpha})$ and the commutator

$$
[\theta, \mathcal{B}\mathcal{U}] \in \mathcal{L}(h^{2-\delta+\alpha}, h^{1+\alpha})
$$

has an extension in $\mathcal{L}(h^{2-\delta+\beta}, h^{1+\alpha})$.

Proof. Fix $\beta \in (0, \alpha)$ and let θ fulfill the hypothesis. As in the proof of the previous lemma it can be shown that $[\theta, \mathcal{A}]$, $[\theta, \mathcal{B}]$, and $[\theta, \mathcal{C}]$ are of lower order, i.e., there exist extension operators of them denoted with the same respective symbol such that

$$
\begin{aligned}
&[\theta, \mathcal{A}] \in \mathcal{L}(buc^{2+\beta}(\Omega), buc^{\alpha}(\Omega)), \quad [\theta, \mathcal{B}] \in \mathcal{L}(buc^{2+\beta}(\Omega), h^{1+\alpha}),\\
&[\theta, \mathcal{C}] \in \mathcal{L}(buc^{2+\beta}(\Omega), h^{2-\delta+\alpha}),
\end{aligned}
$$

which proves the first assertion. Note that also $[\theta, \mathcal{P}]$ and $[\theta, \partial\mathcal{B}[\,\cdot\,, v]]$ are of lower order in the same sense.

We have $(\mathcal{A},\gamma_0,\mathcal{C})[\theta,\mathcal{R}] = ([\mathcal{A},\theta],0,[\mathcal{C},\theta])\mathcal{R}$ and Theorem VI.5 gives the existence of a positive constant C such that

$$\|[\theta,\mathcal{R}](F,g,h)\|_{2+\alpha,\Omega} \le C\big(\|[\mathcal{A},\theta]\mathcal{R}(F,g,h)\|_{\alpha,\Omega} + \|[\mathcal{C},\theta]\mathcal{R}(F,g,h)\|_{2-\delta+\alpha}\big)$$

for all $F \in buc^\alpha(\Omega)$, $g \in h^{2+\alpha}$, and $h \in h^{2-\delta+\alpha}$. Moreover, it holds that

$$\mathcal{R} \in C\big(K, \mathcal{L}(buc^\beta(\Omega) \times h^{2+\beta} \times h^{2-\delta+\beta}, buc^{2+\beta}(\Omega))\big).$$

From these facts and the extension results for $[\theta,\mathcal{A}]$ and $[\theta,\mathcal{C}]$ we infer that

$$\|[\theta,\mathcal{R}](F,g,h)\|_{2+\alpha,\Omega} \le C\big(\|F\|_{\beta,\Omega} + \|g\|_{2+\beta} + \|h\|_{2-\delta+\beta}\big)$$

for all $F \in buc^\alpha(\Omega)$, $g \in h^{2+\alpha}$, and $h \in h^{2-\delta+\alpha}$. Particularly, this implies (for $F=0$, $g=0$, and $h=0$, respectively) that $[\theta,\mathcal{S}]$, $[\theta,\mathcal{T}]$, and $[\theta,\mathcal{U}]$ have extensions denoted with the same respective symbols such that

$$\begin{aligned}&[\theta,\mathcal{S}] \in \mathcal{L}(buc^\beta(\Omega), buc^{2+\alpha}(\Omega)), \quad [\theta,\mathcal{T}] \in \mathcal{L}(h^{2+\beta}, buc^{2+\alpha}(\Omega)),\\ &[\theta,\mathcal{U}] \in \mathcal{L}(h^{2-\delta+\beta}, buc^{2+\alpha}(\Omega)).\end{aligned}$$

These results, the identities

$$\begin{aligned}[\theta,\mathcal{B}\mathcal{T}] &= [\theta,\mathcal{B}]\mathcal{T} + \mathcal{B}[\theta,\mathcal{T}], \quad [\theta,\mathcal{B}\mathcal{U}] = [\theta,\mathcal{B}]\mathcal{U} + \mathcal{B}[\theta,\mathcal{U}],\\ [\theta,\mathcal{B}\mathcal{S}\mathcal{P}] &= [\theta,\mathcal{B}]\mathcal{S}\mathcal{P} + \mathcal{B}[\theta,\mathcal{S}]\mathcal{P} + \mathcal{B}\mathcal{S}[\theta,\mathcal{P}],\end{aligned}$$

and the fact that $[\theta,\mathcal{B}]$, $[\theta,\mathcal{P}]$, and $[\theta,\partial\mathcal{B}[\,\cdot\,,v]]$ are again of lower order imply the second assertion. □

Lemma VI.19. *Assume that θ,χ,ψ satisfy the hypothesis of θ from the preceding lemma such that $\chi|\operatorname{supp}\theta = 1$ and $\psi|\operatorname{supp}\chi = 1$. Then it holds that*

$$\begin{aligned}\chi\mathcal{T} - \mathcal{T}_\pi\chi &= \mathcal{S}_\pi\big([\mathcal{A}_\pi,\chi]\psi\mathcal{T} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{T} + \chi[\mathcal{A},\psi]\mathcal{T}\big),\\ (\chi\mathcal{U} - \mathcal{U}_\pi\chi)\theta &= \mathcal{R}_\pi([\mathcal{A}_\pi,\chi]\psi\mathcal{U} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{U} + \chi[\mathcal{A},\psi]\mathcal{U})\theta,\\ \chi\mathcal{S}\mathcal{P} - \mathcal{S}_\pi\mathcal{P}_\pi\chi &= \mathcal{S}_\pi\big([\mathcal{A}_\pi,\chi]\psi\mathcal{S}\mathcal{P} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{S}\mathcal{P} + \chi[\mathcal{A},\psi]\mathcal{S}\mathcal{P}\\ &\qquad + \chi(\mathcal{P} - \mathcal{P}_\pi) + [\chi,\mathcal{P}_\pi]\big).\end{aligned}$$

Proof. Let $u_1 := (\psi\chi\mathcal{T} - \mathcal{T}_\pi\psi\chi)h$ and $u_2 := (\psi\chi\mathcal{U} - \mathcal{U}_\pi\psi\chi)\theta h$ for $h \in h^{2+\alpha}$ and observe that $u_1 \in h^{2+\alpha}_{\mathbb{H}}$ and $u_2 \in h^{2+\alpha}_{\tilde{\mathbb{H}}}$. It holds that $\gamma_0 u_1 = 0$ thanks to $\gamma_0\mathcal{T} = \mathrm{id}_{h^{2+\alpha}}$ and $\gamma_0\mathcal{T}_\pi = \mathrm{id}_{h^{2+\alpha}_{\mathbb{R}}}$. Moreover, $\mathcal{C}\mathcal{U} = \mathrm{id}_{h^{2-\delta+\alpha}}$, $\mathcal{C}\mathcal{U}_\pi = \mathrm{id}_{h^{2-\delta+\alpha}_{\mathbb{R}}}$, and $\psi\chi = \chi$ imply that $\mathcal{C}u_2 = 0$ since (for $\delta = 0$) it holds that

$$\mathcal{C}u_2 = (\gamma_1\chi\mathcal{C}\mathcal{U} - \mathcal{C}\mathcal{U}_\pi\gamma_1\chi)\gamma_1(\theta h) = 0$$

and (in case $\delta = 1$)

$$\begin{aligned}\mathcal{C}u_2 &= \gamma_1\partial_2[\chi\mathcal{U} - \mathcal{U}_\pi\chi]\gamma_1(\theta h) + \gamma_1[\chi\mathcal{U} - \mathcal{U}_\pi\chi]\gamma_1\partial_2(\theta h)\\ &= (\mathcal{C}\chi\gamma_1\mathcal{U} + \gamma_1\chi\mathcal{C}\mathcal{U} - \mathcal{C}\mathcal{U}_\pi\gamma_1\chi - \gamma_1\mathcal{U}_\pi\mathcal{C}\chi)\gamma_1(\theta h)\\ &\quad + (\gamma_1\chi\gamma_1\mathcal{U} - \gamma_1\mathcal{U}_\pi\gamma_1\chi)\gamma_1\partial_2(\theta h),\end{aligned}$$

which vanishes due to $\mathcal{C}\chi = \gamma_1\partial_2\chi = 0$ on $\operatorname{supp}\theta$. Now, using the identities $(\mu_0^2 + \mathcal{A}_\pi)\mathcal{T}_\pi = 0$ and $\mathcal{A}\mathcal{T} = 0$ as well as $(\mu_0^2 + \mathcal{A}_\pi)\mathcal{U}_\pi = 0$ and $\mathcal{A}\mathcal{U} = 0$, we get

$$
\begin{aligned}
(\mu_0^2 + \mathcal{A}_\pi)u_1 &= (\mu_0^2 + \mathcal{A}_\pi)\psi\chi\mathcal{T}h - (\mu_0^2 + \mathcal{A}_\pi)\mathcal{T}_\pi\psi\chi h \\
&= [\mathcal{A}_\pi, \chi]\psi\mathcal{T}h + \chi(\mu_0^2 + \mathcal{A}_\pi)\psi\mathcal{T}h - \chi\mathcal{A}\psi\mathcal{T}h + \chi\mathcal{A}\psi\mathcal{T}h - \chi\psi\mathcal{A}\mathcal{T}h \\
&= [\mathcal{A}_\pi, \chi]\psi\mathcal{T}h + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{T}h + \chi[\mathcal{A}, \psi]\mathcal{T}h, \\
(\mu_0^2 + \mathcal{A}_\pi)u_2 &= (\mu_0^2 + \mathcal{A}_\pi)\psi\chi\mathcal{U}\theta h - (\mu_0^2 + \mathcal{A}_\pi)\mathcal{U}_\pi\psi\chi\theta h \\
&= [\mathcal{A}_\pi, \chi]\psi\mathcal{U}\theta h + \chi(\mu_0^2 + \mathcal{A}_\pi)\psi\mathcal{U}\theta h - \chi\mathcal{A}\psi\mathcal{U}\theta h \\
&\quad + \chi\mathcal{A}\psi\mathcal{U}\theta h - \chi\psi\mathcal{A}\mathcal{U}\theta h \\
&= [\mathcal{A}_\pi, \chi]\psi\mathcal{U}\theta h + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{U}\theta h + \chi[\mathcal{A}, \psi]\mathcal{U}\theta h.
\end{aligned}
$$

Hence, we may infer from [34, Corollary B.11] and Proposition VI.11 that

$$
\begin{aligned}
u_1 &= \mathcal{S}_\pi\big([\mathcal{A}_\pi, \chi]\psi\mathcal{T} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{T} + \chi[\mathcal{A}, \psi]\mathcal{T}\big)h, \\
u_2 &= \mathcal{R}_\pi\big([\mathcal{A}_\pi, \chi]\psi\mathcal{U} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{U} + \chi[\mathcal{A}, \psi]\mathcal{U}\big)\theta h.
\end{aligned}
$$

Since $\psi\chi = \chi$ the first two assertions follow. Analogously define $u_3 := (\psi\chi\mathcal{S}\mathcal{P} - \mathcal{S}_\pi\mathcal{P}_\pi\psi\chi)h$ and observe that $u_3 \in h^{2+\alpha}_{\mathbb{H}}$. With $(\mu_0^2 + \mathcal{A}_\pi)\mathcal{S}_\pi = \mathrm{id}_{h^{2+\alpha}_{\mathbb{H}}}$ and $\mathcal{A}\mathcal{S} = \mathrm{id}_{buc^{2+\alpha}(\Omega)}$ we obtain

$$
\begin{aligned}
(\mu_0^2 + \mathcal{A}_\pi)u_3 &= (\mu_0^2 + \mathcal{A}_\pi)\psi\chi\mathcal{S}\mathcal{P}h - (\mu_0^2 + \mathcal{A}_\pi)\mathcal{S}_\pi\mathcal{P}_\pi\psi\chi h \\
&= [\mathcal{A}_\pi, \chi]\psi\mathcal{S}\mathcal{P}h + \chi(\mu_0^2 + \mathcal{A}_\pi)\psi\mathcal{S}\mathcal{P}h - \chi\mathcal{A}\psi\mathcal{S}\mathcal{P}h + \chi\mathcal{A}\psi\mathcal{S}\mathcal{P}h \\
&\quad - \chi\psi\mathcal{A}\mathcal{S}\mathcal{P}h + \chi\psi\mathcal{P}h - \mathcal{P}_\pi\psi\chi h \\
&= [\mathcal{A}_\pi, \chi]\psi\mathcal{S}\mathcal{P}h + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{S}\mathcal{P}h + \chi[\mathcal{A}, \psi]\mathcal{S}\mathcal{P}h \\
&\quad + \psi\chi(\mathcal{P} - \mathcal{P}_\pi)h + [\psi\chi, \mathcal{P}_\pi]h.
\end{aligned}
$$

Now the third assertion follows with the same arguments as above. □

Lemma VI.20. *Let $K \subset \mathcal{V}$ be compact and let $\rho \in (0, 1]$. Given $f \in K$, let $w := w_f$ from* (VI.11) *and put, cf.* (VI.10),

$$
\tilde{c} := \tilde{c}_f := \partial_1 v_f(\,\cdot\,, 0) + \frac{2f_x}{c+f}\partial_2 v_f(\,\cdot\,, 0) \quad \text{and} \quad c_\pi := \tilde{c}(x), \tag{VI.16}
$$

where $x \in \mathbb{S}$. There are positive constants C and C_ρ depending on K, where the latter also depends on ρ, such that for $h \in h^{2+\alpha}$ it holds that

$$
\begin{aligned}
\|\mathcal{P}_\pi h - w h_{xx}\|_{0,\mathbb{H}\cap U} + \rho^\alpha[\mathcal{P}_\pi h - w h_{xx}]_{\alpha,\mathbb{H}\cap U} &\le C\rho\|h\|_{2+\alpha}, \\
\|\tilde{c} - c_\pi\|_{0,\mathbb{R}\cap U} + \rho^\alpha[\tilde{c} - c_\pi]_{\alpha,\mathbb{R}\cap U} &\le C\rho, \\
\|b' - b_\pi\|_{0,\mathbb{R}\cap U} + \rho^\alpha[b' - b_\pi]_{\alpha,\mathbb{R}\cap U} &\le C\rho.
\end{aligned}
$$

Proof. The first assertion follows from Lemma 5.2 in [34] and the other ones are consequences of the mean value theorem and the facts that $\tilde{c} \in h^\alpha$ and $b' \in h^\alpha$, respectively. □

Lemma VI.21. *Suppose that $K \subset \mathcal{V}$ is compact and that $\{(U_j, \theta_j) : 1 \le j \le m_\rho\}$ is a ρ-localization sequence, $\rho \in (0,1]$, for $\mathbb{S}$. Furthermore, for $1 \le j \le m_\rho$ let χ_j, ψ_j be as in* (VI.13)–(VI.15). *Then there are positive constants C and C_ρ depending on K, where C_ρ additionally depends on ρ, such that*

$$\|\chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi u\|_{\alpha,\mathbb{H}} + \|\chi(\mathcal{B}_\pi - \mathcal{B})\psi u\|_{1+\alpha,\mathbb{R}} \le C\rho^{1-\alpha}\|u\|_{2+\alpha,\Omega} + C_\rho\|u\|_{1+\alpha,\Omega}$$

for all $u \in buc^{2+\alpha}(\Omega)$.

Proof. Observe that $\|a_{jk}\|_{1+\alpha,\Omega}$, $\|a_2\|_{\alpha,\Omega}$, and $\|b_j\|_{1+\alpha}$, $1 \le j,k \le 2$, are bounded by a positive constant $C = C(K)$. Since the diameter of every U_l, $1 \le l \le m_\rho$, is bounded by ρ, we infer from the mean value theorem that

$$\begin{aligned} &\|a_{jk} - a_{jk}(x)\|_{0,\mathbb{H}\cap U} + \|b_j - b_j(x)\|_{0,\mathbb{R}\cap U} \le C\rho, \\ &[a_{jk} - a_{jk}(x)]_{\alpha,\mathbb{H}\cap U} + [b_j - b_j(x)]_{\alpha,\mathbb{R}\cap U} \le C\rho^{1-\alpha}, \quad 1 \le j,k \le 2, \end{aligned}$$

where $x := x_l \in U_l =: U$ and again $C = C(K)$. These bounds, the bounds on $\chi := \chi_l$ and $\psi := \psi_l$ (see (VI.14), (VI.15)) and the facts that $\rho \in (0,1]$ and that $\psi|\operatorname{supp}\chi = 1$ imply that

$$\begin{aligned} &\|\chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi u\|_{\alpha,\mathbb{H}} \\ &\le \mu_0^2\|\chi\psi u\|_{\alpha,\mathbb{H}\cap U} + \sum_{j,k=1}^{2} \|\chi(a_{jk}(x) - a_{jk})\partial_{jk}(\psi u)\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} \\ &\quad + \|\chi a_2\partial_2(\psi u)\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} \\ &\le \|\chi\|_{\alpha,U}\big(\mu_0^2\|u\|_{\alpha,\mathbb{H}\cap U}\|\psi\|_{\alpha,U} + \|a_2 u\partial_2\psi\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} + \|a_2\psi\partial_2 u\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi}\big) \\ &\quad + \sum_{j,k=1}^{2} \|\chi(a_{jk}(x) - a_{jk})(u\partial_{jk}\psi + \partial_j u\partial_k\psi + \partial_j\psi\partial_k u + \psi\partial_{jk}u)\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} \\ &\le C_\rho\big(\mu_0^2\|u\|_{\alpha,\mathbb{H}\cap U}\|\psi\|_{\alpha,\operatorname{supp}\chi} + C\|\partial_2\psi\|_{\alpha,\operatorname{supp}\chi}\|u\|_{\alpha,\mathbb{H}\cap U} \\ &\qquad + C\|\psi\|_{\alpha,\operatorname{supp}\chi}\|\partial_2 u\|_{\alpha,\mathbb{H}\cap U}\big) \\ &\quad + \sum_{j,k=1}^{2} \big(\|\chi\|_{0,U}\|a_{jk}(x) - a_{jk}\|_{0,\mathbb{H}\cap U} + \|\chi\|_{0,U}[a_{jk}(x) - a_{jk}]_{\alpha,\mathbb{H}\cap U} \\ &\qquad\qquad + [\chi]_{\alpha,U}\|a_{jk}(x) - a_{jk}\|_{0,\mathbb{H}\cap U}\big) \\ &\qquad\qquad \times \big(\|u\|_{\alpha,\mathbb{H}\cap U}\|\partial_{jk}\psi\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} + \|\partial_j u\|_{\alpha,\mathbb{H}\cap U}\|\partial_k\psi\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} \\ &\qquad\qquad + \|\partial_k u\|_{\alpha,\mathbb{H}\cap U}\|\partial_j\psi\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi} + \|\partial_{jk}u\|_{\alpha,\mathbb{H}\cap U}\|\psi\|_{\alpha,\mathbb{H}\cap\operatorname{supp}\chi}\big) \\ &\le C_\rho\big(\|u\|_{\alpha,\mathbb{H}\cap U} + \|\partial_2 u\|_{\alpha,\mathbb{H}\cap U}\big) + C(\rho + \rho^{1-\alpha} + \rho^{-\alpha}\rho)\max_{1\le j,k\le 2}\|\partial_{jk}u\|_{\alpha,\mathbb{H}\cap U} \\ &\le C_\rho\|u\|_{1+\alpha,\Omega} + C\rho^{1-\alpha}\|u\|_{2+\alpha,\Omega} \end{aligned}$$

and

$$\begin{aligned} &\|\chi(\mathcal{B}_\pi - \mathcal{B})\psi u\|_{1+\alpha,\mathbb{R}} \\ &\le \sum_{j=1}^{2}\big(\|\chi(b_j(x) - b_j)\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap U} + \|\partial[\chi(b_j(x) - b_j)\gamma_0\partial_j(\psi u)]\|_{0,\mathbb{R}\cap U} \\ &\qquad \big[\partial[\chi(b_j(x) - b_j)\gamma_0\partial_j(\psi u)]\big]_{\alpha,\mathbb{R}\cap U}\big) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{2} \big(\|\chi(b_j(x)-b_j)\|_{0,\mathbb{R}\cap U} \|\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \|\partial[\chi(b_j(x)-b_j)]\|_{0,\mathbb{R}\cap U} \|\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \|\chi(b_j(x)-b_j)\|_{0,\mathbb{R}\cap U} \|\partial[\gamma_0\partial_j(\psi u)]\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \big[\partial[\chi(b_j(x)-b_j)]\gamma_0\partial_j(\psi u)\big]_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \big[\chi(b_j(x)-b_j)\partial[\gamma_0\partial_j(\psi u)]\big]_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \big) \\
&\leq \sum_{j=1}^{2} \big(\|\chi(b_j(x)-b_j)\|_{1,\mathbb{R}\cap U} \|\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \|\chi\|_{0,U} \|b_j(x)-b_j\|_{0,\mathbb{R}\cap U} \|\partial\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \big[\partial[\chi(b_j(x)-b_j)]\big]_{\alpha,\mathbb{R}\cap U} \|\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \|\partial[\chi(b_j(x)-b_j)]\|_{0,\mathbb{R}\cap U} [\gamma_0\partial_j(\psi u)]_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + [\chi(b_j(x)-b_j)]_{\alpha,\mathbb{R}\cap U} \|\partial\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad + \|\chi\|_{0,U} \|b_j(x)-b_j\|_{0,\mathbb{R}\cap U} [\partial\gamma_0\partial_j(\psi u)]_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \big) \\
&\leq \sum_{j=1}^{2} \Big(2\|\chi(b_j(x)-b_j)\|_{1+\alpha,\mathbb{R}\cap U} \big(\|\psi\|_{\alpha,\operatorname{supp}\chi} \|\gamma_0\partial_j u\|_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad\qquad + \|\gamma_0 u\|_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \|\partial_j\psi\|_{\alpha,\operatorname{supp}\chi} \big) \\
&\qquad + \|\chi\|_{0,U} \|b_j(x)-b_j\|_{0,\mathbb{R}\cap U} \big(\|\partial_1\psi\|_{\alpha,\operatorname{supp}\chi} \|\gamma_0\partial_j u\|_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad\qquad + \|\psi\|_{\alpha,\operatorname{supp}\chi} \|\gamma_0\partial_{1j} u\|_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad\qquad + \|\partial_j\psi\|_{\alpha,\operatorname{supp}\chi} \|\gamma_0\partial_1 u\|_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \\
&\qquad\qquad + \|\partial_{1j}\psi\|_{\alpha,\operatorname{supp}\chi} \|\gamma_0 u\|_{\alpha,\mathbb{R}\cap \operatorname{supp}\chi} \big) \\
&\qquad + \big([\chi]_{\alpha,U} \|b_j(x)-b_j\|_{0,\mathbb{R}\cap U} + \|\chi\|_{0,U} [b_j(x)-b_j]_{\alpha,\mathbb{R}\cap U} \big) \\
&\qquad \times \|\partial\gamma_0\partial_j(\psi u)\|_{0,\mathbb{R}\cap \operatorname{supp}\chi} \Big) \\
&\leq 2\sum_{j=1}^{2} \|\chi(b_j(x)-b_j)\|_{1+\alpha,\mathbb{R}\cap U} \|\gamma_0\partial_j u\|_{\alpha,\mathbb{R}\cap U} \\
&\quad + C(\rho+\rho^\alpha\rho+\rho^{1-\alpha}) \max_{1\leq j\leq 2} \|\gamma_0\partial_{1j} u\|_{\alpha,\mathbb{R}\cap U} \\
&\leq C_\rho \max_{1\leq j\leq 2} \|\gamma_0\partial_j u\|_{\alpha,\mathbb{R}\cap U} + C\rho^{1-\alpha} \max_{1\leq j\leq 2} \|\gamma_0\partial_{1j} u\|_{\alpha,\mathbb{R}\cap U} \\
&\leq C_\rho \|u\|_{1+\alpha,\Omega} + C\rho^{1-\alpha} \|u\|_{2+\alpha,\Omega}
\end{aligned}$$

for all $u \in buc^{2+\alpha}(\Omega)$. □

Lemma VI.22. *Let the hypotheses of the previous lemma hold and suppose that* $h \in h^{2+\alpha}$ *satisfies* $\operatorname{diam}\operatorname{supp} h < 2\pi$. *Then it holds that*

$$\begin{aligned}
&\|\chi(\mathcal{P}-\mathcal{P}_\pi)h\|_{\alpha,\mathbb{H}} + \|\chi(\partial\mathcal{B}[h,v]-A_{\pi,2}h)\|_{1+\alpha,\mathbb{R}} + \|\chi(b'-b_\pi)h\|_{2-\delta+\alpha,\mathbb{R}} \\
&\leq C\rho^{1-\alpha}\|h\|_{2+\alpha} + C_\rho\|h\|_{1+\alpha},
\end{aligned}$$

where the constants C and C_ρ exhibit the same properties as in the preceding lemma.

Proof. It holds that $\mathcal{P}h = wh_{xx} + Nh$, $h \in h^{2+\alpha}$, where $w = w_f$ (see (VI.11)) and N is of lower order in the sense that $N \in C\big(K, \mathcal{L}(h^{1+\alpha}, buc^\alpha(\Omega))\big)$, cf. Lemma VI.3. Using Lemma VI.20 we estimate

$$\begin{aligned}
&\|\chi(\mathcal{P} - \mathcal{P}_\pi)h\|_{\alpha,\mathbb{H}} \\
&\le \|\chi(wh_{xx} - \mathcal{P}_\pi h)\|_{0,\mathbb{H}\cap U} + [\chi(wh_{xx} - \mathcal{P}_\pi h)]_{\alpha,\mathbb{H}\cap U} + \|\chi\|_{\alpha,U}\|Nh\|_{\alpha,\mathbb{H}\cap U} \\
&\le \|\chi\|_{0,U}\|(wh_{xx} - \mathcal{P}_\pi h)\|_{0,\mathbb{H}\cap U} + [\chi]_{\alpha,U}\|(wh_{xx} - \mathcal{P}_\pi h)\|_{0,\mathbb{H}\cap U} \\
&\quad + \|\chi\|_{0,U}[(wh_{xx} - \mathcal{P}_\pi h)]_{\alpha,\mathbb{H}\cap U} + C_\rho\|h\|_{1+\alpha} \\
&\le C(\rho + \rho^\alpha\rho + \rho^{1-\alpha})\|h\|_{2+\alpha} + C_\rho\|h\|_{1+\alpha} \\
&\le C\rho^{1-\alpha}\|h\|_{2+\alpha} + C_\rho\|h\|_{1+\alpha}.
\end{aligned}$$

As before we can write $\partial\mathcal{B}[\,\cdot\,, v] = \tilde{c}h_x + Mh$, $h \in h^{2+\alpha}$, where $\tilde{c}$ is defined in (VI.16) and $M \in C\big(K, \mathcal{L}(h^{1+\alpha})\big)$, see again Lemma VI.3, and using again Lemma VI.20 we obtain

$$\begin{aligned}
&\|\chi(\partial\mathcal{B}[h, v] - A_{\pi,2}h)\|_{1+\alpha,\mathbb{R}} \\
&\le \|\chi(\tilde{c} - c_\pi)h_x\|_{0,\mathbb{R}\cap U} + \|\partial[\chi(\tilde{c} - c_\pi)h_x]\|_{\alpha,\mathbb{R}\cap U} + \|\chi Mh\|_{1+\alpha,\mathbb{R}\cap U} \\
&\le \|\chi\|_{0,U}\|(\tilde{c} - c_\pi)h_x\|_{0,\mathbb{R}\cap U} + \|\partial[\chi(\tilde{c} - c_\pi)]\|_{\alpha,\mathbb{R}\cap U}\|h_x\|_{\alpha,\mathbb{R}\cap U} \\
&\quad + \|\chi(\tilde{c} - c_\pi)\|_{\alpha,\mathbb{R}\cap U}\|h_{xx}\|_{\alpha,\mathbb{R}\cap U} + C_\rho\|h\|_{1+\alpha} \\
&\le C\rho\|h_x\|_{0,\mathbb{R}\cap U} + \big(\|\partial\chi\|_{\alpha,U}\|\tilde{c} - c_\pi\|_{\alpha,\mathbb{R}\cap U} + \|\chi\|_{\alpha,U}\|\partial\tilde{c}\|_{\alpha,\mathbb{R}\cap U}\big)\|h_x\|_{\alpha,\mathbb{R}\cap U} \\
&\quad + \big(\|\chi\|_{0,U}\|\tilde{c} - c_\pi\|_{0,\mathbb{R}\cap U} + [\chi]_{\alpha,U}\|\tilde{c} - c_\pi\|_{0,\mathbb{R}\cap U} \\
&\qquad + \|\chi\|_{0,U}[\tilde{c} - c_\pi]_{\alpha,\mathbb{R}\cap U}\big)\|h_{xx}\|_{\alpha,\mathbb{R}\cap U} + C_\rho\|h\|_{1+\alpha} \\
&\le C\rho\|h\|_{1+\alpha,\mathbb{R}\cap U} + \big(C_\rho\|\chi\|_{1+\alpha,U} + \|\chi\|_{\alpha,U}\|\tilde{c}\|_{1+\alpha,\mathbb{R}\cap U}\big)\|h\|_{1+\alpha,\mathbb{R}\cap U} \\
&\quad + C(\rho + \rho^{-\alpha}\rho + \rho^{1-\alpha})\|h\|_{2+\alpha} + C_\rho\|h\|_{1+\alpha} \\
&\le C_\rho\|h\|_{1+\alpha} + C\rho^{1-\alpha}\|h\|_{2+\alpha}.
\end{aligned}$$

In the third expression we only prove the case $\delta = 0$. The other one is similar. Leibniz' rule and similar arguments as above imply

$$\|\chi(b' - b_\pi)h\|_{2+\alpha,\mathbb{R}} \le C_\rho\|h\|_{1+\alpha,\mathbb{R}\cap U} + \|\chi(b' - b_\pi)h_{xx}\|_{\alpha,\mathbb{R}\cap U}.$$

From Lemma VI.20 we finally infer

$$\begin{aligned}
&\|\chi(b' - b_\pi)h_{xx}\|_{\alpha,\mathbb{R}\cap U} \\
&\le \|\chi\|_{0,U}\|b' - b_\pi\|_{0,\mathbb{R}\cap U}\big(\|h_{xx}\|_{0,\mathbb{R}\cap U} + [h_{xx}]_{\alpha,\mathbb{R}\cap U}\big) \\
&\quad + \big(\|\chi\|_{0,U}[b' - b_\pi]_{\alpha,\mathbb{R}\cap U} + [\chi]_{\alpha,U}\|b' - b_\pi\|_{0,\mathbb{R}\cap U}\big)\|h_{xx}\|_{0,\mathbb{R}\cap U} \\
&\le C\rho\|h_{xx}\|_\alpha + C(\rho^{1-\alpha} + \rho^{-\alpha}\rho)\|h_{xx}\|_0 \\
&\le C\rho^{1-\alpha}\|h\|_{2+\alpha},
\end{aligned}$$

which completes the proof. □

Proof of Lemma VI.16. Pick $\kappa > 0$ and choose θ, χ, ψ according to (VI.13)–(VI.15). We start with the identity

$$\theta\partial\Phi + A_\pi\theta = \chi[\theta, \partial\Phi] + (\chi\partial\Phi + A_\pi\chi)\theta.$$

Inequalities (VI.14) and (VI.15) and Lemma VI.18 imply the existence of a positive constant C_ρ depending on K and ρ such that

$$\|\chi[\theta, \partial\Phi]h\|_{1+\alpha} \leq C_\rho \|h\|_{2+\beta} \quad \text{for all} \quad h \in h^{2+\alpha}.$$

To get an estimate for the second part in the above expression we split this operator and study four single terms. The first is

$$\begin{aligned}\chi\mathcal{B}\mathcal{T} - \mathcal{B}_\pi\mathcal{T}_\pi\chi &= \mathcal{B}_\pi(\chi\mathcal{T} - \mathcal{T}_\pi\chi) - \mathcal{B}_\pi\chi\mathcal{T} + \chi\mathcal{B}\mathcal{T} \\ &= \mathcal{B}_\pi\mathcal{S}_\pi\big([\mathcal{A}_\pi, \chi]\psi\mathcal{T} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{T} + \chi[\mathcal{A}, \psi]\mathcal{T}\big) \\ &\quad + \chi(\mathcal{B} - \mathcal{B}_\pi)\psi\mathcal{T} + \chi[\psi, \mathcal{B}]\mathcal{T} + [\chi, \mathcal{B}_\pi]\psi\mathcal{T},\end{aligned}$$

where we make use of Lemma VI.19. We know that the operators $\mathcal{B}_\pi$, $\mathcal{S}_\pi$, and $\mathcal{T}$ are bounded by a positive constant $C = C(K)$. Moreover, we have estimates for the operators $\chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi$ and $\chi(\mathcal{B} - \mathcal{B}_\pi)\psi$ and the commutators. More precisely, Lemmas VI.17, VI.18, and VI.21 imply

$$\begin{aligned}\|(\chi\mathcal{B}\mathcal{T} - \mathcal{B}_\pi\mathcal{T}_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}} &\leq C\rho^{1-\alpha}\|\mathcal{T}\theta h\|_{2+\alpha,\Omega\cap U} \\ &\quad + C_\rho\big(\|\mathcal{T}\theta h\|_{1+\alpha,\Omega\cap U} + \|\mathcal{T}\theta h\|_{2+\beta,\Omega\cap U}\big) \\ &\leq C\rho^{1-\alpha}\|\mathcal{T}\theta h\|_{2+\alpha,\Omega\cap U} + C_\rho\|\mathcal{T}\theta h\|_{2+\beta,\Omega\cap U} \\ &\leq C\rho^{1-\alpha}\|\theta h\|_{2+\alpha,\mathbb{R}} + C_\rho\|h\|_{2+\beta}\end{aligned}$$

for all $h \in h^{2+\alpha}$. Since $\rho \in (0,1]$ and C is independent of ρ, we can choose ρ sufficiently small such that

$$\|(\chi\mathcal{B}\mathcal{T} - \mathcal{B}_\pi\mathcal{T}_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}} \leq \frac{\kappa}{4}\|\theta h\|_{2+\alpha,\mathbb{R}} + C_\rho\|h\|_{2+\beta}.$$

It holds that

$$\begin{aligned}\chi\mathcal{B}\mathcal{U}b' - \mathcal{B}_\pi\mathcal{U}_\pi b_\pi\chi &= \mathcal{B}_\pi(\chi\mathcal{U} - \mathcal{U}_\pi\chi)b' + \mathcal{B}_\pi\mathcal{U}_\pi(\chi b' - b_\pi\chi) \\ &\quad + \chi(\mathcal{B} - \mathcal{B}_\pi)\psi\mathcal{U}b' + \chi[\psi, \mathcal{B}]\mathcal{U}b' + [\chi, \mathcal{B}_\pi]\psi\mathcal{U}b'.\end{aligned}$$

Given $h \in h^{2+\alpha}$, we obtain from Lemma VI.19 that

$$\begin{aligned}&(\chi\mathcal{B}\mathcal{U}b' - \mathcal{B}_\pi\mathcal{U}_\pi b_\pi\chi)\theta h \\ &= \mathcal{B}_\pi\mathcal{R}_\pi\big([\mathcal{A}_\pi, \chi]\psi\mathcal{U} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{U} + \chi[\mathcal{A}, \psi]\mathcal{U}\big)b'\theta h \\ &\quad + \mathcal{B}_\pi\mathcal{U}_\pi(\chi b' - b_\pi\chi)\theta h + \chi(\mathcal{B} - \mathcal{B}_\pi)\psi\mathcal{U}b'\theta h + \chi[\psi, \mathcal{B}]\mathcal{U}b'\theta h + [\chi, \mathcal{B}_\pi]\psi\mathcal{U}b'\theta h.\end{aligned}$$

Since we also know that the operators $\mathcal{U}$, $\mathcal{U}_\pi$, and $\mathcal{R}_\pi$ are bounded by $C = C(K)$ we analogously estimate using additionally Lemma VI.22

$$\|(\chi\mathcal{B}\mathcal{U}b' - \mathcal{B}_\pi\mathcal{U}_\pi b_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}} \leq C\rho^{1-\alpha}\|\theta h\|_{2+\alpha,\mathbb{R}} + C_\rho\|h\|_{2+\beta}.$$

Hence, choosing ρ sufficiently small we get

$$\|(\chi\mathcal{B}\mathcal{U}b' - \mathcal{B}_\pi\mathcal{U}_\pi b_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}} \leq \frac{\kappa}{4}\|\theta h\|_{2+\alpha,\mathbb{R}} + C_\rho\|h\|_{2+\beta}.$$

Next, we obtain from Lemma VI.19 the identity

$$\begin{aligned}\chi\mathcal{B}\mathcal{S}\mathcal{P} - \mathcal{B}_\pi\mathcal{S}_\pi\mathcal{P}_\pi\chi = \mathcal{B}_\pi\mathcal{S}_\pi\big([\mathcal{A}_\pi,\chi]\psi\mathcal{S}\mathcal{P} + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A})\psi\mathcal{S}\mathcal{P}\\ - \chi[\mathcal{A},\psi]\mathcal{S}\mathcal{P} + \chi(\mathcal{P} - \mathcal{P}_\pi) + [\chi,\mathcal{P}_\pi]\big)\\ + \chi(\mathcal{B} - \mathcal{B}_\pi)\psi\mathcal{S}\mathcal{P} + \chi[\psi,\mathcal{B}]\mathcal{S}\mathcal{P} + [\chi,\mathcal{B}_\pi]\psi\mathcal{S}\mathcal{P}.\end{aligned}$$

Using the same arguments as above (including the boundedness of $\mathcal{P}$) we conclude

$$\begin{aligned}\|(\chi\mathcal{B}\mathcal{S}\mathcal{P} - \mathcal{B}_\pi\mathcal{S}_\pi\mathcal{P}_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}} &\leq C\rho^{1-\alpha}\|\theta h\|_{2+\alpha} + C_\rho\|h\|_{2+\beta}\\ &\leq \frac{\kappa}{4}\|\theta h\|_{2+\alpha,\mathbb{R}} + C_\rho\|h\|_{2+\beta},\end{aligned}$$

where again ρ is chosen sufficiently small.

Finally, we obviously have

$$\chi\partial\mathcal{B}[\,\cdot\,,v] - A_{\pi,2}\chi = \chi(\partial\mathcal{B}[\,\cdot\,,v] - A_{\pi,2}) + [\chi, A_{\pi,2}].$$

Again, if ρ is chosen sufficiently small we infer from Lemmas VI.17 and VI.22 that

$$\|\chi\partial\mathcal{B}[\theta h,v] - A_{\pi,2}\chi\theta h\|_{1+\alpha,\mathbb{R}} \leq \frac{\kappa}{4}\|\theta h\|_{2+\alpha,\mathbb{R}} + C_\rho\|h\|_{2+\beta}.$$

The assertion follows. □

We now use Lemma VI.16 to prove the following generation result, which is the central result in subsequently proving Theorem VI.1.

Theorem VI.23. *Suppose that $f + c \in \mathcal{W}$. Then $-\partial\Phi(f) \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha})$.*

Proof. Let $K \subset \mathcal{W}$ be compact. We show that, given $f + c \in K$ and $t \in [0,1]$, it holds that $-\partial\Phi_t(f) \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha})$. Then the assertion follows with $t = 1$. We show the following equivalent characterization of $-\partial\Phi_t(f) \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha})$, see Propositions III.6 and III.8: There are positive constants λ_* and $C = C(K)$ such that

$$\|h\|_{2+\alpha} + |\lambda|\|h\|_{1+\alpha} \leq C\|(\lambda - \partial\Phi_t(f))h\|_{1+\alpha}, \tag{VI.17}$$
$$\lambda_* - \partial\Phi_t(f) \in \mathcal{L}\mathrm{is}(h^{2+\alpha}, h^{1+\alpha}) \tag{VI.18}$$

for all $h \in h^{2+\alpha}$, $f + c \in K$, and $\operatorname{Re}\lambda \geq \lambda_*$.

Fix $t \in [0,1]$. Given $f + c \in K$ and $x_0 \in \mathbb{S}$, we know from Theorem VI.14 that $A_\pi(f, x_0) \in \mathcal{H}(h^{2+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$. Hence, inequality (VI.17) holds for the operator $A_\pi(f, x_0)$. More precisely, there are positive constants $\tilde{C} = \tilde{C}(K)$ and $\tilde{\lambda}_*$ and a ρ-localization sequence $\{(U_j, \theta_j) : 1 \leq j \leq m_\rho\}$ of $\mathbb{S}$ such that

$$\begin{aligned}\|\theta_j h\|_{2+\alpha,\mathbb{R}} + |\lambda|\|\theta_j h\|_{1+\alpha,\mathbb{R}} &\leq \tilde{C}\|(\lambda + A_\pi(f, x_j))\theta_j h\|_{1+\alpha,\mathbb{R}}\\ &\leq \tilde{C}\big(\|\theta_j(\lambda - \partial\Phi_t(f))h\|_{1+\alpha,\mathbb{R}}\\ &\qquad + \|(A_\pi(f, x_j)\theta_j + \theta_j\partial\Phi_t(f))h\|_{1+\alpha,\mathbb{R}}\big)\end{aligned}$$

for all $h \in h^{2+\alpha}$, $f + c \in K$, $1 \leq j \leq m_\rho$, and $\operatorname{Re}\lambda \geq \tilde{\lambda}_*$. Given $\beta \in (0,\alpha)$ and $\rho \in (0,1]$, we infer from Lemma VI.16 that there is a positive constant $C = C(\tilde{C})$ such that

$$\|(A_{\pi,t}(f,x_j)\theta_j + \theta_j \partial\Phi_t(f))h\|_{1+\alpha,\mathbb{R}} \leq \frac{1}{2\tilde{C}}\|\theta_j h\|_{2+\alpha,\mathbb{R}} + C\|h\|_{2+\beta}$$

for all $h \in h^{2+\alpha}$, $f + c \in K$, and $1 \leq j \leq m_\rho$. Hence, we conclude

$$\begin{aligned}\|\theta_j h\|_{2+\alpha,\mathbb{R}} + |\lambda|\|\theta_j h\|_{1+\alpha,\mathbb{R}} \leq \tilde{C}\|\theta_j(\lambda - \partial\Phi_t(f))h\|_{1+\alpha,\mathbb{R}} \\ + \frac{1}{2}\|\theta_j h\|_{2+\alpha,\mathbb{R}} + \tilde{C}C\|h\|_{2+\beta}\end{aligned}$$

and thus

$$\|\theta_j h\|_{2+\alpha,\mathbb{R}} + |\lambda|\|\theta_j h\|_{1+\alpha,\mathbb{R}} \leq 2\tilde{C}\big(\|\theta_j(\lambda - \partial\Phi_t(f))h\|_{1+\alpha,\mathbb{R}} + C\|h\|_{2+\beta}\big)$$

for all $h \in h^{2+\alpha}$, $f + c \in K$, $1 \leq j \leq m_\rho$, and $\operatorname{Re}\lambda \geq \tilde{\lambda}_*$. Using the facts that $\sup_{1\leq j\leq m_\rho}\|\theta_j \cdot\|_{k+\alpha,\mathbb{R}}$ is an equivalent norm on $h^{k+\alpha}$, $k = 1,2$, and that $h^{2+\beta} = (h^{1+\alpha}, h^{2+\alpha})_{1+\beta-\alpha}$, we deduce that

$$\|h\|_{2+\alpha} + 2(|\lambda| - C_1)\|h\|_{1+\alpha} \leq C\|(\lambda - \partial\Phi_t(f))h\|_{1+\alpha},$$

where $C_1 = C_1(C)$, for all $h \in h^{2+\alpha}$, $f + c \in K$, $1 \leq j \leq m_\rho$, and $\operatorname{Re}\lambda \geq \tilde{\lambda}_*$. Setting $\lambda_* := 2\max\{\tilde{\lambda}_*, C_1\}$ completes the proof of inequality (VI.17).

To prove (VI.18) we choose $f + c \in \mathcal{W}$ and $\lambda_* > 0$ and we show that $\lambda_* - \partial\Phi_0(f) \in \mathcal{L}\mathrm{is}(h^{2+\alpha}, h^{1+\alpha})$. From the a priori estimate (VI.17) we conclude that $\lambda_* - \partial\Phi_t(f) \in \mathcal{L}(h^{2+\alpha}, h^{1+\alpha})$ is one-to-one for all $t \in [0,1]$. In particular, $\lambda_* - \partial\Phi_0(f)$ is one-to-one. To show that it is onto pick $h \in h^{1+\alpha}$. From Theorem VI.5 we infer that, given $f \in \mathcal{V}$ and $\mu > 0$, there is a function $u \in buc^{2+\alpha}(\Omega)$ such that $\big(\mathcal{A}(f), \mu\gamma_0 - \mathcal{B}(f), \mathcal{C}\big)u = (0,h,0)$. Now observe that $\gamma_0 u \in h^{2+\alpha}$ and that

$$\mathcal{T}(f)\gamma_0 u = \big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big)^{-1}(0, \gamma_0 u, 0) = \big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big)^{-1}\big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big)u = u.$$

Put $g := \gamma_0 u$. Then it holds that

$$(\lambda_* - \partial\Phi_0(f))g = (\lambda_* - \mathcal{B}(f)\mathcal{T}(f))g = (\lambda_*\gamma_0 - \mathcal{B}(f))u = h.$$

Since $\lambda_* - \partial\Phi_t(f)$ is one-to-one for all $t \in [0,1]$, the method of continuity (cf. [42, Theorem 5.2]) implies that $\lambda_* - \partial\Phi_t(f) \in \mathcal{L}\mathrm{is}(h^{2+\alpha}, h^{1+\alpha})$ for all $t \in [0,1]$. □

Proof of Theorem VI.1. Due to the equivalence results Lemma VI.4 and Lemma VI.6 we show that, given $f_0 + c \in \mathcal{W}$, the Cauchy problem

$$\partial_t f = \Phi(f), \quad f(0) = f_0, \tag{VI.19}$$

has a unique maximal classical solution.

Fix $c > 0$ and recall that $\beta_c = b(0) = (-c)^\delta \tilde{b}(c) - (1-\delta)c$. Consequently, the hypothesis $\beta_c > -c^3/(1+c^2)$ from Theorem VI.15 is fulfilled if $\delta = 1$ and

$-c\tilde{b}(c) > -c^3/(1+c^2)$ or if $\delta = 0$ and $\tilde{b}(c) - c > -c^3/(1+c^2)$, which is equivalent to $(-1)^\delta \tilde{b}(c) > (-1)^\delta c^{1+\delta}/(1+c^2)$. Thus, if this condition holds, the set $\mathcal{W}$ is an open neighborhood of c in $h^{2+\alpha}$. From Theorem VI.23 and $h_+^{2+\alpha} \subset h_+^{2+\beta}$, $\beta \in (0,\alpha)$, we infer that $-\partial\Phi(f) \in \mathcal{H}(h^{2+\gamma}, h^{1+\gamma})$, $f + c \in \mathcal{W}$, for $\gamma \in [\beta, \alpha]$. This result and the interpolation property $(h^{1+\beta}, h^{2+\beta})_{\alpha-\beta} = h^{1+\alpha}$ imply that $\partial\Phi(f) \in \mathcal{L}(h^{2+\alpha}, h^{1+\alpha})$ is the part in $h^{1+\alpha}$ of the operator $\partial\Phi(f) \in \mathcal{L}(h^{2+\beta}, h^{1+\beta})$ with domain $h^{2+\alpha}$. Now we infer from Proposition III.11 and Theorem III.12 that there exists a unique classical solution of (VI.19) that can be extended to a maximal interval of existence $[0, T)$ for some $T > 0$.

Observe that Theorem III.13 implies

$$\lim_{t\to T}(c + f(t,\,\cdot\,)) \in \partial\mathcal{W} \quad \text{or} \quad T = +\infty$$

provided that the solution f of (VI.19) is uniformly continuous. Furthermore, the solution of (VI.19) generates a local analytic semiflow on $\mathcal{W}$. □

VI.2 Classical Solutions for Newtonian Hele-Shaw Flows with Surface Tension Effects

Assume that $\sigma > 0$ and fix $\alpha \in (0,1)$. Let $h_+^{4+\alpha} := h^{4+\alpha} \cap \mathrm{Ad}_0$ but now $\mathcal{V} := \mathcal{V}_{4+\alpha} := h^{4+\alpha} \cap \mathrm{Ad}_{-c}$. Moreover, let $\beta \in (\alpha, 1)$ and put $\mathcal{W} := \mathcal{W}_{3+\beta} := h^{3+\beta} \cap \mathrm{Ad}_{-c}$. Given $\tilde{f}_0 \in h_+^{4+\alpha}$ and $T > 0$, we are interested in classical solutions of problem (VI.1) having the regularity

$$\begin{aligned} \tilde{f} &\in C([0,T], h_+^{4+\alpha}) \cap C^1([0,T], h^{1+\alpha}), \\ u(\,\cdot\,, t) &\in buc^{2+\alpha}(\Omega(\tilde{f}(t,\,\cdot\,))), \quad 0 \le t \le T. \end{aligned}$$

VI.2.1 The Transformation

Let $c > 0$ be fixed in the following. We transform system (VI.1) on a fixed reference domain in the same way as in the preceding section. Let $\tilde{f} \in h_+^{4+\alpha}$ be given and put

$$\tilde{\phi}_{\tilde{f}}(x,y) := \big(x, 1 - y/\tilde{f}(x)\big), \quad (x,y) \in \Omega(\tilde{f}).$$

The function $\tilde{\phi}_{\tilde{f}}$ is a $C^{4+\alpha}$-diffeomorphism between $\Omega(\tilde{f})$ and $\Omega = \mathbb{S} \times (0,1)$. Put $f := \tilde{f} - c$. Then $f \in \mathcal{V}$. Setting $\phi_f := \tilde{\phi}_{f+c}$ we get the same push forward and pull back operators as before, i.e.,

$$\begin{aligned} \phi_*^f u &= u \circ \phi_f^{-1}, \quad u \in BUC(\Omega(\tilde{f})), \\ \phi_f^* v &= v \circ \phi_f, \quad v \in BUC(\Omega). \end{aligned}$$

The differential operators in the new transformed coordinates read

$$\mathcal{A}(f) = -\phi_*^f \Delta \phi_f^* = -\sum_{j,k=1}^{2} a_{jk}(f)\partial_{jk} + a_2(f)\partial_2,$$

$$\mathcal{B}(f) = -\gamma_0 \phi_*^f \big(\nabla(\phi_f^* \cdot) \cdot n\big) = \sum_{j=1}^{2} b_j(f)\gamma_0 \partial_j,$$

$$\tilde{\mathcal{C}}(f) = (1-\delta)\phi_*^f \gamma_0 \phi_f^* + \delta \phi_*^f \gamma_0 \partial_2(\phi_f^* \cdot) = (1-\delta)\gamma_1 - \frac{\delta}{\tilde{f}}\gamma_1\partial_2,$$

where

$$a_{11}(f) := 1, \quad a_{12}(f) := \frac{\pi f_x}{\tilde{f}}, \quad a_{22}(f) := \frac{1+\pi^2 {f_x}^2}{\tilde{f}^2}, \quad a_2(f) := \frac{\pi}{\tilde{f}}\Big(\frac{2{f_x}^2}{\tilde{f}} - f_{xx}\Big),$$
$$b_1(f) := f_x, \quad b_2(f) := \frac{1+{f_x}^2}{\tilde{f}},$$

and $\pi(x,y) := 1-y$ for $(x,y) \in \overline{\Omega}$. Given $f_0 \in \mathcal{V}$, consider the system

$$\begin{aligned} \mathcal{A}(f)v &= 0 && \text{in } \Omega, \\ v &= -\sigma\mathcal{K}(f)f + f && \text{on } \Gamma_0, \\ \mathcal{C}v &= b(f) && \text{on } \Gamma_1, \\ \partial_t f &= \mathcal{B}(f)v && \text{on } \Gamma_0, \\ f(0) &= f_0 && \text{on } \mathbb{S}, \end{aligned} \tag{VI.20}$$

where $\mathcal{K}(f) := \frac{1}{(1+f_x^2)^{3/2}}\partial^2$, $b(f) := (-f-c)^\delta \tilde{b}(f+c) - (1-\delta)c$, and (recall) $\mathcal{C} = (1-\delta)\gamma_1 + \delta\gamma_1\partial_2$. For $T > 0$ we have the following equivalence result, where a classical solution (f,v) of problem (VI.20) satisfies this set of equations pointwise and has the regularity

$$f \in C([0,T],\mathcal{V}) \cap C^1([0,T],h^{1+\alpha}),$$
$$v(\,\cdot\,,t) \in buc^{2+\alpha}(\Omega), \quad 0 \le t \le T.$$

Lemma VI.24. *If $(\tilde{f},u)$ is a classical solution of problem* (VI.1), *$(\tilde{f}-c, \phi_*^{\tilde{f}-c}u - c)$ is a classical solution of problem* (VI.20) *and if (f,v) is a classical solution of problem* (VI.20), *$(f+c, \phi_f^* v + c)$ is a classical solution of problem* (VI.1).

Given $f \in \mathcal{W}$, recall that $\mathcal{R}(f) = \big(\mathcal{A}(f),\gamma_0,\mathcal{C}\big)^{-1}$ and put

$$\mathcal{S}(f) := \mathcal{R}(f)|buc^\gamma(\Omega) \times \{0\} \times \{0\},$$
$$\mathcal{T}(f) := \mathcal{R}(f)|\{0\} \times h^{2+\gamma} \times \{0\},$$
$$\mathcal{U}(f) := \mathcal{R}(f)|\{0\} \times \{0\} \times h^{2-\delta+\gamma},$$

where $\gamma \in [\alpha,\beta]$ is arbitrary.

Lemma VI.25. *Given* $\gamma \in [\alpha, \beta]$*, it holds that*

$$\begin{aligned} \mathcal{A} &\in C^\omega\big(\mathcal{W}, \mathcal{L}(buc^{2+\gamma}(\Omega), buc^\gamma(\Omega))\big), \\ \mathcal{B} &\in C^\omega\big(\mathcal{W}, \mathcal{L}(buc^{2+\gamma}(\Omega), h^{1+\gamma})\big), \\ \mathcal{T} &\in C^\omega\big(\mathcal{W}, \mathcal{L}(h^{2+\gamma}(\Omega), buc^{2+\gamma})\big), \\ \mathcal{U} &\in C^\omega\big(\mathcal{W}, \mathcal{L}(h^{2-\delta+\gamma}(\Omega), buc^{2+\gamma})\big). \end{aligned}$$

Furthermore,

$$\mathcal{K} \in C^\omega\big(\mathcal{W}, \mathcal{L}(h^{4+\tilde{\gamma}}(\Omega), h^{2+\tilde{\gamma}})\big)$$

for arbitrary $\tilde{\gamma} \in (0, \beta)$.

Proof. This follows immediately from Lemma VI.3, the fact that $\mathcal{W} \subset \mathcal{V}_{2+\gamma}$, and the analyticity of the inversion map. □

Let us conclude this section with the following important isomorphism result, whose proof is analogous to the proof of Theorem VI.5.

Theorem VI.26. *Let* $\gamma \in [\alpha, \beta]$ *and fix* $f \in \mathcal{W}$*. Then it holds that*

$$\big(\mathcal{A}(f), \gamma_0, \mathcal{C}\big) \in \mathcal{L}\mathrm{is}(buc^{2+\gamma}(\Omega), buc^\gamma(\Omega), h^{2+\gamma}, h^{2-\delta+\gamma}).$$

VI.2.2 The Evolution Equation

Observe that the solution of the boundary value problem

$$\mathcal{A}(f)v = 0 \quad \text{in} \quad \Omega, \quad v = -\sigma\mathcal{K}(f)f + f \quad \text{on} \quad \Gamma_0, \quad \mathcal{C}v = b(f) \quad \text{on} \quad \Gamma_1,$$

where $f \in \mathcal{V}$ is fixed, is given by $v = \mathcal{T}(f)(-\sigma\mathcal{K}(f) + \mathrm{id}_{h^{3+\beta}})f + \mathcal{U}(f)b(f)$. This representation and Theorem VI.26 imply the following result.

Lemma VI.27. *The Cauchy problem*

$$\partial_t f + A(f)f = F(f), \quad f(0) = f_0, \tag{VI.21}$$

where

$$A(f) := \sigma\mathcal{B}(f)\mathcal{T}(f)\mathcal{K}(f) \quad \text{and} \quad F(f) := \mathcal{B}(f)(\mathcal{T}(f)f + \mathcal{U}(f)b(f)),$$

and system (VI.20) *are equivalent in the sense that if*

$$f \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha})$$

is a solution of (VI.21)*, then*

$$\big(f, \mathcal{T}(f)(-\sigma\mathcal{K}(f) + \mathrm{id}_{h^{3+\beta}})f + \mathcal{U}(f)b(f)\big)$$

is a classical solution of (VI.20) *and if* (f, v) *is a classical solution of* (VI.20)*, it holds that*

$$v = \mathcal{T}(f)(-\sigma\mathcal{K}(f) + \mathrm{id}_{h^{3+\beta}})f + \mathcal{U}(f)b(f)$$

and f *is a solution of* (VI.21) *having the regularity* $C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha})$.

We call a solution of (VI.21) a classical solution if it satisfies the equation pointwise and if $f \in C([0,T],\mathcal{V}) \cap C^1([0,T],h^{1+\alpha})$ for $T > 0$. Now we can formulate the main result of this section.

Theorem VI.28. *Let $f \in \mathcal{W}$. It holds that $A(f) \in \mathcal{H}(h^{4+\gamma}, h^{1+\gamma})$ for all $\gamma \in [\alpha, \beta]$.*

In order to prove this result we proceed in a similar way as in the preceding section. Fix $f \in \mathcal{W}$, $x_0 \in \mathbb{S}$, $\mu_0 > 0$, and $\gamma \in [\alpha, \beta]$. We recall the following operators:

$$\begin{aligned}
\mathcal{A}_\pi &= -\partial_1^2 - 2a_{12}\partial_{12} - a_{22}\partial_2^2, \\
\mathcal{B}_\pi &= b_1\gamma_0\partial_1 + b_2\gamma_0\partial_2, \\
(\mathcal{T}_\pi g)(x,y) &= (\mathcal{F}^{-1}\mathrm{e}^{-\lambda(\,\cdot\,,\mu_0)y}\mathcal{F}g)(x), \quad g \in h^{2+\alpha}_{\mathbb{R}}, \quad (x,y) \in \mathbb{H} = \mathbb{R} \times (0,\infty),
\end{aligned}$$

where

$$\begin{aligned}
a_{j2} &= a_{j2}(f,x_0) := a_{j2}(f)(x_0,0), \\
b_j &= b_j(f,x_0) := b_j(f)(x_0), \quad 1 \le j \le 2, \\
\lambda(\eta,\mu) &= \frac{a_{12}}{a_{22}}\mathrm{i}\eta + \frac{1}{a_{22}}\sqrt{a_{22}\mu^2 + (a_{22} - a_{12}^2)\eta^2}, \quad (\eta,\mu) \in \mathbb{R} \times (0,\infty).
\end{aligned}$$

Moreover, given $(\eta,\mu) \in \mathbb{R} \times (0,\infty)$, we introduce the symbol $\kappa_\pi(\eta,\mu) := -\mu^2 - \frac{\eta^2}{(1+f_x(x_0)^2)^{3/2}}$ and the operator $\mathcal{K}_\pi := \mathcal{F}^{-1}\kappa_\pi(\,\cdot\,,\mu_0)\mathcal{F}$ on $\mathcal{S}(\mathbb{R})$.

Lemma VI.29. *The operator $A_\pi := \sigma\mathcal{B}_\pi\mathcal{T}_\pi\mathcal{K}_\pi$ belongs to $\mathcal{L}(h^{4+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$ and has the symbol $a(\eta,\mu_0) := \sigma(\mathrm{i}b_1\eta - b_2\lambda(\eta,\mu_0))\kappa_\pi(\eta,\mu_0)$, $\eta \in \mathbb{R}$.*

Proof. We know that $\mathcal{T}_\pi \in \mathcal{L}(h^{2+\alpha}_{\mathbb{R}}, h^{2+\alpha}_{\mathbb{H}})$ and that $\mathcal{B}_\pi \in \mathcal{L}(h^{2+\alpha}_{\mathbb{H}}, h^{1+\alpha}_{\mathbb{R}})$ as well as $\mathcal{B}_\pi\mathcal{T}_\pi = \mathcal{F}^{-1}[\eta \mapsto \mathrm{i}b_1\eta - b_2\lambda(\eta,\mu_0)]\mathcal{F}$. From the definition of the operator $\mathcal{K}_\pi$ we infer that $\mathcal{K}_\pi \in \mathcal{L}(h^{4+\alpha}_{\mathbb{R}}, h^{2+\alpha}_{\mathbb{R}})$. The assertion follows. □

Lemma VI.30. *There is a positive constant α_* such that $a \in \mathcal{E}ll\mathcal{S}^\infty_3(\alpha_*)$, see* (VI.9).

Proof. Obviously, a is smooth and positively homogeneous of degree 3 and all derivatives of a are bounded on $|(\eta,\mu)| = 1$. Note that

$$\mathrm{Re}\, a(\eta,\mu) = \sigma\frac{b_2}{a_{22}}\sqrt{a_{22}\mu^2 + (a_{22} - a_{12}^2)\eta^2}\left(\mu^2 + \frac{\eta^2}{(1+f_x(x_0)^2)^{3/2}}\right)$$

for $(\eta,\mu) \in \mathbb{R} \times (0,\infty)$. We know that $\sigma, b_2, a_{22} > 0$ and that $a_{22} - a_{12}^2 > \alpha_0 := \left(1 + (c + f(x_0))^2 + f_x(x_0)^2\right)^{-1}$, cf. (VI.7). From this we infer that

$$\sqrt{a_{22}\mu^2 + (a_{22} - a_{12}^2)\eta^2} \ge \sqrt{(a_{22} - a_{12}^2)(\mu^2 + \eta^2)} \ge \sqrt{\alpha_0(\mu^2 + \eta^2)}.$$

Since $\sqrt{1 + f_x(x_0)^2} > 1$ the assertion follows with $\alpha_* := \frac{\sigma b_2\sqrt{\alpha_0}}{a_{22}(1+f_x(x_0)^2)^{3/2}}$. □

The next result is an immediate consequence of Lemmas VI.29 and VI.30 and Theorem VI.12.

Corollary VI.31. *It holds that* $A_\pi \in \mathcal{H}(h^{4+\alpha}_{\mathbb{R}}, h^{1+\alpha}_{\mathbb{R}})$.

Lemma VI.32. *Let* $\kappa > 0$, $\gamma \in (0,\alpha)$ *and let* $K \subset \mathcal{W}$ *be compact. There is* $\rho \in (0,1]$, *a* ρ*-localization sequence* $\{(U_j,\theta_j)\colon 1 \le j \le m_\rho\}$ *for* $\mathbb{S}$, *and a positive constant* $C = C(K,\kappa,\rho)$ *such that*

$$\|(\theta_j A(f) - A_\pi(f,x_j)\theta_j)h\|_{1+\alpha,\mathbb{R}} \le \kappa\|\theta_j h\|_{4+\alpha,\mathbb{R}} + C\|h\|_{4+\gamma}$$

for all $h \in h^{4+\alpha}$, $1 \le j \le m_\rho$, *and* $f \in K$.

Proof. Fix $f \in K$ and $j \in \{1,\ldots,m_\rho\}$ and put $U := U_j$, $\theta := \theta_j$, and $x := x_j$. Observe that $x \in U$. Furthermore, choose $\chi, \psi \in \mathcal{D}(U)$ such that (VI.13) and (VI.14) are satisfied and there is a constant $C_\rho > 0$ depending on ρ such that

$$\|\chi\|_{4+\alpha,U} + \|\psi\|_{4+\alpha,U} + \|\theta\|_{4+\alpha,U} \le C_\rho. \tag{VI.22}$$

Then we have

$$\theta A(f) - A_\pi\theta = \chi[\theta, A(f)] + (\chi A(f) - A_\pi\chi)\theta. \tag{VI.23}$$

Moreover,

$$[\theta, A(f)] = [\theta, \sigma\mathcal{B}(f)\mathcal{T}(f)\mathcal{K}(f)] = \sigma[\theta,\mathcal{B}(f)\mathcal{T}(f)]\mathcal{K}(f) + \sigma\mathcal{B}(f)\mathcal{T}(f)[\theta,\mathcal{K}(f)].$$

Pick $\gamma \in (0,\alpha)$. We know from Lemma VI.25 that $\mathcal{B}(f)\mathcal{T}(f) \in \mathcal{L}(h^{2+\alpha}, h^{1+\alpha})$ and that $\mathcal{K}(f) \in \mathcal{L}(h^{4+\gamma}, h^{2+\gamma})$. The commutators are of lower order. More precisely, Lemma VI.18 says that $[\theta, \mathcal{B}(f)\mathcal{T}(f)] \in \mathcal{L}(h^{2+\gamma}, h^{1+\alpha})$ and Leibniz' rule implies $[\theta,\mathcal{K}(f)] \in \mathcal{L}(h^{4+\gamma}, h^{2+\alpha})$. Hence, $[\theta, A(f)] \in \mathcal{L}(h^{4+\gamma}, h^{1+\alpha})$. From this and (VI.22) we infer that

$$\|\chi[\theta, A(f)]h\|_{1+\alpha} \le C_\rho\|h\|_{4+\gamma} \quad \text{for all} \quad h \in h^{4+\alpha}. \tag{VI.24}$$

Choose $\kappa > 0$. Given $h \in h^{4+\alpha}$, let $u_4 := (\psi\chi\mathcal{T}(f)\mathcal{K}(f) - \mathcal{T}_\pi\mathcal{K}_\pi\psi\chi)\theta h$. The proof of Lemma VI.19 shows that

$$\begin{aligned}(\mu_0^2 + \mathcal{A}_\pi)u_4 = {} & [\mathcal{A}_\pi, \chi]\psi\mathcal{T}(f)\mathcal{K}(f)\theta h + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A}(f))\psi\mathcal{T}(f)\mathcal{K}(f)\theta h \\ & + \chi[\mathcal{A}(f),\psi]\mathcal{T}(f)\mathcal{K}(f)\theta h.\end{aligned}$$

Note that thanks to $\gamma_0\mathcal{T}(f) = \mathrm{id}_{h^{2+\alpha}}$, $\gamma_0\mathcal{T}_\pi = \mathrm{id}_{h^{2+\alpha}_{\mathbb{R}}}$, and $\psi\chi = \chi$ it holds that

$$\gamma_0 u_4 = (\gamma_0\chi\mathcal{K}(f) - \mathcal{K}_\pi\gamma_0\chi)\theta h = \gamma_0\chi(\mathcal{K}(f) - \mathcal{K}_\pi)\theta h + [\gamma_0\chi, \mathcal{K}_\pi]\theta h.$$

The unique solution of this boundary value problem in $h^{2+\alpha}_{\mathbb{H}}$ is given by

$$\begin{aligned}u_4 = {} & \mathcal{S}_\pi\big([\mathcal{A}_\pi,\chi]\psi\mathcal{T}(f)\mathcal{K}(f) + \chi(\mu_0^2 + \mathcal{A}_\pi - \mathcal{A}(f))\psi\mathcal{T}(f)\mathcal{K}(f) \\ & + \chi[\mathcal{A}(f),\psi]\mathcal{T}(f)\mathcal{K}(f)\big)\theta h \\ & + \mathcal{T}_\pi\big(\gamma_0\chi(\mathcal{K}(f) - \mathcal{K}_\pi) + [\gamma_0\chi, \mathcal{K}_\pi]\big)\theta h.\end{aligned}$$

We also have

$$\begin{aligned}&\chi\mathcal{B}(f)\mathcal{T}(f)\mathcal{K}(f)-\mathcal{B}_\pi\mathcal{T}_\pi\mathcal{K}_\pi\chi\\ &=\mathcal{B}_\pi\big(\chi\mathcal{T}(f)\mathcal{K}(f)-\mathcal{T}_\pi\mathcal{K}_\pi\chi\big)-\mathcal{B}_\pi\chi\mathcal{T}(f)\mathcal{K}(f)+\chi\mathcal{B}(f)\mathcal{T}(f)\mathcal{K}(f).\end{aligned}$$

These two facts and $\psi\chi=\chi$ give

$$\begin{aligned}&(\chi\mathcal{B}(f)\mathcal{T}(f)\mathcal{K}(f)-\mathcal{B}_\pi\mathcal{T}_\pi\mathcal{K}_\pi\chi)\theta h\\ &=\mathcal{B}_\pi\mathcal{S}_\pi\big([\mathcal{A}_\pi,\chi]\psi\mathcal{T}(f)\mathcal{K}(f)+\chi(\mu_0^2+\mathcal{A}_\pi-\mathcal{A}(f))\psi\mathcal{T}(f)\mathcal{K}(f)\\ &\qquad+\chi[\mathcal{A}(f),\psi]\mathcal{T}(f)\mathcal{K}(f)\big)\theta h\\ &\quad+\mathcal{B}_\pi\mathcal{T}_\pi\big(\gamma_0\chi(\mathcal{K}(f)-\mathcal{K}_\pi)+[\gamma_0\chi,\mathcal{K}_\pi]\big)\theta h\\ &\quad+\chi(\mathcal{B}(f)-\mathcal{B}_\pi)\psi\mathcal{T}(f)\mathcal{K}(f)\theta h+\chi[\psi,\mathcal{B}(f)]\mathcal{T}(f)\mathcal{K}(f)\theta h\\ &\quad+[\chi,\mathcal{B}_\pi]\psi\mathcal{T}(f)\mathcal{K}(f)\theta h.\end{aligned}$$

We know that the operators $\mathcal{B}_\pi$, $\mathcal{T}_\pi$, $\mathcal{S}_\pi$, $\mathcal{T}(f)$, and $\mathcal{K}(f)$ are bounded by a positive constant depending on K. Moreover, due to $\chi|\operatorname{supp}\theta=1$ it holds for all $1\le l\le m_\rho$ that

$$[\gamma_0\chi_l,\mathcal{K}_\pi(f,x_l)]=\big((1+f_x(x_l)^2)^{-3/2}+\mu_0^2\big)\big((\partial^2\gamma_0\chi_l)\theta_l h+2(\partial\gamma_0\chi_l)\partial(\theta_l h)\big)=0$$

on $\operatorname{supp}\theta_l$ and that

$$\|\gamma_0\chi(\mathcal{K}(f)-\mathcal{K}_\pi)\theta h\|_{2+\alpha,\mathbb{R}}=\|(\mathcal{K}(f)-\mathcal{K}_\pi)\theta h\|_{2+\alpha,\mathbb{R}\cap U}.$$

Observe that

$$(\mathcal{K}(f)-\mathcal{K}_\pi(f,x_l))\theta_l h=\big((1+f_x^2)^{-3/2}-(1+f_x(x_l)^2)^{-3/2}\big)\partial^2(\theta_l h)+\mu_0^2\theta_l h,$$

where $1\le l\le m_\rho$. Then we see that there is a positive constant C that depends on K such that

$$\|\gamma_0\chi(\mathcal{K}(f)-\mathcal{K}_\pi)\theta h\|_{2+\alpha,\mathbb{R}}\le C\|h\|_{4+\gamma}.$$

These arguments as well as Lemmas VI.17, VI.18, and VI.21 eventually imply the existence of a positive constant C depending on K but independent of ρ and of a positive constant C_ρ depending on K and ρ such that

$$\|(\chi\mathcal{B}(f)\mathcal{T}(f)\mathcal{K}(f)-\mathcal{B}_\pi\mathcal{T}_\pi\mathcal{K}_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}}\le C\rho^{1-\alpha}\|\theta h\|_{4+\alpha,\mathbb{R}}+C_\rho\|h\|_{4+\gamma}$$

for all $h\in h^{4+\alpha}$. Since $\rho\in(0,1]$ is arbitrary and C does not depend on ρ, we can choose ρ sufficiently small such that

$$\|(\chi A(f)-A_\pi\chi)\theta h\|_{1+\alpha,\mathbb{R}}\le\kappa\|\theta h\|_{4+\alpha,\mathbb{R}}+C_\rho\|h\|_{4+\gamma}.\qquad\text{(VI.25)}$$

Now the assertion follows from (VI.23), (VI.24), and (VI.25). □

We can now prove the following central result of this section. It uses the results collected so far and ideas similar to the proof of Theorem 4.1 in [36].

Theorem VI.33. *Given $f \in \mathcal{W}$, it holds that $A(f) \in \mathcal{H}(h^{4+\gamma}, h^{1+\gamma})$ for all $\gamma \in [\alpha, \beta]$.*

Proof. Pick $f \in \mathcal{W}$ and set $K := K_f := \{tf : t \in [0,1]\}$. Note that K is a compact subset of $\mathcal{W}$. Fix $\gamma \in [\alpha, \beta]$. In the same way as in the proof of Theorem VI.23 we find positive constants λ_* and $C = C(K)$ such that

$$\|h\|_{4+\gamma} + |\lambda| \|h\|_{1+\gamma} \leq C \|(\lambda + A(g))h\|_{1+\gamma} \tag{VI.26}$$

for all $h \in h^{4+\gamma}$, $\operatorname{Re}\lambda \geq \lambda_*$, and $g \in K$. From that proof we also infer that $\lambda_* + \mathcal{B}(g)\mathcal{T}(g) \in \mathcal{L}\mathrm{is}(h^{2+\alpha}, h^{1+\alpha})$ for all $g \in K$. We know that $\lambda_* + \mathcal{B}(0)\mathcal{T}(0) \in \mathcal{L}\mathrm{is}(h^{2+\tau+\alpha}, h^{1+\tau+\alpha})$, where $\tau \in \{0, 2\}$, and that

$$\mathrm{id}_{h^{4+\alpha}} + \mathcal{K}(0) = \mathrm{id}_{h^{4+\alpha}} + \partial^2 \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{2+\alpha}).$$

Consequently,

$$(\lambda_* + \mathcal{B}(0)\mathcal{T}(0))(\mathrm{id}_{h^{4+\alpha}} + \mathcal{K}(0)) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{1+\alpha}).$$

This operator can be equivalently reformulated as

$$\lambda_* + A(0) + \lambda_* \mathcal{K}(0) + \mathcal{B}(0)\mathcal{T}(0). \tag{VI.27}$$

We know that $\lambda_* \mathcal{K}(0) + \mathcal{B}(0)\mathcal{T}(0)$ belongs to $\mathcal{L}(h^{4+\alpha}, h^{2+\alpha})$ and that the imbedding $h^{2+\alpha} \hookrightarrow h^{1+\alpha}$ is compact, see Theorem III.1. Hence, the operator (VI.27) is a compactly perturbed isomorphism in $\mathcal{L}(h^{4+\alpha}, h^{1+\alpha})$, i.e., $\lambda_* + A(0)$ is onto if and only if it is one-to-one. But we already know this last property of the operator $A(0)$, cf. (VI.26). Consequently, $\lambda_* + A(0) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{1+\alpha})$. Since $0 \in K$ and (VI.26) holds for all $g \in K$, it follows that $\lambda_* + A(g) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{1+\alpha})$ for all $g \in K$. In particular, it holds that $\lambda_* + A(f) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}, h^{1+\alpha})$. Now the assertion follows from Propositions III.6 and III.8. □

Theorem VI.34. *Fix $\gamma \in ((2+\beta-\alpha)/3, 1)$ and put $E_\gamma := (h^{1+\alpha}, h^{4+\alpha})_\gamma$ and $\mathcal{X} := \mathcal{W} \cap E_\gamma$. For every $f_0 \in \mathcal{X}$ there exists a unique maximal classical solution of problem* (VI.21) *generating a local analytic semiflow on $\mathcal{X}$.*

Proof. Let $f_0 \in \mathcal{X}$ be given. Observe that $\mathcal{X} \subset \mathcal{W} \subset h^{3+\beta} = (h^{1+\alpha}, h^{4+\alpha})_{(2+\beta-\alpha)/3}$ and that $A \in C^\omega\big(\mathcal{W}, \mathcal{L}(h^{4+\alpha}, h^{1+\alpha})\big)$, see Lemma VI.25. Fix $\gamma \in ((2+\beta-\alpha)/3, 1)$ and note that $h^{3+\beta} \supset E_\gamma \supset h^{4+\alpha}$. From Theorem VI.33 we deduce that $A(f) \in \mathcal{H}(h^{4+\beta}, h^{1+\beta})$ and that $A(f)$ is the part in $h^{1+\alpha}$ of this operator having the domain $h^{4+\alpha}$. Since $h^{1+\alpha} = (h^{1+\beta}, h^{4+\beta})_{(\alpha-\beta)/3}$ we conclude from Proposition III.11 that $A(f) \in \mathcal{M}_\gamma(h^{4+\alpha}, h^{1+\alpha})$ for all $f \in \mathcal{X}$. Finally, observe that $F \in C^\omega(\mathcal{W}, h^{1+\alpha})$. Then we may infer from Theorem III.14 that there exists $T > 0$ and a unique classical solution $f \in C([0,T), \mathcal{V}) \cap C^1([0,T), h^{1+\alpha})$ of (VI.21) with maximal interval of existence $[0,T)$. Another consequence of Theorem III.14 is the semiflow property as asserted. Lemmas VI.24 and VI.27 now complete the proof. □

Chapter VII

Outlook

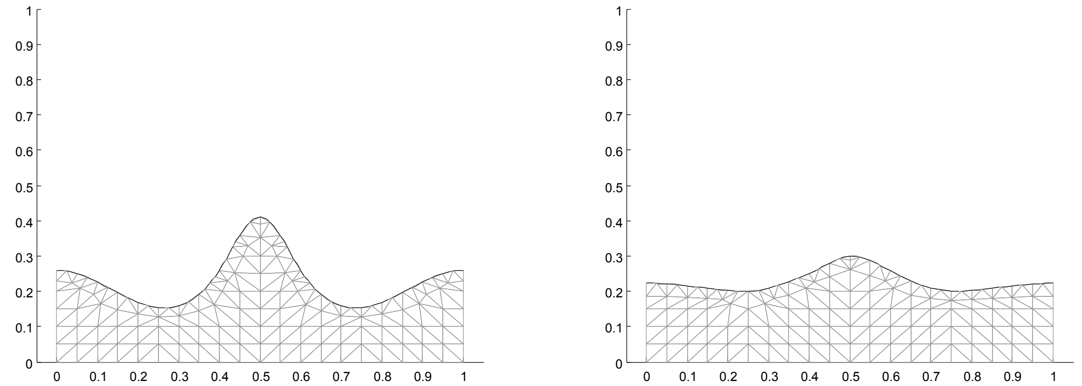
The main results from the last part of this thesis are powerful existence results for periodic Newtonian Hele-Shaw flows. Their main attribute lies in the possibility to determine whether a certain initial condition is admissible in the sense that the respective Hele-Shaw flow has a unique classical solution.

Treating the evolution equation governing the movement of the free surface when surface tension effects are considered as a quasilinear one renders it possible to prove the existence of a unique classical solution in a subspace $E_\gamma = (h^{1+\alpha}, h^{4+\alpha})_\gamma$, $\gamma \in ((2+\beta-\alpha)/3, 1)$, of $h^{4+\alpha}$, where $\alpha \in (0,1)$ and $\beta \in (\alpha, 1)$, cf. Theorem VI.34. The only requirement for an initial condition $f_0 \in E_\gamma$ to be admissible is that it have a lower bound $-c < 0$. But, as c is arbitrary, for *every* function $f_0 \in E_\gamma$ this constant can be chosen large enough so that f_0 is admissible, i.e., $c > -\min_{x\in\mathbb{S}} f(x)$. In this sense this result is optimal, which means that the existence of a unique classical solution is proved for *arbitrary* initial data.

For Hele-Shaw flows without surface tension, where the corresponding evolution equation for the free surface has to be considered as being fully nonlinear, a structure condition for admissible initial conditions in $h^{2+\alpha}$ can be derived, see the set $\mathcal{W}$ in (VI.2). Even this result can be considered as optimal since it can be explicitly verified whether a given initial condition is admissible or not. And since $\mathcal{W}$ is unbounded in $h^{2+\alpha}$, see Lemma VI.15, no a priori restrictions can be assigned to functions in $\mathcal{W}$; specifically to their curvature since second derivatives do not occur in the definition of $\mathcal{W}$. On the other hand, numerical simulations suggest that even for initial conditions that do not belong to $\mathcal{W}$ classical solutions exist, see Figures VII.1a, VII.1b for the evolution of such an initial condition visualized with the numerical scheme from Chapter V.

A natural question that arises is whether it is possible to carry over the techniques from Chapter VI to the problem studied in Chapter IV in order to obtain similar well-posedness results for non-Newtonian Hele-Shaw flows. The derivation of the results in Chapter VI for Newtonian fluids strongly relies on a linear theory for the respective boundary value problem at a fixed time, see, e.g., Theorem VI.5. A similar theory that is not linear anymore would have to be established if one tried to attack this question.

Another probably more feasible question is the well-posedness of the free boundary problem from Chapter IV in an unbounded geometry. For instance,



(a) Non-parabolic initial configuration

(b) Solution at $T = 100$ with $\sigma = \iota = 0$, $\delta = 1$

Figure VII.1: Simulated flow for an initial condition not belonging to $\mathcal{W}$

such a problem can be used to describe the flow of mud (or, of course, some other non-Newtonian fluid) in a (large) porous medium; see also [26], where a problem of water-mud-interaction is studied with the same modeling approach. Such a flow under the influence of gravity obeys the system of equations

$$\begin{aligned} \mathcal{Q}u &= 0 && \text{in} \quad \Omega(f), \\ \partial_y u &= 0 && \text{on} \quad \Gamma_0, \\ u &= f && \text{on} \quad \Gamma(f), \\ \lim_{|y|\to\infty} u &= c && \text{on} \quad \mathbb{R}, \\ \partial_t f &= -\frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla u|^2)}\partial_\nu u && \text{on} \quad \Gamma(f), \\ f(0) &= f_0 && \text{on} \quad \mathbb{R}, \end{aligned} \tag{VII.1}$$

where—for the sake of simplicity—a (non-stabilizing) no-flux condition is imposed on $\Gamma_0 := \mathbb{R} \times \{0\}$ and $c > 0$ is a fixed constant. Note that, of course, the domain covered by the fluid and the free surface are now defined by

$$\begin{aligned} \Omega(f) &:= \{(x,y) \in \mathbb{R}^2 : 0 < y < f(x)\}, \\ \Gamma(f) &:= \{(x, f(x)) : x \in \mathbb{R}\}, \end{aligned}$$

respectively. A reduction of the above problem with the same domain diffeomorphism as in Chapter IV to an abstract evolution equation gives formally the same Cauchy problem as in Chapter IV, which is

$$\partial_t f = \Phi(f) := \mathcal{B}(f, \mathcal{T}(f)), \quad f(0) = f_0, \quad \text{in} \quad h^{1+\alpha}_{\mathbb{R}}, \tag{VII.2}$$

where $\mathcal{B}(f, \cdot\,)$ is the transformed boundary operator $-\frac{\sqrt{1+f_x^2}}{\bar{\mu}(|\nabla\cdot|^2)}\partial_\nu$ and $\mathcal{T}$ is the solution operator of the elliptic boundary value problem consisting of the first three equations in (VII.1) in the transformed coordinates. An inspection of Section IV.2 shows that $\mathcal{T}(c) = c$ and that

$$\partial\Phi(c)h = \frac{1}{c\bar{\mu}(0)}\gamma_0\partial_2\partial\mathcal{T}(c)h \quad \text{for} \quad h \in h^{2+\alpha}_{\mathbb{R}},$$

where $\partial\mathcal{T}(c)h$ solves the system

$$\begin{aligned} c^2\partial_{11}w + \partial_{22}w &= 0 \quad \text{in} \quad \Omega := \mathbb{R} \times (0,1), \\ \partial_2 w &= 0 \quad \text{on} \quad \Gamma_1, \\ w &= h \quad \text{on} \quad \Gamma_0. \end{aligned}$$

This system can be conveniently solved with the Fourier transform: It holds that

$$\mathcal{F}[c^2\partial_{11}w + \partial_{22}w](\eta, y) = (-c^2\eta^2 + \partial_{22})\mathcal{F}w(\eta, y), \quad (\eta, y) \in \mathbb{R} \times (0,1).$$

Given $h \in h^{2+\alpha}_{\mathbb{R}}$, this vanishes and satisfies the required boundary conditions if and only if

$$\mathcal{F}w(\eta, y) = \frac{1}{1 + \mathrm{e}^{-2c\eta}}(\mathrm{e}^{-c\eta y} + \mathrm{e}^{c\eta(y-2)})\mathcal{F}h(\eta), \quad (\eta, y) \in \mathbb{R} \times (0,1),$$

which means

$$(\partial\mathcal{T}(c)h)(x, y) = \mathcal{F}^{-1}\Big[\eta \mapsto \frac{\mathrm{e}^{-c\eta y} + \mathrm{e}^{c\eta(y-2)}}{1 + \mathrm{e}^{-2c\eta}}\Big]\mathcal{F}h(x), \quad (x, y) \in \Omega.$$

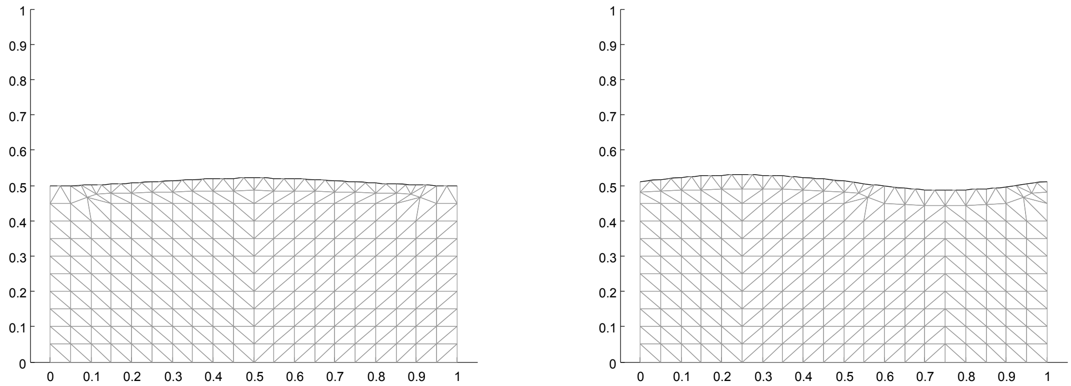
Hence,

$$\partial\Phi(c)h = \mathcal{F}^{-1}\Big[\eta \mapsto \frac{\eta}{\bar{\mu}(0)}\frac{-1 + \mathrm{e}^{-2c\eta}}{1 + \mathrm{e}^{-2c\eta}}\Big]\mathcal{F}h = \mathcal{F}^{-1}a\mathcal{F}h$$

with $a(\eta) := -\frac{1}{\bar{\mu}(0)}\eta\tanh(c\eta)$ for $\eta \in \mathbb{R}$. If the factor $\frac{1}{\bar{\mu}(0)}$ is disregarded, the symbol a is exactly the same as the symbol a_0 from [23, Section 4.1]. Consequently, the same arguments as in [23] apply to show the local well-posedness of the Cauchy problem (VII.2) and thus, the existence of a unique classical solution of problem (VII.1) near flat solutions, cf. [23, Theorem 5.1], provided the equivalence of (VII.1) and (VII.2) can be rigorously established.

By suitable modifications it is possible to reformulate the two-dimensional formulation of the Hele-Shaw free boundary problem considered in this thesis as a multi-dimensional one. For instance, this is done in [35] for Newtonian fluids on an unbounded geometry and in [32] for non-Newtonian fluids on a periodic geometry. It turns out that for multi-dimensional Hele-Shaw flows it is possible to obtain results similar to the main results in Chapter VI for Newtonian fluids on a periodic geometry, see [73].

The stabilizing term b that is a major concern of this thesis is assumed to depend on the free surface only. However, it could be interesting to consider another dependance of b, namely its dependance on the spatial variable. With this approach a fixed stabilizing source could be modeled having a time independent effect on the system, e.g., an approximate point source being nonzero near some



(a) Evolution at $T = 1000$ under the influence of the point source b_1 at Γ_0, $\delta = 1$

(b) Evolution at $T = 1000$ for the prescribed pressure b_0 at Γ_0, $\delta = 0$

Figure VII.2: Simulated flow for a flat initial condition $c = 0.51$ with $\sigma = \iota = 0$

points in S and zero otherwise. Figure VII.2a shows a simulation that involves the point source

$$b_1(x) := \begin{cases} 0.2 & \text{for} \quad x = 0.5, \\ -0.1 & \text{for} \quad x \in \{0.25, 0.75\}, \\ 0 & \text{for} \quad x \in [0, 0.2] \cup [0.3, 0.45] \cup [0.55, 0.7] \cup [0.8, 1] \end{cases}$$

such that—for conservation of mass—$\int_0^1 b_1(x)\, dx = 0$ for the flat initial condition $c = 0.51$ by means of the numerical scheme from Chapter V, which can readily deal with this situation. As a second example, Figure VII.2b shows the effect of the periodic pressure distribution

$$b_0(x) := 0.51 + 0.25 \times \sin(2\pi x), \quad x \in [0, 1],$$

on the lower boundary component depending on the spatial variable. Observe that the form of b_0 can be recognized in the evolved free surface. However, it turns out that in these cases some techniques from this thesis are not appropriate anymore; specifically, it is not possible to determine the linearization about some constant function $c \in (0, 1)$ of the evolution operator from Section IV.2 since the solution operator for the potential problem $\mathcal{T}$—even at the constant c—cannot be expressed explicitly as it is done in (IV.13). Notice that in case $\delta = 0$ and for b depending *only* on the spatial variable a well-posedness result for the corresponding flow problem was proved in [31] for small initial data. However, the function b is allowed to be nonconstant only in the general formulation of the problem. When proving the well-posedness of the problem for an initial datum near some flat solution, the authors suppose that b is a constant function.

A question arising in the numerical treatment of Hele-Shaw free boundary problems is the simulation of non-Newtonian Hele-Shaw flows, which is a much more involved task compared to the Newtonian case since a nonlinear boundary value problem has to be solved at every time step: The elliptic operator $\mathcal{Q}$ in this boundary value problem depends on a modified viscosity function $1/\bar{\mu}$ that, in turn, depends on the gradient of the solution of the boundary value problem. In [39] one approach to this problem is given and applied to a Hele-Shaw problem in a radial geometry for solutions exhibiting certain symmetry properties.

Bibliography

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.

[2] H. Amann. *Linear and Quasilinear Parabolic Problems*, volume I. Birkhäuser-Verlag, Basel, 1995.

[3] A. C. Amar. Ground-water recharge simulation. *J. Hydr. Div.*, 101:1235–1247, 1975.

[4] S. B. Angenent. Nonlinear analytic semiflows. *Proc. Roy. Soc. Edinburgh*, 115 A:91–107, 1990.

[5] W. Arendt and S. Bu. Operator-valued Fourier multipliers on periodic Besov spaces and applications. *Proc. Edinburgh Math. Soc.*, 47:15–33, 2004.

[6] R. Aris. *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*. Prentice-Hall, London, 1962.

[7] N. S. Asaithambi. Computation of free-surface flows. *J. Comp. Phys.*, 73:380–394, 1987.

[8] G. Astarita and G. Marrucci. *Principles of non-Newtonian Fluid Mechanics*. McGraw-Hill, London, 1974.

[9] R. L. Bailey. Lesser known applications of ferrofluids. *J. Magn. Magn. Mat.*, 39:178–182, 1983.

[10] J. Bear and Y. Bachmat. *Introduction to Modeling of Transport Phenomena in Porous Media*. Kluwer Academic Publisher, Dordrecht, 1990.

[11] J. Bergh and J. Löfström. *Interpolation Spaces*. Springer-Verlag, Berlin, 1976.

[12] A. K. Biswas. *History of Hydrology*. North-Holland Publishing Company, Amsterdam, 1970.

[13] R. Bonnerot and P. Jamet. Numerical computation of the free boundary for the two-dimensional Stefan problem by space-time finite elements. *J. Comp. Phys.*, 25:163–181, 1977.

[14] D. Bothe and J. Prüss. L_p-theory for a class of non-Newtonian fluids. *SIAM J. Math. Ann.*, 39(2):379–421, 2007.

[15] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 2008.

[16] X. Chen, J. Hong, and F. Yi. Existence, uniqueness, and regularity of classical solutions of the Mullins–Sekerka problem. *Comm. Part. Diff. Equ.*, 21(11&12):1705–1727, 1996.

[17] M. D. Cowley and R. E. Rosensweig. The interfacial stability of a ferromagnetic fluid. *J. Fluid Mech.*, 30(4):671–688, 1967.

[18] J. Crank. *Free and Moving Boundary Problems*. Oxford University Press, London, 1984.

[19] G. da Prato and P. Grisvard. Equations d'évolution abstraites non linéaires de type parabolique. *Ann. Mat. Pura Appl.*, 120:329–396, 1979.

[20] G. Dore and A. Favini. On the equivalence of certain interpolation methods. *Bol. U.M.I.*, 1-B(7):1227–1238, 1987.

[21] J. Duchon and R. Robert. Evolution d'une interface par capillarité et diffusion de volume I. Existence local en temps. *Ann. Inst. H. Poincaré, Analyse non linéaire*, 1:361–378, 1984.

[22] C. M. Elliot and J. R. Ockendon. *Weak and Variational Methods for Moving Boundary Problems*. Pitman, Boston, 1982.

[23] J. Escher and Z. Feng. Stabilization of flows through porous media. *J. Evol. Equ.*, 7:567–586, 2007.

[24] J. Escher and A.-V. Matioc. Radially symmetric growth of nonnecrotic tumors. *Nonl. Diff. Equ. Appl.*, 17:1–20, 2010.

[25] J. Escher and A.-V. Matioc. Well-posedness and stability analysis for a moving boundary problem modelling the growth of nonnecrotic tumors. *Disc. Cont. Dyn. Sys. Ser. B*, 15:573–596, 2011.

[26] J. Escher and A.-V. Matioc. On the well-posedness of a mathematical model describing water-mud interaction. *Math. Meth. Appl. Sci.*, 2012. DOI: 10.1002/mma.2692.

[27] J. Escher, A.-V. Matioc, and B.-V. Matioc. Analysis of a ferrofluid in a radial magnetic field. *An. Univ. Vest Ser. Mat.-Inform.*, 47:27–44, 2009.

[28] J. Escher, A.-V. Matioc, and B.-V. Matioc. Classical solutions and stability results for Stokesian Hele-Shaw flows. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, 9:325–349, 2010.

[29] J. Escher and B.-V. Matioc. On periodic Stokesian Hele-Shaw flows with surface tension. *Euro. J. Appl. Math.*, 19:717–734, 2008.

[30] J. Escher and B.-V. Matioc. Stability of the equilibria for periodic Stokesian Hele-Shaw flows. *J. Evol. Equ.*, 8:513–522, 2008.

[31] J. Escher and B.-V. Matioc. A moving boundary problem for periodic Stokesian Hele-Shaw flows. *Interfaces and Free Boundaries*, 11:119–137, 2009.

[32] J. Escher and B.-V. Matioc. Multidimensional Hele-Shaw flows modelling Stokesian fluids. *Math. Meth. Appl. Sci.*, 32:577–593, 2009.

[33] J. Escher and G. Prokert. Stability of the equilibria for spatially periodic flows in porous media. *Nonl. Anal.*, 45:1061–1080, 2001.

[34] J. Escher and G. Simonett. Maximal regularity for a free boundary problem. *Nonl. Diff. Equ. Appl.*, 2:463–510, 1995.

[35] J. Escher and G. Simonett. Analyticity of the interface in a free boundary problem. *Math. Ann.*, 305:439–459, 1996.

[36] J. Escher and G. Simonett. Classical solutions for Hele-Shaw models with surface tension. *Adv. Diff. Equ.*, 2:619–642, 1997.

[37] J. Escher and G. Simonett. Classical solutions of multidimensional Hele-Shaw models. *SIAM J. Math. Anal.*, 28(5):1028–1047, 1997.

[38] J. Escher and M. Wenzel. Stabilization of periodic Stokesian Hele-Shaw flows of ferrofluids. *Appl. Anal.*, 92(7):1474-1494, 2013.

[39] P. Fast, L. Kondic, M. J. Shelley, and P. Palffy-Muhoray. Pattern formation in non-Newtonian Hele-Shaw flow. *Phys. Fluids*, 13(5):1191–1212, 2001.

[40] H. O. Fattorini. *The Cauchy Problem*. Addison-Wesley Publishing Company, London, 1983.

[41] A. Freeze. Henry Darcy and the fountains of Dijon. *Ground Water*, 32(1):23–30, 1994.

[42] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 2001.

[43] S. Gross and A. Reusken. *Numerical Methods for Two-Phase Incompressible Flows*. Springer-Verlag, Berlin, 2011.

[44] G. Grubb. *Functional Calculus of Pseudodifferential Boundary Problems*. Birkhäuser-Verlag, Basel, 1996.

[45] H. S. Hele-Shaw. Investigation of the nature of surface resistance of water and of stream-line motion under certain experimental conditions. *Trans. Inst. Nav. Archit.*, 40:21–40, 1898.

[46] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[47] R. Kaiser and R. E. Rosensweig. Study of ferromagnetic liquid. NASA CR-1407, National Aeronautics and Space Administration, Washington, D.C., August 1969.

[48] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin, 1995.

[49] H. Kawarada and H. Koshigoe. Unsteady flow in porous media with a free surface. *Japan J. Indust. Appl. Math.*, 8:41–82, 1991.

[50] L. Kondic, P. Palffy-Muhoray, and M. J. Shelley. Models of non-Newtonian Hele-Shaw flow. *Phys. Rew. E*, 54(5):536–539, 1996.

[51] O. N. Ladyzhenskaya and N. N. Ural'tseva. *Linear and Quasilinear Elliptic Equations*. Academic Press, New York and London, 1968.

[52] G. M. Lieberman. Solvability of quasilinear elliptic equations with nonlinear boundary conditions. *Trans. Amer. Math. Soc.*, 273(2):753–765, 1982.

[53] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser-Verlag, Basel, 1995.

[54] F. Mashayek and N. Ashgriz. A hybrid finite-element-volume-of-fluid method for simulating free surface flows and interfaces. *Int. J. Num. Meth. Fluids*, 20:1363–1380, 1995.

[55] B.-V. Matioc. *Viscous Fingering in Mathematical Fluid Dynamics via Bifurcation*. PhD thesis, Leibniz Universität Hannover, 2009.

[56] A. M. Meirmanov. *The Stefan Problem*. De Gruyter, Berlin, 1992.

[57] J. A. Miranda. Rotating Hele-Shaw cells with ferrofluids. *Phys. Rew. E*, 62(2):2985–2988, 2000.

[58] M. Muskat. *Flow of Homogeneous Fluids*. McGraw-Hill, New York, 1937.

[59] J. L. Neuringer and R. E. Rosensweig. Ferrohydrodynamics. *Phys. Fluids*, 7(12):1927–1937, 1964.

[60] B. E. Petersen. *Introduction to the Fourier Transform and Pseudo-differential Operators*. Pitman, Boston, 1983.

[61] D. Pnueli and C. Gutfinger. *Fluid Mechanics*. Cambridge University Press, Cambridge, 1992.

[62] H. Rasmussen and D. Salhani. Unsteady porous flow with a free surface. *J. Appl. Math.*, 27:307–318, 1981.

[63] M. Reissig. The existence and uniqueness of analytic solutions for a moving boundary problem for Hele-Shaw flows in the plane. *Nonl. Anal., Theo., Meth. & Appl.*, 23(5):565–576, 1994.

[64] R. E. Rosensweig. *Ferrohydrodynamics*. Dover, New York, 1997.

[65] R. E. Rosensweig and R. Kaiser. Study of ferromagnetic liquid. NASA CR-91684, National Aeronautics and Space Administration, Washington, D.C., March 1967.

[66] H.-J. Schmeisser and H. Triebel. *Topics in Fourier Analysis and Function Spaces*. Wiley & Sons, New York, 1987.

[67] E. J. Shaughnessy, I. M. Katz, and J. P. Schaffer. *Introduction to Fluid Mechanics*. Oxford University Press, New York, 2005.

[68] G. Simonett. Quasilinear parabolic equations and semiflows. In *Evolution Equations, Control Theory, and Biomathematics*, page 523–536, New York, 1994. M. Dekker.

[69] J. Stefan. Ueber die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Sitzungsber. der kais. Acad. d. Wiss. in Wien, math.-naturw. Cl.*, 98:269–286, 1889.

[70] J. Stefan. Ueber die Theorie der Eisbildung. *Monatsh. f. Mathematik*, 1:1–6, 1890.

[71] H. Triebel. *Theory of Function Spaces*. Birkhäuser-Verlag, Basel, 1983.

[72] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. J. Ambrosius Barth Verlag, Heidelberg, 1995. 2nd edition.

[73] M. Wenzel. Stabilization of free surface flows in periodic porous media. *Nonl. Anal.*, 2013. submitted.

[74] O. C. Zienkiewicz and J. Z. Zhu. The superconvergent patch recovery (SPR) and adaptive finite element refinement. *Comp. Meth. Appl. Mech. Eng.*, 101:207–224, 1992.

Lebenslauf

Persönliche Daten

Name	Michael Wenzel
Anschrift	Institut für Angewandte Mathematik Leibniz Universität Hannover Welfengarten 1 30167 Hannover E-Mail: wenzel@ifam.uni-hannover.de
Geburtsdatum, -ort	23.04.85 in Celle

Studium und Abitur

11/11 – 04/12	Ecole Normale Supérieure de Cachan, Frankreich Forschungsaufenthalt am Laboratoire de Mécanique et Technologie
12/10 – 11/13	Leibniz Universität Hannover Doktorand in der International Research Training Group 'Virtual Materials and Structures and Their Validation'
10/09 – 11/10	Carl von Ossietzky Universität Oldenburg Master-Studium Mathematik, Nebenfach Chemie
Abschluss	Master of Science in Mathematik
Thema der Arbeit	'Amplitude Equations for Maxwell-Lorentz Systems from Nonlinear Optics'
08/09 – 01/10	Linköping University, Schweden Studienaufenthalt
10/05 – 09/09	Carl von Ossietzky Universität Oldenburg Bachelor-Studium Mathematik, Chemie
Abschluss	Fach-Bachelor Mathematik Zwei-Fächer-Bachelor Mathematik/Chemie
Thema der Arbeit	'Langzeitdynamik der Kuramoto-Sivashinsky Gleichung'
06/04	Hermann-Billung-Gymnasium, Celle Abitur

Berufserfahrung und Zivildienst

seit 12/10	Leibniz Universität Hannover Wissenschaftlicher Mitarbeiter am Institut für Angewandte Mathematik
11/06 – 08/09 mit Unterbrechungen	Carl von Ossietzky Universität Oldenburg Studentische Hilfskraft am Institut für Mathematik
09/04 – 06/05	Stadtkirchengemeinde St. Marien, Celle

Liste wissenschaftlicher Publikationen

- J. Escher and M. Wenzel. Stabilization of periodic Stokesian Hele-Shaw flows of ferrofluids. *Appl. Anal.*, 92(7):1474-1494, 2013.