Invariant Bergman spaces

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades

Doktor der Naturwissenschaften

Dr. rer. nat.

genehmigte Dissertation
von

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2012

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Tag der Promotion:	6. Juli 2012

Real things are sharp and knobbly and complicated and different.
— Miracles, *C.S. Lewis*

Abstract

The goal of this thesis is to realize the space of square integrable functions in an isometric and equivariant way as a space of holomorphic functions. The first example of such a realization was the Segal–Bargmann or heat kernel transform on Euclidean space. Up to normalization, it is nothing else than convolution with the heat kernel. The heat kernel on Euclidean space is replaced by an arbitrary holomorphic function κ on a Riemannian symmetric space G/K.

Under certain conditions on κ, the image of this transform consists entirely of holomorphic functions and there exists a Hilbert structure on the image making the transform a partial isometry. The main result of this thesis is an explicit form of this Hilbert structure: Let $\mathcal{F}\kappa$ denote the Fourier transform of κ. Then, if the inverse Laplace transform of $|\mathcal{F}\kappa|^{-2}$ exists, this image is a weighted Bergman space. Its weight is precisely this inverse Laplace transform and need not be positive.

The method is robust, in the sense that it can be modified to include also Hardy-type spaces. Unfortunately, an impractical technical assumption shows up in the general case. The method applies to Euclidean space and compact Lie groups. Partial results are obtained in the case of non-compact Riemannian symmetric spaces. The remaining questions are studied by example.

Keywords: Segal–Bargmann transform
Bergman spaces
Laplace transform

Kurzfassung

Ziel dieser Dissertation ist, der Raum von quadratisch integrierbaren Funktionen isometrisch und mit Behalt einer Gruppenwirkung als Raum holomorphen Funktionen zu schreiben. Das erste Beispiel solch einer Darstellung war die Segal–Bargmann- oder Wärmeleitungskern-Transformation auf dem Euklidischen Raum. Bis auf Skalierung ist diese Transformation genau eine Faltung mit dem Wärmeleitungskern. Dieser Kern auf dem Euklidischem Raum wird hier ersetzt durch eine beliebige holomorphe Funktion κ auf einem Riemannschen symmetrischen Raum G/K.

Unter gewissen Bedingungen auf κ bildet diese Transformation quadratisch integrierbare Funktionen auf holomorphen Funktionen ab und gibt es eine Hilbertstruktur auf dem Bild die die Transformation zu partieller Isometrie macht. Das Hauptresultat dieser Dissertation ist eine explizite Form dieser Hilbertstruktur: Sei $\mathcal{F}\kappa$ die Fourier-Transformierte von κ. Wenn die inverse Laplace-Transformierte von $|\mathcal{F}\kappa|^{-2}$ existiert, ist das Bild ein gewichteter Bergman-Raum. Das Gewicht ist genau diese inverse Laplace-Transformierte und kann auch negative Werte annehmen.

Diese Methode lässt sich in viele Richtungen erweitern, zum Beispiel zu Hardy-Räume. Eine unpraktische, technische Voraussetzung taucht auf im allgemeinen Fall. Die Methode wird betrachtet für Euklidische Räume, kompakte Lie Gruppen und nichtkompakte Riemannsche symmetrische Räume. Im letzen fall ergibt sich nur ein Teilrestultat. Die offene Fragen werden in einigen Beispiele betrachtet.

Schlagworte: Segal–Bargmann-Transformation
Bergman-Räume
Laplace-Transformation

Contents

Introduction

This thesis is about representations of Lie groups on spaces of holomorphic functions, and in particular unitary representations on weighted Bergman spaces. Shortly after World War II, these representations rapidly gained popularity. A beautiful, but quite general motivation for research in this direction is given by Valentine Bargmann in [Bar47]:

> Apart from possible applications in Mathematical Physics cf. [Dir45] this investigation has an intrinsic mathematical interest as a detailed analysis of the unitary representations of a non-compact group.

The representation theory was picked up by Harish Chandra Mehrotra, whereas the applications in Mathematical Physics were studied by Irving Segal. I will describe the development of these two lines of research and their connection with this thesis.

Harmonic analysis In the quoted paper by Bargmann, he gives a classification of all irreducible representations of the Lorentz group. They come in four types, two of them indexed by an interval and two by an integer. The former are called continuous series representations and the latter discrete series representations, or discrete series for short. These discrete series are realized as unitary representations on spaces of holomorphic functions over the open unit disc, square integrable with respect to a certain weight. Around the same time, similar results were obtained by Israil Gelfand and Mark Naĭmark and by Harish Chandra.

A few years later, Harish Chandra found a generalization of Bargmann's results for certain semisimple Lie groups [Har56]. This was part of his groundbreaking work on representations of semisimple Lie groups. One of his main results is the implementation of André Weil's abstract harmonic analysis: He discovered the Plancherel formula for semisimple Lie groups. See [Har70] for his own survey of this work. Blossoming years for the harmonic analysis on Lie groups and symmetric spaces followed. They culminated in the Plancherel theorem for symmetric spaces, simultaneously described by Erik van den Ban and Henrik Schlichtkrull [vdBS97] and by Patrick Delorme [Del98].

Meanwhile, the study of holomorphic function spaces was taken up by Gelfand and Gindikin. They initiated a program to decompose $L^2(G)$ as a direct sum of function spaces over complex domains with an action of the group G [GG77]. This in turn led to an increased interest in Hardy spaces on domains in complex Lie groups, for example the domains discovered by Robert Stanton and Grigori Olshanski [Sta86, Ols91], also known as Olshanski semigroups.

By the end of the century, the discrete series representations and the Hardy spaces corresponding to them were well understood. Also, after the work of Karl-Hermann Neeb [Nee00], there was a good understanding of Hilbert spaces of holomorphic functions and their reproducing kernels. Bernhard Krötz realized that this theory would extend to Bergman spaces as well, and proved in [Krö98] a Plancherel theorem for the classical Hardy and Bergman space on the Olshanski semigroup: The observation is that these spaces are determined by their kernels, which are the Cauchy-Szegő and Bergman kernel respectively. In order to decompose the space, one only has to decompose the kernel.

Together with Hilgert, Krötz extended this theory to compactly causal symmetric spaces in [HK99]. In the same paper, positive weight functions are considered, leading to the notion of weighted Bergman spaces. Shortly afterwards, the Plancherel theorem is proved [HK01]. Again, the decomposition is done in terms of the reproducing kernel of the Hilbert space.

The focus of this thesis lies on Bergman spaces that can be embedded equivariantly and isometrically in L^2. This type of spaces might play a role in the Gelfand–Gindkin program, as they are natural representations over a domain in the complexification of a Lie group. This application will serve as a motivation for this thesis.

What is contained is a closed theory of invariant Bergman spaces on Euclidean space and compact Lie groups. When these techniques are applied to non-compact Riemannian symmetric spaces, a new condition on the kernel is needed. At the end of this thesis, I propose a method to lift this condition. Although the first example looks promising, it is also slightly misleading; the general case needs a detailed and much more technical approach and is not in the scope of this thesis. The examples provide a motivation and a direction for further research. I feel that a satisfying solution is within reach of present knowledge.

The Segal–Bargmann transform Although harmonic analysis provides a motivation for this research, the main application and leading examples come from a different stock. One of the pioneers in this direction was again Bargmann [Bar61]. Shortly after Irving Segal, he described the transform that is now known as Segal–Bargmann transform or heat kernel transform. In fact, the transform is just a convolution with the heat kernel.

This transform is a unitary map from $L^2(\mathbb{R}^n)$ to a Hilbert space of holomorphic functions on $\mathbb{C}^n$, which is a weighted Bergman space. This space is usually called after Bargmann, Fock or sometimes Fischer. This Bargmann space carries a representation equivalent to the Schrödinger representation on $L^2(\mathbb{R}^n)$. The advantage of the Bargmann space over L^2-space is that both the position and momentum operator act in a natural, although unbounded way on functions in the Bargmann space. For example, if f is analytic, then f' is again analytic, whereas the derivative of a square integrable function is not even a function in general.

The transform takes a prominent place in Segal's work on quantum physics, and he passed his idea to his students. It figures for example in the work of Bertram Kostant on type I groups, and also Bent Ørsted writes about it, amongst others in [ÓØ96]. Under the direction of another of Segal's students, Leonard Gross, the Segal–Bargmann transform made a comeback at the end of the last century. His students Brian Hall, Bruce Driver and later also Jeffrey Mitchell wrote a series of articles on this transform. See [HM08] for a list of references.

Following Hall's paper on the Segal–Bargmann transform for compact Lie groups [Hal94], several authors generalized this transform in their own ways. Probably the most interesting result and a direct motivation for this thesis is the article by Bernhard Krötz, Gestur Ólafsson and Robert Stanton [KÓS05]. After the work of Krötz and Stanton on the Akhiezer–Gindikin crown domain [KS05], they showed that the Segal–Bargmann transform has no direct generalization to this crown domain. They were able to give some formula, and another solution was given by Hall and Mitchell in [HM08].

In this thesis, I replace the heat kernel by an arbitrary function. It explains and sharpens the results of Jacques Faraut in [Far03]. For example, it turns out that there is a direct generalization of convolution transforms to the crown domain as long as the decay rate of the Fourier transform of the convolution kernel is bounded by a certain constant.

Another important example for the theory outlined in this thesis is given in [KTX05]. In this article, the authors study the Segal–Bargmann transform on the Heisenberg group. They find that the image of the transform is a sum of two Bergman spaces. They show that the weights of these two spaces take negative values. This is the main motivation to extend Faraut's theory from positive weights to real-valued weights. Moreover, the example on page 33 suggests that for non-compact Riemannian symmetric spaces, the image consists of countably many Bergman spaces in general. It remains to be seen whether these spaces can be grouped together, resulting in a finite number of Bergman spaces.

For a more physical explanation of this topic, I refer to the book of Folland [Fol89]. He presents the Segal–Bargmann transform on the Heisenberg group and its relation to physics. My understanding of the present results is that in order find an appropriate Fock space for a Riemannian symmetric space, one needs to take into account the monodromy of a certain subspace of the full complexification of this space.

Acknowledgements Properly expressing my gratitude to all the people who have supported me is an impossible task. All I can do is name a few of them. Before that, I would like to thank the sponsors of this research: The Graduate School 1463 for supporting the first two years of my research, the Max Planck Institute Bonn for hosting me during their research activity on Harmonic Analysis and the Universiteit van Amsterdam for providing me with a place to finish this thesis. A special thanks goes to professor Krötz and professor Opdam for sponsoring the last year of my studies.

A big thanks goes to my friends and colleagues from the Institute of Analysis in Hannover. They really were a huge support. I would like to thank Christoph Lienau, Heiko Gimperlein and Job Kuit for answering many of my questions. My friends in Hannover for their kindness. My family for visiting me in Hannover and cheering me up. My supervisor for showing me the way and of course our secretary Susanne Rudolph for her incessant and friendly help with everything. Finally I would like to thank my mother for her belief in me.

The Euclidean and compact case

We will introduce the method by a short example. Let $U = \{u \in \mathbb{C} \mid |\Im u| < 1\}$ be an open strip in the complex plane, and κ the holomorphic function on U defined by

$$\kappa(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \sqrt{\frac{4\xi^2 + \pi^2}{12\xi^2 + \pi^2} \frac{2\xi}{\sinh 2\xi}}\, e^{iu\xi} \, \mathrm{d}\xi. \tag{1}$$

Note that the decay rate of the square root term matches the size of the strip U. It is straightforward to show that this inverse Fourier transform κ is holomorphic on U. In fact, all functions of the form

$$(\psi * \kappa)(u) = \int_{\mathbb{R}} \psi(x)\kappa(u - x) \, \mathrm{d}x$$

with $\psi \in L^2(\mathbb{R})$ are holomorphic on U. The map $T_\kappa : \psi \mapsto \psi * \kappa$ is an injective linear map from $L^2(\mathbb{R})$ to the space $\mathcal{O}(U)$ of holomorphic functions on U and commutes with translations in the real direction. Let $\mathcal{B}_\kappa = T_\kappa(L^2(\mathbb{R}))$ denote its image. The norm on $L^2(\mathbb{R})$ induces a norm on $\mathcal{B}_\kappa$, just denoted by $\|\cdot\|$, and we find that

$$\|\psi\|^2_{L^2(\mathbb{R})} = \|\psi * \kappa\|^2 = \int_U |(\psi * \kappa)(u)|^2 \left(\frac{1}{2} - \cos(\pi\Im u)\right) \mathrm{d}u, \tag{2}$$

$\mathrm{d}u$ being the Lebesgue measure on the strip U. Interestingly, the weight $\frac{1}{2} - \cos\pi\Im u$ is negative on part of its domain, but it still gives a norm.

The relation between (1) and (2) is the following:

$$\int_{-1}^{1} e^{2\xi y} \left(\frac{1}{2} - \cos(\pi y)\right) \mathrm{d}y = \frac{12\xi^2 + \pi^2}{4\xi^2 + \pi^2} \frac{\sinh 2\xi}{2\xi}. \tag{3}$$

We will describe this relation in detail in the case of Euclidean space, compact Lie groups and symmetric spaces of non-compact type. Before considering these cases, let us state a general lemma:

Lemma 1. *Let (X, μ) be a measure space and U a complex manifold. Let $\kappa : X \times U \to \mathbb{C}$ be square integrable on X, holomorphic on U and suppose that $u \mapsto \|\kappa(\cdot, u)\|_{L^2(X)}$ is locally bounded on U. Then $\mathcal{T}_\kappa : L^2(X) \to \mathcal{O}(U)$ given by*

$$\mathcal{T}_\kappa\psi(u) = \int_X \psi(x)\kappa(x, u) \, \mathrm{d}\mu(x)$$

is a continuous linear map.

Proof. Let $\psi \in L^2(X)$. A straightforward application of the theorem on complex differentiation under the integral from [Mat01] gives that $\mathcal{T}_\kappa\psi$ is holomorphic in all of its coordinates. By Hartog's theorem it follows that $\mathcal{T}_\kappa\psi$ is holomorphic. The linearity of $\mathcal{T}_\kappa$ is clear. Since $L^2(X)$ is metric, the only thing left to show is sequential coninuity of $\mathcal{T}_\kappa$. Let $\psi_j \to \psi$ in $L^2(X)$. We have

$$\begin{aligned} |(\psi_j * \kappa)(u) - (\psi * \kappa)(u)| &= \left| \int_X (\psi_j - \psi)(x)\kappa(x, u) \, \mathrm{d}\mu(x) \right| \\ &\leqslant \|\psi_j - \psi\|_{L^2(X)} \, \|\kappa(\cdot, u)\|_{L^2(X)}. \end{aligned}$$

Recall that $\mathcal{O}(U)$ is a Fréchet space. Its semi-norms are the supremum norms on the compact subsets of U. Since $\|\kappa(\cdot, u)\|_{L^2(X)}$ is locally bounded, it is bounded on compact subsets, thus $\mathcal{T}_\kappa\psi_j \to \mathcal{T}_\kappa\psi$ in each of the semi-norms. This proves that $\mathcal{T}_\kappa$ is continuous. □

Euclidean space and compact Lie groups Let G be $\mathbb{R}^n$ or a connected compact Lie group, and $\mathfrak{g}$ its Lie algebra. Then G has a complexification $G_{\mathbb{C}}$, which is identified with $G \times \mathfrak{g}$ by the map $\Phi : (x, y) \mapsto x \exp(iy)$. Here exp is the analytic continuation of the exponential map at the identity. We let G act on $G_{\mathbb{C}}$ from the left. Fix a Haar measure $\mathrm{d}x$ on G and a Lebesgue measure $\mathrm{d}y$ on $\mathfrak{g}$.

Let us recall the Fourier transform on G. Let $\hat{G}$ be the unitary dual of G, which is the set of equivalence classes of irreducible unitary representations of G. We will denote equivalence classes and their representing elements with $\pi \in \hat{G}$. Let V_π denote the representation space of π, and $\langle\cdot,\cdot\rangle_\pi$ the Hilbert–Schmidt inner product $(A, B) \mapsto \operatorname{tr} AB^*$ on $\operatorname{End}(V_\pi)$. The V_π are finite dimensional and equipped with an inner product that makes π unitary. Recall that the representations π extend holomorphically to $G_{\mathbb{C}}$. We find that for $y \in \mathfrak{g}$

$$\pi(\exp iy)^* = \pi(\overline{\exp iy})^{-1} = \pi(\exp iy).$$

The Fourier transform of $f \in L^1(G)$ is

$$\mathcal{F}f(\pi) = \int_G f(x)\pi(x^{-1})\,\mathrm{d}x.$$

According to the Plancherel theorem, there exists a measure μ on $\hat{G}$ such that

$$\int_G f(x)\overline{g(x)}\,\mathrm{d}x = \int_{\hat{G}} \langle\mathcal{F}f(\pi), \mathcal{F}g(\pi)\rangle_\pi\,\mathrm{d}\mu(\pi)$$

for all $f, g \in C_c(G)$, and the Fourier transform extends to an isometry from $L^2(G)$ to the space of measurable sections $\gamma : \hat{G} \to \coprod_{\pi\in\hat{G}} \operatorname{End}(V_\pi)$ satisfying

$$\int_{\hat{G}} \langle\gamma(\pi), \gamma(\pi)\rangle_\pi\,\mathrm{d}\mu(\pi) < \infty.$$

Note that when G is compact, the dual space will be discrete.

Let λ_z, ρ_z denote the left and right regular representations: $\lambda_z f(x) = f(z^{-1}x)$, $\rho_z f(x) = f(xz)$. Then $\mathcal{F}(\lambda_z f)(\pi) = \mathcal{F}f(\pi)\cdot\pi(z^{-1})$ and $\mathcal{F}(\rho_z f)(\pi) = \pi(z)\cdot\mathcal{F}f(\pi)$ whenever defined. Another useful property is that $\mathcal{F}(f * g) = \mathcal{F}g \cdot \mathcal{F}f$ for all $f, g \in L^1(G)$. Here $*$ is the convolution

$$(f * g)(z) = \int_G f(x)g(x^{-1}z)\,\mathrm{d}x.$$

The key to the relation displayed in (3) is given by a combination of these two properties, extended to a neighbourhood of G in its complexification.

From integral kernel to weight We will first give an implicit construction of a weight starting from a kernel. Let Y be an open convex neighbourhood of 0 in $\mathfrak{g}$ and $U = G \times Y$, considered as a submanifold of $G_{\mathbb{C}}$.

Definition 2. *A function $\kappa \in \mathcal{O}(U)$ is called an* admissible kernel on U *provided that $u \mapsto \|\kappa(\cdot u)\|_{L^2(G)}$ is locally bounded on U.*

Remark 3. In the Euclidean case, a function $\kappa \in L^2(\mathbb{R}^n)$ is an admissible kernel on $\mathbb{R}^n \times Y \subset \mathbb{C}^n$ if and only if $\pi \mapsto \pi(\exp -iy)\mathcal{F}\kappa(\pi)$ is square integrable on $\hat{G} \cong \mathbb{R}^n$ for all $y \in Y$. This follows from the classical Fourier–Laplace theory, see for example [RS75, Thm. IX.13].

For an admissible kernel κ on U we define the convolution transform

$$T_\kappa : L^2(G) \to \mathcal{O}(U) : \psi \mapsto \psi * \kappa.$$

This map is a special case of lemma 1, hence it is a well-defined, continuous linear map. This convolution transform will be the main object of study. We note that

Lemma 4. *T_κ is injective if and only if the operators $\mathcal{F}\kappa(\pi)$ have trivial kernel for μ-a.e. $\pi \in \hat{G}$.*

Proof. $T_\kappa\psi = 0$ if and only if $\mathcal{F}\kappa(\pi)\mathcal{F}\psi(\pi) = 0$ for μ-almost all $\pi \in \hat{G}$. □

Lemma 5. *T_κ is G-equivariant.*

Proof. Let $\psi \in L^2(G)$, $x \in G$ and $u \in U$. Then

$$\lambda_x(T_\kappa\psi)(u) = \int_G \psi(y)\kappa(y^{-1}x^{-1}u)\,\mathrm{d}y = \int_G \psi(x^{-1}z)\kappa(z^{-1}u)\,\mathrm{d}z = T_\kappa(\lambda_x\psi)(u).$$

□

Another ingredient we will need is the *Laplace transform* on Y:

$$\mathcal{L}w(\pi) = \int_Y \pi(\exp 2iy)\, w(y)\,\mathrm{d}y. \tag{4}$$

We take $w \in L^1_{\mathrm{loc}}(Y;\mathbb{R})$, i.e. locally integrable and real-valued. In order to make sense of the integral, we impose the condition

$$\langle \mathcal{L}w_-(\pi), \mathcal{L}w_-(\pi)\rangle_\pi < \infty \qquad \text{for } \mu\text{-almost every } \pi \in \hat{G}. \tag{5}$$

Here w_- is the negative part of w, that is $w = w_+ - w_-$ and $w_\pm(y) \geqslant 0$ for all $y \in Y$. Under this condition the integral (4) is defined, but might return infinite values.

We introduce the notation

$$f_y(x) = f(x \exp iy)$$

for $f \in \mathcal{O}(U)$. This enables us to view a function on U as a family of functions on G. We define the Paley–Wiener type space

$$D = \{\psi \in L^2(G) \mid \mathcal{F}\psi \in C_c(\hat{G})\}.$$

The Euclidean case Let us start with the Euclidean case. Then $G = \mathbb{R}^n$, we set $L^2(\hat{G}) = L^2(\mathbb{R}^n, \mu)$, and the Laplace transform reduces to the classical two-sided Laplace transform up to a scaling factor of 2. The notation might seem a bit complicated, but we will see that many of the results established here also hold in the case of compact Lie groups. We find:

Theorem 6. *Let κ be an admissible kernel on U. Suppose that there exists a $w \in L^1_{loc}(\mathfrak{g}; \mathbb{R})$ satisfying condition (5) and such that*

$$(\mathcal{L}w(\pi))^{-1} = |\mathcal{F}\kappa(\pi)|^2 \tag{6}$$

for μ-a.e. $\pi \in \hat{G}$. Then for all $\psi \in D$ we have

$$\int_U |T_\kappa \psi(x \exp iy)|^2 \, w(y) \, \mathrm{d}x\mathrm{d}y \leqslant \|\psi\|^2_{L^2(G)}. \tag{7}$$

Equality holds if and only if $\psi \perp \ker T_\kappa$ in $L^2(G)$.

Proof. The Laplace transform $\mathcal{L}w$ has the following property: It converges absolutely on an open convex set, say $V \subset \hat{G}$, and diverges on the complement of the closure of V. From condition (6) it follows that $\mathcal{F}\kappa(\pi) \neq 0$ for μ-a.e. $\pi \in V$ and vanishes almost everywhere on $\hat{G} \backslash V$. We have

$$\begin{aligned}
\|\psi\|^2_{L^2(G)} &= \int_{\hat{G}} |\mathcal{F}\psi(\pi)|^2 \, \mathrm{d}\mu(\pi) && (8)\\
&\geqslant \int_V |\mathcal{F}\psi(\pi)|^2 \, \mathrm{d}\mu(\pi) && (9)\\
&= \int_V |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa(\pi)|^2 |\mathcal{F}\kappa(\pi)|^{-2} \, \mathrm{d}\mu(\pi) \\
&= \int_V \int_Y |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa(\pi)|^2 \pi(\exp 2iy) \, w(y) \, \mathrm{d}y \, \mathrm{d}\mu(\pi) && (10)
\end{aligned}$$

Let us assume for the moment that this integral converges absolutely. Then we may apply Fubini's theorem to obtain

$$= \int_Y \int_V |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa(\pi)|^2 \pi(\exp 2iy) \, \mathrm{d}\mu(\pi) \, w(y) \, \mathrm{d}y$$

Since $\mathcal{F}\kappa$ vanishes outside of V, this equals

$$= \int_Y \int_{\hat{G}} |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa(\pi)|^2 \pi(\exp 2iy) \, \mathrm{d}\mu(\pi) \, w(y) \, \mathrm{d}y \tag{11}$$

We have seen in remark 3 that $\mathcal{F}\kappa(\pi)\pi(\exp iy)$ is square integrable for any $y \in Y$. It follows that $\mathcal{F}\kappa_y(\pi) = \mathcal{F}\kappa(\pi)\pi(\exp iy)$. Thus (11) equals

$$\begin{aligned}
&= \int_Y \int_{\hat{G}} |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa_y(\pi)|^2 \, \mathrm{d}\mu(\pi) \, w(y) \, \mathrm{d}y \\
&= \int_Y \int_{\hat{G}} |\mathcal{F}(\psi * \kappa_y)(\pi)|^2 \, \mathrm{d}\mu(\pi) \, w(y) \, \mathrm{d}y \\
&= \int_Y \int_G |(\psi * \kappa_y)(x)|^2 \, \mathrm{d}x \, w(y) \, \mathrm{d}y \\
&= \int_{G \times Y} |(\psi * \kappa)(x \exp iy)|^2 \, w(y) \, \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

This proves the desired inequality of norms (7). Note that in the last line we have used the absolute convergence of the double integral (10) another time.

Now we show that the double integral (10) converges absolutely. The difference between (10), which is finite, and

$$\int_V \int_Y |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa(\pi)|^2 \pi(\exp 2iy)\, |w(y)|\, \mathrm{d}y\, \mathrm{d}\mu(\pi)$$

is just twice

$$\int_V |\mathcal{F}\psi(\pi)|^2 |\mathcal{F}\kappa(\pi)|^2 \int_Y \pi(\exp 2iy)\, w_-(y)\, \mathrm{d}y\, \mathrm{d}\mu(\pi), \tag{12}$$

so it suffices to show that (12) is finite. We recognize the inner integral as the Laplace transform of the non-negative function w_-. By assumption (5) this integral is finite for all $\pi \in \hat{G}$. Since this integral is a Laplace transform, the result is in particular continuous, hence bounded on any compact subset of $\hat{G}$. Using that $\psi \in D$, it follows immediately that (12) is finite. This shows that (10) converges absolutely.

Equality of norms holds precisely if equality holds in line (9). This happens whenever $\operatorname{supp} \mathcal{F}\psi \backslash V$ is a null set, or in other words:

$$\langle \psi, \psi_0 \rangle_{L^2(G)} = \langle \mathcal{F}\psi, \mathcal{F}\psi_0 \rangle_{L^2(\hat{G})} = 0$$

for all $\psi_0 \in \ker T_\kappa$. □

This proof hinges on the finiteness of (12). Instead of taking $\psi \in D$ there are of course other conditions that work equally well, such as:

Corollary 7. *If $w_- = 0$, inequality* (7) *holds for all $\psi \in L^2(G)$.*

Proof. This is immediate from the proof, since (12) vanishes. □

On the other hand, the proof of theorem 6 also makes clear what could go wrong:

Corollary 8. *The double integral*

$$\int_U |T_\kappa \psi(x \exp iy)|^2\, w(y)\, \mathrm{d}x\mathrm{d}y \tag{13}$$

converges absolutely for all $\psi \in L^2(G)$ unless

$$\pi \mapsto \int_Y |\mathcal{F}\kappa_y(\pi)|^2\, w_-(y)\, \mathrm{d}y \tag{14}$$

is unbounded on $\hat{G}$.

Proof. Recall that $\mathcal{F}\kappa(\pi)\pi(\exp iy) = \mathcal{F}\kappa_y(\pi)$ for all $y \in Y$ and μ-a.e. $\pi \in \hat{G}$. Thus either (14) is bounded, in which case (12) is finite for all $\psi \in L^2(G)$, or (14) is unbounded. In the latter case, there exists a $\psi \in L^2(G)$ such that the integral (12) does not converge and so (13) does not converge absolutely. Note that it does not matter that we integrate over V instead of $\hat{G}$, since $\mathcal{F}\kappa$ vanishes on $\hat{G}\backslash V$. □

See example 22 below for a natural yet problematic case.

In order to describe the image of T_κ, we need to describe one more property. Let us first recall a basic statment from Lebesgue integration:

Lemma 9. *If* $\chi : \hat{G} \to \mathbb{C}$ *satsifies*

$$(\pi \mapsto \chi(\pi)\pi(\exp iy)) \in L^1(\hat{G})$$

for all $y \in Y$*, then* $g := \mathcal{F}^{-1}\chi$ *is holomorphic on* U *and*

$$\mathcal{F}g_y(\pi) = \chi(\pi)\pi(\exp iy). \tag{15}$$

Proof. First we check that g is holomorphic on U. This follows by Lebesgue dominated convergence once we realize that the exponential function $\pi \mapsto \pi(\exp iy)$ is bounded locally in y by a finite sum of functions $\pi \mapsto \sum_l \pi(\exp iy_l)$ for some $y_l \in Y$. Since $g = \mathcal{F}^{-1}\chi$, this equality extends holomorphically to all of U, that is

$$g(x \exp iy) = \mathcal{F}^{-1}(\pi \mapsto \chi(\pi)\pi(\exp iy))(x)$$

for all $x \exp iy \in U$. Since the g_y are bounded and continuous, they have distributional Fourier transforms $\mathcal{F}g_y$. These functions necessarily agree with the Fourier transform of the right hand side, which is the desired result (15). □

We define

$$\mathcal{A}(U) = \left\{ \mathcal{F}^{-1}\omega \,\middle|\, \pi \mapsto \omega(\pi)\pi(\exp iy) \in L^1(\hat{G}) \text{ for all } y \in Y \right\}.$$

It follows from lemma 9 that $\mathcal{A}(U) \subset \mathcal{O}(U)$.

Lemma 10. *Let* κ *be an admissible kernel on* U*. Then* $T_\kappa(L^2(G)) \subset \mathcal{A}(U)$.

Proof. Let $f \in T_\kappa(L^2(G))$ and write $f = \psi * \kappa$ for some $\psi \in L^2(G)$. Since $\pi \mapsto \mathcal{F}\kappa(\pi)\pi(\exp iy)$ is in $L^2(\hat{G})$ for any $y \in Y$, and $\mathcal{F}f = \mathcal{F}\kappa\mathcal{F}\psi$, it follows that

$$(\pi \mapsto \mathcal{F}f(\pi)\pi(\exp iy)) \in L^1(\hat{G}).$$

This shows that $f \in \mathcal{A}(U)$. □

Our next goal is to endow the image of T_κ with a Hilbert structure. Let $w \in L^1_{\mathrm{loc}}(\mathfrak{g}; \mathbb{R})$. Suppose that it satisfies condition (5) and moreover that

$$\mathcal{L}w(\pi) \in [0, \infty] \qquad \text{for all } \pi \in \hat{G}. \tag{16}$$

We define the map $\mathcal{I}_w : \mathcal{A}(U) \to [0, \infty]$ by

$$\mathcal{I}_w : f \mapsto \int_{\hat{G}} |\mathcal{F}f(\pi)|^2 \mathcal{L}w(\pi)\, \mathrm{d}\mu(\pi).$$

We write $\mathcal{B}(U, w) = \{f \in \mathcal{A}(U) \,|\, \mathcal{I}_w(f) < \infty\}$.

Theorem 11. *Let* κ *be an admissible kernel on* U*. Suppose that there exists a* $w \in L^1_{loc}(\mathfrak{g}; \mathbb{R})$ *satisfying condition* (5) *and such that*

$$(\mathcal{L}w(\pi))^{-1} = |\mathcal{F}\kappa(\pi)|^2 \tag{17}$$

for μ*-a.e.* $\pi \in \hat{G}$*. Then* T_κ *is a partial isometry onto* $\mathcal{B}(U, w)$.

Proof. First note that $|\mathcal{F}\kappa(\pi)|^{-2} \in (0,\infty]$ for μ-a.e. $\pi \in \hat{G}$, so (17) implies (16). This guarantees that $\mathcal{I}_w$ and $\mathcal{B}(U,w)$ are defined. Moreover, it tells us that $\mathcal{L}w$ is nonzero almost everywhere.

From (8–10) we see that for all $\psi \in L^2(G)$

$$\|\psi\|^2 \geqslant \mathcal{I}_w(\psi * \kappa).$$

Together with lemma 10 this shows that $T_\kappa(L^2(G)) \subset \mathcal{B}(U,w)$.

It follows from theorem 6 that $\|\psi\|_0^2 = \mathcal{I}_w(\psi * \kappa)$ if $\|\cdot\|_0$ denotes the norm on $L^2(G)/\ker T_\kappa$. Thus the only thing left to show is that T_κ is surjective on $B(U,w)$. Let $f \in B(U,w)$, and say V is the set where $\mathcal{L}w$ converges absolutely. Then $\mathcal{F}f$ vanishes almost everywhere on $\hat{G}\backslash V$ and we have

$$\begin{aligned}\mathcal{I}_w(f) &= \int_{\hat{G}} |\mathcal{F}f(\pi)|^2 \mathcal{L}w(\pi)\,\mathrm{d}\mu(\pi)\\ &= \int_V |\mathcal{F}f(\pi)|^2 \mathcal{L}w(\pi)\,\mathrm{d}\mu(\pi)\\ &= \int_V |\mathcal{F}f(\pi)|^2 |\mathcal{F}\kappa(\pi)|^{-2}\,\mathrm{d}\mu(\pi) < \infty,\end{aligned}$$

hence $(\mathcal{F}\kappa)^{-1}\cdot\mathcal{F}f \in L^2(V)$. Pick $\psi \in L^2(G)$ such that $\mathcal{F}\psi = (\mathcal{F}\kappa)^{-1}\cdot\mathcal{F}f$ on V and $\mathcal{F}\psi = 0$ on $\hat{G}\backslash V$. Then $\mathcal{F}(\psi * \kappa) = \mathcal{F}\kappa\cdot\mathcal{F}\psi = \mathcal{F}\kappa\cdot(\mathcal{F}\kappa)^{-1}\cdot\mathcal{F}f = \mathcal{F}f$, hence $f = \psi * \kappa$. This completes the proof. □

Remark 12. If the double integral (13) converges absolutely for $f = T_\kappa\psi$, for example if $\psi \in D$, we have

$$\mathcal{I}_w(f) = \int_U |f(x\exp iy)|^2\, w(y)\,\mathrm{d}x\mathrm{d}y. \tag{18}$$

We view $\mathcal{I}_w(f)$ as an extension of the right hand side, so that we can make sense of that integral for arbitrary $f \in \mathcal{A}(U)$.

Remark 13. We see that under the condition (17) the space $\mathcal{B}(U,w)$ with squared norm $\mathcal{I}_w$ turns out to be a Hilbert space.

Compact Lie groups For the rest of the paragraph, assume that G is a compact Lie group. Recall that $Y \subset \mathfrak{g}$ is an open convex set containing 0 and $U = G\exp iY \subset G_{\mathbb{C}}$. Let κ be an admissible kernel on U. For simplicity we will assume that T_κ is injective, i.e. $\mathcal{F}\kappa$ has trivial kernel. The general case can be attacked in the same way as before.

The first simplification we obtain is the following: If $\kappa \in \mathcal{O}(U)$, then κ is an admissible kernel. The next simplification is a counterpart to remark 3:

Lemma 14. *If $\kappa \in \mathcal{O}(U)$, we have for all $y \in Y$ and $\pi \in \hat{G}$ that*

$$\mathcal{F}\kappa_y(\pi) = \pi(\exp iy)\mathcal{F}\kappa(\pi).$$

In particular, $(\pi \mapsto \pi(\exp iy)\mathcal{F}\kappa(\pi)) \in L^1(\hat{G})$ for all $y \in Y$.

Proof. The integrals

$$\int_G \kappa(x \exp iy)\pi(x^{-1})\,\mathrm{d}x = \int_G \kappa(z)\pi(\exp iy \cdot z^{-1})\,\mathrm{d}z$$

are equal since κ is holomorphic on $\{G \exp ity \,|\, t \in [0,1]\}$. Since $\kappa_y \in L^2(G)$, it follows that $(\pi \mapsto \pi(\exp iy)\mathcal{F}\kappa(\pi)) \in L^2(\hat{G})$. This holds for any $y \in Y$, so in particular these functions are integrable. □

Theorem 6 reads in this case:

Theorem 15. *Let $\kappa \in \mathcal{O}(U)$ and assume that T_κ has trivial kernel. Suppose that there exists a $w \in L^1_{loc}(\mathfrak{g};\mathbb{R})$ satisfying condition (5) and such that*

$$\mathcal{L}w(\pi) = (\mathcal{F}\kappa(\pi)\mathcal{F}\kappa(\pi)^*)^{-1}$$

for all $\pi \in \hat{G}$. Then for all $\psi \in D$ we have

$$\int_U |T_\kappa\psi(x \exp iy)|^2\, w(y)\,\mathrm{d}x\mathrm{d}y = \|\psi\|^2_{L^2(G)}.$$

Proof. We will mimic the proof of theorem 6. Let $\|\cdot\|_\pi$ denote the Hilbert-Schmidt norm corresponding to the inner product $(A,B) \mapsto \operatorname{tr} AB^*$ on $\operatorname{End}(V_\pi)$. Assuming absolute convergence in (19), we have

$$\begin{aligned}
\|\psi\|^2_{L^2(G)} &= \int_{\hat{G}} \|\mathcal{F}\psi(\pi)\|^2_\pi\,\mathrm{d}\mu(\pi) \\
&= \int_{\hat{G}}\int_Y \|\pi(\exp iy)\mathcal{F}\kappa(\pi)\mathcal{F}\psi(\pi)\|^2_\pi\, w(y)\,\mathrm{d}y\,\mathrm{d}\mu(\pi) \qquad (19)\\
&= \int_{\hat{G}}\int_Y \|\mathcal{F}(\psi * \kappa_y)(\pi)\|^2_\pi\, w(y)\,\mathrm{d}y\,\mathrm{d}\mu(\pi) \\
&= \int_U \|(\psi * \kappa_y)(x)\|^2\, w(y)\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

The second part of the proof, proving the absolute convergence of (19) for $\psi \in D$, is almost literally the same as in the Euclidean case and will not be given here. □

The third simplification is that we have no need for the space $\mathcal{A}(U)$: Lemma 14 shows that $\mathcal{A}(U) = \mathcal{O}(U)$ in this case. So if $w \in L^1_{\mathrm{loc}}(\mathfrak{g};\mathbb{R})$ satisfies condition (5) and $\mathcal{L}w$ is non-negative, but possibly infinite, we set

$$\mathcal{I}_w : \mathcal{O}(U) \to [0,\infty] : f \mapsto \int_{\hat{G}}\int_Y \|\pi(\exp iy)\mathcal{F}f(\pi)\|^2_\pi\, w(y)\,\mathrm{d}y\,\mathrm{d}\mu(\pi)$$

and $\mathcal{B}(U,w) = \{f \in \mathcal{O}(U) \,|\, \mathcal{I}_w(f) < \infty\}$. With these definitions, theorem 11 goes through almost literally:

Theorem 16. *Let $\kappa \in \mathcal{O}(U)$ and assume that T_κ has trivial kernel. Suppose that there exists a $w \in L^1_{loc}(\mathfrak{g};\mathbb{R})$ satisfying condition (5) and such that*

$$(\mathcal{L}w(\pi))^{-1} = \mathcal{F}\kappa(\pi)\mathcal{F}\kappa(\pi)^* \qquad (20)$$

for all $\pi \in \hat{G}$. Then T_κ is an isometry onto $\mathcal{B}(U,w)$.

The proof is the same as in the Euclidean case, with $V = \hat{G}$ since we assumed here that T_κ is injective.

Remark 17. If we integrate the trace on both sides of equation (20), we see that

$$\int_{\hat{G}} \operatorname{tr}\left[\left(\int_Y \pi(\exp(2iy))\, w(y)\mathrm{d}y\right)^{-1}\right] \mathrm{d}\mu(\pi) < \infty,$$

which is a nontrivial condition on the weight w. Basically it says that all eigenvalues of the inner integral are unbounded as π 'goes to infinity', i.e. the norm of its heighest weight $\|\mu_\pi\|$ goes to infinity. (Compare Theorem 2.1 from [Sai88].)

Some examples We will now give a few examples where the above applies. At first, let $G = \mathbb{R}$ and $Y = (-1, 1)$. We will identify $\hat{G} = \mathbb{R}$ and denote elements in this space with ξ. Say $\mathcal{F}\psi(\xi) = \int_G \psi(x)e^{-ix\xi}\,\mathrm{d}x$, then $\mathcal{F}^{-1}f(x) = \int_{\hat{G}} f(\xi)e^{ix\xi}\mathrm{d}\mu(\xi)$ with $\mathrm{d}\mu(\xi) = \mathrm{d}\xi/2\pi$.

Example 18. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an arbitrary phase function, and let κ^ϕ be defined by its Fourier transform

$$\mathcal{F}\kappa^\phi(\xi) = \sqrt{\frac{\xi}{\sinh 2\xi}}\, e^{i\phi(\xi)}.$$

We have $|\mathcal{F}\kappa^\phi(\xi)|^{-2} = \sinh(2\xi)/\xi$, and we find that w is just the Lebesgue measure on Y. This example illustrates there is a family $\{\kappa^\phi\}$ of holomorphic functions whose convolution transforms T_{κ^ϕ} have the same image. The domain $G + iY$ is a maximal domain of holomorphy for this family: We easily compute that if ϕ is linear, κ^ϕ will have poles at $-\phi' \pm i$. Also, we have that $\kappa^\phi(\cdot + iy)$ is square integrable if and only if $|y| < 1$.

Example 19. Consider now $G = S^1$. Write $\kappa(x) = \sum_m \chi_m(\kappa)x^m$, where $|x| = 1$, $m \in \mathbb{Z}$ and $\chi_m(\kappa)$ are the usual Fourier coefficients. Condition (20) reads

$$|\chi_m(\kappa)|^{-2} = \int_Y e^{-2my}\, w(y)\,\mathrm{d}y.$$

Consider again $Y = (-1, 1)$ and let $w(y) = 1$. This gives

$$\kappa(x) = \sum_{m\in\mathbb{Z}} \sqrt{\frac{m}{\sinh 2m}}\, x^m.$$

This funcion extends to the annulus $S^1 \times (\frac{1}{e}, e)$ in $G_\mathbb{C} \cong \mathbb{C}^*$.

Example 20. Say $G = \mathbb{R}$, $Y = \mathbb{R}$ and $\kappa(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Then $\mathcal{F}\kappa(\xi) = e^{-\xi^2/2}$ and we find that T_κ is injective. Again, according to condition (10), we have

$$e^{\xi^2} = \int_\mathbb{R} e^{-2\xi y}\, w(y)\,\mathrm{d}y.$$

This equality is solved by $w(y) = \frac{1}{\sqrt{\pi}}e^{-y^2}$.

Remark 21. The above example has an interesting feature. Let $w \in L^1_{\mathrm{loc}}(Y, \mathbb{R})$ such that $\mathcal{L}w(\xi) = e^{\xi^2}$. We will show that Y has to be unbounded. In general, the decay rate of $\mathcal{F}\kappa$ determines the size of Y.

Proof. The Laplace transform e^{ξ^2} converges for every $\xi \in \mathbb{R}$, so it converges absolutely everywhere. In particular, $\mathcal{L}w(0) < \infty$, hence $w \in L^1(Y;\mathbb{R})$.

Suppose that Y is bounded from below. Then there exists a $y_0 > 0$ such that $Y + y_0 \subset (0,\infty)$. Let $n \in \mathbb{N}$. Then, on the one hand, we have

$$\lim_{n\to\infty} e^{-2ny_0} e^{n^2} = \infty. \tag{21}$$

On the other hand, we have

$$\lim_{n\to\infty} e^{-2ny_0}\mathcal{L}w(n) = \lim_{n\to\infty}\int_Y e^{-2n(y+y_0)}\, w(y)\mathrm{d}y = 0. \tag{22}$$

This follows easily from Lebesgue dominated convergence, since the integrands are bounded by $|w|$, which is integrable. The left hand sides of (21) and (22) are equal, so we reached a contradiction. We conclude that Y cannot be bounded from below. Similarly, we find that Y cannot be bounded from above. □

Example 22. It may happen that κ is holomorphic, but has an unbounded Fourier transform. As we have seen, allowing negative weights destroys the nice interpretation of the integral (18). We will see what happens for $G = Y = \mathbb{R}^3$: Let $w : Y \to \mathbb{R} : y \mapsto \pi^{-3/2}(\langle y,y\rangle - \frac{3}{2})e^{-\langle y,y\rangle}$, where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^3$. As before, let $\xi \in \hat{G} = \mathbb{R}^3$ be the coordinate on the unitary dual. Then

$$|\mathcal{F}\kappa(\xi)|^{-2} = \langle\xi,\xi\rangle\, e^{\langle\xi,\xi\rangle},$$

and we may choose $\mathcal{F}\kappa(\xi) = \langle\xi,\xi\rangle^{-\frac{1}{2}}e^{-\langle\xi,\xi\rangle/2}$. It is easy to check that the conditions of theorem 11 are fulfilled. However, the pole in $\xi = 0$ violates the condition (14) of corollary 8. Therefore the right hand side of (18) does not converge absolutely for all $f \in \operatorname{im} T_\kappa$ and we have to use the left hand side of this equality instead.

Remark 23. The behaviour sketched in the above example is critical, i.e. it is lost if we perturb w. Set $w_a : Y \to \mathbb{R} : y \mapsto (\langle y,y\rangle - a)e^{-\langle y,y\rangle}$. We have three possibilities:

$a > \frac{3}{2}$: Positivity of $\mathcal{L}w$ is violated and our method does not apply.

$a = \frac{3}{2}$: The right hand side of (18) does not converge absolutely.

$a < \frac{3}{2}$: The right hand side of (18) does converge absolutely.

For other weights w, other types of boundary behaviour may occur.

It is not quite clear what kind of weights we should allow. The possibilities include measures, distributions and hyperfunctions. However, even more general classes appear naturally, as is demonstrated by the following two examples:

Example 24. We consider the Hardy norm

$$\|f\|^2_{H^2} = \sup_{y>0}\int_{\mathbb{R}} |f(x+iy)|^2\,\mathrm{d}x = w\left(\|f_y\|^2_{L^2(\mathbb{R})}\right),$$

setting $w(F) = \sup_{y>0} F(y)$. Adapting property (6) to this setting, it follows that $\mathcal{F}\kappa$ is the Heaviside function, and $\kappa(z) = \frac{i}{2\pi z}$ for $\Im z > 0$.

Example 25. The previous example also has a two-sided version: Let $\kappa(z) = \frac{1}{1+z^2}$. Then $Y = (-1, 1)$ and $\mathcal{F}\kappa(\xi) = \pi e^{-|\xi|}$. We find

$$\frac{1}{\pi^2} e^{2|\xi|} = w(e^{-2\xi y}).$$

The non-linear functional $w(F) = \frac{1}{\pi^2} \sup_{y \in Y} F(y)$ solves this problem.

In the following we will select $w \in L^1_{\text{loc}}$, so that we may think of $\mathcal{I}_w$ as the norm on a weighted Bergman space with weight w. On the other hand, this construction opens up new possibilities and it might be interesting to generalize the weights w to the dual of the space $\{\pi(\exp i\cdot) \mid \pi \in \hat{G}\}$. We will not pursue this possibility here.

A converse approach Theorem 11 could be reformulated in the following way: If κ is an admissible kernel and $|\mathcal{F}\kappa|^{-2}$ is in the image of the Laplace transform, then we can find a weight such that the image of T_κ is a weighted Bergman space. In this paragraph we will study the opposite problem: What kind of weights give rise to a convolution transform?

In order to answer this question, we will study these two steps:

1. Take the Laplace transform of a weight, and compute its inverse square root.
2. Show that the function obtained in this way is square integrable, and its inverse Fourier transform is an admissible kernel.

We will just discuss the Euclidean case. The case of compact Lie groups is not essentially different; the only complicaion is that we would need a detailed description of the $\pi(\exp iy)$ and the Laplace transform in this case. This is not in the scope of this thesis.

Let $w \in L^1_{\text{loc}}(\mathbb{R}^n;\mathbb{R})$. We write for $\xi \in \mathbb{R}^n$ the Laplace transform as

$$\mathcal{L}w(\xi) = \int_Y e^{-2\langle \xi, y\rangle} w(y)\,\mathrm{d}y. \tag{23}$$

Let V be the domain where $\mathcal{L}w$ converges absolutely. We assume that

$$\mathcal{L}w(\xi) > 0 \qquad \text{for almost all } \xi \in V. \tag{24}$$

We define

$$\hat{\kappa}(\xi) = \begin{cases} (\mathcal{L}w(\xi))^{-\frac{1}{2}} & \text{if } \xi \in V \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{\kappa}$ satisfies $|\hat{\kappa}|^2 = (\mathcal{L}w)^{-1}$ almost everywhere on $\mathbb{R}^n$. This completes step 1. For the second step, define $Y \subset \mathbb{R}^n$ by $y \in Y$ if and only if there exists an open neighbourhood $Q \ni y$ such that

$$\sup_{y' \in Q} \|\hat{\kappa}(\cdot)e^{-\langle \cdot, y'\rangle}\|_{L^2(\mathbb{R}^n)} < \infty. \tag{25}$$

Lemma 26. *Let $w \in L^1(\mathbb{R}^n;\mathbb{R})$, suppose that (24) holds and let $\hat{\kappa}$ and Y be as above. Then Y is convex and open, and $\mathcal{F}^{-1}\hat{\kappa}$ is an admissible kernel on $\mathbb{R}^n + iY$.*

Proof. The defining property of Y is an open condition, so Y is open. Since the Laplace transform converges absolutely on a convex set, Y is convex as well. For the last point, just apply remark 3. □

Remark 27. The weight w and the kernel $\mathcal{F}^{-1}\hat{\kappa}$ satisfy condition (6) by construction. Note that we did not recover condition (5). Instead, we assumed absolute integrability of (23) on some set V and put $\hat{\kappa} = 0$ elsewhere.

Under mild conditions on w and its Laplace transform, we can actually determine the set Y that figures in the above lemma. Let $\operatorname{conv} S$ denote the convex hull of a set S and S° its interior.

Lemma 28. *Let $w \in L^1(\mathbb{R}^n;\mathbb{R})$ and suppose that* (24) *holds. Suppose moreover that* $\operatorname{supp} w_-$ *is compact and contained in the interior* $\operatorname{supp} w^\circ$ *of* $\operatorname{supp} w$. *Then, if $(\mathcal{L}w)^{-1}$ is locally integrable,* $Y = \operatorname{conv}(\operatorname{supp} w)^\circ$.

Proof. We will check first that $Y \supset (\operatorname{supp} w)^\circ$, i.e. for any $y \in (\operatorname{supp} w)^\circ$ there is an open neighbourhood Q of y in that set such that (25) holds. Since $|\kappa|^2 = (\mathcal{L}w)^{-1}$ is locally integrable, it is sufficient to check that $\mathcal{L}w$ has a certain growth rate as $\|\xi\| \to \infty$. To be more precise: As $\|\xi\| \to \infty$, we want that $\mathcal{L}w$ grows faster than $e^{-2\langle\xi,y\rangle+\varepsilon\|\xi\|}$ for some $\varepsilon > 0$ locally constant in $y \in \operatorname{supp} w^\circ$. We prove this property pointwise, but it is not hard to see that the proof actually extends locally. So let $y_0 \in \operatorname{supp} w^\circ$.

Consider the unit sphere S in $\mathbb{R}^n$. Let $\xi_0 \in S$. We claim that there is an open neighbourhood P of ξ_0 in S, and $\Xi_P, \varepsilon_P > 0$ such that

$$\int_{\mathbb{R}^n} e^{-2\langle\lambda\xi, y-y_0\rangle} w(y)\mathrm{d}y > e^{\varepsilon_P\lambda} \tag{26}$$

for all $\lambda > \Xi_P$ and all $\xi \in P$. By compactness of S, this claim establishes the desired growth of $\mathcal{L}w$. Let us turn to the proof of the claim.

Since $K = \operatorname{supp} w_- \cup \{y_0\}$ is compact and contained in $\operatorname{supp} w^\circ$, we can find an open subset $V \subset \operatorname{supp} w$ such that

$$\langle\xi_0, y\rangle < \langle\xi_0, y'\rangle$$

for all $y \in \overline{V}$ and $y' \in K$. In particular, we can find a neighbourhood P of ξ_0 and a constant $\varepsilon > 0$ such that $\langle\xi, y\rangle < \langle\xi, y'\rangle - \varepsilon$ for all $\xi \in P$. Taking exponents and absorbing all bounded quantities in the constants ε_P and Ξ_P readily verifies equation (26).

Since Y is convex and $(\operatorname{supp} w)^\circ \subset Y$, it follows that $\operatorname{conv}(\operatorname{supp} w)^\circ \subset Y$.

We will prove that $Y \subset \operatorname{conv}(\operatorname{supp} w)^\circ$ by contradiction. Suppose that there exists a $y \in Y$ which does not lie in the closure of $\operatorname{conv}(\operatorname{supp} w)^\circ$. Then there is $\xi \in \mathbb{R}^n$ such that $e^{-2\langle\xi,y\rangle}$ grows faster than $\mathcal{L}w(\xi)$ in a cone around the direction of ξ. Thus $\hat{\kappa}(\xi)e^{-\langle\xi,y\rangle}$ goes to infinity in this direction. This is however in contradiction with (25), so Y is contained in the closure of $\operatorname{conv}(\operatorname{supp} w)^\circ$. Now we use that Y is open to finish the proof. ☐

Remark 29. The above lemma nicely illustrates what one would expect, namely that w is positive near the boundary of its support. This phenomenon can be phrased and proven in several different ways, one of which is presented here:

Let $a, b \in \mathbb{R}$, $w = w_+ - w_-$ a signed measure on the open interval (a, b) and assume that w_- is finite on (a, b). Let $c_a, c_b > 0$ and set

$$W(\xi) = c_a e^{-2\xi a} + c_b e^{-2\xi b} + \int_a^b e^{-2\xi y}\,\mathrm{d}w(y).$$

Lemma 30. *There exist $\Xi, \varepsilon > 0$ such that*

$$\|\xi\| > \Xi \quad \Rightarrow \quad e^{2\langle\xi,y_0\rangle}\cdot W(\xi) > e^{\varepsilon\|\xi\|}$$

locally for $y_0 \in (a, b)$. If W^{-1} is locally integrable, $\mathcal{F}^{-1}\big(W^{-\frac{1}{2}}\big)$ is an admissible kernel on $\mathbb{R} + i(a, b)$.

Proof. We will have a look at the open unit interval $(0,1)$ and the limit $\xi \to -\infty$, which is a perfectly general setup. Almost by definition, $w_-((\frac{n}{n+1},1)) \to 0$ as $n \to \infty$. Choose N such that $w_-((\frac{N}{N+1},1)) < \frac{1}{2}c_1$. Thus

$$W(\xi) > \frac{1}{2}c_1 e^{-2\xi} - \int_{(0,\frac{N}{N+1}]} e^{-2\xi y}\,\mathrm{d}w_-(y).$$

This proves the desired growth rate, and given the local behaviour, we may insert this in (25) to obtain that $Y = (a,b)$ for the admissible kernel $\hat{\kappa} = W^{-\frac{1}{2}}$. □

This proof generalizes to convex hulls of discrete sets in higher dimensions.

We finish this paragraph with some elementary examples:

Example 31. As a first example, we take $G = \mathbb{R}$ and the Heaviside function as weight, so $Y = (0,\infty)$. Then

$$\mathcal{L}w(\xi) = \int_0^\infty e^{-2\xi y}\,\mathrm{d}y.$$

We see immediately that $\hat{\kappa}(\xi) = \sqrt{2\xi}$ for $\xi > 0$ and $\hat{\kappa}(\xi) = 0$ for $\xi \leqslant 0$. The corresponding kernel $\kappa = \mathcal{F}^{-1}\hat{\kappa}$ is holomorphic on the upper half space $G\exp iY = \{z \in \mathbb{C} \,|\, \Im z > 0\}$ and square integrable on the G-orbits. The kernel of T_κ is given by $\{\psi \in L^2(G) \,|\, \operatorname{supp}\mathcal{F}\psi \subset (-\infty,0]\}$.

Example 32. Suppose the weight is a strictly positive integrable function with (at most) polynomial decay, as for example

$$\mathcal{L}w(\xi) = \int_{-\infty}^\infty e^{2\xi y}\frac{1}{1+y^2}\,\mathrm{d}y.$$

Then $\mathcal{L}w(0) = \pi$, and $\mathcal{L}w$ is infinite everywhere else. Thus $\kappa = 0$ and $Y = \mathbb{R}$. This example shows that w needs serious decay in some directions to define a nontrivial integral kernel.

Example 33. The weight w need not be positive locally, but cannot be too negative either. This would violate the positivity of $\mathcal{L}w$. This is demonstrated in the example

$$\mathcal{L}w(\xi) = \int_{-1}^1 (y^2 - a)e^{2\xi y}\,\mathrm{d}y.$$

We find that

$$\mathcal{L}w(\xi) = \frac{(1+2\xi^2(1-a))\sinh 2\xi - 2\xi\cosh 2\xi}{2\xi^3}$$

and so $\mathcal{L}w(\xi) > 0$ for $a < \frac{1}{3}$ only. Here $Y = (-1,1)$.

The Riemannian symmetric case

We will now consider the situation for Riemannian symmetric spaces of the non-compact type. Again we will start with considering analytic continuation and the Fourier transform in this setting. Once this is done, we will only need to make some small changes to the Euclidean case to obtain our results for the Riemannian symmetric case. We will not discuss the case of compact Riemannian symmetric spaces here.

Let G be a non-compact real semi-simple Lie group. To avoid complications, we will assume that it is linear and connected. Choose a maximal compact subgroup K of G. The Cartan decomposition of the Lie algebra of G is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form on $\mathfrak{g}$.

Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, and write $\Sigma \subset \mathfrak{a}^*$ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $\mathcal{W}$ denote its Weyl group. Fix a positive system $\Sigma^+ \subset \Sigma$, and let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be the corresponding Iwasawa decomposition. Here $\mathfrak{n}$ is the sum of the root spaces $\mathfrak{g}^\alpha$ for $\alpha \in \Sigma^+$. Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. The Iwasawa decomposition for G gives that $N \times A \times K \to G : (n, a, k) \mapsto nak$ is a diffeomorphism onto, so there are maps

$$\begin{aligned} \mathsf{k} &: G \to K : nak \mapsto k \\ \mathsf{a} &: G \to A : nak \mapsto a. \end{aligned}$$

In addition, we define the map $\log : A \to \mathfrak{a}$. It is the inverse of the exponential map of G, which is a diffeomorphism on $\mathfrak{a}$. Let $\mathfrak{a}^+$ denote the positive Weyl chamber $\{y \in \mathfrak{a} \,|\, \alpha(y) > 0 \text{ for all } \alpha \in \Sigma^+\}$ and $\mathfrak{a}^*_+$ its dual cone. Note that the Killing form is definite on $\mathfrak{p}$, so it induces an inner product on $\mathfrak{a}$ and $\mathfrak{a}^*$. Consequently, we may identify these spaces with Euclidean space. The norm on $\mathfrak{a}$ also induces a geodesic distance in G.

Since G is linear, it has a complexification $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{k}_{\mathbb{C}}$ denote the complexification of $\mathfrak{k}$, and $K_{\mathbb{C}}$ the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_{\mathbb{C}}$. Likewise define $\mathfrak{a}_{\mathbb{C}}$, $\mathfrak{n}_{\mathbb{C}}$, $A_{\mathbb{C}}$ and $N_{\mathbb{C}}$. Let M be the centralizer of A in K and set $B = K/M$. Since M normalizes N, the map a is left-M-invariant in addition to being right-K-invariant. Also $\mathsf{k}(mg) = m\mathsf{k}(g)$ for all $m \in M$ and $g \in G$.

The Riemannian symmetric space X associated to G is the quotient G/K. Following [GK02, Krö04] we will introduce a complexification and crown domain for this space. First of all, let $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ denote the complexification of X. Now define

$$\Omega = \left\{ y \in \mathfrak{a} \,\middle|\, |\alpha(y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma \right\}.$$

The complex crown of X is defined as

$$\Xi = G \exp(i\Omega) K_{\mathbb{C}}/K_{\mathbb{C}} \subset X_{\mathbb{C}}.$$

The map $\mathsf{a} : G \to A$ is right-K-invariant, so it is well defined on X. It is even analytic and extends holomorphically to Ξ:

$$\mathsf{a} : \Xi \to T_\Omega : naK_{\mathbb{C}} \mapsto a.$$

Here $T_\Omega = A\exp(i\Omega)$, $\Xi \subset N_{\mathbb{C}}T_\Omega K_{\mathbb{C}}/K_{\mathbb{C}}$, $n \in N_{\mathbb{C}}$ and $a \in T_\Omega$.

Let us do a small computation. Pick $g \in G$ and $y \in \Omega$. We show that

$$\mathsf{a}(g\exp iy) = \mathsf{a}(g)\cdot \mathsf{a}(\mathsf{k}(g)\exp iy).$$

The only thing we will need in addition to the above decompositions of G and Ξ is that A normalizes $N_{\mathbb{C}}$. This however follows immediately from the fact that A normalizes N: For $a \in A$, $\mathrm{Ad}(a) : \mathfrak{n} \to \mathfrak{n}$ extends to a complex linear map. Let us write $g = nak$. Since Ξ is G-invariant, we can write $\mathsf{k}(g)\exp iy = n'a'$ for some $n' \in N_{\mathbb{C}}$ and $a' \in T_\Omega$. Then the formula reduces to $\mathsf{a}(nan'a') = a\cdot \mathsf{a}(n'a')$, which holds since $an'a^{-1} \in N_{\mathbb{C}}$.

We will need some more notation and definitions before we can define the Fourier transform. Set $m_\alpha = \dim \mathfrak{g}^\alpha$, and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. We write for $\xi \in (\mathfrak{a}^*)_{\mathbb{C}}$ and $g \in G$

$$\mathsf{a}(g)^\xi = e^{\xi(\log \mathsf{a}(g))}.$$

Now for some function theory and integrations.

Throughout the paper we will switch continuously between right-K-invariant functions on G and functions on X, usually omitting the identity coset, and similarly for $X_{\mathbb{C}}$.

Let $\mathrm{d}g$ denote a left-invariant Haar measure on G. Let $\mathrm{d}k$ be normalized Haar measure on K. Then there is a unique left invariant measure $\mathrm{d}x$ on X such that

$$\int_G f(g)\,\mathrm{d}g = \int_X \int_K f(xk)\,\mathrm{d}k\,\mathrm{d}x$$

for all $f \in L^1(G)$. We define the Fourier transform of $f \in C_c^\infty(X)$ as

$$\mathcal{F}f(\lambda, b) = \int_G f(g)\,\mathsf{a}(b^{-1}g)^{\rho - i\lambda}\,\mathrm{d}g.$$

Here $(\lambda, b) \in \mathfrak{a}^* \times B$. See for example chapter III of [Hel94] for more details. The Plancherel theorem in this case states that there exists a Weyl group invariant measure μ on $\mathfrak{a}^* \times B$ such that the Fourier transform extends to an isometry

$$\mathcal{F} : L^2(X) \to L^2(\mathfrak{a}^*_+ \times B, \mu).$$

Write $\hat{X} = \mathfrak{a}^*_+ \times B$ for short. The Plancherel measure is explicitly known: $\mathrm{d}\mu(\lambda, b) = |\mathbf{c}(\lambda)|^{-2}\mathrm{d}\lambda\mathrm{d}b$, where $\mathrm{d}b$ is K-invariant normalized Haar measure on B and $\mathrm{d}\lambda$ comes from a suitably normalized Lebesgue measure on $\mathfrak{a}^*$ viewed as Euclidean space. The function $\mathbf{c}(\lambda)$ is Harish-Chanrda's $\mathbf{c}$-function. It will suffice for now that $|\mathbf{c}(\lambda)|^{-2}$ is a tempered distribution on $\mathfrak{a}^*$. See for an exposition of the $\mathbf{c}$-function and its properties for example [GV88]. For later reference, we set $\mathrm{d}\mu(\lambda) = \frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2}$.

A remark on the slightly unconventional definition of the Fourier transform is in place here. In view of our next lemma, we will need to apply the Fourier transform to functions on G rather than functions on X. With this definition, a function on G is made right-K-invariant before applying the standard Fourier transform on G/K. We will need the following property of the Fourier transform on X: If $\psi, \chi \in L^2(X)$ and χ is left-K-invariant, then

$$\mathcal{F}(\psi * \chi)(\lambda, b) = \mathcal{F}\psi(\lambda, b)\cdot \mathcal{F}\chi(\lambda).$$

We define the spherical function with parameter $\lambda \in \mathfrak{a}^*$ for $g \in G$ as

$$\phi_\lambda(g) = \int_K \mathsf{a}(kg)^{\rho+i\lambda}\,\mathrm{d}k.$$

Note that the spherical function is bi-K-invariant. It extends uniquely to a bi-$K_{\mathbb{C}}$-invariant function on Ξ, cf. [Krö04, Prop 2.1.1]. We will denote this extension by the same symbol.

If f is a function on Ξ, we write f_y for the right shift with $\exp iy$, i.e. $f_y(g) = f(g \exp iy)$ for $g \in G$. Generally, f_y will not be right-K-invariant.

We will follow the same steps as in the Euclidean case. Let us start with some definitions. We take $Y \subset \Omega$ an open, convex and Weyl group invariant neighbourhood of 0. We set $U = G \exp iY \subset \Xi$.

We call $\kappa \in \mathcal{O}(U)$ an admissible kernel on U provided that it is bi-K-invariant and $u \mapsto \|\kappa(\cdot u)\|_{L^2(G)}$ is locally bounded on U. The additional left-K-invariance follows naturally from the definition of a convolution transform. Note that κ is determined by its restriction to $\exp \mathfrak{a}^+$ and is locally bi-$K_{\mathbb{C}}$-invariant. The property described in remark 3 is replaced by [Far03, Thm. 4.4]:

$$\|\kappa_y\|^2_{L^2(X)} = \int_{\mathfrak{a}^*_+} |\mathcal{F}\kappa(\lambda)|^2\,\phi_\lambda(\exp 2iy)\,\mathrm{d}\mu(\lambda), \tag{27}$$

for all $y \in Y$. In particular we have $|\mathcal{F}\kappa_y(\lambda)|^2 = |\mathcal{F}\kappa(\lambda)|^2\phi_\lambda(\exp 2iy)$ for almost all λ and all $y \in Y$. We set

$$T_\kappa : L^2(G) \to \mathcal{O}(U) : \psi \mapsto \int_G \psi(g)\kappa(g^{-1}\cdot)\,\mathrm{d}g$$

as before and define the Laplace transform for $w \in L^1_{\mathrm{loc}}(Y;\mathbb{R})$ by

$$\mathcal{L}w(\lambda) = \int_Y \phi_\lambda(\exp 2iy)\,w(y)\,\mathrm{d}y.$$

Since Y and ϕ are Weyl group invariant, we might as well take w to be Weyl group invariant. Instead of carrying condition (5) everywhere with us, we just assume that T_κ is injective and $\mathcal{L}w$ converges absolutely on all of $\mathfrak{a}^*$.

We find the same obstruction as in corollary 8:

Theorem 34. *Let κ be an admissible kernel on U and assume that T_κ is injective. Suppose there exists a $w \in L^1_{\mathrm{loc}}(Y;\mathbb{R})$ such that*

$$\mathcal{L}w(\lambda) = |\mathcal{F}\kappa(\lambda)|^{-2}$$

for almost all $\lambda \in \mathfrak{a}^*$. Then, if

$$\lambda \mapsto |\mathcal{F}\kappa(\lambda)|^2\mathcal{L}w_-(\lambda) \tag{28}$$

is bounded on $\mathfrak{a}^*$ and if $\psi \in L^2(X)$,

$$\|\psi\|^2_{L^2(X)} = \int_U |T_\kappa\psi(x \exp iy)|^2\,w(y)\,\mathrm{d}x\mathrm{d}y$$

and the double integal on the right hand side converges absolutely.

Proof. Since $|\mathcal{F}\kappa|^2\mathcal{L}w = 1$ is bounded, (28) is equivalent to the fact that $|\mathcal{F}\kappa|^2\mathcal{L}|w|$ is bounded. Therefore all integrals below are absolutely convergent and we have for $\psi \in L^2(X)$

$$\begin{aligned}
\|\psi\|^2_{L^2(X)} &= \int_{\hat{X}} |\mathcal{F}\psi(\lambda,b)|^2 \,\mathrm{d}\mu(\lambda,b) && (29)\\
&= \int_{\hat{X}} |\mathcal{F}\psi(\lambda,b)|^2 |\mathcal{F}\kappa(\lambda)|^2 \mathcal{L}w(\lambda)\,\mathrm{d}\mu(\lambda,b) && (30)\\
&= \int_{\hat{X}}\int_Y |\mathcal{F}\psi(\lambda,b)|^2 |\mathcal{F}\kappa(\lambda)|^2 \phi_\lambda(\exp 2iy)\, w(y)\,\mathrm{d}y\,\mathrm{d}\mu(\lambda,b)\\
&= \int_{\hat{X}}\int_Y |\mathcal{F}\psi(\lambda,b)|^2 |\mathcal{F}\kappa_y(\lambda)|^2\, w(y)\,\mathrm{d}y\,\mathrm{d}\mu(\lambda,b)\\
&= \int_{\hat{X}}\int_Y |\mathcal{F}(\psi * \kappa_y)(\lambda,b)|^2\, w(y)\,\mathrm{d}y\,\mathrm{d}\mu(\lambda,b)\\
&= \int_X\int_Y |(\psi * \kappa)(x\exp iy)|^2\, w(y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

□

We proceed along the same lines as in the Euclidean case. We define

$$\mathcal{A}(U) = \left\{\mathcal{F}^{-1}\omega \,\middle|\, (\lambda,b) \mapsto \omega(\lambda,b)\sqrt{\phi_\lambda(\exp 2iy)} \in L^1(\hat{G}) \text{ for all } y \in Y\right\}.$$

It is not clear whether $\mathcal{A}(U)$ is included in $\mathcal{O}(U)$ or not. One could try to prove a result like lemma 9 using the bounds on $\phi_\lambda(\exp iy)$ from [Krö04, KÓS05]. We will not pursue this question because corollary 37 is sufficient for our purpose. We still have

Lemma 35. *Let κ be an admissible kernel on U. Then $T_\kappa(L^2(X)) \subset \mathcal{A}(U)$.*

In view of equation (27), the proof of lemma 10 applies with the obvious modifications.

Choose $w \in L^1_{\mathrm{loc}}(Y;\mathbb{R})$ such that $\mathcal{L}w$ is defined and non-negative almost everywhere on $\mathfrak{a}^*$. Define for $f \in \mathcal{A}(U)$

$$\mathcal{I}_w(f) = \int_{\hat{X}} |\mathcal{F}f(\lambda,b)|^2\, \mathcal{L}w(\lambda)\,\mathrm{d}\mu(\lambda,b)$$

and set $\mathcal{B}(U,w) = \{f \in \mathcal{A}(U) \,|\, \mathcal{I}_w(f) < \infty\}$. Now we can state another version of theorem 11:

Theorem 36. *Let κ be an admissible kernel on U and assume that T_κ is injective. Suppose that there exists a $w \in L^1_{loc}(Y;\mathbb{R})$ such that $\mathcal{L}w = |\mathcal{F}\kappa|^{-2}$ almost everywhere on $\mathfrak{a}^*$. Then T_κ is an isometry onto $\mathcal{B}(U,w)$.*

Proof. Recall that the injectivity of T_κ implies that $\mathcal{F}\kappa$ is non-vanishing. Then (29–30) gives immediately that

$$\|\psi\|^2_{L^2(X)} = \mathcal{I}_w(T_\kappa\psi).$$

Together with lemma 35 this shows that T_κ is an injective linear map from $L^2(X)$ to $\mathcal{B}(U,w)$. Moreover, we see that $\mathcal{I}_w$ induces a norm on $\mathcal{B}(U,w)$ and that T_κ is isometric with respect to this norm. It remains to check surjectivity:

Let $f \in \mathcal{B}(U, w)$. Then

$$\mathcal{I}_w(f) = \int_{\hat{G}} |\mathcal{F}f(\lambda, b)|^2 \mathcal{F}\kappa(\lambda)|^{-2} \, \mathrm{d}\mu(\lambda, b),$$

so there exists a $\psi \in L^2(X)$ such that $\mathcal{F}f(\lambda, b) = \mathcal{F}\psi(\lambda, b)\mathcal{F}\kappa(\lambda)$ and it follows that $f = \psi * \kappa$. □

It follows immediately from lemma 1 that

Corollary 37.

$$\mathcal{B}(U, w) \subset \mathcal{O}(U).$$

We see that the results obtained here strongly resemble the results in the Euclidean case. Still, there are two major problems: The first one is that the spherical functions are not explicitly known in all cases. The second one is that we assumed that $Y \subset \Omega$, the latter set being bounded. In remark 21 we have seen that some kernels require weights with unbounded support. This problem is illustrated in the next example for one of the simplest cases.

Example 38. Let $G = SL(2, \mathbb{C})$ with compact subgroup $K = SU(2)$ and let $\alpha \in \mathfrak{a}^*$ be the positive root. Suppose that $\mathcal{L}w(l\alpha) = e^{l^2}$ for all $l \in \mathbb{R}$. Then, if

$$\mathcal{L}w(l\alpha) = \int_{Y'} \phi_{l\alpha}(\exp 2iy) \, w(y) \, \mathrm{d}y,$$

it follows that Y' is unbounded.

Proof. Let $y = \left(\begin{smallmatrix} \eta & 0 \\ 0 & -\eta \end{smallmatrix}\right)$, $\eta \in \mathbb{R}$. According to [Hel84, p. 433], we have

$$\phi_{l\alpha}(\exp 2iy) = \frac{\sin 4il\eta}{l \sinh 4i\eta} = \frac{\sinh 4l\eta}{l \sin 4\eta}.$$

Just as the classical Laplace transform, this integral converges absolutely everywhere. Therefore we can apply the same trick as in remark 21. It follows that Y' is unbounded. □

Conclusion Recall that the map T_κ is G-equivariant. Thus, in all three cases described above, T_κ intertwines the left regular representation with a representation on a Hilbert space of holomorphic functions. This Hilbert space consists of holomorphic functions satisfying a certain spectral decay and its Hilbert structure is the completion of a Bergman-type inner product. It is remarkable that the weight of this inner product need not be positive. However, the Laplace transform of this weight is positive.

From example 38 we see that the Segal–Bargmann or heat kernel transform on Riemannian symmetric spaces does not fit in the framework established here, cf. [KÓS05]. Generally, the decay rate of the Fourier transform of the kernel function κ is bounded by the size of Ω.

Crownlike domains

In the previous section we have seen that the method presented there only works if the Fourier transform of the kernel κ has bounded exponential decay. The bound depends on the size of the set Ω. In this section we will make the first steps to remove this condition and work through three of the easiest cases. We will assume that $\mathcal{F}\kappa$ decays faster than any exponential, but it is easy to adapt the following method to intermediate cases.

An example of type $\mathbf{A_1}$ Let $G = SL(2,\mathbb{C})$ with maximal compact subgroup $K = SU(2)$ and Iwasawa decomposition $g = \mathsf{n}(g)\mathsf{a}(g)\mathsf{k}(g)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{C})$, we choose

$$\mathsf{n}(g) = \begin{pmatrix} 1 & r^2(a\bar{c} + b\bar{d}) \\ 0 & 1 \end{pmatrix} \qquad \mathsf{a}(g) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \qquad \mathsf{k}(g) = r\begin{pmatrix} \bar{d} & -\bar{c} \\ c & d \end{pmatrix},$$

where $r = (|c|^2 + |d|^2)^{-\frac{1}{2}}$. The root system is $\Sigma = \{\pm\alpha\}$, with positive root

$$\alpha\left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}\right) = 2t$$

and consequently $\mathsf{a}(g)^\rho = r^2$ in the notation deployed above. We compute that

$$\Omega = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \,\middle|\, |t| < \frac{\pi}{4} \right\}.$$

We will be mainly interested in the complexification of A, which is

$$A_{\mathbb{C}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \,\middle|\, z \in \mathbb{C}^\times \right\}.$$

Its Lie algebra is $\mathfrak{a}_{\mathbb{C}} \cong \mathbb{C}$, and the exponential map $\exp : \mathfrak{a}_{\mathbb{C}} \to A_{\mathbb{C}}$ is many-to-one. Let $\Gamma = \exp^{-1}(\{e\}) = 2\pi i\mathbb{Z} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, so that $\log : A_{\mathbb{C}} \to \mathfrak{a}_{\mathbb{C}}/\Gamma$ is a well-defined isomorphism. The problem we will deal with in the following is basically the problem of choosing the right branch of Γ for this logarithm. Therefore we will carefully keep track of the points in $\mathfrak{a}_{\mathbb{C}}$, although this might not seem to make sense. For example, $e^{i\pi} = e^{-i\pi}$, but we will consider them as different points. Alternatively, one could keep track of the branch with an additional variable.

Recall the following definition of the spherical functions

$$\phi_\lambda(g) = \int_K \mathsf{a}(kg)^{\rho+i\lambda}\,\mathrm{d}k = \int_B \mathsf{a}(b^{-1}g)^{\rho+i\lambda}\,\mathrm{d}b.$$

It is clear from our application to the Fourier transform that this definition of a spherical function is the most useful one. On any compact subset of Ω, the spherical functions are bounded by $Ce^{R|\lambda|}$ for some $C, R > 0$. They extend analytically to the sets Ξ and $\exp 2i\Omega$ in $X_{\mathbb{C}}$.

Suppose that κ is bi-K-invariant and that its Fourier transform decays faster than $e^{-R|\lambda|}$ for all $R > 0$. Then for $g \in G$ and $y \in \Omega$, we have

$$\kappa(g \exp iy) = \frac{1}{\#\mathcal{W}} \int_{\mathfrak{a}^*} \mathcal{F}\kappa(\lambda)\,\phi_\lambda(g \exp iy)\,\mathrm{d}\mu(\lambda). \tag{31}$$

We define

$$\mathfrak{a}_r = \Omega + \mathbb{Z}\vartheta, \qquad \vartheta = \begin{pmatrix} \frac{\pi}{2} & 0 \\ 0 & -\frac{\pi}{2} \end{pmatrix}.$$

Note that $\mathfrak{a}_r$ is an open dense subset of $\mathfrak{a}$, and that ϑ satisfies the relation $g \exp i\vartheta = \exp i\vartheta \cdot \tau g$, where τ is an involution of G preserving the Iwasawa decomposition: It fixes A, inverts N and flips K. This allows us to define an extension Φ_λ of ϕ_λ to the set $G \exp i\mathfrak{a}_r$ in the following way: Let $y + l\vartheta \in \mathfrak{a}_r$. Then

$$\begin{aligned} \Phi_\lambda(g \exp i(y + l\vartheta)) &= \int_K \mathsf{a}(kg \exp i(y + l\vartheta))^{\rho+i\lambda} \, \mathrm{d}k \\ &= \int_K \mathsf{a}(k \exp il\vartheta \cdot \tau^l(g) \exp iy)^{\rho+i\lambda} \, \mathrm{d}k \\ &= \int_K \mathsf{a}(\exp il\vartheta \cdot \tau^l(k)\tau^l(g) \exp iy)^{\rho+i\lambda} \, \mathrm{d}k \\ &= \mathsf{a}(\exp il\vartheta)^{\rho+i\lambda} \phi_\lambda(\tau^l(g) \exp iy) \end{aligned}$$

In princple the term $\mathsf{a}(\exp il\vartheta)^{\rho+i\lambda}$ is ill defined, but we might spell this out as

$$= e^{(\rho+i\lambda)(\log \mathsf{a}(\exp il\vartheta))} \phi_\lambda(\tau^l(g) \exp iy)$$

Clearly $\exp il\vartheta \in A_{\mathbb{C}}$, so we might drop the a, and let the log and exp cancel each other, resulting in what we take as definition of $\Phi_\lambda(g \exp i(y + l\vartheta))$:

$$= e^{(\rho+i\lambda)(il\vartheta)} \phi_\lambda(\tau^l(g) \exp iy).$$

Substituting ϕ for Φ, equation (31) reads for $y + l\vartheta \in \mathfrak{a}_r$

$$\kappa(g \exp i(y + l\vartheta)) = \frac{1}{\#\mathcal{W}} \int_{\mathfrak{a}^*} \mathcal{F}\kappa(\lambda) \, e^{(\rho+i\lambda)(il\vartheta)} \, \phi_\lambda(\tau^l(g) \exp iy) \, \mathrm{d}\mu(\lambda).$$

We will also write $\kappa_{l\vartheta}(g \exp iy)$ for the above expression. If we set $y = 0$ we find

$$\begin{aligned} \kappa(g \exp il\vartheta)) &= \frac{1}{\#\mathcal{W}} \int_{\mathfrak{a}^*} \mathcal{F}\kappa(\lambda) \, e^{(\rho+i\lambda)(il\vartheta)} \, \phi_\lambda(\tau^l(g)) \, \mathrm{d}\mu(\lambda) \\ &= \frac{1}{\#\mathcal{W}} \int_{\mathfrak{a}^*} \mathcal{F}\kappa(\lambda) \, e^{(\rho+i\lambda)(il\vartheta)} \, \phi_\lambda(g) \, \mathrm{d}\mu(\lambda). \end{aligned}$$

This follows since an easy computation shows that $\phi_\lambda(g) = \phi_\lambda(\tau(g))$ for $g \in G$. Hence we see that shifting by ϑ causes the Fourier coefficients of κ to pick up a factor of $e^{(\rho+i\lambda)(i\vartheta)}$. This factor pops up because $X_{\mathbb{C}}$ is not simply connected. In particular, since $K = \tau(K)$, we can define the Laplace transform of $w \in L^1_{\mathrm{loc}}(\mathfrak{a}_r; \mathbb{R})$ by

$$\mathcal{L}w(\lambda) = \sum_{l \in \mathbb{Z}} e^{-2l\lambda(\vartheta)} \int_\Omega \phi_\lambda(\exp 2iy) w(y + l\vartheta) \, \mathrm{d}y.$$

It is the sum of the Laplace transforms on the sets $\{\Omega + l\vartheta \mid l \in \mathbb{Z}\}$, weighted with $|e^{(\rho+i\lambda)(il\vartheta)}|^2 = e^{-2l\lambda(\vartheta)}$. This factor appears squared, just as we used $\pi\pi^*$

in the definition of the Laplace transform for compact Lie groups. We also view $\phi_\lambda(\exp 2iy)$ as the square of $\phi_\lambda(\exp iy)$.

Now, if we apply the machinery from the last section on all copies of Ω, we find the following: For $\psi \in L^2(X)$, if the integrals involved converge absolutely, we have the equality of norms

$$\|\psi\|^2_{L^2(X)} = \sum_{l\in\mathbb{Z}} \|\psi * \kappa_{l\vartheta}\|^2_l,$$

with

$$\|f\|^2_l = \int_{G\times\Omega} |f(g\exp iy)|^2\, w(y + l\vartheta)\mathrm{d}g\mathrm{d}y.$$

Remark 39. We may assume that w is Weyl group invariant and reformulate the above in terms of Weyl group symmetrized functions such as

$$\sum_{\sigma\in\mathcal{W}} e^{-2l\lambda(\sigma\vartheta)}.$$

At some points in the computation we have used special features of $SL(2,\mathbb{C})$. We will see in the next examples that the situation becomes more complicated for other symmetric spaces.

A rank two example The method sketched above goes through for $SL(2,\mathbb{R})$, but already for $SL(3,\mathbb{R})$ we encounter a serious problem. Let $\epsilon_j : \mathrm{End}(\mathbb{R}^3) \to \mathbb{R} : a_{mn} \mapsto a_{jj}$, $1 \leqslant j,m,n \leqslant 3$, and choose two positive roots $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = \epsilon_2 - \epsilon_3$. Let $\Sigma = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ be the root system of $SL(3,\mathbb{R})$. Set

$$\theta_\alpha = \begin{pmatrix} \frac{\pi}{2} & 0 & 0 \\ 0 & -\frac{\pi}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \theta_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & -\frac{\pi}{2} \end{pmatrix}$$

and extend θ to a linear map $\theta : \Sigma \to \mathfrak{a} : \gamma \mapsto \theta_\gamma$.

Then with respect to the basis $(\theta_\alpha, \theta_\beta)$, the crown domain

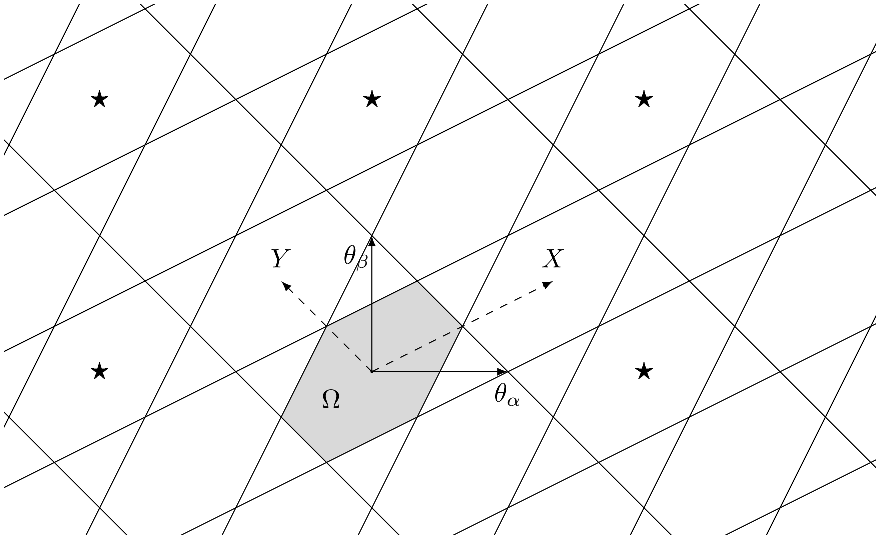
$$\Omega = \left\{ H \in \mathfrak{a} \,\middle|\, \gamma(H) < \frac{\pi}{2} \text{ for all } \gamma \in \Sigma \right\}$$

consists of all the points $\lambda\theta_\alpha + \mu\theta_\beta$ satisfying the equations

$$|2\lambda - \mu| < 1 \qquad |2\mu - \lambda| < 1 \qquad |\lambda + \mu| < 1 \qquad (\lambda, \mu \in \mathbb{R})$$

and is depicted in the figure on the next page. We see that the lines corresponding to the conditions $\gamma(H) \equiv \frac{\pi}{2} \mod \pi$ divide the $\mathfrak{a}$-plane in cells.

We observe that the cells come in two sizes and any two cells of the same size are of the same shape. Let us focus on the hexagonal cells first. We note that some of the cells map to the same region in $X_\mathbb{C}$ under the mapping $H \mapsto \exp iH \cdot K_\mathbb{C}$. Let us say that two cells are equivalent if this is the case. Since $\exp 2i\theta_{\alpha,\beta} \in K$, the cells marked with a ★ are equivalent to Ω. Also for example the cells $\Omega + X$ and $\Omega + Y$ are equivalent. We find that there are three equivalence classes of big hexagonal cells, and we may choose representatives Ω, $\Omega + X$ and $\Omega + 2X$. A short computation yields that $X = \frac{1}{3}(4\theta_\alpha + 2\theta_\beta)$ and $Y = \frac{1}{3}(-2\theta_\alpha + 2\theta_\beta)$.



For $H \in \mathfrak{a}$, define

$$\tau_H : G_{\mathbb{C}} \to G_{\mathbb{C}} : g \mapsto \exp -iH \cdot g \cdot \exp iH.$$

Now we are in good shape to repeat the method sketched earlier. Let $l, m \in \mathbb{Z}$. We define the spherical function on $G \exp i(\Omega + lX + mY)$ as

$$\Phi_\lambda(g \exp i(y + lX + mY)) = e^{i(\rho+i\lambda)(lX+mY)} \cdot \phi_\lambda((\tau_X^l \tau_Y^m)(g) \exp iy).$$

Here the involutions τ_X, τ_Y and $\tau_X\tau_Y$ map G to itself and preserve its Iwasawa decomposition. They are different involutions of G fixing three different copies of $GL(2,\mathbb{R})$ inside G. For this computation it is crucial to forget about right-K-invariance for the moment, so that we can choose the logarithm in an orderly fashion.

This defines a spherical function on the hexagonal cells and we can work our way through the formulae of the previous section. The problem clearly lies in the triangular cells. Still we can say some things about them:

We recover a different type of automorphism if we walk along $Z_\gamma = \frac{2}{3}\theta_\gamma$, $\gamma \in \Sigma$, from the big cell to one of the triangular cells next to it. It is easy to see that τ_{Z_γ} is an automorphism of order 6 of $G_{\mathbb{C}}$. These are the automorphisms I expect to appear in the definition of the spherical functions of the triangular domains. Also, the notion of 'holomorphic in $\tau_{Z_\gamma} z$' for $z \in G_{\mathbb{C}}$ might be relevant. From the picture we see that crossing two walls will restore everything to normal. For example, the automorphism $\tau_{Z_\alpha}^{-1}\tau_{Z_\beta}$ is nothing else than τ_Y since $Y = Z_\beta - Z_\alpha$. Note that all these automorphisms commute, so it is irrelevant how we walk around a crossing of lines.

It is quite interesting to study these automorphisms. For example, if $\gamma \in \Sigma$, τ_γ^3 is the identity on $X_{\mathbb{C}}$, but τ_γ itself does not preserve $K_{\mathbb{C}}$. Instead, it maps $K = SO(3)$ to different compact subgroups of $SU(3)$.

We see that in rank two and higher the structure is already quite complicated and it is unclear whether the method presented in the previous paragraph generalizes to this situation or not.

Rank one revisited The picture becomes a bit more clear if we study the 'intermediate' case of a restricted root system of type BC_1. Let us have a look at the rank one symmetric space $SU(1,p)/S(U(1) \times U(p))$ for $p > 1$.

Taking adjoints and complex conjugates is done with respect to the complex structure of G itself. The symbol i is reserved for the imaginary unit in the way we used it before.

The Lie algebra

$$\mathfrak{g} = \mathfrak{su}(1,p) = \left\{ X \in \mathfrak{sl}(1+p,\mathbb{C}) \,\middle|\, X^* \begin{pmatrix} -1 & 0 \\ 0 & I_p \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & I_p \end{pmatrix} X = 0 \right\}$$

of $G = SU(1,p)$ has ± 1 eigenspaces under the Cartan involution $X \mapsto -X^*$

$$\mathfrak{k} = \left\{ \begin{pmatrix} u & 0 \\ 0 & U \end{pmatrix} \,\middle|\, U = -U^*, u + \operatorname{tr} U = 0 \right\}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix} \,\middle|\, y \in \mathbb{C}^p \right\}.$$

From now on it is convenient to write $1+p \times 1+p$ matrices as $(1,1,p-1) \times (1,1,p-1)$ matrices. We choose a maximal abelian subalgebra

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

of $\mathfrak{p}$. Let H_t denote the above matrix. Define $\alpha \in \mathfrak{a}^*$ by $\alpha(H_t) = t$. Then the restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ are

$$\Sigma = \{\pm\alpha, \pm 2\alpha\}$$

and the corresponding root spaces are given by

$$\mathfrak{g}^{\pm\alpha} = \left\{ \begin{pmatrix} 0 & 0 & z^* \\ 0 & 0 & \pm z^* \\ z & \mp z & 0 \end{pmatrix} \,\middle|\, z \in \mathbb{C}^{p-1} \right\}$$

$$\mathfrak{g}^{\pm 2\alpha} = \left\{ \begin{pmatrix} u & \mp u & 0 \\ \pm u & -u & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, u \in \mathbb{C}, u = -\bar{u} \right\}.$$

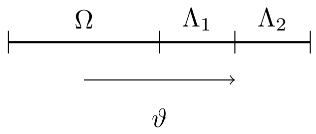
The eigenspace $\mathfrak{g}^0$ is given by $\mathfrak{a} + \mathfrak{m}$, where

$$\mathfrak{m} = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & U \end{pmatrix} \,\middle|\, U = -U^*, 2u + \operatorname{tr} U = 0 \right\}.$$

We will now introduce one by one the notions we used in the case of $SL(2,\mathbb{C})$. We find that

$$\Omega = \left\{ H_t \,\middle|\, |t| < \frac{\pi}{4} \right\}.$$

We set $\vartheta = H_{\frac{\pi}{2}}$. The points $\{H_t \,|\, \gamma(H_t) = \frac{\pi}{2} \mod \pi \text{ for some } \gamma \in \Sigma\}$ divide $\mathfrak{a}$ in a series of intervals. Since $\exp 2i\vartheta \in K$, it suffices to look at a fundamental interval of length 2ϑ:



We see that except for Ω, there are two other connected components or cells, denoted here by $\Lambda_{1,2}$. These two cells are identified with each other under the affine Weyl group, which is the group acting on $\mathfrak{a}$ generated by

$$\rho : H_t \mapsto H_{t+\pi} = H_t + 2\vartheta \qquad \text{and} \qquad \sigma : H_t \mapsto H_{-t}.$$

We define $\mathfrak{a}_r = \mathfrak{a} \backslash \{H_t \,|\, \gamma(H_t) = \frac{\pi}{2} \mod \pi \text{ for some } \gamma \in \Sigma\}$, so $\mathfrak{a}_r$ is dense in $\mathfrak{a}$. We define the automorphism

$$\tau : G_{\mathbb{C}} \to G_{\mathbb{C}} : g \mapsto \exp -i\vartheta \cdot g \cdot \exp i\vartheta.$$

On $\mathfrak{g}$, it acts as

$$\begin{pmatrix} u & \bar{y} & \bar{z} \\ y & v & \bar{w} \\ z & w & U \end{pmatrix} \overset{\tau}{\longmapsto} \begin{pmatrix} v & y & i\bar{w} \\ \bar{y} & u & -i\bar{z} \\ iw & iz & U \end{pmatrix}.$$

In particular, we can read off what τ does to the root spaces:

$$X \overset{\tau}{\longmapsto} \begin{cases} X & \text{if } X \in \mathfrak{m} \\ -X & \text{if } X \in \mathfrak{g}^{\pm 2\alpha} \\ \mp iX & \text{if } X \in \mathfrak{g}^{\pm \alpha} \end{cases}$$

From this it is clear that $\tau(G) \not\subset G$, but $\tau^2(G) = G$. Moreover, we find that $\tau^2(K) = K$, where K is of course the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. It follows that $\tau^3(K) = \tau(K)$.

The results so far point in the following direction:

Let $\xi \in \Xi = G \exp i\Omega$. We have $\phi_\lambda(\xi) = \int_K \mathsf{a}(k\xi)^{\rho+i\lambda} \mathrm{d}k$ and we would like to define its counterpart on $\tau(\Xi)$:

$$\chi_\lambda(\tau(\xi)) = \int_K \mathsf{a}(\tau(k\xi))^{\rho+i\lambda} \, \mathrm{d}k.$$

This definition hinges on the right extension of a to the set $\tau(G)\exp(i\Omega\backslash\{0\})$. Here we have to exclude 0, as we can see from the picture at the top of this page. It should satisfy the Weyl symmetry $\chi_\lambda = \chi_{-\lambda}$ and should extend anlytically to $2\Omega\backslash\{0\}$. It is not clear how this function should be defined, but let us see what it would be good for:

Let $\xi \in \Xi$ and $l \in \mathbb{Z}$. Define the extended spherical function

$$\Phi_\lambda(\xi \exp il\vartheta) = e^{(\rho+i\lambda)(il\vartheta)} \cdot \begin{cases} \phi_\lambda(\tau^l(\xi)) & \text{if } l \text{ is even} \\ \chi_\lambda(\tau^l(\xi)) & \text{if } l \text{ is odd.} \end{cases}$$

With this function we define an extended Fourier transform

$$\kappa(\xi \exp il\vartheta) = \frac{1}{\#\mathcal{W}} \int_{\mathfrak{a}*} \mathcal{F}\kappa(\lambda) \Phi_\lambda(\xi \exp il\vartheta) \, \mathrm{d}\mu(\lambda)$$

and ditto Laplace transform for $w \in L^1_{\mathrm{loc}}(\mathfrak{a}_r;\mathbb{R})$

$$\mathcal{L}w(\lambda) = \sum_{l \text{ even}} e^{-2l\lambda(\vartheta)} \int_\Omega \phi_\lambda(\exp 2iy) w(y + l\vartheta) \mathrm{d}y \\ + \sum_{l \text{ odd}} e^{-2l\lambda(\vartheta)} \int_{\Omega\setminus\{0\}} \chi_\lambda(\exp 2iy) w(y + l\vartheta) \mathrm{d}y.$$

These transforms are adapted to the main problem of this thesis and may allow us to repeat the story for Riemannian symmetric spaces without the lower bound on the spectral decay of the convolution kernel. However it is clear that there is still a lot to do before we could think of proving such theorems.

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