Aspects of analytic representation theory

Der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades
Doktor der Naturwissenschaften
Dr. rer. nat.
genehmigte Dissertation
von
Dipl.-Math. Christoph Lienau
geboren am 19. Dezember 1984 in Essen

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Tag der Promotion: 2. Februar 2012

Zusammenfassung

Diese kumulative Dissertation, basierend auf den drei Arbeiten *Analytic Dirac approximation for real linear algebraic Groups*, *Analytic factorization of Lie group representations* und *The fine structure of Fréchet representations*, behandelt Aspekte der analytischen Darstellungstheorie.
Das erste Kapitel enthält einen konstruktiven Beweis des Satzes von Nelson im Spezialfall von Darstellungen mit moderatem Wachstum von reell linear algebraischen Gruppen. Für eine reell algebraische Gruppe G sei $\mathcal{A}(G)$ die Algebra der analytischen Vektoren der links-regulären Darstellung von G auf dem Raum der superexponentiell abfallenden Funktionen. Wir geben eine explizite Dirac-Folge in $\mathcal{A}(G)$ an. Da $\mathcal{A}(G)$ den Raum E für jede Fréchet-Darstellung (π, E) mit moderatem Wachstum nach E^ω abbildet, liefert dies einen konstruktiven Beweis des Satzes von Nelson, dass der Raum der analytischen Vektoren E^ω dicht in E ist.
Das Hauptresultat des zweiten Kapitels ist ein Faktorisierungssatz für den Raum der analytischen Vektoren E^ω von Darstellungen mit moderatem Wachstum (π, E) einer reellen Lie Gruppe G auf einem Fréchet-Raum E, analog zu dem Faktorisierungssatz von Dixmier und Malliavin für glatte Vektoren. Es sei (Π, E) die integrierte Darstellung von (π, E), dann stimmt der Raum E^ω mit dem linearen Spann der Menge $\{\Pi(\mathcal{A}(G))E^\omega\}$ überein. Als Korollar erhalten wir, dass E^ω mit dem Raum der analytischen Vektoren des Laplace–Beltrami Operators auf G übereinstimmt.
Das dritte Kapitel untersucht das Wachstum von Darstellungen. Wir führen den Begriff der verallgemeinerten Skala ein um das Wachstum einer großen Klasse von Darstellungen zu messen. Dann definieren wir den Begriff der $[S]$-temperierten Darstellung für eine verallgemeinerte Skala $[S]$. Dies verallgemeinert den Begriff der Darstellung vom moderatem Wachstum. Des Weiteren geben wir eine äquivalente Beschreibung der Kategorien von glatten und analytischen $[S]$-temperierten Darstellungen als Kategorien von Algebrendarstellungen von Faltungsalgebren glatter und analytischer Funktionen.
Schlüsselworte: Analytische Vektoren, Analytische Darstellung, Faktorisierung von Darstellungen

Abstract

This cumulative dissertation is based on the three papers *Analytic Dirac approximation for real linear algebraic Groups*, *Analytic factorization of Lie group representations* and *The fine structure of Fréchet representations* and addresses different aspects of analytic representation theory.
The first chapter contains a constructive proof of Nelson's theorem for moderate growth representations of a real linear algebraic group. For a real linear algebraic group G let $\mathcal{A}(G)$ be the algebra of analytic vectors for the left regular representation of G on the space of superexponentially decreasing functions. We present an explicit Dirac sequence in $\mathcal{A}(G)$. Since $\mathcal{A}(G)$ maps E to E^ω for every Fréchet-representation (π, E) of moderate growth, this yields an constructive proof of a result of Nelson that the space of analytic vectors is dense in E.
The main result of the second chapter is an analytic factorization theorem for the space of analytic vectors E^ω of moderate growth representation (π, E) of a real Lie group G on a Fréchet space analogous to the factorization theorem of Dixmier and Malliavin for smooth vectors. Let (Π, E) be the integrated representation of (π, E), then the space E^ω is equal to the linear span of the set $\{\Pi(\mathcal{A}(G))E^\omega\}$. As a corollary we obtain that E^ω coincides with the space of analytic vectors for the Laplace–Beltrami operator on G.
The third chapter studies the growth of representations. We introduce the notion of a generalized scale to measure the growth of a large class of representations. Then we define the notion of a $[S]$-tempered representation for a generalized scale $[S]$. This generalizes the notion of a moderate growth representation. Moreover we describe the categories of smooth and analytic $[S]$-tempered representations equivalently as categories of algebra representations of convolution algebras of smooth and analytic functions.
Keywords: Analytic vectors, Analytic representation, Factorisation of representations

Contents

Introduction

This is a cumulative doctoral thesis. It is based on the paper *Analytic Dirac approximation for real linear algebraic Groups* [1], which has appeared in the Mathematische Annalen vol. 351, the paper *Analytic factorization of Lie group representations* [3] which is joint work with H. Gimperlein and B. Krötz and which has appeared in Journal of Functional Analysis vol. 262 und the preprint *The fine structure of Fréchet representations* [2] which is joint work with B. Krötz.

All papers deal with aspects of analytic representation theory, in fact most effort is devoted to a study of the space of analytic vectors.

The space of analytic vectors was introduced by Harish-Chandra in [8] in order to obtain a well behaved correspondence between representations of a real Lie group and representations of its Lie algebra.

For a representation (π, E) of a compact real Lie group G on a finite dimensional space E, the orbit map $\gamma_v : G \to E, g \mapsto \pi(g)v$ is smooth for every $v \in E$. We thus obtain a Lie algebra representation $(d\pi, E)$ of the Lie algebra $\mathfrak{g}$ of G where the derived representation $d\pi$ is defined by

$$d\pi(X) = \frac{d}{dt}\pi(\exp(tX)v)|_{t=0}, \quad X \in \mathfrak{g}.$$

The correspondence between representations of a compact Lie group and representations of its Lie algebra is a fundamental tool to understand the representations of a compact Lie group.

However, for representations of an arbitrary Lie group on not necessarily finite dimensional spaces the orbit map may not be smooth. Nevertheless it is still possible to obtain a correspondence between representations of the Lie group and representations of its Lie algebra.

Gårding observed in [9] that the space of smooth vectors E^∞, i.e the space of all $v \in E$ such that the orbit map γ_v is smooth, is a dense subset of E. Gårding associated to the representation (π, E) of G the representation $(d\pi, E^\infty)$ of $\mathfrak{g}$.

It was remarked by Harish-Chandra in [8] that this correspondence has the drawback that the closure of a $d\pi(\mathfrak{g})$ invariant subspace may not be G-invariant, since the Taylor series of a smooth function doesn't in general represent the function. Therefore Harish-Chandra restricted the representation further to the space of analytic vectors or, in his own terminology, *well behaved* vectors, i.e. the space of all vectors for which the orbit map is an analytic map.

Harish-Chandra proved that for representations of a semisimple Lie group on a Banach space there are enough analytic vectors: The space of analytic vectors is dense in the representation space. The general result of the denseness of the space of analytic vectors for representations of an arbitrary Lie group on a Banach space is due to Nelson [6]. He obtained a dense set of analytic vectors by convolution with the fundamental solution of the heat-equation.

The goal of the first chapter is to give a constructive proof of Nelson's theorem for the special case of moderate growth representations of real linear algebraic groups on Fréchet spaces.
Let G be an algebraic subgroup of $GL_n(\mathbb{R})$ and let (π, E) be a moderate growth representation of G on the Fréchet space E. We denote by $\mathcal{A}(G)$ the convolution algebra of analytic vectors for the left regular representation on the space of superexponentially decreasing functions on G. Note that the elements of $\mathcal{A}(G)$ are analytic functions. Since the representation is of moderate growth the algebra $\mathcal{A}(G)$ acts on E by

$$\Pi(f)v = \int_G f(x)\pi(x)v \; dx, \qquad (f \in \mathcal{A}(G), v \in E)$$

and furthermore E is mapped to E^ω under this action. Therefore it suffices to construct a Dirac sequence in $\mathcal{A}(G)$, since $\Pi(f_n)v$ tends to v for every Dirac sequence.
We define a positive analytic function on G by $|g| = \mathrm{tr}(gg^t)$. Let $t > 0$ then the family of functions (φ_t) on G defined by

$$\varphi_t : G \to \mathbb{R}, \quad g \mapsto C_t e^{-t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)},$$

with constants $C_t > 0$ such that $\|\varphi_t\|_{\mathrm{L}^1(G)} = 1$ is a Dirac sequence in $\mathcal{A}(G)$.

The main result of the second chapter is an analytic version of the theorem of Dixmier and Malliavin [4]. Their theorem is a strong refinement of the fact that the space of smooth vectors is dense in the representation space. They showed that for a representation (π, E) of a real Lie group G on a Fréchet space every smooth vector is a finite sum of elements of the form $\Pi(f)v$ where v is a smooth vector and $f \in C_c^\infty(G)$.
We show that for a moderate growth representation (π, E) of a Lie group G on a Fréchet space every analytic vector can be written as a finite sum of elements of the form $\Pi(f)v$ where v is an analytic vector and $f \in \mathcal{A}(G)$. However, the method of proof is different. We first proof the result for the case of $G = (\mathbb{R}, +)$.
As a key ingredient we use the following identity of entire functions

$$\alpha_\varepsilon(z)\cosh(\varepsilon z) + \beta_\varepsilon(z) = 1,$$

of functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z\,\mathrm{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z)\cosh(\varepsilon z)$ which belong to the Fourier image of $\mathcal{A}(\mathbb{R})$. This corresponds to an identity of the Fourier multiplication operators $\alpha_\varepsilon(i\partial)$,$\beta_\varepsilon(i\partial)$ and $\cosh(i\varepsilon\partial)$.
The operators $\alpha_\varepsilon(i\partial)$ and $\beta_\varepsilon(i\partial)$ are given by convolution with some $\kappa_\alpha, \kappa_\beta \in \mathcal{A}(\mathbb{R})$. Every analytic vector v lies in the domain of $\cosh(i\varepsilon\partial)$ for sufficiently small $\varepsilon > 0$, hence we may apply $\cosh(i\varepsilon\partial)$ to the orbit map $\gamma_v(g) = \pi(g)v$ and conclude that

$$(\cosh(i\varepsilon\partial)\ \gamma_v) * \kappa_\alpha + \gamma_v * \kappa_\beta = \gamma_v.$$

The theorem follows by evaluating in 0.
Due to the rigid nature of analytic functions it is not possible to lift the result from $(\mathbb{R}, +)$ to an arbitrary Lie group using coordinates as in the paper by Dixmier and Malliavin. The idea is to replace the operator $i\partial$ by the square root of the Laplace-Beltrami operator on a general Lie group and refine the functional calculus of Cheeger, Gromov and Taylor [7] for the Laplace-Beltrami operator in the special case of a Lie group. The general proof then mirrors the arguments for $(\mathbb{R}, +)$.
As a corollary we obtain a generalization of a result of Godman [10] that a vector v is analytic if and only if v is Δ-analytic, i.e there exists a $\varepsilon > 0$ such that for all continuous seminorms p on E one has

$$\sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!}\ p(\Delta^j v) < \infty.$$

The third chapter deals with the notion of growth of a representation. The first two chapters dealt with representations of moderate growth. In the third chapter we introduce the concept of a generalized scale on a Lie group to measure the growth of more general representations. By a generalized scale S on a real Lie group G we understand a set S of locally bounded positive functions on G which satisfies the following submultiplicativity property: For every $s \in S$ there exist functions $s', s'' \in S$ such that

$$s(gh) \leq s'(g)s''(h) \quad \forall g, h \in G.$$

This generalizes the scale structure of [5]. A generalized scale is a scale if it consists of one element.
The set of generalized scales comes equipped which a natural order relation.
Suppose that (π, E) is a representation of G on a locally convex topological vector space E with the property that for every continuous semi-norm p on E there exists a continuous semi-norm q such that the function

$$s_p(g) := \sup_{\substack{v\in E\\ q(v)\neq 0}} \frac{p(\pi(g)v)}{q(v)}, \quad g \in G$$

is locally bounded. Then

$$p(\pi(g)v) \leq s_p(g)q(v) \qquad (g \in G, v \in E).$$

The set of functions s_p is then a generalized scale and if it is dominated by a generalized scale S then the representation (π, E) is called $[S]$-tempered.
In particular, if there exists a scale s such that (π, E) is $[s]$-tempered, then we say that (π, E) is of moderate growth.
We study the categories of smooth and analytic $[S]$-tempered representations on Fréchet spaces.
Using the Dixmier and Malliavin theorem we can describe the category of smooth $[S]$-tempered representations equivalently as a category of algebra representation of a convolution algebra of smooth functions.
Likewise, using the result of the previous chapter, we obtain for scales s an equivalence between the category of analytic $[s]$-tempered representations and a category of algebra representation of a convolution algebra of analytic functions.
Examples of $[S]$-tempered representations include regular representations on spaces of weakly automorphic forms and Schrödinger representations of the Heisenberg group with non-unitary central character.

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Chapter 1

Analytic Dirac approximation for real linear algebraic Groups

1.1 Introduction

In this paper we provide an explicit Dirac sequence of superexponentially decreasing analytic functions on a linear algebraic group. This yields an elementary proof of a theorem of Nelson [2] that the space of analytic vectors is dense. In order to keep the exposition self contained we recall basic constructions from [5, 6].

Let (π, E) be a representation of a Lie group G on a Fréchet-space E. For a vector $v \in E$ we denote by γ_v the corresponding orbit map

$$\gamma_v : \ G \to E, \qquad g \mapsto \pi(g)v.$$

A vector $v \in E$ is called *analytic* if the orbit map γ_v is a real analytic E-valued map. We denote the space of all analytic vectors by E^ω.

Let $\mathbf{g}$ be a left invariant Riemannian metric on G. To $\mathbf{g}$ we associate a Riemannian distance d on G: $d(g)$ is defined as the infimum length of all arcs joining g and 1.

Let $\mathcal{R}(G)$ be the space of superexponentially decreasing smooth functions on G with respect to the distance d, i.e.

$$\mathcal{R}(G) = \left\{ f \in C(G) \mid \forall n \in \mathbb{N} : p_n(f) := \sup_{g \in G} |f(g)| e^{nd(g)} < \infty \right\}.$$

The space $\mathcal{R}(G)$ is a Fréchet algebra under convolution and is independent of the choice of the left invariant metric.

Let us assume that (π, E) is a *F-representation*, i.e. the representation is of moderate growth: For every continuous seminorm q on E exists a continuous seminorm q' and constants $C, c > 0$ such that

$$q\left(\pi(g)v\right) \leq Ce^{cd(g)} q'(v) \quad (\forall g \in G, \forall v \in E).$$

In particular every Banach representation is a F-representation.
Furthermore there exists a constant $r' > 0$ such that $\forall r > r'$

$$\int_G e^{-rd(g)}\ dg < \infty.$$

Hence there is a corresponding algebra representation Π of $\mathcal{R}(G)$ which is given by

$$\Pi(f)v = \int_G f(g)\pi(g)v\ dg \quad (f \in (G), v \in E).$$

We denote the space of analytic vectors $\mathcal{R}(G)^\omega$ for the left regular representation L by $\mathcal{A}(G)$. Let $G_\mathbb{C}$ be the complexification of G. A function $f \in \mathcal{R}(G)$ is in $\mathcal{A}(G)$ if and only if it satisfies the following two conditions.

1. There exists a neighbourhood $U \subset G_\mathbb{C}$ of 1 and a $F \in \mathcal{O}(U^{-1}G)$ with $F|_G = f$.
2. For every compact subset $Q \subset U$ we have $\sup_{k\in Q} p_n\left(L_k(F)\right) < \infty$ for all $n \in \mathbb{N}$.

Throughout this text we refer to these conditions as condition (1) and condition (2).
We define a positive function on $GL_n(\mathbb{R})$ by

$$|g| = \sqrt{\operatorname{tr}\left(g^t g\right)} \quad (g \in GL_n(\mathbb{R})).$$

Let K be the maximal compact subgroup $O(n)$ of $GL_n(\mathbb{R})$ and $K_\mathbb{C}$ its complexification. Then $|\cdot|$ is $K_\mathbb{C}$-bi-invariant and sub-multiplicative. Note that for a matrix $g = (a_{ij})_{1\le i,j\le n}$ we have $|g| = \sqrt{\sum_{1\le i,j\le n} a_{ij}^2}$. Hence $|\cdot|^2$ is holomorphic on $GL_n(\mathbb{C})$.
Let G be a real linear algebraic group, then G is a closed subgroup of some $GL_n(\mathbb{R})$.
We define a norm in the sense of [1] on G by

$$\|g\| = \max\{|g|, |g^{-1}|\}, \qquad (g \in G)\,.$$

For $t > 0$ we consider the function

$$\varphi_t : G \to \mathbb{R}, \quad g \mapsto C_t e^{-t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)},$$

with constants $C_t > 0$ such that $\|\varphi_t\|_{\mathrm{L}^1(G)} = 1$.
Recall that a sequence $(f_k)_{k>0}$ is called a Dirac sequence if it satisfies the following three conditions:

a. $f \geq 0$, $\forall k \in \mathbb{N}$

b. $\int_G f_k(g)\ dg = 1$, $\forall k \in \mathbb{N}$

c. For every $\varepsilon > 0$ and every neighbourhood U of 1 in G exists a $M \geq 1$ such that $\int_{G\setminus U} f_m(g)\ dg < \varepsilon$, $\forall m \geq M$.

We prove the following theorem

Theorem 1.1.1. *a.* $\varphi_t \in \mathcal{A}(G)$ *for all* $t > 0$.

b. The sequence $(\varphi_t)_{t>0}$ *forms a Dirac sequence.*

As a corollary we obtain a result of Nelson [2] for real linear algebraic groups.

Corollary 1.1.2. *Let* (π, E) *be a* F*-representation of a real linear algebraic group* G *on a Fréchet space* E*, then the space of analytic vectors* E^ω *is dense in* E*.*

Remark 1.1.3. In fact [5] every analytic vector is a finite sum of vectors of the form $\Pi(f)v$ with $f \in \mathcal{A}(G)$ and $v \in E$.

Remark 1.1.4. If G is a real reductive group then Theorem 1.1.1 holds even for

$$\varphi'_t : G \to \mathbb{R}, \quad g \mapsto C_t e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)},$$

with constants $C_t > 0$ such that $\|\varphi'_t\|_{\mathrm{L}^1(G)} = 1$.

1.2 Proofs

The function φ_t admits an holomorphic continuation to $G_\mathbb{C}$ which we also denote by φ_t, but φ_t does not satisfy condition (2) on the whole of $G_\mathbb{C}$.
We now describe for $GL_n(\mathbb{R})$ a $K_\mathbb{C} \times GL_n(\mathbb{R})$-invariant domain in $GL_n(\mathbb{C})$ where φ_t satisfies condition (2). It turns out that this domain is a subdomain of the *crown domain* Ξ [3, 4]. Therefore let $\Omega = \{\mathrm{diag}(d_1, \dots, d_n) \ : \ d_k \in \mathbb{R}, |d_k| < \frac{\pi}{4}, \forall\ k = 1 \dots, n\} \subset \mathbb{R}^{n^2}$.

Remark 1.2.1. Note that this Ω is not the same as in [4]. Let us denote by Ω_{ss} the Omega used in [4] for $SL_n(\mathbb{R})$. Then Ω is related to Ω_{ss} in the following way: Ω has the property that $\Omega_{ss} + \mathbb{R}e = \Omega + \mathbb{R}e$ with $e = \mathrm{diag}(1, \dots, 1)$. In other words, up to central shift the Omegas coincide.

We define Ξ_n by $\Xi_n = GL_n(\mathbb{R}) \exp\left(i\frac{1}{n+1}\Omega\right) K_\mathbb{C}$.

Remark 1.2.2. Let us remark that if G is a real reductive group, i.e. a closed subgroup of $GL_n(\mathbb{R})$ which is stable under transposition, then $d(g)$ and $\log\|g\|$ are comparable in the sense that there are constants $c_1, c_2 > 0$ and $C_1, C_2 \in \mathbb{R}$ such that

$$c_1 d(g) + C_1 \leq \log\|g\| \leq c_1 d(g) + C_2.$$

Hence we can give an alternative characterization of the space $\mathcal{R}(G)$ in terms of $\|\cdot\|$:

$$\mathcal{R}(G) = \left\{ f \in C(G) \mid \forall n \in \mathbb{N} : p_n(f) := \sup_{g \in G} \|g\|^n |f(g)| < \infty \right\}.$$

In the proof of the next proposition we need the following notations: For a matrix $g = (a_{ij})_{1\le i,j\le n}$ we denote by g_i the i-th column vector $(a_{1i},\dots,a_{ni})^t$ and for a vector $w\in\mathbb{C}^n$ we denote by $\|w\|_2$ the euclidean norm.

Proposition 1.2.3. *The function φ_t satisfies condition* (2) *on Ξ_n.*

Proof. Let $Q\subset\Xi_n$ be compact. We show that there exists a constant $C>0$ such that

$$|\varphi_t(gq)| \le e^{-C\|g\|^{4n}}, \qquad (\forall g\in G, \forall q\in Q). \tag{1.2.1}$$

There exists a $\Omega'\subset\Omega$ which satisfies the following properties.

a. $Q\subset GL_n(\mathbb{R})\exp(i\frac{1}{n+1}\Omega')K_{\mathbb{C}}$.

b. There exists a constant $C_1'>0$ such that for all $d=\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_n})\in\exp(i\frac{1}{n+1}\Omega')$ we have $\cos(2(\theta_{\alpha_1}+\ldots+\theta_{\alpha_{n+1}}))\ge C_1'$ for all $\alpha_j\in\{1,\dots,n\}$.

This implies that for $k=1,\dots,2n$ there exists a constant $C_1>0$ such that

$$\mathrm{Re}\left(|gq|^{2k}\right)\ge C_1|g|^{2k}, \quad (g\in GL_n(\mathbb{R}), q\in Q)\,. \tag{1.2.2}$$

Therefore, if $d=\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_n})\in\exp(i\Omega')$ and $g'\in GL_n(\mathbb{R})$ then

$$|g'd|^{2k}=\left(e^{2\theta_1 i}\|g_1'\|^2+\cdots+e^{2\theta_n i}\|g_n'\|^2\right)^k \tag{1.2.3}$$

Hence $\mathrm{Re}\left(|g'd|^{2k}\right)\ge C_1'|g'|^{2k}$ according to (b).
Let $q=hdk$ with $h\in GL_n(\mathbb{R})$ and $k\in K_{\mathbb{C}}$.
Then $\mathrm{Re}\left(|gq|^2\right)=\mathrm{Re}\left(|ghdk|^2\right)=\mathrm{Re}\left(|ghd|^2\right)\ge C_1'|gh|^2$. Since Q is compact there exists a constant $C_1>0$ such that $C_1'|gh|^2>C_1|g|^2$ for all $q\in Q$. Thus we obtain (1.2.2).
Likewise we can show that for $k=1,\dots,2n$ there exists a constant $C_2>0$ such that

$$\mathrm{Re}\left(\left|(gq)^{-1}\right|^{2k}\right)\ge C_2|g^{-1}|^{2k}, \quad (g\in GL_n(\mathbb{R}), q\in Q)\,. \tag{1.2.4}$$

Note that for $k=1,\dots,4n$ there exists a constant $C_3>0$ such that

$$\mathrm{Re}\left(\mathrm{tr}\,(gq)^k\right)\le|\,\mathrm{tr}\,(gq)\,|^k\le C_3|g|^k,\ (g\in GL_n(\mathbb{R}), q\in Q)\,. \tag{1.2.5}$$

Since $|g-1|^{4n}=(|g|^2-2\,\mathrm{tr}(g)+n)^{2n}$ we obtain the upper bound (1.2.1) for some $C>0$ by expanding the $2n$-th power and combining the estimates.
Since $GL_n(\mathbb{R})$ is real reductive Remark 3.2.8 implies that φ_t satisfies condition (2) on Ξ. $\square$

Hence $\varphi_t\in\mathcal{A}\,(GL_n(\mathbb{R}))$. Now we show that for every real linear algebraic group $G\subset GL_n(\mathbb{R})$ the functions φ_t are elements of $\mathcal{A}(G)$.

Proposition 1.2.4. *Let $G\subset GL_n(\mathbb{R})$ be a real linear algebraic group then $\varphi_t\in\mathcal{A}(G)$.*

Proof. The set $G_{\mathbb{C}} \cap \Xi_n$ is an open neighborhood of $1 \in G_{\mathbb{C}}$ to which φ_t extends holomorphically.
We give an upper bound for $d(g)$ which implies that φ_t satisfies (2) on this neighborhood.
Every algebraic group G can be decomposed as a semidirect product $G = \operatorname{Rad}_u G \rtimes L$ of a connected unipotent group $\operatorname{Rad}_u G$ and a reductive group L. We write $g = ur$ with u unipotent and r reductive, hence $d(g) = d(ur) \leq d(u) + d(r)$.
Remark 3.2.8 implies that there exists a constant $C > 0$ such that $d(r) \leq C \log(\|r\|) + C$.
Note that the unipotent radical $\operatorname{Rad}_u G$ is connected and u has a real logarithm. The path $\gamma(t) = \exp(t \log(u))$ connects 1 and $\log(u)$ and has length $|\log(u)|$, thus $d(u) \leq |\log(u)|$. Since $\log(u) = \sum_{k=0}^{n} \frac{(-1)^k (u-1)^k}{k}$ and $|u-1|^k \leq 1 + |u-1|^n \leq 1 + |u|^n$ for $k = 0, \ldots, n$ we obtain $|\log(u)| \leq 1 + n + n\|u\|^n$.
Let $J = D + N$ be the Jordan normal form of g with D a diagonal and N a nilpotent matrix and let $P \in GL_n(\mathbb{C})$ be the change of basis matrix. Since the Jordan-Chevalley decomposition is unique, $u = P(1 + D^{-1}N)P^{-1}$ and $r = PDP^{-1}$. Therefore $\|u\| \leq \|P\|^2(\|1\| + \|D^{-1}N\|) \leq \|P\|^2(\|1\| + \|D\|) \leq \|P\|^2(\|1\| + \|g\|)$. The last inequality follows from the fact that the sum of the absolute values of the squares of the eigenvalues is less or equal than the sum of the squares of the singular values. Likewise we obtain $\|r\| \leq \|P\|^2 \|g\|$. Since the column vectors of the matrices P and P^{-1} are chains of generalized eigenvectors of g we obtain $\|P\|^2 \leq n2^n \|g\|^2$.
Combining these estimates we obtain that there exists a constant $R > 0$ such that

$$e^{nd(g)} \leq Re^{R\|g\|^{3n}} \|g\|^R.$$

Hence φ_t satisfies condition (2) on $G_{\mathbb{C}} \cap \Xi_n$. $\square$

Proposition 1.2.5. *The family $(\varphi_t)_{t \geq 1}$ forms for $t \to \infty$ a Dirac sequence.*

Proof. Let V be a neighborhood of 0 in $\mathfrak{g}$ such that the exponential map is a diffeomorphism of V with some neighborhood U of 1 in G. Then

$$\int_G e^{-t^2\left(|g-1|^{4n} + |g^{-1}-1|^{4n}\right)} \, dg \geq \int_U e^{-t^2\left(|g-1|^{4n} + |g^{-1}-1|^{4n}\right)} \, dg.$$

The differential of exp at X is given by

$$dL_{\exp(X)} \circ \tfrac{1 - e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.$$

Therefore

$$\int_U e^{-t^2\left(|g-1|^{4n} + |g^{-1}-1|^{4n}\right)} \, dg = \int_V e^{-t^2\left(|e^X-1|^{4n} + |e^{-X}-1|^{4n}\right)} \left|\det\left(\tfrac{1-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}\right)\right| \, dX$$

There exists a constant $C > 0$ with

$$\left|\det\left(\tfrac{1-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}\right)\right| \geq C, \quad \forall X \in V.$$

Hence

$$\int_G e^{-t^2\left(|g-1|^{4n} + |g^{-1}-1|^{4n}\right)} \, dg \geq C \int_V e^{-t^2\left(|e^X-1|^{4n} + |e^{-X}-1|^{4n}\right)} \, dX$$

There exists a constant $C' > 0$ such that

$$|e^X - 1|^{4n} + |e^{-X} - 1|^{4n} \leq C'|X|^{4n}, \quad \forall X \in V.$$

Thus

$$\begin{aligned}\int_V e^{-t^2\left(|e^X-1|^{4n}+|e^{-X}-1|^{4n}\right)}\, dX &\geq \int_V e^{-t^2C'|X|^{4n}}\, dX \\ &= \int_V e^{-C'|tX|^{4n}}\, dX \\ &= \frac{1}{t^{\dim \mathfrak{h}}} \int_V e^{-C'|X|^{4n}}\, dX\end{aligned}$$

Therefore

$$\int_H e^{-t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)}\, dg \geq \int_V e^{-C'|X|^{4n}}\, dX \geq C_1 t^{-\dim \mathfrak{h}}$$

with $C_1 = C\int_V e^{-C'|X|^{4n}}\, dX < \infty$.
Let U be a neighborhood of 1 in G, there exists a constant $R > 0$ such that

$$|g-1|^{4n} + |g^{-1} - 1|^{4n} \geq R, \quad \forall g \in G\backslash U.$$

Hence

$$e^{-\frac{1}{2}t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} \leq e^{-\frac{1}{2}t^2R}, \quad \forall g \in G\backslash U.$$

Therefore

$$\begin{aligned}&\int_{G\backslash U} e^{-t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} dg \\ &= \int_{G\backslash U} e^{-\frac{1}{2}t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} e^{-\frac{1}{2}t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} dg \\ &\leq e^{-\frac{1}{2}t^2R} \int_{G\backslash U} e^{-\frac{1}{2}t^2\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} dg \\ &\leq e^{-\frac{1}{2}t^2R} \int_{G\backslash U} e^{-\frac{1}{2}\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} dg \\ &= C_2 e^{-\frac{1}{2}t^2R}\end{aligned}$$

with $C_2 = \int_{G\backslash U} e^{-\frac{1}{2}\left(|g-1|^{4n}+|g^{-1}-1|^{4n}\right)} dg < \infty$. Hence

$$\int_{G\backslash U} \varphi_t(g) dh \leq e^{-\frac{1}{2}t^2R} t^{-\dim \mathfrak{g}} \frac{C_2}{C_1}.$$

The expression on the right hand side tends to 0 as t tends to infinity. □

Lemma 1.2.6.

$$\Pi(\mathcal{A}(G))E \subset E^\omega$$

Proof. Let $f \in \mathcal{A}(G), v \in E$. Then the orbit map $\gamma_{\Pi(f)v}$ is given by

$$\begin{aligned}\gamma_{\Pi(f)v}(g) &= \pi(g)\int_H f(x)\pi(x)v\ d\mu(x)\\ &= \int_H f(x)\pi(gx)v\ d\mu(x)\\ &= \int_H f(g^{-1}x)\pi(x)v\ d\mu(x)\\ &= \pi\big(L_g(f)\big)v.\end{aligned}$$

Hence the orbit map is equal to the composition

$$G \to \mathcal{R}(G) \to E$$

Here the first arrow denotes the map $g \mapsto L_g(f)$ and the second the map $\varphi \mapsto \Pi(\varphi)v$. The first map in this composition is analytic and the last is linear. Hence the whole map is an analytic map from G to E. □

Theorem 1.2.7. *For every real linear algebraic group G exists an analytic Dirac sequence, i.e a Dirac sequence which members are elements of $\mathcal{A}(G)$.*

Proof. The sequence of functions $(\varphi_t)_{t\geq 1}$ on G provides a Dirac sequence, as we have seen in Proposition 1.2.5. □

Corollary 1.2.8. *Let (π, E) be a F-representation of a real linear algebraic group G on a Fréchet space E. Then the space E^ω of analytic vectors is dense in E.*

Proof. Let $v \in E$ and let $(\varphi_t)_{t\geq 1}$ be an analytic Dirac sequence. Then $\pi(\varphi_t)v$ is, according to Lemma 1.2.6, a sequence of analytic vectors which tends to v in E. □

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Chapter 2

Analytic factorization of Lie group representations

2.1 Introduction

Consider a category $\mathcal{C}$ of modules over a nonunital algebra $\mathcal{A}$. We say that $\mathcal{C}$ has the *factorization property* if for all $\mathcal{M} \in \mathcal{C}$,

$$\mathcal{M} \;=\; \mathcal{A} \cdot \mathcal{M} \;:=\; \operatorname{span}\{a \cdot m \;\mid\; a \in \mathcal{A},\; m \in \mathcal{M}\}\,.$$

In particular, if $\mathcal{A} \in \mathcal{C}$ this implies $\mathcal{A} = \mathcal{A} \cdot \mathcal{A}$.

Let (π, E) be a representation of a real Lie group G on a Fréchet space E. Then the corresponding space of smooth vectors E^∞ is again a Fréchet space. The representation (π, E) induces a continuous action Π of the algebra $C_c^\infty(G)$ of test functions on E given by

$$\Pi(f)v = \int_G f(g)\pi(g)v\; dg \quad (f \in \mathrm{C}_c^\infty(G), v \in E),$$

which restricts to a continuous action on E^∞. Hence the smooth vectors associated to such representations are a $C_c^\infty(G)$–module, and a result by Dixmier and Malliavin [3] states that this category has the factorization property.

In this article we prove an analogous result for the category of analytic vectors.

For simplicity, we outline our approach for a Banach representation (π, E). In this case, the space E^ω of analytic vectors is endowed with a natural inductive limit topology, and gives rise to a representation (π, E^ω). To define an appropriate algebra acting on E^ω, we fix a left–invariant Riemannian metric on G and let d be the associated distance function. The continuous functions $\mathcal{R}(G)$ on G which decay faster than $e^{-nd(g,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a $G \times G$–module under the left–right regular representation. We define $\mathcal{A}(G)$ to be the space of analytic vectors of this action. Both $\mathcal{R}(G)$ and $\mathcal{A}(G)$ form an algebra under convolution, and the action Π of $C_c^\infty(G)$ extends to give E^ω the structure of an $\mathcal{A}(G)$–module.

In this setting, our main theorem says that the category of analytic vectors for Banach representations of G has the factorization property. More generally, we obtain a result for F–representations:

Theorem 2.1.1. *Let G be a real Lie group and (π, E) an F–representation of G. Then*

$$\mathcal{A}(G) \;=\; \mathcal{A}(G) * \mathcal{A}(G)$$

and

$$E^\omega \;=\; \Pi(\mathcal{A}(G))\; E^\omega \;=\; \Pi(\mathcal{A}(G))\; E.$$

Let us remark that the special case of bounded Banach representations of $(\mathbb{R}, +)$ has been proved by one of the authors in [8]. The above factorization theorem is a crucial tool to understand the minimal analytic globalization of Harish–Chandra modules [5].
As a corollary of Theorem 2.1.1 we obtain that a vector is analytic if and only if it is analytic for the Laplace–Beltrami operator, which generalizes a result of Goodman [6] for unitary representations.
In particular, the theorem extends Nelson's result that $\Pi(\mathcal{A}(G))\, E^\omega$ is dense in E^ω [9]. Gårding had obtained an analogous theorem for the smooth vectors [4]. However, while Nelson's proof is based on approximate units constructed from the fundamental solution $\varrho_t \in \mathcal{A}(G)$ of the heat equation on G by letting $t \to 0^+$, our strategy relies on some more sophisticated functions of the Laplacian.
To prove Theorem 2.1.1, we first consider the case $G = (\mathbb{R}, +)$. Here the proof is based on the key identity

$$\alpha_\varepsilon(z) \cosh(\varepsilon z) + \beta_\varepsilon(z) = 1,$$

for the entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z \,\mathrm{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z) \cosh(\varepsilon z)$ on the complex plane [1]. We consider this as an identity for the symbols of the Fourier multiplication operators $\alpha_\varepsilon(i\partial)$, $\beta_\varepsilon(i\partial)$ and $\cosh(i\varepsilon\partial)$. The functions α_ε and β_ε are easily seen to belong to the Fourier image of $\mathcal{A}(\mathbb{R})$, so that $\alpha_\varepsilon(i\partial)$ and $\beta_\varepsilon(i\partial)$ are given by convolution with some $\kappa_\alpha^\varepsilon, \kappa_\beta^\varepsilon \in \mathcal{A}(\mathbb{R})$. For every $v \in E^\omega$ and sufficiently small $\varepsilon > 0$, we may also apply $\cosh(i\varepsilon\partial)$ to the orbit map $\gamma_v(g) = \pi(g)v$ and conclude that

$$(\cosh(i\varepsilon\partial)\ \gamma_v) * \kappa_\alpha^\varepsilon + \gamma_v * \kappa_\beta^\varepsilon = \gamma_v.$$

The theorem follows by evaluating in 0.
Unlike in the work of Dixmier and Malliavin, the rigid nature of analytic functions requires a global geometric approach in the general case. The idea is to refine the functional calculus of Cheeger, Gromov and Taylor [2] for the Laplace-Beltrami operator in the special case of a Lie group. Using this tool, the general proof then closely mirrors the argument for $(\mathbb{R}, +)$.
The article concludes by showing in Section 2.6 how our strategy may be adapted to solve some related factorization problems.

[1] Some basic properties of these functions and the Gaussian error function erf are collected in the appendix.

2.2 Basic Notions of Representations

For a Hausdorff, locally convex and sequentially complete topological vector space E we denote by $GL(E)$ the associated group of isomorphisms. Let G be a Lie group. By a *representation* (π, E) of G we understand a group homomorphism $\pi : G \to GL(E)$ such that the resulting action

$$G \times E \to E, \quad (g, v) \mapsto \pi(g)v,$$

is continuous. For a vector $v \in E$ we shall denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v,$$

the corresponding continuous orbit map.

If E is a Banach space, then (π, E) is called a *Banach representation.*

Remark 2.2.1. Let (π, E) be a Banach representation. The uniform boundedness principle implies that the function

$$w_\pi : G \to \mathbb{R}_+, \quad g \mapsto \|\pi(g)\|,$$

is a *weight*, i.e. a locally bounded submultiplicative positive function on G.

A representation (π, E) is called an *F-representation* if

- E is a Fréchet space.
- There exists a countable family of seminorms $(p_n)_{n\in\mathbb{N}}$ which define the topology of E such that for every $n \in \mathbb{N}$ the action $G \times (E, p_n) \to (E, p_n)$ is continuous. Here (E, p_n) stands for the vector space E endowed with the topology induced from p_n.

Remark 2.2.2. (a) Every Banach representation is an F-representation.

(b) Let (π, E) be a Banach representation and $\{X_n : n \in \mathbb{N}\}$ a basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra of G. Define a topology on the space of smooth vectors E^∞ by the seminorms $p_n(v) = \|d\pi(X_n)v\|$. Then the representation (π, E^∞) induced by π on this subspace is an F-representation (cf. [1]).

(c) Endow $E = C(G)$ with the topology of compact convergence. Then E is a Fréchet space and G acts continuously on E via right displacements in the argument. The corresponding representation (π, E), however, is not an F-representation.

2.2.1 Analytic vectors

If M is a complex manifold and E is a topological vector space, then we denote by $\mathcal{O}(M, E)$ the space of E-valued holomorphic maps. We remark that $\mathcal{O}(M, E)$ is a topological vector space with regard to the compact-open topology.

Let us denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}_\mathbb{C}$ its complexification. We assume that $G \subset G_\mathbb{C}$ where $G_\mathbb{C}$ is a Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Let us stress that this assumption is superfluous but simplifies notation and exposition. We denote by $\mathcal{U}_\mathbb{C}$ the set of open neighborhoods of $\mathbf{1} \in G_\mathbb{C}$.

If (π, E) is a representation, then we call a vector $v \in E$ *analytic* if the orbit map $\gamma_v : G \to E$ extends to a holomorphic map to some GU for $U \in \mathcal{U}_\mathbb{C}$. The space of all analytic vectors is denoted by E^ω. We note the natural embedding

$$E^\omega \to \lim_{U \to \{\mathbf{1}\}} \mathcal{O}(GU, E), \quad v \mapsto \gamma_v,$$

and topologize E^ω accordingly.

2.3 Algebras of superexponentially decaying functions

We wish to exhibit natural algebras of functions acting on F-representations. For that let us fix a left invariant Riemannian metric $\mathbf{g}$ on G. The corresponding Riemannian measure dg is a left invariant Haar measure on G. We denote by $d(g,h)$ the distance function associated to $\mathbf{g}$ (i.e. the infimum of the lengths of all paths connecting g and h) and set

$$d(g) := d(g, \mathbf{1}) \qquad (g \in G)\,.$$

Here are two key properties of $d(g)$, see [4]:

Lemma 2.3.1. *If $w : G \to \mathbb{R}_+$ is locally bounded and submultiplicative (i.e. $w(gh) \leq w(g)w(h)$), then there exist $c, C > 0$ such that*

$$w(g) \leq Ce^{cd(g)} \qquad (g \in G).$$

Lemma 2.3.2. *There exists $c > 0$ such that for all $C > c$, $\int e^{-Cd(g)}\ dg < \infty$.*

We introduce the space of *superexponentially decaying continuous functions* on G by

$$\mathcal{R}(G) := \left\{ \varphi \in C(G) \mid \forall n \in \mathbb{N} : \sup_{g \in G} |\varphi(g)|\ e^{nd(g)} < \infty \right\}.$$

It is clear that $\mathcal{R}(G)$ is a Fréchet space which is independent of the particular choice of the metric $\mathbf{g}$. A simple computation shows that $\mathcal{R}(G)$ becomes a Fréchet algebra under convolution

$$\varphi * \psi(g) = \int_G \varphi(x)\ \psi(x^{-1}g)\ dx \qquad (\varphi, \psi \in \mathcal{R}(G), g \in G)\,.$$

We remark that the left-right regular representation $L \otimes R$ of $G \times G$ on $\mathcal{R}(G)$ is an F-representation.

If (π, E) is an F-representation, then Lemma 2.3.1 and Remark 2.2.1 imply that

$$\Pi(\varphi)v := \int_G \varphi(g)\ \pi(g)v\ dg \qquad (\varphi \in \mathcal{R}(G), v \in E)$$

defines an absolutely convergent integral. Hence the prescription

$$\mathcal{R}(G) \times E \to E, \quad (\varphi, v) \mapsto \Pi(\varphi)v,$$

defines a continuous algebra action of $\mathcal{R}(G)$ (here continuous refers to the continuity of the bilinear map $\mathcal{R}(G) \times E \to E$).

Our concern is now with the analytic vectors of $(L \otimes R, \mathcal{R}(G))$. We set $\mathcal{A}(G) := \mathcal{R}(G)^\omega$ and record that

$$\mathcal{A}(G) = \lim_{U \to \{\mathbf{1}\}} \mathcal{R}(G)_U,$$

where

$$\mathcal{R}(G)_U = \left\{ \varphi \in \mathcal{O}(UGU) \mid \forall Q \Subset U \ \forall n \in \mathbb{N} : \sup_{g \in G} \sup_{q_1, q_2 \in Q} |\varphi(q_1 g q_2)| \ e^{nd(g)} < \infty \right\}.$$

It is clear that $\mathcal{A}(G)$ is a subalgebra of $\mathcal{R}(G)$ and that

$$\Pi(\mathcal{A}(G)) \ E \subset E^\omega$$

whenever (π, E) is an F-representation.

2.4 Some geometric analysis on Lie groups

Let us denote by $\mathcal{V}(G)$ the space of left-invariant vector fields on G. It is common to identify $\mathfrak{g}$ with $\mathcal{V}(G)$ where $X \in \mathfrak{g}$ corresponds to the vector field $\widetilde{X}$ given by

$$(\widetilde{X} f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)) \qquad (g \in G, f \in C^\infty(G)).$$

We note that the adjoint of $\widetilde{X}$ on the Hilbert space $L^2(G)$ is given by

$$\widetilde{X}^* = -\widetilde{X} - \operatorname{tr}(\operatorname{ad} X).$$

Note that $\widetilde{X}^* = -\widetilde{X}$ in case $\mathfrak{g}$ is unimodular. Let us fix an an orthonormal basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ with respect to $\mathbf{g}$. Then the Laplace–Beltrami operator $\Delta = d^* d$ associated to $\mathbf{g}$ is given explicitly by

$$\Delta = \sum_{j=1}^n (-\widetilde{X_j} - \operatorname{tr}(\operatorname{ad} X_j)) \ \widetilde{X_j}.$$

As $(G, \mathbf{g})$ is complete, Δ is essentially selfadjoint. We denote by

$$\sqrt{\Delta} = \int \lambda \ dP(\lambda)$$

the corresponding spectral resolution. It provides us with a measurable functional calculus, which allows to define

$$f(\sqrt{\Delta}) = \int f(\lambda) \ dP(\lambda)$$

as an unbounded operator $f(\sqrt{\Delta})$ on $L^2(G)$ with domain

$$D(f(\sqrt{\Delta})) = \left\{ \varphi \in L^2(G) \mid \int |f(\lambda)|^2 \, d\langle P(\lambda)\varphi, \varphi\rangle < \infty \right\}.$$

Let $c, \vartheta > 0$. We are going to apply the above calculus to functions in the space

$$\mathcal{F}_{c,\vartheta} = \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N,\vartheta}} |\varphi(z)| \, e^{c|z|} < \infty \right\},$$
$$\mathcal{W}_{N,\vartheta} = \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < N \} \cup \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < \vartheta |\operatorname{Re} z| \}.$$

The resulting operators are bounded on $L^2(G)$ and given by a symmetric and left invariant integral kernel $K_f \in C^\infty(G \times G)$. Hence there exists a convolution kernel $\kappa_f \in C^\infty(G)$ with $\kappa_f(x) = \overline{\kappa_f(x^{-1})}$ such that $K_f(x, y) = \kappa_f(x^{-1}y)$, and for all $x \in G$:

$$f(\sqrt{\Delta}) \; \varphi = \int_G K_f(x, y) \; \varphi(y) \; dy = \int_G \kappa_f(y^{-1}x) \; \varphi(y) \; dy = (\varphi * \kappa_f)(x).$$

A theorem by Cheeger, Gromov and Taylor [2] describes the global behavior:

Theorem 2.4.1. *Let $c, \vartheta > 0$ and $f \in \mathcal{F}_{c,\vartheta}$ even. Then $\kappa_f \in \mathcal{R}(G)$.*

We are going to need an analytic variant of their result.

Theorem 2.4.2. *Under the assumptions of the previous theorem: $\kappa_f \in \mathcal{A}(G)$.*

Proof. We only have to establish local regularity, as the decay at infinity is already contained in [2].

The Fourier inversion formula allows to express κ_f as an integral of the wave kernel:

$$\kappa_f(\cdot) = K_f(\cdot, \mathbf{1}) = f(\sqrt{\Delta}) \; \delta_{\mathbf{1}} = \int_{\mathbb{R}} \hat{f}(\lambda) \; \cos(\lambda\sqrt{\Delta}) \; \delta_{\mathbf{1}} \; d\lambda.$$

As we would like to employ $\| \cos(\lambda\sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} \leq 1$, we cut off a fundamental solution of Δ^k to write

$$\delta_{\mathbf{1}} = \Delta^k \varphi + \psi$$

for a fixed $k > \frac{1}{4}\dim(G)$ and some compactly supported $\varphi, \psi \in L^2$. Hence,

$$\Delta^l \kappa_f(\cdot) = \int_{\mathbb{R}} \hat{f}^{(2k+2l)}(\lambda) \; \cos(\lambda\sqrt{\Delta}) \; \varphi \; d\lambda + \int_{\mathbb{R}} \hat{f}^{(2l)}(\lambda) \; \cos(\lambda\sqrt{\Delta}) \; \psi \; d\lambda.$$

In the appendix we show the following inequality for all $n \in \mathbb{N}$ and some constants $C_n, R > 0$

$$|\hat{f}^{(l)}(\lambda)| \leq C_n \; l! \; R^l e^{-n|\lambda|}.$$

Using $\| \cos(\lambda\sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} \leq 1$ and the Sobolev inequality, we obtain

$$|\Delta^l \kappa_f(\cdot)| \leq C_1 \; (2l)! \; S^{2l}$$

for some $S > 0$. A classical result by Goodman [11] now implies that κ_f extends to holomorphic function on a complex neighborhood U of $\mathbf{1}$. By equivariance, $\kappa_f \in \mathcal{O}(GU)$. Left analyticity follows from $\kappa_f(x) = \overline{\kappa_f(x^{-1})}$, and Browder's theorem (Theorem 3.3.3 in [7]) then implies joint analyticity. The decay at infinity follows from [2]. $\square$

2.4.1 Regularized distance function

In the last part of this section we are going to discuss a holomorphic regularization of the distance function. Later on this will be used to construct certain holomorphic replacements for cut-off functions.
Consider the time–1 heat kernel $\varrho := \kappa_{e^{-\lambda^2}}$ and define $\tilde{d}$ on G by

$$\tilde{d}(g) := e^{-\Delta} d(g) = \int_G \varrho(x^{-1}g)\ d(x)\ dx.$$

Lemma 2.4.3. *There exist $U \in \mathcal{U}_{\mathbb{C}}$ and a constant $C_U > 0$ such that $\tilde{d} \in \mathcal{O}(GU)$ and for all $g \in G$ and all $u \in U$*

$$|\tilde{d}(gu) - d(g)| \leq C_U.$$

Proof. According to Theorem 2.4.2 the heat kernel ϱ admits an analytic continuation to a superexponentially decreasing function on GU for some bounded $U \in \mathcal{U}_{\mathbb{C}}$. This allows to extend $\tilde{d}$ to GU. To prove the inequality, we consider the integral

$$\bar{\varrho}(y) = \int_G \varrho(x^{-1}y)\ dx$$

as a holomorphic function of $y \in GU$. By the left invariance of the Haar measure and the normalization of the heat kernel, $\bar{\varrho} = 1$ on G, and hence on GU. Recall the triangle inequality on G: $|d(x) - d(g)| \leq d(x^{-1}g)$. This implies the uniform bound

$$\begin{aligned} \left|\tilde{d}(gu) - d(g)\right| &= \left|\int_G \varrho(x^{-1}gu)\ (d(x) - d(g))\ dx\right| \\ &\leq \int_G \left|\varrho(x^{-1}gu)\right|\ d(x^{-1}g)\ dx \\ &\leq \sup_{v\in U} \int_G \left|\varrho(x^{-1}v)\right|\ d(x^{-1})\ dx. \end{aligned}$$

□

2.5 Proof of the Factorization Theorem

Let (π, E) be a representation of G on a sequentially complete locally convex Hausdorff space and consider the Laplacian as an element

$$\Delta = \sum_{j=1}^{n} (-X_j - \mathrm{tr}(\mathrm{ad}\, X_j))\ X_j$$

of the universal enveloping algebra of $\mathfrak{g}$. A vector $v \in E$ will be called Δ-*analytic*, if there exists $\varepsilon > 0$ such that for all continuous seminorms p on E one has

$$\sum_{j=0}^{\infty} \frac{\varepsilon^j}{(2j)!}\ p(\Delta^j v) < \infty\,.$$

Lemma 2.5.1. *Let E be a sequentially complete locally convex Hausdorff space and $\varphi \in \mathcal{O}(U, E)$ for some $U \in \mathcal{U}_{\mathbb{C}}$. Then there exists $R = R(U) > 0$ such that for all continuous semi-norms p on E there exists a constant C_p such that*

$$p\left(\left(\widetilde{X_{i_1}} \cdots \widetilde{X_{i_k}}\varphi\right)(\mathbf{1})\right) \le C_p \; k! \; R^k$$

for all $(i_1, \dots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$.

Proof. There exists a small neighborhood of 0 in $\mathfrak{g}$ in which the mapping

$$\Phi : \mathfrak{g} \to E, \;\; X \mapsto \varphi(\exp(X)),$$

is analytic. Let $X = t_1 X_1 + \cdots + t_n X_n$. Because E is sequentially complete, Φ can be written for small X and t as

$$\Phi(X) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \left(\widetilde{X_{\alpha_1}} \cdots \widetilde{X_{\alpha_k}}\varphi\right)(\mathbf{1})\, t^{\alpha}.$$

As this series is absolutely summable, there exists a $R > 0$ such that for every continuous semi-norm p on E there is a constant C_p with

$$p\left(\left(\widetilde{X_{i_1}} \cdots \widetilde{X_{i_k}}\varphi\right)(\mathbf{1})\right) \le C_p \; k! \; R^k$$

for all $(i_1, \dots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$. □

As a consequence we obtain:

Lemma 2.5.2. *Let (π, E) be a representation of G on some sequentially complete locally convex Hausdorff space E. Then analytic vectors are Δ–analytic.*

In Corollary 2.5.6 we will see that the converse holds for F-representations.

Let (π, E) be an F-representation of G. Then for each $n \in \mathbb{N}$ there exists $c_n, C_n > 0$ such that

$$\|\pi(g)\|_n \le C_n \cdot e^{c_n d(g)} \qquad (g \in G),$$

where

$$\|\pi(g)\|_n := \sup_{\substack{p_n(v) \le 1 \\ v \in E}} p_n(\pi(g)v)\,.$$

For $U \in \mathcal{U}_{\mathbb{C}}$ and $n \in \mathbb{N}$ we set

$$\mathcal{F}_{U,n} = \left\{ \varphi \in \mathcal{O}(GU, E) \mid \forall Q \Subset U \; \forall \varepsilon > 0 : \; \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq))\; e^{-(c_n + \varepsilon) d(g)} < \infty \right\}.$$

We are also going to need the subspace of superexponentially decaying functions in $\bigcap_n \mathcal{F}_{U,n}$:

$$\mathcal{R}(GU,E) = \left\{ \varphi \in \mathcal{O}(GU,E) \mid \forall Q \Subset U \ \forall n, N \in \mathbb{N} : \ \sup_{g\in G} \sup_{q\in Q} p_n(\varphi(gq)) \ e^{Nd(g)} < \infty \right\}.$$

We record:

Lemma 2.5.3. *If $\kappa \in \mathcal{A}(G)_V$, then right convolution with κ is a bounded operator from $\mathcal{F}_{U,n}$ to $\mathcal{F}_{V,n}$ for all $n \in \mathbb{N}$.*

We denote by $\mathcal{C}_\varepsilon$ the power series expansion $\sum_{j=0}^{\infty} \frac{\varepsilon^{2j}}{(2j)!}\Delta^j$ of $\cosh(\varepsilon\sqrt{\Delta})$. Note the following consequence of Lemma 2.5.1:

Lemma 2.5.4. *Let $U, V \in \mathcal{U}_{\mathbb{C}}$ such that $V \Subset U$. Then there exists $\varepsilon > 0$ such that $\mathcal{C}_\varepsilon$ is a bounded operator from $\mathcal{F}_{U,n}$ to $\mathcal{F}_{V,n}$ for all $n \in \mathbb{N}$.*

As in the Appendix, consider the functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z \operatorname{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z)\cosh(\varepsilon z)$, which belong to the space $\mathcal{F}_{2\varepsilon,\vartheta}$. We would like to substitute $\sqrt{\Delta}$ into our key identity (2.7.3)

$$\alpha_\varepsilon(z)\cosh(\varepsilon z) + \beta_\varepsilon(z) = 1$$

and replace the hyperbolic cosine by its Taylor expansion.
We denote $\kappa_{\alpha_\varepsilon}$ and $\kappa_{\beta_\varepsilon}$ by $\kappa_\alpha^\varepsilon$ and κ_β^ε.

Lemma 2.5.5. *Let $U \in \mathcal{U}_{\mathbb{C}}$. Then there exist $\varepsilon > 0$ and $V \subset U$ such that for any $\varphi \in \mathcal{F}_{U,n}$, $n \in \mathbb{N}$,*

$$\mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon + \varphi * \kappa_\beta^\varepsilon = \varphi$$

holds as functions on GV.

Proof. Note that $\kappa_\alpha^\varepsilon, \kappa_\beta^\varepsilon \in \mathcal{A}(G)$ according to Theorem 2.4.2. We first consider the case $E = \mathbb{C}$ and $\varphi \in L^2(G)$. With $|\alpha_\varepsilon(z)\cosh(\varepsilon z)|$ being bounded, $\cosh(\varepsilon\sqrt{\Delta})$ maps its domain into the domain of $\alpha_\varepsilon(\sqrt{\Delta})$, and the rules of the functional calculus ensure

$$\varphi - \beta_\varepsilon(\sqrt{\Delta})\varphi = (\alpha_\varepsilon(\cdot)\cosh(\varepsilon\cdot))(\sqrt{\Delta})\varphi = (\cosh(\varepsilon\sqrt{\Delta})\varphi) * \kappa_\alpha^\varepsilon$$

in $L^2(G)$ for all $\varphi \in D(\cosh(\varepsilon\sqrt{\Delta}))$. For such φ, the partial sums of $\mathcal{C}_\varepsilon\varphi$ converge to $\cosh(\varepsilon\sqrt{\Delta})\varphi$ in $L^2(G)$, and hence almost everywhere. Indeed,

$$\begin{aligned}
&\left\| \cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{j=0}^{N} \frac{\varepsilon^{2j}}{(2j)!}\Delta^j\varphi \right\|_{L^2(G)}^2 \\
&= \int \left\langle dP(\lambda) \left(\cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{j=0}^{N} \frac{\varepsilon^{2j}}{(2j)!} \ \Delta^j\varphi \right), \cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{k=0}^{N} \frac{\varepsilon^{2k}}{(2k)!} \ \Delta^k\varphi \right\rangle \\
&= \int \left(\cosh(\varepsilon\lambda) - \sum_{k=0}^{N} \frac{(\varepsilon\lambda)^{2k}}{(2k)!} \right)^2 \ \langle dP(\lambda)\varphi, \varphi\rangle \\
&= \sum_{j,k=N+1}^{\infty} \int \frac{(\varepsilon\lambda)^{2j}}{(2j)!} \ \frac{(\varepsilon\lambda)^{2k}}{(2k)!} \ \langle dP(\lambda)\varphi, \varphi\rangle \ ,
\end{aligned}$$

and the right hand side tends to 0 for $N \to \infty$, because

$$\sum_{j,k=0}^{\infty} \int \frac{(\varepsilon\lambda)^{2j}}{(2j)!} \frac{(\varepsilon\lambda)^{2k}}{(2k)!} \langle dP(\lambda)\varphi, \varphi\rangle = \int \cosh(\varepsilon\lambda)^2 \langle dP(\lambda)\varphi, \varphi\rangle < \infty \ .$$

In particular, given $\varphi \in \mathcal{R}(GU, E)$ and $\lambda \in E'$, we obtain $\mathcal{C}_\varepsilon\lambda(\varphi) = \cosh(\varepsilon\sqrt{\Delta})\lambda(\varphi)$ almost everywhere and

$$\mathcal{C}_\varepsilon(\lambda(\varphi)) * \kappa_\alpha^\varepsilon + \lambda(\varphi) * \kappa_\beta^\varepsilon = \lambda(\varphi)$$

as analytic functions on G for sufficiently small $\varepsilon > 0$.

Since the above identity holds for all $\lambda \in E'$, we obtain

$$\mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon + \varphi * \kappa_\beta^\varepsilon = \varphi$$

on any connected domain GV, $\mathbf{1} \in V \subset U$, on which the left hand side is holomorphic.

Recall the regularized distance function $\tilde{d}(g) = e^{-\Delta} d(g)$ from Lemma 2.4.3, and set $\chi_\delta(g) := e^{-\delta \tilde{d}(g)^2}$ ($\delta > 0$). Given $\varphi \in \mathcal{F}_{U,n}$, $\chi_\delta\varphi \in \mathcal{R}(GU, E)$ and

$$\mathcal{C}_\varepsilon(\chi_\delta\varphi) * \kappa_\alpha^\varepsilon + (\chi_\delta\varphi) * \kappa_\beta^\varepsilon = \chi_\delta\varphi \, .$$

The limit $\chi_\delta\varphi \to \varphi$ in $\mathcal{F}_{U,n}$ as $\delta \to 0$ is easily verified. From Lemma 2.5.3 we also get $(\chi_\delta\varphi) * \kappa_\beta^\varepsilon \to \varphi * \kappa_\beta^\varepsilon$ as $\delta \to 0$. Finally Lemma 2.5.3 and Lemma 2.5.4 imply

$$\mathcal{C}_\varepsilon(\chi_\delta\varphi) * \kappa_\alpha^\varepsilon \to \ \mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon \quad (\delta \to 0).$$

The assertion follows. □

Proof of Theorem 2.1.1. Given $v \in E^\omega$, the orbit map γ_v belongs to $\bigcap_n \mathcal{F}_{U,n}$ for some $U \in \mathcal{U}_\mathbb{C}$. Applying Lemma 2.5.5 to the orbit map and evaluating at $\mathbf{1}$ we obtain the desired factorization

$$v = \gamma_v(\mathbf{1}) = \Pi(\kappa_\alpha^\varepsilon)\,(\mathcal{C}_\varepsilon(\gamma_v)(\mathbf{1})) + \Pi(\kappa_\beta^\varepsilon)\,(\gamma_v(\mathbf{1}))\,.$$

□

Note the following generalization of a theorem by Goodman for unitary representations [6, 11].

Corollary 2.5.6. *Let (π, E) be an F-representation. Then every Δ–analytic vector is analytic.*

Remark 2.5.7. a) A further consequence of our Theorem 2.1.1 is a simple proof of the fact that the space of analytic vectors for a Banach representation is complete.
b) We can also substitute $\sqrt{\Delta}$ into Dixmier's and Malliavin's presentation of the constant function 1 on the real line [3]. This invariant refinement of their argument shows that the smooth vectors for a Fréchet representation are precisely the vectors in the domain of Δ^k for all $k \in \mathbb{N}$.

2.6 Related Problems

We conclude this article with a discussion of how our techniques can be modified to deal with a number of similar questions.
In the context of the introduction, given a nonunital algebra $\mathcal{A}$, a category $\mathcal{C}$ of $\mathcal{A}$–modules is said to have the *strong factorization property* if for all $\mathcal{M} \in \mathcal{C}$,

$$\mathcal{M} = \{am \ \mid \ a \in \mathcal{A},\ m \in \mathcal{M}\}.$$

2.6.1 A Strong Factorization of Test Functions

Our methods may be applied to solve a related strong factorization problem for test functions. On $\mathbb{R}^n$ the Fourier transform allows to write a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ as the convolution $\psi*\Psi$ of two Schwartz functions, and [10] posed the natural problem whether one could demand $\psi, \Psi \in \mathcal{R}(\mathbb{R}^n)$. We are going to prove this in a more general setting.

Theorem 2.6.1. *For every real Lie group G*

$$C_c^\infty(G) \subset \{\psi * \Psi \ \mid \ \psi, \Psi \in \mathcal{R}(G)\}\,.$$

As above, we first regularize an appropriate distance function and set

$$(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * \log(1 + |z|).$$

Lemma 2.6.2. *The function (z) is entire and approximates $\log(1+|z|)$ in the sense that for all $N > 0$, $\vartheta \in (0,1)$ there exists a constant $C_{N,\vartheta}$ such that*

$$|(z) - \log(1 + |z|)| \leq C_{N,\vartheta} \quad (z \in \mathcal{W}_{N,\vartheta}).$$

Let $m \in \mathbb{N}$. We would like to substitute the square root of the Laplacian associated to a left invariant metric G into a decomposition

$$1 = \widehat{\psi}_m(z)\ \widehat{\Psi}_m(z)$$

of the identity. In the current situation we use $\widehat{\psi}_m(z) = e^{-m(z)}$ and $\widehat{\Psi}_m(z) = e^{m(z)}$. Denote the convolution kernels of $\widehat{\psi}_m(\sqrt{\Delta})$ and $\widehat{\Psi}_m(\sqrt{\Delta})$ by ψ_m resp. Ψ_m. The ideas from the proof of Theorem 2.4.2 may be combined with the results of [2] to obtain:

Lemma 2.6.3. *Let $\chi \in C_c^\infty(G)$ with $\chi = 1$ in a neighborhood of $\mathbf{1}$. Then $\chi\Psi_m$ is a compactly supported distribution of order m and $(1-\chi)\Psi_m \in \mathcal{R}(G) \cap C^\infty(G)$. Given $k \in \mathbb{N}$, $\psi_m \in \mathcal{R}(G) \cap C^k(G)$ for sufficiently large m.*

Therefore $\widehat{\Psi}_m(\sqrt{\Delta})$ maps $C_c^\infty(G)$ to $\mathcal{R}(G)$. The functional calculus leads to a factorization

$$\mathrm{Id}_{C_c^\infty(G)} = \widehat{\psi}_m(\sqrt{\Delta})\ \widehat{\Psi}_m(\sqrt{\Delta})$$

of the identity, and in particular for any $\varphi \in C_c^\infty(G)$,

$$\varphi = (\widehat{\Psi}_m(\sqrt{\Delta})\ \varphi) * \psi_m \in \mathcal{R}(G) * \mathcal{R}(G).$$

2.6.2 Strong Factorization of $\mathcal{A}(G)$

It might be possible to strengthen Theorem 2.1.1 by showing that the analytic vectors have the strong factorization property.

Conjecture 2.6.4. *For any F-representation (π, E) of a real Lie group G,*

$$E^\omega = \{\Pi(\varphi)v \mid \varphi \in \mathcal{A}(G),\ v \in E^\omega\}.$$

We provide some evidence in support of this conjecture and verify it for Banach representations of $(\mathbb{R}, +)$ using hyperfunction techniques.

Lemma 2.6.5. *The conjecture holds for every Banach representation of* $(\mathbb{R}, +)$.

Proof. Let (π, E) be a representation of $\mathbb{R}$ on a Banach space $(E, \|\cdot\|)$. Then there exist constants $c, C > 0$ such that $\|\pi(x)\| \leq Ce^{c|x|}$ for all $x \in \mathbb{R}$. If $v \in E^\omega$, there exists $R > 0$ such that the orbit map γ_v extends holomorphically to the strip $S_R = \{z \in \mathbb{C} \mid \operatorname{Im} z \in (-R, R)\}$. Let

$$\begin{aligned} \mathcal{F}_+ (\gamma_v)(z) &= \int_{-\infty}^{0} \gamma_v(t) e^{-itz}\, dt, \quad \operatorname{Im} z > c, \\ -\mathcal{F}_- (\gamma_v)(z) &= \int_{0}^{\infty} \gamma_v(t) e^{-itz}\, dt, \quad \operatorname{Im} z < -c. \end{aligned}$$

Define the Fourier transform $\mathcal{F}(\gamma_v)$ of γ_v by

$$\mathcal{F}(\gamma_v)(x) = \mathcal{F}_+(\gamma_v)(x + 2ic) - \mathcal{F}_-(\gamma_v)(x - 2ic).$$

Note that $\|\mathcal{F}(\gamma_v)(x)\|\, e^{r|x|}$ is bounded for every $r < R$. Let $g(z) := \frac{Rz}{2}\operatorname{erf}(z)$ and write $\mathcal{F}(\gamma_v)$ as

$$\mathcal{F}(\gamma_v) = e^{-g} e^{g} \mathcal{F}(\gamma_v) \tag{2.6.1}$$

Define the inverse Fourier transform $\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))$ for $x \in \mathbb{R}$ by

$$\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))(x) = \int_{\operatorname{Im} t = 2c} \mathcal{F}_+(\gamma_v)(t) e^{itx}\, dt - \int_{\operatorname{Im} t = -2c} \mathcal{F}_-(\gamma_v)(t) e^{itx}\, dt.$$

Applying the inverse Fourier transform to both sides of (2.6.1) and evaluating at 0 yields

$$v = (2\pi)^{-1} \Pi\left(\mathcal{F}^{-1}\left(e^{-g}\right)\right)\left(\mathcal{F}^{-1}\left(e^{g}\mathcal{F}(\gamma_v)\right)(0)\right).$$

The assertion follows because $\mathcal{F}^{-1}\left(e^{-g}\right) \in \mathcal{A}(\mathbb{R})$. $\square$

Strong factorization likewise holds for Banach representations of $(\mathbb{R}^n, +)$. Using the Iwasawa decomposition we are able to deduce from this the conjecture for $SL_2(\mathbb{R})$.

2.7 Appendix

2.7.1 An Identity of Entire Functions

Consider the following space of exponentially decaying holomorphic functions

$$\mathcal{F}_{c,\vartheta} = \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N,\vartheta}} |\varphi(z)| \, e^{c|z|} < \infty \right\},$$
$$\mathcal{W}_{N,\vartheta} = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < N\} \cup \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \vartheta |\operatorname{Re} z|\}.$$

To understand the convolution kernel of a Fourier multiplication operator on $L^2(\mathbb{R})$ with symbol in $\mathcal{F}_{c,\vartheta}$, or more generally functions of $\sqrt{\Delta}$ on a manifold as in Section 2.4, we need some properties of the Fourier transformed functions.

Lemma 2.7.1. *Given $f \in \mathcal{F}_{c,\vartheta}$, there exist $C, R > 0$ such that*

$$|\hat{f}^{(k)}(z)| \leq C_n \; k! \; R^k e^{-n|z|}$$

for all $k, n \in \mathbb{N}$.

Proof. Given $f \in \mathcal{F}_{c,\vartheta}$, the Fourier transform extends to a superexponentially decaying holomorphic function on $\mathcal{W}_{c,\vartheta}$. It follows from Cauchy's integral formula that

$$|\hat{f}^{(k)}(z)| \leq C_n \; k! \; R^k e^{-n|z|}$$

for all $k, n \in \mathbb{N}$. □

Some important examples of functions in $\mathcal{F}_{c,\vartheta}$ may be constructed with the help of the Gaussian error function [12]

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} \; dt.$$

The error function extends to an odd entire function, and $\operatorname{erf}(z) - 1 = O(z^{-1}e^{-z^2})$ as $z \to \infty$ in a sector $\{|\operatorname{Im} z| < \vartheta \operatorname{Re} z\}$ around $\mathbb{R}_+$.

Remark 2.7.2. The function

$$z \operatorname{erf}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * |z| - \frac{1}{\sqrt{\pi}} e^{-z^2}$$

is just one convenient regularization of the absolute value $|z|$, and the basic properties we need also hold for other similarly constructed functions. For example replace the heat kernel $\frac{1}{\sqrt{\pi}} e^{-z^2}$ by a suitable analytic probability density.

For any $\varepsilon > 0$, some algebra shows that the even entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z \operatorname{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z) \cosh(\varepsilon z)$ decay exponentially as $z \to \infty$ in $\mathcal{W}_{N,\vartheta}$ for any $\vartheta < 1$. Hence $\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{F}_{2\varepsilon,\vartheta}$. Our later factorization hinges on a multiplicative decomposition of the constant function 1:

Lemma 2.7.3. *For all $\varepsilon > 0, \vartheta \in (0,1)$, the functions $\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{F}_{2\varepsilon,\vartheta}$ satisfy the identity*

$$\alpha_\varepsilon(z) \cosh(\varepsilon z) + \beta_\varepsilon(z) = 1.$$

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[12] see e.g. http://functions.wolfram.com

Chapter 3

The fine structure of Fréchet representations

3.1 Motivation

A convenient category of representations is the category of moderate growth representations [1]. However, there are naturally arising representations with more general growth behaviour. One important example which we have in mind is the regular representation on spaces of weakly automorphic forms:

Let us consider the basic case $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$.

We denote the image of an element z of the upper half-plane $\mathbb{H}$ under the usual action of an element $g \in G$ by $g(z)$ and the corresponding factor of automorphy by $\mu(g, z)$. For a modular function f of weight m on $\mathbb{H}$ we define a Γ-invariant function F on G by

$$F(g) = \mu(g, i)^{-m} f(g(i)), \quad g \in G.$$

If f is a holomorphic cusp-form then $F \in L^2(\Gamma\backslash G)$. We denote the closure of the set of right translates of F by V_f. Then V_f is an irreducible G-submodule of $L^2(\Gamma\backslash G)$ under the right-regular representation which belongs to the holomorphic discrete series of G.

For $g \in G$ let $\|g\| = \operatorname{tr}(gg^t)^{1/2}$. Note that $\|\cdot\|$ is submultiplicative. Let f be a weakly holomorphic modular form, i.e. it is meromorphic at ∞. There exists a constant $C > 0$ such that for every $\alpha \geq 1$

$$|F(g)| \leq e^{C\|g\|^{2\alpha}} \quad g \in G.$$

Hence F is an element of the space

$$E := \bigcap_{\lambda \in (2,\infty)} L^2(\Gamma\backslash G, e^{-\|g\|^\lambda} dg).$$

We denote the natural norm on $L^2(\Gamma\backslash G, e^{-\|g\|^\lambda} dg)$ by p_λ.

Young's inequality and the fact that the function $\|\cdot\|$ is submultiplicative imply that for every

$\lambda \in (2,\infty)$ there exists a $\lambda' \in (2,\infty)$ and a $\alpha > 0$ such that the right-regular representation (R, V_f) satisfies

$$p_\lambda(R(g)v) \leq e^{\|g\|^\alpha} p_{\lambda'}(v), \quad v \in V_f.$$

The family $\{e^{\|g\|^\alpha} \mid \alpha > 0\}$ describes the growth behaviour of the representation. We say that (R, V_f) is tempered with respect to the generalized scale $\{e^{\|g\|^\alpha} \mid \alpha > 0\}$.
We introduce the notion of a generalized scale which allows us to measure the growth of a large class of representations. Moreover we define the category of tempered representations with respect to a generalized scale.
We want to draw attention to the paper [2] which has introduced a related notion of temperedness for nilpotent groups.

3.2 Scale structures on Lie groups

Throughout this text G shall denote a Lie group.

By a *scale function* on G we understand a locally bounded positive valued function $s : G \to \mathbb{R}^+$ which is submultiplicative, i.e it satisfies $s(gh) \leq s(g)s(h)$ for all $g, h \in G$.
We define an ordering $\prec$ on the set of scale functions by setting $s \prec s'$ provided there exists $C > 0$ and $N \in \mathbb{N}$ such that $s(g) \leq C(s'(g))^N$ for all $g \in G$. We say that s and s' are *equivalent* provided that $s \prec s'$ and $s' \prec s$. By a *scale structure* on a Lie group we understand an equivalence class $[s]$ of a scale function s on G. Here are some standard scale structures on Lie groups.

3.2.0.1 The geometric scale structure

Suppose that G is connected. We endow the Lie group G with a left-invariant Riemannian metric $\mathbf{g}$. Associated to $\mathbf{g}$ we obtain a notion of length on G, that is a function $d : G \to \mathbb{R}_0^+$ with $d(g)$ the infimum of the length of curves joining 1 and $g \in G$. Note that

- $d(g) = d(g^{-1})$ for all $g \in G$.
- $d(gh) \leq d(g) + d(h)$.

It follows in particular that $s_{\text{geo}}(g) := e^{d(g)}$ is a scale structure. We remark that s_{geo} is maximal, i.e. $s \prec s_{\text{geo}}$ for all scale functions s.

3.2.0.2 The algebraic scale structure

Suppose that G is an algebraic real Lie group. We fix a faithful algebraic representation $\iota : G \to GL(n, \mathbb{R})$ and define for $g \in G$

$$\|g\| := \operatorname{tr}(\iota(g)\iota(g)^t) + \operatorname{tr}(\iota(g^{-1})\iota(g^{-1})^t).$$

Note that $\|\cdot\|$ is a scale function on G whose equivalence class is independent of the given faithful representation ι. The resulting scale structure is refered to as the *algebraic scale structure*.

Remark 3.2.1. If G is a connected real reductive group, then the algebraic and maximal scale structure coincide. In general however, this is not the case. For instance if $G = (\mathbb{R}, +)$ then algebraic and maximal scale structure are different, e.g. polynomial versus exponential growth. Similarly, for $G = \mathbb{H}_n$ the $2n + 1$-dimensional Heisenberg group, algebraic and maximal scale structure are different.

3.2.1 Representations of controllable growth

Throughout this text all topological vector spaces considered are assumed to be locally convex, Hausdorff and sequentially complete. By a *representation* of the Lie group G on the topological vector space E we understand a continuous linear action

$$G \times E \to E, \quad (g, v) \mapsto g \cdot v\,.$$

Let us emphasize that continuity is requested in both variables. The continuous action gives rise to a group homomorphism

$$\pi : G \to GL(E);\ \pi(g)v := g \cdot v$$

with $GL(E)$ the group of linear isomorphisms of E. It is common to use the notation (π, E) for representations.

We write $\mathcal{P}_E$ for the convex cone of continuous semi-norms on E.

Definition 3.2.2. We say the representation (π, E) has *controllable growth* provided for all $p \in \mathcal{P}_E$ there exists a $q \in \mathcal{P}_E$ and a locally bounded non-negative function $s_p : G \to \mathbb{R}_0^+$ such that

$$p(\pi(g)v) \leq s_p(g)q(v) \qquad (g \in G, v \in E)\,.$$

Let (π, E) be a representation of controllable growth, then it is no loss of generality to assume that

$$s_p(g) := \sup_{\substack{v \in E \\ q(v) \neq 0}} \frac{p(\pi(g)v)}{q(v)}\,.$$

We denote the set of all s_p by S_π.
We record the following transitivity relation: For every $p \in \mathcal{P}_E$ there are $p', p'' \in \mathcal{P}_E$ such that

$$s_p(gh) \leq s_{p'}(g)s_{p''}(h) \qquad (g, h \in G).$$

This brings us to the notion of a generalized scale structure on a group G.

Definition 3.2.3. Let G be a Lie group. By a generalized scale on G we understand a family S of locally bounded non-negative functions on G such that for all $s \in S$ there exists $s', s'' \in S$ such that for all $g, h \in G$ one has the transitivity relation

$$s(gh) \leq s'(g)s''(h)\,.$$

If (π, E) is a representation of controllable size, then S_π is a generalized scale. We introduce an order of generalized scales: $S \prec S'$ provided for all $s \in S$ exists an $s' \in S'$ and some fixed $C, N > 0$ such that $s(g) \leq C(s'(g))^N$ for all $g \in G$. This leads us to a notion of equivalence and we refer to equivalence classes $[S]$ as generalized scale structures on G.

We say that a generalized scale structure $[S]$ has a *countable core* if there exists $s_1, s_2, \ldots \in S$ such that for all $s \in S$ there exists an $n \in \mathbb{N}$ such that $s \prec s_n$.

Definition 3.2.4. A representation (π, E) of controllable growth of G is $[S]$-tempered if $S_\pi \prec S$.

Example 3.2.5. Let $G = (G, [S])$ be a group with generalized scale structure $[S]$. We define the space $\mathcal{R}_{[S]}(G)$ of rapidly decreasing functions on G with respect to the generalized scale structure $[S]$ by

$$\mathcal{R}_{[\mathcal{S}]}(G) = \Big\{ f \in C(G) \ \mid \ p_{s,n}(f) := \sup_{x \in G} |f(x)| s(x)^N < \infty, \ \forall s \in S \in [S], n \in \mathbb{N} \Big\}.$$

Note that $\mathcal{R}_{[S]}(G)$ does only depend on the equivalence class of $[S]$.
The family $(p_{s,n})_{s \in S \in [S], n \in \mathbb{N}}$ defines a complete locally convex topology on $\mathcal{R}_{[\mathcal{S}]}(G)$. The space $C_c(G)$ injects continuously into $\mathcal{R}_{[S]}(G)$ because $s \in S$ is locally bounded.
In case $[S]$ has a countable core, $\mathcal{R}_{[\mathcal{S}]}(G)$ is a Fréchet space.
The space $\mathcal{R}_{[S]}\,(G)$ is a $[S]$-tempered G-module for the left regular representation.

We denote the category of $[S]$-tempered representations of G by $\mathrm{Rep}_{[S]}(G)$. Furthermore we denote by $\mathrm{FRep}_{[S]}(G)$ the full subcategory of $[S]$-tempered representations of G on Fréchet spaces.

Lemma 3.2.6. *The category* $\mathrm{Rep}_{[S]}(G)$ *is closed under taking subrepresentations and quotient representations.*

Proof. Let (π, E) be a $[S]$-tempered representation of G and $F \subset E$ be a closed G-invariant subspace of E. It is obvious that the $G-$module F is $[S]$-tempered. It remains to show that the corresponding quotient representation $(\widetilde{\pi}, E/F)$ has the same property. Every continuous semi-norm $\widetilde{p}$ on E/F corresponds to a continuous semi-norm p on E via

$$\widetilde{p}(v + F) = \inf_{f \in F} (v + f), \qquad (v + F \in E/F).$$

Let $s_p \in S_\pi$, then we obtain

$$\begin{aligned}\widetilde{p}\left(\widetilde{\pi}(g)\,(v+F)\right) &= \widetilde{p}\left(\pi(g)v+F\right)\\ &= \inf_{f\in F} p\left(\pi(g)\,(v+f)\right)\\ &\leq \inf_{f\in F} s_p(g)q\left(v+f\right)\\ &= s_p(g)\widetilde{q}\left(v+F\right)\end{aligned}$$

This implies that $(\widetilde{\pi}, E/F)$ is $[S]$-tempered. $\square$

3.2.2 Integration of representations

From now on we assume that the space $\mathcal{R}_{[S]}(G)$ injects continuously into $L^1(G)$. Thus $\mathcal{R}_{[\mathcal{S}]}(G)$ becomes a topological algebra under convolution.
For $(\pi, E) \in Rep_{[S]}(G)$ we define an algebra action (Π, E) of $\mathcal{R}_{[S]}(G)$ by

$$\Pi(\varphi)v = \int_G \varphi(g)\pi(g)v\ dg, \qquad (\varphi \in \mathcal{R}_{[S]}(G), v \in E).$$

By a *representation* of a topological algebra $\mathcal{A}$ on the topological vector space E we understand a continuous linear action

$$\mathcal{A} \times E \to E, \quad (g,v) \mapsto g\cdot v\,.$$

Again continuity is requested in both variables.

Lemma 3.2.7. *For* $(\pi, E) \in \mathrm{Rep}_{[\mathrm{S}]}(G)$ *the corresponding algebra action* (Π, E) *of* $\mathcal{R}_{[S]}(G)$ *is a representation.*

Proof. If (π, E) is $[S]-$tempered then the operator $\Pi(f)$ exists for every $f \in \mathcal{R}_{[S]}(G)$. It remains to show that the algebra representation is continuous.
For a continuous semi-norm p on E let $B(p) = \{v \in E \mid p(v) < 1\}$. Let $s_p \in S_\pi$ and $s \in S$ such that $s_p \leq Cs^n$ for $C > 0, n \in \mathbb{N}$. Then

$$\begin{aligned}p\left(\int |f(x)|\pi(x)v\ dx\right) &\leq \int |f(x)|p\left(\pi(x)v\right)\ dx\\ &\leq \int |f(x)|s_p(x)q(v)\ dx\\ &\leq Cq(v)\int |f(x)|s^n(x)\ dx\\ &= Cq(v)p_{s,n}(f),\end{aligned}$$

thus $B(p_{s,n}) \times B(q/C)$ maps into $B(p)$. The action is therefore continuous. $\square$

We denote the functor $\mathrm{Rep}_{[\mathrm{S}]}(G) \to \mathrm{Rep}\left(\mathcal{R}_{[S]}(G)\right)$ which maps the group representation (π, E) to the corresponding algebra representation (Π, E) by I.

3.2.3 Smooth and analytic $[S]$-tempered representations

We call a vector $v \in E$ *smooth* if the orbit map $\gamma_v : G \to E$ defined by $\gamma_v(g) = \pi(g)v$ is a smooth map. We denote the set of all smooth vectors by E^∞. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G acts on E^∞ by the derived representation $d\pi$. We use this action to define a topology on E^∞. For a continuous semi-norm p on E and $u \in \mathcal{U}(\mathfrak{g})$ let p_u be the semi-norm on E^∞ defined by

$$p_u(v) = p\left(d\pi(u)v\right), \qquad (v \in E^\infty).$$

We equip E^∞ by the topology defined by the family of derived semi-norms $(p_u)_{p\in\mathcal{P}_E, u\in\mathcal{U}(\mathfrak{g})}$. The representation (π, E) is called smooth if $E = E^\infty$ as topological vector spaces. We denote the category of smooth $[S]$-tempered representations of G by $\mathrm{Rep}^\infty_{[S]}(G)$ and the subcategory of smooth $[S]$-tempered representations on Fréchet spaces by $\mathrm{FRep}^\infty_{[S]}(G)$.
Likewise we call a vector $v \in E$ *analytic* if the orbit map is analytic. We denote the space of analytic vectors by E^ω. For the definition of the topology on E^ω we refer to [3]. The representation (π, E) is called analytic if $E = E^\omega$ as topological vector spaces. We denote the category of analytic $[S]$-tempered representations of G by $\mathrm{Rep}^\omega_{[S]}(G)$ and the subcategory of analytic $[S]$-tempered representations on Fréchet spaces by $\mathrm{FRep}^\omega_{[S]}(G)$.

Example 3.2.8.

- Define the *Schwartz space* $\mathcal{S}_{[S]}(G)$ as the space of smooth vectors for the left regular representation $(L, \mathcal{R}_{[S]}(G))$ of G. Note that this is a topological subalgebra of $\mathcal{R}_{[S]}(G)$ which contains $C^\infty_c(G)$.
  For a locally convex vector space E the left regular representation $(L, \mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E)$ of G is smooth and $[S]$-tempered.

- Define the algebra $\mathcal{A}_{[S]}(G)$ as the space of analytic vectors for the left regular representation $(L, \mathcal{R}_{[S]}(G))$ of G.
  For a locally convex vector space E the left regular representation $(L, \mathcal{A}_{[S]}(G)\widehat{\otimes}_\pi E)$ of G is analytic and $[S]$-tempered.

We now give a more algebraic description of the category $\mathrm{FRep}^\infty_{[S]}(G)$ using the theorem of Dixmier and Malliavin [4].
We call a representation of an algebra $\mathcal{A}$ on a space V *non-degenerate* if the corresponding map $\mathcal{A} \otimes V \to V$ is surjective. We denote the category of non-degenerate algebra representations of $\mathcal{A}$ on Fréchet spaces by $\mathrm{FRep}_{\mathrm{nd}}(\mathcal{A})$.

Theorem 3.2.9 (Dixmier-Malliavin). *Let (π, E) be a smooth representation of a Lie group G on a Fréchet space E, then the corresponding algebra representation (Π, E^∞) of $C^\infty_c(G)$ is non-degenerate.*

We denote the restriction of the functor I to the subcategory $\mathrm{FRep}^\infty_{[S]}(G)$ by I^∞ and the restriction to $\mathrm{FRep}^\omega_{[S]}(G)$ by I^ω.

Theorem 3.2.10. *The functor I^∞ yields an equivalence of categories*

$$\mathrm{FRep}^\infty_{[S]}(G) \simeq \mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{S}_{[S]}(G)\right).$$

Proof. First note that the functor I^∞ maps $\mathrm{FRep}^\infty_{[S]}(G)$ to $\mathrm{FRep}\left(\mathcal{S}_{[S]}(G)\right)$ because $\mathcal{S}_{[S]}(G)$ injects continuously into $\mathcal{R}_{[S]}(G)$.
Since $\mathcal{S}_{[S]}(G)$ contains $C_c^\infty(G)$ the theorem of Dixmier and Mailliavin implies that the image is in fact contained in $\mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{S}_{[S]}(G)\right)$.
In the next step we construct the quasi-inverse D to I^∞.
For $(\Pi, E) \in \mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{S}_{[S]}(G)\right)$, we denote the left regular action of G on $\mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E$ by π'_o. Let $j : \mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E \to E$ be the continuous linear map which corresponds to the representation (Π, E). Note that j is $\mathcal{S}_{[S]}(G)$ equivariant and surjective. We claim that the kernel J of j is G-invariant. Let $(\delta_k)_{k\geq 1}$ be a Dirac sequence in $\mathcal{S}_{[S]}(G)$, then we obtain for an element $w \in J$

$$\begin{aligned} j\left(\pi'_o(g)w\right) &= \lim_{n\to\infty} j\left(\pi'_o(g)\Pi\left(\delta_n\right)w\right) \\ &= \lim_{n\to\infty} j\left(\Pi\left(L\left(g\right)\delta_n\right)w\right) \\ &= \lim_{n\to\infty} \Pi\left(L\left(g\right)\delta_n\right)j\left(w\right) \\ &= 0. \end{aligned}$$

The quotient representation $(\pi_o, \mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E/J)$ of $(\pi'_o, \mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E)$ is according to Lemma 3.2.6 and Example 3.2.8 a smooth $[S]$-tempered representation of G. We define the functor D by $(\pi, E) \mapsto (\pi_o, \mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E/J)$. Since $\mathcal{S}_{[S]}(G)\widehat{\otimes}_\pi E/J \simeq E$ the functors I^∞ and D are quasi-inverse to each other. □

We give an analogous description of the category $\mathrm{FRep}^\omega_{[s]}(G)$ for a scale s on G using the following theorem from [5].

Theorem 3.2.11. *Let s be a scale on G and (π, E) an analytic $[s]$-tempered representation of G on a Fréchet space E, then the corresponding algebra representation (Π, E^ω) of $\mathcal{A}_{[s]}(G)$ is non-degenerate.*

Using this and the fact that $\mathcal{A}_{[s]}(G)$ contains a Dirac sequence the proof of the next theorem uses the same reasoning as above.

Theorem 3.2.12. *Let s be a scale on G, then the functor I^ω yields an equivalence of categories*

$$\mathrm{FRep}^\omega_{[s]}(G) \simeq \mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{A}_{[s]}(G)\right).$$

Let res_∞ be the functor $(\pi, E) \mapsto (\pi, E^\infty)$ and let $\mathrm{res}_{|\mathcal{S}_{[S]}(G)}$ be the functor $(\Pi, E) \mapsto (\Pi_{|\mathcal{S}_{[S]}(G)}, E^\infty)$. Likewise we denote the functor $(\pi, E) \mapsto (\pi, E^\omega)$ by res_ω and the functor $(\Pi, E) \mapsto (\Pi_{|\mathcal{A}_{[s]}(G)}, E^\omega)$ by $\mathrm{res}_{|\mathcal{A}_{[s]}(G)}$.

Corollary 3.2.13. *The following diagram commutes.*

$$\begin{tikzcd}
\mathrm{FRep}_{[S]}(G) \arrow[r, "I"] \arrow[d, "\mathrm{res}_\infty"] & \mathrm{FRep}\left(\mathcal{R}_{[S]}(G)\right) \arrow[d, "\mathrm{res}_{|\mathcal{S}_{[S]}(G)}"] \\
\mathrm{FRep}^\infty_{[S]}(G) \arrow[r, "\sim", "I^\infty"'] & \mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{S}_{[S]}(G)\right)
\end{tikzcd}$$

Proof. We just need to check, that the functors map to the stated categories.
Therefore note that the image of $\mathrm{FRep}_{\mathrm{nd}}(\mathcal{R}_{[S]}(G))$ under $\mathrm{res}_{|\mathcal{S}_{[S]}(G)}$ is indeed contained in $\mathrm{FRep}_{\mathrm{nd}}(\mathcal{S}_{[S]}(G))$ according to Theorem 3.2.9.
The proof of Theorem 3.2.10 shows that $\mathrm{res}_\infty \simeq D \circ \mathrm{res}_{|\mathcal{S}_{[S]}}$ and that $D \circ \mathrm{res}_{|\mathcal{S}_{[S]}}$ maps to $\mathrm{FRep}^\infty_{[S]}(G)$. Since the property of being $[S]$-tempered only depends on the equivalence class of the representation this implies that res_∞ maps to $\mathrm{FRep}^\infty_{[S]}(G)$. ☐

Corollary 3.2.14. *If s is a scale then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{FRep}_{[s]}(G) \arrow[r, "I"] \arrow[d, "\mathrm{res}_\infty"] & \mathrm{FRep}\left(\mathcal{R}_{[s]}(G)\right) \arrow[d, "\mathrm{res}_{|\mathcal{S}_{[s]}(G)}"] \\
\mathrm{FRep}^\infty_{[s]}(G) \arrow[r, "\sim", "I^\infty"'] \arrow[d, "\mathrm{res}_\omega"] & \mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{S}_{[s]}(G)\right) \arrow[d, "\mathrm{res}_{|\mathcal{A}_{[s]}(G)}"] \\
\mathrm{FRep}^\omega_{[s]}(G) \arrow[r, "\sim", "I^\omega"'] & \mathrm{FRep}_{\mathrm{nd}}\left(\mathcal{A}_{[s]}(G)\right)
\end{tikzcd}$$

In the next section we give evidence that the results for the analytic category are still true if the scale s is replaced by a generalized scale S.

3.3 Examples of tempered representations

In this section we discuss two examples of tempered representations which are not of moderate growth

3.3.1 Two tempered models for the regular representation of $(\mathbb{R}, +)$

Let $\alpha \in (0, 1]$.
For the first model we consider the left regular representation (L, E) of $\mathbb{R}$ on $E = \bigcap_{n>0} L^2(\mathbb{R}, e^{n|x|^{1/\alpha}}dx)$. This is a representation of moderate growth if $\alpha = 1$.
For $\alpha \in (0, 1)$ let $S = \{e^{n|x|^{1/\alpha}} \mid n \in \mathbb{N}\}$ then (L, E) is a $[S]$-tempered representation which is

not of moderate growth.
Note that $E^\omega = S^1_\alpha$ as topological vector spaces. For the definition of the space S^1_α we refer to the appendix 3.4.

Conjecture 3.3.1. *The map*

$$\mathcal{A}_{[S]}(\mathbb{R}) \times E^\omega \to E^\omega, \quad (f,g) \mapsto f * g,$$

is onto.

Remark 3.3.2. We think that it is possible to prove Conjecture 3.3.1 with the Hadamard factorization theorem once one has good estimates of the distribution of zeroes of elements in the space S^1_α.

For the second model let $\alpha \in (1/2, 1]$ and $\beta \in (0, 1/2)$ such that $\alpha + \beta > 1$. We consider the regular representation on $E = S^{\beta,0}_{\alpha,0}$. This is an entire representation which is of moderate growth if $\alpha = 1$ and $[S]$-tempered if $\alpha \in (1/2, 1)$.
This models admits the factorization property.

Theorem 3.3.3. *The map*

$$\mathcal{A}_{[S]}(\mathbb{R}) \times E^\omega \to E^\omega, \quad (f,g) \mapsto f * g,$$

is onto.

Proof. Let $f \in E^\omega$, then $\widehat{f} \in S^{\alpha,0}_{\beta,0}$. Since $e^{-x^2} \in \mathcal{F}\left(\mathcal{A}_{[S]}(\mathbb{R})\right)$ and $e^{x^2}\widehat{f}(x) \in S^{\alpha,0}_{\beta,0}$ the claim follows from the decomposition

$$f(z) = e^{-z^2} f(z) e^{z^2}.$$

□

3.3.2 Non-unitary Schrödinger representations

The *Heisenberg group* $\mathbb{H}_n$ is the group $(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, *)$ where the operation $*$ is defined by

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + xy').$$

Let $|\cdot|$ be the euclidean norm on $\mathbb{R}^n \times \mathbb{R}^n$. The *Korányi metric* $\|\cdot\|_H$ on $\mathbb{H}_n$ is defined by

$$\|(x, y, t)\|_H = \left(|(x, y)|^4 + (t - 1/2xy)^2\right)^{1/4}.$$

Note that it satisfies $\|(x, y, t) * (x', y', t')\|_H \leq \|(x, y, t)\|_H + \|(x', y', t')\|_H$ for all $(x, y, t), (x', y', t') \in \mathbb{H}_n$.
The Korányi metric $\|\cdot\|_H$ and the Riemannian distance d on $\mathbb{H}_n$ are bi-Lipschitz equivalent, i.e there exists constants $c, C > 0$ such that

$$cd((x, y, t)) \leq \|(x, y, t)\|_H \leq Cd((x, y, t)), \quad \forall (x, y, t) \in \mathbb{H}^n.$$

Hence we conclude that the geometric scale structure s_{geo} and the scale structure given by $e^{\|\cdot\|_H}$ are equivalent.

Proposition 3.3.4. *Let (π, E) be a representation of moderate growth of the Heisenberg group $\mathbb{H}_n$ on a locally convex topological Hausdorff vector space E with scalar central character. Then $\pi|_{Z(\mathbb{H}_n)}$ is unitary.*

Proof. Let χ be the scalar central character. The representation (π, E) is $[s_{\text{geo}}]$-tempered and hence $[e^{\|\cdot\|_H}]$-tempered. Let p be a continuous semi-norm on E, then there exists a continuous semi-norm q on E and a constant $N > 0$ such that

$$p\left(\pi\left((x,y,t)\right)v\right) \leq e^{N\|(x,y,t)\|_H} q(v).$$

So we obtain especially

$$|\chi(t)| p(v) \leq e^{N\sqrt{t}} q(v),$$

which is only possible if χ is unitary. □

We now define a model for the (non necessarily unitary) Schrödinger representation.
Let $\alpha \in (0,1)$.
For every $h \in \mathbb{C} \setminus \{0\}$ we define the representation π_h on the space $E = \bigcap_{n>0} L^2(\mathbb{R}, e^{n|x|^{1/\alpha}}dx)$ by

$$\pi_h\left((x,y,t)\right) f(z) = e^{iht} e^{ihyz} f(z-x).$$

Note that π_h has a non-unitary central character if $h \in \mathbb{R} \setminus \{0\}$.
The set $S = \{s_n(x,y,t) = e^{n(|x|^{1/\alpha} + |y|^{1/(1-\alpha)} + |t|)} \mid n \in \mathbb{N}\}$ is a generalized scale on $\mathbb{H}$ and the representation (π_h, E) is $[S]$-tempered.

Lemma 3.3.5. *The representation (π_h, E) of $\mathbb{H}$ is topologically simple for every $h \in \mathbb{C} \setminus \{0\}$, i.e the $\mathbb{H}$-orbit of every non-zero vector is dense.*

Proof. Let $0 \neq f \in E$. It suffices to show that if $g \in L^2\left(\mathbb{R}^n, e^{m|x|^{1/\alpha}}dx\right)$ satisfies $\langle \pi_h\left((x,y,t)\right) f, g\rangle_m = 0$ for all $(x,y,t) \in \mathbb{H}_n$ then $g = 0$. We define an entire function F on $\mathbb{C}^n$ by

$$F(z) = \int_{\xi \in \mathbb{R}^n} e^{iz\cdot\xi} f\left(\xi - x\right) \overline{g\left(\xi\right)} e^{m|\xi|^{1/\alpha}} d\xi.$$

Then $F(hx) = 0$ for all $x \in \mathbb{R}^n$ and thus $F(z) = 0$ for every $z \in \mathbb{C}^n$. Hence $f\left(\xi - x\right) \overline{g\left(\xi\right)} e^{m|\xi|^{1/\alpha}} = 0$ for every $x, y \in \mathbb{R}^n$, i.e $g = 0$. □

Remark 3.3.6. If one assumes that Conjecture 3.3.1 holds then the representation (π_h, E) has the factorization property.

3.4 Appendix

3.4.1 Gelfand-Shilov space S_α^β

In this section we review basic properties of Gelfand-Shilov spaces [6].
For $\alpha, \beta \geq 0$ let S_α^β be the space of smooth functions f which satisfy for some positive

constants A, B, B and all $(n,k) \in \mathbb{N}^2$ the inequalities

$$|x^n D^k f(x)| \leq C A^n B^k n^{n\alpha} k^{k\beta}.$$

Note that S_α^β is nontrivial if and only if $\alpha + \beta \geq 1$.
Let $A, B > 0$. We define $S_{\alpha,A}^{\beta,B}$ as the space of all $f \in S_\alpha^\beta$ such that

$$\|f\|_{\varepsilon,\delta} = \sup_{(n,k)\in\mathbb{N}^2} \sup_{x\in\mathbb{R}} \frac{|x^n D^k f(x)|}{(A+\epsilon)^n (B+\delta)^k n^{n\alpha} k^{k\beta}} < \infty$$

for all $\varepsilon, \delta > 0$. Equipped with this famliy of semi-norms $S_{\alpha,A}^{\beta,B}$ is a Fréchet space and we equip S_α^β with the inductive limit topology of the inductive system $S_{\alpha,A}^{\beta,B} \hookrightarrow S_{\alpha,A'}^{\beta,B'}$ if $A \leq A'$ and $B \leq B'$.

Theorem 3.4.1. *The Fourier transform is a topolocial isomorphism between $S_{\alpha,A}^{\beta,B}$ and $S_{\beta,B}^{\alpha,A}$.*

If $\beta < 1$ then every element of S_α^β is an entire function.
An entire function $f \in \mathcal{O}(\mathbb{C})$ is an element of $S_{\alpha,A}^{\beta,B}$ with $\beta < 1$ if and only if it satisfies the following condition:

- For every $b > \frac{1-\beta}{e}(Be)^{1/(1-\beta)}$ and every $a < \frac{\alpha}{eA^{1/\alpha}}$ there exists a $C > 0$ such that
$$|f(x+iy)| \leq C \exp(-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}).$$

The family $(p_{\varepsilon,\delta})_{\varepsilon,\delta>0}$ of semi-norms defined by

$$p_{\varepsilon,\delta}(f) = \sup_{x+iy\in\mathbb{C}} |f(x+iy)| \exp\left((\tfrac{\alpha}{eA^{1/\alpha}} - \varepsilon)|x|^{1/\alpha} + (\tfrac{1-\beta}{e}(Be)^{1/(1-\beta)} + \delta)|y|^{1/(1-\beta)} \right),$$

defines the same Fréchet topology on $S_{\alpha,A}^{\beta,B}$.
If $\beta = 1$ then every element of S_α^β is holomorphic on a strip domain $\mathbb{R} + i(-R, R)$. In fact $f \in S_{\alpha,A}^{1,B}$ if and only if satisfies the following conditions:

- $f \in \mathcal{O}\left(\mathbb{R} + i\left(-\frac{1}{Be}, \frac{1}{Be}\right)\right)$
- For every $a < \frac{\alpha}{eA^{1/\alpha}}$ there exists a $C > 0$ such that
$$|f(x+iy)| \leq C \exp(-a|x|^{1/\alpha}).$$

We denote by $S_{\alpha,0}^{\beta,0}$ the intersection of the spaces $S_{\beta,B}^{\alpha,A}$.

The Gelfand-Shilov space $S_{\beta,B}^{\alpha,A}$ admits a characterization using Fourier transform [?]:

Theorem 3.4.2. *A function f belongs to $S_{\alpha,A}^{\beta,B}$ if and only if*

$$\sup_{x\in\mathbb{R}} |f(x)| e^{a|x|^{1/\alpha}} < \infty, \qquad \sup_{\xi\in\mathbb{R}} |f(\xi)| e^{b|\xi|^{1/\beta}} < \infty,$$

for every $a < \frac{\alpha}{eA^{1/\alpha}}$ and every $b < \frac{\beta}{eB^{1/\beta}}$.

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Lebenslauf

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