Algebraic independence results for reciprocal sums of Fibonacci and Lucas numbers

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Zusammenfassung

In dieser Arbeit untersuchen wir Werte der Fibonacci-Zetafunktion sowie dreier Varianten dieser Funktion für geradzahlige Argumente auf algebraische Unabhängigkeit über dem Körper $\mathbb{Q}$ der rationalen Zahlen. Wir betrachten die unendliche Menge, die aus den Werten dieser vier Funktionen besteht, und geben eine vollständige Klassifikation ihrer Teilmengen in über $\mathbb{Q}$ algebraisch unabhängige und abhängige Mengen an. Dabei bezeichnen wir in natürlicher Weise eine Menge als algebraisch unabhängig beziehungsweise abhängig über $\mathbb{Q}$, falls die Elemente dieser Menge diese Eigenschaft haben.

Die Unabhängigkeitsergebnisse in dieser Arbeit basieren auf einem Satz von Nesterenko aus dem Jahre 1996 über die Werte der Ramanujan Funktionen P, Q und R an algebraischen Stellen. Zur Anwendung kommt ferner ein Determinantenkriterium für algebraische Unabhängigkeit, das von Elsner, Shiokawa und Shimomura entwickelt wurde. Dieses Kriterium kam bereits in einer im Jahre 2011 erschienenen Publikation zur Anwendung, um erste allgemeine Resultate zur algebraischen Unabhängigkeit der in dieser Arbeit untersuchten Zahlen zu beweisen. Wir greifen die Methode von Elsner, Shiokawa und Shimomura auf und ergänzen ihre Ergebnisse.

Als weiteres Hilfsmittel dienen Laurent-Reihenentwicklungen gewisser Jacobischer elliptischer Funktionen, die in engem Zusammenhang zu den von Ramanujan eingeführten q-Reihen stehen. Dabei werden Identitäten von Zucker (1979) verwendet. Die betrachteten Zetafunktionen lassen sich schließlich als Polynome in drei algebraisch unabhängigen Größen darstellen. Hier spielen vollständige elliptische Integrale eine wesentliche Rolle.

Außerdem beweisen wir Ergebnisse zur linearen Abhängigkeit und Unabhängigkeit über $\mathbb{Q}$ der in dieser Arbeit betrachteten Zahlen.

Abschließend präsentieren wir quantitative Resultate. Wir beweisen ein Lemma, das es gestattet, das Maß für algebraische Unabhängigkeit von einer Zahlenmenge unter gewissen Abschwächungen auf eine andere Menge von Zahlen zu übertragen, wenn die beiden Mengen durch ein quadratisches System von Polynomen verbunden sind. Unter Verwendung eines quantitativen Ergebnisses von Nesterenko aus dem Jahre 1997 leiten wir ein Unabhängigkeitsmaß für die in dieser Arbeit untersuchten Zahlen her.

Schlagwörter: Algebraische Unabhängigkeit, Fibonacci-Zahlen, Nesterenkos Satz über Ramanujan Funktionen

Abstract

In this thesis we investigate values of the Fibonacci zeta function as well as those of three other types of this function at positive even integers with respect to algebraic independence over the field $\mathbb{Q}$ of rational numbers. We study the infinite set consisting of the values of these four functions and give a complete classification for all of its subsets in algebraically independent and dependent sets over $\mathbb{Q}$. In a natural sense we call a set to be algebraically independent or dependent over $\mathbb{Q}$, respectively, if this property holds for the elements of this set.

The independence results in this thesis are based on a theorem of Nesterenko from the year 1996 on the values of Ramanujan's functions P, Q and R at algebraic points. Moreover, we apply a determinant criterion for algebraic independence developed by Elsner, Shiokawa and Shimomura. This criterion was already used in a paper published in 2011 to obtain a first general result on algebraic independence of the numbers studied in this thesis. We pick up the method from Elsner, Shiokawa and Shimomura and complete the results of that paper.

As further auxiliary means we use the Laurent series expansions of certain Jacobian elliptic functions, which are closely connected to the q-series introduced by Ramanujan. Thereby we use some identities found by Zucker in 1979. The zeta functions to be discussed may finally be expressed as polynomials in three algebraically independent quantities. Here the complete elliptic integrals play an essential role.

Furthermore, we prove results on linear dependence and independence over $\mathbb{Q}$ of the numbers treated in this thesis.

At the end of this work we present quantitative results. We prove a lemma, which makes it possible to transcribe the measure of algebraic independence of one number set to another with a certain weakening when these sets are connected by some quadratic polynomial system. Using a quantitative result of Nesterenko from 1997 we derive a measure of algebraic independence for the numbers studied in this thesis.

Keywords: Algebraic independence, Fibonacci numbers, Nesterenko's theorem on Ramanujan functions

Contents

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1. Introduction

Algebraic independence theory is one of the classical branches in analytic number theory. The first result in this area, the Lindemann-Weierstrass theorem about values of the exponential function at algebraic points, was published by Weierstrass [44] in 1885. By this work, he generalized earlier results from Hermite and Lindemann who proved the transcendence of e and π, respectively. With his proof on the transcendence of π Lindemann gave a negative answer to the old question on the possibility of squaring the circle.

Later Siegel and Shidlovskii, in 1929 and 1957, respectively, created a theory for E-functions that contains the Lindemann-Weierstrass theorem as a special case. Siegel introduced E-functions as entire functions whose Taylor series coefficients are algebraic numbers with certain arithmetical properties. In general they are confluent hypergeometric functions and the exponential function is the simplest example. The Siegel-Shidlovskii method is described in [39].

In 1949 Gelfond [23] proved algebraic independence results of values of the exponential function at transcendental points. This was a generalization of Hilbert's seventh problem solved independently by Gelfond and Schneider in 1934 with different methods. In particular, they proved the transcendence of the numbers $\sqrt{2}^{\sqrt{2}}$ and e^{π}. In the 1970's Chudnovsky extended Gelfond's approach to another class of functions, namely elliptic functions. He could prove that the numbers π and $\Gamma(1/4)$ and also π and $\Gamma(1/3)$ are algebraically independent [9].

In the last 30 years there has been further progress in this area, partially based on multiplicity estimates for polynomials in analytic functions. It became possible to study modular functions in view of transcendence questions. In 1996 Nesterenko [32] proved a result on algebraic independence of the values of Ramanujan's functions P, Q and R at algebraic points. As a corollary he obtained the algebraic independence of the numbers π, e^{π} and $\Gamma(1/4)$.

We remark that it is still an open problem if $e+\pi$ is transcendental or even irrational. The same holds for the number $e \cdot \pi$.

In this thesis we study algebraic independence properties of reciprocal sums of Fibonacci numbers F_n and Lucas numbers L_n, defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \qquad (n \geq 0)$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \qquad (n \geq 0)\,,$$

respectively. It is well-known that both sequences satisfy the Binet-type formulas

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}\,, \qquad L_n = \varphi^n + \psi^n\,, \qquad (n \geq 0),$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is the Golden Ratio and

$$\psi = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}\,.$$

The Fibonacci and Lucas numbers have various interesting properties. A wide overview is given in [25].

With the subsequent survey in Section 1.1 to 1.3 on irrationality and transcendence results for series involving reciprocal Fibonacci and Lucas numbers, we follow an unpublished manuscript by Duverney and Shiokawa [15].

1.1. Irrationality results

In 1989 André-Jeannin [2] was the first to prove the irrationality of the series

$$S_1 := \sum_{n=1}^{\infty} \frac{1}{F_n}, \quad S_2 := \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}, \quad S_3 := \sum_{n=1}^{\infty} \frac{1}{L_n}, \quad \text{and} \quad S_4 := \sum_{n=1}^{\infty} \frac{(-1)^n}{L_n}.$$

He used a continued fraction expansion, much inspired by Apéry's proof of the irrationality of $\zeta(3)$ (see [3]).

Five years later Bundschuh and Väänänen [7] used Padé-approximations to the q-exponential function and its derivative to prove that $S_1 \notin \mathbb{Q}(\sqrt{5})$. They also found the following irrationality measure for S_1: For $(p,q) \in \mathbb{Z} \times \mathbb{N}$, with q large enough, we have

$$\left| S_1 - \frac{p}{q} \right| \geq \frac{1}{q^{8.621}}.$$

This measure was improved to 7.893 by Matala-Aho and Väänänen in [28]. In particular, this proves that S_1 is not a Liouville number.

The series S_3 and S_4 have been studied in 1998 by Väänänen [43], who also gave irrationality measures for them. Tachiya [41] found another proof for $S_1 \notin \mathbb{Q}(\sqrt{5})$ and also for $S_2, S_3, S_4 \notin \mathbb{Q}(\sqrt{5})$ by developing Borwein's method from [5]. Eight years after André-Jeannin's first proof had been published, an elementary proof of the irrationality of S_1, using only simple properties of the q-exponential and the q-logarithmic function, was given by Duverney in [11].

All these results have been successively improved by Prévost [37] and Matala-Aho and Prévost [29]: For example, let

$$S_5 := \sum_{n=1}^{\infty} \frac{t^n}{F_n}$$

where $t \in \mathbb{Q}\setminus\{0\}$ satisfying $|t| < (1+\sqrt{5})/2$. Then S_5 is irrational and has an irrationality measure of 2.874. The same result holds if the Fibonacci sequence F_n is replaced by the Lucas sequence L_n. In this case the irrationality measure is 7.652.

Duverney [12] proved in 1997 the irrationality of the series

$$S_6 := \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}.$$

However, his proof does not lead to any irrationality measures. Moreover, the elementary methods developed in [12] does not enable us to prove the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

although these numbers are, in fact, transcendental (and so is S_6), as we will see in the next subsection.

It is an open problem to prove the irrationality of the sum

$$\sum_{n=1}^{\infty} \frac{1}{F_n^3}.$$

1.2. Transcendence results

Seven years after the first irrationality result concerning reciprocal sums of Fibonacci and Lucas numbers, Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa [14] (see also [13]) succeeded in proving the transcendence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^{s}}$$

for any positive integer s. It can also be derived from the methods in [13] and [14] that the series S_6 is transcendental. These results are based on Nesterenko's theorem on Ramanujan functions [32] (see Subsection 1.4).

It follows that the numbers

$$\sum_{n=1}^{\infty} \frac{n}{F_{2n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{F_{2n}}$$

are transcendental because of the following identities which have been proven by Jennings [24]:

$$\sum_{n=1}^{\infty} \frac{n}{F_{2n}} = \sqrt{5}\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = \sqrt{5}\left(\sum_{n=1}^{\infty} \frac{1}{L_n^2} - \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2}\right),$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{F_{2n}} = \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2}.$$

There are many transcendence results for reciprocal sums of Fibonacci and Lucas numbers, which contain subscripts in geometric progressions. For instance, Erdös and Graham [22] asked for the arithmetical nature of the sums

$$S_7 := \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}} \quad \text{and} \quad S_8 := \sum_{n=0}^{\infty} \frac{1}{L_{2^n}}.$$

They were inspired by the well-known identity

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}.$$

This follows from

$$\sum_{n=0}^{k} \frac{1}{F_{2^n}} = 3 - \frac{F_{2^k-1}}{F_{2^k}} \qquad (k \geq 1)$$

which can be proven by induction using the formula

$$\frac{F_{2m}}{F_m} F_{m-1} - (-1)^m = F_{2m-1} \qquad (m \geq 1)$$

in the case of $m = 2^k$. Both of the numbers S_7 and S_8 were proven to be transcendental by Bundschuh and Pethö [6] and Becker and Töpfer [4], respectively. They used a method introduced in 1929 by Mahler in [26] and thereafter known as *Mahler's method*. Basically, it applies to analytic functions f satisfying a functional equation of the form

$$f(x^r) = \Phi(x, f(x))$$

where Φ is a rational function with algebraic coefficients and r is an integer greater than 1.

1.3. Algebraic independence results

Algebraic independence of numbers like S_7 has been established in 1997 by Ku. Nishioka [34] by using an extension of Mahler's method. For example, Nishioka proved that for fixed integers $a \geq 1$ and $d \geq 2$ the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{ad^n+l}} \qquad (l \in \mathbb{N})$$

are algebraically independent over $\mathbb{Q}$.

Later, Nishioka, Tanaka and Toshimitsu [35] obtained more general results: Let again $a \geq 1$ and $d \geq 2$ be fixed integers. Then the numbers

$$\sum_{n=1}^{\infty} \frac{1}{(F_{ad^n+l})^m} \qquad (m, l \in \mathbb{N})$$

are algebraically independent over $\mathbb{Q}$. The same result holds for the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(F_{ad^n+l})^m} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(L_{ad^n+l})^m} \qquad (m, l \in \mathbb{N}).$$

Tanaka [42] proved the algebraic independence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{(F_{F_n+l})^m} \qquad (m \in \mathbb{N}, l \geq 0)$$

over $\mathbb{Q}$. A remarkable special case is the transcendence of the series

$$\sum_{n=1}^{\infty} \frac{1}{F_{F_n}}.$$

In 2007 Elsner, Shimomura and Shiokawa [16, 17, 18, 19, 20] began their joint work on the Fibonacci zeta function

$$\zeta_F(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s} \qquad (\Re(s) > 0)\,,$$

which extends meromorphically to the whole complex plane (cf. [30]), and the related series

$$\zeta_F^*(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^s}\,, \qquad \zeta_L(s) := \sum_{n=1}^{\infty} \frac{1}{L_n^s}\,, \qquad \zeta_L^*(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^s}\,,$$

at positive even integers s. Steuding [40] proved that $\zeta_F(s)$ is hypertranscendental, which means that it satisfies no algebraic differential equation.

In [18] the algebraic independence of the sets

$$\{\zeta_F(2), \zeta_F(4), \zeta_F(6)\}\,, \quad \{\zeta_F^*(2), \zeta_F^*(4), \zeta_F(6)^*\}\,,$$
$$\{\zeta_L(2), \zeta_L(4), \zeta_L(6)\}\,, \quad \{\zeta_L^*(2), \zeta_L^*(4), \zeta_L(6)^*\}\,,$$

over $\mathbb{Q}$ is proven. Moreover, for any integer $s \geq 4$ the authors expressed each of the series $\zeta_F(2s), \zeta_F^*(2s), \zeta_L(2s)$ and $\zeta_L^*(2s)$ as rational functions in the three series of the same type in the above sets, i.e. for $s \geq 4$ we have

$$\zeta_F(2s) \in \mathbb{Q}\big(\zeta_F(2), \zeta_F(4), \zeta_F(6)\big)\,, \qquad \zeta_F^*(2s) \in \mathbb{Q}\big(\zeta_F^*(2), \zeta_F^*(4), \zeta_F^*(6)\big)\,,$$
$$\zeta_L(2s) \in \mathbb{Q}\big(\zeta_L(2), \zeta_L(4), \zeta_L(6)\big)\,, \qquad \zeta_L^*(2s) \in \mathbb{Q}\big(\zeta_L^*(2), \zeta_L^*(4), \zeta_L^*(6)\big)\,.$$

For instance, they obtained

$$\begin{aligned}\zeta_F(8) = \frac{15}{14}\zeta_F(4) + \frac{1}{378(4\zeta_F(2)+5)^2}\big(&256\zeta_F(2)^6 - 3456\zeta_F(2)^5 + 2880\zeta_F(2)^4 \\ &+ 1792\zeta_F(2)^3\zeta_F(6) - 11100\zeta_F(2)^3 + 20160\zeta_F(2)^2\zeta_F(6) - 10125\zeta_F(2)^2 \\ &+7560\zeta_F(2)\zeta_F(6) + 3136\zeta_F(6)^2 - 1050\zeta_F(6)\big)\,.\end{aligned}$$

Similar results were proven in [19] for the series

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s}\,, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_{2n-1}^{2s+1}}\,, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^{2s}}\,,$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)^{2s+1}}{F_{2n-1}}\,, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(2n-1)^{2s+1}}{L_{2n-1}}\,.$$

Other sets containing the values of $\zeta_F(2s), \zeta_F^*(2s), \zeta_L(2s)$ and $\zeta_L^*(2s)$ for $s = 1, 2, 3$ were treated in [20]. Here, Elsner, Shimomura and Shiokawa investigated all subsets of

$$\Gamma := \{\zeta_F(2), \zeta_F(4), \zeta_F(6), \zeta_F^*(2), \zeta_F^*(4), \zeta_F(6)^*, \zeta_L(2), \zeta_L(4), \zeta_L(6), \zeta_L^*(2), \zeta_L^*(4), \zeta_L(6)^*\}$$

and decided on their algebraic independence. They proved that every four numbers in Γ are algebraically dependent, whereas every two numbers in Γ are algebraically independent over $\mathbb{Q}$. Furthermore they could show that 198 of the 220 three-element subsets of Γ are algebraically independent over $\mathbb{Q}$. For the remaining 22 three-element subsets of Γ, explicit algebraic relations were given. Since not all of these 22 relations were published in [20], we put a complete list of the identites in the appendix of this thesis.

In [17] the authors obtained a more general result for the Fibonacci zeta function at positive even integers. By using a new algebraic independence criterion they proved that for positive integers $s_1 < s_2 < s_3$ the series $\zeta_F(2s_1)$, $\zeta_F(2s_2)$ and $\zeta_F(2s_3)$ are algebraically independent over $\mathbb{Q}$ if and only if at least one of the numbers s_i is even.

1.4. Outline of this thesis

In this thesis we study more general problems, which go back to a proposal from Professor Elsner. The main idea is to generalize the results in [20] by using the approach from [17], where actually more general binary recurrences are treated: Let $\alpha, \beta \in \overline{\mathbb{Q}}$ with $|\beta| < 1$ and $\alpha\beta = -1$, where $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers over $\mathbb{Q}$. We define the sequences

$$U_n := \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n := \alpha^n + \beta^n, \qquad (n \geq 0),$$

which satisfy the recurrence formula

$$X_{n+2} = (\alpha + \beta)X_{n+1} + X_n \qquad (n \geq 0)\,.$$

In particular, for $\beta = \psi = (1 - \sqrt{5})/2$ we get the Fibonacci numbers $U_n = F_n$ and the Lucas numbers $V_n = L_n$.

We remark that it is also possible to treat these sequences with $\alpha\beta = +1$ by the same method as presented in this thesis. Algebraic independence results for series involving such sequences were also obtained by Elsner, Shimomura and Shiokawa [16]. In our case, $\alpha\beta = -1$, we treat any sequences U_n and V_n satisfying the second order recurrence formula $X_{n+2} = aX_{n+1} + X_n$ $(n \geq 0)$, where a is an arbitrary algebraic number from the set

$$\overline{\mathbb{Q}} \setminus \big\{\beta - 1/\beta \mid \beta \in \overline{\mathbb{Q}} \setminus \{0\} \,\wedge\, |\beta| = 1\big\}\,.$$

For $s \in \mathbb{N}$ we study the series

$$\Phi_{2s} := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_n^{2s}}, \qquad \Phi_{2s}^* := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{U_n^{2s}},$$

$$\Psi_{2s} := \sum_{n=1}^{\infty} \frac{1}{V_n^{2s}}, \qquad \Psi_{2s}^* := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{V_n^{2s}}.$$

The above mentioned results from Elsner, Shimomura and Shiokawa [18, 20, 17] for the values of $\zeta_F(2s)$, $\zeta_F^*(2s)$, $\zeta_L(2s)$ and $\zeta_L^*(2s)$ with $s \in \mathbb{N}$ are also true for the more general series Φ_{2s}, Φ_{2s}^*, Ψ_{2s} and Ψ_{2s}^*.

We introduce the infinite set

$$\Omega := \{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}, \Psi^*_{2s_4} \mid s_1, s_2, s_3, s_4 \in \mathbb{N}\} \tag{1.1}$$

and investigate which subsets of Ω are algebraically independent over $\mathbb{Q}$.

The results go back to Nesterenko's theorem on Ramanujan functions [32]. As an immediate consequence we will obtain the algebraic independence over $\mathbb{Q}$ of the quantities K/π, E/π, and k under certain conditions. Here, K and E denote the complete elliptic integrals of the first and second kind, respectively, with the modulus $k \in \mathbb{C} \setminus \{0, \pm 1\}$, defined by

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad E = E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}dt\,.$$

To conclude on independence results for subsets of Ω we will use a determinant criterion from Elsner, Shimomura and Shiokawa [17] which is introduced in Section 2 of this thesis.

The sums Φ_{2s}, Φ^*_{2s}, Ψ_{2s} and Ψ^*_{2s} can be written as series of hyperbolic functions. With some identities from Zucker [46] we will be able to express the latter in terms of q-series and then as polynomials in K/π, E/π, and k with rational coefficients.

For $k^2 \in \mathbb{C} \setminus (\{0\} \cup [1, \infty))$, and $K' = K(k')$ with $k^2 + k'^2 = 1$ the equations

$$q = e^{-\pi c}, \qquad c = \frac{K'}{K}$$

give the relation among q and the quantities K/π, E/π, and k. From Zucker [46] we get the identities

$$\Sigma_1 := 2^{-2s} \sum_{n=1}^{\infty} \operatorname{cosech}^{2s}(n\pi c) = \frac{1}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) A_{2j+1}(q), \tag{1.2}$$

$$\Sigma_2 := 2^{-2s} \sum_{n=1}^{\infty} \operatorname{sech}^{2s}(n\pi c) = \frac{(-1)^{s-1}}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) B_{2j+1}(q), \tag{1.3}$$

$$\Sigma_3 := 2^{-2s} \sum_{n=1}^{\infty} \operatorname{sech}^{2s}\left(\frac{(2n-1)\pi c}{2}\right) = \frac{(-1)^{s-1}}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) D_{2j+1}(q), \tag{1.4}$$

$$\Sigma_4 := 2^{-2s} \sum_{n=1}^{\infty} \operatorname{cosech}^{2s}\left(\frac{(2n-1)\pi c}{2}\right) = \frac{1}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) C_{2j+1}(q), \tag{1.5}$$

where the q-series $A_{2j+1}(q)$, $B_{2j+1}(q)$, $C_{2j+1}(q)$, and $D_{2j+1}(q)$ for $j \geq 0$ are defined by

$$A_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1}q^{2n}}{1-q^{2n}}\,, \qquad B_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^{2j+1}q^{2n}}{1-q^{2n}}\,,$$

$$C_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1}q^{n}}{1-q^{2n}}\,, \qquad D_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^{2j+1}q^{n}}{1-q^{2n}}\,.$$

The coefficients $\sigma_i(s)$ are the elementary symmetric functions of the $s-1$ numbers -1, $-2^2, \ldots, -(s-1)^2$ for $s \geq 2$, defined by $\sigma_0(s)=1$ and

$$\sigma_i(s) = (-1)^i \sum_{1 \leq r_1 < \cdots < r_i \leq s-1} r_1^2 \cdots r_i^2 \quad (1 \leq i \leq s-1). \tag{1.6}$$

Now let

$$q = \beta^2 = e^{-\pi c}, \qquad \beta = -e^{-\pi c/2},$$

where $\beta \in \overline{\mathbb{Q}}$ defines the sequences U_n and V_n. Then, with $\alpha = -1/\beta$ we obtain

$$U_{2n} = \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} = \frac{\beta^{-2n}-\beta^{2n}}{\alpha-\beta} = \frac{e^{n\pi c}-e^{-n\pi c}}{\alpha-\beta} = \frac{2}{\alpha-\beta}\sinh(n\pi c) \qquad (n \geq 0).$$

Similar computations give

$$\begin{aligned} U_{2n-1} &= \frac{2}{\alpha-\beta}\cosh\left(\frac{2n-1}{2}\pi c\right) && (n \geq 1), \\ V_{2n} &= 2\cosh(n\pi c) && (n \geq 0), \\ V_{2n-1} &= 2\sinh\left(\frac{2n-1}{2}\pi c\right) && (n \geq 1). \end{aligned}$$

Therefore, by decomposing our reciprocal sums into two parts, we have the following representations of Φ_{2s}, Φ_{2s}^*, Ψ_{2s}, and Ψ_{2s}^* as series of hyperbolic functions:

$$\Phi_{2s} = (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2s}} + (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{2n}^{2s}} = \Sigma_3 + \Sigma_1, \tag{1.7}$$

$$\Phi_{2s}^* = (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{U_{2n-1}^{2s}} + (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{U_{2n}^{2s}} = \Sigma_3 - \Sigma_1, \tag{1.8}$$

$$\Psi_{2s} = \sum_{n=1}^{\infty}\frac{1}{V_{2n-1}^{2s}} + \sum_{n=1}^{\infty}\frac{1}{V_{2n}^{2s}} = \Sigma_4 + \Sigma_2, \tag{1.9}$$

$$\Psi_{2s}^* = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{V_{2n-1}^{2s}} + \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{V_{2n}^{2s}} = \Sigma_4 - \Sigma_2. \tag{1.10}$$

The q-series A_{2j+1}, B_{2j+1}, C_{2j+1} and D_{2j+1} are generated from the Laurent series expansions of the squared Jacobian elliptic functions $\mathrm{ns}^2 z$, $\mathrm{nc}^2 z$, $\mathrm{dn}^2 z$ and $\mathrm{nd}^2 z$. By these expansions we obtain expressions of the corresponding q-series in terms of K/π, E/π and k. For example, in [38] we find the following identities for the well-known Ramanujan functions:

$$\left.\begin{aligned} P(q^2) &:= 1 - 24A_1(q) = \left(\frac{2K}{\pi}\right)^2\left(\frac{3E}{K} - 2 + k^2\right), \\ Q(q^2) &:= 1 + 240A_3(q) = \left(\frac{2K}{\pi}\right)^4\left(1 - k^2 + k^4\right), \\ R(q^2) &:= 1 - 504A_5(q) = \left(\frac{2K}{\pi}\right)^6\frac{1}{2}\left(1+k^2\right)\left(1-2k^2\right)\left(2-k^2\right). \end{aligned}\right\} \tag{1.11}$$

Ramanujan [38] introduced these functions as

$$P(z) = 1-24\sum_{n=1}^{\infty}\sigma_1(n)z^n\,, \qquad Q(z) = 1+240\sum_{n=1}^{\infty}\sigma_3(n)z^n\,, \qquad R(z) = 1-504\sum_{n=1}^{\infty}\sigma_5(n)z^n\,,$$

where $\sigma_r(n)$ is the divisor function, defined by

$$\sigma_r(n) = \sum_{d|n} d^r\,,$$

not to be confused with the elementary symmetric functions from (1.6). He also showed that they satisfy the differential equations

$$z\frac{dP}{dz} = \frac{1}{12}(P^2 - Q)\,, \qquad z\frac{dQ}{dz} = \frac{1}{3}(PQ - R)\,, \qquad z\frac{dR}{dz} = \frac{1}{2}(PR - Q^2)\,.$$

In 1969, Mahler [27] proved that the functions $P(z)$, $Q(z)$, and $R(z)$ are algebraically independent over $\mathbb{C}(z)$. This result is based on the above differential equations. We will use Mahler's result in Section 5.4 of this thesis. In 1996, Nesterenko [32] proved the following theorem on the values of Ramanujan's functions. Its corollary and the resulting lemma play a fundamental role in the proofs of our main theorems.

Theorem 1.1 (Nesterenko [32])**.** *Let $\rho \in \mathbb{C}$ with $0 < |\rho| < 1$. Then we have*

$$\operatorname{tr.deg}\big(\mathbb{Q}(\rho, P(\rho), Q(\rho), R(\rho)) : \mathbb{Q}\big) \geq 3.$$

Corollary 1.1. *Let $\rho \in \overline{\mathbb{Q}}$ with $0 < |\rho| < 1$. Then the numbers $P(\rho)$, $Q(\rho)$, and $R(\rho)$ are algebraically independent over $\mathbb{Q}$.*

Together with (1.11) this corollary implies the following lemma:

Lemma 1.1. *Let $q = e^{-\pi c} \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Then the numbers K/π, E/π, and k are algebraically independent over $\mathbb{Q}$.*

A proof of Lemma 1.1 will be given in Section 2.

Combining the identities (1.2) to (1.10) with the Laurent series expansions of the Jacobian elliptic functions given in Section 3, we will obtain explicit expressions for the reciprocal sums $\Phi_{2s}, \Phi_{2s}^*, \Psi_{2s}$ and Ψ_{2s}^* as polynomials in K/π, E/π and k with rational coefficients. To these polynomials, we will apply an algebraic independence criterion stated in Section 2.

In Section 4 we investigate the one-type three-element subsets of Ω defined in (1.1), namely the sets

$$\{\Phi_{2s_1}^*, \Phi_{2s_2}^*, \Phi_{2s_3}^*\}\,, \qquad \{\Psi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}\,, \qquad \text{and} \qquad \{\Psi_{2s_1}^*, \Psi_{2s_2}^*, \Psi_{2s_3}^*\}$$

for pairwise distinct positive integers s_1, s_2, s_3. The independence properties of the set $\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}\}$ have already been studied in [17].

Section 5 provides results for the mixed subsets of Ω. We will prove that any two numbers in Ω are algebraically independent over $\mathbb{Q}$, whereas any four numbers in Ω are algebraically dependent over $\mathbb{Q}$. To investigate the independence properties of three-element subsets of Ω, a huge number of cases remains to be discussed. Therefore, it will be convenient to classify several cases with the help of some tables.

We also study linear independence properties of numbers in Ω and add results on algebraic independence of the functions $\Phi_{2s}(q)$, $\Phi^*_{2s}(q)$, $\Psi_{2s}(q)$ and $\Psi^*_{2s}(q)$ over $\mathbb{C}(q)$.

In the last section of this thesis we present quantitative results. We prove a general lemma on algebraic independence measures and apply it to Nesterenko's quantitative version of Theorem 1.1. Hence, we obtain algebraic independence measures for three-element subsets of Ω.

2. Algebraic independence criteria

In this section we investigate algebraic independence properties of real number sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ when these sets are connected by some quadratic polynomial system. Lemma 2.3 provides a method to transcribe the algebraic independence property from one set to another under a certain determinant condition. This lemma goes back to Elsner, Shimomura and Shiokawa and can be found in [21]. Corollary 2.1 of Lemma 2.3 will be the main tool in the proofs of algebraic independence results on subsets of Ω.

In the second subsection we prove an analogue criterion to Lemma 2.3 for functions in one variable. This will be applied in Section 5.4.

2.1. A determinant criterion for algebraic independence

Lemma 2.1 (Chain rule for transcendence degrees, [10, Chapter 6.2, Proposition 2])**.** *Let* $\mathbb{K} \subseteq \mathbb{L} \subseteq \mathbb{M}$ *be field extensions. Then*

$$\operatorname{tr.deg}(\mathbb{M} : \mathbb{K}) = \operatorname{tr.deg}(\mathbb{M} : \mathbb{L}) + \operatorname{tr.deg}(\mathbb{L} : \mathbb{K}).$$

The chain rule yields a simple algebraic independence criterion for quadratic polynomial systems:

Lemma 2.2. *Let* $\mathbb{K}$ *be a field satisfying* $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$ *and* $x_1, \dots, x_n \in \mathbb{R}$. *Let* $y_1, \dots, y_n \in \mathbb{K}[x_1, \dots, x_n]$ *be algebraically independent over* $\mathbb{K}$. *Then also the numbers* $x_1, \dots, x_n$ *are algebraically independent over* $\mathbb{K}$.

Proof. We have

$$\mathbb{K} \subseteq \mathbb{K}(y_1, \dots, y_n) \subseteq \mathbb{K}(x_1, \dots, x_n).$$

Therefore, we may apply Lemma 2.1 and obtain

$$\begin{aligned} &\operatorname{tr.deg}(\mathbb{K}(x_1, \dots, x_n) : \mathbb{K}) \\ &\quad = \operatorname{tr.deg}(\mathbb{K}(x_1, \dots, x_n) : \mathbb{K}(y_1, \dots, y_n)) + \operatorname{tr.deg}(\mathbb{K}(y_1, \dots, y_n) : \mathbb{K}). \end{aligned}$$

Since $y_1, \dots, y_n$ are algebraically independent over $\mathbb{K}$, i.e. $\operatorname{tr.deg}(\mathbb{K}(y_1, \dots, y_n) : \mathbb{K}) = n$, we see

$$\operatorname{tr.deg}(\mathbb{K}(x_1, \dots, x_n) : \mathbb{K}) \geq n.$$

On the other hand $\operatorname{tr.deg}(\mathbb{K}(x_1, \dots, x_n) : \mathbb{K}) \leq n$ is obvious. Hence, we conclude on

$$\operatorname{tr.deg}(\mathbb{K}(x_1, \dots, x_n) : \mathbb{K}) = n,$$

and that proves the lemma. □

Now we are able to prove Lemma 1.1 from the preceding section.

Proof of Lemma 1.1. We denote

$$x_1 := k\,, \qquad x_2 := \frac{K}{\pi}\,, \qquad x_3 := \frac{E}{\pi}\,,$$

and

$$y_1 := P(q^2)\,, \qquad y_2 := Q(q^2)\,, \qquad y_3 := R(q^2)\,.$$

From the conditions of Lemma 1.1 we have $\rho := q^2 \in \overline{\mathbb{Q}}$ with $0 < |\rho| < 1$. Therefore, Corollary 1.1 implies that the numbers y_1, y_2, and y_3 are algebraically independent over $\mathbb{Q}$. By (1.11) we have $y_1, y_2, y_3 \in \mathbb{Q}[x_1, x_2, x_3]$. Hence, Lemma 2.2 is applicable with $\mathbb{K} = \mathbb{Q}$. This proves that the numbers x_1, x_2, and x_3 are algebraically independent over $\mathbb{Q}$. □

The main lemma to be applied in this thesis is a modification of Lemma 2.2. In the notation of Lemma 2.2 we will now assume the numbers $x_j \in \mathbb{R}$ to be algebraically independent over $\mathbb{K}$ and ask for independence properties of the numbers y_j defined implicitely for $j = 1, \ldots, n$ as solutions of a certain polynomial system.

Lemma 2.3 (Determinant criterion for algebraic independence, [21])**.** *Let $\mathbb{K}$ be a field satisfying $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$. Let $x_1, \ldots, x_n \in \mathbb{R}$ be algebraically independent over $\mathbb{K}$ and $y_1, \ldots, y_n \in \mathbb{R}$ satisfy the system of equations*

$$f_j(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \qquad (j = 1, \ldots, n),$$

where $f_j(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \in \mathbb{K}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ for $j = 1, \ldots, n$. Assume that

$$\det\left(\frac{\partial f_j}{\partial X_i}(x_1, \ldots, x_n, y_1, \ldots, y_n)\right) \neq 0.$$

Then the numbers $y_1, \ldots, y_n$ are algebraically independent over $\mathbb{K}$.

We remark that the statement from Lemma 2.3 is also true for $x_1, \ldots, x_n \in \mathbb{C}$ (see [21]). We won't need this criterion in the general case but the following slightly weaker corollary, where we restrict the numbers $y_1, \ldots, y_n$ to belong to the ring $\mathbb{K}[x_1, \ldots, x_n]$:

Corollary 2.1. *Let $\mathbb{K}$ be a field satisfying $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$. Let $x_1, \ldots, x_n \in \mathbb{R}$ be algebraically independent over $\mathbb{K}$ and let $y_j = U_j(x_1, \ldots, x_n)$, where $U_j(X_1, \ldots, X_n) \in \mathbb{K}[X_1, \ldots, X_n]$ for $j = 1, \ldots, n$. Assume that*

$$\det\left(\frac{\partial U_j}{\partial X_i}(x_1, \ldots, x_n)\right) \neq 0.$$

Then the numbers $y_1, \ldots, y_n$ are algebraically independent over $\mathbb{K}$.

The key in the proof of Lemma 2.3 is the following proposition, which is a consequence of [45, Ch.II, § 17, Corollary to Theorem 40]:

Proposition 2.1. *Let $\mathbb{L}$ be a field satisfying $\mathbb{Q} \subseteq \mathbb{L} \subseteq \mathbb{R}$ and let $P_j(X_1, \ldots, X_n) \in \mathbb{L}[X_1, \ldots, X_n]$ for $j = 1, \ldots, n$. Assume that $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is an isolated zero of the system of equations*

$$P_j(X_1, \ldots, X_n) = 0 \qquad (j = 1, \ldots, n).$$

Then the numbers $x_1, \ldots, x_n$ are algebraic over $\mathbb{L}$.

We will give an alternative proof for Proposition 2.1, which is based on the concept of semialgebraic sets and the Tarski-Seidenberg theorem.

Definition 2.1. Let $\mathbb{K}$ be a subring of $\mathbb{R}$. A set $S \subset \mathbb{R}^n$ is called *semialgebraic* over $\mathbb{K}$ if S is a Boolean combination (using finitely many intersections, unions, and complements) of sets of the form

$$U(F) := \{a \in \mathbb{R}^n | F(a) > 0\}$$

with $F \in \mathbb{K}[X_1, \ldots, X_n]$.

Remark 2.1 (see [36, p.32]). Every Boolean combination of formulae of the form $F > 0$ (where $F \in \mathbb{K}[X_1, \ldots, X_n]$) is equivalent to a finite disjunction $(\delta_1 \vee \cdots \vee \delta_s)$ of conjunctions δ_j of the form

$$\Big(G = 0 \wedge \bigwedge_{i=1}^{r} H_i > 0\Big),$$

where the new G, H_i are also in $\mathbb{K}[X_1, \ldots, X_n]$.

Lemma 2.4 (Theorem of Tarski-Seidenberg, [36, p.33])**.** *Let $S \subset \mathbb{R}^{n+1}$ be semialgebraic over $\mathbb{K}$. Then the projection*

$$S' := \{a \in \mathbb{R}^n \,|\, \exists\, b \in \mathbb{R} \text{ such that } (a, b) \in S\}$$

of S on $\mathbb{R}^n$ (along the last coordinate) is itself semialgebraic over $\mathbb{K}$.

Proof of Proposition 2.1. We denote by $V(P_1, \ldots, P_n)$ the set of all points $(a_1, \ldots, a_n) \in \mathbb{R}^n$ satisfying

$$P_j(a_1, \ldots, a_n) = 0 \qquad (j = 1, \ldots, n).$$

It is sufficient to consider the polynomial

$$F_1(X_1, \ldots, X_n) := P_1^2(X_1, \ldots, X_n) + \cdots + P_n^2(X_1, \ldots, X_n) \in \mathbb{L}[X_1, \ldots, X_n]\,,$$

because in the real case we have $V(P_1, \ldots, P_n) = V(P_1^2 + \cdots + P_n^2) = V(F_1)$. Since $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is an isolated solution of $F_1(X_1, \ldots, X_n)$, there are rational numbers $r_1, \ldots, r_n, r$ such that the n-dimensional ball B around the center $(r_1, \ldots, r_n)$ with radius $\sqrt{r}$ encloses only the solution $(x_1, \ldots, x_n)$, i.e. the solutions of the system defined by

$$F_1(X_1, \ldots, X_n) = 0 \quad \wedge \quad (X_1 - r_1)^2 + \cdots + (X_n - r_n)^2 < r,$$

satisfy

$$V(F_1) \cap B = \{(x_1, \ldots, x_n)\}\,.$$

It is clear that

$$V(F_1) = \Big(\mathbb{R}^n \setminus U(F_1)\Big) \cap \Big(\mathbb{R}^n \setminus U(-F_1)\Big)\,,$$

and so defining $F_2 \in \mathbb{Q}[X_1, \ldots, X_n]$ by

$$F_2(X_1, \ldots, X_n) := r - (X_1 - r_1)^2 - \cdots - (X_n - r_n)^2$$

we get the Boolean combination

$$\{(x_1, \ldots, x_n)\} = \Big(\mathbb{R}^n \setminus U(F_1)\Big) \cap \Big(\mathbb{R}^n \setminus U(-F_1)\Big) \cap U(F_2).$$

This shows that $S := \{(x_1, \ldots, x_n)\}$ is a semialgebraic set over the field $\mathbb{K} := \mathbb{L}$. Applying Lemma 2.4 $(n-1)$-times, we find that every set $\{x_i\}$ $(i = 1, \ldots, n)$ is also semialgebraic over $\mathbb{L}$.

Then by Remark 2.1 we can express every $\{x_i\}$ by a finite disjunction of conjunctions, e.g.

$$\{x_i\} = \bigcup_{\sigma=1}^{s} \left[V(G_\sigma) \cap \bigcap_{j_\sigma=1}^{r_\sigma} U(H_{\sigma,j_\sigma}) \right]$$

with polynomials $G_\sigma, H_{\sigma,j_\sigma} \in \mathbb{L}[X]$ $(1 \le \sigma \le s; 1 \le j_\sigma \le r_\sigma)$ depending on x_i. If $G_\sigma \equiv 0$ for all σ, then $V(G_\sigma) = \mathbb{R}$ for all σ and

$$\{x_i\} = \bigcup_{\sigma=1}^{s} \bigcap_{j_\sigma=1}^{r_\sigma} U(H_{\sigma,j_\sigma})\,,$$

which is an open set in $\mathbb{R}$ and therefore a contradiction. Hence,

$$G_\sigma \not\equiv 0 \qquad \text{and} \qquad G_\sigma(x_i) = 0$$

for some σ. Since $G_\sigma \in \mathbb{L}[X] \setminus \{0\}$, x_i is algebraic over $\mathbb{L}$. This holds for every $i = 1, \ldots, n$. $\square$

Proof of Lemma 2.3. For $j = 1, \ldots, n$ we set

$$P_j(X_1, \ldots, X_n) := f_j(X_1, \ldots, X_n, y_1, \ldots, y_n) \in \mathbb{K}(y_1, \ldots, y_n)[X_1, \ldots, X_n]\,.$$

The determinant condition in Lemma 2.3 together with the theorem on implicit functions imply that $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is an isolated zero of the system of equations

$$P_j(X_1, \ldots, X_n) = 0 \qquad (j = 1, \ldots, n).$$

Therefore, the conditions of Proposition 2.1 are fulfilled with $\mathbb{L} := \mathbb{K}(y_1, \ldots, y_n)$ and we conclude on

$$\operatorname{tr.deg}(\mathbb{L}(x_1, \ldots, x_n) : \mathbb{L}) = 0\,.$$

By the assumption, we have

$$\operatorname{tr.deg}(\mathbb{K}(x_1, \ldots, x_n) : \mathbb{K}) = n$$

and hence

$$\operatorname{tr.deg}(\mathbb{L}(x_1,\ldots,x_n):\mathbb{K})\geq n\,.$$

Applying the chain rule for transcendence degrees (Lemma 2.1) to the field extensions $\mathbb{K}\subseteq\mathbb{K}(y_1,\ldots,y_n)=\mathbb{L}\subseteq\mathbb{L}(x_1,\ldots,x_n)$, we get

$$\operatorname{tr.deg}(\mathbb{K}(y_1,\ldots,y_n):\mathbb{K})=n\,,$$

as desired. □

2.2. An algebraic independence criterion for functions in one variable

In this subsection we will prove the following lemma, which goes back to an oral communication with Professor Elsner and is referred to Lemma 2.3.

Lemma 2.5. *Let $\mathbb{K}$ be a field extension of $\mathbb{C}(z)$. Let $f_1(z),\ldots,f_n(z)$ be algebraic independent functions over $\mathbb{K}$ and $g_1(z),\ldots,g_n(z)$ satisfy the system of equations*

$$F_j\big(\,f_1(z),\ldots,f_n(z),g_1(z),\ldots,g_n(z)\,\big)\;\equiv\;0\qquad(j=1,\ldots,n)\,,$$

where $F_j(X_1,\ldots,X_n,Y_1,\ldots,Y_n)\in\mathbb{K}[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$ for $j=1,\ldots,n$. Assume that

$$\det\left(\frac{\partial F_j}{\partial X_i}\big(\,f_1(z),\ldots,f_n(z),g_1(z),\ldots,g_n(z)\,\big)\right)\not\equiv 0\,.$$

Then the functions $g_1(z),\ldots,g_n(z)$ are algebraically independent over $\mathbb{K}$.

Remark 2.2. The determinant occurring in Lemma 2.5 is a function in z belonging to the ring

$$\mathbb{K}[f_1(z),\ldots,f_n(z),g_1(z),\ldots,g_n(z)]\,.$$

As an immediate consequence of Lemma 2.5 we obtain the following corollary:

Corollary 2.2. *Let $\mathbb{K}$ be a field extension of $\mathbb{C}(z)$. Let $f_1(z),\ldots,f_n(z)$ be algebraic independent functions over $\mathbb{K}$ and let $g_j(z)=U_j\big(f_1(z),\ldots,f_n(z)\big)$, where $U_j(X_1,\ldots,X_n)\in\mathbb{K}[X_1,\ldots,X_n]$ for $j=1,\ldots,n$. Assume that*

$$\det\left(\frac{\partial U_j}{\partial X_i}\big(\,f_1(z),\ldots,f_n(z)\,\big)\right)\not\equiv 0\,.$$

Then the functions $g_1(z),\ldots,g_n(z)$ are algebraically independent over $\mathbb{K}$.

The main tool for the proof of Lemma 2.5 is an analogue to Proposition 2.1 from the preceding section, which, again, follows from [45, Ch.II, § 17, Corollary to Theorem 40]:

Proposition 2.2. *Let $\mathbb{L}$ be a field extension of $\mathbb{C}(z)$. For $j=1,\ldots,n$ let $P_j(X_1,\ldots,X_n)\in\mathbb{L}[X_1,\ldots,X_n]$. Moreover, let $f_1(z),\ldots,f_n(z)$ be functions satisfying*

$$P_j\big(f_1(z),\ldots,f_n(z)\big)\equiv 0\qquad(j=1,\ldots,n)\,.$$

Assume that

$$\det\left(\frac{\partial P_j}{\partial X_i}\big(\,f_1(z),\ldots,f_n(z)\,\big)\right)\not\equiv 0\,.$$

Then the functions $f_1(z),\ldots,f_n(z)$ are algebraic over $\mathbb{L}$.

Proof of Lemma 2.5. Let $\mathbb{L} = \mathbb{K}\big(g_1(z), \ldots, g_n(z)\big)$. For $j = 1, \ldots, n$ we set

$$P_j(X_1, \ldots, X_n) := F_j\big(X_1, \ldots, X_n, g_1(z), \ldots, g_n(z)\big) \in \mathbb{L}[X_1, \ldots, X_n]\,.$$

By the assumptions in Lemma 2.5 we have

$$P_j\big(f_1(z), \ldots, f_n(z)\big) = F_j\big(f_1(z), \ldots, f_n(z), g_1(z), \ldots, g_n(z)\big) \equiv 0 \qquad (j = 1, \ldots, n)\,,$$

and

$$\det\left(\frac{\partial P_j}{\partial X_i}\big(f_1(z), \ldots, f_n(z)\big)\right) = \det\left(\frac{\partial F_j}{\partial X_i}\big(f_1(z), \ldots, f_n(z), g_1(z), \ldots, g_n(z)\big)\right) \not\equiv 0\,.$$

Therefore, the conditions of Proposition 2.2 are fulfilled and we conclude on

$$\operatorname{tr.deg}\big(\mathbb{L}\big(f_1(z), \ldots, f_n(z)\big) : \mathbb{L}\big) = 0\,.$$

Using the assumption on the algebraic independence over $\mathbb{K}$ of the functions $f_1(z), \ldots, f_n(z)$, we get

$$\operatorname{tr.deg}\big(\mathbb{L}\big(f_1(z), \ldots, f_n(z)\big) : \mathbb{K}\big) \geq n\,.$$

Next, we apply the chain rule for transcendence degrees (Lemma 2.1) to the field extensions $\mathbb{K} \subseteq \mathbb{K}\big(g_1(z), \ldots, g_n(z)\big) = \mathbb{L} \subseteq \mathbb{L}\big(f_1(z), \ldots, f_n(z)\big)$ and obtain

$$\operatorname{tr.deg}\big(\mathbb{K}\big(g_1(z), \ldots, g_n(z)\big) : \mathbb{K}\big) = n\,.$$

This proves the lemma. □

3. Jacobian elliptic functions and the complete elliptic integrals

In this section we study the squares of the Jacobian elliptic functions $\operatorname{ns} z$, $\operatorname{nc} z$, $\operatorname{nd} z$ and $\operatorname{dn} z$. These functions satisfy

$$\operatorname{ns} z = \frac{1}{\operatorname{sn} z}\,, \qquad \operatorname{nc} z = \frac{1}{\sqrt{1-\operatorname{sn}^2 z}}\,,$$
$$\operatorname{dn} z = \sqrt{1-k^2\operatorname{sn}^2 z}\,, \qquad \operatorname{nd} z = \frac{1}{\operatorname{dn} z}\,,$$

where $w = \operatorname{sn} z$ is the inversion of the elliptic integral of the first kind, defined by

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\,.$$

There are a total of twelve Jacobian elliptic functions whereof these four will play a fundamental role in the proofs of the main theorems. We refer to [8] for the reader who is more interested in the theory of elliptic functions and integrals.

In order to express the numbers Φ_{2s}, Φ^*_{2s}, Ψ_{2s} and Ψ^*_{2s} in terms of K/π, E/π and k, it is necessary to compute the Laurent expansions for the functions $\operatorname{ns}^2 z$, $\operatorname{nc}^2 z$, $\operatorname{nd}^2 z$ and $\operatorname{dn}^2 z$.

3.1. Series expansions of the squared Jacobian elliptic functions

The lemmas in this subsection are taken from [18]. We present slightly different proofs anyhow, since some details will be used in the next subsection.

Lemma 3.1 ([18]). *The coefficients of the expansion*

$$\operatorname{ns}^2 z = \frac{1}{z^2} + \sum_{j=0}^{\infty} c_j z^{2j}$$

are given by

$$c_0 = \frac{1}{3}(1+k^2)\,, \quad c_1 = \frac{1}{15}(1-k^2+k^4)\,, \quad c_2 = \frac{1}{189}(1+k^2)(1-2k^2)(2-k^2)\,,$$

$$(j-2)(2j+3)c_j = 3\sum_{i=1}^{j-2} c_i c_{j-i-1} \qquad (j \ge 3)\,.$$

Proof. By [8, (128.01)], the function $w = \operatorname{sn} z$ is a solution of

$$(w')^2 = (1-w^2)(1-k^2w^2)\,, \quad w(0) = 0\,.$$

Hence, the function $u = \operatorname{ns}^2 z = w^{-2}$ satisfies

$$(u')^2 = 4w^{-6}(w')^2 = 4w^{-2}(w^{-2}-1)(w^{-2}-k^2),$$

namely

$$(u')^2 = 4u(u-1)(u-k^2) = 4u^3 - 4(1+k^2)u^2 + 4k^2u\,. \tag{3.1}$$

Differentiation of (3.1) and dividing by $2u'$ leads to

$$u'' = 6u^2 - 4(1+k^2)u + 2k^2.$$

Substituting $u = z^{-2} + \sum_{j=0}^{\infty} c_j z^{2j}$, we get

$$6z^{-4} + \sum_{j=0}^{\infty} 2j(2j-1)c_j z^{2j-2} = 6\left(\sum_{j=0}^{\infty} c_j z^{2j}\right)^2 + 12z^{-2}\sum_{j=0}^{\infty} c_j z^{2j} - 4(1+k^2)\sum_{j=0}^{\infty} c_j z^{2j} + 6z^{-4} - 4(1+k^2)z^{-2} + 2k^2.$$

Equating the coefficients of z^{-2} and the constant terms, we obatin $c_0 = (1+k^2)/3$ and $c_1 = (1-k^2+k^4)/15$. For $j \ge 2$ the coefficients of z^{2j-2} on both sides satisfy

$$2j(2j-1)c_j = 6\sum_{i=0}^{j-1} c_i c_{j-i-1} + 12c_j - 4(1+k^2)c_{j-1}.$$

Since $1+k^2 = 3c_0$, we have

$$(j-2)(2j+3)c_j = 3\sum_{i=0}^{j-1} c_i c_{j-i-1} - 6c_0c_{j-1} = 3\sum_{i=1}^{j-2} c_i c_{j-i-1}.$$

For $j = 2$ both sides vanish and c_2 is not uniquely determined. To compute c_2, substitute $u = z^{-2} + c_0 + c_1z^2 + c_2z^4 + \ldots$ in (3.1) and compare the constant terms. This yields $c_2 = (1+k^2)(1-2k^2)(2-k^2)/189$. Once c_0, c_1 and c_2 are known, the coefficients c_j $(j \ge 3)$ are uniquely determined. □

Lemma 3.2 ([18]). *The coefficients of the expansion*

$$(1-k^2)\,\mathrm{nd}^2 z = 1 - k^2 + \sum_{j=1}^{\infty} d_j z^{2j}$$

are given by

$$d_1 = k^2(1-k^2)\,, \quad d_2 = -\frac{1}{3}k^2(1-k^2)(1-2k^2)\,,$$

$$j(2j-1)d_1d_j = 6d_2d_{j-1} - 3d_1\sum_{i=1}^{j-2} d_i d_{j-i-1} \qquad (j \ge 3).$$

Proof. The function $w = \mathrm{dn}\, z$ satisfies

$$(w')^2 = (1-w^2)(w^2-(1-k^2))\,, \qquad w(0) = 1\,,$$

(see [8, (128.01)]). Then the function $u = (1-k^2)\,\mathrm{nd}^2 z = (1-k^2)w^{-2}$ is a solution of

$$(u')^2 = 4u(u-(1-k^2))(1-u) = -4u^3 + 4(2-k^2)u^2 - 4(1-k^2)u\,,$$

or, equivalently, of

$$u'' = -6u^2 + 4(2-k^2)u - 2(1-k^2), \qquad u(0) = 1-k^2, \qquad u'(0) = 0.$$

Substituting $u = 1 - k^2 + \sum_{j=1}^{\infty} d_j z^{2j}$ yields

$$\sum_{j=1}^{\infty} 2j(2j-1)d_j z^{2j-2} = -6\left((1-k^2)^2 + 2(1-k^2)\sum_{j=1}^{\infty} d_j z^{2j} + \left(\sum_{j=1}^{\infty} d_j z^{2j}\right)^2\right)$$
$$+ 4(2-k^2)\left(1-k^2+\sum_{j=1}^{\infty} d_j z^{2j}\right) - 2(1-k^2).$$

Equating coefficients, we obtain $d_1 = k^2(1-k^2)$ and for $j \geq 2$ we have

$$j(2j-1)d_j = -2(1-2k^2)d_{j-1} - 3\sum_{i=1}^{j-2} d_i d_{j-i-1}.$$

Observing $d_2 = -(1-2k^2)d_1/3$ and multiplying both sides with d_1 we get

$$j(2j-1)d_1 d_j = 6d_2 d_{j-1} - 3d_1 \sum_{i=1}^{j-2} d_i d_{j-i-1} \qquad (j \geq 3).$$

□

Lemma 3.3 ([18]). *The coefficients of the expansion*

$$(1-k^2)(\mathrm{nc}^2\, z - 1) = \sum_{j=1}^{\infty} e_j z^{2j}$$

are given by

$$e_1 = 1-k^2\,, \quad e_2 = \frac{1}{3}(1-k^2)(2-k^2)\,,$$
$$j(2j-1)e_1 e_j = 6e_2 e_{j-1} + 3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1} \qquad (j \geq 3).$$

Proof. In [8, (128.01)] we find that the function $w = \mathrm{cn}\, z$ satisfies

$$(w')^2 = (1-w^2)((1-k^2)+k^2w^2) \qquad w(0) = 1\,.$$

Therefore, the function

$$u = (1-k^2)(\mathrm{nc}^2\, z - 1) = (1-k^2)(w^{-2}-1)$$

is a solution of

$$(u')^2 = 4u(u+1)(u+1-k^2) = 4u^3 + 4(2-k^2)u^2 + 4(1-k^2)u$$

and hence of

$$u'' = 6u^2 + 4(2-k^2)u + 2(1-k^2)\,, \qquad u(0) = u'(0) = 0\,.$$

Substituting $u = \sum_{j=1}^{\infty} e_j z^{2j}$ leads to

$$\sum_{j=1}^{\infty} 2j(2j-1)e_j z^{2j-2} = 6\left(\sum_{j=1}^{\infty} e_j z^{2j}\right)^2 + 4(2-k^2)\sum_{j=1}^{\infty} e_j z^{2j} + 2(1-k^2).$$

Comparing the constant terms yields $e_1 = 1-k^2$. Moreover, for $j \geq 2$ we find

$$2j(2j-1)e_j = 4(2-k^2)e_{j-1} + 6\sum_{i=1}^{j-2} e_i e_{j-i-1}.$$

In particular $e_2 = (2-k^2)e_1/3$. Multiplying both sides with e_1 we obtain

$$j(2j-1)e_1 e_j = 6e_2 e_{j-1} + 3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1} \quad (j \geq 3).$$

□

Lemma 3.4 ([18]). *The coefficients of the expansion*

$$\mathrm{dn}^2\, z = 1 + \sum_{j=1}^{\infty} f_j z^{2j}$$

are given by

$$f_1 = -k^2\,, \quad f_2 = \frac{1}{3}k^2(1+k^2)\,,$$

$$j(2j-1)f_1 f_j = 6f_2 f_{j-1} - 3f_1 \sum_{i=1}^{j-2} f_i f_{j-i-1} \qquad (j \geq 3).$$

Proof. Since $w = \mathrm{dn}\, z$ is a solution of $(w')^2 = (1-w^2)(w^2-(1-k^2))$ with $w(0)=1$ (see [8, (128.01)]), the function $u = \mathrm{dn}^2\, z = w^2$ satisfies $(u')^2 = 4u(1-u)(u-(1-k^2))$ with $u(0) = 1$. Hence, the proof of this lemma is an analogue to that of Lemma 3.2. □

3.2. Expressions of $\Phi_{2s}, \Phi_{2s}^*, \Psi_{2s}$ and Ψ_{2s}^* in terms of $K/\pi, E/\pi$, and k

In addition to the series expansions from the preceding subsection we will need the expansions of the functions $\mathrm{cosec}^2\, z$ and $\sec^2 z$, given by

$$\mathrm{cosec}^2\, z = \frac{1}{z^2} + \sum_{j=0}^{\infty} a_j z^{2j}\,, \qquad a_j = \frac{(-1)^j(2j+1)2^{2j+2}B_{2j+2}}{(2j+2)!} \qquad (j \geq 0), \qquad (3.2)$$

and

$$\sec^2 z = \sum_{j=0}^{\infty} b_j z^{2j}\,, \qquad b_j = \frac{(-1)^j(2j+1)2^{2j+2}(2^{2j+2}-1)B_{2j+2}}{(2j+2)!} \qquad (j \geq 0). \tag{3.3}$$

Here $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, ... denote the Bernoulli numbers. The expansions (3.2) and (3.3) follow from the identities

$$\frac{\mathrm{d}}{\mathrm{d}z}\cot z = -\operatorname{cosec}^2 z\,, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\tan z = \sec^2 z\,,$$

and the formulas (4.3.67) and (4.3.70) in [1].

For brevity we omit the argument q in the notation of the q-series A_{2j+1}, B_{2j+1}, C_{2j+1}, and D_{2j+1} from now on.

With (1.2), (1.4), and (1.7) we obtain

$$\Phi_{2s} = \frac{1}{(2s-1)!}\sum_{j=0}^{s-1}\sigma_{s-j-1}(s)(A_{2j+1} - (-1)^s D_{2j+1}). \tag{3.4}$$

Zucker [46] proved the identities

$$\left(\frac{2K}{\pi}\right)^2 \mathrm{ns}^2\left(\frac{2Kx}{\pi}\right) = \frac{4K(K-E)}{\pi^2} + \operatorname{cosec}^2 x - 8\sum_{j=0}^{\infty}(-1)^j A_{2j+1}\frac{(2x)^{2j}}{(2j)!}\,,$$

$$\left(\frac{2K}{\pi}\right)^2 (1-k^2)\,\mathrm{nd}^2\left(\frac{2Kx}{\pi}\right) = \frac{4KE}{\pi^2} - 8\sum_{j=0}^{\infty}(-1)^j D_{2j+1}\frac{(2x)^{2j}}{(2j)!}\,.$$

Equating coefficients and using Lemma 3.1, Lemma 3.2, and (3.2) we get the following relations among the q-series A_{2j+1} and the coefficients c_j as well as among D_{2j+1} and d_j, respectively:

$$\left(\frac{2K}{\pi}\right)^{2j+2} c_j = a_j - (-1)^j\frac{2^{2j+3}}{(2j)!}A_{2j+1} \quad (j \geq 1),$$

$$\left(\frac{2K}{\pi}\right)^{2j+2} d_j = (-1)^{j+1}\frac{2^{2j+3}}{(2j)!}D_{2j+1} \quad (j \geq 1).$$

For $j = 0$ we have

$$A_1 = \frac{1}{24}\left(1 - \left(\frac{2K}{\pi}\right)^2\left(\frac{3E}{K} - 2 + k^2\right)\right), \quad D_1 = \frac{1}{24}\left(\frac{2K}{\pi}\right)^2\left(\frac{3E}{K} + 3k^2 - 3\right).$$

Equation (3.4) gives for even integers s the expression

$$\begin{aligned}
\Phi_{2s} &= \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)(A_1 - (-1)^s D_1) + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)(A_{2j+1} - (-1)^s D_{2j+1})\right) \\
&= \frac{1}{(2s-1)!}\left[-\frac{(s-1)!^2}{24}\left(1 - \left(\frac{2K}{\pi}\right)^2\left(\frac{6E}{K} - 5 + 4k^2\right)\right)\right. \\
&\quad \left. + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(a_j - \left(\frac{2K}{\pi}\right)^{2j+2}(c_j - d_j)\right)\right],
\end{aligned} \tag{3.5}$$

whereas for odd integers s we obtain

$$\Phi_{2s} = \frac{1}{(2s-1)!}\left[\frac{(s-1)!^2}{24}\left(1-\left(\frac{2K}{\pi}\right)^2(1-2k^2)\right)\right.$$
$$\left.+\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(a_j-\left(\frac{2K}{\pi}\right)^{2j+2}(c_j+d_j)\right)\right]. \tag{3.6}$$

The series Φ^*_{2s} can be expressed using (1.2), (1.4), and (1.8). For even s we find

$$\Phi^*_{2s} = \frac{-1}{(2s-1)!}\sum_{j=0}^{s-1}\sigma_{s-j-1}(s)(A_{2j+1}+(-1)^s D_{2j+1})$$
$$= \frac{1}{(2s-1)!}\left[\frac{(s-1)!^2}{24}\left(1-\left(\frac{2K}{\pi}\right)^2(1-2k^2)\right)\right.$$
$$\left.-\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(a_j-\left(\frac{2K}{\pi}\right)^{2j+2}(c_j+d_j)\right)\right], \tag{3.7}$$

and for odd integers s

$$\Phi^*_{2s} = \frac{1}{(2s-1)!}\left[-\frac{(s-1)!^2}{24}\left(1-\left(\frac{2K}{\pi}\right)^2\left(\frac{6E}{K}-5+4k^2\right)\right)\right.$$
$$\left.-\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(a_j-\left(\frac{2K}{\pi}\right)^{2j+2}(c_j-d_j)\right)\right] \tag{3.8}$$

holds.

Combining (1.3), (1.5), and (1.9) yields

$$\Psi_{2s} = \frac{1}{(2s-1)!}\sum_{j=0}^{s-1}\sigma_{s-j-1}(s)(C_{2j+1}-(-1)^s B_{2j+1}). \tag{3.9}$$

In [46] we find the relations

$$\left(\frac{2K}{\pi}\right)^2(1-k^2)\left(\mathrm{nc}^2\left(\frac{2Kx}{\pi}\right)-1\right) = -\frac{4KE}{\pi^2}+\sec^2 x+8\sum_{j=0}^{\infty}(-1)^j B_{2j+1}\frac{(2x)^{2j}}{(2j)!},$$
$$\left(\frac{2K}{\pi}\right)^2\mathrm{dn}^2\left(\frac{2Kx}{\pi}\right) = \frac{4KE}{\pi^2}+8\sum_{j=0}^{\infty}(-1)^j C_{2j+1}\frac{(2x)^{2j}}{(2j)!}.$$

We equate the coefficients and use the expansions from Lemma 3.3, Lemma 3.4, and (3.3) to get the formulas

$$\left(\frac{2K}{\pi}\right)^{2j+2} e_j = b_j + (-1)^j\frac{2^{2j+3}}{(2j)!}B_{2j+1} \quad (j\ge 1),$$
$$\left(\frac{2K}{\pi}\right)^{2j+2} f_j = (-1)^{j+1}\frac{2^{2j+3}}{(2j)!}C_{2j+1} \quad (j\ge 1).$$

For $j = 0$ we have

$$B_1 = \frac{1}{8}\left(\frac{4KE}{\pi^2} - 1\right), \quad C_1 = \frac{1}{8}\left(\left(\frac{2K}{\pi}\right)^2 - \frac{4KE}{\pi^2}\right).$$

From (3.9) we compute for even integers s the expression

$$\begin{aligned}\Psi_{2s} &= \frac{1}{(2s-1)!}\left(-(s-1)!^2(C_1 - B_1) + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)(C_{2j+1} - B_{2j+1})\right)\\ &= \frac{1}{(2s-1)!}\left[-\frac{(s-1)!^2}{8}\left(1 + \left(\frac{2K}{\pi}\right)^2\left(1 - \frac{2E}{K}\right)\right)\right.\\ &\quad \left.+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(b_j - \left(\frac{2K}{\pi}\right)^{2j+2}(e_j - f_j)\right)\right],\end{aligned} \tag{3.10}$$

and for odd integers s we get

$$\begin{aligned}\Psi_{2s} = \frac{1}{(2s-1)!}&\left[\frac{(s-1)!^2}{8}\left(-1 + \left(\frac{2K}{\pi}\right)^2\right)\right.\\ &\left.+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(\left(\frac{2K}{\pi}\right)^{2j+2}(e_j + f_j) - b_j\right)\right].\end{aligned} \tag{3.11}$$

Similar computations give

$$\begin{aligned}\Psi^*_{2s} = \frac{1}{(2s-1)!}&\left[-\frac{(s-1)!^2}{8}\left(-1 + \left(\frac{2K}{\pi}\right)^2\right)\right.\\ &\left.+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(\left(\frac{2K}{\pi}\right)^{2j+2}(e_j + f_j) - b_j\right)\right]\end{aligned} \tag{3.12}$$

if s is an even integer and

$$\begin{aligned}\Psi^*_{2s} = \frac{1}{(2s-1)!}&\left[\frac{(s-1)!^2}{8}\left(1 + \left(\frac{2K}{\pi}\right)^2\left(1 - \frac{2E}{K}\right)\right)\right.\\ &\left.+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(b_j - \left(\frac{2K}{\pi}\right)^{2j+2}(e_j - f_j)\right)\right]\end{aligned} \tag{3.13}$$

for odd integers s.

For abbreviation we shall introduce

$$\Theta^-_j := c_j - d_j, \quad \Theta^+_j := c_j + d_j, \quad \Lambda^-_j := e_j - f_j, \quad \Lambda^+_j := e_j + f_j, \qquad (j \ge 1), \tag{3.14}$$

where $\Theta^\pm_j, \Lambda^\pm_j \in \mathbb{Q}[k]$ are even polynomials in k. From Lemma 3.1 to Lemma 3.4 it is easily seen that

$$\deg_k \Theta^-_j = 2j + 2, \quad \deg_k \Theta^+_j \le 2j + 2, \quad \deg_k \Lambda^-_j \le 2j, \quad \deg_k \Lambda^+_j \le 2j. \tag{3.15}$$

The following lemma provides exact formulas for the degrees of Θ^+_j, Λ^-_j and Λ^+_j.

Lemma 3.5. *The polynomials defined by (3.14) satisfy*

$$\begin{array}{rcll} \deg_k \Theta_j^+ & = & 2j+2 & (j \geq 1)\,, \\ \deg_k \Lambda_j^- & = & 2j-2 & (j \geq 1)\,, \\ \deg_k \Lambda_j^+ & = & 2j & (j \geq 1)\,. \end{array}$$

Proof. From (3.14) we have

$$\Theta_j^+(k) = c_j(k) + d_j(k), \qquad (j \geq 1),$$

where $c_j(k)$ and $d_j(k)$ are the coefficients from the series expansions of $\mathrm{ns}^2(z,k)$ and $(1-k^2)\,\mathrm{nd}^2(z,k)$, respectively. Let $\lambda(p)$ denote the leading coefficient of a polynomial $p(k)$. Since $\deg_k c_j = \deg_k d_j = 2j+2$, we have to prove that

$$\lambda(c_j) + \lambda(d_j) \neq 0,$$

where $\lambda(c_j)$ and $\lambda(d_j)$ satisfy the recurrence formulas in Lemma 3.1 and Lemma 3.2, respectively. Since the leading coefficients of $c_j(k)$ satisfy the same recurrence formula with also the same initial conditions as the constant terms of $c_j(k)$, we have an explicit formula for $\lambda(c_j)$. Using the identity

$$\mathrm{ns}^2(z,0) = \mathrm{cosec}^2\, z\,,$$

we obtain with (3.2)

$$\lambda(c_j) = a_j = \frac{(-1)^j(2j+1)2^{2j+2}B_{2j+2}}{(2j+2)!} \qquad (j \geq 1)\,.$$

To conclude on $\deg_k \Theta_j^+ = 2j+2$ we will prove that

$$-\lambda(d_j) = b_j \qquad (j \geq 1)\,,$$

where $b_j = (2^{2j+2}-1)a_j$ are defined for $j \geq 1$ by (3.3). According to Lemma 3.2 we have

$$-\lambda(d_j) = \frac{1}{j(2j-1)}\left(-4\lambda(d_{j-1}) + 3\sum_{i=1}^{j-2}\lambda(d_i)\lambda(d_{j-i-1})\right) \qquad (j \geq 3)\,.$$

Moreover, for $k=0$ we have

$$\mathrm{nc}^2(z,0) = \sec^2 z\,,$$

and therefore Lemma 3.3 gives the expansion

$$\sec^2 z = 1 + \sum_{j=1}^{\infty} b_j z^{2j}$$

with

$$b_1 = 1\,, \qquad b_2 = \frac{2}{3}\,, \qquad b_j = \frac{1}{j(2j-1)}\left(4b_{j-1} + 3\sum_{i=1}^{j-2} b_i b_{j-i-1}\right) \qquad (j \geq 3)\,.$$

Hence, the numbers b_j satisfy the same recurrence formula as the numbers $-\lambda(d_j)$. Observing that the initial conditions

$$-\lambda(d_1) = 1 = b_1\,, \qquad -\lambda(d_2) = \frac{2}{3} = b_2$$

are fulfilled, we conclude on $-\lambda(d_j) = b_j$ for $j \geq 1$ and

$$\lambda(c_j) + \lambda(d_j) = a_j - b_j = (2 - 2^{2j+2})a_j \neq 0 \qquad (j \geq 1)\,.$$

This proves the first identity stated in the lemma.

Next, we obtain from Lemma 3.3 and Lemma 3.4 the identities $\deg_k e_j = \deg_k f_j = 2j$ as well as the recurrence formulas

$$\lambda(e_j) = -\frac{2}{j(2j-1)}\lambda(e_{j-1}) \qquad (j \geq 3)$$

and

$$\lambda(f_j) = -\frac{2}{j(2j-1)}\lambda(f_{j-1}) \qquad (j \geq 3)\,.$$

Hence, by induction on j it can be shown that

$$\lambda(e_j) = \lambda(f_j) = \frac{(-1)^j 2^{2j-1}}{(2j)!} \qquad (j \geq 1)\,. \tag{3.16}$$

We conclude on

$$\deg_k \Lambda_j^+ = 2j\,, \qquad \deg_k \Lambda_j^- \leq 2j-2 \qquad (j \geq 1)\,.$$

It remains to prove the second identity stated in the lemma. For this we denote by $\lambda_2(e_j)$ and $\lambda_2(f_j)$ the k^{2j-2}- coefficient of the polynomial e_j and f_j, respectively. From Lemma 3.3 and Lemma 3.4 we obtain the recurrence formulas

$$\lambda_2(e_j) = \frac{1}{j(2j-1)}\left(4\lambda(e_{j-1}) - 2\lambda_2(e_{j-1}) + 3\sum_{i=1}^{j-2}\lambda(e_i)\lambda(e_{j-i-1})\right) \qquad (j \geq 3) \tag{3.17}$$

and

$$\lambda_2(f_j) = \frac{1}{j(2j-1)}\left(-2\lambda(f_{j-1}) - 2\lambda_2(f_{j-1}) - 3\sum_{i=1}^{j-2}\lambda(f_i)\lambda(f_{j-i-1})\right) \qquad (j \geq 3)\,. \tag{3.18}$$

Using (3.16) we have

$$1 + 2\sum_{j=1}^{\infty}\lambda(e_j)z^{2j} = \cos(2z)\,.$$

We define

$$\lambda(e_0) := \frac{1}{2}$$

to obtain

$$\cos(2z) = 2\sum_{j=0}^{\infty} \lambda(e_j) z^{2j} .$$

Now we may compute the Cauchy product:

$$\begin{aligned}\cos^2(2z) &= 4\sum_{j=0}^{\infty}\sum_{i=0}^{j} \lambda(e_i)\lambda(e_{j-i}) z^{2j} = 4\sum_{j=1}^{\infty}\sum_{i=0}^{j-1} \lambda(e_i)\lambda(e_{j-i-1}) z^{2j-2} \\ &= 1 + 4\sum_{j=2}^{\infty}\sum_{i=0}^{j-1} \lambda(e_i)\lambda(e_{j-i-1}) z^{2j-2} . \end{aligned} \tag{3.19}$$

Otherwise we have

$$\cos^2(2z) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{4j-1}}{(2j)!} z^{2j} = 1 + \sum_{j=2}^{\infty} \frac{(-1)^{j-1} 2^{4j-5}}{(2j-2)!} z^{2j-2} , \tag{3.20}$$

which is the well-known power series expansion of $\cos^2(2z)$. Equating the coefficients in the right-hand sides of (3.19) and (3.20) yields

$$4\sum_{i=0}^{j-1} \lambda(e_i)\lambda(e_{j-i-1}) = \frac{(-1)^{j-1} 2^{4j-5}}{(2j-2)!}$$

and then with (3.16)

$$\sum_{i=1}^{j-2} \lambda(e_i)\lambda(e_{j-i-1}) = \frac{(-1)^{j-1} 2^{4j-7}}{(2j-2)!} - \lambda(e_{j-1}) = \frac{(-1)^{j-1} 2^{2j-3}(2^{2j-4}-1)}{(2j-2)!} . \tag{3.21}$$

Since $\lambda(e_j) = \lambda(f_j)$, we may substitute (3.21) into (3.17) as well as into (3.18) and obtain the recurrence formulas

$$\begin{aligned}\lambda_2(e_j) &= -\frac{2\lambda_2(e_{j-1})}{j(2j-1)} + \frac{(-1)^{j-1} 2^{2j-2}(3\cdot 2^{2j-4}+1)}{(2j)!} \qquad (j \geq 3) , \\ \lambda_2(f_j) &= -\frac{2\lambda_2(f_{j-1})}{j(2j-1)} + \frac{(-1)^{j} 2^{2j-2}(3\cdot 2^{2j-4}-1)}{(2j)!} \qquad (j \geq 3) .\end{aligned}$$

For the polynomial $\Lambda_j^- = e_j - f_j$ this gives

$$\lambda_2(\Lambda_j^-) = -\frac{2\lambda_2(\Lambda_{j-1}^-)}{j(2j-1)} + \frac{(-1)^{j-1} 3\cdot 2^{4j-5}}{(2j)!} \qquad (j \geq 3) ,$$

where $\lambda_2(\Lambda_j^-)$ denotes the k^{2j-2}- coefficient of the polynomial Λ_j^-. Hence, by induction on j, one proves

$$\lambda_2(\Lambda_j^-) = \frac{(-1)^{j-1} 2^{4j-3}}{(2j)!} \qquad (j \geq 1) ,$$

which is nonzero for $j \geq 1$. We conclude on

$$\deg_k \Lambda_j^- = 2j-2 \qquad (j \geq 1)$$

and the lemma is proven. □

For the proofs of the main theorems in this thesis we will need explicit formulas for certain coefficients of the polynomials $\Theta^\pm_j$ and $\Lambda^\pm_j$. Although we will only use the first two and the leading coefficients, we compute the third coefficients additionally, since the extra effort is not too large.

Lemma 3.6. *Let the polynomials $\Theta^\pm_j(k), \Lambda^\pm_j(k) \in \mathbb{Q}[k]$ $(j \geq 2)$ defined by (3.14) be written as*

$$\begin{array}{rclcccccccc} \Theta^-_{j-1}(k) &=& \alpha_{j,0} &+& \alpha_{j,1}k^2 &+& \alpha_{j,2}k^4 &+& \cdots &+& \alpha_{j,j}k^{2j}\,, \\ \Theta^+_{j-1}(k) &=& \beta_{j,0} &+& \beta_{j,1}k^2 &+& \beta_{j,2}k^4 &+& \cdots &+& \beta_{j,j}k^{2j}\,, \\ \Lambda^-_{j-1}(k) &=& \gamma_{j,0} &+& \gamma_{j,1}k^2 &+& \gamma_{j,2}k^4 &+& \cdots &+& \gamma_{j,j-2}k^{2j-4}\,, \\ \Lambda^+_{j-1}(k) &=& \delta_{j,0} &+& \delta_{j,1}k^2 &+& \delta_{j,2}k^4 &+& \cdots &+& \delta_{j,j-1}k^{2j-2}\,. \end{array}$$

Using (3.2) and (3.3) we have the following formulas for $j \geq 2$:

$$\begin{array}{llll} \alpha_{j,0} = a_{j-1}\,, & \alpha_{j,1} = \dfrac{(-1)^{j-1}2^{2j-3}}{(2j-2)!} - \dfrac{j}{2}\,a_{j-1}\,, \\ \beta_{j,0} = a_{j-1}\,, & \beta_{j,1} = \dfrac{(-1)^{j}2^{2j-3}}{(2j-2)!} - \dfrac{j}{2}\,a_{j-1}\,, \\ \gamma_{j,0} = b_{j-1}\,, & \gamma_{j,1} = \dfrac{(-1)^{j}2^{2j-3}}{(2j-2)!} - \dfrac{j}{2}\,b_{j-1}\,, \\ \delta_{j,0} = b_{j-1}\,, & \delta_{j,1} = \dfrac{(-1)^{j-1}2^{2j-3}}{(2j-2)!} - \dfrac{j}{2}\,b_{j-1}\,, \end{array}$$

$$\begin{array}{llll} \alpha_{j,2} = \dfrac{(-1)^{j-1}2^{2j-7}(7-8j-2^{2j-1})}{(2j-2)!} + \dfrac{j(4j-7)}{32}\,a_{j-1}\,, & \alpha_{j,j} = 2^{2j}a_{j-1}\,, \\ \beta_{j,2} = \dfrac{(-1)^{j-1}2^{2j-7}(-9+8j+2^{2j-1})}{(2j-2)!} + \dfrac{j(4j-7)}{32}\,a_{j-1}\,, & \beta_{j,j} = (2-2^{2j})a_{j-1}\,, \\ \gamma_{j,2} = \dfrac{(-1)^{j-1}2^{2j-7}(-7+8j-2^{2j-1})}{(2j-2)!} + \dfrac{j(4j-7)}{32}\,b_{j-1}\,, & \gamma_{j,j-2} = \dfrac{(-1)^{j}2^{4j-7}}{(2j-2)!}\,, \\ \delta_{j,2} = \dfrac{(-1)^{j-1}2^{2j-7}(9-8j+2^{2j-1})}{(2j-2)!} + \dfrac{j(4j-7)}{32}\,b_{j-1}\,, & \delta_{j,j-1} = \dfrac{(-1)^{j-1}2^{2j-2}}{(2j-2)!}\,. \end{array}$$

Proof. As in the proof of Lemma 3.1 we use the differential equation for the function $u = \mathrm{ns}^2(z,k)$, namely

$$(u')^2 = 4u(u-1)(u-k^2)\,, \tag{3.22}$$

differentiated with respect to z. For $k = 0$ (3.22) has the solution $u = \mathrm{ns}^2(z,0) = \mathrm{cosec}^2 z$. Therefore, we make the ansatz

$$u = \mathrm{ns}^2(z,k) = \mathrm{cosec}^2 z + f_1(z)k^2 + g_1(z)k^4 + \mathcal{O}(k^6).$$

Near $z = 0$ we have $\mathrm{ns}^2(z,k) = z^{-2} + \mathcal{O}(z)$. Hence, the functions $f_1(z)$ and $g_1(z)$ are analytic at $z = 0$. Substituting this into (3.22) and equating the coefficients of k^2 and k^4

yields

$$\sin z\cos z\cdot f_1' + (1+2\cos^2 z)\cdot f_1 = \cos^2 z \tag{3.23}$$

$$\begin{aligned}\sin z\cos z\cdot g_1' &+ (1+2\cos^2 z)\cdot g_1\\ &= \frac{1}{4}\sin^4 z\cdot (f_1')^2 + (\sin^4 z - 3\sin^2 z)\cdot f_1^2 + (2\sin^2 z-\sin^4 z)\cdot f_1.\end{aligned} \tag{3.24}$$

Since f_1 is bounded at $z=0$ we derive from (3.23)

$$\begin{aligned} f_1(z) &= \frac{\cos z}{2\sin^3 z}(z-\cos z\sin z) = -\frac{z}{4}\left(\frac{1}{\sin^2 z}\right)' + \frac{1}{2}\left(\frac{\cos z}{\sin z}\right)' + \frac{1}{2}\\ &= \frac{1}{2}(z(\cot z)')' - \frac{z}{4}(\cot z)'' + \frac{1}{2} = \frac{z}{4}(\cot z)'' + \frac{1}{2}(\cot z)' + \frac{1}{2}.\end{aligned}$$

We multiply (3.24) with $\tan^2 z$ and substitute

$$f_1(z) = \frac{\cos z}{2\sin^3 z}(z-\cos z\sin z) \quad \text{and} \quad f_1'(z) = \frac{3\sin z\cos z - 2z\cos^2 z - z}{2\sin^4 z}$$

into the resulting equation. Then we obtain

$$\left(\frac{\sin^3 z}{\cos z}g_1\right)' = \frac{1}{16}\left((z^2\tan z)' + 3(z^2\cot z)' + 4\sin^4 z - 3\right).$$

Since g_1 is also bounded at $z=0$ we determine

$$\frac{\sin^3 z}{\cos z}g_1 = \frac{1}{16}\left(z^2\tan z + 3z^2\cot z - \frac{3}{2}z - \frac{3}{2}\sin z\cos z - \sin^3 z\cos z\right).$$

This leads to

$$\begin{aligned} g_1(z) &= \frac{1}{16}\left(\frac{z^2}{\sin^2 z} + \frac{3z^2\cot^2 z}{\sin^2 z} - \frac{3}{2}\cdot\frac{z\cot z}{\sin^2 z} - \frac{3}{2}\cot^2 z - \cos^2 z\right)\\ &= \frac{1}{16}\left(z^2(-\cot z - \cot^3 z)' + \frac{3}{4}z(\cot^2 z)' - \frac{3}{2}\cot^2 z - \cos^2 z\right)\\ &= \frac{1}{16}\left(z^2\left(-\frac{\cot z}{\sin^2 z}\right)' + \frac{3}{4}z\left(\frac{1}{\sin^2 z}\right)' - \frac{3}{2}\left(\frac{1}{\sin^2 z}-1\right) - \cos^2 z\right)\\ &= \frac{1}{16}\left(\frac{z^2}{2}(\cot^2 z)'' - \frac{3}{4}z(\cot z)'' + \frac{3}{2}(\cot z)' + \frac{3}{2} - \cos^2 z\right)\\ &= \frac{1}{16}\left(-\frac{z^2}{2}(\cot z)''' - \frac{3}{4}z(\cot z)'' + \frac{3}{2}(\cot z)' + 1 - \frac{1}{2}\cos 2z\right).\end{aligned}$$

Analogue procedures for $\mathrm{nd}^2(z,k), \mathrm{nc}^2(z,k)$ and $\mathrm{dn}^2(z,k)$ (see also [21]) reveal for the expansions

$$\begin{aligned}\mathrm{nd}^2(z,k) &= 1 + f_2(z)k^2 + g_2(z)k^4 + \mathcal{O}(k^6)\\ \mathrm{nc}^2(z,k) &= \sec^2 z + f_3(z)k^2 + g_3(z)k^4 + \mathcal{O}(k^6)\\ \mathrm{dn}^2(z,k) &= 1 + f_4(z)k^2 + g_4(z)k^4 + \mathcal{O}(k^6)\end{aligned}$$

around $(z,k)=(0,0)$ the functions

$$
\begin{aligned}
f_2(z) &= \sin^2 z = -\frac{1}{2}(\cos 2z - 1)\,,\\
g_2(z) &= \frac{1}{4}\sin^4 z - \frac{1}{16}(8z\cos z - 11\sin z + \sin 3z)\sin z\\
&= \frac{1}{16}(\cos 4z - 4z\sin 2z - 8\cos 2z + 7)\,,\\
f_3(z) &= \frac{1}{2}\tan z(\tan z - z\,\sec^2 z) = -\frac{1}{4}(z(\tan z)'' - 2(\tan z)' + 2)\,,\\
g_3(z) &= \frac{1}{32}\left(2\cos^2 z - 15 + 13\sec^2 z - 4z^2\sec^2 z - 13z\tan z\,\sec^2 z + 6z^2\sec^4 z\right)\\
&= \frac{1}{64}(2z^2(\tan z)''' - 13z(\tan z)'' + 26(\tan z)' + 2\cos 2z - 28)\,,\\
f_4(z) &= -\sin^2 z = \frac{1}{2}(\cos 2z - 1)\,,\\
g_4(z) &= \frac{1}{4}\sin^4 z - \frac{1}{16}(-8z\cos z + 5\sin z + \sin 3z)\sin z\\
&= \frac{1}{16}(\cos 4z + 4z\sin 2z - 1)\,.
\end{aligned}
$$

By (3.14) we have the following identities:

$$
\frac{1}{z^2} + \frac{4k^2-2}{3} + \sum_{j=1}^{\infty}\Theta_j^-(k)z^{2j} = \mathrm{ns}^2(z,k) + (k^2-1)\mathrm{nd}^2(z,k)
$$

$$
= \mathrm{cosec}^2 z - 1 + (f_1(z) - f_2(z) + 1)k^2 + (g_1(z) - g_2(z) + f_2(z))k^4 + \mathcal{O}(k^6)\,, \tag{3.25}
$$

$$
\frac{1}{z^2} + \frac{4-2k^2}{3} + \sum_{j=1}^{\infty}\Theta_j^+(k)z^{2j} = \mathrm{ns}^2(z,k) + (1-k^2)\mathrm{nd}^2(z,k)
$$

$$
= \mathrm{cosec}^2 z + 1 + (f_1(z) + f_2(z) - 1)k^2 + (g_1(z) + g_2(z) - f_2(z))k^4 + \mathcal{O}(k^6)\,, \tag{3.26}
$$

$$
-1 + \sum_{j=1}^{\infty}\Lambda_j^-(k)z^{2j} = (1-k^2)(\mathrm{nc}^2(z,k) - 1) - \mathrm{dn}^2(z,k)
$$

$$
= \sec^2 z - 2 + (f_3(z) - \sec^2 z + 1 - f_4(z))k^2 + (g_3(z) - f_3(z) - g_4(z))k^4 + \mathcal{O}(k^6)\,, \tag{3.27}
$$

$$
1 + \sum_{j=1}^{\infty}\Lambda_j^+(k)z^{2j} = (1-k^2)(\mathrm{nc}^2(z,k) - 1) + \mathrm{dn}^2(z,k)
$$

$$
= \sec^2 z + (f_3(z) - \sec^2 z + 1 + f_4(z))k^2 + (g_3(z) - f_3(z) + g_4(z))k^4 + \mathcal{O}(k^6)\,. \tag{3.28}
$$

Hence, for $k=0$ we get by (3.2) and (3.3) for every $j \geq 2$

$$
\begin{aligned}
\alpha_{j,0} &= a_{j-1}\,,\\
\beta_{j,0} &= a_{j-1}\,,\\
\gamma_{j,0} &= b_{j-1}\,,\\
\delta_{j,0} &= b_{j-1}.
\end{aligned}
$$

Moreover, by equating the k^2-coefficients in (3.25) to (3.28) we obtain the formulas

$$\sum_{j=1}^{\infty} \alpha_{j+1,1}\, z^{2j} = f_1(z) - f_2(z) + 1 - \frac{4}{3}\,, \tag{3.29}$$

$$\sum_{j=1}^{\infty} \beta_{j+1,1}\, z^{2j} = f_1(z) + f_2(z) - 1 + \frac{2}{3}\,, \tag{3.30}$$

$$\sum_{j=1}^{\infty} \gamma_{j+1,1}\, z^{2j} = f_3(z) - \sec^2 z + 1 - f_4(z)\,, \tag{3.31}$$

$$\sum_{j=1}^{\infty} \delta_{j+1,1}\, z^{2j} = f_3(z) - \sec^2 z + 1 + f_4(z)\,. \tag{3.32}$$

The series expansions for the right-hand-sides of (3.29) to (3.32) can be computed as follows:

$$\begin{aligned}
f_1(z) - f_2(z) - \frac{1}{3} &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} 2^{2j-1}(2B_{2j+2} - 1)}{(2j)!}\, z^{2j}\,,\\
f_1(z) + f_2(z) - \frac{1}{3} &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} 2^{2j-1}(2B_{2j+2} + 1)}{(2j)!}\, z^{2j}\,,\\
f_3(z) - \sec^2 z + 1 - f_4(z) &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} 2^{2j-1}((2^{2j+3} - 2)B_{2j+2} + 1)}{(2j)!}\, z^{2j}\,,\\
f_3(z) - \sec^2 z + 1 + f_4(z) &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} 2^{2j-1}((2^{2j+3} - 2)B_{2j+2} - 1)}{(2j)!}\, z^{2j}\,.
\end{aligned}$$

Then, from (3.29) to (3.32) we have for $j \geq 2$

$$\begin{aligned}
\alpha_{j,1} &= \frac{(-1)^j 2^{2j-3}(2B_{2j} - 1)}{(2j-2)!}\,,\\
\beta_{j,1} &= \frac{(-1)^j 2^{2j-3}(2B_{2j} + 1)}{(2j-2)!}\,,\\
\gamma_{j,1} &= \frac{(-1)^j 2^{2j-3}((2^{2j+1} - 2)B_{2j} + 1)}{(2j-2)!}\,,\\
\delta_{j,1} &= \frac{(-1)^j 2^{2j-3}((2^{2j+1} - 2)B_{2j} - 1)}{(2j-2)!}\,.
\end{aligned}$$

Combining these identities with formulas (3.2) and (3.3) we get the desired formulas

$$\begin{aligned}
\alpha_{j,1} &= \frac{(-1)^{j-1}2^{2j-3}}{(2j-2)!} - \frac{j}{2}\,a_{j-1}\,,\\
\beta_{j,1} &= \frac{(-1)^{j}2^{2j-3}}{(2j-2)!} - \frac{j}{2}\,a_{j-1}\,,\\
\gamma_{j,1} &= \frac{(-1)^{j}2^{2j-3}}{(2j-2)!} - \frac{j}{2}\,b_{j-1}\,,\\
\delta_{j,1} &= \frac{(-1)^{j-1}2^{2j-3}}{(2j-2)!} - \frac{j}{2}\,b_{j-1}\,.
\end{aligned}$$

Equating the k^4-coefficients in (3.25) to (3.28) we find

$$\sum_{j=1}^{\infty} \alpha_{j+1,2}\, z^{2j} = g_1(z) - g_2(z) + f_2(z)\,, \tag{3.33}$$

$$\sum_{j=1}^{\infty} \beta_{j+1,2}\, z^{2j} = g_1(z) + g_2(z) - f_2(z)\,, \tag{3.34}$$

$$\sum_{j=1}^{\infty} \gamma_{j+1,2}\, z^{2j} = g_3(z) - f_3(z) - g_4(z)\,, \tag{3.35}$$

$$\sum_{j=1}^{\infty} \delta_{j+1,2}\, z^{2j} = g_3(z) - f_3(z) + g_4(z)\,. \tag{3.36}$$

For the right-hand-sides of (3.33) to (3.36) we compute

$$\begin{aligned}
&g_1(z) - g_2(z) + f_2(z)\\
&= \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-5}(2(4j-3)B_{2j+2} - 1 - 8j - 2^{2j+1})}{(2j)!}\, z^{2j}\,,\\
&g_1(z) + g_2(z) - f_2(z)\\
&= \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-5}(2(4j-3)B_{2j+2} - 1 + 8j + 2^{2j+1})}{(2j)!}\, z^{2j}\,,\\
&g_3(z) - f_3(z) - g_4(z)\\
&= \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-5}((2^{2j+3}-2)(4j-3)B_{2j+2} + 1 + 8j - 2^{2j+1})}{(2j)!}\, z^{2j}\,,\\
&g_3(z) - f_3(z) + g_4(z)\\
&= \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-5}((2^{2j+3}-2)(4j-3)B_{2j+2} + 1 - 8j + 2^{2j+1})}{(2j)!}\, z^{2j}\,.
\end{aligned}$$

From (3.33) to (3.36) it follows for $j \geq 2$

$$\begin{aligned}
\alpha_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}(2(4j-7)B_{2j} + 7 - 8j - 2^{2j-1})}{(2j-2)!}\,, \\
\beta_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}(2(4j-7)B_{2j} - 9 + 8j + 2^{2j-1})}{(2j-2)!}\,, \\
\gamma_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}((2^{2j+1}-2)(4j-7)B_{2j} - 7 + 8j - 2^{2j-1})}{(2j-2)!}\,, \\
\delta_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}((2^{2j+1}-2)(4j-7)B_{2j} + 9 - 8j + 2^{2j-1})}{(2j-2)!}\,.
\end{aligned}$$

Again we use the formulas (3.2) and (3.3) and obtain

$$\begin{aligned}
\alpha_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}(7 - 8j - 2^{2j-1})}{(2j-2)!} + \frac{j(4j-7)}{32}\,a_{j-1}\,, \\
\beta_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}(-9 + 8j + 2^{2j-1})}{(2j-2)!} + \frac{j(4j-7)}{32}\,a_{j-1}\,, \\
\gamma_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}(-7 + 8j - 2^{2j-1})}{(2j-2)!} + \frac{j(4j-7)}{32}\,b_{j-1}\,, \\
\delta_{j,2} &= \frac{(-1)^{j-1}2^{2j-7}(9 - 8j + 2^{2j-1})}{(2j-2)!} + \frac{j(4j-7)}{32}\,b_{j-1}\,.
\end{aligned}$$

The remaining formulas for $\alpha_{j,j}$, $\beta_{j,j}$, $\gamma_{j,j-2}$ and $\delta_{j,j-1}$ can be easily derived from the proof of Lemma 3.5. Hence, the lemma is proven. $\square$

4. Independence results for one-type subsets of Ω

The first general algebraic independence result on particular subsets of Ω can be found in [17]:

Theorem 4.1 ([17]). *Let s_1, s_2, s_3 be pairwise distinct positive integers. Then the numbers Φ_{2s_1}, Φ_{2s_2} and Φ_{2s_3} are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is even.*

According to this theorem we will study the sets $\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}$, $\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}$ and $\{\Psi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}$ in this section.

4.1. Results for the set $\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}$

Theorem 4.2. *Let s_1, s_2, s_3 be pairwise distinct positive integers. Then the numbers $\Phi^*_{2s_1}$, $\Phi^*_{2s_2}$ and $\Phi^*_{2s_3}$ are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is odd.*

The proof of Theorem 4.2 involves the distiction of several cases. At first we will study the case with three odd integers s_1, s_2, s_3.

Lemma 4.1. *Let $s_1 < s_2 < s_3$ be positive odd integers. Assume that*

$$s_3\Theta^-_{s_3-1}(\Theta^-_{s_2-1})' - s_2\Theta^-_{s_2-1}(\Theta^-_{s_3-1})' \neq 0 \tag{4.1}$$

as a polynomial in k. Then the numbers $\Phi^*_{2s_1}$, $\Phi^*_{2s_2}$, and $\Phi^*_{2s_3}$ are algebraically independent over $\mathbb{Q}$.

Remark 4.1. The condition (4.1) is equivalent to

$$\frac{(\Theta^-_{s_2-1})^{s_3}}{(\Theta^-_{s_3-1})^{s_2}} \notin \mathbb{Q}. \tag{4.2}$$

Proof. Assume contrary to the condition (4.1) that

$$s_3\Theta^-_{s_3-1}(\Theta^-_{s_2-1})' - s_2\Theta^-_{s_2-1}(\Theta^-_{s_3-1})' = 0$$

holds, which is equivalent to

$$s_3\frac{(\Theta^-_{s_2-1})'}{\Theta^-_{s_2-1}} = s_2\frac{(\Theta^-_{s_3-1})'}{\Theta^-_{s_3-1}}\,.$$

Since $\Theta^-_{s_2-1} = \Theta^-_{s_2-1}(k)$ and $\Theta^-_{s_3-1} = \Theta^-_{s_3-1}(k)$ are polynomials in k, we may compute the integrals

$$s_3\int_0^k \frac{(\Theta^-_{s_2-1}(t))'}{\Theta^-_{s_2-1}(t)}\,dt = s_2\int_0^k \frac{(\Theta^-_{s_3-1}(t))'}{\Theta^-_{s_3-1}(t)}\,dt\,.$$

This gives

$$\log\left|\frac{(\Theta^-_{s_2-1}(k))^{s_3}}{(\Theta^-_{s_3-1}(k))^{s_2}}\right| = \log\left|\frac{(\Theta^-_{s_2-1}(0))^{s_3}}{(\Theta^-_{s_3-1}(0))^{s_2}}\right| = \log\left|\frac{a^{s_3}_{s_2-1}}{a^{s_2}_{s_3-1}}\right|\,,$$

where the last equation follows from Lemma 3.6. With (3.2) we get

$$\frac{(\Theta^-_{s_2-1}(k))^{s_3}}{(\Theta^-_{s_3-1}(k))^{s_2}} = \frac{a^{s_3}_{s_2-1}}{a^{s_2}_{s_3-1}} \in \mathbb{Q}\,,$$

which is contrary to the condition (4.2). This proves Remark 4.1. □

Proof of Lemma 4.1. In (3.8) we replace the quantities $k, K/\pi, E/\pi$ by the independent variables X_1, X_2, X_3, respectively, and obtain for odd integers s the function

$$\begin{aligned}\Phi^*_{2s}(X_1,X_2,X_3) &= \frac{1}{(2s-1)!}\Bigg[-\frac{(s-1)!^2}{24}\left(1-(2X_2)^2\left(\frac{6X_3}{X_2}-5+4X_1^2\right)\right)\\ &\quad -\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(a_j-(2X_2)^{2j+2}\,\Theta^-_j\right)\Bigg].\end{aligned}$$

Here Θ^-_j, formally a polynomial in k, now denotes the corresponding function from $\mathbb{Q}[X_1]$. We compute the derivatives

$$\frac{\partial\Phi^*_{2s}}{\partial X_1}(k,X_2,E/\pi) = \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{3}(2X_2)^2\,k - \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{j+1}(2j)!}{2^{2j+3}}(2X_2)^{2j+2}\,(\Theta^-_j)'\Bigg], \tag{4.3}$$

$$\frac{\partial\Phi^*_{2s}}{\partial X_2}(k,X_2,E/\pi) = \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{6}\left(\frac{6E}{\pi}+2X_2(4k^2-5)\right) - \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{j+1}(2j)!(j+1)}{2^{2j+1}}(2X_2)^{2j+1}\,\Theta^-_j\Bigg], \tag{4.4}$$

$$\frac{\partial\Phi^*_{2s}}{\partial X_3}(k,X_2,E/\pi) = \frac{(s-1)!^2}{2(2s-1)!}(2X_2)\,. \tag{4.5}$$

We apply Corollary 2.1 with

$$\mathbb{K}=\mathbb{Q}\,,\qquad n=3\,,\qquad x_1=k\,,\qquad x_2=\frac{K}{\pi}\,,\qquad x_3=\frac{E}{\pi}\,,$$

and

$$U_j=\Phi^*_{2s_j}(X_1,X_2,X_3),\qquad y_j=\Phi^*_{2s_j}(k,K/\pi,E/\pi)\qquad (j=1,2,3).$$

For brevity we put

$$\phi^*_i(j)=\phi^*_i(j)(X_1,X_2,X_3):=\frac{\partial\Phi^*_{2s_j}}{\partial X_i}(X_1,X_2,X_3)\qquad (i,j=1,2,3).$$

Then we compute

$$\begin{aligned}
\Delta(X_1, X_2, X_3) :=& \det \begin{pmatrix} \phi_1^*(1) & \phi_1^*(2) & \phi_1^*(3) \\ \phi_2^*(1) & \phi_2^*(2) & \phi_2^*(3) \\ \phi_3^*(1) & \phi_3^*(2) & \phi_3^*(3) \end{pmatrix} \\
&= \Big(\phi_1^*(1)\phi_2^*(2)\phi_3^*(3) + \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) + \phi_1^*(3)\phi_2^*(1)\phi_3^*(2)\Big) \\
&\quad - \Big(\phi_1^*(3)\phi_2^*(2)\phi_3^*(1) + \phi_1^*(1)\phi_2^*(3)\phi_3^*(2) + \phi_1^*(2)\phi_2^*(1)\phi_3^*(3)\Big). \qquad (4.6)
\end{aligned}$$

We have to prove the determinant $\Delta(k, K/\pi, E/\pi)$ to be nonzero. For the following, let $\lambda(2X_2, f)$ denote the leading coefficient of the polynomial $f(X_1, X_2, X_3) \in \mathbb{Q}[X_1, X_2, X_3]$ with respect to the variable $2X_2$. From the formulas (4.3), (4.4) and (4.5), and with $\sigma_0(s) = 1$ we obtain

$$\begin{aligned}
\lambda(2X_2, \phi_1^*(u)) &= \frac{1}{(2s_u - 1)2^{2s_u+1}} (\Theta^-_{s_u-1})' , \\
\lambda(2X_2, \phi_2^*(v)) &= \frac{s_v}{(2s_v - 1)2^{2s_v-1}} \Theta^-_{s_v-1} , \\
\lambda(2X_2, \phi_3^*(w)) &= \frac{(s_w - 1)!^2}{2(2s_w - 1)!} . \qquad (4.7)
\end{aligned}$$

The maximum of

$$\deg_{X_2}(\phi_1^*(u)\phi_2^*(v)\phi_3^*(w)) = 2s_u + (2s_v - 1) + 1 = 2(s_u + s_v)$$

is attained when $(s_u, s_v) = (s_2, s_3)$ and $(s_u, s_v) = (s_3, s_2)$, since $s_1 < s_2 < s_3$. This implies, that the leading coefficient of $\Delta(k, X_2, E/\pi)$ satisfies

$$\begin{aligned}
\lambda(2X_2,&\Delta) \\
&= \lambda(2X_2, \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) - \phi_1^*(3)\phi_2^*(2)\phi_3^*(1)) \\
&= \frac{1}{(2s_2 - 1)2^{2s_2+1}} (\Theta^-_{s_2-1})' \cdot \frac{s_3}{(2s_3 - 1)2^{2s_3-1}} \Theta^-_{s_3-1} \cdot \frac{(s_1 - 1)!^2}{2(2s_1 - 1)!} \\
&\quad - \frac{1}{(2s_3 - 1)2^{2s_3+1}} (\Theta^-_{s_3-1})' \cdot \frac{s_2}{(2s_2 - 1)2^{2s_2-1}} \Theta^-_{s_2-1} \cdot \frac{(s_1 - 1)!^2}{2(2s_1 - 1)!} \\
&= \frac{(s_1 - 1)!^2}{2^{2(s_2+s_3)+1}(2s_2 - 1)(2s_3 - 1)(2s_1 - 1)!} \left(s_3\Theta^-_{s_3-1}(\Theta^-_{s_2-1})' - s_2\Theta^-_{s_2-1}(\Theta^-_{s_3-1})'\right) ,
\end{aligned}$$

which does not vanish as a polynomial in k by the assumption (4.1). Since the numbers $k, K/\pi, E/\pi$ are algebraically independent over $\mathbb{Q}$, we have $\Delta(k, K/\pi, E/\pi) \neq 0$, and therefore Lemma 4.1 is proven. □

In the next lemma, we replace the condition (4.1) by a simpler one. Recall the notation

$$\Theta^-_{j-1}(k) = \alpha_{j,0} + \alpha_{j,1}k^2 + \alpha_{j,2}k^4 + \cdots + \alpha_{j,j}k^{2j} \qquad (j \geq 2)$$

from Lemma 3.6.

Lemma 4.2. *Let $s_1 < s_2 < s_3$ be positive odd integers. Assume that*

$$\frac{s_3}{s_2} \neq \frac{\alpha_{s_2,0}\alpha_{s_3,1}}{\alpha_{s_3,0}\alpha_{s_2,1}} . \tag{4.8}$$

Then the numbers $\Phi^*_{2s_1}$, $\Phi^*_{2s_2}$, and $\Phi^*_{2s_3}$ are algebraically independent over $\mathbb{Q}$.

Proof. Using Lemma 4.1 we have to prove that the condition (4.1) or, equivalently, (4.2) is a consequence of (4.8). Suppose on the contrary, that (4.2) does not hold. Then for some rational number $r \neq 0$ we have

$$\left(\alpha_{s_2,0} + \alpha_{s_2,1}k^2 + \cdots + \alpha_{s_2,s_2}k^{2s_2}\right)^{s_3} = r\left(\alpha_{s_3,0} + \alpha_{s_3,1}k^2 + \cdots + \alpha_{s_3,s_3}k^{2s_3}\right)^{s_2} ,$$

or

$$\alpha^{s_3}_{s_2,0} + s_3\alpha_{s_2,1}\alpha^{s_3-1}_{s_2,0}k^2 + \cdots + \alpha^{s_3}_{s_2,s_2}k^{2s_2s_3} = r\left(\alpha^{s_2}_{s_3,0} + s_2\alpha_{s_3,1}\alpha^{s_2-1}_{s_3,0}k^2 + \cdots + \alpha^{s_2}_{s_3,s_3}k^{2s_2s_3}\right) .$$

Equating coefficients, we obtain

$$\alpha^{s_3}_{s_2,0} = r\alpha^{s_2}_{s_3,0} \qquad \text{and} \qquad s_3\alpha_{s_2,1}\alpha^{s_3-1}_{s_2,0} = rs_2\alpha_{s_3,1}\alpha^{s_2-1}_{s_3,0} .$$

From these equations we derive

$$\frac{s_3}{s_2} = \frac{\alpha_{s_2,0}\alpha_{s_3,1}}{\alpha_{s_3,0}\alpha_{s_2,1}} ,$$

which contradicts our hypothesis (4.8). □

Proposition 4.1. *Let s_1, s_2, s_3 be pairwise distinct odd positive integers. Then the numbers $\Phi^*_{2s_1}$, $\Phi^*_{2s_2}$, and $\Phi^*_{2s_3}$ are algebraically independent over $\mathbb{Q}$.*

For the proof of Proposition 4.1 we have to show that for odd positive integers $s_1 < s_2 < s_3$ the condition (4.8) from Lemma 4.2 is fulfilled. For this, we will use the formulas for $\alpha_{j,0}$ and $\alpha_{j,1}$ given by Lemma 3.6 and the inequalities stated in the following lemma. We remark that the numbers a_j defined by (3.2) are positive for every $j \geq 0$, since $(-1)^j B_{2j+2} = |B_{2j+2}|$.

Lemma 4.3 ([17])**.** *Let $j \geq k+2 \geq 4$ be integers. Then we have*

$$\frac{a_j}{a_k} > 4^{j-k}\frac{(2k)!}{(2j)!} .$$

Moreover, for every $j \geq 1$ we have

$$\frac{a_j}{a_{j-1}} > \frac{j+1}{2\pi^2 j} .$$

Proof. By (3.2) and the following inequalities for Bernoulli numbers (cf. [1, 23.1.15])

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})} \qquad (n \geq 1),$$

we have

$$\frac{(2j+1)2^{2j+3}}{(2\pi)^{2j+2}} \le a_j \le \frac{(2j+1)2^{2j+3}}{(2\pi)^{2j+2}(1-2^{-2j-1})} \qquad (j \ge 0),$$

which yields for any nonnegative integers j and k

$$\frac{a_j}{a_k} \ge \frac{2j+1}{2k+1} 4^{j-k}(2\pi)^{2k-2j}(1-2^{-2k-1})\,. \tag{4.9}$$

For $k = j-1$ we obtain

$$\frac{a_j}{a_{j-1}} > \frac{(2j+1)(1-2^{1-2j})}{(2j-1)\pi^2} \ge \frac{j+1}{2j\pi^2},$$

which is the second inequality stated in the lemma. Now, put $m := j-k \ge 2$, $k \ge 2$. Observing that $2\pi/(2k+3) < 2\pi/7 < 1-2^{-2k-1}$, we have

$$\begin{aligned}
\frac{(2k)!}{(2j)!} &= \frac{1}{(2k+1)\cdots(2j)} \le \frac{(2k+3)^2}{(2k+1)(2k+2)} \cdot \frac{1}{(2k+3)^{2m}} \\
&= \frac{(2k+3)^2}{(2k+2)(2k+2m+1)} \cdot \frac{2j+1}{2k+1} \cdot (2\pi)^{2k-2j} \cdot \left(\frac{2\pi}{2k+3}\right)^{2m} \\
&\le \frac{2j+1}{2k+1}(2\pi)^{2k-2j}\frac{2\pi}{7} < \frac{2j+1}{2k+1}(2\pi)^{2k-2j}(1-2^{-2k-1}).
\end{aligned}$$

Together with (4.9) we obtain the first inequality from the lemma. □

Proof of Proposition 4.1. Since s_1, s_2, s_3 are pairwise distinct we may assume that $s_1 < s_2 < s_3$. From the formulas for $\alpha_{j,0}$ and $\alpha_{j,1}$ given by Lemma 3.6 we derive

$$\begin{aligned}
\left|\frac{s_3}{s_2} - \frac{\alpha_{s_2,0}\alpha_{s_3,1}}{\alpha_{s_3,0}\alpha_{s_2,1}}\right| &= \left|\frac{s_3}{s_2} - \frac{-\frac{s_3}{2} + \frac{2^{2s_3-3}}{a_{s_3-1}(2s_3-2)!}}{-\frac{s_2}{2} + \frac{2^{2s_2-3}}{a_{s_2-1}(2s_2-2)!}}\right| = \left|\frac{s_3}{s_2} - \frac{-s_3 + \frac{2^{2s_3-2}}{a_{s_3-1}(2s_3-2)!}}{-s_2 + \frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}\right| \\
&= \left|\frac{s_3\frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!} - s_2\frac{2^{2s_3-2}}{a_{s_3-1}(2s_3-2)!}}{s_2\Big(-s_2 + \frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\Big)}\right| \\
&= \left|\frac{\frac{s_2}{a_{s_3-1}}\frac{2^{2s_2-2}}{(2s_2-2)!}}{s_2\Big(-s_2 + \frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\Big)}\right| \cdot \left|\frac{s_3}{s_2}\frac{a_{s_3-1}}{a_{s_2-1}} - 2^{2(s_3-s_2)}\frac{(2s_2-2)!}{(2s_3-2)!}\right|
\end{aligned}$$

The conditions of Lemma 4.3 are satisfied for $j = s_3-1$ and $k = s_2-1$, since $s_3 - s_2 \ge 2$ and $s_2 - 1 \ge 3-1 = 2$. Therefore, we conclude on

$$\frac{s_3}{s_2}\frac{a_{s_3-1}}{a_{s_2-1}} - 2^{2(s_3-s_2)}\frac{(2s_2-2)!}{(2s_3-2)!} > \frac{a_{s_3-1}}{a_{s_2-1}} - 4^{s_3-s_2}\frac{(2s_2-2)!}{(2s_3-2)!} > 0.$$

This gives

$$\left| \frac{s_3}{s_2} - \frac{\alpha_{s_2,0}\alpha_{s_3,1}}{\alpha_{s_3,0}\alpha_{s_2,1}} \right| > 0,$$

so that the condition (4.8) is fulfilled. Thus, Proposition 4.1 follows from Lemma 4.2. □

Until now, all the indices s_1, s_2, s_3 were assumed to be odd. For the proof of Theorem 4.2 it remains to discuss the cases in which at least one index is even.

Proof of Theorem 4.2. If s_1, s_2, s_3 are even, then it follows from (3.7) that

$$\Phi^*_{2s_1},\ \Phi^*_{2s_2},\ \Phi^*_{2s_3}\ \in \mathbb{Q}(k, K/\pi),$$

so that these numbers are algebraically dependent over $\mathbb{Q}$. We split the remaining cases into the following two parts:

Case 1: Two indices s_i are even, Case 2: Two indices s_i are odd.

For the conditions of Lemma 4.3 to be fulfilled we will at first investigate the cases with $2 \notin \{s_1, s_2, s_3\}$. Then without loss of generality we have the following two cases:

$$\begin{array}{llll} \text{Case 1:} & s_1 \geq 1 \quad \text{odd}, & 4 \leq s_2 < s_3 & \text{even}, \\ \text{Case 2:} & 1 \leq s_1 < s_2 \quad \text{odd}, & s_3 \geq 4 & \text{even}. \end{array}$$

In (3.7) we replace the numbers $k, K/\pi, E/\pi$ by the independent variables X_1, X_2, X_3, respectively. For even integers s we obtain the function

$$\begin{aligned} \Phi^*_{2s}(X_1, X_2, X_3) &= \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{24}\left(1 - (2X_2)^2\left(1 - 2X_1^2\right)\right) \\ &\quad - \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^j (2j)!}{2^{2j+3}} \left(a_j - (2X_2)^{2j+2}\,\Theta_j^+\right)\Bigg], \end{aligned} \tag{4.10}$$

where Θ_j^+ now denotes an element from $\mathbb{Q}[X_1]$. The derivatives are computed as

$$\begin{aligned} \frac{\partial \Phi^*_{2s}}{\partial X_1}(k, X_2, E/\pi) &= \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{6}(2X_2)^2\, k \\ &\quad - \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^{j+1}(2j)!}{2^{2j+3}} (2X_2)^{2j+2}\,(\Theta_j^+)'\Bigg], \end{aligned} \tag{4.11}$$

$$\begin{aligned} \frac{\partial \Phi^*_{2s}}{\partial X_2}(k, X_2, E/\pi) &= \frac{1}{(2s-1)!}\Bigg[-\frac{(s-1)!^2}{6} 2X_2\left(1 - 2k^2\right) \\ &\quad - \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^{j+1}(2j)!(j+1)}{2^{2j+1}} (2X_2)^{2j+1}\,\Theta_j^+\Bigg]. \end{aligned} \tag{4.12}$$

$$\frac{\partial \Phi^*_{2s}}{\partial X_3}(k, X_2, E/\pi) = 0. \tag{4.13}$$

Case 1: The determinant $\Delta(X_1, X_2, X_3)$ defined by (4.6) is simplified to

$$\Delta(X_1, X_2, X_3) = \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) - \phi_1^*(3)\phi_2^*(2)\phi_3^*(1),$$

since $\phi_3^*(2) = \phi_3^*(3) = 0$ from (4.13). With

$$\deg_{X_2}\big(\phi_1^*(2)\phi_2^*(3)\phi_3^*(1)\big) = \deg_{X_2}\big(\phi_1^*(3)\phi_2^*(2)\phi_3^*(1)\big) = 2(s_2 + s_3)$$

we obtain for even integers s_u, s_v

$$\lambda(2X_2, \phi_1^*(u)) = -\frac{1}{(2s_u - 1)2^{2s_u+1}}(\Theta^+_{s_u-1})',$$
$$\lambda(2X_2, \phi_2^*(v)) = -\frac{s_v}{(2s_v - 1)2^{2s_v-1}}\Theta^+_{s_v-1},$$

and $\lambda(2X_2, \phi_3^*(1))$ was already computed in (4.7). Hence, we get

$$\lambda(2X_2,\Delta) = \frac{(s_1 - 1)!^2}{2^{2(s_2+s_3)+1}(2s_2 - 1)(2s_3 - 1)(2s_1 - 1)!}\left(s_3\Theta^+_{s_3-1}(\Theta^+_{s_2-1})' - s_2\Theta^+_{s_2-1}(\Theta^+_{s_3-1})'\right).$$

Similar as in Remark 4.1, this leading coefficient does not vanish if and only if

$$\frac{(\Theta^+_{s_2-1})^{s_3}}{(\Theta^+_{s_3-1})^{s_2}} \notin \mathbb{Q}. \tag{4.14}$$

We recall the notation

$$\Theta^+_{j-1}(k) \;=\; \beta_{j,0} + \beta_{j,1}k^2 + \beta_{j,2}k^4 + \cdots + \beta_{j,j}k^{2j} \qquad (j \geq 2)$$

from Lemma 3.6. As in the proof of Lemma 4.2 it follows that the condition (4.14) results from

$$\frac{s_3}{s_2} \neq \frac{\beta_{s_2,0}\beta_{s_3,1}}{\beta_{s_3,0}\beta_{s_2,1}}.$$

We use the formulas for $\beta_{j,0}$ and $\beta_{j,1}$ given by Lemma 3.6. Finally, we may apply Lemma 4.3 with $j = s_3 - 1$ and $k = s_2 - 1$, since the conditions are fulfilled by $k \geq 4 - 1 = 3$ and $j - k = s_3 - s_2 \geq 2$. We conclude on (4.14) as in the proof of Proposition 4.1 from

$$\frac{s_3}{s_2} \neq \frac{\beta_{s_2,0}\beta_{s_3,1}}{\beta_{s_3,0}\beta_{s_2,1}} = \frac{s_3 - \dfrac{2^{2s_3-2}}{a_{s_3-1}(2s_3-2)!}}{s_2 - \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}.$$

Hence, we have proved that $\Phi^*_{2s_1}$, $\Phi^*_{2s_2}$ and $\Phi^*_{2s_3}$ are algebraically independent over $\mathbb{Q}$.

Case 2: The determinant $\Delta(X_1, X_2, X_3)$ takes the following form

$$\Delta(X_1, X_2, X_3) = \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) + \phi_1^*(3)\phi_2^*(1)\phi_3^*(2) - \phi_1^*(3)\phi_2^*(2)\phi_3^*(1) - \phi_1^*(1)\phi_2^*(3)\phi_3^*(2),$$

since $\phi_3^*(3) = 0$. We have

$$\begin{aligned} \deg_{X_2}\big(\phi_1^*(2)\phi_2^*(3)\phi_3^*(1)\big) &= \deg_{X_2}\big(\phi_1^*(3)\phi_2^*(2)\phi_3^*(1)\big) = 2(s_2+s_3), \\ \deg_{X_2}\big(\phi_1^*(3)\phi_2^*(1)\phi_3^*(2)\big) &= \deg_{X_2}\big(\phi_1^*(1)\phi_2^*(3)\phi_3^*(2)\big) = 2(s_1+s_3), \end{aligned}$$

hence, by the assumption of Case 2, $\deg_{X_2}\Delta = 2(s_2+s_3)$. This gives

$$\begin{aligned} &\lambda(2X_2,\Delta) \\ &\quad= \lambda(2X_2, \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) - \phi_1^*(3)\phi_2^*(2)\phi_3^*(1)) \\ &\quad= \frac{-(s_1-1)!^2}{2^{2(s_2+s_3)+1}(2s_2-1)(2s_3-1)(2s_1-1)!}\left(s_3\Theta^+_{s_3-1}(\Theta^-_{s_2-1})' - s_2\Theta^-_{s_2-1}(\Theta^+_{s_3-1})'\right). \end{aligned}$$

Assume that the right-hand side vanishes, namely

$$\frac{(\Theta^-_{s_2-1})^{s_3}}{(\Theta^+_{s_3-1})^{s_2}} \in \mathbb{Q}.$$

We express the polynomials $\Theta^-_{s_2-1}$ and $\Theta^+_{s_3-1}$ as in Lemma 3.6 and obtain

$$\frac{s_3}{s_2} = \frac{\alpha_{s_2,0}\beta_{s_3,1}}{\alpha_{s_2,1}\beta_{s_3,0}}. \tag{4.15}$$

Here, we may have $s_2 < s_3$, or $s_3 < s_2$. To handle all possible situations, we distinguish the following four cases:

Case 2.1:	$s_2 \le s_3 - 3$,	Case 2.2:	$s_2 \ge s_3 + 3$,
Case 2.3:	$s_2 = s_3 - 1$,	Case 2.4:	$s_2 = s_3 + 1$.

Case 2.1: As in the proof of Proposition 4.1 we get with odd s_2 and even s_3

$$\left|\frac{s_3}{s_2} - \frac{\alpha_{s_2,0}\beta_{s_3,1}}{\alpha_{s_2,1}\beta_{s_3,0}}\right| = \left|\frac{\dfrac{s_2}{a_{s_3-1}}\dfrac{2^{2s_2-2}}{(2s_2-2)!}}{s_2\Big(-s_2+\dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\Big)}\right|\cdot\left|\frac{s_3}{s_2}\frac{a_{s_3-1}}{a_{s_2-1}} - 2^{2(s_3-s_2)}\frac{(2s_2-2)!}{(2s_3-2)!}\right|.$$

We have $s_3 \ge 6$ and $s_3 - s_2 \ge 3$. Therefore, we may apply Lemma 4.3 with $j = s_3 - 1 \ge 5$ and $k = s_2 - 1 \ge 3 - 1 = 2$ which gives

$$\frac{s_3}{s_2}\frac{a_{s_3-1}}{a_{s_2-1}} - 2^{2(s_3-s_2)}\frac{(2s_2-2)!}{(2s_3-2)!} > \frac{a_{s_3-1}}{a_{s_2-1}} - 4^{s_3-s_2}\frac{(2s_2-2)!}{(2s_3-2)!} > 0\,,$$

and then

$$\left|\frac{s_3}{s_2} - \frac{\alpha_{s_2,0}\beta_{s_3,1}}{\alpha_{s_2,1}\beta_{s_3,0}}\right| > 0\,.$$

This contradicts the assumption (4.15).

Case 2.2: Here, we have $s_2 - s_3 \geq 3$ with $s_3 \geq 4$ and

$$\left| \frac{s_2}{s_3} - \frac{\alpha_{s_2,1}\beta_{s_3,0}}{\alpha_{s_2,0}\beta_{s_3,1}} \right| = \left| \frac{\dfrac{s_3}{a_{s_2-1}} \dfrac{2^{2s_3-2}}{(2s_3-2)!}}{s_3\Big(-s_3 + \dfrac{2^{2s_3-2}}{a_{s_3-1}(2s_3-2)!}\Big)} \right| \cdot \left| \frac{s_2}{s_3} \frac{a_{s_2-1}}{a_{s_3-1}} - 2^{2(s_2-s_3)} \frac{(2s_3-2)!}{(2s_2-2)!} \right| .$$

We apply Lemma 4.3 with $j = s_2 - 1$ and $k = s_3 - 1 \geq 4 - 1 = 3$ and obtain

$$\left| \frac{s_2}{s_3} - \frac{\alpha_{s_2,1}\beta_{s_3,0}}{\alpha_{s_2,0}\beta_{s_3,1}} \right| > 0 \,,$$

a contradiction to (4.15).

Case 2.3: Let $s := s_2 \geq 3$. Then equation (4.15) is written as

$$\frac{s+1}{s} = \frac{\alpha_{s,0}\beta_{s+1,1}}{\alpha_{s,1}\beta_{s+1,0}} = \frac{s + 1 - \dfrac{2^{2s}}{a_s(2s)!}}{s - \dfrac{2^{2s-2}}{a_{s-1}(2s-2)!}},$$

or, equivalently,

$$\frac{a_s}{a_{s-1}} = \frac{2}{(2s-1)(s+1)} \,. \tag{4.16}$$

From the second inequality in Lemma 4.3 we obtain

$$\frac{s+1}{2\pi^2 s} < \frac{2}{(2s-1)(s+1)} \,.$$

Obviously, for $s \geq 4$ this does not hold. Moreover, for $s = 3$ in (4.16) we have

$$\frac{7}{50} = \frac{a_3}{a_2} \neq \frac{2}{(6-1)(3+1)} = \frac{1}{10} \,.$$

Hence, also in this case the assumption (4.15) leads to a contradiction.

Case 2.4: Put $s := s_2 \geq 5$. Again, this leads to (4.16), which is impossible as shown in Case 2.3.

It remains to discuss the two cases with $2 \in \{s_1, s_2, s_3\}$, namely:

$$\begin{aligned} &\text{Case 1:} \quad s_1 \geq 1 \quad \text{odd}, \quad 2 = s_2 < s_3 \quad \text{even}, \\ &\text{Case 2:} \quad 1 \leq s_1 < s_2 \quad \text{odd}, \quad s_3 = 2 \,. \end{aligned}$$

Case 1: By (4.10) the function $\Phi^*_{2s_2} = \Phi^*_4$ takes the form

$$\begin{aligned} \Phi^*_4(X_1, X_2, X_3) &= \frac{1}{6}\left(\frac{1}{24}\big(1 - (2X_2)^2(1 - 2X_1^2)\big) + \frac{2}{2^5}\big(a_1 - (2X_2)^4\Theta_1^+\big)\right) \\ &= \frac{1}{144}\left(1 - (2X_2)^2(1 - 2X_1^2) + \frac{1}{10}\left(1 - (2X_2)^4(1 + 14X_1^2 - 14X_1^4)\right)\right) \,. \end{aligned}$$

The derivatives are written as

$$\frac{\partial \Phi_4^*}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{36}(2X_2)^2\left(k - \frac{1}{10}(2X_2)^2(7k - 14k^3)\right),$$
$$\frac{\partial \Phi_4^*}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{36}\left(2X_2(2k^2-1) - \frac{1}{10}\Big(2(2X_2)^3(1+14k^2-14k^4)\Big)\right),$$
$$\frac{\partial \Phi_4^*}{\partial X_3}(k, X_2, E/\pi) = 0\,.$$

Since s_2 and s_3 are even, we have $\phi_3^*(2) = \phi_3^*(3) = 0$ and get

$$\Delta(X_1, X_2, X_3) = \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) - \phi_1^*(3)\phi_2^*(2)\phi_3^*(1)\,,$$

$$\deg_{X_2}\big(\phi_1^*(2)\phi_2^*(3)\phi_3^*(1)\big) = 4 + (2s_3 - 1) + 1 = 2s_3 + 4\,,$$

$$\deg_{X_2}\big(\phi_1^*(3)\phi_2^*(2)\phi_3^*(1)\big) = 2s_3 + 3 + 1 = 2s_3 + 4\,,$$

$$\lambda(2X_2, \phi_1^*(2)\phi_2^*(3)\phi_3^*(1)) = \frac{7k-14k^3}{360}\cdot\frac{s_3(s_1-1)!^2}{(2s_3-1)2^{2s_3}(2s_1-1)!}\Theta_{s_3-1}^+\,,$$

$$\lambda(2X_2, \phi_1^*(3)\phi_2^*(2)\phi_3^*(1)) = \frac{1+14k^2-14k^4}{180}\cdot\frac{(s_1-1)!^2}{(2s_3-1)2^{2s_3+2}(2s_1-1)!}(\Theta_{s_3-1}^+)'\,.$$

Hence, the leading coefficient of Δ with respect to $2X_2$ satisfies

$$\lambda(2X_2, \Delta)$$
$$= \frac{(s_1-1)!^2}{360(2s_3-1)2^{2s_3}(2s_1-1)!}\left((7k-14k^3)s_3\Theta_{s_3-1}^+ - \left(\frac{1}{2} + 7k^2 - 7k^4\right)(\Theta_{s_3-1}^+)'\right),$$

and therefore vanishes if and only if

$$\frac{(\Theta_{s_3-1}^+)^2}{(1+14k^2-14k^4)^{s_3}} \in \mathbb{Q}\,.$$

We express the polynomial $\Theta_{s_3-1}^+$ as in Lemma 3.6 and obtain $\beta_{s_3,0}^2 = r$ und $\beta_{s_3,0}\beta_{s_3,1} = 7s_3 r$ for some nonvanishing rational number r. This gives

$$7s_3 = \frac{\beta_{s_3,1}}{\beta_{s_3,0}} = \frac{2^{2s_3-3}}{(2s_3-2)!a_{s_3-1}} - \frac{s_3}{2}$$
$$\Longleftrightarrow \qquad s_3 = \frac{2^{2s_3-2}}{15(2s_3-2)!a_{s_3-1}}\,. \tag{4.17}$$

First let $s_3 \geq 6$. Applying Lemma 4.3 with $j = s_3 - 1$ and $k = 3$ we find

$$a_{s_3-1} > a_3 4^{s_3-4}\frac{720}{(2s_3-2)!} = \frac{16}{15}4^{s_3-4}\frac{1}{(2s_3-2)!} = \frac{2^{2s_3-4}}{15(2s_3-2)!}\,.$$

Combining this with (4.17) yields

$$s_3 = \frac{2^{2s_3-2}}{15(2s_3-2)!a_{s_3-1}} < \frac{2^{2s_3-2}15(2s_3-2)!}{15(2s_3-2)!2^{2s_3-4}} = 4\,,$$

which contradicts the condition $s_3 \geq 6$. Hence, equation (4.17) does not hold for any $s_3 \geq 6$. It remains to investigate the case $s_3 = 4$, in which equation (4.17) is fulfilled. But, one easily computes that

$$\frac{(\Theta_3^+)^2}{(1+14k^2-14k^4)^4} \notin \mathbb{Q}\,.$$

Hence, Case 1 is done.

Case 2: We have $s_2 > s_1$. Therefore, with $\phi_3^*(3) = 0$ and

$$\begin{aligned}\deg_{X_2}\big(\phi_1^*(2)\phi_2^*(3)\phi_3^*(1)\big) &= \deg_{X_2}\big(\phi_1^*(3)\phi_2^*(2)\phi_3^*(1)\big) = 2s_2+4\,,\\ \deg_{X_2}\big(\phi_1^*(3)\phi_2^*(1)\phi_3^*(2)\big) &= \deg_{X_2}\big(\phi_1^*(1)\phi_2^*(3)\phi_3^*(2)\big) = 2s_1+4\,,\end{aligned}$$

we conclude on $\deg_{X_2}\Delta = 2s_2+4$. This gives

$$\begin{aligned}&\lambda(2X_2,\Delta)\\ &\quad= \lambda(2X_2, \phi_1^*(2)\phi_2^*(3)\phi_3^*(1) - \phi_1^*(3)\phi_2^*(2)\phi_3^*(1))\\ &\quad= \frac{-(s_1-1)!^2}{360(2s_2-1)2^{2s_2}(2s_1-1)!}\left(\left(\frac{1}{2}+7k^2-7k^4\right)(\Theta^-_{s_2-1})' - s_2\Theta^-_{s_2-1}(7k-14k^3)\right)\,.\end{aligned}$$

As in Case 1 we have $\lambda(2X_2,\Delta) = 0$ if and only if

$$\frac{(\Theta^-_{s_2-1})^2}{(1+14k^2-14k^4)^{s_2}} \in \mathbb{Q}\,.$$

We express $\Theta^-_{s_2-1}$ as in Lemma 3.6 and obtain $\alpha^2_{s_2,0} = r$ und $\alpha_{s_2,0}\alpha_{s_2,1} = 7s_2r$ for some nonvanishing rational number r. This leads to

$$\begin{aligned}7s_2 &= \frac{\alpha_{s_2,1}}{\alpha_{s_2,0}} = \frac{2^{2s_2-3}}{(2s_2-2)!a_{s_2-1}} - \frac{s_2}{2}\\ \iff\qquad s_2 &= \frac{2^{2s_2-2}}{15(2s_2-2)!a_{s_2-1}}.\end{aligned}\tag{4.18}$$

First let $s_2 \geq 7$. Lemma 4.3 with $j = s_2-1$ and $k = 3$ gives

$$a_{s_2-1} > \frac{2^{2s_2-4}}{15(2s_2-2)!}\,.$$

Together with (4.18) we obtain $s_2 < 4$, which contradicts the assumption $s_2 \geq 7$. Lastly it remains to investigate the cases $s_2 = 3$ and $s_2 = 5$. For $s_2 = 3$ equation (4.18) is written as

$$3 = \frac{2^4}{15\cdot 4!a_2} = \frac{2}{45}\cdot\frac{189}{2} = \frac{21}{5}\,,$$

which is a contradiction. For $s_2 = 5$ we have

$$5 = \frac{2^8}{15\cdot 8!a_4} = \frac{2}{4725}\cdot\frac{10395}{2} = \frac{11}{5}\,,$$

which is also wrong. Finally, Theorem 4.2 is proven. □

4.2. Results for the set $\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}$

Theorem 4.3. *Let s_1, s_2, s_3 be pairwise distinct positive integers. Then the numbers Ψ_{2s_1}, Ψ_{2s_2} and Ψ_{2s_3} are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is even.*

At first we shall treat the case when all of the integers s_1, s_2, s_3 are even.

Proposition 4.2. *Let s_1, s_2, s_3 be pairwise distinct even positive integers. Then the numbers Ψ_{2s_1}, Ψ_{2s_2}, and Ψ_{2s_3} are algebraically independent over $\mathbb{Q}$.*

Proof. Without loss of generality we may assume that $s_1 < s_2 < s_3$ holds.

In (3.10) we replace the numbers $k, K/\pi, E/\pi$ by the independent variables X_1, X_2, X_3 and obtain for even integers s the expression

$$\Psi_{2s}(X_1, X_2, X_3) = \frac{1}{(2s-1)!}\Bigg[-\frac{(s-1)!^2}{8}\left(1+(2X_2)^2\left(1-\frac{2X_3}{X_2}\right)\right) + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(b_j-(2X_2)^{2j+2}\Lambda_j^-\right)\Bigg].$$

We derive

$$\frac{\partial\Psi_{2s}}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{j+1}(2j)!}{2^{2j+3}}(2X_2)^{2j+2}(\Lambda_j^-)', \tag{4.19}$$

$$\frac{\partial\Psi_{2s}}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\Bigg[-\frac{(s-1)!^2}{2}\left(2X_2-\frac{2E}{\pi}\right) + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{j+1}(2j)!(j+1)}{2^{2j+1}}(2X_2)^{2j+1}\Lambda_j^-\Bigg], \tag{4.20}$$

$$\frac{\partial\Psi_{2s}}{\partial X_3}(k, X_2, E/\pi) = \frac{(s-1)!^2}{2(2s-1)!}(2X_2). \tag{4.21}$$

For brevity we put

$$\psi_i(j) = \psi_i(j)(X_1, X_2, X_3) := \frac{\partial\Psi_{2s_j}}{\partial X_i}(X_1, X_2, X_3) \qquad (i, j = 1, 2, 3).$$

Then we compute the determinant

$$\begin{aligned}\Delta(X_1, X_2, X_3) :&= \det\begin{pmatrix}\psi_1(1) & \psi_1(2) & \psi_1(3)\\ \psi_2(1) & \psi_2(2) & \psi_2(3)\\ \psi_3(1) & \psi_3(2) & \psi_3(3)\end{pmatrix}\\ &= \Big(\psi_1(1)\psi_2(2)\psi_3(3)+\psi_1(2)\psi_2(3)\psi_3(1)+\psi_1(3)\psi_2(1)\psi_3(2)\Big)\\ &\quad - \Big(\psi_1(3)\psi_2(2)\psi_3(1)+\psi_1(1)\psi_2(3)\psi_3(2)+\psi_1(2)\psi_2(1)\psi_3(3)\Big).\end{aligned} \tag{4.22}$$

We shall prove that this determinant does not vanish as a polynomial in X_1, X_2, X_3. From Corollary 2.1 we will then obtain that the numbers Ψ_{2s_1}, Ψ_{2s_2}, and Ψ_{2s_3} are algebraically independent over $\mathbb{Q}$.

The leading coefficients with respect to the variable $2X_2$ satisfy

$$\begin{aligned}\lambda(2X_2, \psi_1(u)) &= \frac{1}{(2s_u-1)2^{2s_u+1}}(\Lambda^-_{s_u-1})', \\ \lambda(2X_2, \psi_2(v)) &= \frac{s_v}{(2s_v-1)2^{2s_v-1}}\Lambda^-_{s_v-1}, \\ \lambda(2X_2, \psi_3(w)) &= \frac{(s_w-1)!^2}{2(2s_w-1)!}. \end{aligned} \tag{4.23}$$

From $s_1 < s_2 < s_3$ we see that the maximum of

$$\deg_{X_2}(\psi_1(u)\psi_2(v)\psi_3(w)) = 2s_u + (2s_v - 1) + 1 = 2(s_u + s_v)$$

is attained for $(s_u, s_v) = (s_2, s_3)$ and $(s_u, s_v) = (s_3, s_2)$. The leading coefficient of the polynomial $\Delta(k, X_2, E/\pi)$ turns out to be

$$\begin{aligned}\lambda(2X_2,\Delta) &= \lambda(2X_2, \psi_1(2)\psi_2(3)\psi_3(1) - \psi_1(3)\psi_2(2)\psi_3(1)) \\ &= \frac{(s_1-1)!^2}{2^{2(s_2+s_3)+1}(2s_2-1)(2s_3-1)(2s_1-1)!}\left(s_3\Lambda^-_{s_3-1}(\Lambda^-_{s_2-1})' - s_2\Lambda^-_{s_2-1}(\Lambda^-_{s_3-1})'\right),\end{aligned}$$

which is nonzero if and only if

$$s_3\Lambda^-_{s_3-1}(\Lambda^-_{s_2-1})' - s_2\Lambda^-_{s_2-1}(\Lambda^-_{s_3-1})' \neq 0. \tag{4.24}$$

The condition (4.24) is equivalent to

$$\frac{(\Lambda^-_{s_2-1})^{s_3}}{(\Lambda^-_{s_3-1})^{s_2}} \notin \mathbb{Q}, \tag{4.25}$$

which can be deduced in the same way as shown in the proof of Remark 4.1. From Lemma 3.5 we obtain

$$\begin{aligned}\deg_k(\Lambda^-_{s_2-1})^{s_3} &= 2s_2s_3 - 4s_3, \\ \deg_k(\Lambda^-_{s_3-1})^{s_2} &= 2s_2s_3 - 4s_2.\end{aligned}$$

Since $s_2 < s_3$, the degrees of numerator and denominator in (4.25) are different and therefore (4.25) holds. This proves the proposition. □

For the proof of Theorem 4.3 it remains to discuss the cases in which at least one of the indices s_1, s_2, s_3 is odd. We will need the following lemma on the numbers defined by (3.3).

Lemma 4.4. *Let $j \geq k + 2 \geq 4$ be integers. Then we have*

$$\frac{b_j}{b_k} > 4^{j-k}\frac{(2k)!}{(2j)!}.$$

Moreover, for every $j \geq 1$ *we have*

$$\frac{b_j}{b_{j-1}} > \frac{j+1}{2\pi^2 j} \,.$$

Proof. Both of the inequalities follow immediately from Lemma 4.3 by the relation

$$b_j = (2^{2j+2} - 1) a_j \,.$$

For $j > k$ this gives

$$\frac{b_j}{b_k} = \frac{2^{2j+2}-1}{2^{2k+2}-1} \cdot \frac{a_j}{a_k} > \frac{a_j}{a_k} \,,$$

which proves the lemma. □

Proof of Theorem 4.3. If s_1, s_2, s_3 are odd, then it follows immediately from (3.11) that

$$\Psi_{2s_1},\ \Psi_{2s_2},\ \Psi_{2s_3}\ \in \mathbb{Q}(k, K/\pi),$$

so that these numbers are algebraically dependent over $\mathbb{Q}$ in this case. The remaining cases can be splitted in the following two parts:

Case 1: Two indices s_i are odd, Case 2: Two indices s_i are even.

In (3.11) we replace the numbers $k, K/\pi, E/\pi$ by the independent variables X_1, X_2, X_3, respectively, and obtain for odd integers s the function

$$\begin{aligned}\Psi_{2s}(X_1, X_2, X_3) = & \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{8}\left(-1 + (2X_2)^2\right) \\ & + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^j (2j)!}{2^{2j+3}} \left((2X_2)^{2j+2}\, \Lambda_j^+ - b_j\right)\Bigg] .\end{aligned} \tag{4.26}$$

We compute the derivatives

$$\frac{\partial \Psi_{2s}}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{(2s-1)!} \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^j (2j)!}{2^{2j+3}} (2X_2)^{2j+2} (\Lambda_j^+)' \,, \tag{4.27}$$

$$\begin{aligned}\frac{\partial \Psi_{2s}}{\partial X_2}(k, X_2, E/\pi) = & \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{2}(2X_2) \\ & + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^j (2j)!(j+1)}{2^{2j+1}} (2X_2)^{2j+1}\, \Lambda_j^+\Bigg] ,\end{aligned} \tag{4.28}$$

$$\frac{\partial \Psi_{2s}}{\partial X_3}(k, X_2, E/\pi) = 0 \,. \tag{4.29}$$

To fulfill the conditions of Lemma 4.4 we will at first treat the cases where $1 \notin \{s_1, s_2, s_3\}$. Without loss of generality we have the following two cases:

Case 1: $3 \leq s_1 < s_2$ odd, $s_3 \geq 2$ even,
Case 2: $s_1 \geq 3$ odd, $2 \leq s_2 < s_3$ even.

Case 1: Since $\psi_3(1) = \psi_3(2) = 0$ by (4.29) we obtain

$$\Delta(X_1, X_2, X_3) = \psi_1(1)\psi_2(2)\psi_3(3) - \psi_1(2)\psi_2(1)\psi_3(3)\,.$$

For odd integers s_u, s_v we have

$$\begin{aligned}\lambda(2X_2, \psi_1(u)) &= \frac{1}{(2s_u-1)2^{2s_u+1}}(\Lambda^+_{s_u-1})',\\ \lambda(2X_2, \psi_2(v)) &= \frac{s_v}{(2s_v-1)2^{2s_v-1}}\Lambda^+_{s_v-1}.\end{aligned}$$

and $\lambda(2X_2, \psi_3(3))$ was already computed in (4.23). Hence, we get

$$\begin{aligned}&\lambda(2X_2,\Delta)\\ &\quad= \frac{(s_3-1)!^2}{2^{2(s_1+s_2)+1}(2s_1-1)(2s_2-1)(2s_3-1)!}\left(s_2\Lambda^+_{s_2-1}(\Lambda^+_{s_1-1})' - s_1\Lambda^+_{s_1-1}(\Lambda^+_{s_2-1})'\right).\end{aligned}$$

This leading coefficient is nonzero if and only if

$$\frac{(\Lambda^+_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}} \notin \mathbb{Q}\,. \tag{4.30}$$

By Lemma 3.5 we have

$$\begin{aligned}\deg_k(\Lambda^+_{s_1-1})^{s_2} &= 2s_1s_2 - 2s_2\,,\\ \deg_k(\Lambda^+_{s_2-1})^{s_1} &= 2s_1s_2 - 2s_1\,.\end{aligned}$$

Since $s_1 < s_2$, condition (4.30) is fulfilled. This proves that Ψ_{2s_1}, Ψ_{2s_2}, and Ψ_{2s_3} are algebraically independent over $\mathbb{Q}$ in this case.

Case 2: By $\psi_3(1) = 0$ from (4.29) we have

$$\Delta(X_1, X_2, X_3) = \psi_1(3)\psi_2(1)\psi_3(2) + \psi_1(1)\psi_2(2)\psi_3(3) - \psi_1(1)\psi_2(3)\psi_3(2) - \psi_1(2)\psi_2(1)\psi_3(3).$$

Moreover, we compute

$$\begin{aligned}\deg_{X_2}\big(\psi_1(3)\psi_2(1)\psi_3(2)\big) &= \deg_{X_2}\big(\psi_1(1)\psi_2(3)\psi_3(2)\big) = 2(s_1+s_3),\\ \deg_{X_2}\big(\psi_1(1)\psi_2(2)\psi_3(3)\big) &= \deg_{X_2}\big(\psi_1(2)\psi_2(1)\psi_3(3)\big) = 2(s_1+s_2),\end{aligned}$$

and, by the conditions of Case 2, $\deg_{X_2}\Delta = 2(s_1+s_3)$. This gives

$$\begin{aligned}&\lambda(2X_2,\Delta)\\ &\quad= \lambda(2X_2, \psi_1(3)\psi_2(1)\psi_3(2) - \psi_1(1)\psi_2(3)\psi_3(2))\\ &\quad= \frac{(s_2-1)!^2}{2^{2(s_1+s_3)+1}(2s_1-1)(2s_3-1)(2s_2-1)!}\left(s_1\Lambda^+_{s_1-1}(\Lambda^-_{s_3-1})' - s_3\Lambda^-_{s_3-1}(\Lambda^+_{s_1-1})'\right).\end{aligned}$$

The right-hand side is nonzero if and only if

$$\frac{(\Lambda^-_{s_3-1})^{s_1}}{(\Lambda^+_{s_1-1})^{s_3}} \notin \mathbb{Q}\,. \tag{4.31}$$

Here, we apply Lemma 3.5 and obtain

$$\deg_k(\Lambda^-_{s_3-1})^{s_1} = 2s_1s_3 - 4s_1\,,$$
$$\deg_k(\Lambda^+_{s_1-1})^{s_3} = 2s_1s_3 - 2s_3\,.$$

Therefore, (4.31) holds for $s_3 \neq 2s_1$.

Expressing the polynomials $\Lambda^-_{s_3-1}$ and $\Lambda^+_{s_1-1}$ as in Lemma 3.6 and assuming $(\Lambda^-_{s_3-1})^{s_1} = r \cdot (\Lambda^+_{s_1-1})^{s_3}$ for some nonvanishing $r \in \mathbb{Q}$ we obtain

$$\frac{s_3}{s_1} = \frac{\delta_{s_1,0}\gamma_{s_3,1}}{\delta_{s_1,1}\gamma_{s_3,0}}\,.$$

In the case $s_3 = 2s_1$ this is equivalent to

$$2 = \frac{\delta_{s_1,0}\gamma_{2s_1,1}}{\delta_{s_1,1}\gamma_{2s_1,0}}\,. \tag{4.32}$$

We use Lemma 3.6 and derive

$$2 = \frac{b_{s_1-1}\left(\dfrac{2^{4s_1-3}}{(4s_1-2)!} - s_1 b_{2s_1-1}\right)}{b_{2s_1-1}\left(\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}b_{s_1-1}\right)} = \frac{\dfrac{2^{4s_1-3}}{b_{2s_1-1}(4s_1-2)!} - s_1}{\dfrac{2^{2s_1-3}}{b_{s_1-1}(2s_1-2)!} - \dfrac{s_1}{2}}\,,$$

which gives

$$\frac{(4s_1-2)!\,b_{2s_1-1}}{(2s_1-2)!\,b_{s_1-1}} = 2^{2s_1-1}\,. \tag{4.33}$$

Now we may apply Lemma 4.4 with $j = 2s_1 - 1$ and $k = s_1 - 1$ and obtain

$$\frac{(4s_1-2)!\,b_{2s_1-1}}{(2s_1-2)!\,b_{s_1-1}} > 4^{s_1}\,,$$

a contradiction to (4.33). Hence, (4.32) does not hold and this case is also solved.

It remains to treat the two cases with $1 \in \{s_1, s_2, s_3\}$:

Case 1: $s_1 = 1 < s_2$ odd, $s_3 \geq 2$ even,
Case 2: $s_1 = 1\,,\quad 2 \leq s_2 < s_3$ even.

Case 1: By (4.26) the function $\Psi_{2s_1} = \Psi_2$ takes the form

$$\Psi_2(X_1, X_2, X_3) = \frac{1}{8}\big(-1 + (2X_2)^2\big)\,.$$

The derivatives are

$$\frac{\partial \Psi_2}{\partial X_1}(k, X_2, E/\pi) = 0\,,$$
$$\frac{\partial \Psi_2}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{2}(2X_2)\,,$$
$$\frac{\partial \Psi_2}{\partial X_3}(k, X_2, E/\pi) = 0\,.$$

Since $\psi_1(1) = \psi_3(1) = \psi_3(2) = 0$, we get the following simple expression for the determinant Δ:

$$\Delta(X_1, X_2, X_3) = -\psi_1(2)\psi_2(1)\psi_3(3).$$

All the factors $\psi_1(2)$, $\psi_2(1)$, and $\psi_3(3)$ have nonvanishing leading coefficients with respect to the variable $2X_2$. Therefore, the polynomial $\Delta(X_1, X_2, X_3)$ does not vanish and with Corollary 2.1 we conclude on the algebraic independence of Ψ_{2s_1}, Ψ_{2s_2}, and Ψ_{2s_3} over $\mathbb{Q}$.

Case 2: Here we have $\psi_1(1) = \psi_3(1) = 0$, and we find

$$\Delta(X_1, X_2, X_3) = \psi_1(3)\psi_2(1)\psi_3(2) - \psi_1(2)\psi_2(1)\psi_3(3)\,.$$

Moreover, we compute

$$\begin{aligned} \deg_{X_2}\big(\psi_1(3)\psi_2(1)\psi_3(2)\big) &= 2s_3 + 2, \\ \deg_{X_2}\big(\psi_1(2)\psi_2(1)\psi_3(3)\big) &= 2s_2 + 2\,. \end{aligned}$$

Observing $s_3 > s_2$ we find

$$\lambda(2X_2, \Delta) = \lambda(2X_2, \psi_1(3)\psi_2(1)\psi_3(2)) = \frac{(s_2-1)!^2}{2^{2s_3+3}(2s_3-1)(2s_2-1)!}(\Lambda_{s_3-1}^-)'\,.$$

From $\deg_k \Lambda_{s_3-1}^- \geq 4$ we conclude on $(\Lambda_{s_3-1}^-)' \not\equiv 0$. Hence, $\Delta(X_1, X_2, X_3) \not\equiv 0$ and this completes the proof of Theorem 4.3. □

4.3. Results for the set $\{\Psi_{2s_1}^*, \Psi_{2s_2}^*, \Psi_{2s_3}^*\}$

Theorem 4.4. *Let s_1, s_2, s_3 be pairwise distinct positive integers. Then the numbers $\Psi_{2s_1}^*$, $\Psi_{2s_2}^*$ and $\Psi_{2s_3}^*$ are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is odd.*

At first we will study the case when all integers s_1, s_2, s_3 are odd.

Proposition 4.3. *Let s_1, s_2, s_3 be pairwise distinct odd positive integers. Then the numbers $\Psi_{2s_1}^*$, $\Psi_{2s_2}^*$, and $\Psi_{2s_3}^*$ are algebraically independent over $\mathbb{Q}$.*

Proof. Without loss of generality we may assume that $s_1 < s_2 < s_3$ holds.

We replace the numbers $k, K/\pi, E/\pi$ in (3.13) by the independent variables X_1, X_2, X_3, respectively, and obtain for odd integers s the function

$$\begin{aligned} \Psi_{2s}^*(X_1, X_2, X_3) = \frac{1}{(2s-1)!}\Bigg[&\frac{(s-1)!^2}{8}\left(1 + (2X_2)^2\left(1 - \frac{2X_3}{X_2}\right)\right) \\ &+ \sum_{j=1}^{s-1} \sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\left(b_j - (2X_2)^{2j+2}\,\Lambda_j^-\right)\Bigg]\,. \end{aligned}$$

The derivatives are

$$\frac{\partial \Psi^*_{2s}}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\sum_{j=1}^{s-1} \sigma_{s-j-1}(s)\frac{(-1)^{j+1}(2j)!}{2^{2j+3}}\,(2X_2)^{2j+2}\,(\Lambda_j^-)'\,, \tag{4.34}$$

$$\frac{\partial \Psi^*_{2s}}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\left[\frac{(s-1)!^2}{2}\left(2X_2 - \frac{2E}{\pi}\right) + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s)\frac{(-1)^{j+1}(2j)!(j+1)}{2^{2j+1}}\,(2X_2)^{2j+1}\,\Lambda_j^-\right], \tag{4.35}$$

$$\frac{\partial \Psi^*_{2s}}{\partial X_3}(k, X_2, E/\pi) = -\frac{(s-1)!^2}{2(2s-1)!}\,(2X_2)\,. \tag{4.36}$$

We denote

$$\psi_i^*(j) = \psi_i^*(j)(X_1, X_2, X_3) := \frac{\partial \Psi^*_{2s_j}}{\partial X_i}(X_1, X_2, X_3) \qquad (i, j = 1, 2, 3)\,.$$

Then we have

$$\begin{aligned}\Delta(X_1, X_2, X_3) :&= \det\begin{pmatrix} \psi_1^*(1) & \psi_1^*(2) & \psi_1^*(3) \\ \psi_2^*(1) & \psi_2^*(2) & \psi_2^*(3) \\ \psi_3^*(1) & \psi_3^*(2) & \psi_3^*(3)\end{pmatrix} \\ &= \Big(\psi_1^*(1)\psi_2^*(2)\psi_3^*(3) + \psi_1^*(2)\psi_2^*(3)\psi_3^*(1) + \psi_1^*(3)\psi_2^*(1)\psi_3^*(2)\Big) \\ &\quad - \Big(\psi_1^*(3)\psi_2^*(2)\psi_3^*(1) + \psi_1^*(1)\psi_2^*(3)\psi_3^*(2) + \psi_1^*(2)\psi_2^*(1)\psi_3^*(3)\Big)\,. \end{aligned} \tag{4.37}$$

Very similar to the preceding subsection, the leading coefficient of $\Delta(k, X_2, E/\pi)$ with respect to $2X_2$ satisfies

$$\begin{aligned}\lambda(2X_2,&\Delta) \\ &= \lambda(2X_2, \psi_1^*(2)\psi_2^*(3)\psi_3^*(1) - \psi_1^*(3)\psi_2^*(2)\psi_3^*(1)) \\ &= \frac{-(s_1-1)!^2}{2^{2(s_2+s_3)+1}(2s_2-1)(2s_3-1)(2s_1-1)!}\left(s_3\Lambda^-_{s_3-1}(\Lambda^-_{s_2-1})' - s_2\Lambda^-_{s_2-1}(\Lambda^-_{s_3-1})'\right)\,,\end{aligned}$$

which is nonzero if and only if

$$s_3\Lambda^-_{s_3-1}(\Lambda^-_{s_2-1})' - s_2\Lambda^-_{s_2-1}(\Lambda^-_{s_3-1})' \neq 0\,. \tag{4.38}$$

Again, condition (4.38) is equivalent to

$$\frac{(\Lambda^-_{s_2-1})^{s_3}}{(\Lambda^-_{s_3-1})^{s_2}} \notin \mathbb{Q}\,. \tag{4.39}$$

Since $s_2 < s_3$, from Lemma 3.5 we obtain

$$\deg_k(\Lambda^-_{s_2-1})^{s_3} < \deg_k(\Lambda^-_{s_3-1})^{s_2}\,.$$

Hence, condition (4.39) is fulfilled and the proposition is proven. □

For the proof of Theorem 4.4 it remains to discuss the cases in which at least one of the integers s_1, s_2, s_3 is even.

Proof of Theorem 4.4. For three even integers s_1, s_2, s_3 we see from (3.12) that

$$\Psi^*_{2s_1},\ \Psi^*_{2s_2},\ \Psi^*_{2s_3} \in \mathbb{Q}(k, K/\pi)\,,$$

such that these three numbers are algebraically dependent in this case. It remains to investigate the following two cases:

Case 1: Two indices s_i are even, Case 2: Two indices s_i are odd.

In (3.12) we replace the numbers $k, K/\pi, E/\pi$ by the independent variables X_1, X_2, X_3, respectively, and obtain for even integers s the function

$$\Psi^*_{2s}(X_1, X_2, X_3) = \frac{1}{(2s-1)!}\Bigg[\frac{(s-1)!^2}{8}\big(1-(2X_2)^2\big) + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\Big((2X_2)^{2j+2}\,\Lambda_j^+ - b_j\Big)\Bigg]. \tag{4.40}$$

For the derivatives we get

$$\frac{\partial \Psi^*_{2s}}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\,(2X_2)^{2j+2}\,(\Lambda_j^+)'\,, \tag{4.41}$$

$$\frac{\partial \Psi^*_{2s}}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\Bigg[-\frac{(s-1)!^2}{2}(2X_2) + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!(j+1)}{2^{2j+1}}\,(2X_2)^{2j+1}\,\Lambda_j^+\Bigg], \tag{4.42}$$

$$\frac{\partial \Psi^*_{2s}}{\partial X_3}(k, X_2, E/\pi) = 0\,. \tag{4.43}$$

Without loss of generality the two cases can be written as follows:

Case 1: $s_1 \geq 1$ odd, $2 \leq s_2 < s_3$ even,
Case 2: $1 \leq s_1 < s_2$ odd, $s_3 \geq 2$ even.

Case 1: Since $\psi_3^*(2) = \psi_3^*(3) = 0$ we get

$$\Delta(X_1, X_2, X_3) = \psi_1^*(2)\psi_2^*(3)\psi_3^*(1) - \psi_1^*(3)\psi_2^*(2)\psi_3^*(1)\,.$$

By

$$\deg_{X_2}\big(\psi_1^*(2)\psi_2^*(3)\psi_3^*(1)\big) = \deg_{X_2}\big(\psi_1^*(3)\psi_2^*(2)\psi_3^*(1)\big) = 2(s_2+s_3)$$

the leading coefficient of Δ with respect to $2X_2$ satisfies

$$\begin{aligned}\lambda(2X_2,&\Delta)\\ &= \frac{-(s_1-1)!^2}{2^{2(s_3+s_2)+1}(2s_3-1)(2s_2-1)(2s_1-1)!}\left(s_3\Lambda^+_{s_3-1}(\Lambda^+_{s_2-1})' - s_2\Lambda^+_{s_2-1}(\Lambda^+_{s_3-1})'\right) .\end{aligned}$$

This does not vanish if and only if

$$\frac{(\Lambda^+_{s_2-1})^{s_3}}{(\Lambda^+_{s_3-1})^{s_2}} \notin \mathbb{Q} . \tag{4.44}$$

Condition (4.44) follows with Lemma 3.5 and $s_2 < s_3$ from

$$\deg_k(\Lambda^+_{s_2-1})^{s_3} < \deg_k(\Lambda^+_{s_3-1})^{s_2} .$$

This proves the algebraic independence of the numbers $\Psi^*_{2s_1}$, $\Psi^*_{2s_2}$, and $\Psi^*_{2s_3}$ over $\mathbb{Q}$ in this case.

Case 2: We have $\psi^*_3(3) = 0$ and therefore

$$\Delta(X_1,X_2,X_3) = \psi^*_1(2)\psi^*_2(3)\psi^*_3(1)+\psi^*_1(3)\psi^*_2(1)\psi^*_3(2)-\psi^*_1(3)\psi^*_2(2)\psi^*_3(1)-\psi^*_1(1)\psi^*_2(3)\psi^*_3(2) .$$

Moreover, we compute

$$\begin{aligned}\deg_{X_2}\left(\psi^*_1(2)\psi^*_2(3)\psi^*_3(1)\right) &= \deg_{X_2}\left(\psi^*_1(3)\psi^*_2(2)\psi^*_3(1)\right) = 2(s_2+s_3) ,\\ \deg_{X_2}\left(\psi^*_1(3)\psi^*_2(1)\psi^*_3(2)\right) &= \deg_{X_2}\left(\psi^*_1(1)\psi^*_2(3)\psi^*_3(2)\right) = 2(s_1+s_3) ,\end{aligned}$$

which leads to $\deg_{X_2}\Delta = 2(s_2+s_3)$ by the assumption of Case 2. This gives

$$\begin{aligned}\lambda(2X_2,&\Delta)\\ &= \lambda(2X_2, \psi^*_1(2)\psi^*_2(3)\psi^*_3(1) - \psi^*_1(3)\psi^*_2(2)\psi^*_3(1))\\ &= \frac{-(s_1-1)!^2}{2^{2(s_3+s_2)+1}(2s_3-1)(2s_2-1)(2s_1-1)!}\left(s_3\Lambda^+_{s_3-1}(\Lambda^-_{s_2-1})' - s_2\Lambda^-_{s_2-1}(\Lambda^+_{s_3-1})'\right) .\end{aligned}$$

The right-hand side is nonzero if and only if

$$\frac{(\Lambda^-_{s_2-1})^{s_3}}{(\Lambda^+_{s_3-1})^{s_2}} \notin \mathbb{Q} . \tag{4.45}$$

By Lemma 3.5 we have

$$\begin{aligned}\deg_k(\Lambda^-_{s_2-1})^{s_3} &= 2s_2s_3 - 4s_3 ,\\ \deg_k(\Lambda^+_{s_3-1})^{s_2} &= 2s_2s_3 - 2s_2 .\end{aligned}$$

Therefore, (4.45) holds for $s_2 \neq 2s_3$. Since s_2 is odd by the assumption of Case 2, this inequality is fulfilled and we conclude on the algebraic independence of $\Psi^*_{2s_1}$, $\Psi^*_{2s_2}$, and $\Psi^*_{2s_3}$ over $\mathbb{Q}$ from Corollary 2.1. This completes the proof of Theorem 4.4. □

5. Independence results for mixed subsets of Ω

So far we did only study algebraic independence properties of four particular subsets of Ω, namely $\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}\}$, $\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}$, $\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}$, and $\{\Psi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}$. It remains to investigate the sets consisting of at least two different types of numbers in Ω. By the first theorem in this section we treat the two-element subsets of Ω.

5.1. Two-element subsets of Ω

Theorem 5.1. *Any two numbers in Ω are algebraically independent over $\mathbb{Q}$.*

To handle the cases with $s_1 = 1$ or $s_2 = 1$, respectively, we shall extend the definition (3.14) of $\Theta_j^{\pm}$ and $\Lambda_j^{\pm}$ for $j \geq 1$ by suitable quantities in the case of $j = 0$. For odd integers $s \geq 3$ we observe from (3.6), (3.8), (3.11) and (3.13) the following formulas for the leading coefficients with respect to the variable $2X_2$:

$$\begin{aligned}
\lambda(2X_2, \Phi_{2s}) &= \frac{(-1)^s}{2^{2s+1}(2s-1)}\,\Theta^+_{s-1}\,,\\
\lambda(2X_2, \Phi^*_{2s}) &= \frac{(-1)^{s-1}}{2^{2s+1}(2s-1)}\,\Theta^-_{s-1}\,,\\
\lambda(2X_2, \Psi_{2s}) &= \frac{(-1)^{s-1}}{2^{2s+1}(2s-1)}\,\Lambda^+_{s-1}\,,\\
\lambda(2X_2, \Psi^*_{2s}) &= \frac{(-1)^s}{2^{2s+1}(2s-1)}\,\Lambda^-_{s-1}\,.
\end{aligned}$$

Now, we may compute these leading coefficients in the case $s = 1$ and define

$$\begin{aligned}
\lambda(2X_2, \Phi_2) &= \frac{1}{24}(-1+2k^2) &&=: -\frac{1}{8}\,\Theta^+_0\,,\\
\lambda(2X_2, \Phi^*_2) &= \frac{1}{24}(-5+4k^2) &&=: \frac{1}{8}\,\Theta^-_0\,,\\
\lambda(2X_2, \Psi_2) &= \frac{1}{8} &&=: \frac{1}{8}\,\Lambda^+_0\,,\\
\lambda(2X_2, \Psi^*_2) &= \frac{1}{8} &&=: -\frac{1}{8}\,\Lambda^-_0\,.
\end{aligned}$$

This gives

$$\Theta^+_0 = \frac{1}{3} - \frac{2}{3}k^2\,, \qquad \Theta^-_0 = -\frac{5}{3} + \frac{4}{3}k^2\,, \qquad \Lambda^+_0 = 1\,, \qquad \Lambda^-_0 = -1\,. \tag{5.1}$$

Together with Lemma 3.5 we obtain

$$\begin{aligned}
\deg_k \Theta^{\pm}_j &= 2j+2 \quad (j \geq 0)\,,\\
\deg_k \Lambda^{\pm}_j &\leq 2j \qquad\;\; (j \geq 0)\,.
\end{aligned}$$

Proof of Theorem 5.1. For the unmixed sets $\{\Phi_{2s_1}, \Phi_{2s_2}\}$, $\{\Phi^*_{2s_1}, \Phi^*_{2s_2}\}$, $\{\Psi_{2s_1}, \Psi_{2s_2}\}$, and $\{\Psi^*_{2s_1}, \Psi^*_{2s_2}\}$ the statement of Theorem 5.1 follows immediately from the proofs of Theorems 4.1 to 4.4. Therefore, it remains to prove Theorem 5.1 for mixed two-element subsets of Ω. This leaves six cases to be discussed.

We apply Corollary 2.1 by setting

$$\mathbb{K} = \mathbb{Q}(E/\pi), \qquad n = 2, \qquad x_1 = k, \qquad x_2 = \frac{K}{\pi}\,,$$

and use the notation from the preceding section. We remark that we prove the algebraic independence over $\mathbb{Q}(E/\pi)$ in the following six cases, apart from two exceptional sets. For these two exceptions we can only prove the algebraic independence over $\mathbb{Q}$.

Case 1: $\{\Phi_{2s_1}, \Phi^*_{2s_2}\} \subset \Omega$. The determinant from Corollary 2.1 turns out to be

$$\Delta(X_1, X_2) \;:=\; \det\begin{pmatrix} \phi_1(1) & \phi^*_1(2) \\ \phi_2(1) & \phi^*_2(2) \end{pmatrix} \;=\; \phi_1(1)\phi^*_2(2) - \phi^*_1(2)\phi_2(1).$$

For $s_1 \equiv s_2 \pmod 2$ the leading coefficient of Δ with respect to $2X_2$ satisfies

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)} \left(s_2\Theta^{\pm}_{s_2-1}(\Theta^{\mp}_{s_1-1})' - s_1\Theta^{\mp}_{s_1-1}(\Theta^{\pm}_{s_2-1})'\right).$$

Otherwise, for $s_1 \not\equiv s_2 \pmod 2$, we get

$$\lambda(2X_2, \Delta) = \frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)} \left(s_2\Theta^{\mp}_{s_2-1}(\Theta^{\mp}_{s_1-1})' - s_1\Theta^{\mp}_{s_1-1}(\Theta^{\mp}_{s_2-1})'\right).$$

To prove the algebraic independence of Φ_{2s_1} and $\Phi^*_{2s_2}$ over $\mathbb{Q}(E/\pi)$, we shall show that $\lambda(2X_2, \Delta) \neq 0$ holds in any of the four following subcases.

Case 1.1: $s_1 \equiv s_2 \equiv 0 \pmod 2$: Suppose that

$$\frac{(\Theta^{-}_{s_1-1})^{s_2}}{(\Theta^{+}_{s_2-1})^{s_1}} \in \mathbb{Q}\,.$$

This gives

$$\frac{s_2}{s_1} = \frac{\alpha_{s_1,0}\beta_{s_2,1}}{\beta_{s_2,0}\alpha_{s_1,1}} = \frac{a_{s_1-1}\left(\dfrac{2^{2s_2-3}}{(2s_2-2)!} - \dfrac{s_2}{2}a_{s_2-1}\right)}{a_{s_2-1}\left(-\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}a_{s_1-1}\right)} = \frac{s_2 - \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}}.$$

But, we have

$$\frac{s_2}{s_1} - \frac{s_2 - \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}} = \frac{s_2\dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!} + s_1\dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1\left(s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}\right)} > 0,$$

a contradiction.

Case 1.2: $s_1 \equiv 0,\ s_2 \equiv 1 \pmod 2$: Assume that

$$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^-_{s_2-1})^{s_1}} \in \mathbb{Q}\,.$$

First let $s_2 \geq 3$. We conclude on

$$\frac{s_2}{s_1} = \frac{\alpha_{s_1,0}\alpha_{s_2,1}}{\alpha_{s_2,0}\alpha_{s_1,1}} = \frac{a_{s_1-1}\left(\dfrac{2^{2s_2-3}}{(2s_2-2)!} - \dfrac{s_2}{2}a_{s_2-1}\right)}{a_{s_2-1}\left(-\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}a_{s_1-1}\right)} = \frac{s_2 - \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}}.$$

As in Case 1.1 this leads to a contradiction.

Now let $s_2 = 1$. Assuming

$$\frac{\Theta^-_{s_1-1}}{(\Theta^-_0)^{s_1}} \in \mathbb{Q}$$

leads to

$$\begin{aligned}
\frac{1}{s_1} = -\frac{4\alpha_{s_1,0}}{5\alpha_{s_1,1}} &= -\frac{4a_{s_1-1}}{5\cdot\left(-\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}a_{s_1-1}\right)} \\
\iff \qquad s_1 &= \frac{5}{4}\cdot\left(\frac{2^{2s_1-3}}{a_{s_1-1}(2s_1-2)!} + \frac{s_1}{2}\right) \\
\iff \qquad \frac{3s_1}{10} &= \frac{2^{2s_1-3}}{a_{s_1-1}(2s_1-2)!} = \frac{s_1}{4|B_{2s_1}|} \\
\iff \qquad \frac{5}{6} &= |B_{2s_1}|\,,
\end{aligned}$$

which is not fulfilled for any $s_1 \in \mathbb{N}$.

Case 1.3: $s_1 \equiv 1,\ s_2 \equiv 0 \pmod 2$: Assume that

$$\frac{(\Theta^+_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}} \in \mathbb{Q}\,.$$

For $s_1 \geq 3$ this gives

$$\frac{s_2}{s_1} = \frac{\beta_{s_1,0}\beta_{s_2,1}}{\beta_{s_2,0}\beta_{s_1,1}} = \frac{a_{s_1-1}\left(\dfrac{2^{2s_2-3}}{(2s_2-2)!} - \dfrac{s_2}{2}a_{s_2-1}\right)}{a_{s_2-1}\left(-\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}a_{s_1-1}\right)} = \frac{s_2 - \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}},$$

which is false for $s_1, s_2 \in \mathbb{N}$. For $s_1 = 1$ we have

$$
\begin{aligned}
s_2 = -\frac{\beta_{s_2,1}}{2\beta_{s_2,0}} &= -\frac{\left(\frac{2^{2s_2-3}}{(2s_2-2)!} - \frac{s_2}{2}a_{s_2-1}\right)}{2a_{s_2-1}} = -\frac{2^{2s_2-4}}{a_{s_2-1}(2s_2-2)!} + \frac{s_2}{4} \\
\iff \qquad -3s_2 &= \frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!} = \frac{s_2}{2|B_{2s_2}|} \\
\iff \qquad -\frac{1}{6} &= |B_{2s_2}|\,,
\end{aligned}
$$

which is not fulfilled for any $s_2 \in \mathbb{N}$.

Case 1.4: $s_1 \equiv s_2 \equiv 1 \pmod 2$: Assume that

$$
\frac{(\Theta^+_{s_1-1})^{s_2}}{(\Theta^-_{s_2-1})^{s_1}} \in \mathbb{Q}.
$$

First let $s_1, s_2 \geq 3$. Then we have

$$
\frac{s_2}{s_1} = \frac{\beta_{s_1,0}\alpha_{s_2,1}}{\alpha_{s_2,0}\beta_{s_1,1}} = -\frac{a_{s_1-1}\left(\frac{2^{2s_2-3}}{(2s_2-2)!} - \frac{s_2}{2}a_{s_2-1}\right)}{a_{s_2-1}\left(\frac{2^{2s_1-3}}{(2s_1-2)!} + \frac{s_1}{2}a_{s_1-1}\right)} = \frac{s_2 - \frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1 + \frac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}},
$$

which is impossible as shown in Case 1.1.

Now let $s_2 = 1, s_1 \geq 3$. We have

$$
\begin{aligned}
\frac{1}{s_1} = -\frac{4\beta_{s_1,0}}{5\beta_{s_1,1}} &= \frac{4a_{s_1-1}}{5\cdot\left(\frac{2^{2s_1-3}}{(2s_1-2)!} + \frac{s_1}{2}a_{s_1-1}\right)} \\
\iff \qquad s_1 &= \frac{5}{4}\cdot\left(\frac{2^{2s_1-3}}{a_{s_1-1}(2s_1-2)!} + \frac{s_1}{2}\right)
\end{aligned}
$$

which is false for any $s_1 \in \mathbb{N}$ as shown in Case 1.2. For $s_1 = 1, s_2 \geq 3$ we get

$$
\begin{aligned}
s_2 = -\frac{\alpha_{s_2,1}}{2\alpha_{s_2,0}} &= -\frac{\left(\frac{2^{2s_2-3}}{(2s_2-2)!} - \frac{s_2}{2}a_{s_2-1}\right)}{2a_{s_2-1}} = -\frac{2^{2s_2-4}}{a_{s_2-1}(2s_2-2)!} + \frac{s_2}{4} \\
\iff \qquad 3s_2 &= -\frac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\,,
\end{aligned}
$$

which is obviously not fulfilled, since the left-hand side is positive, whereas the right-hand side is negative. Finally, for $s_1 = s_2 = 1$ we see that

$$
\frac{\Theta^+_0}{\Theta^-_0} = \frac{1-2k^2}{-5+4k^2} \notin \mathbb{Q}\,.
$$

Case 2: $\{\Phi_{2s_1}, \Psi_{2s_2}\} \subset \Omega$. We shall prove the nonvanishing of

$$\Delta(X_1, X_2) \quad := \quad \det \begin{pmatrix} \phi_1(1) & \psi_1(2) \\ \phi_2(1) & \psi_2(2) \end{pmatrix} \quad = \quad \phi_1(1)\psi_2(2) - \psi_1(2)\phi_2(1).$$

In the case of $s_1 \equiv 0 \pmod 2$ we have

$$\lambda(2X_2, \Delta) = \frac{1}{2^{2(s_1+s_2)}(2s_1 - 1)(2s_2 - 1)} \left(s_2 \Lambda^{\mp}_{s_2-1} (\Theta^{-}_{s_1-1})' - s_1 \Theta^{-}_{s_1-1} (\Lambda^{\mp}_{s_2-1})' \right) ,$$

whereas for $s_1 \equiv 1 \pmod 2$ we obtain

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1 - 1)(2s_2 - 1)} \left(s_2 \Lambda^{\mp}_{s_2-1} (\Theta^{+}_{s_1-1})' - s_1 \Theta^{+}_{s_1-1} (\Lambda^{\mp}_{s_2-1})' \right) .$$

As before, the algebraic independence of the numbers Φ_{2s_1} and Ψ_{2s_2} over $\mathbb{Q}(E/\pi)$ follows from

$$\frac{(\Theta^{-}_{s_1-1})^{s_2}}{(\Lambda^{\mp}_{s_2-1})^{s_1}} \notin \mathbb{Q} \qquad \text{or} \qquad \frac{(\Theta^{+}_{s_1-1})^{s_2}}{(\Lambda^{\mp}_{s_2-1})^{s_1}} \notin \mathbb{Q}\,,$$

depending on whether s_1 is even or odd. Suppose, on the contrary, that

$$(\Theta^{-}_{s_1-1})^{s_2} = r \cdot (\Lambda^{\mp}_{s_2-1})^{s_1} \qquad \text{or} \qquad (\Theta^{+}_{s_1-1})^{s_2} = r \cdot (\Lambda^{\mp}_{s_2-1})^{s_1}\,, \tag{5.2}$$

for some nonvanishing $r \in \mathbb{Q}$. By Lemma 3.5 we obtain

$$\deg_k (\Theta^{\mp}_{s_1-1})^{s_2} = 2s_1 s_2 > 2s_1 s_2 - 2s_1 \geq \deg_k (\Lambda^{\mp}_{s_2-1})^{s_1}\,,$$

and this contradicts the equations in (5.2).

Case 3: $\{\Phi_{2s_1}, \Psi^*_{2s_2}\} \subset \Omega$. We investigate the determinant

$$\Delta(X_1, X_2) \quad := \quad \det \begin{pmatrix} \phi_1(1) & \psi^*_1(2) \\ \phi_2(1) & \psi^*_2(2) \end{pmatrix} \quad = \quad \phi_1(1)\psi^*_2(2) - \psi^*_1(2)\phi_2(1).$$

If $s_1 \equiv 0 \pmod 2$ we have

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1 - 1)(2s_2 - 1)} \left(s_2 \Lambda^{\mp}_{s_2-1} (\Theta^{-}_{s_1-1})' - s_1 \Theta^{-}_{s_1-1} (\Lambda^{\mp}_{s_2-1})' \right) ,$$

otherwise, for $s_1 \equiv 1 \pmod 2$, we get

$$\lambda(2X_2, \Delta) = \frac{1}{2^{2(s_1+s_2)}(2s_1 - 1)(2s_2 - 1)} \left(s_2 \Lambda^{\pm}_{s_2-1} (\Theta^{+}_{s_1-1})' - s_1 \Theta^{+}_{s_1-1} (\Lambda^{\pm}_{s_2-1})' \right) .$$

The algebraic independence of Φ_{2s_1} and $\Psi^*_{2s_2}$ over $\mathbb{Q}(E/\pi)$ follows from

$$\frac{(\Theta^{-}_{s_1-1})^{s_2}}{(\Lambda^{\pm}_{s_2-1})^{s_1}} \notin \mathbb{Q} \qquad \text{and} \qquad \frac{(\Theta^{+}_{s_1-1})^{s_2}}{(\Lambda^{\pm}_{s_2-1})^{s_1}} \notin \mathbb{Q}\,,$$

which is a consequence of the inequality

$$\deg_k(\Theta^{\pm}_{s_1-1})^{s_2} = 2s_1s_2 > 2s_1s_2 - 2s_1 \geq \deg_k(\Lambda^{\pm}_{s_2-1})^{s_1}\,,$$

similar to Case 2.

Case 4: $\{\Phi^*_{2s_1}, \Psi_{2s_2}\} \subset \Omega$. We investigate

$$\Delta(X_1, X_2) \;:=\; \det\begin{pmatrix} \phi^*_1(1) & \psi_1(2) \\ \phi^*_2(1) & \psi_2(2) \end{pmatrix} \;=\; \phi^*_1(1)\psi_2(2) - \psi_1(2)\phi^*_2(1).$$

For $s_1 \equiv 0 \pmod 2$ we have

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)}\left(s_2\Lambda^{\mp}_{s_2-1}(\Theta^{+}_{s_1-1})' - s_1\Theta^{+}_{s_1-1}(\Lambda^{\mp}_{s_2-1})'\right)\,,$$

whereas for $s_1 \equiv 1 \pmod 2$ we get

$$\lambda(2X_2, \Delta) = \frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)}\left(s_2\Lambda^{\mp}_{s_2-1}(\Theta^{-}_{s_1-1})' - s_1\Theta^{-}_{s_1-1}(\Lambda^{\mp}_{s_2-1})'\right)\,.$$

Again, the algebraic independene of $\Phi^*_{2s_1}$ and Ψ_{2s_2} over $\mathbb{Q}(E/\pi)$ follows from

$$\frac{(\Theta^{+}_{s_1-1})^{s_2}}{(\Lambda^{\mp}_{s_2-1})^{s_1}} \notin \mathbb{Q} \qquad \text{and} \qquad \frac{(\Theta^{-}_{s_1-1})^{s_2}}{(\Lambda^{\mp}_{s_2-1})^{s_1}} \notin \mathbb{Q}$$

by

$$\deg_k(\Theta^{\mp}_{s_1-1})^{s_2} = 2s_1s_2 > 2s_1s_2 - 2s_1 \geq \deg_k(\Lambda^{\mp}_{s_2-1})^{s_1}\,.$$

Case 5: $\{\Phi^*_{2s_1}, \Psi^*_{2s_2}\} \subset \Omega$. Here, we have to prove, that the determinant

$$\Delta(X_1, X_2) \;:=\; \det\begin{pmatrix} \phi^*_1(1) & \psi^*_1(2) \\ \phi^*_2(1) & \psi^*_2(2) \end{pmatrix} \;=\; \phi^*_1(1)\psi^*_2(2) - \psi^*_1(2)\phi^*_2(1)$$

does not vanish. In the case $s_1 \equiv 0 \pmod 2$ the leading coefficient of Δ satisfies

$$\lambda(2X_2, \Delta) = \frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)}\left(s_2\Lambda^{\pm}_{s_2-1}(\Theta^{+}_{s_1-1})' - s_1\Theta^{+}_{s_1-1}(\Lambda^{\pm}_{s_2-1})'\right)\,.$$

For $s_1 \equiv 1 \pmod 2$ we obtain

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)}\left(s_2\Lambda^{\pm}_{s_2-1}(\Theta^{-}_{s_1-1})' - s_1\Theta^{-}_{s_1-1}(\Lambda^{\pm}_{s_2-1})'\right)\,.$$

The numbers $\Phi^*_{2s_1}$ and $\Psi^*_{2s_2}$ are algebraically independent over $\mathbb{Q}(E/\pi)$, since

$$\frac{(\Theta^{+}_{s_1-1})^{s_2}}{(\Lambda^{\pm}_{s_2-1})^{s_1}} \notin \mathbb{Q} \qquad \text{and} \qquad \frac{(\Theta^{-}_{s_1-1})^{s_2}}{(\Lambda^{\pm}_{s_2-1})^{s_1}} \notin \mathbb{Q}\,,$$

which follows from

$$\deg_k(\Theta^{\pm}_{s_1-1})^{s_2} = 2s_1s_2 > 2s_1s_2 - 2s_1 \geq \deg_k(\Lambda^{\pm}_{s_2-1})^{s_1}\,.$$

Case 6: $\{\Psi_{2s_1}, \Psi^*_{2s_2}\} \subset \Omega$. We get

$$\Delta(X_1, X_2) \;:=\; \det\begin{pmatrix} \psi_1(1) & \psi_1^*(2) \\ \psi_2(1) & \psi_2^*(2) \end{pmatrix} \;=\; \psi_1(1)\psi_2^*(2) - \psi_1^*(2)\psi_2(1).$$

For $s_1 \equiv s_2 \pmod 2$ we have to prove the nonvanishing of

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)}\left(s_2\Lambda^{\pm}_{s_2-1}(\Lambda^{\mp}_{s_1-1})' - s_1\Lambda^{\mp}_{s_1-1}(\Lambda^{\pm}_{s_2-1})'\right)\,.$$

Otherwise, for $s_1 \not\equiv s_2 \pmod 2$, we get

$$\lambda(2X_2, \Delta) = -\frac{1}{2^{2(s_1+s_2)}(2s_1-1)(2s_2-1)}\left(s_2\Lambda^{\mp}_{s_2-1}(\Lambda^{\mp}_{s_1-1})' - s_1\Lambda^{\mp}_{s_1-1}(\Lambda^{\mp}_{s_2-1})'\right)\,.$$

Case 6.1: $s_1 \equiv s_2 \equiv 0 \pmod 2$: We assume

$$\frac{(\Lambda^{-}_{s_1-1})^{s_2}}{(\Lambda^{+}_{s_2-1})^{s_1}} \in \mathbb{Q}\,. \tag{5.3}$$

By Lemma 3.5 we have

$$\begin{aligned} \deg_k(\Lambda^{-}_{s_1-1})^{s_2} &= 2s_1s_2 - 4s_2 \\ \deg_k(\Lambda^{+}_{s_2-1})^{s_1} &= 2s_1s_2 - 2s_1\,. \end{aligned}$$

Hence, if $s_1 \neq 2s_2$ assumption (5.3) is false. Let $s_1 = 2s_2$. Then (5.3) leads to

$$\frac{1}{2} = \frac{\gamma_{2s_2,0}\delta_{s_2,1}}{\delta_{s_2,0}\gamma_{2s_2,1}} = \frac{b_{2s_2-1}\left(-\dfrac{2^{2s_2-3}}{(2s_2-2)!} - \dfrac{s_2}{2}b_{s_2-1}\right)}{b_{s_2-1}\left(\dfrac{2^{4s_2-3}}{(4s_2-2)!} - s_2 b_{2s_2-1}\right)} = \frac{\dfrac{s_2}{2} + \dfrac{2^{2s_2-3}}{b_{s_2-1}(2s_2-2)!}}{s_2 - \dfrac{2^{4s_2-3}}{b_{2s_2-1}(4s_2-2)!}},$$

which is equivalent to

$$\frac{2^{4s_2-4}}{b_{2s_2-1}(4s_2-2)!} = -\frac{2^{2s_2-3}}{b_{s_2-1}(2s_2-2)!}\,.$$

The last equation has no solution $s_2 \in \mathbb{N}$, since the numbers b_j are positive for every $j \geq 0$.

Case 6.2: $s_1 \equiv 0$, $s_2 \equiv 1 \pmod 2$: We prove

$$\frac{(\Lambda^{-}_{s_1-1})^{s_2}}{(\Lambda^{-}_{s_2-1})^{s_1}} \notin \mathbb{Q}\,,$$

which is clear for $s_1 \geq 4$ and $s_2 = 1$, since $\Lambda_0^- = -1$ and $\deg_k \Lambda_{s_1-1}^- = 2s_1 - 4 \geq 4$. Let $s_1 \geq 2$ and $s_2 \geq 3$. Assuming $(\Lambda_{s_1-1}^-)^{s_2} = r \cdot (\Lambda_{s_2-1}^-)^{s_1}$ for some nonvanishing $r \in \mathbb{Q}$ leads to $\deg_k(\Lambda_{s_1-1}^-)^{s_2} = \deg_k(\Lambda_{s_2-1}^-)^{s_1}$ and with Lemma 3.5 to $s_1 = s_2$, which is impossible since s_1 is even whereas s_2 is odd. For the remaining case, where $s_1 = 2$ and $s_2 = 1$, we have $\Lambda_1^- = \Lambda_0^+ = 1$. Indeed, Ψ_4 and Ψ_2^* are algebraically dependent over $\mathbb{Q}(E/\pi)$, since $\Psi_4, \Psi_2^* \in \mathbb{Q}[K/\pi, E/\pi]$. Therefore, we apply Corollary 2.1 with

$$\mathbb{K} = \mathbb{Q}, \qquad x_1 = \frac{E}{\pi}, \qquad x_2 = \frac{K}{\pi}\,.$$

We have to prove that the determinant

$$\Delta(X_1, X_2) \;:=\; \det \begin{pmatrix} \psi_1(1) & \psi_1^*(2) \\ \psi_2(1) & \psi_2^*(2) \end{pmatrix} \;=\; \psi_1(1)\psi_2^*(2) - \psi_1^*(2)\psi_2(1).$$

does not vanish. Using the expressions

$$\Psi_4 = -\frac{1}{32} + \frac{1}{6}\left(\frac{K}{\pi}\right)^4 - \frac{1}{12}\left(\frac{K}{\pi}\right)^2 + \frac{1}{6}\frac{KE}{\pi^2}\,,$$
$$\Psi_2^* = \frac{1}{8} + \frac{1}{2}\left(\frac{K}{\pi}\right)^2 - \frac{KE}{\pi^2}\,,$$

given by (3.10) and (3.13), we compute

$$\Delta(X_1, X_2) = \frac{2}{3}X_2^4 \not\equiv 0.$$

That proves that the numbers Ψ_4 and Ψ_2^* are algebraically independent over $\mathbb{Q}$.

Case 6.3: $s_1 \equiv 1$, $s_2 \equiv 0 \pmod 2$: We prove

$$\frac{(\Lambda_{s_1-1}^+)^{s_2}}{(\Lambda_{s_2-1}^+)^{s_1}} \notin \mathbb{Q}\,.$$

Since $\Lambda_0^+ = 1$ and $\deg_k \Lambda_{s_2-1}^+ = 2s_2 - 2 \geq 2$ there is nothing to show in the case $s_1 = 1$. Therefore, let $s_1 \geq 3$. Assuming $(\Lambda_{s_1-1}^+)^{s_2} = r \cdot (\Lambda_{s_2-1}^+)^{s_1}$ for some $r \in \mathbb{Q} \setminus \{0\}$ gives $\deg_k(\Lambda_{s_1-1}^+)^{s_2} = \deg_k(\Lambda_{s_2-1}^+)^{s_1}$, which is not fulfilled.

Case 6.4: $s_1 \equiv s_2 \equiv 1 \pmod 2$: We prove

$$\frac{(\Lambda_{s_1-1}^+)^{s_2}}{(\Lambda_{s_2-1}^-)^{s_1}} \notin \mathbb{Q}\,,$$

which is clear in both of the cases $s_1 = 1$, $s_2 \geq 3$ and $s_2 = 1$, $s_1 \geq 3$, respectively, since $\Lambda_0^\pm = \pm 1$ and $\Lambda_j^\pm \notin \mathbb{Q}$ for $(j \geq 2)$. Therefore, let $s_1 \geq 3$ and $s_2 \geq 3$. Then, the assumption $(\Lambda_{s_1-1}^+)^{s_2} = r \cdot (\Lambda_{s_2-1}^-)^{s_1}$ with $r \in \mathbb{Q} \setminus \{0\}$ leads with application of Lemma 3.5 to $s_1 = s_2$ and then by Lemma 3.6 to

$$1 = \frac{\delta_{s_1,0}\gamma_{s_1,1}}{\gamma_{s_1,0}\delta_{s_1,1}} = \frac{b_{s_1-1}\left(-\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}b_{s_1-1}\right)}{b_{s_1-1}\left(\dfrac{2^{2s_1-3}}{(2s_1-2)!} - \dfrac{s_1}{2}b_{s_1-1}\right)} = \frac{s_1 + \dfrac{2^{2s_1-2}}{b_{s_1-1}(2s_1-2)!}}{s_1 - \dfrac{2^{2s_1-2}}{b_{s_1-1}(2s_1-2)!}}\,,$$

which is not fulfilled for any $s_1 \in \mathbb{N}$.

In the remaining case $s_1 = s_2 = 1$ the numbers Ψ_2 and Ψ_2^* are algebraically dependent over $\mathbb{Q}(E/\pi)$, since $\Psi_2, \Psi_2^* \in \mathbb{Q}[K/\pi, E/\pi]$. We apply Corollary 2.1 with

$$\mathbb{K} = \mathbb{Q}, \qquad x_1 = \frac{E}{\pi}, \qquad x_2 = \frac{K}{\pi}$$

and prove the nonvanishing of the determinant

$$\Delta(X_1, X_2) \;:=\; \det \begin{pmatrix} \psi_1(1) & \psi_1^*(2) \\ \psi_2(1) & \psi_2^*(2) \end{pmatrix} \;=\; \psi_1(1)\psi_2^*(2) - \psi_1^*(2)\psi_2(1).$$

By (3.11) and (3.13) we have

$$\Psi_2 = -\frac{1}{8} + \frac{1}{2}\left(\frac{K}{\pi}\right)^2,$$

$$\Psi_2^* = \frac{1}{8} + \frac{1}{2}\left(\frac{K}{\pi}\right)^2 - \frac{KE}{\pi^2}.$$

This gives

$$\Delta(X_1, X_2) = X_2^2 \not\equiv 0.$$

Finally, this proves Theorem 5.1. □

5.2. Three-element subsets of Ω

As in the proofs so far, the investigation for three-element mixed subsets of Ω also ends up in proving the irrationality of some quotient r, whose numerator and denominator are powers of the polynomials $\Theta_j^{\pm}$ and $\Lambda_j^{\pm}$. In the case where the leading coefficient $\lambda(2X_2, \Delta)$ of the determinant Δ consists of two terms, there are exactly 36 possible forms this quotient r can take. The following table lists the first 20 cases.

Case no.	s_1 mod 2	s_2 mod 2	r	Case no.	s_1 mod 2	s_2 mod 2	r
①	0	0	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^-_{s_2-1})^{s_1}}$	⑪	0	0	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^-_{s_2-1})^{s_1}}$
②	0	1	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^-_{s_2-1})^{s_1}}$	⑫	0	1	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^-_{s_2-1})^{s_1}}$
③	0	0	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑬	0	0	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$
④	0	1	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑭	0	1	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$
⑤	1	1	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^-_{s_2-1})^{s_1}}$	⑮	1	1	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^-_{s_2-1})^{s_1}}$
⑥	1	0	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑯	1	0	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$
⑦	1	1	$\frac{(\Theta^-_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑰	1	1	$\frac{(\Lambda^-_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$
⑧	0	0	$\frac{(\Theta^+_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑱	0	0	$\frac{(\Lambda^+_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$
⑨	0	1	$\frac{(\Theta^+_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑲	0	1	$\frac{(\Lambda^+_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$
⑩	1	1	$\frac{(\Theta^+_{s_1-1})^{s_2}}{(\Theta^+_{s_2-1})^{s_1}}$	⑳	1	1	$\frac{(\Lambda^+_{s_1-1})^{s_2}}{(\Lambda^+_{s_2-1})^{s_1}}$

The remaining 16 possibilities are summarized to the following case:

$$\text{Case no. } ㉑ : \qquad r = \frac{(\Theta^{\pm}_{s_1-1})^{s_2}}{(\Lambda^{\pm}_{s_2-1})^{s_1}}\,.$$

In some cases the leading coefficient $\lambda(2X_2, \Delta)$ consists of four terms. In these cases the quotient r to be investigated can take the following forms:

$$\text{Case no. } \textcircled{22} : \qquad r = \frac{(\Theta^-_{s_1-1})^{s_2}}{(\Lambda^-_{s_2-1} - \Theta^-_{s_2-1})^{s_1}},$$

$$\text{Case no. } \textcircled{23} : \qquad r = \frac{(\Theta^+_{s_1-1})^{s_2}}{(\Lambda^-_{s_2-1} - \Theta^-_{s_2-1})^{s_1}},$$

$$\text{Case no. } \textcircled{24} : \qquad r = \frac{(\Lambda^\pm_{s_1-1})^{s_2}}{(\Lambda^-_{s_2-1} - \Theta^-_{s_2-1})^{s_1}}.$$

Theorem 5.2. *In all of the above listed 24 cases we have $r \notin \mathbb{Q}$ except for trivial cases and the following five nontrivial exceptions:*

$$\textit{Case no. } \textcircled{1} \quad \textit{with} \quad (s_1, s_2) = (2, 4),$$
$$\textit{Case no. } \textcircled{12} \quad \textit{with} \quad (s_1, s_2) = (2, 1),$$
$$\textit{Case no. } \textcircled{14} \quad \textit{with} \quad (s_1, s_2) = (2, 1),$$
$$\textit{Case no. } \textcircled{17} \quad \textit{with} \quad (s_1, s_2) = (1, 1),$$
$$\textit{Case no. } \textcircled{23} \quad \textit{with} \quad (s_1, s_2) = (1, 1).$$

The trivial cases are as follows: $s_1 = s_2$ in ① , ⑤ , ⑧ , ⑩ , ⑪ , ⑮ , ⑱ , and ⑳ .

Proof. We shall frequently use the identity

$$\frac{2^{2s-3}}{a_{s-1}(2s-2)!} = \frac{s}{4|B_{2s}|}. \tag{5.4}$$

Case ①: For $s_1, s_2 \geq 4$ the statement is shown in the proof of Theorem 4.1 (cf. [17]). For the remaining subcase $(s_1, s_2) = (2, 4)$ we have

$$(\Theta^-_1)^4 = 9(\Theta^-_3)^2.$$

This is the first exceptional case stated in Theorem 5.2. Now, without loss of generality, let $s_1 = 2$ and $s_2 \geq 6$. Assuming $r \in \mathbb{Q}$ and applying (5.4) leads to

$$\frac{s_2}{2} = \frac{\alpha_{2,0}\alpha_{s_2,1}}{\alpha_{s_2,0}\alpha_{2,1}} = \frac{a_1\left(-\frac{2^{2s_2-3}}{(2s_2-2)!} - \frac{s_2}{2}a_{s_2-1}\right)}{a_{s_2-1}(-1-a_1)} = \frac{\frac{2^{2s_2-3}}{a_{s_2-1}(2s_2-2)!} + \frac{s_2}{2}}{\frac{1}{a_1}+1}$$

$$\iff \quad \frac{15s_2}{2} = \frac{s_2}{4|B_{2s_2}|} \quad \iff \quad \frac{1}{30} = |B_{2s_2}|,$$

which is only true for $s_2 = 2$ or $s_2 = 4$.

Case ② : The statement follows from the proof of Theorem 5.1, Case 1.2.

Case ③ : The statement follows from the proof of Theorem 5.1, Case 1.1.

Case ④ : Let $s_1 \geq 4$. Then, the statement follows from the proof of Theorem 4.1 (cf. [17]). Now, let $s_1 = 2$. Assuming $r \in \mathbb{Q}$ and applying (5.4) yields

$$\frac{s_2}{2} = \frac{\alpha_{2,0}\beta_{s_2,1}}{\beta_{s_2,0}\alpha_{2,1}} = \frac{a_1\left(-\dfrac{2^{2s_2-3}}{(2s_2-2)!} - \dfrac{s_2}{2}a_{s_2-1}\right)}{a_{s_2-1}\left(-1-a_1\right)} = \frac{\dfrac{2^{2s_2-3}}{a_{s_2-1}(2s_2-2)!} + \dfrac{s_2}{2}}{\dfrac{1}{a_1}+1}$$

$$\iff \quad \frac{15s_2}{2} = \frac{s_2}{4|B_{2s_2}|} \quad \iff \quad \frac{1}{30} = |B_{2s_2}|\,,$$

and this is fulfilled if and only if $s_2 = 2$ or $s_2 = 4$, contrary to the condition $s_2 \equiv 1 \pmod 2$.

Case ⑤ : For $s_1, s_2 \geq 3$ the statement follows from the proof of Proposition 4.1. Now, without loss of generality, let $s_1 = 1$ and $s_2 \geq 3$. Then, by application of (5.4) the assumption $r \in \mathbb{Q}$ gives

$$s_2 = -\frac{5\alpha_{s_2,1}}{4\alpha_{s_2,0}} = -\frac{5}{4}\cdot\left(\frac{2^{2s_2-3}}{a_{s_2-1}(2s_2-2)!} - \frac{s_2}{2}\right)$$

$$\iff \quad -\frac{5}{6} = |B_{2s_2}|$$

which is not fulfilled for any $s_2 \in \mathbb{N}$.

Case ⑥ : For $s_1 \geq 3$ the statement follows from the proof of Theorem 4.2. Therefore, let $s_1 = 1$. The assumption $r \in \mathbb{Q}$ leads to

$$s_2 = -\frac{5\beta_{s_2,1}}{4\beta_{s_2,0}} = -\frac{5}{4}\cdot\left(\frac{2^{2s_2-3}}{a_{s_2-1}(2s_2-2)!} - \frac{s_2}{2}\right),$$

which is impossible, similar to Case ⑤ .

Case ⑦ : The statement follows from the proof of Theorem 5.1, Case 1.4.

Case ⑧ : The statement follows from the proof of Theorem 4.2.

Case ⑨ : The statement follows from the proof of Theorem 5.1, Case 1.3.

Case ⑩ : The statement follows from the proof of Theorem 4.1 (cf. [17]).

Case ⑪ : For $s_1, s_2 \geq 4$ the statement follows from the proof of Proposition 4.2.

Therefore, without loss of generality, let $s_1 = 2$ and $s_2 \geq 4$. Since $\Lambda_1^- = 1$ and $\deg_k(\Lambda_{s_2-1}^-)^2 = 4s_2 - 8 \geq 8$, we conclude on $r \notin \mathbb{Q}$.

Case (12): For $(s_1, s_2) \neq (2, 1)$ the statement follows from the proof of Theorem 5.1, Case 6.2. For $(s_1, s_2) = (2, 1)$ we have $r = 1$, since $\Lambda_0^- = -1$ and $\Lambda_1^- = 1$. This is the second exceptional case stated in Theorem 5.2.

Case (13): The statement follows from the proof of Theorem 5.1, Case 6.1.

Case (14): Let $s_2 \geq 3$ and $s_1 \geq 4$. Then the statement follows from the proof of Theorem 4.3. For $s_2 = 1$ and $s_1 \geq 4$ the statement holds, since $\Lambda_0^+ = 1$ and $\deg_k \Lambda_{s_1-1}^- = 2s_1 - 4 \geq 4$. Similarly, for $s_1 = 2$ and $s_2 \geq 3$ we have $\Lambda_1^- = 1$ and $\deg_k(\Lambda_{s_2-1}^+)^2 = 4s_2 - 4 \geq 8$. Finally let $s_1 = 2$ and $s_2 = 1$. Here we have $r = 1$, since $\Lambda_1^- = \Lambda_0^+ = 1$. This is the third exceptional case stated in Theorem 5.2.

Case (15): In the proof of Proposition 4.3 the statement is shown for $s_1, s_2 \geq 3$. Therefore, without loss of generality, let $s_1 = 1$ and $s_2 \geq 3$. Then, the statement holds, since $\Lambda_0^- = -1$ and $\Lambda_{s_2-1}^- \notin \mathbb{Q}$ by Lemma 3.5.

Case (16): In the proof of Theorem 4.4 the statement is shown for $s_1 \geq 3$. For $s_1 = 1$ and $s_2 \geq 2$ it follows from $\Lambda_0^- = -1$ and $\Lambda_{s_2-1}^+ \notin \mathbb{Q}$, whereas the latter statement is a consequence of Lemma 3.5.

Case (17): For $(s_1, s_2) = (1, 1)$ we have $r = -1$, which is the fourth exceptional case stated in Theorem 5.2. Otherwise, for $(s_1, s_2) \neq (1, 1)$, the statement follows from the proof of Theorem 5.1, Case 6.4.

Case (18): The statement follows from the proof of Theorem 4.4.

Case (19): The statement follows from the proof of Theorem 5.1, Case 6.3.

Case (20): For $s_1, s_2 \geq 3$ the statement follows from the proof of Theorem 4.3. W.l.o.g. let $s_1 = 1$ and $s_2 \geq 3$. Then the statement holds, since $\Lambda_0^+ = 1$ and $\Lambda_{s_2-1}^+ \notin \mathbb{Q}$.

Case (21): We conclude on $r \notin \mathbb{Q}$ from

$$\deg_k(\Lambda_{s_2-1}^\pm)^{s_1} \leq 2s_2 s_1 - 2s_1 < 2s_2 s_1 = \deg_k(\Theta_{s_1-1}^\pm)^{s_2} \qquad (s_1, s_2 \in \mathbb{N}).$$

Case (22): First let $s_1 = s_2 = 1$. Then we have

$$r = \frac{\Theta_0^-}{\Lambda_0^- - \Theta_0^-} = \frac{-5 + 4k^2}{2 - 4k^2} \notin \mathbb{Q}.$$

For $s_1 = 1$ and $s_2 \geq 2$ we assume $(\Theta_0^-)^{s_2} = r(\Lambda_{s_2-1}^- - \Theta_{s_2-1}^-)$. By equating the constant terms this yields

$$r = \frac{(-1)^{s_2} 5^{s_2}}{3^{s_2}(2^{2s_2} - 2)a_{s_2-1}},$$

whereas equation of the leading coefficients gives

$$r = \frac{4^{s_2}}{3^{s_2}2^{2s_2}a_{s_2-1}} = \frac{1}{3^{s_2}a_{s_2-1}} \,.$$

Both equations together mean

$$1 = (-1)^{s_2}\frac{5^{s_2}}{(2^{2s_2}-2)} \,,$$

which has no solution $s_2 \in \mathbb{N}$.

For $s_1 \geq 2$ and $s_2 = 1$ we assume $\Theta^-_{s_1-1} = r(\Lambda^-_0 - \Theta^-_0)^{s_1}$. On the one hand we obtain

$$r = \frac{3^{s_1}a_{s_1-1}}{2^{s_1}} \,,$$

and on the other hand

$$r = (-1)^{s_1}\frac{3^{s_1}2^{2s_1}a_{s_1-1}}{4^{s_1}} = (-1)^{s_1}3^{s_1}a_{s_1-1} \,.$$

Obviously, both equations are not solvable simultaneously with $s_1 \geq 2$.

Finally let $s_1, s_2 \geq 2$. We assume $r \in \mathbb{Q}$. Equating the constant terms and the leading coefficients in

$$\begin{aligned}(\alpha_{s_1,0} + \alpha_{s_1,1}k^2 + \cdots + \alpha_{s_1,s_1}k^{2s_1})^{s_2} &= \\ = r((\gamma_{s_2,0} - \alpha_{s_2,0}) + (\gamma_{s_2,1} - \alpha_{s_2,1})k^2 &+ \cdots + \alpha_{s_2,s_2}k^{2s_2})^{s_1}\end{aligned}$$

we obtain by Lemma 3.6

$$r = \frac{\alpha^{s_2}_{s_1,0}}{(\gamma_{s_2,0} - \alpha_{s_2,0})^{s_1}} = \frac{a^{s_2}_{s_1-1}}{(2^{2s_2}-2)^{s_1}a^{s_1}_{s_2-1}} \,,$$

and

$$r = \frac{\alpha^{s_2}_{s_1,s_1}}{\alpha^{s_1}_{s_2,s_2}} = \frac{a^{s_2}_{s_1-1}}{a^{s_1}_{s_2-1}} \,.$$

Together this yields

$$(2^{2s_2}-2)^{s_1} = 1 \,,$$

which is obviously wrong for $s_1, s_2 \geq 1$.

Case (23): For $s_1 = s_2 = 1$ we compute

$$r = \frac{\Theta^+_0}{\Lambda^-_0 - \Theta^-_0} = \frac{1-2k^2}{2-4k^2} = \frac{1}{2} \,.$$

This is the fifth nontrivial exception stated in Theorem 5.2.

For $s_1 = 1$ and $s_2 \geq 2$ we assume $(\Theta^+_0)^{s_2} = r(\Lambda^-_{s_2-1} - \Theta^-_{s_2-1})$. Equating the constant terms we obtain

$$r = \frac{1}{3^{s_2}(2^{2s_2}-2)a_{s_2-1}} \,,$$

while by comparing the leading coefficients we get

$$r = (-1)^{s_2} \frac{2^{s_2}}{3^{s_2} 2^{2s_2} a_{s_2-1}} = \frac{(-1)^{s_2}}{3^{s_2} 2^{s_2} a_{s_2-1}} .$$

We conclude on

$$\frac{1}{2^{2s_2} - 2} = \frac{(-1)^{s_2}}{2^{s_2}} .$$

which is not solvable for any $s_2 \geq 2$.

For $s_1 \geq 2$ and $s_2 = 1$ we assume $\Theta^+_{s_1-1} = r(\Lambda^-_0 - \Theta^-_0)^{s_1}$. On the one hand we obtain

$$r = \frac{3^{s_1} a_{s_1-1}}{2^{s_1}} ,$$

and on the other hand we have

$$r = (-1)^{s_1} \frac{3^{s_1}(2 - 2^{2s_1}) a_{s_1-1}}{4^{s_1}} .$$

This yields

$$(-1)^{s_1-1}(2^{2s_1} - 2) = 2^{s_1} ,$$

which has no solution $s_1 \geq 2$.

The case $s_1, s_2 \geq 2$ remains to be discussed. We assume $r \in \mathbb{Q}$. Equating the constant terms and the leading coefficients in

$$(\beta_{s_1,0} + \beta_{s_1,1} k^2 + \cdots + \beta_{s_1,s_1} k^{2s_1})^{s_2} =$$
$$= r((\gamma_{s_2,0} - \alpha_{s_2,0}) + (\gamma_{s_2,1} - \alpha_{s_2,1}) k^2 + \cdots + \alpha_{s_2,s_2} k^{2s_2})^{s_1}$$

gives by Lemma 3.6

$$r = \frac{\beta^{s_2}_{s_1,0}}{(\gamma_{s_2,0} - \alpha_{s_2,0})^{s_1}} = \frac{a^{s_2}_{s_1-1}}{(2^{2s_2} - 2)^{s_1} a^{s_1}_{s_2-1}} ,$$

and

$$r = \frac{\beta^{s_2}_{s_1,s_1}}{\alpha^{s_1}_{s_2,s_2}} = \frac{(2 - 2^{2s_1})^{s_2} a^{s_2}_{s_1-1}}{2^{2s_1 s_2} a^{s_1}_{s_2-1}} .$$

Together this yields

$$\frac{(2 - 2^{2s_1})^{s_2} (2^{2s_2} - 2)^{s_1}}{2^{2s_1 s_2}} = 1 ,$$

or, equivalently,

$$(-1)^{s_2} (2^{s_1} - 2^{1-s_1})^{s_2} (2^{s_2} - 2^{1-s_2})^{s_1} = 1 . \tag{5.5}$$

Obviously (5.5) does not hold if s_2 is odd. Therefore, let s_2 be even. We get

$$(2^{s_1} - 2^{1-s_1})^{s_2} (2^{s_2} - 2^{1-s_2})^{s_1} > (2^{s_1} - 1)^{s_2} (2^{s_2} - 1)^{s_1} > 1 \qquad (s_1, s_2 \geq 2)$$

a contradiction to (5.5).

Altogether this proves $r \notin \mathbb{Q}$ for $(s_1, s_2) \neq (1, 1)$.

Case (24) : We conclude on $r \notin \mathbb{Q}$ from

$$\deg_k(\Lambda^{\pm}_{s_1-1})^{s_2} \leq 2s_1 s_2 - 2s_2 < 2s_1 s_2 = \deg_k(\Lambda^-_{s_2-1} - \Theta^-_{s_2-1})^{s_1} .$$

Therefore, Theorem 5.2 is proven. □

In Section 4 we studied the one-type three-element subsets of Ω, namely the sets $\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}\}$, $\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}$, $\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}$, and $\{\Psi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}$ for pairwise distinct positive integers s_1, s_2, s_3. It remains to investigate those three-element subsets of Ω, in which at least two different types of the reciprocal sums $\Phi_{2s}, \Phi^*_{2s}, \Psi_{2s}$, and Ψ^*_{2s} are included. These are the sets of the following 16 forms:

$$\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi^*_{2s_3}\}, \quad \{\Phi_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}, \quad \{\Phi_{2s_1}, \Phi_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\},$$

$$\{\Phi_{2s_1}, \Phi_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Phi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\},$$

$$\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Phi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Psi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Psi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\},$$

$$\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Phi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\}.$$

Each of these sets has to be studied for all possible configurations $(s_1, s_2, s_3) \in \mathbb{N}^3$. Without loss of generality we may assume that $s_1 < s_2$ holds for the sets

$$\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi^*_{2s_3}\}, \quad \{\Phi_{2s_1}, \Phi_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi_{2s_1}, \Phi_{2s_2}, \Psi^*_{2s_3}\},$$

$$\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Psi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\},$$

whereas $s_2 < s_3$ holds for

$$\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}, \quad \{\Phi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\},$$

$$\{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}, \quad \{\Phi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}, \quad \{\Psi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}.$$

In the following we give a complete table for each of the 16 sets, in which the proof of algebraic independence for every configuration $(s_1, s_2, s_3) \in \mathbb{N}^3$ is reduced to one of the 24 cases treated in Theorem 5.2. For this we obtain a clustering of 192 cases into 24 groups.

For brevity we shall write λ instead of $\lambda(2X_2, \Delta)$ in the headers of the tables.

<u>$\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi^*_{2s_3}\}$</u>: For $(s_1, s_2, s_3) \equiv (1,1,0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
1.1	1	1	1		(10)
1.2	1	0	0		(9)
1.3	0	1	0		(9)
1.4	0	0	0		(3)
1.5	0	1	1	$s_1 < s_3$	(7)
1.6	0	1	1	$s_3 < s_1$	(4)
1.7	1	0	1	$s_2 < s_3$	(7)
1.8	1	0	1	$s_3 < s_2$	(4)
1.9	0	0	1	$s_1 < s_3$	(2)
1.10	0	0	1	$s_3 < s_1$	(1)

<u>$\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\}$</u>: For $(s_1, s_2, s_3) \equiv (1,0,0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
2.1	0	0	0		(8)
2.2	1	0	1		(9)
2.3	1	1	0		(9)
2.4	1	1	1		(7)
2.5	0	1	0	$s_2 < s_1$	(3)
2.6	0	1	0	$s_1 < s_2$	(6)
2.7	0	0	1	$s_3 < s_1$	(3)
2.8	0	0	1	$s_1 < s_3$	(6)
2.9	0	1	1	$s_2 < s_1$	(2)
2.10	0	1	1	$s_1 < s_2$	(5)

$\{\Phi_{2s_1}, \Phi_{2s_2}, \Psi_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1,1,1) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
3.1	1	1	0		(10)
3.2	1	0	1		(21)
3.3	0	1	1		(21)
3.4	0	0	1		(21)
3.5	0	1	0	$s_1 < s_3$	(21)
3.6	0	1	0	$s_3 < s_1$	(4)
3.7	0	1	0	$s_1 = s_3$	(23)
3.8	1	0	0	$s_2 < s_3$	(21)
3.9	1	0	0	$s_3 < s_2$	(4)
3.10	1	0	0	$s_2 = s_3$	(23)
3.11	0	0	0	$s_1 < s_3$	(21)
3.12	0	0	0	$s_3 < s_1$	(1)
3.13	0	0	0	$s_1 = s_3$	(22)

$\{\Phi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1,1,1) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
4.1	1	1	0		(21)
4.2	1	0	1		(21)
4.3	0	1	1		(20)
4.4	1	0	0		(21)
4.5	0	1	0	$s_1 < s_3$	(14)
4.6	0	1	0	$s_3 < s_1$	(21)
4.7	0	1	0	$s_1 = s_3$	(24)
4.8	0	0	1	$s_1 < s_2$	(14)
4.9	0	0	1	$s_2 < s_1$	(21)
4.10	0	0	1	$s_1 = s_2$	(24)
4.11	0	0	0	$s_1 < s_2$	(11)
4.12	0	0	0	$s_2 < s_1$	(21)
4.13	0	0	0	$s_1 = s_2$	(24)

$\underline{\{\Phi_{2s_1}, \Phi_{2s_2}, \Psi^*_{2s_3}\}}$: For $(s_1, s_2, s_3) \equiv (1, 1, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
5.1	1	1	1		(10)
5.2	1	0	0		(21)
5.3	0	1	0		(21)
5.4	0	0	0		(21)
5.5	0	1	1	$s_1 < s_3$	(21)
5.6	0	1	1	$s_3 < s_1$	(4)
5.7	1	0	1	$s_2 < s_3$	(21)
5.8	1	0	1	$s_3 < s_2$	(4)
5.9	0	0	1	$s_1 < s_3$	(21)
5.10	0	0	1	$s_3 < s_1$	(1)

$\underline{\{\Phi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}}$: For $(s_1, s_2, s_3) \equiv (1, 0, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
6.1	1	0	1		(21)
6.2	1	1	0		(21)
6.3	0	0	0		(18)
6.4	1	1	1		(21)
6.5	0	0	1	$s_1 < s_3$	(16)
6.6	0	0	1	$s_3 < s_1$	(21)
6.7	0	1	0	$s_1 < s_2$	(16)
6.8	0	1	0	$s_2 < s_1$	(21)
6.9	0	1	1	$s_1 < s_2$	(15)
6.10	0	1	1	$s_2 < s_1$	(21)

$\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (0, 0, 1) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
7.1	0	0	0		(8)
7.2	0	1	1		(21)
7.3	1	0	1		(21)
7.4	1	1	1		(21)
7.5	1	0	0	$s_3 < s_1$	(6)
7.6	1	0	0	$s_1 < s_3$	(21)
7.7	0	1	0	$s_3 < s_2$	(6)
7.8	0	1	0	$s_2 < s_3$	(21)
7.9	1	1	0	$s_3 < s_1$	(5)
7.10	1	1	0	$s_1 < s_3$	(21)

$\{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (0, 1, 1) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
8.1	1	1	1		(20)
8.2	0	0	1		(21)
8.3	0	1	0		(21)
8.4	0	0	0		(21)
8.5	1	1	0	$s_1 < s_3$	(14)
8.6	1	1	0	$s_3 < s_1$	(21)
8.7	1	0	1	$s_1 < s_2$	(14)
8.8	1	0	1	$s_2 < s_1$	(21)
8.9	1	0	0	$s_1 < s_2$	(11)
8.10	1	0	0	$s_2 < s_1$	(21)

$\underline{\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\}}$: For $(s_1, s_2, s_3) \equiv (0,0,0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
9.1	0	0	1		(8)
9.2	0	1	0		(21)
9.3	1	0	0		(21)
9.4	1	1	0		(21)
9.5	1	0	1	$s_1 < s_3$	(21)
9.6	1	0	1	$s_3 < s_1$	(6)
9.7	1	0	1	$s_1 = s_3$	(23)
9.8	0	1	1	$s_2 < s_3$	(21)
9.9	0	1	1	$s_3 < s_2$	(6)
9.10	0	1	1	$s_2 = s_3$	(23)
9.11	1	1	1	$s_1 < s_3$	(21)
9.12	1	1	1	$s_3 < s_1$	(5)
9.13	1	1	1	$s_1 = s_3$	(22)

$\underline{\{\Phi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}}$: For $(s_1, s_2, s_3) \equiv (0,0,0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
10.1	0	0	1		(21)
10.2	0	1	0		(21)
10.3	1	0	0		(18)
10.4	0	1	1		(21)
10.5	1	0	1	$s_1 < s_3$	(16)
10.6	1	0	1	$s_3 < s_1$	(21)
10.7	1	0	1	$s_1 = s_3$	(24)
10.8	1	1	0	$s_1 < s_2$	(16)
10.9	1	1	0	$s_2 < s_1$	(21)
10.10	1	1	0	$s_1 = s_2$	(24)
10.11	1	1	1	$s_1 < s_2$	(15)
10.12	1	1	1	$s_2 < s_1$	(21)
10.13	1	1	1	$s_1 = s_2$	(24)

$\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1, 1, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
11.1	1	1	1		(20)
11.2	1	0	0		(19)
11.3	0	1	0		(19)
11.4	0	0	0		(13)
11.5	0	1	1	$s_1 < s_3$	(17)
11.6	0	1	1	$s_3 < s_1$	(14)
11.7	1	0	1	$s_2 < s_3$	(17)
11.8	1	0	1	$s_3 < s_2$	(14)
11.9	0	0	1	$s_1 < s_3$	(12)
11.10	0	0	1	$s_3 < s_1$	(11)

$\{\Psi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1, 0, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
12.1	1	0	1		(19)
12.2	1	1	0		(19)
12.3	0	0	0		(18)
12.4	1	1	1		(17)
12.5	0	0	1	$s_1 < s_3$	(16)
12.6	0	0	1	$s_3 < s_1$	(13)
12.7	0	1	0	$s_1 < s_2$	(16)
12.8	0	1	0	$s_2 < s_1$	(13)
12.9	0	1	1	$s_1 < s_2$	(15)
12.10	0	1	1	$s_2 < s_1$	(12)

$\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1, 0, 1) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
13.1	1	0	0		(9)
13.2	1	1	1		(21)
13.3	0	0	1		(21)
13.4	0	1	1	$s_1 < s_2$	(21)
13.5	0	1	1	$s_2 < s_1$	(21)
13.6	0	0	0	$s_1 < s_3$	(21)
13.7	0	0	0	$s_3 < s_1$	(3)
13.8	0	0	0	$s_1 = s_3$	(23)
13.9	1	1	0	$s_2 < s_3$	(21)
13.10	1	1	0	$s_3 < s_2$	(7)
13.11	0	1	0	$s_2 < s_1 \wedge s_2 < s_3$	(21)
13.12	0	1	0	$s_2 < s_1 \wedge s_3 < s_2$	(2)
13.13	0	1	0	$s_1 < s_2 \wedge s_1 < s_3$	(21)
13.14	0	1	0	$s_1 < s_2 \wedge s_3 < s_1$	(2)
13.15	0	1	0	$s_1 < s_2 \wedge s_1 = s_3$	(22)

$\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1, 0, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
14.1	1	0	1		(9)
14.2	1	1	0		(21)
14.3	0	0	0		(21)
14.4	0	1	0	$s_1 < s_2$	(21)
14.5	0	1	0	$s_2 < s_1$	(21)
14.6	0	0	1	$s_1 < s_3$	(21)
14.7	0	0	1	$s_3 < s_1$	(3)
14.8	1	1	1	$s_2 < s_3$	(21)
14.9	1	1	1	$s_3 < s_2$	(7)
14.10	1	1	1	$s_2 = s_3$	(23)
14.11	0	1	1	$s_1 < s_2 \wedge s_1 < s_3$	(21)
14.12	0	1	1	$s_1 < s_2 \wedge s_3 < s_1$	(2)
14.13	0	1	1	$s_2 < s_1 \wedge s_2 < s_3$	(21)
14.14	0	1	1	$s_2 < s_1 \wedge s_3 < s_2$	(2)
14.15	0	1	1	$s_2 < s_1 \wedge s_2 = s_3$	(22)

$\{\Phi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (1, 1, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
15.1	1	1	1		(21)
15.2	1	0	0		(21)
15.3	0	1	0		(19)
15.4	1	0	1	$s_2 < s_3$	(21)
15.5	1	0	1	$s_3 < s_2$	(21)
15.6	0	1	1	$s_1 < s_3$	(17)
15.7	0	1	1	$s_3 < s_1$	(21)
15.8	0	0	0	$s_1 < s_2$	(13)
15.9	0	0	0	$s_2 < s_1$	(21)
15.10	0	0	0	$s_1 = s_2$	(24)
15.11	0	0	1	$s_3 < s_1 \wedge s_3 < s_2$	(21)
15.12	0	0	1	$s_3 < s_1 \wedge s_2 < s_3$	(21)
15.13	0	0	1	$s_1 < s_3 \wedge s_1 < s_2$	(12)
15.14	0	0	1	$s_1 < s_3 \wedge s_2 < s_1$	(21)
15.15	0	0	1	$s_1 < s_3 \wedge s_1 = s_2$	(24)

$\{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\}$: For $(s_1, s_2, s_3) \equiv (0, 1, 0) \pmod 2$ we have algebraic dependence.

config. no.	s_1 mod 2	s_2 mod 2	s_3 mod 2	add. cond.	$\lambda \neq 0$ by Case no.
16.1	0	1	1		(21)
16.2	0	0	0		(21)
16.3	1	1	0		(19)
16.4	1	0	0	$s_1 < s_2$	(13)
16.5	1	0	0	$s_2 < s_1$	(21)
16.6	0	0	1	$s_2 < s_3$	(21)
16.7	0	0	1	$s_3 < s_2$	(21)
16.8	1	1	1	$s_1 < s_3$	(17)
16.9	1	1	1	$s_3 < s_1$	(21)
16.10	1	1	1	$s_1 = s_3$	(24)
16.11	1	0	1	$s_2 < s_1 \wedge s_2 < s_3$	(21)
16.12	1	0	1	$s_2 < s_1 \wedge s_3 < s_2$	(21)
16.13	1	0	1	$s_1 < s_2 \wedge s_1 < s_3$	(12)
16.14	1	0	1	$s_1 < s_2 \wedge s_3 < s_1$	(21)
16.15	1	0	1	$s_1 < s_2 \wedge s_1 = s_3$	(24)

According to the exceptional cases of Theorem 5.2 we need to have a closer look at the sets with configuration 1.10, 5.10, 8.5, 11.8, and 14.10:

Configuration no. 1.10: Let $s_1 = 2$, $s_2 = 4$, and $s_3 = 1$, hence, we are dealing with the set $\{\Phi_4, \Phi_8, \Phi_2^*\}$. We have

$$\begin{aligned}\Phi_4 &= \frac{1}{6}\left[-\frac{1}{24}\left(1-\left(\frac{2K}{\pi}\right)^2\left(\frac{6E}{K}-5+4k^2\right)\right)-\frac{1}{16}\left(a_1-\left(\frac{2K}{\pi}\right)^4\Theta_1^-\right)\right],\\ \Phi_8 &= \frac{1}{7!}\left[-\frac{3}{2}\left(1-\left(\frac{2K}{\pi}\right)^2\left(\frac{6E}{K}-5+4k^2\right)\right)-\frac{49}{16}\left(a_1-\left(\frac{2K}{\pi}\right)^4\Theta_1^-\right)\right.\\ &\quad\left.-\frac{21}{8}\left(a_2-\left(\frac{2K}{\pi}\right)^6\Theta_2^-\right)-\frac{45}{32}\left(a_3-\left(\frac{2K}{\pi}\right)^8\Theta_3^-\right)\right],\\ \Phi_2^* &= -\frac{1}{24}\left(1-\left(\frac{2K}{\pi}\right)^2\left(\frac{6E}{K}-5+4k^2\right)\right).\end{aligned}$$

For the determinant Δ from Corollary 2.1 we compute

$$\Delta(X_1,X_2,X_3) = -\frac{8}{4725}X_2^{10}X_1\,,$$

which is obviously not the zero polynomial. Therefore, the numbers Φ_4, Φ_8, and Φ_2^* are algebraically independent over $\mathbb{Q}$ by Corollary 2.1.

Configuration no. 5.10: Let $s_1 = 2$, $s_2 = 4$ and $s_3 = 1$, hence, we are dealing with the set $\{\Phi_4, \Phi_8, \Psi_2^*\}$ where

$$\Psi_2^* = \frac{1}{8}\left(1+\left(\frac{2K}{\pi}\right)^2\left(1-\frac{2E}{K}\right)\right).$$

The determinant Δ satisfies

$$\begin{aligned}\Delta(X_1,X_2,X_3) = \frac{13}{28350}X_2^6X_1 + \frac{16}{2835}X_2^8X_1^3 - \frac{8}{2835}X_2^8X_1 + \frac{128}{14175}X_2^{10}X_1^5\\ - \frac{128}{14175}X_2^{10}X_1^3 + \frac{32}{14175}X_2^{10}X_1\,.\end{aligned}$$

Since $\Delta \not\equiv 0$, we conclude on the algebraic independence of Φ_4, Φ_8, and Ψ_2^* over $\mathbb{Q}$.

Configuration no. 8.5: Let $s_1 = 1$, $s_2 = 1$ and $s_3 = 2$, hence, we are dealing with the set $\{\Phi_2^*, \Psi_2, \Psi_4\}$ where

$$\begin{aligned}\Psi_2 &= \frac{1}{8}\left(-1+\left(\frac{2K}{\pi}\right)^2\right),\\ \Psi_4 &= \frac{1}{6}\left[-\frac{1}{8}\left(1+\left(\frac{2K}{\pi}\right)^2\left(1-\frac{2E}{K}\right)\right)-\frac{1}{16}\left(b_1-\left(\frac{2K}{\pi}\right)^4\Lambda_1^-\right)\right].\end{aligned}$$

We compute

$$\Delta(X_1, X_2, X_3) = \frac{2}{9} X_2^4 X_1 \,,$$

which is not the zero polynomial. Therefore, the numbers Φ_2^*, Ψ_2, and Ψ_4 are algebraically independent over $\mathbb{Q}$.

Configuration no. 11.8: For $s_1 = 1$, $s_2 = 2$ and $s_3 = 1$ the numbers Ψ_2, Ψ_4, and Ψ_2^* are algebraically dependent over $\mathbb{Q}$. We have

$$4 \cdot \Psi_2^2 + \Psi_2 - 6 \cdot \Psi_4 - \Psi_2^* = 0 \,. \tag{5.6}$$

Configuration no. 14.10: For $s_1 = s_2 = s_3 = 1$ the numbers Φ_2, Φ_2^*, and Ψ_2^* are algebraically dependent over $\mathbb{Q}$. Here, we have

$$-2 \cdot \Phi_2 + \Phi_2^* + \Psi_2^* = 0 \,. \tag{5.7}$$

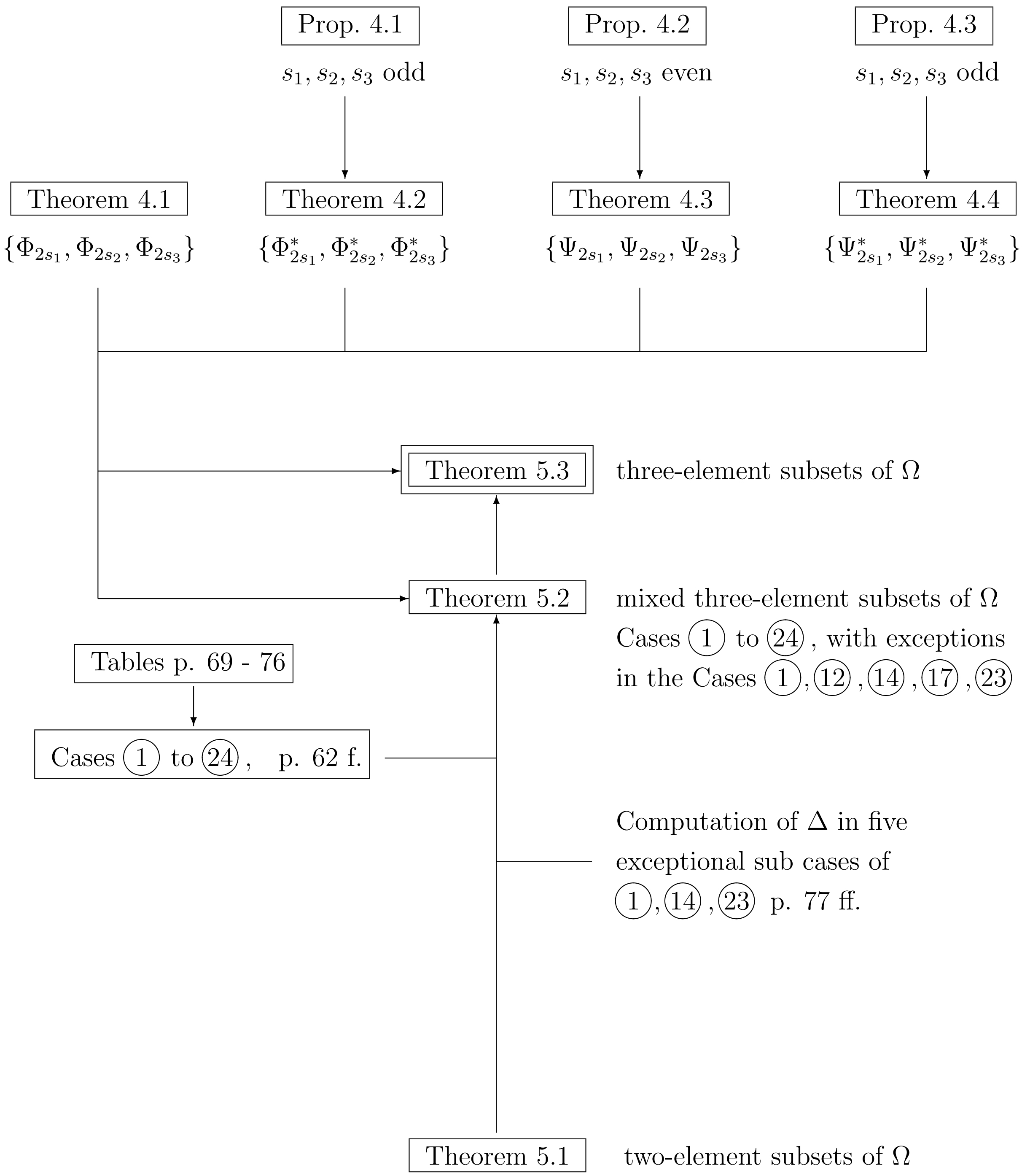
The identities (5.6) and (5.7) can be found in [20].

The results of this and the preceding section will be summarized in the following theorem, which therefore forms the main theorem of this thesis:

Theorem 5.3. *A three-element subset of Ω is algebraically dependent over $\mathbb{Q}$ if and only if it is contained in one of the following sets:*

$$\begin{gathered}
\big\{\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,1) \pmod 2\big\}, \\
\big\{\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,0,0) \pmod 2\big\}, \\
\big\{\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,1) \pmod 2\big\}, \\
\big\{\{\Psi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,0,0) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Phi_{2s_2}, \Phi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,0) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Phi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,0,0) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Phi_{2s_2}, \Psi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,1) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,1) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Phi_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,0) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,0,0) \pmod 2\big\}, \\
\big\{\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,0,1) \pmod 2\big\}, \\
\big\{\{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,1,1) \pmod 2\big\}, \\
\big\{\{\Phi^*_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,0,0) \pmod 2\big\}, \\
\big\{\{\Phi^*_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,0,0) \pmod 2\big\}, \\
\big\{\{\Psi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,0) \pmod 2\big\}, \\
\big\{\{\Psi_{2s_1}, \Psi^*_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,0,0) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,0,1) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Phi^*_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,0,0) \pmod 2\big\}, \\
\big\{\{\Phi_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (1,1,0) \pmod 2\big\}, \\
\big\{\{\Phi^*_{2s_1}, \Psi_{2s_2}, \Psi^*_{2s_3}\} \,:\, s_i \in \mathbb{N} \wedge (s_1, s_2, s_3) \equiv (0,1,0) \pmod 2\big\}, \\
\big\{\{\Psi_2, \Psi_4, \Psi_2^*\}, \{\Phi_2, \Phi_2^*, \Psi_2^*\}\big\}.
\end{gathered}$$

In the following diagram we present a guideline illustrating the relationship between the main results of this thesis ending in Theorem 5.3:



5.3. Larger subsets of Ω

Up to now, we have investigated all the two-element and three-element subsets of Ω and decided on their algebraic independence over the rationals. The results are stated in Theorem 5.1 and Theorem 5.3. It remains to study the subsets of Ω with at least four elements.

Theorem 5.4. *Any four numbers in Ω are algebraically dependent over $\mathbb{Q}$.*

Proof. Let $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$. From the formulas (3.5) to (3.13) we know that

$$\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{Q}\left[\frac{K}{\pi}, \frac{E}{\pi}, k\right] .$$

The chain rule for transcendence degrees (Lemma 2.1) applied to the field extensions

$$\mathbb{Q} \subseteq \mathbb{Q}(\omega_1, \omega_2, \omega_3, \omega_4) =: \mathbb{K} \subseteq \mathbb{Q}\left(\frac{K}{\pi}, \frac{E}{\pi}, k\right) =: \mathbb{L}$$

yields

$$\operatorname{tr.deg}(\mathbb{K} : \mathbb{Q}) = \operatorname{tr.deg}(\mathbb{L} : \mathbb{Q}) - \operatorname{tr.deg}(\mathbb{L} : \mathbb{K}) \leq 3 .$$

Hence, ω_1, ω_2, ω_3, and ω_4 are algebraically dependent over $\mathbb{Q}$. □

Apart from algebraic independencies and dependencies it is interesting to investigate the linear case. For example, formula (5.7) from the preceding subsection shows that the numbers Φ_2, Φ_2^*, and Ψ_2^* are linearly dependent over $\mathbb{Q}$. What can we prove about linear independencies and dependencies for large subsets of Ω? By the following theorem we state that, under certain conditions, arbitrarily many numbers from Ω are linearly independent over the field $\mathbb{Q}(E/\pi, k)$.

For any positive integer s we denote by $W_{2s} \in \Omega$ one of the numbers Φ_{2s}, Φ_{2s}^*, Ψ_{2s} or Ψ_{2s}^*.

Theorem 5.5. *Let $1 \leq s_1 < s_2 < \cdots < s_m \in \mathbb{N}$ for some $m \in \mathbb{N}$. Then the numbers $W_{2s_1}, \dots, W_{2s_m} \in \Omega$ are linearly independent over $\mathbb{Q}(E/\pi, k)$.*

Proof. Let $u_1, \dots, u_m \in \mathbb{Q}(E/\pi, k)$ with

$$u_1 W_{2s_1} + \cdots + u_m W_{2s_m} = 0 . \tag{5.8}$$

By the formulas (3.5) to (3.13) and by Lemma 3.6 we have

$$\deg_{2K/\pi} W_{2s} = 2s .$$

Therefore, with $s_m > s_j$ $(j = 1, \dots, m-1)$, the leading coefficient of the left-hand side of (5.8) satisfies

$$0 = \lambda\left(\frac{2K}{\pi}, u_1 W_{2s_1} + \cdots + u_m W_{2s_m}\right) = \lambda\left(\frac{2K}{\pi}, u_m W_{2s_m}\right) = u_m \lambda\left(\frac{2K}{\pi}, W_{2s_m}\right) .$$

Again, by the formulas (3.5) to (3.13) and by Lemma 3.6, we have

$$\lambda\left(\frac{2K}{\pi}, W_{2s_m}\right) \neq 0\,.$$

Hence, we get $u_m = 0$. Step by step, using the fact that the sequence $(s_j)_{1\le j\le m}$ is strictly increasing, we conclude on

$$u_m = u_{m-1} = \cdots = u_1 = 0\,.$$

□

Next, we consider linear equations in the four numbers Φ_{2s}, Φ_{2s}^*, Ψ_{2s} and Ψ_{2s}^* for any fixed positive integer s.

Theorem 5.6. *For $s \ge 2$ the four numbers Φ_{2s}, Φ_{2s}^*, Ψ_{2s} and Ψ_{2s}^* are linearly independent over $\mathbb{Q}$, i.e. the linear equation*

$$t_s\Phi_{2s} + u_s\Phi_{2s}^* + v_s\Psi_{2s} + w_s\Psi_{2s}^* = 0 \tag{5.9}$$

has no nontrivial solution $t_s, u_s, v_s, w_s \in \mathbb{Q}$ for $s \ge 2$. For $s = 1$ the general solution of (5.9) is

$$-2u_1\Phi_2 + u_1\Phi_2^* + u_1\Psi_2^* = 0 \qquad (u_1 \in \mathbb{Q})\,.$$

Proof. Let $s = 1$ and $t_1, u_1, v_1, w_1 \in \mathbb{Q}$ such that

$$t_1\Phi_2 + u_1\Phi_2^* + v_1\Psi_2 + w_1\Psi_2^* = 0\,.$$

From (3.6), (3.8), (3.11), and (3.13) we obtain

$$t_1 \cdot \frac{1}{24}\left(1 - \left(\frac{2K}{\pi}\right)^2 (1-2k^2)\right) - u_1 \cdot \frac{1}{24}\left(1 - \left(\frac{2K}{\pi}\right)^2 \left(\frac{6E}{K} - 5 + 4k^2\right)\right)$$
$$+ v_1 \cdot \frac{1}{8}\left(-1 + \left(\frac{2K}{\pi}\right)^2\right) + w_1 \cdot \frac{1}{8}\left(-1 + \left(\frac{2K}{\pi}\right)^2 \left(1 - \frac{2E}{K}\right)\right) = 0\,,$$

or, equivalently,

$$\frac{1}{24}\Bigg[(t_1 - u_1 - 3v_1 + 3w_1) + (-t_1 - 5u_1 + 3v_1 + 3w_1)\left(\frac{2K}{\pi}\right)^2$$
$$+ (2t_1 + 4u_1)\left(\frac{2K}{\pi}\right)^2 k^2 + (6u_1 - 6w_1)\frac{2K}{\pi}\frac{2E}{\pi}\Bigg] = 0\,,$$

Since K/π, E/π and k are algebraically independent over $\mathbb{Q}$, this yields

$$\begin{array}{rcrcrcrcl} t_1 & - & u_1 & - & 3v_1 & + & 3w_1 & = & 0\,,\\ -t_1 & - & 5u_1 & + & 3v_1 & + & 3w_1 & = & 0\,,\\ 2t_1 & + & 4u_1 & & & & & = & 0\,,\\ & & 6u_1 & & & - & 6w_1 & = & 0\,, \end{array}$$

with the general solution $(t_1, u_1, v_1, w_1) = u_1 \cdot (-2, 1, 0, 1)$.

Now let $s \geq 2$. The leading coefficient with respect to the quantity $2K/\pi$ on the left-hand side of (5.9) satisfies

$$\lambda\left(\frac{2K}{\pi},\, t_s\Phi_{2s} + u_s\Phi_{2s}^* + v_s\Psi_{2s} + w_s\Psi_{2s}^*\right)$$
$$= \frac{(-1)^{s-1}}{2^{2s+1}(2s-1)}\left(-t_s\Theta_{s-1}^{\mp} + u_s\Theta_{s-1}^{\pm} \mp v_s\Lambda_{s-1}^{\mp} \pm w_s\Lambda_{s-1}^{\pm}\right) = 0\,,$$

depending on whether s is even or odd. If s is even we conclude on

$$-t_s\Theta_{s-1}^{-} + u_s\Theta_{s-1}^{+} - v_s\Lambda_{s-1}^{-} + w_s\Lambda_{s-1}^{+} = 0\,, \tag{5.10}$$

where the left-hand side of (5.10) is a polynomial in k of degree $2s$ with leading coefficient

$$-t_s2^{2s}a_{s-1} + u_s(2 - 2^{2s})a_{s-1} = 0\,,$$

which follows from Lemma 3.5 and Lemma 3.6. Hence, we have

$$t_s = (2^{1-2s} - 1)u_s\,. \tag{5.11}$$

Then, the absolute term with respect to k on the left-hand side of (5.10) satisfies

$$(2 - 2^{1-2s})u_sa_{s-1} - v_sb_{s-1} + w_sb_{s-1} = \big((2 - 2^{1-2s})u_s - (2^{2s} - 1)(v_s - w_s)\big)a_{s-1} = 0\,,$$

which is equivalent to

$$2^{1-2s}u_s = v_s - w_s\,. \tag{5.12}$$

Comparing the k^2-terms in (5.10) we obtain with Lemma 3.6 and (5.12)

$$2^{1-2s}u_s = v_s + w_s\,. \tag{5.13}$$

Both equations (5.12) and (5.13) together give

$$v_s = 2^{1-2s}u_s\,, \qquad w_s = 0\,. \tag{5.14}$$

We substitute the results (5.11) and (5.14) into (5.9) and obtain

$$(2^{1-2s} - 1)u_s\Phi_{2s} + u_s\Phi_{2s}^* + 2^{1-2s}u_s\Psi_{2s} = 0\,.$$

Here, we may compare the KE/π^2-terms. Since s is even, by (3.5), (3.7) and (3.10) we get

$$(2^{1-2s} - 1)u_s + 2^{1-2s}u_s = (2^{2-2s} - 1)u_s = 0\,.$$

With

$$2^{2-2s} - 1 \leq -\frac{3}{4} \qquad (s \geq 2)$$

we conclude on $u_s = 0$ and then on $t_s = v_s = 0$. This proves that the numbers Φ_{2s}, Φ_{2s}^*, Ψ_{2s} and Ψ_{2s}^* are linearly independent over $\mathbb{Q}$ for any positive even integer $s \geq 2$.

If s is odd we have

$$-t_s\Theta^+_{s-1} + u_s\Theta^-_{s-1} + v_s\Lambda^+_{s-1} - w_s\Lambda^-_{s-1} = 0 \tag{5.15}$$

instead of (5.10). Similar as in the previous case we compute the leading coefficients, the constant terms and the k^2-terms in this polynomial and obtain

$$\begin{aligned} u_s &= (2^{1-2s} - 1)t_s\,, \\ 2^{1-2s}t_s &= v_s - w_s\,, \\ 2^{1-2s}t_s &= -v_s - w_s\,, \end{aligned}$$

which leads to

$$v_s = 0\,, \qquad w_s = -2^{1-2s}t_s\,.$$

Then we may compare the KE/π^2-terms in

$$-t_s\Phi_{2s} + (2^{1-2s} - 1)t_s\Phi^*_{2s} + 2^{1-2s}t_s\Psi^*_{2s} = 0\,.$$

Since s is odd, by (3.6), (3.8) and (3.13) we get

$$(2^{1-2s} - 1)t_s - 2^{1-2s}t_s = -t_s = 0\,.$$

We conclude on $u_s = w_s = 0$ and the theorem is proven. □

In the preceding two theorems we stated results on linear independence over $\mathbb{Q}$ for certain subsets of Ω. But we may also find subsets of Ω containing more than three elements, that are linearly dependend over $\mathbb{Q}$. For example, we have

$$(-2u + v)\,\Phi_2 + u\,\Phi^*_2 + (u - v)\,\Psi^*_2 - 7v\,\Phi_4 + 8v\,\Phi^*_4 + v\,\Psi_4 = 0 \qquad (u, v \in \mathbb{Q}) \tag{5.16}$$

and

$$\begin{aligned} &(-u - w)\,\Phi_2 + u\,\Phi^*_2 + w\,\Psi^*_2 + (u - v - w)\,\Phi_4 + v\,\Phi^*_4 + (u - w)\,\Psi_4 \\ &+ \left(\frac{128}{3}u - \frac{16}{3}v - \frac{128}{3}w\right)\Phi_6 + \left(-\frac{124}{3}u + \frac{31}{6}v + \frac{124}{3}w\right)\Phi^*_6 \\ &+ \left(-\frac{4}{3}u + \frac{1}{6}v + \frac{4}{3}w\right)\Psi^*_6 \;=\; 0 \qquad (u, v, w \in \mathbb{Q})\,. \end{aligned} \tag{5.17}$$

The identities (5.16) and (5.17) can be proven by formulas (3.5) to (3.13) for $s = 1, 2, 3$. To prove a general result on such linear dependencies one would need explicit expressions for all the coefficients of the polynomials $\Theta^\pm_j$ and $\Lambda^\pm_j$ as partially given by Lemma 3.6. After computing more examples like (5.16) and (5.17) we may conject the following:

Conjecture 5.1. *For any positive integer m the solutions of*

$$\sum_{s=1}^{m} \left(t_s\Phi_{2s} + u_s\Phi^*_{2s} + v_s\Psi_{2s} + w_s\Psi^*_{2s}\right) = 0$$

with $t_s, u_s, v_s, w_s \in \mathbb{Q}$ $(1 \le s \le m)$ form a $\mathbb{Q}$-vector space of dimension m. Moreover, each solution satisfies

$$\begin{aligned} v_s = 0 \quad (s \equiv 1 \pmod 2)\,, \\ w_s = 0 \quad (s \equiv 0 \pmod 2)\,. \end{aligned}$$

5.4. Results for $\Phi_{2s}(q), \Phi^*_{2s}(q), \Psi_{2s}(q), \Psi^*_{2s}(q)$ as functions of q

In this subsection we will study the reciprocal sums Φ_{2s}, Φ^*_{2s}, Ψ_{2s}, and Ψ^*_{2s} as functions of the independent variable $q = \beta^2$. Therefore, the quantities α and β are not fixed numbers anymore but variables satisfying $\alpha\beta = -1$.

Recall the relation

$$\beta = -e^{-\pi c/2} = -\sqrt{q}, \qquad c = \frac{K'}{K} = \frac{K(k')}{K(k)}$$

from Section 1.4. Hence, the reciprocal sums Φ_{2s}, Φ^*_{2s}, Ψ_{2s}, and Ψ^*_{2s} are written as functions of q in the following way:

$$\begin{aligned}
\Phi_{2s}(q) &= \sum_{n=1}^{\infty} \frac{1}{(q^{-n/2} + (-1)^{n+1} q^{n/2})^{2s}} = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} + q^n + 2(-1)^{n+1})^s}\,, \\
\Phi^*_{2s}(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(q^{-n/2} + (-1)^{n+1} q^{n/2})^{2s}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(q^{-n} + q^n + 2(-1)^{n+1})^s}\,, \\
\Psi_{2s}(q) &= \sum_{n=1}^{\infty} \frac{1}{(q^{-n/2} + (-1)^{n} q^{n/2})^{2s}} = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} + q^n + 2(-1)^{n})^s}\,, \\
\Psi^*_{2s}(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(q^{-n/2} + (-1)^{n} q^{n/2})^{2s}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(q^{-n} + q^n + 2(-1)^{n})^s}\,.
\end{aligned}$$

Let

$$f(k) := e^{-\pi c} = e^{-\pi K(\sqrt{1-k^2})/K(k)}.$$

Then, $q = f(k)$. By [8, formulae 111.02 and 112.01] we have

$$\lim_{k\to 0} f(k) = 0, \qquad \lim_{k\to 1} f(k) = 1,$$

and we derive

$$\begin{aligned}
&\frac{d}{dk} f(k) \\
&\quad = f(k)\frac{\pi}{k(1-k^2)K^2(k)}\left(E(\sqrt{1-k^2})K(k) + E(k)K(\sqrt{1-k^2}) - K(k)K(\sqrt{1-k^2})\right) \\
&\quad = f(k)\frac{\pi^2}{2k(1-k^2)K^2(k)} > 0
\end{aligned}$$

by using Legendre's Relation (see [8, formula 110.10]). Thus,

$$f : [0,1] \to [0,1]$$

is bijective and therefore the inverse function $k = f^{-1}(q)$ does exist. Hence, we may treat k as $k(q)$, such that the elliptic integrals $K(k)$ and $E(k)$ are also functions of q.

It follows from a theorem of Mahler [27] (see also [32]) that the Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$ are algebraically independent over $\mathbb{C}(q)$. Since

$$P(q), Q(q), R(q) \in \mathbb{Q}\left[\frac{K(q)}{\pi}, \frac{E(q)}{\pi}, k(q)\right]$$

by the formulas (1.11), we may apply Lemma 2.1 to the field extensions

$$\mathbb{C}(q) \subseteq \mathbb{C}\big(q, P(q), Q(q), R(q)\big) \subseteq \mathbb{C}\left(q, \frac{K(q)}{\pi}, \frac{E(q)}{\pi}, k(q)\right)$$

and obtain

$$\text{tr.deg}\left(\mathbb{C}\left(q, \frac{K(q)}{\pi}, \frac{E(q)}{\pi}, k(q)\right) : \mathbb{C}(q)\right) = 3\,.$$

We define

$$\Omega(q) := \{\Phi_{2s_1}(q), \Phi^*_{2s_2}(q), \Psi_{2s_3}(q), \Psi^*_{2s_4}(q) \;:\; s_1, \ldots, s_4 \in \mathbb{N}\}\,.$$

The expressions of Φ_{2s}, Φ^*_{2s}, Ψ_{2s}, and Ψ^*_{2s} in terms of K/π, E/π, and k, given in Section 3.2, are valid for any $q = f(k)$. This gives

$$\Phi_{2s}(q), \Phi^*_{2s}(q), \Psi_{2s}(q), \Psi^*_{2s}(q) \in \mathbb{Q}\left[\frac{K(q)}{\pi}, \frac{E(q)}{\pi}, k(q)\right]\,.$$

Theorem 5.7. *Let $g_1(q), g_2(q), g_3(q) \in \Omega(q)$ such that for any algebraic number q with $0 < |q| < 1$ the values $g_1(q)$, $g_2(q)$, and $g_3(q)$ are algebraically independent over $\mathbb{Q}$. Then the functions $g_1(q)$, $g_2(q)$, and $g_3(q)$ are algebraically independent over $\mathbb{C}(q)$.*

Proof. We apply Corollary 2.2 with

$$\mathbb{K} = \mathbb{C}(q)\,, \qquad n = 3\,, \qquad f_1(q) = k(q)\,, \qquad f_2(q) = \frac{K(q)}{\pi}\,, \qquad f_3(q) = \frac{E(q)}{\pi}\,.$$

Then there are polynomials

$$U_j(X_1, X_2, X_3) \in \mathbb{Q}[X_1, X_2, X_3] \subseteq \mathbb{K}[X_1, X_2, X_3] \qquad (j = 1, 2, 3)\,,$$

given in Section 3.2, satisfying

$$\begin{aligned} g_1(q) &= U_1\big(f_1(q), f_2(q), f_3(q)\big)\,,\\ g_2(q) &= U_2\big(f_1(q), f_2(q), f_3(q)\big)\,,\\ g_3(q) &= U_3\big(f_1(q), f_2(q), f_3(q)\big)\,. \end{aligned}$$

The determinant

$$\Delta(X_1, X_2, X_3) \;:=\; \det\left(\frac{\partial U_j}{\partial X_i}\right) \in \mathbb{Q}[X_1, X_2, X_3] \subseteq \mathbb{K}[X_1, X_2, X_3]$$

has been proven to be nonzero in Section 4 or in the proof of Theorem 5.3, respectively, where we have studied independence properties of the values of the functions $g_1(q)$, $g_2(q)$, and $g_3(q)$ at algebraic points. Therefore, by Corollary 2.2 the functions $g_1(q)$, $g_2(q)$, and $g_3(q)$ are algebraically independent over $\mathbb{C}(q)$. □

The following corollary is an analogue to Theorem 5.1.

Corollary 5.1. *Any two functions in $\Omega(q)$ are algebraically independent over $\mathbb{C}(q)$.*

Proof. Let $g_1(q), g_2(q) \in \Omega(q)$. Then there is a function $g_3(q) \in \Omega(q)$ such that for any algebraic number q with $0 < |q| < 1$ the values $g_1(q)$, $g_2(q)$, and $g_3(q)$ are algebraically independent over $\mathbb{Q}$. This is a consequence of Theorem 5.3. Hence, the conditions of Theorem 5.7 are fulfilled and we conclude on the algebraic independence of the functions $g_1(q)$, $g_2(q)$, and $g_3(q)$ over $\mathbb{C}(q)$. This proves the corollary. □

We remark that we did not need Nesterenko's Theorem 1.1 for the proofs in this subsection. All the algebraic independence results for the functions $\Phi_{2s}(q)$, $\Phi^*_{2s}(q)$, $\Psi_{2s}(q)$, and $\Psi^*_{2s}(q)$ over $\mathbb{C}(q)$ go back to Mahler's result. In general, there is no relation between the algebraic independence of functions $f_1(z), \ldots, f_n(z)$ over $\mathbb{C}(z)$ on the one hand, and the algebraic independence of their values at an algebraic point $z = z_0$ over $\mathbb{Q}$ on the other hand.

6. Quantitative results

6.1. An algebraic independence measure for $P(q), Q(q)$, and $R(q)$

In [32] Nesterenko also stated a quantitative version of Theorem 1.1 on Ramanujan's functions $P(q)$, $Q(q)$, and $R(q)$. One year later he improved this result in [33] and gave the following measure of their algebraic independence over $\mathbb{Q}$.

For each polynomial $A(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$, we denote by $\deg A$ the degree of the polynomial A with respect to the totality of variables and by $H(A)$ its height, that is the maximum of moduli of the coefficients of the polynomial A. Moreover, we set $t(A) = \deg A + \log H(A)$.

Theorem 6.1 (Nesterenko [33], 1997)**.** *Let q be an algebraic number, $0 < |q| < 1$, and let $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$ are such that all numbers $P(q), Q(q)$, and $R(q)$ are algebraic over the field $\mathbb{Q}(\omega_1, \omega_2, \omega_3)$. Then there exists a constant γ depending only on the numbers q and ω_i such that the following inequality holds for any polynomial $A \in \mathbb{Z}[X_1, X_2, X_3]$, $A \not\equiv 0$:*

$$|A(\omega_1, \omega_2, \omega_3)| > \exp(-\gamma S d^3 \log^9 S)\,,$$

where S and d are arbitrary numbers satisfying the inequalities

$$S \geq \max\{\log H(A) + \deg A \cdot \log t(A), e\} \qquad d \geq \deg A\,.$$

As an immediate corollary we obtain a first quantitative result for three-element subsets of Ω.

Corollary 6.1. *Let $\omega_1, \omega_2, \omega_3 \in \Omega$ be algebraically independent over $\mathbb{Q}$. Then there exists a constant γ depending only on $q = \beta^4$ and the ω_i such that the following inequality holds for any polynomial $A \in \mathbb{Z}[X_1, X_2, X_3]$, $A \not\equiv 0$:*

$$|A(\omega_1, \omega_2, \omega_3)| > \exp(-\gamma S d^3 \log^9 S)\,,$$

where S and d satisfy the conditions from Theorem 6.1.

Proof. Let $\mathbb{K} := \mathbb{Q}(\omega_1, \omega_2, \omega_3)$ and $\mathbb{L} := \mathbb{Q}(k, K/\pi, E/\pi)$. It suffices to show, that the numbers $P(q), Q(q), R(q) \in \mathbb{L}$ are algebraic over $\mathbb{K}$ for $q = \beta^4 \in \overline{\mathbb{Q}}$. Then the statement follows from Theorem 6.1.

Since ω_1, ω_2, and ω_3 are algebraically independent over $\mathbb{Q}$, we have $\operatorname{tr.deg}(\mathbb{K} : \mathbb{Q}) = 3$. Moreover, from Lemma 1.1 we have $\operatorname{tr.deg}(\mathbb{L} : \mathbb{Q}) = 3$. By the chain rule (Lemma 2.1) we obtain

$$\operatorname{tr.deg}(\mathbb{L} : \mathbb{K}) = \operatorname{tr.deg}(\mathbb{L} : \mathbb{Q}) - \operatorname{tr.deg}(\mathbb{K} : \mathbb{Q}) = 0\,.$$

Applying Lemma 2.1 once more gives

$$0 = \operatorname{tr.deg}(\mathbb{L} : \mathbb{K}) = \operatorname{tr.deg}(\mathbb{L} : \mathbb{K}(P(q), Q(q), R(q))) + \operatorname{tr.deg}(\mathbb{K}(P(q), Q(q), R(q)) : \mathbb{K})$$

and therefore

$$\operatorname{tr.deg}(\mathbb{K}(P(q), Q(q), R(q)) : \mathbb{K}) = 0\,,$$

which proves that the numbers $P(q), Q(q)$, and $R(q)$ are algebraic over $\mathbb{K}$. □

Corollary 6.1 provides an algebraic independence measure for three numbers $\omega_1, \omega_2, \omega_3 \in \Omega$ which depends on an implicit constant γ varying with the choice of the particular set $\{\omega_1, \omega_2, \omega_3\}$. This implicit dependency can be stated more precisely in an explicit way. In the following subsection we prove a general lemma.

6.2. A lemma on algebraic independence measures

The lemma to be proven in this subsection may be noticed as a quantitative supplement to Corollary 2.1. We shall restate the general situation.

Let $x_1, \ldots, x_n \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$. Moreover, let the following algebraic independence measure be given for the numbers $x_1, \ldots, x_n$: For any polynomial $A \in \mathbb{Z}[X_1, \ldots, X_n]$, $A \not\equiv 0$, satisfying $H(A) \leq H$ and $\deg A \leq m$ we have

$$|A(x_1, \ldots, x_n)| > T(m, H) \tag{6.1}$$

for some function $T : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}$.

Now, let $y_1, \ldots, y_n \in \mathbb{Q}[x_1, \ldots, x_n]$ be algebraically independent over $\mathbb{Q}$, where

$$y_j = P_j(x_1, \ldots, x_n), \quad P_j \in \mathbb{Q}[X_1, \ldots, X_n] \quad (j = 1, \ldots, n)\,.$$

Let

$$p_j := \deg P_j \qquad (j = 1, \ldots, n)\,, \qquad \mu := \max_{1 \leq j \leq n} p_j\,.$$

Moreover, for any $j \in \{1, \ldots, n\}$ we denote by d_j the least common multiple of the denominators of the coefficients from the polynomial P_j, such that

$$P_j' := d_j P_j \in \mathbb{Z}[X_1, \ldots, X_n] \quad (j = 1, \ldots, n)\,, \tag{6.2}$$

and

$$D := d_1 \cdots d_n\,.$$

Additionaly, we shall define the *broadness* $b(A)$ of a polynomial A as the number of its monomials.

The following lemma provides an algebraic independence measure for the numbers $y_1, \ldots, y_n$ depending on the measure of $x_1, \ldots, x_n$ and some characteristic quantities concerning the polynomials $P_1, \ldots, P_n$:

Lemma 6.1. *For any polynomial $B \in \mathbb{Z}[X_1, \ldots, X_n]$, $B \not\equiv 0$, satisfying $H(B) \leq H$, $b(B) \leq b$ and $\deg B \leq m$ we have*

$$|B(y_1, \ldots, y_n)| > \frac{1}{D^m} T(\mu m, bH\kappa^m)\,,$$

where

$$\kappa = D^2 \prod_{j=1}^{n} H(P_j)(n+1)^{p_j}\,,$$

and T is the function given by (6.1).

Remark 6.1. It is not usual to take into account the broadness of a polynomial in the theory of algebraic independence measures. It is clear that the number of monomials of a polynomial $B \in \mathbb{Z}[X_1, \ldots, X_n]$, $B \not\equiv 0$, with $\deg B \le m$ is bounded by

$$b(B) \le \binom{m+n}{n}.$$

Therefore, if we omit the condition $b(B) \le b$ in Lemma 6.1, we obtain the weaker result

$$|B(y_1, \ldots, y_n)| > \frac{1}{D^m} T\left(\mu m, \binom{m+n}{n} H\kappa^m\right).$$

Proof of Lemma 6.1. We use the expression (6.2) to find

$$B(y_1, \ldots, y_n) = B\big(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n)\big),$$

where the coefficients of the polynomial

$$B(P_1, \ldots, P_n) = B\left(\frac{P_1'}{d_1}, \ldots, \frac{P_n'}{d_n}\right) \in \mathbb{Q}[X_1, \ldots, X_n]$$

are not integers, in general. Multiplying with $D^{\deg B}$ gives

$$A(X_1, \ldots, X_n) := D^{\deg B} B(P_1, \ldots, P_n) \in \mathbb{Z}[X_1, \ldots, X_n].$$

Then we have

$$B(y_1, \ldots, y_n) = \frac{1}{D^{\deg B}} A(x_1, \ldots, x_n). \tag{6.3}$$

It remains to estimate the degree and the height of A. With the use of (6.1) we will then obtain a lower bound for $|B(y_1, \ldots, y_n)|$. Let

$$B(X_1, \ldots, X_n) = \sum_{(i_1, \ldots, i_n)} \beta_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}, \qquad \beta_{i_1, \ldots, i_n} \in \mathbb{Z}.$$

It is well-known (see [45, Theorem 11, §18, Ch. I]) that

$$\deg(P_1^{i_1} \cdots P_n^{i_n}) = i_1 \deg P_1 + \cdots + i_n \deg P_n = i_1 p_1 + \cdots + i_n p_n.$$

With $i_1 + \cdots + i_n \le m$ and $p_j \le \mu$ for $j = 1, \ldots, n$ this yields

$$\deg A \le \mu m.$$

To estimate the height of A we use [33, Lemma 1.1] and obtain

$$H(P_1^{i_1} \cdots P_n^{i_n}) \le H(P_1)^{i_1} \cdots H(P_n)^{i_n} (n+1)^{i_1 p_1 + \cdots + i_n p_n}.$$

Furthermore, for any polynomials $f_1, f_2 \in \mathbb{Q}[X_1, \ldots, X_n]$ and rational numbers λ_1, λ_2 we have

$$H(\lambda_1 f_1 + \lambda_2 f_2) \le |\lambda_1| H(f_1) + |\lambda_2| H(f_2).$$

We derive

$$\begin{aligned}
H(A) &= H\big(D^{\deg B} B(P_1, \dots, P_n)\big) \\
&= H\left(\sum_{(i_1,\dots,i_n)} \beta_{i_1,\dots,i_n} d_1^{\deg B} P_1^{i_1} \cdots d_n^{\deg B} P_n^{i_n} \right) \\
&\le \sum_{(i_1,\dots,i_n)} |\beta_{i_1,\dots,i_n}| H(d_1^{\deg B} P_1^{i_1} \cdots d_n^{\deg B} P_n^{i_n}) \\
&\le H(B) \sum_{(i_1,\dots,i_n)} d_1^{\deg B - i_1} \cdots d_n^{\deg B - i_n} H(d_1^{i_1} P_1^{i_1} \cdots d_n^{i_n} P_n^{i_n}) \\
&\le H(B) D^{\deg B} \sum_{(i_1,\dots,i_n)} H(d_1 P_1)^{i_1} \cdots H(d_n P_n)^{i_n} (n+1)^{i_1 p_1 + \dots + i_n p_n} \\
&\le H(B) D^{\deg B} b(B) \max_{(i_1,\dots,i_n)} \big(H(d_1 P_1)^{i_1} \cdots H(d_n P_n)^{i_n} (n+1)^{i_1 p_1 + \dots + i_n p_n} \big) \\
&\le bHD^m \prod_{j=1}^{n} d_j^m H(P_j)^m (n+1)^{m p_j} \\
&= bH\kappa^m \,.
\end{aligned}$$

Finally, from (6.1) and (6.3) we conclude on

$$|B(y_1, \dots, y_n)| = \frac{1}{D^{\deg B}} |A(x_1, \dots, x_n)| > \frac{1}{D^m} T(\mu m, bH\kappa^m) \,,$$

as desired. □

Combining Nesterenko's result from Theorem 6.1 and Lemma 6.1, we obtain algebraic independence measures for all three-element subsets of Ω depending only on one implicit constant.

Let $q = \beta^4 = e^{-2\pi c}$ for $c = K'/K$. Then, by formulas (1.11), the numbers $P(q)$, $Q(q)$ and $R(q)$ are algebraic over the field $\mathbb{Q}(k, K/\pi, E/\pi)$. Therefore, we may apply Theorem 6.1 with $(\omega_1, \omega_2, \omega_3) = (k, K/\pi, E/\pi)$, such that the inequality

$$\left| A\left(k, \frac{K}{\pi}, \frac{E}{\pi}\right) \right| > \exp(-\gamma S d^3 \log^9 S) \tag{6.4}$$

holds for any polynomial $A \in \mathbb{Z}[X_1, X_2, X_2]$, $A \not\equiv 0$, where S and d are defined in Theorem 6.1. The constant γ depends only on $q, k, K/\pi$ and E/π. Since $K = K(k)$, $E = E(k)$ and $k = k(q)$ (see Subsection 5.4), γ does actually depend only on $q = \beta^4$. The algebraic independence measures for all three-element subsets of Ω obtained from Lemma 6.1 only depend on this fixed constant γ. By the following example we apply Lemma 6.1 with $(x_1, x_2, x_3) = (k, K/\pi, E/\pi)$ and $(y_1, y_2, y_3) = (\Phi_2, \Phi_4, \Phi_6)$.

Example 6.1. For any polynomial $B \in \mathbb{Z}[X_1, X_2, X_3]$, $B \not\equiv 0$, satisfying $H(B) \le H$, $b(B) \le b$ and $\deg B \le m$ we have

$$|B(\Phi_2, \Phi_4, \Phi_6)| > \frac{1}{4\,180\,377\,600^m} \exp(-1728 \gamma S m^3 \log^9 S) \,,$$

where S is an arbitrary number satisfying

$$S \geq \log(bH) + 72.43m + 12m \log\big(\log(bH) + 84.43m\big)\,.$$

Proof. With (3.5) and (3.6) we compute the polynomials

$$\begin{aligned}
P_1(X_1,X_2,X_3) &= \frac{1}{24} - \frac{1}{6}X_2^2 + \frac{1}{3}X_1^2X_2^2\,,\\
P_2(X_1,X_2,X_3) &= -\frac{11}{1440} + \frac{1}{6}X_2X_3 - \frac{5}{36}X_2^2 + \frac{1}{9}X_1^2X_2^2\\
&\quad + \frac{1}{90}X_2^4 - \frac{8}{45}X_1^2X_2^4 + \frac{8}{45}X_1^4X_2^4\,,\\
P_3(X_1,X_2,X_3) &= \frac{191}{120\,960} - \frac{1}{180}X_2^2 + \frac{1}{90}X_1^2X_2^2\\
&\quad - \frac{1}{360}X_2^4 - \frac{7}{180}X_1^2X_2^4 + \frac{7}{180}X_1^4X_2^4\\
&\quad - \frac{1}{945}X_2^6 + \frac{11}{315}X_1^2X_2^6 - \frac{31}{315}X_1^4X_2^6 + \frac{62}{945}X_1^6X_2^6\,,
\end{aligned}$$

satisfying

$$\Phi_2 = P_1\left(k, \frac{K}{\pi}, \frac{E}{\pi}\right), \qquad \Phi_4 = P_2\left(k, \frac{K}{\pi}, \frac{E}{\pi}\right), \qquad \Phi_6 = P_3\left(k, \frac{K}{\pi}, \frac{E}{\pi}\right).$$

We get

$$\mu = \max\{\deg P_1, \deg P_2, \deg P_3\} = 12\,,$$

$$H(P_1) = \frac{1}{3}\,, \qquad H(P_2) = \frac{8}{45}\,, \qquad H(P_3) = \frac{31}{315}\,,$$

and

$$d_1 = 24\,, \qquad d_2 = 1440\,, \qquad d_3 = 120\,960\,.$$

This yields

$$D = 4\,180\,377\,600\,, \qquad \kappa = D^2 \prod_{j=1}^{3} H(P_j) 4^{\deg P_j} = 28\,686\,540\,327\,344\,554\,955\,144\,114\,995\,200\,.$$

Now, we may apply Lemma 6.1 and obtain the lower bound

$$\begin{aligned}
|B(\Phi_2,\Phi_4,\Phi_6)| &> \frac{1}{4\,180\,377\,600^m} T(\mu m, bH\kappa^m)\\
&= \frac{1}{4\,180\,377\,600^m} T\left(12m, bH \cdot 28\,686\,540\,327\,344\,554\,955\,144\,114\,995\,200^m\right)
\end{aligned}$$

for any polynomial $B \in \mathbb{Z}[X_1,X_2,X_3]$, $B \not\equiv 0$, satisfying $H(B) \leq H$, $b(B) \leq b$ and $\deg B \leq m$, where T is the function on the right-hand side of (6.4). Hence, we need to substitute $\deg A$ by μm and $H(A)$ by $bH\kappa^m$ in Theorem 6.1 which gives

$$|B(\Phi_2,\Phi_4,\Phi_6)| > \frac{1}{4\,180\,377\,600^m} \exp(-1728\gamma S m^3 \log^9 S)$$

where S is an arbitrary number satisfying

$$S \geq \max \{\log(bH) + m \log (28\,686\,540\,327\,344\,554\,955\,144\,114\,995\,200) + 12m \log t_1, e\} \ ,$$

with

$$t_1 := 12m + \log(bH) + m \log (28\,686\,540\,327\,344\,554\,955\,144\,114\,995\,200) \ .$$

Hence, we obtain

$$S \geq \log(bH) + 72.43m + 12m \log \big(\log(bH) + 84.43m \big) \, .$$

□

Remark 6.2. According to Remark 6.1 we use the estimate

$$b(B) \leq \binom{m+3}{3} \qquad (B \in \mathbb{Z}[X_1, X_2, X_3], \ \deg B \leq m)$$

to get rid of the condition $b(B) \leq b$ in the preceding example. This yields

$$\log b \leq 3 \log(m+3) \, .$$

Then, for any polynomial $B \in \mathbb{Z}[X_1, X_2, X_3]$, $B \not\equiv 0$, satisfying $H(B) \leq H$ and $\deg B \leq m$ we have

$$|B(\Phi_2, \Phi_4, \Phi_6)| > \frac{1}{4\,180\,377\,600^m} \exp(-1728\gamma S m^3 \log^9 S)$$

where S is an arbitrary number satisfying

$$S \geq \log(H) + 3\log(m+3) + 72.43m + 12m \log \big(\log(H) + 3\log(m+3) + 84.43m \big) \, .$$

7. Conclusion

In general it is a difficult problem to prove the transcendence of a given number. But, once the transcendence of two numbers α and β is proven, it is even harder to answer the question on their algebraic independence over $\mathbb{Q}$, apart from trivial cases where the algebraic dependence is obvious like $\beta \in \mathbb{Q}[\alpha]$. For instance, it is known for more than 120 years that the numbers e and π are transcendental. The problem on their algebraic independence is still open.

During the last century, a lot of different methods have been established to decide on the algebraic independence of a given set of transcendental numbers, when these numbers belong to particular classes like values of E-functions in the case of Siegel-Shidlovskii. The determinant criterion applied in this thesis is very recent and yet has already led to interesting results (see [21]). More than 20 years after André-Jeannin's result on the irrationality of $\zeta_F(1)$, $\zeta_F^*(1)$, $\zeta_L(1)$, and $\zeta_L^*(1)$ we are able to prove algebraic independence results for values of these zeta functions with the help of that determinant criterion. Theorem 5.3 of this thesis gives a complete answer to the question on algebraic independent subsets of

$$\{\zeta_F(2s_1), \zeta_F^*(2s_2), \zeta_L(2s_3), \zeta_L^*(2s_4) \,|\, s_1, s_2, s_3, s_4 \in \mathbb{N}\}\,.$$

Unfortunately, we cannot prove anything on the algebraic character of the values of the above zeta functions at positive odd integers. This is due to the fact that the identities from Zucker [46] used in this thesis do not cover the odd case. Also such identities seem difficult to find. As mentioned in the introduction, even the arithmetic character of $\zeta_F(3)$ is still unknown. The situation is similar to the Riemann zeta function $\zeta(s)$. While this function is known to take transcendental values at positive even integers, our knowledge of $\zeta(s)$ at positive odd integers s is rather small. In 1979 Apéry [3] showed that $\zeta(3)$ is irrational. About 20 years later, Zudilin [47] could prove, that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is irrational. We are far away from algebraic independence results for such numbers.

By Lemma 6.1 we give a quantitative supplement to the independence criterion from Section 2. If the algebraic independence for a certain set of numbers is proven, it is natural to ask for a measure of their independence. Our lemma provides a method to transcribe the measure from one number set to another, if these sets are connected via polynomials in an *explicit* way. It would also be interesting to have such a lemma for the general *implicit* situation as in Lemma 2.3.

Further research in this area could also be focussed on the linear case. We could not prove Conjecture 5.1 up until now, since our knowledge of the Jacobian elliptic functions investigated in Section 3 is too small. An explicit formula for the Laurent series expansions of these functions would be very helpful.

A. Some identities for algebraically dependent numbers in Ω

In [20] Elsner, Shiokawa and Shimomura showed that 198 of the 220 three-element subsets of

$$\Gamma = \{\Phi_2, \Phi_4, \Phi_6, \Phi_2^*, \Phi_4^*, \Phi_6^*, \Psi_2, \Psi_4, \Psi_6, \Psi_2^*, \Psi_4^*, \Psi_6^*\} \subset \Omega\,.$$

are algebraically independent over $\mathbb{Q}$. For the remaining 22 subsets they computed the following explicit polynomial identities:

1. $\{\Phi_2, \Phi_2^*, \Psi_2^*\}$: total deg = 1,

$$-2\,\Phi_2 + \Phi_2^* + \Psi_2^* = 0$$

2. $\{\Psi_2, \Psi_4, \Psi_2^*\}$: total deg = 2,

$$2 \cdot 2\,\Psi_2^2 + \Psi_2 - 3 \cdot 2\,\Psi_4 - \Psi_2^* = 0$$

3. $\{\Phi_2, \Phi_6, \Phi_4^*\}$: total deg = 3,

$$-2^3\,{\Phi_2}^3 + 3\,{\Phi_2}^2 - 3 \cdot 2^2\,\Phi_4^*\Phi_2 - 3\,\Phi_4^* + 7 \cdot 2\,\Phi_6 = 0$$

4. $\{\Phi_2, \Phi_6, \Psi_2\}$: total deg = 3,

$$\begin{aligned}&-31 \cdot 2^4\,{\Phi_2}^3 + 3 \cdot 2^4\,\Phi_2{\Psi_2}^2 - 9 \cdot 2^2\,{\Phi_2}^2 + 3 \cdot 2^2\,\Phi_2\Psi_2\\ &+3 \cdot 2^2\,{\Psi_2}^2 + 35 \cdot 2^3\,\Phi_6 - 3\,\Phi_2 + 3\,\Psi_2 = 0\end{aligned}$$

5. $\{\Phi_2, \Phi_6, \Psi_6\}$: total deg = 9,

$$\begin{aligned}&-31 \cdot 2^{20}\,{\Phi_2}^9 - 9 \cdot 2^{18}\,{\Phi_2}^8 + 357 \cdot 2^{17}\,\Phi_6{\Phi_2}^6 + 63 \cdot 2^{13}\,{\Phi_2}^7 + 63 \cdot 2^{15}\,\Phi_6{\Phi_2}^5\\ &-9 \cdot 2^{14}\,{\Phi_2}^6 - 5439 \cdot 2^{12}\,{\Phi_6}^2{\Phi_2}^3 - 1071 \cdot 2^9\,\Phi_6{\Phi_2}^4 - 783 \cdot 2^4\,{\Phi_2}^5 + 27 \cdot 2^{12}\,{\Phi_2}^3{\Psi_6}^2\\ &-441 \cdot 2^{10}\,{\Phi_6}^2{\Phi_2}^2 + 693 \cdot 2^8\,\Phi_6{\Phi_2}^3 + 189 \cdot 2^2\,{\Phi_2}^4 + 27 \cdot 2^6\,{\Phi_2}^3\Psi_6 + 81 \cdot 2^{10}\,{\Phi_2}^2{\Psi_6}^2\\ &+1715 \cdot 2^{11}\,{\Phi_6}^3 + 2205 \cdot 2^6\,{\Phi_6}^2\Phi_2 + 189 \cdot 2^5\,\Phi_6{\Phi_2}^2 - 27 \cdot 2^3\,{\Phi_2}^3 + 81 \cdot 2^4\,{\Phi_2}^2\Psi_6\\ &+81 \cdot 2^8\,\Phi_2{\Psi_6}^2 - 3087 \cdot 2^4\,{\Phi_6}^2 - 189 \cdot 2^2\,\Phi_6\Phi_2 + 81 \cdot 2^2\,\Phi_2\Psi_6 + 27 \cdot 2^6\,{\Psi_6}^2\\ &+189\,\Phi_6 + 27\,\Psi_6 = 0\end{aligned}$$

6. $\{\Phi_2, \Phi_6, \Psi_4^*\}$: total deg = 5,

$$\begin{aligned}&-31 \cdot 2^8\,{\Phi_2}^5 + 11 \cdot 2^7\,{\Phi_2}^4 + 35 \cdot 2^7\,\Phi_6{\Phi_2}^2 - 5 \cdot 2^3\,{\Phi_2}^3 + 3 \cdot 2^6\,\Phi_2\Psi_4^{*2}\\ &-35 \cdot 2^5\,\Phi_6\Phi_2 + 3\,{\Phi_2}^2 - 3 \cdot 2^2\,\Phi_2\Psi_4^* + 3 \cdot 2^4\,\Psi_4^{*2} + 35 \cdot 2\,\Phi_6 - 3\,\Psi_4^* = 0\end{aligned}$$

7. $\{\Phi_2, \Phi_4^*, \Psi_2\}$: total deg = 2,

$$7 \cdot 2^2\,{\Phi_2}^2 - 2^2\,{\Psi_2}^2 - 5 \cdot 2^2\,\Phi_4^* + \Phi_2 - \Psi_2 = 0$$

8. $\{\Phi_2, \Phi_4^*, \Psi_6\}$: total deg $= 6$,

$$7 \cdot 2^{11}\, {\Phi_2}^6 - 3 \cdot 2^{13}\, \Phi_4^* {\Phi_2}^4 - 3 \cdot 2^{10}\, {\Phi_2}^5 + 27 \cdot 2^9\, \Phi_4^{*2} {\Phi_2}^2 + 15 \cdot 2^8\, \Phi_4^* {\Phi_2}^3$$
$$+ 9 \cdot 2^6\, {\Phi_2}^4 - 5 \cdot 2^9\, \Phi_4^{*3} - 9 \cdot 2^7\, \Phi_4^{*2} \Phi_2 - 21 \cdot 2^5\, \Phi_4^* {\Phi_2}^2 - 5 \cdot 2^3\, {\Phi_2}^3$$
$$+ 21 \cdot 2^3\, \Phi_4^{*2} + 9 \cdot 2^2\, \Phi_4^* \Phi_2 + 3\, {\Phi_2}^2 - 2^7\, {\Psi_6}^2 - 3\, \Phi_4^* - 2\, \Psi_6 = 0$$

9. $\{\Phi_2, \Phi_4^*, \Psi_4^*\}$: total deg $= 4$,

$$7 \cdot 2^6\, {\Phi_2}^4 - 5 \cdot 2^6\, \Phi_4^* {\Phi_2}^2 - 3 \cdot 2^5\, {\Phi_2}^3 + 5 \cdot 2^4\, \Phi_4^* \Phi_2$$
$$+ 2^2\, {\Phi_2}^2 - 2^4\, \Psi_4^{*2} - 5\, \Phi_4^* + \Psi_4^* = 0$$

10. $\{\Phi_2, \Psi_2, \Psi_6\}$: total deg $= 3$,

$$3 \cdot 2^4\, {\Phi_2}^2 \Psi_2 + 2^4\, {\Psi_2}^3 + 3 \cdot 2\, {\Phi_2}^2 - 3 \cdot 2^3\, \Phi_2 \Psi_2$$
$$+ 3 \cdot 2\, {\Psi_2}^2 - 3\, \Phi_2 + 3\, \Psi_2 - 5 \cdot 2^3\, \Psi_6 = 0$$

11. $\{\Phi_2, \Psi_2, \Psi_4^*\}$: total deg $= 2$,

$$2^3\, \Phi_2 \Psi_2 + \Phi_2 - \Psi_2 - 2^2\, \Psi_4^* = 0$$

12. $\{\Phi_2, \Psi_6, \Psi_4^*\}$: total deg $= 5$,

$$-3 \cdot 2^{10}\, {\Phi_2}^4 \Psi_4^* + 3 \cdot 2^5\, {\Phi_2}^4 + 5 \cdot 2^{10}\, {\Phi_2}^3 \Psi_6 + 9 \cdot 2^8\, {\Phi_2}^3 \Psi_4^* - 2^5\, {\Phi_2}^3$$
$$-15 \cdot 2^7\, {\Phi_2}^2 \Psi_6 - 9 \cdot 2^6\, {\Phi_2}^2 \Psi_4^* - 2^8\, \Psi_4^{*3} + 3\, {\Phi_2}^2 + 15 \cdot 2^4\, \Phi_2 \Psi_6$$
$$+ 15 \cdot 2^2\, \Phi_2 \Psi_4^* + 3 \cdot 2^3\, \Psi_4^{*2} - 5 \cdot 2\, \Psi_6 - 3\, \Psi_4^* = 0$$

13. $\{\Phi_6, \Phi_4^*, \Psi_2\}$: total deg $= 6$,

$$-2^{10}\, {\Psi_2}^6 - 3 \cdot 2^8\, {\Psi_2}^5 - 9 \cdot 2^{12}\, {\Psi_2}^4 \Phi_4^* + 57 \cdot 2^4\, {\Psi_2}^4 - 9 \cdot 2^{11}\, {\Psi_2}^3 \Phi_4^*$$
$$-1581 \cdot 2^8\, {\Psi_2}^2 \Phi_4^{*2} + 67 \cdot 2^3\, {\Psi_2}^3 + 147 \cdot 2^9\, {\Psi_2}^2 \Phi_6 - 531 \cdot 2^4\, {\Psi_2}^2 \Phi_4^* - 1581 \cdot 2^6\, \Psi_2 \Phi_4^{*2}$$
$$-4805 \cdot 2^8\, \Phi_4^{*3} + 69\, {\Psi_2}^2 + 147 \cdot 2^7\, \Psi_2 \Phi_6 - 387 \cdot 2^2\, \Psi_2 \Phi_4^* + 16807 \cdot 2^6\, {\Phi_6}^2$$
$$-1617 \cdot 2^5\, \Phi_6 \Phi_4^* - 507 \cdot 2^5\, \Phi_4^{*2} + 161 \cdot 2\, \Phi_6 - 69\, \Phi_4^* = 0$$

14. $\{\Phi_6, \Phi_4^*, \Psi_6\}$: total deg $= 9$,

$$
\begin{aligned}
&-4805 \cdot 2^{38}\, \Phi_4^{*9} + 302673 \cdot 2^{34}\, \Phi_6{}^2\Phi_4^{*6} - 104139 \cdot 2^{34}\, \Phi_6\Phi_4^{*7} + 84015 \cdot 2^{32}\, \Phi_4^{*8} \\
&-3999 \cdot 2^{34}\, \Phi_4^{*6}\Psi_6{}^2 - 24713493 \cdot 2^{28}\, \Phi_6{}^4\Phi_4^{*3} + 8084853 \cdot 2^{29}\, \Phi_6{}^3\Phi_4^{*4} \\
&-5598495 \cdot 2^{28}\, \Phi_6{}^2\Phi_4^{*5} - 381171 \cdot 2^{29}\, \Phi_6{}^2\Phi_4^{*3}\Psi_6{}^2 + 1788297 \cdot 2^{27}\, \Phi_6\Phi_4^{*6} \\
&+262899 \cdot 2^{29}\, \Phi_6\Phi_4^{*4}\Psi_6{}^2 - 789615 \cdot 2^{25}\, \Phi_4^{*7} - 3999 \cdot 2^{28}\, \Phi_4^{*6}\Psi_6 - 50985 \cdot 2^{28}\, \Phi_4^{*5}\Psi_6{}^2 \\
&-501 \cdot 2^{28}\, \Phi_4^{*3}\Psi_6{}^4 + 40353607 \cdot 2^{26}\, \Phi_6{}^6 - 73060029 \cdot 2^{25}\, \Phi_6{}^5\Phi_4^* \\
&+257860197 \cdot 2^{22}\, \Phi_6{}^4\Phi_4^{*2} - 352947 \cdot 2^{26}\, \Phi_6{}^4\Psi_6{}^2 - 53881527 \cdot 2^{22}\, \Phi_6{}^3\Phi_4^{*3} \\
&+361179 \cdot 2^{27}\, \Phi_6{}^3\Phi_4^*\Psi_6{}^2 + 26384589 \cdot 2^{20}\, \Phi_6{}^2\Phi_4^{*4} - 381171 \cdot 2^{23}\, \Phi_6{}^2\Phi_4^{*3}\Psi_6 \\
&-2500029 \cdot 2^{23}\, \Phi_6{}^2\Phi_4^{*2}\Psi_6{}^2 + 1029 \cdot 2^{26}\, \Phi_6{}^2\Psi_6{}^4 - 589617 \cdot 2^{20}\, \Phi_6\Phi_4^{*5} \\
&+262899 \cdot 2^{23}\, \Phi_6\Phi_4^{*4}\Psi_6 - 95361 \cdot 2^{22}\, \Phi_6\Phi_4^{*3}\Psi_6{}^2 - 3591 \cdot 2^{25}\, \Phi_6\Phi_4^*\Psi_6{}^4 \\
&+126063 \cdot 2^{21}\, \Phi_4^{*6} - 50985 \cdot 2^{22}\, \Phi_4^{*5}\Psi_6 + 284013 \cdot 2^{20}\, \Phi_4^{*4}\Psi_6{}^2 - 501 \cdot 2^{23}\, \Phi_4^{*3}\Psi_6{}^3 \\
&+11493 \cdot 2^{22}\, \Phi_4^{*2}\Psi_6{}^4 - 2^{26}\, \Psi_6{}^6 - 7815255 \cdot 2^{20}\, \Phi_6{}^5 + 453789 \cdot 2^{19}\, \Phi_6{}^4\Phi_4^* \\
&-352947 \cdot 2^{20}\, \Phi_6{}^4\Psi_6 + 10782891 \cdot 2^{16}\, \Phi_6{}^3\Phi_4^{*2} + 361179 \cdot 2^{21}\, \Phi_6{}^3\Phi_4^*\Psi_6 - \\
&252105 \cdot 2^{20}\, \Phi_6{}^3\Psi_6{}^2 - 11714871 \cdot 2^{15}\, \Phi_6{}^2\Phi_4^{*3} - 2500029 \cdot 2^{17}\, \Phi_6{}^2\Phi_4^{*2}\Psi_6 \\
&+56889 \cdot 2^{22}\, \Phi_6{}^2\Phi_4^*\Psi_6{}^2 + 1029 \cdot 2^{21}\, \Phi_6{}^2\Psi_6{}^3 + 1338687 \cdot 2^{13}\, \Phi_6\Phi_4^{*4} \\
&-95361 \cdot 2^{16}\, \Phi_6\Phi_4^{*3}\Psi_6 + 378693 \cdot 2^{16}\, \Phi_6\Phi_4^{*2}\Psi_6{}^2 - 3591 \cdot 2^{20}\, \Phi_6\Phi_4^*\Psi_6{}^3 \\
&+1365 \cdot 2^{21}\, \Phi_6\Psi_6{}^4 - 3400245 \cdot 2^{10}\, \Phi_4^{*5} + 284013 \cdot 2^{14}\, \Phi_4^{*4}\Psi_6 - 43167 \cdot 2^{17}\, \Phi_4^{*3}\Psi_6{}^2 \\
&+11493 \cdot 2^{17}\, \Phi_4^{*2}\Psi_6{}^3 - 6615 \cdot 2^{18}\, \Phi_4^*\Psi_6{}^4 - 3 \cdot 2^{20}\, \Psi_6{}^5 + 3334989 \cdot 2^{14}\, \Phi_6{}^4 \\
&-842751 \cdot 2^{14}\, \Phi_6{}^3\Phi_4^* - 252105 \cdot 2^{14}\, \Phi_6{}^3\Psi_6 + 2117241 \cdot 2^{11}\, \Phi_6{}^2\Phi_4^{*2} \\
&+56889 \cdot 2^{16}\, \Phi_6{}^2\Phi_4^*\Psi_6 - 91287 \cdot 2^{14}\, \Phi_6{}^2\Psi_6{}^2 + 787941 \cdot 2^{8}\, \Phi_6\Phi_4^{*3} \\
&+378693 \cdot 2^{10}\, \Phi_6\Phi_4^{*2}\Psi_6 - 75789 \cdot 2^{12}\, \Phi_6\Phi_4^*\Psi_6{}^2 + 1365 \cdot 2^{16}\, \Phi_6\Psi_6{}^3 + 297837 \cdot 2^{4}\, \Phi_4^{*4} \\
&-85833 \cdot 2^{10}\, \Phi_4^{*3}\Psi_6 + 60705 \cdot 2^{9}\, \Phi_4^{*2}\Psi_6{}^2 - 6615 \cdot 2^{13}\, \Phi_4^*\Psi_6{}^3 + 3417 \cdot 2^{12}\, \Psi_6{}^4 \\
&-431837 \cdot 2^{8}\, \Phi_6{}^3 - 78057 \cdot 2^{7}\, \Phi_6{}^2\Phi_4^* - 23079 \cdot 2^{10}\, \Phi_6{}^2\Psi_6 - 120015 \cdot 2^{3}\, \Phi_6\Phi_4^{*2} \\
&-68607 \cdot 2^{6}\, \Phi_6\Phi_4^*\Psi_6 + 29463 \cdot 2^{7}\, \Phi_6\Psi_6{}^2 - 3429 \cdot 2^{6}\, \Phi_4^{*3} + 37719 \cdot 2^{3}\, \Phi_4^{*2}\Psi_6 \\
&-6615 \cdot 2^{6}\, \Phi_4^*\Psi_6{}^2 + 3427 \cdot 2^{7}\, \Psi_6{}^3 + 168021\, \Phi_6{}^2 + 24003 \cdot 2\, \Phi_6\Psi_6 + 3429\, \Psi_6{}^2 = 0
\end{aligned}
$$

15. $\{\Phi_6, \Phi_4^*, \Psi_4^*\}$: total deg $= 6$,

$$
\begin{aligned}
&8379 \cdot 2^{8}\, \Phi_6\Phi_4^*\Psi_4^* - 8379 \cdot 2^{12}\, \Phi_6\Phi_4^*\Psi_4^{*2} - 26061 \cdot 2^{13}\, \Phi_6\Phi_4^{*2}\Psi_4^* \\
&+19551 \cdot 2^{14}\, \Phi_6{}^2\Phi_4^*\Psi_4^* + 26061 \cdot 2^{17}\, \Phi_6\Phi_4^{*2}\Psi_4^{*2} - 19551 \cdot 2^{18}\, \Phi_6{}^2\Phi_4^*\Psi_4^{*2} \\
&+1519 \cdot 2^{6}\, \Phi_6{}^2 + 48363 \cdot 2^{12}\, \Phi_6{}^3 + 279\, \Psi_4^{*2} + 1319 \cdot 2^{7}\, \Phi_4^{*3} + 279\, \Phi_4^{*2} \\
&+823543 \cdot 2^{16}\, \Phi_6{}^4 - 277 \cdot 2^{5}\, \Psi_4^{*3} - 735 \cdot 2^{8}\, \Phi_6\Phi_4^{*2} - 569037 \cdot 2^{16}\, \Phi_6{}^3\Phi_4^* \\
&-128037 \cdot 2^{9}\, \Phi_6{}^2\Phi_4^* + 205947 \cdot 2^{15}\, \Phi_6{}^2\Phi_4^{*2} - 13699 \cdot 2^{14}\, \Phi_6\Phi_4^{*3} \\
&-235445 \cdot 2^{18}\, \Phi_6{}^2\Phi_4^{*3} + 302715 \cdot 2^{17}\, \Phi_6\Phi_4^{*4} + 34209 \cdot 2^{10}\, \Phi_4^{*4} - 389205 \cdot 2^{14}\, \Phi_4^{*5} \\
&-2^{18}\, \Psi_4^{*6} + 3 \cdot 2^{14}\, \Psi_4^{*5} + 93 \cdot 2^{19}\, \Phi_4^{*2}\Psi_4^{*4} - 8649 \cdot 2^{18}\, \Phi_4^{*4}\Psi_4^{*2} \\
&-20853 \cdot 2^{14}\, \Phi_4^{*3}\Psi_4^{*2} + 8649 \cdot 2^{14}\, \Phi_4^{*4}\Psi_4^* + 21 \cdot 2^{19}\, \Phi_6\Psi_4^{*4} - 93 \cdot 2^{16}\, \Phi_4^{*2}\Psi_4^{*3} \\
&-651 \cdot 2^{4}\, \Phi_6\Psi_4^* - 201 \cdot 2^{15}\, \Phi_4^*\Psi_4^{*4} - 279 \cdot 2\, \Phi_4^*\Psi_4^* - 735 \cdot 2^{9}\, \Phi_6{}^2\Psi_4^* \\
&+735 \cdot 2^{13}\, \Phi_6{}^2\Psi_4^{*2} + 819 \cdot 2^{8}\, \Phi_6\Psi_4^{*2} - 2733 \cdot 2^{6}\, \Phi_4^{*2}\Psi_4^* - 21 \cdot 2^{16}\, \Phi_6\Psi_4^{*3} \\
&+20853 \cdot 2^{10}\, \Phi_4^{*3}\Psi_4^* + 201 \cdot 2^{12}\, \Phi_4^*\Psi_4^{*3} - 525 \cdot 2^{5}\, \Phi_4^*\Psi_4^{*2} \\
&+651 \cdot 2^{4}\, \Phi_6\Phi_4^* + 2919 \cdot 2^{10}\, \Phi_4^{*2}\Psi_4^{*2} + 267 \cdot 2^{8}\, \Psi_4^{*4} = 0
\end{aligned}
$$

16. $\{\Phi_6, \Psi_2, \Psi_6\}$: total deg = 9,

$$
\begin{aligned}
&-3441 \cdot 2^{13}\, {\Psi_2}^3{\Psi_6}^2 + 5481\, \Phi_6 + 783\, \Psi_6 - 1359 \cdot 2^6\, {\Psi_6}^2 - 1323 \cdot 2^4\, {\Phi_6}^2 \\
&-783 \cdot 2^3\, {\Psi_2}^3 - 10503 \cdot 2^2\, {\Psi_2}^4 + 3213 \cdot 2^7\, \Phi_6\Psi_6 + 4805 \cdot 2^{12}\, {\Psi_6}^3 \\
&+837 \cdot 2^5\, \Psi_2\Psi_6 - 189 \cdot 2^{16}\, \Phi_6{\Psi_2}^5 + 3213 \cdot 2^{13}\, \Phi_6{\Psi_2}^2\Psi_6 + 6939 \cdot 2^{10}\, {\Psi_2}^4\Psi_6 \\
&-10323 \cdot 2^{10}\, {\Psi_2}^2{\Psi_6}^2 - 3969 \cdot 2^7\, {\Phi_6}^2\Psi_2 + 16155 \cdot 2^7\, {\Psi_2}^3\Psi_6 - 279 \cdot 2^{12}\, {\Psi_2}^6 \\
&-11097 \cdot 2^5\, {\Psi_2}^5 - 1323 \cdot 2^{13}\, {\Phi_6}^2{\Psi_2}^3 + 34911 \cdot 2^4\, {\Psi_2}^2\Psi_6 + 153 \cdot 2^{16}\, {\Psi_2}^5\Psi_6 \\
&-10287 \cdot 2^9\, \Psi_2{\Psi_6}^2 - 135 \cdot 2^{14}\, {\Psi_2}^7 + 6237 \cdot 2^4\, \Phi_6\Psi_2 + 3213 \cdot 2^{11}\, \Phi_6\Psi_2\Psi_6 \\
&-3969 \cdot 2^{10}\, {\Phi_6}^2{\Psi_2}^2 - 567 \cdot 2^{10}\, \Phi_6{\Psi_2}^3 + 7371 \cdot 2^6\, \Phi_6{\Psi_2}^2 + 51 \cdot 2^{18}\, {\Psi_2}^6\Psi_6 \\
&-945 \cdot 2^{13}\, \Phi_6{\Psi_2}^4 - 9 \cdot 2^{18}\, {\Psi_2}^8 - 2^{21}\, {\Psi_2}^9 = 0
\end{aligned}
$$

17. $\{\Phi_6, \Psi_2, \Psi_4^*\}$: total deg = 5,

$$
\begin{aligned}
&-9 \cdot 2^8\, {\Psi_2}^5 - 3 \cdot 2^{10}\, {\Psi_2}^4\Psi_4^* - 35 \cdot 2^{10}\, \Phi_6{\Psi_2}^3 - 21 \cdot 2^6\, {\Psi_2}^4 - 3 \cdot 2^9\, {\Psi_2}^3\Psi_4^* \\
&-105 \cdot 2^7\, \Phi_6{\Psi_2}^2 - 2^5\, {\Psi_2}^3 + 63 \cdot 2^5\, {\Psi_2}^2\Psi_4^* + 111 \cdot 2^6\, \Psi_2\Psi_4^{*2} + 31 \cdot 2^8\, \Psi_4^{*3} \\
&-105 \cdot 2^4\, \Phi_6\Psi_2 - 3\, {\Psi_2}^2 + 27 \cdot 2^2\, \Psi_2\Psi_4^* + 9 \cdot 2^4\, \Psi_4^{*2} - 35 \cdot 2\, \Phi_6 + 3\, \Psi_4^* = 0
\end{aligned}
$$

18. $\{\Phi_6, \Psi_6, \Psi_4^*\}$: total deg = 9,

$$
\begin{aligned}
&-87885 \cdot 2^{25}\, \Phi_6{\Psi_6}^4\Psi_4^* + 1971\, {\Psi_6}^2 + 96579\, {\Phi_6}^2 - 49049 \cdot 2^7\, {\Phi_6}^3 \\
&+64827 \cdot 2^{12}\, {\Phi_6}^4 + 112995 \cdot 2^{20}\, {\Psi_6}^5 - 1971 \cdot 2^6\, \Psi_4^{*3} + 66885 \cdot 2^{20}\, {\Phi_6}^3{\Psi_6}^2 \\
&+5145 \cdot 2^{15}\, {\Phi_6}^3\Psi_6 + 21 \cdot 2^{21}\, \Phi_6{\Psi_6}^2 + 13797 \cdot 2\, \Phi_6\Psi_6 - 67179 \cdot 2^7\, {\Phi_6}^2\Psi_6 \\
&-61593 \cdot 2^{14}\, {\Phi_6}^2{\Psi_6}^2 - 797475 \cdot 2^{21}\, {\Phi_6}^2{\Psi_6}^3 + 1079841 \cdot 2^{14}\, \Phi_6{\Psi_6}^3 + 87885 \cdot 2^{20}\, \Phi_6{\Psi_6}^4 \\
&+11135 \cdot 2^9\, {\Psi_6}^3 + 139671 \cdot 2^{15}\, {\Psi_6}^4 + 4185 \cdot 2^{20}\, \Psi_4^{*6} - 2672901 \cdot 2^{17}\, {\Phi_6}^2\Psi_6\Psi_4^{*2} \\
&+158697 \cdot 2^7\, \Phi_6\Psi_6\Psi_4^* + 1608831 \cdot 2^{10}\, \Phi_6\Psi_6\Psi_4^{*2} - 77469 \cdot 2^{28}\, {\Phi_6}^2{\Psi_6}^2\Psi_4^{*3} \\
&-80535 \cdot 2^{15}\, \Phi_6\Psi_6\Psi_4^{*3} + 26649 \cdot 2^{26}\, \Phi_6{\Psi_6}^3\Psi_4^{*2} + 41895 \cdot 2^{13}\, {\Phi_6}^2\Psi_6\Psi_4^* \\
&+1203993 \cdot 2^{13}\, \Phi_6{\Psi_6}^2\Psi_4^* + 1528065 \cdot 2^{25}\, {\Phi_6}^3{\Psi_6}^2\Psi_4^* - 56889 \cdot 2^{28}\, {\Phi_6}^2\Psi_6\Psi_4^{*4} \\
&-336393 \cdot 2^9\, \Psi_4^{*5} + 377643 \cdot 2^{22}\, {\Phi_6}^2\Psi_6\Psi_4^{*3} - 1398663 \cdot 2^{16}\, \Phi_6{\Psi_6}^2\Psi_4^{*2} \\
&+147 \cdot 2^{34}\, \Phi_6\Psi_6\Psi_4^{*6} - 194481 \cdot 2^{20}\, \Phi_6{\Psi_6}^3\Psi_4^* + 232407 \cdot 2^{23}\, {\Phi_6}^2{\Psi_6}^2\Psi_4^{*2} \\
&+679413 \cdot 2^{21}\, \Phi_6{\Psi_6}^2\Psi_4^{*3} + 213507 \cdot 2^{20}\, \Phi_6\Psi_6\Psi_4^{*4} + 162477 \cdot 2^{13}\, \Phi_6\Psi_4^{*4} \\
&-441 \cdot 2^{30}\, \Phi_6\Psi_6\Psi_4^{*5} + 46305 \cdot 2^{19}\, {\Phi_6}^3\Psi_6\Psi_4^* + 46305 \cdot 2^{28}\, {\Phi_6}^3\Psi_6\Psi_4^{*2} \\
&+57825 \cdot 2^5\, {\Psi_6}^2\Psi_4^* - 450261 \cdot 2^5\, {\Phi_6}^2\Psi_4^* + 1523655 \cdot 2^9\, {\Phi_6}^2\Psi_4^{*2} \\
&-32679 \cdot 2^{11}\, {\Psi_6}^2\Psi_4^{*2} + 372519 \cdot 2^3\, \Phi_6\Psi_4^{*2} + 215649 \cdot 2^8\, \Phi_6\Psi_4^{*3} - 466965 \cdot 2^4\, \Psi_4^{*4} \\
&-2408301 \cdot 2^{19}\, {\Phi_6}^2{\Psi_6}^2\Psi_4^* - 2^{37}\, \Psi_4^{*9} + 9 \cdot 2^{32}\, \Psi_4^{*8} - 1611 \cdot 2^{24}\, \Psi_4^{*7} \\
&+124497 \cdot 2^{14}\, {\Psi_6}^3\Psi_4^* + 189 \cdot 2^{33}\, \Phi_6\Psi_4^{*7} - 64827 \cdot 2^{27}\, {\Phi_6}^4\Psi_4^{*3} - 30429 \cdot 2^{27}\, {\Phi_6}^2\Psi_4^{*5} \\
&-2499 \cdot 2^{27}\, \Phi_6\Psi_4^{*6} + 837 \cdot 2^{27}\, {\Psi_6}^4\Psi_4^{*3} + 81 \cdot 2^{28}\, {\Psi_6}^3\Psi_4^{*4} - 567 \cdot 2^{27}\, {\Psi_6}^2\Psi_4^{*5} \\
&-1563 \cdot 2^{26}\, \Psi_6\Psi_4^{*6} + 194481 \cdot 2^{22}\, {\Phi_6}^4\Psi_4^{*2} + 265825 \cdot 2^{28}\, {\Phi_6}^3{\Psi_6}^3 \\
&+583443 \cdot 2^{21}\, {\Phi_6}^3\Psi_4^{*3} + 247401 \cdot 2^{21}\, {\Phi_6}^2\Psi_4^{*4} + 40383 \cdot 2^{19}\, \Phi_6\Psi_4^{*5} \\
&-2511 \cdot 2^{22}\, {\Psi_6}^4\Psi_4^{*2} + 189 \cdot 2^{22}\, {\Psi_6}^3\Psi_4^{*3} + 5913 \cdot 2^{21}\, {\Psi_6}^2\Psi_4^{*4} \\
&+8811 \cdot 2^{21}\, \Psi_6\Psi_4^{*5} - 194481 \cdot 2^{17}\, {\Phi_6}^4\Psi_4^* - 268569 \cdot 2^{16}\, {\Phi_6}^3\Psi_4^{*2} \\
&-3157413 \cdot 2^{14}\, {\Phi_6}^2\Psi_4^{*3} + 891 \cdot 2^{21}\, {\Psi_6}^4\Psi_4^* - 243729 \cdot 2^{17}\, {\Psi_6}^3\Psi_4^{*2} \\
&-14781 \cdot 2^{19}\, {\Psi_6}^2\Psi_4^{*3} - 232425 \cdot 2^{12}\, \Psi_6\Psi_4^{*4} + 157437 \cdot 2^{12}\, {\Phi_6}^3\Psi_4^* \\
&-51921 \cdot 2^{11}\, \Psi_6\Psi_4^{*3} - 41391 \cdot 2^3\, \Psi_6\Psi_4^{*2} = 0
\end{aligned}
$$

19. $\{\Phi_4^*, \Psi_2, \Psi_6\}$: total deg $= 6$,

$$
\begin{aligned}
&2^{11}\,{\Psi_2}^6 + 3\cdot 2^{11}\,\Phi_4^*{\Psi_2}^4 + 3\cdot 2^9\,{\Psi_2}^5 + 9\cdot 2^9\,\Phi_4^{*2}{\Psi_2}^2 + 3\cdot 2^{10}\,\Phi_4^*{\Psi_2}^3\\
&+9\cdot 2^6\,{\Psi_2}^4 - 7\cdot 2^{10}\,{\Psi_2}^3\Psi_6 + 9\cdot 2^7\,\Phi_4^{*2}\Psi_2 - 9\cdot 2^6\,\Phi_4^*{\Psi_2}^2 - 21\cdot 2^9\,\Phi_4^*\Psi_2\Psi_6\\
&+9\cdot 2^3\,{\Psi_2}^3 - 21\cdot 2^7\,{\Psi_2}^2\Psi_6 + 9\cdot 2^3\,\Phi_4^{*2} - 81\cdot 2^2\,\Phi_4^*\Psi_2 - 21\cdot 2^6\,\Phi_4^*\Psi_6\\
&+27\,{\Psi_2}^2 - 9\cdot 2^7\,\Psi_2\Psi_6 + 49\cdot 2^7\,{\Psi_6}^2 - 27\,\Phi_4^* - 9\cdot 2\,\Psi_6 = 0
\end{aligned}
$$

20. $\{\Phi_4^*, \Psi_2, \Psi_4^*\}$: total deg $= 4$,

$$
\begin{aligned}
&-2^6\,{\Psi_2}^4 - 5\cdot 2^6\,\Phi_4^*{\Psi_2}^2 - 2^5\,{\Psi_2}^3 - 5\cdot 2^4\,\Phi_4^*\Psi_2 + 2^2\,{\Psi_2}^2\\
&+2^6\,\Psi_2\Psi_4^* + 7\cdot 2^4\,\Psi_4^{*2} - 5\,\Phi_4^* + \Psi_4^* = 0
\end{aligned}
$$

21. $\{\Phi_4^*, \Psi_6, \Psi_4^*\}$: total deg $= 6$,

$$
\begin{aligned}
&-9\cdot 2^{18}\,{\Psi_6}^2\Phi_4^*\Psi_4^{*2} - 33\cdot 2^{17}\,\Psi_6\Phi_4^{*3}\Psi_4^* - 3\cdot 2^4\,\Psi_6\Phi_4^* - 7\cdot 2^5\,\Phi_4^{*3} + 2^6\,{\Psi_6}^2\\
&+9\cdot 2^8\,\Phi_4^{*4} - 7\cdot 2^{16}\,{\Psi_6}^4 + 9\,\Phi_4^{*2} + 2^8\,\Psi_4^{*3} + 9\,\Psi_4^{*2}\\
&-5\cdot 2^{12}\,{\Psi_6}^3 + 2^{18}\,\Psi_4^{*6} - 3\cdot 2^{14}\,\Psi_4^{*5} + 3\cdot 2^{16}\,{\Psi_6}^3\Psi_4^* + 3\cdot 2^{15}\,{\Psi_6}^2\Psi_4^{*2}\\
&-2^{13}\,\Psi_6\Psi_4^{*3} - 3\cdot 2^9\,{\Psi_6}^2\Psi_4^* + 9\cdot 2^8\,\Psi_6\Psi_4^{*2} - 3\cdot 2^{17}\,\Psi_6\Phi_4^*\Psi_4^{*3} + 99\cdot 2^{12}\,\Psi_6\Phi_4^{*2}\Psi_4^*\\
&-45\cdot 2^8\,\Psi_6\Phi_4^*\Psi_4^* + 9\cdot 2^{14}\,{\Psi_6}^2\Phi_4^*\Psi_4^* + 9\cdot 2^{18}\,\Phi_4^{*4}\Psi_4^{*2} + 3\cdot 2^{19}\,\Phi_4^{*2}\Psi_4^{*4}\\
&-35\cdot 2^{18}\,{\Psi_6}^2\Phi_4^{*3} - 9\cdot 2^{14}\,\Phi_4^{*4}\Psi_4^* - 45\cdot 2^{14}\,\Phi_4^{*3}\Psi_4^{*2} - 3\cdot 2^{16}\,\Phi_4^{*2}\Psi_4^{*3}\\
&+9\cdot 2^{14}\,\Phi_4^*\Psi_4^{*4} + 105\cdot 2^{13}\,{\Psi_6}^2\Phi_4^{*2} - 2^{13}\,\Psi_6\Phi_4^{*3} + 3\cdot 2^{12}\,\Phi_4^{*3}\Psi_4^*\\
&+63\cdot 2^{10}\,\Phi_4^{*2}\Psi_4^{*2} - 21\cdot 2^{10}\,\Phi_4^*\Psi_4^{*3} - 57\cdot 2^9\,{\Psi_6}^2\Phi_4^* + 3\cdot 2^8\,\Psi_6\Phi_4^{*2}\\
&-15\cdot 2^5\,\Phi_4^{*2}\Psi_4^* - 9\cdot 2^6\,\Phi_4^*\Psi_4^{*2} + 3\cdot 2^4\,\Psi_6\Psi_4^* - 9\cdot 2\,\Phi_4^*\Psi_4^* = 0
\end{aligned}
$$

22. $\{\Psi_2, \Psi_6, \Psi_4^*\}$: total deg $= 4$,

$$
\begin{aligned}
&-2^5\,{\Psi_2}^4 - 2^4\,{\Psi_2}^3 - 3\,{\Psi_2}^2 + 5\cdot 2^4\,\Psi_2\Psi_6 + 3\cdot 2^2\,\Psi_2\Psi_4^*\\
&-3\cdot 2^3\,\Psi_4^{*2} + 5\cdot 2\,\Psi_6 + 3\,\Psi_4^* = 0
\end{aligned}
$$

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