Heterotic supergravity on manifolds with Killing spinors

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Zusammenfassung

Die vorliegende Arbeit handelt von der Konstruktion heterotischer String-Vakua auf Mannigfaltigkeiten mit reellen Killing Spinoren. Letztere spielen seit Langem eine wichtige Rolle in der Stringtheorie, was durch Bärs Korrespondenz zwischen Killing Spinoren auf einer Mannigfaltigkeit M und parallelen Spinoren auf dem Kegel über M erklärt werden kann. Da parallele Spinoren zu Lösungen der Supergravitations-BPS-Gleichungen führen, werden diese gelöst durch die Kegelmetrik. In der Typ II Stringtheorie sowie der M-Theorie ist es in geeigneter Dimension außerdem möglich eine Bran an der Spitze des Kegels zu platzieren, und die zugehörigen Supergravitationslösungen sind bekannt. Weit entfernt von der Bran approximieren sie die Kegel-Lösung, während sie nahe des Bran-Horizonts die Form einer Freund-Rubin-Lösung annehmen, welche aus dem Produkt eines Anti-de Sitter Raums mit der Basis M besteht.

In der heterotischen Supergravitation hingegen gibt es zwei bekannte branartige Lösungen; die NS5-Bran, welche aus einem $\mathbb{R}^4$-Faktor mit Flüssen (Differentialformen) besteht und einem transversalen Minkowski-Raum, sowie die eich-solitonischen Branen. Diese basieren auf einem Instanton Eichfeld auf dem $\mathbb{R}^p$ und weiteren Flüssen, sowie einem transversalen $(10-p)$-dimensionalen Minkowski-Raum. Die bisher konstruierten Beispiele haben $p = 4, 7$ und 8, und die zugehörigen Instantonen sind die BPST und oktonionischen Instantonen.

Mannigfaltigkeiten mit reellen Killing Spinoren wurden klassifiziert: neben den runden Sphären sind dies 6-dimensionale nearly Kähler, 7-dimensionale fast parallele G_2, Sasaki-Einstein, oder 3-Sasaki Mannigfaltigkeiten. Basierend auf dieser Klassifizierung stelle ich eine Verallgemeinerung der eich-solitonischen Branen auf den Kegel über einer Killing Spinor Mannigfaltigkeit vor. Insbesondere beinhaltet dies die Konstruktion von Instantonen auf dem Kegel. Desweiteren zeige ich daß homogene Killing Spinor Mannigfaltigkeiten Lösungen besitzen die dem Horizont-Limes der NS5-Bran ähneln. Schließlich wird die Instanton Gleichung auf dem Zylinder über dem Kegel einer 6-dimensionalen nearly Kähler Mannigfaltigkeit untersucht. Einige Lösungen existieren, und es bleibt ein interessantes Problem für anschließende Arbeiten diese zu Lösungen der heterotischen Supergravitation zu erweitern.

Neben dem Kegel ist der sogenannte Sinus-Kegel über einer Killing Spinor Mannigfaltigkeit von Bedeutung. Beispielsweise ist bekannt daß der Sinus-Kegel über einer 5-dimensionalen Sasaki-Einstein Mannigfaltigkeit eine nearly Kähler Struktur trägt, und derjenige über einer nearly Kähler Mannigfaltigkeit eine fast parallele G_2-Struktur. Ich beweise eine Verallgemeinerung die besagt, daß der Sinus-Kegel über einer beliebigen Killing Spinor Mannigfaltigkeit wieder einen Killing Spinor besitzt. Insbesondere hat der doppelte Sinus-Kegel über einer Sasaki-Einstein Mannigfaltigkeit wieder eine Sasaki-Einstein Struktur.

Schlagwörter: Heterotische Supergravitation, reelle Killing Spinoren, Yang-Mills Instantonen

Abstract

The present work deals with the construction of heterotic string backgrounds on manifolds with real Killing spinors. The latter have played an important role in string theory for a long time, mainly due to Bär's correspondence between Killing spinors on a manifold M and parallel spinors on the cone over M. Given the fact that parallel spinors always lead to exact supergravity BPS backgrounds, it implies that the cone admits a solution of the BPS equations. Furthermore, in type II string theory and in M-theory it is possible to place a brane at the tip of the cone, in appropriate dimensions, and the resulting supergravity solutions are exactly known. In the limit far away from the brane they converge to the empty space solution, whereas in the near horizon limit one obtains a so-called Freund-Rubin solution, consisting of an anti-de Sitter space times our base manifold M.

In heterotic supergravity on the other hand two types of brane-like solutions are known; the NS5-brane, consisting of an $\mathbb{R}^4$-factor with fluxes and a transverse 6-dimensional Minkowski space, and what is sometimes called the gauge solitonic branes. These come equipped with an instanton gauge field on some Euclidean space $\mathbb{R}^p$, which carries further non-vanishing fluxes, and again a transverse $(10-p)$-dimensional Minkowski space. The possible values for p that appeared in the literature so far are $p = 4, 7$ and 8, and the corresponding instantons are the famous BPST and octonionic instantons.

Manifolds with real Killing spinors have been classified: besides the round spheres they are either 6-dimensional nearly Kähler, 7-dimensional nearly parallel G_2, Sasaki-Einstein, or 3-Sasakian. I present a generalization of the gauge solitonic branes to the cone over any real Killing spinor manifold, based upon this classification. In particular, this involves the construction of instantons on the cone. Additionally, I show that for homogeneous manifolds with real Killing spinors there is a solution similar to the near horizon limit of the NS5-brane. Finally, the instanton equation on the cylinder over the cone over a 6-dimensional nearly Kähler manifold is investigated. Several instanton solutions exist, and to embed these into heterotic supergravity is an interesting problem for future work.

Besides the cone, the so-called sine-cone over a Killing spinor manifold is important. For instance, it is known that the sine-cone over a 5-dimensional Sasaki-Einstein manifold is nearly Kähler, and the one over a nearly Kähler manifold is nearly parallel G_2. I generalize these results by proving that the sine-cone over an arbitrary real Killing spinor manifold has a real Killing spinor again. It is shown in particular, that the iterated sine-cone over a Sasaki-Einstein manifold also carries a Sasaki-Einstein-structure.

Keywords: Heterotic supergravity, real Killing spinors, Yang-Mills instantons

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1 Introduction

1.1 Motivation

Two of the great open problems of current theoretical physics are the construction of a theory of quantum gravity and of a fundamental theory of particle physics that would contain our current standard model of elementary particle physics as a limiting case, but at the same time rely on fewer parameters that need to be fitted to experiments. String theory has been proposed as a candidate to solve both problems at once, which is one of the reasons why it has received considerable attention from the physics community in the past twenty five years.

Gravity is contained in all types of string theories, and the so-called superstrings are described in their low-energy limits by 10-dimensional supergravity theories. These are classical field theories defined on an arbitrary manifold M with typical field content a Lorentzian metric (or graviton in physical terminology), several form fields, and their fermionic superpartners. The supergravity equations of motion can be derived from string theory as the condition for conformal invariance of the quantized sigma model on M, which exhibits a strange duality between classical properties of the supergravity model and quantum properties on the string theory side.

All superstring theories are thought to be related to each other via certain duality transformations, and also to the 11-dimensional hypothetical M-theory. The latter must be approximated by the unique 11-dimensional supergravity at low energies, which is also the highest-dimensional consistent supergravity. Typically, the duality transformations relate the weak coupling limit of one theory with the strong coupling limit of another string theory, so that in particular the supergravity approximation cannot be valid for all of them at the same time. Several proposals have been made how to extract the standard model of elementary particle physics from the different string theories, in type II theory for instance one considers intersecting brane scenarios [68]. But the most straight-forward one is heterotic supergravity, since it contains a gauge field with non-abelian gauge group SO(32) or $E_8 \times E_8$, both of which contain the standard model gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and also several groups proposed for grand unified theories.

To obtain a realistic model for our universe in the supergravity approximation to string theory it is necessary to reduce the dimensions of the model from ten to four. In the heterotic setting this is usually done by compactification: the space time is chosen in the form $M = M^4 \times M^6$, where M^4 is any Lorentzian 4-dimensional manifold, and M^6 a compact Euclidean manifold of sufficiently small size to be unobservable. Then one can perform a Kaluza-Klein reduction to obtain an effectively 4-dimensional model. This approach is being applied rather successfully in the case where M^6 is Ricci-flat, i.e. a Calabi-Yau 3-fold. Several models have been found which approximate the standard model very closely (see e.g. [6] and references therein), and the most serious problem seems to be that there is no simple principle that would select one out of them. Another problem that plagues many Calabi-Yau backgrounds is the appearance of moduli; the Kähler structure and complex structure of M^6 typically admit deformations, which should appear as massless scalar fields in the 4-dimensional theory. Since the standard model does not contain such fields, they spoil the phenomenological properties of the models.

The moduli problem may be circumvented by so-called flux-compactifications. In this case the internal manifold M^6 is no longer Ricci-flat, and to solve the supergravity equations of motion one has to allow for non-vanishing form fields, the so-called fluxes. In the heterotic case the bosonic form fields are a 3-form H, a function ϕ (the 'dilaton'), and a gauge field A. The 3-form appears as torsion in a covariant derivative, which is why these flux compactifications are also called 'compactifications with torsion' [78]. Despite their relevance to phenomenological applications and a long history of investigations, comparatively few flux vacua of heterotic supergravity have been found, and it seems much harder to reproduce the desired particle physics properties within this framework, compared to the Calabi-Yau approach.

In the last years, the AdS/CFT correspondence has added another strong motivation to study supergravity theories. Its core feature is a duality between a theory of gravity on an $(n+1)$-dimensional manifold M, or some compactified gravitational theory on $M \times X$ with X compact, and a conformal field theory on the n-dimensional boundary of M. In most examples M is some anti-de Sitter space, and the conformal boundary is Minkowski space in one dimension lower. Maldacena's original work treats the duality between string theory on $\mathrm{AdS}_5 \times S^5$ and super-Yang-Mills on $\mathbb{R}^{3,1}$ [69].

An important property of the AdS/CFT correspondence is that it relates the strong coupling limit of one theory to the weak coupling limit of another, which allows one to gain information about a strongly coupled quantum field theory in a semiclassical approximation of the dual theory. Applications are under investigation to the quark-gluon plasma [65] and to condensed matter physics, for instance to explain certain exotic superconductors [55]. To find realistic dual gravity models for real condensed matter systems one needs to have a broad panoply of solutions to gravitational theories at one's disposal. In the context of heterotic supergravity solutions have been found that interpolate between two different AdS_3-backgrounds and that are dual to the renormalization group flow of a 2-dimensional conformal field theory [45]. The Callan-Harvey-Strominger model with a linear dilaton, on the other hand, describes a decoupling limit of NS5-branes that has no interpretation in terms of a local quantum field theory [5, 48].

For the reasons explained above it is desirable to get a better understanding of torsionful heterotic supergravity backgrounds, and in particular to construct further solutions. In the present work I will analyze the heterotic supergravity equations from a mathematical point of view, instead of focussing on solutions with predefined physical properties, like a compact 6-dimensional constituent. The focus is on supersymmetric solutions, which solve a set of first order BPS equations that (partly) imply the equations of motion of the theory.

To motivate the choice of manifolds to work with in this thesis let me briefly recapitulate some well-known solutions of 10D type II supergravity and 11D supergravity. To begin with, there is a common sector of all 10-dimensional supergravity theories, whose BPS equation is simply

$$\nabla \epsilon = 0, \tag{1.1}$$

where ϵ is a Majorana-Weyl spinor, and ∇ is the Levi-Civita connection of a Lorentzian manifold. This common sector is obtained by setting all other fields to zero, except for the gauge field in heterotic supergravity which has to coincide with the Levi-Civita connection. A similar sector exists in 11-dimensional supergravity, with the only obvious difference that spacetime is 11-dimensional there. Let us assume a solution of the form $\mathbb{R}^{p-1,1} \times X^n$, with $p + n = 10$ or 11 as appropriate. Equation (1.1) tells us that the holonomy of X must be contained in the stabilizer subgroup of $\mathrm{Spin}(n)$ of a spinor, and the possible groups that occur have been listed by Wang [82]; for irreducible X they are given in Table 1.1.

$\dim X$	$\mathrm{Hol}(X)$
$2k$	$\mathrm{SU}(k)$
$4k$	$\mathrm{Sp}(k)$
7	G_2
8	$\mathrm{Spin}(7)$

Table 1.1: Irreducible holonomy groups admitting parallel spinors.

An interesting class of manifolds with parallel spinors consists of the conical ones. It was shown by Bär that the cone over a Riemannian manifold M has a parallel spinor if and only if the base M has a so-called real Killing spinor [8]. Therefore we get solutions for the common sector BPS equations with $X = c(M)$, the cone over a manifold M with real Killing spinors. In the type II and 11-dimensional theories it is possible to obtain further solutions which describe a brane placed at the tip of the cone [1, 64]. In the limit far from the brane they approach the Ricci-flat $\mathbb{R}^{p-1,1} \times c(M)$-background described above, whereas the near horizon limit of the brane is of the form

$$\mathrm{AdS}_{p+1} \times M. \tag{1.2}$$

It should be emphasized that these are not solutions of the common sector, but that additional form fields appear which are specific to the theory at hand. The possible p-values of interest to us are

11D: $p = 3$ and M either nearly parallel G_2, (7-dimensional) Sasaki-Einstein or 3-Sasakian. There is a 4-form turned on, and the solutions are called M2-branes.

IIB: $p = 4$ and M 5-dimensional Sasaki-Einstein, with 5-form flux. This gives rise to the D3-brane.

IIA: $p = 3$ and M 6-dimensional nearly Kähler. Here 2-, 3- and 4-form flux appears.

The common sector of the 10-dimensional supergravities extends beyond the Ricci-flat solutions presented above. The BPS equations involve only the dilaton and the 3-form, and are given by

$$\nabla^{-}\epsilon = 0, \qquad \text{and} \qquad \gamma(d\phi - \tfrac{1}{2}H)\epsilon = 0, \tag{1.3}$$

where γ is the map from forms to the Clifford algebra and ∇^{-} is a connection with torsion proportional to H. For the heterotic theory one has to impose additionally that

$$\gamma(F)\epsilon = 0, \tag{1.4}$$

where F is the curvature 2-form of the Yang-Mills field, and the H-Bianchi identity is modified. Instead of $dH = 0$ as for type II theories, we must have

$$dH = \frac{\alpha'}{4}\mathrm{tr}\Big(R^{+} \wedge R^{+} - F \wedge F\Big), \tag{1.5}$$

where R^{+} is the curvature form of the connection ∇^{+}, whose torsion is minus the one of ∇^{-}. There is another brane solution shared by all 10-dimensional theories, the NS5-brane, which will be explained in Section 4.3, where the right hand side of the heterotic Bianchi identity (1.5) vanishes. Its asymptotic form is $\mathbb{R}^{5,1} \times \mathbb{R} \times S^3$, with non-vanishing 3-form flux on S^3 and a linear dilaton on $\mathbb{R}$ [22, 5].

Let us collect some important properties of heterotic BPS vaua. We need a 3-form H, which does not have to be closed, in contrast to most other supergravity theories, and there must be a metric-compatible connection with torsion equal to H and reduced holonomy. Additionally, a gauge field appears, which has to solve the so-called gaugino or instanton equation (1.4). It turns out that a manifold M with real Killing spinors can supply us with all of these ingredients. There is a so-called canonical 3-form P on M, as well as a canonical connection ∇^P whose torsion equals P. For nearly Kähler, nearly parallel G_2 and Sasaki-Einstein manifolds this is well known [40], and I will show in Section 3.3.4 that it is also the case for a 3-Sasakian manifold. Additionally, ∇^P solves the instanton equation, as will be shown in the text as well. Since P is not closed when $\dim(M) > 3$, we will not get solutions to the common sector equations, but intrinsically heterotic ones. The full solution will be of the form $\mathbb{R} \times M$, the metric conformal to the product metric, the dilaton a function on $\mathbb{R}$ only, and H will be proportional to P.

Furthermore we need the gauge field. In [61] instantons on the tangent bundle over a large class of non-symmetric homogeneous spaces have been constructed, and in fact many of those spaces carry a real Killing spinor. I use a similar ansatz to obtain instantons on every real Killing spinor manifold, and show that they can be extended to solutions of heterotic supergravity. These instantons coincide with those of [61] only in the case that M is a round sphere, and they generalize the BPST instanton on $\mathbb{R}^4$ [12], the octonionic instantons on $\mathbb{R}^7$ and $\mathbb{R}^8$ [35, 41, 62, 51], and the quaternionic instantons on $\mathbb{R}^{4m+4}$ [28, 20]. Their embedding into heterotic string theory generalizes a method introduced in [77, 56, 51]; I call this type of solutions 'gauge solitonic branes'. It is also possible to choose the gauge field to coincide with the canonical connection on M, at least for homogeneous Killing spinor manifolds; in this case we get a linear dilaton supergravity solution that resembles the near horizon limit of the NS5-brane.

An interesting relation between the different Killing spinor manifolds has been discovered in [15] and [36]. The so-called sine-cone over a 5-dimensional Sasaki-Einstein manifold is nearly Kähler, and the sine-cone over a nearly Kähler manifold is nearly parallel G_2. I prove an extension of this result: the sine-cone over an arbitrary manifold with real Killing spinors has again a real Killing spinor. As a special case of this proposition I show explicitly that the iterated sine-cone over a Sasaki-Einstein manifold is again Sasaki-Einstein. One can expect a similar result for 3-Sasakian spaces: the fourth sine-cone over a 3-Sasakian manifold should be 3-Sasakian again, but this has not yet been proven.

It is well-known that the cone over the sine-cone over a Riemannian manifold M is the same space as the cylinder over the cone over M, and if M has a real Killing spinor then this space has reduced holonomy. A particularly interesting case is when M is nearly Kähler. Then the cone over the sine-cone has holonomy Spin(7), and it contains several submanifolds with non-integrable G_2-structures. I show that solutions to the G_2-instanton equations on the submanifolds can be lifted to Spin(7)-instantons on the full space, and review some known solutions. The general embedding of these instantons into heterotic supergravity is left for future work.

An outline of the thesis is as follows. In Chapter 2 heterotic supergravity is introduced, with special attention given to the instanton equation. Chapter 3 develops the relevant geometry background, in particular the Killing spinor equation. I explain why cones, sine-cones and cylinders are important in the study of manifolds with real Killing spinors, give some detailed review of the four types of geometries that occur, and sketch why many non-symmetric homogeneous spaces carry real

Killing spinors. In Section 3.5 I review the explicit construction of G-structures on the cone and the sine-cone over the four types of real Killing spinor manifolds, and show that the iterated sine-cone over a Sasaki-Einstein manifold is again Sasaki-Einstein. In Chapter 4 the BPST instanton on the cylinder over the 3-sphere is introduced, as well as its embedding into supergravity (the 'gauge solitonic brane'), and the NS5-brane solution. This serves as a guideline for the following two chapters; the generalization of the BPST instanton and the gauge solitonic brane to the cone over a real Killing spinor manifold is presented in Chapter 5. This is based on the article [53] with D. Harland. Chapter 6 contains a brief summary of the construction of linear dilaton supergravity solutions, where the gauge field is identified with the canonical connection, as presented in [73]. This generalizes the near horizon limit of the NS5-brane. Finally, the Spin(7)-instanton equation on the cone over the sine-cone on a nearly Kähler manifold is studied in Chapter 7, based on joint work with K.-P. Gemmer, O. Lechtenfeld and A.D. Popov [46].

1.2 Conventions

Frame. Given a Riemannian manifold (M, g) we will always work with an orthonormal frame $\{e^\kappa\}$ for the cotangent bundle, where $\kappa, \lambda, \mu, \nu, \ldots$ are generic indices; the dual frame of vector fields will be denoted $\{I_\kappa\}$. We will adopt the shorthand $e^{\mu\nu} = e^\mu \wedge e^\nu$ etc. Latin indices $a, b, c, \ldots$ typically cover only a subset of all coordinates, they are explained where they occur.

Connections. A connection ∇ on the tangent bundle TM of a manifold M is determined locally by the choice of frame $\{e^\mu\}$ and a 1-form Γ with values in the endomorphisms of TM. The 1-form can be expanded in a basis as

$$\Gamma = \Gamma^\mu_{\nu\lambda} e^\nu (I_\mu \otimes e^\lambda), \tag{1.6}$$

where we identify $TM \otimes T^*M \simeq \mathrm{End}(TM)$. The $\Gamma^\mu_{\nu\lambda}$ are the usual connection coefficients. In the following, the symbol ∇ will always refer to the Levi-Civita connection of a Riemannian manifold M, whereas other connections come equipped with additional labels. The curvature R of a connection on TM is a 2-form with values in $\mathrm{End}(TM)$, accordingly it can be expanded as

$$R = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda} e^{\kappa\lambda} (I_\mu \otimes e^\nu). \tag{1.7}$$

The symbol R will be reserved for the Riemannian curvature, i.e. the curvature 2-form of the Levi-Civita connection, whereas other curvature forms get additional labels.

Clifford algebra. The Clifford algebra generators are denoted by γ_μ, and they satisfy the Clifford relation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{1.8}$$

The quantization map γ from forms to the Clifford algebra is

$$\gamma\Big(\frac{1}{p!}\omega_{\mu_1\dots\mu_p}e^{\mu_1\dots\mu_p}\Big) = \frac{1}{p!}\omega_{\mu_1\dots\mu_p}\gamma^{\mu_1\dots\mu_p}, \tag{1.9}$$

where

$$\gamma^{\mu_1\dots\mu_p} = \frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sign}(\sigma)\gamma^{\mu_{\sigma(1)}}\dots\gamma^{\mu_{\sigma(p)}}. \tag{1.10}$$

The orthogonal Lie algebra $\mathfrak{so}(n)$ can be identified with the space of two-forms Λ^2 via

$$\mathfrak{so}(n) \to \Lambda^2,\ A \mapsto \frac{1}{4}g_{\mu\lambda}A^\lambda{}_\nu e^{\mu\nu}, \tag{1.11}$$

and this map gives rise to the isomorphism between $\mathfrak{so}(n)$ and the spin algebra $\mathfrak{spin}(n) \subset \mathrm{Cl}(n)$:

$$\mathfrak{so}(n) \to \mathfrak{spin}(n),\ A \mapsto \frac{1}{4}g_{\mu\lambda}A^\lambda{}_\nu\gamma^{\mu\nu}. \tag{1.12}$$

Gauge theory. Let ∇^A be a connection on a vector bundle $E \to M$, given in a local trivialization by a 1-form A with values in the endomorphism bundle of E, and let s be a section of $\Lambda T^*M \otimes \mathrm{End}(E)$. The exterior covariant derivative d_A acts as exterior derivative on the form part of s, and by ∇^A on the endormorphism part. If s is an r-form, then

$$d_A s = ds + A\wedge s + (-1)^{r-1}s\wedge A. \tag{1.13}$$

The curvature F of ∇^A is a section of $\Lambda^2T^*M \otimes \mathrm{End}(E)$,

$$F = dA + A\wedge A. \tag{1.14}$$

It satisfies the Bianchi identity

$$d_AF = dF + A\wedge F - F\wedge A = 0, \tag{1.15}$$

whereas the Yang-Mills equation

$$d_A * F = d*F + A\wedge *F + (-1)^{n-1} *F\wedge A = 0 \tag{1.16}$$

is a true restriction. Here $n = \dim M$ and $*$ is the Hodge dual defined in terms of a Riemannian metric g on M. The Yang-Mills operator can be expressed in a frame as

$$*\, d_A * F = (-1)^n(\nabla^{\Gamma\otimes A}_\mu F)_{\nu\lambda}g^{\mu\nu}e^\lambda, \tag{1.17}$$

where $\nabla^{\Gamma\otimes A}$ is the product connection of the Levi-Civita connection acting on two-forms and ∇^A acting on $\mathrm{End}(E)$. As usual, $g^{\mu\nu}$ denotes the components of the induced metric on the cotangent bundle.

Trace. It is a widely adopted convention in the string community to denote a positive definite inner product on a simple Lie algebra by 'tr'; this is just minus the ordinary trace

$$\mathrm{tr}(AB) = -A^{\mu}{}_{\nu} B^{\nu}{}_{\mu} \tag{1.18}$$

for $A, B \in \mathfrak{so}(n) \subset \mathrm{End}(\mathbb{R}^n)$.

2 Heterotic supergravity

2.1 Supergravity

The low-energy limit of heterotic string theory is given by 10D $\mathcal{N}=1$ supergravity coupled to super Yang-Mills, with bosonic field content [50]

- a metric g,
- two-form B, the Kalb-Ramond field,
- function ϕ, the dilaton,
- gauge field A, with gauge group contained in either SO(32) or $E_8 \times E_8$.

Their fermionic superpartners will be set to zero throughout the text. The curvature forms $F = dA + A \wedge A$ and

$$H = dB + \frac{\alpha'}{4}\Big(\mathrm{CS}(\Gamma^+) - \mathrm{CS}(A)\Big) \tag{2.1}$$

play an important role as well. Here the 3-forms $\mathrm{CS}(\Gamma^+)$ and $\mathrm{CS}(A)$ are the Chern-Simons forms of Γ^+ and A:

$$\mathrm{CS}(A) = \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A\Big). \tag{2.2}$$

Furthermore, Γ^+ and Γ^- are connections on the tangent bundle, related to the Levi-Civita connection Γ of g by

$$(\Gamma^{\pm})^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} \mp \frac{1}{2} H^{\kappa}{}_{\mu\nu}. \tag{2.3}$$

The definition (2.1) of H is therefore recursive. The inverse string tension α' has the dimension of a length squared, but we will mostly consider it simply as a (small) positive real number. The bosonic part of the effective action, truncated at order α', is [13, 11]

$$S = \int_M \Big[\mathrm{Scal}^g + 4|d\phi|^2 - \frac{1}{2}|H|^2 + \frac{\alpha'}{4}\mathrm{tr}\Big(|R^+|^2 - |F|^2\Big)\Big] e^{-2\phi}\, \mathrm{Vol}^g. \tag{2.4}$$

It leads to the following field equations, to order α':

$$\begin{aligned} \mathrm{Ric}_{\mu\nu} + 2(\nabla d\phi)_{\mu\nu} - \frac{1}{4} H_{\mu\kappa\lambda} H_{\nu}{}^{\kappa\lambda} + \frac{\alpha'}{4}\Big[R^+_{\mu\kappa\lambda\sigma} R^{+\kappa\lambda\sigma}_{\nu} - \mathrm{tr}\big(F_{\mu\kappa} F_{\nu}{}^{\kappa}\big)\Big] &= 0, \\ \mathrm{Scal} + 4\Delta\phi - 4|d\phi|^2 - \frac{1}{2}|H|^2 + \frac{\alpha'}{4}\mathrm{tr}\Big[|R^+|^2 - |F|^2\Big] &= 0, \\ e^{2\phi} d * (e^{-2\phi} F) + A \wedge *F - *F \wedge A + *H \wedge F &= 0, \\ d * (e^{-2\phi} H) &= 0. \end{aligned} \tag{2.5}$$

The full action with fermions is invariant under $\mathcal{N} = 1$ supersymmetry, and this gives rise to a set of first-order BPS equations

$$0 = \nabla^- \epsilon, \tag{2.6a}$$
$$0 = \gamma\big(d\phi - \tfrac{1}{2}H\big)\epsilon, \tag{2.6b}$$
$$0 = \gamma(F)\epsilon, \tag{2.6c}$$

where ϵ is a Majorana-Weyl spinor. They are called gravitino, dilatino and gaugino equation, in this order. If we consider H as a fundamental field instead of B, then we have to impose the Bianchi identity

$$dH = \frac{\alpha'}{4}\mathrm{tr}\Big(R^+ \wedge R^+ - F \wedge F\Big), \tag{2.7}$$

which follows from (2.1), in addition to the equations of motion or supersymmetry equations. Strictly speaking, equation (2.7) is weaker than (2.1), since it does not imply global existence of the 2-form B, but from a string theoretical point of view (2.7) is indeed sufficient, at least if the third real cohomology group of the 10-dimensional spacetime vanishes. This is more transparent in type II string theory, where the definition $H = dB$ can be replaced by the Bianchi identity $dH = 0$ plus the requirement that H defines an integral cohomology class. A geometric interpretation can be given in terms of bundle gerbes [24]. The most prominent example with non-exact H is the SU(2)-Wess-Zumino-Witten model, which appears as an ingredient in the Callan-Harvey-Strominger model, itself a limit of the NS5-brane explained in Section 4.3.

The equations presented above should be thought of as truncated α'-expansions, where the higher order terms are stringy corrections. The BPS equations (2.6) together with the Bianchi identity (2.7) are expected to imply the field equations (2.5), but this is only partly true. Suppose first that the 10-dimensional spacetime is a metric product of the form $\mathbb{R}^{1,1} \times M$, with M some 8-dimensional Riemannian manifold. If $d\phi, H$ and A only have components in M, then the BPS equations plus Bianchi identity do imply the equations of motion, but only up to terms of order $(\alpha')^2$. If, additionally, R^+ is replaced by another curvature form $\tilde{R}$ on the tangent bundle that also satisfies the gaugino equation $\gamma(\tilde{R})\epsilon = 0$, both in the Bianchi identity and the equations of motion, then the equations of motion at order α' are implied without higher order terms. This is a theorem of Ivanov [59]. From a string theoretical point of view it seems that the original set of equations is preferred over Ivanov's modified ones [11], but it has even been argued that they are equivalent via field redefinitions [58]. On the other hand, only the modified set of equations gives rise to a complete consistent supergravity theory, independently of any string theory extension.

The simplest solutions with non-vanishing H of heterotic supergravity solve the modified equations [77, 56, 60, 51], but solutions to the original Bianchi identity and

BPS equations do exist as well [22, 37, 11, 79]. A particularly important solution is the NS5-brane or 'symmetric solution' of [22], where R^+ itself solves the gaugino equation, so Ivanov's theorem applies without any modifications. I will present solutions to the modified equations in Chapter 5, and to the originial ones in Chapter 6. If the 10-dimensional manifold does not split into a 2-dimensional Minkowski space plus a complement, then the time-like components of the equations of motion are independent of the BPS equations and Bianchi identity, and have to be checked as well [42]. We will not consider this situation here.

The most well-known class of solutions consists of Riemannian manifolds with vanishing H and $d\phi$. The gravitino equation reduces to $\nabla\varepsilon = 0$, which means that the holonomy group of the Levi-Civita connection must be contained in the stabilizer of a spinor. From Wang's theorem [82], which is based on Berger's famous list, it follows that on simply connected spaces the possible irreducible holonomy groups are

- $\mathrm{SU}(n)$ (Calabi-Yau) in dimension $2n$,
- $\mathrm{Sp}(n)$ (hyperkähler) in dimension $4n$,
- G_2 in dimension 7, or
- Spin(7) in dimension 8.

Every manifold of this type gives rise to a solution of (2.6) and (2.7), by setting the gauge field equal to the Levi-Civita connection. The most important examples for phenomenological applications are compact Calabi-Yau 3-folds (six real dimensions), as the transverse space in this case is four-dimensional Minkowski space. We will be concerned with so-called flux compactifications, i.e. solutions with non-vanishing H and/or $d\phi$. These are far less well-understood, and instead of focusing on phenomenological requirements we will take a more geometric approach to understand the structure of the equations.

2.2 The gauge sector

2.2.1 Instanton equations

Heterotic supergravity contains a gauge field, whose first-order equation is

$$\gamma(F)\epsilon = 0. \tag{2.8}$$

This is a generalization of the usual four-dimensional instanton equation

$$*F = \pm F. \tag{2.9}$$

Recall that the Lie-algebra $\mathfrak{so}(4)$ splits as

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{2.10}$$

Now, the space of two-forms over a point $x \in M$ on a Riemannian manifold (M, g) is isomorphic to the Lie-algebra $\mathfrak{so}(T_xM, g_x)$, via the map (1.11). The instanton condition (2.9) can then be understood as the requirement that the 2-form F be contained in one of the two $\mathfrak{su}(2)$-subalgebras of $\mathfrak{so}(4)$. Suppose that M is a n-dimensional spin manifold which admits a nowhere vanishing spinor ϵ. Assume furthermore that the stabilizer group $G \subset \mathrm{Spin}(n)$ of the spinor is constant. The condition (2.8) means that F as a two-form is contained in the Lie algebra $\mathfrak{g}$ of G.

There is another useful way of rewriting the instanton equation (2.8). Suppose that (M, g) has reduced structure group $G \subset \mathrm{SO}(n)$, and that G is simple. Then its Lie-algebra $\mathfrak{g}$ comes equipped with the Killing form, a positive-definite bilinear form, whose inverse we can consider as a G-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$. Invariance implies that it gives rise to a globally defined, nowhere vanishing section of $\mathrm{End}(TM) \otimes \mathrm{End}(TM)$, which maps under the isomorphism (1.11) to a section of $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$. Applying the wedge product then gives rise to a globally defined four-form $Q \in \Gamma(\Lambda^4 T^*M)$. We are not guaranteed that Q is non-zero, however, as the wedge product is not injective. It turns out that Q vanishes for $G = \mathrm{SO}(n)$, but is non-zero for every true subgroup.

Given the four-form Q on M, we can construct an operator $T_Q : \Lambda^2 T^*M \to \Lambda^2 T^*M$ as

$$T_Q \eta = *(*Q \wedge \eta), \qquad \eta \in \Lambda^2 T^*M. \tag{2.11}$$

It commutes with the action of the structure group, and therefore acts as a constant on irreducible subrepresentations of Λ^2. The eigenvalue on the adjoint representation turns out to be non-degenerate in any case, and we normalize it it to -1:

$$\mathfrak{g} \cong \left\{ \eta \in \Lambda^2 \;\middle|\; *(*Q \wedge \eta) = -\eta \right\}. \tag{2.12}$$

The other eigenvalues can be determined by linear algebra, and are listed below for some Lie groups.

- structure group $\mathrm{SU}(n)$: here $\mathfrak{so}(2n)$ splits as

$$\mathfrak{so}(2n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}, \tag{2.13}$$

  where $\mathfrak{m}$ is spanned by $(2,0)$- and $(0,2)$-forms, and $\mathfrak{u}(1)$ by the Kähler form, when considered as subspaces of Λ^2. The eigenvalues are

representation	$\mathfrak{su}(n)$	$\mathfrak{u}(1)$	$\mathfrak{m}$
eigenvalue of T_Q	-1	$n-1$	1

- structure group $\mathrm{Sp}(n)$: this gives rise to the splitting

$$\mathfrak{so}(4n) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}, \tag{2.14}$$

  and the eigenvalues are

representation	$\mathfrak{sp}(n)$	$\mathfrak{sp}(1)$	$\mathfrak{m}$
eigenvalue of T_Q	-1	$(2n+1)/3$	$1/3$

- structure group G_2: in this case we have $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$, with $\mathfrak{m}$ being the 7-dimensional irreducible representation of $\mathfrak{g}_2$. The eigenvalues are

representation	$\mathfrak{g}_2$	$\mathfrak{m}$
eigenvalue of T_Q	-1	2

- structure group Spin(7): here $\mathfrak{so}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}$, with $\mathfrak{m}$ the 7-dimensional irreducible representation of $\mathfrak{spin}(7)$. The eigenvalues are

representation	$\mathfrak{spin}(7)$	$\mathfrak{m}$
eigenvalue of T_Q	-1	3

The instanton equation can thus be recast in the form

$$* F = - * Q \wedge F, \tag{2.15}$$

which looks similar to the original four-dimensional equation (2.9). Furthermore, by taking a covariant derivative it immediately implies the Yang-Mills equation

$$d_A * F = -(d * Q) \wedge F. \tag{2.16}$$

If Q is coclosed this reduces to the ordinary Yang-Mills equation, otherwise we will call it the torsionful Yang-Mills equation, with torsion $d * Q$. Note that instead of imposing that F be in the adjoint representation of the structure group, we could consider a modified instanton equation corresponding to any of the additional eigenvalues of the operator T_Q. For $d*Q = 0$ the Yang-Mills equation follows again. It appears much more difficult to find solutions to these modified equations, however, if any are known at all [70]. Whereas the Yang-Mills equation is invariant under conformal transformations in dimension four only, the instanton equation is always conformally invariant.

Examples of higher-dimensional instantons can be constructed as follows. Suppose that (M, g) has reduced holonomy contained in the stabilizer of a spinor ϵ, then $d * Q = 0$ and the Riemann tensor (the curvature two-form of the Levi-Civita connection) satisfies

$$R_{\mu\nu\kappa\lambda}\gamma^{\mu\nu}\epsilon = 0, \tag{2.17}$$

where the first two indices are Lie-algebra indices, and the last two differential form indices. Due to the exchange symmetry $R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu}$ it also satisfies the instanton condition

$$\gamma(R)\epsilon = 0 \qquad \Leftrightarrow \qquad R_{\mu\nu\kappa\lambda}\gamma^{\kappa\lambda}\epsilon = 0. \tag{2.18}$$

Note that this exchange symmetry is special for the Levi-Civita connection; in general a connection on the tangent bundle with reduced holonomy will not satisfy the instanton equation. In the next chapter I will prove that on manifolds with geometric Killing spinors one can always find a connection on the tangent bundle which has both reduced holonomy and the exchange symmetry, and therefore solves the instanton equation, despite having torsion. Furthermore, every instanton on these spaces satisfies the Yang-Mills equation without torsion, although $d*Q \neq 0$. Therefore, from a gauge-theoretic point of view they are almost as good as integrable geometries with parallel spinors, and as we shall see later they are also useful for heterotic supergravity.

2.2.2 Chern-Simons flow

Consider an oriented Riemannian manifold X equipped with a reduction of the structure group to a simple subgroup of $\mathrm{SO}(n)$, with fundamental 4-form Q. Let A be a gauge field on the vector bundle $E \to X$, with simple gauge group and curvature form F. For any codimension one submanifold Σ of X define the Chern-Simons functional

$$S_\Sigma(A) = -\frac{1}{2}\int_\Sigma \mathrm{CS}(A) \wedge *Q \tag{2.19}$$

on gauge fields, where $\mathrm{CS}(A)$ is the Chern-Simons 3-form (2.2). If $d*Q\big|_\Sigma = 0$, then S_Σ is a locally defined function on the space $\mathcal{A}/\mathcal{G}$ of gauge fields mod gauge equivalence [32], so that dS_Σ is a closed (but in general non-exact) 1-form on $\mathcal{A}/\mathcal{G}$. Gauge invariance of dS_Σ can be seen from the explicit expression below. If Σ and Σ' are cobordant submanifolds, so that one finds $V \subset X$ with $\partial V = \Sigma - \Sigma'$, then

$$S_\Sigma(A) - S_{\Sigma'}(A) = -\frac{1}{2}\,\mathrm{tr}\int_V F \wedge F \wedge *Q + \frac{1}{2}\int_V \mathrm{CS}(A) \wedge d*Q, \tag{2.20}$$

which is explicitly gauge invariant in the case $d*Q = 0$. The gradient flow equation for S is defined as

$$\mathrm{tr}\int_\Sigma \big(\xi \wedge *F\big) = dS_\Sigma(A)\cdot\xi \tag{2.21}$$

for all Σ of codimension one without boundary, and $\xi \in \Omega^1(\mathrm{End}(E))$. The notation is as follows: we consider S_Σ as a function on the space of gauge fields, and $dS_\Sigma(A)\cdot\xi$ is the total derivative of S_Σ in the point A applied to the tangent vector ξ. Concretely, we obtain after a partial integration

$$dS_\Sigma(A)\cdot\xi = -\mathrm{tr}\int_\Sigma \Big\{\xi \wedge \big(dA + A\wedge A\big) \wedge *Q + \tfrac{1}{2}\xi \wedge d*Q \wedge A\Big\}. \tag{2.22}$$

Thus, the flow equation (2.21) is equivalent to

$$*\,F = -F \wedge *Q - \tfrac{1}{2} d*Q \wedge A. \tag{2.23}$$

If in particular Q is coclosed, this reduces to the instanton equation (2.15).

Example 2.1 (the cylinder)**.** As an example consider the case where X is a Riemannian product $X = \mathbb{R} \times M$, and M carries a nowhere vanishing 3-form P' and 4-form Q', such that $d *_M P' = 0$. Then we can define a 4-form Q_{cyl} on X as

$$Q_{cyl} = d\tau \wedge P' + Q', \tag{2.24}$$

with τ the linear coordinate on $\mathbb{R}$. Let A be a gauge field on X with the property that $d\tau \lrcorner A = 0$, which is simply a choice of gauge. The instanton equation on the cylinder splits into the two equations

$$*\dot{A} = - * P' \wedge F \tag{2.25a}$$

$$*F = - * Q' \wedge F - \dot{A} \wedge *P', \tag{2.25b}$$

where $*$ is the Hodge dual on M. Since the 4-form Q_{cyl} is not coclosed, the Chern-Simons gradient flow is not equivalent to the instanton equation in general. If we restrict attention to submanifolds of the form

$$\Sigma = \{\tau_0\} \times M \tag{2.26}$$

for fixed $\tau_0 \in \mathbb{R}$, however, then due to $d * Q_{cyl}\big|_\Sigma = 0$ the action functional becomes

$$S_M = -\frac{1}{2} \int_M \mathrm{CS}(A) \wedge *P', \tag{2.27}$$

and the gradient flow equation is equation (2.25a). In certain special cases (2.25a) already imples (2.25b), so that the restricted gradient flow becomes equivalent to the instanton equation, but this is not the general case. It does happen if M is either 3-dimensional, or a 7-dimensional nearly parallel G_2-manifold. In the latter case we have a canonical 3-form P' and $Q' = *P'$, and every 2-form F satisfies an equation

$$Q' \wedge *(Q' \wedge F) = P' \wedge F + *F. \tag{2.28}$$

It follows that (2.25a) implies (2.25b).

Any real Killing spinor manifold M of dimension n comes equipped with a 3-form P' and 4-form Q' which satisfy (see Section 3.2.2 below)

$$dP' = 4Q', \qquad d * Q' = (n-3) * P'. \tag{2.29}$$

For $n > 3$ the Chern-Simons functional can then be written as

$$S_M = -\frac{1}{2(n-3)} \int_M \mathrm{Tr}(F \wedge F) \wedge *Q', \tag{2.30}$$

which is explicitly gauge-invariant.

3 Geometry

3.1 Cones, sine-cones and cylinders

For (M, g) a Riemannian manifold we define the following higher-dimensional spaces:

1. $c(M) = (\mathbb{R}_{\geq 0} \times M, \bar{g})$ with $\bar{g} = dr^2 + r^2 g$ as the *(metric) cone* on M,
2. $sc(M) = ((0, \pi) \times M, \bar{g})$ with $\bar{g} = d\theta^2 + \sin^2(\theta)\, g$ as the *sine-cone* on M, and
3. $cyl(M) = (\mathbb{R} \times M, \bar{g})$ with $\bar{g} = d\tau^2 + g$ as the *cylinder* on M.

The cone is conformal to the cylinder, because under the substitution $r = e^\tau$ its metric becomes $\bar{g} = e^{2\tau}(d\tau^2 + g)$. Furthermore, the cone over the sine-cone equals the cylinder over the cone: $c(cs(M)) = cyl(c(M))$. This is illustrated by the following diagram

$$\begin{array}{ccc} M & \xrightarrow[+\theta]{\text{sine-cone}} & sc(M) \\ {\scriptstyle\text{cone}}\big\downarrow{\scriptstyle +r} & & {\scriptstyle\text{cone}}\big\downarrow{\scriptstyle +\rho} \\ c(M) & \xrightarrow[+\tau]{\text{cylinder}} & c(sc(M)) \end{array} \tag{3.1}$$

where the two pairs of coordinates are related as

$$(\tau, r) = (\rho \cos\theta, \rho \sin\theta). \tag{3.2}$$

The sine-cone over a compact manifold can be extended to a compact manifold by adding two points corresponding to $\theta = 0$ and $\theta = \pi$, whereas the cone can be extended by adding one more point, the apex, corresponding to $r = 0$. In general the metric becomes singular at these additional points, but if M is a properly normalized round sphere then so will be its sine-cone, and the cone becomes Euclidean space:

$$\begin{array}{ccc} S^n & \xrightarrow{\text{sine-cone}} & S^{n+1} \\ {\scriptstyle\text{cone}}\big\downarrow & & {\scriptstyle\text{cone}}\big\downarrow \\ \mathbb{R}^{n+1} & \xrightarrow{\text{cylinder}} & \mathbb{R}^{n+2} \end{array} \tag{3.3}$$

Suppose that (M, g) is Einstein, then its sine-cone is Einstein and the cone Ricci-flat if and only if the Einstein constant of M is $\dim M - 1$, i.e. $\mathrm{Ric}^g = (\dim M - 1)g$. Furthermore, this normalization is preserved under taking sine-cones, in other words the Einstein constant of $sc(M)$ is $\dim M$. If we start with a different normalization then we need to modify the definition of the sine-cone and cone metrics to make them again Einstein and Ricci-flat, but this will not be necessary in the present work.

3.2 Real Killing spinors

3.2.1 Classification

Let (M, g) be a Riemannian spin manifold with (Dirac-)spinor bundle $\mathcal{S} \to M$. If $\epsilon \in \Gamma(\mathcal{S})$ satisfies the equation

$$\big(\nabla_\mu - i\lambda\gamma_\mu\big)\epsilon = 0 \qquad (\lambda \in \mathbb{R} \cup i\mathbb{R}), \tag{3.4}$$

then it is called a Killing spinor. The notion of a Killing spinor is sometimes used in the literature more generally for spinors that solve BPS equations for supergravity theories, like equations (2.6) in the heterotic setting, and to distinguish the two concepts solutions to (3.4) are called 'geometric Killing spinors', but I will not adopt this convention. For real λ one talks about 'real Killing spinors',[1] and these are the ones of relevance to us. Real Killing spinors are eigenspinors of the Dirac operator with smallest possible eigenvalue [38], and complete manifolds with real Killing spinors have been classified [8]:

Theorem 3.1 (Bär)**.** *Suppose that M is complete and carries a real Killing spinor. Then it is either a round sphere, or one of the following:*

- *six-dimensional nearly Kähler,*
- *seven-dimensional nearly parallel G_2,*
- $(2n+1)$*-dimensional Sasaki-Einstein,*
- $(4n+3)$*-dimensional 3-Sasakian.*

This result is based on a 1-1 correspondence between real Killing spinors on M and parallel spinors on the cone over M. The cone over a round sphere S^n is Euclidean space $\mathbb{R}^{n+1}$, whereas the four types of spaces listed above have cones with holonomy group $G = G_2$, Spin(7), SU$(n+1)$ (Calabi-Yau), and Sp$(n+1)$ (hyperkähler), respectively. On the other hand, the Killing spinors induce a reduction of the structure group to $K =$ SU(3), G_2, SU(n) and Sp(n) on the base manifold, which does not coincide with the holonomy group of the Levi-Civita connection, however. The relation between Killing spinors and parallel spinors plays an important role in type II string theory and M-theory as well, as explained in the introduction, and it is also relevant for the AdS/CFT correspondence [1, 72].

[1] my Clifford algebra convention differs from most of the mathematical literature by a sign in the defining relation $\{\gamma^\mu, \gamma^\nu\} = +2g^{\mu\nu}$, which is why the imaginary unit i occurs in (3.4) for real Killing spinors. Switching the sign in the Clifford relation eliminates the i.

Real Killing spinor manifolds are necessarily Einstein, and the preferred normalization $\mathrm{Ric} = (\dim M - 1)g$ is equivalent to $\lambda = \pm\frac{1}{2}$. We will henceforth always assume this normalization for these manifolds. It has been shown in [15] that the sine-cone over a nearly Kähler manifold is nearly parallel G_2, and the sine-cone over a five-dimensional Sasaki-Einstein manifold is nearly Kähler [36]. This suggests that in general the sine-cone over a real Killing spinor manifold has again a real Killing spinor, and this is the first result I want to prove. The method is very similar to the proof of Bär's correspondence between Killing and parallel spinors, therefore I also include this one. The essential step is a relation between gamma-matrices in a given dimension and those in one dimension higher.

Denote by $\mathcal{S}_n$ spinor space in n-dimensions, by $\mathcal{S}_n^+$ the space of positive-chirality spinors for even n, by γ_μ the generators of the Clifford algebra $\mathrm{Cl}(n)$ and by $\tilde{\gamma}_\mu \in \mathrm{Cl}(n+1)$ the generators in one dimension higher.

Lemma 3.2. *There is an isomorphism $\iota : \mathcal{S}_n \to \mathcal{S}_{n+1}$ for even n, and $\iota : \mathcal{S}_n \to \mathcal{S}_{n+1}^+$ for odd n, such that*

$$\tilde{\gamma}_{\mu\nu} \circ \iota = \iota \circ \begin{cases} \gamma_{\mu\nu} & \text{for } 1 \le \mu, \nu \le n \\ -i\gamma_\mu & \text{for } \mu < \nu = n+1. \end{cases} \tag{3.5}$$

In particular ι is Spin(n)-equivariant.

The proof can be found in Appendix A.

Proposition 3.3 (Bär). *A spin manifold M^n carries a real Killing spinor if and only if its cone has a parallel spinor.*

Proof. The cone over M is $\mathbb{R} \times M$ with metric $g = e^{2\tau}(d\tau^2 + g_M)$, where τ is the logarithm of the radial coordinate, $r = e^\tau$. Let e^a be a local orthonormal frame of 1-forms on M with dual vector fields I_a, then the Levi-Civita connection in the orthonormal frame $\{e^\tau d\tau, e^\tau e^a\}$ on $c(M)$ reads

$$\Gamma^c = \Gamma^M + e^a(I_a \otimes d\tau - \partial_\tau \otimes e^a), \tag{3.6}$$

where we consider the tensor product of a vector field with a 1-form as an endomorphism of the tangent bundle, and Γ^M is the Levi-Civita connection on M in the frame $\{e^a\}$. The corresponding spin connection is

$$\nabla^c = d\tau\,\partial_\tau + \nabla^M + \frac{1}{2} e^a \tilde{\gamma}_a \tilde{\gamma}_0, \tag{3.7}$$

where $\tilde{\gamma}_0$ corresponds to the 1-form $e^\tau d\tau$ and $\tilde{\gamma}_a$ to $e^\tau e^a$. Suppose there is a parallel spinor $\tilde{\epsilon}$ on $c(M)$, and assume for simplicity that n is even. Then $\tilde{\epsilon}$ is in the image under ι of the pull-back of spinor bundle on M, and we can write $\tilde{\epsilon} = \iota(\epsilon)$ for a possibly τ-dependent spinor ϵ on M. Furthermore, ϵ satisfies

$$\Big(d\tau\,\partial_\tau + \nabla^M - \frac{i}{2} e^a \gamma_a\Big)\epsilon = 0. \tag{3.8}$$

But then ϵ has to be τ-independent, hence a Killing spinor on M. It is clear that the converse statement is true as well. If n is odd one has to take care of chirality, but in principle the argument remains the same. □

Proposition 3.4. *A spin manifold M^n carries a real Killing spinor if and only if its sine-cone has a real Killing spinor.*

Proof. Here we start with the other direction. Assume that ϵ is a Killing spinor on M, i.e.

$$\Big(\nabla^M - \frac{i}{2}e^a\gamma_a\Big)\epsilon = 0. \tag{3.9}$$

We identify ϵ with $\iota(\epsilon)$, denote the gamma-matrix corresponding to $d\theta$ by $\tilde\gamma_0$, and consider on $sc(M)$ the spinor

$$\tilde\epsilon = e^{\frac{i}{2}\theta\tilde\gamma_0}\epsilon = \cos(\theta/2)\epsilon + i\sin(\theta/2)\tilde\gamma_0\epsilon. \tag{3.10}$$

The Killing property on M implies in one dimension higher that

$$\nabla^M\epsilon = -\frac{1}{2}e^a\tilde\gamma_a\tilde\gamma_0\epsilon, \tag{3.11}$$

where e^a is again a local orthonormal frame of 1-forms for M, and $\tilde\gamma_a$ corresponds to $\sin(\theta)e^a$. Introduce the orthonormal 1-forms $\sigma^a = \sin(\theta)e^a$, then the Levi-Civita connection on the sine-cone in the frame $\{\sigma^a, d\theta\}$ is given by

$$\Gamma^{sc} = \Gamma^M + \cos(\theta)e^a\big(I_a\otimes d\theta - \partial_\theta\otimes\sigma^a\big), \tag{3.12}$$

where I_a is dual to σ^a. The corresponding spin connection is

$$\nabla^{sc} = d\theta\,\partial_\theta + \nabla^M + \frac{1}{2}\cos(\theta)e^a\tilde\gamma_a\tilde\gamma_0, \tag{3.13}$$

and thus we obtain

$$\begin{aligned}\nabla^{sc}\tilde\epsilon &= -\frac{1}{2}\Big(\cos(\theta/2) + i\sin(\theta/2)\tilde\gamma_0\Big)e^a\tilde\gamma_a\tilde\gamma_0\epsilon\\ &\quad + \frac{1}{2}\cos(\theta)e^a\tilde\gamma_a\Big(\cos(\theta/2)\tilde\gamma_0 + i\sin(\theta/2)\Big)\epsilon\\ &\quad + \frac{i}{2}d\theta\,\tilde\gamma_0\,e^{\frac{i}{2}\theta\tilde\gamma_0}\epsilon\\ &= -\frac{1}{2}\sigma^a\tilde\gamma_a\frac{1}{\sin(\theta)}\Big(\cos(\theta/2)\tilde\gamma_0 - i\sin(\theta/2)\Big)\epsilon\\ &\quad + \frac{1}{2}\sigma^a\tilde\gamma_a\frac{\cos(\theta)}{\sin(\theta)}\Big(\cos(\theta/2)\tilde\gamma_0 + i\sin(\theta/2)\Big)\epsilon\\ &\quad + \frac{i}{2}d\theta\,\tilde\gamma_0\,\tilde\epsilon.\end{aligned} \tag{3.14}$$

Now make use of the addition theorems in the form

$$\begin{aligned}\frac{\cos(\theta/2)}{\sin(\theta)}\big(\cos(\theta)-1\big) &= -\sin(\theta/2)\\ \frac{\sin(\theta/2)}{\sin(\theta)}\big(\cos(\theta)+1\big) &= \cos(\theta/2)\end{aligned} \tag{3.15}$$

to obtain

$$\nabla^{sc}\tilde{\epsilon} = \frac{i}{2}\Big(\sigma^a\tilde{\gamma}_a + d\theta\,\tilde{\gamma}_0\Big)\tilde{\epsilon}, \tag{3.16}$$

i.e. the Killing spinor equation on $sc(M)$. The other direction follows as in the proof of Proposition 3.3. □

Proposition 3.4 seems to contradict Theorem 3.1, as there aren't any even-dimensional manifolds with real Killing spinors except in dimension six, or round spheres. However, as sine-cones are incomplete in general, they do not satisfy the assumption of Theorem 3.1 and therefore are not covered by Bär's classification. Of the four spaces in diagram (3.1) the upper two have a Killing spinor and the lower two have a parallel spinor, and extending the diagram to the right by taking sine-cones in the upper row and cylinders in the lower row gives two sequences of incomplete manifolds with Killing spinors and parallel spinors, respectively.

3.2.2 Forms and connections

Recall from Section 2.2 that every manifold M^n with reduced structure group $K \subset \mathrm{SO}(n)$ carries a nowhere vanishing 4-form Q. The Killing spinor manifolds are special in that their canonical 4-form is always closed, i.e. there is a 3-form P such that $dP = 4Q$. Obvious candidates for P and Q are the spinor bilinears

$$\begin{aligned}P' &= -\frac{i}{3!}\big\langle\epsilon, \gamma_{\mu\nu\lambda}\epsilon\big\rangle e^{\mu\nu\lambda},\\ Q' &= -\frac{1}{4!}\big\langle\epsilon, \gamma_{\mu\nu\kappa\lambda}\epsilon\big\rangle e^{\mu\nu\kappa\lambda},\end{aligned} \tag{3.17}$$

which satisfy

$$dP' = 4Q', \qquad d*Q' = (n-3)*P', \tag{3.18}$$

if $\dim M = n$. Indeed, they are equal to the canonical 3- and 4-forms P and Q on nearly Kähler, nearly parallel G_2, and Sasaki-Einstein spaces. This is not true for 3-Sasakian manifolds, however.

Note that the Killing connection

$$\nabla^K = \nabla - \frac{i}{2}e^\mu\gamma_\mu \tag{3.19}$$

is defined on the spinor bundle, but is not induced from a connection on the tangent bundle. It turns out, however, that if M admits a real Killing spinor, then one can also find a connection with parallel spinors that is induced from a connection on the tangent bundle. We will call this the canonical connection, and denote it by ∇^P, as opposed to the Levi-Civita connection ∇. For nearly Kähler and nearly parallel G_2-manifolds the torsion of ∇^P is proportional to P, whereas for Sasaki-Einstein and 3-Sasakian manifolds the torsion is not totally skew-symmetric. But the original metric can be deformed in such a way that the torsion becomes skew-symmetric, and proportional to P again. The connection ∇^P is compatible with a whole family of deformed metrics in these cases.

While the Killing connection ∇^K is important in type II and eleven-dimensional supergravity, the canonical connection appears in the spinor equation of heterotic supergravity. The canonical connection of nearly Kähler, nearly parallel G_2, and Sasaki-Einstein manifolds and its role in supergravity theories has been explored in [40], whereas the observation that 3-Sasakian manifolds possess a canonical connection as well appears to be new. In Section 3.3 I review the geometries appearing in Bär's list, and give in particular an explicit realization of the two canonical forms P and Q and the connection ∇^P.

The concept of a canonical connection introduced here in general differs from the characteristic connection of Agricola and Friedrich [3, 4]. A characteristic connection must have antisymmetric torsion for the Einstein metric, but satisfies a weaker holonomy property. For instance, the characteristic connection on a Sasaki-Einstein manifold of dimension $2n+1$ has holonomy group $\mathrm{U}(n)$, and hence no parallel spinor, whereas the canonical connection has holonomy group $\mathrm{SU}(n)$. Furthermore, there is no characteristic connection on a 3-Sasakian manifold. For nearly Kähler and nearly parallel G_2-manifolds the two concepts coincide.

3.2.3 Gauge theory

Concerning gauge theory, manifolds with real Killing spinors are quite similar to manifolds with parallel spinors. Given a Killing spinor ϵ we can consider the instanton equation

$$\gamma(F)\epsilon = 0, \tag{3.20}$$

and this can be rewritten as $*F = - * Q \wedge F$, as explained in Section 2.2. Now, Q is not coclosed and therefore any instanton gauge field satisfies a torsionful Yang-Mills equation. It turns out, however, that the torsion in the Yang-Mills operator vanishes when applied to instantons:

Proposition 3.5. *Suppose that M carries a real Killing spinor ϵ, and A solves the instanton equation on M. Then A satisfies the ordinary Yang-Mills equation without torsion.*

This result has first been obtained for nearly Kähler manifolds by Xu [84]. I reproduce his original argument in Section 3.3.1 below. For the case of abelian gauge groups Hijazi has proven that the converse also holds: every solution of the Yang-Mills equation solves the instanton equation [57].

Proof. Denote by $\mathcal{D} = \gamma^\mu \nabla_\mu^{\Gamma\otimes A}$ the Dirac operator acting on sections of the tensor product of the spin bundle with the endomorphisms of the gauge bundle, then we have [57, 10]

$$\mathcal{D}\gamma(F) = \gamma(d_A F) + (-1)^n \gamma(*d_A * F) + \gamma^\mu \gamma(F) \nabla_\mu^{\Gamma\otimes A}, \tag{3.21}$$

and the first term on the right hand side vanishes due to the Bianchi identity. Denote by $\mathbf{1}$ the identity operator acting on the gauge bundle E. We identify the spinor ϵ with $\epsilon \otimes \mathbf{1}$, and apply $\mathcal{D}\gamma(F)$ to it:

$$0 = \mathcal{D}\gamma(F)\epsilon = (-1)^n \gamma(*d_A * F)\epsilon - i\lambda\gamma^\mu\gamma(F)\gamma_\mu\epsilon. \tag{3.22}$$

Using the relation $\gamma^\mu\gamma(F)\gamma_\mu = (\dim M - 4)\gamma(F)$, which holds for 2-forms in general, and again $\gamma(F)\epsilon = 0$, we end up with

$$\gamma(*d_A * F)\epsilon = 0. \tag{3.23}$$

But the Clifford action of a non-vanishing 1-form is invertible, and we conclude that $*d_A * F = 0$, which is the Yang-Mills equation. □

In the following section I explain the construction of the canonical connection ∇^P on the tangent bundle of a real Killing spinor manifold. ∇^P satisfies the instanton condition, which follows from

Lemma 3.6. *Let ∇^t be a family of metric-compatible connections on the tangent bundle of M, with totally antisymmetric torsion $T^\mu = te^\mu \lrcorner P$ for some 3-form P. Suppose that P is parallel for $t = 1$, i.e. $\nabla^1 P = 0$. Then the curvature R^t has the interchange symmetry $R^t_{\mu\nu\kappa\lambda} = R^t_{\kappa\lambda\mu\nu}$ for all values of t.*

The important case $t = 1$ has also been proven in [3].

Proof. In a basis where P and the metric assume standard form the condition for P to be parallel with respect to ∇^1 reads:

$$\Gamma^\rho_{\mu[\nu} P_{\kappa\lambda]\rho} = -\frac{1}{2} P^\rho_{[\mu\nu} P_{\kappa\lambda]\rho}, \tag{3.24}$$

where $\Gamma^\kappa_{\lambda\sigma}$ are the coefficients of the Levi-Civita connection, and the bracket [ ] denotes antisymmetrization. Consider then the connection ∇^t with torsion $T^\lambda_{\mu\nu} = tP^\lambda{}_{\mu\nu}$. Its curvature is quite generally

$$R^t = R + \frac{t}{4} g^{\sigma\mu} \Big[6\Gamma^\rho_{\kappa[\lambda} P_{\mu\nu]\rho} + tP_{\mu\kappa\rho} P^\rho{}_{\lambda\nu} \Big] e^{\kappa\lambda} (I_\sigma \otimes e^\nu), \tag{3.25}$$

and due to (3.24) this becomes

$$R^t = R + \frac{t}{4} g^{\sigma\mu} \Big[-3P^\rho{}_{[\kappa\lambda} P_{\mu\nu]\rho} + tP_{\mu\kappa\rho} P^\rho{}_{\lambda\nu} \Big] e^{\kappa\lambda} (I_\sigma \otimes e^\nu). \tag{3.26}$$

Thus the coefficients of R^t are

$$R^t_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \frac{t}{2}\Big[3P^\rho{}_{[\kappa\lambda}P_{\mu\nu]\rho} - tP_{\rho\mu[\kappa}P_{\lambda]\nu}{}^\rho\Big], \tag{3.27}$$

and they have the desired interchange symmetry for any value of t. □

Corollary 3.7. *Under the assumptions of Lemma 3.6 assume further that ∇^t has a parallel spinor ϵ for some value $t = t_0$. Then the curvature form R^{t_0} satisfies the instanton equation $\gamma(R^{t_0})\epsilon = 0$. In particular this applies to the canonical connection on a real Killing spinor manifold.*

3.3 Examples of real Killing spinor manifolds

In this section I review the geometries that appear in Bär's list of real Killing spinor manifolds [8]. A more detailed account of all four geometries with emphasis on the spinorial point of view can be found in [10, 18]. A useful reference for nearly Kähler manifolds is [21], for nearly parallel G_2 manifolds see [39], and Sasakian geometry is treated in [19, 76, 4].

3.3.1 Nearly Kähler manifolds

Consider a six-dimensional manifold with a Majorana real Killing spinor ϵ. It carries the following non-vanishing forms

$$\begin{aligned} \omega &\propto \frac{i}{2}\big\langle\epsilon, \gamma_{\mu\nu}\Gamma\epsilon\big\rangle e^{\mu\nu}, \\ \omega\wedge\omega &\propto \frac{1}{4!}\big\langle\epsilon, \gamma_{\mu\nu\rho\sigma}\epsilon\big\rangle e^{\mu\nu\rho\sigma}, \\ \Omega &\propto \frac{1}{6}\Big[\big\langle\epsilon, \gamma_{\mu\nu\rho}\epsilon\big\rangle + \big\langle\epsilon, \gamma_{\mu\nu\rho}\Gamma\epsilon\big\rangle\Big]e^{\mu\nu\rho}, \end{aligned} \tag{3.28}$$

where Γ is the chirality operator. The 2-form ω defines an almost hermitian structure on M, with almost complex structure J defined in the usual way: $g(X, J(Y)) = \omega(X, Y)$. We have $\omega \in \Omega^{(1,1)}$ and $\Omega \in \Omega^{(3,0)}$ with respect to J. The canonical 3- and 4-forms P and Q are given by

$$P = \operatorname{Re}\Omega, \qquad Q = \frac{1}{2}\omega\wedge\omega. \tag{3.29}$$

One can choose an orthonormal frame such that the differential forms assume their standard form

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \Omega &= (e^1 + ie^2)\wedge(e^3 + ie^4)\wedge(e^5 + ie^6). \end{aligned} \tag{3.30}$$

The Killing spinor equation implies that

$$\begin{aligned} d\omega &= 3\operatorname{Im}\Omega = 3 * P, \\ d\Omega &= d\overline{\Omega} = 2\omega\wedge\omega, \end{aligned} \tag{3.31}$$

and these equations are often used to define a nearly Kähler manifold. The canonical connection is

$$\nabla^P - \nabla = \frac{1}{2}(\mathrm{Re}\,\Omega)^{\mu}{}_{\nu\rho}e^{\nu}(I_\mu \otimes e^\rho), \tag{3.32}$$

and the torsion of ∇^P equals P. The Fierz identity implies

$$(\mathrm{Re}\ \Omega)_{\mu\nu\lambda}\gamma^{\nu\lambda} \cdot \epsilon = 4i\gamma_\mu \cdot \epsilon, \tag{3.33}$$

and from the Killing spinor equation it follows that $\nabla^P\epsilon = 0$. Then ∇^P must have holonomy SU(3).

SU(3)-instantons. The instanton equation

$$* F = - * Q \wedge F = -\omega \wedge F \tag{3.34}$$

is equivalent to

$$F \in \Omega^{(1,1)} \qquad \text{and} \qquad \omega \lrcorner F = 0. \tag{3.35}$$

In the form (3.34) it implies the Yang-Mills equation with torsion

$$d_A * F + 3F \wedge \mathrm{Im}\,\Omega = 0, \tag{3.36}$$

but the torsion term vanishes due to $F \in \Omega^{(1,1)}$ and $\mathrm{Im}\,\Omega \in \Omega^{(3,0)} \oplus \Omega^{(0,3)}$. Thus F satisfies the ordinary Yang-Mills equation. This argument is due to Xu [84].

Examples. There are the following four homogeneous examples:

$$\begin{array}{cc} S^6 = G_2/\mathrm{SU}(3), & S^3 \times S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)_{\mathrm{diag}}, \\ \mathrm{SU}(3)/\mathrm{U}(1)^2, & \mathrm{Sp}(2)/\mathrm{Sp}(1)\times \mathrm{U}(1). \end{array} \tag{3.37}$$

Homogeneous nearly Kähler manifolds G/K are so-called 3-symmetric spaces: there exists an automorphism s of G of order three, i.e. $s^3 = \mathrm{id}_G$, such that the Lie algebra of K is the fixed point set of the differential of s at the identity [21]. Currently, complete non-homogeneous examples are not known, but there exists a nearly Kähler structure with two conical singularities on the sine-cone over every 5-dimensional Sasaki-Einstein manifold, giving rise to incomplete non-homogeneous examples [36].

3.3.2 Nearly parallel G_2 manifolds.

On a seven-manifold with a Majorana real Killing spinor we can define a 3-form Φ, with dual 4-form $*\Phi$, as

$$\begin{aligned} \Phi &\propto \frac{i}{6}\langle\epsilon, \gamma_{\mu\nu\rho}\epsilon\rangle e^{\mu\nu\rho}, \\ *\Phi &\propto \frac{1}{4!}\langle\epsilon, \gamma_{\mu\nu\rho\sigma}\epsilon\rangle e^{\mu\nu\rho\sigma}, \end{aligned} \tag{3.38}$$

and the Killing spinor equation implies $d\Phi = 4 * \Phi$. The canonical 4-form is $Q = *\Phi$, whereas $P = \Phi$. The standard form is

$$\begin{aligned} \Phi &= e^{127} + e^{347} + e^{567} + e^{136} + e^{145} + e^{235} - e^{246}, \\ *\Phi &= e^{1234} + e^{1256} + e^{3456} - e^{1357} + e^{1467} + e^{2457} + e^{2367}. \end{aligned} \tag{3.39}$$

The canonical connection ∇^P is given by

$$\nabla^P - \nabla = \frac{1}{3}\Phi^\mu{}_{\nu\rho} e^\nu (I_\mu \otimes e^\rho), \tag{3.40}$$

so that its torsion is proportional to P again. It follows from the Fierz identity that

$$\Phi_{\mu\nu\lambda}\gamma^{\nu\lambda} \cdot \epsilon = 6i\gamma_\mu \cdot \epsilon, \tag{3.41}$$

and this together with the Killing spinor equation implies that ϵ is parallel with respect to ∇^P, so ∇^P has holonomy G_2.

G_2-instantons. The instanton equation becomes

$$* F = -\Phi \wedge F. \tag{3.42}$$

The 2-forms decompose as $\Lambda^2 \simeq \underline{14} \oplus \underline{7}$ under G_2, where $\underline{14}$ is the adjoint and $\underline{7}$ the fundamental representation. As explained above, the instanton equation is equivalent to F being in the adjoint representation. Now Φ and $*\Phi$ are G_2-invariant which implies that $*\Phi \wedge F \in \Lambda^6$ must be in the same representation as F, and from $\Lambda^6 \simeq \underline{7}$ it follows that (3.42) is equivalent to

$$F \wedge *\Phi = 0. \tag{3.43}$$

Applying the Yang-Mills operator to (3.42) leads to

$$d_A * F + 4 * \Phi \wedge F = 0, \tag{3.44}$$

but the torsion $*\Phi \wedge F$ vanishes due to (3.43), confirming Proposition 3.5.

Examples. Simply connected nearly parallel G_2-manifolds with two Killing spinors are Sasaki-Einstein, and those with three Killing spinors are 3-Sasakian. More than three Killing spinors exist only on the round sphere S^7. The following examples with exactly one Killing spinor are known [39]. First of all, the Aloff-Wallach spaces $N(k,l) = \mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$, where U(1) embeds into SU(3) as

$$z \mapsto \mathrm{diag}(z^k, z^l, z^{-(k+l)}) \tag{3.45}$$

for $z \in S^1$ and positive integers k, l, each carry two homogeneous metrics with at least one Killing spinor. For $(k,l) \neq (1,1)$ they both have exactly one Killing spinor, whereas for $(k,l) = (1,1)$ one of the two metrics is 3-Sasakian. Another homogeneous example is the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)_{\mathrm{max}}$, where SO(3) acts on the tangent space by its unique irreducible 7-dimensional representation. Additionally, every 3-Sasakian manifold in dimension 7 has a second Einstein metric with exactly one Killing spinor, which gives some further examples. This metric will be described in Section 3.3.4 below. In particular, this construction gives rise to an additional nearly parallel G_2-structure on S^7, the so called squashed seven-sphere.

The Aloff-Wallach spaces, Berger space and the squashed 7-sphere all have positive sectional curvature [85, 81], and this seems to be true for many of the nearly parallel G_2 metrics obtained from 3-Sasakian manifolds as well [31, 19].

3.3.3 Sasaki-Einstein manifolds

A Sasaki-Einstein manifold M^{2n+1} comes equipped with a contact 1-form η and a 2-form $2\omega = d\eta$, satisfying $\eta \wedge \omega^n \neq 0$, and a metric which is compatible with these forms in an appropriate sense. The structure group is $\mathrm{SU}(n)$, which leaves two Dirac spinors invariant. Both of them are Killing spinors, but the relative sign of their Killing constants depends on the dimension. Denote the two Killing spinors by ϵ and $\tilde{\epsilon}$, and assume that ϵ has Killing constant $\lambda = 1/2$. Again, the forms can be constructed as spinor bilinears:

$$\begin{aligned} \eta &\propto \langle \epsilon, \gamma_\mu \epsilon \rangle e^\mu, \\ \omega &\propto \langle \epsilon, \gamma_{\mu\nu} \epsilon \rangle e^{\mu\nu}. \end{aligned} \tag{3.46}$$

The canonical 3- and 4-forms are then given by

$$P = \eta \wedge \omega \qquad \text{and} \qquad Q = \frac{1}{2}\omega \wedge \omega, \tag{3.47}$$

and in standard form we have

$$\begin{aligned} \eta &= e^1 \\ \omega &= e^{23} + e^{45} + \cdots + e^{2n,2n+1}. \end{aligned} \tag{3.48}$$

The metric assumes the form

$$g = g_Z + \eta \otimes \eta, \tag{3.49}$$

where g_Z is a possibly singular Kähler metric on the leaves of the foliation defined by the 1-form η. In particular $\eta \lrcorner \omega = 0$. There is also an n-form

$$\Omega \propto \langle \epsilon^*, \gamma_{\mu_1 \dots \mu_n} \epsilon \rangle e^{\mu_1 \dots \mu_n}, \tag{3.50}$$

where ϵ^* is the charge conjugate of ϵ, satisfying

$$\begin{aligned} d\Omega &= i(n+1)\eta \wedge \Omega, \\ \Omega \wedge \overline{\Omega} &= (-1)^{\frac{1}{2}n(n+1)} \frac{(2i)^n}{n!} \omega^n, \\ \eta \lrcorner \Omega &= 0. \end{aligned} \tag{3.51}$$

The canonical connection ∇^P is related to the Levi-Civita connection ∇ of g by

$$\nabla^P = \nabla + \omega_{ab} e^a \big(\xi \otimes e^b - I_b \otimes \eta\big) - \frac{1}{n} \eta \otimes J, \tag{3.52}$$

where ξ is the vector field dual to η, e^a is an orthonormal basis of 1-forms orthogonal to η with dual vector fields I_a, and $J \in \mathrm{End}(TM)$ is the almost complex structure on the vanishing space of η, determined by ω and g_Z. From the identities,

$$\begin{aligned} \gamma(\omega) \cdot \epsilon &= 2ni\gamma_1 \cdot \epsilon \\ \omega_{ab}\gamma^{1b} \cdot \epsilon &= -i\gamma_a \cdot \epsilon, \end{aligned} \tag{3.53}$$

and the Killing spinor equation, it follows that ϵ is parallel with respect to ∇^P, and hence that ∇^P has holonomy $\mathrm{SU}(n)$. Here γ_1 is the gamma matrix corresponding to ξ. Then ∇^P has a second parallel spinor $\tilde{\epsilon}$ as discussed above. From the identities,

$$\begin{aligned} \gamma(\omega) \cdot \tilde{\epsilon} &= (-1)^{n-1} 2ni\gamma_1 \cdot \tilde{\epsilon} \\ \omega_{ab}\gamma^{1b} \cdot \tilde{\epsilon} &= -(-1)^{n-1} i\gamma_a \cdot \tilde{\epsilon}, \end{aligned} \tag{3.54}$$

it follows that $\tilde{\epsilon}$ is a Killing spinor as well, with the same Killing constant as ϵ if and only if n is odd. ∇^P is compatible with the whole family of metrics

$$g_h = e^{2h} g_Z + \eta \otimes \eta \tag{3.55}$$

on M, where $h \in \mathbb{R}$. The torsion of ∇^P (which is metric-independent) can be calculated using the Cartan equation $T^\mu = de^\mu + (\Gamma^P)^\mu{}_\nu \wedge e^\nu$. Thus,

$$T^1 = 2\omega, \qquad T^a = -\tfrac{n+1}{n} \omega_{ab}\, \eta \wedge e^b, \tag{3.56}$$

where T^1 is the ξ-component. There are two special values of the parameter h. The metric with $e^{2h} = 1$ is special, because the Levi-Civita connection has a Killing spinor and the cone has special holonomy. On the other hand, the torsion of the connection ∇^P is antisymmetric exactly when

$$e^{2h} = \frac{2n}{n+1}. \tag{3.57}$$

SU(n)-instantons. The instanton condition

$$* F = - * Q \wedge F = -\frac{\eta \wedge \omega^{n-2}}{(n-2)!} \wedge F \tag{3.58}$$

is equivalent to $F \in \mathfrak{su}(n)$, which implies in particular $\eta \lrcorner F = \omega \lrcorner F = 0$. Differentiating the instanton equation leads to the Yang-Mills equation

$$d_A * F + \frac{2\omega^{n-1}}{(n-2)!} \wedge F = 0, \tag{3.59}$$

whose torsion term is proportional to $F \lrcorner (\eta \wedge \omega)$, and thus vanishes.

Examples. In dimension 3 the only simply connected Sasaki-Einstein manifold is the sphere S^3, but already in dimension 5 a complete classification is missing. Many examples in arbitrary dimensions, including all homogeneous ones, can be obtained from the following construction. Let (Z, g_Z) be a $2n$-dimensional Kähler-Einstein manifold with positive Ricci curvature $\mathrm{Ric}_Z = 2ng_Z$. Then there exists a principal U(1)-bundle on Z whose total space carries a Sasaki-Einstein structure. Sasaki-Einstein manifolds obtained in this way are called regular; a generalization of this construction to Kähler-Einstein orbifolds gives rise to quasi-regular Sasaki-Einstein manifolds. Homogeneous Sasaki-Einstein manifolds are regular and can be obtained as circle bundles over generalized flag manifolds, including Hermitian symmetric spaces. Examples are

- odd-dimensional spheres $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$,
- Stiefel manifolds $V_2(\mathbb{R}^{n+1}) = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ (dimension $2n-1$),
- $\mathrm{SO}(2n)/\mathrm{SU}(n)$ (dimension $n^2 - n + 1$),
- $\mathrm{Sp}(n)/\mathrm{SU}(n)$ (dimension $n^2 + n + 1$),
- $E_6/\mathrm{SO}(10)$ (dimension 33) and E_7/E_6 (dimension 55).

They are U(1)-bundles over irreducible compact Hermitian symmetric spaces, at least for n large enough. Additional homogeneous examples are obtained by allowing for a reducible base. Low-dimensional Sasaki-Einstein manifolds of this type are the 7-dimensional spaces

$$Q(1,1,1) = \frac{SU(2)^3}{U(1)^2}, \tag{3.60}$$

with the $\mathrm{U}(1)^2$-embedding orthogonal to the diagonal U(1)-subgroup, fibred over $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, and

$$M(3,2) = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)}, \tag{3.61}$$

fibred over $\mathbb{C}P^2 \times \mathbb{C}P^1$ [34]. The precise embedding of the subgroup for $M(3,2)$ is explained in [25].

Many non-regular and even irregular (non-quasi-regular) Sasaki-Einstein manifolds exist in dimension ≥ 5 [19, 76]. For instance, S^5 and the Stiefel manifold $S^2 \times S^3$ carry several distinct quasi-regular non-regular Sasaki-Einstein structures. The same is true for the connected sums $k(S^2 \times S^3)$, where $k \geq 1$. Regular structures exist only up to $k = 8$, where the values $3 \leq k \leq 8$ can be realized in terms of U(1)-bundles over del-Pezzo surfaces P_k [10], and irregular structures have been constructed on $S^2 \times S^3$ [43].

In higher dimensions an interesting class of examples consists of exotic spheres. For instance, all 28 smooth structures on S^7 admit several Sasaki-Einstein metrics

[17]. Families of Sasaki-Einstein manifolds in every odd dimension ≥ 5 have been constructed in [16, 44, 30, 67].

3.3.4 3-Sasakian manifolds

A $(4n+3)$-dimensional 3-Sasakian manifold M is a particular Sasaki-Einstein manifold, with a further reduction of the structure group from $\mathrm{SU}(2(n+1)) \times 1$ to $\mathrm{Sp}(n) \times 1^3$. Since the number of Dirac spinors stabilized by $\mathrm{Sp}(n)$ is $2n+2$ in dimension $4n+3$, any connection with holonomy $\mathrm{Sp}(n)$ has $2n+2$ parallel spinors. On the other hand we have established a 1-1 correspondence between Killing spinors on M and parallel spinors on its cone $c(M)$. The holonomy group of the cone is $\mathrm{Sp}(n+1)$, which leaves invariant $n+2$ spinors. Thus, not all of the invariant spinors on M can be Killing spinors.

The centraliser of $\mathrm{Sp}(n)$ is a subgroup $\mathrm{Sp}(1)_1 \times \mathrm{Sp}(1)_2 \subset \mathrm{Spin}(4n+3)$, where $\mathrm{Sp}(1)_1, \mathrm{Sp}(n) \subset \mathrm{Spin}(4n)$, and $\mathrm{Sp}(1)_2 = \mathrm{Spin}(3)$. The $2n+2$ spinors transform in the irreducible representations $\underline{2}$ of $\mathrm{Sp}(1)_1$ and $\underline{n+1}$ of $\mathrm{Sp}(1)_2$. Of particular interest to us will be the diagonal subgroup $\mathrm{Sp}(1)_d$; the $2n+2$ spinors transform in the representation

$$\underline{2} \otimes \underline{n+1} \cong \underline{n} \oplus \underline{n+2} \tag{3.62}$$

of this subgroup. An orthonormal basis for $\underline{n+2}$ will be labelled ϵ_A, and for $\underline{n}$ $\tilde{\epsilon}_A$, where A runs from 1 to n or $n+2$ as appropriate. By a direct calculation using the expression (3.71) below for the canonical connection one can show that the ϵ_A are Killing spinors, whereas the $\tilde{\epsilon}_A$ are not, see Lemma A.4 in the appendix.

The stabiliser of a highest weight vector in the representation $\underline{n+2}$ of $\mathrm{Sp}(1)_d$ is $\mathrm{SU}(2n+1)$. It follows that any 3-Sasakian manifold admits a 2-sphere's worth of Sasaki-Einstein structures, which are rotated by the group $\mathrm{Sp}(1)_d$. Thus there exist three 1-forms η^α and three 2-forms ω^α, $\alpha = 1,2,3$. They satisfy the differential identities

$$\begin{aligned} d\eta^\alpha &= 2\omega_\alpha - \varepsilon^\alpha{}_{\beta\gamma}\eta^\beta \wedge \eta^\gamma \\ d\omega_\alpha &= -2\varepsilon_{\alpha\beta}{}^\gamma \eta^\beta \wedge \omega_\gamma. \end{aligned} \tag{3.63}$$

The family of Sasaki-Einstein structures (η, ω) is parametrized by $x \in \mathbb{R}^3$ with $|x|^2 = 1$ as

$$\eta = x_\alpha \eta^\alpha \qquad \text{and} \qquad \omega = x_\alpha \Big(\omega_\alpha - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\eta^\beta \wedge \eta^\gamma\Big). \tag{3.64}$$

The metric is of the form

$$g = g_Z + \eta^\alpha \otimes \eta^\alpha, \tag{3.65}$$

where g_Z is a $4n$-dimensional metric on the vanishing space of the ηs. The canonical 3-form is

$$P = \frac{1}{3}\Big(\eta^\alpha \wedge \omega_\alpha - \frac{1}{6}\varepsilon_{\alpha\beta\gamma}\eta^{\alpha\beta\gamma}\Big), \tag{3.66}$$

and it satisfies

$$dP = \frac{2}{3}\omega_\alpha \wedge \omega_\alpha =: 4Q. \tag{3.67}$$

We can introduce a frame $e^j_1, e^j_2, e^j_3, e^j_4$, $j = 1, \dots, n$ and e^α, $\alpha = 1, 2, 3$, which brings the forms to the following standard form

$$\begin{aligned} \eta^\alpha &= e^\alpha, \\ \omega_1 &= -e^j_1 \wedge e^j_3 - e^j_2 \wedge e^j_4, \\ \omega_2 &= e^j_1 \wedge e^j_4 - e^j_2 \wedge e^j_3, \\ \omega_3 &= -e^j_1 \wedge e^j_2 + e^j_3 \wedge e^j_4. \end{aligned} \tag{3.68}$$

Collectively, we write e^a for an arbitrary basis element of the form $e^j_{1,2,3,4}$. Both P and Q are $\mathrm{Sp}(n) \times \mathrm{Sp}(1)_d$-invariant, and the same is true for the following forms:

$$\begin{aligned} P' &= \frac{1}{3}\Big(\eta^\alpha \wedge \omega_\alpha - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\eta^{\alpha\beta\gamma}\Big) \\ Q' &= \frac{1}{6}\Big(\omega_\alpha \wedge \omega_\alpha - \varepsilon_{\alpha\beta\gamma}\eta^{\alpha\beta} \wedge \omega_\gamma\Big); \end{aligned} \tag{3.69}$$

they satisfy $dP' = 4Q'$. The forms P' and Q' can be constructed as bilinears in the $n+2$ spinors ϵ_A, whereas Q is constructed out of the full set of spinors:

$$\begin{aligned} P' &\propto \sum_{A=1}^{n+2} \big\langle \epsilon_A, \gamma_{\mu\nu\kappa}\epsilon_A \big\rangle e^{\mu\nu\kappa} \\ Q' &\propto \sum_{A=1}^{n+2} \big\langle \epsilon_A, \gamma_{\mu\nu\kappa\lambda}\epsilon_A \big\rangle e^{\mu\nu\kappa\lambda} \\ Q &\propto \left(\sum_{A=1}^{n+2} \big\langle \epsilon_A, \gamma_{\mu\nu\kappa\lambda}\epsilon_A \big\rangle + \sum_{A=1}^{n} \big\langle \tilde\epsilon_A, \gamma_{\mu\nu\kappa\lambda}\tilde\epsilon_A \big\rangle \right) e^{\mu\nu\kappa\lambda} \\ \eta^{123} &\propto \left(\sum_{A=1}^{n+2} \big\langle \epsilon_A, \gamma_{\mu\nu\kappa}\epsilon_A \big\rangle + \sum_{A=1}^{n} \big\langle \tilde\epsilon_A, \gamma_{\mu\nu\kappa}\tilde\epsilon_A \big\rangle \right) e^{\mu\nu\kappa}. \end{aligned} \tag{3.70}$$

The Levi-Civita connection ∇ and the canonical connection ∇^P are related by

$$\nabla^P - \nabla = \omega_{\alpha,ab} e^a (\xi_\alpha \otimes e^b - I_b \otimes \eta^\alpha) + \varepsilon_{\alpha\beta\gamma}\eta^\alpha(\xi_\beta \otimes \eta^\gamma), \tag{3.71}$$

where ξ_α is the vector field dual to η^α, and the I_a are dual to the orthonormal 1-forms e^a. Indices a, b, c run from 4 to $4n+3$, and $\alpha, \beta, \gamma = 1, 2, 3$. It can be proven that the Killing spinor equation for the spinors ϵ_A is equivalent to the condition $\nabla^P \epsilon_A = 0$. The canonical connection is compatible with a 1-parameter family of metrics

$$g_h = e^{2h} g_Z + \eta^\alpha \otimes \eta^\alpha, \tag{3.72}$$

and its torsion is given by

$$\begin{aligned} T^\alpha &= 2\omega_\alpha - \varepsilon_{\alpha\beta\gamma}\eta^{\beta\gamma}, \\ T^a &= -\omega_{\alpha,ab}\eta^\alpha \wedge e^b. \end{aligned} \tag{3.73}$$

With respect to the metric (3.72) the torsion tensor with only lower indices is totally antisymmetric if and only if $e^{2h} = 2$, and then proportional to P (3.66). There is another special h-value; for $e^{2h} = 2n+3$ the metric is Einstein again, but not 3-Sasakian.

The n remaining ∇^P-parallel spinors $\tilde{\epsilon}_A$ do not play any important role in 3-Sasakian geometry, except in dimension seven. In this case ($n = 1$) there are three Killing spinors for $h = 0$ plus one additional ∇^P-parallel spinor $\tilde{\epsilon}$. It turns out that for $e^{2h} = 5$ the spinor $\tilde{\epsilon}$ satisfies the Killing spinor equation, whereas the original Killing spinors do not. Therefore, M equipped with this deformed metric carries a nearly parallel G_2-structure [4, 39].

Sp(n)-instantons. Again there is no torsion in the Yang-Mills equation obeyed by Sp(n)-instantons. The derivative of the instanton equation

$$* F = -\frac{1}{6} * (\omega_\alpha \wedge \omega_\alpha) \wedge F \tag{3.74}$$

gives

$$d_A * F \;\propto\; F \wedge *(\eta^\alpha \wedge \omega_\alpha). \tag{3.75}$$

Due to $\eta^\alpha \lrcorner F = \omega_\alpha \lrcorner F = 0$ for $F \in \mathfrak{sp}(n)$ the right hand side vanishes.

Examples. Homogeneous, simply connected 3-Sasakian manifolds are in a 1-1 correspondence with compact simple Lie groups:

$$\begin{gathered} S^{4n+3} = \frac{Sp(n+1)}{Sp(n)}, \quad \frac{SU(n)}{S\big(U(n-2)\times U(1)\big)}, \quad \frac{SO(n)}{SO(n-4)\times Sp(1)}, \\ \frac{G_2}{Sp(1)}, \quad \frac{F_4}{Sp(3)}, \quad \frac{E_6}{SU(6)}, \quad \frac{E_7}{\mathrm{Spin}(12)}, \quad \frac{E_8}{E_7}. \end{gathered} \tag{3.76}$$

Furthermore, there is only one family of non-simply connected homogeneous examples, given by the real projective spaces $\mathbb{R}P^{4n+3} = S^{4n+3}/\mathbb{Z}_2$. Non-homogeneous 3-Sasakian manifolds can be constructed through a reduction procedure [19], and some examples are obtained as follows. Let $p \in \mathbb{Z}^{m+1}$ be such that

$$0 < p_1 \leq \cdots \leq p_{m+1}, \qquad \text{and} \qquad \gcd(p_i, p_j) = 1 \quad \forall i \neq j. \tag{3.77}$$

Define an action of U(1)×U($m-1$) on U($m+1$) through

$$(z, A) \cdot S = \mathrm{diag}\big(z^{p_1}, \ldots, z^{p_{m+1}}\big) \cdot S \cdot \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & A \end{pmatrix} \tag{3.78}$$

for $z \in S^1$, $A \in \mathrm{U}(m-1)$ and $S \in \mathrm{U}(m+1)$. Then the bi-quotient

$$S^m(p) = \mathrm{U}(m+1)\big/\big(\mathrm{U}(1)_p \times \mathrm{U}(m-1)\big) \qquad (3.79)$$

carries a 3-Sasakian structure. The dimension of $S^m(p)$ is $4m-1$, and for every m the $S^m(p)$ give infinitely many homotopy inequivalent simply connected compact inhomogeneous 3-Sasakian manifolds. In 7 dimensions the $S^2(p)$ carry a second metric of positive sectional curvature. Equipped with this positive metric they are examples of Eschenburg spaces [33]. Whether or not the Eschenburg metric coincides with the second nearly parallel G_2 metric that exists on every 3-Sasakian manifold is not known to me, but it seems at least plausible, given the fact that the standard examples of nearly parallel G_2-manifolds all have positive sectional curvature.

Similarly to the Sasaki-Einstein case, 3-Sasakian manifolds can be obtained as fibrations. Let (Z, g_Z) be a positive quaternionic Kähler manifold of dimension $4n$ and Ricci curvature $\mathrm{Ric}_Z = 4(n+2)g_Z$. Then there exists a principal SO(3)- or Sp(1)-bundle over Z carrying a 3-Sasakian structure, which is regular by definition. A generalization of this construction to quaternionic Kähler orbifolds gives rise to quasi-regular 3-Sasakian manifolds, and it turns out that every 3-Sasakian manifold is quasi-regular. Based on the LeBrun-Salamon conjecture that every positive quaternionic Kähler manifold is symmetric [66], there is a conjecture that every regular 3-Sasakian manifold is homogeneous.

3.4 Homogeneous spaces

Many Killing spinor manifolds are homogeneous, so in this section I collect some well-known results on homogeneous spaces [63, 2]. A homogeneous space is a space of cosets $M = G/K$, where $K \subset G$ are two Lie groups. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. M is called *reductive*, if there exists a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, such that

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}. \qquad (3.80)$$

Hence $\mathfrak{m}$ is a $\mathfrak{k}$-module for a reductive homogeneous space. M is called *symmetric*, if it is reductive and

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \qquad (3.81)$$

We will be mostly interested in non-symmetric spaces. Suppose that M is reductive, and let $\langle \cdot, \cdot \rangle$ be a K-invariant metric on $\mathfrak{m}$. There is an induced metric on M, and one says that M is naturally reductive (with respect to G and $\langle \cdot, \cdot \rangle$) if

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0 \qquad \forall X, Y, Z \in \mathfrak{m}. \qquad (3.82)$$

Suppose instead that one is given a G-invariant metric on all of $\mathfrak{g}$, and that $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal. Then the restriction of the metric to $\mathfrak{m}$ is naturally reductive. This is a good source for naturally reductive metrics: if G is simple, then the Killing form is the unique G-invariant metric on $\mathfrak{g}$, and if G is semi-simple then one has a multi-parameter family of G-invariant metrics, given by linear combinations of the Killing forms on simple factors.

Note that for a naturally reductive G/K we have that $\mathfrak{k} \subset \mathfrak{so}(m)$, and there is a reduction of the structure group from $\mathrm{O}(\mathfrak{m})$ to K. In fact, there is a K'-structure on G/K for every subgroup K' such that $K \subset K' \subset \mathrm{O}(\mathfrak{m})$. For instance, a 6-dimensional homogeneous nearly Kähler manifold G/K has necessarily $K \subset \mathrm{SU}(3)$.

We adopt the following index convention. Basis elements of $\mathfrak{k}$ are denoted by $I_i, I_j, I_k, \dots$, those of $\mathfrak{m}$ by $I_a, I_b, \dots$, and the full basis of $\mathfrak{g}$ gets Greek indices, $I_\mu, I_\nu, \dots$. The dual basis of left-invariant 1-forms on G is denoted e^μ, or e^k and e^a for those dual to I_k and I_a. The pull-backs of these forms to G/K under a local section $G/K \to G$ will be denoted by the same symbols, and they satisfy the Maurer-Cartan equations

$$\begin{aligned} de^k &= -\frac{1}{2} f^k_{lm} e^l \wedge e^m - \frac{1}{2} f^k_{ab} e^a \wedge e^b, \\ de^a &= -\frac{1}{2} f^a_{bc} e^b \wedge e^c - f^a_{bk} e^b \wedge e^k, \end{aligned} \tag{3.83}$$

where $f^\lambda_{\mu\nu}$ are the structure constants of $\mathfrak{g}$, defined by $[I_\mu, I_\nu] = f^\lambda_{\mu\nu} I_\lambda$.

3.4.1 Homogeneous vector bundles

Assume that $M = G/K$ is a naturally reductive space, where $\mathfrak{g}$ is equipped with a G-invariant metric $\langle \cdot, \cdot \rangle$ which induces the metric on M. Let (V, ρ) be a representation of K, and $E = G \times_K V$ the associated vector bundle over G/K, which consists of equivalence classes $[g, v]$ with $g \in G$ and $v \in V$, and identification $[g, v] = [gk^{-1}, \rho(k)v]$ for all $k \in K$. The sections of E are in a 1-1 correspondence with maps $f : G \to V$ satisfying

$$f(gk) = \rho(k)^{-1} f(g), \qquad \forall k \in K. \tag{3.84}$$

G acts on the space of sections $\Gamma(G/K, E)$ through $(g \cdot f)(h) = f(g^{-1}h)$. For instance, the tangent bundle of M is homogeneous:

$$TM \cong G \times_K \mathfrak{m}. \tag{3.85}$$

The set of G-invariant sections of E (also called homogeneous sections) is given by the constant functions, and hence is in a 1-1 correspondence to the K-invariant elements of V:

Lemma 3.8. *Let V carry a representation of K, then*

$$\Gamma(G/K, G \times_K V)^G \simeq V^K.$$

One example of a homogeneous section is the metric; it corresponds to a K-invariant element of $\mathfrak{m}^* \otimes \mathfrak{m}^*$.

3.4.2 The 3-form and 4-form.

Further important examples of homogeneous sections are the following. Define $P \in \Lambda^3\mathfrak{m}^*$ through

$$P(X,Y,Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle, \qquad \forall X,Y,Z \in \mathfrak{m}, \tag{3.86}$$

and $Q \in \Lambda^4\mathfrak{m}^*$ as the image under the wedge product of $\tilde{Q} \in \Lambda^2\mathfrak{m}^* \otimes \Lambda^2\mathfrak{m}^*$, where

$$\tilde{Q}(W,X,Y,Z) = \langle [W,X]_{\mathfrak{k}}, [Y,Z]_{\mathfrak{k}}\rangle. \tag{3.87}$$

In coordinates

$$\begin{aligned} P &= -\frac{1}{6} f_{abc} e^{abc}, \\ Q &= \frac{1}{4!} f_{kab} f^k_{cd} e^{abcd} = -\frac{1}{4!} f_{abe} f^e_{cd} e^{abcd}. \end{aligned} \tag{3.88}$$

The definition (3.86) makes sense on a general Riemannian homogeneous space, but for P to be totally antisymmetric we need M to be naturally reductive. On a symmetric space on the other hand, P and Q vanish. One easily checks that P and Q are K-invariant, so that they give rise to a 3- and 4-form on G/K. In case K is chosen trivial and G is simple, connected and compact, P becomes a generator of $H^3(G,\mathbb{Z}) = \mathbb{Z}$ upon proper normalization of the metric. Q is the canonical 4-form associated to the reduction of the structure group of G/K to K. For supergravity applications we need the derivative and coderivative of P [2]:

Lemma 3.9. *We have*

$$dP = -6Q, \tag{3.89}$$

whereas $d*P = 0$.

Proof. From the Maurer-Cartan equation we have

$$dP = \frac{1}{4} f_{abe} f^e_{cd} e^{abcd} + \frac{1}{4} f_{abc} f^a_{dk} e^{dkbc}.$$

Consider the last term. It follows from the Jacobi identity that $f_{abc} f^a_{dk}$ splits into a part which is symmetric in b and d, and another part symmetric in c and d. Therefore this term vanishes. Using again a Jacobi identity, we conclude that

$$f_{abe} f^e_{cd} e^{abcd} = -f_{kab} f^k_{cd} e^{abcd}.$$

$d*P$: We assume the I_μ to form an othonormal basis, such that $f_{\mu\nu\lambda}$ is totally antisymmetric. Furthermore we will not keep track of whether an index is up or down, but rather sum over any index appearing more than once. Then we have

$$*P = -\frac{1}{6(n-3)!} \varepsilon^{a_1 \ldots a_n} f_{a_1 a_2 a_3} e^{a_4 \ldots a_n},$$

with derivative

$$
\begin{aligned}
d*P &= \frac{1}{12(n-4)!}\varepsilon^{a_1\dots a_n} f_{a_1a_2a_3} f^{a_4}_{bc}\, e^{bca_5\dots a_n} \\
&+ \frac{1}{6(n-4)!}\varepsilon^{a_1\dots a_n} f_{a_1a_2a_3} f^{a_4}_{bk}\, e^{bka_5\dots a_n}.
\end{aligned}
\tag{3.90}
$$

The first term is easily seen to vanish: b and c only run over the values of a_1, a_2 and a_3, giving contributions of the type

$$
\varepsilon^{a_1\dots a_n} f_{a_1a_2a_3} f_{a_4a_1a_2} e^{a_1a_2a_5\dots a_n},
$$

where the two f factors are symmetric in a_3 and a_4, and thus vanish. Now let us consider the second contribution in (3.90). We have

$$
\begin{aligned}
d*P &= \frac{1}{2(n-4)!}\varepsilon^{a_1\dots a_n} f_{a_1a_2a_3} f^{a_4}_{a_3k} e^{a_3ka_5\dots a_n} \\
&= \frac{1}{2(n-3)!}\varepsilon^{a_1\dots a_n} f_{a_1a_2\mu} f^{a_4}_{\mu k} e^{a_3ka_5\dots a_n},
\end{aligned}
$$

which vanishes, again due to the Jacobi identity. □

3.4.3 Connections.

Due to the identification $T^*(G/K) = G \times_K \mathfrak{m}^*$, a connection on a homogeneous vector bundle $G \times_K V$ can be considered as a map

$$
\nabla : C^\infty(G, V)^K \to C^\infty(G, V \otimes \mathfrak{m}^*)^K,
$$

satisfying additional properties. The simplest example is the so-called canonical connection ∇^P (the coincidence of name and notation with the canonical connection on a real Killing spinor manifolds is intended), acting as

$$
\nabla^P_X f = X_L(f) \qquad \forall X \in \mathfrak{m},\ f \in C^\infty(G, V)^K. \tag{3.91}
$$

Here X_L is the left-invariant vector field on G corresponding to X. In a frame of left-invariant 1-forms on G/K its connection form is given by

$$
\Gamma^P = d\rho_e(I_k)e^k, \tag{3.92}
$$

with $d\rho_e$ the differential of $\rho : K \to \mathrm{Aut}(V)$ at the identity. As is clear from the definition, the parallel sections of ∇^P correspond to constant functions, and thus to K-invariant elements of V:

Lemma 3.10. *Let V carry a K-representation, then the parallel sections of $G\times_K V$ with respect to ∇^P are in a 1-1 correspondence with K-invariant elements of V, and by Lemma 3.8 are precisely the homogeneous sections.*

This is a special case of the so-called general holonomy principle [3]; parallel sections of a vector bundle are in a 1-1 correspondence with elements invariant under the holonomy group. Starting from the canonical connection ∇^P we can define a 1-parameter family of homogeneous connections by adding a multiple of the 3-form P:

$$\nabla^t = \nabla^P - tP^a{}_{bc}e^b(a \otimes e^c). \tag{3.93}$$

Apparently, $t = 0$ gives back the canonical connection. For $t = \frac{1}{2}$ we obtain the Levi-Civita connection of the metric induced by the Killing form, and there are further t-values of geometric significance [2, 3]. The torsion of ∇^t is

$$T^a = \frac{1}{2}(2t-1)f^a_{bc}e^{bc}. \tag{3.94}$$

From the point of view of heterotic supergravity the most important connections are ∇^P and $\nabla^{t=1}$; in Chapter 6 we will identify ∇^P with ∇^-, the connection appearing in the gravitino equation (2.6a), then $\nabla^{t=1}$ gets identified with ∇^+, which appears in the Bianchi identity. The curvature of ∇^t is

$$\begin{aligned} R^t &= -\tfrac{1}{2}\left\{tf^e_{ab}f^c_{ed} + f^k_{ab}f^c_{kd} - 2t^2 f^c_{ae}f^e_{bd}\right\} e^a \wedge e^b(I_c \otimes e^d), \\ \mathrm{Ric}^t &= -\left\{(t-t^2)f^d_{ac}f^c_{bd} + f^k_{ac}f^c_{bk}\right\}e^a \otimes e^b. \end{aligned} \tag{3.95}$$

The holonomy group for these connections is generically $\mathrm{SO}(\mathfrak{m})$, but ∇^P has holonomy group $K \subset \mathrm{SO}(\mathfrak{m})$. The curvatures for $t = 0$ and $t = 1$ are particularly simple, as elements of $\mathrm{End}(\mathfrak{m}) \otimes \Lambda^2\mathfrak{m}^*$ and in coordinates they are given by

$$\begin{aligned} R^{t=1} &= -\mathrm{ad}(I_a) \circ \pi_{\mathfrak{k}} \circ \mathrm{ad}(I_b)e^a \wedge e^b, \qquad & (R^1)^c{}_{dab} &= 2f^c_{k[a}f^k_{b]d}, \\ R^{t=0} &= -\frac{1}{2}f^k_{ab}\,\mathrm{ad}(I_k)e^a \wedge e^b, & (R^0)^c{}_{dab} &= -f^k_{ab}f^c_{kd}, \end{aligned} \tag{3.96}$$

with $\pi_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ the orthogonal projection and 'ad' the adjoint representation of $\mathfrak{g}$ on itself. Suppose now that G is simple and the metric induced by the Killing form. The heterotic Bianchi identity contains the following expressions

$$\begin{aligned} \mathrm{tr}_{\mathfrak{m}}\big(R^0 \wedge R^0\big) &= \frac{1}{4}\langle I_k, I_l\rangle_{\mathfrak{m}}\, f^k_{ab}f^l_{cd}e^{abcd}, \\ \mathrm{tr}_{\mathfrak{m}}\big(R^1 \wedge R^1\big) &= -\frac{1}{4}\langle I_k, I_l\rangle_{\mathfrak{k}}\, f^k_{ab}f^l_{cd}e^{abcd}, \end{aligned} \tag{3.97}$$

where we introduced the negative Killing form $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ of the subalgebra $\mathfrak{k}$, and similarly

$$\langle I_k, I_l\rangle_{\mathfrak{m}} = \mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}(I_k) \circ \mathrm{ad}(I_l)\big). \tag{3.98}$$

Thus we have

$$\mathrm{tr}\Big(R^1 \wedge R^1 - R^0 \wedge R^0\Big) = -\frac{1}{4}\langle I_k, I_l\rangle_{\mathfrak{g}}\, f^k_{ab}f^l_{cd}e^{abcd}. \tag{3.99}$$

Using the result of Lemma 3.9 we conclude that

$$\mathrm{tr}\big(R^1 \wedge R^1 - R^0 \wedge R^0\big) \propto dP, \tag{3.100}$$

which is very similar to the Bianchi identity (2.7). Note that the right hand side of this equation can be rescaled arbitrarily by changing the metric on G, whereas the left-hand side is metric-independent.

Many examples of non-symmetric homogeneous spaces turn out to have a real Killing spinor. This is somewhat surprising, given that they are not even Einstein in general for the metric induced by the Killing form, according to (3.95). In fact, the Killing form does not usually coincide with the Killing spinor metric, but rather with the metric that makes the canonical torsion totally antisymmetric. On homogeneous nearly Kähler and nearly parallel G_2-manifolds, however, the Killing form is indeed the Einstein metric.

3.4.4 Spinors

The spin bundle on a homogeneous manifold is constructed as follows. G-invariance of the metric implies that $\mathfrak{k}$ acts orthogonally on $\mathfrak{m}$, giving rise to an embedding

$$\mathrm{ad}_{\mathfrak{m}} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{m}), \tag{3.101}$$

which can be composed with the spin representation $dS : \mathfrak{so}(\mathfrak{m}) \to \mathfrak{spin}(\mathfrak{m})$, to give $\widetilde{\mathrm{ad}} := dS \circ \mathrm{ad}_{\mathfrak{m}}$. We assume that this lifts to a representation of K, a sufficient condition for this is simply-connectedness of K. Denoting the spinor space over $\mathfrak{m}$ by $S(\mathfrak{m})$, we get an associated bundle

$$\mathcal{S} = G \times_K S(\mathfrak{m}), \tag{3.102}$$

which is the spinor bundle over G/K. The homogeneous connections considered above induce connections on $\mathcal{S}$, and the parallel sections with respect to ∇^P correspond to K-invariant elements of S. To determine whether there exist parallel spinors we thus need to know whether the trivial representation of K occurs in the decomposition of the spinor representation $S(\mathfrak{m})$ into irreducibles, which is a purely algebraic task.

Proposition 3.11. *Suppose that K leaves invariant a spinor $\epsilon \in S(\mathfrak{m})$. Then the curvature form R^P of the canonical connection satisfies the instanton equation $\gamma(R^P)\epsilon = 0$.*

Proof. Since ϵ is K-invariant it gives rise to a ∇^P-parallel spinor on G/K, which implies $R^P_{abcd}\gamma^{ab}\epsilon = 0$. But according to Lemma (3.6) or our explicit calculation above, R^P has the interchange symmetry $R^P_{abcd} = R^P_{cdab}$, and hence satisfies the instanton condition. □

The following commutation relation between the quantized 3-form and elements of $\mathfrak{k}$ acting on spinors over $\mathfrak{m}$ will be useful:

Lemma 3.12. *For $X \in \mathfrak{k}$ we have*

$$[\gamma(P), \widetilde{ad}(X)] = 0. \tag{3.103}$$

Proof. A simple calculation in the Clifford algebra shows that

$$\big[\gamma(P), \widetilde{\mathrm{ad}}(X)\big] = \gamma\big(\mathrm{ad}(X) \cdot P\big), \tag{3.104}$$

where $\cdot$ denotes the action of $\mathfrak{so}(\mathfrak{m})$ on $\Lambda^3\mathfrak{m}^*$, and we know that P is invariant under this action of $\mathfrak{k}$. □

This means that $\gamma(P)$ leaves the set of K-invariant elements in $S(\mathfrak{m})$ invariant, so if there is only one invariant spinor then $\gamma(P)$ maps it to a multiple of itself.

3.4.5 Examples

Four classes of non-symmetric naturally reductive homogeneous spaces are of interest to us, corresponding to the four types of real Killing spinor manifolds. In this section Greek variables $\lambda, \mu, \ldots$ are used to denote roots of a simple Lie algebra, instead of coordinates on a manifold.

Sasaki-Einstein manifolds. It has been mentioned already that every homogeneous Sasaki-Einstein manifold is the total space of a principal U(1)-bundle over a so-called generalized flag manifold. These latter spaces are by definition coset spaces of a Lie group G by the centralizer of a torus, and they carry a Kähler-Einstein structure [7]. The classification and an exhaustive list of examples for non-exceptional G can also be found in Arvanitogeórgos' book [7]. We now give an explicit method to obtain the U(1)-bundles over generalized flag manifolds for simple G, in terms of the root space decomposition of its Lie algebra $\mathfrak{g}$.

Let G be a connected compact simple simply connected real Lie group with Lie algebra $\mathfrak{g}$, and Cartan subalgebra $\mathfrak{h}$. Choose the positive roots, and let μ be the highest simple root. Denote by $\mathcal{R}^+$ the set of positive roots, and by $\mathcal{S}^+ \subset \mathcal{R}^+$ the set of positive roots which are linear combinations of all the simple roots except μ. Then $\mathfrak{g}$ has a root space decomposition of the form

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{h} \bigoplus_{\lambda \in \mathcal{R}^+} \big(\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}\big), \tag{3.105}$$

and we define the subalgebras $\mathfrak{t} \subset \mathfrak{c} \subset \mathfrak{g}$:

$$\begin{aligned} \mathfrak{t} &= \big\{H \in \mathfrak{h} \mid \lambda(H) = 0 \ \forall \lambda \in \mathcal{S}^+\big\} \cap \mathfrak{g}, \\ \mathfrak{c} &= \Big[\mathfrak{h} \bigoplus_{\lambda \in \mathcal{S}^+} \big(\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}\big)\Big] \cap \mathfrak{g}. \end{aligned} \tag{3.106}$$

Furthermore we need the space

$$\mathfrak{m} = \Big[\mathfrak{t} \bigoplus_{\lambda\in\mathcal{R}^+\setminus\mathcal{S}^+} \big(\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}\big)\Big] \cap \mathfrak{g}. \tag{3.107}$$

These definitions imply the following commutation relations:

$$[\mathfrak{c},\mathfrak{c}] \subset \mathfrak{c}, \qquad [\mathfrak{c},\mathfrak{t}] = 0, \qquad [\mathfrak{c},\mathfrak{m}] \subset \mathfrak{m}. \tag{3.108}$$

Denote by $C, T \subset G$ the corresponding Lie groups, then C is the centralizer of $T \simeq \mathrm{U}(1)$ in G, and $T \subset C$ is a normal subgroup. Thus $K := C/T$ is a group again, and its Lie algebra is the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{c}$ with respect to the Killing form. Due to the relation $(G/K)/T \simeq G/C$ we get a $U(1)$-fibration

$$\pi : G/K \to G/C, \tag{3.109}$$

where the base space carries a homogeneous Kähler structure, with complex structure induced by the choice of positive roots. The tangent bundle of G/K can be represented as $T(G/K) \cong G \times_K \mathfrak{m}$. Let h be a generator of $\mathfrak{t}^*$; it is K-invariant and hence gives rise to a globally defined 1-form η on G/K. For $\lambda \in \mathcal{R}^+ \setminus \mathcal{S}^+$ let e^λ be a generator of the dual root space $\mathfrak{g}^*_\lambda$, and $e^{-\lambda}$ a generator of $\mathfrak{g}^*_{-\lambda}$. Furthermore, let $H_\mathfrak{t} \in \mathfrak{t}$ be dual to h. Then

$$\sum_{\lambda\in\mathcal{R}^+\setminus\mathcal{S}^+} \lambda(H_t) e^\lambda \wedge e^{-\lambda} \tag{3.110}$$

defines a K-invariant element of $\Lambda^2\mathfrak{m}^*$, which induces a globally defined 2-form ω on G/K. In fact, (3.110) is even C-invariant, and it induces the Kähler form on G/C. It can be shown that η, ω and a certain homogeneous metric (not induced by the Killing form of $\mathfrak{g}$) define a Sasaki-Einstein structure on G/K. The second important metric on Sasaki-Einstein spaces, which makes the canonical torsion totally antisymmetric, is the metric induced by the Killing form on $\mathfrak{g}$. Examples of this type have been listed already in Section 3.3.3.

3-Sasakian manifolds. The construction is very similar to the one for Sasaki-Einstein manifolds [83]. Let $\mathfrak{h} \subset \mathfrak{g} \otimes \mathbb{C}$ be a Cartan subalgebra, $\mathcal{R}^+$ the set of positive roots, and μ the highest root. Define a Cartan generator $H_\mu \in \mathfrak{h}$ through $\langle H_\mu, H\rangle = \mu(H)$ for all $H \in \mathfrak{h}$, where $\langle\cdot,\cdot\rangle$ is the Killing form. Let $\mathcal{S}^+$ be the set of positive roots which are linear combinations of all positive roots except μ. Define subalgebras

$$\begin{aligned}\mathfrak{k} &= \Big[\big\{H \in \mathfrak{h} \mid \mu(H) = 0\big\} \bigoplus_{\lambda\in\mathcal{S}^+} (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})\Big] \cap \mathfrak{g} \\ \mathfrak{a} &= \mathbb{R}\cdot H_\mu \oplus (\mathfrak{g}_\mu \oplus \mathfrak{g}_{-\mu}) \cap \mathfrak{g}.\end{aligned} \tag{3.111}$$

Then $\mathfrak{k}$ is the centralizer of $\mathfrak{a} \simeq \mathfrak{sp}(1)$ in $\mathfrak{g}$. Let $K \subset G$ be the subgroup generated by $\mathfrak{k}$ and $K\cdot \mathrm{Sp}(1)$ the subgroup generated by $\mathfrak{k} \oplus \mathfrak{a}$. Then $G/K\cdot \mathrm{Sp}(1)$ is quaternionic

Kähler, and G/K is 3-Sasakian. The tangent bundle of G/K is given by $T(G/K) \cong G \times_K \mathfrak{m}$, where

$$\mathfrak{m} = \Big[\mathbb{R} \cdot H_\mu \bigoplus_{\lambda \in \mathcal{R}^+ \setminus \mathcal{S}^+} (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})\Big] \cap \mathfrak{g}. \tag{3.112}$$

The three (K-invariant) generators of $\mathfrak{a}^*$ give rise to three 1-forms η^1, η^2, η^3 on G/K, and the three 2-forms ω_j can be obtained as follows. Let I_1, I_2, I_3 be a basis of $\mathfrak{a}$, and for $\lambda \in \mathcal{R}^+ \setminus (\mathcal{S}^+ \cup \{\mu\})$ denote by X_λ the generator of $\mathfrak{g}_\lambda$, by e^λ the dual generator of $\mathfrak{g}_\lambda^*$ and by $e^{-\lambda}$ the complex conjugate of e^λ. Then

$$\omega_\alpha \propto \sum_{\lambda \in \mathcal{R}^+ \setminus (\mathcal{S}^+ \cup \{\mu\})} \langle X_\lambda, I_\alpha \rangle e^\lambda \wedge e^{-\lambda}, \tag{3.113}$$

for $\alpha = 1, 2, 3$, define K-invariant elements of $\Lambda^2 \mathfrak{m}^*$ and hence give rise to homogeneous 2-forms on G/K. An exhaustive list of examples has been given in (3.76). The Killing form makes the torsion of ∇^P antisymmetric, and the Einstein metric has been constructed in [14].

Nearly Kähler manifolds. There is no general construction for homogeneous nearly Kähler or nearly parallel G_2-manifolds, although the root space decomposition proves useful when dealing with homogeneous nearly Kähler manifolds as well. In this case the almost complex structure is not obtained directly from the decomposition of the roots into positive and negative ones, but gives rise to an alternative splitting of the set of roots. I will illustrate this for the special case $S^6 = G_2/\mathrm{SU}(3)$. The six positive roots of $\mathfrak{g}_2$ split into two groups λ_j, μ_k, $j,k = 1,2,3$, of three roots each, where the λ_j-root spaces together with the Cartan algebra generate an $\mathfrak{su}(3)$-subalgebra, and the μ_k-root spaces generate the orthogonal complement of $\mathfrak{su}(3)$. They are ordered as follows:

$$\lambda_1 > \lambda_2 > \mu_3 > \mu_2 > \lambda_3 > \mu_1, \tag{3.114}$$

and satisfy the relations

$$\begin{gathered} \lambda_1 = \lambda_2 + \lambda_3, \qquad \lambda_1 = \mu_2 + \mu_3, \qquad \lambda_2 = \mu_1 + \mu_3, \\ \mu_2 = \lambda_3 + \mu_1, \qquad \mu_3 = \mu_1 + \mu_2. \end{gathered} \tag{3.115}$$

Denote the generator of the root space $\mathfrak{g}_{\mu_j}$ by X_j, for $j = 1,2,3$. An SU(3)-invariant complex structure on $\mathfrak{m} = \mathrm{span}\{X_1, X_2, X_3, \overline{X}_1, \overline{X}_2, \overline{X}_3\} \cap \mathfrak{g}_2$ is obtained by defining the $\pm i$-eigenspaces as

$$+i: \quad X_1, X_2, \overline{X}_3, \qquad -i: \quad \overline{X}_1, \overline{X}_2, X_3. \tag{3.116}$$

This lifts to a homogeneous almost-complex structure on $G_2/\mathrm{SU}(3)$, and similar constructions must be possible for the other three homogeneous nearly Kähler manifolds.

3.5 Sine-cones over manifolds with Killing spinors

I mentioned already that the sine-cone over a five-dimensional Sasaki-Einstein manifold is nearly Kähler and that the sine-cone over a nearly Kähler manifold is nearly parallel G_2. Their cones are a Calabi-Yau 3-fold, a G_2-manifold in dimension seven, and an eight-dimensional Spin(7)-manifold, respectively. Furthermore, the cone over the sine-cone agrees with the cylinder over the cone, as illustrated by the following 'commutative' diagram:

$$\begin{array}{ccccc} \mathrm{SE}^5 & \xrightarrow{sc} & \mathrm{nK} & \xrightarrow{sc} & \mathrm{np}\ G_2 \\ {\scriptstyle c}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle c}\downarrow \\ \mathrm{CY} & \xrightarrow{cyl} & G_2 & \xrightarrow{cyl} & \mathrm{Spin}(7). \end{array} \tag{3.117}$$

This section contains a review of these constructions, as well as a proof that the iterated sine-cone over a Sasaki-Einstein manifold is again Sasaki-Einstein. Similarly, one should expect that the fourth sine-cone over a 3-Sasakian manifold is 3-Sasakian again, but I will not prove that.

3.5.1 5D Sasaki-Einstein diagram

5D Sasaki-Einstein to nearly Kähler. Let $(M, \eta, \tilde{\omega}, \tilde{\Omega})$ be a 5-dimensional Sasaki-Einstein manifold. On the sine-cone $sc(M)$ with angular coordinate θ define the following forms [36]

$$\begin{aligned} \omega &= \sin^2(\theta)\cos(\theta)\tilde{\omega} + \sin^3(\theta)\mathrm{Re}\big(e^{i\alpha}\tilde{\Omega}\big) + \sin(\theta) d\theta \wedge \eta, \\ \Omega &= \sin^2(\theta)\Big[\cos(\theta)\mathrm{Re}\big(e^{i\alpha}\tilde{\Omega}\big) + i\mathrm{Im}\big(e^{i\alpha}\tilde{\Omega}\big) - \sin(\theta)\tilde{\omega}\Big] \wedge \big(-\sin(\theta)\eta + i d\theta\big). \end{aligned} \tag{3.118}$$

Here $\alpha \in \mathbb{R}$ is a parameter. One can show that ω and Ω together with the sine-cone metric satisfy the defining relations of a nearly Kähler manifold.

Sasaki-Einstein to Calabi-Yau. On the cone over a Sasaki-Einstein manifold M^{2n+1} define the forms

$$\begin{aligned} \omega' &= r^2\tilde{\omega} + r dr \wedge \eta, \\ \Omega' &= r^n\tilde{\Omega} \wedge \big(dr + ir\eta\big). \end{aligned} \tag{3.119}$$

Together with the cone metric they define a Calabi-Yau structure.

6D Calabi-Yau to parallel $\mathbf{G_2}$. Let (M, ω', Ω') be a 6-dimensional Calabi-Yau, so that ω' and Ω' satisfy the same algebraic relations as for nearly Kähler manifolds, but are both closed and coclosed. On the cylinder $cyl(M)$ define the 3-form Φ and its dual as

$$\begin{aligned} \Phi &= d\tau \wedge \omega' + \mathrm{Re}\big(e^{i\alpha}\Omega'\big), \\ *\Phi &= \frac{1}{2}\omega' \wedge \omega' - d\tau \wedge \mathrm{Im}\big(e^{i\alpha}\Omega'\big), \end{aligned} \tag{3.120}$$

for some real parameter α. Both forms are closed, and they define a G_2-holonomy reduction.

Nearly Kähler to integrable $\mathbf{G_2}$. Start with a nearly Kähler manifold (M, ω, Ω), and consider on the cone $c(M)$, with radial variable ρ, the 3-form

$$\begin{aligned} \Phi &= \rho^2 d\rho \wedge \omega + \rho^3 \mathrm{Im}(\Omega), \\ *\Phi &= \frac{1}{2}\rho^4 \omega \wedge \omega + \rho^3 d\rho \wedge \mathrm{Re}(\Omega). \end{aligned} \tag{3.121}$$

We have $d\Phi = d * \Phi = 0$, and they define a G_2-holonomy reduction on $c(M)$.

The constructions of the last four paragraphs give rise to the diagram

$$\begin{array}{ccc} \text{5D Sasaki-Einstein} & \xrightarrow[(+\theta)]{sc} & \text{nearly Kähler} \\ c\big\downarrow(+r) & & c\big\downarrow(+\rho) \\ \text{Calabi-Yau} & \xrightarrow[(+\tau)]{cyl} & \text{parallel } G_2, \end{array} \tag{3.122}$$

which reads in terms of forms:

$$\begin{array}{ccc} (\eta, \tilde{\omega}, \tilde{\Omega}) & \xrightarrow{sc} & (\omega, \Omega) \\ c\big\downarrow & & c\big\downarrow \\ (\omega', \Omega') & \xrightarrow{cyl} & \Phi. \end{array} \tag{3.123}$$

One can check that the definition of Φ does not depend on the path chosen, if the coordinates are related by $r = \rho \sin(\theta)$ and $\tau = \rho \cos(\theta)$.

3.5.2 Nearly Kähler diagram

Nearly Kähler to nearly parallel $\mathbf{G_2}$. Start with a nearly Kähler manifold (M, ω, Ω). Then

$$\begin{aligned} \tilde{\Phi} &= \sin^2(\theta) d\theta \wedge \omega + \sin^3(\theta) \mathrm{Im}\big(e^{i\theta}\Omega\big), \\ *\tilde{\Phi} &= \frac{1}{2}\sin^4(\theta)\omega \wedge \omega + \sin^3(\theta) d\theta \wedge \mathrm{Re}\big(e^{i\theta}\Omega\big) \end{aligned} \tag{3.124}$$

define a nearly parallel G_2-structure on the sine-cone $sc(M)$ [15].

Nearly parallel $\mathbf{G_2}$ to Spin(7). Let $(M, \tilde{\Phi})$ be a nearly parallel G_2-manifold with $d\tilde{\Phi} = 4 * \tilde{\Phi}$. On the cone $c(M)$ consider the self-dual 4-form

$$\Psi = (1 + *_8)\rho^3 d\rho \wedge \tilde{\Phi} = \rho^3 d\rho \wedge \tilde{\Phi} + \rho^4 *_7 \tilde{\Phi}. \tag{3.125}$$

Ψ is closed and defines a parallel Spin(7)-structure.

Integrable $\mathbf{G_2}$ to Spin(7). Consider the seven-manifold (M, Φ) with $d\Phi = d * \Phi = 0$, and the cylinder $cyl(M)$. The self-dual 4-form

$$\Psi = (1 + *)d\tau \wedge \Phi = d\tau \wedge \Phi + *_7\Phi \tag{3.126}$$

satisfies $d\Psi = d * \Psi = 0$ and defines a Spin(7)-structure.

From the last four paragraphs we arrive at the commuting diagram

$$\begin{array}{ccc} \text{nearly Kähler} & \xrightarrow[(+\theta)]{sc} & \text{nearly parallel } G_2 \\ c\Big\downarrow(+r) & & c\Big\downarrow(+\rho) \\ \text{parallel } G_2 & \xrightarrow[(+\tau)]{cyl} & \text{parallel Spin}(7), \end{array} \tag{3.127}$$

which reads in terms of forms:

$$\begin{array}{ccc} (\omega, \Omega) & \xrightarrow{sc} & \tilde{\Phi} \\ c\Big\downarrow & & c\Big\downarrow \\ \Phi & \xrightarrow{cyl} & \Psi. \end{array} \tag{3.128}$$

According to Proposition 3.4 the sine-cone over a Killing spinor manifold has a Killing spinor again, so one may wonder what happens if we extend the diagram to the right by taking the sine-cone over a nearly parallel G_2-manifold. It turns out that the structure group G_2 cannot be enlarged in any interesting way in this case. This can be understood for instance by considering the stabilizer of a Majorana Killing spinor in dimension eight. A chiral Majorana spinor has stabilizer Spin(7), whereas the common stabilizer of two Majorana spinors of different chirality is G_2. Since Killing spinors are non-chiral, it follows that their stabilizer is G_2, which is then the canonical structure group for an eight-dimensional manifold with real Killing spinors. Contrary to the cases considered above the structure group in seven dimensions cannot be enlarged in the transition to eight dimension.

A different perspective is offered by the lower row of (3.127). An eight-dimensional Spin(7)-manifold is defined in terms of a single 4-form Ψ. Adding another cylinder the only non-trivial parallel differential form we could construct is the 5-form $d\tau \wedge \Psi$, but if this one is parallel then the same is true for the 4-form $\Psi = *(d\tau \wedge \Psi)$, and the holonomy group reduces to Spin(7) again. Therefore nothing interesting happens if the diagram (3.127) is extended to higher dimensions. Of course, this is also reflected in Bär's theorem which forbids the existence of any complete eight-dimensional manifold with real Killing spinors besides the sphere.

3.5.3 Sasaki-Einstein diagram

Sasaki-Einstein to Sasaki-Einstein: iterated sine-cones. Let (M, ω, Ω) be a Sasaki-Einstein manifold, and denote the coordinate of the first sine-cone by θ_1 and the other one by θ_2. On $sc(sc(M))$ we define the 1-forms

$$\begin{aligned}
\widehat{\eta} &= \sin^2(\theta_1)\sin^2(\theta_2)\eta - \sin(\theta_1)\sin(\theta_2)\cos(\theta_2)d\theta_1 + \cos(\theta_1)d\theta_2,\\
\lambda &= -\cos(\theta_1)\sin(\theta_2)\cos(\theta_2)d\theta_1 - \sin(\theta_1)d\theta_2 + \sin(\theta_1)\cos(\theta_1)\sin^2(\theta_2)\eta,\\
\mu &= \sin^2(\theta_2)d\theta_1 + \sin(\theta_1)\sin(\theta_2)\cos(\theta_2)\eta.
\end{aligned} \tag{3.129}$$

The $\mathrm{SU}(n+1)\times 1$-structure on $sc(sc(M))$ is given by

$$\begin{aligned}
\widehat{g} &= \sin^2(\theta_1)\sin^2(\theta_2)g_K + \widehat{\eta}\otimes\widehat{\eta} + \lambda\otimes\lambda + \mu\otimes\mu\\
&= d\theta_2^2 + \sin^2(\theta_2)\big(d\theta_1^2 + \sin^2(\theta_1)g_M\big),\\
\widehat{\eta} &= \sin^2(\theta_1)\sin^2(\theta_2)\eta - \sin(\theta_1)\sin(\theta_2)\cos(\theta_2)d\theta_1 + \cos(\theta_1)d\theta_2,\\
\widehat{\omega} &= \sin^2(\theta_1)\sin^2(\theta_2)\omega + \mu\wedge\lambda\\
&= \sin^2(\theta_1)\sin^2(\theta_2)\omega - \sin(\theta_1)\sin^2(\theta_2)d\theta_1\wedge d\theta_2\\
&+ \sin(\theta_1)\cos(\theta_1)\sin^2(\theta_2)d\theta_1\wedge\eta + \sin^2(\theta_1)\sin(\theta_2)\cos(\theta_2)d\theta_2\wedge\eta,\\
\widehat{\Omega} &= \sin^n(\theta_1)\sin^n(\theta_2)\Omega\wedge(\lambda - i\mu)\\
&= -\Omega\wedge\Big[\sin^n(\theta_1)\sin^{n+1}(\theta_2)\big(\cos(\theta_1)\cos(\theta_2) + i\sin(\theta_2)\big)d\theta_1\\
&\quad + \sin^{n+1}(\theta_1)\sin^n(\theta_2)d\theta_2\\
&\quad + \sin^{n+1}(\theta_1)\sin^{n+1}(\theta_2)\big(-\cos(\theta_1)\sin(\theta_2) + i\cos(\theta_2)\big)\eta\Big].
\end{aligned} \tag{3.130}$$

As required, the forms satisfy

$$\begin{aligned}
d\widehat{\eta} &= 2\widehat{\omega},\\
d\widehat{\Omega} &= i(n+2)\widehat{\eta}\wedge\widehat{\Omega},\\
\widehat{\eta}\wedge *\widehat{\omega} &= \widehat{\eta}\wedge *\widehat{\Omega} = \widehat{\omega}\wedge\widehat{\Omega} = 0,\\
\widehat{\Omega}\wedge\overline{\widehat{\Omega}} &= -(-1)^{\frac{1}{2}n(n+1)}\frac{(2i)^{n+1}}{(n+1)!}\,\widehat{\omega}^{n+1},\\
\widehat{\omega}^{n+1}\wedge\widehat{\eta} &= -(n+1)\sin^{2n+1}(\theta_1)\sin^{2n+2}(\theta_2)\omega^n\wedge\eta\wedge d\theta_1\wedge d\theta_2 \propto \mathrm{Vol},
\end{aligned} \tag{3.131}$$

and therefore define a Sasaki-Einstein structure.

Calabi-Yau to Calabi-Yau: iterated cylinders. Let (M, ω', Ω') be a Calabi-Yau. The iterated cylinder $cyl(cyl(M))$ with extra coordinates τ_1, τ_2 has a Calabi-Yau structure as well, given by

$$\begin{aligned}
\omega'' &= d\tau_2\wedge d\tau_1 + \omega',\\
\Omega'' &= \Omega'\wedge(d\tau_2 + i d\tau_1).
\end{aligned} \tag{3.132}$$

Sasaki-Einstein to $\mathbf{1\times SU(n+1)}$: cone over sine-cone. Let (M, ω, Ω) be Sasaki-Einstein and consider the cone over its sine-cone, $c(cs(M))$, with angular variable

θ_1 and radial variable ρ_2. Define the 1-forms

$$\begin{aligned} \eta' &= \cos(\theta_1)d\rho_2 - \rho_2\sin(\theta_1)d\theta_1, \\ \lambda &= \sin(\theta_1)d\rho_2 + \rho_2\cos(\theta_1)d\theta_1, \\ \mu &= 2\rho_2\sin(\theta_1)\eta, \end{aligned} \tag{3.133}$$

and a $\mathbf{1}\times \mathrm{SU}(n+1)$-structure through

$$\begin{aligned} g &= \rho_2^2\sin^2(\theta_1)g_K + \eta'\otimes\eta' + \lambda\otimes\lambda + \mu\otimes\mu \\ &= d\rho_2^2 + \rho_2^2\big(d\theta_1^2 + \sin^2(\theta_1)g_M\big), \\ \eta' &= \cos(\theta_1)d\rho_2 - \rho_2\sin(\theta_1)d\theta_1, \\ \omega' &= \rho_2^2\sin^2(\theta_1)\omega + \lambda\wedge\mu \\ &= d\big(\rho_2^2\sin^2(\theta_1)\eta\big) \\ &= \rho_2^2\big(\sin^2(\theta_1)\omega + 2\sin(\theta_1)\cos(\theta_1)d\theta_1\wedge\eta\big) + 2\rho_2\sin^2(\theta_1)d\rho_2\wedge\eta, \\ \Omega' &= \rho_2^n\sin^n(\theta_1)\Omega\wedge\big(\lambda + i\mu\big) \\ &= \rho_2^n\sin^n(\theta_1)\Omega\wedge\Big(\sin(\theta_1)d\rho_2 + \rho_2\cos(\theta_1)d\theta_1 + 2i\rho_2\sin(\theta_1)\eta\Big). \end{aligned} \tag{3.134}$$

We have

$$d\eta' = d\omega' = d\Omega' = 0, \tag{3.135}$$

hence the structure is integrable.

Summarizing the results of the last paragraphs we obtain the following diagram of structure groups

$$\begin{array}{ccccccc} \mathbf{1}\times\mathrm{SU}(n) & \xrightarrow[(+\theta_1)]{sc} & \mathbf{1}^2\times\mathrm{SU}(n) & \xrightarrow[(+\theta_2)]{sc} & \mathbf{1}\times\mathrm{SU}(n+1) & \xrightarrow{sc} & \cdots \\ c\big\downarrow(+\rho_1) & & c\big\downarrow(+\rho_2) & & c\big\downarrow(+\rho_3) & & \\ \mathrm{SU}(n+1) & \xrightarrow[(+\tau_1)]{cyl} & \mathbf{1}\times\mathrm{SU}(n+1) & \xrightarrow[(+\tau_2)]{cyl} & \mathrm{SU}(n+2) & \xrightarrow{cyl} & \cdots \end{array} \tag{3.136}$$

where, as usual, the manifolds in the upper row carry real Killing spinors and a non-integrable reduction of structure groups, whereas the ones in the lower row have parallel spinors and their structure groups coincide with the respective holonomy groups. In terms of forms the diagram reads

$$\begin{array}{ccccc} (\eta,\omega,\Omega) & \xrightarrow{sc} & (\eta,d\theta_1,\omega,\Omega) & \xrightarrow{sc} & (\widehat{\eta},\widehat{\omega},\widehat{\Omega}) \\ c\big\downarrow & & c\big\downarrow & & c\big\downarrow \\ (\omega',\Omega') & \xrightarrow{cyl} & (d\tau_1,\omega',\Omega') & \xrightarrow{cyl} & (\omega'',\Omega''). \end{array} \tag{3.137}$$

4 Examples in dimension 4

4.1 The BPST instanton

The anti-self-duality equation on $\mathbb{R}^4$ can be understood as the requirement that the curvature F of a gauge field A be contained in one of the two $\mathfrak{su}(2)$-subalgebras of $\mathfrak{so}(4)$. Instead of $\mathbb{R}^4$ we consider the conformally equivalent cylinder $\mathbb{R} \times S^3$, which does not make a difference at the level of instanton equations. The metric is simply $d\tau^2 + g_{S^3}$, and the canonical 4-form is

$$Q = d\tau \wedge \mathrm{Vol}_{S^3} = \mathrm{Vol}_{\mathbb{R}\times S^3}. \tag{4.1}$$

Let $\{e^a\}$, $a = 1, 2, 3$, be a basis of SU(2)-left-invariant orthonormal 1-forms, so that $\mathrm{Vol}_{S^3} = e^{123}$, and $e^0 = d\tau$. They satisfy the Maurer-Cartan equation

$$de^a = \varepsilon_{abc} e^{bc}, \qquad de^0 = 0. \tag{4.2}$$

A basis of anti-self-dual 2-forms is given by

$$e^{0a} - \frac{1}{2}\varepsilon_{abc} e^{bc}. \tag{4.3}$$

Using the metric-induced isomorphism $\mathfrak{so}(4) = \Lambda^2$ we can translate these 2-forms into a set of generators for one of the $\mathfrak{su}(2)$-subalgebras:

$$(I_a)^b{}_0 = -(I_a)^0{}_b = \delta_{ab}, \qquad (I_a)^b{}_c = \varepsilon_{abc}, \tag{4.4}$$

their commutation relations are[1]

$$[I_a, I_b] = -2\varepsilon_{abc} I_c. \tag{4.5}$$

Define a gauge field A on $\mathbb{R} \times S^3$ through

$$A = \psi(\tau) e^a I_a, \tag{4.6}$$

its curvature is

$$F = \left[\dot\psi e^{0a} - \psi(\psi - 1)\varepsilon_{abc} e^{bc}\right] I_a. \tag{4.7}$$

Comparison with (4.3) shows that F is an instanton if and only if

$$\dot\psi = 2\psi(\psi - 1), \tag{4.8}$$

which is solved by

$$\psi = \left(1 + e^{2(\tau - \tau_0)}\right)^{-1}, \tag{4.9}$$

for some constant $\tau_0 \in \mathbb{R}$. This is an example of a BPST-instanton [12].

[1] note that the I_a are considered as sections of $\mathrm{End}(T\mathbb{R}^4)$ instead of TS^3 here. In fact, one can view (4.4) as defining a map from TS^3 to $\mathrm{End}(T\mathbb{R}^4)$.

4.2 The gauge-solitonic brane

The BPST instanton has been lifted to heterotic supergravity in [77], and I briefly review the construction, which serves as a guiding principle for the next chapter. The metric is chosen conformal to the cylinder metric

$$\tilde{g} = e^{2f(\tau)}(d\tau^2 + g), \tag{4.10}$$

and we introduce an orthonormal basis $\tilde{e}^\mu = \exp(f)e^\mu$, $\mu = 0, \dots, 3$.

The BPS equations. The gaugino equation has been solved above already, now we focus on the dilatino and gravitino equations. There are two SU(2)-invariant Weyl spinors $\epsilon, \tilde{\epsilon}$ on $\mathbb{R} \times S^3$, of the same chirality. Since the volume form $\tilde{e}^{0123}$ is SU(2)-invariant, it maps the space of invariant spinors to itself. In fact, $\gamma(\tilde{e}^{0123})$ is minus the chirality operator, and therefore acts as the identity on ϵ and $\tilde{\epsilon}$. Hence, the following relations hold:

$$\gamma(\tilde{e}^{123}) \cdot \epsilon = -\gamma(\tilde{e}^0) \cdot \epsilon, \qquad \gamma(\tilde{e}^{123}) \cdot \tilde{\epsilon} = -\gamma(\tilde{e}^0) \cdot \tilde{\epsilon}. \tag{4.11}$$

We can then solve the dilatino equation (2.6b) for arbitrary $H = a(\tau)\tilde{e}^{123}$ by setting

$$\phi(\tau) = -\frac{1}{2} \int_{\tau_0}^{\tau} e^{f(t)} a(t) dt. \tag{4.12}$$

To solve the gravitino equation (2.6a) we make an ansatz for ∇^- similar to the instanton ansatz in the last section:

$$\nabla^- = d + s(\tau) e^a I_a. \tag{4.13}$$

This has automatically holonomy SU(2), so that ϵ and $\tilde{\epsilon}$ are parallel. Its torsion can be calculated from the Cartan equation $T^\mu = d\tilde{e}^\mu + (\Gamma^-)^\mu{}_\nu \wedge \tilde{e}^\nu$, and is given by $T^0 = 0$ and

$$T^a = e^f \Big\{ (\dot{f} - s) e^{0a} - \varepsilon_{abc}(s - 1) e^{bc} \Big\}. \tag{4.14}$$

To solve the gaugino equation we need this to be of the form $T^\mu = \tilde{e}^\mu \lrcorner H$ for some 3-form H, which is satisfied if and only if

$$s = \dot{f}. \tag{4.15}$$

The 3-form is then given by

$$H = 2e^{2f}(1 - \dot{f}) e^{123}, \tag{4.16}$$

and the dilaton by

$$\phi(\tau) = \phi_0 + f - \tau. \tag{4.17}$$

The Bianchi identity. Since we are looking for a solution to order $O(\alpha')$ to the Bianchi identity, we can replace the curvature form R^+ by the Riemann tensor of flat $\mathbb{R}^4$, which vanishes. Therefore, the Bianchi identity reduces to

$$dH = -\frac{\alpha'}{4}\mathrm{tr}(F \wedge F). \tag{4.18}$$

We normalize the trace such that $\mathrm{tr}(I_a I_b) = \delta_{ab}$, and obtain the condition

$$dH = \frac{\alpha'}{2}\dot{\psi}\psi(\psi - 1)\varepsilon_{abc}e^{0abc} = \frac{\alpha'}{2}d\Big((1-\psi)^2(1+2\psi)e^{123}\Big). \tag{4.19}$$

Now use the instanton equation (4.8) as well as (4.16) to get the equation

$$e^{2f}(1-\dot{f}) = \frac{\alpha'}{4}\big(\psi\dot{\psi} - \psi^2 + 1\big). \tag{4.20}$$

This is solved by

$$e^{2f} = e^{2\tau} + \frac{\alpha'}{4}(1-\psi^2). \tag{4.21}$$

We have thus obtained a perturbative solution of heterotic string theory, or an exact solution of the modified heterotic supergravity, as discussed in Section 2.1.

4.3 The NS5-brane

This is an alternative solution of the original heterotic supergravity on $\mathbb{R} \times S^3$ or $\mathbb{R}^4$ (i.e. R^+ is kept in the equations), which has enhanced supersymmetry and does not receive any α'-corrections. It was found by Callan, Harvey and Strominger in 1991 [22], who called it 'the symmetric solution'.

We consider again the metric $\tilde{g} = e^{2f}(d\tau^2 + g)$, and make the same ansatz for ∇^- as above in (4.13). We know already that the dilatino equation is solved by (4.12) if the 3-form H is chosen in the form $H = a(\tau)\tilde{e}^{123}$. One can show that in this particular situation the interchange relation

$$(R^+)_{\mu\nu\kappa\lambda} = (R^-)_{\kappa\lambda\mu\nu} \tag{4.22}$$

holds. Since ∇^- has a parallel spinor, it follows that R^+ satisfies the instanton condition, and we can solve the gaugino equation by setting $F = R^+$. Then the Bianchi identity reduces to $dH = 0$, which is consistent with (4.16) if and only if

$$e^{2f}(1-\dot{f}) = q \tag{4.23}$$

for some constant $q \geq 0$. This equation is solved by

$$e^{2f} = q + e^{2\tau}. \tag{4.24}$$

The resulting fields are

$$\begin{aligned} \tilde{g} &= \big(q + e^{2\tau}\big)\big(d\tau^2 + g\big), \\ H &= 2qe^{123} \\ \phi &= \phi_0 + \frac{1}{2}\log\big(1 + qe^{-2\tau}\big) \\ F &= R^+. \end{aligned} \tag{4.25}$$

A thorough discussion shows that q is quantized, and depends linearly on α' [22, 23]. It can be thought of as a brane charge. The limiting behaviour of the solution for $\tau \to -\infty$ is as follows

$$\begin{aligned} \tilde{g} &= q(d\tau^2 + g_{S^3}) \\ H &= 2q\,\mathrm{Vol}_{S^3} \\ \phi &= \phi_0 - \frac{1}{2}\log q - \tau \\ F &= R^+ = R^- = 0. \end{aligned} \tag{4.26}$$

The metric becomes cylindrical, and the dilaton grows linearly with τ. The connection ∇^- in this limit becomes identified with the canonical connection ∇^P on S^3, which is flat and has vanishing connection coefficients in the frame defined by the SU(2)-invariant 1-forms. The connection with opposite torsion ∇^+ is flat as well, so that $F = 0$. Notice that the choice $F = R^-$ is also admissible in this limit, because R^- trivially satisfies the instanton equation. In Chapter 6 below I show that cylindrical solutions similar to the cylinder limit (4.26) exist on many non-symmetric homogeneous spaces. Probably this is true for all real Killing spinor manifolds.

5 Gauge solitonic branes

5.1 Instantons on the cone

In this section I present a generalization of the 4-dimensional BPST-instantons to the cone over a real Killing spinor manifold. For the special cases of a cone over S^6 and S^7 these instantons have been found in [35, 41, 62, 51]. Similar instantons on homogeneous spaces have been constructed in [61, 52], but they coincide with the ones considered here only on $\mathbb{R}^7$ and $\mathbb{R}^8$ [46].

It will actually prove more convenient to study the instanton equation on the cylinder $cyl(M)$, equipped with metric

$$g_{cyl} = d\tau^2 + g_h \tag{5.1}$$

(with g_h the h-dependent metric in the Sasaki-Einstein and 3-Sasakian cases, and the Einstein metric in the other cases). The cylinder inherits a K-structure from M, and this can be lifted to a G-structure, where $G = \mathrm{Spin}(7)$, G_2, $\mathrm{SU}(n{+}1)$ or $\mathrm{Sp}(n{+}1)$ when M is nearly parallel G_2, nearly Kähler, Sasaki-Einstein or 3-Sasakian. The orthogonal complement with respect to the Killing form of $\mathfrak{k}$ in $\mathfrak{g}$ will be denoted by $\mathfrak{m}$. The instanton equation on the cylinder is $F \in \mathfrak{g}$, or equivalently

$$* F = - * Q_{cyl} \wedge F, \tag{5.2}$$

where Q_{cyl} is the Casimir 4-form associated to the G-structure on the cylinder. Since the instanton equations are conformally invariant, and the cylinder metric is conformal to the the cone metric, instantons on the cylinder will also be instantons on the cone.

There are two obvious examples of instantons on the cylinder (or cone): the Levi-Civita connection ∇^c on the cone is an instanton, because the cone is a manifold of special holonomy, and the canonical connection ∇^P on M lifts to an instanton on the cylinder. Both of these connections have holonomy contained in the structure group G of the cylinder. The instantons constructed in this section also have holonomy group G. They interpolate between the Levi-Civita and canonical connections: at the apex $\tau = -\infty$ they agree with the Levi-Civita connection, and at the boundary $\tau = \infty$ they agree with the canonical connection. The instantons depend on a single parameter τ_0: this is a translational parameter from the point of view of the cylinder, or a scale parameter from the point of view of the cone. In the case where $M = S^3$ the construction reproduces the BPST instanton on $\mathbb{R}^4$ [12].

5.1.1 Nearly Kähler and nearly parallel G_2

Nearly parallel G_2-manifolds and nearly Kähler 6-manifolds are sufficiently similar to be treated in a unified way. In both cases the Casimir 4-form on the cylinder is

$$Q_{cyl} = d\tau \wedge P + Q, \tag{5.3}$$

where as above the normalization is $dP = 4Q$. The canonical connection lifts to a connection on the tangent bundle of the cylinder, with holonomy group $K = G_2$ or SU(3). Our ansatz for a connection on the cylinder $\mathbb{R} \times M$ will be a perturbation of the canonical connection, chosen in such a way that the gauge group of the perturbed connection will be $G = \mathrm{Spin}(7)$ or G_2.

We now choose a local frame e^a for T^*M so that the 3-form P takes its standard form, and extend this to a local frame for the cylinder by defining $e^0 = d\tau$. Then there is an associated basis I_a, $a = 1, \dots, n = \dim M$ for $\mathfrak{m} \subset \mathfrak{g} \subset \mathfrak{so}(n+1)$. Since these are $n+1$-dimensional matrices we can attach matrix indices $\mu, \nu = 0 \dots n$. Explicitly, these matrices can be written as follows:

$$-\left(I_a\right)^0{}_b = \left(I_a\right)^b{}_0 = \delta_{ab}, \quad \left(I_a\right)^c{}_b = -\frac{1}{\rho} P_{abc}, \tag{5.4}$$

where $\rho = 2, 3$ in the cases $n = 6, 7$. One way to see that these matrices belong to $\mathfrak{g} \subset \mathfrak{so}(n+1)$ is to note that the 2-forms,

$$e^{0a} - \frac{1}{2\rho} P_{abc} e^{bc}, \tag{5.5}$$

solve the instanton equation (5.2) on the cylinder, so belong to $\mathfrak{g} \subset \Lambda^2$. The generators I_a are the images of these 2-forms under the metric-induced isomorphism $\Lambda^2 \cong \mathfrak{so}(n+1)$. The matrices I_a are orthonormal with respect to a multiple of the Cartan-Killing form on $\mathfrak{g}$, and we extend them to a basis for $\mathfrak{g}$ using an orthonormal basis I_i for $\mathfrak{k}$. Clearly $(I_i)^0{}_a = -(I_i)^a{}_0 = 0$. The structure constants satisfy

$$f^a_{ib} = I^a_{ib}, \quad f^a_{bc} = -\frac{2}{\rho} P_{abc}. \tag{5.6}$$

Here the first equality merely expresses the fact that $\mathfrak{m}$ and $\mathbb{R}^n$ are isomorphic as $\mathfrak{k}$ representations. The ansatz for a connection on the cylinder may now be written

$$\nabla^A = \nabla^P + \psi(\tau) e^a I_a. \tag{5.7}$$

When $\psi(\tau) = 1$, ∇^A is in fact the Levi-Civita connection ∇^c on the cone. This can be easily proven by a direct calculation. Alternatively, it is enough to show that the connection with $\psi(\tau) = 1$ is torsion-free when acting on an orthonormal frame for the cone metric. This will be done in the next section. In the nearly Kähler case we could add another term propotional to $e^a \omega_{ab} I_b$ to the gauge field; this possiblity will be explored in Chapter 7 below.

The calculation of the curvature of ∇^A involves the torsion of the canonical connection, which is proportional to P. This leads to the relation

$$d(e^a I_a) + \Gamma^P \wedge I_a e^a + I_a e^a \wedge \Gamma^P = \frac{1}{\rho} P_{abc} I_a e^{bc}. \tag{5.8}$$

Therefore the curvature of ∇^A is

$$F = R^P + \frac{1}{2}\psi^2 f^i_{ab} e^{ab} I_i + \dot{\psi} e^{0a} I_a + \frac{1}{\rho}(\psi - \psi^2) P_{abc} e^{bc} I_a, \tag{5.9}$$

with R^P the curvature of ∇^P. Now we consider the instanton equation for F. We already know that R^P solves it. It is also not hard to see that the term $f^i_{ab} e^{ab}$ solves the instanton equation. The map $I_i \mapsto f^b_{ia}$ describes the embedding $\mathfrak{k} \mapsto \mathfrak{so}(n)$, so for each i, the 2-form $f^i_{ab} e^{ab}$ lies in the subspace $\mathfrak{k} \subset \Lambda^2$. Alternatively, one only needs to note that $R^P + \frac{1}{2} f^i_{ab} e^{ab} I_i$ is the curvature of the Levi-Civita connection on the cone, and hence an instanton. Thus F is an instanton if and only if the I_a terms solve the instanton equation, and from equation (5.5) one sees that this happens exactly when

$$\dot{\psi} = 2\psi(\psi - 1), \tag{5.10}$$

the same equation as in 4 dimensions (4.8). The solution is again

$$\psi = \left(1 + e^{2(\tau - \tau_0)}\right)^{-1}. \tag{5.11}$$

The limit $\tau_0 \to -\infty$ is the original connection ∇^P on the cylinder, and the limit $\tau_0 \to \infty$ is the Levi-Civita connection on the cone.

5.1.2 Sasaki-Einstein

The Casimir 4-form on the cylinder over a Sasaki-Einstein manifold depends on the metric parameter h, which is promoted to a function of τ:

$$Q_{cyl} = e^{2h(\tau)} d\tau \wedge P + e^{4h(\tau)} Q \tag{5.12}$$

We construct instantons on Sasaki-Einstein manifolds by the same method as in the nearly Kähler and nearly parallel G_2-cases. Again, we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g} = \mathfrak{su}(n+1)$, $\mathfrak{k} = \mathfrak{su}(n)$, and $\mathfrak{m}$ is the $2n+1$-dimensional space orthogonal to $\mathfrak{k}$. Once again, $\mathfrak{m}^*$ is isomorphic to the $2n+1$-dimensional orthogonal representation of $\mathfrak{su}(n)$ that defines the cotangent bundle of M. Assume that a local orthonormal basis e^1, e^a for T^*M has been chosen so that the parallel forms take their standard forms, and also set $e^0 = d\tau$. The generators of $\mathfrak{k}$ will be denoted I_i and the additional generators of $\mathfrak{m}$ associated to the frame e^1, e^a will be denoted I_1, I_a. Written as matrices, these have the following non-vanishing elements

$$\begin{aligned} (I_i)^b{}_a =& f^b_{ia}, \\ (I_1)^b{}_a =& -\tfrac{1}{n}\omega_{ab}, \qquad -(I_1)^0{}_1 = (I_1)^1{}_0 = 1, \\ -(I_a)^0{}_b =& (I_a)^b{}_0 = \delta_{ab}, \quad -(I_a)^1{}_b = (I_a)^b{}_1 = \omega_{ab}. \end{aligned} \tag{5.13}$$

In particular, the matrices I_1, I_a are the images in $\mathfrak{so}(2m+2)$ of the anti-self-dual 2-forms

$$e^{01} - \frac{e^{2h}}{n}\omega, \quad e^h(e^{0a} + \omega_{ab}e^{1b}). \tag{5.14}$$

With this choice of basis, the structure constants satisfy

$$f^1_{ab} = -2P_{ab1}, \quad f^c_{ab} = 0, \quad f^b_{1a} = -\frac{n+1}{n}P_{1ab}. \tag{5.15}$$

Notice the similarity with the formulae (3.56) for the torsion. There are three sections that could be used to perturb the canonical connection, given by $e^a I_a$, $e^a\omega_{ab}I_b$ and $e^1 I_1$, so the natural ansatz for ∇^A is

$$\nabla^A = \nabla^P + \chi(\tau)e^1 I_1 + \psi(\tau)e^a I_a + \tilde{\psi}(\tau)e^a\omega_{ab}I_b, \tag{5.16}$$

with $\chi, \psi, \tilde{\psi}$ real functions of τ. It can be shown that the instanton equation implies that the argument of $\psi + i\tilde{\psi}$ is constant and that the instanton equation is invariant under phase rotations of this complex variable. To simplify the equations we will fix $\tilde{\psi} = 0$. Then the curvature of ∇^A is

$$\begin{aligned} F \;=\; & R^P + \frac{1}{2}\psi^2 f^i_{ab}e^{ab}I_i + \dot{\chi}e^{01}I_1 + \dot{\psi}e^{0a}I_a \\ & + (\chi - \psi^2)P_{ab1}e^{ab}I_1 + \frac{n+1}{n}(\psi - \psi\chi)P_{1ba}e^{1b}I_a. \end{aligned} \tag{5.17}$$

Once again, R^P is the curvature of ∇^P so solves the instanton equation. The term $\psi^2 f^i_{ab}e^{ab}I_i$ also solves the instanton equation, as can be shown either by a direct argument, or by using the fact (to be proved in the next section) that the connection with $\psi = \chi = 1$ is the Levi-Civita connection on the cone. Therefore the instanton equation is equivalent to

$$\dot{\chi} = 2ne^{-2h}(\psi^2 - \chi) \tag{5.18}$$

$$\dot{\psi} = \frac{n+1}{n}\psi(\chi - 1). \tag{5.19}$$

The ansatz (5.16) and the associated instanton equations (5.18), (5.19) are equivalent to those given in [26].

The flow equations (5.18), (5.19) have two fixed points at $(\psi, \chi) = (0, 0)$ and $(1, 1)$ corresponding to the instantons ∇^P and ∇^c. The first critical point is stable and the second semi-stable, so one expects solutions interpolating from the second to the first to exist for reasonable choices of $h(\tau)$. In fact, when $e^{2h} = \frac{2n^2}{n+1}$ one can consistently set $\chi = \psi$, and a solution similar to (5.11) can be written down:

$$\psi = \chi = \left(1 + e^{2(\tau-\tau_0)}\right)^{-1}. \tag{5.20}$$

In the particular case $n = 1$ this is just the BPST instanton. A similar class of solutions for general n has been discovered on non-symmetric homogeneous spaces (including homogeneous Sasaki-Einstein manifolds) in [61], although the instanton equation used there appears to differ from ours.

However, the most interesting choice for h is $h = 0$, corresponding to the Einstein metric. For $n > 1$ solutions can be found only numerically [53]. There is however an exact solution in the $n \to \infty$ limit: in this limit, equation (5.18) simplifies to $\psi^2 = \chi$ and equation (5.19) becomes

$$\dot{\psi} = 2\psi(\psi^2 - 1). \tag{5.21}$$

This is solved by $\psi = (1 + \exp(2(\tau - \tau_0)))^{-1/2}$.

Of particular interest are the cases where $M = S^{2n+1}$. Then the cone metric extends smoothly over the apex $\tau = -\infty$ to form the manifold $\mathbb{R}^{2n+2}$. The instantons also extend over the apex, since at $\tau = -\infty$ they coincide with the Levi-Civita connection on $\mathbb{R}^{2n+2}$, which does extend over the apex. Thus the solutions to (5.18) and (5.19) give rise to a new family of instantons on even-dimensional Euclidean spaces.

5.1.3 3-Sasakian

The Casimir 4-form on the cylinder is

$$Q_{cyl} = \frac{1}{6}\Big(e^{4h}\omega_\alpha \wedge \omega_\alpha - e^{2h}\varepsilon_{\alpha\beta\gamma}\omega_\alpha \wedge \eta^{\beta\gamma} + 2e^{2h}d\tau \wedge \eta^\alpha \wedge \omega_\alpha - 6d\tau \wedge \eta^{123}\Big), \tag{5.22}$$

where once again we allow h to depend on τ. As above, we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, with $\mathfrak{g} = \mathfrak{sp}(n+1)$, $\mathfrak{k} = \mathfrak{sp}(n)$, and $\mathfrak{m}$ is the $4n+3$-dimensional space orthogonal to $\mathfrak{k}$. Once again, $\mathfrak{m}^*$ is isomorphic to the $(4n+3)$-dimensional orthogonal representation of $\mathfrak{sp}(n)$ that defines the cotangent bundle of M.

We assume that a local orthonormal basis e^α, e^a for T^*M has been chosen so that the parallel forms take their standard forms, and also set $e^0 = d\tau$. The generators of $\mathfrak{k}$ will be denoted I_i and the additional generators of $\mathfrak{m}$ associated to the frame e^α, e^a will be denoted I_α, I_a. The non-vanishing components of these as matrices are

$$\begin{aligned}
(I_i)^b{}_a &= f^b_{ia}, \\
(I_\alpha)^\beta{}_0 &= -\delta_{\alpha\beta}, \quad (I_\alpha)^\gamma{}_\beta = \varepsilon_{\alpha\beta\gamma}, \\
(I_a)^b{}_0 &= \delta_{ab}, \qquad (I_a)^\alpha{}_b = -\omega^\alpha_{ab}.
\end{aligned} \tag{5.23}$$

In particular, the matrices I_α, I_a are the images in $\mathfrak{so}(4n+4)$ of the anti-self-dual 2-forms

$$e^{0\alpha} + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}e^{\beta\gamma}, \quad e^h(e^{0a} + \omega^\alpha_{ab}e^{\alpha b}). \tag{5.24}$$

The Lie algebra structure constants satisfy

$$f^\alpha_{\beta\gamma} = 2\varepsilon_{\alpha\beta\gamma}, \quad f^\alpha_{ab} = 2\omega^\alpha_{ab}, \quad f^b_{\alpha a} = \omega^\alpha_{ab}, \tag{5.25}$$

which should be compared to the torsion (3.73). There are 2 matrix-valued forms which are parallel with respect to connections with holonomy $\mathrm{Sp}(1)_d \times \mathrm{Sp}(m)$. We use both to make an ansatz for a connection:

$$\nabla^A = \nabla^P + \chi(\tau)e^\alpha I_\alpha + \psi(\tau)e^a I_a, \tag{5.26}$$

with χ, ψ real functions of τ. Using the above result (3.73) for the canonical torsion, we obtain for the curvature of the connection:

$$\begin{aligned} F = R^P &+ \frac{1}{2}\psi^2 f^k_{ab}e^{ab}I_k \\ &+ \Big(\dot\chi e^{0\alpha} + 2(\chi-\psi^2)\omega_\alpha - \chi(1-\chi)\varepsilon_{\alpha\beta\gamma}e^{\beta\gamma}\Big)I_\alpha \\ &+ \Big(\dot\psi e^{0a} - \psi(1-\chi)\omega^\alpha_{ab}e^{\alpha b}\Big)I_a. \end{aligned} \tag{5.27}$$

Once more, the connection with $\chi = \psi = 1$ is the Levi-Civita connection on the cone. The first two terms solve the instanton equation, using the fact that the canonical connection and the Levi-Civita connection on the cone are instantons. Thus the instanton equation reduces to

$$0 = \chi - \psi^2, \tag{5.28}$$
$$\dot\chi = 2\chi(\chi-1), \tag{5.29}$$
$$\dot\psi = \psi(\chi-1), \tag{5.30}$$

which are independent of $h(\tau)$. Note that these are 3 equations for 2 unknown functions, so naively one would not expect to find any solutions. However, in the case at hand the condition (5.28) is conserved by the flow described by the other two equations, so solutions can be found. They are:

$$\begin{aligned} \chi &= \Big(1 + e^{2(\tau-\tau_0)}\Big)^{-1}, \\ \psi &= \pm\Big(1 + e^{2(\tau-\tau_0)}\Big)^{-1/2}. \end{aligned} \tag{5.31}$$

For $M = S^{4n+3}$ we obtain an instanton on Euclidean space $\mathbb{R}^{4n+4}$. In the case $n = 0$ this is again the BPST instanton, for $n \geq 1$ it seems likely that our instantons coincide with those constructed in [28, 20].

5.2 Heterotic string theory

We want to solve the heterotic BPS equations (2.6) together with the Bianchi identity (2.7). In the previous section we have constructed instantons on the cones over manifolds with real Killing spinor(s). These are solutions of the gaugino equation (2.6c). In the present section we will extend these solutions to the full set of BPS equations. Our procedure generalises constructions given in [77, 56, 60, 51] in the cases where $M = S^3$, S^6 and S^7. Like in those references, we work perturbatively in α'. At $O(1)$, the BPS equations are solved by the cone metric, with $H = 0$ and ϕ constant. At $O(\alpha')$ the gauge field comes into the game, which opens up the possibility of further solutions in which H and $d\phi$ no longer vanish.

Working perturbatively in α', we will replace the curvature R^+ in the Bianchi identity by the Riemann curvature of the cone. As explained in Section 2.1 this perturbative approach can also be interpreted as a modified heterotic supergravity where the BPS equations, Bianchi identity and equations of motion are simultaneously satisfied.

We work using the following metric and G-structure on the cone:

$$\tilde{g} = e^{2f(\tau)} g_{cyl}, \quad \tilde{Q} = e^{4f(\tau)} Q_{cyl}, \tag{5.32}$$

where g_{cyl} and Q_{cyl} are the metric and 4-form on the cylinder introduced in (5.1), (5.3), (5.12), (5.22). The cone metric is of course $f = \tau$ (and $h = 0$, where appropriate). Throughout this section, the Clifford action γ and the contraction operator $\lrcorner$ will be assumed to be taken with respect to this metric $\tilde{g}$. The dilatino equation on a cylinder is discussed in Section A.3 of the appendix, here we shall make use of these results.

5.2.1 Nearly Kähler and nearly parallel G_2

Dilatino equation. If the base is either nearly Kähler or nearly parallel G_2, then there are two SU(3) or G_2-invariant spinors $\epsilon, \tilde{\epsilon}$ on the cone, respectively. But only one of them, ϵ say, is invariant under the full holonomy group of the cone, i.e. G_2 or Spin(7). The spinors satisfy

$$e^{2f}\gamma(P)\cdot\epsilon = -\kappa\gamma(d\tau)\cdot\epsilon, \qquad e^{2f}\gamma(P)\cdot\tilde{\epsilon} = \kappa\gamma(d\tau)\cdot\tilde{\epsilon} \tag{5.33}$$

where $\kappa = 4$ in dimension 7 or 7 in dimension 8. Thus for any $\phi = \phi(\tau)$, the dilatino equation for ϵ is solved by

$$H = -\frac{2}{\kappa} d\phi\lrcorner\tilde{Q} = -\frac{2\dot{\phi}}{\kappa} e^{2f} P. \tag{5.34}$$

Gravitino equation. To solve the gravitino equation, we make an ansatz for the connection ∇^- similar to the ansatz (5.7) for the gauge field:

$$\nabla^- = \nabla^P + s(\tau)e^a I_a. \tag{5.35}$$

This connection always solves equation (2.6a), since by construction its holonomy group is contained in G_2 or Spin(7), but we still need to check that its torsion is given by H. The torsion is calculated by choosing an orthonormal basis

$$\tilde{e}^0 = \exp(f)e^0, \quad \tilde{e}^a = \exp(f)e^a, \tag{5.36}$$

and employing the Cartan equation $T^\mu = d\tilde{e}^\mu + {}^-\Gamma^\mu_\nu \wedge \tilde{e}^\nu$. We find that $T^0 = 0$ and

$$T^a = \exp(f)\left((\dot{f} - s)e^{0a} + \frac{1-s}{\rho}P_{abc}e^{bc}\right). \tag{5.37}$$

On the other hand, the torsion should be $T^\mu = \tilde{e}^\mu \lrcorner H$, where H is the solution (5.34) to the dilatino equation. Thus we must set $s = \dot{f}$, and the gravitino and dilatino equations are equivalent to

$$\dot{f} - 1 \ = \ \frac{\rho}{\kappa}\dot{\phi}. \tag{5.38}$$

The general solution is

$$\phi = \phi_0 + \frac{\kappa}{\rho}(f - \tau), \tag{5.39}$$

where ϕ_0 is the asymptotic value of ϕ.

Notice that the torsion of ∇^- vanishes when $s = \dot{f} = 1$. So this connection is a torsion-free metric-compatible connection on the cone. This justifies the earlier claim that the connection (5.7) with $\psi = 1$ is the Levi-Civita connection on the cone.

The Bianchi identity. The solution (5.7), (5.10) of the instanton equation is valid for arbitrary scale factor f, since the instanton equations are conformally invariant. Thus to complete our solution we only need to solve the Bianchi identity. Since we are only working to leading order in α', the curvature R^+ appearing in the Bianchi identity (2.7) can be replaced by the curvature $R = R^i I_i$ of the Levi-Civita connection on the cone. The trace appearing in the Bianchi identity will be taken using the quadratic form that makes I_i, I_a orthonormal, so that

$$\mathrm{tr}(F \wedge F - R^+ \wedge R^+) = F^i \wedge F^i - R^i \wedge R^i + F^a \wedge F^a. \tag{5.40}$$

These terms will be evaluated separately. First, using the identity,

$$\frac{1}{4}P_{abc}P_{ade}e^{bcde} = 2\rho\, Q, \tag{5.41}$$

we find that

$$F^a \wedge F^a = \left(\dot\psi e^{0a} + \frac{\psi-\psi^2}{\rho}P_{abc}e^{bc}\right) \wedge \left(\dot\psi e^{0a} + \frac{\psi-\psi^2}{\rho}P_{ade}e^{de}\right) \quad (5.42)$$

$$= \frac{8}{\rho}(\psi-\psi^2)^2 Q + \frac{12}{\rho}\dot\psi(\psi-\psi^2)e^0 \wedge P. \quad (5.43)$$

To evaluate the remaining terms, we note that the Riemann curvature $R^a_b = R^i f^a_{ib}$ satisfies the first Bianchi identity, $\exp(\tau)e^b \wedge R^a_b = 0$, and it follows that

$$f^i_{ab}e^{ab} \wedge R^i = 0. \quad (5.44)$$

In addition we note that the 4-form Q can be expressed as the Casimir for the structure group K:

$$\frac{1}{4}f^i_{ab}f^i_{cd}e^{abcd} = -\frac{8}{\rho}Q. \quad (5.45)$$

It follows from the above that

$$F^i \wedge F^i = \left(R^i + \frac{\psi^2-1}{2}f^i_{ab}e^{ab}\right) \wedge \left(R^i + \frac{\psi^2-1}{2}f^i_{cd}e^{cd}\right) \quad (5.46)$$

$$= -\frac{8}{\rho}(\psi^2-1)^2 Q + R^i \wedge R^i. \quad (5.47)$$

Thus, the Bianchi identity is

$$dH = -\frac{\alpha'}{\rho}\left(3\dot\psi(\psi-\psi^2)e^0 \wedge P + 2(-1+3\psi^2-2\psi^3)Q\right) \quad (5.48)$$

$$= \frac{\alpha'}{2\rho}\, d\left((1-\psi)^2(1+2\psi)P\right). \quad (5.49)$$

Comparing with (5.34), the Bianchi identity is equivalent to

$$\frac{\dot\phi}{\kappa}\exp(2f) = \frac{\alpha'}{4\rho}(1-\psi)^2(1+2\psi). \quad (5.50)$$

Using (5.10) and (5.38) gives the equation

$$(\dot f - 1)e^{2f} = \frac{\alpha'}{4}(-1+\psi^2-\psi\dot\psi), \quad (5.51)$$

which can in fact be integrated exactly:

$$e^{2f} = e^{2\tau} + \frac{\alpha'}{4}(1-\psi^2). \quad (5.52)$$

Together with (5.10), (5.39), this gives a solution of the gaugino, gravitino and dilatino equations and the Bianchi identity. The constant ϕ_0 is the background value of the dilaton field and τ_0 is a parameter controlling the instanton size. In the cases $M = S^6, S^7$ this reproduces solutions constructed in [56, 51].

5.2.2 Sasaki-Einstein

Dilatino equation. We have that

$$e^{2(f+h)}\gamma(P)\cdot\epsilon = -n\gamma(d\tau)\cdot\epsilon. \tag{5.53}$$

Thus for any $\phi = \phi(\tau)$, the dilatino equation is solved by

$$H = -\frac{2}{n}d\phi\lrcorner\tilde{Q} = -\frac{2\dot{\phi}}{n}e^{2(f+h)}P. \tag{5.54}$$

Gravitino equation. To solve the gravitino equation, we make an ansatz for the connection ∇^- similar to (5.16)

$$\nabla^- = \nabla^P + t(\tau)e^1 I_1 + s(\tau)e^a I_a. \tag{5.55}$$

This has holonomy $\mathrm{SU}(n+1)$, so solves (2.6a). To calculate the torsion of ∇^-, we choose an orthonormal basis

$$\tilde{e}^0 = \exp(f)e^0, \quad \tilde{e}^a = \exp(f+h)e^a, \quad \tilde{e}^1 = \exp(f)e^1, \tag{5.56}$$

and employ the Cartan equation $T^\mu = d\tilde{e}^\mu + {}^-\Gamma^\mu_\nu \wedge \tilde{e}^\nu$. We find that $T^0 = 0$ and

$$\begin{aligned} T^1 &= e^f\left((\dot{f}-t)e^{01} + (1-e^h s)P_{1ab}e^{ab}\right) \\ T^a &= e^{f+h}\left\{(\dot{f}+\dot{h}-e^{-h}s)e^{0a} + ((1-e^{-h}s) + (1-t)/n)P_{ab1}e^{b1}\right\}. \end{aligned} \tag{5.57}$$

On the other hand, the torsion should be $T^\mu = \tilde{e}^\mu\lrcorner H$, with H given by (5.54). Thus we set $t = \dot{f}$, $s = e^h(\dot{f}+\dot{h})$, so that the gravitino and dilatino equations are equivalent to

$$\frac{2}{n}\dot{\phi} = \frac{n+1}{n}(\dot{f}-1) + \dot{h} \tag{5.58}$$

$$\frac{n-1}{n}\dot{f} + \dot{h} = 2e^{-2h} - \frac{n+1}{n}. \tag{5.59}$$

Equation (5.58) can be integrated to give ϕ in terms of f and h:

$$\phi = \phi_0 + \frac{n+1}{2}(f-\tau) + \frac{n}{2}h. \tag{5.60}$$

Notice that the torsion of ∇^- vanishes when $s = t = \dot{f} = 1$, $h = 0$. So this connection is a torsion-free metric-compatible connection on the cone. This justifies the earlier claim that the connection (5.16) with $\chi = \psi = 1$ is the Levi-Civita connection on the cone.

The Bianchi identity. The trace appearing in the Bianchi identity will be normalised so that the I_a are orthonormal. This convention implies that $\mathrm{tr}(I_1^2) = (n+1)/2n$. The I_i will be taken to be orthonormal also. Thus,

$$\mathrm{tr}(F \wedge F - R^+ \wedge R^+) = F^i \wedge F^i - R^i \wedge R^i + F^a \wedge F^a + \frac{n+1}{2n} F^1 \wedge F^1. \tag{5.61}$$

Here as above R^+ has been replaced by the curvature $R = R^i I_i$ of the Levi-Civita connection on the cone, since we are working only to leading order in α'. These terms will be evaluated separately. First, using the identity,

$$P_{1ab} P_{1cd} e^{abcd} = 8Q, \tag{5.62}$$

we find that

$$\begin{aligned} F^a \wedge F^a &= 4\frac{n+1}{n} \dot{\psi}\psi(1-\chi) e^0 \wedge P, & (5.63)\\ \frac{n+1}{2n} F^1 \wedge F^1 &= 2\frac{n+1}{n} \dot{\chi}(\chi - \psi^2) e^0 \wedge P + 4\frac{n+1}{n}(\chi - \psi^2)^2 Q. & (5.64) \end{aligned}$$

From the first Bianchi identity for R it follows that

$$F^i \wedge F^i = -4\frac{n+1}{n}(\psi^2 - 1)^2 Q + R^i \wedge R^i. \tag{5.65}$$

Thus, the Bianchi identity is

$$\begin{aligned} dH &= -\frac{\alpha'(n+1)}{4n}\Big(4\dot{\psi}\psi(1-\chi) + 2\dot{\chi}(\chi - \psi^2)\Big) e^0 \wedge P \\ &\quad -\frac{\alpha'(n+1)}{n}\left(\chi^2 - 2\chi\psi^2 + 2\psi^2 - 1\right) Q & (5.66)\\ &= -\frac{\alpha'(n+1)}{4n} d\left((\chi^2 - 2\chi\psi^2 + 2\psi^2 - 1)P\right). & (5.67) \end{aligned}$$

Comparing with (5.54), (5.58), (5.59) the Bianchi identity reduces to

$$(\dot{f} + \dot{h} - e^{-2h}) e^{2(f+h)} = \frac{\alpha'(n+1)}{8n}\left(\chi^2 - 2\chi\psi^2 + 2\psi^2 - 1\right). \tag{5.68}$$

We have reduced the heterotic supergravity equations to 4 equations (5.18), (5.19), (5.59), (5.68). In the case $n = 1$ these are solved exactly [77] by (5.20), $h = 0$, and

$$e^{2f} = e^{2\tau} + \frac{\alpha'}{4}(1 - \chi^2). \tag{5.69}$$

For $n > 1$ solutions may be obtained only numerically. We assume that the solutions can be expanded in α':

$$\begin{aligned} \chi &= \chi_0 + \alpha'\chi_1, \quad f = f_0 + \alpha' f_1, \\ \psi &= \psi_0 + \alpha'\psi_1, \quad h = h_0 + \alpha' h_1. \end{aligned} \tag{5.70}$$

The functions χ_0, ψ_0, f_0, h_0 are solutions at $O(1)$ in α'. The unique $O(1)$ solution of (5.59), (5.68) for which h_0 does not blow up at $\tau = -\infty$ is $f_0 = \tau$, $h_0 = 0$. Then ψ_0, χ_0 must solve (5.18), (5.19) with $h = 0$. As discussed in Section 5.1, there is a 1-parameter family of solutions which do not blow up, with a translational parameter τ_0. At $O(\alpha')$, equations (5.18), (5.19), (5.59), (5.68) reduce to the following differential equations for χ_1, ψ_1, f_1, h_1:

$$\begin{aligned} \dot{h}_1 &= -2(n+1)h_1 - \frac{n^2-1}{8n}e^{-2\tau}\left(\chi_0^2 - 2\chi_0\psi_0^2 + 2\psi_0^2 - 1\right) &(5.71)\\ \dot{f}_1 &= 2nh_1 + \frac{n+1}{8}e^{-2\tau}\left(\chi_0^2 - 2\chi_0\psi_0^2 + 2\psi_0^2 - 1\right) &(5.72)\\ \dot{\chi}_1 &= -4nh_1(\psi_0^2 - \chi_0) + 4n\psi_0\psi_1 - 2n\chi_1 &(5.73)\\ \dot{\psi}_1 &= \frac{n+1}{n}(\chi_0 - 1)\psi_1 + \frac{n+1}{n}\psi_0\chi_1. &(5.74) \end{aligned}$$

We assume that solutions of these equations exist for all τ. Solutions h_1, χ_1, ψ_1 of (5.71), (5.73), (5.74) may blow up as $\tau \to -\infty$, and for each τ_0 there is a unique solution which does not. Then equation (5.72) has a unique solution satisfying $f_1 \to 0$ as $\tau \to \infty$. So, the supergravity equations have a 1-parameter family of solutions to $O(\alpha')$. These solutions have the following asymptotics:

$$\begin{aligned} &1-\chi_0, 1-\psi_0, h_1, \dot{f}_1 \sim e^{2\tau}, \quad \chi_1, \psi_1 \sim e^{4\tau} \quad \text{as } \tau \to -\infty; \\ &\psi_0, \psi_1 \sim e^{-\frac{n+1}{n}\tau}, \quad \chi_0, \chi_1 \sim e^{-2\frac{n+1}{n}\tau}, \quad h_1, f_1 \sim e^{-2\tau} \quad \text{as } \tau \to \infty. \end{aligned} \tag{5.75}$$

Numerical solutions can be found in [53].

5.2.3 3-Sasakian

Gravitino equation. Our ansatz for ∇^- is similar to (5.26):

$$\nabla^- = \nabla^P + t(\tau)e^\alpha I_\alpha + s(\tau)e^a I_a. \tag{5.76}$$

This solves the gravitino equation (2.6a), for the $n+2$ Sp(n)-invariant spinors ϵ. Introducing the orthonormal basis

$$\tilde{e}^0 = \exp(f)d\tau, \quad \tilde{e}^\alpha = \exp(f)e^\alpha, \quad \tilde{e}^a = \exp(f+h)e^a \tag{5.77}$$

and using the Cartan equation $T^\mu = d\tilde{e}^\mu + {}^-\Gamma^\mu_\nu \wedge \tilde{e}^\nu$ we find that $T^0 = 0$ and

$$\begin{aligned} T^\alpha &= e^f\Big\{(\dot{f} - t)e^{0\alpha} + 2(1 - e^h s)\omega^\alpha - (1-t)\varepsilon_{\alpha\beta\gamma}e^{\beta\gamma}\Big\}, \\ T^a &= e^{f+h}\Big\{(\dot{f} + \dot{h} - e^{-h}s)e^{0a} + (1 - e^{-h}s)\omega^\alpha_{ab}e^{b\alpha}\Big\}. \end{aligned} \tag{5.78}$$

Skew-symmetry of the torsion requires it to be of the form $T^\mu = \tilde{e}^\mu \lrcorner H$ for some 3-form H. This means in particular that we must set $t = \dot{f}$ and $s = e^h(\dot{f} + \dot{h})$. Then the torsion will be skew-symmetric if and only if

$$\dot{f} + \dot{h} = 2e^{-2h} - 1. \tag{5.79}$$

Assuming that (5.79) holds, the 3-form H is given by

$$H = 2e^{2f}(\dot{f}-1)\eta^{123} - e^{2(f+h)}\left(\dot{f}+\dot{h}-1\right)\eta^\alpha\wedge\omega^\alpha. \tag{5.80}$$

Notice that when $s = t = \dot{f} = e^h = 1$ the torsion vanishes. This justifies our earlier claim that the connection (5.26) with $\psi = \chi = 1$ is the Levi-Civita connection on the cone.

Dilatino equation. The Sp(n)-invariant spinors come in two groups of $n+2$ spinors ϵ and n spinors $\tilde{\epsilon}$, which satisfy the relations (see Lemma A.3 in the appendix)

$$\begin{aligned}
\gamma(\eta^{123})\cdot\epsilon &= e^{-2f}\gamma(d\tau)\cdot\epsilon, \qquad & \gamma(\eta^\alpha\wedge\omega_\alpha)\cdot\epsilon &= -e^{-2(f+h)}2n\gamma(d\tau)\cdot\epsilon,\\
\gamma(\eta^{123})\cdot\tilde{\epsilon} &= e^{-2f}\gamma(d\tau)\cdot\tilde{\epsilon}, \qquad & \gamma(\eta^\alpha\wedge\omega_\alpha)\cdot\tilde{\epsilon} &= e^{-2(f+h)}2(n+2)\gamma(d\tau)\cdot\tilde{\epsilon}.
\end{aligned} \tag{5.81}$$

Since only the $n+2$ spinors ϵ solve the gravitino equation, we need to adjust ϕ such that they solve the dilatino equation as well. If ϕ is a function of τ and H is given by (5.80), the dilatino equation (2.6b) for ϵ is equivalent to

$$\dot{\phi} = (n+1)(\dot{f}-1) + n\dot{h}. \tag{5.82}$$

Clearly, this is solved by

$$\phi = \phi_0 + (n+1)(f-\tau) + nh, \tag{5.83}$$

where the integration constant ϕ_0 may be interpreted as the background value of the dilaton field.

The Bianchi identity. The instanton that we constructed in the previous section solves the gaugino equation on the cone for all possible choices of the functions f, h. Thus it remains to solve the Bianchi identity (2.7). Working to leading order in α', we shall replace R^+ by $R = R^i I_i$, the Riemann curvature form of the cone. We have

$$F^i = R^i + \frac{1}{2}(\psi^2-1)f^i_{ab}e^{ab}. \tag{5.84}$$

Now we can calculate the terms occurring in the Bianchi identity (without at this point assuming that ψ, χ solve the instanton equation):

$$\begin{aligned}
\frac{1}{2}F^\alpha\wedge F^\alpha &= 2\dot{\chi}(\chi-\psi^2)e^0\wedge\eta^\alpha\wedge\omega^\alpha - 6\dot{\chi}(\chi-\chi^2)e^0\wedge\eta^{123}\\
&\quad - 2(\chi-\psi^2)(\chi-\chi^2)\epsilon_{\alpha\beta\gamma}\eta^{\alpha\beta}\wedge\omega^\gamma + 2(\chi-\psi^2)^2\omega^\alpha\wedge\omega^\alpha\\
F^a\wedge F^a &= 4\dot{\psi}\psi(1-\chi)e^0\wedge\eta^\alpha\wedge\omega^\alpha + 2\psi^2(1-\chi)^2\epsilon_{\alpha\beta\gamma}\eta^{\alpha\beta}\wedge\omega^\gamma\\
F^i\wedge F^i &= R^i\wedge R^i - 2(\psi^2-1)^2\omega^\alpha\wedge\omega^\alpha
\end{aligned} \tag{5.85}$$

Here we have used the first Bianchi identity $R^i f^b_{ia}\tilde{e}^a = 0$, as well as the following formula for the Casimir 4-form

$$f^i_{ab}f^i_{cd}e^{abcd} = -8\omega^\alpha\wedge\omega^\alpha. \tag{5.86}$$

We assume that the trace has been normalised so that I_a, I_i are orthonormal, then we must have that $\mathrm{tr}(I_\alpha I_\beta) = 1/2$. Thus the Bianchi identity is

$$\begin{aligned} dH &= -\frac{\alpha'}{4}\left(F^i \wedge F^i - R^i \wedge R^i + F^a \wedge F^a + \frac{1}{2}F^\alpha \wedge F^\alpha\right) && (5.87) \\ &= \frac{\alpha'}{4} d\left[(\chi - 1)\left\{(1+\chi-2\chi^2)\eta^{123} - (1+\chi-2\psi^2)\eta^\alpha \wedge \omega_\alpha\right\}\right]. && (5.88) \end{aligned}$$

Now we assume that the gauge field is an instanton, so that in particular $\chi = \psi^2$ (cf. (5.28)). The Bianchi identity is solved by

$$H = \frac{\alpha'}{4}\left[-(1+2\chi)(1-\chi)^2\eta^{123} + (1-\chi)^2\eta^\alpha \wedge \omega^\alpha\right]. \quad (5.89)$$

Comparing with our earlier solution (5.80) of the gravitino equation, we see that the Bianchi identity, gravitino equation, and gaugino equation are equivalent to

$$\begin{aligned} e^{2f}\left(\dot{f} - 1\right) &= -\frac{\alpha'}{8}(1+2\chi)(1-\chi)^2, && (5.90) \\ e^{2(f+h)}\left(\dot{f} + \dot{h} - 1\right) &= -\frac{\alpha'}{4}(1-\chi)^2, && (5.91) \end{aligned}$$

together with equation (5.79), where χ is given by the solution (5.31) to the differential equation (5.29). Note that once again we have more equations than unknowns, so naively one would not expect this system to have any solutions. In spite of this, an analytic solution can be found: it is

$$\begin{aligned} e^{2f} &= e^{2\tau} + \frac{\alpha'}{8}(1-\chi^2) && (5.92) \\ e^{2(f+h)} &= e^{2\tau} + \frac{\alpha'}{4}(1-\chi). && (5.93) \end{aligned}$$

Thus we have obtained a 1-parameter family of solutions of the gaugino, gravitino and dilatino equations and the Bianchi identity, with the functions χ, ψ, f, h, ϕ given in equations (5.31), (5.92), (5.93), (5.83).

In the limits $\tau \to \pm\infty$ we get

$$\begin{aligned} &\tau \to -\infty: \quad h \to 0, \quad \chi, \psi \to 1, \quad e^{2f} \to e^{2\tau}\left(1 + \frac{\alpha'}{4}\right) \\ &\tau \to +\infty: \quad h \to 0, \quad \chi, \psi \to 0, \quad e^{2f} \to e^{2\tau}. \end{aligned} \quad (5.94)$$

The limiting behaviour is very similar to the nearly Kähler and nearly parallel G_2 cases. In particular, the metric equals the Ricci-flat cone metric in both limits, and the instanton approaches the canonical connection for $\tau \to \infty$ and the Levi-Civita connection on the cone for $\tau \to -\infty$. In the particular case $M = S^{4n+3}$ the solution extends over the apex of the cone: thus the quaternionic instanton of [28, 20] lifts to heterotic supergravity on $\mathbb{R}^{4(n+1)}$.

6 Near horizon solutions

In the last chapter I presented a class of solutions to the heterotic supergravity BPS equations which generalize the gauge solitonic branes of [77, 56, 60, 51]. Here an alternative approach will be taken, which leads to a generalization of the near horizon solution of the NS5-brane. The main difference is that we do not replace the curvature form R^+ in the Bianchi identity by an instanton in this chapter, so that we obtain an exact solution of the BPS equations and the Bianchi identity, but the equations of motion will be satisfied only up to terms of higher order in α'.

For simplicity we restrict attention to homogeneous real Killing spinor manifolds, although it appears very likely that the solution exists on non-homogeneous manifolds as well. It is only in the Bianchi identity that the homogeneity assumption considerably simplifies the calculation. We use the notation and results of Section 3.4. Recall that a homogeneous naturally reductive non-symmetric space $(M = G/K, g)$ comes equipped with a 3-form P and a 4-form Q, and a canonical connection ∇^P whose torsion is

$$T^a = \frac{1}{2} P^a{}_{bc} e^{bc}. \tag{6.1}$$

Moreover, we have a splitting of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and $\mathfrak{m}$ is a $\mathfrak{k}$-module. In fact, $\mathfrak{k} \subset \mathfrak{so}(\mathfrak{m})$. We want to identify ∇^P with the connection ∇^- that occurs in the gravitino equation. This is possible only if ∇^P has a parallel spinor, and we found the condition for this to be that $\mathfrak{k}$ as a subalgebra of $\mathfrak{spin}(\mathfrak{m})$ leaves a spinor invariant. But this is indeed the case for a real Killing spinor manifold if the metric is chosen such that the torsion of ∇^P becomes totally antisymmetric. Since the torsion of ∇^P is equal to P, and the torsion of ∇^- is the supergravity 3-form H, we have to identify $H = P$. The dilaton ϕ is then obtained from the general method explained in Section A.3; it becomes a linear function of an additional cylinder variable τ.

To gaugino equation imposes the instanton equation on the gauge field. The simplest solution is the canonical connection on the tangent bundle of M, so we set $F = R^- = R^P$. Now the Bianchi identity turns into the equation

$$dH = \frac{\alpha'}{4} \mathrm{tr}\Big(R^+ \wedge R^+ - R^- \wedge R^- \Big), \tag{6.2}$$

with $H = P$. From (3.100) we know that left and right hand side are proportional with a positive proportionality, and due to the fact that the right hand side of this equation is completely independent of the overall scale of the metric whereas the left hand side scales in the same way as the metric, we can conclude that a solution exists and that it fixes the volume of M in terms of α'.[1]

To summarize, the solution consists of the following ingredients:

- the cylinder over a non-symmetric naturally reductive homogeneous space $M = G/K$;
- the metric on M is induced by the Cartan-Killing form on G;
- the 3-form H coincides with the canonical 3-form P on M;
- both the gauge field and the connection ∇^- are given by the canonical connection ∇^P on M;
- the dilaton is a linear function of the cylinder variable τ;
- the supersymmetry generator ϵ is obtained from a K-invariant spinor on M. This restricts the set of possible base spaces M, since an invariant spinor need not exist on an arbitrary non-symmetric coset space. All homogeneous Killing spinor manifolds do satisfy this criterion, however.

In the simplest case where $M = S^3 = \mathrm{SU}(2)$ the solution reproduces the near-horizon limit of the NS5-brane [22]. In the general case it would be interesting to find an interpolating solution between the cone over G/K and the above solution, similarly to the NS5-brane.

[1] the tensor $P^{\mu}{}_{\nu\lambda}$ must be metric-independent because it appears in the canonical connection; therefore the 3-form P transforms as $P' = \alpha^2 P$ under a rescaling of the metric in the form $g' = \alpha^2 g$.

7 Instantons on cones and sine-cones over nearly Kähler manifolds

7.1 Instantons and submanifolds

Recall the diagram (3.127):

$$\begin{array}{ccc} \text{nearly Kähler} & \xrightarrow[(+\theta)]{sc} & \text{nearly parallel } G_2 \\ c\Big\downarrow(+r) & & c\Big\downarrow(+\rho) \\ \text{parallel } G_2 & \xrightarrow[(+\tau)]{cyl} & \text{parallel Spin(7)} \end{array} \tag{7.1}$$

Let (M^6, ω, Ω) be a 6-dimensional nearly Kähler manifold and M^8 the Spin(7)-manifold $c(sc(M^6))$. Its Spin(7)-structure is determined by the 4-form

$$\Psi = \frac{1}{2}r^4\omega \wedge \omega + r^2 d\tau \wedge dr \wedge \omega + r^3 d\tau \wedge \operatorname{Im}\Omega + r^3 dr \wedge \operatorname{Re}\Omega. \tag{7.2}$$

Besides the cone and the sine-cone over M^6, it contains the cylinder $cyl(M^6) = \{r = 1\}$ as a submanifold. The G_2-structure on $cyl(M^6)$ induced by the Spin(7)-structure of M^8 is determined by the 3-form

$$\Phi^2 = \omega \wedge d\tau - \operatorname{Re}\Omega, \tag{7.3}$$

which is coclosed but not closed. It is a so-called cocalibrated G_2-structure. On the other hand, there is a G_2-structure on the cylinder which is conformally equivalent to the integrable cone structure, with 3-form given by

$$\Phi^1 = \omega \wedge d\tau + \operatorname{Im}\Omega. \tag{7.4}$$

In this section we consider the Spin(7)-instanton equation on M^8, its reduction to the submanifolds $c(M^6)$, $sc(M^6)$ and $cyl(M^6)$ and the relation to the G_2-instanton equations on these submanifolds. Finally, we consider the reduction of these equations to M^6 and compare it to the SU(3)-instanton equation.

7.1.1 Reduction to seven dimensions

Consider the cone $M^7 = c(M^6)$ and a connection ∇^A on M^8. We can decompose the curvature form of ∇^A as $F + F_{\tau a} d\tau \wedge e^a$, where e^a is a basis of orthonormal 1-forms for $c(M^6)$ with the cone metric, and F is a two-form on M^7. The Spin(7)-instanton equation is equivalent to the set of equations

$$\begin{aligned} *F &= -\Phi \wedge F - *\Phi \wedge F_{\tau a} e^a \\ F \wedge *\Phi &= - * F_{\tau a} e^a, \end{aligned} \tag{7.5}$$

where all Hodge stars are taken with respect to the cone metric on M^7 and Φ is the 3-form (3.121), defining the integrable G_2-structure on M^7. In fact the two equations are equivalent, which can be proven by decomposing F according to the splitting $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}'$ into its $\mathfrak{g}_2$ and $\mathfrak{m}'$-components, and using the fact that the $\mathfrak{g}_2$-instanton operator $F \mapsto *(\Phi \wedge F)$ has eigenvalues -1 on $\mathfrak{g}_2$ and 2 on $\mathfrak{m}'$. Additionally one needs the property that $F \in \mathfrak{g}_2$ is equivalent to $F \lrcorner \Phi = 0$. Like the G_2-instanton equation, (7.5) does not restrict the $\mathfrak{g}_2$-part of F, but it allows for a non-vanishing $\mathfrak{m}'$-component as well. In particular, if $F_{\tau a} = 0$ then the Spin(7)-instanton equation on M^8 becomes equivalent to the G_2-instanton equation on the submanifold M^7.

We can perform a similar reduction to $M^7 = sc(M^6)$, with curvature decomposition $F + F_{\rho a} d\rho \wedge e^a$. The Spin(7)-instanton equation becomes

$$\begin{aligned} *F &= -\tilde{\Phi} \wedge F - \rho * \tilde{\Phi} \wedge F_{\rho a} e^a \\ F \wedge *\tilde{\Phi} &= -\rho * F_{\rho a} e^a, \end{aligned} \tag{7.6}$$

where now $\tilde{\Phi}$ is the 3-form (3.124) on the sine-cone. Again, the two equations are equivalent, and the case $F_{\rho a} = 0$ brings us back to the G_2-instanton equation on the sine-cone.

The metric on the cylinder $\{r = r_0 = \text{const}\}$ assumes the form $g = d\tau^2 + r_0^2 g^{(6)}$, and the appearance of r_0 is somewhat unconvenient. Therefore I will not work out the general reduction, but consider only a special case below.

7.1.2 Reduction to six dimensions

Similarly, we can reduce the Spin(7)-instanton equation down to six dimensions. Depending on which set of coordinates we use, we have the two possible decompositions

$$\begin{aligned} &F + F_{\tau a} d\tau \wedge e^a + F_{ra} dr \wedge e^a + F_{\tau r} d\tau \wedge dr, \\ &F + F_{\rho a} d\rho \wedge e^a + F_{\theta a} d\theta \wedge e^a + F_{\rho\theta} d\rho \wedge d\theta, \end{aligned} \tag{7.7}$$

with F a 2-form on M^6, and e^a a basis of orthonormal 1-forms on M^6. Introducing the complex 1-forms

$$\begin{aligned} \hat{F} &= (F_{ar} + iF_{a\tau}) e^a, \\ \tilde{F} &= (F_{a\theta} + i\rho F_{a\rho}) e^a, \end{aligned} \tag{7.8}$$

we can write the instanton equation either as

$$\begin{aligned} F &= (F \lrcorner \omega)\omega - *(\omega \wedge F) + 2r\,\mathrm{Re}(\hat{F} \lrcorner \overline{\Omega}) \\ F \lrcorner \Omega &= -r(g_{ab} - i\omega_{ab})\hat{F}_b e^a \\ F \lrcorner \omega &= -r^2 F_{\tau r}, \end{aligned} \tag{7.9}$$

or, equivalently, in the coordinates (θ, ρ):

$$\begin{aligned} F &= (F \lrcorner \omega)\omega - *(\omega \wedge F) + 2\sin(\theta)\,\mathrm{Re}\big(e^{i\theta}\tilde{F} \lrcorner \overline{\Omega}\big) \\ F \lrcorner \Omega &= -\sin(\theta)e^{-i\theta}(g_{ab} - i\omega_{ab})\tilde{F}_b e^a \\ F \lrcorner \omega &= -\rho\sin^2(\theta)F_{\rho\theta}. \end{aligned} \tag{7.10}$$

As usual, the contraction operator $\lrcorner$ and Hodge dual $*$ are taken with respect to the 6-dimensional metric. It turns out that the first equation implies the other two. We may decompose F as

$$F = F^{2,0} + F^{0,2} + \mathring{F}^{1,1} + F^{\omega}\,\omega\,, \tag{7.11}$$

where $F^{2,0}$ and $F^{0,2} = \overline{F^{2,0}}$ are (2,0)- and (0,2)-forms with respect to the almost complex structure J while $\mathring{F}^{1,1}$ is a (1,1)-form with zero ω-trace. Equivalently, we can decompose the first equation of (7.10) into two equations determining the (2,0)$\oplus$(0,2)- and ω-part of F. The (1,1)-part orthogonal to ω is unrestricted, as in the usual SU(3)-instanton equation. The (2,0)$\oplus$(0,2) component of F is determined by the second equation of (7.10), and the third one governs the ω-part.

7.2 Explicit ansätze for the connection

A number of instanton solutions on cylinders or cones over nearly Kähler-manifolds are known [61, 52, 54, 9]. Here I give a brief review of these constructions. As in Section 5.1 the starting point is the canonical connection ∇^P on the nearly Kähler manifold M^6, which has holonomy SU(3) and satisfies the SU(3)-instanton equation.

7.2.1 Reduced instanton equations for gauge group G_2

As before, let M^6 be a nearly Kähler manifold and denote by M^7 either the cone, sine-cone, or cylinder over M^6. We define a gauge field on the tangent bundle over M^7 with gauge group G_2 by using a similar - but slightly more general - ansatz as in Section 5.1:

$$\nabla^A = \nabla^P + e^a\psi_{ab}I_b, \tag{7.12}$$

with e^a a basis of 1-forms on M^6, I_b the local section defined in (5.4) and ψ_{ab} given by

$$\psi_{ab} = \psi_1\, g_{ab} + \psi_2\, \omega_{ab}, \tag{7.13}$$

where $\psi_{1,2} \in \mathbb{R}$. The curvature of this connection is given by

$$F = R^p + \frac{1}{2}|\psi|^2 f^i_{ab}I_i e^{ab} - \frac{1}{2}\mathrm{Re}\Big[\big(\overline{\psi}^2 - \psi\big)\big(g_{cd} + i\omega_{cd}\big)\Big]P_{abd}I_c e^{ab}, \tag{7.14}$$

where $\psi := \psi_1 + i\psi_2$, and the $\{I_i\}$ are a set of $\mathfrak{su}(3)$-generators. For this ansatz, the SU(3)-instanton equation is equivalent to

$$-\psi + \overline{\psi}^2 = 0, \tag{7.15}$$

which has four solutions: $\psi = 0$, $\psi = 1$ and $\psi = \exp\left(\pm i\frac{2\pi}{3}\right)$. If we consider ψ as a function of the additional coordinate r, τ or θ on cone, cylinder or sine-cone, respectively, then the curvature (7.14) acquires additional contributions of the form

$$\partial_\tau \psi_1 \, d\tau \wedge e^a I_a + \partial_\tau \psi_2 \, d\tau \wedge e^a \omega_{ab} I_b \,, \tag{7.16}$$

and the G_2-instanton equation yields a differential equation for ψ:

$$\frac{1}{2}\frac{d\psi}{d\tau} = -\psi + \overline{\psi}^2 \qquad \text{(conformally parallel cylinder)}, \tag{7.17a}$$

$$\frac{i}{2}\frac{d\psi}{d\tau} = -\psi + \overline{\psi}^2 \qquad \text{(cocalibrated cylinder)}, \tag{7.17b}$$

$$\frac{r}{2}\frac{d\psi}{dr} = -\psi + \overline{\psi}^2 \qquad \text{(metric cone)}\,, \tag{7.17c}$$

$$\frac{1}{2}\sin(\theta)\, e^{-i\theta}\frac{d\psi}{d\theta} = -\psi + \overline{\psi}^2 \qquad \text{(sine-cone)}. \tag{7.17d}$$

As explained before, the cone equation (7.17c) is equivalent to the standard instanton equation (7.17a) on the cylinder, via the substitution $r = e^\tau$. It can be considered as a gradient flow equation, whereas the cylinder equation (7.17b) admits an interpretation as a Hamiltonian flow equation [52]. The equations (7.17) appear more naturally from an eight-dimensional point of view, if we consider $M^8 = cyl(c(M^6)) = c(sc(M^6))$ with the metric

$$g^{(8)} = d\tau^2 + dr^2 + r^2\, g^{(6)}\,. \tag{7.18}$$

Choose ∇^A as above, with ψ a complex function of r and τ. By defining a complex coordinate $z = r - i\tau$ on $\mathbb{R}^+ \times \mathbb{R}$, the Spin(7)-instanton equation for this ansatz can be written as

$$\mathrm{Re}(z)\frac{d\psi}{dz} = -\psi + \overline{\psi}^2\,. \tag{7.19}$$

This equation reduces to the differential equation for the cone, the sine-cone or the cocalibrated cylinder if we restrict it to a particular path in the complex half-plane spanned by z. One chooses

- $z = r + i\tau_0$ for some constant $x_0 \in \mathbb{R}$ to obtain the metric cone and (7.17c),
- $z = -i\rho_0 e^{i\theta}$ for some constant $\rho_0 > 0$ to obtain the sine-cone and (7.17d),
- $z = r_0 - i\tau$ for some constant $r_0 \in \mathbb{R}^+$ to obtain the cocalibrated cylinder with metric $\bar{g} = d\tau^2 + r_0^2 g^{(6)}$. The instanton equation reads

$$\frac{ir_0}{2}\frac{d\psi}{d\tau} = -\psi + \overline{\psi}^2, \tag{7.20}$$

which is a slight generalization of (7.17b). Contrary to the two cases above, a solution of (7.20) on the submanifold $\{r = r_0 = \text{const}\}$ does not extend trivially to a solution on the full eight-dimensional space. This can be accomplished by choosing instead the parametrization $z = s(1 - it)$. For s-independent ψ the instanton equation becomes

$$\frac{1}{2}(i - t)\frac{d\psi}{dt} = -\psi + \overline{\psi}^2, \tag{7.21}$$

and the slices $\{s = s_0 = \text{const}\}$ carry the cylinder metric $s_0^2(dt^2 + g^{(6)})$. Both (7.20) and (7.21) can be obtained from the Spin(7)-instanton equation restricted to the cylinder. In the first case one obtains exactly the cylindrical G_2-instanton equation by imposing $dr \lrcorner F = 0$, where dr is the 1-form normal to the cylinder in M^8. This condition is not satisfied for solutions of (7.21) however, so that these do not solve the G_2-instanton equation. Substituting $t = \cot(\theta)$ in (7.21) brings us back to the sine-cone equation (7.17d).

Of course, other foliations of M^8 are possible, and they lead to additional instanton equations.

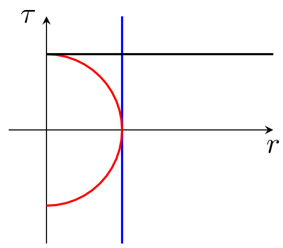

Figure 7.1: The complex z-plane, $z = r - i\tau$. M^8 is a twisted product of M^6 with the right halfplane $\{r > 0\}$. Embedded into M^8 are the sine-cone (red half-circle), cylinder (vertical blue line), and cone (horizontal black line). M^8 is foliated either by cylinders or cones, corresponding to the foliation of the half-plane by translations of the black and blue lines. A foliation by sine-cones is obtained through variation of the radius of the red half-circle. Upon a good parametrization of the three submanifolds the G_2-instanton equation on one of them becomes invariant under these shifts, so that a solution on a submanifold trivially extends to a Spin(7)-instanton on all of M^8.

7.2.2 Solutions

Here I collect the known finite-action solutions to the instanton equations (7.17). The conformally parallel cylinder, and thus the cone, was discussed in Section 5.1 and in a similar setting in [61, 52, 54, 9]:

$$\psi(\tau) = c\Big(1 + e^{2(\tau - \tau_0)}\Big)^{-1} \qquad \text{with} \quad c = 1 \quad \text{or} \quad c = \exp\big(\pm i\tfrac{2\pi}{3}\big)\,. \tag{7.22}$$

The solution (7.22) interpolates between the stationary SU(3)-instantons $\psi \to c$ for $\tau \to -\infty$ and $\psi \to 0$ for $\tau \to \infty$, and it is represented by the black edges in Figure 7.2. Solutions to (7.17b) on the cocalibrated cylinder have been found in [52] and are given by

$$\psi(\tau) = -\frac{c}{2}\Big(1 + i\sqrt{3}\tanh\big[\sqrt{3}(\tau-\tau_0)\big]\Big)\,, \tag{7.23}$$

with c as above being equal to one of the three non-trivial fixed points. These solutions interpolate between the fixed points $c\cdot\exp(-2\pi i/3)$ and $c\cdot\exp(2\pi i/3)$, as is illustrated by the blue edges in Figure 7.2. We also found solutions for the sine-cone equation (7.17d), namely

$$\psi(\theta) = c\left(\cos(\theta) - \tfrac{i}{3}\sin(\theta)\right)e^{i\theta/3}\,. \tag{7.24}$$

These are drawn in red in Figure 7.2 and interpolate between c for $\theta \to 0$ and $c\cdot\exp(-2\pi i/3)$ for $\theta \to \pi$. Moreover, they give rise to solutions of the Spin(7)-instanton equation (7.21) restricted to the cylinder, upon substituting $\theta = \mathrm{arccot}(t)$. Explicitly, we get

$$\psi(t) = c\,\frac{(t-\frac{i}{3})(t+i)^{1/3}}{(t^2+1)^{2/3}}\,. \tag{7.25}$$

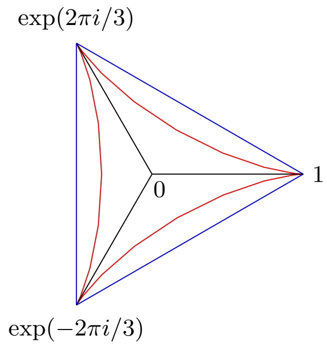

Figure 7.2: Instantons in the complex ψ plane. The nodes correspond to the four SU(3)-instantons on the nearly Kähler manifold M^6, whereas the edges are interpolating Spin(7)-instantons on $M^8 = cyl(c(M^6))$ or submanifolds thereof. The blue edges can be realized as G_2-instantons on the cocalibrated cylinder $cyl(M^6)$, the red ones solve the nearly parallel G_2-instanton equation on the sine-cone $sc(M^6)$, and the black edges are solutions on the cone $c(M^6)$. The 3-symmetry of homogeneous nearly Kähler manifolds is reflected in the permutation symmetry of the diagram.

7.2.3 Interpretation

Consider again the diagram (7.1). The four spaces with reduced structure group SU(3), G_2 or Spin(7) are related by certain geometric operations, and each carries a distinguished instanton. As a gauge field, the Levi-Civita connection on a Riemannian manifold can be identified with the one on its cylinder, because their connection 1-forms (or gauge fields) are the same. Thus we end up with four geometries giving rise to three different instantons. They correspond to the canonical connection on the base M for $\psi = 0$, the Levi-Civita connection on the cone $c(M)$ for $\psi = 1$, and additionally the canonical G_2-connection on the sine-cone $sc(M)$, which is gauge-equivalent to the non-stationary solution (7.24).

To make the identification of (7.24) with the canonical connection on the sine-cone more precise, we consider an explicit realization of the generators I_a. The canonical connection ∇^P on a nearly Kähler or nearly parallel G_2-manifold is obtained from its Levi-Civita connection by adding a suitable multiple of the canonical 3-form P. For the sine-cone over a nearly Kähler manifold this recipe leads to the following expression:

$$\nabla^{P,7} = \nabla^{P,6} + \left[\cos(\theta)\delta_{ab} - \tfrac{1}{3}\sin(\theta)\omega_{ab}\right]e^a I_b - \tfrac{1}{3}d\theta\, J\,, \tag{7.26}$$

where J is the almost complex structure acting non-trivially only on the tangent space to M^6 and the I_a are skew-symmetric 7×7-matrices

$$(I_a)_{b0} = \delta_{ab} \qquad \text{and} \qquad (I_a)_{bc} = \tfrac{1}{2}\mathrm{Re}(e^{i\theta}\Omega)_{abc}\,. \tag{7.27}$$

Again, indices a, b, c run from 1 to 6. Upon a gauge transformation

$$\nabla^{P,7} \;\mapsto\; e^{-\theta J/3}\nabla^{P,7}\, e^{\theta J/3}\,, \tag{7.28}$$

the connection form is transferred to the form of our ansatz (7.13), with ψ given by (7.24) and the I_a by (5.4), i.e.

$$(I_a)_{b0} = \delta_{ab} \qquad \text{and} \qquad (I_a)_{bc} = \tfrac{1}{2}\mathrm{Re}(\Omega)_{abc}. \tag{7.29}$$

A more general ansatz for the gauge field on the tangent bundle of M^8 is possible, with gauge group Spin(7) instead of G_2. So far we made use of the decomposition $\mathfrak{g}_2 = \mathfrak{su}(3)\oplus\mathfrak{m}$, and our gauge field consisted of the canonical SU(3)-connection plus a 1-form with values in $\mathfrak{m}$. We have a similar decomposition

$$\mathfrak{spin}(7) = \mathfrak{su}(3) \oplus 2\mathfrak{m} \oplus \underline{1}, \tag{7.30}$$

where $\mathfrak{m}$ is the same $\mathfrak{su}(3)$-module as in the decomposition of $\mathfrak{g}_2$, and $\underline{1}$ is the trivial $\mathfrak{su}(3)$-representation. We could therefore introduce a second set $\{\tilde{I}_a\}$ of $\mathfrak{m}$-generators, a $\underline{1}$-generator I, and consider the gauge field

$$\nabla^A = \nabla^{P,6} + \psi_{ab}e^a I_b + \tilde{\psi}_{ab}e^a \tilde{I}_b + f\mu\, I, \tag{7.31}$$

where f is a function and μ some arbitrary 1-form on M^8, for instance $\mu = d\tau, dr, d\theta$, or $d\rho$. The Spin(7)-instanton equation reduces to a partial differential equation for

$\psi, \tilde{\psi}$ and f, and the solutions found above are special cases where $\tilde{\psi} = f = 0$. A particular solution where neither ψ nor $\tilde{\psi}$ vanish is the Spin(7)-instanton (5.11), which interpolates between the canonical G_2-connection $\hat{A}^7$ on $sc(M^6)$ and the Levi-Civita connection on M^8.

The ansatz for the gauge field relies on the splitting $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$. For a homogeneous nearly Kähler manifold G/K there is another decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively, and $\mathfrak{m}$ is the same $\mathfrak{k} \subset \mathfrak{su}(3)$-module as in the $\mathfrak{g}_2$-decomposition. In fact, the canonical SU(3)-connection on G/K takes values in $K \subset \mathrm{SU}(3)$, and depending on whether we consider $\mathfrak{m}$ as a subset of $\mathfrak{g}_2$ or $\mathfrak{g}$, the gauge field

$$\nabla^A = \nabla^{P,6} + \psi_{ab} e^a I_b \tag{7.32}$$

has gauge group G_2 or G. It turns out that the instanton equation for ψ is the same for both approaches, and whereas we have concentrated on gauge group G_2 here, the gauge group G case has mostly been considered in the literature [61, 52, 54, 9]. For $M^6 = S^6 = G_2/\mathrm{SU}(3)$ the two alternatives are equivalent.

In Chapter 5 we have shown that the conical instantons (7.22) for $c = 1$ embed into heterotic supergravity, similarly for the solution interpolating between the canonical connection on the sine-cone and the Levi-Civita connection on the cylinder over the cone. It would be interesting to find a general supergravity extension for every solution of the instanton equation (7.19), or even every instanton constructed from the generalized ansatz (7.31).

8 Conclusions and outlook

I have shown that manifolds M with real Killing spinors play an important role in heterotic string theory, similarly to the type II string theories and M-theory. A family of instantons on the Ricci-flat cone over M has been constructed and embedded into supergravity. These solutions have been found analytically for nearly Kähler, nearly parallel G_2, and 3-Sasakian manifolds, whereas the case of Sasaki-Einstein manifolds has been solved numerically by D. Harland [53]. Another solution of the supergravity equations, similar to the near horizon limit of the NS5-brane, is given by the cylinder metric over M, with gauge field equal to the canonical connection on M and a linear dilaton. In this case the H-Bianchi identity has been proven only for homogeneous base manifolds M, but the BPS equations are solved for arbitrary Killing spinor manifolds.

One important ingredient of the heterotic BPS equations is the gravitino equation $\nabla^- \epsilon = 0$ for some Majorana-Weyl spinor ϵ, which can be restated as the requirement that the holonomy group of the connection ∇^- must be contained in the stabilizer subgroup of Spin(9,1) of a spinor. In this thesis I constructed several examples with compact stabilizer groups, like Spin(7), G_2, SU(n), and Sp(n). It is known that some of these groups are contained in non-compact groups which stabilize the same number of spinors [49]: there is one invariant spinor for $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$, two for $\mathrm{SU}(4) \ltimes \mathbb{R}^8$, and three for $\mathrm{Sp}(2) \ltimes \mathbb{R}^8$. It appears likely that one can generalize the ansatz for the supergravity fields to construct solutions with any of these non-compact holonomy groups. An example of this type can be found in [75], where the holonomy group is contained in $\mathrm{SU}(2) \times \mathbb{R}^4$, and the solution consists of a flat 4-dimensional Euclidean space without fluxes times a 6-dimensional Lorentzian manifold with non-vanishing fluxes.

Another possible extension of the results could involve resolutions of the conical singularity. The cone metric over a non-spherical manifold becomes singular at the tip of the cone, and although there is no general algorithm to resolve this singularity, many explicit resolutions for special base manifolds are known [74, 47, 71]. In type II and 11-dimensional supergravity one can extend the brane solutions on the cone to the resolution [29], and this should be possible in heterotic supergravity as well. The instanton equation on the resolved cone over a regular Sasaki-Einstein manifold has been studied in [27].

Since the sine-cone over a manifold with real Killing spinors has real Killing spinors again, one can consider the instanton and supergravity equations on the cone over the sine-cone over M. Several solutions of the former are known to exist, due to the fact that the cone over the sine-cone is the same space as the cylinder over the cone, and both the cone and the sine-cone over a real Killing spinor manifold come equipped with canonical instanton gauge fields. I have illustrated this for the case of a nearly Kähler manifold, but it seems likely that the full moduli space of instantons contains more solutions than those presented here. Also the embedding into supergravity has not been worked out yet, and the case of other base manifolds, for instance with a Sasaki-Einstein structure, is another interesting project for future work.

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A Spinors

A.1 Clifford algebras

Let (V, g) be a real vector space endowed with a metric. Define a new product on the tensor algebra $\mathcal{T}V$ over V by imposing the relation

$$\{v, w\} := vw + wv = 2g(v, w) \tag{A.1}$$

for all $v, w \in V$. The resulting algebra $\mathrm{Cl}(V, g)$ is called the Clifford algebra over V. If (V, g) is $\mathbb{R}^{p,g}$ we denote its Clifford algebra by $\mathrm{Cl}(p, q)$, and if $q = 0$ then we write $\mathrm{Cl}(p)$ for $\mathrm{Cl}(p, 0)$. Given an orthonormal basis $\{I_\mu\}$ for V the corresponding elements in the Clifford algebra will be denoted by γ_μ. There is an induced metric g^{-1} on the dual space V^*, which gives rise to the Clifford algebra $\mathrm{Cl}(V^*, g^{-1})$. Its generators corresponding to the dual basis $\{e^\mu\}$ are denoted by γ^μ. The metric has a unique complex linear extension to $V \otimes \mathbb{C}$, and there is also a complex Clifford algebra, satisfying $\mathrm{Cl}(V \otimes \mathbb{C}, g) = \mathrm{Cl}(V, g) \otimes \mathbb{C}$. We write $\mathrm{Cl}(n, \mathbb{C}) = \mathrm{Cl}(\mathbb{C}^n)$.

A.1.1 Even dimensions

A representation of $\mathrm{Cl}(2n, \mathbb{C})$ can be defined in terms of a maximal complex subspace $W \leq \mathbb{C}^{2n}$ such that

$$g(w^1, w^2) = 0 \tag{A.2}$$

for all $w^1, w^2 \in W$. The representation space is the exterior algebra $\mathcal{S} := \Lambda W$, with action $\gamma : \mathrm{Cl}(2n, \mathbb{C}) \to \mathrm{End}(\mathcal{S})$ determined by

$$\begin{aligned} \gamma(w)w^1 \wedge \cdots \wedge w^r &= w \wedge w^1 \wedge \cdots \wedge w^r, \\ \gamma(\overline{w})w^1 \wedge \cdots \wedge w^r &= \sum_{i=1}^{r} (-1)^{i-1} 2g(\overline{w}, w^i) w^1 \wedge \cdots \wedge \check{w}^i \wedge \cdots \wedge w^r, \end{aligned} \tag{A.3}$$

for $w \in W$ and $\overline{w} \in \overline{W}$, where $\check{w}^i$ means leaving out w^i. To give a complex subspace W of $V \otimes \mathbb{C}$ which satisfies (A.2) is equivalent to the choice of a metric-compatible complex structure on V, and we shall choose W to be generated by holomorphic elements $I_{2j-1} + iI_{2j}$ for $j = 1, \ldots, n$.

Concretely, this means the following. Consider $\mathrm{Cl}(2n)$ with generators $\gamma_1, \ldots, \gamma_{2n}$. In the complexified Clifford algebra we can introduce the creation and annihilation operators $\zeta_j, \overline{\zeta}_j$ (often denoted by $a_j^\dagger$ and a_j instead), $j = 1, \ldots, n$, as

$$\zeta_j = \tfrac{1}{2}(\gamma_{2j-1} + i\gamma_{2j}), \qquad \overline{\zeta}_j = \tfrac{1}{2}(\gamma_{2j-1} - i\gamma_{2j}). \tag{A.4}$$

They satisfy the relations

$$\{\overline{\zeta}_j, \overline{\zeta}_k\} = \{\zeta_j, \zeta_k\} = 0, \qquad \{\zeta_j, \overline{\zeta}_k\} = \delta_{jk}. \tag{A.5}$$

Let $w^1, \ldots, w^n$ be a basis of $\mathbb{C}^n \simeq W$, then the ζ_j and $\overline{\zeta}_j$ act on $\mathcal{S}$ as

$$\begin{aligned} \zeta_j \cdot w^{k_1} \wedge \cdots \wedge w^{k_r} &= w^j \wedge w^{k_1} \wedge \cdots \wedge w^{k_r}, \\ \overline{\zeta}_j \cdot w^{k_1} \wedge \cdots \wedge w^{k_r} &= \sum_{i=1}^{r} (-1)^{i-1} \delta_{j,k_i} w^{k_1} \wedge \cdots \wedge \check{w}^{k_i} \wedge \cdots \wedge w^{k_r}. \end{aligned} \tag{A.6}$$

It is known that $\mathrm{Cl}(2n,\mathbb{C})$ is isomorphic to $\mathrm{End}(\mathcal{S})$, so that we have constructed the unique irreducible representation of $\mathrm{Cl}(2n,\mathbb{C})$, up to equivalence [80]. Since $\mathrm{Cl}(2n,\mathbb{C})$ contains the real Clifford algebra $\mathrm{Cl}(2n)$ as a subspace, we also obtain a representation of the latter, but it is unclear whether this representation is irreducible. This question is related to the concept of charge conjugation, to be explained below.

Inside the real Clifford algebra $\mathrm{Cl}(2n)$ we find the so-called spin algebra, which is a copy of the orthogonal Lie algebra, under the map

$$\mathfrak{so}(2n) \to \mathrm{Cl}(2n), \ A \mapsto \frac{1}{4} g_{\mu\kappa} A^{\kappa}{}_{\nu} \gamma^{\mu\nu}. \tag{A.7}$$

Hence, $\mathfrak{spin}(2n)$ is generated by elements $\gamma^{\mu\nu}$ of degree two. Spinor space splits into the sets of positive and negative chirality spinors, or *Weyl spinors*,

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- := \Lambda^+ W \oplus \Lambda^- W, \tag{A.8}$$

where $\Lambda^{\pm} W$ denotes the even and odd degree components of the exterior algebra. A single gamma matrix interchanges $\mathcal{S}^+$ and $\mathcal{S}^-$, so that the action of $\mathfrak{spin}(2n)$ respects the splitting (A.8). This can be restated in terms of the so-called chirality operator Γ, which acts through eigenvalue $+1$ on $\mathcal{S}^+$ and by -1 on $\mathcal{S}^-$. The complexified spin algebra commutes with Γ, and by Schur's lemma the representation on $\mathcal{S}$ must be reducible. It turns out that $\mathcal{S}^+$ and $\mathcal{S}^-$ are always irreducible representations of $\mathfrak{spin}(2n) \otimes \mathbb{C}$, whereas the representation of the real spin algebra sometimes can be further reduced.

There is an automorphism α_C of the Clifford algebra, determined by

$$\gamma_{2j+1} \mapsto \gamma_{2j+1}, \qquad \gamma_{2j} \mapsto -\gamma_{2j}. \tag{A.9}$$

Since the representation of $\mathrm{Cl}(2n,\mathbb{C})$ on $\mathcal{S}$ is unique, there must exist two maps $\tilde{C}_1, \tilde{C}_2 \in \mathrm{Aut}(\mathcal{S})$ which implement α_C and $-\alpha_C$, in the sense that

$$\begin{aligned} \alpha_C(\eta) &= \tilde{C}_1^{-1} \eta \tilde{C}_1 \\ -\alpha_C(\eta) &= \tilde{C}_2^{-1} \eta \tilde{C}_2 \end{aligned} \tag{A.10}$$

for all $\eta \in \mathrm{Cl}(2n, \mathbb{C})$. One can check that

$$\tilde{C}_1 = \begin{cases} \gamma_1\gamma_3 \cdots \gamma_{2n-1} & \text{if } n \text{ is odd} \\ \gamma_2\gamma_4 \cdots \gamma_{2n} & \text{if } n \text{ is even} \end{cases}$$
$$\tilde{C}_2 = \begin{cases} \gamma_2\gamma_4 \cdots \gamma_{2n} & \text{if } n \text{ is odd} \\ \gamma_1\gamma_3 \cdots \gamma_{2n-1} & \text{if } n \text{ is even} \end{cases} \tag{A.11}$$

do the job. Let us introduce a real structure on spinor space $\mathcal{S}$, by considering the basis elements $w^{j_1} \wedge \cdots \wedge w^{j_r}$ as real. Then complex conjugation acts on gamma matrices as

$$\gamma^*_{2j-1} = \gamma_{2j-1}, \qquad \gamma^*_{2j} = -\gamma_{2j}. \tag{A.12}$$

We can also introduce a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}$ by demanding that the basis elements $w^{j_1} \wedge \cdots \wedge w^{j_r}$ are orthonormal. Then the spin representation is unitary. Furthermore, we define the complex antilinear maps $C_1, C_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{S})$ by

$$C_k\psi = (\tilde{C}_k\psi)^* \tag{A.13}$$

for all $\psi \in \mathcal{S}$ and $k = 1, 2$. A simple calculation shows that

$$C_1^{-1}\gamma_\mu C_1 = -C_2^{-1}\gamma_\mu C_2 = \gamma_\mu \tag{A.14}$$

for all $\mu = 1, \ldots, 2n$. From (A.14) and antilinearity of C_1 and C_2 it follows that the embedding of the real spin algebra into its complexification is determined by C_1 or C_2:

$$\mathfrak{spin}(2n) = \big\{\eta \in \mathfrak{spin}(2n) \otimes \mathbb{C} \;\big|\; C_k^{-1}\eta C_k = \eta\big\} \tag{A.15}$$

for both $k = 1$ or $k = 2$, where $\mathfrak{spin}(2n) \otimes \mathbb{C}$ is considered as a subalgebra of $\mathrm{Cl}(2n, \mathbb{C})$. This property explains the relevance of the so-called *charge conjugations* C_1 and C_2.

To summarize, we have constructed a representation space $\mathcal{S}$ of $\mathfrak{spin}(2n)$, and three operators $C_1, C_2, \Gamma \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{S})$ which commute with the action of $\mathfrak{spin}(2n)$. One of them, Γ, is indeed complex linear, and Schur's lemma implies that the representation space $\mathcal{S}$ splits into the eigenspaces $\mathcal{S}^+ \oplus \mathcal{S}^-$ of Γ. Since Schur's lemma does not hold over the real numbers, we cannot draw the same conclusion for C_1 and C_2.

It is not difficult to show that C_1 and C_2 both square to 1 or -1, depending on the dimension. Whenever C_k squares to the identity we can define the *Majorana spinors*

$$\mathcal{S}_M = \big\{\psi \in \mathcal{S} \;\big|\; C_k\psi = \psi\big\}. \tag{A.16}$$

If, in addition, C_k commutes with Γ, it makes sense to define *Majorana-Weyl spinors*

$$\mathcal{S}_M^\pm = \mathcal{S}_M \cap \mathcal{S}^\pm. \tag{A.17}$$

Whenever Majorana-Weyl spinors exist they form an irreducible $\mathfrak{spin}(2n)$-representation. When there are no Majorana-Weyl spinors, then Weyl spinors are irreducible, and so are Majorana spinors if they exist [80]. To determine the existence of Majorana and Majorana-Weyl spinors we simply have to check whether the relations $C_k^2 = 1$ and $[C_k, \Gamma] = 0$ hold in a given dimension. The result is

$$\begin{aligned} C_1^2 &= (-1)^{\frac{1}{2}n(n-1)} \\ C_2^2 &= (-1)^{\frac{1}{2}n(n+1)} \\ [C_k, \Gamma] &= 0 \quad \text{if } n \text{ is even} \\ \{C_k, \Gamma\} &= 0 \quad \text{if } n \text{ is odd.} \end{aligned} \tag{A.18}$$

Hence, there are Majorana spinors in dimensions $0, 2$ and 6 mod 8, and Majorana-Weyl spinors in dimensions 0 mod 8. Elements of $\mathcal{S}$ that are neither Weyl nor Majorana are usually called *Dirac spinors*.

A.1.2 Odd dimensions

The Clifford algebra in dimension $2n+1$ has a nontrivial center:

$$Z\big(\mathrm{Cl}(2n+1, \mathbb{C})\big) = \langle \{1, \gamma^{1\ldots 2n+1}\} \rangle,$$

and one has

$$\mathrm{Cl}(2n+1, \mathbb{C}) = \mathrm{Cl}^+(2n+1, \mathbb{C}) \otimes Z\big(\mathrm{Cl}(2n+1, \mathbb{C})\big), \tag{A.19}$$

where $\mathrm{Cl}^+(2n+1, \mathbb{C})$ is the subalgebra consisting of elements with an even number of gamma matrices. A representation can be defined as follows. Single out one gamma matrix, say γ^{2n+1}. Recall that $\mathrm{Cl}(0, 2n)$ denotes the Clifford algebra for a purely timelike metric, i.e. $g_{\mu\nu} = -\delta_{\mu\nu}$. We denote the generators of $\mathrm{Cl}(0, 2n)$ by $\tilde{\gamma}_a$, $a = 1, \ldots, 2n$. The map $\mathrm{Cl}^+(2n+1) \to \mathrm{Cl}(0, 2n)$,

$$\gamma^{a,2n+1} \mapsto -\tilde{\gamma}^a, \qquad \gamma^{ab} \mapsto -\tilde{\gamma}^{ab} \tag{A.20}$$

$a = 1, \ldots, 2n$, defines an algebra isomorphism. Instead of working with time-like gamma matrices $\tilde{\gamma}^a$ we can work in the complexified algbebra $\mathrm{Cl}(2n, \mathbb{C})$, and replace $\tilde{\gamma}^a$ by $i\gamma^a_{\mathrm{ev}}$, where the γ^a_{ev} are space-like generators of $\mathrm{Cl}(2n)$. The isomorphism (A.20) can be extended to a map $\mathrm{Cl}(2n+1, \mathbb{C}) \to \mathrm{Cl}(2n, \mathbb{C})$ by demanding that the central element $i^n \gamma^{1\ldots 2n+1}$ be mapped to the identity. The map is then determined by the following correspondences

$\mathrm{Cl}(2n+1,\mathbb{C})$	$\mathrm{Cl}(2n,\mathbb{C})$
1	1
γ^a	$-i\gamma_{\mathrm{ev}}^a\Gamma$
γ^{2n+1}	Γ
γ^{ab}	$\gamma_{\mathrm{ev}}^{ab}$
$\gamma^a\gamma^{2n+1}$	$-i\gamma_{\mathrm{ev}}^a$

Table A.1: Mapping gamma matrices in odd dimension to even-dimensional ones.

where $a = 1, \ldots, 2n$, and $\Gamma = i^n \gamma_{\mathrm{ev}}^{1\ldots 2n}$ is the usual chirality operator in $\mathrm{Cl}(2n,\mathbb{C})$. The spin representation of $\mathrm{Cl}(2n,\mathbb{C})$ then defines a representation of $\mathrm{Cl}(2n+1,\mathbb{C})$ as well, but the question of irreducibility has to be reconsidered. Again, the spin algebra $\mathfrak{spin}(2n+1)$ is contained in $\mathrm{Cl}^+(2n+1)$, but it does not respect spinor chirality, since generators $\gamma^a\gamma^{2n+1} \simeq -i\gamma_{\mathrm{ev}}^a$ change chirality. For the same reason, the charge conjugation must commute not only with even degree elements in $\mathrm{Cl}(0,2n)$ but with the full real Clifford algebra. In the notation used above only C_1 is an admissible candidate for the charge conjugation. It is determined in time-like dimensions by the complex linear map

$$\tilde{C} = \begin{cases} \tilde{\gamma}_1\tilde{\gamma}_3\ldots\tilde{\gamma}_{2n-1} & \text{if } n \text{ is even,} \\ \tilde{\gamma}_2\tilde{\gamma}_4\ldots\tilde{\gamma}_{2n} & \text{if } n \text{ is odd,} \end{cases} \tag{A.21}$$

and satisfies

$$C^2 = (-1)^{\frac{n}{2}(n+1)}. \tag{A.22}$$

Hence, Majorana spinors exist in dimensions 1 and 7 mod 8.

A.1.3 Summary: Majorana spinors

We found that Majorana and Majorana-Weyl spinors exist in the following dimensions, mod 8:

n	Majorana	Majorana-Weyl
0	✓	✓
1	✓	
2	✓	-
3	-	
4	-	-
5	-	
6	✓	-
7	✓	

Table A.2: Existence of Majorana(-Weyl) spinors for the spin algebra $\mathfrak{spin}(n)$.

The case of non-Euclidean signatures was not considered, but it is of relevance to supergravity which is always defined for Lorentzian signature. In this case a similar analysis shows that the dimensions are shifted by two relative to the Euclidean situation: Majorana-Weyl spinors exist in dimension 2 mod 8, and so on. The Clifford algebras for different signatures in a fixed dimension can all be embedded into the same complex Clifford algebra, $\mathrm{Cl}(p,q) \subset \mathrm{Cl}(p+q,\mathbb{C})$, and the information about the precise embedding is encoded in the charge conjugation $C^{(p,q)}$.

A.1.4 Embedding spinors into higher dimensions

In the previous sections we have constructed the spin representation $\mathcal{S}_n$ of $\mathfrak{spin}(n)$ for any dimension n. We found that $\mathcal{S}_{2n+1} = \mathcal{S}_{2n}$, the dimension is $\dim_{\mathbb{C}} \mathcal{S}_{2n} = 2^n$, and that $\mathcal{S}_{2n}$ splits into the subrepresentations $\mathcal{S}_{2n}^+ \oplus \mathcal{S}_{2n}^-$.

Lemma A.1. *There is a $\mathfrak{spin}(2n-1)$-equivariant isomorphism*

$$\iota : \mathcal{S}_{2n-1} \to \mathcal{S}_{2n}^+. \tag{A.23}$$

It satisfies

$$\begin{aligned} \gamma_{ev}^{ab} \circ \iota &= \iota \circ \gamma^{ab} \\ \gamma_{ev}^{a,2n} \circ \iota &= \iota \circ (-i\gamma^a), \end{aligned} \tag{A.24}$$

for $a = 1, \ldots, 2n-1$, where $\gamma_{ev}^a, \gamma_{ev}^{2n}$ are gamma matrices in dimension $2n$, and γ^a those in dimension $2n-1$.

Proof. Let $\psi \in \mathcal{S}_{2n-1}$ be a Dirac spinor. Then we can decompose ψ into its positive and negative chirality components $\psi = \psi^+ + \psi^-$.[1] Recall that spinor space in dimension $2n$ is obtained from the one in dimension $2n-2$ by the introduction of one more complex generator w, and adding to $\mathcal{S}_{2n-2}$ all possible elements $w \wedge \phi$ for $\phi \in S_{2n-2}$. The gamma matrices γ_{2n-1} and γ_{2n} combine into a generator and annihilator for w, as in (A.4). Now we can define $\iota(\psi) \in \mathcal{S}_{2n}^+$ by

$$\iota(\psi) = \psi^+ - iw \wedge \psi^-. \tag{A.25}$$

Equivariance under $\mathfrak{spin}(2n-2)$ is straightforward, whereas for $\mathfrak{spin}(2n-1)$-generators of the form $\gamma^a\gamma^{2n-1}$, $a = 1, \ldots, 2n-2$, a short calculation gives

$$\gamma_{\mathrm{ev}}^a \gamma_{\mathrm{ev}}^{2n-1} \iota(\psi) = -i\gamma_{\mathrm{ev}}^a \psi^- + \gamma_{\mathrm{ev}}^a w \wedge \psi^+ = \iota\big(\gamma^a \gamma^{2n-1} \psi\big). \tag{A.26}$$

Furthermore, we obtain

$$\begin{aligned} \gamma_{\mathrm{ev}}^{a,2n} \iota(\psi) &= \gamma_{\mathrm{ev}}^a \psi^- - i\gamma_{\mathrm{ev}}^a w \wedge \psi^+ = \iota(-i\gamma^a \psi) \\ \gamma_{\mathrm{ev}}^{2n-1,2n} \iota(\psi) &= -i\psi^+ + w \wedge \psi^- = \iota(-i\gamma^{2n-1}\psi). \end{aligned} \tag{A.27}$$

This proves the claim. □

As Table A.1 shows, the property (A.24) also holds for the identity map $\mathcal{S}_{2n} \to \mathcal{S}_{2n+1}$, which proves Lemma 3.2.

[1] although the splitting is respected only by the action of the subalgebra $\mathfrak{spin}(2n-2)$ of $\mathfrak{spin}(2n-1)$, the map ι will indeed be $\mathfrak{spin}(2n-1)$-equivariant.

A.2 Stabilizers

In geometric applications it is of interest to know the stabilizer group $G_\psi \subset \mathrm{Spin}(n)$ of a given spinor $\psi \in \mathcal{S}$. If, for instance, the holonomy group of a connection on a Riemannian manifold coincides with G_ψ, then there exists a parallel spinor. A general classification of possible stabilizer groups exists only in low dimensions.

A.2.1 Pure spinors

The simplest case is that of pure spinors. Recall that spinor space in dimension $2n$ is of the form $\mathcal{S} = \Lambda W$, with $W \simeq \mathbb{C}^n$. By definition, a pure spinor is one that can be rotated to $\Lambda^0 W \simeq \mathbb{C}$ by a $\mathrm{Spin}(2n)$-transformation. Pure spinors are then necessarily chiral and non-Majorana. The restriction of the spin representation to the subgroup $\mathrm{SU}(n) \subset \mathrm{Spin}(2n)$ splits into the irreducible representations $\mathcal{S}^j = \Lambda^j \mathbb{C}^n$, $j = 0, \ldots, n$, and in particular pure spinors are invariant under $\mathrm{SU}(n)$. In fact, $G_\psi = \mathrm{SU}(n)$ if ψ is pure. The concept of pure spinors makes sense in odd dimensions as well, and the stabilizer remains $\mathrm{SU}(n)$ in dimension $2n+1$. A $2n$-dimensional manifold with holonomy group $\mathrm{SU}(n)$ is called a Calabi-Yau. In $4n$ dimensions the subgroup $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$ has one invariant spinor in every even degree component $\mathcal{S}^{2j}$, $j = 0, \ldots, n$, so a total of $n+1$ invariant spinors, and a manifold with holonomy group $\mathrm{Sp}(n)$ is called hyperkähler. In $4n+3$ dimensions the group $\mathrm{Sp}(n)$ embedded into a $\mathrm{Spin}(4n)$ subgroup of $\mathrm{Spin}(4n+3)$ leaves $2n+2$ spinors invariant, one in every degree $\mathcal{S}^j$, $j = 0, \ldots, 2n+2$. They are constructed explicitly in the proof of Lemma A.4 below.

A.2.2 Some exceptional stabilizers

Consider n-dimensional spinor space, with n chosen such that Majorana spinors exist, and let ψ be a pure spinor. We can build a Majorana spinor out of ψ by adding its charge conjugate: $\tilde{\psi} = \psi + C\psi$. What is the stabilizer of $\tilde{\psi}$?

This question is most conveniently analyzed in the Lie algebra setting. The condition for $\tilde{\psi}$ to be annihilated by $\mathfrak{g} \subset \mathfrak{so}(n)$ is that

$$X_{\mu\nu}\gamma^{\mu\nu}\tilde{\psi} = 0 \qquad \forall X \in \mathfrak{g}, \tag{A.28}$$

where $X_{\mu\nu} = g_{\mu\kappa}X^\kappa{}_\nu$. The answer is then simple in dimensions greater than 5 (there are no Majorana spinors in dimensions 4 and 5 anyway) and different from 7, 8 and 9: $G_{\tilde{\psi}}$ conincides with G_ψ, i.e. $G_{\tilde{\psi}} = \mathrm{SU}(k)$ with $k = [\frac{n}{2}]$. First we have $G_{C\psi} = G_\psi$, since the charge conjugation commutes with $\mathfrak{spin}(n)$. Furthermore, it follows immediately that $\mathfrak{g}$ must annihilate ψ and $C\psi$ separately, unless the dimension is 7,8 or 9. This is because our explicit construction of the charge conjugation shows that if $\psi \in \Lambda^0 W$, then $C\psi \in \Lambda^k W$, and in even dimension $2k$ the spin algebra maps Λ^0 to $\Lambda^0 \oplus \Lambda^2$, and Λ^k to $\Lambda^{k-2} \oplus \Lambda^k$. In odd dimension $2k+1$ it maps Λ^0 to $\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2$

and Λ^k to $\Lambda^{k-2} \oplus \Lambda^{k-1} \oplus \Lambda^k$.

In dimension 7 a Majorana spinor is given by $\tilde{\psi} = 1 - iw^{123}$, where the w^j for $j = 1, 2, 3$ form a basis of $\mathbb{C}^3$. In the explicit spin representation the condition (A.28) translates into algebraic equations on the components $X_{\mu\nu}$ of $X \in \mathfrak{g}$:

$$\begin{aligned} X_{17} &= -X_{36} - X_{45}, & X_{27} &= X_{46} - X_{35}, \\ X_{37} &= X_{25} + X_{16}, & X_{47} &= X_{15} - X_{26}, \\ X_{57} &= -X_{14} - X_{23}, & X_{67} &= X_{24} - X_{13}, \\ & X_{12} + X_{34} + X_{56} = 0, \end{aligned} \tag{A.29}$$

which are just the defining equations of the Lie algebra $\mathfrak{g}_2$ inside $\mathfrak{so}(7)$ [39]. Hence, the stabilizer of a Majorana spinor in 7 dimensions is the exceptional Lie group G_2.

In dimension 8 a Majorana spinor is of the form $\tilde{\psi} = 1 + w^{1234}$, where w^j for $j = 1, \ldots, 4$ are a basis of $\mathbb{C}^4$. Equation (A.28) gives the following equations:

$$X_{12} + X_{34} + X_{56} + X_{78} = 0 \tag{A.30}$$

and

$$\begin{aligned} X_{13} - X_{24} - X_{57} + X_{68} &= 0 \\ X_{14} + X_{23} + X_{58} + X_{67} &= 0 \\ X_{15} - X_{26} + X_{37} - X_{48} &= 0 \\ X_{16} + X_{25} - X_{38} - X_{47} &= 0 \\ X_{17} - X_{28} - X_{35} + X_{46} &= 0 \\ X_{18} + X_{27} + X_{36} + X_{45} &= 0. \end{aligned} \tag{A.31}$$

These are the defining equations of $\mathfrak{spin}(7)$ inside $\mathfrak{spin}(8)$, and the stabilizer of a Majorana spinor in dimension 8 is Spin(7). The situation remains unchanged in dimension 9; the stabilizer of $\tilde{\psi}$ is Spin(7) embedded into a Spin(8)-subgroup of Spin(9).

A.3 The dilatino equation on a cylinder

In Chapters 5 and 6 we need to solve the dilatino equation $\gamma(d\phi - \frac{1}{2}H)\epsilon = 0$ on the cylinder over a real Killing spinor manifold M. In Chapter 5 the metric is only conformal to the cylinder metric, but the conformal factor can easily be taken into account separately, so here we focus exclusively on the cylinder. We solve the equation in a case by case treatment of the different Killing spinor geometries, using the notation of Section 3.3. For nearly Kähler, nearly parallel G_2 and Sasaki-Einstein manifolds the 3-form H is chosen in the form

$$H = a(\tau)P, \tag{A.32}$$

where τ is the cylinder variable, and P the canonical 3-form on M. On a 3-Sasakian manifold we consider a more general ansatz

$$H = a(\tau)\eta^{123} + b(\tau)\eta^{\alpha} \wedge \omega_{\alpha}. \tag{A.33}$$

The dilaton ϕ will be a function of τ only. Let $\mathfrak{k}$ be the Lie algebra of the structure group of M.

Lemma A.2. *For $X \in \mathfrak{k}$ acting on spinors we have*

$$[\gamma(P), X] = 0. \tag{A.34}$$

In the Sasaki-Einstein case the result extends to $X \in \mathfrak{u}(n)$, and in the 3-Sasakian case to $X \in \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_d$. Moreover, if M is 3-Sasakian, then the two quantized 3-forms $\gamma(\eta^{123})$ and $\gamma(\eta^{\alpha} \wedge \omega_{\alpha})$ separately commute with $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_d$.

Proof. A simple calculation in the Clifford algebra shows that

$$\big[\gamma(P), X\big] = \gamma\big(X \cdot P\big), \tag{A.35}$$

where $\cdot$ denotes the action of $\mathfrak{so}(n)$ on Λ^3, and we know that P is invariant under this action restricted to $\mathfrak{k}$. On a Sasaki-Einstein manifold we have $P = \eta \wedge \omega$, which is also $\mathfrak{u}(1)$-invariant, since $\omega \in \Lambda^2$ corresponds to the $\mathfrak{u}(1)$-generator under the isomorphism $\Lambda^2 \simeq \mathfrak{so}(2n+1)$. In the 3-Sasakian case we know that η^{123} and $\eta^{\alpha} \wedge \omega_{\alpha}$ are $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_{\mathrm{d}}$-invariant. □

Nearly parallel $\mathbf{G_2}$. Suppose first that M is a nearly parallel G_2-manifold. Then there is exactly one Majorana Killing spinor ϵ on M, which is G_2-invariant. It lifts to two G_2-invariant Majorana-Weyl spinors ϵ_1, ϵ_2 on the cylinder, of opposite chiralities. In fact, we can set $\epsilon_2 = \gamma(d\tau)\epsilon_1$. From the property that $\gamma(P)$ changes chirality and from Lemma A.2 we conclude that $\gamma(P)$ interchanges the two 8-dimensional spinors:

$$\gamma(P) \cdot \epsilon_j \;=\; \kappa_j \, \gamma(d\tau) \cdot \epsilon_j \tag{A.36}$$

for $j = 1, 2$ and $\kappa_j \in \mathbb{C}$. Applying the charge conjugation to (A.36) shows that the κ_j must be real. Furthermore, $\gamma(P)$ and $\gamma(d\tau)$ anticommute, which implies

$$\kappa_1 = -\kappa_2. \tag{A.37}$$

If $H = a(\tau)P$ then we can solve the dilatino equation either for ϵ_1 or ϵ_2 by defining

$$\phi_j(\tau) = \frac{\kappa_j}{2} \int_{\tau_0}^{\tau} a(t)dt. \tag{A.38}$$

Since $\kappa_1 \neq \kappa_2$ there is at most one spinor solving all BPS-equations, and the amount of supersymmetry preserved in the two-dimensional space orthogonal to the cylinder $cyl(M)$ can be at most $\mathcal{N} = 1$. An explicit calculation shows that $\kappa_j = \pm 7$.

Nearly Kähler. Let M be 6-dimensional nearly Kähler. There are two invariant Weyl spinors and both $\gamma(P)$ and $\gamma(d\tau)$ commute with $\mathfrak{su}(3)$. In seven dimensions there are still only two SU(3)-invariant Dirac spinors $\epsilon, \tilde{\epsilon}$, of opposite U(1)-charges, and we can set $\tilde{\epsilon}$ equal to the charge conjugate of ϵ. The fact that P is in a non-trivial $\mathfrak{u}(1)$-representation implies that $\gamma(P)$ interchanges ϵ and $\tilde{\epsilon}$, whereas $\gamma(d\tau)$ maps them to multiples of themselves. Since $\gamma(d\tau)$ squares to 1 and anticommutes with the charge conjugation, we must have

$$\gamma(d\tau) \cdot \epsilon = \sigma\epsilon, \qquad \gamma(d\tau) \cdot \tilde{\epsilon} = -\sigma\tilde{\epsilon}, \tag{A.39}$$

with $\sigma = \pm 1$. There are two SU(3)-invariant Majorana spinors in 7 dimensions:

$$\epsilon_1 = \epsilon + \tilde{\epsilon}, \qquad \epsilon_2 = i(\epsilon - \tilde{\epsilon}). \tag{A.40}$$

The operator $\gamma(P)$ anticommutes with both $\gamma(d\tau)$ and the charge conjugation, which implies that either again (A.36) holds, with $\kappa_1 = -\kappa_2 \in \mathbb{R}$, or

$$\gamma(P) \cdot \epsilon_1 = \kappa\gamma(d\tau) \cdot \epsilon_2 \qquad \text{and} \qquad \gamma(P) \cdot \epsilon_2 = \kappa\gamma(d\tau) \cdot \epsilon_1, \tag{A.41}$$

with the same constant κ. Contrary to the 8-dimensional case above we cannote use a chirality argument in favour of one or the other alternative here, but an explicit computation reveals that the first option is realized. One finds that $\kappa_1 = -\kappa_2 = \pm 4$, and the dilatino equation is solved by (A.38) for either ϵ_1 or ϵ_2, but not both at once.

Sasaki-Einstein. There are two Dirac spinors invariant under the complexification of the structure group SU(n), which are charge conjugate to each other. The commutant of $\mathfrak{su}(n)$ inside $\mathfrak{so}(2n+1)$ is $\mathfrak{u}(1)$, and the two spinors transform in dual $\mathfrak{u}(1)$-representations. Furthermore, the canonical 3-form $P = \eta \wedge \omega$ is invariant under all of $\mathfrak{u}(n)$, according to Lemma A.2. Hence $\gamma(P)$ commutes with $\mathfrak{u}(n)$, and the same is true for $\gamma(d\tau)$, so that both of them map the spinors to multiples of themselves. A direct calculation shows that (A.36) holds again with a real constant $\kappa_1 = \kappa_2$ (a priori the κ_j could also be purely imaginary, which would rule out solutions to the dilatino equation). In terms of Majorana spinors, where appropriate, there are two solutions to the dilatino equation. If the metric is given by

$$g = d\tau \otimes d\tau + \eta \otimes \eta + e^{2h(\tau)} g_Z \tag{A.42}$$

and the 3-form by $H = a(\tau)e^{2h(\tau)}\eta \wedge \omega$, then

$$\gamma(H) \cdot \epsilon = -na\gamma(d\tau) \cdot \epsilon, \tag{A.43}$$

and a solution of the dilatino equation is given by

$$\phi(\tau) = -\frac{n}{2} \int_{\tau_0}^{\tau} a(t) dt. \tag{A.44}$$

3-Sasakian. The situation is more complicated for a 3-Sasakian manifold M. There are $2n+2$ $\mathrm{Sp}(n)$-invariant spinors in dimension $4n+3$, which split into a $(n+2)$-dimensional and an n-dimensional irreducible representation of the subalgebra $\mathfrak{sp}(1)_d$ of $\mathfrak{so}(4n+3)$. The quantized 3-forms $\gamma(\eta^{123})$ and $\gamma(\eta^\alpha \wedge \omega_\alpha)$ commute with the action of $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)_d$ on spinor space, which by Schur's lemma implies that in $4n+3$ dimensions they act as scalars on the irreducible subrepresentation of $\mathfrak{sp}(1)_d$. From this one can deduce that in $4n+4$ dimensions the action of the two 3-forms is proportional to $\gamma(d\tau)$, at least on irreducible subrepresentation. More precisely, one finds the following

Lemma A.3. *The action of $\eta^\alpha \wedge \omega_\alpha$ and η^{123} on $\mathfrak{sp}(n)$-invariant spinors is as follows. For ϵ in $\underline{n+2}$:*

$$\begin{aligned} \gamma(\eta^\alpha \wedge \omega_\alpha) \cdot \epsilon &= -2n\gamma(d\tau) \cdot \epsilon \\ \gamma(\eta^{123}) \cdot \epsilon &= \gamma(d\tau) \cdot \epsilon, \end{aligned} \tag{A.45}$$

whereas for $\widetilde{\epsilon}$ in the $\underline{n}$:

$$\begin{aligned} \gamma(\eta^\alpha \wedge \omega_\alpha) \cdot \widetilde{\epsilon} &= 2(n+2)\gamma(d\tau) \cdot \widetilde{\epsilon} \\ \gamma(\eta^{123}) \cdot \widetilde{\epsilon} &= \gamma(d\tau) \cdot \widetilde{\epsilon}. \end{aligned} \tag{A.46}$$

The proof can be found in the following section. For a general $\mathfrak{sp}(n)$-invariant 3-form on $\mathbb{R} \times M$

$$H = a(\tau) e^{2h(\tau)} \eta^\alpha \wedge \omega_\alpha + b(\tau)\eta^{123} \tag{A.47}$$

and metric

$$g = d\tau \otimes d\tau + \eta^\alpha \otimes \eta^\alpha + e^{2h(\tau)} g_Z \tag{A.48}$$

on the cylinder, we can therefore define two functions

$$\begin{aligned} \phi_{n+2}(\tau) &= -n \int_{\tau_0}^{\tau} a(t)dt + \frac{1}{2}\int_{\tau_0}^{\tau} b(t)dt, \\ \phi_n(\tau) &= (n+2) \int_{\tau_0}^{\tau} a(t)dt + \frac{1}{2}\int_{\tau_0}^{\tau} b(t)dt, \end{aligned} \tag{A.49}$$

which satisfy

$$\begin{aligned} \gamma\big(d\phi_{n+2} - \tfrac{1}{2}H\big) \cdot \epsilon &= 0, \\ \gamma\big(d\phi_n - \tfrac{1}{2}H\big) \cdot \tilde{\epsilon} &= 0, \end{aligned} \tag{A.50}$$

for all ϵ in the image of $\underline{n+2}$ and $\tilde{\epsilon}$ in the image of $\underline{n}$. Unless $a = 0$ we cannot satisfy the dilatino equation $(d\phi - \frac{1}{2}H) \cdot \epsilon = 0$ for all $\mathfrak{sp}(n)$-invariant spinors simultaneously. Instead one has a choice whether to solve it for the $n+2$ spinors ϵ or the n spinors $\tilde{\epsilon}$.

A.4 Spinors on 3-Sasakian manifolds

This section contains the proofs of two statements made earlier, based on an explicit construction of the $\mathrm{Sp}(n)$-invariant spinors in dimension $4n+3$. Recall that the

$(4n+3)$-dimensional spin group has a subgroup $\mathrm{Sp}(n)\times\mathrm{Sp}(1)_d$, and that the $\mathrm{Sp}(n)$-invariant spinors transform in the $\underline{n+2}\oplus\underline{n}$-representation of $\mathrm{Sp}(1)_d$.

Lemma A.4. *Sp(n)-invariant spinors in the $\underline{n+2}$-representation of $Sp(1)_d$ are Killing spinors on M.*

Proof. $\mathrm{Sp}(n)$-invariance implies that the spinors are ∇^P-parallel, so we only have to show that they are annihilated by the difference of the canonical connection ∇^P and the Killing connection $\nabla - \frac{i}{2}e^\mu\gamma_\mu$:

$$\left(\nabla^P - \nabla + \tfrac{i}{2}e^\mu\gamma_\mu\right)\epsilon = 0. \tag{A.51}$$

Choose the following embedding of $\mathfrak{sp}(n)$ into $\mathfrak{so}(4n)$:

$$\mathfrak{s}p(n) = \left\{ \begin{pmatrix} A_x & -A_k & A_j & -A_i \\ A_k & A_x & -A_i & -A_j \\ -A_j & A_i & A_x & -A_k \\ A_i & A_j & A_k & A_x \end{pmatrix} \right\} \subset \mathfrak{so}(4n), \tag{A.52}$$

where $A = A_x + iA_i + jA_j + kA_k \in \mathbb{H}^{n\times n}$ satisfies $A^\dagger = -A$, the defining relation for $\mathfrak{sp}(n)$. The additional generators

$$I = \begin{pmatrix} 0 & -\mathbf{1}_{2n} \\ \mathbf{1}_{2n} & 0 \end{pmatrix},\quad J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix},\quad K = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}. \tag{A.53}$$

span the $\mathfrak{sp}(1)$-commutant of $\mathfrak{sp}(n)$. Furthermore we choose generators

$$\tilde{I} = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\quad \tilde{J} = 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad \tilde{K} = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.54}$$

of the $\mathfrak{so}(3)$ algebra orthogonal to $\mathfrak{so}(4n)$. The diagonal subalgebra $\mathfrak{sp}(1)_d$ is then generated by

$$\hat{I} = I + \tilde{I}, \qquad \hat{J} = J + \tilde{J}, \quad \hat{K} = K + \tilde{K}. \tag{A.55}$$

Now we group the $4n$ basis 1-forms of $\mathbb{R}^{4n}$ into 4 groups of n forms $e_1^j, e_2^j, e_3^j, e_4^j$, where $j = 1,\ldots,n$, denote again by η^1,η^2,η^3 the 3-dimensional 1-forms, and introduce complex coordinates using the complex structure $\hat{I}$ in the $4n+2$ dimensions orthogonal to η^1:

$$z_1^j = \frac{1}{2}(e_1^j + ie_3^j),\quad z_2^j = \frac{1}{2}(e_2^j + ie_4^j),\quad z = \frac{1}{2}(\eta^2 + i\eta^3). \tag{A.56}$$

We use the map from the odd-dimensional Clifford algebra $\mathrm{Cl}(4n+3,\mathbb{C})$ to $\mathrm{Cl}(4n+2,\mathbb{C})$ given in Table A.1, where we single out γ_1 to be mapped to the chirality operator Γ. Recall the splitting of spinor space into homogeneous components $\mathcal{S} =$

$\mathcal{S}^0 \oplus \mathcal{S}^1 \oplus \cdots \oplus S^{2n+1}$. There is one $\mathfrak{sp}(n)$-invariant spinor in every degree, explicitly given by

$$\begin{aligned}\widehat{\epsilon}^A &= \frac{1}{A!}\Big(\sum_{j=1}^n w_1^j \wedge w_2^j\Big)^A \qquad (\in \mathcal{S}^{2A}) \\ &= \sum_{1\le j_1<\cdots<j_A\le n} w_1^{j_1}\wedge w_2^{j_1}\wedge w_1^{j_2}\wedge\cdots\wedge w_2^{j_A} \\ \widehat{\epsilon}^{A,w} &= w\wedge\widehat{\epsilon}^A \qquad (\in \mathcal{S}^{2A+1}),\end{aligned} \tag{A.57}$$

where $A = 0, \ldots, n$, the $w_{1,2}^j$ corresponds to the 1-forms $z_{1,2}^j$, and w corresponds to z. Using the explicit spin representation one can show that the Cartan generator $\hat{I}$ of our diagonal $\mathfrak{sp}(1)$ algebra acts as

$$\hat{I}\widehat{\epsilon}^{A+1} = 4i(n-2A-1)\widehat{\epsilon}^{A+1}, \qquad \hat{I}\widehat{\epsilon}^{A,w} = 4i(n-2A-1)\widehat{\epsilon}^{A,w}, \tag{A.58}$$

so that the weight vectors of the irreducible $\underline{n}$ and $\underline{n+2}$-representations must be linear combinations of the type $a\epsilon^{A+1} + b\epsilon^{A,w}$. The precise form of these basis elements can be determined from the ladder operators

$$J^+ = \hat{J} + i\hat{K} = 8(\zeta - i\zeta_1^j\zeta_2^j), \qquad J^- = \hat{J} - i\hat{K} = 8(-\overline{\zeta} + i\overline{\zeta}_1^j\overline{\zeta}_2^j). \tag{A.59}$$

Here ζ, ζ_1^j and ζ_2^j are creation operators for w, w_1^j and w_2^j respectively, and $\overline{\zeta}, \overline{\zeta}_1^j$ and $\overline{\zeta}_2^j$ the corresponding annihilators. Let

$$\epsilon^A = i\widehat{\epsilon}^{A,w} + \widehat{\epsilon}^{A+1}, \qquad \tilde{\epsilon}^A = (n-A)\widehat{\epsilon}^{A,w} + i(A+1)\widehat{\epsilon}^{A+1}, \tag{A.60}$$

then we obtain

$$\begin{aligned}J^+\epsilon^A &= -8i(A+2)\epsilon^{A+1} \\ J^+\tilde{\epsilon}^A &= -8i(A+1)\tilde{\epsilon}^{A+1},\end{aligned} \tag{A.61}$$

using $\zeta_1^j\zeta_2^j\widehat{\epsilon}^A = (A+1)\widehat{\epsilon}^{A+1}$ and $\overline{\zeta}_1^j\overline{\zeta}_2^j\widehat{\epsilon}^A = -(n-A+1)\widehat{\epsilon}^{A-1}$. Of particular interest are the 'empty and filled state' spinors $1 \in \mathcal{S}^0$ and $\widehat{\epsilon}^{n,w} \in \mathcal{S}^{2n+1}$. They satisfy

$$J^+1 = -8i\epsilon^0, \qquad J^+\epsilon^{n-1} = 8(n+1)\widehat{\epsilon}^{n,w}, \tag{A.62}$$

whereas J^+ annihilates $\tilde{\epsilon}^{n-1}$. Thus the ϵ^A together with 1 and $\widehat{\epsilon}^{n,w}$ span the $\underline{n+2}$-representation, whereas the $\tilde{\epsilon}^A$ span the $\underline{n}$. As required, elements of the different irreps are orthogonal, which follows from $|\widehat{\epsilon}^A|^2 = |\widehat{\epsilon}^{A,w}|^2 = \binom{n}{A}$.

Now that we determined which spinors are in the $\underline{n+2}$- and $\underline{n}$-representations, we can go ahead and determine the condition on a ∇^P-parallel spinor to solve the Killing equation. Due to (3.71) this is equivalent to the two equations

$$e^b\big(J^a_{\alpha b}\gamma_a\gamma^\alpha + i\gamma_b\big)\epsilon = \eta^\alpha\big(\varepsilon_{\alpha\beta\gamma}\gamma^{\beta\gamma} + 2i\gamma_\alpha\big)\epsilon = 0, \tag{A.63}$$

where as usual latin indices a, b, c are $4n$-dimensional, and greek indices α, β take values $1, 2, 3$, corresponding to the η^α. In terms of creation and annihilation operators we have

$$\begin{aligned} e^b J^a_{\alpha b}\gamma_a\gamma^\alpha = e^j_a\Big\{\overline{\zeta}^j_1 - \zeta^j_1 - 2i\overline{\zeta}^j_2\zeta + 2i\zeta^j_2\overline{\zeta}\Big\} + e^j_b\Big\{\overline{\zeta}^j_2 - \zeta^j_2 + 2i\overline{\zeta}^j_1\zeta - 2i\zeta^j_1\overline{\zeta}\Big\} \\ + e^j_c\Big\{i(\zeta^j_1 + \overline{\zeta}^j_1) - 2\zeta^j_2\overline{\zeta} - 2\overline{\zeta}^j_2\zeta\Big\} + e^j_d\Big\{i(\zeta^j_2 + \overline{\zeta}^j_2) + 2\zeta^j_1\overline{\zeta} + 2\overline{\zeta}^j_1\zeta\Big\}, \end{aligned} \tag{A.64}$$

and finally

$$\begin{aligned} e^b\Big(J^a_{\alpha b}\gamma_a\gamma^\alpha + i\gamma_b\Big) = e^j_a\Big\{\overline{\zeta}^j_1(1+\Gamma) + \zeta^j_1(\Gamma-1) - 2i\overline{\zeta}^j_2\zeta + 2i\zeta^j_2\overline{\zeta}\Big\} \\ + e^j_b\Big\{\overline{\zeta}^j_2(1+\Gamma) + \zeta^j_2(\Gamma-1) + 2i\overline{\zeta}^j_1\zeta - 2i\zeta^j_1\overline{\zeta}\Big\} \\ + e^j_c\Big\{i\zeta^j_1(1-\Gamma) + i\overline{\zeta}^j_1(\Gamma+1) - 2\zeta^j_2\overline{\zeta} - 2\overline{\zeta}^j_2\zeta\Big\} \\ + e^j_d\Big\{i\zeta^j_2(1-\Gamma) + i\overline{\zeta}^j_2(1+\Gamma) + 2\zeta^j_1\overline{\zeta} + 2\overline{\zeta}^j_1\zeta\Big\} \end{aligned} \tag{A.65}$$

It is not too hard to check that this annihilates the n+2-generators ϵ^A, 1 and $\widehat{\epsilon}^{n,w}$, but not the n-generators $\tilde{\epsilon}^A$. The same is true for

$$\begin{aligned} \frac{1}{2}\eta^\alpha\varepsilon_{\alpha\beta\gamma}\gamma^{\beta\gamma} &= ie^1\big(\zeta\overline{\zeta} - \overline{\zeta}\zeta\big) + e^2\big(\overline{\zeta} - \zeta\big) + ie^3\big(\zeta + \overline{\zeta}\big), \\ \frac{1}{2}\eta^\alpha\Big(\varepsilon_{\alpha\beta\gamma}\gamma^{\beta\gamma} + 2i\gamma_\alpha\Big) &= ie^1\big(\zeta\overline{\zeta} - \overline{\zeta}\zeta + \Gamma\big) \\ &\quad + e^2\big(\overline{\zeta}(1+\Gamma) + \zeta(\Gamma-1)\big) + ie^3\big(\zeta(1-\Gamma) + \overline{\zeta}(1+\Gamma)\big), \end{aligned} \tag{A.66}$$

which proves the lemma. □

The explicit construction of a basis of the $\underline{n+2}$- and $\underline{n}$-representations of $\mathrm{Sp}(1)_d$ allows us to prove Lemma A.3 as well:

Proof of Lemma A.3. Spinor space in $4n+3$ dimensions embeds into $4n+4$ dimensions as described in Section A.1.4. In addition to the $w^j_{1,2}$ and w there is one more complex variable w^0 corresponding to $\frac{1}{2}(\eta^1 + id\tau)$. The spinors are lifted to $(4n+4)$-dimensional spinor space according to (A.25):

$$\widehat{\epsilon}^A \mapsto \widehat{\epsilon}^A, \qquad \widehat{\epsilon}^{A,w} \mapsto -iw^0 \wedge \widehat{\epsilon}^{A,w}. \tag{A.67}$$

In particular the n+2 and n-generators are

$$\begin{aligned} \dim 4n+3:\quad \epsilon^A &= i\widehat{\epsilon}^{A,w} + \widehat{\epsilon}^{A+1}, \\ \tilde{\epsilon}^A &= (n-A)\widehat{\epsilon}^{A,w} + i(A+1)\widehat{\epsilon}^{A+1} \\ \dim 4n+4:\quad \epsilon^A &= w^0 \wedge \widehat{\epsilon}^{A,w} + \widehat{\epsilon}^{A+1}, \\ \tilde{\epsilon}^A &= -i(n-A)w^0 \wedge \widehat{\epsilon}^{A,w} + i(A+1)\widehat{\epsilon}^{A+1}. \end{aligned} \tag{A.68}$$

We have the usual complex gamma matrices in $4n$ dimensions, additionally $\zeta = \frac{1}{2}(\gamma(\eta^2) + i\gamma(\eta^3))$ with corresponding spinor-variable w, and in $4n+4$ dimensions also $\zeta^0 = \frac{1}{2}(\gamma(\eta^1) + i\gamma(d\tau))$ with variable w^0. Let us introduce the abbreviations

$$P_1 = \eta^\alpha \wedge \omega_\alpha, \qquad P_2 = \eta^{123}. \tag{A.69}$$

The images of P_1 and P_2 in the Clifford algebra are (we consider $\mathrm{Cl}(4n+3)$ as a subset of $\mathrm{Cl}(4n+2,\mathbb{C})$)

$$\begin{aligned}
\dim 4n+3: \quad & \gamma(P_1) = i\Big(\overline{\zeta}_1^j\zeta_1^j + \overline{\zeta}_2^j\zeta_2^j - \zeta_1^j\overline{\zeta}_1^j - \zeta_2^j\overline{\zeta}_2^j\Big)\Gamma + 4\Big(\zeta\overline{\zeta}_1^j\overline{\zeta}_2^j - \overline{\zeta}\zeta_1^j\zeta_2^j\Big)\Gamma \\
& \gamma(P_2) = i\big(\zeta\overline{\zeta} - \overline{\zeta}\zeta\big)\Gamma \\
\dim 4n+4: \quad & \gamma(P_1) = i(\zeta^0 + \overline{\zeta}^0)\Big(\overline{\zeta}_1^j\zeta_1^j\overline{\zeta}_2^j\zeta_2^j - \zeta_1^j\overline{\zeta}_1^j - \zeta_2^j\overline{\zeta}_2^j\Big) + 4i\Big(\zeta\overline{\zeta}_1^j\overline{\zeta}_2^j - \overline{\zeta}\zeta_1^j\zeta_2^j\Big) \\
& \gamma(P_2) = i(\zeta^0 + \overline{\zeta}^0)\big(\zeta\overline{\zeta} - \overline{\zeta}\big)
\end{aligned} \tag{A.70}$$

In particular the action on the basis elements $\widehat{\epsilon}^A$ and $\widehat{\epsilon}^{A,w}$ is as follows:

$$\begin{aligned}
\dim 4n+3: \quad \gamma(P_1)\widehat{\epsilon}^A &= 2i(n-2A)\widehat{\epsilon}^A - 4(n-A+1)\widehat{\epsilon}^{A-1,w} \\
\gamma(P_1)\widehat{\epsilon}^{A,w} &= -2i(n-2A)\widehat{\epsilon}^{A,w} + 4(A+1)\widehat{\epsilon}^{A+1} \\
\gamma(P_2)\widehat{\epsilon}^A &= -i\widehat{\epsilon}^A, \qquad \gamma(P_2)\widehat{\epsilon}^{A,w} = -i\widehat{\epsilon}^{A,w} \\
\dim 4n+4: \quad \gamma(P_1)\widehat{\epsilon}^A &= 2i(n-2A)w^0 \wedge \widehat{\epsilon}^A - 4i(n-A+1)\widehat{\epsilon}^{A-1,w} \\
\gamma(P_1)\cdot w^0 \wedge \widehat{\epsilon}^{A,w} &= 2i(n-2A)\widehat{\epsilon}^{A,w} + 4i(A+1)w^0 \wedge \widehat{\epsilon}^{A,w} \\
\gamma(P_2)\widehat{\epsilon}^A &= -iw^0 \wedge \widehat{\epsilon}^A, \qquad \gamma(P_2)\cdot w^0 \wedge \widehat{\epsilon}^{A,w} = i\widehat{\epsilon}^{A,w}.
\end{aligned} \tag{A.71}$$

After all, we can conclude that

$$\begin{aligned}
\dim 4n+3: \quad & \gamma(P_1)\epsilon^A = 2in\epsilon^A \\
& \gamma(P_1)\tilde{\epsilon}^A = -2i(n+2)\tilde{\epsilon}^A \\
& \gamma(P_2)\epsilon^A = -i\epsilon^A, \qquad \gamma(P_2)\tilde{\epsilon}^A = -i\tilde{\epsilon}^A \\
\dim 4n+4: \quad & \gamma(P_1)\epsilon^A = -2n\gamma(d\tau)\epsilon^A \\
& \gamma(P_1)\tilde{\epsilon}^A = 2(n+2)\gamma(d\tau)\tilde{\epsilon}^A \\
& \gamma(P_2)\epsilon^A = \gamma(d\tau)\epsilon^A, \qquad \gamma(P_2)\tilde{\epsilon}^A = \gamma(d\tau)\tilde{\epsilon}^A.
\end{aligned} \tag{A.72}$$

Thus $\gamma(P_1)$ and $\gamma(P_2)$ are proportional to the identity on the irreducible representations of $\mathfrak{sp}(1)_d$ on $(4n+3)$-dimensional spinor space, in accordance with Schur's lemma. □

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