Essays on PD-LGD Dependencies

Modeling Issues, Estimation Procedures, and Predictive Accuracy

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Abstract

Credit risk is the predominant source of risk for a financial institution. It is measured by the expected loss (EL), which is the product of a claims default probability (PD) and the expected loss rate given default (ELGD). Dating back to the structural approach by Merton (1974), several types of models were developed in the past decades in order to quantify both numbers. Common feature of the modeling approaches, particularly of those used in practise, is that ELGD is assumed to be independent from PD. According to empirical evidence this assumption does not hold. One approach developed by Pykhtin (2003) relates the processes underlying PD and ELGD by an unobservable systematic risk factor as well as by unobservable idiosyncratic risk driving both quantities simultaneously.

This thesis analysis the economic model by Pykhtin (2003) in several ways. First of all, it is extended to an econometric model by implementing observable co-variates. Secondly, a Maximum Likelihood estimation (MLE) procedure for a simultaneous derivation of all model parameters is provided. Since LGDs are only observable in the event of default, a sample selection bias would arise from a separate estimation of the parameters driving PD and LGD. This sample selection problem is the subject of a subsequent empirical and simulation based analysis. The empirical analysis with Moody's bond data shows that the processes underlying PD and LGD are highly correlated with respect to the common idiosyncratic risk. Thus, a separate estimation is most likely to yield biased parameter estimates. A performance comparison based on the same data underlines that an OLS estimation yields less precise LGD forecasts than the MLE procedure accounting for the correlation. The simulation study compares the impact of the sample selection bias on PD, EL, ELGD, and economic capital forecasts for different data generation processes. It is shown that biased parameter estimates lead to a systematic underestimation of economic capital charges, whereas the simultaneous MLE procedure neither underestimates nor overestimates economic capital in a systematic manner.

Keywords: Asset Value, Loss Given Default, Sample Selection Bias

Zusammenfassung

Kreditrisiken bilden die für das Tagesgeschäft von Finanzinstituten vorherrschende Risikoart. Gemessen wird es anhand des erwarteten Verlustes (EL), welcher als Produkt aus Ausfallwahrscheinlichkeit (PD) und erwarteter Verlustquote bei Ausfall (ELGD) berechnet wird. Ausgehend von dem Strukturansatz von Merton (1974) wurden in den vergangen Jahrzehnten zahlreiche Modelltypen entwickelt. Diesen und insbesondere den in der Praxis verwendeten Ansätzen ist gemein, dass ELGD und PD als voneinander unabhängig betrachtet werden. Vor dem Hintergrund empirischer Forschungsergebnisse ist diese Annahme nicht haltbar. Ein von Pykhtin (2003) entwickelter Ansatz modelliert die Prozesse für PD und ELGD sowohl in Abhängigkeit eines nicht beobachtbaren systematischen Risikofaktors, als auch von idiosynkratischen Risiken abhängig, die beide Risikoparameter simultan beeinflussen.

Diese Dissertation analysiert das ökonomische Modell von Pykhtin (2003) in vielerlei Hinsicht. Zunächst wird es durch die Implementierung von beobachtbaren Variablen zu einem ökonometrischen Modell erweitert. Anschließend wird eine Maximum Likelihood Schätzmethode (MLE) zu der simultanen Bestimmung aller Modellparameter vorgestellt. Da LGDs nur bei Eintritt eines Ausfallereignisses beobachtbar sind, würde bei einer separaten Schätzung der Parameter, die PD und ELGD beeinflussen, ein Selektionsbias auftreten. Dieser Selektionsbias wird in einer empirischen sowie in einer simulationsbasierten Studie analysiert. Die empirische Analyse anhand von Moody's Anleihedaten zeigt, dass die Prozesse, welche PD und LGD zu Grunde liegen, in Bezug auf das idiosynkratische Risiko hochgradig korreliert sind. Basierend auf den selben Daten unterstreicht ein Perfomancevergleich, dass separate Schätzungen weniger präzise LGD Prognosen liefern als die MLE Methode, welche die Korrelation mit berücksichtigt. Die Simulationsstudie vergleicht die Auswirkung des Selektionbiases für verschiedene Daten generierende Prozesse auf die Prognosen für PD, EL, ELGD und ökonomisches Kapital. Es zeigt sich, dass verzerrte Parameterschätzungen zu einer systematischen Unterschätzung des benötigten ökonomischen Kapitals führen, während die simultane MLE Methode dieses weder systematisch unter- noch überschätzt.

Schlagwörter: Firmenwert, Sample Selection Bias, Verlustquote

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Chapter 1

Credit Risk – an Introductory Review

1.1 The Importance of Credit Risk for Financial Institutions

Managing risks effectively is one of the most important tasks of a financial institution's (FI) daily business. In particular, five risk categories have a severe magnitude on the sustainable success of a FI: credit risk, operational risk, market risk, liquidity risk, and systemic risk (compare Duffie & Singleton (2003), p. 3).

The first pillar of the Basel II framework defines minimum capital requirements for the first three risk categories (compare Basel Committee on Banking Supervision (2006), pp. 12–203). The purpose of these capital requirements is to cover the financial institution against an unexpected loss, i.e., a negative deviation from the expected loss associated with, e.g., granting a loan.

Liquidity risk and systemic risk, however, have not been adequately considered in the Basel II framework, as highlighted by the Global Financial Crisis (GFC, compare Bank for International Settlements (2009), pp. 23–31). Particularly, the breakdown of the overnight interbank market due to a loss of confidence in the aftermath of the Lehman Brothers insolvency on September 15, 2008 resulted in liquidity troubles for many FIs. As a consequence, one of the Basel Committee on Banking Supervision's (BCBS) focal points in extending the Basel II framework[1] to Basel III, was the introduction of a global liquidity standard.[2] Furthermore, rules, which account for the systemic risks, resulting from procyclical effects of the capital requirements and from the interconnectedness of large banks, were provided.[3]

Nevertheless, the GFC was in first place caused by wrong assessments of credit risk (compare Gasha et al. (2009), p. 3), i.e., of 'the risk that the value of a portfolio changes due to unexpected changes in the credit quality of issuers or trading partners' (McNeil et al. (2005), p. 327), inherent in credit derivatives (compare Bank for International Settlements (2009), pp. 8–10). This is why the BCBS put forth rules on dealing with counterparty credit risk and external ratings. Both of them were crucial reasons behind the misspecified risk assessments of derivative products prior to the crisis. In addition, a supplemental maximum leverage ratio is part of the Basel III framework (compare Basel Committee on Banking Supervision (2011), p. 4).

Yet, not just since the turmoils on the markets for credit derivatives in the emergence of the GFC, one could imagine the relevance of credit risk for the weal and woe of a FI. Figure 1.1 shows this by presenting the average quarterly amount of loans and leases outstanding by US commercial banks from 1985 to 2011. Furthermore, the figure presents the relation of loans and leases to total assets for the same data base. On the one hand, one can see the vast and almost constant growth of the outstanding loan amount, from about \$1.5 trillion in the first quarter of 1985 up to about \$6.9 trillion in the fourth quarter of 2008. On the other hand, the outstanding loan amount to total assets ranges from 59% to 64% during this period. Thus, credit exposures typically are not only a rising quantity, but they also build the largest block of a financial institution's assets. Consequently, the credit risk taken by FIs are enormous and underline the need for an adequate measurement and management of these risks.

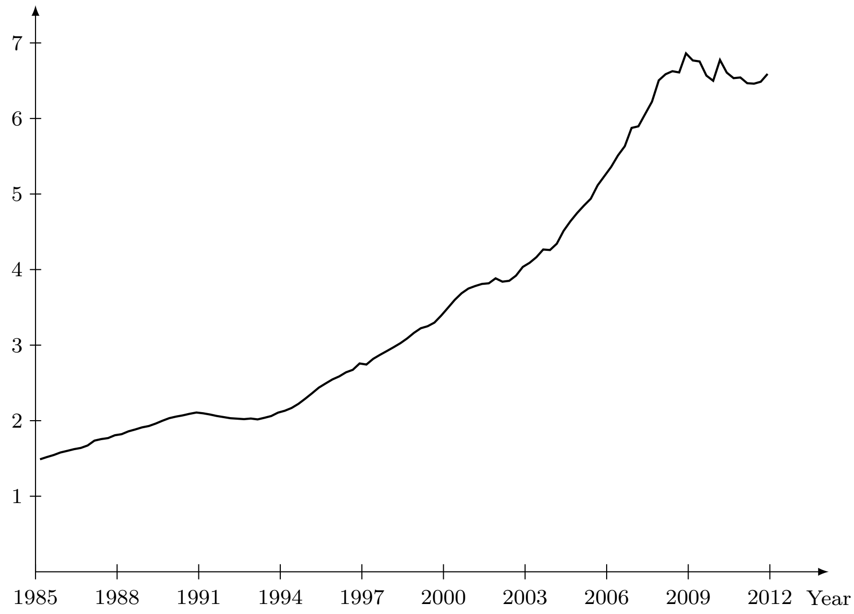
1 Basel II's single note on liquidity risk: 'Each bank must have adequate systems for measuring, monitoring and controlling liquidity risk.' Basel Committee on Banking Supervision (2006), p. 208.

2 Among other requirements, financial institutions now have to fulfil two minimum standards to account for liquidity risk. In order to survive a severe stress scenario for one month, a Liquidity Coverage Ratio > 1 has to be satisfied. In order to maintain a sustainable structure of assets and liabilities, a Net Stable Fundings Ratio also has to be satisfied with a value > 1. Compare Basel Committee on Banking Supervision (2011), pp. 8–9.

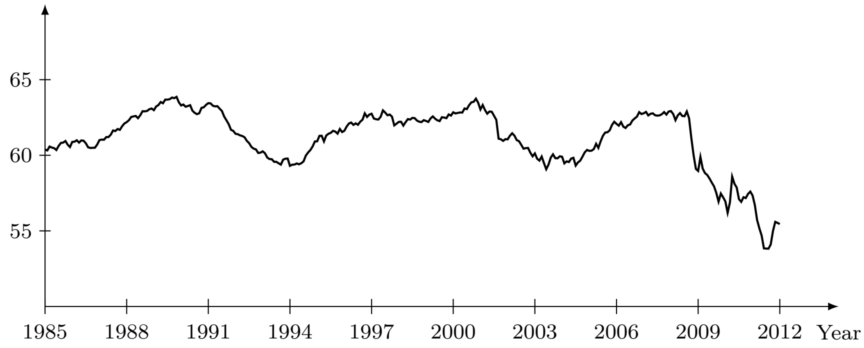
3 These rules include, e.g., a countercyclical buffer for capital requirements, and raising the risk weights on exposures to financial institutions relative to the non-financial corporate sector. Compare Basel Committee on Banking Supervision (2011), p. 5–8.

Figure 1.1: Loans and leases outstanding by US commercial banks from 1985 to 2011

(a) LaL per Quarter (in \$ TN)



(b) $\frac{\text{LaL}}{\text{TA}}$ per Month (in %)



This figure shows in panel (a) that the quarterly average amount of loans and leases (LaL) outstanding by US commercial banks increased substantially from 1985 to the end of 2008, where the huge value of write-offs due to the losses during the GFC stopped the constant growth of loan amounts. In panel (b) the figure shows the monthly relation between loans and leases outstanding and total assets (TA) of US commercial banks for the same period of time. Up to the same cut-off point the relation ranges between about 60% and 65%, while afterwards it decreases sharply to a low point of about 54%. Source: http://www.federalreserve.gov

1.2 Credit Risk Models in Academic Literature and Practise

1.2.1 Measuring and Modeling Credit Risk

According to the definition of credit risk from the previous section, the main objective of measuring credit risk is the prognosis of future value reductions for a portfolio of defaultable claims $i = 1, ..., N$ over a given period of time. Since, in general, these value reductions, i.e., losses, are stochastic, the challenge is to determine their distribution or statistic measures of their distribution (compare McNeil et al. (2005), p. 4).

Principally, this portfolio loss distribution is modeled in two stages. On the first stage, the monetary loss for one defaultable claim i is defined as the product of three components:

1. The exposure at default, EAD_i, which measures the outstanding amount of a loan when a borrower defaults.
2. A Bernoulli-variable, D_i, which is set to 1 if a claim defaults and to 0 otherwise.
3. The loss rate given default, LGD_i, which determines the loss severity of a defaulted claim.

Each component represents a stochastic variable, such that a single claim's loss distribution is defined as the joint distribution of the three components. On the second stage, the monetary losses are aggregated to the portfolio level by modeling their joint distribution (compare McNeil et al. (2005), p. 4).

In order to model the process behind each stochastic variable, it is necessary to make assumptions on its distribution, the underlying parameters of this distribution, and the functional relation to the respective two other variables. Historical data may provide information regarding the distributional aspects, the dependency between the three components and borrower-specific, economic or other potential risk drivers.

In most applications, the exposure at default is assumed to be deterministic. It is set to the face value of debt or according to a predefined repayment scheme.[4] In contrast, for modeling the default event in a credit risk model,

[4] As an exception, Kupiec (2008) builds a credit risk model allowing for stochastic EAD, which may be applicated to revolving lines of credit.

there exists a large number of suggestions, which may be categorized mainly in two ways. First of all, the scope of application of a credit risk model defines the data requirements for its calibration as well as the time horizon and the interpretation of the model results. The second categorization is defined by the formulation of the model (compare McNeil et al. (2005), p. 328). Modeling LGD_i by a stochastic or empirical process is comparably recent in this respect with the major number of contributions provided in the last decade.

Basically, two credit risk model categories are distinguished according to the scope of application: credit risk management models and credit pricing models (compare McNeil et al. (2005), p. 327). Credit risk management models are typically used to compute the loss distribution of a loan or bond portfolio and to assess the credit risk in terms of loss severity for a predefined probability of occurrence within a period of one year. Thus, according to McNeil et al. (2005), one can refer to these models as static models. Risk capital, e.g., according to the Basel II framework, may be allocated on the basis of static models. In order to calibrate credit risk management models, historical default and loss data are required, which are typically scarce since they are not publicly available (at least for loans) and rare events within a FI's portfolio. In opposition to this, credit pricing models are calibrated to market prices of bonds, credit default swaps or credit derivatives (compare McNeil et al. (2005), p. 401). Thus, they rely on data, which is publicly available in a high frequency for liquid markets. These models are typically dynamic in the sense of evaluating the payoff scheme until maturity and not for a given time horizon. Consequently, the timing of default, which has a major influence on the payoff scheme, is substantially more important than in a static model (compare McNeil et al. (2005), p. 385–386). Furthermore, it is state of the art to build credit pricing models incorporating so-called risk-neutral measures for PD_i and LGD_i and discounting the expected payoffs under these measures by the risk-free interest rate. The models are calibrated, such that the price of the expected payoffs under the real-world measures, discounted by a risk-adequate interest rate, accounting for the credit risk, and the price under the risk-neutral measure are equivalent.

According to the second broad categorization, one can distinguish between structural models and reduced-form models. Structural models motivate the default event economically by a stochastic asset or firm value process. Whenever the asset value falls below a threshold, representing the firm's liabilities, default is triggered (compare Altman (2009), p. 3). Reduced-form models do not model the default event as an economic process. They do not incorporate an explicit dependence of the default event on the characteristics of a firm like its value or its capital structure (compare Altman (2009), p. 7). Instead, the default event is modeled as a sudden event with a non-zero probability of occurrence even for an infinitesimal time interval (compare Altman (2009), p. 7). Following Jarrow & Protter (2004), the choice of the two modeling approaches mainly differ in the information available to the modeler. Structural models assume management level information to the modeler, i.e., 'complete knowledge of the processes of all firm's assets and liabilities' (Gasha et al. (2009), p. 10). Reduced-form models assume market-level information available to the modeler, i.e., the incomplete information level of other market participants concerning the economic processes, driving default and recovery (compare Jarrow & Protter (2004), p. 2).

The categorizations by scope of application and by model formulation cut across each other. Static structural models may be referred to as threshold models. Static reduced-form models, so-called mixture models, depend the default risk of a claim 'on a set of common economic factors, such as macroeconomic variables' (McNeil et al. (2005), p. 352). Since the purpose of this thesis is to investigate credit risk in a management and not in a pricing context, the focus for the remainder of this chapter is on static applications of credit risk models.

1.2.2 Modeling Credit Risk for Management Purposes

Starting point for the evolution of credit risk models is the structural model framework by Merton (1974). In a static context of the model framework, the face value of a zero bond with maturity of one year, B_i determines the debt value and, hence, the default threshold exogenously. The firm value process is assumed to be lognormally distributed. Consequently, the default probability, i.e., the probability that the firm value at maturity, V_i, falls below the default threshold, can be evaluated as a quantile of the standard normal distribution (compare McNeil et al. (2005), p. 332):

$$PD_i = P\left(D_i = 1\right) = P\left(V_i < B_i\right) = \Phi\left(-DD_i\right). \tag{1.1}$$

DD_i measures the so-called distance-to-default, which is interpreted as standardized equity value of the firm (compare Gasha et al. (2009), p. 5). $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. LGD_i is calculated as the difference between liabilities and asset value.

Although being very simplistic, the Merton-model serves as benchmark and tool for credit risk analysis in practice (compare McNeil et al. (2005), p. 331). Several extensions, especially, for applications in the dynamic pricing context, were provided in the aftermath of Merton's work. These include the possibility of default before maturity, i.e., within a given period (so-called first-passage-time models, compare McNeil et al. (2005), p. 336), models with an endogenous default threshold, which is a result of management's obstacle to maximize the equity value (compare Leland (2004), p. 5) and structural models with incomplete information, i.e., with a partial or non-observability of the firm value or liabilities (compare McNeil et al. (2005), p. 336). More of interest for practical risk management purposes was the development of the KMV model by Kealhofer, McQuown and Vasicek in the 1990s (compare McNeil et al. (2005), p. 336). This industry model accounts for various real-world issues in calculating default probabilities, which are called expected default frequencies in the KMV model. An iterative procedure for determining the current asset value, which is generally not directly observable, the implementation of a more complex capital structure and the estimation of an empirical distribution for DD_i instead of assuming a normal distribution are key aspects that differ from the assumptions of the Merton-model (compare McNeil et al. (2005), pp. 336–337).

As pointed out in the last section, defining the default event and the loss given default of a single claim is only the first stage of assessing the credit risk of a portfolio. For the second step of aggregating the individual risk to portfolio risk, several approaches extend a firm value model by an unobservable systematic risk factor representing, e.g., the state of the economy that influences the firm value of each defaultable claim within a portfolio. Most popular in this respect is to use the so-called Gaussian factor model, firstly applied to credit risk models by Vasicek (1987). A simple version of this model describes the asset value return R_i as a linear combination of an average asset value return, which can be defined as the distance-to-default, a systematic risk factor F and an idiosyncratic risk factor U_i. Both risk factors are standard normally as well as independent and identically distributed (i.i.d.) and weighted, such that the linear combination of both factors is standard normally distributed. Consequently, R_i is normally distributed with mean DD_i and variance of one:

$$R_i = DD_i + \sqrt{\rho}F + \sqrt{1-\rho}Z_i. \tag{1.2}$$

In this model setup, defining the average asset value return as distance-to-default has the consequence that the default threshold becomes 0. $0 \leq \rho \leq 1$ measures the asset correlation, i.e., the correlation between the asset returns of two different borrowers i and j. As simplification, ρ is often assumed to be homogeneous across all borrowers. For a given realization of $F = f$, the asset value returns of different borrowers are independent (compare Schönbucher (2001), p. 50). Furthermore, a conditional distance-to-default, CDD_i, may be defined given such a realization. As a result, the default probability becomes a conditional default probability (compare Gordy (2003), p. 203) explicitly depending on the common factor:

$$\begin{aligned} CPD_i(f) &= P(D_i = 1|f) = P(R_i < 0|, f) \\ &= \Phi(-CDD_i) = \Phi\left(-\frac{DD_i + \sqrt{\rho}f}{\sqrt{1-\rho^V}}\right). \end{aligned} \tag{1.3}$$

In line with the Merton-framework, DD_i may also be written as the inverse of the probability that claim i will not default, independent from a realization of the systematic risk factor:

$$DD_i = \Phi^{-1}(1 - PD_i) = -\Phi^{-1}(PD_i), \tag{1.4}$$

since PD_i is the expected value of CPD_i over all realizations of F (compare Gordy (2003)):

$$PD_i = \int_{-\infty}^{\infty} CPD_i(f)\,\phi(f)\;df = \Phi(-DD_i). \tag{1.5}$$

ϕ represents the probability density function of the standard normal distribution. Capturing the requirements for a mixture model formulation, Expression (1.3) can be utilized to formulate a Bernoulli mixture model representation of the actual factor model (compare McNeil et al. (2005), pp. 352 and 360–361). A simple empirical extension of the model decomposes DD_i into a linear combination of observable covariates (compare McNeil et al. (2005), p. 355). The resulting Probit-model may be estimated by Maximum-Likelihood (compare, e.g., Gordy & Heitfield (2000)).

Examples for so-called 'Merton-based' factor models (compare Koyluoglu & Hickman (1998), p. 1) comparable to the model presented above are Moody's KMV's CreditPortfolioManager (compare Vasicek (2002)) or J.P. Morgan's CreditMetrics (compare Gupton et al. (1997)). The Basel II Internally Ratings Based (IRB) approach for calculating the regulatory capital for credit risk also applies this approach (compare Basel Committee on Banking Supervision (2005)).[5]

'Default Mode' (Altman (2009), p. 9) factor models, like the KMV model or the IRB approach, define a credit loss only in the event of default. The so-called 'Mark-to-Market' models, however, like CreditMetrics and CreditPortfolioView, further define a credit loss for the event of a (downward) rating migration (compare Gordy (2003), p. 210).[6]

Concerning LGD_i, most credit risk management models assume it to be a constant or a stochastic variable independent from D_i (compare Altman (2009), p. 10). As a result of the simplifying assumption that LGD_i is independent from default, basic statistic measures may easily be expressed by closed formulas. For example, the expected loss rate of a defaultable claim is written as the product of default probability and the expected value of LGD_i in this case (compare Casella & Berger (2002) , p. 183).

1.2.3 The Relation between Default Probability and Loss Given Default

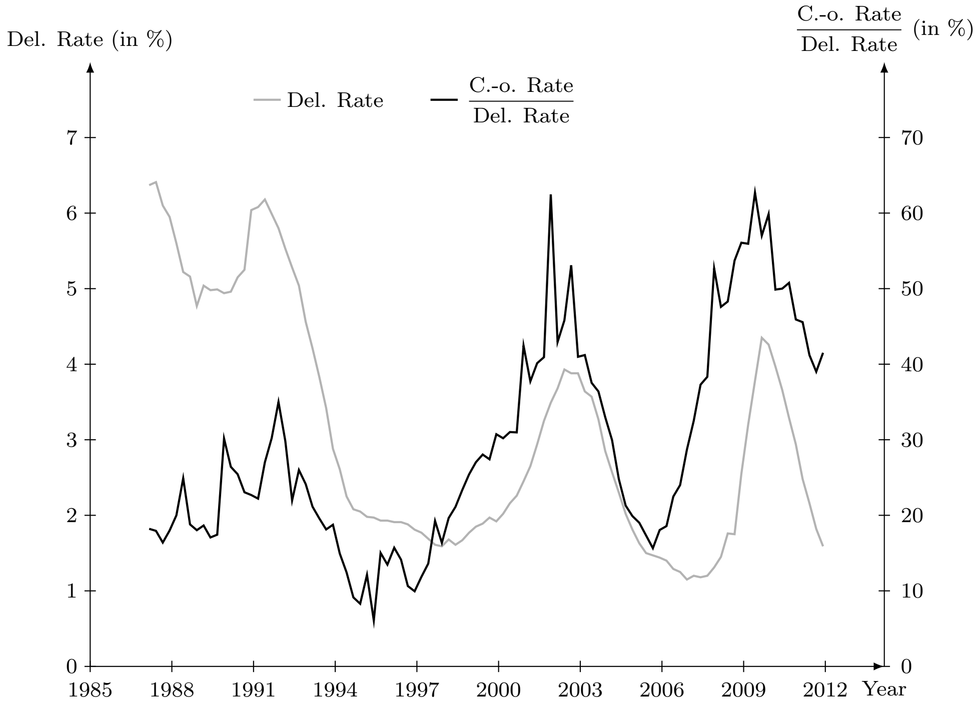
Figure 1.2 presents quarterly delinquency rates of business loans, outstanding by US commercial banks in the period from 1987 to 2011. These delinquency rates are the fraction of delinquent loans to all loans outstanding. They are used as a proxy for the default probability of each loan within the portfolio

[5] In contrast to these portfolio models, McKinsey's 'econometric' (compare Koyluoglu & Hickman (1998), p. 1) linear factor model CreditPortfolioView transforms the distance-to-default into a conditional default probability by a logit function (compare Crouhy et al. (2000), p. 114), i.e., it implicitly assumes a logistic distribution for Z_i. Credit Suisse Financial Product's 'actuarial' (compare Koyluoglu & Hickman (1998), p. 1) model CreditRisk$^+$ uses a linear combination of gamma distributed systematic risk factors as a multiplicative scaling factor for the unconditional default probabilities in order to calculate conditional default probabilities (compare Gordy (2000), p. 122).

[6] For comparing surveys of different industry factor models, see, e.g., Koyluoglu & Hickman (1998), Crouhy et al. (2000) or Gordy (2000).

of all outstanding loans. As a proxy for the loss given default of each loan, Figure 1.2 further presents the relation between charge-off rates on all loans and delinquency rates for the same data. The figure shows that, in addition to the default probability, loss given default is time variant, too. Furthermore, both time series tend to move together. Empirical evidence on this observation is provided by several authors, e.g., Altman et al. (2004), Altman et al. (2005), Altman (2009), Frye (2000*b*), Frye (2005), Cantor & Varma (2005), Hu & Perraudin (2006), and Moody's (2009). Especially, the latter observation contrasts the assumption of PD_i and LGD_i being independent widely used in credit risk models.

Figure 1.2: Credit risk figures of US commercial banks from 1987 to 2011



This figure shows that delinquency (Del.) rates and the relation between charge-off (C.-o.) rates and delinquency rates as proxy for LGD vary over time and tend to move together. Source: http://www.federalreserve.gov

In response to this contrast, Frye (2000*a*) suggested to extend the Gaussian factor model for the default process by a second process for the value of an underlying collateral. Starting with his suggestion, an emerging strand in the academic literature tries to incorporate a dependency between default probability and loss given default into credit risk models. The main feature of Frye's approach is to relate the collateral value, which is assumed to be normally distributed with expectation μ_i and standard deviation σ_i, on the same systematic risk factor as the firm value (compare Frye (2000*a*), p. 91):

$$C_i = \mu_i + \sigma_i \left(\sqrt{\rho^C} F + \sqrt{1 - \rho^C} Z_i^C \right). \tag{1.6}$$

$0 \leq \rho^C \leq 1$ represents the correlation of the collateral value between different claims. Like the asset correlation, it is assumed to be homogeneous across all borrowers and time. Z_i^C is a standard normally distributed idiosyncratic risk factor, driving the collateral value. For decreasing values of the systematic risk factor F, PD_i and $LGD_i = \max(1 - C_i; 0)$ jointly increase. $\max(1 - C_i; 0)$ defines a lower boundary of 0 and an upper boundary of 1 for LGD_i.

Two further approaches rely on the same process definition for the collateral value, but use different transformations of this value in order to calculate LGD_i and bound it between 0 and 1. The first one is the logit transformation (compare Düllmann & Trapp (2005), p. 7, and Schönbucher (2003), pp. 148–150):

$$LGD_i^{Logit} = \frac{1}{1 + \exp(C_i)}, \tag{1.7}$$

and the second one is the probit transformation (compare Andersen & Sidenius (2005), p. 41):

$$LGD_i^{Probit} = \Phi(-C_i). \tag{1.8}$$

Both are frequently used in literature and practise, especially, for regressing realized values of LGD_i on observable covariates (compare, e.g., Bastos (2010) or Bellotti & Crook (2012)).

Besides these approaches, solely differing in the distributional assumptions of the loss given default, a suggestion by Pykhtin (2003) takes one further step in modeling the dependence between PD_i and LGD_i. In addition to explaining this dependence by a single systematic unobservable risk factor, he allows the idiosyncratic risk factors Z_i and Z_i^C to be correlated as well. The economic reasoning behind this approach is that the values of tangible assets, which can be pledged as collateral, are positively related to maintenance expenses, made by firms. Since firms in financial distress, i.e., firms that are most likely to default, often reduce these expenses, a higher default risk coincides with a higher loss rate given default (compare Pykhtin (2003), p. 74). Pykhtin assumes the collateral value to be lognormally distributed. Based on this assumption, he derives closed-form expressions for expected loss and quantiles of the loss distribution, which allow the calculation of economic capital. Thus, the model offers a plausible economic intuition, mathematical elegance, and, due to the less restrictive assumption of a possibly non-zero correlation between default process and recovery process, a gain in flexibility compared with the other models.

Nevertheless, two challenges arise for its practical implementation and calibration. Firstly, the model is a pure economic one by not including any observable covariates. Secondly, the model requires a simultaneous estimation of its parameters for a complete parametrization. According to Pykhtin, this is only possible if the critical assumption of a homogeneous portfolio of borrowers, with respect to default probability and expected loss given default, is met (compare Pykhtin (2003), pp. 76–77).

1.3 Focus and Structure of this Thesis

The initial contribution, in Chapter 2 of this thesis, is to provide an extension for Pykthin's model by observable co-variates. This extension allows to relate PD_i and LGD_i on qualitative and quantitative empirical risk factors. In order to estimate the parameters of this econometric extension of Pykthin's model consistently, one has to deal with the fact that empirical values of LGD_i are only observable in the event of default. Such a sample selection problem was firstly addressed by Heckman (1979), who built an econometric model, similar to the model in Chapter 2. The parameters of this sample selection model may be estimated via Maximum-Likelihood estimation (MLE). Bierens (2007) provides a derivation of the respective likelihood function. All model parameters are determined simultaneously when applying the MLE procedure. Generally, a separate estimation of the parameters for the default process and the recovery process assumes an error term correlation of zero. Therefore, it yields biased estimators for the model parameters. The model presented in Chapter 2 extends Heckman's empirical model by an unobservable systematic risk factor. Thus, in addition to the sample selection bias, the simultaneous estimation method has to account for the unobservable systematic risk. For this purpose, the MLE procedure for the basic model is extended by this factor. Besides these theoretical contributions, Chapter 2 provides empirical evidence on the model parameters. The results of an empirical study with Moody's bond data on nonfinancial companies show that default and recovery process are highly correlated, with respect to unobservable idiosyncratic risk. The high correlation estimates underline the danger of biased parameter estimators when choosing separate estimation methods for the default process and the recovery process.

The main purpose of Chapter 3 is to measure the quantitative impact of the selection bias on the estimation quality of LGD_i. Using the same data set as the empirical study in Chapter 2, the absolute and the relative performance of several empirical models are compared. It is shown that accounting for a correlation between the idiosyncratic unobservable risk factors is advantageous. Compared to models, based on stricter assumptions (predetermined correlation of zero or one), the predictive accuracy for LGD_i estimates is higher.

The purpose of Chapter 4 is to provide a more in-depth performance comparison between the model, based on Pykhtin's suggestion, and Ordinary Least Square (OLS) models. The latter models estimate the parameters of the recovery process separately from the parameters of the default process. In a simulation study, default and recovery data for a given number of borrowers are generated. Then, parameter estimates, based on a large subsample of the data, are derived for the competing modeling approaches. In a first step of the analysis, these estimates are compared to their data generating counterparts. Contrary to the real world, the latter are known quantities in a simulation study, enabling the identification of biased estimators. It is shown that the MLE procedure for Pykthin's model is able to provide consistent estimators for the model parameters, whereas the parameters derived by the OLS models are biased. α and β errors are relatively low compared to the estimates derived by the OLS models. In the second step of the analysis, the parameter estimates are used to calculate PD_i, EL_i, and expected LGD_i, which are subsequently compared to their data generating counterparts as well. A relative performance measure of each model's predictive quality allows for a ranking of it with respect to the competitors. This part of the analysis shows that biased parameter estimators propagate to the level of the risk parameters and, thus, yield a wrong assessment of a defaultable claim's credit risk. In addition to this qualitative model comparison, the impact of the estimation bias on economic capital charges is investigated quantitatively. This final part of the analysis reveals that in the precedent model setup a wrong assessment of credit risk yields a systematic underestimation of the capital charges. The capital charges, calculated under the MLE procedure for Pykthin's model, however, do not predict the required economic capital with such a systematic error. The results hold for a repetition of the simulation study under alternative assumptions concerning the data generating process and, in particular, the value of the correlation parameter.

Chapter 5 gives a brief outlook on possible extensions to the research topics analyzed in Chapters 2, 3, and 4. Chapter 6 concludes this thesis by an excursus to Discounted Cash Flow (DCF) theory. A major contribution to this theory, which aims at determining the risk adequate price of cash flows generated by a firm or a security, was the development of the so-called WACC (Weighted Average Cost of Capital) textbook formula, firstly presented in Solomon (1963) (compare Arnold (2005), p. 900). Commenting on a proposal by Miller (2009) for the calculation of the WACC, which differs from the standard text book approach, it is shown that - per se - a model is not right or wrong, since a model always is based on simplifying assumptions. Instead, it is necessary to bear these assumptions in mind when interpreting the model outcome. Comparable to the approach by Pykhtin (2003) for modeling credit risk, the comment on the WACC calculation gives a further example that the flexibility to account for mutually exclusive assumptions (in the WACC context: financing with predetermined debt amount vs. financing with predetermined debt-to-value ratios) may be more suitable to yield superior results over a model based on more restrictive assumptions than vice versa.

Chapter 2

Default and Recovery Risk Dependencies in a Simple Credit Risk Model

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2.1 Introduction

The measurement of credit default and recovery risk has enjoyed an unprecedented popularity in literature and practice. The rapid growth of credit derivatives in the past decade highlights the challenge to correctly aggregate individual risks for credit portfolios. The global financial crisis (GFC) has forcefully shown that the implementation of risk models and regulatory requirements has not matched this popularity (compare Jorion (2009), Gorton (2009)).

Risk of credit portfolio losses is generally described by three variables: the default event, the exposure at default, and the loss rate in the event of default. Modeling and estimating the parameters of the processes and dependencies of these variables is the important challenge. The level of exposure at default is commonly treated as the face or notional value of debt and assumed to be known *ex ante* and is therefore deterministic.

Research on recovery and loss rates given default (LGDs) is fairly recent, mainly due to scarcity of data, as recoveries can be observed only after the realization of (rare) default events. Here there are two streams of literature: the first stream provides theoretical models for recoveries that incorporate correlations between recoveries and incorporates correlations between defaults and recoveries (see Pykhtin (2003), Jokivuolle & Peura (2003)). Pykhtin (2003) follows the Merton (1974) model and defines two processes – one for the asset value return of a firm that drives the default event by falling below a threshold, and one for the recovery given such an event. The contribution of this framework is the differentiation between asset value and recovery value and the conditioning of the latter on the realization of the default event. The second stream uses simple and independent ordinary least square regression models for observed recoveries in order to identify risk factors that drive recoveries or losses given default. Examples include Carey (1998), Citron et al. (2003), Dermine & de Carvalho (2006), Acharya et al. (2007), Qi & Yang (2009) and Grunert & Weber (2009).

These two streams in the literature on recovery constitute a gap, as links between them do not exist. Theoretical models are not tested and analyzed empirically – Pykhtin (2003) himself states that in his model '[The average loss rate given default] is impossible to estimate'. From the opposite viewpoint, empirical approaches do not relate to the theoretical models.

Hence, the first contribution of this paper is to provide a bridge over this gap. A tractable version of the Pykhtin (2003) approach is developed and used to show how the parameters may be empirically estimated. The model is applied to a large sample of bond default and recovery histories using bond-specific and macroeconomic variables for the explanation of both PDs and recoveries.

Second, in accordance with the current literature (see Frye (2000*a*), Altman et al. (2005)), this paper confirms that the default and recovery processes are highly correlated and concludes that a model approach that addresses this relationship is needed. Heckman (1979) shows that isolated models for two variables lead to biased parameter estimates if i) the two variables are correlated, and ii) one variable can be observed only if the first variable exceeds a particular threshold. This matches the case where realized recoveries are estimated, as they can be observed only in the event of default. Empirical evidence suggests that PDs and recoveries are negatively related. We propose a simultaneous model in line with Heckman (1979) and argue that previous approaches in the literature that use isolated models for recoveries are subject to this bias.

Finally, this paper suggests an econometric extension of the Heckman (1979) model that accounts for an unobserved systematic factor in both PDs and recoveries, and therefore captures the exposure to macroeconomic fluctuations. This approach allows the measurement of a system of correlations. To our best knowledge this is the first paper that applies such a model in the context of credit risk measurement.

The remainder of this paper is organized as follows. In Section 2.2 we develop the theoretical framework, introduce the estimation procedure. Section 2.3 describes the data set and presents the empirical results. In Section 2.4, we show the implications of applying the models to credit portfolios. Section 2.5 summarizes the paper and sets forth our conclusions.

2.2 The credit default and recovery model

2.2.1 A joint model for default and recovery

Following the seminal firm value model accredited to Merton (1974), a borrower is assumed to default when the value of assets falls below the value of debt at maturity.[7] In the original model, the asset value is modeled as a log-normally distributed variable that implies normally distributed log-returns on the asset value. Let V_{it} be the log-return of borrower or bond issuer i's assets in time period t ($i = 1, ..., N_t; t = 1, ..., T$). We express this return by the following factor model

$$V_{it} = \beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} F_t + \sqrt{1-\rho^V} Z_{it}^V \tag{2.1}$$

where $x_{it}^V = \left(x_{it1}^V, \ldots, x_{itK}^V\right)'$ are K observable and deterministic firm-specific, industry-specific, or macroeconomic risk factors that influence the asset value return. $\beta = (\beta_1, \ldots, \beta_K)'$ are the sensitivities with respect to these factors and β_0 is a constant. F_t is a systematic random variable driving all asset returns in time period t jointly and Z_{it}^V is a random idiosyncratic variable driving the return of borrower i's assets in time period t. $\sqrt{\rho^V}$ and $\sqrt{1-\rho^V}$ are weighting parameters for the risk factors. All random variables are assumed

[7] Other modern credit risk applications that are based on a similar framework include Gordy (2000), the Basel II Internal Ratings Based Approach for the calculation of regulatory capital charges, as well as the CreditMetrics framework by Gupton et al. (1997).

to be independent from each other and across time.

A borrower defaults when the asset return crosses a threshold. Generally speaking, the market value of assets is not observable for the majority of banks' loan portfolios. The asset return in Equation (2.1) is treated as an unobserved continuous variable linking an observable default event with the observable covariates x_{it}^V. We implement two assumptions common in the discipline. Firstly, and consistent with Merton (1974), we assume that F_t and Z_{it}^V are normally distributed and, implicitly, that the unobservable asset return V_{it} is also normally distributed with variance equal to unity as F_t and Z_{it}^V are standard normally distributed. Secondly, the default threshold is set to zero, i.e., it is absorbed by the parameter β_0.

The correlation between the latent return of borrowers i and j in time period t $(i \neq j)$ is given by ρ^V, which is known as 'asset (return) correlation'.

We define the default event as the Bernoulli random variable

$$D_{it} = \begin{cases} 1 & \text{borrower } i \text{ defaults in } t, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The PD is then given as the probability that V_{it} falls below zero (given the observable covariates), and under the normality assumption we obtain

$$\begin{aligned} PD_{it} &= P(D_{it} = 1|x_{it}^V) = P(V_{it} < 0|x_{it}^V) \\ &= 1 - \Phi\left(\beta_0 + \beta' x_{it}^V\right) \end{aligned} \tag{2.3}$$

where $\Phi(.)$ denotes the standard normal cumulative distribution function. In line with the IRB Approach in Basel II, the conditional probability of default (CPD) on the realization of a systematic risk factor $F_t = f_t$ is given by

$$\begin{aligned} CPD_{it}\left(f_t\right) &= P(D_{it} = 1|x_{it}^V, f_t) = P\left(V_{it} < 0|x_{it}^V, f_t\right) \\ &= 1 - \Phi\left(\frac{\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t}{\sqrt{1-\rho^V}}\right). \end{aligned} \tag{2.4}$$

The expectation of the CPD gives the unconditional PD in Equation (2.3) (compare Gordy (2003)).

For modeling the recovery process, we follow Pykhtin (2003) and apply a linear factor model

$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} F_t + \sigma U_{it} \tag{2.5}$$

where $x_{it}^Y = \left(x_{it1}^Y, \ldots, x_{itL}^Y\right)'$ are L deterministic observable risk factors driving the recovery, $\gamma = (\gamma_1, \ldots, \gamma_L)'$ represent the loadings of these factors, and γ_0 is a constant. The unobservable random risk factor is weighted by $\sqrt{\rho^Y}$. U_{it} is a standard normally distributed error term and σ is a constant parameter.

Following Pykhtin (2003), we assume that the recovery rate is given as

$$RR_{it} = \exp\left(Y_{it}\right) \tag{2.6}$$

which implies a log-normal distribution and avoids negative recoveries while permitting values greater than 1. Please note that we derive recovery rates from bond prices. This implies that theoretical values are between zero and infinity. A recovery greater than 1 may result if the bond price after default exceeds the notional value. Cumulative density functions such as the probit or logistic function may be taken in other applications where the range of values is limited to between zero and 1.

Economically, Y_{it} is interpretable as the (potential) return on the exposure at default (EAD) of the debt holder:

$$Y_{it} = \ln\left(RR_{it}\right) = \ln\left(\frac{LV_{it}}{EAD_{it}}\right) = \ln\left(LV_{it}\right) - \ln\left(EAD_{it}\right), \tag{2.7}$$

where LV_{it} is the liquidation value of firm assets. Empirical calibration to real data, as in the present paper, ensures realistic values of the recoveries as observed in practice.

The correlation between the firm value process and the recovery process (and therefore between default and recovery) is introduced by the joint exposure to the systematic random factor F_t in Equation (2.1) and (2.5). We now allow for additional correlation via firm-specific random errors. This is done by splitting U_{it} of Equation (2.5) into

$$U_{it} = \rho^U Z_{it}^V + \sqrt{1 - \rho^{U2}} Z_{it}^Y \tag{2.8}$$

where Z_{it}^Y is a standard normally distributed random variable that is independent from all other random variables and results in

$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y - \sqrt{\rho^Y} F_t + \sigma \rho^U Z_{it}^V + \sigma \sqrt{1 - \rho^{U2}} Z_{it}^Y . \tag{2.9}$$

Since Z_{it}^V is now in Equations (2.1) and (2.9), the parameter ρ^U is the correlation between both firm-specific errors. Furthermore, ρ^U is the conditional correlation between the asset return and the log-recovery process given the observable covariates and given the systematic random factor $F_t = f_t$.

2.2.2 Impacts of ρ^U on the empirical distribution of the log-recoveries

Model (2.5) looks like an ordinary linear regression model if the systematic random factor is dropped or its loading is assumed to be zero and the log-recoveries are calculated from the observed recoveries. However, measurement of the correlation between the dependent variable in Equation (2.5) and (2.1) is complicated by the fact that the recovery can be observed only in the event of an obligor's default. The isolated estimation of the recovery process yields biased estimates for γ_0 and γ if Equations (2.5) and (2.1) are correlated. This suggests application of a model where the default and the recovery processes should be estimated jointly in order to avoid biased parameter estimates, unless their correlation is zero. Such isolated empirical models are generally applied both in literature and in practice.

In order to show the effect of correlated error terms between the default and recovery processes on the distribution of the recovery rates conditional on default, i.e., the observable distribution, we simulate portfolio defaults and recoveries for 100,000 homogeneous obligors with $\rho^U = 0$ and with $\rho^U = -0.99$. β_0 is set to -1.6449, corresponding to a PD of 5%, γ_0 is set to four and σ to 2.5. All other parameters are set to zero for transparency. The parameters approximate the empirical values for a B rating and the results are robust for other rating classes.

The conditional distributions of the observed log-recoveries are shown in Figure 2.1. For uncorrelated errors ($\rho^U = 0$) the distribution matches the normal distribution, since default and recovery are independent from each other. However, for almost perfectly negatively correlated error terms ($\rho^U = -0.99$), the conditional distribution shifts to the left and resembles a truncated normal distribution. Thus, the more negative the correlation, the more the default threshold translates into a threshold for the observed log-recoveries.

As a consequence, a separate estimation of both processes, ignoring the distributional effects of the correlation parameter, results in a bias of the estimates, which becomes larger the more default and recovery process are correlated.

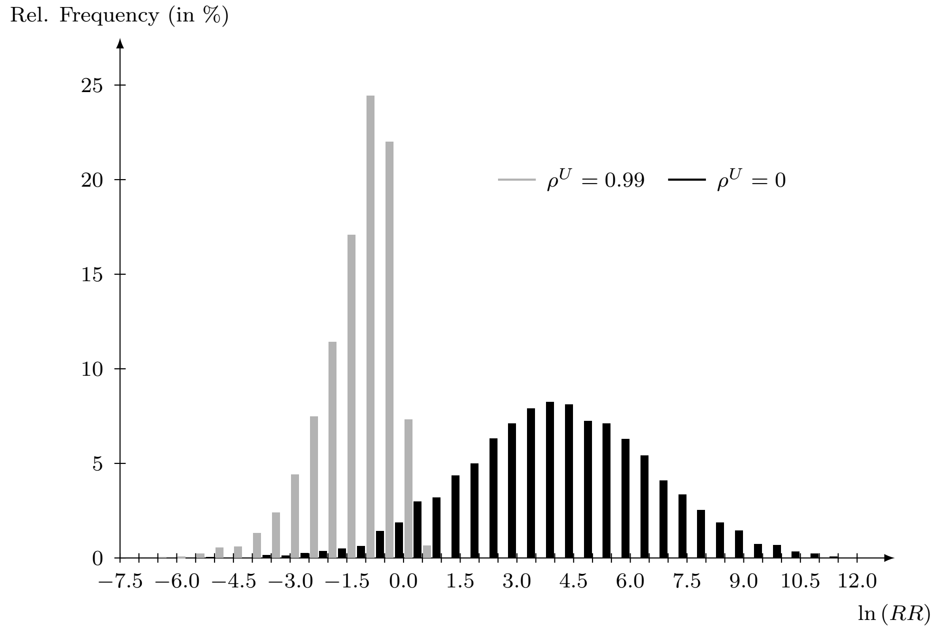

Figure 2.1: Conditional distributions of log-recoveries for a sample portfolio of 100,000 obligors

This figure presents conditional distributions of log-recoveries for a sample portfolio of 100,000 obligors and different ρ^U (0.99 and zero). The underlying parameters of the simulations are $\beta_0 = -1.6449$ (which corresponds to a PD of 5%), $\gamma_0 = 4$ and $\sigma = 2.5$.

2.2.3 Further model properties

The correlation between two log-recoveries can easily be calculated from Equation (2.9) as

$$\rho^{ln(RR)} = \frac{\rho^Y}{\rho^Y + \sigma^2}, \tag{2.10}$$

and the correlation between the recovery rates can be derived as

$$\rho^{RR} = \frac{\exp\left(\rho^Y\right) - 1}{\exp\left(\rho^Y + \sigma^2\right) - 1}, \tag{2.11}$$

compare Kotz (1972).

The correlation ρ^{VY} between the asset return and log-recovery is computed as follows:

$$\rho^{VY} = \frac{\sqrt{\rho^V \rho^Y} + \sigma\rho^U\sqrt{1-\rho^V}}{\sqrt{\rho^Y + \sigma^2}}. \tag{2.12}$$

For the correlation between the asset value return and the recovery rate

$$\rho^{V,RR} = \frac{\sqrt{\rho^V \rho^Y} + \sigma\rho^U\sqrt{1-\rho^V}}{\sqrt{\exp\left(\rho^Y + \sigma^2\right) - 1}} \tag{2.13}$$

applies.

The introduced model setup allows, in accordance with Pykhtin (2003), the derivation of a closed-form expression for the contribution of a borrower to the expected portfolio loss. The unconditional expected loss (EL_{it}) of a borrower is calculated as:

$$\begin{aligned} EL_{it} =& \mathbb{E}\left(L_{it}|x_{it}^V, x_{it}^Y\right) \\ =& \Phi_2\left[-\left(\beta_0 + \beta' x_{it}^V\right), -\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma}, \rho^{VY}\right] \\ &- \exp\left(\gamma_0 + \gamma' x_{it}^Y + \frac{\sigma^2}{2}\right) \\ &\cdot \Phi_2\left[-\left(\beta_0 + \beta' x_{it}^V\right) - \sigma\rho^{VY}, -\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma} - \sigma, \rho^{VY}\right]. \end{aligned} \tag{2.14}$$

$\Phi_2[\cdot,\cdot,\cdot]$ represents the distribution function of the bivariate normal distribution function.

In the case of infinitely granular portfolios with fully diversified idiosyncratic risks (compare Gordy (2003)), the expected loss of borrower i conditional on the systematic risk factor (CEL_{it}) is given by:

$$
\begin{aligned}
& CEL_{it} \\
=& \mathbb{E}\left(L_{it}|x_{it}^{V}, x_{it}^{Y}, f_t\right) \\
=& \Phi_2\left[-\frac{\beta_0+\beta' x_{it}^{V}+\sqrt{\rho^V} f_t}{\sqrt{1-\rho^V}}, -\frac{\gamma_0+\gamma' x_{it}^{Y}+\sqrt{\rho^Y} f_t}{\sigma}, \rho^U\right] \\
& - \exp\left(\gamma_0+\gamma' x_{it}^{Y}+\sqrt{\rho^Y} f_t+\frac{\sigma^2}{2}\right) \\
& \cdot \Phi_2\left[-\frac{\beta_0+\beta' x_{it}^{V}+\sqrt{\rho^V} f_t}{\sqrt{1-\rho^V}} - \sigma\rho^U, -\frac{\gamma_0+\gamma' x_{it}^{Y}+\sqrt{\rho^Y} f_t}{\sigma} - \sigma, \rho^U\right]. \qquad (2.15)
\end{aligned}
$$

Equation (2.15) allows us to compute the portfolio loss distribution of an infinitely granular portfolio. Further, the value at risk (VaR) of such a portfolio can be calculated by setting F_t to the 99.9^{th} percentile of the standard normal distribution (see Section 2.4.1).

Since EL_{it} is measured relative to the exposure, the expected recovery rate ER_{it} is calculated as $1 - EL_{it}$. The expected recovery rate given default $ERGD_{it}$ results from taking $1 - \frac{EL_{it}}{PD_{it}}$. While the expected loss measures loss independent of default status, ERGD determines the recovery rate conditional on the appearance of default. Thus, observed recovery rates of single bonds should be compared to recoveries given default, since they are observable only in the instance of default.

However, observed portfolio loss rates should preferably be compared to expected losses since portfolios generally consist of both defaulted and solvent obligors and are thus created independently of default. The relation between PD, EL, and ERGD holds for the conditional values, i.e., $CERGD = 1 - \frac{CEL}{CPD}$. Both, expected and conditional expected loss rates may be used for calculating regulatory capital (e.g., Basel II) in settings where the recovery is deterministic. Thus, the presented model provides a valuable extension to Basel II, as recovery rates may now be modeled as random variables.

2.2.4 Joint estimation of the model parameters via expected maximum likelihood

The estimation procedure for the Heckman model, which accounts for the unobservability of recovery rates for non-defaulted obligors, as well as a possible correlation between both processes, is suggested by Bierens (2007). In our case, the introduction of an unobserved systematic random factor provides a further extension to this model. This leads to a Heckman model with a time-specific random effect as suggested by Gordy (2000) and Gordy (2003) for the default process and as implemented in the Basel II framework. The model has never been applied in the context of credit risk despite its intuitiveness.

Following Bierens (2007), we construct the likelihood function by first conditioning on a given realization f_t of the time-specific random effect. In the instance of a non-default ($D_{it} = 0$) the conditional likelihood is

$$\begin{aligned} & P\left(D_{it} = 0, D_{it}Y_{it} = 0 | x_{it}^V, x_{it}^Y, f_t\right) \\ = & P\left(D_{it} = 0 | x_{it}^V, x_{it}^Y, f_t\right) \\ = & 1 - CPD_{it}. \end{aligned}$$

In the instance of default ($D_{it} = 1$) the conditional likelihood is given by:

$$\begin{aligned} & \frac{d}{dy} P\left(D_{it} = 1, D_{it}Y_{it} \leq y_{it} | x_{it}^V, x_{it}^Y, f_t\right) \\ = & \frac{d}{dy} P\left(Y_{it} \leq y_{it} | D_{it} = 1, x_{it}^V, x_{it}^Y, f_t\right) \cdot P\left(D_{it} = 1 | x_{it}^V, x_{it}^Y, f_t\right) \\ = & h\left(y_{it} | x_{it}^Y, x_{it}^V, f_t, \beta_0, \beta, \gamma_0, \gamma, \rho^U, \rho^V, \rho^Y, \sigma\right) \cdot CPD_{it}. \end{aligned}$$

$h(\cdot)$ is the conditional density of the observed log-recoveries:

$$\begin{aligned} & h\left(y_{it} | x_{it}^Y, x_{it}^V, f_t, \beta_0, \beta, \gamma_0, \gamma, \rho^U, \rho^V, \rho^Y, \sigma\right) \\ = & \frac{\phi\left(\left(y_{it} - \left(\gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} f_t\right)\right) / \sigma\right)}{\left(\sigma / \sqrt{1 - \rho^V}\right) \Phi\left(-\left(\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t\right) / \sqrt{1 - \rho^V}\right)} \\ & \cdot \Phi\left[-\frac{\frac{\rho^U}{\sigma}\left(y_{it} - \left(\gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} f_t\right)\right) + \left(\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t\right) / \sqrt{1 - \rho^V}}{\sqrt{1 - \rho^{U2}}}\right] \end{aligned}$$

where $\phi(\cdot)$ specifies the density function of the standard normal distribution.

Then, the log-likelihood function is constructed in a four step process:

- Step 1: Calculation of the conditional likelihood function for a given period:

$$\begin{aligned}\mathcal{L}_t\left(f_t\right) = \prod_{i=1}^{n_t} \Bigg\{ & \left[\Phi\left(\left(\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t\right) / \sqrt{1-\rho^V}\right)\right]^{1-d_{it}} \\ & \cdot \left[\Phi\left(-\left(\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t\right) / \sqrt{1-\rho^V}\right)\right]^{d_{it}} \\ & \cdot \left[h\left(y_{it} | x_{it}^Y, x_{it}^V, f_t, \beta_0, \beta, \gamma_0, \gamma, \rho^U, \rho^V, \rho^Y, \sigma\right)\right]^{d_{it}} \Bigg\}.\end{aligned} \tag{2.16}$$

- Step 2: Calculation of the expectation of $\mathcal{L}_t$ since the realizations of F_t are not observable:

$$\mathbb{E}\left[\mathcal{L}_t\left(f_t\right)\right] = \int_{-\infty}^{\infty} \mathcal{L}_t\left(f_t\right) \phi\left(f_t\right) df_t. \tag{2.17}$$

- Step 3: Calculation of the log-likelihood for a time series of T periods:

$$\ell = \ln\left(\prod_{t=1}^{T} \mathbb{E}\left[\mathcal{L}_t\left(f_t\right)\right]\right) = \sum_{t=1}^{T} \ln \mathbb{E}\left[\mathcal{L}_t\left(f_t\right)\right]. \tag{2.18}$$

- Step 4: Numerical optimization of the log-likelihood function in Equation (2.18).

The parameter estimates are consistent, asymptotically existent, and normally distributed (compare Davidson & MacKinnon (1993)).

2.3 Empirical Analysis

2.3.1 Default and recovery data

The data sample underlying the empirical analysis is provided by Moody's credit rating agency. The data set contains the annual ratings of regular US bond issues, as well as default dates and recovery rates given default. Moody's records a default event if: i) interest or principal payments are missed or delayed; ii) Chapter 11 or Chapter 7 bankruptcy is filed; or iii) a distressed exchange, such as a reduction in a financial obligation, occurs. The recovery rate is equal to the price of a defaulted bond measured 30 days after a default event in relation to the face value of the bond.

Table 2.1 summarizes important descriptive statistics for the data set, which consists of 187,638 observations for regular US bond issues of non-financial institutions from 1982 to 2009. Coincident with a change in Moody's rating methodology in 1982 and the role of ratings in the subsequent analysis, earlier observations are excluded from this empirical study.

During the observation period a total of $1,659$ defaults occurred, which yields a default rate (DR) of 0.884%. The mean recovery rate for all defaulted bonds is 37.541%; the median recovery rate is 32%.

Table 2.1 shows the descriptive statistics per rating category: all bond issues with a rating higher than Ba are aggregated to an investment grade (IG) rating; and all bond issues with a rating lower than B are aggregated to rating C. This categorization addresses the limited number of default events in the subcategories. The table shows that – as one may expect – the default rate increases from rating IG to C. The mean recovery rate decreases from rating IG to C, except for grades Ba (48.607%) and IG (46.823%), which may be due to the small number of defaults, and hence the small number of recovery events in both grades.

Table 2.1: Number of observations, default rate, and mean recovery

This table reports descriptive statistics on defaults and recoveries of non-financial bonds from 1982 to 2009. The data set provided by Moody's is split up into four rating categories: investment grade (IG), containing all observations with a Moody's rating higher than Ba, Ba, B and C, containing all observations with a Moody's rating worse than B. $N_{obs.}$ is the number of observations. $N_{def.}$ is the number of defaults. DR (default rate) is the ratio of the number of defaults to the number of observations in each rating grade. $RRGD_{\emptyset}$ is the mean recovery rate of the defaulted bonds in each rating grade. Recovery rate is the ratio of the price of defaulted debt obligations after 30 days of the occurrence of a default event to par value.

Rating	$N_{obs.}$	% of all obs.	$N_{def.}$	% of all def. obs.	DR (in %)	$RRGD_{\emptyset}$ (in %)
IG	146,582	78.120	51	3.074	0.035	46.823
Ba	15,262	8.134	87	5.244	0.570	48.607
B	20,132	10.729	530	31.947	2.633	39.890
C	5,662	3.018	991	59.735	17.503	34.836
Total	187,638	100.000	1,659	100.000	0.884	37.541

Figure 2.2 shows that default and recovery rates vary over time. The default rate varies from 0.085% in 2007 to 2.157% in 2002 and the median recovery varies from 16% in 2001 to 75% in 2007. Due to the GFC we observe increasing default rates and declining recoveries in 2008 and 2009.

The figure also shows the negative relationship between default and recovery rates and identifies the years 1990 and 1991 (First Gulf War), 2001 and 2002 (period following the US terrorist attacks and general downturn in the US technology industry) and 2008 and 2009 (GFC) as periods of economic downturn.

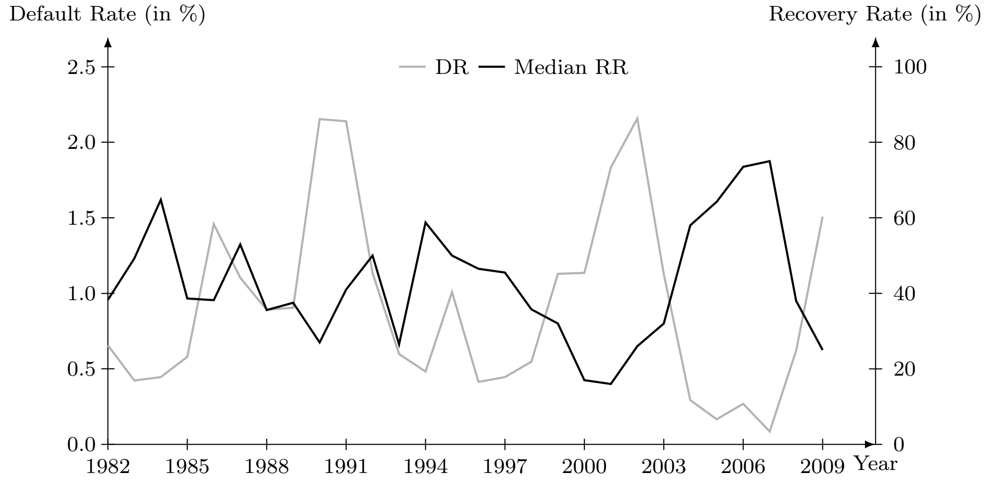

Figure 2.2: Default rates and recovery rates of non-financial bond issues from 1982 to 2009

This figure shows that default and recovery rates vary over time and are negatively related. Default rate is the ratio of defaulted bond issues to total bond issues per year. Recovery rate is the ratio of the price of defaulted debt obligations after 30 days of the occurrence of a default event to the par value.

The distribution of the log-recoveries of the defaulted non-financial bond issues in the Moody's data set is shown in Figure 2.3 and resembles the conditional distribution of log-recoveries for $\rho^U = -0.99$ in Figure 2.1. Thus, graphical analysis of the data set indicates that default and recovery processes are highly correlated. This strongly supports our suggested estimation procedure using the Heckman approach.

2.3.2 Summary of models

The univariate analysis above shows that the rating grade is an important characteristic to distinguish between degrees of idiosyncratic credit risk. Thus we include three dummy variables for rating grades (x_{Ba}, x_B, x_C) in our multivariate analysis. An IG rating is implied if all three dummies take the value of zero; another rating grade is implied if the respective dummy takes the value of 1.

Furthermore, we include an ordinal variable measuring a shift (change) in the rating class in the year before the default status is observed $(x_{\Delta Rat.})$. This variable relates to the rating classes Aaa to C. For example, a downgrade from Aaa to Baa would be measured as a rating shift of +3, and an upgrade from

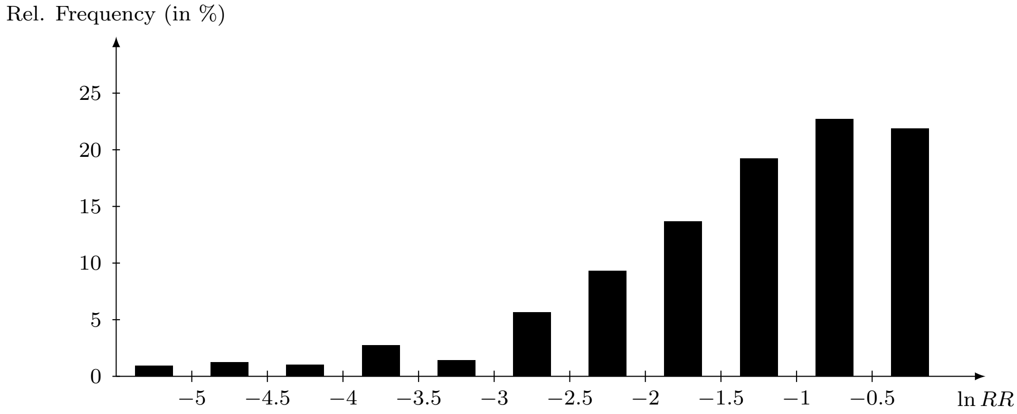

Figure 2.3: Distribution of the log-recoveries from 1982 to 2009

This figure presents a histogram of log-recoveries of defaulted non-financial bond issues in the Moody's data set from 1982 to 2009. The data suggest a high positive correlation between the default and recovery processes (compare Figure 2.2).

Baa to A would be measured as a rating shift of -1. This variable may be interpreted as the momentum of credit quality changes.

The percentage change in gross private domestic investment (GPDI) lagged by 1 year ($x_{\Delta GPDI}$) is included in the subsequent analysis to account for the impact of the macro-economy.

Based on these factors we analyze four different empirical models with the following explanatory variables:

- Model (1): ratings;
- Model (2): ratings and rating shift;
- Model (3): ratings and lagged GPDI change; and
- Model (4): ratings, rating shift and lagged GPDI change.

2.3.3 Results

Tables 2.2 and 2.3 show the estimated parameters for the models where $\rho^V = \rho^Y = 0$ is assumed. The rating grades are significant and have similar exposures for the default process as well as for the recovery process in all models. This confirms that a lower rating results in a lower return on firm value, and thus a higher PD. The same plausible relation applies to the log-recoveries.

A shift in rating grade in the year prior to default results in a significant impact on the asset value returns and the log-recoveries. The more severe a downgrade, the more likely a default event becomes and the lower the recovery rate (compare Models (2) and (4)). Note that for Models (2) and (4), all observations with a rating history of at least two years had to be included to account for the shift in rating in the preceding year. Thus, the number of observations is reduced to 124,648 for estimating these particular models. In other words, the first year of observation of a rating is dropped.

The lagged percentage change in GPDI is highly significant in both Model (3) and Model (4). As one would expect, the stronger the economic investment growth in the year preceding the observation of the default status, the higher the asset and recovery value returns, and the lower the default and recovery risk. In other words, if a downgrade (upgrade) occurred in the past, a default event is more (less) likely and the magnitude of a recovery given such an event is lower (higher).

The standard deviation of the log-recoveries is reduced most by accounting for rating shifts. This underlines the high explanatory power of rating shifts with respect to changes in credit risk.

As indicated by the graphical analysis in Section 2.3.1, asset value return and log-recovery are nearly perfectly correlated. From an economic perspective this is not surprising, since the liquidation value of bond issuer assets is generally taken as part of the asset value (compare Jokivuolle & Peura (2003)). Thus, the recovery process is at least partially driven by the same idiosyncratic component as the default process.

Table 2.3 sets forth the various correlation parameters. In particular, the correlation between asset return and recovery rate is substantially lower, with a range between 6.4% and 9.1%.

Tables 2.4 and 2.5 present estimation results for those models that include the systematic risk factor. The parameters assigned to the rating grades remain largely unchanged compared to the models in Table 2.2; in comparison to these models, the rating shift parameters and the change in GPDI parameters are smaller. Thus, ignoring unobservable systematic risk would yield

Table 2.2: Parameter estimates for the Heckman models without systematic risk

This table reports the parameter estimates for the Moody's data set using the Heckman approach for the empirical model

$$V_{it} = \beta_0 + \beta' x_{it}^V + Z_{it}^V,$$
$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y + \sigma \rho^U Z_{it}^V + \sigma \sqrt{1 - {\rho^U}^2} Z_{it}^Y.$$

The rating grade at the beginning of (all four models) and the rating shift of the year prior to the year of the observed default status ($\Delta Rat.$, Model (2) and (4)) as well as the change in U.S. GPDI (in %) lagged by one year ($\Delta GPDI$, Model (3) and (4)) serve as explanatory variables. For Models (2) and (4), all observations with a rating history of at least two years (124,648 observations) are used. Standard deviations are reported in parentheses. All parameters are significant at the 1%-level.

Parameter	Model (1)	Model (2)	Model (3)	Model (4)
β_0	3.380	3.342	3.359	3.320
	(0.038)	(0.042)	(0.038)	(0.043)
β_{Ba}	-0.849	-0.774	-0.881	-0.794
	(0.053)	(0.063)	(0.054)	(0.063)
β_B	-1.444	-1.446	-1.503	-1.482
	(0.042)	(0.048)	(0.043)	(0.049)
β_C	-2.438	-2.344	-2.459	-2.361
	(0.042)	(0.049)	(0.043)	(0.049)
$\beta_{\Delta Rat.}$		-0.186		-0.168
		(0.017)		(0.017)
$\beta_{\Delta GPDI}$			0.021	0.018
			(0.002)	(0.002)
γ_0	9.203	8.642	9.055	8.562
	(0.273)	(0.275)	(0.270)	(0.274)
γ_{Ba}	-2.361	-2.042	-2.427	-2.090
	(0.168)	(0.182)	(0.169)	(0.183)
γ_B	-4.048	-3.859	-4.174	-3.950
	(0.157)	(0.165)	(0.160)	(0.167)
γ_C	-6.710	-6.112	-6.704	-6.153
	(0.204)	(0.207)	(0.204)	(0.208)
$\gamma_{\Delta Rat.}$		-0.491		-0.437
		(0.046)		(0.047)
$\gamma_{\Delta GPDI}$			0.059	0.049
			(0.005)	(0.006)

Table 2.3: Correlation and standard deviation estimates for the Heckman models without systematic risk

This table reports the estimated correlations and standard deviations of Models (1) to (4) for the Moody's data set using the Heckman approach for the empirical model

$$V_{it} = \beta_0 + \beta' x_{it}^V + Z_{it}^V,$$
$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y + \sigma\rho^U Z_{it}^V + \sigma\sqrt{1-{\rho^U}^2} Z_{it}^Y.$$

All parameters are significant at the 1%-level. The correlation between the error terms of the default and recovery processes (ρ^U) results from the same maximum likelihood estimation as in Table 2.2. The correlation between asset return and recovery rate ($\rho^{V,RR}$) is calculated according to

$$\rho^{V,RR} = \frac{\sigma\rho^U}{\sqrt{\exp(\sigma^2) - 1}}.$$

Parameter	Model (1)	Model (2)	Model (3)	Model (4)
σ	2.744	2.598	2.715	2.588
	(0.070)	(0.072)	(0.069)	(0.072)
ρ^U	99.843%	99.870%	99.841%	99.878%
	(0.000)	(0.000)	(0.001)	(0.000)
$\rho^{V,RR}$	6.355%	8.876%	6.795%	9.094%

an overestimation of the influence of two explanatory variables that capture observable systematic (GPDI) and idiosyncratic (rating shift) risk.

The asset correlation ρ^V varies between 3.1% and 3.5%, in line with other empirical studies.[8] The estimate declines when the model is extended by the change in GPDI (compare Model (1) with (3) and Model (2) with (4)). The same applies, with a slightly higher magnitude, for correlation of the log-recoveries, varying between 3.995% and 4.257%. The high sensitivities of the log-recoveries to the systematic risk factor in all four models ($\sqrt{\rho^Y} \approx 0.5$ in each case) underline that liquidation values, i.e., the values of a firm's securities, are highly volatile with respect to business cycle fluctuations, as pointed out theoretically, due to fire sales, by Shleifer & Vishny (1992) and empirically by Acharya et al. (2007). Almost no correlations can be observed between recoveries. While the correlation between asset value return and log-recovery remains nearly unchanged compared to models without a systematic risk factor, the correlation between asset value return and recovery rate increases from 9.4% to 11.8%.

[8] For an overview of empirical studies on asset correlations, see Chernih et al. (2006).

Table 2.4: Parameter estimates for the Heckman models with systematic risk

This table reports the parameter estimates for the Moody's data set using the Heckman approach for the empirical model

$$V_{it} = \beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} F_t + \sqrt{1-\rho^V} Z_{it}^V,$$
$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} F_t + \sigma \rho^U Z_{it}^V + \sigma \sqrt{1-\rho^{U2}} Z_{it}^Y.$$

The rating grade at the beginning of (all four models) and the rating shift at the year prior to the year of the observed default status ($\Delta Rat.$, Model (2) and (4)), as well as the change in U.S. GPDI (in %) lagged by one year ($\Delta GPDI$, Model (3) and (4)) serve as explanatory variables. For Models (2) and (4), all observations with a rating history of at least two years (124,648 observations) are used. Standard deviations are reported in parentheses. All parameters are significant at the 1%-level.

Parameter	Model (1)	Model (2)	Model (3)	Model (4)
β_0	3.451	3.394	3.407	3.349
	(0.053)	(0.058)	(0.055)	(0.060)
β_{Ba}	-0.854	-0.785	-0.858	-0.788
	(0.054)	(0.064)	(0.054)	(0.064)
β_B	-1.495	-1.492	-1.503	-1.497
	(0.044)	(0.049)	(0.044)	(0.050)
β_C	-2.501	-2.427	-2.504	-2.430
	(0.044)	(0.051)	(0.044)	(0.051)
$\beta_{\Delta Rat.}$		-0.158		-0.157
		(0.018)		(0.018)
$\beta_{\Delta GPDI}$			0.014	0.014
			(0.005)	(0.005)
γ_0	8.863	8.407	8.690	8.256
	(0.297)	(0.291)	(0.301)	(0.292)
γ_{Ba}	-2.263	-1.985	-2.254	-1.985
	(0.163)	(0.176)	(0.163)	(0.176)
γ_B	-3.987	-3.824	-3.969	-3.823
	(0.156)	(0.161)	(0.158)	(0.162)
γ_C	-6.522	-6.107	-6.482	-6.092
	(0.206)	(0.205)	(0.208)	(0.206)
$\gamma_{\Delta Rat.}$		-0.376		-0.373
		(0.045)		(0.045)
$\gamma_{\Delta GPDI}$			0.034	0.036
			(0.013)	(0.013)

Table 2.5: Correlation estimates for the Heckman models with systematic risk

This table reports the estimated correlations of Models (1) to (4) for the Moody's data set using the Heckman approach for the empirical model

$$V_{it} = \beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} F_t + \sqrt{1-\rho^V} Z_{it}^V,$$
$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} F_t + \sigma \rho^U Z_{it}^V + \sigma \sqrt{1-{\rho^U}^2} Z_{it}^Y.$$

All parameters are significant at the 1%-level. The correlation between the error terms of the default and recovery process (ρ^U), the asset correlation (ρ^V), and ρ^Y, result from the same Maximum-Likelihood estimation as in Table 2.4. The log-recovery correlation ($\rho^{ln(RR)}$) is calculated according to

$$\rho^{ln(RR)} = \frac{\rho^Y}{\rho^Y + \sigma^2},$$

and the correlation between the recovery rates (ρ^{RR}) is calculated according to

$$\rho^{RR} = \frac{\exp\left(\rho^Y\right) - 1}{\exp\left(\rho^Y + \sigma^2\right) - 1}.$$

The correlation between the default process and the recovery process (ρ^{VY}) is calculated according to

$$\rho^{VY} = \frac{\sqrt{\rho^V \rho^Y} + \sigma \rho^U \sqrt{1-\rho^V}}{\sqrt{\rho^Y + \sigma^2}},$$

and the correlation between asset return and recovery rate ($\rho^{V,RR}$) is calculated according to

$$\rho^{V,RR} = \frac{\sqrt{\rho^V \rho^Y} + \sigma \rho^U \sqrt{1-\rho^V}}{\sqrt{\exp\left(\rho^Y + \sigma^2\right) - 1}}.$$

Parameter	Model (1)	Model (2)	Model (3)	Model (4)
σ	2.518	2.422	2.507	2.417
	(0.070)	(0.070)	(0.070)	(0.070)
ρ^U	99.810%	99.875%	99.806%	99.870%
	(0.001)	(0.001)	(0.001)	(0.001)
ρ^V	3.411%	3.496%	3.129%	3.250%
	(0.012)	(0.014)	(0.012)	(0.014)
ρ^Y	27.186%	26.079%	26.158%	24.527%
	(0.035)	(0.040)	(0.037)	(0.040)
$\rho^{ln(RR)}$	4.112%	4.257%	3.995%	4.029%
ρ^{RR}	0.042%	0.065%	0.043%	0.063%
ρ^{VY}	99.800%	99.865%	99.789%	99.853%
$\rho^{V,RR}$	9.418%	11.568%	9.677%	11.756%

2.4 Applications of the results to credit portfolios

2.4.1 Sensitivity analysis of single obligation credit risk

In this section, we investigate the quantitative impact of the estimation results on the credit risk of a single bond. We provide a sensitivity analysis for default probability, as calculated in Equation (2.3), for expected loss, as calculated in Equation (2.14), and for expected recovery given default, calculated as $1 - \frac{EL}{PD}$.

Furthermore, we calculate values for the conditional expected loss by Equation (2.15), by setting the realization of the systematic risk factor to 3.09, which corresponds to the 99.9^{th} percentile of the standard normal distribution. This is consistent with the current Basel II assumption in relation to the Internal Ratings Based Approach (compare Basel Committee on Banking Supervision (2006)). The resulting conditional expected loss is referred to as VaR, i.e., the loss that will not be exceeded with a probability of 99.9% within one year. For these calculations, the parameters of Model (4) in Tables 2.4 and 2.5 are used. The rating shift is varied from an upgrade of two rating categories (-2) to a downgrade of two rating categories ($+2$). The change in GPDI varies from -20% to 20%.

Table 2.6 shows the sensitivity analysis for the default probability. In the base-case scenario ($x_{\Delta Rat.} = x_{\Delta GPDI} = 0$), the PD of a bond issue with investment grade rating is 0.041%. The PD changes disproportionately for each 10% change of GPDI for each rating grade and shift in the rating grade. For example, a 10% decrease of GPDI yields a 0.025 percentage point higher PD of 0.066%, and a 20% decrease yields a 0.064 percentage point higher PD of 0.105% compared to the base-case PD of an IG-rated bond issue.

As indicated by the higher parameter values estimated for Model (4), the impact of an upgrade or downgrade is more severe. For example, an upgrade by one rating grade approximately halves the PD from 0.041% to 0.023%. The higher the value of $x_{\Delta Rat.}$, the higher the change in PD when the GPDI changes. With respect to the other rating categories, the PD grows substantially compared to the PD in the IG category. For a Ba-rated bond issue, the base case PD is 0.522%, and thus about 13 times higher than an IG-rated bond issue. A bond issue rated B has a PD of 3.203% in the base case (80 times as high), and a C rated bond issue has a PD of 17.905% (about 442 times as high). The lower the rating and thus the higher the level of the PD, the stronger the effect of an upgrade or downgrade and the effect of the GPDI variation. For example, a downgrade by two rating grades would raise the PD of a C-rated bond issue by nearly 10 percentage points.

Table 2.6: PD sensitivity analysis for each rating grade

This table reports the results of a sensitivity analysis of the Probability of Default (PD) calculated by

$$PD_{it} = 1 - \Phi\left(\beta_0 + \beta' x_{it}^V\right)$$

for each rating grade using the parameter estimates of Model (4). $x_{\Delta Rat.}$ varies from -2 to 2 and $x_{\Delta GPDI}$ from -20% to 20%.

		Rating IG			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	0.035%	0.021%	0.013%	0.007%	0.004%
-1	0.061%	0.038%	0.023%	0.014%	0.008%
0	0.105%	0.066%	0.041%	0.025%	0.015%
1	0.175%	0.112%	0.070%	0.044%	0.026%
2	0.287%	0.187%	0.120%	0.075%	0.047%
		Rating Ba			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	0.465%	0.310%	0.203%	0.130%	0.082%
-1	0.726%	0.493%	0.329%	0.216%	0.139%
0	1.108%	0.767%	0.522%	0.349%	0.230%
1	1.654%	1.167%	0.810%	0.553%	0.371%
2	2.416%	1.738%	1.229%	0.855%	0.585%
		Rating B			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	2.929%	2.128%	1.521%	1.068%	0.738%
-1	4.139%	3.064%	2.231%	1.598%	1.126%
0	5.725%	4.317%	3.203%	2.338%	1.679%
1	7.754%	5.956%	4.502%	3.348%	2.450%
2	10.288%	8.046%	6.194%	4.693%	3.499%
		Rating C			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	16.888%	13.669%	10.897%	8.555%	6.611%
-1	21.125%	17.392%	14.108%	11.272%	8.868%
0	25.927%	21.702%	17.905%	14.555%	11.655%
1	31.239%	26.572%	22.288%	18.427%	15.012%
2	36.972%	31.943%	27.226%	22.883%	18.959%

The sensitivity analysis of the expected loss presented in Table 2.7 shows the same qualitative results as the PD analysis. In the base-case scenario, a bond issue with IG rating has an expected loss of 0.012%, a bond issue with Ba rating has an EL of 0.208% (about 16 times higher), a bond issue with B rating has an EL of 1.652% (more than 133 times higher), and a C rated bond issue has an EL of 10.977% (about 891 times higher).

Table 2.7: EL sensitivity analysis for each rating grade

This table reports the results of a sensitivity analysis of the Expected Loss (EL) calculated by

$$EL_{it} = \Phi_2\left[-\left(\beta_0 + \beta' x_{it}^V\right), -\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma}, \rho^{VY}\right] - exp\left(\gamma_0 + \gamma' x_{it}^Y + \frac{\sigma^2}{2}\right)$$
$$\cdot \Phi_2\left[-\left(\beta_0 + \beta' x_{it}^V\right) - \sigma\rho^{VY}, -\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma} - \sigma, \rho^{VY}\right]$$

for each rating grade using the parameter estimates of Model (4). $x_{\Delta Rat.}$ varies from -2 to 2 and $x_{\Delta GPDI}$ from -20% to 20%.

		Rating IG			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	0.012%	0.007%	0.004%	0.002%	0.001%
-1	0.021%	0.012%	0.007%	0.004%	0.002%
0	0.036%	0.021%	0.012%	0.007%	0.004%
1	0.062%	0.037%	0.022%	0.013%	0.007%
2	0.104%	0.064%	0.038%	0.023%	0.013%
		Rating Ba			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	0.199%	0.125%	0.077%	0.047%	0.028%
-1	0.316%	0.203%	0.128%	0.079%	0.048%
0	0.494%	0.324%	0.208%	0.131%	0.081%
1	0.754%	0.504%	0.331%	0.213%	0.134%
2	1.128%	0.769%	0.514%	0.338%	0.217%
		Rating B			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	1.584%	1.102%	0.753%	0.504%	0.331%
-1	2.282%	1.619%	1.127%	0.770%	0.516%
0	3.221%	2.328%	1.652%	1.151%	0.786%
1	4.455%	3.281%	2.373%	1.685%	1.174%
2	6.042%	4.533%	3.340%	2.417%	1.716%
		Rating C			
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	10.673%	8.317%	6.371%	4.794%	3.544%
-1	13.654%	10.828%	8.442%	6.470%	4.871%
0	17.149%	13.834%	10.977%	8.563%	6.566%
1	21.159%	17.355%	14.008%	11.121%	8.681%
2	25.656%	21.389%	17.555%	14.177%	11.262%

Table 2.8 shows that in the base-case scenario, the expected recovery given default decreases by about 10 percentage points per rating category, as shown in Table 2.8. It is striking that the sensitivity of the expected recovery given default is less volatile, while PD and EL have a relatively high sensitivity to a rating shift and changes in GPDI. The difference between a very good realization (upgrade by two rating grades and 20% growth of GPDI) and a very bad realization (downgrade by two rating grades and 20% decrease of GPDI) is lowest for an IG-rated bond issue (11.636%) and highest for a C-rated bond issue (15.788%). The omission of the rating shift and GPDI growth would result in miscalculations of PD and EL, but still quite precise calculations of ERGD. The relevance of expected loss on a portfolio level supports the proposed models.

Table 2.8: ERGD sensitivity analysis for each rating grade

This table reports the results of a sensitivity analysis of the Expected Recovery Given Default (ERGD) calculated as $1 - \frac{EL}{PD}$ for each rating grade using the parameter estimates of Model (4). $x_{\Delta Rat.}$ varies from -2 to 2 and $x_{\Delta GPDI}$ from -20% to 20%.

Rating IG					
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	66.789%	68.930%	71.016%	73.037%	74.987%
-1	66.076%	68.240%	70.349%	72.394%	74.368%
0	65.313%	67.502%	69.636%	71.707%	73.708%
1	64.496%	66.713%	68.874%	70.973%	73.002%
2	63.624%	65.869%	68.060%	70.189%	72.249%
Rating Ba					
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	57.344%	59.650%	61.935%	64.190%	66.403%
-1	56.438%	58.770%	61.081%	63.362%	65.602%
0	55.469%	57.829%	60.168%	62.477%	64.745%
1	54.434%	56.823%	59.192%	61.531%	63.830%
2	53.329%	55.749%	58.149%	60.521%	62.853%
Rating B					
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	45.916%	48.194%	50.495%	52.811%	55.136%
-1	44.863%	47.168%	49.495%	51.838%	54.188%
0	43.740%	46.072%	48.427%	50.798%	53.176%
1	42.545%	44.905%	47.288%	49.687%	52.095%
2	41.275%	43.662%	46.075%	48.504%	50.942%
Rating C					
$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
-2	36.800%	39.153%	41.540%	43.957%	46.396%
-1	35.366%	37.743%	40.157%	42.602%	45.071%
0	33.855%	36.254%	38.693%	41.166%	43.665%
1	32.268%	34.686%	37.148%	39.647%	42.175%
2	30.608%	33.040%	35.522%	38.045%	40.600%

The VaRs in Table 2.9 are between 2.5 (C rating) and 9 times (IG rating) higher than the corresponding expected losses for the base-case scenario. Furthermore, the variation of the VaRs with respect to sensitivity to a rating shift and the change in GPDI is stronger within each rating category than for the ELs. The difference between a very bad realization and the base-case is as much as 21 percentage points. Thus, the omission of these inputs may result in a large miscalculation of the VaR.

Table 2.9: VaR sensitivity analysis for each rating grade

This table reports the results of a sensitivity analysis of the Conditional Expected Loss (CEL) calculated by

$$CEL_{it} = \Phi_2\left[-\frac{\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t}{\sqrt{1-\rho^V}}, -\frac{\gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} f_t}{\sigma}, \rho^U\right] - \exp\left(\gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} f_t + \frac{\sigma^2}{2}\right)$$
$$\cdot \Phi_2\left[-\frac{\beta_0 + \beta' x_{it}^V + \sqrt{\rho^V} f_t}{\sqrt{1-\rho^V}} - \sigma\rho^U, -\frac{\gamma_0 + \gamma' x_{it}^Y + \sqrt{\rho^Y} f_t}{\sigma} - \sigma, \rho^U\right]$$

for each rating grade using the parameter estimates of Model (4). $x_{\Delta Rat.}$ varies from -2 to 2 and $x_{\Delta GPDI}$ from -20% to 20%. f_t is set to 3.09, corresponding to the 99.9^{th} percentile of the standard normal distribution. The CEL may hence be interpreted as Value-at-Risk.

$x_{\Delta Rat.} \diagdown x_{\Delta GPDI}$	-20%	-10%	0%	10%	20%
Rating IG					
-2	0.103%	0.064%	0.038%	0.023%	0.013%
-1	0.170%	0.107%	0.066%	0.040%	0.024%
0	0.275%	0.176%	0.111%	0.068%	0.041%
1	0.433%	0.283%	0.182%	0.114%	0.070%
2	0.668%	0.446%	0.291%	0.187%	0.118%
Rating Ba					
-2	1.112%	0.761%	0.512%	0.338%	0.218%
-1	1.638%	1.144%	0.784%	0.527%	0.348%
0	2.363%	1.682%	1.175%	0.806%	0.542%
1	3.338%	2.421%	1.724%	1.205%	0.827%
2	4.620%	3.413%	2.477%	1.765%	1.235%
Rating B					
-2	5.903%	4.434%	3.272%	2.373%	1.690%
-1	7.895%	6.039%	4.540%	3.355%	2.435%
0	10.352%	8.061%	6.172%	4.645%	3.436%
1	13.311%	10.549%	8.223%	6.302%	4.748%
2	16.795%	13.541%	10.741%	8.381%	6.429%
Rating C					
-2	25.544%	21.309%	17.507%	14.158%	11.267%
-1	30.533%	25.877%	21.608%	17.769%	14.383%
0	35.890%	30.885%	26.200%	21.898%	18.023%
1	41.517%	36.256%	31.227%	26.513%	22.178%
2	47.299%	41.888%	36.609%	31.557%	26.815%

2.4.2 Portfolio loss distributions

In order to investigate the magnitude of the systematic risk factor on portfolio risk, we simulate portfolio default and loss distributions. Our sample portfolio consists of the 13,392 non-financial bond issues of the Moody's data set that did not default in 2009. Along with the rating information on each bond issue, we take the rating shift and the change in GPDI in 2009 into account. The effect of the two latter covariates on the sample portfolio is analyzed by comparison of the default and loss distributions drawn for Models (1) and (4). The random sampling of these models is executed 100,000 times for the sample portfolio.

Table 2.10 shows default probabilities, expected losses, and expected recoveries given default calculated according to these models. For Model (1), the average PD is 0.518% and the average EL is 0.298%. For Model (4) $x_{\Delta GPDI} = -23.1\%$ and the rating shift is set to zero for the calculations of each rating category. The average PD and the average EL are calculated by averaging the PD and EL for each observation in the sample portfolio. The rating shift is taken into account in these values. The average PD (with a value of 1.036%) is nearly doubled in comparison to Model (1). The expected loss of 0.617% is even higher in relation to Model (1). The ERGDs are calculated by taking $1 - \frac{EL_{av.}}{PD_{av.}}$. The average ERGD in Model (4) is reduced by two percentage points compared to Model (1).

Table 2.10: PD, EL and ERGD for a bond in the sample portfolio

This table reports the PD, EL and ERGD for each rating grade using the parameters of Models (1) and (4). In Model (4) $x_{\Delta Rat.} = 0$ and $x_{\Delta GPDI} = -23.1\%$ are assumed for each rating grade. For the portfolio average, PD and EL are computed for each bond in the sample portfolio, then aggregated by weighting the value for each bond with its portfolio share. The ERGDs are calculated by taking $1 - \frac{EL_{av.}}{PD_{av.}}$.

	Model (1)		
Rating	PD	EL	ERGD
IG	0.028%	0.008%	69.911%
Ba	0.471%	0.195%	58.592%
B	2.524%	1.306%	48.262%
C	17.111%	10.536%	38.424%
Mean	0.518%	0.298%	42.558%
	Model (4)		
Rating	PD	EL	ERGD
IG	0.121%	0.012%	89.807%
Ba	1.238%	0.209%	83.141%
B	6.228%	1.671%	73.163%
C	27.318%	11.059%	59.517%
Mean	1.036%	0.617%	40.407%

Figure 2.4 shows the distributions of the simulated portfolio default rates for Model (1) and Model (4). Comparing the two distributions yields two observations: the latter distribution shifts to the right and it has much heavier

tails. The first observation is in line with the higher PD estimate for Model (4). The average default rate is 0.518% for the first distribution and 1.036% for the second distribution. Both values are quite similar to their estimates. The second observation underlines that the distribution of the default rate is more volatile for similar asset correlations the higher is the PD (compare Gersbach & Lipponer (2003)). Both distributions are positively skewed, since they are bounded on the left at zero.

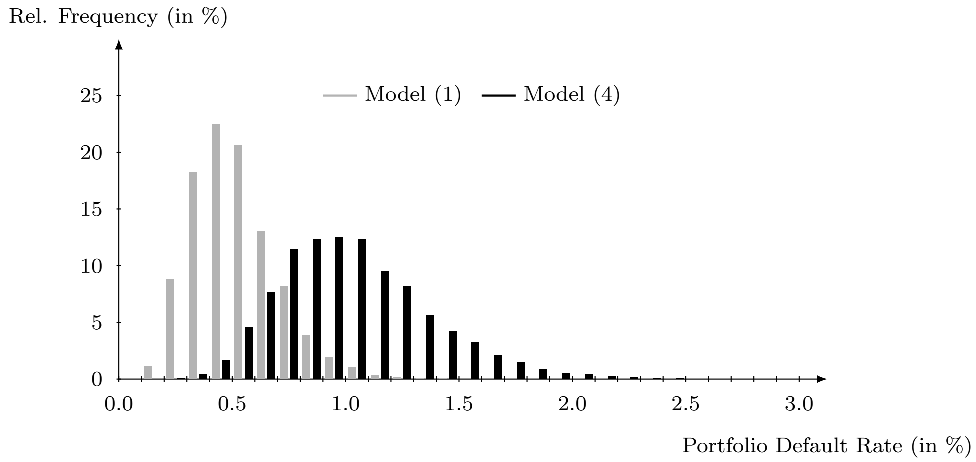

Figure 2.4: Portfolio default rate distributions, Model (1) and Model (4)

This figure shows the distributions of the simulated portfolio default rates for Model (1) and Model (4). Defaults and recoveries for 2010 are randomly sampled for a portfolio consisting of the 13,392 observations for non-defaulted, non-financial bonds of 2009 by applying the parameters of Model (1) and Model (4) with the systematic risk factor. The above distribution for the portfolio default rate is generated using 100,000 iterations.

The same conclusions can be drawn for the portfolio loss rate distributions presented in Figure 2.4. Contradicting the portfolio default rate, the average portfolio loss rate is not as close to the expected loss calculated in Table 2.10 as are the PDs and average default rates. The average loss rate for Model (1) is 0.314%, which is slightly higher than the expected loss of 0.298%. For Model (4), the portfolio loss is 0.647%, which is also slightly higher than the expected loss of 0.617%. It is notable that this number is equal to the expected portfolio loss only in the instance of an infinitely granular portfolio, i.e., a portfolio with an infinite number of debt obligations. The sample portfolio consists of only 13,392 observations, so that idiosyncratic risk is not fully diversified and the expected portfolio loss is underestimated.

The large difference between Model (1) and Model (4) in both expected losses and portfolio loss distributions again supports the importance of accounting, especially for rating shifts. In other words, calculating the expected portfolio loss for Model (4) by simply weighting the expected loss of each rating grade with its portfolio weight and aggregating these values results in a loss of 0.327%. This number is slightly higher than in Model (1) and much lower than the average portfolio loss resulting from the simulation study.

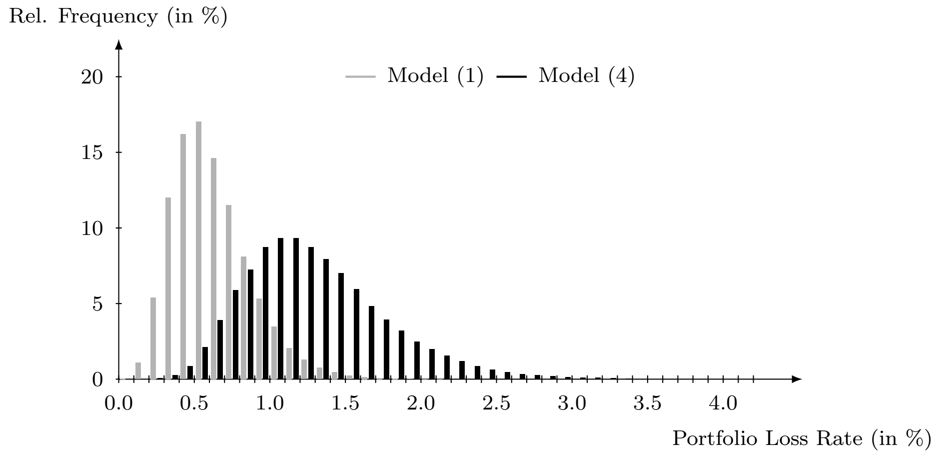

Figure 2.5: Portfolio loss rate distributions, Model (1) and Model (4)

This figure shows the distributions of the simulated portfolio loss rates for Model (1) and Model (4). Defaults and recoveries for 2010 are randomly sampled for a portfolio consisting of the 13,392 observations for non-defaulted, non-financial bonds of 2009 by applying the parameters of Model (1) and Model (4) with the systematic risk factor. The above distribution for the portfolio default rate is generated using 100,000 iterations.

2.5 Summary

The literature and current banking practice do not address various properties of credit risk. Default probabilities, recovery rates, and correlations are often modeled as constant and deterministic over time. Secondly, conditional parameters such as recoveries that are conditional on the occurrence of default are modeled using unconditional OLS regression models, which do not take conditionality into account and lead to bias in the estimated parameters.

In response to these shortcomings, this paper provides an approach for estimating time-varying default probabilities and recovery rates that are conditional on default and based on observable information. By extension, the residual correlation between the default and recovery processes of a portfolio of borrowers is included via systematic random effects. The dependence structure is modeled via correlated error terms.

Empirical analysis links bond recoveries with credit ratings and business cycle information and provides evidence for the relationship between credit quality, recovery rate, and correlation. We investigate default and recovery data provided by Moody's with respect to the sensitivities of rating grades as proxies for specific bond issue characteristics as well as macroeconomic factors.

The main findings are as follows. The rating grade and rating shift provide a highly significant explanation for default risk and recovery risk of US bond issues. Further, macroeconomic factors add explanatory value. Finally, the default and recovery processes are highly correlated, which underlines the importance of the stipulated conditional relationship.

The Global Financial Crisis reveals that current credit portfolio risk models exhibit a low degree of transparency. Addressing the dependence between the default and recovery processes is one dimension along which credit portfolio risk models should improve.

Chapter 3

Empirical Performance of LGD Prediction Models

The content of this chapter was originally published as Bade, B., Rösch, D. & Scheule, H. (2011b), 'Empirical performance of LGD prediction models', *Journal of Risk Model Validation* 5(2), pp. 25–44.

3.1 Introduction

Calculating an accurate measurement of the credit risk underlying defaultable obligations such as loans or bonds is probably one of the most challenging tasks involved in the risk management of a financial institution. The trade-off between complying with the Basel capital requirements and the opportunity costs of tying up too much capital makes this task even more challenging. Appropriate models for the probability of a default event (PD), the exposure at the time of default (EAD) and the loss given a default event (LGD) have to be defined and calibrated by empirical data. In particular, the test of modeling PD and LGD deals with a high level of uncertainty.

Looking at the theoretical and empirical realization of this task in theory as well as in practice, several gaps are identifiable. First of all, there is wide literature on analyzing the drivers of either PD (see, e.g., Leland (1994), Jarrow & Turnbull (1995), Longstaff & Schwartz (1995), Madan & Unal (1995), Leland & Toft (1996), Jarrow et al. (1997), Duffie & Singleton (1999), Shumway (2001), McNeil & Wendin (2007), Duffie et al. (2007)) or LGD (see, e.g., Carey (1998), Citron et al. (2003), Dermine & de Carvalho (2006), Acharya et al.

(2007), Altman (2009), Qi & Yang (2009), Grunert & Weber (2009), Calabrese & Zenga (2010)). Many industry credit portfolio risk models are also based on isolated modules for default probabilities and recoveries in the event of default. In contrast, approaches to joint modeling and estimation are scarce (exceptions are, e.g., Pykhtin (2003), Rösch & Scheule (2005), Kupiec (2008), Bruche & González-Aguado (2010), Rösch & Scheule (2010)), although empirical data shows that default and recovery rates jointly deteriorate during economic downturns. Figure 3.1 highlights this stylized fact for the recession years 1990 and 1991 (the time of the Persian Gulf War), 2001 and 2002 (the period following the September 11, 2001 terrorist attacks and the general downturn in the US technology industry) as well as 2008 and 2009 (the Global Financial Crisis).

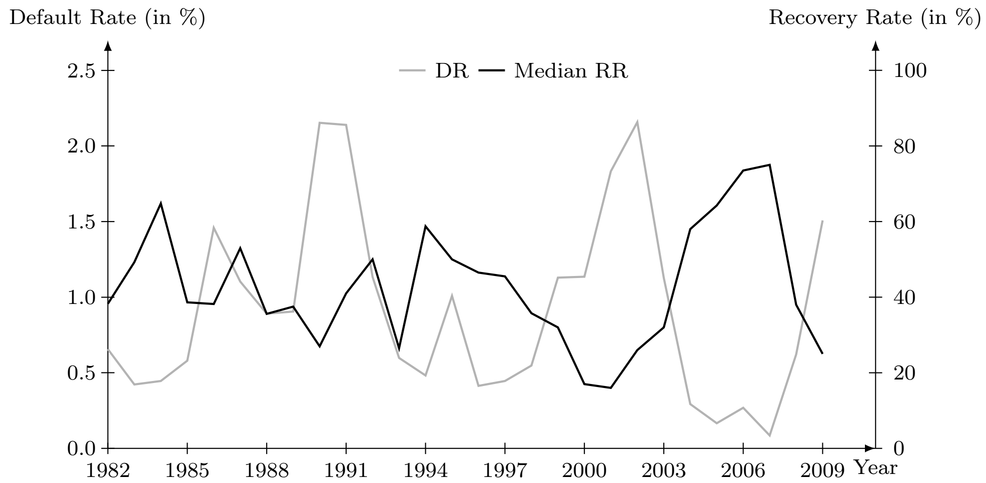

Figure 3.1: Default rates and recovery rates of non-financial bond issues from 1982 to 2009

This figure shows that default and recovery rates vary over time and are negatively related. Default rate is the ratio of defaulted bond issues to total bond issues per year. Recovery rate is the ratio of the price of defaulted debt obligations after 30 days of the occurrence of a default event to the par value. Source: Moody's. For a more detailed description of the data see Section 3.3.

Bade et al. (2011*a*) provide empirical evidence that indeed default process and recovery process are highly correlated by applying US nonfinancial corporate bond data to an econometric extension of the economic model introduced by Pykhtin (2003).

The second gap in the literature is performance comparisons among the several different approaches to PD and LGD forecasting. Besides the most recent contribution of Qi & Zhao (2011), one exception is Bastos (2010) who compares simple ordinary least squares (OLS) estimation procedures of LGD with a nonparametric regression tree approach on the basis of root mean squared errors (RMSEs) and relative absolute errors (RAEs). Nevertheless, the authors of both papers use data solely from defaulted obligations, as do their predecessors from this strand of literature (see, for example, Caselli et al. (2008)).

This paper addresses these weaknesses by comparing predictions derived from the model by Bade et al. (2011*a*) with a quick and dirty mean prediction, a simple OLS model and a model incorporating a perfect correlation between default and recovery process as proposed by Rösch & Scheule (2009). Following Bastos (2010) we do this by calculating RMSEs and RAEs for the recovery rate estimates of defaulted bonds. In addition, we apply these measures to the portfolio level: namely the difference between portfolio default rate and PD as well as between portfolio loss rate and expected loss (EL).

The paper proceeds as follows. Section 3.2 briefly introduces the models used, including their estimation and the calculation procedures of the required risk measures based on the derived parameter estimates. In Section 3.3 we describe the empirical data and the framework of our analysis. The results are presented in Section 3.4. Section 3.5 concludes the paper.

3.2 Theoretical framework

3.2.1 The general default and recovery process specification

Generally, we assume that the default process of a single borrower or bond issuer i in time period t $(i = 1, ..., N_t, t = 1, ..., T)$ is driven by a normally distributed asset value return V_{it} as introduced by Merton (1974). A default event occurs if the asset value return, specified by:

$$V_{it} = \beta_0 + \beta' x_{it}^V + Z_{it}^V \tag{3.1}$$

crosses a threshold, generally assumed to be zero. $x_{it}^V = \left(x_{it1}^V, \ldots, x_{itK}^V\right)'$ are K observable and deterministic firm-specific, industry-specific, or macroeconomic risk factors that influence the asset value return. $\beta = (\beta_1, \ldots, \beta_K)'$ are the sensitivities with respect to these factors and β_0 is a constant. Z_{it}^V is an idiosyncratic i.i.d. $N \sim (0,1)$ random variable driving the return of borrower i's assets in time period t.

Following Bade et al. (2011*a*) we specify the recovery process by

$$Y_{it} = \gamma_0 + \gamma' x_{it}^Y + \sigma \rho^U Z_{it}^V + \sigma \sqrt{1 - {\rho^U}^2} Z_{it}^Y, \tag{3.2}$$

where Y_{it} is the logarithm of the recovery rate and thus interpretable as (potential) return on the debt amount outstanding. $x_{it}^Y = \left(x_{it1}^Y, \ldots, x_{itL}^Y\right)'$ are L deterministic observable risk factors driving the recovery, $\gamma = (\gamma_1, \ldots, \gamma_L)'$ represent the loadings of these factors, and γ_0 is a constant. Z_{it}^Y is i.i.d. $N \sim (0,1)$ and σ is a constant parameter. Yet, since Z_{it}^V is part of Equations (3.1) and (3.2), the parameter ρ^U is the correlation between both firm-specific errors as well as the conditional correlation between the asset return and the log-recovery process given the observable covariates.

Besides the possible correlation of the default process and the recovery process introduced in the model presented above, the second feature, which we would like to introduce is that, in general, the recovery rate of a debt obligation is only observable in the case of default. In order to account for this fact, Bierens (2007) derives a maximum likelihood procedure to simultaneously estimate the parameters for such a statistical model firstly introduced by Heckman (1979). The log-likelihood for a single observation i in period t takes the following form:

$$\begin{aligned} \mathcal{L}_{it} = {} & (1 - d_{it}) \cdot \ln \Phi \left(\beta_0 + \beta' x_{it}^V\right) + d_{it} \cdot \ln \left(1 - \Phi \left(\beta_0 + \beta' x_{it}^V\right)\right) \\ & + d_{it} \cdot \ln \frac{\phi \left(\left(y_{it} - \left(\gamma_0 + \gamma' x_{it}^Y\right)\right) / \sigma\right)}{\sigma \left(1 - \Phi \left(\beta_0 + \beta' x_{it}^V\right)\right)} \\ & + d_{it} \cdot \ln \left(1 - \Phi \left[\frac{\frac{\rho^U}{\sigma} \left(y_{it} - \left(\gamma_0 + \gamma' x_{it}^Y\right)\right) + \left(\beta_0 + \beta' x_{it}^V\right)}{\sqrt{1 - {\rho^U}^2}}\right]\right). \end{aligned} \tag{3.3}$$

$\phi(\cdot)$ specifies the density function and $\Phi(\cdot)$ the cumulative distribution function of the standard normal distribution. d_{it} indicates whether the observed obligation defaults ($d_{it} = 1$) or not ($d_{it} = 0$). Thus, all parameters may be estimated without the knowledge of values for V_{it}. Equation (3.3) is then maximized over n_t observations per period and T periods:

$$\ell = \sum_{t=1}^{T} \sum_{i=1}^{n_t} \mathcal{L}_{it}. \tag{3.4}$$

3.2.2 Model assumptions and consequences

For the general framework presented above especially two restrictive assumptions are of particular interest. The first one is the assumption that, conditional on given realizations of the observable risk-factors, both processes are uncorrelated, i.e., $\rho^U = 0$. In this case the observed log-recoveries are normally distributed (see the dark bars in Figure 3.2). The assumption of uncorrelated error terms allows a separate estimation of the parameters underlying both processes in the model, since $\mathcal{L}_{it}$ simplifies to:

$$\begin{aligned}\mathcal{L}_{it}^{uncorr.} = &\underbrace{(1-d_{it}) \cdot \ln \Phi\left(\beta_0 + \beta' x_{it}^V\right) + d_{it} \cdot \ln\left(1 - \Phi\left(\beta_0 + \beta' x_{it}^V\right)\right)}_{\mathcal{L}_{it}^{Probit}} \\ &+ \underbrace{d_{it} \cdot \ln \frac{\phi\left(\left(y_{it} - \left(\gamma_0 + \gamma' x_{it}^Y\right)\right)/\sigma\right)}{\sigma}}_{\mathcal{L}_{it}^{Recovery}}.\end{aligned} \tag{3.5}$$

The parameters of $\mathcal{L}_{it}^{Probit}$ are estimated by a standard probit procedure via maximum likelihood (see, e.g., Gordy & Heitfield (2000), Gordy & Heitfield (2002) or Hamerle et al. (2003)):

$$\ell^{Probit} = \sum_{t=1}^{T} \sum_{i=1}^{n_t} \mathcal{L}_{it}^{Probit}. \tag{3.6}$$

Due to the independence of the recovery process from the default process, the parameters of $\mathcal{L}_{it}^{Recovery}$ need not necessarily be estimated via maximum likelihood. For convenience, a simple OLS regression of the observed log-recoveries may be run.[9]

The second possible restrictive assumption to the model is that default and recovery processes are perfectly positive correlated, i.e., $\rho^U = 1$, and that $\beta_0 = \frac{\gamma_0}{\sigma}$ as well as $\beta = \frac{\gamma}{\sigma}$. In other words, both processes are driven by

[9] Please note that many other transformations of the recovery rates, such as logit or probit, are possible (see, e.g., Dermine & de Carvalho (2006) or Bastos (2010)), but in order to ensure that results are comparable to the unrestricted model we focus on the logarithmic transformation.

the same explanatory variables and each variable has the same standardized exposure in both processes. Thus, the default barrier translates into a cut-off point for the observed log-recoveries. Their distribution equals a truncated normal distribution (see the lighter bars in Figure 3.2).

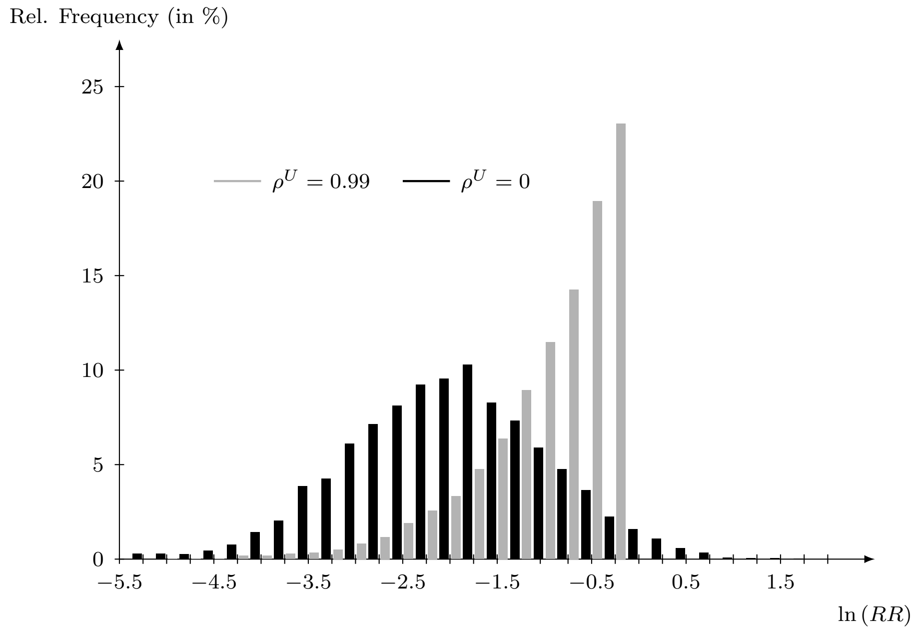

Figure 3.2: Distributions of observable log-recoveries for a sample portfolio of 100,000 obligors and differently correlated error terms

This figure presents distributions of log-recoveries for defaulted obligors in a sample portfolio of 100,000 obligors under different assumptions concerning the correlation between default and recovery process. The underlying parameters of the simulation for uncorrelated error terms (dark pattern), i.e., $\rho^U = 0$, are $\beta_0 = 1.6449$ (which corresponds to a PD of 5%), $\gamma_0 = -2.3551$ and $\sigma = 1$. The underlying parameters of the simulation for perfectly correlated error terms (pale pattern), i.e., $\rho^U = 1$, are $\beta_0 = 1.6449$ (which corresponds to a PD of 5%), $\gamma_0 = 2.46735$ and $\sigma = 1.5$.

The log-likelihood for a single observation under this restriction simplifies to the log-likelihood of a Tobit model:[10]

$$\begin{aligned} \mathcal{L}_{it}^{Tobit} &= \left(1 - d_{it}^{Tobit}\right) \cdot \ln \Phi \left(\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma} \right) \\ &\quad + d_{it}^{Tobit} \cdot \ln \frac{\phi\left(\left(y_{it} - \left(\gamma_0 + \gamma' x_{it}^Y\right)\right)/\sigma\right)}{\sigma}. \end{aligned} \tag{3.7}$$

Since the default barrier generally is assumed to be zero, the truncation of

[10] For the derivation of such a likelihood, see Bierens (2004). For an empirical application for bond defaults and recoveries, see Rösch & Scheule (2009).

the log-recoveries is made at zero too. Nevertheless, real data may contain recovery rates greater than one, i.e., log-recoveries greater than zero. These observations should be treated as non-defaults, such that $d_{it}^{Tobit} \neq d_{it}$ in these cases. The maximum likelihood function is:

$$\ell^{Tobit} = \sum_{t=1}^{T} \sum_{i=1}^{n_t} \mathcal{L}_{it}^{Tobit}. \tag{3.8}$$

3.2.3 Calculation of risk measures

In order to predict the risk of a debt obligation, the parameters derived by the methods presented above are only of secondary interest. The primary risk measures of importance are the PD, the EL and the recovery rate in the case where such an obligation defaults (expected recovery given default (ERGD)). Generally, these three ratios are linked by:

$$ERGD_{it} = 1 - \frac{EL_{it}}{PD_{it}}. \tag{3.9}$$

Since we assume an asset value process for the default event, the PD is given as the probability that V_{it} falls below zero (given the observable covariates). Under the normality assumption we obtain:

$$PD_{it} = 1 - \Phi\left(\beta_0 + \beta' x_{it}^V\right). \tag{3.10}$$

For EL and ERGD, respectively, the assumptions concerning the link between default and recovery process have to be considered. In the general case the parameter estimates of Equation (3.4) are used to calculate the expected loss by:

$$\begin{aligned} EL_{it}^{general} =& \Phi_2\left[-\left(\beta_0 + \beta' x_{it}^V\right), -\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma}, \rho^U\right] \\ & - \exp\left(\gamma_0 + \gamma' x_{it}^Y + \frac{\sigma^2}{2}\right) \\ & \cdot \Phi_2\left[-\left(\beta_0 + \beta' x_{it}^V\right) - \sigma\rho^U, -\frac{\gamma_0 + \gamma' x_{it}^Y}{\sigma} - \sigma, \rho^U\right]. \end{aligned} \tag{3.11}$$

$\Phi_2[\cdot,\cdot,\cdot]$ represents the distribution function of the bivariate normal distribution. For the more restrictive case of uncorrelated error terms it is most convenient to calculate the expected recovery given default at first and the expected loss afterward by applying the parameter estimates of Equation (3.6) to the PD and rearranging Equation (3.9). If the parameters of the recovery process with log-recoveries as dependent variable are estimated by simple OLS, ERGD is calculated by:

$$ERGD_{it}^{OLS} = \exp\left(\gamma_0 + \gamma' x_{it}^{Y} + 0.5\right). \tag{3.12}$$

With the parameters derived under the assumptions of the Tobit approach in Equation (3.7) we obtain EL by:

$$\begin{aligned} EL_{it}^{Tobit} = {} & \Phi\left(-\frac{\gamma_0 + \gamma' x_{it}^{Y}}{\sigma}\right) \\ & - \exp\left(\gamma_0 + \gamma' x_{it}^{Y} + 0.5\sigma^2\right) \cdot \Phi\left(-\frac{\gamma_0 + \gamma' x_{it}^{Y} + \sigma^2}{\sigma}\right). \end{aligned} \tag{3.13}$$

Please note that in the Tobit case the PD is computed as:

$$PD_{it}^{Tobit} = \Phi\left(-\frac{\gamma_0 + \gamma' x_{it}^{Y}}{\sigma}\right). \tag{3.14}$$

3.3 Data and Methodology of the performance comparison

3.3.1 Default and recovery data

The data sample underlying the empirical analysis is provided by Moody's credit rating agency and is the same as the one used by Bade et al. (2011*a*). The data set contains the annual ratings of regular US bond issues, as well as default dates and recovery rates given default. Moody's records a default event if interest or principal payments are missed or delayed, Chapter 11 or Chapter 7 bankruptcy is filed or a distressed exchange, such as a reduction in a financial obligation, occurs. The recovery rate is equal to the price of a defaulted bond measured thirty days after a default event in relation to the face value of the bond.

Table 3.1 summarizes important descriptive statistics for the data set, which consists of $187,638$ observations for regular US bond issues of non-financial institutions from 1982 to 2009. Coincident with a change in Moody's rating methodology in 1982 and the role of ratings in the subsequent analysis, earlier observations are excluded from this empirical study.

During the observation period, a total of $1,659$ defaults occurred, which yields a default rate (DR) of 0.884%. The mean recovery rate for all defaulted bonds is 37.541%; the median recovery rate is 32%.

Table 3.1 also shows the descriptive statistics per rating category: all bond issues with a rating higher than Ba are aggregated to an investment grade (IG) rating, and all bond issues with a rating lower than B are aggregated to rating C. This categorization addresses the limited number of default events in the subcategories. The table shows that, as one may expect, the default rate increases from rating IG to C. The mean recovery rate decreases from rating IG to C, except for grades Ba (48.607%) and IG (46.823%), which may be due to the small number of defaults, and hence the small number of recovery events in both grades.

Since the rating grade as well as the rating shift in the year prior to the observed rating status ($rating_{it} - rating_{it-1}$) are statistically and economically significant for the data set, we include rating dummies as well as an ordinal variable for the rating shift as explanatory variables in the empirical study.[11]

Table 3.1: Number of observations, default rate, and mean recovery

This table reports descriptive statistics on defaults and recoveries of non-financial bonds from 1982 to 2009. The data set provided by Moody's is split up into four rating categories: investment grade (IG), containing all observations with a Moody's rating higher than Ba, Ba, B and C, containing all observations with a Moody's rating worse than B. $N_{obs.}$ is the number of observations. $N_{def.}$ is the number of defaults. DR (default rate) is the ratio of the number of defaults to the number of observations in each rating grade. $RRGD_{\emptyset}$ is the mean recovery rate of the defaulted bonds in each rating grade. Recovery rate is the ratio of the price of defaulted debt obligations after 30 days of the occurrence of a default event to par value.

Rating	$N_{obs.}$	% of all obs.	$N_{def.}$	% of all def. obs.	DR (in %)	$RRGD_{\emptyset}$ (in %)
IG	146,582	78.120	51	3.074	0.035	46.823
Ba	15,262	8.134	87	5.244	0.570	48.607
B	20,132	10.729	530	31.947	2.633	39.890
C	5,662	3.018	991	59.735	17.503	34.836
Total	187,638	100.000	1,659	100.000	0.884	37.541

In order to account for the time series variation of default and recovery rates shown in Figure 3.1, we include the lagged change of GPDI as further explanatory variable in the study.

[11] Since, for bonds originated in the year of observation, $rating_{it} - rating_{it-1}$ yields a missing value (MV), we include a dummy variable for these observations and set $rating_{it} - rating_{it-1} = 0$. Through this, we are able to keep these observations in the data set and differentiate between observations with $rating_{it} - rating_{it-1} = 0$ and $rating_{it} - rating_{it-1} = MV$.

Since all explanatory variables are lagged by one year, they can be treated as known quantities when predicting PD, EL and ERGD.

3.3.2 Model validation framework

In the empirical study we compare four banks with competing approaches to the projection of future defaults and losses:

- Bank 1 simply estimates PD, EL and ERGD by historical averages, which is probably the most convenient but most likely also the least accurate method to predict future default or recovery rates.
- Bank 2 follows the restrictive approach of Equation (3.5), i.e., it estimates the PD with the Probit approach of Equation (3.6), which allows an incorporation of firm-specific, industry-specific and macroeconomic covariates and an explanation of the marginal effect of each considered variable on the likelihood of a default. With regard to LGD forecasts, Bank 2 uses an OLS regression with the natural logarithm of the recovery rate of defaulted bonds as dependent variable.
- Bank 3 uses the Tobit approach of Equation (3.7) to obtain the relevant parameters from the historical data.
- Bank 4 uses the general Heckman approach of Equation (3.3) to forecast PD and LGD simultaneously.

In detail, the model validation framework for our performance comparison consists of five steps, which are repeated 10,000 times in order to exclude sample effects:

Step 1: we select 90% of the data as a random sample and treat the remaining 10% of the data as out-of-sample.

Step 2: with the in-sample data we estimate the relevant parameters of the models underlying the banks' prediction techniques. For each model, we investigate four different specifications containing the following explanatory variables.

- Specification 1: ratings.
- Specification 2: ratings and lagged GPDI change.

- Specification 3: ratings and rating shift.
- Specification 4: ratings, rating shift and lagged GPDI change.

Step 3: these parameters are incorporated to estimate PD, ERGD and EL for each observation of the in-sample data set as well as for the out-of-sample data set.

Step 4: on the single borrower level we follow the approach by Bastos (2010) and compare the realized recovery rates of the defaulted bonds RR_{jt}, where $j = 1, \ldots, n_t^{def.}$, in each data subset with their estimates via RMSE:

$$RMSE_{RR} = \sqrt{\left(\sum_{t=1}^{T} n_t^{def.}\right)^{-1} \cdot \sum_{t=1}^{T} \sum_{j=1}^{n_t^{def.}} \left(RR_{jt} - ERGD_{jt}^{model}\right)^2}, \tag{3.15}$$

and RAE:

$$RAE_{RR} = \frac{\sum_{t=1}^{T} \sum_{j=1}^{n_t^{def.}} \left|RR_{jt} - ERGD_{jt}^{model}\right|}{\sum_{t=1}^{T} \sum_{j=1}^{n_t^{def.}} \left|RR_{jt} - ERGD_{jt}^{simple}\right|} \cdot 100. \tag{3.16}$$

RMSE measures the accuracy of the estimates in absolute terms while RAE measures the accuracy relative to a benchmark estimator. For convenience we use the arithmetic mean of the realized recovery rates calculated by Bank 1 for the corresponding rating grade of each observation as simple predictor.

Step 5: on the portfolio level we aggregate the PDs and ELs of the borrowers in the both subsamples to portfolio PDs and ELs by:

$$PD^{PF} = \left(\sum_{t=1}^{T} n_t\right)^{-1} \sum_{t=1}^{T} \sum_{i=1}^{n_t} PD_{it} \tag{3.17}$$

and:

$$EL^{PF} = \left(\sum_{t=1}^{T} n_t\right)^{-1} \sum_{t=1}^{T} \sum_{i=1}^{n_t} EL_{it}. \tag{3.18}$$

Since we only get one value per risk measure and portfolio that is compared with the realized portfolio default rate and portfolio loss rate, respectively, we have to calculate RMSE and RAE over the 10,000 iterations of this random sampling procedure. We do this for the out-of-sample portfolio.

3.4 Results

3.4.1 Single borrower level

Table 3.2 shows the RMSEs in-sample and out-of-sample by Bank and specification. On average, the least accurate predictive power is reached by using the arithmetic mean of observed recovery rates (Bank 1) as forecast for ERGD in-sample as well as out-of-sample. Despite the highest average RMSE, the standard deviation for the in-sample RMSEs of Bank 1 (0.0006) is the lowest of all four banks in each specification. In contrast, the out-of-sample standard deviation of Bank 1's RMSEs is the highest. Thus, Bank 1 not only has the least accurate method to predict future (i.e., out-of-sample) recovery rates for defaulted bonds on average, but also the most insecure method.

Using a simple OLS regression and calculating ERGDs on the basis of the regression results yields improved results compared to Bank 1's approach. The RMSEs are reduced on average and for the out-of-sample data in standard deviation, too. The more elaborate the model specification, the lower the average $RMSE_{RR}^{in-sample}$. Out-of-sample, adding GPDI to the regression model, i.e., switching from Specification 1 to 2 or from 3 to 4 reduces the predictive accuracy. With the exception of switching from Specification 1 to 2 for the out-of-sample data, the standard deviation of the RMSEs increases with the number of variables taken into account in both subsamples.

The Tobit procedure used by Bank 3 yields a further improvement of the results compared with Banks 1 and 2. Nevertheless, some qualitative differences are apparent. In contrast to Bank 2, the incorporation of GPDI yields more accurate recovery rate forecasts; RMSE decreases on average as well as its standard deviation. On the other hand, incorporating the rating shift increases RMSE as well as its standard deviation and thus lowers the predictive power of the Tobit model for recovery rates. Unusually, the average $RMSE_{RR}^{out-of-sample}$ is lower than the average $RMSE_{RR}^{in-sample}$ for each model specification.

The Heckman model implemented by Bank 4 yields the best results for the average RMSE. The predictive power rises by adding more explanatory variables to the model specification. The standard deviation of RMSE, though, is higher than for the other banks' predictions in-sample and higher than Bank 2's predictions out-of-sample. Due to the computational complexity of the likelihood function, too little recovery data in the sample might be an explanation for a higher number of outliers for the recovery rate estimates compared with the other models. Since such outliers have a higher loading in a quadratic measure like RMSE than for a measure based on the absolute value like RAE, the distribution of RMSE itself is more sensitive to these. Thus, a higher standard deviation of RMSEs, which itself is the square root of a quadratic measure, is likely to be caused by this connection.

Table 3.2: Results for RMSE on the recovery rate level

This table reports the average $RMSE_{RR}^{in-sample}$ and average $RMSE_{RR}^{out-of-sample}$ of the survey. RMSE is calculated by Equation (3.15). Standard deviations are reported in parentheses.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RMSE_{RR}^{in-sample}$		
Specification 1	0.28623	0.27280	0.26855	0.26632
	(0.00060)	(0.00130)	(0.00126)	(0.00134)
Specification 2	0.28623	0.27281	0.26821	0.26592
	(0.00059)	(0.00130)	(0.00123)	(0.00133)
Specification 3	0.28623	0.27231	0.26864	0.26553
	(0.00060)	(0.00133)	(0.00127)	(0.00137)
Specification 4	0.28623	0.27224	0.26833	0.26520
	(0.00060)	(0.00134)	(0.00126)	(0.00138)
		$RMSE_{RR}^{out-of-sample}$		
Specification 1	0.28633	0.27380	0.26842	0.26665
	(0.01386)	(0.01195)	(0.01167)	(0.01258)
Specification 2	0.28631	0.27402	0.26808	0.26636
	(0.01394)	(0.01152)	(0.01141)	(0.01241)
Specification 3	0.28629	0.27373	0.26851	0.26617
	(0.01406)	(0.01202)	(0.01171)	(0.01281)
Specification 4	0.28630	0.27388	0.26823	0.26598
	(0.01413)	(0.01218)	(0.01167)	(0.01279)

The results for $RAE_{RR}^{in-sample}$ and $RAE_{RR}^{out-of-sample}$ presented in Table 3.3 broadly confirm the results above. Relative to the results of Bank 1, the Heckman model performs best, followed by the Tobit approach and the OLS approach, which only performs a little better than the historical average. Notable is that in-sample the standard deviation of the RAEs increases with decreasing average RAE, while out-of-sample the result is the opposite.

In order to check whether the results are data specific, we excluded all observations which have a missing value (62,990 observations) for $rating_{it} - rating_{it-1}$ and repeated the study. Tables 3.4 and 3.5 present the results of this robustness check for RMSE and RAE. Qualitatively, the previous results remain unchanged. In absolute terms the data reduction has contrary effects.

Table 3.3: Results for RAE on the recovery rate level

This table reports the average $RAE_{RR}^{in-sample}$ and average $RAE_{RR}^{out-of-sample}$ of the survey. RAE is calculated by Equation (3.16). Standard deviations are reported in parentheses.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RAE_{RR}^{in-sample}$		
Specification 1	100	99.067	97.275	95.026
	-	(0.417)	(0.433)	(0.590)
Specification 2	100	99.121	97.121	94.897
	-	(0.413)	(0.424)	(0.571)
Specification 3	100	98.367	96.856	94.271
	-	(0.418)	(0.437)	(0.605)
Specification 4	100	98.376	96.841	94.240
	-	(0.420)	(0.437)	(0.621)
		$RAE_{RR}^{out-of-sample}$		
Specification 1	100	99.657	97.477	95.323
	-	(7.272)	(6.525)	(5.712)
Specification 2	100	99.785	97.322	95.217
	-	(7.142)	(6.399)	(5.600)
Specification 3	100	99.114	97.070	94.654
	-	(7.236)	(6.451)	(5.537)
Specification 4	100	99.214	97.065	94.664
	-	(7.385)	(6.523)	(5.611)

While RMSE for Bank 1 rises on average, it decreases for almost every specification of the other three banks. Relative to Bank 1, each of the other three banks performs better than for the whole data set, as Table 3.5 shows.

Table 3.4: Robustness check results for RMSE on the recovery rate level

This table reports the average $RMSE_{RR}^{in-sample}$ and average $RMSE_{RR}^{out-of-sample}$ of the robustness check. RMSE is calculated by Equation (3.15). Standard deviations are reported in parentheses.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RMSE_{RR}^{in-sample}$		
Specification 1	0.28625	0.27267	0.26810	0.26618
	(0.00066)	(0.00145)	(0.00143)	(0.00151)
Specification 2	0.28625	0.27251	0.26761	0.26596
	(0.00065)	(0.00144)	(0.00142)	(0.00151)
Specification 3	0.28625	0.27167	0.26700	0.26482
	(0.00066)	(0.00145)	(0.00145)	(0.00152)
Specification 4	0.28625	0.27152	0.26674	0.26487
	(0.00066)	(0.00145)	(0.00143)	(0.00153)
		$RMSE_{RR}^{out-of-sample}$		
Specification 1	0.28638	0.27407	0.26793	0.26649
	(0.01579)	(0.01324)	(0.01318)	(0.01400)
Specification 2	0.28638	0.27421	0.26746	0.26639
	(0.01566)	(0.01329)	(0.01315)	(0.01390)
Specification 3	0.28638	0.27335	0.26685	0.26527
	(0.01582)	(0.01359)	(0.01339)	(0.01416)
Specification 4	0.28637	0.27351	0.26658	0.26543
	(0.01553)	(0.01355)	(0.01328)	(0.01396)

Table 3.5: Robustness check results for RAE on the recovery rate level

This table reports the average $RAE_{RR}^{in-sample}$ and average $RAE_{RR}^{out-of-sample}$ of the robustness check. RAE is calculated by Equation (3.16). Standard deviations are reported in parentheses.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RAE_{RR}^{in-sample}$		
Specification 1	100	98.641	96.450	94.778
	-	(0.470)	(0.490)	(0.617)
Specification 2	100	98.575	96.225	94.611
	-	(0.469)	(0.494)	(0.612)
Specification 3	100	97.864	95.648	93.879
	-	(0.474)	(0.494)	(0.606)
Specification 4	100	97.826	95.576	93.984
	-	(0.473)	(0.500)	(0.618)
		$RAE_{RR}^{out-of-sample}$		
Specification 1	100	99.440	96.700	95.121
	-	(8.342)	(7.253)	(6.517)
Specification 2	100	99.483	96.485	94.989
	-	(8.381)	(7.310)	(6.615)
Specification 3	100	98.778	95.916	94.264
	-	(8.516)	(7.365)	(6.573)
Specification 4	100	98.830	95.838	94.389
	-	(8.407)	(7.290)	(6.577)

3.4.2 Portfolio level

The results of the performance analysis for the portfolio default rate are presented in Table 3.6.[12] The Probit approach of Bank 2 and the Heckman approach of Bank 4 yield almost identical RMSEs, which are lower than for Bank 1 and Bank 3. Bank 3's Tobit approach yields the worst predictions of all four banks for the first three model specifications. Specification 4 shows a slightly higher $RMSE_{DR}$ for Bank 1. The draw between Bank 2 and Bank 4 is confirmed by RAE_{DR}. For Specifications 1 and 2 the Heckman approach yields a lower value for RAE_{DR} and for the remaining two specifications the Probit approach is advantageous.

The results for the portfolio loss rate shown in Table 3.7 are much more widespread. Here, the simple prediction by historical average is the best predictor for future portfolio loss rates, followed closely by the Heckman approach of Bank 4. The Tobit approach performs rather poorly with a 20% worse loss estimation against the historical average, indicating that default and recovery process are not perfectly correlated. The worst performance is reached

[12] Note that the portfolio loss rate may also be compared with the value-at-risk of the LGD models considered in this paper. This would require accounting for an unobservable systematic risk factor to capture the comovement of default and recovery processes. Bade et al. (2011*a*) introduce such models.

Table 3.6: Results for RMSE and RAE of the portfolio default rate
This table reports RMSE and RAE of the survey for the portfolio default rate.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RMSE_{DR}$		
Specification 1	0.000668	0.000668	0.000674	0.000669
Specification 2	0.000679	0.000675	0.000680	0.000675
Specification 3	0.000681	0.000676	0.000684	0.000677
Specification 4	0.000684	0.000675	0.000683	0.000676
		RAE_{DR}		
Specification 1	100	100.000	100.591	99.941
Specification 2	100	99.256	99.837	99.192
Specification 3	100	99.099	100.207	99.227
Specification 4	100	98.668	99.652	98.746

by Bank 2, with more than 35% fewer accurate loss rate predictions. Thus, an estimation of two separate models for PD and LGD followed by a calculation of the expected loss based on the parameters derived from both models is not suitable. It results in a high degree of misspecification, since the possible correlation between the processes is ignored.

Table 3.7: Results for RMSE and RAE of the portfolio loss rate
This table reports RMSE and RAE of the survey for the portfolio loss rate.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RMSE_{LR}$		
Specification 1	0.000462	0.000616	0.000550	0.000463
Specification 2	0.000466	0.000618	0.000554	0.000466
Specification 3	0.000463	0.000626	0.000557	0.000465
Specification 4	0.000470	0.000629	0.000560	0.000469
		RAE_{LR}		
Specification 1	100	136.607	120.479	100.912
Specification 2	100	135.682	120.010	99.967
Specification 3	100	137.819	121.082	100.214
Specification 4	100	137.341	120.604	100.197

We provide the same robustness check on the portfolio level as on the single borrower level. Table 3.8 shows the results. Due to the data reduction, $RMSE_{DR}$ and $RMSE_{LR}$ deteriorate for all four banks. The draw between Bank 2 and Bank 4 concerning the default rate forecast switches to a marginal advantage for the Probit approach. The portfolio loss rate predictions of Banks 2 to 4 relative to Bank 1 improve compared to the primary results. Yet the Heckman approach yields the best predictions if more explanatory variables than the rating grade are taken into consideration.

Table 3.8: Robustness check results for RMSE and RAE on the portfolio level
This table reports RMSE and RAE of the robustness check for the portfolio default rate and the portfolio loss rate.

	Bank 1	Bank 2	Bank 3	Bank 4
		$RMSE_{DR}$		
Specification 1	0.000874	0.000874	0.000878	0.000874
Specification 2	0.000862	0.000859	0.000863	0.000859
Specification 3	0.000875	0.000868	0.000874	0.000870
Specification 4	0.000876	0.000867	0.000872	0.000868
		RAE_{DR}		
Specification 1	100	100.000	100.306	99.966
Specification 2	100	99.628	99.988	99.631
Specification 3	100	99.263	99.804	99.374
Specification 4	100	98.882	99.366	98.975
		$RMSE_{LR}$		
Specification 1	0.000595	0.000777	0.000664	0.000599
Specification 2	0.000587	0.000766	0.000652	0.000586
Specification 3	0.000600	0.000782	0.000666	0.000595
Specification 4	0.000591	0.000773	0.000658	0.000587
		RAE_{LR}		
Specification 1	100	131.485	110.730	100.292
Specification 2	100	131.982	110.707	99.698
Specification 3	100	132.489	111.229	99.334
Specification 4	100	133.427	111.760	99.252

3.5 Conclusion

Various work in the literature on default rates and recovery rates, as well as recent contributions suggesting a joint modeling of both variables, show the high complexity of these quantities and the challenge involved in obtaining an accurate measurement. While many previous contributions focused on the qualitative and quantitative drivers to both variables, this paper compares the predictive performance of several modeling approaches. RMSEs and RAEs are calculated for four banks, with each bank using a different approach to forecast future defaults and losses. In order to check their contribution to the predictive power of each bank's approach, four different combinations of explanatory variables are investigated.

The results show that a disjunct consideration of default and recovery ignoring the high correlation between both quantities yields not only biased parameter estimates, but also a worse predictive power for future losses than the general approach applied by Bade et al. (2011*a*). Especially on the portfolio loss level, the relative inaccuracy is severe.

While the portfolio default rate estimates may not be considered as significantly differing among the four banks, the portfolio loss rate as well as the recovery rate of a single borrower is predicted best with the general model allowing default and recovery to be correlated. Nevertheless, the quick and dirty solution also yields a relatively accurate measure of the future portfolio loss rate.

Thus, accounting for the high correlation of default and recovery rates highlighted during past economic downturns – most recently by the Global Financial Crisis – is a necessary condition for a suitable credit risk model. This paper provides further evidence that the model suggested by Pykhtin (2003) and adopted by Bade et al. (2011*a*) is a suitable model fulfilling this requirement.

Chapter 4

Statistical Methods of LGD Estimation – a Comparison

The content of this chapter was submitted to *European Journal of Operational Research* and is currently under revision.

4.1 Motivation

Measuring the credit risk of a defaultable obligation is a challenging task for most financial institutions. Especially, the tradeoff between meeting the regulatory requirements of the Basel II framework and the opportunity costs of tying up too much capital when adopting an Internal Ratings Based (IRB) approach emphasize the importance of this task. Namely, internal models require an appropriate predictive power for the probability that an obligor defaults (Probability of Default – PD) and the expected loss rate in the case of default (Loss Given Default – LGD).[13]

Reviewing the academic research on credit risk models, one finds a vast literature on determining the drivers behind either PD (see, e.g., Leland (1994), Jarrow & Turnbull (1995), Longstaff & Schwartz (1995), Madan & Unal (1995), Leland & Toft (1996), Jarrow et al. (1997), Duffie & Singleton (1999), Shumway (2001), McNeil & Wendin (2007), Duffie et al. (2007), Tong et al. (2012)) or LGD (see, e.g., Carey (1998), Citron et al. (2003), Dermine & de Carvalho (2006), Acharya et al. (2007), Altman (2009), Qi & Yang (2009), Grunert &

[13] Furthermore, the outstanding obligation of a borrower in the case of default (Exposure At Default – EAD) has to be estimated. However, since the EAD may be assumed as deterministic, we will not consider it further in this paper.

Weber (2009), Calabrese & Zenga (2010)). Particularly, the LGD parameter has received an increasing popularity in the past decade. Despite the growing attention practitioners and theoreticians paid to default risk and loss risk in the past twenty years, the Global Financial Crisis highlighted the need for further research.

First of all, there exist only very few modeling approaches controlling for the dependence between default rates and recovery rates (i.e., $1 - LGD$). Examples are, e.g., Pykhtin (2003), Rösch & Scheule (2005), Kupiec (2008), Bruche & González-Aguado (2010), Rösch & Scheule (2010), Jacobs Jr. & Karagozoglu (2011). Bade et al. (2011*a*) provide empirical evidence that the default process and recovery process are highly correlated by applying US nonfinancial corporate bond data to an econometric extension of the economic model introduced by Pykhtin (2003).

The second for a long time unattended but evolving stream in the literature is performance comparisons among the several different approaches to PD and LGD forecasting. Examples are Hlawatsch & Ostrowski (2011), who compare different estimation procedures based on simulated credit portfolios, and Bellotti & Crook (2012), who compare several approaches to modeling and transforming LGD on the basis of mean squared errors (MSEs) and mean absolute differences. Zhang & Thomas (2012) compare linear regression models with survival analysis models for LGD and find a better performance of the former. Other examples are Qi & Zhao (2011), Bastos (2010), and Caselli et al. (2008). All authors despite Hlawatsch & Ostrowski (2011) rely on empirical data. However, Bade et al. (2011*b*) adapt a performance comparison to the model and the data of Bade et al. (2011*a*). They show that the econometric version of the model by Pykhtin (2003), which considers defaulted and non-defaulted claims too, performs better than naive OLS models in predicting LGDs on the single borrower level. Like Bellotti & Crook (2012), they extend the comparison to the portfolio level and find a better performance of the sophisticated model as well. Anyway, due to the scarce set of explanatory variables the model does not outperform the simple historical average on portfolio level. This result is supported by further recent contributions of Frye (2010) and Frye & Jacobs Jr. (2010). They find that, based on time-series data of portfolio default rates and recovery rates, more sophisticated credit risk models – one of them is the model by Pykhtin (2003) – do not necessarily produce statistically significant better predictions for credit portfolio losses.

Considering these empirical findings, we identify four minimum standards for a well-conceived credit risk model. The first one is that the model is able to account for the co-movement of default rates and realized LGDs. A second basic requirement is that at least the qualitative influence of potential credit risk factors should be revealed during the parametrization of the model. That is, significant parameter estimates should be assigned only to variables, which are included in the data generating process. A theoretic foundation of the model may help to increase the plausibility and reliability of the estimates in this respect. Thirdly, and even more importantly, the LGD forecasts derived from the model parameters should fit realized LGDs precisely. Since LGD forecasting models hardly do so in absolute terms (compare Bellotti & Crook (2012)), we alleviate this requirement to an improvement in the predictive quality relative to other models. As Frye (2010) points out, data dependence in terms of the size of the data set, the homogeneity of the portfolio, and the credit risk of each borrower is a concern and a good model should reduce this dependence. Finally, a modern credit risk model, in addition, should be able to yield good predictions for PD, expected LGD, and EL, which are used for pricing and capital allocation purposes.

In order to address the four requirements in this paper, we investigate whether Bade et al. (2011*a*)'s econometric extension of the model by Pykhtin (2003) is able to account for these standards. Initially, we analyze the PD-LGD relation based on changing model properties graphically. Then, in the main part of this paper, we elaborate on the other requirements the model should meet. Unfortunately, the second and the fourth requirement cannot be tested with real world data, because the data generating processes are unknown. Hence, the parameters that enter the formulae for calculating the risk measures mentioned above are unknown as well. Consequently, we need to create a controlled environment, where each parameter is known, by providing a simulation study. We calibrate the model to generate defaults and LGDs dependent on realizations of simulated economic covariates weighted by given parameters. Afterwards, we estimate the parameters based on the resulting data set and besides the forecasts for LGD realizations, we calculate PD, EL, and expected LGD. All these estimates are compared either to the realized values (in the case of LGDs) or to the calculated values based on the parameters underlying the data generating process (in the case of credit risk measures) via a relative performance measure (in-sample and out-of-sample). We repeat this process iteratively and are thus able to analyze the data dependence and the reliability of the competing models. Further, we vary the assumptions underlying the data generating processes and hereby check the adaptability and robustness of each model. In the end, we test the forecasting precision of the model qualitatively and also quantitatively. In addition, we extend the analysis to measuring the absolute accuracy of economic capital allocations.

The remainder of the paper is organized as follows. Section 4.2 introduces the different models we consider for our performance comparison. Section 4.3 provides a sensitivity analysis of the data generating model with respect to important parameters. Section 4.4 presents the results of the simulation study concerning the forecasting performance of the models compared. In Section 4.5, we extend the simulation study to the predictive power in allocating economic capital and present the respective results. Section 4.6 concludes the paper.

4.2 Introduction of the competing credit risk models

4.2.1 Model setups and parameter estimation

In many applications (e.g., collateralized lending), default events and recovery rates are driven by separate but dependent economic processes. Implementing a Merton (1974) based framework similar to the Basel II IRB approach and industry credit portfolio models, such as Credit Metrics, we assume a default event to be triggered by a latent asset value return V_{it} crossing a threshold, generally assumed to be zero. V_{it} is driven by K observables $x_{it}^V = \left(x_{it1}^V, \ldots, x_{itK}^V\right)'$ and a non-observable idiosyncratic i.i.d. $N \sim (0,1)$ random variable Z_{it}^V. $i = 1, ..., N_t$ and $t = 1, ..., T$ identify different borrowers at different dates. The x_{it}^V are weighted by factor loadings $\beta = (\beta_1, \ldots, \beta_K)'$, such that

$$V_{it} = \beta_0 + \beta' x_{it}^V + Z_{it}^V. \tag{4.1}$$

β_0 is a constant.

Concerning the recovery process, we compare different links of the recovery process to the default process as well as different functional transformations of the recovery rate RR_{it}. Recovery rates generally range between zero and one (for bond recoveries values > 1 are also possible). Since most estimation methods, such as OLS regression models, require the dependent variable Y_{it} to be unrestricted, i.e., between $-\infty$ and ∞, the transformation of recovery rates is necessary to restrict forecasts of the dependent variable to reasonable values. Namely, we consider the logistic transformation

$$Y_{it}^{log} = \ln\left(\frac{RR_{it}}{1 - RR_{it}}\right), \tag{4.2}$$

and the probit transformation

$$Y_{it}^{pro} = \Phi^{-1}\left(RR_{it}\right), \tag{4.3}$$

which are both commonly used in practice, as well as a simple logarithmic transformation

$$Y_{it}^{ln} = \ln\left(RR_{it}\right). \tag{4.4}$$

In this respect, $\Phi^{-1}(\cdot)$ represents the inverse cumulative standard normal distribution.While the former two transformations require recovery rates between zero and one and thus a truncation of values greater than one, the latter transformation allows for these values.

The structure of the recovery process resembles the default process:

$$Y_{it} = \gamma_0 + \gamma' x_{it}^{Y} + U_{it}. \tag{4.5}$$

The L observable risk factors $x_{it}^{Y} = \left(x_{it1}^{Y}, \ldots, x_{itL}^{Y}\right)'$ weighted by their respective loadings $\gamma = (\gamma_1, \ldots, \gamma_L)'$ and the constant parameter γ_0 drive the transformed recovery rates as well as an idiosyncratic error term U_{it} does.

Next to the transformation of recovery rates, the error term correlation of Process (4.1) and Process (4.5) are of interest in order to derive proper estimates for the parameters γ_0 to γ_L from observed data. If we assume Z_{it}^{V} and U_{it} to be uncorrelated, the respective processes are conditionally uncorrelated with respect to the observable covariates. Consequently, the parameters β_0 to β_K and γ_0 to γ_L may be derived by separate estimation procedures. For the default process parameters the Maximum-Likelihood estimation (MLE) of a Probit model is applied, whereas the recovery process parameters may be estimated by a simple OLS regression. For the case of an OLS regression,

U_{it} is assumed to be standard normally distributed, irrespective of the chosen recovery rate transformation.

A more general approach of Bade et al. (2011*a*) inspired by Pykhtin (2003) is to model U_{it} as

$$U_{it} = \sigma \cdot \left(\rho^U Z_{it}^V + \sqrt{1 - {\rho^U}^2} Z_{it}^Y \right) \tag{4.6}$$

and thus to allow Z_{it}^V and U_{it}, in this case being $N \sim (0, \sigma)$ distributed, to be correlated by ρ^U. Z_{it}^Y is i.i.d. standard normally distributed. Conditional on the observables x_{it}^V and x_{it}^Y, default and recovery process are correlated by ρ^U, too. The economic reasoning behind this modeling approach, which we will refer to as correlation model throughout the remainder of this paper, is that collateral values, which define the recovery to a large extend, depend on idiosyncratic information next to systematic information. E.g., firms in financial distress often reduce maintaining expenses and consequently the value of collateral. Therefore, Z_{it}^V is incorporated in Equation (4.1) as well as in Equation (4.6) to capture this effect.

Due to this modeling approach, a joint estimation procedure for all model parameters is required, which further accounts for the fact that recovery rates are solely observable in the case of default. Ignoring the possible correlation of both processes and the limited observability of Y_{it} yields biased estimates for γ_0 to γ_L as shown by Heckman (1979). In particular, the selection bias of an OLS estimation is severe if the relation between selected and non-selected observations (in our case defaulted and non-defaulted borrowers) is small (compare Wu & Hand (2007)), which is a typical feature of defaults. A MLE procedure for such a sample selection problem is provided by Bierens (2007) and adopted to the case of bond recoveries by Bade et al. (2011*a*). Following Pykhtin (2003), we restrict the approach of (possibly) correlated error terms to the logarithmic transformation of the recovery rates, which allows us to derive closed form expressions, e.g., for the expected loss.

4.2.2 Forecasting of fundamental risk measures

Deriving the parameters of the models presented in the previous section from observed data is just the first step of our study. Given the parameters and values for the explanatory variables under consideration, important risk measures, namely PD, EL, and expected LGD (ELGD), may be calculated for each model setup of Section 4.2.1. In order to predict the values of each risk measure, we assume that the factors driving the default risk and loss risk are quantities known ex ante.[14] We indicate these observable variables by $x^V_{i,t-1}$ and $x^Y_{i,t-1}$, respectively.

The PD is calculated as

$$PD_{it} = 1 - \Phi\left(\beta_0 + \beta' x^V_{i,t-1}\right), \tag{4.7}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

For the approaches implying uncorrelated error terms, this paper calculates the expected LGD at first and the expected loss afterwards via the relation

$$EL_{it} = ELGD_{it} \cdot PD_{it}. \tag{4.8}$$

For each possible transformation in Equation (4.2) to Equation (4.4), a different measure of ELGD applies:

$$ELGD^{log}_{it} = 1 - \int_{-\infty}^{\infty} \frac{\exp\left(\gamma_0 + \gamma' x^Y_{i,t-1} + U_{it}\right)}{1 + \exp\left(\gamma_0 + \gamma' x^Y_{i,t-1} + U_{it}\right)} \cdot \phi\left(U_{it}\right) \, \mathrm{d}U_{it}, \tag{4.9}$$

$$ELGD^{pro}_{it} = 1 - \Phi\left(\frac{\gamma_0 + \gamma' x^Y_{i,t-1}}{\sqrt{2}}\right), \tag{4.10}$$

$$ELGD^{ln}_{it} = 1 - \exp\left(\gamma_0 + \gamma' x^Y_{i,t-1} + 0.5\right). \tag{4.11}$$

$\phi(\cdot)$ denotes the probability density function of the standard normal distribution.

For the correlation model, however, the expected loss should be computed

[14] Consequently, we do not have to predict these measures, which reduces the uncertainty of the forecasts and allows to focus on the predictive accuracy of the competing model approaches.

in a first step by

$$\begin{aligned} EL_{it}^{corr.} = \Phi_2 & \left[-\left(\beta_0 + \beta' x_{i,t-1}^V\right), -\frac{\gamma_0 + \gamma' x_{i,t-1}^Y}{\sigma}, \rho^U \right] \\ & - \exp\left(\gamma_0 + \gamma' x_{i,t-1}^Y + \frac{\sigma^2}{2}\right) \\ & \cdot \Phi_2 \left[-\left(\beta_0 + \beta' x_{i,t-1}^V\right) - \sigma\rho^U, -\frac{\gamma_0 + \gamma' x_{i,t-1}^Y}{\sigma} - \sigma, \rho^U \right] \end{aligned} \tag{4.12}$$

and the expected LGD in the second step by rearranging Equation (4.8). $\Phi_2 [\cdot, \cdot, \cdot]$ represents the cumulative distribution function of the bivariate normal distribution.

We validate the estimated values for PD, ELGD, and EL by calculating relative absolute errors, measuring the accuracy of the estimates relative to a benchmark estimator. We focus on this relative performance measure, since an absolute performance measure like the mean squared error or root mean squared error generally takes a relatively high level of values, especially for LGD forecasts (compare Bellotti & Crook (2012)). In other words, LGD forecasts generally show a low accuracy in absolute terms. For real world data and on the single borrower level, solely a comparison of estimated LGDs with realized LGDs of defaulted obligations would be possible (see, e.g., Bastos (2010), Bade et al. (2011*b*) or Bellotti & Crook (2012) for contributions on this topic). This is the case, because no one knows either the actual default and loss generating processes or the real qualitative and quantitative influences of potential risk factors. Hence, PDs, expected LGDs, and ELs may be estimated based on assumptions and empirical results in relation to the underlying risk processes and parameters, but it is impossible to compare the estimates to their data generating counterparts. The simulation approach adopted in this paper purports the correlation model and the underlying parameters as the data generating framework (see Sections 4.3.2 and 4.4.1). By doing so, we overcome the limitations of real world data. Now, it is possible to include the risk measures derived by this model via Equation (4.7), Equation (4.8), and Equation (4.12) in the performance analysis. This allows us to analyze whether the compared models perform relatively well in predicting future losses and, moreover, to judge the adequacy of each model setup for pricing and capital allocation purposes (see for the latter Section 4.5).

For PD, ELGD, and EL the relative absolute error is calculated by

$$RAE_{RM} = \frac{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\left|RM_{it}^{dgp} - RM_{it}^{estimate}\right|}{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\left|RM_{it}^{dgp} - RM_{it}^{corr.}\right|} \cdot 100. \tag{4.13}$$

RM_{it}^{dgp} represents a value of one of the considered risk measures following from the data generating process. $RM_{it}^{estimate}$ is the corresponding estimated value from one of the models calculated by the equations presented in this section, and $RM_{it}^{corr.}$ is the corresponding predictor using the parameter estimates derived by the correlation model. Please note that RAE_{RM} is aggregated over all observations $i = 1, \ldots, n_t$ and periods T. In contrast, for the realized LGD_{jt} the error measure is solely aggregated over all defaulted obligations $j = 1, \ldots, n_t^{def.}$, simply because there are no realizations for LGD of non-defaulted obligations. Furthermore, the realized LGDs are compared to the expected LGDs derived by the respective estimation models in the numerator and the estimated expected LGD for the correlation model in the denominator.

$$RAE_{LGD} = \frac{\sum_{t=1}^{T}\sum_{j=1}^{n_t^{def.}}\left|LGD_{jt} - ELGD_{jt}^{estimate}\right|}{\sum_{t=1}^{T}\sum_{j=1}^{n_t^{def.}}\left|LGD_{jt} - ELGD_{jt}^{corr.}\right|} \cdot 100. \tag{4.14}$$

The relative performance measure is strictly positive. Values below 100 indicate a better performance of a model's predictions compared to the benchmark predictions. Values above 100 indicate the opposite, whereas values of 100 indicate an equal performance of both models, which are compared.

4.3 Sensitivity analysis of important model parameters

4.3.1 Graphical analysis on the functional relation of PD and expected LGD

Empirical findings suggest that PD and expected LGD are positively correlated (see, e.g., Frye (2000*b*), Altman (2009) or Bade et al. (2011*a*)). Following the suggestion of Frye (2000*a*), Pykhtin (2003)'s correlation model links the two quantities by the choice of an unobservable systematic risk factor that jointly drives the default-process as well as the recovery-process of an underlying collateral. The simpler econometric version of the model introduced in this paper does not imply such an unobservable systematic factor. However, it allows for firm-specific, industry-specific or macroeconomic observables as well as the correlation parameter ρ^U connecting Process (4.1) and Process (4.5) by Assumption (4.6).

Figure 4.1 shows the relation between PD, the linear predictor of the recovery process (γ), and ELGD for different choices of ρ^U and σ. The graphs for $\rho^U = 0$ in the first row show that independent processes for default and recovery yield expected LGDs independent from PD. The higher the choice of γ, the lower the expected LGD. A higher fluctuation of log-recovery rates yields higher ELGDs in the case of $\gamma > 0$ and lower ELGDs for $\gamma < 0$. Especially for a positive γ, raising the correlation parameter affects the level of ELGD and its functional relation to the default probability. Interestingly, an increasing default probability coincides with a decreasing expected LGD. This counterintuitive result is even more evident in the case of a very high correlation of default- and recovery-process. The empirical results of Bade et al. (2011*a*) suggest such a high correlation. Nevertheless, they also show that PD and expected LGD both increase even if neither macroeconomic covariates nor unobservable systematic risk are implied in the model specification. The usage of rating dummies as firm-specific characteristics has a highly significant explanatory power for both processes within each model specification presented in Bade et al. (2011*a*). As a consequence, β, the linear predictor of the default process, and γ jointly decrease for a decreasing rating grade. Hence, the positive relation between PD and expected LGD is induced by the co-movement of the linear predictors.

Figure 4.2 assumes that $\gamma = \beta + Z$, with the variable Z varying from -10 to 10. The figure reveals that, generally, the co-movement of the linear predictors is no sufficient condition for the positive relation between PD and expected LGD. If $\rho^U = 0$ and Z relatively close to zero, so that ELGD does not reach its boundaries of either zero or one, then a positive relation of both quantities is observable. However, for almost perfectly correlated errors and growing σ, both quantities have a negative relationship.

Finally, Figure 4.3 shows a positive monotone relationship between PD and expected LGD for various combinations of ρ^U and σ. In order to derive this result, we specify both β and γ in dependence of Z. If we interpret Z as an

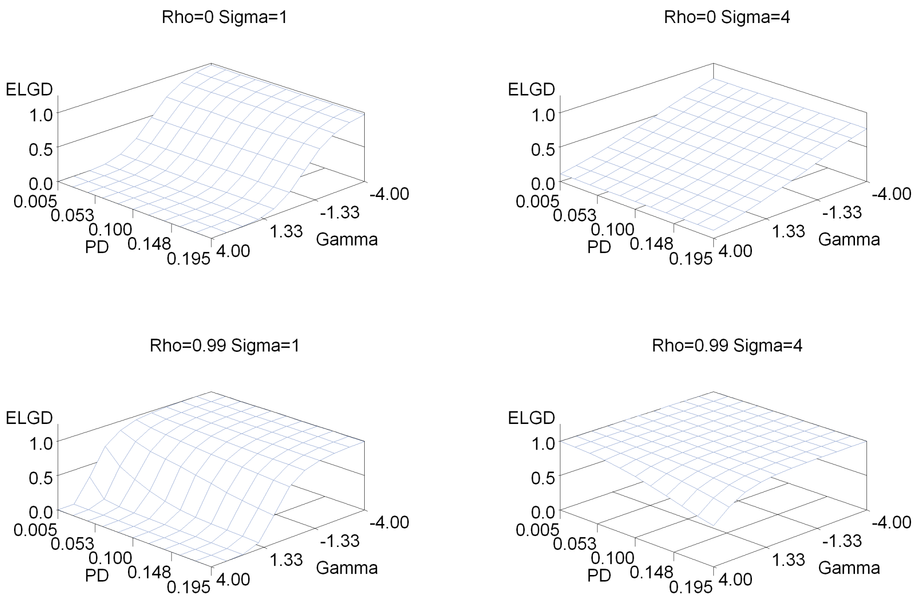

Figure 4.1: PD-ELGD dependencies for the correlation model

This figure shows that, dependent on γ, ρ, and σ, PD and ELGD are either uncorrelated or have a negative relation if β and γ have no functional relationship.

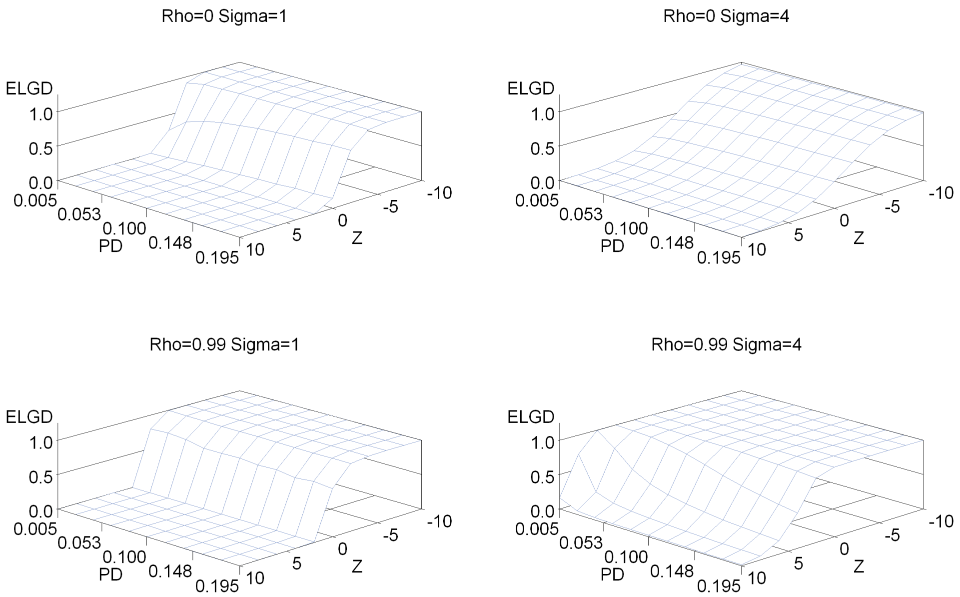

Figure 4.2: PD-ELGD dependencies for the correlation model if $\gamma = \beta + Z$

This figure shows that if γ is a linear function of β, then a higher value for ρ changes the sign of the relation between PD and ELGD from positive to negative.

observable quantity, which is known in advance, e.g., lagged GDP-growth, we can see how important the distinction between observable and unobservable (or unobserved) information driving default and recovery is. Especially, if the correlation between the processes is high, adding an observed quantity to the model specification alters the relation of default probability and ELGD dramatically from negative to positive. Furthermore, this result explains past empirical findings: the main factor of the positive PD-ELGD-correlation is observable co-variates that drive both quantities simultaneously.

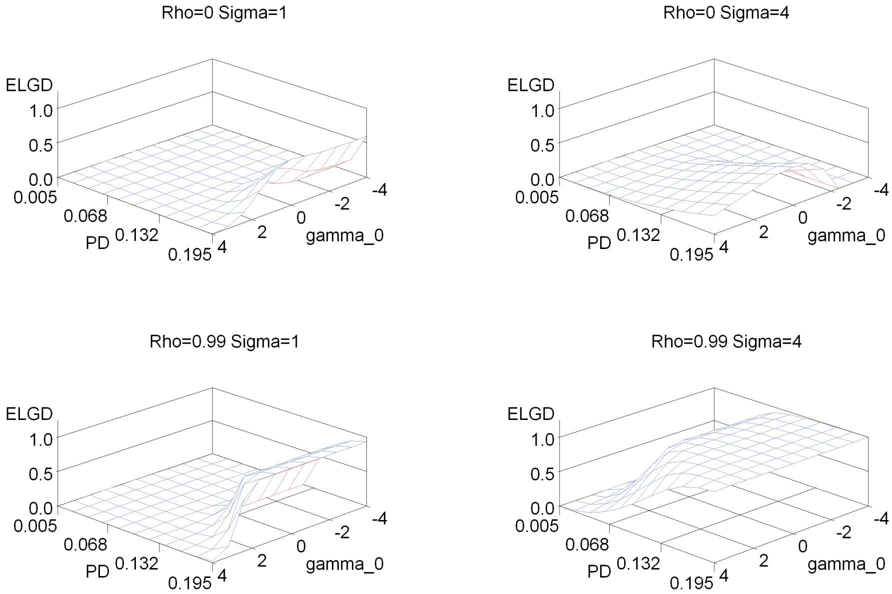

Figure 4.3: PD-ELGD dependencies for the correlation model if $\beta = 1 + 0.1Z$ and $\gamma = \gamma_0 + Z$

This figure shows that if β and γ both are linear functions of an observable factor Z, the functional relationship is positive, irrespective of the chosen values for ρ, σ, and γ_0.

4.3.2 The influence of the correlation parameter on credit risk forecasts

Why is it important to account for a correlation ρ^U if we are able to explain the positive correlation of default probability and expected LGD by implementing observable factors in both underlying processes? In order to answer this question, we construct a simple simulation study for a homogeneous portfolio of borrowers. In a first step, we calibrate the correlation model to three different combinations of PD and expected LGD. The first combination refers to a scenario with relatively low credit risk, where $PD = 1\%$ and $ELGD = 30\%$. The second scenario is referred to as an intermediate risk environment with $PD = 5\%$ and $ELGD = 45\%$, whereas the third scenario assumes a high level of credit risk with $PD = 10\%$ and $ELGD = 70\%$.[15] We choose $\sigma = 2$ as well as different combinations of ρ^U and γ_0, including the extreme cases of $\rho^U = 0$ (basic assumption of most credit portfolio models) and $\rho^U = 0.99$. The parameter assumptions are based on the empirical findings of Bade et al. (2011*a*). Therefore, we are able to test the relative performance of the models and recovery rate transformations presented in Section 4.2 depending on the underlying credit risk as well as on the data generating dependence of default process and recovery process.

For each parameter combination, we randomly draw Z^V_{it} and Z^Y_{it} for a portfolio size of 100,000 borrowers. The resulting defaults and recovery rates are randomly divided into two sub-samples containing either 90% or 10% of the total sample, respectively. We use the larger sub-sample as in-sample data and the remaining sub-sample as out-of-sample data. In the next step, the in-sample data are used to derive parameter estimates for the models presented in Section 4.2. Afterwards, each parameter estimate is plugged into the respective formulae for PD, expected LGD, and EL. These estimates are compared for the competing models to their benchmark values via the relative absolute error measures calculated by Equation (4.13). Since the portfolio is homogeneous, i.e., each borrower has the same PD, expected LGD, and EL, there is no distinction between in-sample and out-of-sample for the RAEs, which are calculated for these risk measures. However, the expected losses conditional on default events further are compared to the realized in-sample LGDs and to the realized out-of-sample LGDs by Equation (4.14), potentially yielding different results for the training data and the validation data. The results for each step of this sampling procedure are saved and repeated 10,000 times.

The median[16] RAE_{PD} differs from 100 at the most in the second decimal

[15] Altman (2009) finds average LGDs that vary from about 40% for senior secured bonds or loans to 75% for discount bonds.

[16] Since there is only one estimate for each risk measure per iteration, each corresponding RAE is susceptible to outliers and so is the mean of the 10,000 RAEs. Thus, we use the median RAE in order not to be forced into using a more or less arbitrary correction of such values.

place for each underlying PD. Thus, under the current setting of default risk solely driven by β_0 and the idiosyncratic factor Z_{it}^V, the relative performances of the probit approach and the correlation model for predicting the PD are very similar.

Figure 4.4 presents the median RAE_{ELGD} of the respectively best performing OLS model in dependence of ρ^U. Only one of the OLS models incorporating Transformation (4.2), Transformation (4.3) or Transformation (4.4) enters the figure per scenario. The graph reveals that, irrespective of the correlation and the scenario, the relative performance of the OLS models to the correlation model is far worse. The level of the RAE_{ELGD} depends on the underlying credit risk and the estimation method. The relatively best performing OLS model for the low as well as for the high risk setting is the one incorporating probit transformed recovery rates with RAE_{ELGD} levels of about 3,700. However, for intermediate credit risk, the logarithmic transformation yields the lowest RAE_{ELGD} of the three OLS models. With values between 6,800 and 7,900, although, on a much higher level and with a much higher fluctuation compared to the two other scenarios. As a result, not accounting for the correlation parameter during the estimation procedure yields seriously biased estimates for the expected LGD compared to the sophisticated approach accounting for the correlation parameter. The results for RAE_{EL} are qualitatively the same and quantitatively very similar to the results for expected LGD. Thus, they are not reported here.

Figure 4.5 shows the same comparison for the realized out-of-sample LGDs. It reveals two striking observations. The first is that the relative performance of the OLS models to the correlation model in predicting realized LGDs declines in ρ^U. Especially, for higher correlations exceeding 0.7, the RAE_{ELGD}^{OLS} increase sharply. Secondly, the general advantage of the simultaneous estimation procedure accounting for ρ^U in forecasting ELGD and EL does not hold for realized LGDs. Solely, for the intermediate risk environment, each OLS model performs weaker than the correlation model, irrespective of the preset value for the correlation parameter. The median values for RAE_{LGD} derived under the low risk setting are very close to 100 until the preset correlation exceeds 0.9. However, the values derived under the high risk setting show a higher gap between the two modeling approaches. For lower and intermediate values of ρ^U, the OLS model incorporating logarithmically transformed recovery rates better fits the realized LGDs. Similar to the low risk environment,

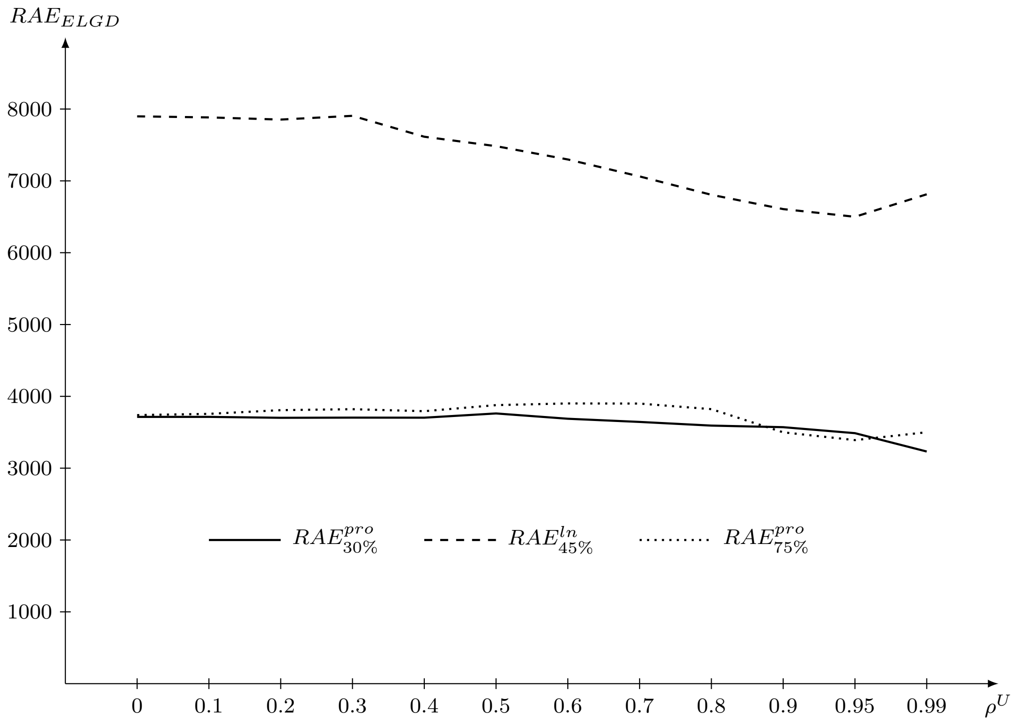

Figure 4.4: Sensitivity of RAE_{ELGD} in dependence of ρ^U

This figure compares the relative absolute errors of the ELGD estimates (RAE_{ELGD}) for a homogeneous portfolio of borrowers. For three different credit risk environments, the estimates by the respectively best performing OLS model are compared to Pykhtin (2003)'s correlation model. It is shown that, regardless of ρ^U, none of the OLS models has a higher predictive accuracy than the correlation model.

the forecast performance of the correlation model is not superior over the simple model's performance before $\rho^U \geq 0.9$. For the in-sample predictions of LGD, qualitatively equal and quantitatively similar results apply.

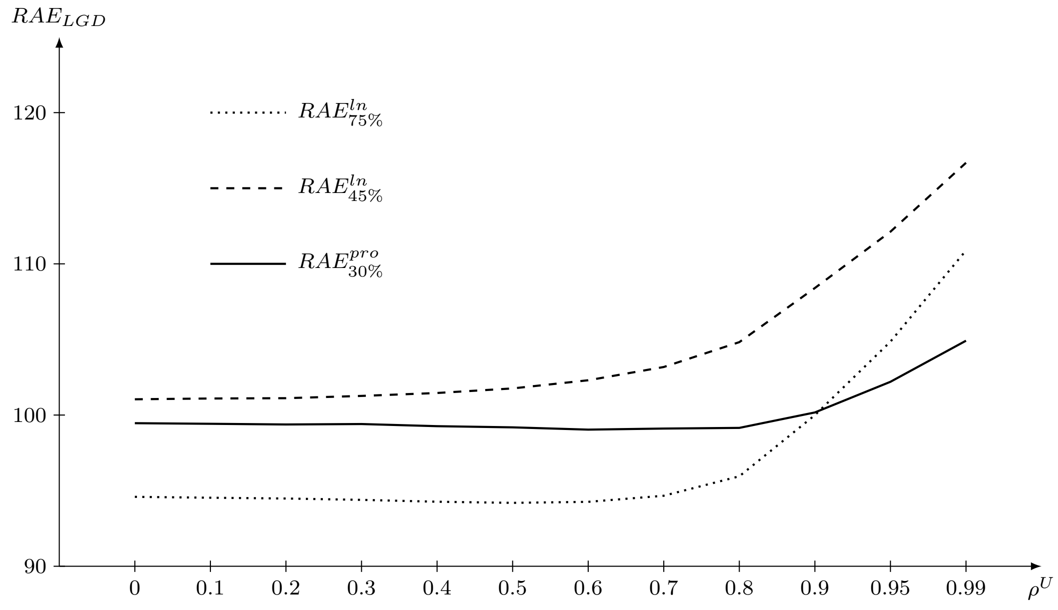

Figure 4.5: Sensitivity of $RAE_{LGD}^{out-of-sample}$ in dependence of ρ^U

This figure compares the relative absolute errors of the LGD estimates (RAE_{LGD}) for a homogeneous portfolio of borrowers. For three different credit risk environments, the estimates by the respectively best performing OLS model are compared to Pykhtin (2003)'s correlation model. It is shown that, in this setting, the OLS models have a lower predictive accuracy than the correlation model if the preset correlation parameter is high.

Therefore, the major advantage of the correlation model in forecasting ELGD and EL, which are used for rating or pricing purposes, cannot be confirmed for the prediction of realized LGDs. However, if default and recovery are simultaneously driven to a high extent by factors that cannot be observed or are neglected during the estimation procedure, i.e., if the correlation of the two processes is high, a simplifying modeling approach may yield biased predictions for LGD. The magnitude of this inaccuracy may not be overcome by the slight better relative performance in the case of low or intermediate correlations. Furthermore, the simple construction of the simulation engine (no incorporation of explanatory variables, the homogeneity of the portfolio, a relatively high portfolio size, which increases the likelihood of precise estimations) might be the cause of the relatively good fit of the OLS models in these cases.

4.4 Simulation study on the model performances

4.4.1 Default and recovery data generation and sampling

In order to verify the last statement concerning the simplicity of the model assumptions, we extend the simulation engine by simulated observables. Namely, we include five explanatory variables:

1. A macroeconomic variable x_{t-1}^{macro}, which once per period is randomly drawn i.i.d. $N \sim (4, 8.8)$ for all observations. It may be interpreted as the percentage change in gross domestic product (GDP).

2. A balance sheet variable $x_{i,t-1}^{bal.}$, which is drawn randomly from a uniform distribution with bounds 20 and 80 once per period and obligor. We may interpret this variable as liquidity or leverage in percent.

3. $x_{i,t-1}^{size}$ represents the natural logarithm of firm size (e.g., asset value), uniformly distributed between $\ln(1,000)$ and $\ln(1,000,000)$. Due to this transformation, bigger firm sizes become less likely.

4. $x_{i,t-1}^{CFROI}$ represents the cash flow return on investment in percent, which is also randomly drawn $N \sim (15, 30)$ once per period and obligor.

Further idiosyncratic information (e.g., management ability and behavior) is captured by the unobservables Z_{it}^{V} and Z_{it}^{Y}. Due to the higher computational complexity of the simulation and validation engine, we focus on two extreme cases for the correlation parameter. The first one refers to a high error correlation of $\rho^{U} = 0.95$ and yields the data generating process

$$
\begin{aligned}
V_{it} &= 0.847 + 0.02\, x_{t-1}^{macro} + 0.01\, x_{i,t-1}^{bal.} + 0.025\, x_{i,t-1}^{size} + 0.003\, x_{i,t-1}^{CFROI} \\
&\quad + Z_{it}^{V}, \qquad (4.15) \\
Y_{it} &= 1 + 0.03\, x_{t-1}^{macro} + 0.02\, x_{i,t-1}^{bal.} + 0.05\, x_{i,t-1}^{size} + 0.005\, x_{i,t-1}^{CFROI} \\
&\quad + 2 \cdot \left(0.95\, Z_{it}^{V} + \sqrt{1 - 0.95^{2}}\, Z_{it}^{Y}\right). \qquad (4.16)
\end{aligned}
$$

The second extreme case incorporates $\rho^U = 0$ and yields

$$\begin{aligned} V_{it} &= 0.847 + 0.02\, x_{t-1}^{macro} + 0.01\, x_{i,t-1}^{bal.} + 0.025\, x_{i,t-1}^{size} + 0.003\, x_{i,t-1}^{CFROI} \\ &\quad + Z_{it}^{V}, \qquad (4.17) \\ Y_{it} &= -3.5 + 0.03\, x_{t-1}^{macro} + 0.02\, x_{i,t-1}^{bal.} + 0.05\, x_{i,t-1}^{size} + 0.005\, x_{i,t-1}^{CFROI} \\ &\quad + Z_{it}^{Y}, \qquad (4.18) \end{aligned}$$

where $Y_{it} = \ln(RR_{it})$ in both cases.[17] Besides the correlation parameter, we recalibrate the recovery-processes by adjusting γ_0 from 1 to -3.5 and σ from 2 to 1.

Using these data generation processes, we produce time series of 20 periods for a portfolio size of 5,000 borrowers. From each time series containing 100,000 observations, we treat the observations of the first 19 periods (i.e., 95% of the data) as in-sample and the last period as out-of-sample. Then, like in the previous section, the in-sample data are used to estimate the parameters of the various models presented in Section 4.2.1. In order to determine the influence of the explanatory variables in each model, we specify four different combinations of these:

- Specification 1: x_{t-1}^{macro};
- Specification 2: $x_{i,t-1}^{bal.}$, $x_{i,t-1}^{size}$, and $x_{i,t-1}^{CFROI}$;
- Specification 3: x_{t-1}^{macro}, $x_{i,t-1}^{bal.}$, $x_{i,t-1}^{size}$, and $x_{i,t-1}^{CFROI}$, plus a beta distributed pseudo-variable $x_{i,t-1}^{pseudo}$ that is not incorporated in the data generating process and thus should have no statistical significant power in explaining the realized default events and recovery rates; and
- Specification 4: x_{t-1}^{macro}, $x_{i,t-1}^{bal.}$, $x_{i,t-1}^{size}$, and $x_{i,t-1}^{CFROI}$.

Especially, the third specification allows us to test whether the parameter estimates are biased and their influence thus might be neglected despite its significance in the data generating process. Furthermore, this specification allows us to check whether the pseudo-variable erroneously proves to have an explanatory power. All parameter estimates are saved and in a next step

[17] The default processes in (4.15) and (4.17) are calibrated to an average PD of about 4.17%. The recovery process in (4.16) is calibrated to an average expected LGD of about 64.4%. The recovery process in (4.18) however is calibrated to an average expected LGD of 74.6% in order to generate most of the realized LGDs within the range of zero and one.

utilized to calculate the default probability, the expected loss given default, and the expected loss, which are compared via RAE to PD, expected LGD, and EL derived from the original parameters. In contrast to the analysis in Section 4.3.2, each risk measure is borrower specific and thus the relative errors may be calculated in-sample as well as out-of-sample. The comparison of realized and estimated recovery rates for the defaulted obligations is performed as well. We repeat all steps 10,000 times.

4.4.2 Economic and statistical significance of the parameter estimates

Table 4.1 presents the parameter estimates derived by the simultaneous MLE approach for both data generation engines. It is shown that, irrespective of the assumptions underlying the data generating process, the parameters of the explanatory variables in the default process are economically correct specified on average. Statistically, the model needs to be fully specified (Specification 4) or overspecified (Specification 3) in order to yield average estimates that do not differ significantly from the preset parameter values. Solely, the intercept β_0 is not correctly specified if the estimation model does not contain all observable information. On average, $\beta_{pseudo} < 0.0005$ and neither is economically nor statistically significant.

If the preset correlation is 0.95, the economically correct Specification 4 does not yield as good parameter estimates for the recovery process as the overspecified Specification 3. γ_{macro}, $\gamma_{bal.}$, and γ_{size} are slightly underestimated if the pseudo-variable is not included in the model specification, whereas the influence of these variables on average is measured correctly if this actually redundant variable is included. For γ_0 and ρ^U, the average predictions are improved as well if we estimate Specification 3 instead of Specification 4. The influence of the pseudo-variable on the recovery-rate neither economically nor statistically differs from zero with significance.

Considering the case of conditionally independent processes for default and recovery, ρ^U and γ_0 are both either overestimated (Specifications 1, 3 and 4) or underestimated (Specification 2). Nevertheless, the data generating parameters for the model intercept as well as for the correlation are in the range of a half standard error around their respective estimates, corresponding to relatively narrow confidence intervals and a low probability of declining the null-hypothesis of $\gamma_0 = \widehat{\gamma_0}$ or $\rho^U = \widehat{\rho^U}$, respectively.

Table 4.1: Average parameter estimates for the correlation model

This table reports the average parameter estimates for different specifications of Pykhtin (2003)'s correlation model derived during the simulation study of Section 4.4.1. In order to produce data for the different estimation procedures, the default process was parameterized as

$$V_{it} = \beta_0 + \beta_{macro}\, x_{t-1}^{macro} + \beta_{bal.}\, x_{i,t-1}^{bal.} + \beta_{size}\, x_{i,t-1}^{size} + \beta_{CFROI}\, x_{i,t-1}^{CFROI} + Z_{it}^{V}$$

and the recovery process as

$$Y_{it} = \gamma_0 + \gamma_{macro}\, x_{t-1}^{macro} + \gamma_{bal.}\, x_{i,t-1}^{bal.} + \gamma_{size}\, x_{i,t-1}^{size} + \gamma_{CFROI}\, x_{i,t-1}^{CFROI} + \sigma \left(\rho^U Z_{it}^{V} + \sqrt{1 - {\rho^U}^2}\, Z_{it}^{Y} \right).$$

OV (original value) refers to the preset parameters of these data generating processes. The average standard errors are reported in parentheses. All average parameter estimates with an * are not statistically different from the preset values in the data generating processes at the 5%-level.

	$\rho^U = 0.95$					$\rho^U = 0$				
Parameter	OV	Sp. 1	Sp. 2	Sp. 3	Sp. 4	OV	Sp. 1	Sp. 2	Sp. 3	Sp. 4
β_0	0.847	1.619	0.915	0.848	0.848*	0.847	1.619	0.915	0.848	0.848
		(0.008)	(0.042)	(0.047)	(0.042)		(0.008)	(0.042)	(0.047)	(0.043)
β_{macro}	0.020	0.020		0.020*	0.020*	0.020	0.020		0.020*	0.020*
		(0.001)		(0.001)	(0.001)		(0.001)		(0.001)	(0.001)
$\beta_{bal.}$	0.010		0.010	0.010*	0.010*	0.010		0.010	0.010*	0.010*
			(0.000)	(0.000)	(0.000)			(0.000)	(0.000)	(0.000)
β_{size}	0.025		0.025	0.025*	0.025*	0.025		0.025	0.025	0.025
			(0.004)	(0.004)	(0.004)			(0.004)	(0.004)	(0.004)
β_{CFROI}	0.003		0.003	0.003*	0.003*	0.003		0.003	0.003*	0.003*
			(0.000)	(0.000)	(0.000)			(0.000)	(0.000)	(0.000)
β_{pseudo}	0.000			0.000*		0.000			0.000*	
				(0.033)					(0.033)	
γ_0	1.000	2.560	1.119	0.992	0.779	-3.500	-1.798	-3.831	-3.189	-3.186
		(0.122)	(0.131)	(0.140)	(0.163)		(1.517)	(0.882)	(0.744)	(0.743)
γ_{macro}	0.030	0.030		0.030	0.027	0.030	0.031		0.034	0.034
		(0.003)		(0.003)	(0.003)		(0.013)		(0.009)	(0.009)
$\gamma_{bal.}$	0.020		0.020*	0.020	0.019	0.020		0.017	0.022	0.022
			(0.001)	(0.001)	(0.001)			(0.005)	(0.005)	(0.005)
γ_{size}	0.050		0.050*	0.050	0.046	0.050		0.044	0.055	0.055
			(0.009)	(0.009)	(0.009)			(0.016)	(0.014)	(0.014)
γ_{CFROI}	0.005		0.005	0.005	0.005	0.005		0.004	0.006	0.006
			(0.001)	(0.001)	(0.001)			(0.002)	(0.002)	(0.002)
γ_{pseudo}	0.000			0.001*		0.000			0.001*	
				(0.083)					(0.073)	
ρ^U	0.950	0.945	0.950*	0.947	0.866	0.000	0.111	-0.222	0.194	0.196
		(0.008)	(0.005)	(0.006)	(0.039)		(0.622)	(0.509)	(0.456)	(0.456)
σ	2.000	2.028	2.015	1.995	1.912	1.000	1.172	1.153	1.107	1.107
		(0.051)	(0.051)	(0.050)	(0.048)		(0.138)	(0.136)	(0.126)	(0.126)

Table 4.2 presents the results for a separate estimation of the model parameters. On average, the estimates derived for the default-process by the Probit model do not differ from those derived by the estimation procedure for the correlation model. Concerning the recovery-process, the pattern for a preset correlation of $\rho^U = 0.95$ clearly shows a significant bias for the predictions from γ_0 to γ_{CFROI} in relation to the data generating parameters. The influence of these variables is strongly underestimated or even erroneously shown to be negative in the case of the macro-variable. However, the parameters derived

under the favorable data generating assumptions of $\rho^U = 0$ and $\sigma = 1$ prove to be relatively precise and even more precise than the parameters derived by the simultaneous estimation procedure. The non-existent influence of the pseudo-variable is correctly revealed on average for both data generating processes. Due to the different transformation of the recovery-rates, the results for the parameter estimates of the other OLS regression models lack of quantitative comparability and thus are not reported here.

Table 4.2: Average parameter predictions of separate estimation procedures for default- and recovery-process

This table reports the average parameter estimates for the default-process, using the Probit approach, and the recovery-process, using an OLS regression of log-transformed recovery rates, derived during the simulation study of Section 4.4.1. OV (original value) refers to the preset parameters of the data generating processes shown in Table 4.1. The average standard errors are reported in parentheses. All average parameter estimates with an * are not statistically different from the preset values in the data generating processes at the 5%-level. Estimates for ρ^U and σ are not provided by this approach, since $\rho^U = 0$ and $\sigma = 1$ are preset assumptions of the estimation procedure.

	$\rho^U = 0.95$					$\rho^U = 0$				
Parameter	OV	Sp. 1	Sp. 2	Sp. 3	Sp. 4	OV	Sp. 1	Sp. 2	Sp. 3	Sp. 4
β_0	0.847	1.619	0.915	0.848	0.848*	0.847	1.619	0.915	0.848	0.848
		(0.008)	(0.042)	(0.047)	(0.043)		(0.008)	(0.042)	(0.047)	(0.043)
β_{macro}	0.020	0.020		0.020*	0.020*	0.020	0.020		0.020*	0.020*
		(0.001)		(0.001)	(0.001)		(0.001)		(0.001)	(0.001)
$\beta_{bal.}$	0.010		0.010	0.010*	0.010*	0.010		0.010	0.010*	0.010*
			(0.000)	(0.000)	(0.000)			(0.000)	(0.000)	(0.000)
β_{size}	0.025		0.025	0.025*	0.025*	0.025		0.025	0.025	0.025
			(0.004)	(0.004)	(0.004)			(0.004)	(0.004)	(0.004)
β_{CFROI}	0.003		0.003	0.003*	0.003*	0.003		0.003	0.003*	0.003*
			(0.000)	(0.000)	(0.000)			(0.000)	(0.000)	(0.000)
β_{pseudo}	0.000			0.000*		0.000			0.000*	
				(0.033)					(0.033)	
γ_0	1.000	-1.366	-1.595	-1.591	-1.591	-3.500	-2.063	-3.443	-3.502	-3.502
		(0.015)	(0.056)	(0.058)	(0.056)		(0.017)	(0.091)	(0.089)	(0.088)
γ_{macro}	0.030	-0.002		-0.002	-0.002	0.030	0.029		0.030*	0.030*
		(0.001)		(0.001)	(0.001)		(0.002)		(0.002)	(0.002)
$\gamma_{bal.}$	0.020		0.004	0.004	0.004	0.020		0.020	0.020*	0.020*
			(0.001)	(0.001)	(0.001)			(0.001)	(0.001)	(0.001)
γ_{size}	0.050		0.006	0.006	0.006	0.050		0.049	0.050*	0.050*
			(0.003)	(0.003)	(0.003)			(0.008)	(0.008)	(0.008)
γ_{CFROI}	0.005		0.000	0.000	0.000	0.005		0.005	0.005*	0.005*
			(0.000)	(0.000)	(0.000)			(0.001)	(0.001)	(0.001)
γ_{pseudo}	0.000			0.000*		0.000			0.000*	
				(0.007)					(0.007)	

For each estimate and model specification per iteration, a p-value is calculated. This p-value corresponds to the probability of the null-hypothesis that the estimated parameter value is zero being not rejected, i.e., that the estimate is not statistically different from zero. Thus, for preset parameter values that differ from zero, the fraction of statistically significant estimates to total estimates should be maximized, whereas for preset parameter values equal to zero this fraction should be minimized. In other words: for actually significant parameters, the fraction measures the β error, while for actually insignificant parameters the fraction measures the α error with respect to the null-hypothesis.

The parameter estimates for the default-process all are able to fulfill this requirement, irrespective of the two estimation procedures considered in this paper. The parameter estimates derived for the recovery-process are likely to be biased if the preset correlation exceeds zero, as already shown by Table 4.2. This result gets further support by looking at Table 4.3, which reports the fraction between significant estimates at $\alpha = 5\%$ and all estimates for each parameter. On the one hand, it is shown that if $\rho^U = 0.95$, then the OLS regression models have obvious problems to identify the significance of actually significant variables, i.e., these models produce estimates with a high β error. Especially, the influences of x^{size} and x^{CFROI} are hardly of statistic significance in more than 25% of all iterations. Yet, the simultaneous estimation yields statistically significant estimates for each variable, exempt from the pseudo-variable, in at least 90% of all iterations. The fraction with respect to this variable exceeds 5% for each model, albeit by just a few percentage points for the correlation model, the OLS model with logarithmic recovery rates, and the OLS model with probit-transformed recovery rates. Only the OLS model with logit-transformed recovery rates shows a relatively high fraction of more than 10% for the pseudo-variable. On the other hand, the table shows that if $\rho^U = 0$, the OLS models do not have the identification problem that occurs for the high correlation. As long as the estimation model is not underspecified, like for Specification 1, then the MLE approach for the correlation model does not have an identification problem concerning the explanatory variables. Anyway, the problems concerning a correct estimation of ρ^U shown in Table 4.1 also reveal themselves when looking at the relatively high fraction of correlation estimates significantly $\neq 0$, which may be due to the short data sets.

Considering the accuracy of parameter estimates, the previous results suggest a tradeoff between the flexibility to account for a possible correlation > 0 and overestimating this correlation if the data generating correlation is zero on

Table 4.3: Fraction of the parameter estimates with a p-value < 5% for the different estimation procedures deriving parameters for the recovery process

This table reports the fraction of estimates with a p-value < 5% to all estimates, derived for each parameter of the recovery process by the competing models presented in Section 4.2 during the simulation study of Section 4.4.1.

	$\rho^U = 0.95$				$\rho^U = 0$			
				Correlation model				
Parameter	Spec. 1	Spec. 2	Spec. 3	Spec. 4	Spec. 1	Spec. 2	Spec. 3	Spec. 4
γ_0	99.48%	100.00%	99.96%	99.48%	16.87%	93.72%	97.02%	97.04%
γ_{macro}	99.32%		99.70%	91.90%	68.00%		89.76%	89.57%
$\gamma_{bal.}$		100.00%	99.70%	91.92%		91.49%	96.54%	96.69%
γ_{size}		99.95%	99.65%	92.17%		83.60%	94.49%	94.39%
γ_{CFROI}		100.00%	99.70%	91.89%		78.96%	92.35%	92.32%
γ_{pseudo}			5.09%				4.51%	
ρ^U	99.32%	100.00%	99.70%	91.89%	32.49%	40.44%	37.41%	37.20%
σ	100.00%	100.00%	100.00%	100.00%	99.86%	99.86%	99.88%	99.92%
				OLS^{ln}				
Parameter	Spec. 1	Spec. 2	Spec. 3	Spec. 4	Spec. 1	Spec. 2	Spec. 3	Spec. 4
γ_0	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
γ_{macro}	35.15%		31.70%	31.70%	100.00%		100.00%	100.00%
$\gamma_{bal.}$		99.03%	98.92%	98.93%		100.00%	100.00%	100.00%
γ_{size}		26.02%	25.80%	25.80%		100.00%	100.00%	100.00%
γ_{CFROI}		5.75%	5.68%	5.68%		100.00%	100.00%	100.00%
γ_{pseudo}			5.37%				5.09%	
				OLS^{log}				
Parameter	Spec. 1	Spec. 2	Spec. 3	Spec. 4	Spec. 1	Spec. 2	Spec. 3	Spec. 4
γ_0	21.50%	58.63%	50.16%	53.98%	100.00%	100.00%	100.00%	100.00%
γ_{macro}	65.11%		64.68%	64.66%	100.00%		100.00%	100.00%
$\gamma_{bal.}$		78.63%	79.20%	78.64%		100.00%	100.00%	100.00%
γ_{size}		17.48%	17.24%	17.50%		98.59%	98.94%	98.94%
γ_{CFROI}		7.25%	7.54%	7.61%		100.00%	100.00%	100.00%
γ_{pseudo}			11.57%				4.92%	
				OLS^{pro}				
Parameter	Spec. 1	Spec. 2	Spec. 3	Spec. 4	Spec. 1	Spec. 2	Spec. 3	Spec. 4
γ_0	100.00%	99.45%	98.74%	99.35%	100.00%	100.00%	100.00%	100.00%
γ_{macro}	69.61%		67.62%	67.62%	100.00%		100.00%	100.00%
$\gamma_{bal.}$		87.76%	86.54%	86.56%		100.00%	100.00%	100.00%
γ_{size}		16.00%	16.09%	16.01%		99.86%	99.93%	99.93%
γ_{CFROI}		6.26%	6.58%	6.60%		100.00%	100.00%	100.00%
γ_{pseudo}			5.16%				4.98%	

the one hand or simplifying the estimation approach and deriving biased parameter estimates if the data generating correlation exceeds zero on the other hand.

4.4.3 The predictive power of the competing approaches

Table 4.4 provides the average $RAE_{RM}^{out-of-sample}$ derived for the four different estimation model specifications under the assumptions of Section 4.4.1. Hereby, the fully specified correlation model serves as benchmark model for calculating the $RAE_{RM}^{out-of-sample}$, which is indicated by the value 100 in the respective cells of the table.[18] First of all, the table underlines the equivalence of the Probit approach and the MLE approach, which accounts for the correlation parameter, when considering the estimation of parameters for the default-process. Irrespective of the underlying correlation, both approaches yield very similar RAEs for each Specification. Relative to the full model in Specification 4, especially, the model Specifications 1 and 2 perform far worse in predicting PDs and the expected loss. The relation $EL = PD \cdot ELGD$ explains this result, since the interactions of these risk measures lead to interactions in the predictive power of the different approaches underlying their estimation. Thus, a notable improvement in one of these measures necessarily improves one of the other measures due to the underlying improvement in estimating the process-parameters, e.g., β_0.

The estimates for expected LGD are by far more precise if the correlation parameter is estimated together with the loadings of the explanatory variables. Each OLS regression model misspecifies the influence of observable covariates, as already shown in the previous section, and yields predictions for expected LGD that are severely outperformed by the correlation model. If the preset correlation is zero, the quality of the estimates depends more on the specification than in the case of $\rho^U = 0.95$. Particularly, for the correlation model, the predictions of the simultaneous approach improve significantly, while the predictions of the simple models do not in the case of $\rho^U = 0$.

Generally, the correct Specification 4 yields the best estimates for each of the three risk measures under consideration. Overspecifying the estimation

[18] The predictive quality of the competing models relative to the benchmark does not differ much between in-sample and out-of-sample. Neither the ranking among the specifications of each model nor among the models are altered by the choice of the data sample. Thus, we do not report the in-sample results here.

model by the pseudo-variable does not influence the predictive power as much as the underspecified estimation models 1 and 2. As shown in the previous section, the insignificance of the pseudo-variable is correctly revealed in most of the cases and thus its influence on the forecasts is negligible.

In contrast to these results, the estimates for the realized LGDs show only little variation in RAE, underlining the difficulties of a precise LGD estimation. None of the competing estimation procedures shows a notable improvement among the four Specifications. Contrarily, for logistically transformed recovery rates, more explanatory variables yield higher RAEs.

Table 4.4: Out-of-sample performance of the competing models

This table reports the average out-of-sample relative absolute errors calculated for the competing models introduced in Section 4.2. The mean RAEs are computed from 10,000 iterations of the simulation engine presented in Section 4.4.1.

	$\rho^U = 0.95$				$\rho^U = 0$			
Model	Sp. 1	Sp. 2	Sp. 3	Sp. 4	Sp. 1	Sp. 2	Sp. 3	Sp. 4
				$RAE_{PD}^{out-of-sample}$				
Correlation model	1360.5	1031.5	109.9	100.0	1341.7	1005.4	110.0	100.0
Probit	1360.5	1031.5	110.4	100.3	1341.7	1005.4	110.0	100.0
				$RAE_{EL}^{out-of-sample}$				
Correlation model	1315.4	929.8	109.8	100.0	1605.5	1176.2	109.9	100.0
OLS^{ln}	1300.8	898.8	384.6	384.1	1601.2	1162.8	172.0	165.0
OLS^{log}	1416.5	1134.1	942.3	939.1	1680.5	1370.4	669.0	667.6
OLS^{pro}	1305.7	903.6	443.6	443.3	1581.8	1125.4	358.6	357.7
				$RAE_{ELGD}^{out-of-sample}$				
Correlation model	202.2	148.4	107.8	100.0	1192.1	695.4	108.4	100.0
OLS^{ln}	828.1	981.9	979.5	979.2	1175.4	781.6	615.2	614.2
OLS^{log}	2042.7	2541.3	2506.4	2499.0	2090.0	2488.0	2721.3	2719.5
OLS^{pro}	991.8	1105.6	1093.0	1092.9	1268.2	1205.9	1315.3	1315.2
				$RAE_{LGD}^{out-of-sample}$				
Correlation model	100.2	100.5	100.0	100.0	106.2	101.1	100.0	100.0
OLS^{ln}	109.1	109.5	109.1	109.1	107.6	103.5	104.6	104.6
OLS^{log}	124.6	132.9	128.5	128.4	108.7	115.5	119.9	119.9
OLS^{pro}	110.6	111.6	110.5	110.5	121.5	115.3	116.5	116.4

Summarizing the results of this section, the suggested trade-off concerning the correlation parameter and the choice of an estimation procedure reveals itself as non-existent when looking at the predictive power of the competing approaches. The joint estimation outperforms the other models in the case of assumptions that are not fulfilled by simple OLS estimation procedures (high correlation), and it does so in the case of assumptions leading to some estimation problems for the joint estimation procedure (low correlation).

4.5 Forecasting economic capital requirements

4.5.1 The measurement of unexpected loss

A precise prediction of PD, expected LGD, and EL are basic requirements for a proper measurement of the credit risk underlying a loan, bond or other defaultable claim. The more precise the measurement of credit risk, the closer the approximation of the risk adequate credit spread for a financial claim. However, the risk adequate pricing as well as the coverage against the downside risk of financial obligations is a key determinant for the sustainable success of a financial institution.

Generally, this downside risk is calculated as the unexpected loss, i.e., the difference between Value-at-Risk (VaR) and expected loss:

$$\begin{aligned} EC_{it} &= VaR_{it} - EL_{it} \\ &= LGD_{it}^{down} \cdot CPD_{it} - ELGD_{it} \cdot PD_{it}. \end{aligned} \tag{4.19}$$

This fraction of the bank's exposure should be held as economic capital. The VaR corresponds to the 99.9% quantile of a borrower's loss distribution with respect to the realization of systematic risk. It either may be computed or simulated directly or obtained by the product of a downturn LGD, LGD_{it}^{down}, and a default probability conditional on a negative realization of systematic risk, CPD_{it}. For the purpose of deriving distributions for these three measures, credit portfolio models are often extended by a standard normal distributed unobservable systematic risk factor, as Pykhtin (2003) does theoretically and Bade et al. (2011*a*) do empirically for the correlation model of Equation (4.6). Estimating such a model and, especially, the correlation model discussed in this paper is computationally challenging. Thus, we simplify this approach by plugging in the 0.1 percentile of the distribution of the observable macroeconomic variable instead of its respective realization from $t-1$ into the formulae for PD, expected LGD, and EL, presented in Section 4.2.2, to derive estimates for CPD, downturn LGD, and for VaR.

Extending the validation framework of Section 4.4.1, we are able to calculate relative absolute errors, in-sample and out-of-sample, by Equation (4.13), for the economic capital. Furthermore, we are able to calculate the fraction of iterations with underestimated capital requirements and overestimated capital requirements respective to the total number of iterations as well as the average difference between actual and estimated capital needs conditional on over- or underestimation.

4.5.2 The adequacy of economic capital allocations by the competing models

Table 4.5 reports the results of the framework extension with respect to the out-of-sample data.[19] It shows that, basically, the previous results concerning the order of the competing models hold for the capital requirements as well. Interestingly, the RAEs calculated for the economic capital with the parameters of Specification 2 are much higher on average for each model than those for the other specifications and do not differ among the models. The reason behind this is the assumption of $\beta_{macro} = \gamma_{macro} = 0$. That is, systematic influence on the fluctuation of default rates and loss rates is neglected in this specification. Consequently, the unexpected loss calculated from the parameter estimates is zero, resulting in identical RAEs for each model. The fraction of underestimates for the economic capital by this specification is high compared to the other specifications as further shown in this table.

Focussing on the results for the correct Specification 4 shows that the application of the correlation model is the only one of the models compared that does not systematically understate the economic capital requirements. The fraction of underestimations is about 50% and the average difference between actual economic capital and predicted economic capital conditional on either under- or overestimation is nearly equal for both cases, at least for $\rho^U = 0$. But recalling the improvement of the correlation estimates when using Specification 3 instead of Specification 4 for $\rho^U = 0.95$ supports the conclusion that this result holds irrespective of the underlying assumptions for the correlation parameter. In this case, the overspecified model yields almost equal averages for under- and overestimations, like the correct model does in the case of $\rho^U = 0$.

Regarding the other models, a relatively high correlation underlying the data generating process yields severely biased predictions of the economic capital. For Specification 4, at least 85% of the estimates are below the actual requirements, corresponding to an average underestimation of 0.455% with respect to the credit exposure. The relatively weak performance shown by

[19] Once more, there is no notable difference between the in-sample and the out-of-sample results.

the OLS model incorporating logistically transformed recovery rates in Section 4.4.3 holds with an average underestimation of economic capital by about 2.9% of the exposure and about 98.6% underestimated economic capital requirements in the case of $\rho^U = 0.95$. In the case of $\rho^U = 0$, this model shows the highest fraction of overestimated capital requirements (91.6%) with average overestimations of about 1.1% for economic capital. The OLS model with correctly transformed recovery rates tends to overestimate the economic capital needs ($\sim 40 : 60$) if $\rho^U = 0$. The estimation procedure with probit transformed recovery rates understates the capital allocation systematically in the case of $\rho^U = 0.95$. In fact, it tends to do so in the case of $\rho^U = 0$. Quantitatively, however, the biases are not as dramatic as the biases by the OLS model with logistically transformed recovery rates.

Comparing the relative and the absolute performance of the models under consideration with respect to adequate capital allocations reveals an important issue: although the RAEs are seemingly in an acceptable range, the absolute performance of the OLS models in the case of $\rho^U = 0.95$ is far away from being adequate for capital allocation purposes. Thus, the relative ranking of the model's predictive performance should not serve as a single decision criterion. A joint consideration with the absolute performance is necessary in order to determine the predictive quality of a model.

4.6 Conclusion

A well-conceived credit risk model needs to fulfil four standards in order to suitably forecast future defaults and losses. This paper analyzes whether a simplified version of Bade et al. (2011*a*)'s econometric extension to Pykhtin (2003)'s correlation model generally may fulfil these requirements.

First of all, the model has to account for the empirical observation that default rates (as realization of PD) and LGD (as realization of expected LGD) move jointly. We show graphically that the positive relation of both quantities may solely be explained by observed factors[20] that drive both quantities simultaneously, irrespective of the values chosen for ρ^U. Thus, the underlying

[20] Either macroeconomic, e.g., GDP or firm-specific, e.g., ratings, compare Bade et al. (2011*a*).

Table 4.5: Relative absolute errors and absolute accuracy of economic capital estimates

This table reports the average relative absolute errors of economic capital calculated for the competing models introduced in Section 4.2. The mean RAEs are computed from 10,000 iterations of the simulation engine presented in Section 4.4.1. Furthermore, the fraction of underestimates to total estimates as well as the average amounts of over- and underestimation for the economic capital are presented.

	$\rho^U = 0.95$				$\rho^U = 0$			
	Sp. 1	Sp. 2	Sp. 3	Sp. 4	Sp. 1	Sp. 2	Sp. 3	Sp. 4
	$RAE_{EC}^{out-of-sample}$							
Correlation model	439.2	1688.6	101.3	100.0	548.4	2108.7	101.5	100.0
OLS^{ln}	451.3	1688.6	181.9	181.3	550.0	2108.7	103.7	102.3
OLS^{log}	1112.6	1688.6	1101.6	1101.4	599.4	2108.7	328.8	328.5
OLS^{pro}	488.1	1688.6	308.0	307.8	548.7	2108.7	114.0	112.7
	Fraction of underestimates for $EC^{out-of-sample}$							
Correlation model	47.0%	99.9%	50.7%	49.7%	47.8%	99.9%	50.5%	50.4%
OLS^{ln}	57.2%	99.9%	85.3%	85.5%	46.5%	99.9%	42.7%	42.5%
OLS^{log}	91.8%	99.9%	98.5%	98.6%	34.6%	99.9%	8.5%	8.4%
OLS^{pro}	66.9%	99.9%	95.2%	95.3%	47.8%	99.9%	61.3%	61.4%
	Average amount of underestimation for $EC^{out-of-sample}$							
Correlation model	0.594%	4.520%	0.133%	0.126%	0.949%	7.155%	0.170%	0.168%
OLS^{ln}	0.821%	4.520%	0.456%	0.455%	0.905%	7.155%	0.137%	0.135%
OLS^{log}	2.844%	4.520%	2.871%	2.871%	0.547%	7.155%	0.029%	0.028%
OLS^{pro}	1.057%	4.520%	0.805%	0.804%	0.955%	7.155%	0.237%	0.234%
	Average amount of overestimation for $EC^{out-of-sample}$							
Correlation model	0.598%	0.001%	0.135%	0.138%	0.951%	0.001%	0.172%	0.169%
OLS^{ln}	0.401%	0.001%	0.030%	0.029%	1.001%	0.001%	0.213%	0.210%
OLS^{log}	0.058%	0.001%	0.005%	0.005%	1.556%	0.001%	1.122%	1.122%
OLS^{pro}	0.257%	0.001%	0.009%	0.008%	0.946%	0.001%	0.152%	0.149%

processes are not necessarily required to be linked directly by a correlation parameter to explain the co-movement of PD and expected LGD.

Secondly, at least the qualitative impact of potential drivers for the default risk and loss risk should be revealed correctly. In a simulation study, we show that simple OLS models, even if correctly specified, have difficulties to reveal this impact if the correlation parameter preset in the data generating processes is high. However, a MLE approach for the simultaneous estimation of the parameters from the default process and the recovery process, which accounts for the possible correlation, is able to identify variables from pseudo-variables properly. Due to the short data sets, the correlation parameter is not estimated correctly if its data generating counterpart is low.

Even a 100 percent precise estimation of the model parameters has no major advantage if an uncorrect model yields similar or maybe even better forecasts with less effort. Thus, the third requirement is the LGD forecasting quality of a credit risk model relative to other modeling approaches. This quality should neither depend on the underlying risk environment nor on the attributes of the data generating process. In a simulation based analysis of three different scenarios (low credit risk, intermediate credit risk, and high credit risk), we show that the flexible model yields comparable forecasts for lower correlation values preset in the data generating process and also outperforms simple OLS models for high correlations. In a more sophisticated setting, implementing explanatory covariates in the simulation engine, we show that, even in the case of assumptions in favor of the OLS models ($\rho^U = 0$), the flexible model outperforms the OLS models. Consequently, Pykhtin (2003)'s modeling approach pays off at the latest in a more realistic setting, where data are generated by observable risk factors and not simply by unobservable idiosyncratic risk.

The fourth requirement is that a credit risk model may be adopted for pricing purposes and the allocation of economic or regulatory capital. Thereto, we need to investigate the relative performance in predicting realized LGDs and, additionally, the underlying risk measures, namely PD, EL, and expected LGD. Like the second one, this requirement can only be tested in a simulation-based environment, where the data generating model and its underlying parameters are known quantities. Thus, we extend the simple and the sophisticated simulation study to the analysis of these risk measures. We show that, apart from the PD forecasts, none of the competing models is able to perform as well as the correlation model. For high preset correlation values, they perform even far worse than the benchmark, while the correlation model performs far better than the benchmark. Concerning the absolute performance of the forecasts for economic capital, the simple models seriously underestimate the capital needs if the preset correlation is high, while the flexible model neither systematically under- nor systematically overestimates the economic capital.

To conclude, the flexibility to account for the different assumptions of the data generating process, most importantly the correlation parameter, reveals the superiority of the correlation model over less sophisticated approaches, neglecting the possible correlation of default- and recovery-process, with respect to forecasts of future LGDs. Even more importantly, it reveals that a risk-adequate pricing and a sufficiently but not excessively high allocation of economic capital are complementary features of the credit risk modeling approach analyzed in this paper.

Chapter 5

A Brief Outlook to Further Research Topics

5.1 Basic Results of this Thesis

A review of the previous chapters yields one basic insight: implementing a dependence structure between default probability and loss given default is a major requirement for modeling credit risk suitably. The empirical result of co-moving default probabilities and losses given default, derived by Altman (compare Altman et al. (2004), Altman et al. (2005), Altman (2009)), Frye (compare Frye (2000*b*), Frye (2005)) and many other authors, is supported for nonfinancial companies by analyzing Moody's bond data in Chapter 2. Moreover, the results of previous authors are extended by strong evidence that, in addition to systematic risk, idiosyncratic risk factors – either observable or not – play a key role in explaining this co-movement. Given that observable idiosyncratic risk factors, like the rating grade, or systematic risk factors, like the change in gross private domestic investment, influence the processes underlying default and recovery simultaneously, the positive relation of PD and LGD may already be explained by these factors. Nevertheless, the empirical analysis shows that the unobservable idiosyncratic risk factors, driving both processes, are almost perfectly correlated. Hence, the positive relation of PD and LGD is even stronger than the relation explained by the observable factors.

The empirical evidence on the high correlation suggests to estimate all parameters of a joint model for PD and LGD, including the correlation parameter, simultaneously, using the MLE method. Following statistical theory, a separate estimation of disjunct models for both risk measures, excluding the correlation parameter, yields biased estimators for the factor loadings of the recovery process. As a consequence of the high correlation estimates, this sample selection bias, which is due to the non-observability of LGD in the case of a non-defaulted claim, is likely to be severe. When looking at the accuracy of the two modeling approaches for predicting LGD in Chapter 3, the estimation bias may be assessed to be an economic issue as well. Accounting for the correlation parameter in the estimation procedure leads to a higher predictive accuracy for LGD than a separate estimation of the process parameters. Thus, the estimation bias is statistically and economically relevant, since accounting for this bias allows for a more precise assessment of a defaultable claim's credit risk.

The empirical findings get strong support by the results of the simulation study in Chapter 4. First of all, it is shown that, if default and loss data for a portfolio of borrowers are generated by a high correlation, a simultaneous estimation procedure for all model parameters, including the correlation parameter, yields consistent parameter estimators. A separate estimation of the model parameters, excluding the correlation parameter, however, yields statistically biased parameter estimators, as predicted by statistical theory. Additionally, realizations of these biased estimators may even have a wrong sign, yielding a misleading economic interpretation of the underlying risk factor. Based on the parameter estimates, important credit risk parameters, especially, the economic capital, are predicted and compared to their data generating counterparts. It is shown that, if the estimators for the model parameters are biased, the capital requirements, covering a financial institution against an unexpected loss, are systematically underestimated. The simultaneous MLE procedure corrects for the estimation bias. Consequently, the credit risk parameters and, in particular, the economic capital charges are neither underestimated nor overestimated in a systematic manner.

5.2 Suggestions for Further Analyses

Albeit the supportive results from the empirical and simulation based analysis, for implementing the modeling approach by Pykhtin (2003), there remain several possibilities for extending the analysis. First and foremost, the empirical analysis of Chapter 2 and Chapter 3 should be repeated with an alternative data base to bond data, e.g., bank data with a greater set of explanatory variables. In this respect, it would be interesting to see whether some of these variables add explanatory power solely to one of the two processes, as opposed to both processes. Certainly, the magnitude of the correlation parameter is also of key importance. It determines the degree of a possible sample selection bias when choosing separate procedures for estimating the model parameters. In order to derive more reliable predictions for credit risk, improving the data quality is also an important issue.

A further concern, with regard to empirical data, is the bimodality of LGD distributions, which often can be observed, although not explained by any variables (compare Schuermann (2005), p. 4). A comparison between estimation methods, trying to account for the bimodality (see, e.g., Bellotti & Crook (2012) or Hlawatsch & Ostrowski (2011)), and the estimation method, suggested in this thesis, may contribute to a more comprehensive analysis of different modeling approaches. Additionally, nonparametric approaches like regression trees (see, e.g., Bastos (2010)) or density estimators (see, e.g., Calabrese & Zenga (2010)) as well as survival analysis (see, e.g., Zhang & Thomas (2012)) may be part of future model comparisons.

The design of the model comparison, especially in a simulation study, may be altered as well. On the one hand, the impact of the study settings on the results is an interesting topic for further analysis. Particularly, the portfolio size and composition, the number and distribution of explanatory variables, the relation between in-sample and out-of-sample portfolio, and the definition of the data generating model may be altered. On the other hand, different performance measures to RAE and RMSE may be applied. E.g., Hlawatsch & Reichling (2010) apply two criterions from PD validation, the area under curve and the accuracy ratio, to validating LGD estimates.

Finally, a possible contribution of further research is the extension of the model. Particularly, the implementation of different distributional assumptions for the unobservable systematic and idiosyncratic risk factors, like the fatter tailed t-distribution, is an interesting concern. The normal distribution is viewed as understating the loss, associated with rare events, like default (compare Frey et al. (2001), p. 1). For contributions on implementing t-copulas in factor models, see, e.g., Frey et al. (2001), Frey & McNeil (2003) or Hamerle & Rösch (2005). Another imaginable direction for extending the model is to apply it for pricing purposes, e.g., of risky bonds or credit derivatives. In this context, however, one problem, when calibrating the pricing model to observed spreads, is to identify the implied recovery rate (compare Schönbucher (2003), p. 159).

Regarding the results of this thesis and the future research topics, suggested in the previous paragraphs, it is obvious that there are still a lot of challenges remaining. Both in the field of credit risk, in general, and the dependence between default and recovery, in particular, empirical work is the main source of improving and extending existing models and estimation methods, with respect to their predictive accuracy.

Chapter 6

Comment on "The weighted average cost of capital is not quite right

The content of this chapter was originally published as Bade, B. (2009), 'Comment on "The weighted average cost of capital is not quite right"', *The Quarterly Review of Economics and Finance* 49, pp. 1476–1480.

6.1 Introduction

For years finance literature deals with the question how future cash flows have to be discounted in order to obtain their adequate present value. Many special cases and the corresponding valuation equations including adequate adjustments for the cost of capital were investigated. Names like Gordon-Shapiro, Modigliani-Miller and Miles-Ezzell are just the tip of the iceberg of large contributions to corporate financial theory. The textbook formula for the Weighted Average Cost of Capital ($WACC$) accounting for the capital structure and the resulting tax consequences is taught if not in all but at least in most finance courses around the world.

In his article "The weighted average cost of capital is not quite right", Richard A. Miller questions if the $WACC$ yield correct normal profits to both equity and debt holders and if tax consequences should be accounted for in the denominator of a valuation equation at all. He derives a so called "non-linear $WACC$" which in his opinion is superior to the standard approach of calculating the $WACC$.

The purpose of this paper is to show that the traditional $WACC$ are able to yield sufficient normal profits to satisfy both investor groups and that the $NLWACC$ are only valid for a very restrictive set of assumptions while the $WACC$ textbook formula is always valid.

The paper is organized as follows: section 6.2 addresses the question of the correct normal profit and section 6.3 the question of the correct tax treatment. Section 6.4 concludes the paper.

6.2 Conclusion 1: The traditional WACC are inadequate to produce a sufficient normal profit

6.2.1 The fundamental Principle: No Arbitrage

In finance the correctness of every valuation model is based on the fundamental principle of no arbitrage. That means that cash flows of equivalent risk have to be priced equivalently. In terms of project valuation the value of the project cash flows at time t (V_t) is obtained by discounting the cash flow and the value of the project cash flows of time $t+1$ ($CF_{t+1} + V_{t+1}$) with the appropriate cost of capital r_t representing the return on an alternative investment with identical risk:

$$V_t = \frac{CF_{t+1} + V_{t+1}}{1 + r_t}. \tag{6.1}$$

Assuming a project generating cash flows until time T yields a present value of the project of

$$V_t = \sum_{s=t+1}^{T} \frac{CF_s}{\prod_{i=t}^{s-1} (1 + r_i)}. \tag{6.2}$$

If the cost of capital and the cash flows are constant over time, equation 6.2 simplifies to

$$V_t = \frac{CF}{r} \cdot \left(1 - \frac{1}{(1+r)^{T-t}}\right). \tag{6.3}$$

In order to compare the results derived by Miller with the ones in this article the same example is used. For a project with a lifetime $T = 8$ and an investment $I_0 = 200,000$ the break-even cash flow yielding a net present value of zero (normal profit) will be derived under various financing assumptions. It is assumed that if the project would be financed by equity as well as by debt the debt-to-value ratio would be $\frac{1}{4}$ at time 0. The risk adequate cost of capital at this ratio are for the debt holders $r^D = 6\%$ and for the equity holders $r^E = 12\%$ respectively. The $WACC$ are calculated as follows:

$$WACC = \frac{3}{4} \cdot 12\% + \frac{1}{4} \cdot 6\% = 10.5\%. \tag{6.4}$$

For simplification it is assumed that the tax rate equals zero.

6.2.2 The WACC under Autonomous Financing

"A firm is autonomously financed exactly then when it's future amount of debt D_t is already a certain quantity today" (see Kruschwitz & Löffler (2005), p. 66). For the considered project this definition means that an amortization schedule for $D_0 = 50,000$ has to be fixed. Since not only the total cash flows to the investors but also the payments to each group of them should be constant over time, an annuity schedule is appropriate. The annuity payment is

$$CF^D = 50,000 \cdot \left(\frac{1}{6\%} \cdot \left(1 - \frac{1}{1.06^8} \right) \right)^{-1} = 8,051.80, \tag{6.5}$$

which is the same amount as calculated by Miller.

In order to obtain a time constant payment to the equity holders, too, it is necessary to make the assumption that the risk of these payments is not affected by the amount paid to debt holders so that $r^E = 12\%$ is constant as well. Then the following payment is obtained:

$$CF^E = 150,000 \cdot \left(\frac{1}{12\%} \cdot \left(1 - \frac{1}{1.12^8} \right) \right)^{-1} = 30,195.43, \tag{6.6}$$

which is also the same amount as calculated by Miller.

In the next step debt values D_t and equity values E_t are calculated for each period using Equation (6.1) (see Table 6.1). Based on these values the total value of the project, the debt-to-value ratio d_t and the $WACC$ for each period are derived. As the last two columns of Table 6.1 show the debt-to-value ratio and the $WACC$ at first are the same as in the article of Miller. But in the subsequent years the debt-to-value ratios decline and consequently the $WACC$ grow. So, the $WACC$ are not constant over time. Hence, for calculating the present value of CF_1 to CF_8 it is necessary to take Equation (6.2) and not Equation (6.3) as did in Miller's article.

Table 6.1: Cash flows, equity and debt values under autonomous financing
The table shows the cash flows to equity holders and debt holders as well as the total cash flows for every period. Furthermore it shows the values of equity and debt, the total value of the project, the debt-to-value ratio and the $WACC$ for every period.

Period	CF_t^E	E_t	CF_t^D	D_t
0	-150,000.00	150,000.00	-50,000.00	50,000.00
1	30,195.43	137,804.57	8,051.80	44,948.20
2	30,195.43	124,145.70	8,051.80	39,593.30
3	30,195.43	108,847.75	8,051.80	33,917.10
4	30,195.43	91,714.06	8,051.80	27,900.33
5	30,195.43	72,524.32	8,051.80	21,522.55
6	30,195.43	51,031.81	8,051.80	14,762.11
7	30,195.43	26,960.20	8,051.80	7,596.04
8	30,195.43	0.00	8,051.80	0.00
Period	$CF_t = CF_t^E + CF_t^D$	$V_t = E_t + D_t$	d_t	$WACC_t$
0	-200,000.00	200,000.00	25.00%	10.50%
1	38,247.22	182,752.78	24.60%	10.52%
2	38,247.22	163,738.99	24.18%	10.55%
3	38,247.22	142,764.85	23.76%	10.57%
4	38,247.22	119,614.39	23.33%	10.60%
5	38,247.22	94,046.87	22.88%	10.63%
6	38,247.22	65,793.92	22.44%	10.65%
7	38,247.22	34,556.24	21.98%	10.68%
8	38,247.22	0.00	-	-

The result of an autonomous financing policy is in line with the result taking the "nonlinear $WACC$ ($NLWACC$)" by Miller since the $NLWACC = 10,5533\%$ are simply the internal rate of return (IRR) of an investment of 200,000 yielding a yearly cash flow of 38.247,22 for 8 years. But calculating EVAs or the project values for $t > 0$ based on these would lead to wrong results because the IRR changes with declining investment horizon.

6.2.3 The WACC under Financing based on Market Values

Taking Equation (6.3) to calculate the present value of time constant cash flows requires the assumption that the debt-to-value ratio and as a consequence the

$WACC$ remain constant over time. Such a financing policy with at present already certain future debt-to-value ratios is referred as financing based on market values (compare Kruschwitz & Löffler (2005), p. 70).

In the example the required cash flow with constant $WACC$ is

$$CF = 200,000 \cdot \left(\frac{1}{10.5\%} \cdot \left(1 - \frac{1}{1.105^8}\right)\right)^{-1} = 38,173.86, \qquad (6.7)$$

which is lower than the total cash flow under the assumption of a fixed amortization schedule. Table 6.2 shows debt and equity values and also the cash flows to each investor group. Regarding the latter it is remarkable that both are not constant over time any more. As a result of the constant debt-to-value ratio the CF_t^E decline form period 1 to period 8 and the CF_t^D grow during that time.

Are these cash flows sufficient to satisfy the providers of debt financing and also the providers of equity financing? The answer is: yes they are. Since the cash flows to debt holders are calculated using the required interest rate of 6% and all of the debt is repaid after period 8 none of their financial claims are outstanding after the project has finished. The cash flows to the equity holders are calculated as the residual of the total cash flows and the debt service. In order to fulfill the financial claims of the equity holders as well, the residuals have to return the equity cost of capital of 12% in every period. Rearranging Equation (6.1), r_t^E easily can be obtained as the required 12% for every period. Hence, the statement that "by not calculating separately the normal profit amounts for debt and equity financing, the $WACC$ understates the necessary CF" (see Miller (2009), p. 132) is doubtable.

The IRR in this case equals the $WACC$ and not the $NLWACC$. Hence, the $NLWACC$ are a special case of the IRR under the assumption of an autonomous financing policy which yields time constant cash flows to debt holders and to equity holders as well.

Table 6.2: Cash flows, equity and debt values under financing based on market values

The table shows the cash flows to equity holders and debt holders as well as the total cash flows for every period. Furthermore it shows the values of equity, debt and the total value of the project.

Period	$CF_t^E = CF_t - CF_t^D$	$E_t = V_t - D_t$	$CF_t^D = D_{t-1} \cdot (1 + r^D) - D_t$	$D_t = 0.25 \cdot V_t$
0	-150,000.00	150,000.00	-50,000.00	50,000.00
1	30,880.39	137,119.61	7,293.46	45,706.54
2	30,687.19	122,886.78	7,486.67	40,962.26
3	30,473.69	107,159.50	7,700.16	35,719.83
4	30,237.78	89,780.85	7,936.07	29,926.95
5	29,977.10	70,577.45	8,196.75	23,525.82
6	29,689.05	49,357.69	8,484.80	16,452.56
7	29,370.76	25,909.86	8,803.10	8,636.62
8	29,019.04	0.00	9,154.82	0.00
Period	CF_t	V_t		
0	-200,000.00	200,000.00		
1	38,173.86	182,826.14		
2	38,173.86	163,849.03		
3	38,173.86	142,879.33		
4	38,173.86	119,707.80		
5	38,173.86	94,103.27		
6	38,173.86	65,810.25		
7	38,173.86	34,546.48		
8	38,173.86	0.00		

6.3 Conclusion 2: Including the tax rate in the WACC seems misplaced

Of course, ignoring taxes would be the wrong choice since they yield an interest tax shield of $\tau \cdot r_d \cdot D_{t-1}$. For valuing these tax shields different approaches exist:

- Calculate the cash flows and the cost of capital as if no debt is raised. Then value the unlevered project and add the present value of the tax shields to obtain the value of the levered project.[21]

- Calculate the cash flows of the project as if no debt is raised. Account for the tax shields by adjusting the debt cost of capital by a tax factor $(1 - \tau)$ and calculate the $WACC$ by the textbook formula

$$WACC_t = \frac{E_t}{V_t} \cdot r_t^E + \frac{D_t}{V_t} \cdot r_t^D \cdot (1 - \tau) .^{22} \tag{6.8}$$

[21] This method is referred to as the "adjusted present value APV" method. Compare Berk & DeMarzo (2007), pp. 581-585.

[22] This method is reffered to as the $WACC$ method. Compare Berk & DeMarzo (2007), pp. 577-579.

- Calculate the cash flows of the project taking into account the interest payments. No tax adjustment of the cost of capital is required.

Which of the three methods is superior in the sense of computational complexity depends on the assumptions made on the financing policy and the sensitivities which should be analyzed. But under the same assumptions all three methods yield the same result (compare also Berk & DeMarzo (2007), p. 605).

In his conclusion, Miller states that "the stockholders still require r^E and the bondholders still require r^D, regardless of the tax deductability of interest payments." Although this statement is not wrong it neglects that from the point of view of the project the required cash flows for zero normal profit are lower with taxes than without. The quantitative impact of the tax advantage is best obtained by comparing these. Using the $NLWACC$ then means a loss of information.

6.4 Summary

The valuation of a firm or a project is not quite easy. Besides the uncertainty concerning the estimation of cash flows and the risk adequate cost of capital one mistake is avoidable: the formulation of a valuation model not considering the assumptions on the financing policy and the resulting tax advantages correctly. Hence, the same cash flows may yield different project values and accordingly different cash flows may yield the same project values.

As shown in the previous sections, concluding that the $WACC$ textbook formula is inadequate and that taxes rather should be implemented in the income statement, seems to be premature, since the $WACC$ formula is far away from being inadequate. In fact it is always valid. It simply requires to be calculated with the correct debt-to-value ratios which are time varying unless a time constant ratio is assumed. Furthermore the tax adjustment in the denominator of the valuation equation allows for easier comparisons of different tax rates since no new computation of the income statement is required for each tax rate. The $NLWACC$ not only yield incorrect project values for most assumptions according to the financing policy. When used to derive a time constant cash flow yielding a $NPV = 0$ they furthermore do only provide an answer to the question which cash flows the investors receive but not which cash flows have to be generated by the project.

Though, stating that the $NLWACC$ are superior to the $WACC$ is inappropriate, since rather it is the other way around.

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Lebenslauf des Verfassers

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Wissenschaftliche Publikationen

2011	Empirical Performance of LGD Prediction Models, *The Journal of Risk Model Validation*, 5(2), 25-44 (zusammen mit Daniel Rösch und Harald Scheule)
2011	Default and recovery risk dependencies in a simple credit risk model, *European Financial Management* 17(1), 120-144 (zusammen mit Daniel Rösch und Harald Scheule)
2009	Comment on "The weighted average cost of capital is not quite right", *The Quarterly Review of Economics and Finance*, 49, 1476-1480