Geometry of moduli spaces of spin and prym curves of small genus

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Kurzzusammenfassung

Diese Arbeit beschäftigt sich mit geometrischen Eigenschaften der Modulräume $\overline{S}_{g,n}$ und $\overline{R}_{g,n}$ von Spin- bzw. Prymkurven von Geschlecht g mit n markierten Punkten. Hauptsächlich werden kohomologische Eigenschaften dieser Räume für kleines g untersucht, beispielsweise der Kohomologie-Ring oder der Chow-Ring mit Koeffizienten in $\mathbb{Q}$ berechnet. Da $\overline{S}_{g,n}$ und $\overline{R}_{g,n}$, ähnlich wie der Modulraum $\overline{M}_{g,n}$ von stabilen punktierten Kurven, in natürlicher Weise als Orbifolds bzw. glatte Deligne-Mumford-Stacks aufgefasst werden können, besitzen sie auch einen Chen-Ruan Orbifold-Kohomologie-Ring. Auch dieser ist ein Gegenstand der Arbeit. Der Inhalt gliedert sich thematisch in vier Teile:

Im ersten Teil werden die hyperelliptischen Orte $\overline{HS}_{g,n} \subseteq \overline{S}_{g,n}$ und $\overline{HR}_{g,n} \subseteq \overline{R}_{g,n}$ untersucht. Mit Hilfe der Ergebnisse des ersten Teils, wird im zweiten Teil der Kohomologie-Ring von $\overline{R}_2$ und $\overline{S}_2$ als $\mathbb{Q}$-Algebra durch Angabe von Erzeugern und Relationen zwischen diesen bestimmt. Der Kohomologie-Ring ist, wie sich zeigt, für diese beiden Räume isomorph zum Chow-Ring. Der dritte Teil beschäftigt sich mit der Geometrie der Räume $\overline{R}_{1,n}$ für kleines n. Es wird für $n \leq 6$ gezeigt, dass die $\overline{R}_{1,n}$ rationale Varietäten sind, und dass der Chow-Ring $A^*(\overline{R}_{1,n})$ von den sogenannten Randklassen erzeugt wird. Für $n \leq 4$ wird die Struktur der $\mathbb{Q}$-Algebra $A^*(\overline{R}_{1,n})$ bestimmt und gezeigt dass sie zum Kohomologie-Ring $H^*(\overline{R}_{1,n})$ isomorph ist. Zusätzlich wird die Kodaira Dimension von $\overline{R}_{1,11}$ berechnet. Da $\overline{S}_{1,n} \cong \overline{M}_{1,n} \uplus \overline{R}_{1,n}$ (als Varietäten), decken diese Ergebnisse für $\overline{R}_{1,n}$ auch den Fall $\overline{S}_{1,n}$ ab. Im letzten Teil der Arbeit geht es um die Chen-Ruan Orbifold-Kohomologie der Orbifolds/Stacks $\overline{R}_{1,n}$ für beliebiges $n \in \mathbb{N}$. Dabei wird der Chen-Ruan-Kohomologie-Ring $H^*_{CR}(\overline{R}_{1,n})$ als Algebra über dem üblichen Kohomologie-Ring $H^*(\overline{R}_{1,n})$ behandelt. Die Ergebnisse dieses Teils für allgemeines n beschreiben die (additive und multiplikative) Struktur der Chen-Ruan Kohomologie daher im wesentlichen relativ zur Struktur der üblichen Kohomologie. Nur in den Fällen in welchen die letztere Struktur bekannt ist (wie für $n \leq 4$ nach dem dritten Teil dieser Arbeit), bestimmen diese Ergebnisse die Struktur der Chen-Ruan Kohomologie als $\mathbb{Q}$-Vektorraum bzw. als $\mathbb{Q}$-Algebra.

Schlagworte: Modulräume, Spinkurven, Prymkurven.

Abstract

This thesis is concerned with geometric properties of the moduli spaces $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$ of spin- respectively prym curves of genus g with n marked points. Primarily cohomological properties of these spaces for small values of g are investigated. In particular the cohomology ring and the Chow ring with coefficients in $\mathbb{Q}$ are calculated. Since $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$, like the moduli space $\overline{M}_{g,n}$ of stable pointed curve, are orbifolds or smooth Deligne-Mumford stacks in a natural way, they have a Chen-Ruan orbifold cohomology ring. We also study this ring. Thematically the content of this thesis can be divided into four parts:

In the first part the hyperelliptic loci $\overline{HS}_{g,n} \subseteq \overline{S}_{g,n}$ and $\overline{HR}_{g,n} \subseteq \overline{R}_{g,n}$ are investigated. Applying results from the first part, in the second part the cohomology ring of $\overline{R}_2$ and $\overline{S}_2$ is determined as a $\mathbb{Q}$-algebra in terms of generators and relations between these generators. The cohomology ring turns out to be isomorphic to the Chow ring for these two spaces. The third part is concerned with the geometry of the spaces $\overline{R}_{1,n}$ for small n. It is shown, for $n \leq 6$, that the spaces $\overline{R}_{1,n}$ are rational varieties, and that the Chow ring $A^*(\overline{R}_{1,n})$ is generated by the so called boundary cycle classes. For $n \leq 4$ the structure of the $\mathbb{Q}$-algebra $A^*(\overline{R}_{1,n})$ is determined, and $A^*(\overline{R}_{1,n})$ is shown to be isomorphic to the cohomology ring $H^*(\overline{R}_{1,n})$. Furthermore the Kodaira dimension of $\overline{R}_{1,11}$ is calculated. Since, as varieties, $\overline{S}_{1,n} \cong \overline{M}_{1,n} \uplus \overline{R}_{1,n}$, these results for $\overline{R}_{1,n}$ also cover the case of $\overline{S}_{1,n}$. In the last part of the thesis, the Chern-Ruan orbifold cohomology of the orbifolds/stacks $\overline{R}_{1,n}$ for arbitrary $n \in \mathbb{N}$ is studied. The Chen-Ruan cohomology ring $H^*_{CR}(\overline{R}_{1,n})$ is treated as algebra over the usual cohomology ring $H^*(\overline{R}_{1,n})$. Consequently the results of this part for arbitrary n describe the (additive and multiplicative) structure of the Chen-Ruan cohomology mainly relative to the structure of the usual cohomology. Only in those cases in which the latter structure is known (like for $n \leq 4$ by the third part of the thesis), our results determine the structure of the Chen-Ruan cohomology as a $\mathbb{Q}$-vector space respectively as a $\mathbb{Q}$-algebra.

Key words: Moduli spaces, spin curves, prym curves.

Contents

Introduction

The objects studied in this thesis are the compact moduli spaces of spin curves and of prym curves of a given (arithmetic) genus g. These spaces, $\overline{S}_g$ respectively $\overline{R}_g$, are normal projective varieties, which compactify the moduli spaces S_g resp.[1] R_g of smooth spin- resp. prym curves of genus g, akin to the way the moduli space of stable curves $\overline{M}_g$ compactifies the moduli space of smooth curves M_g. A smooth spin curve is a pair of a smooth curve C and a line bundle $\mathcal{L}$ on C such that $\mathcal{L}^{\otimes 2} \cong \omega_C$, where ω_C denotes the canonical bundle. Such a line bundle is called a theta characteristic. A smooth prym curve can be defined analogously by requiring instead that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C$, where $\mathcal{O}_C$ is the trivial bundle on C (the case $\mathcal{L} \cong \mathcal{O}_C$ is excluded). Equivalently (and more classically) a smooth prym curve can be seen as a smooth curve C together with an unramified degree 2 cover $Y \to C$. The compactification $\overline{S}_g$ was constructed by Maurizio Cornalba in [Cor89] as a moduli space of quasi-stable curves X, together with a line bundle $\mathcal{L}$ on X and a homomorphism $b : \mathcal{L}^{\otimes 2} \to \omega_X$ with certain properties. If one views smooth prym curves as curves plus unramified double covers, the natural way to compactify this space is by allowing stable curves with admissible double covers. In this way $\overline{R}_g$ was constructed in [Bea77] by Arnaud Beauville. The interpretation of smooth prym curves as curve plus line bundle, allows also to construct a compactification of R_g analogous to the compactification $\overline{S}_g$ constructed by Cornalba. This construction was carried out in [BCF04], and it was shown that the resulting compactification is isomorphic to Beauville's compactification $\overline{R}_g$ as varieties. In this thesis we work with the definition of prym curves as introduced in [BCF04], and will not use Beauville's description involving admissible double covers. There is a third way to compactify S_g (and also R_g). Instead of letting the compactification parametrise certain quasi-stable curves with line bundles, as in Cornalba's construction, one can also restrict to stable curves, but allow torsion-free sheaves as extra structure. This approach was taken by Tyler J. Jarvis in [Jar98], [Jar00], where also more general moduli spaces of curves with roots of line bundles are constructed. Again the compactification obtained for S_g and R_g is isomorphic to those obtained following the other aproaches.

Like in the case of stable curves, one can also introduce n-pointed spin or prym curves, i.e. let the underlying quasi-stable curve carry n ordered pairwise different smooth marked points. The compact moduli spaces parametrising these objects will be denoted by $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$. In this thesis we investigate the geometry and especially cohomological properties of

[1]From here on the abbreviation "resp." is used for the often needed "respectively", although this may not be a common abbreviation in English.

the spaces $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$ for certain (small) values of g and n. In particular the cohomology ring and the Chow ring with coefficients in $\mathbb{Q}$ is calculated. $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$, like the moduli space $\overline{M}_{g,n}$ of stable pointed curve, are orbifolds or smooth Deligne-Mumford stacks in a natural way, so they have a Chen-Ruan orbifold cohomology ring, as introduced in [CR04]. We also study this ring. (Since we nearly always work with coefficients in $\mathbb{Q}$ we denote the rational Chow ring and rational cohomology ring of a variety X by $A^*(X)$ resp. $H^*(X)$ instead of $A^*_{\mathbb{Q}}(X)$ and $H^*_{\mathbb{Q}}(X)$, and write $A^*_{\mathbb{Z}}(X)$ resp. $H^*_{\mathbb{Z}}(X)$ if we for once need integer coefficients.)

Before describing the content of this thesis in more detail, we give a very short overview of some general results known about the geometry of $\overline{R}_{g,n}$ and $\overline{S}_{g,n}$. Much more about this, and also about the historic development of the study of spin resp. prym curves and their moduli can be found in the survey articles [Far12] and [Far11]. There are morphisms $\tau_{\overline{S}_{g,n}} : \overline{S}_{g,n} \to \overline{M}_{g,n}$ and $\tau_{\overline{R}_{g,n}} : \overline{R}_{g,n} \to \overline{M}_{g,n}$, which correspond to forgetting the line bundle on a spin/prym curve and sending the underlying quasi-stable curve X to its stable model C. These forgetful morphisms are finite of degree 2^{2g} resp. $2^{2g} - 1$, reflecting that there are 2^{2g} theta characteristics on a smooth curve, and $2^{2g} - 1$ points of order 2 on its Jacobian. These morphisms are important in investigating $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$, since they relate these spaces to the more extensively studied $\overline{M}_{g,n}$. One basic geometric property of $\overline{S}_{g,n}$ is that the space is not connected, but the disjoint union of the spaces $\overline{S}^+_{g,n}$ and $\overline{S}^-_{g,n}$ of even resp. odd spin curves. This means spin curves with theta characteristics whose space of global sections is even resp. odd dimensional. The restricted forgetful morphisms $\tau_{\overline{S}^+_{g,n}} : \overline{S}^+_{g,n} \to \overline{M}_{g,n}$ resp. $\tau_{\overline{S}^-_{g,n}} : \overline{S}^-_{g,n} \to \overline{R}_{g,n}$ are of degree $2^{g-1}(2^g + 1)$ resp. $2^{g-1}(2^g - 1)$. That $\overline{S}^+_{g,n}$ and $\overline{S}^-_{g,n}$ are not connected to each other follows from the fact that even and odd theta characteristics do never both appear in one family of spin curves over a connected basis, as shown by David Mumford in [Mum71]. (For families of possibly singular spin curves it was shown in [Cor89].) The singularities of the normal varieties $\overline{S}_g$ and $\overline{R}_g$ have been studied in [Lud10] and [FL10] and it was shown that global pluricanonical forms lift to the desingularisations of these spaces, which is an important ingredient in computing Kodaira dimensions. By work of Gavril Farkas and Alessandro Verra the Kodaira dimension of $\overline{S}^+_g$ and $\overline{S}^-_g$ is known for all g ([Far10], [FV10]), and the Kodaira dimension of $\overline{R}_g$ is known for all $g \leq 7$ and all $g \geq 14$ ([FL10]). The homology groups of the space of smooth spin curves S_g have been investigated and its Picard group has been computed by J. Harer in [Har90], [Har93].

This thesis is structured as follows:

Chapter 1, "'General Preliminaries", mainly provides definitions and summarizes known results which will be used in the later chapters.

In chapter 2 the hyperelliptic loci $\overline{HS}_{g,n} \subseteq \overline{S}_{g,n}$ and $\overline{HR}_{g,n} \subseteq \overline{S}_{g,n}$ are investigated. These are the closures of the subvarieties of $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$ whose points parametrise spin resp. prym curves supported on smooth hyperelliptic curves X, such that the n marked points on X are fixed by the hyperelliptic involution. We construct and study finite surjective degree 1 morphisms from quotients of $\overline{M}_{0,2g+2}$ to the irreducible components of $\overline{HS}_{g,n}$ and

$\overline{HR}_{g,n}$, which factor through isomorphisms to the normalisations of these components. The existence of these morphisms is certainly known and they were applied in many special cases before, although the explicit description of them over the boundary of the moduli spaces we give may be new. The results of this chapter are applied in the third and fifth chapter.

In chapter 3 the cohomology ring with rational coefficients of $\overline{R}_2$ and $\overline{S}_2$ is computed as a $\mathbb{Q}$-algebra in terms of generators and relations between these generators. The cohomology ring turns out to be isomorphic to the Chow ring for these two spaces, via the cycle map. In this chapter we follow the approach of the article [BF09a] by G. Bini and C. Fontanari, in which these computations are done for $\overline{S}_2$, and also correct some mistakes made in this article. In addition to the methods of [BF09a] we also apply the morphisms constructed in chapter 2 to obtain new relations in the cohomology ring. (Note that $\overline{HS}_2 = \overline{S}_2$ and $\overline{HR}_2 = \overline{R}_2$, since all genus 2 curves are hyperelliptic.)

Chapter 4 is concerned with properties of the varieties $\overline{R}_{1,n}$ and $\overline{S}_{1,n}$ for small n. We follow the PhD-thesis of Pavel Belorousski ([Bel98]) in which he computed the rational Chow ring $A^*(\overline{M}_{1,n})$ for $n \leq 4$ and showed that $\overline{M}_{1,n}$ is rational for $n \leq 10$. We compute the Chow ring $A^*(\overline{R}_{1,n})$ for $n \leq 4$ and show rationality for $n \leq 6$. Since as varieties $\overline{S}_{1,n} \cong \overline{R}_{1,n} \uplus \overline{M}_{1,n}$, these results together with Belorousski's also cover the case of $\overline{S}_{1,n}$. Later (in chapter 5) we show that for $n \leq 4$ again $A^*(\overline{R}_{1,n}) \cong H^*(\overline{R}_{1,n})$ via the cycle map.

In Chapter 5, the Chern-Ruan orbifold cohomology of the orbifolds/stacks $\overline{R}_{1,n}$ for arbitrary $n \in \mathbb{N}$ is studied. Here we use many results and ideas from Nicola Pagani's article [Pag08], in which the Chen-Ruan cohomology of $\overline{M}_{1,n}$ is computed. The Chen-Ruan cohomology ring $H^*_{CR}(\overline{R}_{1,n})$ is treated as algebra over the usual cohomology ring $H^*(\overline{R}_{1,n})$. Consequently the results of this part for arbitrary n describe the (additive and multiplicative) structure of the Chen-Ruan cohomology mainly relative to the structure of the usual cohomology. Only in those cases in which the latter structure is known (like for $n \leq 4$ by chapter 4 of the thesis), our results determine the structure of the Chen-Ruan cohomology as a $\mathbb{Q}$-vector space respectively as a $\mathbb{Q}$-algebra. Since the spaces are isomorphic as varieties, $H^*(\overline{R}_{1,n}) \cong H^*(\overline{S}_{1,n})$. But the moduli stacks/orbifolds for the two moduli problems are not isomorphic, and $H^*_{CR}(\overline{R}_{1,n})$ is not isomorphic to $H^*_{CR}(\overline{S}_{1,n})$. After treating $H^*_{CR}(\overline{R}_{1,n})$ we sketch what is different for $H^*_{CR}(\overline{S}_{1,n})$. Using the information gathered in this chapter about automorphisms of pointed genus 1 prym curves, we also analyse the singularities of the varieties $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$ and $\overline{M}_{1,n}$ in the style of [Lud10], and see that $\overline{R}_{1,n}$ has only canonical singularities. Furthermore the Kodaira-Dimension $\kappa(\overline{R}_{1,11})$ of $\overline{R}_{1,11} \cong \overline{S}^+_{1,11}$ is computed. (All $\kappa(\overline{R}_{1,n})$ for $n \neq 11$ have been computed in [BF06].)

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Chapter 1

General Preliminaries

In this chapter we give basic definitions and results, needed in our thesis, and fix notation. Some general notation and conventions first:

Notation 1.1 (Notation and Conventions applied in the whole thesis)

- In the whole thesis we work with varieties over the field $\mathbb{C}$, and the word "variety" always stands for "variety over $\mathbb{C}$". More precisely we mean by a *variety* a reduced separated scheme of finite type over $\mathbb{C}$. So varieties are not required to be irreducible. A *curve* means a projective one dimensional variety. By the genus of a curve we will always mean the arithmetic genus, unless stated otherwise.

- For any ring B and any group G acting on B we denote by B^G the subring of invariants under the action of G.

- For any $n \in \mathbb{N}$ we denote the set $\{1, 2, ..., n\}$ by $\underline{n}$. Let N be a finite set, then a *partition* of N is a set $\{I_1, ..., I_m\}$ of sets $I_i \subseteq N$, such that N is the disjoint union of theses sets. An *ordered partition* is a tuple $(I_1, ..., I_m)$ of sets fulfilling the same condition. Unless explicitly stated otherwise, we require all the sets I_i of an (ordered) partition to be non-empty.

- As already mentioned in the introduction, for a variety X, by $A^*(X)$ resp. $H^*(X)$ we denote the Chow group resp. singular cohomology group of X with coefficients in $\mathbb{Q}$. We often call these the *rational Chow group* resp. the *rational cohomology* of X. If M is a variety which is a moduli space and V a closed subvariety, then denote by $[V]$ the usual cycle class of V in $A^*(M)$. The "Q-class" of V can be seen as $[V]_Q := \frac{1}{r}[V]$, where r is the number of automorphism of the objects parametrised by general points of V. (In Summary 2.6 the sense of this definition will be explained.)

- In these Preliminaries we will distinguish in our notation strictly between moduli stacks and their coarse moduli spaces, and between morphisms of stacks and morphisms of the coarse moduli spaces. In the later chapters we will not do so. For example we denote the moduli stack of prym curves by $\overline{\mathcal{R}}_{g,n}$ here and the coarse moduli spaces by $\overline{R}_{g,n}$. Later $\overline{R}_{g,n}$ can also stand for the moduli stack, in cases in which this is explicitly stated or should be clear from the context. Since we always work on the Chow ring of a coarse moduli space with the multiplication " $\cdot$ " induced by the multiplication on the Chow ring of the moduli stack, we will also always work with the pullback along morphisms of stacks, or if we have a morphism f of coarse moduli spaces, work with the adjusted pullback $f^{\circledast}$ (cf. Summary 1.34, below for the definitions). Since we do never use the unadjusted pullback f^* from chapter 2 on, we will denote $f^{\circledast}$ instead by f^* everywhere except in chapter 1.

- If O is an object of the kind parametrised by a moduli space M, then we denote the point in M parametrising O as $[O]$. For example if $(X;\mathcal{L};b)$ is a prym curve of genus g, then $[(X;\mathcal{L};b)]$ is the corresponding point in $\overline{R}_g$.
- If we have on a family $\mathcal{X} \to S$ sections $\sigma_1, ..., \sigma_n$, $\sigma_i : S \to \mathcal{X}$, we will sometimes also denote by the symbols σ_i their images $\sigma_i(S)$ on $\mathcal{X}$. In particular for a family of curves, $\sum_{i=1}^n \sigma_i$ can denote the divisor on $\mathcal{X}$ which is the sum of the images of the sections σ_i.
- If $X \to S$ and $Y \to S$ are schemes over a scheme S, then $\mathrm{Isom}_S(X,Y)$ denotes the set of isomorphisms from X to Y over S.

1.1 Moduli problems and Moduli spaces.

Definition 1.2 (i) An *(n-pointed)* nodal curve $(X;p_1,...,p_n)$ is a tuple of a connected curve X having only nodes as singularities, and distinct non-singular points $p_1,....p_n \in X$. (We allow $n=0$. We also often call such a curve a *nodal curve with n ordered marked points.*) An isomorphism $\varphi : (X;p_1,...,p_n) \to (X',p_1',...,p_n')$ of nodal curves with marked points is an isomorphism $\varphi : X \to X'$ such that $\varphi(p_i) = p_i'$ for all $i \in \underline{n}$.

(ii) An (n-pointed) *stable curve* $(C;p_1,...,p_n)$ is an (n-pointed) nodal curve having a finite group of automorphisms. Having a finite automorphism group is equivalent to the following stability condition: When we consider as "special points" on an irreducible component of C the marked points as well as the points in which the component meets the rest of C, then every component of genus 0 must carry at least three special points, and every component of genus 1 must carry at least one special point. Shorter: For each component C_i of C, $2g(C_i) - 2 + \nu(C_i) > 0$, where $g(C_i)$ the genus and $\nu(C_i)$ the number of special points.

We often denote a pointed stable curve $(C;p_1,...,p_n)$ by $\mathfrak{C}$.

Denote by $Sch/\mathbb{C}$ the category of schemes over $\mathrm{Spec}\,\mathbb{C}$.

A moduli problem over $Sch/\mathbb{C}$ is given by the following data

(1) Specify which objects the moduli space is supposed to parametrise. (For example (A) nodal curves or (B) stable curves, both for a fixed genus g and a fixed number of marked points n.)

(2) Specify what a family of the chosen objects over a scheme $S \in Sch/\mathbb{C}$ is. Or said somewhat differently, for each $S \in Sch/\mathbb{C}$ specify the set $\mathscr{F}(S)$ of families $\mathcal{X}/S$ of objects of the moduli problem, in such a way that $\mathscr{F}(\operatorname{Spec}\mathbb{C})$ corresponds to the set of objects specified in (1). [1]

(3) Specify when two families in $\mathscr{F}(S)$ are to be considered equivalent, i.e. declare an equivalence relation $\sim_S$ on each $\mathscr{F}(S)$. Furthermore specify a notion of pullback along morphisms for these families: I.e. for every morphism $f : S' \to S$ of schemes and every family $\mathcal{X}/S \in \mathscr{F}(S)$, define a family $f^*(\mathcal{X}/S) \in \mathscr{F}(S')$. We require that the map $f^* : \mathscr{F}(S) \to \mathscr{F}(S')$ defined such is compatible with the equivalence relations. [2]

We continue our examples (A) and (B) of moduli problems by:

Definition 1.3 (i) A *family of nodal curve* (with n marked points) $(\varphi : \mathcal{X} \to S; \sigma_1, ..., \sigma_n)$ is a tuple of

(a) A proper surjective flat morphism $\varphi : \mathcal{X} \to S$ of schemes over $\mathbb{C}$, such that every geometric fibre is a nodal curve, and

(b) Sections $\sigma_i : S \to \mathcal{X}$ of φ, such that the images of the σ_i are pairwise disjoint, and do not meet any singularities (i.e. nodes) of the fibres. (One interprets the image of each section on a fibre as a marked point.)

An isomorphism ψ of families $(\varphi : \mathcal{X} \to S; \sigma_1, ..., \sigma_n)$ and $(\varphi' : \mathcal{X}' \to S; \sigma'_1, ..., \sigma'_n)$ of nodal curves over a fixed basis S is a $\psi \in \operatorname{Isom}_S(\mathcal{X}, \mathcal{X}')$ such that for all $i \in \underline{n}$, $\psi \circ \sigma_i = \sigma'_i$.

(Families of nodal curves over analytic spaces S are defined completely analogously, by replacing everywhere in the definition "scheme(s)" by "analytic space(s)".)

(ii) A *family of stable curves* is a family of nodal curves, all whose geometric fibres are stable curves. The notion of isomorphisms over a fixed S is the same as for stable curves.

This finishes step (2) in the definition of the moduli problems (A) and (B). For step (3), we consider families of nodal or stable curves over a given S as equivalent, if they are isomorphic in the sense of Def. 1.3. We define pullbacks of families of nodal and stable curves via the fibre product (also cf. Def. 1.5 below).

Now if we have defined a moduli problem, this specifies a moduli functor

$$\mathbf{F} : Sch/\mathbb{C} \to Sets$$

from $Sch/\mathbb{C}$ to the category of sets: On the level of objects, for each $S \in Sch/\mathbb{C}$, we set $\mathbf{F}(S) := \mathscr{F}(S)/\sim_S$, the set of equivalence classes of families over S. And for each morphism $f : S' \to S$, $\mathbf{F}(f) : \mathscr{F}(S)/\sim_S \to \mathscr{F}(S')/\sim_{S'}$ is the map induced by the pullback f^*.

[1] Strictly speaking one can omit (1) and begin directly by defining the families of the moduli problem.

[2] To obtain a moduli functor as below, it would also suffice to define pullbacks f^* of eqivalence classes of families instead of defining them individually for each family.

A "solution" to a moduli problem is a moduli space $M \in Sch/\mathbb{C}$, which means that M fulfils one of the following conditions:

Definition 1.4 (i) $M \in Sch/\mathbb{C}$ is called a *fine moduli* space for the given moduli problem if it represents the functor $\mathbf{F}$.

(ii) $M \in Sch/\mathbb{C}$ is called a *coarse moduli* space for the given moduli problem, if: There is a natural transformation Ψ_M from the functor $\mathbf{F}$ to the functor of points Mor_M of M, such that:

1. The map $\Psi_{M,\mathrm{Spec}\,\mathbb{C}} : \mathbf{F}(\mathrm{Spec}\,\mathbb{C}) \to M(\mathbb{C}) = \mathrm{Mor}(\mathrm{Spec}\,\mathbb{C}, M)$ is a bijection of sets. [3]
2. Given another scheme M' and a natural transformation $\Psi_{M'}$ from $\mathbf{F}$ to $\mathrm{Mor}_{M'}$, there is a unique morphism $\pi : M \to M'$ such that the associated natural transformation $\Pi : \mathrm{Mor}_M \to \mathrm{Mor}_{M'}$ satisfies $\Psi_{M'} = \Pi \circ \Psi_M$.

Every fine moduli space is a coarse moduli space, and the moduli space for a moduli problem is unique if it exists. It is well known that the moduli problem (B) of stable curves of genus g we considered has a coarse moduli space $\overline{M}_{g,n}$, which is a projective variety, for all pairs $g, n \in \mathbb{N}_0$ with $2g + n \geq 3$. But only for large n, (B) has a fine moduli space (in $Sch/\mathbb{C}$). The moduli problem (A) has at least no coarse moduli space which is a variety.[4]

A slightly different approach to moduli problems is via moduli groupoids. For our two examples we first introduce the following notion of morphisms of families of nodal/stable curves, which is more general than the one introduced in Def. 1.3:

Definition 1.5 A *morphism* between two families of pointed nodal (or stable) curves $(\alpha : \mathcal{X} \to S, \sigma_1, ..., \sigma_n)$ and $(\alpha' : \mathcal{X}' \to S', \sigma'_1, ..., \sigma'_n)$ is a pullback square in the following sense: The morphism is a pair (H, h) of morphisms of schemes, such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathcal{X}' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ S & \xrightarrow{h} & S' \end{array}$$

is cartesian, i.e. the diagram of a fibre product, and such that $H \circ \sigma_i = \sigma'_i \circ h$ for all $i \in \underline{n}$. (One gets the isomorphisms for fixed S, introduced in Def. 1.3, if one requires h to be the identity on S.)

With this notion of morphisms one can define the categories (say $\mathcal{A}$ and $\mathcal{B}$) of families of n-pointed nodal resp. stable curves (over $Sch/\mathbb{C}$) of genus g. There are obvious functors from these two categories to the category $Sch/\mathbb{C}$, of passing from families to their bases. The categories $\mathcal{A}$ and $\mathcal{B}$ together with these functors both fulfil the definition of a *category fibred in groupoids over $Sch/\mathbb{C}$*:

[3] I.e. it defines a bijection between the equivalence classes of objects of the moduli problem and the closed points of M.

[4] Actually I do not know whether there is a scheme which is a coarse moduli space for (A).

Definition 1.6 (i) A category fibered in groupoids over a category $\mathcal{S}$ is a category $\mathcal{M}$ together with a functor $p : \mathcal{M} \to \mathcal{S}$ satisfying the following conditions:

(1) For every morphism $f : T' \to T$ in $\mathcal{S}$ and a object η of $\mathcal{M}$ such that $p(\eta) = T$, there exists a unique morphisms $\varphi : \xi \to \eta$, such that $p(\varphi) = f$.

(2) Every morphisms $\varphi : \xi \to \eta$ in $\mathcal{M}$ is cartesian in the following sense. Given any other morphism $\varphi' : \xi' \to \eta$ and a morphisms $h : p(\xi) \to p(\xi')$ such that $p(\varphi') \circ h = p(\varphi)$, there exists a unique morphism $\psi : \xi \to \xi'$ such that $p(\psi) = h$ and $\varphi' \circ \psi = \varphi$.

We often call a category fibered in groupoids over $\mathcal{S}$ shorter a *groupoid over* $\mathcal{S}$, and sometimes call a groupoid over $Sch/\mathbb{C}$ just a *groupoid*.

(ii) A *morphism* between two groupoids $\mathcal{M} \xrightarrow{p} S$, $\mathcal{M}' \xrightarrow{p'} S$ over S is a functor $q : \mathcal{M} \to \mathcal{M}'$ such that $p = p' \circ q$. Such a morphism q is called an *isomorphism* if it is an equivalence of categories between $\mathcal{M}$ and $\mathcal{M}'$ (i.e. not only if it is an isomorphism of categories).

We call $\mathcal{A}$ and $\mathcal{B}$ the moduli groupoids of nodal resp. stable curves. $\mathcal{B}$ is usually denoted as $\overline{\mathcal{M}}_{g,n}$. Instead of setting up a moduli problem as describe above, one can also define families of the moduli problem and morphisms between them first, in such a way that they constitute a *moduli groupoid* $\mathcal{M} \xrightarrow{p} Sch/C$. Declaring two families over a given scheme S to be equivalent if they are isomorphic via a morphism φ with $p(\varphi) = id_S$, the moduli groupoid defines a moduli problem and a moduli functor as above. If this moduli functor has a coarse moduli space M, we also call M a coarse moduli space of the groupoid. To pass from the moduli groupoid to the moduli functor is in general loosing information. Passing from the moduli functor to a coarse but not fine moduli space one looses information again. Accordingly one can say the following about morphisms:

Lemma & Definition 1.7 *For any two moduli problems (A) and (B) with corresponding moduli functors F_A, F_B:*

(i) A natural transformation $\Phi : F_A \to F_B$ we also call a morphism of moduli functors*. Such a Φ is an assignment as follows: If we denote for $S \in Sch/\mathbb{C}$ families of the problem (A) over S in the form $\mathcal{X}/S$ and families of the problem (B) by $\mathcal{Y}/S$, then for every $S \in Sch/\mathbb{C}$, Φ assigns to every equivalence class of families $[\mathcal{X}/S]$ an equivalence class of families $[\mathcal{Y}/S]$ which we denote by $\Phi([\mathcal{X}/S])$. This assignment is compatible with pullbacks, i.e. for every morphisms of schemes $f : S' \to S$, we have $f^*\Phi([\mathcal{X}/S)]) = \Phi(f^*[\mathcal{X}/S])$.*

(ii) If the moduli problems have moduli spaces M_A and M_B, then Φ induces by the defining property of coarse moduli spaces a unique morphism of schemes $\varphi : M_A \to M_B$.

(iii) If there are moduli groupoids $\mathcal{A} \xrightarrow{p_\mathcal{A}} Sch/\mathbb{C}$ and $\mathcal{B} \xrightarrow{p_\mathcal{B}} Sch/\mathbb{C}$ which induce the moduli functors F_A resp. F_B, then every morphism of groupoids $q : \mathcal{A} \to \mathcal{B}$ over $Sch/\mathbb{C}$ induces uniquely a morphism of moduli functors $\Phi : F_A \to F_B$.

(iv) If q is an isomorphism then Φ is a natural equivalence of functors, and if there are moduli spaces the induced $\varphi : M_A \to M_B$ is an isomorphism of schemes.

An example of a morphism between the moduli groupoids $\mathcal{A}$ of nodal curves and $\mathcal{B} = \mathcal{M}_{g,n}$ of stable curves is the forming of the stable model:

Remark 1.8 If $(X, p_1, ..., p_n)$ is an n-pointed nodal curve, then there is a unique morphism $\beta : X \to C$, such that β contracts to a point each component of X which does not fulfil the stability condition of Def. 1.2, and β is an isomorphism on all other components of X. Then $(C, \beta(p_1), ..., \beta(p_n))$ is an n-pointed stable curve. We call $(C, \beta(p_1), ..., \beta(p_n))$ (and also β) the *stable model of* $(X, p_1, ..., p_n)$. We often denote the marked points on C again by $p_1, ..., p_n$. One can simultaneously form the stable model of every fibre of a family of nodal curves $(\mathcal{X} \to S, \sigma_1, ..., \sigma_n)$ to obtain a morphism $\beta : \mathcal{X} \to \mathcal{C}$ over S, for which $(\mathcal{C} \to S, \beta \circ \sigma_1, ..., \beta \circ \sigma_n)$ is a family of pointed stable curves, which we again call the *stable model*. (Cf. section 6 of chapter 10 of [ACG11]. Especially cf. Remark (6.9) for the fact that forming the stable model is a functor from the category of families of nodal curves to the category of families of stable curves, which implies (since it obviously does not affect the base of a family) that it is a morphism of moduli groupoids.

In case of this morphism of moduli groupoids we are not interested in the induced morphisms of moduli spaces, since the moduli problem (A) does at least have no nice coarse moduli space in $Sch/\mathbb{C}$. But the gluing morphisms to boundary cycles we will discuss in section 1.3, and use later, are examples of morphisms of coarse moduli spaces induced by morphisms of moduli groupoids, namely by the clutching functors.

Instead of working with the coarse moduli space of a moduli groupoid, it is also possible to show, in some cases, that the groupoid itself behaves similar to a scheme, and to consider the groupoid as a fine moduli space for the moduli problem. This means showing that the groupoid is an *algebraic stack*. In this thesis an algebraic stack will mean a Deligne-Mumford stack. We do not define this notion here but refer to the appendix of [Vis89] or to chapter 12 of [ACG11] for a treatment in the context of moduli spaces of curves. We only remark that the definition of a (Deligne-Mumford) stack just requires a groupoid to have certain properties, but does not add any extra structure to the groupoid. Accordingly a morphism between (Deligne-Mumford) stacks is just a morphism between two groupoids, which happen to be (Deligne-Mumford) stacks. It is known that the groupoid $\overline{\mathcal{M}}_{g,n}$ fulfils the definition of a smooth Deligne-Mumford stack, and is a fine moduli space (in the category of stacks).

Most of what was said in this section can be found (often in more detail) in chapter 1 of [HM98] or in chapter 10 and 12 of [ACG11]. In most parts of the thesis we will work more with the coarse moduli spaces of our moduli problems than with the moduli stacks. But the fact that the moduli groupoids of spin/prym curves are smooth Deligne-Mumford stacks will be used to apply the intersection theory existing for such stacks.

In the category of analytic spaces there is a description of families of nodal curves which is equivalent to Definition 1.2, and which we will also use sometimes (cf. Proposition 2.1. in chapter X of [ACG11]):

Proposition 1.9 *A proper surjective morphism $\pi : X \to S$ of analytic spaces is a family* *of nodal curves if and only if the following holds. For any $p \in X$, either π is smooth at p with one dimensional fibre, or else, setting $s = \pi(p)$, there is a neighbourhood of p which is isomorphic, as space over S, to a neighbourhood of $(0, s)$ in the analytic subspace of $\mathbb{C}^2 \times S$, with equation*

$$xy = f,$$

where f is a function on a neighbourhood of s in S whose germ at s belongs to the maximal ideal of $\mathcal{O}_{S,s}$.

1.2 Spin- and prym curves and their moduli spaces.

References for the definition of spin resp. prym curves and the facts about them we collect here are for example [Cor89] resp. [BCF04]. In case of spin curves also cf. [Lud07], for a sometimes more detailed discussion. All these references however deal with spin/prym curves without marked points, but one can check that everything carries over to the case of pointed spin/prym curves. Jarvis, who gave an alternative description of spin curves also treated the pointed case (cf. [Jar00]). Also cf. [CCC07].

Definition 1.10 (i) A *semistable curve* $(X; p_1, ..., p_n)$ is a nodal curve, such that every connected component of genus 1 carries at least one special point, and every component of genus 0 carries at least two special points.

(ii) A component of genus 0 (i.e. isomorphic to $\mathbb{P}^1$) of a semistable curve $(X, p_1, ..., p_n)$ meeting the rest of X in exactly two points and carrying no marked points is called an *exceptional component* of X.

(iii) A semistable curve $(X, p_1, ..., p_n)$ is called *quasistable*, if all components of X not fulfilling the stability condition of Def. 1.2 (ii) are exceptional components, and if no two of these exceptional components intersect each other. Families of quasistable curves are families of nodal curves all whose fibres are quasistable curves.

(iv) The *non-exceptional subcurve* $\widetilde{X}$ of a quasistable curve X is the closure of the complement of all exceptional components of X.

Definition 1.11 (i) A *spin curve* resp. *prym curve* of genus g with n marked points is a tuple $\mathfrak{X} = (X; p_1, ..., p_n; \mathcal{L}; b)$, where $(X; p_1, ..., p_n)$ is a quasistable curve with n ordered marked points, and with stable model $\beta : X \to C$, $\mathcal{L}$ is a line bundle on X, such that the restriction of $\mathcal{L}$ to any exceptional component E is isomorphic to $\mathcal{O}_E(1)$. For a spin curve, b is a homomorphism $b : \mathcal{L}^{\otimes 2} \to \omega_X$ and is not zero at general points of each non-exceptional component of $(X, p_1, ..., p_n)$. For a prym curve replace ω_X by $\mathcal{O}_X$ in the above definition, and additionally forbid the case $\mathcal{L} \cong \mathcal{O}_X$. The curve $(X, p_1, ..., p_n)$ is called the *support* of the spin- resp. prym curve $\mathfrak{X}$, the pair $(\mathcal{L}; b)$ the spin- resp. prym structure on $\mathfrak{X}$. A spin- resp. prym curve is called smooth if X is smooth. If we speak about the stable model $\mathfrak{C}$ of a spin resp. prym curve $\mathfrak{X}$, we mean the stable model $\mathfrak{C} = (C; p_1, ..., p_n)$ of the support $(X; p_1, ..., p_n)$.

In case of a spin curve, one calls $\mathfrak{X}$ *even* resp. *odd*, if the number $\dim H^0(X, \mathcal{L})$ is even resp. odd.

(ii) An isomorphism $\varphi : (X; p_1, ..., p_n; \mathcal{L}; b) \to (X'; p_1, ..., p_n; \mathcal{L}'; b')$ of spin- resp. prym curves is an isomorphism $\varphi : X \to X'$ of the underlying n-pointed nodal curves, such that there is an isomorphism $\gamma : \varphi^* \mathcal{L}' \to \mathcal{L}$ which is compatible with b and b'. This means:

$$\begin{array}{ccc} (\varphi^*\mathcal{L}')^{\otimes 2} & \xrightarrow{\gamma^{\otimes 2}} & \mathcal{L}^{\otimes 2} \\ \downarrow{\scriptstyle \varphi^* b'} & & \downarrow{\scriptstyle b} \\ \varphi^*\omega_{X'} & \xrightarrow{\delta} & \omega_X \end{array} \quad \text{resp.} \quad \begin{array}{ccc} (\varphi^*\mathcal{L}')^{\otimes 2} & \xrightarrow{\gamma^{\otimes 2}} & \mathcal{L}^{\otimes 2} \\ \downarrow{\scriptstyle \varphi^* b'} & & \downarrow{\scriptstyle b} \\ \varphi^*\mathcal{O}_{X'} & \xrightarrow{\delta'} & \mathcal{O}_X \end{array}$$

commutes, where δ resp. δ' are the natural isomorphisms induced by φ. Note that γ is determined by φ up to multiplication by -1.

(iii) A family of spin resp. prym curves $(\mathcal{X} \to S; \sigma_1, ..., \sigma_n; \mathbf{L}, \mathbf{b})$ is a family of pointed nodal curves $(\mathcal{X} \to S; \sigma_1, ..., \sigma_n)$ together with a line bundle $\mathbf{L}$ on $\mathcal{X}$ and a homomorphism $\mathbf{b} : \mathbf{L}^{\otimes 2} \to \omega_{\mathcal{X}/S}$ [5] resp. $\mathbf{b} : \mathbf{L}^{\otimes 2} \to \mathcal{O}_{\mathcal{X}}$, such that the restriction to each fibre is a spin resp. prym curve. Isomorphisms of spin resp. prym curves over a fixed S are isomorphisms of the underlying families of nodal curves (cf. Def. 1.3), which are compatible with $\mathbf{b}$ and $\mathbf{b}'$ as above. In the same way one defines morphism of families of spin resp. prym curves analogously to Def. 1.5.

We define $\overline{\mathcal{S}}_{g,n}$ and $\overline{\mathcal{R}}_{g,n}$ to be the groupoids over $Sch/\mathbb{C}$, which have as their objects families of n-pointed spin resp. prym curves of genus g, and as morphisms the morphism between families of pointed spin resp. prym curves just defined. This, as explained in section 1.1, also defines the moduli problems/functors of n-pointed spin resp. prym curves of genus g.

(iv) For a given quasistable curve $(X; p_1, ..., p_n)$ we call every line bundle (i.e. invertible sheaf) $\mathcal{L}$ that fits into the definition of a spin curve or prym curve with support $(X; p_1, ...p_n)$ a *spin sheaf* resp. a *prym sheaf* of $(X; p_1, ..., p_n)$. We sometimes also call the trivial sheaf a prym sheaf, and speak of non-trivial prym sheaves if we want to exclude it.

(v) Let $\mathfrak{X} := (X; p_1, ..., p_n; \mathcal{L}; b)$, $\mathfrak{X}' := (X'; p_1', ..., p_n'; \mathcal{L}'; b')$ be two spin- or two prym curves, Let $\mathfrak{C} := (C, p_1, ..., p_n)$, $\mathfrak{C}' := (C', p_1', ..., p_n')$ be the stable models of X resp. X', let N, N' be the sets of nodes of C resp. C', to which exceptional components are contracted ("exceptional nodes"). Then there is a surjective homomorphism of isomorphism groups

$$\psi' : \mathrm{Isom}((X; p_1, ..., p_n), (X'; p_1', ..., p_n')) \to \mathrm{Isom}((C; p_1, ..., p_n; N), (C'; p_1', ..., p_n'; N'))$$

which can of course be restricted to a group homomorphism

$$\psi : \mathrm{Isom}(\mathfrak{X}, \mathfrak{X}') \to \mathrm{Isom}((C; p_1, ..., p_n; N), (C'; p_1', ..., p_n'; N'))$$

The isomorphisms lying in the kernel of ψ are called *inessential isomorphisms*. In case of $\mathfrak{X}' = \mathfrak{X}$ we speak of *inessential automorphisms*. We denote the subgroup of inessential automorphisms of a spin/prym curve $\mathfrak{X}$ by $\mathrm{Aut}_0(\mathfrak{X})$.

[5] With $\omega_{\mathcal{X}/S}$ the relative dualizing sheaf of the family of nodal curves $\mathcal{X} \to S$.

The moduli spaces $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$: For every pair $g, n \in \mathbb{N}_0$ with $2g+n \geq 3$ there exist coarse moduli spaces $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$ for n-pointed spin curves resp. prym curves of genus g. They are projective algebraic varieties of dimension $3g-3+n$, are normal and (which is a stronger property) have only finite quotient singularities. [6] Hence they are $\mathbb{Q}$-cartier, i.e. for every Weil divisor D on them, there is an $m \in \mathbb{N}$ such that mD is a Cartier divisor. The open subsets parametrising smooth spin- resp. prym curves are denoted by $S_{g,n}$ and $R_{g,n}$. The variety $\overline{S}_{g,n}$ consists of two connected components $\overline{S}^+_{g,n}$ and $\overline{S}^-_{g,n}$ parametrising the even resp. the odd spin curves. All $\overline{S}^+_{g,n}$, $\overline{S}^-_{g,n}$ and $\overline{R}_{g,n}$ are irreducible.

Remark 1.12 (i) The definition of isomorphisms of spin/prym curves given in Def. 1.11 (ii) coincides with the definition as for example given in [Cor89], [BCF04] and [FL10]. But for example in [Cor91] and [Lud10], the isomorphisms of spin curves are pairs of (φ, γ), i.e. they include an isomorphism γ of sheaves which is only required to exist in the definition we use. This choice of definition influences the number of automorphisms of spin/prym curves. More precisely if we denote for a spin/prym curve $\mathfrak{X}$ by $\mathrm{Aut}(\mathfrak{X})$ the automorphism group according to our definition, and by $\mathrm{Aut}'(\mathfrak{X})$ the one according to the other definition, there is an exact sequence

$$0 \to \mu_2 \to \mathrm{Aut}'(\mathfrak{X}) \to \mathrm{Aut}(\mathfrak{X}) \to 0,$$

with μ_2 the group of second roots of unity. The image of -1 in $\mathrm{Aut}'(\mathfrak{X})$ is the inessential automorphism (id, γ_0), where $\gamma_0 : \mathcal{L} \to \mathcal{L}$ acts as multiplication by -1 on all fibres (Cf. [Cor91]). In particular $|\mathrm{Aut}'(\mathfrak{X})| = 2 \cdot |\mathrm{Aut}(\mathfrak{X})|$. Which of these definitions one chooses does not seem to matter for most questions about spin and prym curves. In particular the coarse moduli spaces $\overline{R}_{g,n}$ and $\overline{S}_{g,n}$ remain the same, since (id, γ) acts trivially on the local universal deformation space of each spin/prym curve (cf. section 1.5).

(ii) If one uses the definition of isomorphisms which includes the isomorphism γ of the spin/prym sheaves, then one can describe generators of the group of inessential automorphisms $\mathrm{Aut}'_0(\mathfrak{X})$ as follows. Let $\widetilde{X}$ be the non-exceptional subcurve of the support of $\mathfrak{X}$, $\widetilde{X}_1, ..., \widetilde{X}_r$ its connected components. Then there are unique automorphisms $(\varphi_{\widetilde{X}_i}, \gamma_{\widetilde{X}_i})$ for $i \in \underline{r}$, where $\gamma_{\widetilde{X}_i}$ acts by multiplying by -1 on the fibres of the spin/prym sheaf $\mathcal{L}$ over $\widetilde{X_i}$, and $\gamma_{\widetilde{X}_i}$ is the identity restricted to each component $\widetilde{X}_j$ with $j \neq i$. The automorphism $\varphi_{\widetilde{X}_i}$ is of order 2 and acts non-trivially restricted to each exceptional components of X meeting $\widetilde{X}_i$ and acts trivially on all other components of X. These $(\varphi_{\widetilde{X}_i}, \gamma_{\widetilde{X}_i})$ are of order 2 and generate $\mathrm{Aut}'_0(\mathfrak{X})$. Furthermore $|\mathrm{Aut}'_0(\mathfrak{X})| = 2^r$. I.e. the inessential automorphism (φ, γ) in $\mathrm{Aut}'_0(\mathfrak{X})$ correspond to tuples $(a_1, a_2, ..., a_r)$ with all $a_i \in \{1, -1\}$, where a_i is the number by which γ multiplies each fibre of $\mathcal{L}$ over $\widetilde{X}_i$. For our choice of definition of isomorphisms however $(a_1, a_2, ..., a_r)$ and $(-a_1, -a_2, ..., -a_r)$ define the same automorphism, since the automorphism φ of X is the same in both cases. Hence $|\mathrm{Aut}_0(\mathfrak{X})| = 2^{r-1}$ with our definition.

Now we summarize some facts about spin and prym curves

[6] This follows form the fact that they are locally quotients of the smooth local universal deformation spaces of spin/prym curves, as we will see in section 1.5.

Summary 1.13 *Let $\mathfrak{X} = (X; p_1, ..., p_n; \mathcal{L}; b)$ be a spin resp. prym curve. Let $\widetilde{X}$ be the non-exceptional subcurve of X, and let D be the divisor on $\widetilde{X}$, which is the sum of all points in which $\widetilde{X}$ meets an exceptional component of X. Let $\mathcal{L}_{|\widetilde{X}}$ be the restriction of $\mathcal{L}$ to $\widetilde{X}$. Then:*

(i) For a spin curve, $\mathcal{L}^{\otimes 2}_{|\widetilde{X}} \cong \omega_{\widetilde{X}}$, while for a prym curve $\mathcal{L}^{\otimes 2}_{|\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(-D)$.

(ii) Let X_i be an irreducible component of X. If X_i is exceptional then $\mathcal{L}_{|X_i} \cong \mathcal{O}_{X_i}(-1)$. If X_i is non-exceptional, then it may be singular. Let $h : Y_i \to X_i$ be the normalisation of X_i and $\varphi_i : Y_i \to X$ the composition of h with the inclusion $X_i \hookrightarrow X$. Let M_i be the set of all points on Y_i which are preimages of nodes on X under φ. We write $M_i = M_{i,E} \uplus M_{i,N}$ where $M_{i,E}$ are the preimages of exceptional nodes [7] *and $M_{i,N}$ the preimages of non-exceptional nodes. Then, for $[M_{i,E}]$ resp. $[M_{i,N}]$ the divisor which sums up all points in $M_{i,N}$ resp. $M_{i,E}$:*

$$\varphi_i^*\mathcal{L}^{\otimes 2} = \omega_{Y_i}([M_{i,N}]) \quad \textit{if } \mathfrak{X} \textit{ a spin curve,} \quad \varphi_i^*\mathcal{L}^{\otimes 2} = \mathcal{O}_{Y_i}(-[M_{i,E}]) \quad \textit{if } \mathfrak{X} \textit{ a prym curve.}$$

(iii) Let $(X, p_1, ..., p_n)$ be an n-pointed quasistable curve, $\beta : (X; p_1, ..., p_n) \to (C; p_1, ..., p_n)$ the stable model. Let normalisations Y_i of the non-exceptional components of X, and sets of points $M_{i,E}$, $M_{i,N}$ on Y_i be defined as in (ii). Then:

There is a prym structure on $(X, p_1, ..., p_n)$ if and only if for each non-exceptional component X_i of X the number $|M_{i,E}|$ is even.

There is a spin structure on $(X, p_1, ..., p_n)$ if and only if for each non-exceptional component X_i of X the number $|M_{i,N}|$ is even.

(iv) The image of the group homomorphism ψ (defined in Def. 1.11 (v)) in the group $\mathrm{Isom}((C; p_1, ..., p_n; N), (C'; p'_1, ..., p'_n; N'))$ *can be described as follows: Denote by $\widetilde{X}$ and $\widetilde{X}'$ the non-exceptional subcurves of X resp. X', by $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$ the restrictions of the spin/prym sheaves to these subcurves, and for each $\varphi \in$* $\mathrm{Isom}((C; p_1, ..., p_n; N), (C'; p'_1, ..., p'_n; N'))$ *denote by $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}'$ the induced isomorphism. Then φ is in the image of ψ if and only if $\widetilde{\varphi}^*\widetilde{\mathcal{L}}' \cong \widetilde{\mathcal{L}}$. (cf. Prop. 2.2.11 in [Lud07])*

Remark 1.14 We sometimes also work with the more general moduli spaces of *twisted* spin resp. prym curves $\overline{S}^{(r_1,...,r_n)}_{g,n}$ resp. $\overline{R}^{(r_1,...,r_n)}_{g,n}$, for $r_1, ..., r_n \in \mathbb{Z}$ such that $\sum_{i=1}^n r_i$ is even. For a given $(r_1, ..., r_n)$ such twisted spin resp. prym curves are defined varying the definition of a spin- resp. prym curve as follows: If $(p_1, ..., p_n)$ are the marked points on X, then the line bundle $\mathcal{L}$ on X is a square root of $\omega_X(\sum_{i=1}^n r_i p_i)$ resp. $\mathcal{O}_X(\sum_{i=1}^n r_i p_i)$, instead of ω_X resp. $\mathcal{O}_X$. So for $(r_1, ..., r_n) = (0, ..., 0)$ one obtains the usual pointed spin resp. prym curves. Proceeding completely analogously to the definitions for usual pointed spin/prym curves above one defines families of twisted spin/prym curves and morphisms between such families, and thereby defines moduli groupoids $\overline{\mathcal{S}}^{(r_1,...,r_n)}_{g,n}$ and $\overline{\mathcal{R}}^{(r_1,...,r_n)}_{g,n}$ and corresponding moduli problems/functors. The coarse moduli spaces $\overline{S}^{(r_1,...,r_n)}_{g,n}$ resp. $\overline{R}^{(r_1,...,r_n)}_{g,n}$ to these moduli problems can be shown to exist as projective varieties finite over $\overline{M}_{g,n}$ in the same

[7] I.e. nodes in which X_i meets an exceptional component

way this is done for $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$. For this cf. section 4.2. of [CCC07], where also higher spin curves, i.e. curves with r-th roots of the canonical bundle for $r \geq 2$ are considered.

Proposition 1.15 *The moduli groupoids $\overline{\mathcal{S}}_{g,n}$ and $\overline{\mathcal{R}}_{g,n}$ are (for all $2g+n-3 \geq 0$) smooth Deligne-Mumford stacks. (This holds more generally also for the moduli groupoids $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)}$ and $\overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)}$.)*

This proposition seems to be some kind of folklore knowledge. At least, I do not know of a published proof of it for the definition of $\overline{\mathcal{S}}_{g,n}$ and $\overline{\mathcal{R}}_{g,n}$ given by Cornalba (and used in this thesis). But there is an alternative treatment of spin (and prym) curves by T. Jarvis in (for example) [Jar98] and [Jar00]. There the moduli problem of (higher twisted) smooth spin resp. prym curves is compactified using torsion-free sheaves on stable curves instead of line bundles on quasi-stable curve. In particular in section 2.4 of [Jar00] moduli groupoids $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K})$ are defined and shown to be smooth Deligne-Mumford stacks. Here $\mathcal{K}$ is any line bundle on the universal curve $\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, which exists as a stack. Denote by $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)'}$ and $\overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)'}$ the groupoids of twisted spin/prym curves one obtains by defining objects as we did above, but defining the morphisms instead to include an isomorphism of the (twisted) spin/prym sheaves, as discussed in Remark 1.12 (i). In section 4.2 of [CCC07] it is stated (in a more general form) that the moduli functors defined by $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)'}$ and $\overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)'}$ [8] are equivalent to the moduli functors defined by certain $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K})$, [9] and it is remarked that this is easy to prove using Proposition 4.2.2. from that section. In the following we will sketch a proof, using 4.2.2., for the somewhat stronger fact that

$$\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)'} \cong \overline{\mathfrak{S}}_{g,n}^{1/2}\Big(\omega_{\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sum_{i=1}^{n} r_i\sigma_i)\Big), \quad \text{and} \quad \overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)'} \cong \overline{\mathfrak{S}}_{g,n}^{1/2}\Big(\mathcal{O}_{\mathcal{C}_{g,n}}(\sum_{i=1}^{n} r_i\sigma_i)\Big), \tag{1.1}$$

as categories fibred in groupoids (i.e. moduli groupoids). [10] This together with Jarvis' proof that all $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K})$ are smooth Deligne-Mumford stacks of course implies Proposition 1.15. [11] Our proof is not self contained but uses several results from articles by Jarvis to

[8] Which coincide with the moduli functors induced by $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)}$ and $\overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)}$, as is easy to see.

[9] Actually the more general moduli functors there are stated to be equivalent to the functor of $\mathrm{Root}_{g,n}^{1/r}(\mathcal{K})$, but in our more special situation this agrees with $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K})$ (see below).

[10] This compatibility is one reason to prefer the alternative definition of isomorphisms of spin/prym curves to the one we use in this thesis. Of course it is also possible to change the definition of isomorphisms in Jarvis' construction in order to make it compatible with the definition we use, but contrary to Cornalba's constructions, in which both definitions of isomorphisms seem quite natural, in Jarvis construction such a definition would be artificial. One further such reason is that it seems that the "alternative" moduli groupoid $\overline{\mathcal{R}}'_g$ of prym curves is isomorphic to the moduli groupoid of unramified admissible double covers of stable genus g curves, with a natural definition of isomorphism for such covers, like in Def. 2.6 below. But we will not show this here. (In [BCF04] it is shown that the coarse moduli spaces for both moduli problems are isomorphic. Looking at the proof there it would seem that $\overline{\mathcal{R}}'_g$ is not isomorphic to the groupoid of double covers. But this is because an inappropriate definition of isomorphisms of double covers is chosen, which also does not work for the proof given there.) So probably the alternative stacks $\overline{\mathcal{S}}'_{g,n}$ and $\overline{\mathcal{R}}'_{g,n}$ are all in all preferable as stack structures for spin and prym curves to the stacks $\overline{\mathcal{S}}_{g,n}$ and $\overline{\mathcal{R}}_{g,n}$ we use.

[11] One checks that if $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)'}$ and $\overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)'}$ and $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)'}$ are smooth Deligne-Mumford stacks, then so are $\overline{\mathcal{S}}_{g,n}^{(r_1,\dots,r_n)}$ and $\overline{\mathcal{R}}_{g,n}^{(r_1,\dots,r_n)}$. (If one does not want to check that all the defining properties of

which we refer but which we do not quote.

Sketch of Proof: (Also cf. section 2.2.2. of [Jar01] for a discussion for more general higher twisted spin curves, which includes parts of what follows next.) We show (1.1) for $\overline{\mathcal{S}}^{(r_1,\ldots,r_n)'}_{g,n}$, for $\overline{\mathcal{R}}^{(r_1,\ldots,r_n)'}_{g,n}$ it works analogously. For $1/r = 1/2$ it can be seen directly from the definitions that the moduli groupoids $\mathrm{Root}^{1/2}_{g,n}\big(\omega_{\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sum_{i=1}^n r_i\sigma_i)\big)$ and $\mathfrak{S}^{1/2}_{g,n}\big(\omega_{\mathcal{C}_{g,n}/\mathcal{M}_{g,n}}(\sum_{i=1}^n r_i\sigma_i)\big)$ as defined in sections 2.2.3. resp. 2.4. of [Jar00] are isomorphic. [12] We define a functor $\Psi : \overline{\mathcal{S}}^{(r_1,\ldots,r_n)'}_{g,n} \to \mathrm{Root}^{1/2}_{g,n}\big(\omega_{\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sum_{i=1}^n r_i\sigma_i)\big)$ and show that it is an isomorphism of groupoids. On the level of objects, for a family

$$\mathbf{X} = (f : \mathcal{X} \to S; \sigma_1, ..., \sigma_n; \mathbf{L}, \mathbf{b})$$

of twisted spin curves, set, for $\pi : \mathcal{X} \to \mathcal{C}$, $\bar{f} : \mathcal{C} \to S$ the stable model of $\mathcal{X} \to S$:

$$\Psi(\mathbf{X}) := (\bar{f} : \mathcal{C} \to S; \pi \circ \sigma_1, ..., \pi \circ \sigma_n; \pi_*\mathbf{L}, \pi_*\mathbf{b}),$$

where $\pi_*\mathbf{b} : \pi_*\mathbf{L} \to \omega_{\mathcal{C}/S}$, since $\pi_*\omega_{\mathcal{X}/S} = \omega_{\mathcal{C}/S}$. [13]

$$\text{For} \quad \mathbf{X}_1 = (\mathcal{X}_1 \to S_1; \sigma_1, ..., \sigma_n; \mathbf{L}_1, \mathbf{b}_1), \quad \mathbf{X}_2 = (\mathcal{X}_2 \to S_2; \sigma'_1, ..., \sigma'_n; \mathbf{L}_2, \mathbf{b}_2),$$

let $(\Phi, \phi, \gamma) : \mathbf{X}_1 \to \mathbf{X}_2$ be a morphism with $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$, $\phi : S_1 \to S_2$, $\gamma : \mathbf{L}_1 \overset{\cong}{\to} \Phi^*\mathbf{L}_2$. Then set $\Psi((\Phi, \phi, \gamma)) = (\overline{\Phi}, \phi, \overline{\gamma})$, where $\overline{\Phi} : \mathcal{C}_1 \to \mathcal{C}_2$ is the morphism between the stable models induced by Φ. Let $\pi_1 : \mathcal{X}_1 \to \mathcal{C}_1$, $\pi_2 : \mathcal{X}_2 \to \mathcal{C}_2$ denote the contractions to the stable models. Then π_1, π_2, Φ and $\overline{\Phi}$ form a fibre square. To be able to define $\overline{\gamma}$ we first note that the natural morphism $\rho : \pi_{1*}\Phi^*\mathbf{L}_2 \to \overline{\Phi}^*\pi_{2*}\mathbf{L}_2$ (cf. [Har77] Remark 9.3.1. in chapter III), is an isomorphism in this case, which follows from Proposition 3.1.2. of [Jar98]. Now $\overline{\gamma}$ is obtained from the isomorphism $\pi_{1*}\gamma : \pi_{1*}\mathbf{L}_1 \to \pi_{1*}\Phi^*\mathbf{L}_2$ by composing with the isomorphism ρ. The defined Ψ is a functor and a morphism of groupoids. To prove that Ψ is an isomorphism, it suffices to show that it is an equivalence of categories on the fibres of the groupoids over every fixed $S \in Sch/\mathbb{C}$ (cf. Lemma (5.1.) of chapter 12 of [ACG11]). By Proposition 4.2.2. of [CCC07], Ψ is clearly essentially surjective over S. It remains to show that Ψ is full and faithful. Here it is not difficult to show, using again Proposition 4.2.2. (III), that it suffices to check that Ψ is full and faithful on the *inessential automorphisms* for each given twisted spin curve over $\operatorname{Spec}\mathbb{C}$. But this follows

smooth Deligne-Mumford stacks carry over, one can use that, as we will see in section 1.6, from general results on stacks it follows that each of the smooth Deligne-Mumford stacks $\overline{\mathcal{S}}^{(r_1,\ldots,r_n)'}_{g,n}$ and $\overline{\mathcal{R}}^{(r_1,\ldots,r_n)'}_{g,n}$ is isomorphic to a quotient stacks $[X/G]$ where X is a smooth variety and G is a linear algebraic group acting with finite stabilisers on X. By the discussion in Remark 1.12, G has to contain a (central) subgroup $\mathbb{S}_2$ which acts trivially on all of X, and whose generator corresponds to the inessential automorphism (id, γ_0). Then the grupoid $\overline{\mathcal{S}}^{(r_1,\ldots,r_n)}_{g,n}$ resp. $\overline{\mathcal{R}}^{(r_1,\ldots,r_n)}_{g,n}$ is isomorphic to the smooth quotient stack $[X/(G/\mathbb{S}_2)]$, which is Deligne-Mumford since $G/\mathbb{S}_2$ again acts with finite stabilisers.)

[12] The definition for $\overline{\mathfrak{S}}^{1/r\ \mathbf{m}}_{g,n}(\mathcal{K})$ is developed through large parts of the article, but as one can check, many conditions put on coherent nets of sheaves are empty in case $r = 2$.

[13] This is clear on the fibres over each point of S since the part of a quasistable curve which is contracted by the stable model consists of several disjoint $\mathbb{P}^1$'s, hence the canonical sheaf of a quasistable curve is trivial restricted to this subcurve. For the relative dualizing sheaves on the families this implies the same, using for example the results from section 3.1.2. of [Jar98].

from the description of inessential automorphisms in Remark 1.12 above together with the description of inessential automorphisms of Jarvis' spin curves [14] in Proposition 4.1.11. (+ proof) of [Jar98].

1.3 Generalities on boundary strata and cycles of $\overline{M}_{g,n}$

In this section we first will explain the stratification of $\overline{M}_{g,n}$ by topological type and the corresponding notions of boundary strata and boundary cycles. Then some properties of these objects will be shown. Most of the material of this section stems from [ACG11]. The material there is inspired by the appendix of [GP03]. Our notation and definitions are somehow a compromise between the ones in the two mentioned texts, but closer to [ACG11]. Some language from graph theory is used.

We want to introduce the dual graph of an n-pointed stable curve. The strata of the stratification by topological type will correspond to the different dual graphs that are possible. Cf. example 1.24 below to get an idea of how the dual graph of a stable curve looks like. We will work with an abstract notion of graphs here, in contrast to the usual "geometric" graphs. (A geometric graph can be seen as a CW-complex of dimension 1.) Using this abstract notion of graphs, we define so called stable graphs. Later we will see that each dual graph of a stable curve is a stable graph, and vice versa.

Definition 1.16 (i) An (abstract) *graph* is a tuple

$$\Gamma = (V, H, a : H \to V, i : H \to H)$$

with the following properties:

(1) V is a finite set, called the set of vertices.

(2) H is a finite set, called the set of half-edges. Each half-edge is assigned to a vertex by the map a.

(3) The map i is an involution on H, which may have fixed points. This map defines a set E, called the set of edges, and a set $L \subseteq H$ called the set of legs: $L := \mathrm{Fix}(i)$, $E := \{\{h, h'\} \mid h, h' \in H,\ i(h) = h',\ h \neq h'\}$.

Note that this data defines a (geometric) graph $[\Gamma]$ if we interpret V as the vertices of a graph and E as the edges of this graph, and say that each $e \in E$, connects the two vertices v and v' to which the two half-edges constituting e are assigned by a. ($e = \{h, i(h)\}$ for some $h \in H$.) By the definition above, $v = v'$ is possible, in which case e is called a self-edge of v (or a loop). If we allow a geometric graph to have legs, i.e. edges with one free end, we can also define a geometric graph $|\Gamma|$ by starting with $[\Gamma]$, and then for every

[14] Jarvis does not call them inessential automorphisms. The proposition there is formulated for objects over $\operatorname{Spec} k$ of $\mathrm{QSPIN}_{r,g}$ with is just $\mathrm{ROOT}^{1/r}_{g,0}(\omega_{\mathcal{C}_g/\mathcal{M}_g})$. But is is clear that the proof works for any $\mathrm{ROOT}^{1/r}_{g,n}(\mathcal{K})$ with $\mathcal{K}$ a line bundle on $\mathcal{C}_{g,n}$.

$h \in L$ attaching such a leg to the vertex $v = a(h)$. This geometric graph $|\Gamma|$ determines the abstract graph Γ and vice versa.

(ii) A *stable graph*, is an (abstract) graph Γ as above together with a "genus map" $g : V \to \mathbb{Z}_{\geq 0}$, such that

(4) The geometric graph $[\Gamma]$ is connected.

(5) For each vertex v, the stability condition (compare to Def. 1.2 (ii)) holds:

$$2g(v) - 2 + n(v) > 0$$

where $n(v)$ is the so called *valence* of Γ at v. It is the number of half-edges attached to v, i.e. $n(v) = |a^{-1}(v)|$.

The genus $g(\Gamma)$ of a stable graph Γ is defined as

$$g(\Gamma) = \sum_{v \in V} g(v) + h^1(\Gamma)$$

where $h^1(\Gamma)$ is the first Betti number of the connected geometric graph $[\Gamma]$ defined by Γ.

Denote by $v(\Gamma)$, $e(\Gamma)$, $n(\Gamma)$ the cardinality of V, E and L respectively. (In general $n(\Gamma) \neq \sum_{v \in V} n(v)$.) For a given stable graph Γ we often write $V(\Gamma)$, $H(\Gamma)$ and so on, to denote the set V of vertices resp. H of half-edges, and so on, belonging to Γ.

(iii) For a finite set P a *P-marked graph* is a graph Γ together with an injective map $p : P \to H$ with image $L = \mathrm{Fix}(i)$, called a *marking*.

For $g, n \in \mathbb{Z}_{\geq 0}$: A *stable (g, P)-graph* is a P-marked stable graph of genus g. A *stable (g, n)-graph* is an $\underline{n}$-marked stable graph of genus g.

Definition 1.17 (i) For a graph $\Gamma = (V, H, a, i)$ each pair (V', H') of subsets $V' \subseteq V$, $H' \subseteq H$ defines a *subgraph* $\Gamma(V', H') = (V', H', a', i')$, if the condition $a(H') \subseteq V'$ is fulfilled: Then define $\Gamma(V', H')$ by setting $a' := a_{|H'}$, and for each $h \in H'$, $i'(h) := i(h)$ if $i(h) \in H'$ and $i'(h) := h$ otherwise (i.e. if we include one half of an edge in H' but not the other half, then this half-edge becomes a leg in the subgraph.)

(ii) For a stable graph $\Gamma = (V, H, a, i, g)$, a stable subgraph is a subgraph $\Gamma(V', H')$ which is stable with respect to the restricted genus map $g' := g_{|V'}$.

(iii) If a graph Γ is P-marked, with marking $p : P \to H$, the subgraph has a natural structure as a P'-marked graph, where $P' := p^{-1}(H') \cup \{h \in H' | i(h) \notin H'\}$.

(iv) If $\Gamma = (V, H, a, i, g)$ is a stable graph, then each subgraph of the form $\Gamma(v) := \Gamma(\{v\}, a^{-1}(v))$ for $v \in V$ is stable and of genus $g(\Gamma(v)) = g(v)$. $\Gamma(v)$ consists of the vertex v and all half-edges attached to v. We call the $\Gamma(v)$ the *smooth cells* [15] of Γ. In a sense Γ is the disjoint union of its smooth cells. At least V is the disjoint union of the vertices of all the smooth cells of Γ, and H is the disjoint union of the sets of half-edges of all the smooth cells.

[15] We call them smooth cells because they are the maximal subgraphs of Γ which are dual graphs of smooth curves, cf. Definition 1.21.

Definition 1.18 Let $\Gamma = (V, H, a, i, g, p)$ and $\Gamma' = (V', H', a', i', g', p')$ be two P-marked stable graphs.

(i) An *isomorphism* $\varphi : \Gamma \to \Gamma'$ is a pair $\varphi = (\varphi_V, \varphi_H)$ of bijections $\varphi_V : V \to V'$, $\varphi_H : H \to H'$, such that , $a' \circ \varphi_H = \varphi_V \circ a$, $i' \circ \varphi_H = \varphi_H \circ i$, $\varphi_H \circ p = p'$, $g' \circ \varphi_V = g$.

Accordingly we define *automorphisms* of a graph Γ and the automorphism group $\mathrm{Aut}(\Gamma)$.

(ii) A *contraction* $c : \Gamma \rightsquigarrow \Gamma'$ is a pair $c = (c_V, c_H)$ of a surjection $c_V : V \to V'$ and a map $c_H : H \to H' \cup V'$, fulfilling the following conditions: The diagrams

$$\begin{array}{ccc} H & \xrightarrow{c_H} & H' \cup V' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a' \cup id_{V'}} \\ V & \xrightarrow[c_V]{} & V' \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{c_H} & H' \cup V' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i' \cup id_{V'}} \\ H & \xrightarrow[c_H]{} & H' \cup V' \end{array}$$

commute, $c_H \circ p = p'$. [16] These conditions imply that the preimage under c of every smooth cell $\Gamma(v')$ of Γ' is a subgraph of Γ. More precisely: for each $v' \in V'$ the pair $(c_V^{-1}(v'), c_H^{-1}(a'^{-1}(v'))) \subseteq (V, H)$ defines a subgraph of Γ, which we denote by $c^{-1}(\Gamma(v'))$. Now $c^{-1}(\Gamma(v'))$ is a union of smooth cells of Γ, hence a stable graph if connected. The last conditions on c are: For every $v' \in V$, $c^{-1}(\Gamma(v'))$ is connected, and is of genus $g(c^{-1}(\Gamma(v'))) = g'(v')$ and of valence $n(c^{-1}(\Gamma(v'))) = n(v')$.

Note that an isomorphism of graphs is an example of a contraction.

In [ACG11], page 313-314, contractions of graphs are introduced in a more geometric way: If one looks at the geometric graphs $|\Gamma'|$ and $|\Gamma|$, each contraction corresponds to a continuous map between these geometric graphs, contracting certain subgraphs of Γ into vertices of Γ'. By our definition above $(c_H)_{|c_H^{-1}(H')} : c_H^{-1}(H') \to H'$ is a bijection between the set of the half-edges of Γ which are not contracted into vertices with the set of all half-edges of Γ'. So this yields an inclusion $H' \hookrightarrow H$, which induces an inclusion of the edges $E' \hookrightarrow E$. We will write the image of these inclusions as $c^{-1}(E') \subseteq E$ resp. $c^{-1}(H') \subseteq H$, in accordance with the geometric meaning of a contraction just explained [17].

(iii) We say that Γ is a *specialisation* of Γ' if there exists a contraction $c : \Gamma \rightsquigarrow \Gamma'$.

Remark 1.19 Let Γ be a stable (g, P)-graph. For $v \in V(\Gamma)$, set $P(v) := p^{-1}(\Gamma(v)) := p^{-1}(a^{-1}(v))$ and let $\widetilde{L}(v)$ be the set of legs of $\Gamma(v)$ which are not in the image of p. Then $L(\Gamma(v)) = p(P(v)) \cup \widetilde{L}(v)$, and the smooth cell $\Gamma(v)$ is $P(v) \cup \widetilde{L}(v)$-marked in an obvious way. Assume we are given for each $v \in V(\Gamma)$ a stable $(g(v), P(v) \cup \widetilde{L}(v))$-graph Γ_v. We want to show that this defines a specialisation $\bar{\Gamma}$ of Γ: The set $\biguplus_{v \in V} \widetilde{L}(v)$ contains exactly those half-edges of Γ which are glued by i to become edges. The $\widetilde{L}(v)$-part of the markings on the Γ_v identify these half-edges with half edges of the Γ_v. It thus allows us to define a stable (g, P)-graph $\bar{\Gamma}$ which arises from Γ as follows: Replace each smooth cell $\Gamma(v)$ by

[16] We say that a half-edge $h \in H$ is contracted into a vertex v' if $c_H(h) = v' \in V'$. The condition $c_H \circ p = p'$ tells us that legs of Γ are mapped bijectively to the legs of Γ' by c_H. An edge $\{h_1, h_2\}$ of Γ, between vertices v_1, v_2 is either mapped to an edge between $c_V(v_1)$ and $c_V(v_2)$ or, if $c_V(v_1) = c_V(v_2) = v'$, it may be contracted into the vertex v'. This information is contained in the two commutative diagrams.

[17] This notation is also used in [ACG11].

the graph Γ_v, and glue the $\widetilde{L}(v)$-marked legs of the graphs Γ_v just like the legs of the cells $\Gamma(v)$ are glued to each other in Γ. [18] Thanks to the $P(v) \cup \widetilde{L}(v)$-marking, for each v there is a *unique* contraction $c_v : \Gamma_v \rightsquigarrow \Gamma(v)$ of stable $(g(v), P(v) \cup \widetilde{L}(v))$-graphs. The union of maps $\bar{c} := \biguplus_{v \in V} c_v$ is a contraction $\bar{c} : \bar{\Gamma} \rightsquigarrow \Gamma$, so $\bar{\Gamma}$ is a specialisation of Γ.

If $c : \widetilde{\Gamma} \rightsquigarrow \Gamma$ is a contraction, then it is naturally identified with the contraction $\bar{c} : \bar{\Gamma} \rightsquigarrow \Gamma$ one obtains by the construction just described when setting $\Gamma_v := c^{-1}(\Gamma(v))$.

Remark 1.20 Usually the marked points on an n-pointed curve are indexed by the elements of $\underline{n}$. But of course this is arbitrary and one can use any set P with n elements as index set. We call such a curve a P-pointed curve. If P is such a non-standard index set, we for example write $\overline{M}_{g,P}$ for the moduli space of stable P-pointed genus g curves.

Definition 1.21 Let $\mathfrak{C} = (C; (p_i)_{i \in P})$ be a stable P-pointed curve of genus g. Let $\pi : \widetilde{C} \to C$ be the normalisation of C.

(i) The *dual graph*, $\Gamma(\mathfrak{C})$ of this curve is the stable (g, P)-graph $\Gamma(\mathfrak{C}) = (V, H, a, i, g, p)$ defined by:

(1) V is the set of irreducible components of C, and g is the map assigning to every such component its geometric genus, i.e. the genus of its normalisation.

(2) H is the union of two sets: The set H' consisting of all the points of $\widetilde{C}$ which are mapped to nodes of C by π, and of the set $\{p_1, ..., p_n\}$.

(3) The involution $i : H \to H$ fixes the elements of $\{p_1, ..., p_n\}$, and swaps the two points in H' belonging to each node. Thus the edges E correspond to the nodes of C. Self-edges correspond to nodes in which one irreducible component of C meets itself.

(4) The map $p : \underline{n} \to \{p_1, ..., p_n\},\ i \mapsto p_i$, makes A into an $\underline{n}$-marked graph.

(ii) C consists of the irreducible components C_v corresponding to the $v \in V$. Then $\widetilde{C}$ is the disjoint union of smooth curves $\widetilde{C}_v$, where $\widetilde{C}_v$ is the normalisation of C_v. On each $\widetilde{C}_v$ we consider some "special points" as marked: First there may be some of the marked points $p_1, ..., p_n$ on C_v. We denote the set of indices of these points by $P(v) \subseteq \underline{n}$. We denote the preimage on $\widetilde{C}_v$ of each p_i with $i \in P(v)$ again by p_i. Furthermore denote by $\widetilde{L}(v)$ the set of points q on $\widetilde{C}_v$ which are preimages of nodes of C. Then $\widetilde{\mathfrak{C}}_v := (\widetilde{C}_v; (p_i)_{i \in P(v)}, (q)_{q \in \widetilde{L}(v)})$ is a smooth stable curve which is $P(v) \cup \widetilde{L}(v)$-pointed. We call the collection $\widetilde{\mathfrak{C}}$ of the $\widetilde{\mathfrak{C}}_v$ the *pointed normalisation* of $\mathfrak{C}$.

[18] More precisely: $\bar{\Gamma} = (\bar{V}, \bar{H}, \bar{a}, \bar{i}, \bar{p}, \bar{g})$, where $\bar{V}$, $\bar{H}$, $\bar{a}$, $\bar{g}$ are obtained by just taking the union over the corresponding sets/maps of the graphs Γ_v. Let $\pi_v : P(v) \cup \widetilde{L}(v) \to L(\Gamma_v)$ be the $P(v) \cup \widetilde{L}(v)$ marking of Γ_v, and let $p_v := \pi_{v|P(v)}$ be the restriction. Then $\bar{p} : P \to \bar{H}$ is the union over the p_v. Finally $\bar{i}$ is a bit more complicated to define since it has to glue together the $\widetilde{L}(v)$-marked legs of the Γ_v, although they are still fixed by the involutions i_v of the Γ_v. Identify these legs of Γ_v with $\widetilde{L}(v)$, and let i'_v be the restriction of i_v to $H(\Gamma_v) \smallsetminus \widetilde{L}(v)$ and let i' be the restriction of i to $\bigcup_{v \in V} \widetilde{L}(v) \subseteq \bar{H}$. Then $\bar{i} = i' \cup \bigcup_{v \in V} i'_v$.

Now note that a smooth cell $\Gamma(v)$ of the dual graph $\Gamma(\mathfrak{C})$ can naturally be identified with the dual graph $\Gamma(\widetilde{\mathfrak{C}}_v)$. In particular $P(v) = p^{-1}(\Gamma(v)) := p^{-1}(a^{-1}(v))$ and $\widetilde{L}(v) = L(\Gamma(v)) \setminus P(v) = H(\Gamma(v)) \setminus P(v)$.

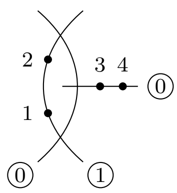
Remark 1.22 For a stable pointed curve $\mathfrak{C}$ let $\Gamma := \Gamma(\mathfrak{C})$ be its dual graph. An automorphism $\varphi \in \mathrm{Aut}(\mathfrak{C})$ permutes the nodes and irreducible components of C, while it fixes the marked points. Each φ lifts uniquely to the normalisation $\widetilde{C}$. The lifted automorphism $\varphi_{\widetilde{C}}$ then accordingly permutes the connected components, fixes the preimages of all marked points $p_1, ..., p_n$, and permutes the points of $\widetilde{C}$ which are preimages of nodes of C: Let ν_1, ν_2 be nodes of C, such that $\varphi(\nu_1) = \nu_2$, let $\bullet_i, \circ_i$ be the two preimage points on $\widetilde{C}$ of the node ν_i. Then $\varphi_{\widetilde{C}}$ restricts to a bijection $\{\bullet_1, \circ_1\} \to \{\bullet_2, \circ_2\}$.

So it is easy to see that φ induces via $\varphi_{\widetilde{C}}$ a $\varphi_\Gamma = (\varphi_V, \varphi_H) \in \mathrm{Aut}(\Gamma)$ (cf. Def. 1.18 (i)), where φ_V permutes the vertices $V(\Gamma)$ like $\varphi_{\widetilde{C}}$ permutes the corresponding components of $\widetilde{C}$, while φ_H acts on $H(\Gamma)$ like $\varphi_{\widetilde{C}}$ acts on the preimage points of the p_i and ν_i.

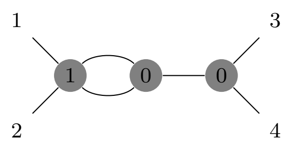
Definition 1.23 If Γ is a stable graph, a connected subgraph Γ' of Γ fulfilling the following conditions is called a *rational tree*: Γ' is connected to the rest of the graph only by one (disconnecting) edge, the graph Γ' contains no non-disconnecting edges, i.e. $h^1(\Gamma') = 0$, and all vertices of Γ' have genus 0.

If $(C, p_1, ..., p_n)$ is a stable curve with dual graph Γ then we call a subcurve of $(C, p_1, ..., p_n)$ a rational tree, if its dual graph Γ' as subgraph of Γ is a rational tree.

Example 1.24 We consider a stable genus 2 curve $\mathfrak{C} = (C; p_1, ..., p_4)$ with 4 marked points of the following type: C consists of 3 irreducible components C_1, C_2, C_3, which all are smooth. C_1 is of genus 1, C_2, C_3 are of genus 0. Component C_1 meets component C_2 in two nodes, C_2 meets C_3 in one node. There are no other nodes. The marked points with indices 1 and 2 lie on C_1, those with indices 3 and 4 on C_3. We symbolize a curve of this kind by the picture



The encircled number is the *geometric* genus of the irreducible component is stands close to. We will usually use pictures of this kind to explain how a curve looks like. Now the dual graph $\Gamma = \Gamma(\mathfrak{C})$ of this genus 2 curve looks as follows:



Here we write the genus of each vertex into the gray dot, standing for this vertex. The vertex on the right hand side with its two legs and the disconnecting edge connecting it to the rest of the graph is an example of a (small) rational tree. The graph has one non-trivial automorphism, exchanging the two edges that connect the genus 1 vertex to the genus 0 vertex in the middle.

Definition 1.25 (i) For a given stable (g, n)-graph Γ, let U_Γ be the subset of $\overline{M}_{g,n}$ parametrising curves with dual graph Γ.

(ii) The U_Γ are all non-empty, and the collection of the U_Γ for all stable (g, n)-graphs, forms a stratification of $\overline{M}_{g,n}$. It is called the *stratification by topological type*. The largest of these strata is the one belonging to the simplest graph, consisting of one vertex of genus g, no edges, and n legs attached to the vertex. This stratum is $M_{g,n}$. (For $n = 0$, the possible stable graphs Γ correspond to the classes of stable curves up to homeomorphism, therefore the name of this stratification.)

(iii) All smaller strata U_Γ are contained in the boundary of $\overline{M}_{g,n}$ and are usually called *boundary strata* of $\overline{M}_{g,n}$. For simplicity we shall call all strata U_Γ, including $M_{g,n}$, boundary strata. The closures Δ_Γ of these U_Γ will be called *boundary cycles*. The Δ_Γ are of codimension $e(\Gamma)$ in $\overline{M}_{g,n}$. The Δ_Γ of codimension 1 will be called *boundary divisors*.

The boundary $\overline{M}_{g,n} \setminus M_{g,n}$ is the union of these boundary divisors.

(iv) The Q-classes $\delta_\Gamma := [\Delta_\Gamma]_Q$ in $A^*(\overline{M}_{g,n})$ and $H^*(\overline{M}_{g,n})$ will be called *boundary cycle classes* or shorter *boundary classes*. Sometimes they are also called boundary stratum classes.

The geometry of the boundary cycles Δ_Γ can be investigated using the following *gluing morphisms*. They play an important role in computing Chow- and cohomology rings of $\overline{M}_{g,n}$:

Proposition 1.26 *(i) Let $\Gamma = (V, H, a, i, g, p)$ be a stable (g, P)-graph. Define a moduli space $\overline{M}_\Gamma$ by the product*

$$\overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{g(v),a^{-1}(v)} \cong \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)}.^{19}$$

Then there is a finite gluing morphism

$$\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,P}$$

surjecting onto Δ_Γ. (ξ_Γ is also a representable morphism of stacks.) It corresponds to taking all pairs of marked points $p_{h'}$, $p_{h'}$ indexed by elements $h, h' \in H$, such that h and h' are swapped by i, and gluing p_h and $p_{h'}$ together. "Gluing together" here means identifying the two points in such a way that the resulting curve obtains a simple node. (This can be made precise on families of curves using the clutching functor introduced in [Knu83], also cf. [ACG11] chapter 10, section 8. These clutching functors define ξ_Γ as a morphism of stacks, which then induces a morphism of the coarse moduli spaces, which we call by the same name.)

[19] So we have $\overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{\Gamma(v)}$, for $\Gamma(v)$ the smooth cells of Γ.

If $\nu_\Gamma : \widetilde{\Delta}_\Gamma \to \Delta_\Gamma$ is the normalisation, then ξ_Γ factors as $\xi_\Gamma : \overline{M}_\Gamma \xrightarrow{\xi'_\Gamma} \widetilde{\Delta}_\Gamma \xrightarrow{\nu_\Gamma} \Delta_\Gamma$. Here ξ'_Γ can be identified with the quotient morphism $\overline{M}_\Gamma \to [\overline{M}_\Gamma / \operatorname{Aut}(\Gamma)]$. In particular as morphisms of stacks, ξ'_Γ and ξ_Γ have degree $|\operatorname{Aut}(\Gamma)|$.

(ii) It follows that all boundary cycles Δ_Γ are irreducible.

(iii) Every contraction $c : \Gamma \rightsquigarrow \Gamma'$ of stable (n,P)-graphs induces a morphism of stacks $\xi_c : \overline{M}_\Gamma \to \overline{M}_{\Gamma'}$, which we call a partial gluing morphism*. It corresponds to gluing those marked points which belong to edges which are contracted by c. In this sense the gluing morphisms of (i) corresponds to the contraction of Γ to the stable (g,P)-graph consisting of one vertex and $|P|$ legs.*

(iv) For each $v \in V(\Gamma)$, let Δ_v be some boundary cycle of $\overline{M}_{g(v),a^{-1}(v)}$. Then the image of the subset $\prod_{v\in V(\Gamma)} \Delta_v \subseteq \prod_{v \in V(\Gamma)} \overline{M}_{g(v),a^{-1}(v)}$ under ξ_Γ is a boundary cycle of $\overline{M}_{g,n}$.

(v) For two stable graphs Γ_1 and Γ_2, we have $\Delta_{\Gamma_2} \subseteq \Delta_{\Gamma_1}$ if and only if Γ_2 is a specialisation of Γ_1.

Proof: For (i), cf. the appendix of [GP03], or for more details [ACG11], chapter 12, section 10.

(iii): Note that $\overline{M}_{\Gamma'} = \prod_{v' \in V(\Gamma')} \overline{M}_{\Gamma(v')}$ and $\overline{M}_\Gamma = \prod_{v\in V(\Gamma)} \overline{M}_{\Gamma(v)}$, where $\Gamma(v')$ and $\Gamma(v)$ are the smooth cells. Now Γ is the disjoint union of the stable subgraphs $c^{-1}(\Gamma(v'))$ for $v' \in V(\Gamma')$ (cf. Definition 1.18 (ii)). We have

$$\overline{M}_{c^{-1}(\Gamma(v'))} = \prod_{v \in c^{-1}(\Gamma(v'))} \Gamma(v) \quad \text{and} \quad \overline{M}_\Gamma = \prod_{v' \in \Gamma'} \overline{M}_{c^{-1}(\Gamma(v'))}.$$

Let $p : P \to H(\Gamma)$, $p' : P \to H(\Gamma')$ be the P-markings. Set $P(v') := {p'}^{-1}(\Gamma(v')) := {p'}^{-1}(a^{-1}(v'))$, and let $\widetilde{L}(v')$ be the set of legs of $\Gamma(v')$ which are not in the image of p'. Then $L(\Gamma(v')) = p'(P(v')) \cup \widetilde{L}(v')$ and the stable graph $c^{-1}(\Gamma(v'))$ is $P(v') \cup \widetilde{L}(v')$-marked in a natural way. So by (i), there are gluing morphisms

$$\xi_{v'} := \xi_{c^{-1}(\Gamma(v'))} : \overline{M}_{c^{-1}(\Gamma(v'))} \to \overline{M}_{g(v'),P(v')\cup \widetilde{L}(v')} = \overline{M}_{\Gamma(v')}.$$

The partial gluing morphism ξ_c is $\prod_{v' \in V(\Gamma')} \xi_{v'}$.

It is quite clear that $\xi_\Gamma = \xi_{\Gamma'} \circ \xi_c$, when considering how these morphism correspond to gluing marked points on curves.

(iv): By definition of a boundary cycle, each $\Delta_v \subseteq \overline{M}_{g(v),a^{-1}(v)}$ corresponds to a stable $(g(v), a^{-1}(v))$-graph Γ_v. Moreover $a^{-1}(v)$ can be identified with $P(v) \cup \widetilde{L}(v)$ (as defined in Remark 1.19) in an obvious way. Now let $c : \bar{\Gamma} \rightsquigarrow \Gamma$ be the contraction defined by this collection of $(g(v), P(v) \cup \widetilde{L}(v))$-graphs Γ_v, as in Remark 1.19. Since the partial gluing morphism $\xi_{\bar{c}}$ of (iii) corresponding to $\bar{c}$ is just the product over the gluing morphisms $\xi_{\Gamma_v} : \overline{M}_{\Gamma_v} \to \overline{M}_{g(v),a^{-1}(v)}$, the image of ξ_c is $\prod_{v\in V(\Gamma)} \Delta_v \subseteq \prod_{v\in V(\Gamma)} \overline{M}_{g(v),a^{-1}(v)}$. With $\xi_{\bar{\Gamma}} = \xi_\Gamma \circ \xi_{\bar{c}}$, we get that the image of $\prod_{v \in V(\Gamma)} \Delta_v$ under ξ_Γ is $\Delta_{\bar{\Gamma}}$.

(v): By the discussion for (iii) and (iv) it is clear that $\Delta_{\Gamma_2} \subseteq \Delta_{\Gamma_1}$, if there is a contraction $c : \Gamma_2 \rightsquigarrow \Gamma_1$.

To show the "only if" direction: For any stable graph Γ, we call $\Delta_\Gamma \smallsetminus U_\Gamma$ the boundary of Δ_Γ. Index the elements of $V(\Gamma)$ as $v_1, ..., v_r$. We call *boundary divisors of* Δ_Γ, the images under ξ_Γ of loci of the form

$$\overline{M}_{g(v_1),n(v_1)} \times ... \times D_{v_i} \times ... \times \overline{M}_{g(v_r),g(v_r)} \subset \overline{M}_{g(v_1),n(v_1)} \times ... \times \overline{M}_{g(v_r),g(v_r)} = \overline{M}_\Gamma,$$

where D_{v_i} is a boundary divisor of $\overline{M}_{g(v_i),n(v_i)}$. Each boundary divisor of Δ_Γ can be written as $\Delta_{\Gamma'}$ for some specialisation Γ' of Γ, by the proof of (iv). Now the boundary of Δ_Γ is the union of the boundary divisors of Δ_Γ, since the boundary of each $\overline{M}_{g(v_i),n(v_i)}$ is the union of the boundary divisors of $\overline{M}_{g(v_i),n(v_i)}$. The latter fact follows from deformation theory, which tells us that each point of $\overline{M}_{g(v_i),n(v_i)}$ parametrising a nodal curve, lies in the closure of the locus in $\overline{M}_{g(v_i),n(v_i)}$, parametrising curves with exactly one node. (Cf. Summary 1.30 (vii)).

As a boundary cycle, Δ_{Γ_2} is irreducible. If $\Delta_{\Gamma_2} \subsetneqq \Delta_{\Gamma_1}$, then Δ_{Γ_2} must be contained in the boundary of Δ_{Γ_1}, and hence, by irreducibility, in one boundary divisor $\Delta_{\Gamma_1'}$ of Δ_{Γ_1}. Γ_1' is a specialisation of Γ_1, as we have seen, and $\dim \Delta_{\Gamma_1'} = \dim \Delta_{\Gamma_1} - 1$. Now either $\Delta_{\Gamma_2} = \Delta_{\Gamma_1'}$, or we can iterate the argument until we arrive at a specialisation Γ_1'' of Γ_1 such that $\Delta_{\Gamma_2} = \Delta_{\Gamma_1''}$. □

Notation: (i) We will often use non-standard index sets (cf. Remark 1.20) of the following type when defining gluing morphisms: We use indices of the form $\bullet_i$ and $\circ_i$ to indicate which pairs of marked points will be identified by the gluing morphism. For example we would denote the gluing morphism corresponding to the graph Γ of Example 1.24 by

$$\xi_\Gamma : \overline{M}_{1,\{1,2,\bullet_1,\bullet_2\}} \times \overline{M}_{0,\{\circ_1,\circ_2,\bullet_3\}} \times \overline{M}_{0,\{3,4,\circ_3\}} \to \overline{M}_{1,\{1,2,3,4\}} = \overline{M}_{1,4}$$

In this notation one can reconstruct the graph Γ just by looking at the indices used. The notation is very similar to the one used in the articles by Nicola Pagani.

(ii) If Δ is some boundary stratum we often write ξ_Δ for the gluing morphism surjecting to it.

1.4 Generalities on boundary strata of $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$

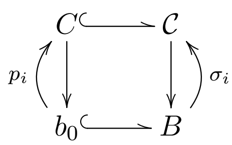
Definition 1.27 (i) Let $\overline{X}_{g,n}$ be either $\overline{S}_{g,n}$ or $\overline{R}_{g,n}$ or a space of twisted spin resp. prym curves $\overline{S}_{g,n}^{(r_1,...,r_n)}$ resp. $\overline{R}_{g,n}^{(r_1,...,r_n)}$. Let $\pi : \overline{X}_{g,n} \to \overline{M}_{g,n}$ be the forgetful morphism. If U_Γ is a stratum of the stratification of $\overline{M}_{g,n}$ by topological type, then $\pi^{-1}(U_\Gamma)$ may have several irreducible components (all of the same dimension). We define the *stratification by topological type* of $\overline{X}_{g,n}$ to be the collection of these irreducible components of the $\pi^{-1}(U_\Gamma)$ for all the possible stable (g,n)-graphs Γ.

(ii) Boundary strata, boundary cycles, boundary divisors, boundary cycle classes and so on for $\overline{X}_{g,n}$ are then defined analogously to the case of $\overline{M}_{g,n}$.

1.5 Deformation spaces of pointed spin and prym curves

In this section we give a short summary of the results about local universal deformation spaces of pointed stable curves, and pointed spin- or prym curves we will need in this thesis. The moduli spaces $\overline{M}_{g,n}$, $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$ locally are quotients of these deformation spaces, by the automorphism groups of their central fibres. We will be interested in how these automorphism groups act on the deformation spaces. We take these results mainly from [ACG11] and [Lud07]. More details can be found there. As in [ACG11], we will describe deformations in the complex analytic category, but we will call "local universal deformation" what is called a "Kuranishi family" in [ACG11], and so stay closer to the terminology of algebraic geometry.

Definition 1.28 (i) A *deformation of an n-pointed nodal curve* $\mathfrak{C} = (C; p_1, ..., p_n)$, is a family of n-pointed nodal curves $(\mathcal{C} \to B; \sigma_1, ..., \sigma_n)$ together with a closed point $b_0 \in B$ and a closed embedding $C \hookrightarrow \mathcal{C}$, fulfilling the following condition: For $C \to b_0$ the constant morphism of C to b_0, denote by $p_i : b_0 \to C$ the section having the point p_i as its image. With $b_0 \hookrightarrow B$ the inclusion, the following diagram commutes for all $i \in \underline{n}$:



[20]

We often denote such a deformation by $(C \hookrightarrow \mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$.

(ii) A *deformation of an n-pointed stable curve* $\mathfrak{C} = (C, p_1, ..., p_n)$ is defined analogously, replacing the family of pointed nodal curves by a family of pointed stable curves.

(iii) A *deformation of an n-pointed spin or prym curve* $\mathfrak{X} = (X; p_1, ..., p_n; \mathcal{L}, b)$ is a family $(\mathcal{X} \to S; \sigma_1, ..., \sigma_n; \mathbf{L}, \mathbf{b})$ together with a closed point $s_0 \in S$ and an isomorphisms between $\mathfrak{X}$ and the fibre of the family over s_0.

(iv) A morphism between two deformations of one fixed nodal curves or spin/prym curve $\mathfrak{X}$ over two bases (S, s_0) and (S', s_0') is a morphisms of the underlying families in the sense of Def. 1.5 (i.e. a pullback square), such that s_0 is sent to s_0' and such that the restriction to the central fibre induces the identity on $\mathfrak{X}$, via the given isomorphisms of $\mathfrak{X}$ to the central fibres of each deformation.

(v) A deformation $(C \hookrightarrow \mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$ of a stable curve $\mathfrak{C}$, is called a *local universal deformation*, if every deformation $(C \hookrightarrow \mathcal{C}' \to (B', b_0'); \sigma_1', ..., \sigma_n')$, is, after restricting it from B'' to an open analytic neighbourhood $\widehat{B}'$ of b_0', the pullback of $(C \hookrightarrow \mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$ via a *unique* morphism $(\widehat{B}', b_0') \to (B, b_0)$. I.e. let $\widehat{\mathcal{C}}'$ be the open subvariety of $\mathcal{C}'$ lying over $\widehat{B}'$, let $\widehat{\sigma}_i'$ be the restriction of σ_i' to $\widehat{B}'$. Then there is a morphism $\widehat{B}' \hookrightarrow B$, sending b_0' to b_0 and inducing a commutative diagram as follows, such

[20]I.e. the curve C is identified in an explicit way with the fibre of $\mathcal{C} \to B$ over b_0 (called the *central fibre*), and one further requires that the image of σ_i restricted to the central fibre C is the point p_i.

that the square in the middle is *cartesian* [21] :

$$\begin{array}{ccc} \widehat{\mathcal{C}}' & \hookrightarrow & \mathcal{C} \\ {\scriptstyle \widehat{\sigma}'_i} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle \sigma_i} \\ \widehat{B}' & \hookrightarrow & B \end{array}$$

(vi) A local universal deformation of a spin or prym curve is defined analogously.

(vii) If we have a deformation, and speak about an automorphism φ of $(\mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$ or $(\mathcal{X} \to (S, s_0); \sigma_1, ..., \sigma_n; \mathbf{L}, \mathbf{b})$, we mean a automorphism of these underlying families of pointed curves or spin/prym curves, such that $\varphi(b_0) = b_0$ resp. $\varphi(s_0) = s_0$. We do not require that φ is an automorphism of the deformation in the sense of (iii). We call such an automorphism an *automorphism of the centred family underlying the deformation.*

Notation 1.29 (i) If B is a the n-dimensional unit ball $B = \{z \in \mathbb{C}^n \mid |z| < 1\}$, we will speak about *linear subspaces of* B, meaning subsets of the form $W = B \cap V \subseteq B$, where V is a sub vector space of $\mathbb{C}^n$. By a *basis* of such a linear subspace W we will mean a basis of V. If $x_1, ..., x_r$ are vectors in $\mathbb{C}^n$, we use the notation $\operatorname{span}_B(x_1, ..., x_n) := B \cap \operatorname{span}(x_1, ..., x_n)$. A *linear action* of a group G on B will be the restriction of a linear action of G on $\mathbb{C}^n$ such that every group element acts as a bijection on B. B is said to be a *direct sum* of linear subspaces $B = W_1 \oplus ... \oplus W_m$, with $W_i = B \cap V_i$, if $\mathbb{C}^n = V_1 \oplus ... \oplus V_m$.

(ii) The next two summaries use the notation introduced in Definition 1.21 and Remark 1.22 for dual curves, pointed normalisations, and the automorphism induced on the dual graph by an automorphism of the curve. If $\Gamma = \Gamma(\mathfrak{C})$ is the dual graph of a stable curve, $e \in E(\Gamma)$ an edge, we know that to e belongs a node of $\mathfrak{C}$. We will often also name this node e, or directly call $e \in E(\Gamma)$ a node.

Summary 1.30 *For* $\mathfrak{C} := (C; p_1, ..., p_n)$ *a stable* n *-pointed curve of genus* g*, there exists a family* $(\mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$*, which is a local universal deformation of* $\mathfrak{C}$*. It has, possibly after restricting* B *to a smaller open neighbourhood of* b_0*, the following properties:*

(i) The total space $\mathcal{C}$ *is smooth and* B *is isomorphic to an open ball in* $\mathbb{C}^{3g-3+n}$*.*

(ii) The deformation is a local universal deformation not only for the fibre over b_0*, but for each of its fibres.*

(iii) Every $\varphi \in \operatorname{Aut}(\mathfrak{C})$ *on the central fibre extends uniquely to an automorphism (in the sense of Def 1.28 (vii)) of* $(\mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$*.*

(iv) For any isomorphism (of n-pointed curves) between two fibres of the family, there is a unique $\varphi \in \operatorname{Aut}(\mathfrak{C})$*, such that extension of* φ *to* $\mathcal{C}$ *restricts on the two fibres to this isomorphism. So with (iii), we can in particular make the identification* $\operatorname{Aut}(\mathfrak{C}) = \operatorname{Aut}((\mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n))$*.*

(v) Hence, locally analytically around the point $[\mathfrak{C}] \in \overline{M}_{g,n}$*,* $\overline{M}_{g,n}$ *is isomorphic to the quotient* $B/\operatorname{Aut}(\mathfrak{C})$*. More precisely, the classification map* $B \to \overline{M}_{g,n}$*, induced by the family over* B*, factors through an open embedding* $B/\operatorname{Aut}(\mathfrak{C}) \hookrightarrow \overline{M}_{g,n}$*.*

[21] This means that the square is the diagram of a fibre product

We can identify B with the open unit ball in $\mathbb{C}^{3g-3+n}$ in such a way that $b_0 = 0 \in \mathbb{C}^{3g-3+n}$, and such that (with $\Gamma := \Gamma(\mathfrak{C})$ the dual graph) all the following properties hold:

(vi) The action of $Aut(\mathfrak{C})$ on B is a linear action, in the sense of Notation 1.29 (i).

(vii) There are linear subspaces $W_v \subseteq B$ for $v \in V(\Gamma)$ and linearly independent vectors $\vec{x}_e$ for $e \in E(\Gamma)$ such that

$$B = \bigoplus_{v \in V(\Gamma)} W_v \oplus \operatorname{span}_B(\{\vec{x}_e\}_{e \in E(\Gamma)})$$

and such that, over each W_v all the nodes of C are retained, and only the irreducible component of C corresponding to v is deformed, and actually W_v is isomorphic to the local universal deformation space of $\widetilde{\mathfrak{C}}_v$. Denote by x_e the coordinate in direction $\vec{x}_e$. Let B' be a 2-dimensional complex ball with coordinates z_1, z_2, then locally analytically around the node e on C, the morphism $\mathcal{C} \to (B, b_0)$ is isomorphic to the projection from $\{z_1 \cdot z_2 = x_e\} \subset B' \times B$ to the second factor. So the node e is smoothed in direction $\vec{x}_e$, and is retained over the subspace $\{x_e = 0\} \subset B$.

(viii) The $3g(v) - 3 + n(v)$-dimensional subspace W_v can be further analysed as follows: $W_v = W_{v,Pt} \oplus W_{v,Sch}$, with

$$\dim_{\mathbb{C}} W_{v,Sch} = \begin{cases} 3g(v) - 3, & g(v) \geq 2 \\ 1, & g(v) = 1 \\ 0, & g(v) = 0 \end{cases}, \quad \dim_{\mathbb{C}} W_{v,Pt} = \begin{cases} n(v), & g(v) \geq 2 \\ n(v) - 1, & g(v) = 1 \\ n(v) - 3, & g(v) = 1 \end{cases}.$$

The deformations in $W_{v,Pt}$ only move the marked points p_i, $i \in P(v)$ and $\bullet_h$, $h \in \widetilde{L}(v)$, but keep unchanged the underlying curve $\widetilde{C}_v$. The space $W_{v,Sch}$ is generated by so called Schiffer variations, at general points of $\widetilde{C}_v$. A Schiffer variation deforms the complex structure of $\widetilde{C}_v$ locally around some point. (More precisely one obtains generators of $W_{v,Sch}$ by integrating such Schiffer variations, which are actually first order deformations, cf. [ACG11], chapter 11, section 2.)

Order the elements of V as $(v_1, ..., v_{|V|})$ and of E as $(e_1, ..., e_{|E|})$ in any way. Relative to this fixed order, for any $\varphi \in \operatorname{Aut}(\mathfrak{C})$, the permutations φ_V and φ_E correspond to permutation matrices, which we call $\mathbb{E}'_{\varphi_V}$ and $\mathbb{E}'_{\varphi_E}$. Now choose a basis $\vec{x}_{v_i,1}, ..., \vec{x}_{v_i,d(v_i)}$ for each space W_{v_i}, where $d(v_i) := \dim_{\mathbb{C}} W_{v_i} = 3g(v_i) - 3 + n(v_i)$. Then fix the basis

$$\left(\vec{x}_1, ..., \vec{x}_{3g-3+n}\right) := \left((\vec{x}_{v_i,1}, ..., \vec{x}_{v_i,d(v_i)})_{i=1,...,|V|}, (\vec{x}_{e_i})_{i=1,...,|E|}\right)$$

of B. We call such a basis of B a standard basis. *For each $\varphi \in \operatorname{Aut}(\mathfrak{C})$, the induced linear automorphism on B, restricts to isomorphisms $W_v \xrightarrow{\cong} W_{\varphi_V(v)}$ and maps each $\vec{x}_e$ to $\alpha \vec{x}_{\varphi_E(e)}$ for some $\alpha \in \mathbb{C}^*$. Hence:*

(ix) Relative to the chosen basis of B, an automorphism $\varphi \in \operatorname{Aut}(\varphi)$ is represented by a matrix $M(\varphi)$ of the form:

$$M(\varphi) = \begin{pmatrix} M_V \mathbb{E}_{\varphi_V} & 0 \\ 0 & M_E \mathbb{E}_{\varphi_E} \end{pmatrix}$$

here $\mathbb{E}_{\varphi_E} := \mathbb{E}'_{\varphi_E}$, while $\mathbb{E}_{\varphi_V}$ is the "block permutation matrix" obtained by replacing in the permutation Matrix $\mathbb{E}'_V$, for every $1 \leq i \leq |V|$, the entry 1 in the i-th column, by an identity matrix $\mathbb{1}_{v_i}$ of the size $d(v_i) \times d(v_i)$. M_E is a diagonal matrix, while M_V is a block diagonal matrix, whose $i-th$ block is of the size $d(v_i) \times d(v_i)$.

Summary 1.31 *For an n-pointed spin or prym-curve $\mathfrak{X} := (X; p_1, ..., p_n; \mathcal{L}, b)$ of genus g, there exists a local universal deformation $(\mathcal{X} \to (S, s_0); \sigma_1, ..., \sigma_n; \mathbf{L}; \mathbf{b})$. For the stable model $\mathfrak{C} = (C; p_1, ..., p_n)$ of $\mathfrak{X}$ let $(\mathcal{C} \to (B, b_0); \sigma_1, ..., \sigma_n)$ be the local universal deformation of $\mathfrak{C}$, let $\Gamma = \Gamma(\mathfrak{C})$ be the dual graph. We have, possibly after restricting S and B to smaller open neighbourhoods of s_0 resp. b_0, the following properties:*

(i) For $(\mathcal{X} \to (S, s_0); \sigma_1, ..., \sigma_n; \mathbf{L}; \mathbf{b})$, analogs of the properties (i)-(v) listed in Summary 1.30 hold.

(ii) The functor of passing from a family of spin resp. prym curves to the stable model, induces a morphisms $\widetilde{\pi} : \mathcal{X} \to \mathcal{C}$ and $\pi : (S, s_0) \to (B, b_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\widetilde{\pi}} & \mathcal{C} \\ \downarrow & & \downarrow \\ (S, s_0) & \xrightarrow[\pi]{} & (B, b_0) \end{array}$$

The morphisms in the diagram also commute with the sections σ_i of the two families. We have already indicated this by giving them the same names for both families.

(iii) For every $\varphi \in \operatorname{Aut}(\mathfrak{X})$, if we denote by $\varphi_{\mathfrak{C}} \in \operatorname{Aut}(\mathfrak{C})$ the induced automorphism on $\mathfrak{C}$, then the action of $\varphi_{\mathfrak{C}}$ on B is compatible with the action of φ on S via π. Furthermore, let $\bar{\pi} : S/\operatorname{Aut}(\mathfrak{X}) \to B/\operatorname{Aut}(\mathfrak{C})$ be the morphisms induced by π, and let $\tau : \overline{R}_{g,n} \to \overline{M}_{g,n}$ [22] *be the forgetful morphism on the moduli spaces, $B/\operatorname{Aut}(\mathfrak{X}) \hookrightarrow \overline{M}_{g,n}$, $S/\operatorname{Aut}(\mathfrak{X}) \hookrightarrow \overline{R}_{g,n}$ be the closed embeddings from 1.30 (v) and its analogue. Then following diagram commutes:*

$$\begin{array}{ccc} S/\operatorname{Aut}(\mathfrak{X}) & \hookrightarrow & \overline{R}_{g,n} \\ {\scriptstyle\bar{\pi}}\downarrow & & \downarrow{\scriptstyle\tau} \\ B/\operatorname{Aut}(\mathfrak{C}) & \hookrightarrow & \overline{M}_{g,n} \end{array}$$

We write $E(\Gamma) = E_N \uplus E_\Delta$, where E_N contains the edges corresponding to nodes which are blown up when passing from C to X, while E_Δ contains the others.

One can simultaneously identify (S, s_0) and (B, b_0) with unit balls in $\mathbb{C}^{3g-3+n}$, such that for (B, b_0) all the properties (vi)-(viii) of Summary 1.30 hold, and such that:

(iv) $\operatorname{Aut}(\mathfrak{X})$ acts linearly on S.

(v) There are linear subspaces $U_v \subseteq S$ for $v \in V(\Gamma)$ and linearly independent vectors $\vec{y}_e$ for $e \in E(\Gamma)$ such that

$$S = \bigoplus_{v \in V(\Gamma)} U_v \oplus \operatorname{span}_S(\{\vec{y}_e\}_{e \in E_\Delta}) \oplus \operatorname{span}_S(\{\vec{y}_e\}_{e \in E_N})$$

[22]Replace $\overline{R}_{g,n}$ by $\overline{S}_{g,n}$ everywhere in (iii) if $[\mathfrak{X}] \in \overline{S}_{g,n}$

and such that, over each U_v all the nodes (and exceptional components) of X are retained, and only the irreducible non-exceptional component of X, corresponding to v is deformed. Furthermore if we denote by y_e the coordinate in direction $\vec{y}_e$, then $\{y_e = 0\} \subseteq B$ is the locus over which the node resp. exceptional component of X corresponding to e is retained. I.e. this node is smoothed in direction $\vec{y}_e$, resp. the two nodes connecting the exceptional component to the rest of the curve are smoothed in direction $\vec{y}_e$.

Order the sets V, E_Δ and E_N as $(v_1, ..., v_{|V|})$, $(e_1, ..., e_{|E_\Delta|})$, resp. $(e_{|E_\Delta|+1}, ..., e_{|E|})$ in any way. For any $\varphi \in \mathrm{Aut}(\mathfrak{X})$ let $\varphi_{\mathfrak{C}} \in \mathrm{Aut}(\mathfrak{C})$ be the induced automorphism. It induces permutations φ_V, φ_E on $V(\Gamma)$ resp. $E(\Gamma)$ (cf. Remark 1.22). Now φ_E respects the partition of $E(\Gamma)$ into E_Δ and E_N and so splits into permutations φ_{E_Δ} and φ_{E_N} on these sets. Relative to the order on V, E_Δ and E_N fixed above, they correspond to permutation matrices $\mathbb{E}'_{\varphi_V}$ $\mathbb{E}'_{\varphi_{E_\Delta}}$ and $\mathbb{E}'_{\varphi_{E_\Delta}}$. Choose a basis $\vec{y}_{v_i,1}, ..., \vec{y}_{v_i,d(v_i)}$ for each space U_{v_i} ($d(v_i) := \dim_{\mathbb{C}} U_{v_i} = 3g(v_i) - 3 + n(v_i)$). Then fix the basis

$$\big(\vec{y}_1, ..., \vec{y}_{3g-3+n}\big) := \big((\vec{y}_{v_i,1}, ..., \vec{y}_{v_i,d(v_i)})_{i=1,...,|V|}, (\vec{y}_{e_i})_{i=1,...,|E|}\big)$$

of S. By (vi), setting $(\vec{x}_1, ..., \vec{x}_{3g-3+n}) := (\pi(\vec{y}_1), ..., \pi(\vec{y}_{3g-3+n}))$ gives us a basis of B. We call such simultaneously defined bases of (S, s_0) and (B, b_0) a pair of standard bases.

(vi) The forgetful morphism $\pi : (S, s_0) \to (B, b_0)$ restricts to isomorphisms $U_v \xrightarrow{\cong} W_v$. If we rearrange the basis such that $\vec{y}_1, ..., \vec{y}_{|E_N|}$ are the basis vectors of the form $\vec{y}_e$ with $e \in E_N$, we can describe π by

$$\pi\left(\sum_{i=1}^{3g-3+n} \alpha_i \vec{y}_i\right) = \sum_{i=1}^{|E_N|} \alpha_i^2 \vec{x}_i + \sum_{i=|E_N|+1}^{3g-3+n} \alpha_i \vec{x}_i, \quad \textit{for every } (\alpha_1, ..., \alpha_{3g-3+n}) \in \mathbb{C}^{3g-3+n}.$$

In particular π is a finite map of degree $2^{|E_N|}$ which is simply ramified at each subspace $\{y_e = 0\}$ for $e \in E_N$ and not ramified anywhere else. (Here we again denoted by y_e the coordinate in direction $\vec{y}_e$.)

For each $\varphi \in \mathrm{Aut}(\mathfrak{C})$, the induced linear automorphism on B, restricts to isomorphisms $W_v \xrightarrow{\cong} W_{\varphi_V(v)}$ and maps each $\vec{x}_e$ to $\alpha \vec{x}_{\varphi_E(e)}$ for some $\alpha \in \mathbb{C}^*$. Hence:

(vii) Relative to the chosen basis of S, an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{X})$ is represented by a matrix $N(\varphi)$ of the form:

$$N(\varphi) = \begin{pmatrix} N_V \mathbb{E}_{\varphi_V} & 0 & 0 \\ 0 & N_{E_\Delta} \mathbb{E}_{\varphi_{E_\Delta}} & 0 \\ 0 & 0 & N_{E_N} \mathbb{E}_{\varphi_{E_N}} \end{pmatrix}$$

here $\mathbb{E}_{\varphi_{E_\Delta}} := \mathbb{E}'_{\varphi_{E_\Delta}}$ and $\mathbb{E}_{\varphi_{E_N}} := \mathbb{E}'_{\varphi_{E_N}}$, while $\mathbb{E}_{\varphi_V}$ is the "block permutation matrix" obtained by replacing in the permutation Matrix $\mathbb{E}'_V$, for every $1 \leq i \leq |V|$, the entry 1 in the i-th column, by an identity matrix $\mathbb{1}_{v_i}$ of the size $d(v_i) \times d(v_i)$. N_{E_Δ} and N_{E_N} are diagonal matrices, while N_V is a block diagonal matrix, whose $i-th$ block is of the size $d(v_i) \times d(v_i)$. Then the induced automorphisms $\varphi_{\mathfrak{C}} \in \mathrm{Aut}(\mathfrak{C})$ is relative to the basis

$\vec{x}_1, ..., \vec{x}_{3g-3+n}$ *represented by the matrix*

$$M(\varphi_{\mathfrak{C}}) = \begin{pmatrix} N_V \mathbb{E}_{\varphi_V} & 0 & 0 \\ 0 & N_{E_\Delta} \mathbb{E}_{\varphi_{E_\Delta}} & 0 \\ 0 & 0 & N_{E_N}^2 \mathbb{E}_{\varphi_{E_N}} \end{pmatrix}$$

References/Sketches of Proof: All claims of Summary 1.30 (i)-(viii) can be found in chapter 11 of [ACG11] or follow directly from discussion there. In particular, cf. Theorem 6.5 and the discussion following it. Also cf. section 3.2.1. of [Lud07]. For Summary 1.30 (ix) cf. [Lud07] Corollary 3.2.14. The claims of Summary 1.31 can be found (for the case of spin curves without marked points) in section 3.2.2. of [Lud07] (for prym curves also cf. section 6 of [FL10]). They follow relatively directly from the claims of Summary 1.30 and from the way in which the local universal deformation of a spin curve $\mathfrak{X}$ is constructed in [Cor89] starting from the local universal deformation of the stable model $\mathfrak{C}$ of $\mathfrak{X}$, and results proved there. (The case of prym curves is analogous, cf. [BCF04]). If one reads [Cor89] one will find that this construction goes though in the case of pointed spin curves completely analogously, so that the claims of Summary 1.31 also hold in this case. What one may also find is a mistake which affects the proof of the analogue of Summary 1.30 (iv) for spin curves (this is Lemma (5.1) in [Cor89]). We give a short explanation of this mistake and sketch a way of how to repair the proof. (This is probably only understandable if one reads [Cor89] parallely. We also use the notation introduced there, which does not coincide with the one in the two summaries above.): Section 4 of [Cor89] contains two incorrect short sequences:

$$1 \to H \to G' \to \Gamma' \to 1, \quad \text{and} \quad 1 \to H \to G \to \mathrm{Aut}(\overline{C}) \to 1.$$

The latter sequence is called (4.5). Actually the image of G in $\mathrm{Aut}(\overline{C})$ is only the (in general proper) subgroup $\mathrm{Aut}_{bl}(\overline{C}) \subseteq \mathrm{Aut}(\overline{C})$, of automorphisms which map all nodes of the stable curve $\overline{C}$ which are blown up in passing to the quasi-stable curve C again to nodes of this kind [23]. This is exactly the subgroup of automorphisms of $\overline{C}$ which lift to C. Now in the proof of Lemma (5.1) there appears a $\overline{\sigma} \in \mathrm{Aut}(\overline{C})$, and it is claimed that $\overline{\sigma}$ lifts to a $\sigma \in G$. This would follow from sequence (4.5), but now requires to show that $\overline{\sigma} \in \mathrm{Aut}_{bl}(\overline{C})$. This one can prove as follows: Note, to prepare the proof, that each automorphism of the centred family underlying any deformation of a (spin) curve (cf. Def. 1.28 (vii)), is locally induced by a unique automorphism of the centred family underlying the local universal deformation of this (spin) curve. By Proposition (4.6) of [Cor89] one already knows that the $\mathcal{U} = (\rho : \mathcal{D} \to B, \zeta_{\mathcal{U}}, \alpha_{\mathcal{U}})$ constructed there is a local universal deformation of the spin curve X, and it is easy to see that $\mathcal{U}$ is also a local universal deformation of each of its fibres $\rho^{-1}(a)$. This implies that the isomorphism $\gamma : \rho^{-1}(a) \to \rho^{-1}(b)$ [24] of Lemma (5.1.) extends locally uniquely to an isomorphism $\gamma' : \rho^{-1}(\mathcal{U}_a) \to \rho^{-1}(\mathcal{U}_b)$ of neighbourhoods on $\mathcal{U}$ of our two fibres. Using that also $\overline{\mathcal{D}} \to \overline{B}$ is the local universal deformation of each of its fibres, and forming of the stable model of $\mathcal{D} \to B$, we obtain that γ' descends to some $\overline{\gamma}$ on

[23] $\mathrm{Aut}_{bl}(\overline{C})$ is new notation we introduce here.

[24] Note that γ is meant to be an isomorphism of the spin curves, which are obtained by restricting the spin structure of $\mathcal{U}$ to the quasi-stable curves $\rho^{-1}(a)$ and $\rho^{-1}(b)$, not only of the quasi-stable curves.

$\overline{\mathcal{D}} \to \overline{B}$ (maybe after restricting to smaller neighbourhoods). Choose $c \in \mathcal{U}_a$, $d \in \mathcal{U}_b$ such that $\rho^{-1}(c)$ and $\rho^{-1}(d)$ are smooth and $\gamma'(\rho^{-1}(c)) = \rho^{-1}(d)$. Let $\overline{c}$ and $\overline{d}$ be the images on $\overline{B}$. If we choose standard bases of B and $\overline{B}$ as in Summary 1.31 above, then we can write in coordinates

$$c = (c_1, ..., c_m, c_{m+1}, ..., c_{3g-3}), \quad \overline{c} = (c_1^2, ..., c_m^2, c_{m+1}, ..., c_{3g-3})$$

where m is the number of exceptional nodes of C, analogously for d, $\overline{d}$. Let $L_{\overline{c}}$, $L_{\overline{d}}$ be the (segments of) complex lines which pass through 0 and $\overline{c}$ resp. through 0 and $\overline{d}$ on $\overline{B}$. Define subsets of B:

$$S_c := \{(tc_1, .., tc_m, t^2 c_{m+1}, ..., t^2 c_{3g-3}) \mid t \in \mathbb{C}\} \cap B,$$

define S_d analogously. Then S_c and S_d are isomorphic to complex unit discs, and

$$H : S_c \to S_d, \quad (tc_1, .., tc_m, t^2 c_{m+1}, ..., t^2 c_{3g-3}) \mapsto (td_1, .., td_m, t^2 d_{m+1}, ..., t^2 d_{3g-3})$$

$$h : L_{\overline{c}} \to L_{\overline{d}}, \quad (tc_1^2, .., tc_m^2, tc_{m+1}, ..., tc_{3g-3}) \mapsto (td_1^2, .., td_m^2, td_{m+1}, ..., td_{3g-3})$$

are isomorphisms which form, together with the restrictions of the cover $\pi : B \to \overline{B}$, a commutative diagram

$$\begin{array}{ccc} S_c & \xrightarrow{H} & S_d \\ \downarrow{\scriptstyle \pi_c} & & \downarrow{\scriptstyle \pi_d} \\ L_{\overline{c}} & \xrightarrow{h} & L_{\overline{d}} \end{array}$$

Since all automorphisms of $\overline{C}$ act linearly on $\overline{B}$, h is the restriction of the action of $\overline{\sigma}$. Set $S_c' := S_c \setminus \{0\}$, $S_d' := S_d \setminus \{0\}$, then the families of smooth curves $\rho^{-1}(S_c') \to S_c'$ and $\rho^{-1}(S_d') \to S_d'$ are pullbacks of the family $\overline{\mathcal{D}} \to \overline{B}$ via π_c resp. π_d. Hence there is an isomorphism $\gamma'' : \rho^{-1}(S_c') \to \rho^{-1}(S_d')$ of families of curves, which is compatible with $H_{|S_c'}$. Since γ' locally lifts the action of $\overline{\sigma}$, we see that γ'' and γ' agree everywhere they are both defined. So over $S_c' \cap \mathcal{U}_a$, γ'' is an isomorphism of families of spin curves. Since spin sheaves extend over families of curves uniquely (cf. Remark 3.0.6. of [CCC07]), γ'' is even an isomorphism of families of spin curves over S_c'. But then by Lemma (5.3) of [Cor89], which is proven without using (5.1.), γ'' extends to an isomorphism of centred families of spin curves $\gamma''' : \rho^{-1}(S_c) \to \rho^{-1}(S_d)$. Now if we call σ the restriction of γ''' to the central fibre X, the automorphism σ induces an automorphism of $\mathcal{U}$ which coincides with γ''' over $\rho^{-1}(S_c)$. But then σ must be a lifting of the automorphism $\overline{\sigma}$ which induces the isomorphism $\overline{\gamma}'$. □

Lemma & Definition 1.32 *Let (B, b_0) be the local universal deformation space of a stable curve $\mathfrak{C}$ and assume, that we have identified (B, b_0) with the unit ball in $\mathbb{C}^{3g-3+n}$ and chosen a standard basis as in Summary 1.30. For $\varphi \in \mathrm{Aut}(\mathfrak{C})$ we say that φ* extends into a direction $\vec{z}$ *of a vector $\vec{z} \in \mathbb{C}^n$ if $\mathrm{span}_B(\vec{z}) \subseteq \mathrm{Fix}(\varphi) := \{b \in B \mid \varphi(b) = b\}$. Then:*

(i) Assume that φ fixes the node of C belonging to an $e \in E$. We will also call the node e. Let α_1 and α_2 be the weights with which φ acts on the tangent spaces to the two branches

of C meeting in e. Let N be the order of φ. Then φ extends in the direction $\vec{x}_e$ if and only if $N|(\alpha_1+\alpha_2)$. If N does not divide $\alpha_1+\alpha_2$ then we even have $\mathrm{Fix}(\varphi) \subseteq \{x_e = 0\}$.

Let $\mathfrak{X}$ be a spin or prym curve, $\mathfrak{C}$ be the stable model, (S, b_0) and (B, b_0) the local universal deformation spaces, already suitably identified with the unit ball in $\mathbb{C}^{3g-3+n}$ as in Summary 1.31. Let $\varphi \in \mathrm{Aut}(\mathfrak{X})$, $\varphi_{\mathfrak{C}} \in \mathrm{Aut}(\mathfrak{C})$ the induced automorphism.

(ii) If $\varphi_{\mathfrak{C}}$ is of order 2 then we can choose a pair of standard bases of (S, s_0) and (B, b_0) in such a way that for each pair of nodes $e_1, e_2 \in E$ of C which are swapped by $\varphi_{\mathfrak{C}}$, one has $\varphi_{\mathfrak{C}}(\vec{x}_{e_1}) = \vec{x}_{e_2}$, $\varphi_{\mathfrak{C}}(\vec{x}_{e_2}) = \vec{x}_{e_1}$.

(iii) The group of inessential automorphisms $\mathrm{Aut}_0(\mathfrak{X})$ (cf. Def. 1.11 (v), Remark 1.12) acts on (S, s_0) as follows: Let $(a_1, ..., a_r) \in \{-1, 1\}^r$ be the tuple (unique up to multiplying all entries by -1) which belongs to a $\varphi \in \mathrm{Aut}_0(\mathfrak{X})$. Then φ acts on (S, s_0) by $\varphi(\vec{y}_e) = -\vec{y}_e$ for all $e \in E_N$ with the property that e connects two components $\widetilde{X}_i$ and $\widetilde{X}_j$ of the non-exceptional subcurve $\widetilde{X}$, such that $a_i \neq a_j$. All other vectors of the standard basis are fixed by φ.

Proof: (i): By Summary 1.30 (vii), in particular the local description of the deformation around the node e by $z_1 \cdot z_1 = x_e$, we see that φ acts on the coordinate x_e by

$$x_e = z_1 \cdot z_2 \mapsto \nu_N^{\alpha_1} z_1 \cdot \nu_N^{\alpha_2} z_2 = \nu_N^{\alpha_1+\alpha_2} x_e,$$

where ν_N is a primitive N-th root of unity. (Also cf. [Pag09].)

(ii) Choose an arbitrary pair of standard bases first. We use that $M(\varphi_{\mathfrak{C}})$ is of the form of Summary 1.31 (vii). This tells us, since e_1 and e_2 are swapped and $\varphi_{\mathfrak{C}}$ has order 2, that $\varphi_{\mathfrak{C}}$ acts on $\mathrm{span}_B(\vec{x}_{e_1}, \vec{x}_{e_2})$ by a matrix

$$M = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}, \quad \text{with} \quad a_1 a_2 = 1.$$

Now we can for example replace $\vec{x}_{e_1}$ by $\frac{1}{a_2}\vec{x}_{e_1}$ in the base of B, and $\varphi_{\mathfrak{C}}$ will act on the new basis as claimed. To still retain a pair of standard bases we also replace $\vec{y}_{e_1}$ by $\frac{1}{a_2}\vec{y}_{e_1}$ if $e \in E_\Delta$ or by $\frac{1}{\sqrt{a_2}}\vec{y}_{e_1}$ if $e_1 \in E_N$. It is furthermore clear that this base-change can be done for all pairs of swapped nodes simultaneously.

(iii): cf. page 10 of [Lud10] □

Lemma 1.33 *Let Δ_Γ, $\Delta_{\Gamma'}$ be two boundary cycles of $\overline{M}_{g,n}$ defined by stable graphs Γ, Γ'. Let D and D' be two boundary cycles of $\overline{S}_{g,n}$ or of $\overline{R}_{g,n}$. Then:*

(i) The irreducible components of the set-theoretic intersection $\Delta_\Gamma \cap \Delta_{\Gamma'}$ are all of the form Δ_Λ for some stable graph Λ which is a specialisation of Γ and Γ'. Also the irreducible components of $D \cap D'$ are boundary cycles of $\overline{S}_{g,n}$ resp. of $\overline{R}_{g,n}$.

(ii) Assume that there is a $\Delta_\Lambda \subseteq \Delta_\Gamma \cap \Delta_{\Gamma'}$ such that for $m := \mathrm{codim}(\Delta_\Gamma, \overline{M}_{g,n})$, $m' := \mathrm{codim}(\Delta_{\Gamma'}, \overline{M}_{g,n})$, $\mu := \mathrm{codim}(\Delta_\Lambda, \overline{M}_{g,n})$ we have $m + m' = \mu$, i.e. Δ_Γ and $\Delta_{\Gamma'}$ "intersect properly in Δ_Λ". Then let $[\mathfrak{C}] \in \Gamma_\Lambda$ be any point, and let V, V' and W be the preimages of Δ_Γ, $\Delta_{\Gamma'}$ and Δ_Λ on the local universal deformation space (B, b_0) of $\mathfrak{C}$, and choose a

standard basis on (B, b_0) as defined in the summaries above. Denote the sets of irreducible components of V, V' resp. W by $\{V_i\}_{i\in I}$, $\{V'_j\}_{j\in J}$ resp. $\{W_k\}_{k\in K}$. All these irreducible components are then linear subspaces of (B, b_0) of codimension m, m' resp. μ. For every $k \in K$ there is exactly one $i(k) \in I$ and exactly one $j(k) \in J$ such that $W_k \subseteq V_{i(k)}$ and $W_k \subseteq V'_{j(k)}$, furthermore for these $i(k)$, $j(k)$: $W_k = V_{i(k)} \cap V'_{j(k)}$.

(iii) Also if D'' is a boundary cycle of $\overline{S}_{g,n}$ resp. $\overline{R}_{g,n}$ with $D'' \subseteq D \cap D'$ in which D and D' intersect properly, then on the local universal deformation space (S, s_0) of any $[\mathfrak{X}] \in D''$ the analogue of (ii) holds.

Proof: (i) is easy to check. For (ii) let $\Gamma(\mathfrak{C})$ be the dual graph of $\mathfrak{C}$, E be its set of edges. For every $F \subseteq E$ set $S(F) := \bigcap_{e\in F}\{x_e = 0\}$ (for the coordinates x_e as in the summaries above). Then for each subset $F \subseteq E$ such that the stable graph obtained from $\Gamma(\mathfrak{C})$ by contracting all edges in $E \setminus F$ is isomorphic to Γ, the linear subspace $S(F) \subset B$ is one of the V_i. Furthermore each V_i is of this form. Analogously for the V'_j and W_k. Denote by $F(V_i)$, $F(V'_j)$, $F(W_k)$ the subsets of E corresponding to the irreducible components in this way. It is clear that there must be at least one $i(k)$ and one $j(k)$ such that $W_k \subseteq V_{i(k)} \cap V'_{j(k)}$. Also for such $i(k)$, $j(k)$ one must have $F(V_{i(k)}) \cap F(V'_{j(k)}) = \emptyset$, since otherwise $\operatorname{codim}(V_{i(k)} \cap V'_{j(k)}, B) < m + m'$ and hence W_k would be contained in a larger irreducible component of $W = V \cap V'$. In particular this implies $W_k = V_{i(k)} \cap V'_{j(k)}$, hence $F(W_k) = F(V_{i(k)}) \cup F(V'_{j(k)})$. Now assume there is another $i'(k) \in I$ such that $W_k \subseteq V_{i'(k)}$. Then $F(W_k) = F(V_{i(k)}) \cup F(V_{i'(k)}) \cup F(V'_{j(k)})$, from which by what we already discussed it follows that $F(V_{i(k)}) = F(V_{i'(k)})$, so $i'(k) = i(k)$. One can see (iii) using (ii) as follows: Say we are on $\overline{R}_{g,n}$, let $\tau : \overline{R}_{g,n} \to \overline{M}_{g,n}$ be the forgetful morphisms and set $\tau(D) = \Delta$, $\tau(D') = \Delta'$, $\tau(D'') = \Delta''$. Then Δ, Δ' intersect properly in Δ'', and for $\mathfrak{C}$ the stable model of $\mathfrak{X}$, (ii) holds on the deformation space (B, b_0) of $\mathfrak{C}$. Now one obtains (iii) by the description of the forgetful morphism $\pi : (S, s_0) \to (B, b_0)$ from Summary 1.31 (vi) and by 1.31 (iii), and the definition of boundary cycles of $\overline{R}_{g,n}$. □

1.6 Rational cohomology and rational Chow ring for smooth Deligne-Mumford stacks.

We will work with the rational Chow ring as well as with the rational cohomology of our moduli spaces. Every variety X has a Chow group $A_*(X)$ and a (singular) cohomology group $H^*(X)$. But since $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$ are in general singular one might suspect that there is a problem with the multiplicative structure on $A_*(X)$, i.e. the intersection product, and that $A^*(...)$ may not be isomorphic to $A_*(...)$. But there is an intersection theory (with rational coefficients) for smooth Deligne-Mumford stacks and for their coarse moduli spaces, which has more or less the same properties as the analogous theories for smooth varieties. Since $\overline{\mathcal{S}}_{g,n}$ and $\overline{\mathcal{R}}_{g,n}$ are such stacks by Proposition 1.15, we can apply this theory.

In [Mum83], D. Mumford introduced the rational Chow ring of Q-varieties and Q-stacks with global Cohen-Macaulay cover. More generally intersection theory with rational coefficients on smooth Deligne-Mumford stacks was developed in [Vis89] by A. Vistoli. Earlier,

H. Gillet in [Gil84] had introduced such an intersection theory, under the assumption that the stack was of finite type over a field, using higher K-theory. We compile some results about the Chow ring of smooth Deligne-Mumford stacks and their coarse moduli spaces. References for this are [Gil84], [Vis89] and for some points [Ful98]. Also much of the following is taken from section 2 of [AGV08], which is a compilation of facts about Chow rings and cohomology of stacks. We choose the conditions on the stacks in our following Summaries in such a way that also H. Gillet's intersection theory and the one introduced for quotient stacks via equivariant Chow rings in [EG98] apply, and are known to coincide with the one introduced in [Vis89]. [25] So we can use results proven for any of these intersection theories. We also fix some notation in the Summaries:

Summary 1.34 *Let $\mathcal{M}$ and $\mathcal{M}'$ be a smooth proper integral Deligne-Mumford stack of finite type over $\mathbb{C}$. They then have coarse moduli spaces M and M', which are complete irreducible varieties having only finite quotient singularities. We furthermore assume that these varieties are projective.* [26] *Then:*

(i) There is a natural proper surjective morphism of stacks $\pi : \mathcal{M} \to M$ which has degree $\frac{1}{m}$ where m is the number of automorphisms of the general objects of $\mathcal{M}$.

(ii) There is a Chow group with rational coefficients $A_*(\mathcal{M})$ defined in [Vis89], such that $A_k(\mathcal{M})$ is the group of $\mathbb{Q}$-linear combinations of closed integral substacks of $\mathcal{M}$ of dimension k, modulo a rational equivalence defined in [Vis89]. There is a pushforward $\pi_* : A_*(\mathcal{M}) \to A_*(M)$ and a pullback $\pi^* : A_*(M) \to A_*(\mathcal{M})$ which are isomorphisms of graded $\mathbb{Q}$-vector spaces. If $\mathcal{V}$ is a closed integral substack of $\mathcal{M}$ then it has a coarse moduli space V, and V is in a natural way a closed irreducible subvariety of M. If $[\mathcal{V}] \in A_*(\mathcal{M})$ resp. $[V] \in A_*(M)$ are the cycle classes, then $\pi_*[\mathcal{V}] = \frac{1}{r}[V]$, where r is the number of automorphisms of a general object of $\mathcal{V}$.

Notation: *We usually identify $A_*(\mathcal{M})$ with $A_*(M)$ via π_*. Under this identification we usually denote the class $[\mathcal{V}]$ in $A_*(M)$ as $[V]_Q$. Hence, for V irreducible, $[V] = r[V]_Q$, where r is the number of automorphisms of almost all objects parametrised by points of V.*

(iii) On $A_*(\mathcal{M})$ an intersection product is defined in [Vis89] which has more or less the same properties as the intersection product on smooth varieties. In particular the properties described in Proposition 8.1.1 of [Ful98] all hold for this intersection product. For $\alpha, \beta \in A_*(\mathcal{M})$ we denote the product by $\alpha \cdot \beta$. An intersection product on $A_*(M)$ is defined by the identification with $A_*(\mathcal{M})$ via π_*. This product is dependent on $\mathcal{M}$, not only on M.

[25] This works since the conditions on $\mathcal{M}$ in the following summaries, which are obviously fulfilled for $\overline{\mathcal{S}}_{g,n}$ and $\overline{\mathcal{R}}_{g,n}$, imply that the stack $\mathcal{M}$ is a quotient stack: By Theorem 4.4 of [Kre09] every smooth separated Deligne-Mumford stack over a field of characteristic 0 with quasi-projective coarse moduli space is a quotient stack. So $\mathcal{M} \cong [X/G]$ for some smooth irreducible variety X and some linear algebraic group G acting with finite and reduced stabilisers on X. (G acts with finite reduced stabilisers since $\mathcal{M}$ is Deligne-Mumford, X is a smooth irreducible variety since the natural morphism $X \to [X/G]$ is smooth for a quotient stack, and since we assume $\mathcal{M} = [X/G]$ to be smooth and integral). Hence $A^*(\mathcal{M})$ can be identified with the G-equivariant Chow ring of X (cf. [EG98] or [Edi10]). Furthermore the intersection theory defined on the quotient stack in this way coincides with the intersection theory on smooth Deligne-Mumford stacks by Vistoli as well as with the one by Gillet. (This is Proposition 11 of [EG98].)

[26] Most results on $\mathcal{M}$ listed here, also hold without some or any of these assumed properties, cf. [Vis89].

One can define for $\alpha, \beta \in A_*(M)$ *a product* $\alpha \bullet \beta := m\pi_*(\pi^*\alpha \cdot \pi^*\beta)$, *which is independent of* $\mathcal{M}$, *where* m *as in (i).*

Then $\alpha \bullet \beta = \frac{1}{m}\alpha \cdot \beta$ *for all* $\alpha, \beta \in A_*(M)$. *In particular* $[M]$ *is the neutral element of the multiplication* $\bullet$, *while for* $\cdot$ *the neutral element is* $[M]_Q$. *The map* $A_*(M) \xrightarrow{m\cdot} A_*(M)$, *multiplying every element by the number* m, *is an isomorphism of graded* $\mathbb{Q}$-*algebras from* $A_*(M)$ *with the multiplicative structure given by the product* $\cdot$ *to* $A_*(M)$ *with the multiplicative structure given by* $\bullet$.

Via bivariant intersection theory the ring $A^*(\mathcal{M})$ *is defined and turns out to be isomorphic to* $A_*(\mathcal{M})$. *We will usually just interpret* $A^*(\mathcal{M})$ *resp.* $A^*(M)$ *as* $A_*(\mathcal{M})$ *resp.* $A_*(M)$ *with reversed grading (i.e.* $A^r(\mathcal{M}) = A_{n-r}(\mathcal{M})$, *where* n *is the dimension of* $\mathcal{M}$*).*

Convention: *For* $M \in \{\overline{R}_{g,n}, \overline{S}_{g,n}, \overline{M}_{g,n}\}$, *when talking about the Chow ring* $A_*(M)$ *or* $A^*(M)$, *we will always use the multiplication* $\cdot$ *induced by the identification with* $A_*(\mathcal{M})$ *for the corresponding* $\mathcal{M} \in \{\overline{\mathcal{R}}_{g,n}, \overline{\mathcal{S}}_{g,n}, \overline{\mathcal{M}}_{g,n}\}$, *not the "intrinsic" multiplication* $\bullet$. [27] *An advantage of this choice can be seen in (v) below, a disadvantage in (iv).*

(iv) For all morphisms of stacks $g : \mathcal{M} \to \mathcal{M}'$ *(with* $\mathcal{M}$, $\mathcal{M}'$ *as above), there is a pullback* $g^* : A_*(\mathcal{M}') \to A_*(\mathcal{M})$ *and if* g *is proper there is a pushforward* $g_* : A_*(\mathcal{M}) \to A_*(\mathcal{M}')$, *such that* g^* *is a homomorphisms of graded rings and* g_* *is a homomorphism of graded* $\mathbb{Q}$-*vector spaces. Furthermore the projection formula* $g_*(g^*(\alpha) \cdot \beta) = \alpha \cdot g_*(\beta)$ *holds for all* $\alpha \in A_*(\mathcal{M}')$, $\beta \in A_*(\mathcal{M})$, *where " $\cdot$ " denotes the intersection product on* $\mathcal{M}$ *resp.* $\mathcal{M}'$.

If $f : M \to M'$ *is any morphism of schemes for* M, M' *as above, then there is a pullback* $f^* : A_*(M') \to A_*(M)$, *and if* f *proper there is the usual pushforward* $f_* : A_*(M) \to A_*(M')$, *with the following properties:* f^* *coincides with the usual flat pullback if* f *is flat, and is a homomorphism of graded* $\mathbb{Q}$-*algebras* for the ring structures on $A_*(M)$ and $A_*(M')$ defined by their "intrinsic" intersection products $\bullet$. *Also the projection formula holds for these intrinsic products:* $f_*(f^*(\alpha) \bullet \beta) = \alpha \bullet f_*(\beta)$ *. Since we work with the products " $\cdot$ " on* $A_*(M)$ *and* $A_*(M')$ *depending on* $\mathcal{M}$ *resp.* $\mathcal{M}'$, *we usually adjust the pullback: Let* m, m' *be the number of automorphisms of the general objects of* $\mathcal{M}$ *resp.* $\mathcal{M}'$, *then define the* adjusted pullback $f^{\circledast}$ *by* $f^{\circledast}(\alpha) := \frac{m'}{m} f^*(\alpha)$ *for all* $\alpha \in A^*(M')$. *Now* $f^{\circledast}$ *is a homomorphism of graded* $\mathbb{Q}$-*algebras for the induced multiplications* $\cdot$ *we use, and the projection formula* $f_*(f^{\circledast}(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$ *holds. Furthermore, if* f *is induced by a morphism of stacks* $g : \mathcal{M} \to \mathcal{M}'$, *then* $f^{\circledast} = g^*$, *using the identification of* $A^*(M)$ *with* $A^*(\mathcal{M})$ *and* $A^*(M')$ *with* $A^*(\mathcal{M}')$ *introduced above. (We later almost exclusively use the adjusted pullback* $f^{\circledast}$, *and thus will denote* $f^{\circledast}$ *instead by* f^*, *in every chapter except these preliminaries.)*

(v) If V *and* V' *are closed irreducible subvarieties of codimensions* d *resp.* d' *in* M, *which intersect properly, i.e. all components* $W_1, ..., W_k$ *of the set theoretic intersections* $V \cap V'$ *are of the expected codimension* $d + d'$, *then*

$[\mathcal{V}]_Q \cdot [\mathcal{V}']_Q = \sum_{j=1}^k i_j [W_j]_Q$, *and the multiplicity* $i_j \neq 0$ *can be calculated locally on étale*

[27] With our definition of automorphisms of prym/spin curves, $\cdot$ and $\bullet$ only differ by a factor 2 in the cases $(g, n) = (2, 0)$ and $(g, n) = (1, 1)$. For all other values of (g, n), $\cdot$ and $\bullet$ agree. (This holds for all of $\overline{M}_{g,n}$, $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$.)

sheets, in the following sense: Let U be a scheme, $f : U \to \mathcal{M}$ be an étale morphism of stacks, whose image contains the generic points of those W_j with $j \in L$ for some set $L \subseteq \underline{k}$. Let $f^{-1}(V)$, $f^{-1}(V')$ and $f^{-1}(W_j)$ be the reduced preimages on U. Then $f^{-1}(V)$ and $f^{-1}(V')$ intersect properly, and $[f^{-1}(V)] \cdot [f^{-1}(V')] = \sum_{j \in L} i_j [f^{-1}(W_j)]$. (Cf. the paragraph before Theorem 6.9. of [Gil84]) For our moduli spaces of (spin/prym) curves this means that we can calculate the intersection multiplicity for a W_j on the local universal deformation space of an object parametrised by a general point of W_j.

(Since the morphism from the deformation space to the moduli stack, induced by the universal family over the deformation space, is étale, as is easy to check.) In particular, if D and D' are boundary cycles of $\overline{M}_{g,n}$, $\overline{S}_{g,n}$ or $\overline{R}_{g,n}$ which intersect properly, then, with Lemma 1.33, their Q-classes intersect transversally, i.e. $[D]_Q \cdot [D']_Q = [D \cap D']_Q$ where $D \cap D'$ is the (reduced) set-theoretic intersection.

Remark 1.35 (i) Analogously to the intersection multiplicities, also flat pullbacks of Q-classes can locally be computed on the étale sheets. Hence with Summary 1.31 (vi):

If Δ_Γ is a boundary cycle of $\overline{M}_{g,n}$, $\tau_{\overline{S}_{g,n}} : \overline{S}_{g,n} \to \overline{M}_{g,n}$ the forgetful morphism. Note that $\tau_{\overline{S}_{g,n}}$ is induced by the forgetful morphism of stacks $\tau_{\overline{\mathcal{S}}_{g,n}} : \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Let $D_1, ..., D_k$ be the irreducible component of the (reduced) preimage $\tau^{-1}_{\overline{S}_{g,n}}(\Delta_\Gamma)$, then in particular each D_i is a boundary cycle of $\overline{S}_{g,n}$ and:

$$\tau^*_{\overline{\mathcal{S}}_{g,n}}([\Delta]_Q) = \sum_{i=1}^{k} 2^{r_i} [D_i]_Q, \tag{$*$}$$

where r_i is the number of exceptional components of a general spin curve parametrised by D_i. Furthermore $\tau^*_{\overline{\mathcal{S}}_{g,n}} = \tau^{\circledast}_{\overline{S}_{g,n}} = \tau^*_{\overline{S}_{g,n}}$ for our definition of isomorphisms of spin/prym curves, since $m = m'$ for m resp. m' the number of automorphisms of a general object of $\overline{\mathcal{S}}_{g,n}$ resp. of $\overline{\mathcal{M}}_{g,n}$. (For pullbacks along $\tau_{\overline{R}_{g,n}}$ the same holds.)

(ii) Because of the way we identified $A^*(M)$ with $A^*(\mathcal{M})$ and $A^*(M')$ with $A^*(\mathcal{M}')$ we have $g_* = f_*$ for $g : \mathcal{M} \to \mathcal{M}'$ a proper morphism of stacks, and $f : M \to M'$ the induced proper morphism of the coarse moduli spaces. If V is a closed irreducible subvariety of M then $f_*([V]) = \deg(f_{|V}) \cdot [f(V)]$, where $f(V)$ is the image (cf. section 1.4. of [Ful98]). The according formula for Q-classes is hence $f_*([V]_Q) = \frac{r'}{r} \deg(f_{|V}) \cdot [f(V)]_Q$, where r resp. r' is the number of automorphisms of objects parametrised by general points of V resp. of $f(V)$. There is a notion of degree for proper morphisms of D-M-stacks and with the conditions put on the stacks in the above summary, we have $\deg(g) = \frac{m'}{k} \deg(f)$, where k is the number of automorphisms of general objects parametrised by $f(M')$. So for the (reduced) preimage $\mathcal{V}$ of V on $\mathcal{M}$: $\deg(g_{|\mathcal{V}}) = \frac{r'}{r} \deg(f_{|V})$. So $g_*([\mathcal{V}]) = \deg(g_{|\mathcal{V}}) \cdot [g(\mathcal{V})]$ or equivalently $f_*([V]_Q) = \deg(g_{|\mathcal{V}}) \cdot [f(V)]_Q$.

Concerning the homology and cohomology with rational coefficients of smooth Deligne-Mumford stacks and their coarse moduli spaces, we compile the following results, mainly taken from section 2 of [AGV08].

Summary 1.36 *Let $\mathcal{M}$, $\mathcal{M}'$, M, M' be as in Summary 1.34. Then:*

(i) One can define $H_*(\mathcal{M})$ *and* $H^*(\mathcal{M})$ *to be just* $H_*(M)$ *resp.* $H^*(M)$*. For a morphism* $g : \mathcal{M} \to \mathcal{M}'$ *with* $f : M \to M'$ *the induced morphism of coarse moduli spaces, one then defines* $g_* : H_*(\mathcal{M}) \to H_*(\mathcal{M}')$ *resp.* $g^* : H^*(\mathcal{M}') \to H^*(\mathcal{M})$ *to be* $f_* : H_*(M) \to H_*(M')$ *resp.* $f^* : H^*(M') \to H^*(M)$*. Then* H_* *is a covariant functor from the* 2*-category of smooth proper integral Deligne-Mumford stacks of finite type over* $\mathbb{C}$ *to the category of graded* $\mathbb{Q}$*-vector spaces, and* H^* *a contravariant functor from the same* 2*-category to the category of graded commutative* $\mathbb{Q}$*-algebras.*

One also defines a cap product $\cap : H^*(\mathcal{M}) \times H_*(\mathcal{M}) \to H_*(\mathcal{M})$ *by carrying over the cap product* $\cap : H^*(M) \times H_*(M) \to H_*(M)$*. The projection formula* $g_*(g^*\alpha \cap \beta) = \alpha \cap g_*\beta$ *holds for any morphism of stacks* $g : \mathcal{M} \to \mathcal{M}'$, $\alpha \in H^*(\mathcal{M}')$, $\beta \in H_*(\mathcal{M})$.

(ii) By chapter 19 of [Ful98] for every M *there is a cycle map* $\mathrm{cyc}_M : A_*(M) \to H_*(M)$*, which is a morphism of graded vector spaces, and compatible with pushforward via proper morphisms. (One can see the collection of the* cyc *as a natural transformation between the two functors* A_* *and* H_* *which go to the category of graded vector spaces.) One defines a cycle map* $\mathrm{cyc}_{\mathcal{M}} : A_*(\mathcal{M}) \to H_*(\mathcal{M})$ *with the same properties, by composing* cyc_M *with the isomorphism* $\pi_* : A_*(\mathcal{M}) \to A_*(M)$.

Notation: *For a closed substack* $\mathcal{V}$ *of* $\mathcal{M}$ *and* $[\mathcal{V}] \in A^*(\mathcal{M})$ *its cycle class, denote* $\mathrm{cyc}_{\mathcal{M}}([\mathcal{V}]) \in H_*(\mathcal{M})$ *again by* $[\mathcal{V}]$*. For* V *a subvariety of* M *denote* $\mathrm{cyc}_M([V])$ *resp.* $\mathrm{cyc}_M([V]_Q)$ *again by* $[V]$ *resp.* $[V]_Q$*. For* $\mathrm{cyc}^{\mathcal{M}}$ *and* cyc^M*, introduced below, we apply the same convention.*

(iii) Via the cap product $\cap$ *one defines homomorphisms*

$$\mathrm{PD}_M : H^*(M) \to H_*(M), \quad \alpha \mapsto \alpha \cap [M], \quad \mathrm{PD}_{\mathcal{M}} : H^*(\mathcal{M}) \to H_*(\mathcal{M}), \quad \alpha \mapsto \alpha \cap [\mathcal{M}].$$

PD_M *and* $\mathrm{PD}_{\mathcal{M}}$ *are isomorphisms and are called the Poincaré duality for* M *resp. for* $\mathcal{M}$.

With $i_M : A^*(M) \to A_*(M)$ *and* $i_{\mathcal{M}} : A^*_{\mathcal{M}} \to A_*(\mathcal{M})$ *the natural isomorphisms inverting the grading, we define cycle maps* $\mathrm{cyc}^{\mathcal{M}} : A^*(\mathcal{M}) \to A^*(\mathcal{M})$ *and* $\mathrm{cyc}^M : A^*(M) \to H^*(M)$ *by* $\mathrm{cyc}^{\mathcal{M}} := \mathrm{PD}_{\mathcal{M}}^{-1} \circ \mathrm{cyc}_{\mathcal{M}} \circ i_{\mathcal{M}}$ *resp.* $\mathrm{cyc}^M := \mathrm{PD}_M^{-1} \circ \mathrm{cyc}_M \circ i_M$*. These cycle maps are homomorphism of graded vector spaces compatible with pullback: For* $g : \mathcal{M} \to \mathcal{M}'$, $f : M \to M'$ *morphisms of stacks resp. of varieties, we have* $\mathrm{cyc}^{\mathcal{M}} \circ g^* = g^* \circ \mathrm{cyc}^{\mathcal{M}}$*, and* $\mathrm{cyc}^{\mathcal{M}} \circ f^{\circledast} = f^{\circledast} \circ \mathrm{cyc}^{\mathcal{M}}$ *and* $\mathrm{cyc}^M \circ f^* = f^* \circ \mathrm{cyc}^M$*. Furthermore for the multiplicative structures defined by the cup product on* $H^*(\mathcal{M})$ *and* $H^*(M)$*, and for the multiplication* $\cdot$ *on* $A^*(\mathcal{M})$ *resp. the intrinsic multiplication* $\bullet$ *on* $A^*(M)$*, the maps* cyc^M *resp.* $\mathrm{cyc}^{\mathcal{M}}$ *are homomorphisms of graded* $\mathbb{Q}$*-algebras.* [28]

[28] That cyc^M and $\mathrm{cyc}^{\mathcal{M}}$ are homomorphisms of graded $\mathbb{Q}$-algebras can probably most easily be seen using the definition of the Chow ring of $\mathcal{M}$ via equivariant Chow rings from [EG98], or [Edi10]. Recall from a previous footnote that $\mathcal{M}$ is isomorphic to a quotient stack $[X/G]$ with X a smooth variety. Now, by 3.16. and 3.26 of [Edi10], $H^*([X/G])$ and $A^*([X/G])$ can be identified with the equivariant cohomology/Chow rings $H^*_G(X)$ resp. $A^*_G(X)$. Furthermore $H^*_G(X) = H^*((X \times U)/G)$ and $A^*_G(X) = A^*((X \times U)/G)$, where U is a smooth algebraic variety which "approximates" in some sense close enough the total space EG of the universal principal G-bundle. Since G acts freely on U, the quotient $(X \times U)/G$ is smooth. The cycle map $\mathrm{cyc}^{\mathcal{M}} : A^*(\mathcal{M}) \to H^*(\mathcal{M})$ then coincides with the usual cycle map $A^*((X \times U)/G) \to H^*((X \times U)/G)$ of smooth varieties (as one can check looking at the definitions in [Edi10]). But this is a homomorphism

Notation: *Since we work on $A^*(M)$ with the product $\cdot$ and the adjusted pullbacks $f^{\circledast}$, we will also adjust our cycle map cyc^M accordingly, so that it is compatible with this multiplication and adjusted pullback. Hence instead of cyc^M we use $\dot{\mathrm{cyc}}^M := \mathrm{PD}_{\mathcal{M}}^{-1} \circ \mathrm{cyc}_M \circ i_M$ as our cycle map.* [29] *Because of the multiplicativity of the cycle maps we usually denote the cup products on $H^*(\mathcal{M})$ and $H^*(M)$ like the intersection products by " $\cdot$ ".*

Furthermore we have the following two results from [Ste77]:

(iv) The hard Lefschetz theorem holds, i.e.: Let $L \in H^2(M)$ be the class of an ample divisor on M. Then for all $q \in \mathbb{N}$ the map $\omega \mapsto L^q \cup \omega$ induces an isomorphism between $H^{n-q}(M)$ and $H^{n+q}(M)$. ([Ste77] Thm. 1.13)

(v) The canonical Hodge structure of $H^k(M)$, that would be mixed for an arbitrary singular variety, is pure of weight k for all $k \geq 0$. ([Ste77] Cor. 1.5)

This allows us to speak of the pure Hodge structure on our moduli spaces, and especially to define Hodge numbers.

The following Lemmas will be used sometimes:

Lemma 1.37 *Let X be a smooth algebraic variety, let G be a finite group acting algebraically on X and let $Y = X/G$ be the quotient. Then*
(i) $H^*(Y) = \left(H^*(X)\right)^G$ (Cf. [Bre72] Page 120.)
(ii) $A^*(Y) = \left(A^*(X)\right)^G$ (Cf. [Ful98], Example 1.7.6.)

Lemma 1.38 (Faber, [Fab90]) *Let $f : X \to Y$ be a finite surjective morphism of varieties. If $A^k(X) = 0$ then $A^k(Y) = 0$ as well.*

Lemma 1.39 ([Ful98], Proposition 1.8.) *If X is a variety, Y a closed subvariety and $U = X \setminus Y$, then for every $k \in \mathbb{N}_0$ there is an exact sequence*

$$A_k(Y) \to A_k(X) \to A_k(U) \to 0$$

We define certain subspaces of the cohomology and Chow rings of our moduli spaces:

Definition 1.40 For $\overline{X}_{g,n} \in \{\overline{M}_{g,n}, \overline{S}_{g,n}, \overline{R}_{g,n}\}$ we denote by $H^*_{Div}(\overline{X}_{g,n})$ resp. $A^*_{Div}(\overline{X}_{g,n})$ the sub-$\mathbb{Q}$-algebra of $H^*(\overline{X}_{g,n})$ resp. $A^*(\overline{X}_{g,n})$ generated by all divisor classes (not only boundary divisor classes). $H^*_{BCl}(\overline{X}_{g,n})$ resp. $A^*_{BCl}(\overline{X}_{g,n})$ denotes the sub-algebra generated by all boundary cycle classes (not only divisors).

of graded $\mathbb{Q}$-algebras by Corollary 19.2. of [Ful98]. The claimed compatibility with pullbacks can also be inferred in this way form the compatibility in case of smooth varieties.

[29] Probably it would by somewhat better to just work with the moduli stacks instead of the coarse moduli spaces throughout the whole thesis, instead of making all these adjustments. But firstly I do not like to rewrite all the following chapters because of this late insight, and secondly we also work with morphisms between coarse moduli spaces which are not obviously induced by morphisms of the moduli stacks, so for them one would have to apply the adjusted pullbacks anyway.

1.7 Calculating excess intersections between boundary cycles of $\overline{M}_{g,n}$.

We will sometimes need to calculate excess intersections between boundary cycles. We take the formulas needed for this from [ACG11] or the Appendix A.4. of [GP03]. [30]

Using the notation from the previous subsection, to compute in $\overline{M}_{g,n}$ an intersection of a boundary cycle class $\delta_\Gamma := [\Delta_\Gamma]_Q$ (Γ a stable (g,n)-graph) with any other class δ', is almost the same as computing $(\xi_\Gamma)_*(\xi_\Gamma^*(\delta'))$. More exactly, because of Proposition 1.26 (i), we have

$$\delta_\Gamma\delta' = \frac{1}{|\operatorname{Aut}(\Gamma)|}(\xi_\Gamma)_*(\xi_\Gamma^*(\delta')).$$

Calculating the pushforward $(\xi_\Gamma)_*$ often is no problem, since ξ_Γ is a finite morphism which can be described quite explicitly.

In case $\delta' = [\Delta_{\Gamma'}]_Q =: \delta_{\Gamma'}$ is a boundary cycle class too, there is a recipe how to calculate $\xi_\Gamma^*(\delta_{\Gamma'}) = \frac{1}{|\operatorname{Aut}(\Gamma')|}\xi_\Gamma^*\big((\xi_{\Gamma'})_*([\overline{M}_{\Gamma'}]_Q)\big)$.

First we will describe the normal bundle N_{ξ_Γ} for the gluing morphisms $\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}$ introduced in the last section. These bundles will be needed to compute our excess intersections. Cf. [ACG11], chapter 13, section 3, page 344-346 for more details.

For any smooth Deligne-Mumford stack M it makes sense to talk about its *tangent bundle* T_M. For a definition cf. [ACG11]. Like for smooth schemes, the *normal sheaf to a morphism* $f : M \to N$ of smooth Deligne-Mumford stacks can be defined as

$$N_f := f^*T_N/T_M$$

In the case $f = \xi_\Gamma$, the sheaf N_{ξ_Γ} is actually a vector bundle (cf. [ACG11], page 345).

Definition 1.41 (i) For $\Gamma = (V, H, a, i, g, p)$ a stable (g,n)-graph, and $\overline{M}_\Gamma$ as in Proposition 1.26 (i), and $v_0 \in V(\Gamma) =: V$ a vertex, we denote by

$$\eta_{\Gamma,v_0} : \overline{M}_\Gamma := \prod_{v\in V(\Gamma)} \overline{M}_{g(v),a^{-1}(v)} \to \overline{M}_{g(v_0),a^{-1}(v_0)}$$

the projection to the factor belonging to the vertex v_0.

(ii) For any g and P a finite set, we define for any $i \in P$ a line bundle $\mathbb{L}_i$ on the stack $\overline{M}_{g,P}$, called the i-th point bundle: Let $\pi : \overline{M}_{g,P\cup\{\bullet\}} \to \overline{M}_{g,P}$ be the forgetful morphism, that forgets the marked point $\bullet$. Considered as a morphism of stacks, π is the universal family over $\overline{M}_{g,P}$. Let ω_π be the relative dualizing sheaf, and let s_i resp. be the section of π corresponding to the marked point with index i. Then on $\overline{M}_{g,P}$, $\mathbb{L}_i$ is the pullback $s_i^*(\omega_\pi)$. [31] Informally one can say that the fibre of $\mathbb{L}_i$ at a point $[(C;(p_j)_{j\in P})] \in \overline{M}_{g,P}$ is the cotangent space to C at the point p_i.

[30] In [ACG11] the derivation of the excess intersection formula contains a small mistake, which leads to a slightly incorrect formula. This will be explained in a later footnote. This mistake is not present in the derivation of the same formula in Appendix A.4. of [GP03] (Formula (11)). Still one may prefer [ACG11] to [GP03] as a reference, since the definitions involved are more precise there.

[31] One can also define $\mathbb{L}_i \in \operatorname{Pic}_{fun}(\overline{M}_{g,P})$ by describing for each family $f : C \to B$ of $\overline{M}_{g,P}$ the line bundle $(L_i)_f$ as $s_i^*(\omega_f)$ for s_i the i-th section of the family.

(iii) We define $\psi_i := c_1(\mathbb{L}_i) \in \mathrm{Pic}_{\mathbb{Q}}(\overline{M}_{g,P})$. These tautological classes play an important role for the intersection theory on moduli spaces of curves.

Recall that $\overline{M}_\Gamma = \prod_{v\in V(\Gamma)} \overline{M}_{g(v),a^{-1}(v)}$. With the notation just introduced we have

$$N_{\xi_\Gamma} = \sum_{\{h,h'\}\in E(\Gamma)} \eta^*_{\Gamma,a(h)}\mathbb{L}_h^\vee \otimes \eta^*_{\Gamma,a(h')}\mathbb{L}_{h'}^\vee.$$

Now let Γ and Γ' be two stable (g,n)-graphs. Then look at the following fibre product *of stacks*:

$$\begin{array}{ccc} \overline{M}_{\Gamma\Gamma'} := \overline{M}_\Gamma \times_{\overline{M}_{g,n}} \overline{M}_{\Gamma'} & \xrightarrow{\xi'} & \overline{M}_{\Gamma'} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_{\Gamma'}} \\ \overline{M}_\Gamma & \xrightarrow{\xi_\Gamma} & \overline{M}_{g,n} \end{array} \tag{1.2}$$

By $G_{\Gamma\Gamma'}$ denote a set obtained by choosing [32] one representative of each of the isomorphism classes of triples (Λ, c, c'), where $c : \Lambda \rightsquigarrow \Gamma$, $c' : \Lambda \rightsquigarrow \Gamma'$ are contractions of (g,n)-graphs, with the property that $E(\Lambda) = c^{-1}(E(\Gamma)) \cup (c')^{-1}(E(\Gamma'))$ (cf. Def. 1.18 (ii)). Here an isomorphism $(\Lambda_1, c_1, c_1') \overset{\cong}{\to} (\Lambda_2, c_2, c_2')$ of such triples is an isomorphism $\Lambda_1 \overset{\cong}{\to} \Lambda_2$ of (g,n)-graphs, compatible with the contractions.

Then by Prop. XII. 10.24 of [ACG11], $\overline{M}_{\Gamma\Gamma'}$ is isomorphic to the disjoint union

$$\overline{M}_{\Gamma\Gamma'} \cong \coprod_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} \overline{M}_\Lambda \tag{†}$$

Let $\xi_c : \overline{M}_\Lambda \to \overline{M}_\Gamma$ and $\xi_{c'} : \overline{M}_\Lambda \to \overline{M}_{\Gamma'}$ be the partial gluing morphisms (cf. Proposition 1.26 (iii)). Use the isomorphism of (†), to identify in the diagram (1.2) the space $\overline{M}_{\Gamma\Gamma'}$ with $\coprod_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} \overline{M}_\Lambda$. Then we can write ξ and ξ' in (1.2) as

$$\xi = \coprod_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} \xi_c, \qquad \text{resp.} \qquad \xi' = \coprod_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} \xi_{c'}.$$

In analogy to the excess intersection formula for regular embeddings in smooth varieties, there is an excess intersection bundle $E_{\Gamma\Gamma'}$ on $\overline{M}_{\Gamma\Gamma'} = \coprod_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} \overline{M}_\Lambda$, such that

$$\xi_\Gamma^*\big((\xi_{\Gamma'})_*([\overline{M}_{\Gamma'}]_Q)\big) = \xi_*(c_{top}(E_{\Gamma\Gamma'})). \tag{1.3}$$

Where, again analogous to the case of smooth varieties, we have $E_{\Gamma\Gamma'} = (\xi)^*(N_{\xi_\Gamma})/N_{\xi'}$, where N_{ξ_Γ} and $N_{\xi'}$ are the normal bundles of the maps as explained before. [33] (By c_{top} we denote the top Chern class.)

[32] The results of the later formulas are independent of this choice

[33] In [ACG11], equation (1.3) is erroneously assumed to hold with $\xi_\Gamma^*(\delta_{\Gamma'})$ on the left hand side instead. Since $\delta_{\Gamma'} = \frac{1}{|\operatorname{Aut}(\Gamma')|}(\xi_{\Gamma'})_*([\overline{M}_{\Gamma'}])$, the resulting excess intersection formula (4.33) in chapter 17 misses a factor $\frac{1}{|\operatorname{Aut}(\Gamma')|}$ on the right hand side.

It suffices to describe $E_{\Gamma\Gamma'}$ on every connected component of $\overline{M}_{\Gamma\Gamma'} = \coprod_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} \overline{M}_\Lambda$. We denote the restriction of $E_{\Gamma\Gamma'}$ to the (Λ, c, c')-component by $E_{(\Lambda,c,c)} = (\xi_c)^*(N_{\xi_\Gamma})/N_{\xi_{c'}}$. But as we have seen above

$$N_{\xi_\Gamma} = \bigoplus_{\{h,h'\}\in E(\Gamma)} \eta^*_{\Gamma,a(h)}\mathbb{L}^\vee_h \otimes \eta^*_{\Gamma,a(h')}\mathbb{L}^\vee_{h'}$$

Similarly one obtains

$$N_{\xi_{c'}} = \bigoplus_{\{h,h'\}\in E(\Lambda)\smallsetminus(c')^{-1}(E(\Gamma'))} \eta^*_{\Lambda,a(h)}\mathbb{L}^\vee_h \otimes \eta^*_{\Lambda,a(h')}\mathbb{L}^\vee_{h'},$$

Putting this together yields

$$E_{(\Lambda,c,c')} = \bigoplus_{\{h,h'\}\in c^{-1}(E(\Gamma))\cap(c')^{-1}(E(\Gamma'))\subseteq E(\Lambda)} \eta^*_{\Lambda,a(h)}\mathbb{L}^\vee_h \otimes \eta^*_{\Lambda,a(h')}\mathbb{L}^\vee_{h'}. \tag{1.4}$$

Inserting this into the formula (1.3) gives us, with CE $:= c^{-1}(E(\Gamma)) \cap (c')^{-1}(E(\Gamma'))$,

$$\begin{aligned} \xi^*_\Gamma(\delta_{\Gamma'}) &= \frac{1}{|\operatorname{Aut}(\Gamma')|}\xi^*_\Gamma\big((\xi_{\Gamma'})_*([\overline{M}_{\Gamma'}]_Q)\big) \\ &= \frac{1}{|\operatorname{Aut}(\Gamma')|}\sum_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} (\xi_c)_*\left(\prod_{\{h,h'\}\in \mathrm{CE}} \big(-\eta^*_{\Lambda,a(h)}(\psi_h) - \eta^*_{\Lambda,a(h')}(\psi_{h'})\big)\right) \end{aligned} \tag{1.5}$$

Here we have to interpret the empty product in the case CE $= \emptyset$ as $1 = [\overline{M}_\Lambda]_Q$.

By projection formula $(\xi_\Gamma)_*\xi^*_\Gamma(\delta_{\Gamma'}) = |\operatorname{Aut}(\Gamma)|\delta_\Gamma\delta_{\Gamma'}$. Inserting this into 1.5 we get:

$$\delta_\Gamma\delta_{\Gamma'} = \frac{1}{|\operatorname{Aut}(\Gamma)|\cdot|\operatorname{Aut}(\Gamma')|}\sum_{(\Lambda,c,c')\in G_{\Gamma\Gamma'}} (\xi_\Lambda)_*\left(\prod_{\{h,h'\}\in \mathrm{CE}} \big(-\eta^*_{\Lambda,a(h)}(\psi_h) - \eta^*_{\Lambda,a(h')}(\psi_{h'})\big)\right) \tag{1.6}$$

where $\xi_\Lambda : \overline{M}_\Lambda \to \overline{M}_{g,n}$ is the gluing morphism.

The following formulas can be helpful to calculate the ψ-classes that appear in the excess intersection formula. For small g they even suffice to express the ψ's as a linear combination of boundary divisors:

Summary 1.42 *By $\psi_{g,n,i}$ denote the class ψ_i on the moduli space $\overline{M}_{g,n}$ as defined in Definition 1.41 (iii). Then:*

(i) For $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$, and $i \in \underline{n}$, the following recursion formula holds,

$$\psi_{g,n+1,i} = \pi^*(\psi_{g,n,i}) + \delta_{\{i,n+1\}}$$

where $\delta_{\{i,n+1\}}$ denotes the Q-class of the boundary divisor $\Delta_{\{i,n+1\}}$ of $\overline{M}_{g,n+1}$, whose general points parametrise pointed curves $(C, p_1, ..., p_{n+1})$ such that C has two smooth irreducible components, one of which is of genus 0 (i.e. a rational tail) and carries exactly the marked points p_i and p_{n+1}, while the other component, of genus g, carries all the other marked points.

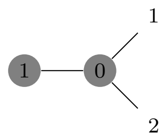
(ii) For any $i \in \underline{4}$, $\psi_{0,4,i} = [p]$*, the class of any point* $p \in \overline{M}_{0,4} \cong \mathbb{P}^1$.

(iii) $\psi_{1,1,1} = \frac{1}{24}[p]$*, where* p *is any point in* $\overline{M}_{1,1} \cong \mathbb{P}^1$.

(iv) On $\overline{M}_{0,\underline{n}\cup\{\bullet\}} \cong \overline{M}_{0,n+1}$*, this yields, already using Notation 1.47 in the last terms,*

$$\psi_{0,\underline{n}\cup\{\bullet\},\bullet} = \sum_{\emptyset \neq I \subseteq \underline{n} \smallsetminus \{1,2\}} \delta_{I\cup\{\bullet\}} = \sum_{\emptyset \neq I \subseteq \underline{n} \smallsetminus \{1,2\}} [\bullet, I] = \sum_{\{1,2\} \subseteq I \subsetneq \underline{n}} [I]$$

Example 1.43 As a simple example we use formula (1.5) to calculate the self intersection $\delta^2_{\{1,2\}}$ of the boundary divisor class $\delta_{\{1,2\}} \in A^1(\overline{M}_{1,2})$, where $\delta_{\{1,2\}}$ is defined as in Summary 1.42 (i). Here we have $\Gamma = \Gamma'$ and the graph looks like



The gluing morphism is

$$\xi_\Gamma : \overline{M}_\Gamma = \overline{M}_{1,\{\bullet_1\}} \times \overline{M}_{0,\{1,2,\circ_1\}} \to \overline{M}_{1,2}.$$

In this case it is easy to see, that $G_{\Gamma\Gamma'} = G_{\Gamma\Gamma}$ only has one element, namely (Γ, c, c), where $c : \Gamma \rightsquigarrow \Gamma$ is the trivial contraction, i.e. the identity: If we had $(\Lambda, c, c') \in G_{\Gamma\Gamma}$ for a graph $\Lambda \neq \Gamma$, then Λ would have to have 2 edges e_1 and e_2, such that c would map e_1 to the only edge e of Γ, while c' would map e_2 to e. But this is impossible, since for any specialisation Λ of Γ, Λ will be of the following form: There is one rational tree with two legs, connected by a disconnecting node e' to a graph which arises as a specialisation of the genus 1 vertex. It is clear that the contraction has to identify e with e'. So c has to be an automorphism of Γ, and it is clear that the only automorphism of Γ is the identity.

Hence if we denote the two half-edges of Γ, constituting the edge e, by $\bullet_1$, $\circ_1$, then formula (1.5), reads

$$\xi^*_{\delta_{\{1,2\}}}(\delta_{\{1,2\}}) = \xi^*_c\left(-\eta^*_{\Gamma,a(\bullet_1)}(\psi_{\bullet_1}) - \eta^*_{\Gamma,a(\circ_1)}(\psi_{\circ_1})\right)$$

Since ξ_c is just the identity and since $\psi_{\circ_1} = 0$ (because $\overline{M}_{0,3}$ is a point), this simplifies to

$$\xi^*_{\delta_{\{1,2\}}}(\delta_{\{1,2\}}) = -\eta^*_{\Gamma,a(\bullet_1)}(\psi_{\bullet_1}) = -\frac{1}{24}[p],$$

where for the second equation we used Summary 1.42 (iii), and where $[p]$ denotes the class of any point of $\overline{M}_\Gamma \cong \mathbb{P}^1$. If we push this forward by the closed embedding $\xi_{\delta_{\{1,2\}}}$ we obtain

$$\delta^2_{\{1,2\}} = -\frac{1}{24}[p],$$

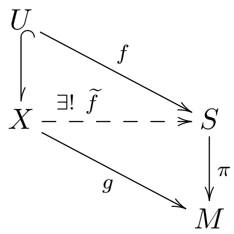
where now $[p]$ denotes the class of any point on the rational variety $\overline{M}_{1,2}$.

1.8 Some lemmas for extending morphisms

We call a morphism of complex analytic spaces *finite* if it is proper and has finite fibres. The following lemmas can be proven quite easily using basic theorems from complex analysis and commutative algebra.

Lemma 1.44 *Let X, Y be complex analytic spaces, X normal, and U a dense open subset of X. If $f : U \to Y$ is a holomorphic map, and $\widetilde{f} : X \to Y$ is a continuous map extending f, then $\widetilde{f}$ is holomorphic.*

Lemma 1.45 *(i) Let X, S and M be complex analytic spaces, X normal, $U \subseteq X$ an open subset. Let $\pi : S \longrightarrow M$ be a finite holomorphic map, and let $g : X \longrightarrow M$ and $f : U \longrightarrow S$ be holomorphic maps, such that the following diagram commutes:*



Then f extends uniquely to a holomorphic map $\widetilde{f} : X \longrightarrow S$, compatible with the diagram.

(ii) If furthermore g is finite, then $\widetilde{f}$ is finite too.

Lemma 1.46 *Let X, Y be algebraic varieties, Y normal. Let $f : X \to Y$ be a finite morphism of degree 1, then f is an isomorphism.*

1.9 Some properties of $\overline{M}_{0,n}$

The moduli spaces $\overline{M}_{0,n}$ ($n \geq 3$) of stable genus 0 curves with ordered marked points where examined by S. Keel in [Kee92]. Among other things he computed their cohomology ring (and, what is the same for these spaces, the Chow ring) for all $n \geq 3$. We summarize some facts about these spaces we are going to use.

Notation 1.47 Recall that the boundary divisors of $\overline{M}_{0,n}$ correspond to stable $(0, n)$-graphs with one edge by section 1.3. Denote by Δ_J the boundary divisor which generically parametrises curves consisting of two $\mathbb{P}^1$'s meeting in one node, one of which carries exactly the marked points with indices in J. So, denoting $J^c := \underline{n} \setminus J$, $\Delta_J = \Delta_{J^c}$. It is clear that we must have $2 \leq |J| \leq n - 2$ for stability reasons, and that all boundary divisors of $\overline{M}_{0,n}$ are of this form.

We introduce the following further abbreviation for the boundary divisors of $\overline{M}_{0,n}$: $[J] := \Delta_J$. Furthermore for $i_1, ..., i_m \in \underline{n}$, we write $[i_1, ..., i_m] := [\{i_1, ..., i_m\}] = \Delta_{\{i_1,...,i_m\}}$, and $[i_1, ..., i_m, J] := [\{i_1, ..., i_m\} \cup J]$ for $J \subset \underline{n}$. Since the objects of $\overline{M}_{0,n}$ have no automorphisms and $\overline{M}_{0,n}$ is smooth (see below) there is no need to distinguish Q-classes and usual cycle classes of subvarieties, in the sense of Summary 1.34. We denote by $[J]$ also the class of $[J]$ in the Chow or cohomology ring.

We also apply this notation for boundary divisors of $\overline{M}_{0,N}$, where N is any finite index-set.

Summary 1.48 (S. Keel)

For all $n \geq 3$:

(i) $\overline{M}_{0,n}$ is a smooth rational projective variety of dimension $n-3$.

(ii) The cohomology ring of $\overline{M}_{0,n}$ is generated by the boundary divisors $[J]$, for $J \subset \underline{n}$ with $2 \leq |J| \leq n-2$, as described in Notation 1.47, and is isomorphic to the Chow ring via the cycle map.

(iii) In more detail:

$$H^*(\overline{M}_{0,n}) \cong A^*(\overline{M}_{0,n}) \cong \frac{\mathbb{Z}\big[\{[J] | J \subset \underline{n},\ |J| \geq 2,\ |J^c| \geq 2\}\big]}{\{\textit{the following relations}\}}.$$ [34]

The relations in the Chow ring are:

(1) *For all $J \subset \underline{n}$ such that $2 \leq |J| \leq n-2$: $[J] = [J^c]$*

(2) *For all pairwise different $i, j, k, l \in \underline{n}$:*

$$\sum_{\substack{J \subset \underline{n}, \\ i,j \in J,\ k,l \notin J}} [J] = \sum_{\substack{J \subset \underline{n}, \\ i,k \in J,\ j,l \notin J}} [J] = \sum_{\substack{J \subset \underline{n}, \\ i,l \in J,\ j,k \notin J}} [J] \tag{1.7}$$

(3) *For all $J, K \subset \underline{n}$ such that $|J|, |K|, |J^c|, |K^c| \geq 2$: $[J] \cdot [K] = 0$ unless one of the following conditions holds:*

$$J \subseteq K, \quad K \subseteq J, \quad J \subseteq K^c, \quad J^c \subseteq K$$

(iv) $H^m(\overline{M}_{0,n})$ is generated as $\mathbb{Q}$ vector space by products of boundary divisors $[J_1] \cdot ... \cdot [J_m] \neq 0$, such that the $[J_k]$ are pairwise different. Furthermore such $[J_1], ..., [J_m]$ intersect transversally, and every codimension m boundary cylce Z of $\overline{M}_{0,n}$ can be written in the form $Z = [J_1] \cap ... \cap [J_m]$.

Proof: (i)-(iii) can all be found in the introduction of [Kee92]. Much (maybe all) of (iv) can also be found in [Kee92], but can also be shown as follows: $M_{0,n} \subseteq \overline{M}_{0,n}$ is isomorphic to $\big((\mathbb{P}^1 \smallsetminus \{0,1,\infty\}) \times .... \times (\mathbb{P}^1 \smallsetminus \{0,1,\infty\})\big) \smallsetminus \Delta$, where Δ denotes the diagonal and $n-3$ factors $(\mathbb{P}^1 \smallsetminus \{0,1,\infty\})$ appear. Hence $M_{0,n}$ is isomorphic to an open subset of $\mathbb{A}^{n-3}$ and hence $A^*(M_{0,n}) = 0$ by the exact sequence of Lemma 1.39. By Proposition 1.26, each boundary divisor of $\overline{M}_{0,n}$ is isomorphic to some $\overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1}$ where $n_i + 1 < n$ for $i \in \underline{2}$. Hence using the exact sequence of Lemma 1.39 and Proposition 1.26 (iv), we can show by induction on n that $A^*(\overline{M}_{0,n})$ is generated by the boundary cycle classes of $\overline{M}_{0,n}$. The stable graph belonging to a codimension m boundary cycle Z is a rational tree, and

[34] $\mathbb{Z}\big[\{[J] | J \subset \underline{n},\ |J| \geq 2,\ |J^c| \geq 2\}\big]$ denotes the polynomial ring over $\mathbb{Z}$ generated by the $[J]$.

it is easy to check that hence Z is of the form $Z = [J_1] \cap ... \cap [J_m]$ for pairwise different $[J_k]$. By Summary 1.34 (v) the class of Z is equivalent to $[J_1] \cdot ... \cdot [J_m]$, hence $A^*(\overline{M}_{0,n})$ is generated by such products. □

1.10 Some notions from birational geometry

Definition 1.49 (i) A variety X is called rational if it is birational to some $\mathbb{P}^n$

(ii) X is called unirational if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$ with $n = \dim X$.

(iii) X is called uniruled if there is a dominant rational map $Y \times \mathbb{P}^1 \to X$, where Y is an irreducible variety with $\dim Y = \dim X - 1$.

(iv) X is called rationally connected if any two sufficiently general points $x_1, x_2 \in X$ lie on a rational curve $C \subseteq X$.

We have: X rational $\Rightarrow$ X unirational $\Rightarrow$ X uniruled, and in general no implication in the opposite direction holds. But for complex varieties X of dimension ≤ 2 it is known that X unirational implies X rational. X rational or unirational implies that X is rationally connected, and for a complex variety, X rationally connected implies X uniruled. (cf. [Hui08])

For a smooth variety X over $\mathbb{C}$, rational connectedness is equivalent to the following apriori stronger condition: Any two points $x_1, x_2 \in X$ lie on a rational curve $C \subseteq X$. (Cf. Corollary 6.8 in [Hui08], as you can see there, one can additionally even require C to be "very free" but we do not want to introduce this notion here.) This implies that (except of possibly the "very free" assumption) the same holds on a singular rationally connected variety X over $\mathbb{C}$, since one can use a desingularisation and then push rational curves down by the desingularisation morphism.

Lemma 1.50 *If X is a rationally connected variety over $\mathbb{C}$, we have $A_0(X) = \mathbb{Q}$.*

Proof: As mentioned above every two points $x_1, x_2 \in X$ are connected by a, possibly singular, rational curve C. But for any rational curve $A_0(C) = \mathbb{Q}$. Hence $x_1 \sim \alpha x_2$ on X for some $\alpha \in \mathbb{Q}$. This implies $A_0(X) = \mathbb{Q}$. □

Definition 1.51 (i) Let D be a Cartier divisor on a normal variety X then the Iitaka dimension $\kappa(X, D)$ is defined as follows: In case $\dim H^0(X, \mathcal{O}(nD)) = 0$ for all n one sets $\kappa(X, D) = -\infty$. Otherwise define $\kappa(X, D)$ in one of the following equivalent ways:

1. $\kappa(X, D)$ is the minimal number $r \in \mathbb{N}_0$ such that the sequence $\dim H^0(X, \mathcal{O}_X(nD))/n^r$ is bounded.

2. $\kappa(X, D)$ is the Krull-dimension of the ring $\bigoplus_{n \in \mathbb{N}_0} H^0(X, \mathcal{O}(nD))$, minus 1.

3. $\kappa(X, D)$ is $\max\{\dim \varphi_n(X) \mid n \in \mathbb{N}\}$, where $\varphi_n : X \dashrightarrow \mathbb{P}^N$ is the birational map induced by nD.

From the last characterisation it is clear that $\kappa(X,\, D) \in \{-\infty, 0, 1, ..., \dim X\}$.

(ii) Let $f : \widetilde{X} \to X$ be a desingularisation of X and let $K_{\widetilde{X}}$ be canonical divisor of $\widetilde{X}$, then the Kodaira dimension $\kappa(X)$ of X is defined to be the Iitaka dimension $\kappa(X) := \kappa(\widetilde{X},\, K_{\widetilde{X}})$.

The Kodaira dimension is a birational invariant.

Chapter 2

The hyperelliptic loci of $\overline{S}_{g,n}$ and $\overline{R}_{g,n}$

This chapter will be concerned with the following spaces:

Definition 2.1 For each pair of moduli space and compactification

$$(X_{g,n}, \overline{X}_{g,n}) \in \{(M_{g,n}, \overline{M}_{g,n}), (R_{g,n}, \overline{R}_{g,n}), (S_{g,n}, \overline{S}_{g,n}), (S^+_{g,n}, \overline{S}^+_{g,n}), (S^-_{g,n}, \overline{S}^-_{g,n})\},$$

denote by $HX_{g,n}$ the following subvariety of $X_{g,n}$: For $g \geq 2$ it is the space parametrising curves $(C; p_1, ..., p_n; ...)$ such that C is smooth, hyperelliptic, and such that $p_1, ..., p_n$ are fixed points of the hyperelliptic involution. For $g = 1$ we have instead the condition that the elliptic involution on C fixing p_1 also fixes all other marked points $p_2, ..., p_n$. Note that $X_{g,n} = \emptyset$ if and only if $n > 2g + 2$.

By $\overline{HX}_{g,n}$ denote the closure of $HX_{g,n}$ in $\overline{X}_{g,n}$. We call this locus the *hyperelliptic locus* of $\overline{X}_{g,n}$. We also call the $\overline{HX}_{g,n}$ the moduli spaces of stable hyperelliptic curves resp. hyperelliptic spin/prym curves (with marked points). [1]

From an analysis of the locus of stable hyperelliptic curves on the local deformation spaces as is for example carried out in Lemma 6.15. of Chapter XI of [ACG11], together with the local description of $\overline{M}_{g,n}$ as a quotient of these deformation spaces (cf. Summary 1.30), it follows that:

Fact 2.2 *The space $\overline{HM}_{g,n}$ is for all $g \geq 1$ and $n \leq 2g + 2$, an irreducible subvariety of $\overline{M}_{g,n}$ of dimension $2g - 1$, which has finite quotient singularities, so in particular is normal.*

We will see that the normal varieties $HS^+_{g,n}$, $HS^-_{g,n}$, $HR_{g,n}$, in general have several connected components, so the compactifications $\overline{HS}^+_{g,n}$, $\overline{HS}^-_{g,n}$, $\overline{HR}_{g,n}$, are not irreducible. Furthermore not even the irreducible components of these compactifications are normal, in general (cf. Remark 2.11).

[1] With this definition an elliptic curve is also hyperelliptic.

We show that the normalizations of these compact moduli spaces, are isomorphic to certain disjoint unions of several of what we call moduli spaces of stable genus 0 curves with *sorted* marked points (cf. Definition 2.4). These moduli spaces can be described as quotients by finite groups acting on moduli spaces $\overline{M}_{0,2g+2}$ of stable genus 0 curves with $2g+2$ *ordered* marked points. [2] The cohomology rings of the latter moduli spaces are known by work of S. Keel ([Kee92]).

To construct the isomorphisms we will use the following fact (cf. [GH94] p. 254):

Fact 2.3 *For every set B of $2g+2$ distinct points in $\mathbb{P}^1$ there is a (unique up to isomorphism) degree 2 cover $f : C \longrightarrow \mathbb{P}^1$ ramified exactly over the given points, where C is a genus g smooth hyperelliptic curve. Moreover for every smooth hyperelliptic curve C there is such a finite degree 2 morphism $f : C \to \mathbb{P}^1$ ramified over $2g+2$ points. The hyperelliptic involution h on C swaps the two sheets of this cover, so f can be seen as the quotient map to $C/h = \mathbb{P}^1$.*

The spin- resp. prym sheaves on C can then be recovered as the invertible sheaves corresponding to certain divisors that are linear combinations of the ramification points. Using admissible double covers of stable genus 0 curves with $2g+2$ marked points, one can extend this correspondence to the asserted isomorphisms.

By our construction we at first only know the existence of the isomorphisms and how they act on the interior of the moduli spaces (Proposition 2.14) . In a second step their behaviour on the boundary will be determined more explicitly (Proposition 2.19). This description will then be used to compare the automorphism group of an object parametrised by a point $p \in \overline{HX}_{g,n}$ to the automorphism groups of the objects parametrised by the preimage of p, on the corresponding moduli space of stable genus 0 curves with sorted marked points.

The results of this chapter will play an important role in computing the cohomology rings of $\overline{R}_2 = \overline{HR}_2$ and $\overline{S}_2 = \overline{HS}_2$, in the following chapter, and also in dealing with the hyperelliptic loci of $\overline{R}_{1,n}$ in chapter 5.

Surely most of what is proven in this chapter is somehow known. In the special cases of $\overline{S}_2^+$ and $\overline{S}_2^-$ morphisms from $\overline{M}_{0,6}$, which factor through the isomorphisms constructed here are are constructed in [BF09a]. The idea of how to construct the isomorphisms in the general case is quite the same. The hyperelliptic locus HR_g is discussed in section 4.2 of [Ver11], where the rationality of most of its connected components is shown.

2.1 Preliminaries

2.1.1 Curves with sorted marked points and admissible (double) covers

Definition 2.4 (i) For us a *sorting* of depth 1 of a finite set M is a tuple $\mathscr{P}$ of non-empty subsets of M, such that M is the disjoint union of these non-empty subsets (i.e. an ordered

[2] In the case of $\overline{HM}_g$ this result can be found in [AL02], where it is shown that $\overline{HM}_g$ is isomorphic to $\overline{M}_{0,[2g+2]}$, the moduli space of stable genus 0 curves with $2g+2$ unordered marked points.

partition of M). A sorting of depth $d > 1$ of M is a tuple of sets of tuples of sets of ... of non-empty subsets of M, such that M is the disjoint union of these non-empty subsets, and such that the word "of" appears d times in the above enumeration. We call these non empty-subsets of M "lying at the bottom of $\mathscr{P}$", the *ground sets* of $\mathscr{P}$.

We will actually allow sortings $\mathscr{P}$ to take a somewhat less strict form, in order to simplify notation: If there is a tuple which contains only one set, we will replace it by just the set. If there is a set containing only one tuple, we only write down the tuple. E.g. we give the sorting (of depth 3) of the set $\underline{10}$,

$$\mathscr{P} = \Big(\big\{(\{1,3\},\{2,5\}),(\{4,6\},\{7,8\})\big\},\big\{(\{9\})\big\},\big\{(\{10\})\big\}\Big) \quad \text{instead as}$$

$$\mathscr{P} = \Big(\big\{(\{1,3\},\{2,5\}),(\{4,6\},\{7,8\})\big\},9,10\Big).$$

A *sorted set* is a finite set M with a sorting $\mathscr{P}$ of M. Usually we will only write down $\mathscr{P}$ if we speak of a sorted set, since loosely speaking $\mathscr{P}$ determines M. [3] By an element of a sorted set $\mathscr{P}$ we mean an element of the underlying set M.

(ii) An *isomorphism* $\varphi : \mathscr{P} \to \mathscr{P}'$ of sorted sets is a bijection $\varphi : M \to M'$ of the underlying sets, respecting the sorting. (Respecting the sorting means: $\varphi(\mathscr{P}) = \mathscr{P}'$, where $\varphi(\mathscr{P})$ denotes the sorting of M' one obtains by applying φ to all the elements in the ground sets of $\mathscr{P}$.)

(iii) We call a *sorting label* an expression

$$\mathbf{label} := (n,[n_1],...,[n_s],[[m_{1,1}],...,[m_{1,t_1}]],...,[[m_{u,1}],....,[m_{u,t_u}]])$$

where the n, n_i and $m_{j,k}$ are all in $\mathbb{N}_0$, as well as the s, t_i and u. Define $|\mathbf{label}| := n + \sum_{i=1}^{s} n_s + \sum_{j=1}^{u}\sum_{k=1}^{t_j} m_{j,k}$.

For a given such **label**, a **label**-*sorted set* is a tuple

$$\mathscr{P} = \big(a_1,...,a_n,(A_1,...A_s),\{B_{1,1},...,B_{1,t_1}\},...,\{B_{u,1},...,B_{u,t_u}\}\big) \qquad (*)$$

consisting of a tuple $(a_1,...,a_n)$ of elements a_i, a tuple $(A_1,...,A_s)$ of sets A_i, and u sets $\{B_{j,1},...,B_{j,t_j}\}$ of sets $B_{j,k}$, such that for all A_i, $|A_i| = n_i$, and for all $B_{j,k}$, $|B_{j,k}| = m_{j,k}$. [4]

Remark: Later, in special cases, we will also use brackets of the form $\langle ... \rangle$ in sorting labels. Such brackets will have the following meaning: They are to be read as (...) in case $n \geq 1$, and as [....] in case $n = 0$. We will also denote sorted sets in the form $\mathscr{P} = (I,(A_1,...,A_s),...)$ where I stand for the tuple $(a_1,...,a_n)$ of elements of M. Compatible with our use in case of sorting labels, brackets of the form $\langle .... \rangle$ in sorted sets are to be read as (....) if $I \neq \emptyset$ and as {....} if $I = \emptyset$.

[3] Strictly speaking $\mathscr{P}$ does not determine M, but only does so if one sticks to write sortings $\mathscr{P}$ strictly as they are defined above without allowing our simplified notation, and additionally specifies the depth d of $\mathscr{P}$. (Since set-theoretically also the elements of M will be sets (of sets of ...).)

[4] These is are of course special sorted sets (of depth ≤ 2). One could also define sorting labels for arbitrary sorted sets, if one allows for more nested round and square brackets in the labels. All lemmas we prove later for our sorting labels would also hold for these general sorting labels, but we will not need this.

(iv) A *family of nodal curves with (***label***-)sorted marked points* is a family of nodal curves $\mathcal{X} \to S$, together with a set $\{\sigma_1, ..., \sigma_\nu\}$ of $\nu := |\mathbf{label}|$ disjoint sections σ_i, each meeting no singular points of the fibres of $\mathcal{X} \to S$, and a (**label**-)sorting $\mathscr{P}$ of the set $\{\sigma_1, ..., \sigma_\nu\}$. We can write such family as $(\mathcal{X} \to S, \mathscr{P})$ since the sorted set $\mathscr{P}$ determines $\{\sigma_1, ..., \sigma_\nu\}$. Note that the "extreme" special cases of this definition are families of ν-pointed nodal curves (with $n = \nu$, $s = 0$, $t = 0$) and families with ν *unordered marked points* (with e.g. $r = 0$, $s = 0$, $t = 1$ and $|B_1| = \nu$) [5] . The condition on a nodal curve with sorted marked points to be called *stable* is the same as for ν-pointed nodal curves. [6]

(v) An *isomorphism* of two families of nodal curves over S with **label**-sorted marked points $(\mathcal{X} \to S, \mathscr{P})$ and $(\mathcal{X}' \to S, \mathscr{P}')$, is an isomorphism φ of the underlying families of nodal curves, such that the induced bijection $\mathscr{P} \to \mathscr{P}'$ of the sets of marked sections, is an isomorphism of sorted sets.

(vi) Let us denote by $\overline{M}_{g,\mathbf{label}}$ the moduli space of stable curves of genus g with **label**-sorted marked points.

To shorten notation we will cancel the appropriate parts of the label if numbers n, s, u or t are 0. Furthermore we will often write $\overline{M}_{g,[m_1,...,m_t]}$ for $\overline{M}_{g,([[m_1],...,[m_t]])}$ in applications.

Remark 2.5 One can construct the moduli space $\overline{M}_{g,\mathbf{label}}$ as a quotient of $\overline{M}_{g,\nu}$ for $\nu := |\mathbf{label}|$ as follows: Let

$$\big((a_1, ..., a_n), (A_1, ...A_s), \{B_{1,1}, ..., B_{1,t_1}\}, ..., \{B_{u,1}, ..., B_{u,t_u}\}\big)$$

be a sorting of the set $\{1, 2, ..., \nu\}$. Write **label** as in the definition above, and set $\mathbf{label}^* := (n, [n_1], ..., [n_s], [m_{1,1}], ..., [m_{1,t-1}], ..., [m_{u,1}], ..., [m_{u,t_u}])$. One obtains $\overline{M}_{g,\mathbf{label}^*}$ as the quotient of $\overline{M}_{g,\nu}$ by the action of $\mathbb{S}_{n_1} \times .. \times \mathbb{S}_{n_s} \times \mathbb{S}_{m_{1,1}} \times ... \times \mathbb{S}_{m_{u,t_u}}$ permuting the indices inside the sets $A_1, ..., A_s, B_{1,1}, ..., B_{u,t_u}$. Finally $\overline{M}_{g,\mathbf{label}}$ can be constructed as the quotient of $\overline{M}_{g,\mathbf{label}^*}$ by the action permuting for each $j \in \underline{u}$ the indices in $\underline{t_j}$ of those of the sets $B_{j,1}, ..., B_{j,t_j}$ having the same cardinality. [7]

Definition 2.6 (i) Let $\mathscr{D}/S := (\mathcal{D} \to S; \{\sigma_1, ..., \sigma_\nu\})$ be a family of stable genus 0 curves with ν unordered marked points, over a basis S. For us a family of admissible double

[5] Here note that different sorting labels can define the same class of families. For example one can omit the n and replace it by n sets A_i with one element each. The n is only introduced to shorten notation later.

[6] It may be a more natural definition of families of nodal curves with sorted marked points, to allow the marked points from one set A_i to form a n_i-multi-section (i.e. a finite unramified cover of S of degree n_i, not necessarily connected), and to replace the sections belonging to the sets $B_{j,1} \uplus ... \uplus B_{j,t_j}$ by $(m_{j,1} + ... + m_{j,t_j})$-multi-section together with compatible partitions of the $m_1 + ... + m_t$ points coming from each multi-section on all fibres. But note that on the level of coarse moduli spaces, and also for local deformations this would not make any difference. Since this is the level we are concerned with in this chapter, and since the alternative definition would complicate the notation in the proofs, we gave a definition allowing no multi-sections. However it seems to me that the definition allowing multi-sections would be appropriate if one wanted to study how the (iso)morphisms of coarse moduli spaces constructed in section 2.2 relate to morphisms of stacks. Remark: Even with this alternative definition the morphisms $a_{...}$ and $b_{...}$ of Proposition 2.14 would not be induced by morphisms of stacks.

[7] The isomorphism of $\overline{M}_{g,\mathbf{label}}$ with the described quotient of $\overline{M}_{g,\nu}$ also holds on the level of stacks.

covers [8] of $\mathscr{D}/S$ is a finite surjective degree 2 morphism $\mathbf{f} : \mathcal{Y} \to \mathcal{D}$ over S such that Y is a family of connected nodal curve, and $\mathbf{f}$ is étale except over the following loci of $\mathcal{D}$.

1. Over the images of the sections of marked points σ_i, $\mathbf{f}$ is simply branched, i.e. locally analytically at such a point one can describe $\mathbf{f}$ by $x^2 = u$, where x is a local coordinate of $\mathcal{Y}$ over S and u is a local coordinate of $\mathcal{D}$ over S. [9]

2. Let $\gamma \in \mathcal{D}$ be a point on $\mathcal{D}$ which is a node of the fibre of $\mathcal{D}$ over S which contains γ. Then $\mathbf{f}$ may or may not be étale over γ: For $\gamma' \in \mathbf{f}^{-1}(\gamma)$, there is a local coordinate a of S and a $p \in \{1, 2\}$ such that locally analytically around γ' resp. γ, one can describe $\mathcal{Y} \to S$ resp. $\mathcal{D} \to \mathcal{S}$ by $xy = a$ resp. $uv = a^p$, and $\mathbf{f} : \mathcal{Y} \to \mathcal{D}$ is locally described by $x^p = u$ and $y^p = v$. [10] (We have $p = 2/|\mathbf{f}^{-1}(\gamma)|$.)

We often write such a family of admissible covers as $\mathcal{Y} \xrightarrow{\mathbf{f}} \mathscr{D} \to S$.

(ii) An morphism between two families of admissible covers $\mathcal{Y} \xrightarrow{\mathbf{f}} \mathscr{D} \to S$ and $\mathcal{Y}' \xrightarrow{\mathbf{f}'} \mathscr{D}' \to S'$ is a pair of morphisms (φ, Φ) with $\varphi : \mathcal{Y} \to \mathcal{Y}'$, $\Phi : S \to S'$, such that (φ, Φ) is a morphism between the families of nodal curves $\mathcal{Y} \to S$ and $\mathcal{Y}' \to S'$, in the sense of Def. 1.5, and such that there is a morphism $\psi : \mathscr{D} \to \mathscr{D}'$ with $\psi \circ \mathbf{f} = \mathbf{f}' \circ \varphi$, such that also (ψ, Φ) is a morphism of families of nodal curves.

(iii) For a $\mathscr{D}/S$ having **label**-sorted marked points instead (in particular for ν-pointed curves), we define admissible covers of $\mathcal{D}$ and isomorphisms of such covers analogously [11]

We compile some facts about admissible double covers, which we mostly take from [HM82] and [AL02]:

Proposition 2.7 *For each sorting label* **label** *with* $|\mathbf{label}| =: \nu \geq 4$ *even:*

(i) There is a normal variety $\overline{H}_{2,\mathbf{label}}$ *which is the coarse moduli space of admissible double covers of curves* $\mathfrak{D}$ *with* $[\mathfrak{D}] \in \overline{M}_{0,\mathbf{label}}$*, and there is a finite surjective forgetful morphism*

$$\rho : \overline{H}_{2,\mathbf{label}} \to \overline{M}_{0,\mathbf{label}},$$

which is an isomorphism of varieties (but not of stacks).

(ii) For any admissible double cover $f : Y \to \mathfrak{D}$*, with* $[\mathfrak{D}] \in \overline{M}_{0,\nu}$ *there is a local universal deformation of admissible covers* $\mathscr{Y} \xrightarrow{\mathbf{f}} \mathscr{D} \to (\mathscr{T}, t_0)$*, where* $(\mathscr{T}, t_0)$ *is a complex* $\nu - 3$ *dimensional ball. We denote this deformation by* Def*. It has the property that* $\overline{H}_{2,\nu}$ *locally*

[8] There are also admissible covers of higher degree, defined analogously, and they also have moduli spaces (cf. [HM82]), but we will not need them in this thesis.

[9] A local coordinate of $\mathcal{Y}$ over S at a point $p \in Y$, means a local coordinate of $\mathcal{Y}$ at p which is tangent to the fibre of $\mathcal{Y} \to S$ which contains p. The same for $\mathcal{D}$. This definition implies that for $s_0 \in S$ the morphisms $f_0 : Y_0 \to D_0$ on the fibres over s_0, is simply branched over the marked points on D_0.

[10] Hence, for every node γ on a fibre D_0, every point in $\gamma' \in f^{-1}(\gamma)$ is a node of Y_0 and for every such γ' the two branches of Y_0 at γ' are mapped to the two branches of D_0 at γ, both with the same ramification index $p \in \{1, 2\}$.

[11] The sorting of the marked points does not enter into the conditions on the covering curve $\mathcal{Y}$. For the isomorphisms one requires that $\psi : \mathscr{D} \to \mathscr{D}'$ is an isomorphism of curves with **label**-sorted marked points (Def. 2.4 (ii).

around the point $[f : Y \to \mathfrak{D}]$ *is the quotient* $(\mathscr{T}, t_0)/\operatorname{Aut}(f : Y \to \mathfrak{D})$. [12] *Furthermore: Let* $\mathfrak{D}'$ *be a curve with* **label**-*sorted marked points obtained by partially forgetting the information about the ordering of the marked points on* $\mathfrak{D}$*, then* $f : Y \to \mathfrak{D}'$ *belongs to* $\overline{H}_{2,\mathbf{label}}$. [13] *If now we define* $\mathscr{D}'$ *by partially forgetting the ordering of the sections of marked points on* $\mathscr{D}$ *in the same way, then* $\mathscr{Y} \xrightarrow{\mathbf{f}} \mathscr{D}' \to (\mathscr{T}, t_0)$*, which we call* Def′*, is the local universal deformation of* $f : Y \to \mathfrak{D}'$.

(iii) The covering space Y *of an admissible double cover* $f : Y \to \mathfrak{D}$ *is a semistable curve, all whose irreducible components are smooth. Furthermore every exceptional component of* Y *meets the rest of* Y *in exactly two points* q_1 *and* q_2*, such that* $f(q_1) = f(q_2)$.

Proof: (i): In [HM82] Theorem 4 this (except the normality) is shown in the case of n ordered marked points, i.e. for $\overline{H}_{2,\nu}$ (and also for the space of degree d admissible covers $\overline{H}_{d,\nu}$). $\overline{H}_{2,\nu}$ is normal by (ii), which even implies that it has only finite quotient singularities. But $\overline{H}_{2,\mathbf{label}}$ can be constructed as a quotient of $\overline{H}_{2,\nu}$ in exactly the same way that $\overline{M}_{0,\mathbf{label}}$ is constructed as a quotient of $\overline{M}_{0,\nu}$ in Remark 2.5 (i) (also cf. [AL02]). It is clear that the finite forgetful morphism $\rho' : \overline{H}_{2,\nu} \to \overline{M}_{0,\nu}$ is compatible with forming these two quotients, hence induces ρ.

To show that ρ is an isomorphism of varieties, by Lemma 1.46 it suffices to show that ρ has degree 1. But over the dense open $M_{0,\mathbf{label}}$, ρ is clearly bijective. (Since a $\mathfrak{D}$ parametrised by this open set is a $\mathbb{P}^1$ with an even number of sorted marked points, this follows from the definition of admissible double covers and Fact 2.3.)

(ii): In the case of $\overline{H}_{2,\nu}$ this follows from the discussion on pages 61-62 of [HM82]. There a local universal deformation for families of degree d admissible covers is constructed, such that $\overline{H}_{d,\nu}$ is locally the quotient of this deformation space by the automorphism group of the central fibre. A criterion for smoothness of the deformation space is given on page 62, and this criterion is always fulfilled for $d = 2$.

Let $\mathbf{f}'' : \mathscr{Y}'' \to \mathscr{D}'' \to (T'', t_0)$, called def, be any deformation of $f : Y \to \mathfrak{D}'$. Reorder the points on $\mathfrak{D}'$ and extend this order to the sections of marked points on $\mathscr{D}''$, to make def into a deformation $\widetilde{\text{def}}$ of $Y \to \mathfrak{D}$. Then $\widetilde{\text{def}}$ can be locally pulled back from Def. By partially forgetting the ordering again we see that def locally is a pull back of Def′ over the base $(\mathscr{T}, t_0)$. It remains to show that the local morphism $(T'', t_0) \to (\mathscr{T}, t_0)$ over which this pull back happens is unique. But if we had two such morphisms, we could again reorder the marked points, and obtain that $\widetilde{\text{def}}$ does not pull back from Def locally uniquely.

(iii): Cf. [AL02] Lemma 2.3. (Or Lemma 2.10 (iii) below.) □

2.1.2 Families of stable hyperelliptic (spin/prym) curves and of admissible double covers.

In the following we say that a (pointed) stable curve is hyperelliptic if it is parametrised by a point of the hyperelliptic locus (in the sense of Definition 2.1) of the appropriate $\overline{M}_{g,n}$.

[12] Cf. section 1.5 about local universal deformations.

[13] It is clear that every admissible cover with sorted marked points can be obtained in this way.

We define hyperelliptic (pointed) spin/prym curves analogously. Then clearly a spin/prym curve is hyperelliptic if and only if its stable model is.

Summary 2.8 *(i) A stable pointed curve* $\mathfrak{C} = (C; \sigma_1, ..., \sigma_n)$ *is hyperelliptic if and only if there is a* $h \in \mathrm{Aut}(\mathfrak{C})$ *of order* 2*, such that the fixed points of* h *are isolated, and such that* C/h *is a curve of arithmetic genus* 0*. Such an automorphism is unique and we call it the* hyperelliptic involution *on* $\mathfrak{C}$*.*

(ii) If $\mathcal{C} \to S$ *is a family of stable curves with sorted marked points, all of whose fibres are hyperelliptic, then there is an* $h \in \mathrm{Aut}_S(\mathcal{C})$ *restricting on each fibre to the hyperelliptic involution.*

(iii) For a hyperelliptic spin or prym curve $\mathfrak{X}$*, the hyperelliptic involution* h *on its stable model* $\mathfrak{C}$ *lifts to an automorphism of order* 2 *on* $\mathfrak{X}$*. This lifting is unique if* $\mathfrak{X}$ *is smooth. Also on each family* $\mathcal{X} \to S$ *of hyperelliptic spin/prym cures, over an irreducible basis* S*, the subgroup* $\mathrm{Aut}_S(\mathcal{X}) \subseteq \mathrm{Aut}(\mathcal{X} \to S)$ *contains (at least one) lifting of the hyperelliptic involution on the stable model* $\mathcal{C} \to S$*. If the family furthermore has any smooth fibres, the lifting is unique.*

Proof: (i): Follows from Lemma 3.5 in Chapter X and Lemmas 6.14. and 6.15 in Chapter XI of [ACG11]. (Except that the definition of stable pointed hyperelliptic curves in [ACG11] requires genus $g \geq 2$, while we also allowed $g = 1$ with $n \geq 1$. But one can check by reading the proofs there, that everything also works for our slightly more general definition.)

(ii): This is true, by the proof of the Lemma 6.15 just mentioned, for the universal deformation of each stable hyperelliptic curve. From this it follows over local charts on T. But by the uniqueness claim in (i) these involutions over the local charts glue together.

(iii): We show this later in the proof of Lemma 2.12. □

Lemma 2.9 *(i) For each family* $\mathcal{Y} \xrightarrow{\mathbf{f}} \mathscr{D} \to S$ *of admissible double covers there is an automorphism* $h \in \mathrm{Aut}_S(\mathcal{Y})$*, exchanging the two sheets of the degeree* 2 *cover* $\mathbf{f} : \mathcal{Y} \to \mathcal{D}$*. We call* h *the* hyperelliptic involution *on* $\mathcal{Y} \to \mathcal{D} \to S$*. Then* $\mathcal{D}$ *is isomorphic over* S *to the geometric quotient* $\mathcal{Y}/h$ *and* $\mathbf{f}$ *can be identified with the quotient morphism.*

(ii) For any $g \geq 1$ *and any* $n \leq 2g + 2$ *the following assignment is a morphism of moduli functors (in the sense of Def. 1.7): Send (families of) double covers* $Y \xrightarrow{f} \mathfrak{D}$ *with* $\mathfrak{D} = (D; q_1, ..., q_n; \{q'_1, ..., q'_{2g+2-n}\})$ *to the stable model of the pointed curve* $(Y, p_1, ..., p_n)$*, where* p_i *is the point (resp. section)* $f^{-1}(q_i)$ *for each* $i \in \underline{n}$*. Thus the assignment induces a morphism* $c_{\overline{M}_{g,n}} : \overline{H}_{2,(n,[2g+2-n])} \to \overline{M}_{g,n}$ *of coarse moduli spaces. As morphism of varieties,* $c_{\overline{M}_{g,n}}$ *is a closed embedding with image* $\overline{HM}_{g,n}$*.* [14]

The hyperelliptic involution on the resulting family of hyperelliptic curves is induced by the hyperelliptic involution on the family $f : Y \to \mathfrak{D}$*.*

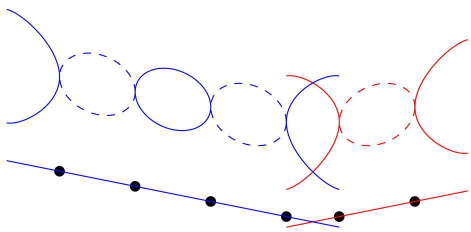
[14] The assignment even is a morphism of stacks, but this is no embedding of stacks, due to differences in the automorphism groups of the objects.

Proof: (i): By the local description of the admissible cover in Definition 2.6 it is clear that the map exchanging the two sheets is holomorphic and hence a morphism of varieties. That the defining properties of a geometrical quotient are fulfilled is easy to check.

(ii): It is easy to check that the assignment is a morphism of moduli functors. It is also clear by Fact 2.3 that $c_{\overline{M}_{g,n}}$ maps the interior $H_{2,(n,[2g+2-n])}$ to $HM_{g,n}$ and is $1:1$ on this locus. Since $\overline{H}_{2,(n,[2g+2-n])}$ is compact, this implies that $c_{\overline{M}_{g,n}}$ is finite of degree 1. By Lemma 1.46, and since $\overline{HM}_{g,n}$ is normal, it follows that $c_{\overline{M}_{g,n}}$ is a closed embedding. □

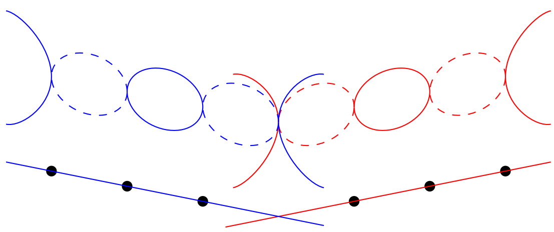
Lemma 2.10 *Let $\mathfrak{D}$ be a stable genus 0 curve with $2g+2$ sorted marked points, such that the underlying curve D has two irreducible components D_1 and D_2 meeting each other in one node γ. Let $2 \leq \mu \leq 2g+2$ be the number of marked points on D_1, and let $Y \xrightarrow{f} \mathcal{D}$ be an admissible double cover. Then:*

(i) If μ even, $Y \xrightarrow{f} \mathfrak{D}$ looks as follows: For $Y_i := f^{-1}(D_i)$ $(i \in \underline{2})$, $f_{|Y_i} : Y_i \to D_i$ is the unique double cover of D_i branched exactly over all the marked points on D_i. The fibre $f^{-1}(\gamma)$ consists of two points, and Y_1 and Y_2 meet in each of these two points in simple nodes:


[15]

Set $h := \frac{\mu-2}{2}$, then the genus of Y_1 is h and the genus of Y_2 is $g-h-1$.

(ii) For μ odd: Here $f_{|Y_i} : Y_i \to D_i$ is the unique double cover of D_i branched exactly over all the marked points on D_i and over the point γ. The fibre $f^{-1}(\gamma)$ consists of one point in which Y_1 and Y_2 meet in a simple node:


[16]

Set $h := \frac{\mu-1}{2}$, then Y_1 is of genus h and Y_2 of genus $g-h$.

(iii) Now allow $\mathfrak{D}$ to have arbitrarily many irreducible components, and let D_i be one of them, then $f_{|Y_i} : Y_i \to D_i$ is the (unique) double cover branched over all marked points

[15] We will several times use pictures like this to symbolize admissible double covers. Here we have an underlying genus 0 curve $\mathfrak{D}$ with 6 marked points, consisting of two $\mathbb{P}^1$'s meeting in one node, one of which (the red line), carries 4 marked points, while the other one (the blue line) carries 2 marked points. Above them one sees the covering curve Y which is ramified exactly over the marked points of $\mathfrak{D}$ in this case, and has two components one mapping to the blue resp. red part of $\mathfrak{D}$ each. The dashed parts indicate that the covering curve Y is complex and connected. If one would draw only the real points of Y, one would get something like the non-dashed part.

[16] This picture is somewhat misleading, since it looks like the two irreducible components of Y would meet in a tacnode, not a simple node.

on D_i and over exactly those nodes γ of D on D_i with the following property: The tree of rational curves attached to D_i at γ carries a even number of marked points. (In particular we see that Y is unique up to isomorphism.)

Proof: This can be found in [AL02]. But also: By Proposition 2.7 (iii) each Y_i is smooth and $f_{|Y_i} : Y_i \to D_i$ is finite of degree 2, branched over the marked points, possibly branched over γ, unbranched everywhere else. Since $f_{|Y_i}$ must be branched over an even number of points by the Hurwitz formula, (i) and (ii) follow.

(iii): Consider the local universal deformation $\mathscr{Y} \xrightarrow{\mathbf{f}} \mathscr{D} \to (\mathscr{T}, t_0)$ of $f : Y \to \mathfrak{D}$. Let $\mathscr{T}(\gamma) \subset \mathscr{T}$ be the subspace over which the node γ is retained. Let $\mathbf{f} : \mathscr{Y}(\gamma) \to \mathscr{D}(\gamma) \to \mathscr{T}(\gamma)$ be the restriction of the family over $\mathscr{T}(\gamma)$. Using the local description of families of admissible covers at nodes from Definition 2.6 (i) we see: On every fibre D_{s_1} for $s_1 \in \mathscr{T}(\gamma)$ close enough to s_0, and for γ_1 the node on D_{s_1} to which γ deforms, we have $|\mathbf{f}^{-1}(\gamma_1)| = |\mathbf{f}^{-1}(\gamma)|$. But almost all fibres over $\mathscr{T}(\gamma)$ have only one node, so for them (iii) holds by (i) and (ii). □

2.1.3 The hyperelliptic local universal deformation of a hyperelliptic (spin/prym) curve, and automorphisms

Now we describe the locus of (stable) hyperelliptic (spin/prym) curves on the local universal deformation spaces of such curves [17]. For this we use the notation introduced in section 1.5, and the Summaries 1.30 and 1.31, without further mentioning it:

Let $\mathfrak{X}$ be a pointed spin or prym curve which is hyperelliptic in the sense of definition 2.1, let $\mathfrak{C}$ be the stable model of $\mathfrak{X}$ which is then a stable hyperelliptic curve. Let $\mathcal{X} \to (S, s_0)$ be the local universal deformation of $\mathfrak{X}$ and $\mathcal{C} \to (B, b_0)$ the local universal deformations of $\mathfrak{C}$ [18]. Let $\mathscr{B} \subseteq B$ and $\mathscr{S} \subseteq S$ be the sub-loci of the two deformation spaces parametrising stable hyperelliptic curves, resp. hyperelliptic spin/prym curves, and let

$$\mathscr{X} \to (\mathscr{S}, s_0), \quad \mathscr{C} \to (\mathscr{B}, b_0)$$

be the restrictions of the universal families. Then it is easy to check that these two families are the local universal deformations of $\mathfrak{X}$ resp. $\mathfrak{C}$ in the category of deformations of *hyperelliptic* stable curves, resp. *hyperelliptic* prym/spin curves. We call these families the *hyperelliptic local universal deformations* of $\mathfrak{X}$ resp. $\mathfrak{C}$. Most properties of the usual local universal deformations described in section 1.5 carry over to the hyperelliptic ones. In particular it is clear that $\overline{HM}_{g,n}$ is locally around $[\mathfrak{C}]$ isomorphic to $\mathscr{B}/\operatorname{Aut}(\mathfrak{C})$, and locally at $[\mathfrak{X}] \in \overline{HX}_{g,n}$, $\overline{HX}_{g,n}$ is isomorphic to $\mathscr{S}/\operatorname{Aut}(\mathfrak{X})$.

Let $h \in \operatorname{Aut}(\mathfrak{C})$ be the hyperelliptic involution. Define a partition of the set of nodes $E = E_1 \uplus E_2$ of $\mathfrak{C}$, such that E_1 contains those nodes which are fixed by h while those in E_2 are exchanged with an other node by h. $E_{N,i} := E_N \cap E_i$, $E_{\Delta,i} := E_\Delta \cap E_i$, for $i \in \underline{2}$.

[17] For stable hyperelliptic curves this can all be found in chapter XI of [ACG11], Lemma 6.15. (+proof).

[18] We suppress the sections σ_i of marked points, as well as the spin/prym structure $(\mathbf{L}, \mathbf{b})$ on $\mathcal{X}$ in the notation here.

Choose a pair of standard bases $(\vec{y}_1, ..., \vec{y}_{3g-3+n})$, $(\vec{x}_1, ..., \vec{x}_{3g-3+n})$, such that for each node e which is not fixed by h, $h(\vec{x}_e) = \vec{x}_{h(e)}$ (cf. Lemma 1.32 (ii)). By Summary 2.8 (i)+(ii), we have that on (B, b_0), $\mathscr{B} = \mathrm{Fix}(h)$. Hence, for suitable linear subspaces $H_v \subseteq W_v$,

$$\mathscr{B} = \bigoplus_{v \in V(\Gamma)} H_v \oplus \mathrm{span}_B(\{\vec{x}_e\}_{e \in E_1}) \oplus \mathrm{span}_B(\{\vec{x}_e + \vec{x}_{h(e)}\}_{e \in E_2}).$$

So $\mathscr{B} \subseteq (B, b_0)$ is a linear subspace. By chapter XI of [ACG11], Lemma 6.15, $\mathscr{B}$ is of dimension $2g-1$.

Using the explicit description of the forgetful morphism $\pi : S \to B$ from Summary 1.31 (vi), one now determines $\mathscr{S} \subseteq (S, s_0)$. Set

Part $:= \{(E^+_{N,2}, E^-_{N,2}) \mid E^+_{N,2} \uplus E^-_{N,2} = E_{N,2}$ and $h(E^+_{N,2}) = E^+_{N,2},\ h(E^-_{N,2}) = E^-_{N,2}\}$. Then:

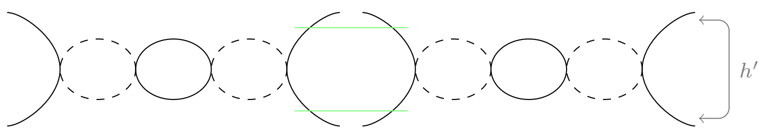
$$\mathscr{S} = \pi^{-1}(\mathscr{B}) = \bigcup_{(E^+_{N,2}, E^-_{N,2}) \in \mathbf{Part}} \mathscr{S}^{(E^+_{N,2}, E^-_{N,2})}, \quad \text{where } \mathscr{S}^{(E^+_{N,2}, E^-_{N,2})} \text{ is:}$$

$$\bigoplus_{v \in V(\Gamma)} H'_v \oplus \mathrm{span}_S(\{\vec{y}_e\}_{e \in E_1}) \oplus \mathrm{span}_S(\{\vec{y}_e + \vec{y}_{h(e)}\}_{e \in E_{\Delta,2} \cup E^+_{N,2}}) \oplus \mathrm{span}_S(\{\vec{y}_e - \vec{y}_{h(e)}\}_{e \in E^-_{N,2}}).$$

Here $H'_v := \pi^{-1} H_v$ for each v. Note that on each H'_v, $\pi_{|H'_v}$ is an isomorphism.

So we see that $\mathscr{S}$ is the union of $l := \sum_{k=0}^{|E_{N,2}|/2} \binom{|E_{N,2}|/2}{k}$ linear subspaces $\mathscr{S}^{(E^+_{N,2}, E^-_{N,2})}$ of (S, s_0), each of dimension $2g-1$. [19]

Remark 2.11 From this we can conclude that while $\overline{HM}_g$ is a normal variety for all g and n (since it is locally of the form $\mathscr{B}/\mathrm{Aut}(\mathfrak{C})$), the spaces $\overline{HS}^+_g$, $\overline{HS}^-_g$ and $\overline{HR}_g$ in general are not. Take for example the point $[\mathfrak{X}] \in \overline{S}^-_3$ of a spin curve $\mathfrak{X} = (X; \mathcal{L}; b)$ with X consisting of two disjoint smooth genus 1 curves X_1, X_2, and two exceptional components, such that each exceptional component meets each genus 1 component in exactly one point. Such a curve is hyperelliptic. Call $\mathfrak{C}$ its stable model.

X :


[20]

It is clear that the hyperelliptic involution h on $\mathfrak{C}$ swaps the two nodes e_1, e_2, so $|E_{N,2}| = 2$ for $\mathfrak{X}$ and hence $l = 2$. More precisely

$$\mathscr{S} = (U_{v_1} \oplus U_{v_2} \oplus \mathrm{span}_S(\vec{y}_{e_1} + \vec{y}_{e_2})) \cup (U_{v_1} \oplus U_{v_2} \oplus \mathrm{span}_S(\vec{y}_{e_1} - \vec{y}_{e_2}))$$

where U_{v_1}, U_{v_2} are the 2 dimensional deformation spaces of the components X_1 resp. X_2 with their two special points. If X_1, X_2 are sufficiently general, $\mathrm{Aut}(\mathfrak{C}) = \{id, h\}$. As stated

[19] We will see in Remark 2.28 that l not necessarily equals the number of irreducible components of the local analytic neighbourhood of $[\mathfrak{X}]$ in $\overline{HX}_{g,n}$, since there can be automorphisms which permute components of $\mathscr{S}$.

[20] The two exceptional components have been coloured light green here.

above, h acts on the pair of vectors $(\vec{x}_{e_1}, \vec{x}_{e_2})$, by $(\vec{x}_{e_1}, \vec{x}_{e_2}) \mapsto (\vec{x}_{e_2}, \vec{x}_{e_1})$. By Summary 1.31 (vii), a lifting h' of h to $\mathfrak{X}$ has only the following four options how to act:

$$(\vec{y}_{e_1}, \vec{y}_{e_2}) \mapsto (\vec{y}_{e_2}, \vec{y}_{e_1}), \quad (\vec{y}_{e_1}, \vec{y}_{e_2}) \mapsto (-\vec{y}_{e_2}, -\vec{y}_{e_1}),$$

$$(\vec{y}_{e_1}, \vec{y}_{e_2}) \mapsto (-\vec{y}_{e_2}, \vec{y}_{e_1}), \quad (\vec{y}_{e_1}, \vec{y}_{e_2}) \mapsto (\vec{y}_{e_2}, -\vec{y}_{e_1}).$$

But since h' has order 2 by Summary 2.8 (iii), the options in the second line can be excluded. Finally the only inessential automorphism of $\mathfrak{X}$ acts by $(\vec{x}_{e_1}, \vec{x}_{e_2}) \mapsto (-\vec{x}_{e_1}, -\vec{x}_{e_2})$ (cf. Lemma 1.32 (iii)), so we see that the two components of $\mathscr{S}$ are not swapped by $\mathrm{Aut}(\mathfrak{X})$. So a local analytic neighbourhood of $[\mathfrak{X}] \in \overline{HS_3^-}$ has two irreducible components, hence is not normal.

As we shall see in Proposition 2.14, $\overline{HS_3^-}$ is irreducible. So, in general, not even the irreducible components of the $\overline{HX}_{g,n}$ are normal varieties.

The next Lemma provides some properties of automorphisms of hyperelliptic spin/prym curves we use later. We also prove Summary 2.8 (iii), already used in the previous remark.

Lemma 2.12 *(i) For every (pointed) stable hyperelliptic curve $\mathfrak{C} = (C, p_1, ..., p_n)$, every $\varphi \in \mathrm{Aut}(\mathfrak{C})$ commutes with the hyperelliptic involution h.*

*(ii) **Definition:** For a (pointed) stable hyperelliptic curve $\mathfrak{C}$, let $\mathrm{Aut}_{hyp}(\mathfrak{C}) \subseteq \mathrm{Aut}(\mathfrak{C})$ be the subgroup of* partial hyperelliptic involutions*, i.e. of automorphisms which on each component of C act either like the hyperelliptic involution, or like the identity.*

(iii) Let $\widetilde{\gamma}$ be a disconnecting node of C. Then $\widetilde{\gamma}$ is fixed by the hyperelliptic involution h. Let s be the number of such nodes on C. Write $C = C_1 \cup C_2$ such that $C_1 \cap C_2 = \widetilde{\gamma}$. Then there are involutions $h_1, h_2 \in \mathrm{Aut}(\mathfrak{C})$ such that each h_i acts on C_i like h, and as the identity on the rest of C.

$\mathrm{Aut}_{hyp}(\mathfrak{C})$ *is generated by these involutions for all $\widetilde{\gamma}$, and has order $|\mathrm{Aut}_{hyp}(C)| = 2^{s+1}$.*

(iv) If $\mathfrak{X}$ is a hyperelliptic prym/spin curve with stable model $\mathfrak{C}$, then all elements of $\mathrm{Aut}_{hyp}(\mathfrak{C})$ *lift to $\mathfrak{X}$.*

(v) If $Y \to \mathfrak{D}$ is a admissible double cover from $\overline{H}_{2,\mathbf{label}}$, i.e. $[\mathfrak{D}] \in \overline{M}_{0,\mathbf{label}}$, for any sorting label with even $|\mathbf{label}| \geq 4$, then every element of $\mathrm{Aut}(\mathfrak{D})$ *lifts to* $\mathrm{Aut}(Y \to \mathfrak{D})$ *(not uniquely).*

Proof: (i): First let $\mathfrak{C}'$, $\mathfrak{C}''$ be two smooth (pointed) hyperelliptic curves, let $f' : C' \to D'$, $f'' : C'' \to D''$ be the quotient maps form the underlying curves to the quotients $D' = C'/h'$, $D'' = C''/h''$, where h', h'' are the hyperelliptic involutions of $\mathfrak{C}'$ resp. $\mathfrak{C}''$. Then every isomorphism $\varphi : \mathfrak{C}' \to \mathfrak{C}''$ induces a unique $\overline{\varphi} : D' \to D''$ such that $\overline{\varphi} \circ f' = \varphi \circ f''$. We refer to this by $(*)$. In case $g(\mathfrak{C}') \geq 2$, $(*)$ is shown in [GH94] p. 254-255 [21]. For $g(\mathfrak{C}') = 1$ the curves have at least one marked point and then the same holds (Cf. [Har77], Chapt. IV,4.). From this (i) follows for smooth curves. One can show (i) using the description of the admissible

[21] Shown for the unpointed case there, but obviously this implies the same for pointed smooth hyperelliptic curves

double cover Y for a given $\mathfrak{D}$ from Lemma 2.10, and analysing how the automorphisms act on each component. We instead argue using the hyperelliptic universal deformation $\mathscr{C} \to (\mathscr{B}, b_0)$ of $\mathfrak{C}$, and the fact that $\mathrm{Aut}(\mathscr{C} \to (\mathscr{B}, b_0)) = \mathrm{Aut}(\mathfrak{C})$ (cf. Summary 1.30 (iv), and recall that we speak about automorphisms of the centred family underlying the deformation here, cf. Def. 1.28 (vii)). By Lemma 2.9 (i), $h \in \mathrm{Aut}(\mathfrak{C})$ is even contained in $\mathrm{Aut}_{\mathscr{B}}(\mathscr{C}) \subseteq \mathrm{Aut}(\mathscr{C} \to (\mathscr{B}, b_0))$ and restricts to the hyperelliptic involution on each fibre. Now a $\varphi \in \mathrm{Aut}(\mathfrak{C})$ not commuting with h would thus induce an isomorphism between two smooth fibres $\mathfrak{C}'$ and $\mathfrak{C}''$ of $\mathscr{C} \to (\mathscr{B}, b_0)$ that would violate $(*)$. So such a φ does not exist.

(iii): The existence of the h_i is clear, and also that they generate $\mathrm{Aut}_{hyp}(\mathfrak{C})$. We have $h_1 h = h_2$. So $\mathrm{Aut}_{hyp}(C)$ is generated by any set containing h plus for each of the s nodes $\widetilde{\gamma}$ one of the h_i, which we call $h_{\widetilde{\gamma}}$. For a collection $\widetilde{\gamma}_1, ..., \widetilde{\gamma}_m$ of distinct nodes, it is impossible for $h_{\widetilde{\gamma}_1} \cdot h_{\widetilde{\gamma}_2} \cdot ... \cdot h_{\widetilde{\gamma}_m}$ to be the identity or the hyperelliptic involution h. So we can conclude $|\mathrm{Aut}_{hyp}(C)| = 2^{s+1}$.

(iv): We argue using the hyperelliptic local universal deformation $(\mathscr{X} \to (\mathscr{S}, s_0), \mathscr{L}, \mathfrak{b})$ and the fact that $\mathrm{Aut}(\mathfrak{X}) = \mathrm{Aut}((\mathscr{X} \to (\mathscr{S}, s_0), \mathscr{L}, \mathfrak{b}))$ which follows from Summary 1.31 (i). Let $h_{\widetilde{\gamma}_i} \in \mathrm{Aut}_{hyp}(\mathfrak{C})$ be as in (iii), call the subcurve of C on which it acts non-trivially C_1, the other part C_2. Denote by ν the node or exceptional component on X corresponding to $\widetilde{\gamma}_i$, and denote the two components into which ν divides X by X_j ($j \in \underline{2}$), such that each X_j stabilises to C_j. Chose a fibre $\mathfrak{X}'$ of the universal deformation, on which the node ν from the central fibre $\mathfrak{X}$ persists (as node ν'), but all other nodes are smoothed. Then ν' divides X' in two smooth hyperelliptic curves X'_1, X'_2 to which X_1, X_2 deform. The spin/prym sheaf on X' restricts to spin/prym sheaves on X_1, X_2 since ν' is disconnecting. Classes of spin and prym sheaves on smooth hyperelliptic curves correspond to certain divisors supported on the fixed points of the hyperelliptic involution (cf. Lemma 2.13). So letting the hyperelliptic involution act on X'_1 and the identity on X'_2 defines an automorphism of the non-exceptional subcurve of X' respecting the prym/spin structure. It extends to an automorphism φ' of $\mathfrak{X}'$ (cf. Summary 1.13 (iv)), which again extends to a $\varphi \in \mathrm{Aut}((\mathscr{X} \to (\mathscr{S}, s_0), \mathscr{L}, \mathfrak{b}))$ (Summary 1.30 (iv)). Now φ acts on the central fibre $\mathfrak{X}$ as a lifting of $h_{\widetilde{\gamma}_i}$.

The only element of $\mathrm{Aut}_{hyp}(\mathfrak{C})$ we have not shown to lift yet is the hyperelliptic involution h on the whole curve. But this can be shown completely analogously by choosing $\mathfrak{X}'$ to be a smooth fibre. (This finishes the proof of (iv).) Furthermore in this case φ' has order 2, which implies that φ and its restriction to the central fibre also have order 2. This proves the first two sentences of Summary 2.8 (iii). The rest can then be shown arguing as in the proof of Summary 2.8 (ii).

(v): This can either be checked over the irreducible components of $\mathfrak{D}$ using the descriptions of admissible double covers from Lemma 2.10, or can be proven similar to (iv) using the local universal deformation of $Y \to \mathfrak{D}$ and its map to the local universal deformation of $\mathfrak{D}$, which is described in [HM82] page 61-62. (It should also follow from the local description of $Y \to \mathfrak{D}$ in Definition 2.6 (and Lemma 2.10), via analytic continuation.) □

2.2 Construction of the isomorphisms

Lemma 2.13 *For $g \geq 1$, let $q_1, ..., q_{2g+2}$ be distinct points in $\mathbb{P}^1$, and $f : Y \longrightarrow \mathbb{P}^1$ the (unique) degree 2 cover of $\mathbb{P}^1$ ramified exactly over these points. Then Y is a smooth genus g hyperelliptic curve. For $i = 1, ..., 2g+2$, define $Q_i := f^{-1}(q_i)$. Let M be the set of all Q_i and denote by P_m the set of possible partitions of M into a set of m elements and a set of $m' := 2g+2-m$ elements. I.e.:*

$$P_m := \{\{A, B\} \mid A, B \subseteq M,\ A \uplus B = M,\ |A| = m,\ |B| = m' := 2g+2-m\}$$

Let $J_R(Y)$, $J_S(Y)$, $J_+(Y)$, $J_-(Y)$ be the sets of isomorphism classes of non-trivial prym sheaves, resp. spin sheaves, resp. even spin sheaves, resp. odd spin sheaves on Y. [22] *(Of course $J_S(Y) = J_+(Y) \uplus J_-(Y)$.) Then we have:*

For any $\{A, B\} \in P_m$ and $R_1, ..., R_m$ the points in A, $R'_1, ..., R'_{m'}$ the points of B, Q any of the points Q_i:

(i) For all even $2 \leq m \leq 2g$:

1. *$\phi_{R,m}(\{A, B\}) := \mathcal{O}_Y(-m \cdot Q + \sum_{i=1}^{m} R_i)$ is a non-trivial prym sheaf of Y, whose isomorphism class is independent of the choice of Q. Furthermore $\phi_{R,m}(\{A, B\}) \cong \phi_{R,m'}(\{A, B\}) = \mathcal{O}_Y(-m' \cdot Q + \sum_{i=1}^{m'} R'_i)$.*
2. *The map $\phi_{R,m} : P_m \to J_R(Y)$, $\{A, B\} \mapsto \phi_{R,m}(\{A, B\})$ is injective.*
3. *The map $\phi_R : \biguplus_{\substack{2 \leq m \leq g+1,\\ m \text{ even}}} P_m \to J_R(Y)$, obtained as union of the maps $\phi_{R,m}$ with $m \leq g+1$ is a bijection.*

(ii) Analogously for spin structures:

1. *If g is even, then for all $0 \leq m \leq 2g+2$, with m odd:*
   $\phi_{S,m}(\{A, B\}) := \mathcal{O}_Y((g-1-m) \cdot Q + \sum_{i=1}^{m} R_i)$ is a spin sheaf of Y.
2. *If g is odd, then for all $0 \leq m \leq 2g+2$, with m even:*
   $\phi_{S,m}(\{A, B\}) := \mathcal{O}_Y((g-1-m) \cdot Q + \sum_{i=1}^{m} R_i)$ is a spin sheaf of Y.
3. *In both cases the isomorphism class of $\phi_{S,m}(\{A, B\})$ is independent of the choice of Q. Thus the map $\phi_{S,m} : P_m \to J_S(Y)$, $\{A, B\} \mapsto \phi_{R,m}(\{A, B\})$ is well defined. It is injective, and the map $\phi_S : \biguplus_{\substack{1 \leq m \leq g+1,\\ m \equiv g+1 \bmod 2}} P_m \to J_S(Y)$, obtained as union of the maps $\phi_{S,m}$ with $m \leq g+1$ is a bijection. Again $\phi_{S,m}(\{A, B\}) \cong \phi_{S,m'}(\{A, B\})$.*

(iii) For every $g \geq 2$ the bijection ϕ_S splits into two bijections $\phi_+ : \phi_S^{-1}(J_+(Y)) \to J_+(Y)$ and $\phi_- : \phi_S^{-1}(J_-(Y)) \to J_-(Y)$. They can also be written (by describing $\phi_S^{-1}(J_+(Y))$ and

[22] We are talking about isomorphism classes of sheaves on a fixed curve Y here. For two non isomorphic (spin/prym) sheaves $\mathcal{L}, \mathcal{L}'$ on Y, the (spin/prym) curves $(Y, \mathcal{L})$, $(Y, \mathcal{L}')$ may still be isomorphic.

$\phi_S^{-1}(J_-(Y))$ *explicitly) as:*

$$\phi_+ : \biguplus_{\substack{1\leq m\leq g+1,\\ m\equiv g+1 \; mod\, 4}} P_m \to J_+(Y)$$

and

$$\phi_- : \biguplus_{\substack{1\leq m\leq g+1,\\ m\equiv g-1 \; mod\, 4}} P_m \to J_-(Y)$$

Proof: It is easy to show that, for all $i, j \in \{1, ..., 2g + 2\}$, $2p_i - 2p_j \sim 0$. I.e. all $2p_i$ are equivalent.

Using this, all claims of part (i) follow from what is shown in section be 5.2.2. in [Dol10].

All assertions of (ii) follow from the fact that the canonical sheaf of Y is equivalent to $(2g - 2)Q_i$ for any $i \in \{1, ..., 2g + 2\}$ and the corresponding assertions of part (i) of the Lemma. (Can also be found in section 5.2.3. of [Dol10])

For (iv): From Lemma 5.2.1. in [Dol10] it follows that $h^0(\phi_{S,m}(\{A, B\}))$ is even if $g-m+1 \equiv 0 \mod 4$ and odd if $g - m + 1 \equiv 2 \mod 4$. This proves part (iv) of the Lemma. □

Proposition 2.14 *Fix as in Definition 2.1 some* $\overline{X}_{g,n} \in \{\overline{M}_{g,n}, \overline{R}_{g,n}, \overline{S}_{g,n}\}$*. We say that* $[CM]$ *if* $\overline{X}_{g,n} = \overline{M}_{g,n}$*, that* $[CR]$ *if* $\overline{X}_{g,n} = \overline{R}_{g,n}$ *and that* $[CS]$*, if* $\overline{X}_{g,n} = \overline{S}_{g,n}$*.*

We set $s := 0$ *if* $[CM]$ *or* $[CR]$*,* $s := g - 1$ *if* $[CS]$*, and we set* $\mu := 0$ *if* $[CM]$*, and* $\mu := 2$ *otherwise. Set* $u := 2$ *if* $[CR]$*,* $u := 0$ *otherwise.*

Denote by $\overline{HX}^{\sim}_{g,n}$ *the normalisation of* $\overline{HX}_{g,n}$*. Then:*

(i) For each k *with* $u \leq k \leq 2g + 2 - u$*,* $k \equiv s \mod \mu$ *and each choice of a subset* $T \subseteq \underline{n}$ *with* $t := |T| \leq k$ *and* $0 \leq 2g + 2 - k - (n - t) =: \tau$*, define a map*

$$a'_{\overline{X}_{g,n},k,T} : M_{0,(n,\langle [k-t],[\tau]\rangle)} \to HX_{g,n},$$

by setting for every $[\mathfrak{D}] := [(\mathbb{P}^1, p_1, ..., p_n, \{q_1, ..., q_{k-t}\}, \{q'_1, ..., q'_\tau\})] \in M_{0,(n,\langle [k-t],[\tau]\rangle)}$*:*

$$a'_{\overline{X}_{g,n},k,T}([\mathfrak{D}]) = [(C; P_1, ..., P_n, \mathcal{O}_C(B))] \in HX_{g,n}, \quad \textit{where:}$$

$f : C \to \mathbb{P}^1$ *is the unique degree* 2 *cover, branched exactly over all the* $2g + 2$ *points* p_i*,* q_i *and* q'_i*. The marked points on* C *are* $P_i := f^{-1}(p_i)$ *for* $i \in \underline{n}$*. Denote by* $r_1, .., r_k$ *those of the points* p_i *with indices in* T *together with all the points* $q_1, .., q_{k-t}$*; the ordering does not mater here. Set* $\bar{s} := s - k$*, then* $\bar{s}$ *is even. Set* $R_i := f^{-1}(r_i)$ *(each* R_i *a point), let* ξ *be the divisor class of any point of* $\mathbb{P}^1$ *and* $\Xi := f^*(\xi)$ [23]*. Then* B *is the divisor*

$$B := \frac{\bar{s}}{2}\Xi + \sum_{i=1}^{k} R_i.$$

In case $[CM]$*, ignore* $\mathcal{O}_C(B)$*, which is the just* $\mathcal{O}_C$ *then.*

[23] In particular we may choose $\Xi = 2R_i$ for any R_i if $k > 0$.

Then the map $a'_{\overline{X}_{g,n},k,T}$ is a morphism of varieties, which is an isomorphism to one of the connected components of $HX_{g,n}$. (cf. Definition 2.4 (iii) for the notation $M_{0,(n,\langle[k-t],[\tau]\rangle)}$.)

In case $[CS]$, $a'_{\overline{X}_{g,n},k,T}$ maps to a component of $HS^+_{g,n}$ if $k \equiv g+1$ mod 4, and to a component of $HS^-_{g,n}$ if $k \equiv g-1$ mod 4.

Two maps $a'_{\overline{X}_{g,n},k_1,T_1}$ and $a'_{\overline{X}_{g,n},k_2,T_2}$ have the same image if and only if either $k_1 = k_2$ and $T_1 = T_2$ or $k_1 + k_2 = 2g+2$ and $\underline{n} = T_1 \uplus T_2$. Furthermore every connected component of the normal variety $HX_{g,n}$ is the image of one of these morphisms.

(ii) The morphism $a'_{\overline{X}_{g,n},k,T}$ extends to a morphism

$$b_{\overline{X}_{g,n},k,T} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}_{g,n}.$$

which surjects onto one of the irreducible components of $\overline{HX}_{g,n}$. It factors through a morphism

$$a_{\overline{X}_{g,n},k,T} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}^{\sim}_{g,n}$$

to the normalisation. This $a_{\overline{X}_{g,n},k,T}$ is an isomorphism to one of the connected components of $\overline{HX}^{\sim}_{g,n}$.

Restricted to the interiors of the moduli spaces, which are normal varieties, the morphisms $a'_{\overline{X}_{g,n},k,T}$, $a_{\overline{X}_{g,n},k,T}$ and $b_{\overline{X}_{g,n},k,T}$ coincide.

(iii) Hence the number of irreducible components of $\overline{HX}_{g,n}$ is

$$\frac{1}{2} \sum_{\substack{k\in\mathbb{N}_0,\ u\le k\le 2g+2-u \\ k\equiv s \text{ mod } m}} \left(\sum_{\substack{t\in\mathbb{N}_0,\ s.\ th.\ 0\le t\le k \\ and\ n+k-2g-2\le t}} \binom{n}{t} \right).$$

Note that if $[CM]$, the only possible value of k is 0, so there is only one component of $\overline{HM}^{\sim}_{g,n}$. Also $\overline{HM}^{\sim}_{g,n} = \overline{HM}_{g,n}$.

Proof: Obviously the conditions on k and T in case [CM] imply $k = 0$ and $T = \emptyset$. In this proof we use the notation $a'_{\overline{M}_{g,n}} := a'_{\overline{M}_{g,n},0,\emptyset}$, $a_{\overline{M}_{g,n}} := a_{\overline{M}_{g,n},0,\emptyset}$, $b_{\overline{M}_{g,n}} := a_{\overline{M}_{g,n},0,\emptyset}$. By Fact 2.2, $\overline{HM}_{g,n}$ is normal, so $b_{\overline{M}_{g,n}} = a_{\overline{M}_{g,n}}$. Let $\rho : \overline{H}_{2,(n,[2g+2-n])} \to \overline{M}_{0,(n,[2g+2-n])}$ be the forgetful morphism, which is an isomorphism by Lemma 2.9 (iv) and $c_{\overline{M}_{g,n}} : \overline{H}_{2,(n,[2g+2-n])} \to \overline{HM}_{g,n}$ the isomorphism introduced in 2.9 (iii). Then define $a_{\overline{M}_{g,n}} := c_{\overline{M}_{g,n}} \circ \rho^{-1}$, let $a'_{\overline{M}_{g,n}}$ be the restriction of $a_{\overline{M}_{g,n}}$ to the interior $M_{0,(n,[2g+2-n])}$. Now it is easy to check, using 2.9, that these isomorphisms fulfil all claims of our proposition for the case [CM]. For the other cases:

(i): Let $\widetilde{\rho} : H_{2,(n,\langle[k-t],[\tau]\rangle)} \to M_{0,(n,\langle[k-t],[\tau]\rangle)}$ be the restriction of the isomorphism from Proposition 2.7 (i) to the interior of the moduli spaces. Then set $a'_{\overline{X}_{g,n},k,T} := a'' \circ \widetilde{\rho}^{-1}$, where $a'' : H_{2,(n,\langle[k-t],[\tau]\rangle)} \to HX_{g,n}$ is the closed embedding which is sending a point $[C \xrightarrow{f} \mathfrak{D}] \in H_{2,(n,\langle[k-t],[\tau]\rangle)}$ to the point $[(C; P_1, ..., P_n, \mathcal{O}_C(B))] \in HX_{g,n}$, as defined in (i). The image of a'' is in $HX_{g,n}$ by Lemma 2.9 (ii) and Lemma 2.13. That a'' is indeed a morphisms of varieties, one sees as follows: The assignment defining a'', can be caried over to the level of families. Here one starts with a double cover $\mathcal{C} \xrightarrow{\mathbf{f}} \mathscr{D} \to S$, with $\mathscr{D} = (\mathcal{D}, \sigma_1, ..., \sigma_n, \{\xi_1, ..., \xi_{k-t}\}, \{\xi'_1, ..., \xi'_\tau\})$, $\mathcal{D} \to S$ a family of $\mathbb{P}^1$'s, and assigns to it the family $\mathscr{X} \to S$, where $\mathscr{X} := (\mathcal{C}, \Sigma_1, ..., \Sigma_n, \mathcal{O}_{\mathcal{C}}(B))$ with $B = \bar{s}\Omega + \sum_{i=1}^k \Omega_i$. Here Σ_i are the liftings of the sections σ_i to $\mathcal{C}$, the Ω_i are also liftings of sections and are defined analogously to the R_i in (i). Ω is the lifting of any of the sections of marked points belonging to $\mathscr{D}$. It is clear that this assignment is compatible with base change, and so defines a morphism of moduli functors [24], and the morphism of coarse moduli spaces induced by this is a''. If follows from Lemma 2.9 (ii) and the injectivity of the maps $\phi_{...}$ from Lemma 2.13, that a'' is a bijection to one of the components of $HX_{g,n}$. Hence (with Lemma 1.46) a'' is a closed embedding since $HX_{g,n}$ is normal (which follows from the description of the hyperelliptic local universal deformation in section 2.1.3).

The claims of the last two paragraphs of (i) follow from Lemma 2.13, in particular the claim for [CS] follows from part (iii) of that Lemma.

(ii): Since $HX_{g,n}$ is normal, it embeds into $\overline{HX}^{\sim}_{g,n}$. We have a commutating diagram

$$\begin{array}{ccc}
M_{0,(n,\langle[k-t],[\tau]\rangle)} & \xrightarrow{a'_{\overline{X}_{g,n,k,T}}} & HX_{g,n} \\
\downarrow & & \downarrow \\
\overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} & \overset{b_{\overline{X}_{g,n,k,T}}}{\dashrightarrow} & \overline{HX}_{g,n} \\
\pi \downarrow & & \downarrow \tau' \\
\overline{M}_{0,(n,[2g+2-n])} & \xrightarrow{a_{\overline{M}_{g,n}}} & \overline{HM}_{g,n}
\end{array}$$

where π is the morphism forgetting the partition on the $2g+2-n$ marked points which are not ordered, while τ' is the restriction of the finite forgetful morphism $\tau : \overline{X}_{g,n} \to \overline{M}_{g,n}$. The "dashed" finite morphism $b_{\overline{X}_{g,n,k,T}}$ exists by Lemma 1.45. Since $\overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)}$ is normal $b_{\overline{X}_{g,n,k,T}}$ factors through an $a_{\overline{X}_{g,n,k,T}} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}^{\sim}_{g,n}$. Now $a_{\overline{X}_{g,n,k,T}}$ has degree 1 by (i), thus is an isomorphism by Lemma 1.46.

(iii) is implied by (i) and (ii). □

2.2.1 Conclusions from the Proposition

Corollary 2.15 *For all $g \geq 2$ and every $\overline{Q} \in \{\overline{HM}_{g,n}, (\overline{HS}^+_{g,n})^\sim, (\overline{HS}^-_{g,n})^\sim, (\overline{HR}_{g,n})^\sim\}$ we have:*

(i) Every connected component of $\overline{Q}$ is unirational.

(ii) $A^*(\overline{Q}) \cong H^*(\overline{Q})$, as graded $\mathbb{Q}$-algebras, via the cycle map. In particular $H^n(\overline{Q}) = 0$ for all odd n.

(iii) $Pic_{\mathbb{Q}}(\overline{Q}) \cong A^1(\overline{Q})$

(iv) $A^1(\overline{Q})$ is generated by the boundary divisors of $\overline{Q}$. (Meaning the preimages of the boundary divisors of the moduli space on its normalization.)

[24] It is easy to check that it is even a morphism of moduli groupoids.

(v) $h^{p,0}(\overline{Q}) = 0$ *for* $p > 0$.

Proof: For all these claims it suffices to show them for every connected component of $\overline{Q}$. Let $\overline{Y}$ be such a component, Y its Interior. Then, by Proposition 2.14 and the Remark 2.5, $\overline{Y} \cong \overline{M}_{0,2g+2}/G$ for some subgroup G of $\mathbb{S}_{2g+2}$.

(i): $\overline{Y} \cong \overline{M}_{0,2g+2}/G$ is of course covered by $\overline{M}_{0,2g+2}$, and all spaces $\overline{M}_{0,n}$ are rational (Summary 1.48 (i)).

(ii): By Summary 1.48 (ii), $A^*(\overline{M}_{0,2g+2}) \cong H^*(\overline{M}_{0,2g+2})$. Using Lemma 1.37 we get:

$$A^*(\overline{Y}) \cong A^*(\overline{M}_{0,2g+2}/G) \cong (A^*(\overline{M}_{0,2g+2}))^G$$
$$\cong (H^*(\overline{M}_{0,2g+2}))^G \cong H^*(\overline{M}_{0,2g+2}/G) \cong H^*(\overline{Y})$$

(iii): $\overline{Y}$ is normal, so the Picard group is in a natural way a subgroup of the divisor class group, cf. [Har77] Remark 6.11.2. and Prop. 6.15. Thus there is an injection

$$Pic_{\mathbb{Q}}(\overline{Y}) \longrightarrow A^1(\overline{Y})$$

Since $\overline{Y} \cong \overline{M}_{0,2g+2}/G$ has only finite quotient singularities, it is $\mathbb{Q}$-factorial, i.e. every Weil-divisor is $\mathbb{Q}$-Cartier. Thus the map is also surjective.

(iv): By Summary 1.48, $A^1(\overline{M}_{0,2g+2}) = A_{(2g-1)-1}(\overline{M}_{0,2g+2})$ is generated by the boundary divisor classes, i.e. the map $A_{(2g-1)-1}(\overline{M}_{0,2g+2} \smallsetminus M_{0,2g+2}) \longrightarrow A_{(2g-1)-1}(\overline{M}_{0,2g+2})$ is surjective. The exact sequence

$$A_{(2g-1)-1}(\overline{M}_{0,2g+2} \smallsetminus M_{0,2g+2}) \longrightarrow A_{(2g-1)-1}(\overline{M}_{0,2g+2}) \longrightarrow A_{(2g-1)-1}(M_{0,2g+2}) \longrightarrow 0$$

then yields $A_{(2g-1)-1}(M_{0,2g+2}) = A_{(2g-1)-1}(M_{0,2g+2}) = 0$. By Lemma 1.37, then

$$A_{(2g-1)-1}(Y) \cong A_{(2g-1)-1}(M_{0,2g+2}/G) \cong (A_{(2g-1)-1}(M_{0,2g+2}))^G = 0$$

Again using an exact sequence like the one above we conclude that $A_{(2g-1)-1}(\overline{Y} \smallsetminus Y) \longrightarrow A_{(2g-1)-1}(\overline{Y})$ is surjective, i.e. that $A_{(2g-1)-1}(\overline{Y}) \cong A^1(\overline{Y})$ is generated by the boundary divisor classes.

(v): According to [Kee92], every $\overline{M}_{0,2g+2}$, is rational. Thus $H^{p,0}(\overline{M}_{0,2g+2}) \cong H^{p,0}(\mathbb{P}^{n-3}) = 0$ for all $p > 0$, since all $h^{p,0}$ are birational invariants (cf. [GH94] p. 494). This implies $H^{p,0}(\overline{Y}) = (H^{p,0}(\overline{M}_{0,2g+2}))^G = 0$. □

2.3 Description of the morphisms $b_{\overline{X}_{g,n},k,T}$ on the boundary.

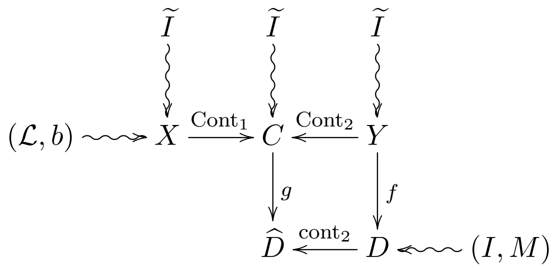
In Proposition 2.14 we constructed morphisms

$$b_{\overline{X}_{g,n},k,T} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}_{g,n}.$$

By the construction we know these morphisms explicitly only on the interior of the moduli spaces, i.e. on classes of smooth curves. In this section we investigate the behaviour of $b_{\overline{X}_{g,n},k,T}$ on the boundary. But first we fix a lot of notation, which will be used in this and also the next section.

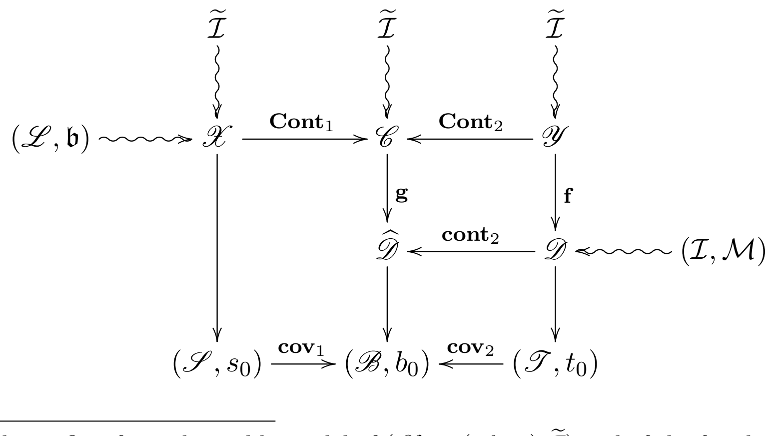
Notation 2.16 (i) Fix an $\overline{X}_{g,n} \in \{\overline{S}_{g,n}, \overline{R}_{g,n}\}$ and an $\mathfrak{X} = (X; p_1, ..., p_n; \mathcal{L}, b)$ with $[\mathfrak{X}] \in \overline{HX}_{g,n}$. Let $\mathrm{Cont}_1 : \mathfrak{X} \to \mathfrak{C}$ be the stable model of $\mathfrak{X}$ ($\mathfrak{C} = (C; p_1, ..., p_n)$). Choose a $\mathfrak{D}'' = (D; (I, M))$ with $[\mathfrak{D}''] \in \overline{M}_{0,(n,[2g+2-n])}$, $I = (q_1, ..., q_n)$, $M = \{q'_1, ..., q'_{2g+2-n}\}$, such that for the (unique up to isomorphism) admissible double cover $f : Y \to \mathfrak{D}''$ we have that $\mathfrak{C}$ is the stable model of the pointed nodal curve $(Y; f^{-1}(q_1), ..., f^{-1}(q_n))$. Denote by $\mathrm{Cont}_2 : (Y; f^{-1}(q_1), ..., f^{-1}(q_n)) \to \mathfrak{C}$ the morphism to the stable model, contracting the exceptional components.

(ii) Let $h \in \mathrm{Aut}(\mathfrak{C})$ be the hyperelliptic involution, and let $C \to \widehat{D} := C/h$ be the quotient morphism. Then since Cont_2 is compatible with h and the hyperelliptic involution on Y (cf. Lemma 2.9), it induces a morphism $\mathrm{cont}_2 : D \to \widehat{D}$. So we have a commutative diagram



Here the same symbol $\widetilde{I}$ is used to denote the tuple $(p_1, ..., p_n)$ of marked points on X as well as on C, and also the tuple $(f^{-1}(q_1), ..., f^{-1}(q_n))$ on Y, since these tuples of marked points are "identified" by the morphisms Cont_1 resp. Cont_2. Here, and in the following, we indicate by curly arrows that extra structures are attached to some varieties.

(iii) Let $\mathscr{X} \to (\mathscr{S}, s_0)$, $\mathscr{C} \to (\mathscr{B}, b_0)$ be the hyperelliptic local universal deformations of $\mathfrak{X}$ resp. $\mathfrak{C}$ (cf. end of section 2.1.2), and let $\mathscr{Y} \xrightarrow{\mathbf{f}} \mathscr{D} \to (\mathscr{T}, t_0)$ be the local universal deformation of the admissible double cover $f : Y \to \mathfrak{D}''$. Then (possibly after shrinking $\mathscr{S}$, $\mathscr{T}$, $\mathscr{B}$ appropriately) by forming the stable model one induces morphisms of the two other families to $\mathscr{C} \to (\mathscr{B}, b_0)$, which can be seen in the commutative diagram [25]



[25] More precisely we first form the stable model of $(\mathscr{X} \to (\mathscr{S}, s_0), \widetilde{\mathcal{I}})$ and of the family $(\mathscr{Y} \to (\mathscr{T}, t_0), \widetilde{\mathcal{I}})$ where $\widetilde{\mathcal{I}}$ are the preimages on $\mathscr{Y}$ of the n ordered sections of marked points on $\mathscr{D}$. The resulting families of hyperelliptic pointed stable curves are (possibly after reducing the radius of the complex balls $\mathscr{S}$, $\mathscr{T}$) pullbacks from the local universal deformation $\mathscr{C} \to (\mathscr{B}, b_0)$ via finite surjective maps $\mathbf{cov}_1 : (\mathscr{S}, s_0) \to (\mathscr{B}, b_0)$ and $\mathbf{cov}_2 : (\mathscr{T}, t_0) \to (\mathscr{B}, b_0)$.

Here $\widehat{\mathscr{D}}$ is the quotient $\mathscr{C}/h$, where h is the hyperelliptic involution on $\mathscr{C} \to (\mathscr{B}, b_0)$. [26] If we restrict everything in this diagram to the central fibres over s_0, b_0, t_0, we get back to the diagram of (ii).

(iv) If for example $\mathcal{T}$ is a set of sections of some family $\mathcal{X} \to B$ we denote by $[\mathcal{T}]$ the divisor class in $A^1(\mathcal{X})$ which is the sum of all the images of the sections in $\mathcal{T}$.

Now for a given $\mathfrak{X} \in \overline{HX}_{g,n}$ we would like to use the diagram of local universal deformations defined in (iii), to relate the hyperelliptic deformation of $\mathfrak{X}$ to the local universal deformation of curves $\mathfrak{D}$ with $[\mathfrak{D}] \in b^{-1}_{\overline{X}_{g,n},k,T}([\mathfrak{X}]) \subset \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)}$. This will be possible on these dense open subsets of the deformations which parametrise smooth curves, since for smooth curves we already know $b^{-1}_{\overline{X}_{g,n},k,T}([\mathfrak{X}])$ explicitly. This relation on the open parts, will be used to obtain the description of $b_{\overline{X}_{g,n},k,T}$ on singular curves (in this section), and also to compare the automorphism groups of the central fibres over s_0 and t_0 (in the next section).

Lemma & Definition 2.17 *We use Notation 2.16 and also the notation of Proposition 2.14.*

(i) Fix one of the morphisms

$$b_{\overline{X}_{g,n},k,T} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}_{g,n}$$

such that $[\mathfrak{X}]$ lies in the image of $b_{\overline{X}_{g,n},k,T}$. Denote by $\mathbf{H}_{k,T}$ the irreducible component of $\overline{HX}_{g,n}$ which is the image of $b_{\overline{X}_{g,n},k,T}$. Let $\mathscr{S}^{k,T}$ be the preimage of $\mathbf{H}_{k,T}$ on $\mathscr{S}$. Then $\mathscr{S}^{k,T}$ may have several irreducible components $\mathscr{S}^{(1)}, ..., \mathscr{S}^{(r)}$. Pick one of these components $\mathscr{S}^{(j)}$, and call the restriction of the local universal deformation to it $\mathscr{X}^{(j)} \to \mathscr{S}^{(j)}$

(ii) Let $\mathscr{X}^{(j)\prime} \to \mathscr{S}^{(j)\prime}$, $\mathscr{C}' \to \mathscr{B}'$ and $\mathscr{Y}' \xrightarrow{\mathbf{f}'} \mathscr{D}' \to \mathscr{T}'$ be the open subfamilies of our deformations containing all smooth fibres. Then over these sets the diagram of Notation 2.16 restricts to a cartesian diagram [27]:

$$\begin{array}{ccccccc}
 & & (\widetilde{\mathcal{I}'}, \widetilde{\mathcal{M}'})_{\mathscr{X}^{(j)}} & & (\widetilde{\mathcal{I}'}, \widetilde{\mathcal{M}'})_{\mathscr{C}} & & (\widetilde{\mathcal{I}'}, \widetilde{\mathcal{M}'})_{\mathscr{Y}} & & \\
 & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\
(\mathscr{L}, \mathfrak{b}) & \rightsquigarrow & \mathscr{X}^{(j)\prime} & \xrightarrow{\mathbf{Cont}'_1} & \mathscr{C}' & \xleftarrow{\mathbf{Cont}'_2} & \mathscr{Y}' & & \\
 & & \downarrow & & \downarrow \mathbf{g}' & & \downarrow \mathbf{f}' & & \\
 & & & & \widehat{\mathscr{D}}' & \xleftarrow{\mathbf{cont}'_2} & \mathscr{D}' & \leftsquigarrow & (\mathcal{I}', \mathcal{M}') \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathscr{S}^{(j)\prime} & \xrightarrow{\mathbf{cov}'_1} & \mathscr{B}' & \xleftarrow{\mathbf{cov}'_2} & \mathscr{T}' & &
\end{array}$$

Here we also refined the extra structures: On each of $\mathscr{X}^{(j)\prime}$, $\mathscr{C}'$ and $\mathscr{Y}'$ there is a unique hyperelliptic involution. They are compatible with each other via the morphisms in the diagram. For each of the three families let $\widetilde{\mathcal{F}}'$ denote the set of $2g+2$ sections which are

[26] $\widehat{\mathscr{D}} \to (\mathscr{B}, b_0)$ is a family of nodal curves of genus 0 (cf. [ACG11], page 210).

[27] I.e. the squares in the diagram are squares of fibre products.

fixed by the hyperelliptic involution. Among these sections are the n sections of ordered marked points, i.e. $\widetilde{\mathcal{I}}' \subseteq \widetilde{\mathcal{F}}'$. Set for each family $\widetilde{\mathcal{M}}' := \widetilde{\mathcal{F}}' \setminus \widetilde{\mathcal{I}}'$ to define on each a sorting $(\widetilde{\mathcal{I}}', \widetilde{\mathcal{M}}')_{\dots}$ of $\widetilde{\mathcal{F}}'$. Then the above diagram also commutes in the sense that these sections on the families and their sortings are compatible via the morphisms in the diagram.

Denote by $\mathcal{T} \subseteq \mathcal{I}$, resp. $\widetilde{\mathcal{T}} \subseteq \widetilde{\mathcal{I}}$ the set of sections with indices in $T \subseteq \underline{n}$ (T from $b_{\overline{X}_{g,n},k,T}$).

(iii) On $\mathscr{X}^{(j)'}$ there is a set of sections,

$$\widetilde{\mathcal{A}}' \subseteq \widetilde{\mathcal{F}}' \quad \textit{with} \quad |\widetilde{\mathcal{A}}'| = k, \quad \widetilde{\mathcal{A}}' \cap \widetilde{\mathcal{I}}' = \widetilde{\mathcal{T}}', \quad \textit{such that} \quad \mathscr{L}' \cong \mathcal{O}_{\mathscr{X}^{(j)'}}(\overline{s}\Omega + [\widetilde{\mathcal{A}}']),$$

where $\Omega \in \widetilde{\mathcal{F}}'$ arbitrary, and where $\overline{s} = -k$ if [CR] and $\overline{s} = g - 1 - k$ if [CS]. This $\widetilde{\mathcal{A}}'$ is unique unless $n = 0$ and $k = g + 1$, in which case the only other possible choice is $\widetilde{\mathcal{F}}' \setminus \widetilde{\mathcal{A}}'$.

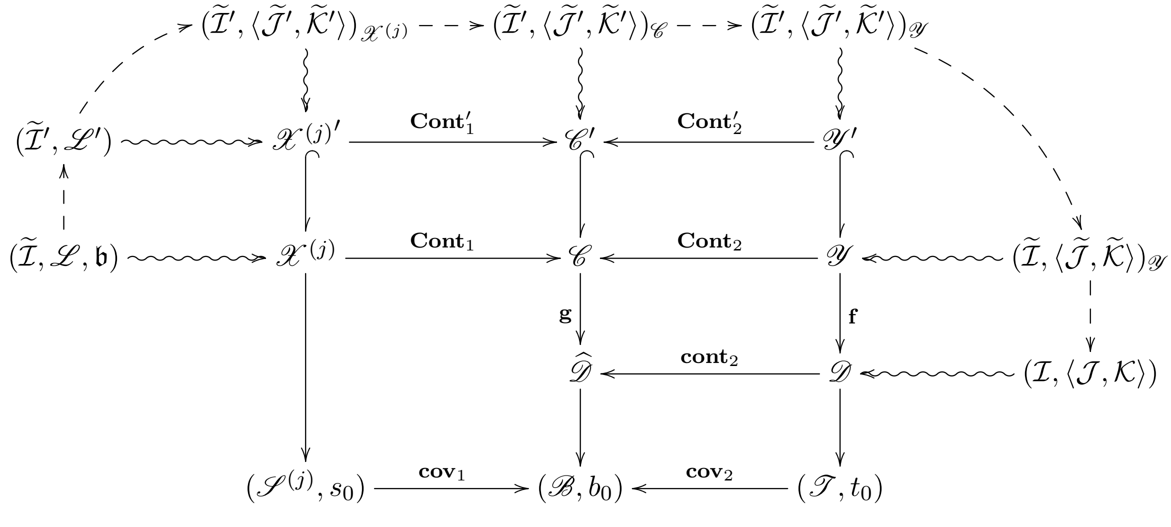
Use this to define a sorting $(\widetilde{\mathcal{I}}', \langle \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}' \rangle)_{\mathscr{X}^{(j)}}$ [28] on $\widetilde{\mathcal{F}}'$, where we set $\widetilde{\mathcal{J}}' := \widetilde{\mathcal{A}}' \cap \widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{K}}' = \widetilde{\mathcal{M}}' \setminus (\widetilde{\mathcal{A}}' \cap \widetilde{\mathcal{M}}')$. Note that $|\widetilde{\mathcal{J}}'| = k - t$ and $|\widetilde{\mathcal{K}}'| = \tau$.

(iv) Denote the induced sortings $\mathrm{Cont}_1'\big((\widetilde{\mathcal{I}}', \langle \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}' \rangle)_{\mathscr{X}^{(j)}}\big)$ [29] on the sections $\mathcal{F}'$ of $\mathscr{C}'$ by $(\widetilde{\mathcal{I}}', \langle \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}' \rangle)_{\mathscr{C}}$. Transfer this sorting to $\mathscr{Y}$ by setting

$$(\widetilde{\mathcal{I}}', \langle \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}' \rangle)_{\mathscr{Y}} := \mathrm{Cont}_2'^{-1}\big((\widetilde{\mathcal{I}}', \langle \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}' \rangle)_{\mathscr{C}}\big).$$

The hyperelliptic involution of $\mathscr{Y}$ has a set of $2g + 2$ fixed point sections $\widetilde{\mathcal{F}}$, and these sections are disjoint and only meet smooth points of each fibre of $\mathscr{Y} \to (\mathscr{T}, t_0)$. They are the $\mathbf{f}$-preimages of the $2g+2$ sections of marked points in $\mathcal{I} \cup \mathcal{M}$ on $\mathscr{D}$. Extend the sorting $(\widetilde{\mathcal{I}}', \langle \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}' \rangle)_{\mathscr{Y}}$ to a sorting $(\widetilde{\mathcal{I}}, \langle \widetilde{\mathcal{J}}, \widetilde{\mathcal{K}} \rangle)_{\mathscr{Y}}$ on $\widetilde{\mathcal{F}}$. Set $(\mathcal{I}, \langle \mathcal{J}, \mathcal{K} \rangle) := \mathbf{f}\big((\widetilde{\mathcal{I}}, \langle \widetilde{\mathcal{J}}, \widetilde{\mathcal{K}} \rangle)_{\mathscr{Y}}\big)$.

What we have constructed so far is shown in the following diagram, where dashed arrows point from one extra structure to an extra structure constructed from this one.



(v) Restrict the sorting of sections $(\mathcal{I}, \langle \mathcal{J}, \mathcal{K} \rangle)$ to a sorting of the set of marked points on the central fibre D and denote it as $(I, \langle J, K \rangle)$. Set $\mathfrak{D}^{(j)} := (D, (I, \langle J, K \rangle))$. Then $[\mathfrak{D}^{(j)}] \in \overline{M}_{0,(n,\langle [k-t],[\tau] \rangle)}$, and

$$b_{\overline{X}_{g,n},k,T}\big([\mathfrak{D}^{(j)}]\big) = [\mathfrak{X}] \in \overline{HX}_{g,n}.$$

[28] cf. Definition 2.4 (iii) for the notation.

[29] By this we mean: Cont_1' is applied to every one of the sets $\widetilde{\mathcal{I}}', \widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}'$.

The $\mathfrak{D}^{(j)}$ we obtained in general depends on the choice of $\mathscr{S}^{(j)}$ made in (i), and if $\mathfrak{D}^{(1)},...,\mathfrak{D}^{(r)}$ are the $\mathfrak{D}^{(j)}$ obtained from the different $\mathscr{S}^{(j)}$, then

$$b^{-1}_{\overline{X}_{g,n},k,T}([\mathfrak{X}]) = \{[\mathfrak{D}^{(1)}],...,[\mathfrak{D}^{(r)}]\}.$$ [30]

(vi) On $\mathfrak{X}^{(j)}$ resp. $\mathfrak{C}$ define $\widetilde{\mathcal{F}}$, $(\widetilde{\mathcal{I}},\langle\widetilde{\mathcal{J}},\widetilde{\mathcal{K}}\rangle)_{\mathscr{X}^{(j)}}$ resp. $(\widetilde{\mathcal{I}},\langle\widetilde{\mathcal{J}},\widetilde{\mathcal{K}}\rangle)_{\mathscr{C}}$ by uniqely continuing the sections from $\widetilde{\mathcal{F}}'$, and hence those from $\widetilde{\mathcal{J}}'$, $\widetilde{\mathcal{K}}'$, to the whole family. The resulting sections in $\widetilde{\mathcal{J}}$ and $\widetilde{\mathcal{K}}$ on $\mathfrak{X}^{(j)}$ resp. $\mathfrak{C}$ are not necessarily disjoint and may contain singular points of fibres. These sorted sets of continued sections are again compatible via $\mathbf{Cont}_1$ *and* $\mathbf{Cont}_2$.

(vii) The varieties $\mathscr{X}^{(j)}$, $\mathscr{C}$, $\mathscr{Y}$ are normal, and hence forming the first Chern class of line bundles induces inclusions $\mathrm{Pic}_{\mathbb{Z}}(\mathscr{X}^{(j)}) \subseteq A^1_{\mathbb{Z}}(\mathscr{X}^{(j)})$, and so on. [31] *If $\mathscr{L}$ is a line bundle on $\mathscr{X}^{(j)}$, which is the pullback of a line bundle from the usual local universal deformation $\mathcal{X} \to (S,s_0)$, and $A \in A^1_{\mathbb{Z}}(\mathscr{X}^{(j)})$ is a divisor such that $\mathcal{O}_{\mathscr{X}^{(j)}}(A) \cong \mathscr{L}$ or equivalently $c_1(\mathscr{L}) = A$, then for any smooth variety Z and any morphism $\psi : Z \to \mathscr{X}^{(j)}$ we have*

$$\psi^*\mathscr{L} = \mathcal{O}_Z(\psi^*A).$$

Proof: Large parts are just definitions or are quite obvious. (i) follows from the description of the hyperelliptic local universal deformation spaces at the end of section 2.1.2. (ii) is clear by the construction of $\mathbf{Cont}_1$ and $\mathbf{Cont}_2$, and by Summary 2.8 (iii) and Lemma 2.9 (ii).

(iii): By Lemma 2.13, on every fibre X_{s_1} of $(\mathscr{X}' \to \mathscr{S}^{(j)\prime}, \mathscr{L}')$ we can write $\mathscr{L}'_{|X_{s_1}}$ in the form $\mathcal{O}_{X_{s_1}}(\overline{s}\omega + [A_{s_1}])$, where $\{p_1,...,p_{2g+2}\}$ are the fixed points of the hyperelliptic involution on X_{s_1}, and A_{s_1} is a certain subset of these and ω is any of the points p_i. We can choose A_{s_1} such that $T_{s_1} = A_{s_1} \cap I_{s_1}$ and such that $|A_{s_1}| = k$, since the class of X_{s_1} is in the image of $b_{\overline{X}_{g,n},k,T}$, like it is for every fibre over $\mathscr{S}^{(j)}$. Let $\widetilde{\mathcal{A}}'$ be the subset of the set of sections $\widetilde{\mathcal{F}}'$, which restricts to A_{s_1} on X_{s_1}, Ω the section which restricts to ω. Then $\mathcal{O}_{\mathscr{S}^{(j)\prime}}(\overline{s}\Omega + [\widetilde{\mathcal{A}}'])$ is a prym sheaf on the family $\mathscr{X}^{(j)\prime} \to \mathscr{S}^{(j)\prime}$ which agrees with $\mathscr{L}'$ on X_{s_1}. Since over a families of smooth curves a prym sheaf can locally be deformed in only one way, it follows that $\mathscr{L}' \cong \mathcal{O}_{\mathscr{S}^{(j)\prime}}(\overline{s}\Omega + [\widetilde{\mathcal{A}}'])$.

(iv): From (i) and by definition of families of admissible covers.

(v): By the discussion in the proof of (iii) along with the description of $b_{\overline{X}_{g,n},k,T}$ on classes of smooth curves in Proposition 2.14, if $s_1 \in \mathscr{S}^{(j)\prime}$ and $t_1 \in \mathscr{T}'$ lie over the same point b_1 of $\mathscr{B}'$, then the class of fiber $[(D_{t_1},(I_{t_1},\langle J_{t_1},K_{t_1}\rangle))] \in \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)}$ is mapped to $[(X_{s_1},\mathscr{L}_{|X_{s_1}})] \in \overline{HX}_{g,n}$ by $b_{\overline{X}_{g,n},k,T}$. So by continuity $b_{\overline{X}_{g,n},k,T}([\mathfrak{D}]) = [\mathfrak{X}]$. By construction of $b_{\overline{X}_{g,n},k,T}$, the preimage $b^{-1}_{\overline{X}_{g,n},k,T}([\mathfrak{X}])$ has one element for each branch of the local analytic neighbourhood of $[\mathfrak{X}]$ in $\overline{HX}_{g,n}$. Furthermore by section 2.1.3, forming the quotient of $\mathscr{S}$ by $\mathrm{Aut}(\mathfrak{X})$ maps each $\mathscr{S}^{(j)}$ surjectively to one of these branches. This implies the second claim of (v).

[30] In some cases $[\mathfrak{D}^{(i)}] = [\mathfrak{D}^{(j)}]$ for some $i \neq j$ in $\underline{r}$.

[31] To be able to use the Chow group here we should strictly speaking switch from the analytic to the algebraic category, and work with local universal deformations over the spectrum of complete local rings instead of complex balls.

(vi) is clear. For (vii): We know from section 1.5 that the local universal deformations $\mathcal{X} \to (S, s_0)$ and $\mathcal{C} \to (B, b_0)$ are smooth. The subspaces $\mathscr{S}^{(j)} \subseteq S$ and $\mathscr{B} \subseteq B$ are both linear, hence complete intersections. So also $\mathscr{X}^{(j)} \subseteq \mathcal{X}$ and $\mathscr{C} \subseteq \mathcal{C}$ are complete intersections. Since they are also regular in codimension 1 they are normal varieties by Proposition 8.23. of Chapter II of [Har77]. The last equation of (vii) is well known for smooth varieties (cf. appendix A 3.) of [Har77]). Since $\mathcal{X}$ is smooth, our claim follows from the fact that $\mathscr{X}^{(j)}$ is a complete intersection in $\mathcal{X}$ and Proposition 2.6 (e) of [Ful98]. □

Fix any $\mathscr{X}^{(j)}$. Consider a node γ of D, let $\mathscr{T}(\gamma) \subseteq \mathscr{T}$ be the codimension 1 linear subspace over which γ is retained (i.e. not smoothed). Set $\mathscr{B}(\gamma) := \mathbf{cov}_2(\mathscr{T}(\gamma))$, $\mathscr{S}(\gamma) := \mathbf{cov}_1^{-1}(\mathscr{B}(\gamma))$, let $\mathscr{Y}(\gamma) \to \mathscr{D}(\gamma) \to \mathscr{T}(\gamma)$, $\mathscr{X}(\gamma) \to \mathscr{S}(\gamma)$, be the restrictions of the families and let $\Gamma \subset \mathscr{D}$ be (the image of) the section to which γ extends on $\mathscr{D}(\gamma)$. Then the divisor $\mathscr{D}(\gamma)$ on $\mathscr{D}$ consists of two smooth irreducible components $\mathscr{D} = \mathscr{D}_1(\gamma) \cup \mathscr{D}_2(\gamma)$ such that $\mathscr{D}_1(\gamma) \cap \mathscr{D}_2(\gamma) = \Gamma$. Set for $i \in \underline{2}$, with *clo*(...) standing for the closure:

$$\widehat{\mathscr{X}_i}(\gamma) := \mathbf{Cont}_1^{-1}\Big(\mathbf{Cont}_2\big(\mathbf{f}^{-1}(\mathscr{D}_i(\gamma))\big)\Big), \quad \mathscr{E}(\gamma) := \widehat{\mathscr{X}_1}(\gamma) \cap \widehat{\mathscr{X}_2}(\gamma),$$

$$\mathscr{X}_i(\gamma) := clo(\widehat{\mathscr{X}_i}(\gamma) \smallsetminus \mathscr{E}(\gamma)) \quad \text{if } \mathscr{E}(\gamma) \subsetneqq \widehat{\mathscr{X}_i}(\gamma). \text{ Otherwise:} \quad \mathscr{X}_i(\gamma) := \widehat{\mathscr{X}_i}(\gamma) = \mathscr{E}(\gamma).$$

Each $\mathscr{X}_i(\gamma) \subset \mathscr{X}^{(j)}$ and $\mathscr{E}(\gamma) \subset \mathscr{X}^{(j)}$ is either a divisor of $\mathscr{X}$ or of codimension 2. ($\mathscr{E}(\gamma)$ may have one or two components. The $\mathscr{Y}_i(\gamma) := \mathbf{f}^{-1}(\mathscr{D}_i(\gamma))$ are always divisors, while $\mathbf{f}^{-1}(\Gamma)$ is always of codimension 2. But $\mathscr{Y}_i(\gamma)$ may be contracted by $\mathbf{Cont}_2$, and $\mathbf{Cont}_2(\mathbf{f}^{-1}(\Gamma))$, may be blown up by $\mathbf{Cont}_1^{-1}$.) Now denote by $[\mathscr{X}_i(\gamma)]$ and $[\mathscr{E}(\gamma)]$ the divisor classes in $A^1(\mathscr{X})$. For those which are of codimension 2 set $[\mathscr{X}_i(\gamma)] = 0$ resp. $[\mathscr{E}(\gamma)] = 0$. For a fixed γ, write $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$, $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$, $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, such that $\mathcal{I}_1$ contains those sections in $\mathcal{I}$ which meet $\mathscr{D}_1(\gamma)$, $\mathcal{I}_2$ those that meet $\mathscr{D}_2(\gamma)$, and so on. Define on $\mathscr{X}^{(j)}$, $\widetilde{\mathcal{T}}_i$, $\widetilde{\mathcal{I}}_i$, $\widetilde{\mathcal{J}}_i$, $\widetilde{\mathcal{K}}_i$ analogously.

By T_i, I_i, J_i, K_i, denote the sets of points in which the sets of sections $\mathcal{T}_i$, $\mathcal{I}_i$, $\mathcal{J}_i$, $\mathcal{K}_i$ meet the central fibre D. Then, using the notation of 2.16, 2.17 and of Proposition 2.14:

Lemma 2.18 *For the first Chern class $c_1(\mathscr{L}) \in A^1(\mathscr{X}^{(j)})$, of the spin/prym sheaf $\mathscr{L}$ of $\mathscr{X}^{(j)}$, and for some $z_1(\gamma), z_2(\gamma), z_E(\gamma) \in \mathbb{Z}$, and $\Omega \in \widetilde{\mathcal{F}}$ arbitrary :*

$$c_1(\mathscr{L}) = \overline{s}\Omega + [\widetilde{T} \cup \widetilde{J}] + \sum_{\gamma \in \mathrm{sing}(D)} z_1(\gamma)[\mathscr{X}_1(\gamma)] + z_2(\gamma)[\mathscr{X}_2(\gamma)] + z_E(\gamma)[\mathscr{E}(\gamma)] \qquad (\dagger)$$

Now fix a γ, and set $b_i := |I_i| + |J_i| + |K_i|$, $z_i := z_i(\gamma)$, $z_E := z_E(\gamma)$. [32] *Then:*

(i) $\mathscr{X}_i(\gamma)$ is of codimension 2 in $\mathscr{X}$ if and only if $b_i = 2$ and furthermore $|I_i = 0|$ and $|J_i| = 0$ or $|K_i| = 0$. In this case $[\mathscr{X}_1(\gamma)] = [\mathscr{X}_2(\gamma)] = [\mathscr{E}(\gamma)] = 0$.

(ii) If both $\mathscr{X}_i(\gamma)$ are divisors, then $\mathscr{E}(\gamma)$ has two components if and only if b_1 (and hence also b_2) is even.

(iii) Assume b_1 is odd. Then $\mathscr{E}(\gamma)$ is blown up, i.e. a divisor, if and only if [CS]. Furthermore if [CS], $z_1 + z_2 - 2z_E \equiv 1$ mod 2, and $-z_i + z_E \equiv \frac{1}{2}(b_i - 1) - 1 - |T_i| - |J_i|$ mod 2. If [CR], $z_2 - z_1 \equiv |T_1| + |J_1|$ mod 2.

[32] $b_1, b_2 \geq 2$, since $\mathfrak{D}$ is stable.

(iv) If b_1 is even, either both components of $\mathscr{E}(\gamma)$ or none are blown up. They are blown up, if and only if $[CS]$ and $|T_1| + |J_1| \equiv \frac{1}{2}b_1$ mod 2 or if $[CR]$ and $|T_1| + |J_1| \equiv 1$ mod 2.

Proof: We will use the previous Lemma 2.17 without further mentioning it. "(ii)" is clear by Lemma 2.10. In this poof $\equiv$ will always stand for $\equiv$ modulo 2.

To show (i) and (iii)-(iv), let $s^\bullet \in S(\gamma)$ be a point whose fibre $X^\bullet$ is general for the family $\mathscr{X}(\gamma) \to S(\gamma)$, in the sense that the set theoretic intersections $X_i^\bullet := X^\bullet \cap \mathscr{X}_i(\gamma)$, and $E^\bullet := X^\bullet \cap \mathscr{E}(\gamma)$ are of codimension 0 in $X^\bullet$ if and only if $\mathscr{X}_i(\gamma)$ resp. $\mathscr{E}(\gamma)$ are divisors, and $\operatorname{sing} X^\bullet = (X_1^\bullet \cap E^\bullet) \cup (X_2^\bullet \cap E^\bullet)$. Let $\check{X}_1^\bullet$, $\check{X}_2^\bullet$ and $\check{E}^\bullet$ be the normalisations of these components, and let $\mathcal{L}_1^\bullet$, $\mathcal{L}_2^\bullet$, $\mathcal{L}_E^\bullet$ be the pullbacks of $\mathscr{L}$ to these normalisations. Let μ be the number of components of $E^\bullet = \check{E}^\bullet$. Denote by $T_i^\bullet$, $I_i^\bullet$, $J_i^\bullet$, $K_i^\bullet$ the sets of points in which the sets of sections from $\widetilde{\mathcal{T}}_i, \widetilde{\mathcal{I}}_i, \widetilde{\mathcal{J}}_i$ and $\widetilde{\mathcal{K}}_i$ meet $X^\bullet$. Note $|T_i^\bullet| = |\widetilde{\mathcal{T}}_i| = |T_i|$, $|I_i^\bullet| = |\widetilde{\mathcal{I}}_i| = |I_i|$, and so on. Denote by $\Gamma_i \subset X_i^\bullet$ the set of one or two points in which $X_i^\bullet$ meets the rest of $X^\bullet$. Let $\check{\Gamma}_i \subset \check{X}_i^\bullet$ be the preimages on the normalisations. If $E^\bullet$ is 1 dimensional, set $\Gamma_{E,1} := E^\bullet \cap X_1^\bullet$, $\Gamma_{E,2} := E^\bullet \cap X_2^\bullet$.

We know by 2.17 (iii) that in $A^1(\mathscr{X}^{(j)\prime})$, $c_1(\mathscr{L}_{|\mathscr{X}^{(j)\prime}}) = \overline{s}\Omega' + [\widetilde{\mathcal{T}}' \cup \widetilde{\mathcal{J}}']$. Since

$$\mathscr{X} = \mathscr{X}' \uplus \bigcup_{\gamma \in \operatorname{sing}(D)} \mathscr{D}_1(\gamma) \cup \mathscr{D}_2(\gamma) \cup \mathscr{E}(\gamma),$$

equation (†) follows mostly from the exact sequence of Lemma 1.39. The only thing that remains to show is that if $\mathscr{E}(\gamma)$ is a divisor and has two components, then the classes of both divisors appear in $c_1(\mathscr{L})$ with the same coefficient z_E. This is shown below.

A fact which we will use again and again is that in $A^1(\mathscr{X})$

$$[\mathscr{X}(\gamma)] = [\mathscr{X}_1(\gamma)] + [\mathscr{X}_2(\gamma)] + [\mathscr{E}(\gamma)] = 0, \tag{$*$}$$

since it is the pullback of the class of the divisor $S(\gamma)$ from the open ball $\mathscr{S}^{(j)}$.

Now assume $\mathscr{E}(\gamma)$ has two components, call $E_a^\bullet$, $E_b^\bullet$ the two corresponding components of $E^\bullet$, and for $i \in \underline{2}$ set $\Gamma_{E,i,a} := \Gamma_{E,i} \cap E_a^\bullet$, $\Gamma_{E,i,b} := \Gamma_{E,i} \cap E_b^\bullet$. Note that all of the four sets defined by this consist of exactly one point. Let $\mathcal{L}_{E,a}^\bullet$, $\mathcal{L}_{E,b}^\bullet$ be the pullbacks of $\mathscr{L}$ to the two components. Then by (†) and ($*$) we have

$$c_1(\mathcal{L}_E^\bullet) = z_1[\Gamma_{E,1}] + z_2[\Gamma_{E,2}] - z_{E,a}([\Gamma_{E,1,a}] + [\Gamma_{E,2,a}]) - z_{E,b}([\Gamma_{E,1,b}] + [\Gamma_{E,2,b}]). \quad \text{hence:}$$

$$c_1(\mathcal{L}_{E,x}^\bullet) = (z_1 - z_{E,x})[\Gamma_{E,1,x}] + (z_2 - z_{E,x})[\Gamma_{E,2,x}], \quad \text{for} \quad x \in \{a, b\},$$

and so $\deg c_1(\mathcal{L}_{E,a}^\bullet) = z_1 + z_2 - 2z_{E,a}$, $\deg c_1(\mathcal{L}_{E,b}^\bullet) = z_1 + z_2 - 2z_{E,b}$. But since $E_a^\bullet$ and $E_b^\bullet$ are exceptional we know that $\mathcal{L}_{E,a}^\bullet = \mathcal{O}_{E_a^\bullet}(1)$, $\mathcal{L}_{E,b}^\bullet = \mathcal{O}_{E_b^\bullet}(1)$. So both are of degree 1 and we must have $z_{E,a} = z_{E,b} =: z_E$. This finishes the proof of (†).

Assume that both $X_1^\bullet$ and $X_2^\bullet$ are of codimension 0 in $X^\bullet$. Set $\epsilon(\gamma) = 1$ if $E^\bullet$ is of codimension 0 and $\epsilon(\gamma) = 0$ if it is of codimension 1. For $i \in \underline{2}$, set $\delta_i = 1$ if Ω meets X_i, $\delta_i = 0$ otherwise, let $\omega^\bullet$ be the point in which Ω meets $X^\bullet$. By (†) and ($*$) we have:

$$c_1(\mathcal{L}_1^\bullet) = [T_1^\bullet] + [J_1^\bullet] + \delta_1 \overline{s}[\omega^\bullet] + \big(\epsilon(\gamma) z_E + (1 - \epsilon(\gamma)) z_2 - z_1\big)[\widetilde{\Gamma}_1], \tag{$\diamondsuit$}$$

$$c_1(\mathcal{L}_2^\bullet) = [T_2^\bullet] + [J_2^\bullet] + \delta_2\overline{s}[\omega^\bullet] + \big(\epsilon(\gamma)z_E + (1-\epsilon(\gamma))z_1 - z_2\big)[\widetilde{\Gamma}_2], \tag{$\clubsuit$}$$

$$c_1(\mathcal{L}_E^\bullet) = z_1[\Gamma_{E,1}] + z_2[\Gamma_{E,2}] - 2z_E([\Gamma_{E,1}] + [\Gamma_{E,2}]). \tag{$\spadesuit$}$$

Now we show (i). $X_i^\bullet$ may only be a point if the corresponding component $Y_i^\bullet$ is contracted when stabilising $Y^\bullet$. By Lemma 2.10 this implies that $b_i = 2$ and $|I_i| = 0$. By stability this condition can be fulfilled for at most one $i \in \underline{2}$. For the rest of this paragraph assume it is fulfilled for $i = 1$. First assume that $X_1^\bullet$ is not contracted. Then $X_1^\bullet$ is an exceptional component, hence $E^\bullet$ is 0-dimensional, since if it would be exceptional, then $X^\bullet$ could not be quasistable (but only semistable), furthermore $\mu = 2$, i.e. $E^\bullet$ consists of two points. Since $X_1^\bullet$ is exceptional, $\deg c_1(\mathcal{L}_1^\bullet) = 1$ in this case. But by ($\diamondsuit$), $\deg c_1(\mathcal{L}_1) = |J_1| + \delta_1\overline{s} + 2(z_2 - z_1)$. Hence $|J_1| \equiv 1 \bmod 2$, and with $2 = b_1 = |I_1| + |J_1| + |K_1| = |J_1| + |K_1|$ we obtain $|J_1| = 1$ and $|K_1| = 1$. If on the other hand $X_1^\bullet$ is contracted, then $[\mathscr{X}(\gamma)] = [\mathscr{X}_2] = 0$, the two points of $\breve{\Gamma}_2 \subset \breve{X}_2^\bullet$ map to one node of $X_2^\bullet$, and the sections of $\widetilde{\mathcal{J}_1}$ and $\widetilde{\mathcal{K}_1}$ run into this node, meeting both branches of $X_1^\bullet$ there transversally. With this compute that (†) pulls back to

$$c_1(\mathcal{L}_2^\bullet) = [T_2^\bullet] + [J_2^\bullet] + \delta_2\overline{s}[\omega^\bullet] + (|J_1| + \delta_1\overline{s})[\widetilde{\Gamma}_2], \tag{$\star$}$$

i.e. since $T_2 = T$, $\deg c_1(\mathcal{L}_2^\bullet) = |T| + |J| + |J_1| + \delta_2\overline{s} + 2\delta_1\overline{s}$. Hence modulo 2, $\deg c_1(\mathcal{L}_2^\bullet) - (|T| + |J|) \equiv |J_1|$. In case $[CR]$, $(\mathcal{L}_2^\bullet)^{\otimes 2} = \mathcal{O}_{\breve{X}_2^\bullet}$ by Summary 1.13 (ii), so we get $0 \equiv |J_1|$ in this case. In case $[CS]$ instead $(\mathcal{L}_2^\bullet)^{\otimes 2} = \omega_{\breve{X}_2^\bullet}([\breve{\Gamma}_2])$, hence $\deg c_1(\mathcal{L}_2^\bullet) = \frac{1}{2}(2g(\breve{X}_2^\bullet) - 2 + 2) = g - 1$. Since $|T| + |J| \equiv g - 1$ also in this case $0 \equiv |J_1|$. So if $X_1^\bullet$ is contracted $|J_1| = 0$ and $|K_1| = 2$ or $|J_1| = 2$ and $|K_1| = 0$. This finishes the proof of (i).

From now on, we always assume that both $X_i^\bullet$ are 1-dimensional. To show (iii) assume b_1 is odd. That then $E^\bullet$ is a point if $[CR]$ and an exceptional component if $[CS]$ is clear by Summary 1.13 (iii). By 1.13 (ii) we know that in this case for $[CR]$, $\deg c_1(\mathcal{L}_1^\bullet) = \deg c_1(\mathcal{L}_2^\bullet) = 0$ and thus with ($\diamondsuit$), $|T_1| + |J_1| + z_2 - z_1 + \delta_1\overline{s} = 0$, hence $z_2 - z_1 \equiv |T_1| + |T_2|$. If $[CR]$, by Lemma 2.10 $g(X_i^\bullet) = \frac{1}{2}(b_i + 1 - 2)$ hence by 1.13 (ii) $\deg c_1(\mathcal{L}_i^\bullet) = \frac{1}{2}(b_i - 1) - 1$. From this and $\deg c_1(\mathcal{L}_E^\bullet) = 1$, we obtain the remaining claims of (iii) with ($\diamondsuit$), ($\clubsuit$), ($\spadesuit$).

Now we assume that b_1 is even, and show (iv). First check, using ($\diamondsuit$), ($\clubsuit$) and (ii), that in these cases $\deg c_1(\mathcal{L}_i^\bullet) \equiv |T_i| + |J_i|$. Then note that by 2.10, $g(X_i^\bullet) = \frac{1}{2}(b_i - 2) = \frac{1}{2}b_i - 1$. Furthermore by 1.13 (ii), if $E^\bullet$ is 0 dimensional then $\deg c_1(\mathcal{L}_i^\bullet) = g(X_i^\bullet) - 1 + 1 = g(X_i^\bullet)$ if [CS], and $\deg c_1(\mathcal{L}_i^\bullet) = 0$ if [CR]. If $E^\bullet$ is exceptional, $\deg c_1(\mathcal{L}_i^\bullet) = g(X_i^\bullet) - 1$ if [CS], and $\deg c_1(\mathcal{L}_i^\bullet) = -1$ if [CR]. Putting this information together the claims of (iv) follow. □

The next Proposition refines Proposition 2.14 by describing the finite degree 1 morphisms

$$b_{\overline{X}_{g,n},k,T} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}_{g,n}$$

more explicitly on the boundary of these moduli spaces. We continue to use the notation introduced in this section and in 2.14.

Proposition 2.19 *Choose a* $\mathfrak{D} = (D, p_1, ..., p_n, \{q_1, ..., q_{k-t}\}, \{q_1', ..., q_\tau'\})$ *such that* $[\mathfrak{D}] \in \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)}$. *Then* D *is a tree of irreducible components* $D_1, ..., D_M$ *for some* $M \in$

$\mathbb{N}$, which are all isomorphic to $\mathbb{P}^1$. Now $b_{\overline{X}_{g,n},k,T}([\mathfrak{D}]) \in \overline{HX}_{g,n}$ parametrises an object $(X; P_1, ..., P_n; \mathcal{L})$ (ignore the $\mathcal{L}$, in case $[CM]$). We first describe the quasistable curve X:

Let $f : Y \to \mathfrak{D}$ be the (unique) admissible double cover. Then in particular, for each D_i, $Y_i := f^{-1}(D_i)$ is smooth. Let $I_i \subseteq \underline{n}$, $J_i \subseteq \underline{(k-t)}$, $K_i \subseteq \underline{\tau}$ be the sets of indices of the points p_h, q_j, q'_k which lie on D_i. Set $T_i := I_i \cap T$. Let $L_i \subseteq \underline{M}$ be the set of indices m such that D_m meets D_i in a common node $\gamma_{i,m}$. Every node $\gamma_{i,m}$ divides D into two rational trees meeting in this node. Denote by $D_{i,m}$ the one of those two rational trees containing D_i but not D_m. Let $I_{i,m} \subseteq \underline{n}$, $J_{i,m} \subseteq \underline{(k-t)}$, $K_{i,m} \subseteq \underline{\tau}$ be the sets of indices of the points p_h, q_j, q'_k which lie on $D_{i,m}$. Set $T_{i,m} := I_{i,m} \cap T$.

Divide L_i into two sets $G_{i,1}$ and $G_{i,2}$, such that $m \in G_{i,1}$ means that $|I_{i,m}|+|J_{i,m}|+|K_{i,m}|$ is odd. Then:

(1) *The restriction of f, $f_i : Y_i \to D_i$, is the unique degree 2 cover of $D_i \cong \mathbb{P}^1$ branched over the points p_h, q_i, q'_i for $h \in I_i$, $j \in J_i$, $k \in K_i$, and over exactly those $\gamma_{i,m}$ for which $m \in G_{i,1}$. This means that f_i is ramified in $Ram\,(i) := |I_i|+|J_i|+|K_i|+|G_{i,1}|$ points. Hence Y_i has genus $g(Y_i) = \frac{1}{2}(Ram\,(i) - 2)$.*

Y_i meets Y_m in the one or two points contained in $\Gamma'_{i,m} := f^{-1}(\gamma_m)$. We denote by $\widehat{P}_1, ..., \widehat{P}_n$, $\widehat{Q}_1, ..., \widehat{Q}_{k-t}$, $\widehat{Q}'_1, ..., \widehat{Q}'_\tau$ the preimages of the p_i, q_j, q'_k under f (each of these preimages is a point). We call D_i an extremity of D if $|L_i| = 1$, $|I_i| = 0$ and $|J_i|+|K_i| = 2$. D_i is an extremity if and only if Y_i is an exceptional component of Y.

Now X is the curve obtained from Y by:

(2) *Contract all those exceptional components Y'_i for which $|J_i| = 0$ or $|K_i| = 0$.* [33]

(3) *Blow up all the nodes contained in sets $\Gamma'_{i,m}$ with $m \in G_{i,1}$ if $[CS]$, do not blow them up otherwise.*

(4) *Blow up the two nodes contained in a set $\Gamma'_{i,m}$ with $m \in G_{i,2}$, if*

$$[CS] \quad and \quad |T_{i,m}| + |J_{i,m}| \equiv \frac{1}{2}(|I_{i,m}| + |J_{i,m}| + |K_{i,m}|)\; mod\, 2, \text{ [34]}$$

$$or\ if \quad [CR] \quad and \quad |T_{i,m}| + |J_{i,m}| \equiv 1\; mod\, 2.$$

The marked points $P_1, ..., P_n$ on X are the ones corresponding to the points $\widehat{P}_1, ..., \widehat{P}_n$ on Y.

We know $(X; P_1, ..., P_n)$ now, so we are done in case $[CM]$. In the other cases we still do not know $\mathcal{L}$. What we will do is to describe the pullback of $\mathcal{L}$ to every component of the normalisation $X^\sim$ of X. On any exceptional component E of X, $\mathcal{L}_{|E} \cong \mathcal{O}_E(1)$. By what we have seen so far, the normalisation of each non-exceptional component of X is one of

[33] So all exceptional components are contracted in case $[CM]$

[34] Recall that $|I_{i,m}| + |J_{i,m}| + |K_{i,m}|$ is even iff $m \in G_{i,2}$. Note that in this case, if we call $X_{i,m}$ the part of X coming from the part of Y lying over $D_{i,m}$, then $\frac{1}{2}(|I_{i,m}| + |J_{i,m}| + |K_{i,m}| - 2)$ is the genus of $X_{i,m}$.

the Y_i. Let $\varphi_i : Y_i \to X$ be the morphism expressing Y_i as such a component. We want to describe $\varphi_i^* \mathcal{L}$.

Divide each set L_i into $L_{i,a}$ and $L_{i,b}$, such that $m \in L_{i,a}$ means that on X, $X_i := \varphi_i(Y_i)$ meets an exceptional component in the points of $\Gamma_{i,m}$. I.e. if $m \in L_{i,a}$, either $X_m := \varphi_m(Y_m)$ is an exceptional component not contracted in Step (2), or the nodes in $\Gamma_{i,m}$ are blown up in step (3).

Let ξ be the class of any point of D_i and let Ξ be the divisor class $f_i^* \xi$ on Y_i. Let $R_1, ..., R_{k_i}$ be the collection of the following points on Y_i (the ordering does not matter): All points P_h with $h \in T_i$, all points Q_j (coming from the $\widehat{Q}_j$ on Y) with $j \in J_i$, and in addition, for each $m \in G_{i,1}$, the point $\Gamma'_{i,m}$, if m has the property that:

$$[CS] \quad \textit{and} \quad |T_{i,m}| + |J_{i,m}| \equiv \frac{1}{2}(|I_{i,m}| + |J_{i,m}| + |K_{i,m}| - 1) \; mod\, 2, \text{ [35]} \quad \textit{or}$$

$$[CR] \quad \textit{and} \quad |T_{i,m}| + |J_{i,m}| \equiv 1 \; mod\, 2.$$

Define

$$\overline{s}_i := \begin{cases} \frac{1}{2}(Ram(i) - 2) - 1 + |G_{i,2} \cap L_{i,b}| - k_i, & \textit{if } [CS] \\ -|G_{i,2} \cap L_{i,a}| - k_i, & \textit{if } [CR] \end{cases} .$$

Then $\overline{s}_i$ is an even integer, and :

$$\varphi_i^* \mathcal{L} \cong \mathcal{O}_{X_i'}(B_i), \quad \textit{where} \quad B_i = \frac{\overline{s}_i}{2} \Xi + \sum_{j=1}^{k_i} R_j.$$

Proof: All claims for [CM] follow from Lemma 2.10. Up to the point at which blowing up of nodes and contraction of components are described, the proposition consists of definitions and things which follow immediately from 2.10. The description of X compared to Y in (2)-(4) follows from Lemma 2.18, if we set for the pair i, m we are interested in $\gamma = \gamma_{i,m}$ and let $\mathscr{D}_1(\gamma)$ resp. $\mathscr{D}_2(\gamma)$ be the components of $\mathscr{D}(\gamma)$ restricting to $D_{i,m}$ resp. $D_{m,i}$. To see this, note that $\mathscr{X}^{(j)} \to \mathscr{S}^{(j)}$ is a family of nodal curves, and recall the local description of such families from Proposition 1.9. By this description it is clear that over $S(\gamma)$, a blown up node γ can not deform into a node which is not blown up, or the other way around.

For $i \in \underline{M}$, set $\delta_i = 1$ if Ω meets X_i, $\delta_i = 0$ otherwise, let ω be the point in which Ω meets X. Then, similar to (♠) and so on, in the proof of Lemma 2.18, using $(*)$, $(\dagger)$ from the mentioned proof, we see that:

$$c_1(\varphi_i^*(\mathcal{L})) = [T_i] + [J_i] + \delta_i \overline{s}[\omega] + \sum_{m \in L_i} (\epsilon_{i,m} z_{E,i,m} + (1 - \epsilon_{i,m}) z_{m,i} - z_{i,m})[\Gamma'_{i,m}]$$

where the $\epsilon_{i,m}$ and z are the $\epsilon(\gamma)$ and z from the proof of Lemma 2.18, except if $Y_{m,i} = Y_m$ is an exceptional component which is contracted in passing to X. In this case we have to

[35] $|I_{i,m}| + |J_{i,m}| + |K_{i,m}|$ is odd iff $m \in G_{i,1}$. In this case $\frac{1}{2}(|I_{i,m}| + |J_{i,m}| + |K_{i,m}| - 1)$ is the genus of $X_{i,m}$.

set $z_{i,m} = |J_m| + \delta_m \overline{s}$ (compare to equation ($\star$) in the proof of Lemma 2.18). Note that in this latter case $m \in G_{i,2}$. We can continue the above equation with

$$= [T_i] + [J_i] + \sum_{m \in G_{i,1}} (\epsilon_{i,m} z_{E,i,m} + (1 - \epsilon_{i,m}) z_{m,i} - z_{i,m})[\Gamma_{i,m}] + \frac{s'_i}{2} \cdot \Xi, \quad \text{where}$$

$$s'_i := \delta_i \overline{s} + 2 \sum_{m \in G_{i,2}} \epsilon_{i,m} z_{E,i,m} + (1 - \epsilon_{i,m}) z_{m,i} - z_{i,m}.$$

Ξ is defined in the Proposition. Here we used that for $m \in G_{i,2}$, $[\Gamma_{i,m}] = f_i^*[\gamma_{i,m}]$ and that $\Xi = f^*\xi$, and $\xi \sim \gamma_{i,m}$ on $D_i \cong \mathbb{P}^1$. Let par : $\mathbb{Z} \to \{0, 1\}$ be the map sending all odd numbers to 1 and all even numbers to 0. With this, and noting that by (1) for $m \in G_{i,1}$, $2[\Gamma_{i,m}] = f^*[\gamma_i, m]$,

$$c_1(\varphi_i^*(\mathcal{L})) = [T_i] + [J_i] + \sum_{m \in G_{i,1}} (\text{par}(\epsilon_{i,m} z_{E,i,m} + (1 - \epsilon_{i,m}) z_{m,i} - z_{i,m}))[\Gamma_{i,m}] + \frac{\overline{s}_i}{2} \cdot \Xi \quad (\heartsuit)$$

where: $\overline{s}_i = s'_i + \sum_{m \in G_{i,1}} \big((\epsilon_{i,m} z_{E,i,m} + (1-\epsilon_{i,m}) z_{m,i} - z_{i,m}) - \text{par}(\epsilon_{i,m} z_{E,i,m} + (1-\epsilon_{i,m}) z_{m,i} - z_{i,m})\big).$

Using that for $m \in G_{i,1}$ by (3), $\epsilon_{i,m} = 1$ if and only if $[CS]$, and Lemma 2.18 (iii), we find that $\text{par}(\epsilon_{i,m} z_{E,i,m} + (1 - \epsilon_{i,m}) z_{m,i} - z_{i,m})$ is 1 for $[CR]$ if and only if $|T_{i,m}| + |J_{i,m}| \equiv 1$. For $[CS]$, it is 1 if and only if $|T_{i,m}| + |J_{i,m}| \equiv \frac{1}{2}(|I_{i,m}| + |J_{i,m}| + |K_{i,m}| - 1)$. Comparing this with the definition of the points $R_1, ..., R_{k_i}$ in our proposition we obtain from ($\heartsuit$):

$$c_1(\varphi_i^*(\mathcal{L})) = \sum_{l=1}^{k_i} R_l + \frac{\overline{s}_i}{2} \cdot \Xi, \qquad (\ddagger)$$

which is of the form claimed in the proposition. To compute $\overline{s}_i$, use that by Summary 1.13 (ii) (and by (1)-(4)),

$$\deg c_1(\varphi_i^*(\mathcal{L})) = -\frac{1}{2}(|G_{i,1} \cap L_{i,a}| + 2|G_{i,2} \cap L_{i,a}|) = -|G_{i,2} \cap L_{i,a}|, \quad \text{if} \quad [CR],$$

$$\text{and if} \quad [CS] \quad \text{then:} \quad \deg c_1(\varphi_i^*(\mathcal{L})) =$$

$$\frac{1}{2}(Ram(i) - 2) - 1 + \frac{1}{2}(|G_{i,1} \cap L_{i,b}| + 2|G_{i,2} \cap L_{i,b}|) = \frac{1}{2}(Ram(i) - 2) - 1 + |G_{i,2} \cap L_{i,b}|.$$

For the last line, note that for $Ram(i) \geq 2$, $\frac{1}{2}(Ram(i) - 2) = g(Y_i)$ and that for $Ram(i) = 0$, Y_i is the disjoint union of two $\mathbb{P}^1$'s and hence $\deg \omega_{Y_i} = -4$. By ($\ddagger$) we have $\overline{s}_i = \deg c_1(\varphi_i^*(\mathcal{L})) - k_i$, so $\overline{s}_i$ is as claimed in the proposition. □

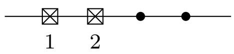
Remark Part (iii) describes the morphism $b_{\overline{X}_{g,n},k,T}$ only "almost explicitly" in the cases $[CS]$ and $[CR]$, since $\mathcal{L}$ is not always completely determined by its pullbacks to all components of the normalisation of X. The bundle $\mathcal{L}$ is obtained by gluing together the fibres of the bundles $\varphi_i^*\mathcal{L}$ over the nodes of X, and there can be several non-isomorphic permitted ways to do this.

Example 2.20 As a first example of an application of Propositions 2.14 and 2.19, we examine the hyperelliptic locus $\overline{HR}_{1,2} \subset \overline{R}_{1,2}$ and determine the boundary of its components. The results will also be used later in this thesis. Firstly by Proposition 2.14, we know that $\overline{HR}_{1,2}$ has two components (lets call them $\overline{A}_{2,a}$ and $\overline{A}_{2,b}$ [36]) which are the images of the two morphisms

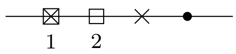
$$a_{\overline{R}_{1,2},2,\{1,2\}} : \overline{M}_{0,(2,[0],[2])} \xrightarrow{\cong} \overline{A}_{2,a}, \qquad a_{\overline{R}_{1,2},2,\{1\}} : \overline{M}_{0,(2,[1],[1])} \xrightarrow{\cong} \overline{A}_{2,b}.$$

(Of course $\overline{M}_{0,(2,[0],[2])} = \overline{M}_{0,(2,[2])}$ and $\overline{M}_{0,(2,[1],[1])} = \overline{M}_{0,4}$, but we keep the general notation of the Summary here.)

We introduce diagrams to symbolise the "topological type" (cf. section 1.3) of the genus 0 curves with pointed marked curves involved, and also take into account the distribution of marked points belonging to T, which is the set defining the morphisms. For a general point of $\overline{M}_{0,(2,[0],[2])}$ resp. $\overline{M}_{0,(2,[1],[1])}$ the curves are symbolised by:



resp.



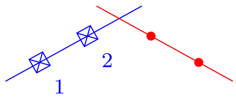
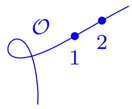
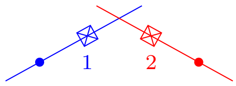
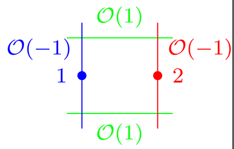
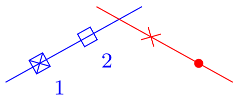
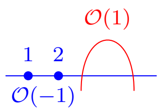
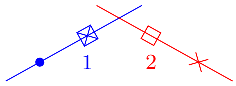
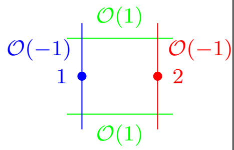
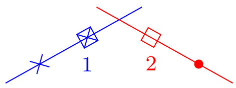
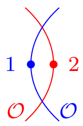
The diagrams are to be read as follows : For both moduli spaces the objects have 4 marked points, two of which form an ordered pair, while on the other two there is an ordered partition. In case of $\overline{M}_{0,(2,[0],[2])}$ this partition consist of one set containing both points, in the case $\overline{M}_{0,(2,[1],[1])}$ of two sets containing one point each. The boxes with indices 1 and 2 stand for the two ordered marked points. These boxes contain a cross if the marked point is contained in T. The dots and crosses without boxes stand for the remaining marked points, where crosses belong to one set of the partition, and points to the other set.

The interpretation of the symbols with regard to Summary 2.14 (i) is: The rational curve symbolised by such a diagram is mapped to the pair $(C, p_1, p_2, \mathcal{L})$ where $C \to \mathbb{P}^1$ is the degree 2 cover ramified over the four marked points on $\mathbb{P}^1$, and p_1, p_2 are the preimages of the marked points symbolised by the boxes. The crosses (regardless whether boxed or not) indicate the partition of the ramification points, which defines $\mathcal{L}$: Let q and q' be the preimages of the two marked points symbolised by crosses, then $\mathcal{L} \cong \mathcal{O}_C(q - q')$.

Now on the boundary of $\overline{M}_{0,(2,[0],[2])}$, $\overline{M}_{0,(2,[1],[1])}$ we find the curves which correspond to the possible stable degenerations of the general diagrams. Using the notation of 2.14 and 2.19 every such degenerated curve D consist of two components D_1 and D_2, each of which carries two of the marked points. They meet in the node $\gamma_{1,2} = \gamma_{2,1}$, and in this case $D_{1,2} = D_1$ and $D_{2,1} = D_2$. The table below lists all of these possible degenerated curves resp. their diagrams. In the way we defined the symbols, boxes with or without marked points stand for points in I, boxed crosses for points in $T \subseteq I$, crosses without boxes for point in J, and dots for points in K. We coloured the subcurve $D_{1,2}$ red in the diagrams and the subcurve $D_{2,1}$ blue. The table lists information about the sets of marked points used in the Proposition 2.19, and with this information the Proposition determines the type of $\mathfrak{X}$ such that $[\mathfrak{D}]$ is mapped to $[\mathfrak{X}]$. The last column shows the quasistable curve X underlying $\mathfrak{X}$. Here we coloured the part of X coming from $Y_{1,2}$ red and the part coming from $Y_{2,1}$ blue. Exceptional components of X which arise from blowing up nodes of Y are coloured green. All components of all X appearing in the table have arithmetic genus 0.

[36] To be compatible with the notation in section 5.2.

Each normalisation of a connected component is hence isomorphic to $\mathbb{P}^1$. One can use the summary to show that pullback of $\mathcal{L}$ to the normalisation of a component is then either $\mathcal{O}(-1) := \mathcal{O}_{\mathbb{P}}(-1)$, $\mathcal{O} := \mathcal{O}_{\mathbb{P}}$ or $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^1}(1)$. The latter is the case if and only if the component is exceptional.

Diagram of $\mathfrak{D}$	$I_{1,2}$	$T_{1,2}$	$\lvert J_{1,2}\rvert$	$\lvert K_{1,2}\rvert$	$I_{2,1}$	$T_{2,1}$	$\lvert J_{2,1}\rvert$	$\lvert K_{2,1}\rvert$	Sketch of $\mathfrak{X}$
For $a_{\overline{R}_{1,2,2,\{1,2\}}} : \overline{M}_{0,\langle 2,[0,2]\rangle} \xrightarrow{\cong} \overline{A}_{2,a}$:									
	$\{1,2\}$	$\{1,2\}$	0	0	$\emptyset$	$\emptyset$	0	2	
	$\{1\}$	$\{1\}$	0	0	$\{2\}$	$\{2\}$	0	1	
For $a_{\overline{R}_{1,2,2,\{1\}}} : \overline{M}_{0,\langle 2,[1,1]\rangle} \xrightarrow{\cong} \overline{A}_{2,b}$:									
	$\{12\}$	$\{1\}$	0	0	$\emptyset$	$\emptyset$	1	1	
	$\{1\}$	$\{1\}$	0	1	$\{2\}$	$\emptyset$	1	0	
	$\{1\}$	$\{1\}$	1	0	$\{2\}$	$\emptyset$	0	1	

2.4 Comparison of automorphisms

Fix for this whole section one of the morphisms $b_{\overline{X}_{g,n},k,T} : \overline{M}_{0,(n,\langle[k-t],[\tau]\rangle)} \to \overline{HX}_{g,n}$ as described in the Propositions 2.14 and 2.19, a $\mathfrak{D} = (D, (I, \langle J, K\rangle))$ with class $[\mathfrak{D}] \in \overline{M}_{0,(n,\langle[k-1],[\tau]\rangle)}$ together with a (pointed) hyperelliptic (spin/prym) curve $\mathfrak{X} = (X, \widetilde{I}, \mathcal{L}, b)$ such that $[\mathfrak{X}] = b_{\overline{X}_{g,n},k,T}([\mathfrak{D}]) \in \overline{HX}_{g,n}$.

Then one may ask how the automorphism groups $\mathrm{Aut}(\mathfrak{D})$ and $\mathrm{Aut}(\mathfrak{X})$ fit together. Here we will give an answer to this question, for all cases fulfilling the following condition

Condition 2.21 Using the notation of Lemma&Definition 2.17: Choose the numbering of the irreducible components $\mathscr{S}^{(j)}$ of the hyperelliptic deformation space $\mathscr{S}$ of $\mathfrak{X}$ in such a way that $\mathfrak{D}$ belongs to $\mathscr{S}^{(1)}$, in the sense of 2.17 (v). Then our condition is that the action of every $\varphi \in \mathrm{Aut}(\mathfrak{X})$ on $\mathscr{S}$ maps $\mathscr{S}^{(1)}$ again to $\mathscr{S}^{(1)}$. [37]

[37] This condition, by Lemma&Definition 2.17, implies in particular that $b^{-1}_{\overline{X}_{g,n},k,T}([\mathfrak{X}])$ has only one element $[\mathfrak{D}]$, but is not equivalent to it. We mainly apply our result for $\overline{S}_2 = \overline{HS}_2$ and $\overline{R}_2 = \overline{HR}_2$, for which the condition is fulfilled trivially since in this case $(\mathscr{S}, s_0) = (S, s_0)$ always has only one component.

The results from this section will later make it more easy to push forward stack classes from the Chow groups along $b_{\overline{X}_{g,n},k,T}$ in some special cases. (Recall, from Remark 1.35 (ii), that when pushing forward Q-classes one has to take into account automorphism numbers.)

In this section we continue to use the large amount of notation introduced in section 2.3.

Definition 2.22 If $\mathfrak{D}$ is a stable genus 0 curve with sorted marked points, then we call those irreducible components of D the *extremities of* $\mathfrak{D}$ which meet the rest of D only in one point and which carry only two of the marked points.

First we note that by Lemma 2.10 (or Proposition 2.19):

Lemma 2.23 *In the situation described in Notation 2.16:*

(i) The preimage $Y_i := f^{-1}(D_i)$ *under* $f : Y \to \mathfrak{D}$, *of a irreducible component* D_i *of* D *is an exceptional component of* $(Y, \widetilde{I})$ *if and only if* D_i *is an extremity of* $\mathfrak{D}$ *and carries none of the* n *ordered marked points from* I*. We call such an extremity a* genuine extremity*.*

(ii) Cont $: X \to C$ *resp.* $\mathrm{Cont}_2 : Y \to C$ *contracts exactly the exceptional components of* X *resp.* Y, *and* $\mathrm{cont}_2 : D \to \widehat{D}$ *contracts exactly the genuine extremities of* $\mathfrak{D}$*.*

Notation 2.24 Set $\widehat{I} := (\mathrm{cont}_2(p_1), ..., \mathrm{cont}_2(p_n))$, and let $\widehat{J}$, $\widehat{K}$ be the set of those points on $\widehat{D}$ that come from those marked points in J resp. K that lie on components of D not contracted by cont_2. By H we will denote the set of points of $\widehat{D}$ to which extremities of $\mathfrak{D}$ are contracted by cont_2. We set $\widehat{M} := \widehat{J} \cup \widehat{K}$.

To retain more information about the extremities contracted to points of H, divide this set into H_{JK}, H_J, H_K, where H_J contains the points to which extremities carrying only marked points of J are contracted, H_K contains those coming from extremities with marked point only from K, while to the points of H_{JK} extremities that carry one point of J and one point of K are contracted. Then sort the marked points by $(\widehat{I}, H_{JK}, \langle(\widehat{J}, H_J), (\widehat{K}, H_K)\rangle)$ (cf. Def. 2.4 (iii), again $\langle ... \rangle$ is to be read as $\{...\}$ if $\widehat{I} = \emptyset$, and as $(...)$ otherwise.).

Lemma 2.25 *Using Notation 2.16 and 2.24:*

(i) There are (unique) group homomorphisms

$$\mathrm{Aut}((X, \widetilde{I})) \xrightarrow{\chi_1} \mathrm{Aut}((C, \widetilde{I})) \xrightarrow{\chi_2} \mathrm{Aut}((\widehat{D}; (\widehat{I}, \widehat{M}, H))) \xleftarrow{\psi_2'} \mathrm{Aut}((D; (I, M))),$$

which make commutative the following diagrams for all $\varphi_1 \in \mathrm{Aut}((X, \widetilde{I}))$, $\varphi_2 \in \mathrm{Aut}((C, \widetilde{I}))$ *and* $\varphi_3 \in \mathrm{Aut}((D; (I, M)))$*:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi_1} & X \\ \downarrow{\scriptstyle \mathrm{Cont}_1} & & \downarrow{\scriptstyle \mathrm{Cont}_1} \\ C & \xrightarrow{\chi_1(\varphi_1)} & C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\varphi_2} & C \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ \widehat{D} & \xrightarrow{\chi_2(\varphi_2)} & \widehat{D} \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{\varphi_3} & D \\ \downarrow{\scriptstyle \mathrm{cont}_2} & & \downarrow{\scriptstyle \mathrm{cont}_2} \\ \widehat{D} & \xrightarrow{\psi_2'(\varphi_3)} & \widehat{D} \end{array}$$

Furthermore χ_2 *and* ψ_2' *are surjective.*

(ii) The kernel $\ker \chi_1$ *consist of those automorphisms that are non-trivial only on the exceptional components of* X*. The kernel* $\ker \psi_2'$ *consists of those* $\varphi \in \mathrm{Aut}((D;(I,M)))$ *that are non-trivial only on the (genuine) extremities of* $(D;(I,M))$*.*

(iii) $\ker \chi_2 = \mathrm{Aut}_{hyp}((C,\widetilde{I}))$*, with* $\mathrm{Aut}_{hyp}((C,\widetilde{I}))$ *as defined in Lemma 2.12 (iii).*

Proof: (i): χ_1 exists, since forming of the stable model is a functor (cf. section 1.1).

For χ_2: Every $\varphi \in \mathrm{Aut}((C,\widetilde{I}))$ uniquely induces a compatible automorphism φ^* on the quotient $D = C/h$, since it commutes with the hyperelliptic involution h by Lemma 2.12 (i). We have to check that φ^* respects $(\widehat{I},\widehat{M},H)$: Considering that $(C,\widetilde{I})$ is obtained by stabilising $(Y,\widetilde{I})$, we see with Lemma 2.23 that the points in H are exactly the images of those nodes γ of C with the property: γ is fixed by the hyperelliptic involution, and the two branches of C meeting in γ are swapped by the hyperelliptic involution. Since φ commutes with the hyperelliptic involution, φ^* respects H. The points in $\widehat{I} \cup \widehat{M}$ are the images of smooth fixed points of the hyperelliptic involution, and $\widehat{I}$ is the image of $\widetilde{I}$, so φ^* also respects these two sets.

The morphism ψ_2' obviously exists and is surjective.

That χ_2 is surjective follows from Lemma 2.12 (v), and the fact that $\mathfrak{C}$ is the stable model of $(Y, f^{-1}(q_1), ..., f^{-1}(q_n))$ (cf. Notation 2.16), together with the surjectivity of ψ_2'.

(ii): Follows from Lemma 2.23 (ii).

(iii): The kernel of χ_2 consists of all $\varphi \in \mathrm{Aut}(C)$ such that $g(\varphi(a)) = g(a)$ for all $a \in C$. $\square$

Definition 2.26 (i) A nodal curve *with n sorted marked points and sorted nodes*, is a tuple $(X;\mathscr{R})$ of a nodal curve X, a sorted set $\mathscr{R}$ whose underlying set consists of n pairwise different smooth points of X together with all nodal points of X.

(ii) The automorphism group $\mathrm{Aut}((X,\mathscr{R}))$, is the subgroup of $\mathrm{Aut}(X)$ of automorphisms respecting the sorted set $\mathscr{R}$, like in Definition 2.4 (ii).

Lemma 2.27 *Using Notation 2.16, and the notation introduced in this section:*

(i) $\mathrm{Aut}(\mathfrak{X})$ *is a subgroup of* $\mathrm{Aut}((X,\widetilde{I}))$*. We call the restriction of* $\chi_2 \circ \chi_1$ *to this subgroup*

$$\psi_1 : \mathrm{Aut}(\mathfrak{X}) \to \mathrm{Aut}((\widehat{D};(\widehat{I},\widehat{M},H))).$$

$\mathrm{Aut}(\mathfrak{D})$ *is a subgroup of* $\mathrm{Aut}((D;(I,M)))$ *and we call the restriction of the morphism* ψ_2' *of Lemma 2.25*

$$\psi_2 : \mathrm{Aut}(\mathfrak{D}) \to \mathrm{Aut}((\widehat{D};(\widehat{I},\widehat{M},H))).$$

(ii) $\mathrm{Aut}((\widehat{D};(\widehat{I},H_{J,K},\langle(\widehat{J},H_J),(\widehat{K},H_K)\rangle)))$ *is a subgroup of* $\mathrm{Aut}((\widehat{D};(\widehat{I},\widehat{M},H)))$*, and:*

$$\psi_1(\mathrm{Aut}(\mathfrak{D})) = \mathrm{Aut}((\widehat{D};(\widehat{I},H_{J,K},\langle(\widehat{J},H_J),(\widehat{K},H_K)\rangle))).$$

If $\mathfrak{X}$ *fulfils condition 2.21 then also*

$$\psi_2(\mathrm{Aut}(\mathfrak{X})) = \mathrm{Aut}((\widehat{D};(\widehat{I},H_{J,K},\langle(\widehat{J},H_J),(\widehat{K},H_K)\rangle))).$$

(iii) $\ker\psi_1$ *is the subgroup of* $\mathrm{Aut}(\mathfrak{D})$ *of automorphisms acting nontrivially only on extremities of* $\mathfrak{D}$*. We have* $\chi_2^{-1}(\mathrm{Aut}_{hyp}(C)) \subseteq \mathrm{Aut}(\mathfrak{X})$ *and* $\ker\psi_2 = \chi_2^{-1}(\mathrm{Aut}_{hyp}(C))$.

(iv) Assume Condition 2.21 holds. Set $N := |\mathrm{Aut}((\widehat{D}; (\widehat{I}, H_{J,K}, \langle(\widehat{J}, H_J), (\widehat{K}, H_K)\rangle)))|$*. Let* l' *be the number of those extremities of* $\mathfrak{D}$*, whose two marked points either lie both in* J *or lie both in* K*. We have to distinguish a special case: If all marked points from* $J \cup K$ *on* $\mathfrak{D}$ *lie on extremities, and in addition all these extremities carry one point from* J *and one point from* K*, set* $l := l' + 1 = 1$*. In all other cases set* $l := l'$*. Let* s *be the number of nodes* γ *on* D *dividing* D *into two parts* D_1*,* D_2*, both of which carry an odd number of marked points. Let* $\mathrm{Aut}_0(\mathfrak{X}) \subseteq \mathrm{Aut}(\mathfrak{X})$ *be the subgroup of inessential automorphisms. Then:*

$$|\mathrm{Aut}(\mathfrak{D})| = 2^l \cdot N \ \text{[38]}, \qquad |\mathrm{Aut}(\mathfrak{X})| = |\mathrm{Aut}_{hyp}(\mathfrak{C})| \cdot |\mathrm{Aut}_0(\mathfrak{X})| \cdot N$$

and thus with $|\mathrm{Aut}_{hyp}(\mathfrak{C})| = 2^{s+1}$*,*

$$|\mathrm{Aut}(\mathfrak{X})| = 2^{(s+1-l)} \cdot |\mathrm{Aut}_0(\mathfrak{X})| \cdot |\mathrm{Aut}(\mathfrak{D})|.$$

One can also write $|\mathrm{Aut}_0(\mathfrak{X})| = 2^{u-1}$ *where* u *is the number of connected components of* $\widetilde{X}$*, the non-exceptional subcurve of* X*.*
[39]

Proof: The different assertions that one automorphism group is a subgroup of another one, made in parts (i) and (ii), are all quite obvious.

The first things we prove are the two equations of part (ii).

We start with the commutative diagram of Lemma&Definition 2.17 (iii), with $k = 1$ (and $\mathfrak{D} = \mathfrak{D}^{(1)}$). Recall the whole notation introduced in 2.17.

For $(\widetilde{\mathcal{I}}, \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{K}}\rangle)_{\mathscr{Y}}$, we know by 2.17 (iv) that the (images of the) sections are disjoint and only contain smooth points of the fibres. Since we know that $\mathbf{Cont}_1$ resp. $\mathbf{Cont}_2$ act on the central fibres only by contracting some exceptional components, we can conclude from this, that on X two sections from $(\widetilde{\mathcal{I}}, \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{K}}\rangle)_{\mathscr{X}^{(1)}}$ can only meet in nodal points of X. Furthermore, since exceptional components of Y are met by exactly two sections, if a node of X is contained in at least one section it is contained in exactly two sections. By (2) of Proposition 2.19 such nodes are either contained in two sections from $\widetilde{\mathcal{J}}$ or two from $\widetilde{\mathcal{K}}$.

Using $(\widetilde{\mathcal{I}}, \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{K}}\rangle)_{\mathscr{X}^{(1)}}$, we give the central fibre X the structure of a nodal curve with sorted marked points and nodes: Let $\widetilde{I} = (p_1, ..., p_n)$ be the tuple of marked points, belonging to the data of $\mathfrak{X}$ already. $\widetilde{I}$ coincides with the tuple of smooth points in which the sections from $\widetilde{\mathcal{I}}$ meet X. Let $\widetilde{J}$ resp. $\widetilde{K}$ be the sets of smooth points of X which are contained in a section from $\widetilde{\mathcal{J}}$ resp. from $\widetilde{\mathcal{K}}$. Denote by G_J, G_K the sets of those nodes

[38] This equation also holds without condition 2.21.

[39] It is possible to use Proposition 2.19 to describe u in terms of properties of $\mathfrak{D}$. But this requires to distinguish cases. To apply the resulting formula would not be much simpler then to determine for a given $\mathfrak{D}$ the underlying curve X of $\mathfrak{X}$ directly by Prop. 2.19, and then to count the components of $\widetilde{X}$. So we omit it.

of X which are contained in two sections from $\widetilde{\mathcal{J}}$ resp. from $\widetilde{\mathcal{K}}$. Let $G_\emptyset$ be the set of all remaining nodes. They are contained in none of the sections of $(\widetilde{\mathcal{I}}, \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{K}}\rangle)_{\mathscr{X}^{(1)}}$. Then $\big(X, \big(\widetilde{I}, G_\emptyset, \langle(\widetilde{J}, G_J), (\widetilde{K}, G_K)\rangle\big)\big)$ is a nodal curve with sorted marked points and nodes. Define, using $(\widetilde{\mathcal{I}}, \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{K}}\rangle)_{\mathscr{C}}$, an analogous sorting of marked points and nodes on C, which we denote by $\big(\widetilde{I}, G^*_\emptyset, G^*_{JK}, \langle(\widetilde{J^*}, G^*_J), (\widetilde{K^*}, G^*_K)\rangle\big)$. Here G^*_{JK} is the set of nodes contained in one section from $\widetilde{\mathcal{J}}$ and one from $\widetilde{\mathcal{K}}$.

By Summary 1.31 (i), and the construction of $\mathscr{X} \to \mathscr{S}$, $\mathrm{Aut}(\mathfrak{X})$ can be identified with the subset of $\mathrm{Aut}((\mathscr{X} \to (\mathscr{S}, s_0))$ consisting of automorphisms which respect $(\widetilde{I}, \mathscr{L})$. Under condition 2.21 it even implies that $\mathrm{Aut}(\mathfrak{X})$ can be identified with the analogous subset of $\mathrm{Aut}((\mathscr{X}^{(1)} \to (\mathscr{S}^{(1)}, s_0)))$. But this is just the subset respecting the sorted sections $(\widetilde{\mathcal{I}}', \langle\widetilde{\mathcal{J}}', \widetilde{\mathcal{K}}'\rangle)$, which is equivalent to respecting $(\widetilde{\mathcal{I}}, \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{K}}\rangle)$. We conclude

$$\mathrm{Aut}(\mathfrak{X}) = \mathrm{Aut}\Big(\big(X, \big(\widetilde{I}, G_\emptyset, \langle(\widetilde{J}, G_J), (\widetilde{K}, G_K)\rangle\big)\big)\Big).$$

Now Cont_1 maps the sorted points and nodes

$$\big(\widetilde{I}, G_\emptyset, \langle(\widetilde{J}, G_J), (\widetilde{K}, G_K)\rangle\big) \quad \text{to} \quad \big(\widetilde{I}, G^*_\emptyset, G^*_{JK}, \langle(\widetilde{J^*}, G^*_J), (\widetilde{K^*}, G^*_K)\rangle\big)$$

in the following sense: An exceptional component of X carrying one point from $\widetilde{J}$ and one point from $\widetilde{K}$ is contracted to a node, so these two points are both mapped to one node in G^*_{JK}. Using the description of X in Proposition 2.19 (1)-(4), we see that nodes from G_J resp. G_K are mapped 1:1 to nodes of G^*_J resp. G^*_K. Furthermore $G_\emptyset$ is mapped surjectively to $G^*_\emptyset \cup G^*_{JK}$: If two nodes from $G_\emptyset$ are adjacent to the same exceptional component, they are mapped to the same node in G_{JK}, the nodes not adjacent to exceptional components are maped bijectively to $G^*_\emptyset$. The subsets of points of $\widetilde{J}$ resp. $\widetilde{K}$ which do not lie on exceptional components map bijectively to $\widetilde{J^*}$ resp. $\widetilde{K^*}$. Hence

$$\chi'_1\Big(\mathrm{Aut}\Big(\big(X; \big(\widetilde{I}, G_\emptyset, \langle(\widetilde{J}, G_J), (\widetilde{K}, G_K)\rangle\big)\big)\Big)\Big)$$

$$\subseteq \mathrm{Aut}\Big(\big(C; \big(\widetilde{I}, G^*_\emptyset, G^*_{JK}, \langle(\widetilde{J^*}, G^*_J), (\widetilde{K^*}, G^*_K)\rangle\big)\big)\Big)$$

We want to show that the $\subseteq$ can be replaced by $=$. From the discussion above we conclude that it suffices to show that any automorpism $\varphi \in \mathrm{Aut}(C)$ which is contained in the second group fulfils: φ maps all those nodes of C which are blown up in passing to X again to such nodes. But this follows from Proposition 2.19 (2)-(4), which characterizes such nodes.

Now nodes of C belonging to G^*_{JK}, G^*_J or G^*_K arise by contracting components of Y. Firstly this shows that the hyperelliptic involution swaps the two branches of such nodes. Hence they are mapped to smooth points of the quotient $\widehat{D}$. Furthermore, with Lemma 2.23 and considering how the sorted sets of sections on $\mathscr{X}^{(i)}$, $\mathscr{C}$, $\mathscr{Y}$ and $\mathscr{D}$ fit together in the diagram of 2.17 (iii), it implies that g maps $\big(\widetilde{I}, G^*_\emptyset, G^*_{JK}, \langle(\widetilde{J^*}, G^*_J), (\widetilde{K^*}, G^*_K)\rangle\big)$ to $(\widehat{I}, H_{J,K}, \langle(\widehat{J}, H_J), (\widehat{K}, H_K)\rangle)$ in the following sense:

$$g(\widetilde{I^*}) = \widehat{I}, \quad g(\widetilde{J^*}) = \widehat{J}, \quad g(\widetilde{K^*}) = \widehat{K}, \quad g(G^*_J) = H_J, \quad g(G^*_K) = H_K, \quad g(G^*_{JK}) = H_{JK},$$

while the set of nodes of $G^*_\emptyset$ is mapped surjectively to the set of all nodes of $\widehat{D}$ [40] . This implies (with Lemma 2.25 (i)):

$$\chi_2(\mathrm{Aut}((C;(\widetilde{I},G^*_\emptyset,G^*_{JK},\langle(\widetilde{J}^*,G^*_J),(\widetilde{K^*},G^*_K)\rangle)))) = \mathrm{Aut}((\widehat{D};(\widehat{I},H_{J,K},\langle(\widehat{J},H_J),(\widehat{K},H_K)\rangle)))$$

and hence the second equation of (ii). The first equation of (ii) is clear by Lemma 2.23 (ii).

(iii): Follows from Lemma 2.25 and Lemma 2.12 (iv).

(iv): Let $D^* \subset D$ be the union of all components of D which are no genuine extremities of $\mathfrak{D}$. For the first equation, by (ii) it suffices to determine $|\ker\psi_2|$: By Lemma 2.25 (ii), $\ker\psi_2$ consist of the $\varphi \in \mathrm{Aut}(\mathfrak{D})$ which are trivial restricted to D^*. In the special case described in (iv), for which we have set $l := l'+1 = 1$, there is only one nontrivial such φ. It acts on all extremities carrying a point from J and a point from K simultaneously by swapping the marked point from J with the marked point from K. In all cases except this special one, we have: For each genuine extremity $E \subset D$ there is a φ acting nontrivially on E, but trivially on D^*, if and only if the two marked point on E both lie in J or both lie in K. Indeed, for such an extremity there is a unique automorphism φ_E swapping the two marked points and acting trivially on all other components of D. On extremities not of this type, there is a point from J and a point from K. But if these two are exchanged by an automorphism φ, all points of J must be exchanged with all points of K by φ. This is only possible (while still fixing D^*) if we are in the special case treated earlier. In all other cases $\ker\psi_2$ is generated by these automorphisms φ_E, hence has 2^l elements. For the second equation: By (iii), and the fact that $\ker\chi_1 \cap \mathrm{Aut}(\mathfrak{X}) = \mathrm{Aut}_0(\mathfrak{X})$ (more or less by definition of inessential automorphisms) we see $|\ker\psi_1| = |\mathrm{Aut}_0(\mathfrak{X})| \cdot |\mathrm{Aut}_{hyp}(\mathfrak{C})|$. Hence $|\mathrm{Aut}(\mathfrak{X})| = |\mathrm{Aut}_{hyp}(\mathfrak{C})| \cdot |\mathrm{Aut}_0(\mathfrak{X})| \cdot N$ by (ii). □

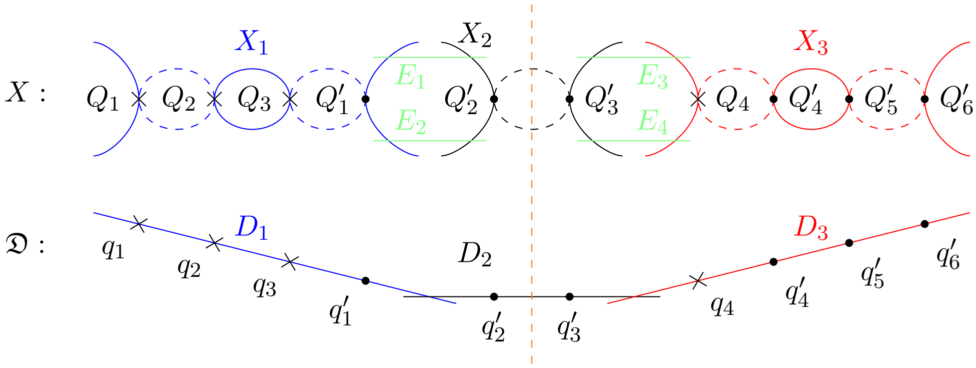
Remark 2.28 How do the formulas of part (iv) of the Lemma change if Condition 2.21 does not hold? This means there are $r > 1$ components $\mathscr{S}^{(j)}$ of $\mathscr{S}$, contained in the orbit of $\mathscr{S}^{(1)}$ under the action of $\mathrm{Aut}(\mathfrak{X})$. Write them as $\mathscr{S}^{(1)}, \mathscr{S}^{(2)}, ..., \mathscr{S}^{(r)}$. The first equation of (iv) remains the same. Concerning the second equation: The proof of the Lemma still yields the same equation with $\mathrm{Aut}(\mathfrak{X})$ replaced by $\mathrm{Aut}((j)) := \mathrm{Aut}((\mathscr{X}^{(j)} \to \mathscr{S}^{(j)}, \widetilde{\mathcal{I}}, \mathscr{L}^{(j)}))$, for all $j \in \underline{r}$. By definition of r there are automorphisms $\varphi_1, \varphi_2, ..., \varphi_r$, such that $\varphi_j(\mathscr{S}^{(1)}) = \mathscr{S}^{(j)}$ for all $j \in \underline{r}$. Now it is easy to check that $\varphi_j \circ \mathrm{Aut}((1)) \subset \mathrm{Aut}((\mathscr{X} \to \mathscr{S}, \widetilde{\mathcal{I}}, \mathscr{L})) = \mathrm{Aut}(\mathfrak{X})$ is the subset of all automorphisms which map $\mathscr{S}^{(1)}$ to $\mathscr{S}^{(j)}$. Hence as a set: $\mathrm{Aut}(\mathfrak{X}) = \mathrm{Aut}((1)) \uplus \varphi_2 \circ \mathrm{Aut}((1)) \uplus ... \uplus \varphi_r \circ \mathrm{Aut}((1))$. So if condition 2.21 does not hold, we have to multiply the right hand sides of the second equation by r to get the correct result. The same holds for the third formula.

However, if one wants to apply these formulas to compute automorphism numbers, they will of course only be of use if one has a way to determine r for a given $\mathfrak{D}$. We will not

[40] Since a node of $G^*_\emptyset$ corresponds to a node of Y which is not adjacent to exceptional components, it is either exchanged with another node by the hyperelliptic involution of C, or is fixed and the hyperelliptic involution maps each branch at the node to itself. In both cases the node is mapped to a node on the quotient $\widehat{D}$.

provide such a result in this thesis [41], and in all cases in which we apply the Lemma it will be obvious that condition 2.21 holds.

But here we give an **example** of a case with $r > 1$: By Proposition 2.14 $\overline{HR}_4$ has two components corresponding to the choice of $k \in \{2, 4\}$. For $k = 4$ look at the following pair of $[\mathfrak{D}] \in \overline{M}_{0,[4,6]}$ and $[\mathfrak{X}] = b_{\overline{R}_4,4}([\mathfrak{D}])$, where like in Example 2.20 we denote the 4 point $q_1, ..., q_4 \in J$ by crosses, and the 6 points $q'_1, ..., q'_6 \in K$ by dots:



Exceptional components of X are drawn in light green. Here we require that, as curve with 4 *ordered* marked points, $(D_1, q'_1, q_3, q_2, q_1)$ is isomorphic to $(D_3, q_4, q'_4, q'_5, q'_6)$. Then there will be automorphisms

$$\psi \in \mathrm{Aut}(D, \{q_1, ..., q_4, q'_1, ..., q'_6\}), \qquad \varphi \in \mathrm{Aut}(X, \{Q_1, ..., Q_4, Q'_1, ..., Q'_6\})$$

such that φ is a lifting of ψ, and such that both act, loosely speaking, as the reflection on the dashed orange axis in the image above. Now ψ is not an automorphism of $\mathfrak{D} = (D, \{J, K\}) = (D, \{\{q_1, ..., q_4\}, \{q'_1, ..., q'_6\}\})$ since it does not respect the sorting of the marked points. So φ is not the lifting of an automorphism of $\mathfrak{D}$. Note that in this example $\mathfrak{D} = (\widehat{D}; (\widehat{I}, H_{J,K}, \langle(\widehat{J}, H_J), (\widehat{K}, H_K)\rangle))$ using the notation of the Lemma. But φ is an automorphism of $\mathfrak{X}$: By Proposition 2.19, for $\mathcal{L}$ the prym sheaf on X, $\mathcal{L}_{|X_1} \cong \mathcal{O}_{X_1}(-4Q_1 + Q_1 + Q_2 + Q_3)$ and $\mathcal{L}_{|X_3} = \mathcal{O}_{X_3}(-2Q_4 + Q_4) \cong \mathcal{O}_{X_3}(-Q_4)$. We have $-4Q_1 + Q_1 + Q_2 + Q_3 \sim -Q'_1$, since $Q_1 + Q_2 + Q_3 + Q'_1 \sim 4Q'_1$ and $2Q_1 \sim 2Q'_1$ on X_1. Hence

$$\varphi^* \mathcal{L}_{|X_1} \cong \varphi^* \mathcal{O}_{X_1}(-Q'_1) \cong \mathcal{O}_{X_3}(-\varphi^{-1}(Q'_1)) = \mathcal{O}_{X_3}(-Q_4) \cong \mathcal{L}_{|X_3}.$$

Analogously $\varphi^* \mathcal{L}_{|X_3} \cong \mathcal{L}_{|X_1}$. So the second equation of 2.27 (ii) does not hold in this example.

Now the induced automorphism $\varphi_{\mathfrak{C}}$ on the stable model $\mathfrak{C}$ of $\mathfrak{X}$ swaps the two pairs of nodes e_1, e_2 and e_3, e_4 which are blown up to obtain E_1, E_2, E_3, E_4. If $\vec{x}_{e_1}, ..., \vec{x}_{e_4}$ are the corresponding base vectors of the deformation spaces (B, b_0) of $\mathfrak{C}$ (compare to section 2.1.3), then $(\varphi_{\mathfrak{C}}(\vec{x}_{e_1}), \varphi_{\mathfrak{C}}(\vec{x}_{e_2}), \varphi_{\mathfrak{C}}(\vec{x}_{e_3}), \varphi_{\mathfrak{C}}(\vec{x}_{e_4})) = (\vec{x}_{e_3}, \vec{x}_{e_4}, \vec{x}_{e_1}, \vec{x}_{e_2})$. And φ acts, possibly after multiplying with inessential automorphisms, by $(\varphi(\vec{y}_{e_1}), \varphi(\vec{y}_{e_2}), \varphi(\vec{y}_{e_3}), \varphi(\vec{y}_{e_4})) = (\vec{y}_{e_3}, \vec{y}_{e_4}, \vec{y}_{e_1}, \vec{y}_{e_2})$. (This is not difficult to prove, but we do not show it here.) Hence

[41] To give a way to determine r might be slightly interesting, since this would for example allow to compute the number of irreducible component of the local analytic neighbourhood of any given point of $\overline{HX}_{g,n}$, using also the description of the hyperelliptic local universal deformation from section 2.1.3.

φ swaps the two components of the hyperelliptic local universal deformation of $\mathfrak{X}$ corresponding to the partitions $(E^+_{2,N}, E^-_{2,N}) = (\{E_1, E_2\}, \{E_3, E_4\})$ and $(E^+_{2,N}, E^-_{2,N}) = (\{E_3, E_4\}, \{E_2, E_1\})$. (Cf. section 2.1.3, and recall that we defined $E_{2,N}$ to denote a certain set of edges resp. exceptional components there.) These are the two components lying in the image of $b_{\overline{R}_{4,4}}$ on $\mathscr{S}$, and we see that they form one orbit, so $r = 2$. There are two further components of $\mathscr{S}$, corresponding to $(E^+_{2,N}, E^-_{2,N}) = (\{E_1, E_2, E_3, E_4\}, \emptyset)$ and $(E^+_{2,N}, E^-_{2,N}) = (\emptyset, \{E_1, E_2, E_3, E_4\})$, and they belong to the image of $b_{\overline{R}_{4,2}}$. A local analytic neighbourhood of $[\mathfrak{X}]$ in $\overline{HR}_4$ has three irreducible components, one belonging to the image of $b_{\overline{R}_{4,4}}$ and two belonging to the image of $b_{\overline{R}_{4,2}}$. (Globally of course $\overline{HR}_4$ has two irreducible components, namely the images of $b_{\overline{R}_{4,4}}$ and $b_{\overline{R}_{4,2}}$.)

2.5 Application to $\overline{S}_2$ and $\overline{R}_2$

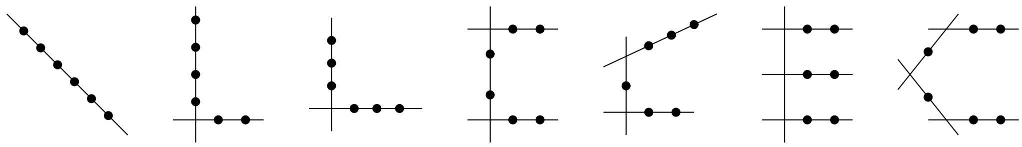
In this section we apply the results of the Chapter to $\overline{M}_2$, $\overline{S}_2$ and $\overline{R}_2$. Since all smooth genus 2 curves are hyperelliptic we have $\overline{M}_2 = \overline{HM}_2$, $\overline{S}_2 = \overline{HS}_2$ and $\overline{R}_2 = \overline{HR}_2$. These spaces are normal and (except $\overline{S}_2 = \overline{S}_2^+ \uplus \overline{S}_2^-$) irreducible. Hence by Proposition 2.14 we have isomorphisms from moduli spaces with $2 \cdot 2 + 2 = 6$ partitioned marked points to each of $\overline{M}_2$, $\overline{S}_2^+$, $\overline{S}_2^-$, $\overline{R}_2$. We call these isomorphisms:

$$b : \overline{M}_{0,[6]} \xrightarrow{\cong} \overline{M}_2 \quad \text{resp.}$$

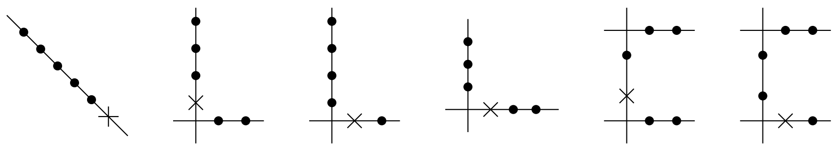
$$a_R : \overline{M}_{0,[2,4]} \xrightarrow{\cong} \overline{R}_2 \quad \text{resp.} \quad a_+ : \overline{M}_{0,[3,3]} \xrightarrow{\cong} \overline{S}_2^+ \quad \text{resp.} \quad a_- : \overline{M}_{0,[1,5]} \xrightarrow{\cong} \overline{S}_2^-$$

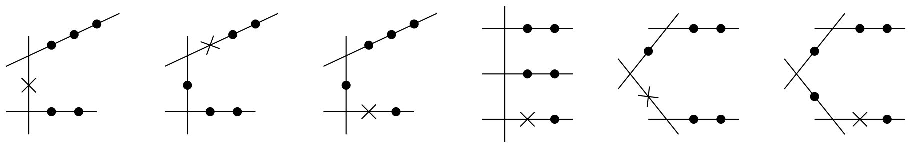
We know that they map boundary points to boundary points.

We now use these isomorphisms to gain information about the boundary cycles (cf. sections 1.3 and 1.4) of $\overline{M}_2$, $\overline{S}_2$ and $\overline{R}_2$. It is easy to list all boundary cycles of the spaces $\overline{M}_{0,[6]}$, $\overline{M}_{0,[2,4]]}$, ..., by writing down the diagrams of the rational curves they generally parametrise. This is since the stable genus 0 curves are just trees of $\mathbb{P}^1$'s, each carrying 3 special points (i.e. marked points or intersection points with other components) to make them stable. For $\overline{M}_{0,[6]}$ we e.g. have the possibilities:

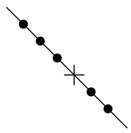
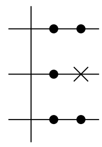
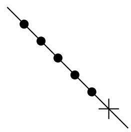
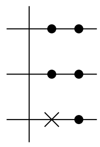


For the cases with partitioned marked points e.g. $\overline{M}_{0,[1,5]}$, we have the same underlying curves, but have to distinguish the different possibilities to partition the marked points into two sets of the given sizes. We indicate this partition by symbolising marked points in one set of the partition by crosses, and from the other set by dots. In case of $\overline{M}_{0,[1,5]}$ e.g. we find the possibilities:



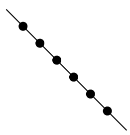
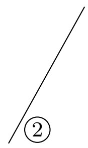
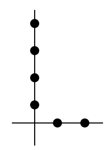
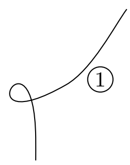
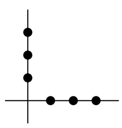
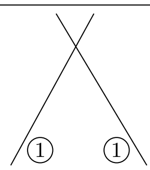


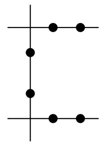
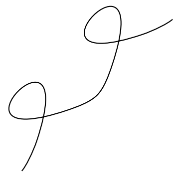
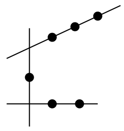
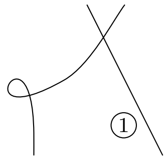
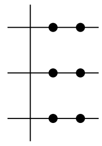
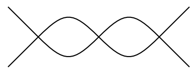
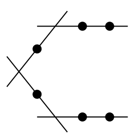
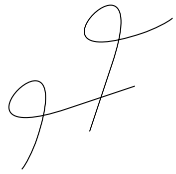
Note that, since we are only interested in listing the boundary cycles, it does not matter “which” of the points on one component are dots or crosses, but only how many of each kind there are. Also we can reorder the sub-trees hanging on each line segment of our diagrams. Thus the list above given for $\overline{M}_{0,[1,5]}$ is complete, and for example

 ,  not define new cycles, but are equivalent to  ,  .

After generating these lists, we can apply Proposition 2.19 which tells us to what kind of (spin/prym) curve $\mathfrak{X}$, a genus 0 curve $\mathfrak{D}$ belonging to such a diagram is mapped by b, a_-, a_+ resp. a_R. Since the diagrams describe the general curves of each boundary cycle, this gives us a list of all boundary cycles of the corresponding space $\overline{M}_2$, $\overline{S}_2^-$, $\overline{S}_2^+$, and tells us how the general object $\mathfrak{X}$ parametrised by each cycle looks like. Furthermore we use Lemma 2.27 to compute the number of automorphisms of each such general object. How the latter is done is explained after providing the results in the following tables. We also give each boundary cycle of $\overline{M}_2$, and so on, a name in these tables, which will be used in the next chapter.

For $\overline{M}_2$, we obtain:

Codim.	**Cycle**	$\mathfrak{D}$	$\mathfrak{X}$	$\lvert \mathrm{Aut}(\mathfrak{X}) \rvert$
0	M_2			2
1	Δ_0			2
1	Δ_1			4

2	Δ_{00}			4
2	Δ_{01}			4
2	C_{000}			12
2	C_{001}			8

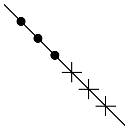
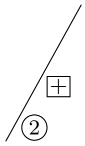
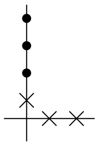
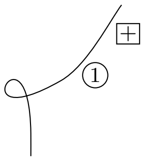
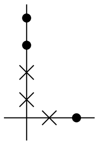
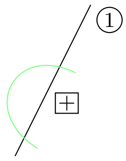
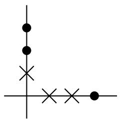
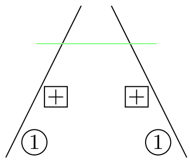
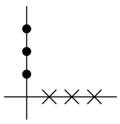
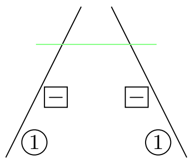
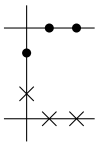
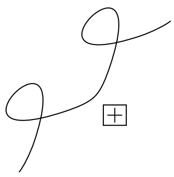
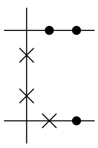
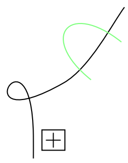
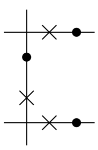
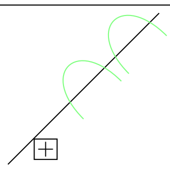
Like in example 1.24 the encircled numbers in the sketch of X, denote the geometric genus of the component of the curve they stand close to. All components without such an encircled number have geometric genus 0. Of course all information provided in this table is known from [Mum83]. We also used the names introduced there for the cycles.

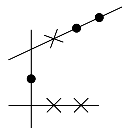
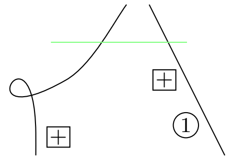
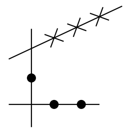
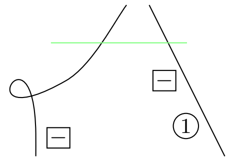
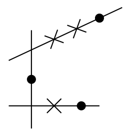
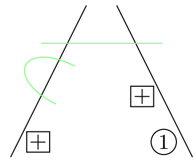
The next tables provide the same for $\overline{S}_2$, but with an additional column, showing to which class in $A^*(\overline{M}_2)$ the Q-class of each cycle is pushed forward by π_+. [42] The column for $\mathfrak{X}$ contains a sketch of the underlying quasi-stable curve X. To each of the (non-exceptional) components into which X is divided by disconnecting nodes, we attached a label ⊞ or ⊟, indicating whether the restriction of the spin sheaf of $\mathfrak{X}$ to this component is an even spin sheaf or an odd spin sheaf. For $\overline{S}_2$ and $\overline{R}_2$ some X will have exceptional components, which we draw in green in our pictures. A list of the boundary strata of $\overline{S}_2$ is contained in the appendix of [BF09a], and we continue to use the names introduced for them there.

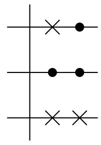
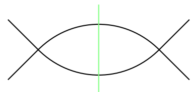
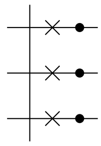
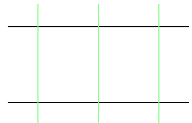
In the table for $\overline{R}_2$ coming later, to every (non-exceptional) irreducible component of X we attached a label with one or two entries (for example $\boxed{N|t}$), giving the following information: The first entry of the label at a component X_i is T if the restriction $\mathcal{L}_{|X_i}$ of the spin sheaf $\mathcal{L}$ of $\mathfrak{X}$ is the trivial sheaf, and is N if $\mathcal{L}_{|X_i}$ is a nontrivial prym sheaf. If $\mathcal{L}_{|X_i}$ is nontrivial and X_i is not normal, the label will contain a second entry, which describes the pull-back $\mathcal{L}'_{|X_i}$ of $\mathcal{L}_{|X_i}$ to the normalisation of X_i. It is t, if $\mathcal{L}'_{|X_i}$ is trivial, and n if $\mathcal{L}'_{|X_i}$ is a nontrivial prym sheaf. There is also the possibility that $\mathcal{L}'_{|X_i}$ is a twisted prym sheaf,

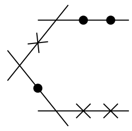
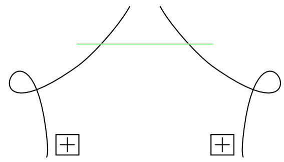
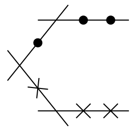
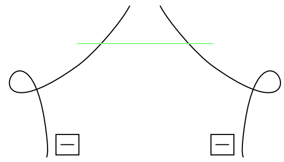
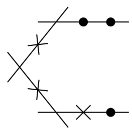
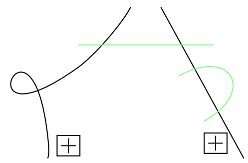
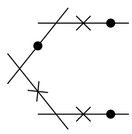
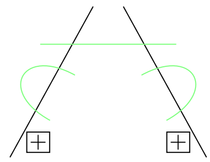
[42] This is computed as follows: Determine the degree m of the forgetful morphism π_+ on the given cycle D (as morphism of varieties), by counting (using the diagrams in the table) the number of *non-isomorphic* possibilities to put a sorting on the marked points of a given $\mathfrak{D}$ (i.e. to distribute the dots and crosses) so that it still belongs to the given cycle (cf. section 3.1.1 for similar countings). Then $(\pi_+)_*[D] = m[\Delta]$, where Δ is the boundary cycle of $\overline{M}_2$ which is the image of D. To express this in Q-classes instead, one uses the automorphism numbers for general $\mathfrak{X}$ for $\mathfrak{X} \in D$ resp. $\mathfrak{X} \in \Delta$, which can be found in the tables (cf. Summary 1.34 (ii)).

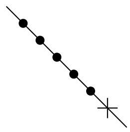
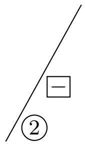
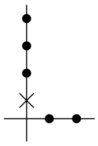
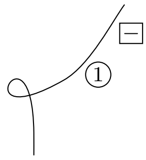
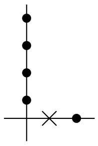
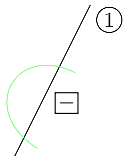
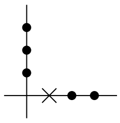
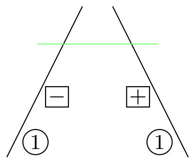
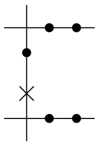
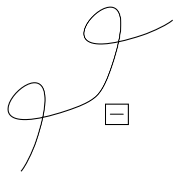
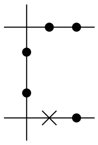
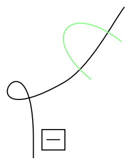
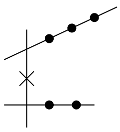
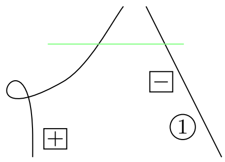
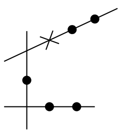
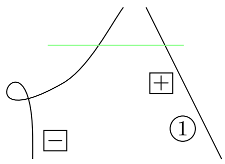
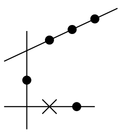
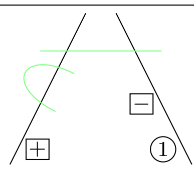
if X_i meets any exceptional components (cf. Summary 1.13 (ii)). Then $\mathcal{L}'_{|X_i}$ is a square root of a sheaf of degree $-r$, where r is the (even) number of points in which X_i meets exceptional components, and the label will contain $-r$ as its second entry.

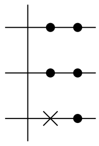
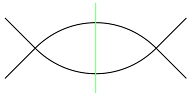
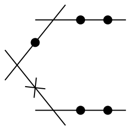
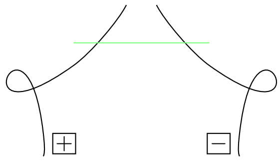
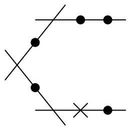
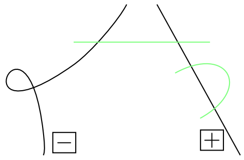
The boundary cycles of $\overline{S}_2^+$					
Codim.	**Cycle**	$\mathfrak{D}$	$\mathfrak{X}$	$\lvert\mathrm{Aut}(\mathfrak{X})\rvert$	$(\pi_+)_*([...]_Q)$
0	$\overline{S}_2^+$			2	$10[\overline{M}_2]_Q$
1	A_0^+			2	$4\delta_0$
1	B_0^+			2	$3\delta_0$
1	A_1^+			8	$\frac{9}{2}\delta_1$
1	B_1^+			8	$\frac{1}{2}\delta_1$
2	C^+			4	$2[\Delta_{00}]_Q$
2	D^+			2	$2[\Delta_{00}]_Q$
2	E			4	$[\Delta_{00}]_Q$

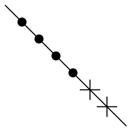
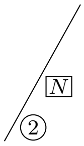
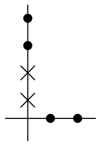
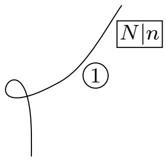
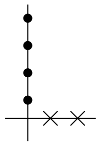
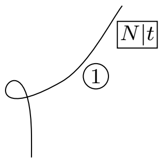
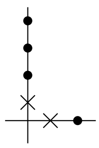
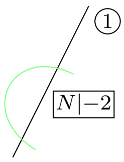
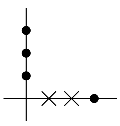
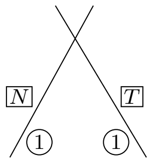
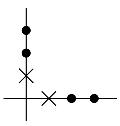
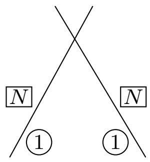
2	X^+			8	$\frac{3}{2}[\Delta_{01}]_Q$
2	Y^+			8	$\frac{1}{2}[\Delta_{01}]_Q$
2	Z^+			8	$\frac{3}{2}[\Delta_{01}]_Q$

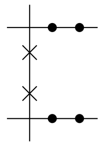
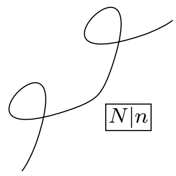
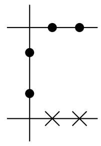
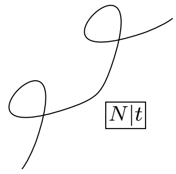
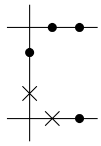
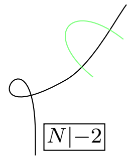
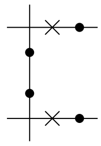
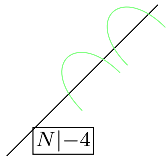
3	L^+			4	$3[\Delta_{000}]_Q$
3	M			24	$\frac{1}{2}[\Delta_{000}]_Q$

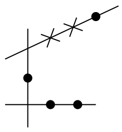
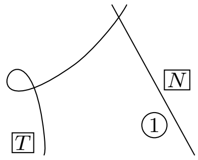
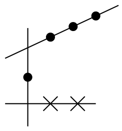
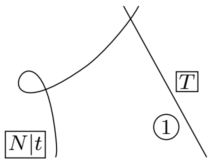
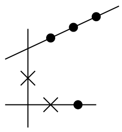
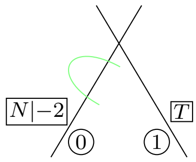
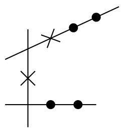
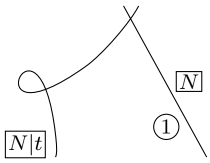
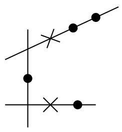
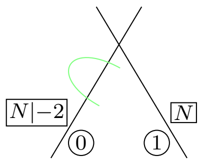
3	P^+			16	$\frac{1}{2}[\Delta_{001}]_Q$
3	Q^+			16	$\frac{1}{2}[\Delta_{001}]_Q$
3	U^+			8	$[\Delta_{001}]_Q$
3	R			16	$\frac{1}{2}[\Delta_{001}]_Q$

The boundary cycles of $\overline{S}_2^-$					
Codim.	**Cycle**	$\mathfrak{D}$	$\mathfrak{X}$	$\lvert\mathrm{Aut}(\mathfrak{X})\rvert$	$(\pi_-)_*([\ldots]_Q)$
0	$\overline{S}_2^-$			2	$6[\overline{M}_2]_Q$
1	A_0^-			2	$4\delta_0$
1	B_0^-			2	δ_0
1	A_1^-			8	$3\delta_1$
2	C^-			2	$2[\Delta_{00}]_Q$
2	D^-			2	$2[\Delta_{00}]_Q$
2	X^-			8	$\frac{1}{2}[\Delta_{01}]_Q$
2	Y^-			8	$\frac{3}{2}[\Delta_{01}]_Q$
2	Z^-			8	$\frac{1}{2}[\Delta_{01}]_Q$

3	L^-			4	$3[\Delta_{000}]_Q$
3	P^-			8	$[\Delta_{001}]_Q$
3	U^-			8	$[\Delta_{001}]_Q$

The boundary cycles of $\overline{R}_2$					
Codim.	**Cycle**	$\mathfrak{D}$	$\mathfrak{X}$	$\lvert\mathrm{Aut}(\mathfrak{X})\rvert$	$(\pi_R)_*([...]_Q)$
0	$\overline{R}_2$			2	$15[\overline{M}_2]_Q$
1	D_0'			2	$6\delta_0$
1	D_0''			2	δ_0
1	D_0^r			2	$4\delta_0$
1	D_1			4	$6\delta_1$
1	$D_{1:1}$			4	$9\delta_1$

2	$E^{\prime,\prime}$			4	$[\Delta_{00}]_Q$
2	$E^{\prime,\prime\prime}$			2	$2[\Delta_{00}]_Q$
2	$E^{\prime,r}$			2	$4[\Delta_{00}]_Q$
2	$E^{r,r}$			4	$[\Delta_{00}]_Q$

2	$F_1^{\prime}$			4	$3[\Delta_{01}]_Q$
2	$F_1^{\prime\prime}$			4	$[\Delta_{01}]_Q$
2	F_1^{r}			4	$[\Delta_{01}]_Q$
2	$F_{1:1}^{\prime}$			4	$3[\Delta_{01}]_Q$
2	$F_{1:1}^{r}$			4	$[\Delta_{01}]_Q$

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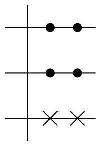
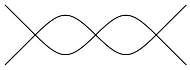
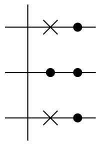
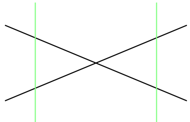
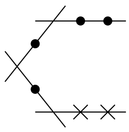
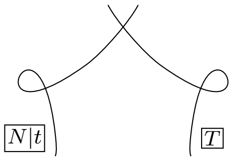
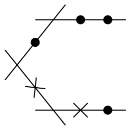
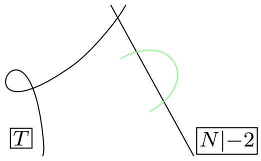
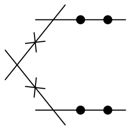
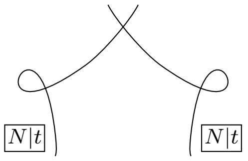
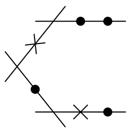
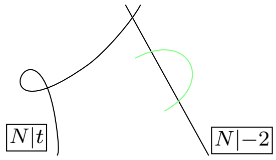
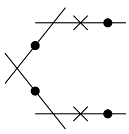
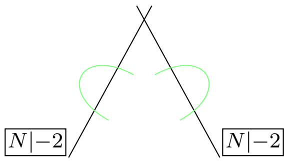
<tr><td>3</td><td>G'</td><td></td><td></td><td>4</td><td>$3[\Delta_{000}]_Q$</td></tr>
<tr><td>3</td><td>G^r</td><td></td><td></td><td>4</td><td>$3[\Delta_{000}]_Q$</td></tr>
<tr><td>3</td><td>H'_1</td><td></td><td></td><td>4</td><td>$2[\Delta_{001}]_Q$</td></tr>
<tr><td>3</td><td>H^r_1</td><td></td><td></td><td>4</td><td>$2[\Delta_{001}]_Q$</td></tr>
<tr><td>3</td><td>$H'_{1:1}$</td><td></td><td></td><td>8</td><td>$[\Delta_{001}]_Q$</td></tr>
<tr><td>3</td><td>$H^r_{1:1}$</td><td></td><td></td><td>4</td><td>$2[\Delta_{001}]_Q$</td></tr>
<tr><td>3</td><td>$H^{r,r}_{1:1}$</td><td></td><td></td><td>8</td><td>$[\Delta_{001}]_Q$</td></tr>
</table>

2.5.1 Automorphism numbers

Here we explain how the automorphism numbers in the previous tables where computed. One ingredient is:

Lemma 2.29 *Let $p_1, ..., p_n$ be n distinct points of $\mathbb{P}^1$ in general position. We describe, for different $n \in \mathbb{N}$, the group $A := \mathrm{Aut}(\mathbb{P}^1; \{p_1, ..., p_n\})$ of automorphisms of $\mathbb{P}^1$ that map points of the set $\{p_1, ..., p_n\}$ again to points of this set.*

(i) For $n \leq 2$, A is an infinite group.

(ii) For $n = 3$, A has 6 elements corresponding to the permutations of the 3 points.

(iii) For $n = 4$, A has 4 elements, one is the identity, the others correspond to choosing two disjoint pairs of the points, and interchanging the points in each pair.

(iv) For $n \geq 5$, A consists only of the identity.

Proof: The automorphisms of $\mathbb{P}^1$ are the Möbius transformations $x \mapsto \frac{Ax+B}{Cx+D}$ where $A, B, C, D \in \mathbb{C}$. Using this one checks that the assertions of the Lemma are true. $\square$.

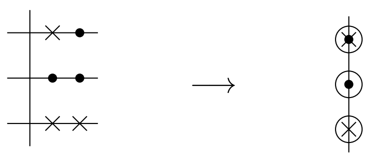
We use Lemma 2.29 together with Lemma 2.27 (iv) to compute the number of automorphisms of a general prym- or spin curve $\mathfrak{X}$ of the cycles appearing in the previous tables. In our case $\overline{HS}_2^+ = \overline{S}_2^+$, $\overline{HS}_2^- = \overline{S}_2^-$, $\overline{HR}_2 = \overline{R}_2$, the hyperelliptic local universal deformation of a prym/spin curve is the whole usual local universal deformation, so the deformation space $(\mathscr{S}, s_0) = (S, s_0)$ has only one irreducible component. Hence Condition 2.21 is necessarily fulfilled. By Lemma 2.27 (iv) we have

$$|\operatorname{Aut}(\mathfrak{X})| = 2^{s+u} \cdot N.$$

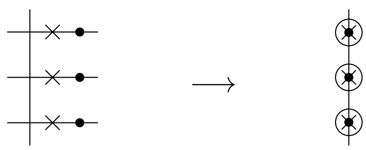
The numbers s of non-disconnecting nodes of X and u of connected components of the non-exceptional subcurve of X can be counted at the sketch of X included in the table. (How X looks like was determined using Proposition 2.19.)

It remains to determine $N = |\operatorname{Aut}((\widehat{D}; (H_{JK}, \{(\widehat{J}, H_J), (\widehat{K}, H_K)\})))|$ (note that $I = \emptyset$ in our case).

Example: We take the diagram of the general object $\mathfrak{D} = (D; \{J, K\})$ from the table, and reduce it to a diagram of $(\widehat{D}; (H_{JK}, \{(\widehat{J}, H_J), (\widehat{K}, H_K)\}))$ as follows: We keep the markings that do not lie on extremities, and we introduce for every point to which an extremity is contracted a circle, in the centre of which we insert a dot if the extremity carried two dots, a cross if the extremity carried two crosses, a cross and a dot if the extremity carried one cross and one dot. For example, in the case of the cycle L^+ of $\overline{S}_2^+$ we obtain



For M of $\overline{S}_2^+$ we obtain



An automorphism must either take all symbols to symbols of the same kind (i.e. dots to dots, crosses to crosses, circled dots to circled dots,...), or it must take all dots to crosses and vice versa and all circled dots to circled crosses and vice versa. Now, using Lemma 2.29 (ii), it is clear that $N = 2$ for L^+ (it is possible to swap the cross and the dot), and $N = 6$ for M^+. From the diagrams of $\mathfrak{D}$, one sees directly that $s = 0$ in both cases. From the sketch of X in the table, we see that the non-exceptional subcurve $\widetilde{X} \subset X$ in case of L^+ has one connected components, so here $u = 1$. For M^+, $\widetilde{X}$ has two connected components, so there $u = 2$. Putting all this together the automorphism number is $2 \cdot N = 4$ for L^+ and $4 \cdot N = 24$ for M^+.

Chapter 3

Rational cohomology of $\overline{R}_2$ and $\overline{S}_2$

In this chapter we determine the rational Chow ring $A^*(\overline{R}_2)$ [1] of $\overline{R}_2$, as a $\mathbb{Q}$-algebra, in terms of generators and relations (Theorem 3.14). We also show that it is isomorphic to the rational cohomology ring $H^*(\overline{R}_2)$ of this space via the cycle map (Thm. 3.12). Gilberto Bini and Claudio Fontanari did the same for $\overline{S}_2$, the moduli space of spin curves of genus 2, in [BF09a]. In computing the cohomology of $\overline{R}_2$ we follow their approach in large parts. As a new ingredient, we also apply the isomorphism $a_R : \overline{M}_{0,[2,4]} \to \overline{R}_2$, which is a special case of the isomorphisms constructed in the previous chapter, to compute additional relations in the Chow/cohomology ring, by pushing forward Keel relations. (As explained in the introduction of this thesis, using a_R to obtain relations was an idea suggested to me by Orsola Tommasi.)

Concerning $\overline{S}_2$, we correct some errors made in [BF09a]. It turns out that, contrary to what is stated in [BF09a], the classes of the boundary divisors of $\overline{S}_2^+$ are not independent in the Picard group, and as a consequence the first Betti number $h^1(\overline{S}_2^+)$ is 3, not 4. Also some of the relations in the cohomology rings computed there are not correct. To obtain new relations to replace them, we use the isomorphisms $a_+ : \overline{M}_{0,[3,3]} \to \overline{S}_2^+$ and $a_- : \overline{M}_{0,[1,5]} \to \overline{S}_2^+$, also known from the previous chapter. Similar morphisms (from $\overline{M}_{0,6}$) to $\overline{S}_2^+$ and $\overline{S}_2^-$ are constructed in [BF09a], but are not used to obtain relations.

Remarks and Notation: Strictly speaking, what we compute in this chapter is the rational Chow ring and the rational cohomology of the stacks $\overline{\mathcal{S}}_2$ and $\overline{\mathcal{R}}_2$. Or putting it differently, we compute $H^*(\overline{S}_2)$ and $A^*(\overline{R}_2)$ with the multiplication " $\cdot$ " induced by push-forward from the stacks, as explained in Summary 2.6, not with the intrinsic multiplication " $\bullet$ ". Since the number of automorphisms of a generic spin/prym curve parametrised by $\overline{S}_2$ (or $\overline{R}_2$) is 2, the map $A^*(\overline{S}_2) \xrightarrow{2\cdot} A^*(\overline{R}_2)$, multiplying each class by 2, becomes an isomorphism of $\mathbb{Q}$-algebras, if on the left hand side the multiplicative structure is given by $\cdot$ and on the right hand side by $\bullet$. The same holds for $H^*(\overline{S}_2)$ and for $A^*(\overline{R}_2)$, $H^*(\overline{R}_2)$. Like in the whole thesis, we work with the adjusted pullbacks introduced in Summary 1.34 (iv), and denote them by f^* instead of $f^{\circledast}$.

The names for the boundary cycles of $\overline{S}_2$ and $\overline{R}_2$ introduced in the tables of section 2.5

[1] I.e. the Chow ring with coefficients in $\mathbb{Q}$.

will often be used in this chapter, without explicitly referring to these tables again.

3.0.2 Some remarks and notation

Definition 3.1 We denote by $\pi_R : \overline{R}_2 \longrightarrow \overline{M}_2$, $\pi_+ : \overline{S}_2^+ \longrightarrow \overline{M}_2$ and $\pi_- : \overline{S}_2^- \longrightarrow \overline{M}_2$ the "forgetful morphisms", which corresponds to discarding the additional prym or spin structure, and passing from the underlying curve X to its stable model C.

We know all boundary cycles of $\overline{R}_2$ and $\overline{S}_2$ from the tables of section 2.5, which also tell us which kind of spin or prym curves each cycle parametrises generically.

Recall that the boundary divisors of $\overline{R}_2$ are D_1, $D_{1:1}$, D_0', D_0'' and D_0^r. We assign *boundary divisor classes* in $A^1(\overline{R}_2)$ by taking Q-classes:

$$d_1 := [D_1]_Q, \quad d_{1:1} := [D_{1:1}]_Q, \quad d_0' := [D_0']_Q, \quad d_0'' := [D_0'']_Q, \quad d_0^r := [D_0^r]_Q.$$

Equivalently one defines the boundary divisor classes δ_0 and δ_1 of $\overline{M}_2$.

The forgetful map $\pi_R : \overline{R}_2 \longrightarrow \overline{M}_2$, is ramified in codimension 1 only at D_0^r i.e. is branched over Δ_0. So the boundary divisor classes of $\overline{M}_2$ pull back to $\overline{R}_2$ as follows (also cf. Remark 1.35 (i)):

$$\pi^*(\delta_0) = d_0' + d_0'' + 2d_0^r \qquad \text{and} \qquad \pi^*(\delta_1) = d_1 + d_{1:1}$$

The boundary divisors $A_0^+, B_0^+, A_1^+, B_1^+$ of $\overline{S}_2^+$ and A_0^-, B_0^-, A_1^- of $\overline{S}_2^-$ we again know from section 2.5, and define corresponding classes:

$$\alpha_0^+ := [A_0^+]_Q, \quad \beta_0^+ := [B_0^+]_Q, \quad \alpha_1^+ := [A_1^+]_Q, \quad \beta_1^+ := [B_1^+]_Q,$$

$$\alpha_0^- := [A_0^-]_Q, \quad \beta_0^- := [B_0^-]_Q, \quad \alpha_1^- := [A_1^-]_Q$$

The pullbacks of δ_0 and δ_1 to these spaces are:

$$\pi_+^*(\delta_0) = \alpha_0^+ + 2\beta_0^+, \quad \pi_+^*(\delta_1) = 2\alpha_1^+ + 2\beta_1^+,$$

$$\pi_-^*(\delta_0) = \alpha_0^- + 2\beta_0^-, \quad \pi_-^*(\delta_1) = 2\alpha_1^-$$

3.1 Morphisms to $\overline{S}_2$ and $\overline{R}_2$.

In this section we introduce several finite morphisms from other moduli spaces to $\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$. They will later be used to determine relations between cycle classes on our moduli spaces, by pushing forward known relations, or by using the projection formula.

3.1.1 Surjections from moduli spaces of genus 0 curves with 6 marked points.

Recall from the beginning of section 2.5 that:

Lemma 3.2 *(& **Definition**) There are isomorphisms*

$$b : \overline{M}_{0,[6]} \xrightarrow{\cong} \overline{M}_2 \quad \textit{resp.}$$

$$a_R : \overline{M}_{0,[2,4]} \xrightarrow{\cong} \overline{R}_2 \quad \textit{resp.} \quad a_+ : \overline{M}_{0,[3,3]} \xrightarrow{\cong} \overline{S}_2^+ \quad \textit{resp.} \quad a_- : \overline{M}_{0,[1,5]} \xrightarrow{\cong} \overline{S}_2^-$$

These isomorphisms map the boundary of $\overline{M}_{0,6}$ onto the boundary of the images.

By composing every one of the isomorphisms above with the appropriate quotient morphism out of $\pi_{0,[6]} : \overline{M}_{0,6} \to \overline{M}_{0,[6]}$, $\pi_{0,[4,6]} : \overline{M}_{0,6} \to \overline{M}_{0,[4,6]}$, $\pi_{0,[3,3]} : \overline{M}_{0,6} \to \overline{M}_{0,[3,3]}$, and $\pi_{0,[1,5]} : \overline{M}_{0,6} \to \overline{M}_{0,[1,5]}$, we define surjective finite morphisms:

$$g : \overline{M}_{0,6} \xrightarrow{720:1} \overline{M}_2,$$

$$f_R : \overline{M}_{0,6} \xrightarrow{48:1} \overline{R}_2, \quad f_+ : \overline{M}_{0,6} \xrightarrow{72:1} \overline{S}_2^+ \quad \textit{and} \quad f_- : \overline{M}_{0,6} \xrightarrow{120:1} \overline{S}_2^-.$$

(The symbols $d:1$ over the arrows indicate that a morphism is finite of degree d.)

Proof: Everything except the degrees of the finite surjective morphisms is just a special case of Proposition 2.14, as explained in section 2.5. The degrees equal those of the forgetful morphisms $\pi_{0,[6]}$, $\pi_{0,[2,4]}$, $\pi_{0,[3,3]}$, $\pi_{0,[1,5]}$, which can easily be counted. □

By the previous Lemma we know:

$$H^*(\overline{R}_2) \cong (H^*(\overline{M}_{0,6}))^{S_2 \times S_4}, \quad H^*(\overline{S}_2^+) \cong (H^*(\overline{M}_{0,6}))^{S_3 \times S_3 \times S_2},$$

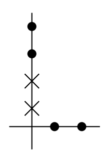
$$H^*(\overline{S}_2^-) \cong (H^*(\overline{M}_{0,6}))^{S_1 \times S_5}$$

where the group actions are those of Remark 2.5. As the cohomology of $\overline{M}_{0,6}$ is known (cf. Summary 1.48), a computer algebra program could at least compute this invariant cohomology as graded vector spaces. It was checked that these computation yields the Betti numbers we obtained by hand in Theorem 3.13.

For our computation of the rational cohomology of $\overline{R}_2$ and $\overline{S}_2$ as $\mathbb{Q}$-algebras, we need some more information about the isomorphism a_R, a_+ and a_-, and the finite surjective maps f_R, f_+, and f_- defined from them.

By the tables of section 2.5 we know which boundary divisors get identified by the isomorphisms a_R, a_+ and a_-.

Now we can determine how f_R, f_+ and f_- behave on the boundary divisors of $\overline{M}_{0,6}$. Using Notation 1.47, all these boundary divisors are of the form $[i_1, i_2]$ or $[j_1, j_2, j_3]$ $(i_1, i_2, j_1, j_2, j_3 \in \underline{6})$. To which component a boundary divisor of $\overline{M}_{0,6}$ is mapped, can be seen using the tables of section 2.5. The degree of the map on a given boundary divisor one gets as in the following example: The boundary divisor $[3,4]$ is mapped to D_0'. A general point of $[3,4]$ is thus mapped by f_R to a point of $D_0' \subset \overline{R}_2$ corresponding in $\overline{M}_{0,[2,4]}$ to a diagram of the form



One gets that the degree of f_R on $[3,4]$ is 4 by counting how many non-isomorphic possibilities there are to assign indices $1,...,6$ to the marked points of the diagram, such that the dots get $3,4,5,6$, the crosses get $1,2$ and such that 3 and 4 go to the component with only two marked points. There are 8 possibilities, but swapping 3 and 4 yields isomorphic objects.

Behaviour of $f_R : \overline{M}_{0,6} \xrightarrow{48:1} \overline{R}_2$: For arbitrary $b_1, b_2 \in \{3,4,5,6\}$ we have,

- Boundary divisors of the form $[b_1, b_2]$ are mapped $4:1$ (each) onto D_0'.
- The boundary divisor $[1,2]$ is mapped $24:1$ onto D_0''.
- Boundary divisors of the form $[1, b_1]$ or $[2, b_1]$ are mapped $6:1$ (each) onto D_0^r.
- Boundary divisors of the form $[1,2,b_1]$ are mapped $12:1$ (each) onto D_1.
- Boundary divisors of the form $[1, b_1, b_2]$ (or equivalently $[2, b_1, b_2]$) are mapped $8:1$ (each) onto $D_{1:1}$.

Behaviour of $f_+ : \overline{M}_{0,6} \xrightarrow{72:1} \overline{S}_2^+$: For arbitrary $a_1, a_2 \in \{1,2,3\}$ and $b_1, b_2 \in \{4,5,6\}$ we have,

- Boundary divisors of the form $[a_1, a_2]$ or $[b_1, b_2]$ are mapped $6:1$ (each) onto A_0^+.
- Boundary divisors of the form $[a_1, b_1]$ are mapped $8:1$ (each) onto B_0^+.
- Boundary divisors of the form $[a_1, a_2, b_1]$ (or equivalently $[a_1, b_1, b_2]$) are mapped $8:1$ (each) onto A_1^+.
- The boundary divisor $[1,2,3]$ is mapped $72:1$ onto B_1^+.

Behaviour of $f_- : \overline{M}_{0,6} \xrightarrow{120:1} \overline{S}_2^-$: For arbitrary $b_1, b_2 \in \{2,3,4,5,6\}$,

- Boundary divisors of the form $[1, b_1]$ are mapped $24:1$ (each) onto B_0^-.
- Boundary divisors of the form $[b_1, b_2]$ are mapped $6:1$ (each) onto A_0^-.
- Boundary divisors of the form $[1, b_1, b_2]$ are mapped $12:1$ (each) onto A_1^-.

We now use this to compute:

Lemma 3.3 *There are the following relations between boundary divisor classes:*
(i) In $A^1(\overline{R}_2)$*:* $d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1} = 0$
(ii) In $A^1(\overline{S}_2^+)$*:* $3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+ = 0$

Proof: (i): Using equation (1.7) from Summary 1.48 with $i,j,k,l := 1,2,3,4$ we get

$$[1,2] + [1,2,5] + [1,2,6] + [1,2,5,6] = [1,3] + [1,3,5] + [1,3,6] + [1,3,5,6]$$

which is the same as

$$0 = [1,2] + [1,2,5] + [1,2,6] + [3,4] - [1,3] - [1,3,5] - [1,3,6] - [2,4]$$

Pushing this relation forward by f_R we get:

$$0 = 24[D_0''] + 12[D_1] + 12[D_1] + 4[D_0'] - 6[D_0^r] - 8[D_{1:1}] - 8[D_{1:1}] - 6[D_0^r]$$

$$= 4[D_0'] + 24[D_0''] - 12[D_0^r] + 24[D_1] - 16[D_{1:1}]$$

Using the automorphism numbers from the tables of section 2.5, this can be written as

$$0 = 8d_0' + 48d_0'' - 24d_0^r + 96d_1 - 64d_{1:1}$$

$$\Leftrightarrow 0 = d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1}$$

(ii): Using equation (1.7), this time with $i,j,k,l := 1,2,4,5$, we get

$$[1,2] + [1,2,3] + [1,2,6] + [1,2,3,6] = [1,4] + [1,3,4] + [1,4,6] + [1,3,4,6]$$

Pushing this relation forward by f_+, and proceeding like in part (i) we get:

$$0 = 24\alpha_0^+ - 32\beta_0^+ - 64\alpha_1^+ + 576\beta_1^+$$

$$\Leftrightarrow 0 = 3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+$$

□

3.1.2 Morphisms to the boundary divisors of $\overline{R}_2$ and $\overline{S}_2$

Now we come to several finite surjective morphisms from other moduli spaces to different boundary divisors of $\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$. Later they will be used to determine relations between intersection products of boundary divisors via the projection formula.

Morphisms from $\overline{M}_{0,5}$

First we define a Morphism $c : \overline{M}_{0,5} \times \overline{M}_{0,3} \to [5,6] \subset \overline{M}_{0,6}$. ($[5,6]$ is one of the boundary divisors of $\overline{M}_{0,6}$, cf. Notation 1.47.) To the pair of $[(C;(q_0,...,q_4))] \in \overline{M}_{0,5}$ and $[(C';(q_0',...,q_2'))] \in \overline{M}_{0,3}$ the morphism c assigns $[D;(p_1,...,p_6)] \in [5,6] \subset \overline{M}_{0,6}$, where D is the curve obtained from C and C' by gluing the points q_0 and q_0', and where $p_1,...,p_4$ are defined as the images of $q_1,...,q_4$ at D, and p_5 resp. p_6 are defined as the images of q_1' resp. q_2'. $\overline{M}_{0,3}$ is just a point, so there is an isomorphism $i : \overline{M}_{0,5} \to \overline{M}_{0,5} \times \overline{M}_{0,3}$. The composed map $c \circ i$ is a finite degree 1 morphism onto $[5,6]$. We compose this morphism with f_R and get a finite Morphism:

$$h_0' : \overline{M}_{0,5} \xrightarrow{4:1} D_0'$$

h_0' is $4:1$ because that is the degree of f_R on $[5,6]$ (cf. section 3.1.1).

By composing $c \circ i$ with f_- one gets a morphism

$$h_0^\alpha : \overline{M}_{0,5} \xrightarrow{6:1} A_0^-$$

Similar to what was done in section 2.2 for f_R, f_+ and f_-, one can determine the behaviour of these two morphisms on the boundary of $\overline{M}_{0,5}$. For each boundary divisor of $\overline{M}_{0,5}$ we describe to which boundary cycle of $\overline{R}_2$ resp. $\overline{S}_2^-$ (cf. section 2.5) it is mapped by h_0' resp. h_0^α. The boundary divisors of $\overline{M}_{0,5}$ are (for our choice of the indices of the marked points) all of the form $[i_1, i_2]$ $(i_1, i_2 \in \{0, 1, 2, 3, 4\})$.

Behaviour of $h_0' : \overline{M}_{0,5} \xrightarrow{4:1} D_0' \subset \overline{R}_2$. For arbitrary $a \in \{1, 2\}$ and $b \in \{3, 4\}$:

- The boundary divisor $[1, 2]$ is mapped $2 : 1$ onto $E^{','',} = D_0' \cap D_0''$.
- Boundary divisors of the form $[a, b]$ are mapped $1 : 1$ (each) onto $E^{',r} = D_0' \cap D_0^r$.
- The boundary divisor $[3, 4]$ is mapped $2 : 1$ onto $E^{','}$.
- Boundary divisors of the form $[0, a]$ are mapped $2 : 1$ (each) onto $F_{1:1}' = D_0' \cap D_{1:1}$.
- Boundary divisors of the form $[0, b]$ are mapped $2 : 1$ (each) onto $F_1' = D_0' \cap D_1$.

Behaviour of $h_0^\alpha : \overline{M}_{0,5} \xrightarrow{6:1} A_0^- \subset \overline{S}_2^-$. For arbitrary $b_1, b_2 \in \{2, 3, 4\}$:

- Boundary divisors of the form $[b_1, b_2]$ are mapped $2 : 1$ (each) onto C^-. ($2 : 1$ because two non-isomorphic diagrams of $\overline{M}_{0,5}$ are assigned to two different but isomorphic diagrams of $\overline{M}_{0,[1,5]} \cong \overline{S}_2^-$.)
- Boundary divisors of the form $[1, b_1]$ are mapped $2 : 1$ (each) onto $D^- = A_0^- \cap B_0^-$.
- The boundary divisor $[0, 1]$ is mapped $6 : 1$ onto X^-.
- Boundary divisors of the form $[0, b_1]$ are mapped $2 : 1$ (each) onto Y^-.

We use this to compute:

Lemma 3.4 *There are the following relations between cycle classes in the Chow ring of our moduli spaces:*

(i) In $A^2(\overline{R}_2)$*:* $2d_0'd_0'' + 4d_0'd_1 - 4d_0'd_{1:1} - d_0'd_0^r = 0$

(ii) In $A^2(\overline{S}_2^-)$*:* $16[X^-]_Q + [C^-]_Q - 4\alpha_0^-\alpha_1^- - \alpha_0^-\beta_0^- = 0$

(iii) In $A^2(\overline{R}_2)$*:* $[E^{',r}]_Q = 2[E^{','}]_Q + [E^{','',}]_Q$

Proof: (i): Using equation (1.7) with $i, j, k, l := 0, 1, 2, 3$ we get

$$[0, 1] + [2, 3] = [0, 3] + [1, 2]$$

Pushing this relation forward by h_0' we get:

$$0 = 2[D_0' \cap D_1] + 2[D_0' \cap D_0''] - 2[D_0' \cap D_{1:1}] - [D_0' \cap D_0^r]$$

Using the automorphism numbers from section 2.5 this can be written as

$$0 = 8d_0'd_1 + 4d_0'd_0'' - 8d_0'd_{1:1} - 2d_0'd_0^r$$
$$\Leftrightarrow 2d_0'd_0'' + 4d_0'd_1 - 4d_0'd_{1:1} - d_0'd_0^r = 0$$

(ii): We again use the equation

$$0 = [0,3] + [1,2] - [0,1] - [2,3]$$

and now push it forward by h_0^α. Then proceeding as above, we arrive at

$$0 = 12[X^-]_Q + [C^-]_Q - 4[Y^-]_Q - \alpha_0^-\beta_0^-$$

Now we use that $A_0^- \cap A_1^- = X^- \cup Y^-$ is a proper intersection. We can treat all proper intersections of Q-classes of boundary cycles as transversal, since those cycles meet transversally on the deformation space (cf. Summary 1.34 (v)). Thus $\alpha_0^-\alpha_1^- = [X^-]_Q + [Y^-]_Q$. Using this one can rewrite the equation as

$$0 = 16[X^-]_Q + [C^-]_Q - 4\alpha_0^-\alpha_1^- - \alpha_0^-\beta_0^-$$

(iii) Using equation (1.7) with $i, j, k, l := 1, 2, 3, 4$ we get

$$[1,2] + [3,4] = [1,3] + [2,4]$$

Pushing this relation forward by h_0' and using the automorphism numbers from section 2.5 we get:

$$4[E^{','' }]_Q + 8[E^{','}]_Q = 2[E^{',r}]_Q$$
$$\Leftrightarrow [E^{','' }]_Q + 2[E^{','}]_Q = [E^{',r}]_Q$$

□

Gluing morphisms whose images are boundary divisors

For $\overline{R}_2$ and $\overline{S}_2$ we introduce the following gluing morphisms whose images are boundary divisors. They are defined similar to the general gluing morphisms to boundary cyles of $\overline{M}_{g,n}$ as described in Proposition 1.26 (i). For $\overline{S}_2$ they are introduced and used in [BF09a], but have different names there. We describe how they behave on general points. [2]

[2]To prove that these gluing morphism exist, one should strictly speaking check that the gluing procedure of spin/prym curves described for each gluing morphism below, can also be applied to families of such spin/prym curves. Then one should proof that this induces morphisms of moduli functors (or even of moduli groupoids/stacks). Here one would use that families of nodal curves can be glued along sections of marked points, and that this is a functor (the clutching functor, cf. Prop. 1.26 (i)), and a morphism of groupoids. Then one would show that also the fibres over the sections of marked points of the spin/prym bundles of the families can be glued consistently. The morphism of moduli functors obtained then induces a morphism of the coarse moduli spaces as explained in section 1.1. In section 1.7.1. and 1.7.2. of [JKV01] such gluing procedures are examined in general for (higher) twisted spin curves in the sense of Jarvis. It is shown in which cases they define morphisms of stacks. Since our coarse moduli spaces of spin curves are isomorphic to the moduli spaces of certain of these stacks, the discussion there implies that all the gluing morphisms below to boundary divisors of $\overline{S}_2$ exist. For $\overline{R}_2$ one could show the existence analogously.

For $\overline{R}_2$:

$$\tau_1 : \overline{M}_{1,1} \times \overline{R}_{1,1} \xrightarrow{1:1} D_1$$

The image of a pair of $[(X;p)] \in \overline{M}_{1,1}$ and $[(Y;q;\mathcal{L},b)] \in \overline{R}_{1,1}$ is the point in D_1 parametrising the following prym curve $(X';\mathcal{L}')$: The quasistable curve X' is generated by gluing the points p and q on the curves X and Y. The prym bundle $\mathcal{L}'$ is obtained from the trivial bundle on X and the prym bundle $\mathcal{L}$ on Y, by identifying the fibres over p resp. q. All possible choices of identification yield isomorphic prym bundles.

$$\tau_{1:1} : \overline{R}_{1,1} \times \overline{R}_{1,1} \xrightarrow{2:1} D_{1:1}$$

This morphism is defined analogously to τ_1 . It is of degree 2 since a pair

$$([(X;p;\mathcal{L},b)],[(X';p';\mathcal{L}',b')]) \in \overline{R}_{1,1} \times \overline{R}_{1,1}$$

and the transposed pair are mapped to the same point in $D_{1:1}$.

$$\tau_0'' : \overline{M}_{1,2} \xrightarrow{1:1} D_0''$$

A point $[(X;p,q)] \in \overline{M}_{1,2}$ is mapped to the point parametrising the following prym curve $(X';\mathcal{L})$: The underlying quasistable curve X' is obtained by gluing the points p and q. There are two ways to glue the fibres of the trivial bundle of X over the points p and q such that a prym bundle on X' is obtained. One way yields the trivial bundle on X', the other one yields the non-trivial prym bundle $\mathcal{L}$.

$$\tau_0^r : \overline{R}_{1,2}^{(-1,-1)} \xrightarrow{1:1} D_0^r$$

A point $[(X;\mathcal{L};p,q)] \in \overline{R}_{1,2}^{(-1,-1)}$ is mapped to the point parametrising the following prym curve $(X';\mathcal{L}')$: The underlying quasistable curve X' is obtained by gluing the points p and q, and then blowing up the node. $\mathcal{L}'$ is the prym bundle on X, such that if $\widetilde{X}$ is the non-exceptional subcurve of X and E the exceptional component, $\mathcal{L}'_{|\widetilde{X}} \cong \mathcal{L}$ and $\mathcal{L}'_{|E} \cong \mathcal{O}_E(1)$.

$$\tau_0' : \overline{M}_{0,([2],[2],[1])} \xrightarrow{1:1} D_0'$$

The morphism $h_0' : \overline{M}_{0,5} \to D_0'$ factors through the moduli space of genus 0 curves with sorted marked points $\overline{M}_{0,([2],[2],[1])}$ (cf. Def. 2.4 for this notation), and we use this to define τ_0'.

For $\overline{S}_2^+$ we will use the following morphisms.

$$\rho_0^\alpha : \overline{S}_{1,2}^{(1,1)} \xrightarrow{1:1} A_0^+$$

A point $[(X;p,q;\mathcal{L},b)] \in \overline{S}_{1,2}^{1,1}$ is mapped to the point parametrising the following spin curve $(X';\mathcal{L}')$: The underlying quasistable curve X' is obtained by gluing the points p and q. There are two ways to glue the fibres of the bundle $\mathcal{L}$ of X over the points p and q such that a spin bundle on X' is obtained. One way yields an odd bundle, the other one the even bundle $\mathcal{L}'$. (This is implicit in [Cor89], Example 3.2)

$$\rho_0^\beta : \overline{S}^+_{1,2} \xrightarrow{1:1} B_0^+$$

Defined analogously to τ_0^r.

$$\rho_1^\alpha : \overline{S}^+_{1,1} \times \overline{S}^+_{1,1} \xrightarrow{2:1} A_1^+$$

Defined analogously to τ_1, but the node is blown up.

$$\rho_1^\beta : \overline{S}^-_{1,1} \times \overline{S}^-_{1,1} \xrightarrow{2:1} B_1^+$$

Defined analogously to ρ_1^α

For $\overline{S}_2^-$ there are the following morphisms.

$$\eta_0^\alpha : \overline{S}^{(1,1)}_{1,2} \xrightarrow{1:1} A_0^-$$

Defined analogously to ρ_0^α.

$$\eta_0^\beta : \overline{S}^-_{1,2} \xrightarrow{1:1} B_0^-$$

Defined analogously to ρ_0^β.

$$\eta_1^\alpha : \overline{S}^+_{1,1} \times \overline{S}^-_{1,1} \xrightarrow{1:1} A_1^-$$

Defined analogously to ρ_1^α.

Now we gather facts about some of the moduli spaces of pointed curves that the domains of the morphisms just defined consist of. Especially this will be facts about the rational Chow groups of these spaces.

1. $\overline{M}_{1,1}$ has only one boundary divisor: $\widetilde{\Delta}_0$. It parametrises curves with one node. The corresponding Q-class we call $\widetilde{\delta}_0 := [\widetilde{\Delta}_0]_Q$.

2. $\overline{R}_{1,1}$ has boundary divisors $\widetilde{D_0''}$ and $\widetilde{D_0^r}$, defined analogously to D_0'' and D_0^r. The corresponding Q-classes we call $\widetilde{d_0''}$ and $\widetilde{d_0^r}$. $\overline{R}_{1,1}$ is isomorphic to $\mathbb{P}^1$, thus $\widetilde{d_0''} = \widetilde{d_0^r}$ in the Chow group.

3. $\overline{M}_{1,2}$ has boundary divisors $\widehat{\Delta}_0$ and $\widehat{\Delta}_1$. A curve parametrised by a general point of $\widehat{\Delta}_0$ is irreducible with one node. A general curve parametrised by $\widehat{\Delta}_1$ consists of two irreducible components, one smooth elliptic curve and one smooth rational curve with two marked points. The corresponding Q-classes we call $\widehat{\delta}_0$ and $\widehat{\delta}_1$.

4. $\overline{R}_{1,2}$ has boundary divisors $\widehat{D}_0''$, $\widehat{D}_0^r$ and $\widehat{D}_1$. Where $\widehat{D}_0''$ and $\widehat{D}_0^r$ are defined analogously to D_0'' and D_0^r. For a prym curve $(X; p, q; \mathcal{L}, b)$ parametrised by a general point of $\widehat{D}_1$, X consists of two irreducible components, one smooth elliptic curve and one

smooth rational curve with two marked points. The prym sheaf $\mathcal{L}$ is non-trivial restricted to the elliptic curve and (necessarily) trivial restricted to the rational curve. The Q-classes $\widehat{d_0''}$ and $\widehat{d_0^r}$ are equivalent in the Chow group, because they are the pullbacks of the corresponding classes on $\overline{R}_{1,1}$.

5. $\overline{S}^-_{1,1}$ and $\overline{S}^-_{1,2}$ are just $\overline{M}_{1,1}$ respectively $\overline{M}_{1,2}$ because an odd prym sheaf on a genus 1 curve is trivial. **In later computations, we will usually replace $\overline{S}^-_{1,1}$ and $\overline{S}^-_{1,2}$ by $\overline{M}_{1,1}$ respectively $\overline{M}_{1,2}$ without further mentioning it**.

6. $\overline{S}^+_{1,1}$: The boundary divisors are $\widetilde{A}^+_0$ and $\widetilde{B}^+_0$. Defined analogously to A^+_0 and B^+_0. The corresponding Q-classes $\widetilde{\alpha}^+_0$ and $\widetilde{\beta}^+_0$ are equivalent in the Chow group, since $\overline{S}^+_{1,1} \cong \mathbb{P}^1$.

7. $\overline{S}^+_{1,2}$: The boundary divisors are $\widehat{A}^+_0$, $\widehat{B}^+_0$ and $\widehat{A}^+_1$. The Q-classes $\widehat{\alpha}^+_0$ and $\widehat{\beta}^+_0$ are equivalent in the Chow group, since they are the pullbacks of the corresponding classes on $\overline{S}_{1,1}$.

8. $\overline{S}^{(1,1)}_{1,2}$: There are, among others, the boundary divisors $\check{A}_0$ and $\check{B}_0$ whose general points parametrise irreducible curves with one node that is blown up in the case of $\check{B}_0$. The Q classes $\check{\alpha}_0$ and $\check{\beta}_0$ are not equivalent.

The facts listed above are probably all known (for some of them cf. [BF09a], Page 8, and [BF09b]). One way of proving them is to use that the moduli spaces of curves with one marked points which appear in the list are all isomorphic to certain quotients of $\overline{M}_{0,4}$. The moduli spaces of curves with two marked points appearing are, after forgetting the order of the two marked points, isomorphic to certain quotients of $\overline{M}_{0,5}$. For an example look at Part (ii) of the following Lemma. Forgetting the order of the two marked points on the genus 1 curves does not change the coarse moduli spaces.

Lemma 3.5 *(i) Define the morphism*

$$\pi_{(2,2,1)} : \overline{M}_{0,5} \xrightarrow{4:1} \overline{M}_{0,([2],[2],[1])}, \quad [(X;(p_1,....,p_4,p_0))] \mapsto [(X;(\{p_1,p_2\},\{p_3,p_4\},\{p_0\}))],$$

and let $a \in \{1,2\}$ and $b \in \{3,4\}$ be arbitrary. We define

$$C'' := \pi_{(2,2,1)}([1,2]), \quad C' := \pi_{(2,2,1)}([3,4]), \quad C^r := \pi_{(2,2,1)}([a,b]),$$

$$C_{1:1} := \pi_{(2,2,1)}([a,0]), \quad C_1 := \pi_{(2,2,1)}([b,0])$$

These images are independent of the choice of a and b, which implies that the moduli space $\overline{M}_{0,([2],[2],[1])}$ has exactly the five boundary divisors C', C'', C^r, C_1 and $C_{1:1}$. Denote the Q-classes by by $c', c'', c^r, c_1, c_{1:1}$.

(ii) There is an isomorphism $\overline{M}_{0,[4,1]} \to \overline{M}_{1,[2]} \cong \overline{M}_{1,2}$. By combining this with the forgetful morphism $\overline{M}_{0,([2],[2],[1])} \to \overline{M}_{0,[4,1]}$ we define a finite surjective morphism θ : $\overline{M}_{0,([2],[2],[1])} \xrightarrow{6:1} \overline{M}_{1,2}$

Proof: (i): Easy to check. (For the notation used, cf. 1.47.)

(ii): To a point $[(D;\{q_1,...,q_4\},p)] \in \overline{M}_{0,[4,1]}$, let $f : Y \to D$ be the admissible double cover of $(D;\{q_1,...,q_4\})$, and let Q be the set $f^{-1}(p)$. Then $[(D;\{q_1,...,q_4\},p)] \mapsto [(Y;Q)]$ defines a morphism $\theta' : \overline{M}_{0,[4,1]} \to \overline{M}_{1,[2]} \cong \overline{M}_{1,2}$. It is easy to check that it is 1:1 on the locus of smooth curves. Since both moduli spaces are normal projective varieties this suffices to prove that θ' is an isomorphism. □

Lemma 3.6 *The following table shows the pushforwards of several classes by the morphisms defined in this section.*

Morphism	***class***	***Pushforward***
τ_0'	1	$2d_0'$
τ_0'	c'	$2[E^{\prime,\prime}]_Q$
τ_0'	c''	$d_0'd_0''$
τ_0'	c^r	$2d_0'd_0^r$
τ_0'	c_1	$4d_0'd_1$
τ_0'	$c_{1:1}$	$4d_0'd_{1:1}$
τ_0''	1	$2d_0''$
τ_0''	$\widehat{\delta}_0$	$2d_0'd_0''$
τ_0''	$\widehat{\delta}_1$	$2d_0'd_1$
τ_1	$\widetilde{d}_0'' \otimes 1$	$d_0''d_1$
τ_1	$\widetilde{d}_0'' \otimes 1$	$d_0''d_1$
τ_1	$\widetilde{d}_0^r \otimes 1$	$d_0^r d_1$
τ_1	$1 \otimes \widetilde{\delta}_0$	$d_0'd_1$
$\tau_{1:1}$	$\widetilde{d}_0'' \otimes 1$	$d_0'd_{1:1}$
$\tau_{1:1}$	$1 \otimes \widetilde{d}_0''$	$d_0'd_{1:1}$
$\tau_{1:1}$	$\widetilde{d}_0^r \otimes 1$	$d_0^r d_{1:1}$
$\tau_{1:1}$	$1 \otimes \widetilde{d}_0^r$	$d_0^r d_{1:1}$

Morphism	***class***	***Pushforward***
ρ_0^α	1	$2\alpha_0^+$
ρ_0^α	$\breve{\alpha}_0$	$4[C^+]_Q$
ρ_0^α	$\breve{\beta}_0$	$2\alpha_0^+\beta_0^+$
ρ_0^β	1	$2\beta_0^+$
ρ_0^β	$\widehat{\alpha}_0^+$	$2\alpha_0^+\beta_0^+$
ρ_0^β	$\widehat{\beta}_0^+$	$4[E]_Q$
ρ_1^α	$\widetilde{\alpha}_0^+ \otimes 1$	$2\alpha_0^+\alpha_1^+$
ρ_1^α	$1 \otimes \widetilde{\alpha}_0^+$	$2\alpha_0^+\alpha_1^+$
ρ_1^α	$\widetilde{\beta}_0^+ \otimes 1$	$2\beta_0^+\alpha_1^+$
ρ_1^α	$1 \otimes \widetilde{\beta}_0^+$	$2\beta_0^+\alpha_1^+$
ρ_1^β	$\widetilde{\delta}_0 \otimes 1$	$2\alpha_0^+\beta_1^+$
ρ_1^β	$1 \otimes \widetilde{\delta}_0$	$2\alpha_0^+\beta_1^+$

η_0^α	1	$2\alpha_0^-$
η_0^α	$\breve{\alpha}_0$	$4[C^-]_Q$
η_0^α	$\breve{\beta}_0$	$2\alpha_0^-\beta_0^-$
η_0^β	1	$2\beta_0^-$
η_0^β	$\widehat{\delta}_0$	$2\alpha_0^+\beta_0^+$
η_1^α	$\widetilde{\alpha}_0^+ \otimes 1$	$2[X^-]_Q$
η_1^α	$\widetilde{\beta}_0^+ \otimes 1$	$2\beta_0^-\alpha_1^-$
η_1^α	$1 \otimes \widetilde{\delta}_0$	$2[Y^-]_Q$

Proof: By counting the degree of the given morphism on the given cycle, and comparing the automorphism number of an object parametrised by a general point of the cycle, with the automorphism number of the object parametrised by the image of such a point, under the given morphism. □

3.1.3 Hodge classes

Another type of cycle classes used in our computation, beside boundary cycle classes, are first Chern classes of the Hodge bundles on moduli spaces, and their pullbacks.

Definition 3.7 Let $\widetilde{\pi}_R : \overline{R}_{1,1} \longrightarrow \overline{M}_{1,1}$, $\widetilde{\pi}^+ : \overline{S}_{1,1}^+ \longrightarrow \overline{M}_{1,1}$, $\widehat{\pi}^+ : \overline{S}_{1,2}^+ \longrightarrow \overline{M}_{1,2}$, and $\breve{\pi} : \overline{S}_{1,2}^{(1,1)} \longrightarrow \overline{M}_{1,2}$ be the usual forgetful morphisms, and let $\theta : \overline{M}_{0,([2],[2],[1])} \to \overline{M}_{1,2}$ be the morphism of Lemma 3.5 (ii). Let λ, $\widetilde{\lambda}$ resp. $\widehat{\lambda}$ be the first Chern class of the Hodge bundle on $\overline{M}_2$, $\overline{M}_{1,1}$ resp. $\overline{M}_{1,2}$.

We define classes:

$$l := (\pi_R)^*\lambda, \quad l^+ := (\pi_+)^*\lambda, \quad l^- := (\pi_-)^*\lambda, \quad \widetilde{l} := (\widetilde{\pi}_R)^*\widetilde{\lambda},$$

$$\widetilde{l}^+ := (\widetilde{\pi}^+)^*\widetilde{\lambda}, \quad \widehat{l}^+ := (\widehat{\pi}^+)^*\widehat{\lambda}, \quad \breve{l} := (\breve{\pi})^*\widehat{\lambda}, \quad \bar{l} := \theta^*\widehat{\lambda}$$

Lemma 3.8 *We can describe the pullbacks of l, l^+ and l^- by the boundary morphisms in the following way*

(i) $(\tau_1)^*l = \widetilde{\lambda} \otimes 1 + 1 \otimes \widetilde{l}$

(ii) $(\tau_{1:1})^*l = \widetilde{l} \otimes 1 + 1 \otimes \widetilde{l}$

(iii) $(\tau_0')^*l = \bar{l}$

(iv) $(\tau_0'')^*l = \widehat{\lambda}$

(v) $(\rho_0^\alpha)^*l^+ = \breve{l}$

(vi) $(\rho_0^\beta)^*l^+ = \widehat{l}^+$

(vii) $(\rho_1^\alpha)^*l^+ = \widetilde{l}^+ \otimes 1 + 1 \otimes \widetilde{l}^+$

(viii) $(\rho_1^\beta)^*l^+ = \widetilde{\lambda} \otimes 1 + 1 \otimes \widetilde{\lambda}$

(ix) $(\eta_0^\alpha)^*l^- = \breve{l}$

(x) $(\eta_0^\beta)^*l^- = \widehat{\lambda}$

(xi) $(\eta_1^\alpha)^* l^- = \widetilde{\lambda} \otimes 1 + 1 \otimes \widetilde{l}^+$

Proof: First consider the commutative diagram

$$\begin{array}{ccc} \overline{S}_{1,2}^{(1,1)} & \xrightarrow{\rho_0^\alpha} & \overline{S}_2^+ \\ \downarrow{\scriptstyle \check{\pi}} & & \downarrow{\scriptstyle \pi_+} \\ \overline{M}_{1,2} & \xrightarrow{f} & \overline{M}_2 \end{array}$$

where f is the morphism corresponding to gluing the two marked points on a curve. Because of the way l^+ and $\check{l}$ are defined, it suffices to show $\widehat{\lambda} = f^*\lambda$ in order to prove (v). That this equation indeed is true, is shown in [Mum83], section 10. [3] The assertions (iii), (iv), (vi), (ix) and (x) can be proved in the same way.

Now we consider the commutative diagram:

$$\begin{array}{ccc} \overline{R}_{1,1} \times \overline{R}_{1,1} & \xrightarrow{\tau_{1:1}} & \overline{R}_2 \\ \downarrow{\scriptstyle \widetilde{\pi}_R \times \widetilde{\pi}_R} & & \downarrow{\scriptstyle \pi_R} \\ \overline{M}_{1,1} \times \overline{M}_{1,1} & \xrightarrow{g} & \overline{M}_2 \end{array}$$

Where g is the morphism corresponding to gluing two genus 1 curves, each with one marked point, together at those marked points. In [Mum83], section 10, $g^*\lambda = \widetilde{\lambda} \otimes 1 + 1 \otimes \widetilde{\lambda}$ is proven (there the notation is slightly different). From this (i) follows. (ii), (vii), (viii) and (xi) can be proved analogously.

□

If λ is the first Chern class of the Hodge bundle on a $\overline{M}_{1,n}$, $n \geq 1$ arbitrary, then for δ_0 the Q class of the divisor of $\overline{M}_{1,n}$ parametrising irreducible curves with one node, $\lambda = \frac{1}{12}\delta_0$ (cf. [BF09a] Page 8). By pulling these relations back one obtains the following equations:

Lemma 3.9

(i) $\widetilde{\lambda} = \frac{1}{12}\widetilde{\delta_0}$

(ii) $\widehat{\lambda} = \frac{1}{12}\widehat{\delta_0}$

(iii) $\bar{l} = \frac{1}{12}\theta^*\widehat{\delta_0} = \frac{1}{12}(2c' + 2c'' + 2c^r)$, *with* c', c'', c^r *as defined in Lemma 3.5 (ii).*

(iv) $\widetilde{l} = \frac{1}{12}\widetilde{\pi}_R^*\widetilde{\delta_0} = \frac{1}{12}(\widetilde{d_0''} + 2\widetilde{d_0^r}) = \frac{1}{4}\widetilde{d_0^r}$

(v) $\check{l} = \frac{1}{12}\check{\pi}^*\widehat{\delta_0} = \frac{1}{12}(\check{\alpha}_0 + 2\check{\beta}_0)$

(vi) $\widetilde{l}^+ = \frac{1}{12}(\widetilde{\pi}^+)^*\widetilde{\delta_0} = \frac{1}{12}(\widetilde{\alpha}_0^+ + 2\widetilde{\beta}_0^+) = \frac{1}{4}\widetilde{\alpha}_0^+$

(vii) $\widehat{l}^+ = \frac{1}{12}(\widehat{\pi}^+)^*\widehat{\delta_0} = \frac{1}{12}(\widehat{\alpha}_0^+ + 2\widehat{\beta}_0^+) = \frac{1}{4}\widehat{\alpha}_0^+$

Lemma 3.10 *All the following products are equal to* 0 *in the rational Chow rings they are contained in.*

[3] In [Mum83], Mumford works with morphisms of stacks, so the pullbacks computed there coincide with the adjusted pullbacks we use (cf. Summary 1.34).

$$l^2 d_0', \quad l^2 d_0'', \quad l^2 d_0^r, \quad (l^+)^2\alpha_0^+, \quad (l^+)^2\beta_0^+, \quad (l^-)^2\alpha_0^-, \quad (l^-)^2\beta_0^-$$

Proof: Take for example $(l^+)^2\alpha_0^+$. Using the boundary morphism $\rho_0^\alpha : \overline{S}_{1,2}^{(1,1)} \xrightarrow{1:1} A_0^+$ and the fact that $\alpha_0^+ = \frac{1}{2}(\rho_0^\alpha)_*(1)$ we can write $(l^+)^2\alpha_0^+$ by the projection formula as $\frac{1}{2}(\rho_0^\alpha)_*(\rho_0^\alpha)^*(l^+)^2$. According to Lemma 3.8 $(\rho_0^\alpha)^*(l^+) = \check{l}$, thus $(l^+)^2\alpha_0^+ = \frac{1}{2}(\rho_0^\alpha)_*(\check{l})^2$. By definition $\check{l} = (\check{\pi})^*\widehat{\lambda}$. But $\widehat{\lambda}$ is, as shown in [Mum83] section 10, equal to the pullback of $\widetilde{\lambda}$ from $\overline{M}_{1,1}$ to $\overline{M}_{1,2}$. $\overline{M}_{1,1}$ is one dimensional, thus $(\widetilde{\lambda})^2 = 0$. This implies $(\check{l})^2 = 0$, which pushed forward by ρ_0^α yields $(l^+)^2\alpha_0^+ = 0$. That the other products listed in the Lemma are equal to 0 can be proved analogously. □

3.2 Computation of the rational cohomology

3.2.1 The rational Picard group

Lemma 3.11 *The rational Chow groups $A^1(\overline{R}_2)$, $A^1(\overline{S}_2^+)$ and $A^1(\overline{S}_2^-)$, are isomorphic to the rational Picard groups $\mathrm{Pic}_{\mathbb{Q}}(\overline{R}_2)$, $\mathrm{Pic}_{\mathbb{Q}}(\overline{S}_2^+)$ resp. $\mathrm{Pic}_{\mathbb{Q}}(\overline{S}_2^-)$, and they are generated by the boundary divisors of the moduli spaces. Furthermore the linear relations of Lemma 3.3 are the only ones. Thus:*

(i) $A^1(\overline{R}_2) = (d_0'\mathbb{Q} \oplus d_0''\mathbb{Q} \oplus d_0^r\mathbb{Q} \oplus d_1\mathbb{Q} \oplus d_{1:1}\mathbb{Q})/(d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1})\mathbb{Q}$

(ii) $A^1(\overline{S}_2^+) = (\alpha_0^+\mathbb{Q} \oplus \beta_0^+\mathbb{Q} \oplus \alpha_1^+\mathbb{Q} \oplus \beta_1^+\mathbb{Q})/(3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+)\mathbb{Q}$

(iii) $A^1(\overline{S}_2^-) = \alpha_0^-\mathbb{Q} \oplus \beta_0^-\mathbb{Q} \oplus \alpha_1^-\mathbb{Q}$

Proof: That the Chow groups of codimension 1 cycles are generated by boundary divisors and are isomorphic to the rational Picard groups is a special case of Corollary 2.15 (iv) resp. (iii).

It remains to show that there are no linear relations between the boundary divisor classes other than those of lemma 3.3.

To do this we compute the intersection numbers of all boundary divisor classes with all classes of codimension 2 boundary cycles. The latter are the cycles lying above the cycles Δ_{00} and Δ_{01} of $\overline{M}_2$ with respect to the forgetful morphisms. Look at the tables in section 2.5 for a list of them. For a codimension 1 cycle D and a codimension 2 cycle E we take the intersection number to be the number n such that $D \cdot E = n[x]$ where x is a general point of the moduli space. Note that in the definition we use the class $[x]$, not $[x]_Q$, to be consistent with [Mum83]. For $\overline{R}_2$ we get the intersection numbers:

Underlying cyle of $\overline{M}_2$	cycle class	d_0'	d_0''	d_0^r	d_1	$d_{1:1}$
Δ_{00}	$[E^{\prime,\prime}]_Q$	$-\frac{1}{2}$	$\frac{1}{4}$	0	0	$\frac{1}{8}$
Δ_{00}	$[E^{\prime,\prime\prime}]_Q$	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	0
Δ_{00}	$[E^{\prime,r}]_Q$	-1	0	0	$\frac{1}{4}$	$\frac{1}{4}$
Δ_{00}	$[E^{r,r}]_Q$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$\frac{1}{8}$
Δ_{01}	$[F_1']_Q$	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{48}$	0
Δ_{01}	$[F_1'']_Q$	$\frac{1}{4}$	0	0	$-\frac{1}{48}$	0
Δ_{01}	$[F_1^r]_Q$	$\frac{1}{4}$	0	0	$-\frac{1}{48}$	0
Δ_{01}	$[F_{1:1}']_Q$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{3}{48}$
Δ_{01}	$[F_{1:1}^r]_Q$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{3}{48}$

If we have a linear relation $\alpha_1 d_0' + \alpha_2 d_0'' + \alpha_3 d_0^r + \alpha_4 d_1 + \alpha_5 d_{1:1} = 0$ between the boundary components, the vector $\alpha = (\alpha_1, ..., \alpha_5)$ has to lie in the kernel of the 9×5 matrix formed by the intersection numbers in the table above. One can check, that this matrix has rank 4 and thus has 1-dimensional kernel, and that the relation $d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1}$ indeed lies in its kernel.

For $\overline{S}_2^+$ the intersection numbers are:

Underlying cycle of $\overline{M}_2$	cycle class	α_0^+	β_0^+	α_1^+	β_1^+
Δ_{00}	$[C^+]_Q$	-1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$
Δ_{00}	$[D^+]_Q$	0	$-\frac{1}{4}$	$\frac{1}{8}$	0
Δ_{00}	$[E]_Q$	0	$-\frac{1}{8}$	$\frac{1}{16}$	0
Δ_{01}	$[X^+]_Q$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{192}$	0
Δ_{01}	$[Y^+]_Q$	$\frac{1}{8}$	0	0	$-\frac{1}{192}$
Δ_{01}	$[Z^+]_Q$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{192}$	0

One can check that the 6×4 matrix formed by the intersection numbers, has rank 3, and that $3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+$ lies inside the kernel.

For $\overline{S}_2^-$ the intersection numbers are:

Underlying cycle of $\overline{M}_2$	cycle class	α_0^-	β_0^-	α_1^-
Δ_{00}	$[C^-]_Q$	-1	$\frac{1}{4}$	$\frac{1}{8}$
Δ_{00}	$[D^-]_Q$	0	$-\frac{1}{4}$	$\frac{1}{8}$
Δ_{01}	$[X^-]_Q$	$\frac{1}{8}$	0	$-\frac{1}{192}$
Δ_{01}	$[Y^-]_Q$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{192}$
Δ_{01}	$[Z^-]_Q$	$\frac{1}{8}$	0	$-\frac{1}{192}$

The 5×3 matrix formed by the intersection numbers has rank 3.

As examples we will compute some intersection numbers from the tables above. The other numbers can be computed analogously. From [Mum83], Theorem 10.1, we know that

$$\delta_0[\Delta_{00}]_Q = -\frac{1}{4}p, \quad \delta_1[\Delta_{00}]_Q = \frac{1}{8}p, \quad \delta_1[\Delta_{01}]_Q = -\frac{1}{48}p, \quad \delta_0[\Delta_{01}]_Q = \frac{1}{4}p, \qquad (\dagger)$$

where p is the class $[y]$ of a general point $y \in \overline{M}_2$.

For $\overline{X} \in \{\overline{R}_2, \overline{S}_2^+, \overline{S}_2^-\}$ let S be one of the codimension 2 cycles on $\overline{X}$ listed in the tables above. If $\pi : \overline{X} \to \overline{M}_2$ is the forgetful morphism, then $\pi_* S = mD$ for some $m \in \mathbb{Q}$, and for D the Q-class of the image of S under π, thus $D = [\Delta_{00}]_Q$ or $D = [\Delta_{01}]_Q$. The number m is listed for all cycles S in the tables of section 2.5. Thus one can compute the intersection number n of S with the pullback of δ_i $(i = 0, 1)$ by using the forgetful map π and the projection formula:

$$\pi^* \delta_i S = n[x] \quad \Leftrightarrow \quad \delta_i \pi_* S = n[y] = np$$

$$\Leftrightarrow \quad m\delta_i D = np$$

Where $\delta_i D$ is one of the four intersections on $\overline{M}_2$ known from (†) above.

For the example $E^{\prime,\prime}$ we have $(\pi_R)_*[E^{\prime,\prime}]_Q = [\Delta_{00}]_Q$, thus

$$\pi^* \delta_0 [E^{\prime,\prime}]_Q = -\frac{1}{4}[x] \quad \text{and} \quad \pi^* \delta_1 [E^{\prime,\prime}]_Q = \frac{1}{8}[x].$$

We also have $D_0^r \cap E^{\prime,\prime} = D_1 \cap E^{\prime,\prime} = \emptyset$ (as one can show using the description of these cycles from section 2.5), so the corresponding intersection numbers are 0. Using $(\pi_R)^* \delta_0 = d_0' + d_0'' + 2d_0^r$ and $(\pi_R)^* \delta_1 = d_1 + d_{1:1}$, we get $d_1[E^{\prime,\prime}]_Q = \frac{1}{8}[x]$ and

$$(d_0' + d_0'')[E^{\prime,\prime}]_Q = -\frac{1}{4}[x] \tag{3.1}$$

The intersection $D_0'' \cap E^{\prime,\prime} = G'$ is proper (use description of these cycles form section 2.5), so by Summary 1.34 (v) we can treat the intersection as transversal and we get $d_0''[E^{\prime,\prime}]_Q = [G']_Q$. Now G' consists of one point, and the corresponding prym curve has 4 automorphisms (cf. section 2.5), thus $d_0''[E^{\prime,\prime}]_Q = \frac{1}{4}[x]$. By plugging this into equation (3.1) we obtain the last intersection number $d_0'[E^{\prime,\prime}]_Q = -\frac{1}{2}[x]$.

All rows in the above tables can be computed in this way, except for the ones containing the intersection numbers of $E^{\prime,\prime\prime}$, $E^{\prime,r}$ and D^-. In computing the first two one has to use additionally the relation $[E^{\prime,r}]_Q = 2[E^{\prime,\prime}]_Q + [E^{\prime,\prime\prime}]_Q$. For the intersections with $[D^-]_Q$ one uses the relation $12[X^-]_Q + [C^-]_Q - 4[Y^-]_Q = [D^-]_Q$. Both relations are proven in Lemma 3.4. □

Remark: In [BF09a], Page 5-6, it is claimed that the boundary divisors of S_2^+ (and S_2^-) are independent, which results in wrong Betti (and Hodge) numbers computed for S_2^+. It is claimed that Cornalba's proof of independence of the boundary classes for genus $g \geq 3$ in [Cor89], can also be applied to $g = 2$. Cornalba's proof works similar to the proof of the lemma above by computing intersections of the boundary divisor classes with various test curves. The proof does not extend to genus 2, because some of the families used do not yield test curves in the genus 2 case but only points. (For example one family is constructed by attaching a fixed elliptic curve to a moving point on a fixed $g - 1$ curve. For genus $g = 2$ all the curves in the family are isomorphic.).

3.2.2 Hodge numbers

Theorem 3.12 *For every $\overline{X} \in \{\overline{R}_2, \overline{S}_2^+, \overline{S}_2^-\}$, the rational cohomology of $\overline{X}$ is algebraic, i.e. all odd cohomology groups vanish, and for all $n \in \mathbb{N}$ we have $H^{2n}(\overline{X}) \cong A^n(\overline{X})$ via the cycle map. Furthermore:*

(i) The boundary divisor classes generate the $\mathbb{Q}$-vector space $H^2(\overline{X})$.

(ii) There is an ample divisor L which is a linear combination of the boundary divisor classes of $\overline{X}$, such that $LH^2(\overline{X}) = H^4(\overline{X})$. Thus the products of L with the boundary divisor classes generate the $\mathbb{Q}$-vector space $H^4(\overline{X})$.

Hence the boundary divisor classes generate the $\mathbb{Q}$-algebras $H^*(\overline{X})$ and $A^*(\overline{X})$.

Proof: All except part (ii) follows as a special case from Corollary 2.15 (ii) and (iv).

Proof of (ii): $\overline{X}$ being projective, there is an ample divisor on this space. Like every divisor, according to lemma 3.11, it is equivalent to a linear combination L of boundary divisor classes. Of course L is also ample. According to the Hard Lefshetz Theorem, multiplication with L induces an isomorphism from $H^2(\overline{X})$ to $H^4(\overline{X})$. The Hard Lefshetz Theorem holds for our moduli spaces according to Summary 1.36 (iv) □

Theorem 3.13 *$\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$ all have Hodge diamonds of the following form*

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & n & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & n & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

with $n = 4$ for $\overline{R}_2$ and $n = 3$ for $\overline{S}_2^+$ as well as $\overline{S}_2^-$.

Proof: For every $\overline{X} \in \{\overline{R}_2, \overline{S}_2^+, \overline{S}_2^-\}$, $h^{2,0}(\overline{X}) = 0$ by Corollary 2.15 (v), thus, due to the symmetries of the Hodge diamond, also $h^{0,2}(\overline{X}) = 0$, $h^{1,3}(\overline{X}) = 0$ and $h^{3,1}(\overline{X}) = 0$. Theorem 3.12 then yields $h^{1,1}(\overline{X}) = h^{2,2}(\overline{X})$, and the value for $n = h^{1,1}(\overline{X})$ is given by Lemma 3.11. □

3.2.3 The cohomology rings in terms of generators and relations.

By Theorem 3.12 we know that for our moduli spaces the Chow ring and the rational cohomology ring coincide, and that they are generated by the boundary divisor classes. Now we determine the graded ring structures:

Theorem 3.14 *(i) The rational Chow ring $A^*(\overline{R}_2)$ is as a graded $\mathbb{Q}$-Algebra isomorphic to the quotient $\mathbb{Q}[d_0', d_0'', d_0^r, d_1, d_{1:1}]/I$, where I is the homogeneous ideal generated by the following (independent) elements:*

$$d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1},$$

$$d_0''d_{1:1}, \qquad d_0''d_0^r, \qquad d_1 d_{1:1},$$

$$d_1(d_0'' - d_0^r), \qquad d_{1:1}(d_0' - d_0^r), \qquad 4(d_{1:1})^2 + d_0^r d_{1:1},$$

$$2d_0'd_0'' + 4d_0'd_1 - 4d_0'd_{1:1} - d_0'd_0^r,$$

$$d_0'(d_0^r)^2, \qquad (d_0')^2 d_0''$$

(ii) $A^*(\overline{S}_2^+) \cong \mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]/J$, *where* J *is the homogeneous ideal generated by the following (independent) elements:*

$$3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+,$$

$$\alpha_1^+\beta_1^+, \qquad \beta_0^+\beta_1^+, \qquad \alpha_0^+\alpha_1^+ - \beta_0^+\alpha_1^+,$$

$$(\alpha_0^+)^2\beta_0^+, \qquad (\alpha_0^+)^2(\alpha_1^+ - \beta_1^+)$$

(iii) $A^*(\overline{S}_2^-) \cong \mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-]/K$, *where* K *is the homogeneous ideal generated by the following (independent) elements:*

$$24(\alpha_1^-)^2 + \alpha_0^-\alpha_1^- + 2\beta_0^-\alpha_1^-, \qquad 12(\beta_0^-)^2 + 24\beta_0^-\alpha_1^- + \alpha_0^-\beta_0^-,$$

$$3(\alpha_0^-)^2 - 4\alpha_0^-\beta_0^- - 8\alpha_0^-\alpha_1^- + 80\beta_0^-\alpha_1^-$$

Proof: The general idea of the proof and many of its steps are adopted from [BF09a].

The rational Chow rings of our moduli spaces are generated by the boundary divisors according to Theorem 3.12. Thus there is a surjective morphism from the quotient algebras of our Theorem to these Chow rings, if only the elements listed above as generators of the ideals of relations I, J and K, indeed are equal to zero in the rational Chow ring.

If this is shown, the following fact implies that the morphisms are even isomorphisms: The homogeneous components of the algebra $\mathbb{Q}[d_0', d_0'', d_0^r, d_1, d_{1:1}]/I$ have $\mathbb{Q}$-vector space dimensions $1, 4, 4, 1, 0, 0, \ldots$, whereas the homogeneous components of $\mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]/J$ and $\mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-]/K$ have dimensions $1, 3, 3, 1, 0, 0, \ldots$, as one can check using a computer algebra system like Macaulay 2. These are exactly the vector space dimensions of the homogeneous components of the rational Chow rings (according to theorem 3.13).

To prove most of the relations, we will use the finite morphisms onto boundary divisors described in section 3.1.2. By these morphisms we will push forward classes and relations. Many of the relations we will push forward are already described in section 3.1.2. Push-forwards of boundary cycles are listed in the tables of Lemma 3.6. In the computations we will use these facts without mentioning that we take them from section 3.1.2.

First we prove the relations for $\overline{R}_2$.

The linear relation

$$d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1} = 0 \tag{3.2}$$

holds by Lemma 3.3.

A prym curve corresponding to a point in D_0'' can not correspond to a point in $D_{1:1}$. The preimage of such a point under $\tau_0'' : \overline{M}_{1,2} \longrightarrow D_0''$, would have to correspond to a reducible curve. Such a curve is of the following form: It consists of a component D of genus 1, and a component $E \cong \mathbb{P}^1$ with two marked points on it. D and E meet in one node. The prym curve generated by gluing the marked points has a genus 1 component corresponding to D. Restricted to this component its prym sheaf is trivial. The prym curve can thus not correspond to a point in $D_{1:1}$. So $D_0'' \cap D_{1:1} = \emptyset$, and:

$$d_0'' d_{1:1} = 0 \tag{3.3}$$

Similarly one can prove

$$d_0'' d_0^r = 0 \tag{3.4}$$

and

$$d_1 d_{1:1} = 0 \tag{3.5}$$

Now we use the morphism $\tau_1 : \overline{M}_{1,1} \times \overline{R}_{1,1} \longrightarrow D_1$. In $A^1(\overline{R}_{1,1})$ the relation $\widetilde{d_0''} = \widetilde{d_0^r}$ holds. Thus we also have $1 \otimes \widetilde{d_0''} = 1 \otimes \widetilde{d_0^r}$ in $A^1(\overline{M}_{1,1} \times \overline{R}_{1,1})$. Pushing this forward by τ_1 one gets:

$$(\tau_1)_*(1 \otimes \widetilde{d_0''}) = (\tau_1)_*(1 \otimes \widetilde{d_0^r})$$
$$\Leftrightarrow \quad d_1 d_0'' = d_1 d_0^r$$

$$\Leftrightarrow \quad d_1(d_0'' - d_0^r) = 0 \tag{3.6}$$

Similarly, but using $\tau_{1:1} : \overline{R}_{1,1} \times \overline{R}_{1,1} \longrightarrow D_{1:1}$, we get:

$$d_{1:1}(d_0' - d_0^r) = 0 \tag{3.7}$$

According to [Mum83], page 321, in $A^*(\overline{M}_2)$ the relation $10\lambda = \delta_0 + 2\delta_1$ holds. Pulling this back by π_R to $\overline{R}_2$ one gets:

$$l = \frac{1}{10}(d_0' + d_0'' + 2d_0^r + 2d_1 + 2d_{1:1}) \tag{3.8}$$

Multiplying equation (3.8) with $d_{1:1}$ and using equations (3.5), (3.3) and (3.7) yields:

$$d_{1:1} l = \frac{1}{10}(3d_{1:1}d_0^r + 2(d_{1:1})^2) \tag{3.9}$$

On the other hand, because of $d_{1:1} = \frac{1}{2}(\tau_{1:1})_*(1))$ we can write $d_{1:1}l = \frac{1}{2}(\tau_{1:1})_*((\tau_{1:1})^*l)$ by the projection formula. According to the Lemmas 3.8 and 3.9

$$(\tau_{1:1})^*l = \widetilde{l} \otimes 1 + 1 \otimes \widetilde{l} = \frac{1}{4}(\widetilde{d_0^r} \otimes 1) + \frac{1}{4}(1 \otimes \widetilde{d_0^r})$$

We use $d_{1:1}d_0^r = (\tau_{1:1})_*(\widetilde{d_0^r} \otimes 1) = (\tau_{1:1})_*(1 \otimes \widetilde{d_0^r})$ and get:

$$\begin{aligned} d_{1:1}l &= \frac{1}{2}(\tau_{1:1})_*((\tau_{1:1})^*l) = \frac{1}{2}(\tau_{1:1})_*(\frac{1}{4}(\widetilde{d_0^r} \otimes 1) + \frac{1}{4}(1 \otimes \widetilde{d_0^r})) \\ &= \frac{1}{2}\frac{1}{4}(d_{1:1}d_0^r + d_{1:1}d_0^r) = \frac{1}{4}d_{1:1}d_0^r \end{aligned}$$

By subtracting the equation $d_{1:1}l = \frac{1}{4}d_{1:1}d_0^r$ from equation (3.9), and multiplying by 20, one gets:

$$4(d_{1:1})^2 + d_0^r d_{1:1} = 0 \tag{3.10}$$

The last codimension 2 relation

$$2d_0'd_0'' + 4d_0'd_1 - 4d_0'd_{1:1} - d_0'd_0^r \tag{3.11}$$

we have proven earlier (Lemma 3.4).

To obtain the codimension 3 relations we use that $l^2d_0' = l^2d_0'' = l^2d_0^r = 0$ (cf. Lemma 3.10).

Because of $d_0'' = \frac{1}{2}(\tau_0'')_*1$ we can write $d_0''l = \frac{1}{2}(\tau_0'')_*((\tau_0'')^*l)$. According to Lemma 3.8 and 3.9 one has

$$(\tau_0'')^*l = \widehat{\lambda} = \frac{1}{12}\widehat{\delta_0}$$

By using $d_0'd_0'' = \frac{1}{2}(\tau_0'')_*\widehat{\delta_0}$ we get

$$d_0''l = \frac{1}{2}(\tau_0'')_*(\frac{1}{12}\widehat{\delta_0}) = \frac{1}{12}d_0'd_0''$$

Thus $0 = l^2d_0'' = \frac{1}{12}ld_0'd_0'' = \frac{1}{144}(d_0')^2d_0''$, and so

$$(d_0')^2d_0'' = 0 \tag{3.12}$$

Using $d_0' = \frac{1}{2}(\tau_0')_*1$ we can write $d_0'l = \frac{1}{2}(\tau_0')_*((\tau_0')^*l)$. According to Lemma 3.8 and 3.9 one has

$$(\tau_0')^*l = \bar{l} = \frac{1}{6}(c' + c'' + c^r)$$

By using the pushforwards of Lemma 3.6 we get

$$d_0'l = \frac{1}{2}(\tau_0')_*(\frac{1}{6}(c' + c'' + c^r)) = \frac{1}{12}(2[E'^{,\prime}]_Q + d_0'd_0'' + 2d_0'd_0^r)$$

Together with the relation $2[E'^{,\prime}]_Q + d_0'd_0'' = d_0'd_0^r$ of Lemma 3.4 (iii), this yields

$$d_0'l = \frac{1}{4}d_0'd_0^r$$

Thus $0 = l^2d_0' = \frac{1}{4}ld_0'd_0^r = \frac{1}{16}d_0'(d_0^r)^2$, and so

$$d_0'(d_0^r)^2 = 0 \tag{3.13}$$

We have proven that the generators of the ideal I are indeed equal to 0 in the rational Chow ring of $\overline{R}_2$.

Now we prove the relations on $\overline{S}_2^+$. The linear relation

$$3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+ = 0 \tag{3.14}$$

holds by Lemma 3.3.

Similar to what was done for $\overline{R}_2$ above, one can show that $A_1^+ \cap B_1^+ = \emptyset$ and $B_0^+ \cap B_1^+ = \emptyset$, so we have the relations

$$\alpha_1^+\beta_1^+ = 0 \tag{3.15}$$

$$\beta_0^+\beta_1^+ = 0 \tag{3.16}$$

Proceeding like in the proof of equation (3.6) and using the morphism $\rho_1^\alpha : \overline{S}_{1,1}^+ \times \overline{S}_{1,1}^+ \longrightarrow A_1^+$ we get:

$$\alpha_1^+(\alpha_0^+ - \beta_0^+) = 0 \tag{3.17}$$

To obtain the codimension 3 relations, similar to the case of $\overline{R}_2$ we use that $\alpha_0^+(l^+)^2 = \beta_0^+(l^+)^2 = 0$ (cf. Lemma 3.10).

Because of $\beta_0^+ = \frac{1}{2}(\rho_0^\beta)_*1$ we can write $\beta_0^+l^+ = \frac{1}{2}(\rho_0^\beta)_*((\rho_0^\beta)^*l^+)$. According to Lemma 3.8 and 3.9 one has

$$(\rho_0^\beta)^*l^+ = \widehat{l}^+ = \frac{1}{4}\widehat{\alpha}_0^+$$

By using $\alpha_0^+\beta_0^+ = \frac{1}{2}(\rho_0^\beta)_*\widehat{\alpha}_0^+$ we get

$$\beta_0^+l^+ = \frac{1}{2}(\tau_0')_*(\frac{1}{4}\widehat{\alpha}_0^+) = \frac{1}{4}\alpha_0^+\beta_0^+$$

Thus $0 = \beta_0^+(l^+)^2 = \frac{1}{4}\alpha_0^+\beta_0^+l^+ = \frac{1}{16}(\alpha_0^+)^2\beta_0^+$, and so

$$(\alpha_0^+)^2\beta_0^+ = 0 \tag{3.18}$$

We would also like to make use of $\alpha_0^+(l^+)^2 = 0$, by expressing $\alpha_0^+(l^+)^2$ in a non-trivial way as a product of boundary divisor classes, but the morphism ρ_0^α does not help. We instead use equation (3.14) to write $3\alpha_0^+$ as $4\beta_0^+ + 8\alpha_1^+ - 72\beta_1^+$ and to get $0 = (4\beta_0^+ + 8\alpha_1^+ - 72\beta_1^+)(l^+)^2$. Because of $\beta_0^+(l^+)^2 = 0$ this simplifies to

$$(\alpha_1^+ - 9\beta_1^+)(l^+)^2 = 0 \tag{3.19}$$

We can write $\alpha_1^+ l^+ = \frac{1}{4}(\rho_1^\alpha)_*((\rho_1^\alpha)^* l^+)$, and here the Lemmas 3.8 and 3.9 yield

$$(\rho_1^\alpha)^* l^+ = \widetilde{l}^+ \otimes 1 + 1 \otimes \widetilde{l}^+ = \frac{1}{4}(\widetilde{\alpha}_0^+ \otimes 1 + 1 \otimes \widetilde{\alpha}_0^+)$$

By using $\alpha_0^+\alpha_1^+ = \frac{1}{2}(\rho_1^\alpha)_*(\widetilde{\alpha}_0^+ \otimes 1) = \frac{1}{2}(\rho_1^\alpha)_*(1 \otimes \widetilde{\alpha}_0^+)$ we get

$$\alpha_1^+ l^+ = \frac{1}{4}(\rho_1^\alpha)_*(\frac{1}{4}(\widetilde{\alpha}_0^+ \otimes 1 + 1 \otimes \widetilde{\alpha}_0^+)) = \frac{1}{4}\alpha_0^+\alpha_1^+$$

Analogously, from $\beta_1^+ l^+ = \frac{1}{4}(\rho_1^\beta)_*((\rho_1^\beta)^* l^+)$ we get to

$$\beta_1^+ l^+ = \frac{1}{4}(\rho_1^\beta)_*(\frac{1}{12}(\widetilde{\alpha}_0^+ \otimes 1 + 1 \otimes \widetilde{\alpha}_0^+)) = \frac{1}{12}\alpha_0^+\beta_1^+$$

By using $\alpha_1^+ l^+ = \frac{1}{4}\alpha_0^+\alpha_1^+$ and $\beta_1^+ l^+ = \frac{1}{12}\alpha_0^+\alpha_1^+$ one can now rewrite equation (3.19)

$$0 = (\alpha_1^+ - 9\beta_1^+)(l^+)^2 = \alpha_0^+(\frac{1}{4}\alpha_1^+ - 9\frac{1}{12}\beta_1^+)l^+ = (\alpha_0^+)^2(\frac{1}{16}\alpha_1^+ - 9\frac{1}{144}\beta_1^+)$$

Thus

$$(\alpha_0^+)^2(\alpha_1^+ - \beta_1^+) = 0 \tag{3.20}$$

(The codimension 3 relations computed in [BF09a], except of $(\alpha_0^+)^2\beta_0^+ = 0$, are incompatible with our results.)

Now we come to the relations on $\overline{S}_2^-$.

The relation $12(\delta_1)^2 + \delta_0\delta_1 = 0$ holds on $\overline{M}_2$ as follows directly from Theorem 10.1. of [Mum83]. Pulling this relation back by π_- yields the first relation

$$24(\alpha_1^-)^2 + \alpha_0^-\alpha_1^- + 2\beta_0^-\alpha_1^- = 0 \tag{3.21}$$

Pulling back the relation $10\lambda = \delta_0 + 2\delta_1$ by π_- one gets:

$$l^- = \frac{1}{10}(\alpha_0^- + 2\beta_0^- + 4\alpha_1^-) \tag{3.22}$$

Multiplication by β_0^- yields:

$$l^-\beta_0^- = \frac{1}{10}(\alpha_0^-\beta_0^- + 2(\beta_0^-)^2 + 4\beta_0^-\alpha_1^-) \tag{3.23}$$

On the other hand, because of $\beta_0^- = \frac{1}{2}(\eta_0^\beta)_*(1))$, we can write $\beta_0^- l^- = \frac{1}{2}(\eta_0^\beta)_*((\eta_0^\beta)^* l)$. According to the Lemmas 3.8 and 3.9

$$(\eta_0^\beta)^* l^- = \widehat{\lambda} = \frac{1}{12}\widehat{\delta}_0$$

We use $\alpha_0^-\beta_0^- = \frac{1}{2}(\eta_0^\beta)_*\widehat{\delta}_0$ and get:

$$l^-\beta_0^- = \frac{1}{2}(\eta_0^\beta)_*(\frac{1}{12}\widehat{\delta}_0) = \frac{1}{12}\alpha_0^-\beta_0^- \quad (*)$$

By subtracting the equation $\beta_0^- l^- = \frac{1}{12}\alpha_0^-\beta_0^-$ from equation (3.23), and multiplying by 60, one gets:

$$12(\beta_0^-)^2 + 24\beta_0^-\alpha_1^- + \alpha_0^-\beta_0^- \tag{3.24}$$

(In [BF09a] it is claimed that $l^-\beta_0^- = \frac{1}{6}\alpha_0^-\beta_0^-$ instead of $(*)$, from this then follows $3(\beta_0^-)^2 + 6\beta_0^-\alpha_1^- - \alpha_0^-\beta_0^-$ instead of equation (3.24).)

To get the last relation we first compute three relations containing classes that can not immediately be written as products of boundary cycle classes (for the description of the boundary cycles, cf. the tables of section 2.5). The first of these relations we take from Lemma 3.4:

$$16[X^-]_Q + [C^-]_Q - 4\alpha_0^-\alpha_1^- - \alpha_0^-\beta_0^- = 0 \tag{3.25}$$

In $A^1(\overline{S}_{1,1}^+)$ the relation $\widetilde{\alpha}_0^+ = \widetilde{\beta}_0^+$ holds, which implies for $A^1(\overline{S}_{1,1}^+ \times \overline{M}_{1,1})$ the relation $\widetilde{\alpha}_0^+ \otimes 1 = \widetilde{\beta}_0^+ \otimes 1$. Pushing this forward by the morphism $\eta_1^\alpha : \overline{S}_{1,1}^+ \times \overline{M}_{1,1} \longrightarrow A_0^- \subset \overline{S}_2^-$ yields:

$$[X^-]_Q = \beta_0^-\alpha_1^- \tag{3.26}$$

(In [BF09a] the authors claim, that one can get the equation $\alpha_0^-\alpha_1^- = \beta_0^-\alpha_1^-$ instead of equation (3.26). Using the projection formula and the morphism η_1^α they obtain the equation $\alpha_0^-\alpha_1^- - (\eta_0^\alpha)_*(1 \otimes \delta_0) = \beta_0^-\alpha_1^-$. Then they claim that $(\eta_1^\alpha)_*(1 \otimes \delta_0) = \frac{1}{2}\alpha_0^-\alpha_1^-$, from which their equation would follow. If I understand them correctly, they assume that $\overline{S}_{1,1}^+ \times \Delta_0$ is mapped $1:1$ onto $A_0^- \cap A_1^-$ by η_1^α. This would be wrong. $\overline{S}_{1,1}^+ \times \Delta_0$ is only mapped onto Y^-, which is one of the two irreducible components of $A_0^- \cap A_1^-$, the other being X^-. There is no a priori reason for $[Y^-]_Q$ and $[X^-]_Q$ to be equivalent, so their equation does not follow. As one can check after computing all relations, the equation does not hold.)

By multiplying equation (3.22) with α_0^- one gets

$$l^-\alpha_0^- = \frac{1}{10}((\alpha_0^-)^2 + 2\alpha_0^-\beta_0^- + 4\alpha_0^-\alpha_1^-) \tag{3.27}$$

On the other hand, because of $\alpha_0^- = \frac{1}{2}(\eta_0^\alpha)_*(1))$, we can write $\alpha_0^- l^- = \frac{1}{2}(\eta_0^\alpha)_*((\eta_0^\alpha)^* l)$. According to the Lemmas 3.8 and 3.9

$$(\eta_0^\alpha)^* l^- = \breve{l} = \frac{1}{12}(\breve{\alpha}_0 + 2\breve{\beta}_0)$$

We use $[C^-]_Q = \frac{1}{4}(\eta_0^\alpha)_*\breve{\alpha}_0$ and $\alpha_0^-\beta_0^- = \frac{1}{2}(\eta_0^\alpha)_*\breve{\beta}_0$ to get :

$$l^-\alpha_0^- = \frac{1}{2}(\eta_0^\alpha)_*(\frac{1}{12}(\breve{\alpha}_0 + 2\breve{\beta}_0)) = \frac{1}{6}([C^-]_Q + \alpha_0^-\beta_0^-)$$

By subtracting the equation $l^-\alpha_0^- = \frac{1}{6}([C^-]_Q + \alpha_0^-\beta_0^-)$ from equation (3.27), and multiplying by 30, one gets:

$$5[C^-]_Q = 3(\alpha_0^-)^2 + \alpha_0^-\beta_0^- + 12\alpha_0^-\alpha_1^- \tag{3.28}$$

Plugging equation (3.26) into equation (3.25) yields:

$$16\beta_0^-\alpha_1^- + [C^-]_Q - 4\alpha_0^-\alpha_1^- - \alpha_0^-\beta_0^- = 0$$

By multiplying this with 5 and plunging in equation (3.28) we get

$$3(\alpha_0^-)^2 - 4\alpha_0^-\beta_0^- - 8\alpha_0^-\alpha_1^- + 80\beta_0^-\alpha_1^- = 0 \tag{3.29}$$

This is the last relation we had to check. □

Remarks: (i) One can test these relations by pulling the known relations $\delta_0\delta_1 + 12(\delta_1)^2 = 0$ and $528(\delta_1)^3 + (\delta_0)^3 = 0$ (known from [Mum83]) back from $\overline{M}_2$ to our moduli spaces and check whether they are fulfilled in the rings that Theorem 3.14 claims to be to the rational Chow rings.

(ii) While the cohomology rings of $\overline{S}_2^+$ and $\overline{S}_2^-$ have, according to our computation, the same Betti numbers, they are still non-isomorphic: Otherwise there would have to be a commutating diagram of homomorphisms of graded $\mathbb{Q}$-algebras

$$\begin{array}{ccc}
\mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-] & \longrightarrow & \mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+] \\
\downarrow & \overset{\psi}{\searrow} & \downarrow g \\
 & & \mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]/(3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+) \\
 & & \downarrow h \\
\mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-]/K & \xrightarrow{\varphi} & \mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]/J
\end{array}$$

with φ and ψ isomorphisms. This would imply that in $\mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-]$: $K = \psi^{-1}h^{-1}(0)$. But since $J = g^{-1}h^{-1}(0)$ in $\mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]$ is not generated by its elements of degree ≤ 2, the same must hold for $h^{-1}(0)$. Hence K could not be generated in degree ≤ 2 either, which would contradict our Theorem.

Chapter 4

Geometry of $\overline{R}_{1,n}$ (and $\overline{S}_{1,n}$) for small n

This chapter is concerned with properties of the coarse moduli spaces $\overline{R}_{1,n}$ and $\overline{S}_{1,n}$ for small n. We follow the PhD-thesis of Pavel Belorousski ([Bel98]) in which he computed the rational Chow ring $A^*(\overline{M}_{1,n})$ for $n \leq 4$ and showed that $\overline{M}_{1,n}$ is rational for $n \leq 10$. We will also compute the Chow ring of our moduli spaces for $n \leq 4$ and show rationality for $n \leq 6$.

Let us first remark that as varieties $\overline{S}^+_{1,n} \cong \overline{R}_{1,n}$ and $\overline{S}^-_{1,n} \cong \overline{M}_{1,n}$. This can be seen as follows: On a smooth genus 1 curve C, we have $\mathcal{O}_C = \omega_C$, and in general any invertible sheaf of degree 0 on a smooth curve is trivial if it has non-zero global sections. Hence $R_{1,n} = S^+_{1,n}$ and $S^-_{1,n} = M_{1,n}$. This identity on the interiors can be extended to the claimed isomorphisms of normal projective varieties by applying Lemma 1.45.

From now on we will only speak about $\overline{R}_{1,n}$ in this chapter, knowing that this case together with Belorousski's results on $\overline{M}_{1,n}$, also covers the case of $\overline{S}_{1,n} \cong \overline{R}_{1,n} \uplus \overline{M}_{1,n}$. But when properties of the orbifolds or stacks $\overline{R}_{1,n}$ and $\overline{S}_{1,n}$ are concerned, like in the next chapter, we have to treat both spaces separately, since the isomorphisms mentioned above do not hold on the level of orbifolds/stacks.

Notation: We always work with the Chow ring and the cohomology with rational coefficient in this chapter. $A^*(...)$ will denote the rational Chow ring, $H^*(...)$ the rational cohomology ring. We will use the shorthand $\underline{n}$ to denote the set $\{1, ..., n\}$.

Let $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ be the forgetful morphism. Since τ_n is finite and surjective, the pullback $\tau_n^* : A^*(\overline{M}_{1,n}) \to A^*(\overline{R}_{1,n})$ is injective, and we can regard $A^*(\overline{R}_{1,n})$ as an algebra over the ring $A^*(\overline{M}_{1,n})$. Now we can express the main results of this chapter as follows:

- $\overline{R}_{1,n}$ is rational for $n \leq 6$ (Corollary 4.25). (Already in [BF06], Lemma 2, it was shown, using Belorousski's results, that $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$ is uniruled for $n \leq 10$. As also shown in [BF06] this result is sharp since the Kodaira dimension of $\overline{R}_{1,n}$ is ≥ 0 for $n = 11$ and is 1 for $n \geq 12$.)

- The Kodaira dimension $\kappa(\overline{R}_{1,11})$ is 1, in contrast to $\kappa(\overline{M}_{1,11}) = 0$. For all $n \neq 11$, $\kappa(\overline{R}_{1,n})$ is computed in [BF06], and is equal to $\kappa(\overline{M}_{1,n})$, also computed there. (This result is actually part of the next chapter 5 since we use information provided there to derive it. But thematically it would better fit into this chapter.)

- $A^*(\overline{R}_n)$ as $\mathbb{Q}$ vector space is spanned by the boundary cycle classes for $n \leq 6$. (Prop. 4.26)

- We compute the $\mathbb{Q}$-algebra $A^*(\overline{R}_{1,n})$ for $n \leq 4$, in terms of generators and relations (Cor. 4.29, Thm. 4.32), and obtain in particular that:

- For $n \leq 3$ the pullback $\tau_n^* : A^*(\overline{M}_{1,n}) \to A^*(\overline{R}_{1,n})$ is an isomorphism

- The pullback $\tau_4^* : A^*(\overline{M}_{1,4}) \to A^*(\overline{R}_{1,4})$ is not surjective, and unlike $A^*(\overline{M}_{1,4})$, $A^*(\overline{R}_{1,4})$ is not generated by the boundary divisors.

Remark: The case $n = 1$ is quite trivial, since $\overline{R}_{1,1} \cong \overline{M}_{0,(2,1,1)}$, and thus $\overline{R}_{1,1}$ is a normal curve covered by $\overline{M}_{0,4} \cong \mathbb{P}^1$. Hence $\overline{R}_{1,1} \cong \mathbb{P}^1$, and we do not need to treat this case in the rest of this chapter (cf. Proposition 4.15 (i)).

We give a short sketch of the approach in this chapter:

- The rationality of $\overline{R}_{1,n}$ is obtained by constructing isomorphisms from open parts of rational parameter spaces of certain plain cubic curves to open parts of $R_{1,n}$.

- These open parts of the parameter spaces will be shown to have trivial Chow ring, which will be a main ingredient in the proof that the Chow ring $A^*(\overline{R}_{1,n})$ is generated by boundary cycle classes for $n \leq 6$.

- By Belorousski's work we know $A^*(\overline{M}_{1,n})$, for $n \leq 4$, and thus also the subspace $\tau_n^* A^*(\overline{M}_{1,n}) \subseteq A^*(\overline{R}_{1,n})$. We investigate how many boundary cycles of $\overline{R}_{1,n}$ lie above a given cycle of $\overline{M}_{1,n}$ and conclude that only special boundary cycle classes, called banana cycle classes, can possibly contribute to $A^*(\overline{R}_{1,n}) \smallsetminus \tau_n^* A^*(\overline{M}_{1,n})$.

- Then we compute relations in $A^*(\overline{R}_{1,n})$ for $n \leq 4$ involving these banana cycle classes, again using finite gluing morphisms to boundary components. For $n \leq 3$ these relations suffice to show that also all banana cycles lie in $\tau_n^* A^*(\overline{M}_{1,n})$. For $n = 4$ these relations do not suffice to put all banana cycle classes inside $\tau_4^* A^*(\overline{M}_{1,4})$, and we compute a matrix of intersection numbers to check that these relations, together with those pulled back from $\overline{M}_{1,4}$, are basically all that exist in $A^*(\overline{R}_{1,n})$.

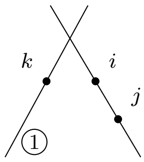
4.1 The boundary cycles, and other preliminaries

4.1.1 Boundary cycles of $\overline{M}_{1,n}$

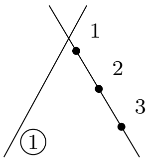
First we will introduce a notation for all the boundary cycles of $\overline{M}_{1,n}$ of dimension > 0, for $n \in \underline{4}$. This is the notation used in [Bel98], except for the few cycles which were not given a name there. First for any $n \in \mathbb{N}$, $\overline{M}_{1,n}$ has exactly the following boundary divisors: Δ_0 which is the closure of the locus of curves with one non-disconnecting node, and no other nodes. Furthermore divisors Δ_I for every subset $I \subseteq \underline{n}$ with $|I| \geq 2$, where Δ_I is the closure of the locus of curves consisting of a smooth genus 1 component and a smooth genus 0 component meeting in one node, such that the genus 0 component carries exactly those marked points with indices in I.

For $n = 2$ all boundary cycles of dimension > 0 are of course divisors. For $n = 3, 4$, we now describe each boundary cycle by a picture showing how a general curve parametrised by this cycle looks like. The kind of pictures we use here was explained in Example 1.24, except that here we apply the convention that every component without a genus number near to it is of geometric genus 0. The number in brackets behind the name of a cycle indicates how many cycles of this type exist. Note that many symbols are used for several cycles. Such a symbol only fixes a unique cycle if also the number $n \in \underline{4}$ is specified.

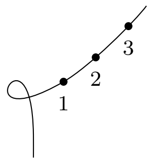
Codimension 1 boundary cycles of $\overline{M}_{1,3}$:



$\Delta_{\{ij\}}$ (3)

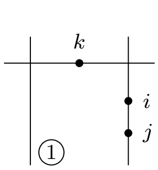


Δ_3

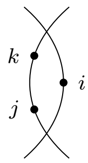


Δ_0

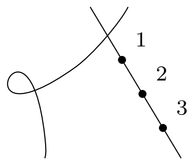
Codimension 2 boundary cycles of $\overline{M}_{1,3}$:



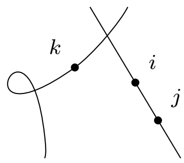
$\Delta_{\{k\{ij\}\}}$ (3)



$\Delta_{\alpha,\{i\}}$ (3)

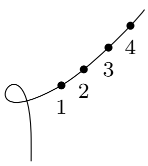


$\Delta_{0,3}$

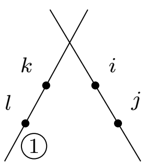


$\Delta_{0,\{ij\}}$ (3)

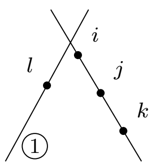
Codimension 1 boundary cycles of $\overline{M}_{1,4}$:



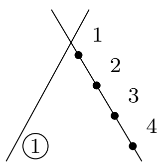
Δ_0



$\Delta_{\{ij\}}$ (6)

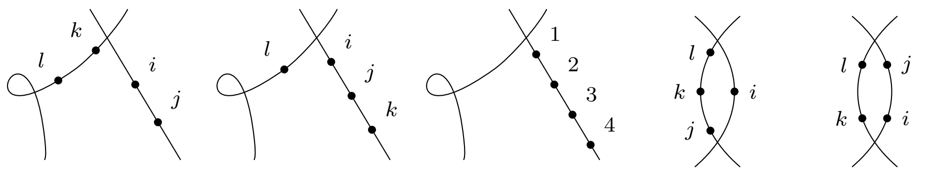


$\Delta_{\{ijk\}}$ (4)



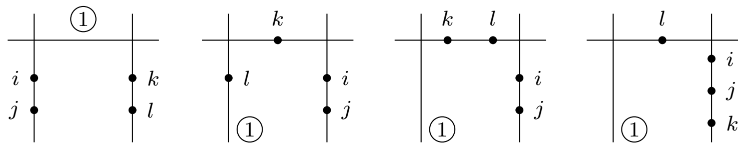
Δ_4

Codimension 2 boundary cycles of $\overline{M}_{1,4}$:



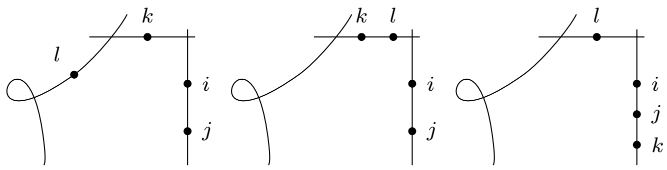
$\Delta_{0,\{ij\}}$ (6) $\Delta_{0,\{ijk\}}$ (4) $\Delta_{0,4}$ $\Delta_{\alpha,\{i\}}$ (4) $\Delta_{\beta,\{ij\}}$ (3)

Further codimension 2 boundary cycles of $\overline{M}_{1,4}$:



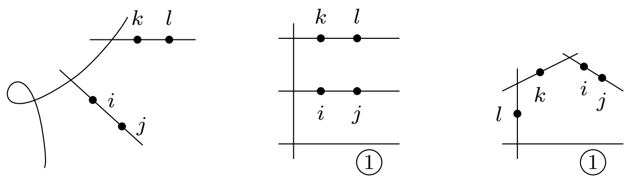
$\Delta_{\{ij\},\{kl\}}$ (3) $\Delta_{\{k\{ij\}\}}$ (12) $\Delta_{\{kl\{ij\}\}}$ (6) $\Delta_{\{l,\{ijk\}\}}$ (4)

Codimension 3 boundary cycles of $\overline{M}_{1,4}$:



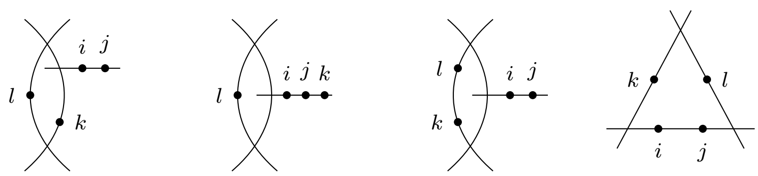
$\Delta_{0,\{k,\{ij\}\}}$ (12) $\Delta_{0,\{kl\{ij\}\}}$ (6) $\Delta_{0,\{l\{ijk\}\}}$ (4)

Further codimension 3 boundary cycles of $\overline{M}_{1,4}$:



$\Delta_{0,\{ij\},\{kl\}}$ (3) $\Delta_{4,\{ij\},\{kl\}}$ (3) $\Delta_{\{l\{k\{ij\}\}\}}$ (12)

Further codimension 3 boundary cycles of $\overline{M}_{1,4}$:



$\Delta_{\alpha,\{l\},\{ij\}}$ (12) $\Delta_{\alpha,\{l\},\{ijk\}}$ (4) $\Delta_{\beta,\{ij\},\{ij\}}$ (6) $\Delta_{\gamma,\{ij\}}$ (6)

Definition 4.1 (i) The boundary cycles of $\overline{M}_{1,n}$ parametrising curves with at least two non-disconnecting nodes, are called banana cycles. These are boundary cycles Δ_Γ belonging to a graph Γ with $h^1(\Delta) = 1$ and without self edges. It is clear that the codimension of a banana cycle is ≥ 2. Examples of banana cycles are $\Delta_{\alpha,\{i\}}$, $\Delta_{\gamma,\{ij\}}$ and $\Delta_{\beta,\{ij\},\{ij\}}$.

(ii) We call a boundary cycle Δ_Γ a *simple banana cycle* if Γ has at least two non-disconnecting edges, and has no disconnecting edges.

Let $(I_1, ...I_r)$ be a partition of $\underline{n}$. We define $B_{I_1,...,I_r}$ to be Δ_Γ, where Γ is the following stable graph: Γ has vertices $v_1, ..., v_r$. To each v_i legs with indices in I_i are attached. The graph has the form of a circuit. I.e. consider the indices $1, ..., r$ as elements of $\mathbb{Z}/r\mathbb{Z}$. Then each v_i is connected to v_{i-1} and v_{i+1} by one edge each. There are no other edges.

Every simple banana cycle is of the form $B_{I_1,...,I_r}$ for some partition of $\underline{n}$. For example $B_{\{1\},\{2\}} = \Delta_{\alpha,\{1\}} \subset \overline{M}_{1,2}$ and $B_{\{1,2\},\{3\},\{4\}} = \Delta_{\gamma,\{12\}} \subset \overline{M}_{1,4}$. $\Delta_{\beta,\{ij\},\{ij\}}$ is an example of a non-simple banana cycle.

Proposition 4.2 *Let $Z \subset \overline{M}_{1,n}$ be a boundary cycle of $\overline{M}_{1,n}$ of codimension m.*

(i) If Z is not a banana cycle, then Z is contained in exactly m different boundary divisors $D_1, ..., D_m$. Furthermore Z is the proper intersection $Z = D_1 \cap ... \cap D_m$.

(ii) If Z is a banana cycle, then there is a smallest simple *banana cycle $B_{I_1,...,I_r}$ containing Z, and except Δ_0 there are exactly $m - r$ other boundary divisors $D_1, ..., D_{m_r}$ containing Z. Furthermore Z is the proper intersection $Z = B_{I_1,...,I_r} \cap D_1 \cap ... \cap D_{m-r}$.*

(iii) In particular the subalgebra $A^*_{BCl}(\overline{M}_{1,n}) \subseteq A^*(\overline{M}_{1,n})$ (cf. Def. 1.40) is generated as $\mathbb{Q}$-algebra by the classes of boundary divisors together with the classes of simple banana cycles, for all $n \in \mathbb{N}$.

Proof: (iii) is a direct consequence of (i) together with (ii). Let $(C, p_1, .., p_n)$ be a general pointed curve parametrised by the cycle Z. Let Γ be the dual graph of this curve, i.e. $Z = \Delta_\Gamma$.

The parts (i) and (ii) are implied by:

Let r be the number of non-disconnecting nodes of C, let M be the set containing as elements all simple banana cycles of $\overline{M}_{1,n}$, and the divisor Δ_0 and $\overline{M}_{1,n}$. Then:

1. There is a smallest cycle $B \in M$ containing Z. B is of codimension r.

2. Z is contained in exactly $m-r$ different boundary divisors $D'_1, ..., D'_{m-r}$, none of which is Δ_0. $Z = B \cap D'_1 \cap ... \cap D'_{m-r}$.

We show this by induction on the codimension m. For $m = 0$, we have $Z = \overline{M}_{1,n}$, so 1. and 2. hold. For $m \geq 1$, first recall that all boundary divisors except Δ_0 are of the form Δ_I for some $I \subseteq \underline{n}$. We have $\Delta_I = \Delta_{\Gamma_I}$, where Γ_I is the following stable graph: It consists of two vertices, one of genus 1 the other of genus 0. The vertices are connected by one edge, the legs with indices in I are attached to the genus 0 vertex, the others to the genus 1 vertex. Also note that $m-r$ is the number of disconnecting nodes of C and of disconnecting edges of Γ.

We distinguish two cases. The first possible case is $r = m$. But then Z itself is an element of M, so 1. is clear. Also such a cycle can not be contained in any D_I, since the graph Γ_I

contains a disconnecting edge.

In the second case, $r < m$, Γ contains at least one vertex v connected to the rest of the graph by only one edge e. Let I be the set of indices of the legs attached to v. Then Z is contained in Δ_I. Now let $\widetilde{\Gamma}$ be the graph obtained from Γ by contracting e and melting v with the vertex it is connected to by e. Then $Z \subseteq \Delta_{\widetilde{\Gamma}}$, since Γ is a specialisation of $\widetilde{\Gamma}$. Let m' be the codimension of $\Delta_{\widetilde{\Gamma}}$, r' the number of $\widetilde{\Gamma}$'s non-disconnecting edges. Then $r' = r$ and $m' = m - 1$. By induction hypothesis $\Delta_{\widetilde{\Gamma}} = B \cap D'_1 \cap ... \cap D'_{m-r-1}$, where B is of codimension r. It is clear that Z is not contained in a cycle from M smaller than B, since such a cycle would correspond to a graph with at least $r + 1$ non-disconnecting edges. This shows 2. in this case. Also $\Delta_{\widetilde{\Gamma}}$ is not contained in Δ_I, so Δ_I is not among $D'_1, ..., D'_{m-r-1}$. Set $D'_{m-r} = \Delta_I$ then it is clear that $Z \subseteq \Delta_{\widetilde{\Gamma}} \cap \Delta_I = B \cap D'_1 \cap ... \cap D'_{m-r}$. It remains to show $\Delta_{\widetilde{\Gamma}} \cap \Delta_I \subseteq Z = \Delta_\Gamma$. But it is easy to see that every stable graph Γ' that is simultaneously a specialisation of $\widetilde{\Gamma}$ and Γ_I is also a specialisation of Γ. So the claim follows by Proposition 1.26 (iv). □

Remark: Since every boundary cycle contained in a banana cycle is also a banana cycle in our use of the word, the proposition implies that $A^*_{BCl}(\overline{M}_{1,n})$ is as a $\mathbb{Q}$ vector space spanned by products of boundary divisors together with the banana cycle classes. (This is more or less (2.12) of [Pag08].)

Lemma 4.3 *Let Z be a boundary cycle of $\overline{M}_{1,n}$, which we write in a unique way as as $Z = B \cap D'_1 \cap ... \cap D'_{m-r}$, like in the proof of Prop 4.2. Then if $B \neq \Delta_0$, Z is a normal variety.*

Proof: Let $\mathfrak{C}$ be a pointed stable curve such that $[\mathfrak{C}] \in Z \subset \overline{M}_{1,n}$. It suffices to prove that locally around any such point $[\mathfrak{C}]$, the preimage of Z on the local universal deformation space of $\mathfrak{C}$ is normal, since Z is the quotient of this preimage by a finite automorphism group (cf. Summary 1.30). We will show more by proving that this preimage is actually a linear subspace of the deformation space. Since the preimage of Z is the intersection of the preimage of B and the preimages of the D'_i on the deformation space, it will suffice to show the claim for boundary cycles Δ which are simple banana cycles, like B, or of the form Δ_I, like the D'_i. Let $\Delta = \Delta_\Gamma$ be such a boundary cycle.

Let $\Gamma(\mathfrak{C})$ be the dual curve of $\mathfrak{C} \in \Delta_\Gamma$. It is a specialisation of Γ. If we are able to show that for all contractions $c : \Gamma(\mathfrak{C}) \rightsquigarrow \Gamma$, the subset $c^{-1}(E(\Gamma)) \subseteq E(\Gamma(\mathfrak{C}))$ is the same, then our lemma will follow: If there are exactly the contractions $c_1, ..., c_r$ with $c_i : \Gamma(\mathfrak{C}) \rightsquigarrow \Gamma$, then, using the notation of Summary 1.30, the preimage of Δ is the union $\bigcup_{i=1}^r \bigcap_{e \in c^{-1}(E(\Gamma))} \{x_e = 0\}$. To see this, note that a local deformation can change the dual graph of a curve only by smoothing nodes, which on the dual graph corresponds to contracting the corresponding edges. Now $\{x_e = 0\}$ is the locus in which the node corresponding to the edge e is retained, and a deformation of $\mathfrak{C}$ leads to curves whose dual graphs are still specialisations of Γ iff it retains all nodes in at least one of the sets of nodes $c_i^{-1}(E(\Gamma))$. But such curves are exactly those parametrised by $Z = \Delta_\Gamma$. Since $\bigcap_{e \in c_i^{-1}(E(\Gamma))} \{x_e = 0\}$ is a linear subspace of the universal deformation space, the preimage of Δ is normal (and smooth) if and only if all the $c_i^{-1}(E(\Gamma))$ coincide.

What we want to show, is equivalent to showing that the sets $E_i' := E(\Gamma(\mathfrak{C})) \smallsetminus c_i^{-1}E(\Gamma)$ of edges which are contracted by the c_i are the same for all $i \in \underline{r}$. If $\Delta = \Delta_I$, then Γ has a genus 0 vertex v_0 to which legs with indices in I are attached and a genus 1 vertex v_1 to which legs with indices in $\underline{n} \smallsetminus I$ are attached. The two vertices are connected by one edge. Now each vertex v of $\Gamma(\mathfrak{C})$ either carries legs itself, or there is a rational tree hanging on v, carrying such legs, or v is of genus 1. Since contractions respect the marked legs, if the mentioned legs belong to I, v is contracted into v_0 by every c_i. If the legs belong to $\underline{n} \smallsetminus I$ v is contracted into v_1, the same if v is of genus 1. [1] Hence all the c_i act the same on the vertices of $\Gamma(\mathfrak{C})$. Since Γ contains no self-edges, an edge of $\Gamma(\mathfrak{C})$ becomes contracted by c_i, i.e. belongs to E_i', if and only if it connects two vertices, which are contracted to the same vertex of Γ by c_i. This shows that all E_i' are the same.

If Δ is a simple banana cycle instead, Γ only has vertices $v_1, ..., v_r$ of genus 0, and each v_i carries legs with indices in a subset I_i with $\emptyset \neq I_i \subset \underline{n}$. One shows that all E_I' are equal in this case analogously. □

4.1.2 Boundary cycles of $\overline{R}_{1,n}$

In this section we gather some facts about the boundary cycles of $\overline{R}_{1,n}$ and their relation to the boundary cycles of $\overline{M}_{1,n}$.

We will show later that the Chow ring of $\overline{R}_{1,n}$ for $n \leq 6$ is generated as a $\mathbb{Q}$-vector space by boundary cycle classes. We know that the same is true for $\overline{M}_{1,n}$ by Belorousski's thesis, in which also $A^*(\overline{M}_{1,4})$ for $n \leq 4$ is computed. So we already know the sub-algebra $\tau_n^* A^*(\overline{M}_{1,n})$ of $A^*(\overline{R}_{1,n})$ for $n \leq 4$. (Where $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ the forgetful morphism.)

By definition each boundary cycle of $\overline{R}_{1,n}$ lies above one boundary cycle of $\overline{M}_{1,n}$ with respect to τ_n. Only in cases where there is more than one boundary cycle of $\overline{R}_{1,n}$ lying over a given cycle of $\overline{M}_{1,n}$ we can get a contribution to $A^*(\overline{R}_{1,n})$ that does not lie inside $\tau_n^* A^*(\overline{M}_{1,n})$. So for the purpose of computing $A^*(\overline{R}_{1,n})$, we would like to know how many boundary cycles are there lying over a given cycle $\Delta = \Delta_\Gamma$ of $\overline{M}_{1,n}$. We can distinguish 3 cases, according to the type of the stable graph Γ.

Lemma 4.4 *(i) If Γ has only disconnecting edges, $\tau_n^{-1}\Delta$ is irreducible. (Examples: Δ_3, $\Delta_{\{3\{12\}\}}$)*

(ii) If Γ has exactly one non-disconnecting edge, then $\tau_n^{-1}\Delta$ has two irreducible components D'' and D^r. Here D'' parametrises prym curves supported on a stable curve C, while D^r parametrises prym curves supported on a semi-stable curve X obtained by blowing up the non-disconnecting node of a stable curve C. If we denote by δ, d'' and d^r the corresponding Q-classes, then $\tau_n^*\delta = d'' + 2d^r$. But $d'' = d^r$ in $\mathbb{A}^*(\overline{R}_{1,n})$, and thus d'' and d^r both lie in $\tau_n^* A^*(\overline{M}_{1,n})$. (Examples: Δ_0, $\Delta_{\{3,\{12\}\}}$)

(iii) In the last case Γ has two or more non-disconnecting nodes, i.e. Δ is a banana cycle*. Also in this case there are two irreducible components of $\tau_n^{-1}\Delta$. The prym curves*

[1] It is impossible that these legs at v come from I as well as $\underline{n} \smallsetminus I$, for otherwise $\Gamma(\mathfrak{C})$ could not be a specialisation of Γ.

parametrised by one component are supported on stable curves C, while the other component parametrises prym curves supported on the quasi-stable curve X obtained from some stable C by blowing up all its non-disconnecting nodes. Like in the case (ii), we call the first component D'' and the second one D^r, and the Q-classes d'' resp. d^r. Here $\tau_n^*\delta = d'' + 2^l d^r$, where l is the number of non-disconnecting nodes on C, but it is not any more true in general that $d'' = d^r$. So we do not know a priori whether d'' and d^r are contained in $\tau_n^* A^*(\overline{M}_{1,n})$. (Examples: $\Delta_{\alpha,\{1\}}$, $\Delta_{\beta,\{12\},\{12\}}$)

Before proving the Lemma, we use it to introduce a notation for the boundary cycles of $\overline{R}_{1,n}$ for $n \geq 4$.

Definition 4.5 (i) All the boundary cycles of $\overline{M}_{1,n}$, $n \leq 4$, are denoted by symbols of the form Δ_{index}, and the corresponding class is denoted by δ_{index} (cf. beginning of section 4.1.1). If $\tau_n^{-1}\Delta_{\text{index}}$ is irreducible (i.e. in case (i) of the Lemma) we denote this boundary cycle by D_{index}. If $\tau_n^{-1}\Delta_{\text{index}}$ has two irreducible components (i.e. in case (ii) and (iii) of the lemma) we call them D''_{index} and D^r_{index}, defined as in the Lemma above. For the corresponding Q-classes, we replace δ by d in the same way. For example $\tau_4^*\Delta_{\beta,\{12\}} = D''_{\beta,\{12\}} \cup D^r_{\beta,\{12\}}$ and $\tau_4^*\,\delta_{\beta,\{12\}} = d''_{\beta,\{12\}} + 4d^r_{\beta,\{12\}}$. We do the same for the simple banana cycles $B_{I_1,\dots,I_m}$, calling the two components of the preimage $B''_{I_1,\dots,I_m}$ and $B^r_{I_1,\dots,I_m}$. For the Q-classes then $\tau_n^*\,b_{I_1,\dots,I_m} = b''_{I_1,\dots,I_m} + 2^m b^r_{I_1,\dots,I_m}$ holds.

(ii) We call the boundary cycles of $\overline{R}_{1,n}$ lying over (simple) banana cycles of $\overline{M}_{1,n}$ (simple) banana cycles too.

Proof (of the Lemma): In the case (i), Γ consists of one vertex v_1 of genus $g(v_1) = 1$, to which some rational trees may be attached. By Proposition 1.26 (i) there is a finite gluing morphism $\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{1,n}$ with image $\Delta \subset \overline{M}_{1,n}$. In this case $\overline{M}_\Gamma$ can be written as

$$\overline{M}_\Gamma = \overline{M}_{1,a^{-1}(v_1)} \times \overline{M}_{rest}$$

Here $\overline{M}_{rest}$ is some product of moduli spaces of stable pointed genus 0 curves, which parametrises the rational trees. We can define a morphism

$$\zeta_\Gamma : \overline{R}_{1,a^{-1}(v_1)} \times \overline{M}_{rest} \to \overline{R}_{1,n},$$

corresponding to the following procedure: First apply the same gluing procedure on the underlying curves as for ξ_Γ. The genus 1 component of the resulting curve comes from $\overline{R}_{1,a^{-1}(v_1)}$ and is thus equipped with a non-trivial prym bundle. Endow the genus 0 components with the trivial bundle. Identify those fibres of the bundles on the different components, which lie over points that are glued together.

The image of ζ_Γ is an irreducible component of $\pi_n^{-1}\Delta$. But if $(C, p_1, ..., p_n)$ is a general curve parametrised by Δ, C consists of a smooth genus 1 component D and rational trees. Then all prym curves having $(C, p_1, ..., p_n)$ as stable model, must be of the form $[(C, p_1, ..., p_n; \mathcal{L})]$, where the prym sheaf $\mathcal{L}$ restricts to a non-trivial prym sheaf on D and to the trivial sheaf on the rest of C. (By Summary 1.13 (i), no node can be blown up, and on a rational curve no non-trivial prym sheaf exists.) Thus ζ_Γ surjects on $\pi_n^{-1}(\Delta)$, which hence is an irreducible variety.

In the cases *(ii)* and *(iii)*: Let r be the number of non-disconnecting edges ($r = 1$ is case (ii)). Then Γ contains r vertices $v_1, ..., v_r$ forming a circuit as described in Definition 4.1 (ii). This has to be so, since otherwise Γ would have to contain more than one such circuit, and would thus be of genus ≥ 2 . (In case $r = 1$ this means that there is one self edge attached to v_1.) The rest of Γ again consists of rational trees attached to the vertices $v_1, ..., v_r$. Let $(C, p_1, ..., p_n)$ be any curve parametrised by Δ. It consist of one genus 1 subcurve C_1, which only has non-disconnecting nodes, and of rational trees attached to C_1.

We again use Summary 1.13 (i). It implies that for a prym curve $(X, p_1, .., p_n, \mathcal{L}, b)$, having $(C, p_1, ..., p_n)$ as stable model, either $X = C$, or $X = C'$, where C' is obtained by blowing up all the non-disconnecting nodes of C. All irreducible non-exceptional components of X are $\mathbb{P}^1$'s. If D is a non-exceptional component meeting no exceptional component then $\mathcal{L}_{|D} = \mathcal{O}_D$. Otherwise D meets two exceptional components in points $a, b \in D$ and $\mathcal{L}_{|D}$ is a square-root of $\mathcal{O}_D(-a-b)$, i.e. $\mathcal{L}_{|D} \cong \mathcal{O}_D(-1)$. This means that once $X = C$ or $X = C'$ is fixed, then $\mathcal{L}_{|D}$ is fixed on any irreducible component. So $\mathcal{L}$ beyond that only depends on the way the bundles $\mathcal{L}_{|D}$ are glued together over the nodes of X. On the rational trees all possible ways to glue yield the trivial bundle.

In the case $X = C$ there are two non-isomorphic ways to glue the bundles on components of C_1. One yields the trivial bundle, in which case the whole bundle $\mathcal{L}$ would be trivial, which is not allowed. The other yields a non-trivial prym sheaf. Hence there is only one isomorphism class of prym curves lying over $[(C, p_1, ..., p_n)]$, with $X = C$.

In the case $X = C'$ there is only one isomorphism class of prym curves too: The only interesting part is here C'_1, the subcurve of C' obtained by blowing up all the non-disconnecting nodes of C_1. But the non-exceptional components of C'_1 are connected with each other only via exceptional components E equipped with the bundles $\mathcal{O}_E(1)$. Hence every two different ways to glue together the bundles on the components of C'_1, yield prym curves isomorphic to each other by inessential isomorphisms.

The unique prym curve supported by C is parametrised by a point of the boundary cycle D'' and the one supported on C' is parametrised by a point of D^r. Thus the morphism $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ restricted to D'' resp. D^r yields a bijective morphisms $D'' \to \Delta$ and $D^r \to \Delta$. Hence D'' and D^r must be irreducible. We get $\tau_n^* \delta = d'' + 2^l d^r$, where l is the number of non-disconnecting nodes on C, by Remark 1.35.

The discussion above also shows that there are finite gluing morphisms

$$\zeta''_\Gamma : \overline{M}_\Gamma \to D'' \subset \overline{R}_{1,n}, \quad \text{and} \quad \zeta^r_\Gamma : \overline{M}_\Gamma \to D^r \subset \overline{R}_{1,n},$$

surjecting on D'' resp. D^r. They correspond to: First glue together tuples of curves parametrised by $\overline{M}_\Gamma$ by the same procedure defining the morphism ξ_Γ (Prop. 1.26 (i)). Then, in case of ζ^r_Γ blow up all the non-disconnecting nodes of the resulting curve, in case of ζ''_Γ do nothing. Finally endow the resulting curve with the only non-trivial prym structure existing on it.

In case (ii), d'' and d^r are equivalent for the following reason. Let Γ' be the graph that is obtained by replacing in Γ the genus 0 vertex v_1 and the self-edge attached to it, by a vertex v' with $g(v') = 1$ and without a self-edge. Then Γ' is of type (i). Like in the proof of (i) there is a gluing morphism

$$\zeta_{\Gamma'} : \overline{R}_{1,a^{-1}(v')} \times \overline{M}_{rest} \to \overline{R}_{1,n}$$

with image $\tau_n^{-1}(\Delta_{\Gamma'})$. We see that d'' and d^r are the pushforwards under $\zeta_{\Gamma'}$ of the boundary divisor classes d_0'' and d_0^r of $\overline{R}_{1,a^{-1}(v')}$. Thus pushing forward the relation $\delta_0'' = \delta_0^r$ (cf. Lemma 4.8 (ii)) by $\zeta_{\Gamma'}$ gives us $d'' = d^r$. □

Remark 4.6 (i) Using Lemma 4.4 it is easy to see that an analogue of Proposition 4.2 holds for $\overline{R}_{1,n}$ as well, i.e. every boundary cycle is the proper intersection of some boundary divisors of the form D_I, and possibly one of the divisors D_0'' and D_0^r, or one of the simple banana cycles of $\overline{R}_{1,n}$. The $\mathbb{Q}$-algebra $A^*_{BCl}(\overline{R}_{1,n}) \subseteq A^*(\overline{R}_{1,n})$ (cf. Def. 1.40) is generated by boundary divisor classes and the classes of simple banana cycles.

(ii) Also one can show that, similar to the boundary strata of $\overline{M}_{g,n}$, which correspond to stable graphs, also the boundary strata of $\overline{R}_{1,n}$ correspond to graphs. These are stable graphs of genus 1 with the additional data of a map

$$c : H \to \{0, -1\},$$

satisfying the following conditions: For all $h \in H$, $c(h) = c(i(h))$, and for all $v \in V$, $\sum_{h \in a^{-1}(v)} c(h)$ is even. The interpretation of this map on the dual graph of a curve is that $c(h) = -1$ means the node the branch h belongs to is blown up, while $c(h) = 0$ means it is not blown up. One can then also show a complete analogue of Proposition 1.26, for $\overline{R}_{1,n}$.

(iii) In particular the proof of Lemma 4.4 tells us how to define for every boundary cycle D of $\overline{R}_{1,n}$ a finite surjective gluing morphism

$$\zeta_D : \overline{R}_D \to D \subset \overline{R}_{1,n}$$

where $\overline{R}_D$ is a certain product of possibly a $\overline{R}_{1,m}$ $(1 \leq m \leq n)$ with moduli spaces of pointed stable genus 0 curves. And Proposition 1.26 (iii) together with the definition of the boundary strata of $\overline{R}_{1,n}$ then quite obviously implies, that the image of a boundary cycle of $\overline{R}_D$ under ζ_D is a boundary cycle of $\overline{R}_{1,n}$.

The analogue of Proposition 4.2 together with Lemma 4.4 also implies:

Corollary 4.7 *For all $n \in \mathbb{N}$, $A^*_{BCl}(\overline{R}_{1,n})$ is generated as $\tau_n^* A^*_{BCl}(\overline{M}_{1,n})$-algebra by the classes of simple banana cycles. It also suffices to take as generators only those of type $b''_{I_1,..,I_m}$ or only those of type $b^r_{I_1,...,I_m}$.*

Lemma 4.8 *For any fixed $n \geq 1$ let D_0'' and D_0^r as usual denote the two boundary divisors of $\overline{R}_{1,n}$ parametrising prym curves with non-disconnecting nodes. Let $D^{(1)} \subseteq D_0''$ and $D^{(2)} \subseteq D_0^r$ be closed subvarieties. Then* *(i) D_0'' and D_0^r are disjoint.*

(ii) $d_0'' = d_0^r$ and $d_0'' d^{(1)} = d_0^r d^{(2)} = 0$.

Proof: For $n = 1$, D_0'' and D_0^r are two different points in $\overline{R}_{1,1}$, each one parametrising a prym curve with two automorphisms. So for $n = 1$ all the claims are true. For the rest of the proof we denote these two points in $\overline{R}_{1,1}$ by $(D_0'')_1$ and $(D_0^r)_1$, and their Q-classes by $(d_0'')_1$ and $(d_0^r)_1$. For $n > 1$, the boundary divisors D_0'', D_0^r of $\overline{R}_{1,n}$ are the preimages of the points $(D_0'')_1$ and $(D_0^r)_1$ under the morphisms $\pi : \overline{R}_{1,n} \to \overline{R}_{1,1}$ forgetting all marked points but the first one. So (i) holds for general n. The morphism π is flat, and the boundary divisors D_0'', D_0^r of $\overline{R}_{1,n}$ both in general parametrise curves with one automorphism. Thus

$$d_0'' = [\pi^{-1}(D_0'')_1]_Q = \pi^*(d_0'')_1 = \pi^*(d_0^r)_1 = [\pi^{-1}(D_0^r)_1]_Q = d_0^r,$$

and (ii) also holds for all n. □

4.1.3 Summary of Belorousski's results

In this section we summarize some results from [Bel98].

Summary 4.9 *(i) For $n \leq 10$, the varieties $\overline{M}_{1,n}$ are rational. (This result is sharp: $\overline{M}_{1,n}$ has Kodaira dimension 0 for $n = 11$ and 1 for $n \geq 12$, by [BF06], Thm. 3.)*

(ii) For $n \leq 10$, $A^*(M_{1,n}) = \mathbb{Q}$.

(iii) For $n \leq 10$, the Chow ring $A^*(\overline{M}_{1,n})$ is as $\mathbb{Q}$-vector space generated by boundary cycle classes.

(vi) For $n \leq 5$ the Chow ring $A^*(\overline{M}_{1,n})$ is as $\mathbb{Q}$-algebra generated by boundary divisors. For $n \geq 6$ it is not [2].

For $n \geq 4$ Belorousski computes the ring $A^*(\overline{M}_{1,n})$ in terms of generators, which are classes of boundary divisors, and relations.

Summary 4.10 *(i) The Chow ring of $\overline{M}_{1,2}$ is given by*

$$A^*(\overline{M}_{1,2}) = \mathbb{Q}[\delta_0, \delta_{\{12\}}]/I$$

where I is the ideal generated by the two independent codimension 2 relations:

$$\delta_0^2 = 0, \quad \delta_{\{12\}}^2 = -\frac{1}{12}\delta_0\delta_{\{12\}}$$

(ii) The Chow ring of $\overline{M}_{1,3}$ is given by

$$A^*(\overline{M}_{1,3}) = \mathbb{Q}[\delta_0, \delta_3, \delta_{\{12\}}, \delta_{\{13\}}, \delta_{\{23\}}]/J$$

[2] This assertion for $n \geq 6$ is proven only under the assumption that a certain claim by E. Gezler holds, which was not proven yet. (Cf. Claim 5.1 in chapter 5.)

where J is an ideal described below. The dimensions of the homogeneous parts of $A^*(\overline{M}_{1,3})$ are $1, 5, 5, 1$. The pairing

$$A^k(\overline{M}_{1,3}) \times A^{3-k}(\overline{M}_{1,3}) \to \mathbb{Q}$$

is perfect.

(iii) The ideal J is generated by the following 10 independent codimension 2 relations:

$$\delta_0^2 = 0, \quad \delta_3^2 = -\frac{1}{12}\delta_0\delta_3 - \delta_3\delta_{\{12\}}, \quad \delta_{\{12\}}^2 = -\frac{1}{12}\delta_0\delta_{\{12\}} - \delta_3\delta_{\{12\}}$$

$$\delta_{\{13\}}^2 = -\frac{1}{12}\delta_0\delta_{\{13\}} - \delta_3\delta_{\{12\}}, \quad \delta_{\{23\}}^2 = -\frac{1}{12}\delta_0\delta_{\{23\}} - \delta_3\delta_{\{12\}}$$

$$\delta_{\{12\}}\delta_{\{13\}} = 0, \quad \delta_{\{12\}}\delta_{\{23\}} = 0, \quad \delta_{\{13\}}\delta_{\{23\}} = 0$$

$$\delta_3\delta_{\{13\}} = \delta_3\delta_{\{12\}}, \quad \delta_3\delta_{\{23\}} = \delta_3\delta_{\{12\}}$$

(iv) The Chow ring $A^*(\overline{M}_{1,4})$ is given by

$$\mathbb{Q}[D_1, ..., D_{12}]/K$$

where $D_1, ..., D_{12}$ are meant to be the 12 classes of boundary divisors and K is an ideal described below. The dimensions of the homogeneous parts of $A^*(\overline{M}_{1,4})$ are $1, 12, 23, 12, 1$. The pairing

$$A^k(\overline{M}_{1,4}) \times A^{4-k}(\overline{M}_{1,4}) \to \mathbb{Q}$$

is perfect.

(v) The generators of K are not written down completely explicit in [Bel98], but: K is generated by 55 independent codimension 2 relations and one codimension 3 relation. They arise as follows: 30 relations are of the form $D_i \cdot D_j = 0$, coming from the 30 pairs of disjoint boundary divisors. 12 are of the form $D_i^2 = ...$, and are obtained by calculating the self intersection of each boundary divisor. The other 13 codimension 2 relations are:

$$\forall\ i \neq j \neq k \in \underline{4} \quad \delta_{\{ijk\}}\delta_{\{jk\}} = \delta_{\{ijk\}}\delta_{\{ik\}} = \delta_{\{ijk\}}\delta_{\{ij\}} \quad \textit{(8 relations)}$$

$$\forall\ \{i,j,k,l\} = \underline{4} \quad \delta_4(\delta_{\{kl\}} + \delta_{\{jkl\}}) = \delta_4(\delta_{\{il\}} + \delta_{\{ijl\}})$$

The latter relations form a 5 dimensional space. The codimension 3 relation can be taken to be

$$6\delta_0\delta_{2,2} - 2\delta_0\delta_{2,3} - \delta_0\delta_{2,4} + 3\delta_0\delta_{3,4} = 0$$

where $\delta_{2,2}$, $\delta_{2,3}$, $\delta_{2,4}$, $\delta_{3,4}$ are the $\mathbb{S}_4$-invariant classes

$$\delta_{2,2} := \sum_{\substack{\{\{i,j\},\{k,l\}\},\\ s.th.\ \{i,j,k,l\}=\underline{4}}} \delta_{\{ij\}\{kl\}}, \quad \delta_{2,3} := \sum_{\substack{i\in\underline{4},\{j,k\}\subset\underline{4},\\ s.th.\ |\{i,j,k\}|=3}} \delta_{\{i\{jk\}\}}$$

$$\delta_{2,4} := \sum_{\substack{(\{i,j\},\{k,l\}),\\ s.th.\ \{i,j,k,l\}=\underline{4}}} \delta_{\{ij\{kl\}\}}, \quad \delta_{3,4} := \sum_{\substack{i\in\underline{4},\{j,k,l\}\subset\underline{4},\\ s.th.\ \{i,j,k,l\}=\underline{4}}} \delta_{\{i\{jkl\}\}}.$$

Remark: Actually I did not find (i), i.e. the description of the Chow ring of $A^*(\overline{M}_{1,2})$ in [Bel98], but anyway it is easy to compute. (The first relation follows from the fact that δ_0 is the pullback of a point in $\overline{M}_{1,1}$, by the forgetful morphism. To obtain the second one, one can compute explicitly the proper intersection $\delta_0\delta_{\{12\}} = \frac{1}{2}$ and the excess intersection $\delta^2_{\{12\}} = -\frac{1}{24}$ using the excess intersection formula, cf. Example 1.43.) The parts (ii) and (iii) of the Summary come from Thm. 3.3.2. and its proof in [Bel98]. Part (iv) and (v) are from Thm. 3.5.1. and its proof.

Next we cite some Lemmas shown in [Bel98] we will also use

Lemma 4.11 (0.1.3. in [Bel98]) *Let $f : X \to Y$ be a bijective morphism between varieties over an algebraically closed field of characteristic zero, and assume that Y is normal. Then f is an isomorphism.*

Lemma 4.12 ((i) is 0.1.5. in [Bel98])

(i) For $a_1, ..., a_n \in \mathbb{Z}$, such that $\sum a_i = 0$, let $\{\sum a_ip_i \sim 0\}$ be the subset of $M_{1,n}$ of pointed elliptic curves $(C; p_1, ..., p_n)$ such that $\sum a_ip_i \sim 0$ holds in the divisor class group of C. Then this subset is a closed algebraic subvariety.

We use the same notation to denote the subset of $R_{1,n}$ consisting of smooth pointed prym curves $(C; p_1, ..., p_n, \mathcal{L})$ such that $\sum a_ip_i \sim 0$. It is a closed subvariety too.

(ii) If we denote by $\{\sum a_ip_i \sim \text{prym}\}$ the subset of $R_{1,n}$ of pointed smooth prym curves $(C; p_1, ..., p_n; \mathcal{L})$ with prym sheaf $\mathcal{L}$, such that $\mathcal{L}(-\sum a_ip_i) \cong \mathcal{O}_C$, then this is a closed algebraic subvariety of $R_{1,n}$.

Proof: Let $M_{1,n}[N]$ be the moduli spaces of smooth n-pointed elliptic curves with full level N structure for some $N \geq 3$. In contrast to $M_{1,n}$ this space carries a universal family $\mathcal{C} \to M_{1,n}[N]$ with n sections $\sigma_i : M_{1,n}[N] \to \mathcal{C}$ corresponding to the n marked points. Analogously let $R_{1,n}[N]$ be a moduli space of smooth prym curves together with a full level N structure. It carries a universal family $\mathcal{C}' \to R_{1,n}[N]$ with n sections $\sigma'_i : R_{1,n}[N] \to \mathcal{C}'$ and a universal prym sheaf $\mathbb{L}$ on $\mathcal{C}'$. I.e. $\mathbb{L}$ is a square root of the sheaf $\mathcal{O}_{\mathcal{C}'}$ such that if $p \in R_{1,n}[N]$ is a point parametrising a prym curve $(C; p_1, ..., p_n; \mathcal{L})$ with some level N structure, then the restriction of $\mathbb{L}$ to the fibre $\mathcal{C}'_p = C$ is isomorphic to $\mathcal{L}$. Define the line bundles $\mathcal{F} := \mathcal{O}_\mathcal{C}(\sum a_i\bar{\sigma}_i)$ on $\mathcal{C}$ and $\mathcal{F}' := \mathbb{L}(-\sum a_i\bar{\sigma}')$ on $\mathcal{C}'$, where $\bar{\sigma}_i$ resp. $\bar{\sigma}'_i$ are the images of σ_i resp. σ'_i. Set

$$D := \{p \in M_{1,n}[N] \,|\, \mathcal{F}_{|\mathcal{C}_p} = \mathcal{O}_{\mathcal{C}_p}\} = \{p \in M_{1,n}[N] \,|\, \dim H^0(\mathcal{C}_p, \mathcal{F}_{|\mathcal{C}_p}) \geq 1\}, \quad \text{and}$$

$$D' := \{p \in R_{1,n}[N] \,|\, \mathcal{F}'_{|\mathcal{C}'_p} = \mathcal{O}_{\mathcal{C}'_p}\} = \{p \in R_{1,n}[N] \,|\, \dim H^0(\mathcal{C}'_p, \mathcal{F}'_{|\mathcal{C}'_p}) \geq 1\}.$$

Then by the semi-continuity theorem ([Har77], Thm. 12.8) D and D' are closed subvarieties of $M_{1,n}[N]$ resp. $R_{1,n}[N]$. But $\{\sum a_ip_i \sim 0\}$ resp. $\{\sum a_ip_i \sim \text{prym}\}$ are just the images of D resp. D' under the finite forgetful morphisms $M_{1,n}[N] \to M_{1,n}$ resp. $R_{1,n}[N] \to R_{1,n}$. $\square$

Lemma 4.13 ((i) is 2.1.3. in [Bel98]) *Suppose that $a_1, ..., a_{n+1}$ are integers, such that $\sum a_i = 0$ and $|a_i| = 1$ for some i. Then using the notation of Lemma 4.12:*

(i) The closed subvariety $\{\sum a_i p_i \sim 0\} \subset M_{1,n+1}$ *is irreducible and of codimension* 1. *It is isomorphic to an open subvariety of* $M_{1,n}$.

(ii) Also $\{\sum a_i p_i \sim 0\}, \{\sum a_i p_i \sim prym\} \subset R_{1,n+1}$ *are irreducible and of codimension* 1. *They are both isomorphic to open subvarieties of* $R_{1,n}$.

Proof: We show (ii), the proof of (i) is analogous. Set $D_1 := \{\sum a_i p_i \sim 0\}$, $D_2 := \{\sum a_i p_i \sim prym\}$. Assume WLOG that $a_{n+1} = -1$. Let $f : R_{1,n+1} \to R_{1,n}$ be the morphism forgetting the point p_{n+1}. We show that $f_{|D_1}$, $f_{|D_2}$ are open embeddings, from which all the assertions of the Lemma follow. Set $U_1 := R_{1,n} \smallsetminus \biguplus_{j=1}^n \{p_j \sim \sum_{i=1}^n a_i p_i\}$, $U_2 := R_{1,n} \smallsetminus \biguplus_{j=1}^n \{p_j - \sum_{i=1}^n a_i p_i \sim prym\}$. By Lemma 4.12 these are open subvarieties of $R_{1,n}$. If $(C; p_1, ..., p_n; \mathcal{L})$ is a prym curve from U_1 resp. U_2 there is a unique point p_{n+1} on C such that $(C; p_1, ..., p_n, p_{n+1}; \mathcal{L})$ corresponds to a point in D_1 resp. D_2. This is because every given divisor of degree 1 on an elliptic curve C is equivalent to a unique point on C. Thus the morphisms $f_{|D_i} : D_i \to U_i$ is bijective. By Lemma 4.11 it is an isomorphism. □

4.2 The rational Picard group of $\overline{M}_{1,n}$ and $\overline{R}_{1,n}$

Surely the rational Picard group of $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$ is known, but I did not find an explicit reference. The structure of the Picard group follows quite directly from results of [BF09b].

Proposition 4.14 *For all* $n \in \mathbb{N}$:

(i) $Pic_{\mathbb{Q}}\,\overline{M}_{1,n} = A^1(\overline{M}_{1,n}) = H^2(\overline{M}_{1,n})$ *and* $Pic_{\mathbb{Q}}\,\overline{R}_{1,n} = A^1(\overline{R}_{1,n}) = H^2(\overline{R}_{1,n})$.

(ii) The classes of boundary divisors form a basis of the $\mathbb{Q}$ *vector space* $A^1(\overline{M}_{1,n})$.

(iii) The classes of boundary divisors of $\overline{R}_{1,n}$ *span* $A^1(\overline{M}_{1,n})$ *with the single relation* $d_0'' = d_0^r$. *Hence the pullback* $\tau_n^* : A^1(\overline{M}_{1,n}) \to A^1(\overline{R}_{1,n})$ *is an isomorphism.*

(iv) Consequently also

$$\tau_n^* : A^*_{Div}(\overline{M}_{1,n}) \to A^*_{Div}(\overline{R}_{1,n}), \quad \tau_n^* : H^*_{Div}(\overline{M}_{1,n}) \to H^*_{Div}(\overline{R}_{1,n})$$

are isomorphisms. ($A^*_{Div}(...)$, $H^*_{Div}(...)$ *as in Definition 1.40.)*

Proof: (i): $Pic_{\mathbb{Q}} = A^1$ holds for every variety having only finite quotient singularities. (cf. the proof of Cor. 2.15 (iii)) For the equality to the second cohomology group cf. the proof of part (iii).

(ii) Cf. Theorem (4.1) in chapter 19 of [ACG11], for the same statement for $H^2(\overline{M}_{1,n})$. From this (ii) follows by (i).

(iii): The pullback τ_n^* is injective since τ_n is finite and surjective. By Lemma 4.4 and Lemma 4.8 (ii) the pullbacks of the boundary divisors of $\overline{M}_{1,n}$ generate the same subspace of $A^1(\overline{R}_{1,n})$ that is generated by the boundary divisors of $\overline{R}_{1,n}$. Thus it suffices to show that $A^1(\overline{R}_{1,n})$ is generated by boundary divisors of $\overline{R}_{1,n}$. By Thm. 1 of [BF09b], $H^2(\overline{R}_{1,n})$ is generated by the boundary divisors, and by the same theorem $H^1(\overline{R}_{1,n}) = 0$. Since

$\overline{R}_{1,n}$ has only finite quotient singularities its cohomology with coefficient in $\mathbb{C}$ has a pure canonical Hodge structure (cf. Summary 1.36 (v)). Using this we get

$$H^1(\overline{R}_{1,n}, \mathcal{O}_{\overline{R}_{1,n}}) = H^{0,1}(\overline{R}_{1,n}) \subseteq H^1(\overline{R}_{1,n}, \mathbb{C}) = H^1(\overline{R}_{1,n}) \otimes \mathbb{C} = 0$$

Insert $H^1(\overline{R}_{1,n}, \mathcal{O}_{\overline{R}_{1,n}}) = 0$ into the long exact sequence

$$... \to H^1(\overline{R}_{1,n}, \mathcal{O}_{R_{1,n}}) \otimes \mathbb{Q} \to H^1(\overline{R}_{1,n}, \mathcal{O}^*_{R_{1,n}}) \otimes \mathbb{Q} \xrightarrow{c_1} H^2(\overline{R}_{1,n}, \mathbb{Z}) \otimes \mathbb{Q} \to ...$$

which is obtained by tensoring the standard exponential sequence with $\mathbb{Q}$. This tells us that $Pic_{\mathbb{Q}}\,\overline{R}_{1,n} = H^1(\overline{R}_{1,n}, \mathcal{O}^*_{R_{1,n}}) \otimes \mathbb{Q}$ injects into $H^2(\overline{R}_{1,n})$ by the Chern class map c_1. Since $H^2(\overline{R}_{1,n})$ is generated by boundary divisors, this implies that the same holds for $Pic_{\mathbb{Q}}\,\overline{R}_{1,n}$, and also that $Pic_{\mathbb{Q}}\,\overline{R}_{1,n} = H^2(\overline{R}_{1,n})$.

(iv): The two pullback morphisms are surjective by (iii) and the definition of A^*_{Div}, H^*_{Div} (Def. 1.40). They are injective since τ_n is finite surjective. □

4.3 Rationality of $\overline{R}_{1,n}$, and $A^*(R_{1,n}) = \mathbb{Q}$, for $n \leq 6$.

Proposition 4.15 *With $\cong$ standing for isomorphism of varieties:*

(i) $\overline{M}_{1,1} \cong \overline{M}_{0,(1,[3])}$ and $\overline{R}_{1,1} \cong \overline{M}_{0,(2,[2])}$, and hence $\overline{R}_{1,1} \cong \mathbb{P}^1 \cong \overline{M}_{1,1}$. (Cf. Def. 2.4 for the notation used for moduli spaces of genus 0 curves with sorted marked points.)

(ii) There is an isomorphism $f : \overline{M}_{0,[1,4]} \xrightarrow{\cong} \overline{M}_{1,2}$ mapping $M_{0,[1,4]}$ onto $M_{1,2}$.

(iii) There is an isomorphism $g : \overline{M}_{0,[1,2,2]} \xrightarrow{\cong} \overline{R}_{1,2}$ mapping $M_{0,[1,2,2]}$ onto $R_{1,2}$.

(iv) Hence $\overline{R}_{1,2}$ is rational, $A^*(R_{1,2}) = \mathbb{Q}$, and $A^{2*}(\overline{R}_{1,2}) \cong H^*(\overline{R}_{1,2})$.

Proof: We constructed similar isomorphisms quite detailed in Proposition 2.14, the proofs will be kept shorter here.

(i): Let $\overline{H}_{2,4}$ be the moduli space of admissible double covers of stable genus 0 curves, ramified over the 4 ordered marked points of the genus 0 curve. Write the objects as $(\pi : X \to D; p_1, ..., p_4)$ where the p_i are the marked point on the genus 0 curve D. We have $\overline{H}_{2,4} \cong \overline{M}_{0,4}$. Define a finite surjective morphism $\varphi : \overline{H}_{2,4} \to \overline{M}_{1,1}$, corresponding to keeping only the cover with one marked point $(X; \pi^{-1}(p_1))$ and forming the stable model. It factors through the claimed isomorphism $\overline{M}_{0,(3,1)} \cong \overline{H}_{2,(3,1)} \to \overline{M}_{1,1}$.

Now it suffices to construct a morphism $H_{2,4} \to R_{1,1}$, compatible with φ, on the interior of the moduli spaces. To define it, like for φ we keep $(X; \pi^{-1}(p_1))$, but include the prym sheaf $\mathcal{O}_X(\pi^{-1}(p_1) - \pi^{-1}(p_2))$ in the data (forming the stable model of X is not necessary here, since X is smooth). The extended morphism $\overline{M}_{0,4} \cong \overline{H}_{2,4} \to \overline{R}_{1,1}$ factors through the claimed isomorphism.

Now we know that the smooth curve $\overline{R}_{1,1}$ is covered by $\overline{M}_{0,4} \cong \mathbb{P}^1$, and hence $\overline{R}_{1,1} \cong \mathbb{P}^1$ (Hurwitz formula).

(ii): Let $\overline{H}_{2,4,1}$ be the moduli space of 1-pointed admissible double covers of $4+1$-pointed genus 0 curves: By this we mean the moduli space parametrising objects $(\pi : X \to$

$D; p_1, ..., p_4; q; q')$, where $(D; p_1, ..., p_4, q)$ is a 5-pointed stable genus 0 curve, where π is the admissible double cover of the 4 pointed curve $(D; p_1, ..., p_4)$ (cf. Def. 2.6) and q' is one of the two points in $\pi^{-1}(q)$ (also cf. Def. 2.1.6. of [Bel98] and the discussion following it, for the existence of this space). We have $\overline{H}_{2,4,1} \cong \overline{M}_{0,5}$. There is a finite surjective morphism $\varphi : \overline{H}_{2,4,1} \to \overline{M}_{1,2}$ corresponding to keeping only the stable model of the two pointed curve $(X; q', q'')$ where q'' is the other point in $\pi^{-1}(q)$. It factors through the claimed isomorphism $\overline{M}_{0,[4,1]} \cong \overline{H}_{2,[4],1} \to \overline{M}_{1,2}$.

(iii): Define a morphism $H_{2,4,1} \to R_{1,2}$ corresponding to again keeping $(X; q', q'')$ and including the prym sheaf $\mathcal{O}_X(\pi^{-1}(p_1) - \pi^{-1}(p_2))$ in the data. The extended morphism $\overline{H}_{2,4,1} \to \overline{R}_{1,2}$ factors through the claimed isomorphism $\overline{M}_{0,[2,2,1]} \cong \overline{H}_{2,[2,2],1} \to \overline{R}_{1,2}$.

(vi): We know that $\overline{M}_{0,5}$ is rational, $A^*(M_{0,5}) = \mathbb{Q}$, and $A^{2*}(\overline{M}_{0,5}) = H^*(\overline{M}_{0,5})$ by Summary 1.48. By (iii), $\overline{R}_{1,2}$ is isomorphic to a quotient $\overline{M}_{0,5}/\mathbb{S}_2 \times \mathbb{S}_2$. So the second two claims of (iv) follow with Lemma 1.37. Unirationality of $\overline{R}_{1,2}$ follows directly from $\overline{R}_{1,2} \cong \overline{M}_{0,5}/\mathbb{S}_2 \times \mathbb{S}_2$. But $\overline{R}_{1,2}$ is a complex surface, so unirationality implies rationality here. (One can also proof rationality of $\overline{R}_{1,2}$ by constructing a birational map $f_2 : \Phi_2 \dashrightarrow \overline{R}_{1,2}$ very similar to $f_3 : \Phi_3 \dashrightarrow \overline{R}_{1,3}$ we will construct soon. $\Phi_2 \cong \mathbb{P}^2$ would, like Φ_3, be a certain linear subspace of the space of plane cubics.) $\square$

Next we will, for any $3 \leq n \leq 6$, construct birational maps $f_n : \Phi_n \dashrightarrow R_{1,n}$, where the Φ_n are rational parameter spaces. The maps f_n will be isomorphisms on their domain of definition, and thus will provide the rationality of $R_{1,n}$ for $3 \leq n \leq 6$. They will also be used to prove that $A^*(\overline{R}_{1,n})$ is generated by the boundary cycle classes, for $3 \leq n \leq 6$. The construction of these morphisms will work quite similar to the construction of the birational morphisms to $M_{1,n}$ in chapter 1 of [Bel98]. We also use a similar notation for the most part.

Definition 4.16 Let G be the 10-dimensional $\mathbb{C}$-vector space of homogeneous polynomials of degree three in three variables. I.e.,

$$G = \{f = \sum_{i+k+j=3} a_{ijk} x^i y^j z^k \,|\, a_{ijk} \in \mathbb{C}\}.$$

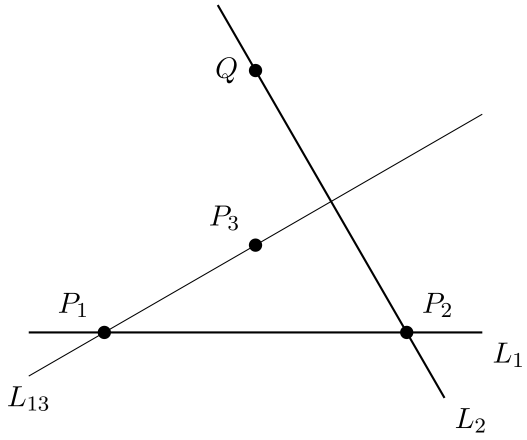
We can view G as the space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. $\mathbb{P}(G) \cong \mathbb{P}^9$ is the parameter space of cubics in $\mathbb{P}^2$. For

$$\mathcal{C} := \{((a:b:c), [f]) \in \mathbb{P}^2 \times \mathbb{P}(G) \,|\, f(a,b,c) = 0\},$$

the projection $\mathcal{C} \to \mathbb{P}(G)$ is the universal family over the parameter space $\mathbb{P}(G)$. It is flat.

Provide $\mathbb{P}^2$ with homogeneous coordinates $(x : y : z)$. We fix a configuration of points and lines in $\mathbb{P}^2$.

- Points: $P_1 := (1:0:0)$, $P_2 := (0:1:0)$, $P_3 := (1:1:1)$, $Q := (0:0:1)$
- Lines: L_1 the line through P_1 and P_2, L_2 the line through P_2 and Q, L_{13} is the line through P_1 and P_3.



Definition 4.17 Using this configuration we define a subset of $\mathbb{P}(G)$:

Φ_3 is the set of cubics C in $\mathbb{P}^2$ passing through P_1, P_2, P_3 and Q such that C is tangent to L_1 in P_1 and to L_2 in Q.

Lemma 4.18 *(i) The defining conditions of* Φ_3 *constitute* 6 *independent linear conditions on the space of plane cubics (i.e. linear conditions on the coefficients of homogeneous polynomials defining such cubics). So* $\Phi_3 \cong \mathbb{P}^3$.

(ii) Almost all cubics of Φ_3 *are smooth.*

(iii) If $\widetilde{\Phi}_3 \subset \Phi_3$ *is the dense open subset parametrising smooth cubics, then there is an isomorphism*

$$f_3 : \widetilde{\Phi}_3 \to R_{1,3} \setminus (B_1^{(3)} \cup B_2^{(3)} \cup B_3^{(3)}).$$

Here, using the notation introduced in Lemma 4.12, $B_1^{(3)}$, $B_2^{(3)}$, $B_3^{(3)}$ *are the closed subvarieties of* $R_{1,3}$ *defined by* $B_1^{(3)} := \{p_1 - p_2 \sim prym\}$, $B_2^{(3)} := \{p_1 - p_3 \sim prym\}$, *and* $B_3^{(3)} := \{p_2 - p_3 \sim prym\}$.

Proof: The defining conditions of Φ_3 impose the following conditions on the coefficient of a polynomial $f = \sum_{i+j+k} a_{i,j,k} x^i y^j z^k$, defining a cubic C:

$$P_1 \in C \Leftrightarrow a_{3,0,0} = 0, \qquad P_2 \in C \Leftrightarrow a_{0,3,0} = 0, \qquad Q \in C \Leftrightarrow a_{0,0,3} = 0,$$

$$P_3 \in C \Leftrightarrow \sum_{i+j+k=3} a_{i,j,k} = 0, \qquad C \text{ tangent to } L_1 \text{ at } P_1 \Leftrightarrow a_{210} = 0$$

$$C \text{ tangent to } L_2 \text{ at } Q \Leftrightarrow a_{012} = 0.$$

It is easy to check that these linear equations are independent.

To prove (ii) it is enough to show that there is one smooth cubic belonging to Φ_3, since smoothness is an open condition. We use $\sim$ to denote equivalence of two sums of points on an elliptic curve.

Let $(C; p_1, p_2, p_3; \mathcal{L})$ be a smooth genus 1 prym curve with three marked points p_1, p_2, p_3 and prym sheaf $\mathcal{L}$, and let $c \in R_{1,3}$ be the point parametrising $(C; p_1, p_2, p_3; \mathcal{L})$. Choose $(C; p_1, p_2, p_3; \mathcal{L})$ such that $c \in R_{1,3} \setminus (B_1^{(3)} \cup B_2^{(3)} \cup B_3^{(3)})$. Let q the unique point on C such that $\mathcal{L} \cong \mathcal{O}_C(p_1 - q)$. Embed C into $\mathbb{P}^2$ by the linear system $|2p_1 + p_2|$. We denote the image of C and the points p_1, p_2, p_3, q in $\mathbb{P}^2$ by the same symbols again. We denote by l_1 the tangent of C at p_1. By the choice of the embedding, p_2 also lies on l_1. Let l_2 be the tangent of C at q. Since $\mathcal{L} \cong \mathcal{O}_C(p_1 - q)$, $2p_1 \sim 2q$ on C and thus $2q + p_2 \sim 2p_1 + p_2$. C is embedded by $|2p_1 + p_2|$, so this implies that p_2 also lies on l_2. We have $q \neq p_2$ by $c \notin B_1^{(2)}$, $q \neq p_3$ by $c \notin B_1^{(2)}$, and $q \neq p_1$ by definition of q. Hence the points p_1, p_2, p_3, q are distinct. It is impossible that $p_3 \in l_1$ or $p_3 \in l_2$, for otherwise C would intersect the line in more than 3 points, counted with multiplicity. If furthermore p_1, q, and p_3 are not collinear then the points p_1, p_2, p_3, q are in general position. But this is guaranteed by $c \notin B_3^{(3)}$. Now by Lemma 4.19 below, there is a unique projective transformation T of $\mathbb{P}^2$ which maps the points p_1, p_2, p_3, q which are in general position, to the points P_1, P_2, P_3, Q which are in general position too. This T has to map l_1 resp. l_2 to L_1 resp. L_2 automatically. Hence the image $T(C)$ of C is a smooth cubic fulfilling all the defining conditions of Φ_3. Thus we have proven (ii).

For later use, note that the resulting smooth cubic $T(C)$ does only depend on c and not on the representative $(C; p_1, p_2, p_3; \mathcal{L})$.

Now we show (iii). Several times we will use the following fact: If C is a smooth cubic from Φ_3 then the inclusion $C \hookrightarrow \mathbb{P}^2$ can be regarded as induced by the linear system $|2P_1 + P_2|$ (since the tangent to C at P_1 cuts out the divisor $2P_1 + P_2$).

To define the morphism $f_3 : \widetilde{\Phi}_3 \to R_{1,3}$, first restrict to $\widetilde{\Phi}_3$ the universal family of plane cubics, which lies over the space $\mathbb{P}(G)$ of cubics in $\mathbb{P}^2$ (see above). So we get a flat family of smooth curves $\mathcal{C}_3 \to \widetilde{\Phi}_3$, with $\mathcal{C}_3$ smooth. Let $\mathcal{P}_i$, $i = 1, 2, 3$, and $\mathcal{Q}$, be the sections on $\mathcal{C}_3$ corresponding to the points P_i resp. Q in $\mathbb{P}^2$. We denote the divisors on $\mathcal{C}_3$ that are the images of these section by the same symbols. Then the invertible sheaf $\mathcal{O}_{\mathcal{C}_3}(\mathcal{P}_1 - \mathcal{Q})$ is a prym sheaf: $\mathcal{O}_{\mathcal{C}_3}(\mathcal{P}_1 - \mathcal{Q})$ restricted to an arbitrary fibre C of the family $\mathcal{C}_3 \to \widetilde{\Phi}_3$ yields the sheaf $\mathcal{O}_C(P_1 - Q)$. This is a prym sheaf, since $2P_1 + P_2 \sim 2Q + P_2$ on C by the definition of Φ and thus $2(P_1 - Q) \sim 0$. Thus

$$(\mathcal{C}_3 \to \widetilde{\Phi}_3; \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3; \mathcal{O}_{\mathcal{C}_3}(\mathcal{P}_1 - \mathcal{Q}))$$

is a family of smooth prym curves with 3 marked points over $\widetilde{\Phi}_3$. Call the morphism this family induces $f_3 : \widetilde{\Phi}_3 \to R_{1,3}$.

The image of f_3 lies inside $R_{1,3} \setminus (B_1^{(3)} \cup B_2^{(3)} \cup B_3^{(3)})$: If C is a smooth cubic fulfilling the defining conditions of Φ, so that its image under f_3 lies in ($B_1^{(3)}$ resp. $B_2^{(3)}$ resp. $B_3^{(3)}$), this would imply $P_2 = Q$ resp. $P_3 = Q$ resp. Q, P_1 and P_3 are collinear, contradicting the definition of these points.

In the proof of (ii) we described a construction. It starts with any point c in $R_{1,3} \setminus (B_1^{(3)} \cup B_2^{(3)} \cup B_3^{(3)})$ and yields a smooth cubic in $\mathbb{P}^2$, belonging to $\widetilde{\Phi}_3$. If we compare this construction with the definition of f_3, we see that the point in $\widetilde{\Phi}_3$ we obtain, is mapped by f_3 to the point c we started with. Thus f_3 is surjective.

Furthermore for every c as above, the preimage point of c under f_3 that is given by the construction is the only preimage points that exist: Let C be a cubic from Φ_3. The corresponding point in Φ_3 is mapped by f_3 to the point $c := [(C; P_1, P_2, P_3; \mathcal{O}_C(P_1 - Q))] \in$

$R_{1,2}$. If we apply the construction from the proof of (ii) to c and choose as a representative $(C; P_1, P_2, P_3; \mathcal{O}_C(P_1 - Q))$, then the cubic $C' \subset \mathbb{P}^2$ we get has the following properties: It arises by embedding C by the linear system $|2P_1 + P_2|$. On $|2P_1 + P_2|$ the unique system of coordinates is chosen, such that the embedding maps each point P_i on C to the point P_i in $\mathbb{P}^2$, and such that Q on C is mapped to Q in $\mathbb{P}^2$. These properties determine the cubic C' uniquely. But C has the same properties, thus $C' = C$.

So now we know that f_3 is bijective. Thus it is an isomorphism by Lemma 4.11. □

We used the following well known fact:

Lemma 4.19 *There is a unique projective transformation T on $\mathbb{P}^2$ mapping a given configuration of 4 points $p_1, ..., p_4$ in general position (i.e. no three points collinear) to any other given such configuration $p'_1, ..., p'_4$. (By this we mean that $T(p_i) = p'_i$ for all $i \in \underline{4}$).*

Definition 4.20 For $m \geq 0$ define:

(i) $\Phi_{3+m} \subset \Phi_3 \times (\mathbb{P}^2)^m$ is the set of tuples $(C; R_1, ..., R_m)$ such that C is a cubic from Φ_3 and such that the points $R_1, ..., R_m$ in $\mathbb{P}^2$ lie on C.

(ii) Let H be the sub vector space of G (cf. Definition 4.16), such that H consists of all homogeneous polynomials of degree 3 which define cubics parametrised by points of Φ_3, and the 0-polynomial. Then we have $\mathbb{P}(H) = \Phi_3$.

Lemma 4.21 *For all $m \geq 0$ the projection $\Phi_{3+m} \to \Phi_3$ is flat, and Φ_{3+m} is a irreducible projective variety.*

Proof: (cf. [Bel98] p. 14-15.) By definition of Φ_{3+m} there are projections

$$\begin{array}{ccc} \Phi_{3+m} & \xrightarrow{\nu_{3+m}} & (\mathbb{P}^2)^m \\ {\scriptstyle \rho_{3+m}}\downarrow & & \\ \Phi_3 & & \end{array}$$

Now $\rho_4 : \Phi_4 \to \Phi_3$ is the natural *flat* family of cubics over Φ. As subvariety of $\mathbb{P}(H) \times (\mathbb{P}^2)^m$, Φ_{3+m} is defined by m equations

$$f(x_1, y_1, z_1) = ... = f(x_m, y_m, z_m) = 0$$

where $f \in H$ and $(x_i : y_i : z_i)$ are homogeneous coordinates on the i-th $\mathbb{P}^2$-factor. Thus the homogeneous coordinate ring of Φ_{3+m} is the m-th tensor power of the coordinate ring of Φ_4 over the coordinate ring of $\mathbb{P}(H) = \Phi_3$ From this we conclude that Φ_{3+m} is the m-fold fibre product $\Phi_{3+m} = \Phi_4 \times_{\Phi_3} ... \times_{\Phi_3} \Phi_4$ (with respect to ρ_4). Since flatness is preserved under base change, the projections $\Phi_{3+m+1} \to \Phi_{3+m}$ we obtain from this fibre product are flat. So ρ_m, which is the composition of the projections

$$\Phi_{3+m} \to ... \to \Phi_4 \to \Phi_3$$

is flat too. Thus, and since $\Phi_3 \cong \mathbb{P}^3$ is irreducible, every irreducible component of Φ_{3+m} is mapped dominantly to Φ_3. This follows from the fact that every flat morphism between varieties is open (cf. [Har77], Exercise III.9.1). If Φ_{3+m} had more than one irreducible component, this would now imply that almost all fibres of ρ_{3+m} are reducible. But we know that almost all fibres are smooth, since almost all fibres of the family of cubics $\Phi_4 \to \Phi_3$ are smooth by Lemma 4.18 (ii), and since $\Phi_{3+m} = \Phi_4 \times_{\Phi_3} ... \times_{\Phi_3} \Phi_4$. □

Lemma 4.22 *(i) For all $m \geq 1$ there are open subsets $U_{3+m} \subseteq \Phi_{3+m}$ (defined in the proof), and morphisms $f_{3+m} : U_{3+m} \to R_{1,3+m}$, which are open embeddings.*

(ii) The images of these morphisms are $f_n(U_n) = R_{1,n} \smallsetminus (B_1^{(n)} \cup B_2^{(n)} \cup B_3^{(n)})$, Where in $R_{1,n}$, $B_1^{(n)} := \{p_1 - p_2 \sim prym\}$, $B_2^{(n)} := \{p_1 - p_3 \sim prym\}$, and $B_3^{(n)} := \{p_2 - p_3 \sim prym\}$.

Proof: Let again $\widetilde{\Phi}_3 \subseteq \Phi_3$ be the subset parametrising smooth cubics. Define subsets $V_{3+m} \subseteq (\mathbb{P}^2)^m$ by:

$$V_{3+m} := \{(R_1, ..., R_m) \mid R_i \neq R_j \text{ for } i \neq j;\ R_i \neq P_j \text{ for all } i, j;\ R_i \notin L_1 \cup L_2\}$$

where, as above, L_1, L_2 are the lines through P_1 and P_2 resp. through P_1 and Q. Define $U_{3+m} := \Phi_{3+m} \cap (\widetilde{\Phi}_3 \times V_{3+m})$. Pull back to U_{3+m} the natural family of plane cubics lying over Φ_3. The resulting flat family $\mathcal{C} \to U_{3+m}$ is a family of smooth cubics by definition of $\widetilde{\Phi}_3$. Like in the proof of Lemma 4.18 (iii) we define sections $\mathcal{P}_i$ resp. $\mathcal{Q}$ corresponding to the points P_i and Q. Furthermore the sections $\mathcal{R}_i : U_{3+m} \to \mathbb{P}^2 \times U_{3+m}$, corresponding to the points R_i, are (by restricting the target spaces) also sections of the families $\mathcal{C} \to U_{3+m}$. Similar to what is done in the proof of Lemma 4.18 (iii) we get families of pointed smooth prym curves

$$(\mathcal{C} \to U_{3+m}; \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{R}_1, ..., \mathcal{R}_m; \mathcal{O}_{\mathcal{C}}(\mathcal{P}_1 - \mathcal{Q})),$$

These families induce the morphisms $f_{3+m} : U_{3+m} \to R_{1,3+m}$.

To see that f_n is dominant, we proceed analogously to the proof of Lemma 4.18 (ii): Embed any prym curve with class

$$[(C; p_1, p_2, p_3, r_1, ..., r_m; \mathcal{L})] \in R_{1,3+m} \smallsetminus (B_1^{(3+m)} \cup B_2^{(3+m)} \cup B_3^{(3+m)})$$

in $\mathbb{P}^2$ by the linear system $|2p_1 + p_2|$. Then move, by a (unique) projective transformation, the resulting smooth pointed plane cubic into one fulfilling the defining conditions of U_n. The point in U_n corresponding to this cubic is mapped to $[(C; p_1, p_2, p_3, r_1, ..., r_m; \mathcal{L})]$ by f_{3+m}.

As in the proof of Lemma 4.18 (ii) we see that the preimage point of a point in $R_{1,3+m}$ under f_{3+m}, that we obtain by this construction, is the only preimage point that exists.

Thus f_{3+m} is bijective onto its image. So, by Lemma 4.11, f_{3+m} is an isomorphism onto its image.

To prove (ii), it only remains to show that the image of f_{3+m} is contained in $R_{1,3+m} \smallsetminus (B_1^{(3+m)} \cup B_2^{(3+m)} \cup B_3^{(3+m)})$. This goes just like the proof of the analogous part of Lemma 4.18 (iii). □

Lemma 4.23 *Using the notation introduced in the proof of Lemma 4.22:*

For any $m \in \underline{3}$ and for any tuple of points $\mathbf{R} := (R_1, ..., R_m) \in V_{3+m}$, the subset $S(\mathbf{R}) \subseteq \Phi_3$ consisting of those cubics which pass through all the points $R_1, .., R_m$ is a linear subspace of $\Phi_3 \cong \mathbb{P}^3$. Define

$$S'(\mathbf{R}) := \{(C, \mathbf{R}) \mid C \in S(\mathbf{R})\} \subset \Phi_{3+m}.$$

Then for every $\mathbf{R} \in V_{3+m}$ at least one of the following three conditions is fulfilled:

(a) $S(\mathbf{R})$ is of codimension m. (I.e. of dimension $3 - m$.)

(b) All cubics which are elements of $S(\mathbf{R})$ are singular. In particular $S'(\mathbf{R}) \cap U_{3+m} = \emptyset$.

(c) $m = 3$ and $f_6(S'(\mathbf{R}) \cap U_6) \subseteq \{2p_1 + 2p_2 - p_3 - r_1 - r_2 - r_3 \sim 0\}$, where the set on the right side is a subset of $R_{1,6}$ as defined in Lemma 4.12 (ii).

Furthermore for each $m \in \underline{3}$ the subset $W_{m+3} \subset V_{m+3}$ of points fulfilling (a) is open and dense.

Proof: The cubics in Φ_3 are exactly those defined by non-zero polynomials of the form:

$$a(x^2z - xz^2) + b(xy^2 - xz^2) + c(xyz - xz^2) + d(y^2z - xz^2), \qquad a, b, c, d \in \mathbb{C} \qquad (4.1)$$

This can be shown using the explicit linear conditions on the coefficients listed in the proof of Lemma 4.18. Furthermore the condition to pass through any given point, translates into a linear condition on the coefficients of a cubic. Hence $S(\mathbf{R})$ is a linear subspace of the $\mathbb{P}^9$ of all plane cubics, as well as of Φ_3. For $m \leq 2$ we show that one of (a) and (b) has to be fulfilled, using results from chapter V.4. of [Har77], similar as in the proof of Lemma 2.3.2 in [Bel98].

The condition on a plane cubic C to be contained in $S(\mathbf{R})$ is: C passes through the $4 + m$ points $P_1, P_2, P_3, Q, R_1, ..., R_m \in \mathbb{P}^2$ and the tangents to C in P_1 and Q both pass through P_2. The condition on the two tangents can be translated into a condition that C passes through certain points P_1' and Q' which are infinitely near to P_1 resp. Q (cf. Chapter V.3. of [Har77] for the definition of infinitely near points on a surface). Hence $S(\mathbf{R})$ can be seen as the linear system of plane cubic curves with assigned base points $P_1, P_2, P_3, Q, R_1, ..., R_m, P_1', Q'$, in the language of chapter V.4. of [Har77].

Assume that $m \leq 2$ and (b) is not fulfilled. We would like to say that then (a) is fulfilled according to Corollary V.4.4. (a) from [Har77]. Firstly under our assumption, there is a non-singular cubic passing through the $6 + m$ assigned base points. Hence, as required in that corollary, no four of the points lie on a line and no seven lie on a conic (Bézout). But in the formulation of Corollary V.4.4. only one of the points is allowed to be an infinitely near point, while we have two such points. However looking at the proofs in [Har77] one realizes that this is because the hypotheses in Corollary V.4.4 are carried over from Proposition V.4.3., and that the hypotheses can be weakened for Corollary V.4.4. (a) to allow two infinitely near points: Among the $5 + m$ points $P_1, P_2, P_3, Q, R_1, ..., R_m, P_1'$ there is only one infinitely near point, so Proposition V.4.3. says that the linear system of plane cubics $\mathfrak{d}$ defined by these points has no unassigned base points, and Corollary V.4.4. (a) says that $\dim \mathfrak{d} = 9 - (5 + m)$. So in particular Q' is no unassigned base point of $\mathfrak{d}$, hence, by Remark V.4.0.2. of [Har77], $\dim S(\mathbf{R}) = \dim \mathfrak{d} - 1 = 3 - m$. This implies condition (a) (and shows that in general two infinitely near points can be allowed in V.4.4. (a)).

In case $m = 3$, we show that $\neg(a) \wedge \neg(b)$ implies (c). For every $(C, \mathbf{R}) \in S'(\mathbf{R}) \cap U_6$, by definition of U_6, C is smooth. Now $\neg(a) \wedge \neg(b)$ implies that $\dim S(\mathbf{R}) \geq 1$, and since smoothness is an open condition, that there is a $(C', \mathbf{R}) \in S'(\mathbf{R}) \cap U_6$ with $C' \neq C$. For L_1, as before, the line through P_1, P_2 in $\mathbb{P}^2$, $3L_1 - C' \sim 0$ in $\operatorname{Pic} \mathbb{P}^2$. For $i : C \hookrightarrow \mathbb{P}^2$ the inclusion, $i^*L_1 = 2P_1 + P_2$ and $i^*C' = 2P_1 + P_2 + P_3 + 2Q + R_1 + R_2 + R_3$. With $2P_1 \sim 2Q$ hence $2P_1 + 2P_2 - P_3 - R_1 - R_2 - R_3 \sim 0$ in $\operatorname{Pic} C$. So $f_6((C, \mathbf{R})) \in \{2p_1 + 2p_2 - p_3 - r_1 - r_2 - r_3 \sim 0\}$.

Let $\nu_{3+m} : \Phi_{3+m} \to (\mathbb{P}^2)^m$ be the morphism from the proof of Lemma 4.21. Set

$$W'_{3+m} := \{\mathbf{R} \in (\mathbb{P}^2)^m \mid \dim \nu_{3+m}^{-1}(\mathbf{R}) = 3 - m\}. \quad (3 - m = \dim \Phi_{3+m} - \dim(\mathbb{P}^2)^m)$$

Since Φ_{3+m} is projective, and ν_{3+m} is surjective for $m \leq 3$, we obtain that $W'_{3+m} \subset (\mathbb{P}^2)^m$ is open (and dense) by upper semicontinuity of the fibre dimension. For every $\mathbf{R} \in V_{m+3}$, one has $\nu_{3+m}^{-1}(\mathbf{R}) = S'(\mathbf{R}) \cong S(\mathbf{R})$. From this we conclude that $W_{3+m} = W'_{3+m} \cap V_{3+m}$, which implies that W_{3+m} is open and dense too. $\square$

Lemma 4.24 *Set $D := \{2p_1 + 2p_2 - p_3 - r_1 - r_2 - r_3 \sim 0\} \subset R_{1,6}$. Define the following subsets of Φ_n*

$$O_3 := U_3 := \widetilde{\Phi}_3, \quad O_4 := U_4, \quad O_5 := U_5, \quad O_6 := U_6 \smallsetminus f_6^{-1}(D)$$

Then, for $3 \leq n \leq 6$, by definition we have inclusions

$$O_n \subseteq U_n \subseteq \Phi_n.$$

These inclusions are all open and dense. Furthermore:

(i) The O_n, and thus also the U_n, Φ_n, are rational varieties.

(ii) O_n has trivial Chow ring (i.e. $A^*(O_n) = \mathbb{Q}$).

Proof: (i): The case $n = 3$ is clear by Lemma 4.18 (i).

The following is similar to the proof of Lemma 1.2.3. in [Bel98]. Recall the definition of H from Definition 4.20, and note that $H \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. If we denote by $\mathcal{O}_{\mathbb{P}^2_i}(3)$ the pullback to $(\mathbb{P}^2)^m$ of the vector bundle $\mathcal{O}_{\mathbb{P}^2}(3)$ living on the i-th factor of $(\mathbb{P}^2)^m$, then we can define a morphism of (geometric) vector bundles

$$H \times (\mathbb{P}^2)^m \xrightarrow{ev} \oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^2_i}(3)$$

by sending a point $(f; R_1, ..., R_m) \in H \times (\mathbb{P}^2)^m$ to the point $(f(R_1), ..., f(R_m))$ in the fibre of $\oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^2_i}(3)$ over the point $(R_1, ..., R_m)$, where by $f(R_i)$ we denote the value of the global section $f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ in the fibre of $\mathcal{O}_{\mathbb{P}^2}(3)$ at the point $R_i \in \mathbb{P}^2$.

We define K'_{3+m} to be the kernel of the evaluation morphism ev, i.e. the preimage of the 0-section of the bundle $\oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^2_i}(3)$.

The fibre of K'_{3+m} over a point $\mathbf{R} = (R_1, ..., R_m) \in (\mathbb{P}^2)^m$ we denote by $K'(\mathbf{R})$. It corresponds to the sub vector space of H which consists of 0 and all those elements of H which define a cubic C which passes through all the points $R_1, ..., R_m$. Hence $S(\mathbf{R})$ from Lemma 4.23 is the projectivisation of $K'(\mathbf{R})$.

The restriction $K_{3+m} := (K'_{3+m})_{|W_{3+m}}$ is a vector bundle of rank $4-m$ over W_{3+m}, since $\dim K'(\mathbf{R}) = \dim S(\mathbf{R})+1 = 4-m$ for all $\mathbf{R} \in W_{3+m}$. Let $\mathbb{P}(K_{3+m})$ be the projectivisation of this bundle.

For $0 \leq m \leq 6$, we have $O_{3+m} \subseteq U_{3+m} \subseteq \Phi_{3+m}$ by definition, and it is easy to check that

$$O_{3+m} = U_{3+m} \cap (\Phi_3 \times W_{3+m}) \qquad (*)$$

The inclusions $O_{3+m} \subseteq U_{3+m} \subseteq \Phi_{3+m}$ are open because O_{3+m} and U_{3+m} are defined by intersecting Φ_{3+m} with open subsets of $\Phi_3 \times (\mathbb{P}^2)^6$. Clearly O_{3+m}, U_{3+m} are non-empty.

Now $\mathbb{P}(K_{3+m}) \to W_{3+m}$ is a sub-bundle of the projective bundle $\Phi_3 \times W_{3+m} = \mathbb{P}(H) \times W_{3+m}$, and $\mathbb{P}(K_{3+m}) = \Phi_{3+m} \cap (\Phi_3 \times W_{3+m})$ as a subset of $\Phi_3 \times (\mathbb{P}^2)^6$. (For this, recall that the fibre of $\mathbb{P}(K_{3+m})$ over any $\mathbf{R} \in W_{3+m}$ is $S(\mathbf{R})$.) Then by $(*)$, O_{m+3} is contained in $\mathbb{P}(K_{3+m})$. As we have seen O_{3+m} is open and dense in Φ_{3+m}, hence also in $\mathbb{P}(K_{3+m})$.

But as projective bundle over the rational variety $W_{3+m} \subset (\mathbb{P}^2)^m$, $\mathbb{P}(K_{3+m})$ is a rational variety, hence the same is true for the open subvariety O_{3+m}.

(ii): This goes very similar to the proof of Prop. 2.3.1. in [Bel98].

$O_3 = \widetilde{\Phi}_3$ is the open subset of smooth cubics in Φ_3. But $\Phi_3 = \mathbb{P}(H)$, and as stated in the proof of Lemma 4.23, H can be described as the set of polynomials of the form

$$a(x^2z - xz^2) + b(xy^2 - xz^2) + c(xyz - xz^2) + d(y^2z - xz^2), \qquad a, b, c, d \in \mathbb{C}$$

If $d = 0$, the defined cubic is reducible, thus $\widetilde{\Phi}_3$ lies inside the complement of the hyperplane $\{d = 0\}$ in $\Phi_3 \cong \mathbb{P}^3$. Since $\Phi_3 \smallsetminus \{d = 0\} \cong \mathbb{A}^3$, O_3 is an open subvariety of an $\mathbb{A}^3$ and thus $A^*(O_3) = \mathbb{Q}$

As shown above for $m \in \underline{3}$, O_{3+m} is an open subvariety of the projective bundle $\bar{K}_{3+m} := \mathbb{P}(K_{3+m})$ over W_{3+n}. But $W_{3+m} \subseteq V_{3+m} \subset (\mathbb{P}^2 \smallsetminus L_1)^m \cong \mathbb{A}^{2m}$, thus $A^*(W_{m+3}) = \mathbb{Q}$. This implies that $A^*(\bar{K}_{m+3})$ is generated as $\mathbb{Q}$-algebra by the first Chern class $c_1(\mathcal{O}_{\bar{K}_{3+m}}(1))$, by Thm. 3.3. in [Ful98]. If $h : O_{3+m} \to \bar{K}_{3+m}$ is the open embedding it thus suffices to show that $h^*c_1(\mathcal{O}_{\bar{K}_{3+m}}(1)) = 0$ to proof (ii). Since this pullback is equal to $c_1(\mathcal{O}_{\bar{K}_{3+m}}(1)_{|O_{3+m}})$, it suffices to show that $\mathcal{O}_{\bar{K}_{3+m}}(1)$ has a global section vanishing nowhere on O_{3+m}. Choose a linear form on H that vanishes only on the codimension-1 subspace $S = \{d = 0\}$ of H, i.e. choose the linear form d. It gives rise to a global section of $\mathcal{O}_{\bar{K}_{3+m}}(1)$. This section vanishes nowhere on O_{3+m} since O_{3+m} is contained inside the complement $\mathbb{P}(H) \times W_{3+m} \smallsetminus \mathbb{P}(S) \times W_{3+m}$. □

Proposition 4.15 for $n = 1, 2$, and Lemma 4.22 (i) together with Lemma 4.24 (i) for $3 \leq n \leq 6$ immediately imply:

Corollary 4.25 *For $n \leq 6$, $\overline{R}_{1,n}$ is rational.*

Proposition 4.26 *For $n \leq 6$:*

(i) $A^*(R_{1,n}) = \mathbb{Q}$

(ii) The Chow ring $A^*(\overline{R}_{1,n})$ is spanned as $\mathbb{Q}$-vector space by the boundary cycle classes.

Proof: (i): For $n = 2$ we know this by Proposition 4.15 (ii). We proceed by "induction" on n, although the reason for doing so may only become apparent later.

For $3 \leq n \leq 6$, we call f'_n the restriction of the open embedding $f_n : U_n \to R_{1,n}$, to the open subsets $O_n \subseteq U_n$. By Lemma 4.24 (ii) the images of these f'_n have trivial Chow ring. For $3 \leq n \leq 5$, $O_n = U_n$ and thus the image of f'_n is $R_{1,n} \smallsetminus (B_1^{(n)} \cup B_2^{(n)} \cup B_3^{(n)})$ by Lemma 4.22 (ii). For $n = 6$ the image is $R_{1,6} \smallsetminus (B_1^{(6)} \cup B_2^{(6)} \cup B_3^{(6)} \cup D)$ (D defined in Lemma 4.24).

By the exact sequence of Lemma 1.39, to get $A^*(R_{1,n}) = \mathbb{Q}$ it now suffices to show:

1. For all $i \in \underline{3}$, $A^*(B_i^{(n)}) = \mathbb{Q}$. If $n = 6$ also show $A^*(D) = \mathbb{Q}$.
2. The classes $[B_i^{(n)}]$ in $A^*(R_{1,n})$ are all equivalent to 0. If $n = 6$ show the same for $[D]$.

To show the first, note that by Lemma 4.13, for $i \in \underline{3}$ and $n \geq 3$, $B_i^{(n)}$ is isomorphic to an open subvariety of $R_{1,n-1}$, and D is isomorphic to an open subvariety of $R_{1,5}$. Now we apply our induction hypothesis, and get "1.". The part "2." will be shown in Lemma 4.27 below.

(ii): We know from Summary 1.48, that $A^*(\overline{M}_{0,n})$ is spanned by boundary cycle classes. Using this, we show (ii) by "induction" on n. For $n = 1$, by Prop. 4.15, $\overline{R}_{1,1} \cong \mathbb{P}^1$ and so (ii) holds here. Denote by Y_n the boundary $\overline{R}_{1,n} \smallsetminus R_{1,n}$. Then by the exact sequences

$$A_k(Y_n) \to A_k(\overline{R}_{1,n}) \to A_k(R_{1,n}) \to 0 \qquad (k \in \mathbb{N}_0)$$

and by (i) we have $A^r(\overline{R}_{1,n}) = A^{r-1}(Y_n)$ for $r \geq 1$ and $A^0(\overline{R}_{1,n}) = \mathbb{Q}$. Now Y_n is the union of boundary divisors $D_1, ..., D_m$ and by the proof of Lemma 4.4 each D_i is the image of a finite gluing morphisms $\zeta_{D_i} : \overline{R}_{D_i} \to \overline{R}_{1,n}$. Here $\overline{R}_{D_i}$ is of the form $\overline{R}_{1,(\underline{n} \smallsetminus I) \cup \{\bullet\}} \times \overline{M}_{0,I \cup \{\circ\}}$ if $D_i = D_I$ with $I \subseteq \underline{n}$, $|I| \geq 2$. If D_i is D''_0 or D^r_0, then $\overline{R}_{D_i} = \overline{M}_{0,\underline{n} \cup \{\bullet,\circ\}}$. The $\mathbb{Q}$-vector space $A^*(Y_n)$ is generated by the subspaces $(\zeta_{D_i})_* A^*(\overline{R}_{D_i})$, and the same holds for all $A^r(\overline{R}_{1,n})$ $(n \geq 1)$. But all the moduli spaces the $\overline{R}_{D_i}$ are products of, have Chow groups generated by their boundary cycle classes, by the induction hypothesis and Summary 1.48 mentioned above. Furthermore if one pushes forward by an ζ_{D_i} a boundary cycle class, then the result is a boundary cycle class of $\overline{R}_{1,n}$ (cf. Remark 4.6 (iii)). Thus all $A^r(\overline{R}_{1,n})$ for $r \geq 0$ are generated by boundary cycle classes. □

Lemma 4.27 *(i) For all $3 \leq n \leq 6$, $i \in \underline{3}$ the classes $[B_i^{(n)}] \in A^*(R_{1,n})$ are equivalent to 0.*

(ii) The class $[D]$ is equivalent to 0 in $A^*(R_{1,6})$.

Proof: (i): First note that it suffices to show that the class of $B = \{p_1 - p_2 \sim prym\}$ is equivalent to zero on $R_{1,2}$, since every class $[B_i^{(n)}]$ is obtained by pulling back $[B]$ via the forgetful morphism $\pi : R_{1,n} \to R_{1,2}$, and renaming indices if necessary. But we already know by Proposition 4.15 (ii), that $A^*(R_{1,2}) = A^0(R_{1,2}) \cong \mathbb{Q}$. Since B is of codimension 1, thus $[B] = 0$ in $A^*(R_{1,2})$.

(ii) For $\tau_6 : R_{1,6} \to M_{1,6}$ the forgetful morphism and for D' the set $\{2p_1 + 2p_2 - p_3 - r_1 - r_2 - r_3 \sim 0\}$ in $M_{1,6}$ (cf. Lemma 4.12), we have $D = \tau_6^{-1}(D')$ and hence $[D] = \tau_6^*[D']$ in $A^*(R_{1,6})$. But $A^*(M_{1,6}) = \mathbb{Q}$ according to Theorem 2.0.1. of [Bel98], hence $[D'] = 0$. □

Remark: In [Bel98] isomorphisms from rational varieties with trivial Chow ring onto open subvarieties of $M_{1,n}$ are constructed for $n \leq 10$, similar to our embeddings $O_n \to R_{1,n}$ for $n \leq 6$. The complements of these open subvarieties of $M_{1,n}$ are composed of subvarieties of the form $\{\sum a_i p_i \sim 0\}$. Belorousski shows that these closed subvarieties define classes that are equivalent to 0 in the Chow ring $A^*(M_{1,n})$. From this, as in the Lemma above, $A^*(M_{1,n}) = \mathbb{Q}$ follows.

To show that the classes are 0 in $A^*(M_{1,n})$, moduli spaces of pointed admissible covers are utilized. We denote by $\overline{H}_{2,b,n}$ the moduli space of n-pointed admissible double covers of stable $b+n$ pointed genus 0 curves, defined like in the proof of Proposition 4.15 (i). The covering curves in such a cover are of genus $g = \frac{1}{2}b - 1$. Usually one denotes this moduli space by $\overline{H}_{2,g,n}$ instead.

If we choose $b = 4$ always, the covering curves are of genus 1. Now one can define a surjective morphism $\lambda : \overline{H}_{2,4,n} \to \overline{M}_{1,n}$ corresponding to only keeping the covering genus 1 curve with the n marked points, and forming the stable model. This λ is a proper morphisms with fibre-dimension 1. Also there is the finite surjective morphism $\pi : \overline{H}_{2,4,n} \to \overline{M}_{0,4+n}$ corresponding to forgetting the cover and only retaining the underlying rational curve with its marked points.

Denote by D a closed subvariety of $M_{1,n}$, that Belorousski wants to show to have class $[D] = 0$ in $A^*(M_{1,n})$. The boundary of $\overline{H}_{2,4,n}$ consist exactly of those points lying over the boundary of $\overline{M}_{0,n+4}$ with respect to π. But on the other hand the images of some of the boundary cycles of $\overline{H}_{2,4,n}$ under λ meet the interior of $\overline{M}_{1,n}$. Usually D will be the image of such a boundary cycle B of $\overline{H}_{2,4,n}$ under λ (or more precisely, it will be the intersection of such an image with $M_{1,n}$). One can pull back Keel relations from $\overline{M}_{0,n+4}$ to $\overline{H}_{2,4,n}$ via π, and use them to express B in $A^*(\overline{H}_{2,4,n})$ as linear combination of other boundary cycles $B_1, ..., B_r$ of $\overline{H}_{2,4,n}$, such that $\lambda(B_1), ..., \lambda(B_r)$ all do not meet $M_{1,n}$. This will then prove $[D] = 0$ in $A^*(M_{1,n})$.

It is possible to define a morphism $\lambda' : \overline{H}_{2,4,n} \to \overline{R}_{1,n}$ and to use it to apply Belorousski's method directly to $R_{1,n}$. (This morphism is constructed similar to the one in Proposition 4.15 (iii).)

How do boundary cycles $B \subset \overline{H}_{2,4,n}$, such that $\lambda(B)$ meets $M_{1,n}$ look like? There are for example boundary divisors which generally parametrise covers $X \to D$ with the following properties: The covering curve X has one smooth rational component X_0 and one smooth genus 1 component X_1. X_0 meets X_1 in only one point and X_0 carries exactly one of the b ramification points p_i and one of the n marked points q_j. It is also possible that X_0 consists of two disjoint components $X_0^{(1)}$ and $X_0^{(2)}$ which are mapped to the same component of D, and meet X_1 in two different points that are mapped to the same point on D. In this case $X_0^{(1)}$ carries a marked point q_j and $X_0^{(2)}$ carries a marked point q_k, and X_0 contains none of the b ramification points. These covers arise as limits: In the first case let the marked point q_j approach the ramification point p_i. In the second case denote by q_j' the second point in the fibre of the admissible cover that contains the point q_j. Then let q_k approach q_j'. In both cases the stable model of X is a smooth genus 1 curve. But the marked points on this resulting curve are in a special position.

We also remark that it is possible to define a finite surjective morphism $\overline{H}_{2,4,n} \to \overline{M}_{1,n+1}$ (and $\overline{H}_{2,4,n} \to \overline{R}_{1,n+1}$), by interpreting the first of the 4 ramification points as a marked point on the cover. But this morphism may be less useful than λ here, since it maps fewer boundary divisors of $\overline{H}_{2,4,n}$ to the interior of $\overline{M}_{1,n+1}$.

4.4 The Chow rings $A^*(\overline{R}_{1,n})$ for $n \leq 4$

First we prove relations involving the banana cycle classes of $\overline{R}_{1,n}$, which are the only boundary cycle classes we do not already know to lie inside $\tau_n^* A^*(\overline{M}_{1,n})$ by Lemma 4.4.

Lemma 4.28 *(i) In $A^2(\overline{R}_{1,2})$: $d''_{\alpha,\{1\}} = 2d^r_{\alpha,\{1\}}$.*

(ii) In $A^2(\overline{R}_{1,3})$: $d''_{\alpha,\{i\}} = 2d^r_{\alpha,\{i\}}$ for $i = 1,2,3$. Thus $d''_{\alpha,\{i\}}, d^r_{\alpha,\{i\}} \in \tau_3^* A^2(\overline{M}_{1,3})$ for $i = 1,2,3$.

(iii) In $A^2(\overline{R}_{1,3})$: For all possible $\{i,j,k\} = \underline{3}$

$$2d''_{\alpha,\{i\}} = d''_{0,\{ij\}} + d''_{0,\{ik\}} - d''_{0,\{jk\}} + d''_{0,3}.$$

(iv) In $A^2(\overline{R}_{1,4})$:

$$(d''_{\alpha,\{i\}} - d''_{\alpha,\{j\}}) = 2(d^r_{\alpha,\{i\}} - d^r_{\alpha,\{j\}}) \quad \text{for all} \quad i,j \in \underline{4}. \tag{4.2}$$

$$(d''_{\beta,\{1i\}} - d''_{\beta,\{1j\}}) = 2(d^r_{\beta,\{1i\}} - d^r_{\beta,\{1j\}}) \quad \text{for all} \quad i,j \in \{2,3,4\}. \tag{4.3}$$

And for all $\{i,j,k,l\} = \underline{4}$:

$$d''_{\alpha,\{i\}} + d''_{\alpha,\{k\}} + d''_{\beta,\{ij\}} + d''_{\beta,\{il\}} = d''_{0,\{ik\}} + d''_{0,\{ijk\}} + d''_{0,\{ikl\}} + d''_{0,4} \tag{4.4}$$

$$= 4(d^r_{\alpha,\{i\}} + d^r_{\alpha,\{k\}} + d^r_{\beta,\{ij\}} + d^r_{\beta,\{il\}}). \tag{4.5}$$

(v) In $A^3(\overline{R}_{1,4})$, for all possible $\{i,j,k,l\} = \underline{4}$:

$$d''_{\alpha,\{l\},\{ij\}} = 2d^r_{\alpha,\{l\},\{ij\}}, \quad d''_{\alpha,\{l\},\{ijk\}} = 2d^r_{\alpha,\{l\},\{ijk\}}, \quad d''_{\beta,\{ij\},\{ij\}} = 2d^r_{\beta,\{ij\},\{ij\}}.$$

So $d''_{\alpha,\{l\},\{ij\}}, d^r_{\alpha,\{l\},\{ij\}}, d''_{\alpha,\{l\},\{ijk\}}, d^r_{\alpha,\{l\},\{ijk\}}, d''_{\beta,\{ij\},\{ij\}}$ and $d^r_{\beta,\{ij\},\{ij\}}$ lie in $\tau_4^* A^*(\overline{M}_{1,4})$. Furthermore

$$d''_{\gamma,\{ij\}} = d''_{\beta,\{kl\}\{kl\}}, \quad d^r_{\gamma,\{ij\}} = d^r_{\beta,\{kl\}\{kl\}}.$$

Hence also $d''_{\gamma,\{ij\}}, d^r_{\gamma,\{ij\}} \in \tau_4^* A^*(\overline{M}_{1,4})$.

(iv) In $A^3(\overline{R}_{1,4})$, for all possible $\{i,j,k,l\} = \underline{4}$:

$$2d''_{\beta,\{ij\},\{ij\}} = d''_{0,\{k,\{ij\}\}} + d''_{0,\{l,\{ij\}\}} - d''_{0,\{ij\},\{kl\}} + d''_{0,\{kl,\{ij\}\}}$$

Proof: (i): The subvarieties $D''_{\alpha,\{1\}}$ and $D^r_{\alpha,\{1\}}$ of the rational variety $\overline{R}_{1,2}$, are points, and thus $[D''_{\alpha,\{1\}}] = [D^r_{\alpha,\{1\}}]$. But due to the different number of inessential automorphisms of the prym curves parametrised by these points, for the Q-classes we get

$$d''_{\alpha,\{1\}} = [D''_{\alpha,\{1\}}]_Q = 2[D^r_{\alpha,\{1\}}]_Q = 2d^r_{\alpha,\{1\}}.$$

(ii): For $i \in \underline{3}$ let $\pi_i : \overline{R}_{1,3} \to \overline{R}_{1,2}$ be the morphism forgetting the i-th marked point. We have: For $i \neq j \in \underline{3}$, $\pi_i^* d''_{\alpha,\{j\}} = d''_{\alpha,\{j\}} + d''_{\alpha,\{k\}}$ and $\pi_i^* d^r_{\alpha,\{j\}} = d^r_{\alpha,\{j\}} + d^r_{\alpha,\{k\}}$, [3] where k is the unique element of $\underline{3} \setminus \{i,j\}$. Pulling back the relation proven in (i) by the different forgetful morphisms π_1, π_2 and π_3 we obtain

$$d''_{\alpha,\{2\}} + d''_{\alpha,\{3\}} = 2(d^r_{\alpha,\{2\}} + d^r_{\alpha,\{3\}}) \tag{4.6}$$

$$d''_{\alpha,\{1\}} + d''_{\alpha,\{3\}} = 2(d^r_{\alpha,\{1\}} + d^r_{\alpha,\{3\}}) \tag{4.7}$$

$$d''_{\alpha,\{1\}} + d''_{\alpha,\{2\}} = 2(d^r_{\alpha,\{1\}} + d^r_{\alpha,\{2\}}) \tag{4.8}$$

Combining these equations we get the equations of (ii). They in turn imply that $d''_{\alpha,\{i\}}$ and $d^r_{\alpha,\{i\}}$ are both rational multiples of the class $\tau_3^*\delta_{\alpha,\{i\}} = d''_{\alpha,\{i\}} + 2d^r_{\alpha,\{i\}}$.

(iii) First we prove for all $i \neq j \in \underline{3}$:

$$d''_{\alpha,\{i\}} + d''_{\alpha,\{j\}} = d''_{0,\{ij\}} + d''_{0,3}$$

From this the equations of (iii) follow directly. The proof is analogous to the proof of Lemma 3.2.1. in [Bel98]. The cycles involved are all contained in the divisor $D''_0 \subset \overline{R}_{1,3}$. We use the finite surjective gluing morphism

$$\zeta_{D''_0} : \overline{M}_{0,\{1,2,3,\bullet,\circ\}} \to D''_0 \subset \overline{R}_{1,3}.$$

(Cf. Remark 4.6.) Choose any i,j,k with $\{i,j,k\} = \underline{3}$. On $\overline{M}_{0,\{1,2,3,\bullet,\circ\}} \cong \overline{M}_{0,5}$ we have the Keel-relation (cf. Summary 1.48, also for the notation used)

$$[i\bullet] + [j\circ] = [ij] + [\bullet\circ].$$

Pushing this relation forward by $\zeta_{D''_0}$ gives the equation

$$2(d''_{\alpha,\{i\}} + d''_{\alpha,\{j\}}) = 2(d''_{0,\{ij\}} + d''_{0,3}).$$

($\zeta_{D''_0}$ is $2:1$ on any of the divisors involved, except on $[\bullet\circ]$, where it is $1:1$. This is compensated by the fact that the general curve parametrised by $D_{0,3} = \zeta_{D''_0}([\bullet\circ])$ has two automorphisms.)

(iv): This time, for $j \in \underline{4}$, let π_j be the morphism $\overline{R}_{1,4} \to \overline{R}_{1,3}$ forgetting the j-th point. For $i \in \underline{3}$ and $j \in \underline{4} \setminus \{i\}$ we have $\pi_j^* d''_{\alpha,\{i\}} = d''_{\alpha,\{i\}} + d''_{\beta,\{ij\}}$ and $\pi_j^* d^r_{\alpha,\{i\}} = d^r_{\alpha,\{i\}} + d^r_{\beta,\{ij\}}$. Pulling back the equations of (ii) by all the possible π_j we obtain

$$d''_{\alpha,\{i\}} + d''_{\beta,\{ij\}} = 2(d^r_{\alpha,\{i\}} + d^r_{\beta,\{ij\}}) \quad \text{for all} \quad i \in \underline{3},\ j \in \underline{4} \setminus \{i\}.$$

[3] Here $d''_{\alpha,\{j\}}$ denotes two different classes on the right and on the left side, since our notation is uniqe only if also the n of $\overline{R}_{1,n}$ is given.

forming several different combinations of these equations we get the equations (4.2) and (4.3) of (iii).

Equation (4.4) of (iii) is proven analogous to Lemma 3.4.1. of [Bel98]: The cycles involved are all contained in the divisor $D_0'' \subset \overline{R}_{1,4}$. We use the gluing morphism

$$\zeta_{D_0''} : \overline{M}_{0,\{1,\dots,4,\bullet,\circ\}} \to D_0'' \subset \overline{R}_{1,4}$$

existing by Remark 4.6. On $\overline{M}_{0,\{1,\dots,4,\bullet,\circ\}}$ we have the Keel-relation

$$[ik] + [ijk] + [ikl] + [ijkl] = [i\bullet] + [ij\circ] + [il\bullet] + [ijl\circ].$$

Pushing this relation forward by $\zeta_{D_0''}$ gives equation (4.4) multiplied by 2. (Like in the proof of (iii) we have to take into account automorphism numbers.)

Pushing forward the same Keel-relation by the gluing morphism

$$\zeta_{D_0^r} : \overline{M}_{0,\{1,\dots,4,\bullet,\circ\}} \to D_0^r \subset \overline{R}_{1,4}$$

instead of $\zeta_{D_0^r}$, and then applying Lemma 4.4 (ii), yields equation (4.5).

To prove most of the equations in (v) and (vi) we use for $\{i,j,k,l\} = \underline{4}$ the gluing morphisms

$$\zeta_{D_{\{ij\}}} : \overline{R}_{1,\{k,l,\bullet\}} \times \overline{M}_{0,\{i,j,\circ\}} \to D_{\{ij\}} \subset \overline{R}_{1,4},$$
$$\zeta_{D_{\{ijk\}}} : \overline{R}_{1,\{l,\bullet\}} \times \overline{M}_{0,\{i,j,k,\circ\}} \to D_{\{ijk\}} \subset \overline{R}_{1,4}.$$

By (ii) we have the equation $d''_{\alpha,\{l\}} = 2d^r_{\alpha,\{l\}}$ in $A^*(\overline{R}_{1,3})$. Pushing $d''_{\alpha,\{l\}} \otimes 1 = 2d^r_{\alpha,\{l\}} \otimes 1$ forward by $\zeta_{D_{\{i,j\}}}$ yields $d''_{\alpha,\{l\},\{ij\}} = 2d^r_{\alpha,\{l\},\{ij\}}$. Also by (ii) we know $d''_{\alpha,\{\bullet\}} \otimes 1 = 2d^r_{\alpha,\{\bullet\}} \otimes 1$, which, pushed forward by $\zeta_{D_{\{ij\}}}$ gives $d''_{\beta,\{ij\},\{ij\}} = 2d^r_{\beta,\{ij\},\{ij\}}$. In $A^*(\overline{R}_{1,\{l,\bullet\}})$ the equation $d''_{\alpha,\{l\}} = 2d^r_{\alpha,\{l\}\}}$ holds by (i). We push $d''_{\alpha,\{l\}} \otimes 1 = 2d^r_{\alpha,\{l\}} \otimes 1$ forward by $\zeta_{D_{\{ijk\}}}$ to obtain $d''_{\alpha,\{l\},\{ijk\}} = 2d^r_{\alpha,\{l\},\{ijk\}}$. By (iii) on $\overline{R}_{1,\{k,l,\bullet\}}$ we have the equation:

$$2d''_{\alpha,\{\bullet\}} = d''_{0,\{\bullet k\}} + d''_{0,\{\bullet l\}} - d''_{0,\{kl\}} + d''_{0,3}$$

We push this forward by $\zeta_{D_{\{ij\}}}$ and obtain the equation of (vi).

It only remains to show the equations in (v) involving $d''_{\gamma,\{ij\}}$ and $d^r_{\gamma,\{ij\}}$. They are proved using the boundary morphisms

$$\zeta_{D''_{\beta,\{kl\}}} : \overline{M}_{0,\{i,j,\bullet_1,\bullet_2\}} \times \overline{M}_{0,\{k,l,\circ_1,\circ_2\}} \to D''_{\beta,\{kl\}} \subset \overline{R}_{1,4},$$
$$\text{and} \quad \zeta_{D^r_{\beta,\{kl\}}} : \overline{M}_{0,\{i,j,\bullet_1,\bullet_2\}} \times \overline{M}_{0,\{k,l,\circ_1,\circ_2\}} \to D^r_{\beta,\{kl\}} \subset \overline{R}_{1,4}.$$

Now $d''_{\gamma,\{ij\}} = (\zeta_{D''_{\beta,\{kl\}}})_* (1 \otimes [k,\bullet_1])$ and $d''_{\beta,\{kl\},\{kl\}} = (\zeta_{D''_{\beta,\{kl\}}})_* (1 \otimes [k,l])$. But the Keel-relation $[k,\circ_1] = [k,l]$ holds in $A^1(\overline{M}_{0,\{k,l,\circ_1,\circ_2\}})$, thus $d''_{\gamma,\{ij\}} = d''_{\beta,\{kl\}\{kl\}}$. The relation involving $d^r_{\gamma,\{ij\}}$ is proven analogously, using $\zeta_{D^r_{\beta,\{kl\}}}$ instead of $\zeta_{D''_{\beta,\{kl\}}}$. □

Corollary 4.29 *(i) For $n = 1,2,3$, the pullback $\tau_n^* : A^*(\overline{M}_{1,n}) \to A^*(\overline{R}_{1,n})$ is an isomorphism of $\mathbb{Q}$-algebras.*

(ii) The $\mathbb{Q}$-vector space $A^*(\overline{R}_{1,4})$ is spanned by the subspace $\tau_4^*\, A^*(\overline{M}_{1,4})$ together with the class $d''_{\beta,\{12\}} \in A^2(\overline{R}_{1,4})$.

Proof: The pullback τ_n^* is injective for arbitrary n since τ_n is finite and surjective. We know by Proposition 4.26 that the Chow rings $A^*(\overline{R}_{1,n})$ for $n \le 4$ are generated by boundary cycle classes, and by Lemma 4.4, among these classes only the banana cycle classes can fail to lie in $\tau_n^* A^*(\overline{M}_{1,n})$. One can list easily all the banana cycles that exist on $\overline{R}_{1,n}$ for $n \le 4$ using the list of boundary cycles of $\overline{M}_{1,n}$ in Section 4.1.1 and Lemma 4.4 (iii).

(i): The banana classes $d''_{\alpha,\{i\}}, d^r_{\alpha,\{i\}} \in A^2(\overline{R}_{1,3})$ lie in $\tau_3^* A^*(\overline{M}_{1,3})$ by part (ii) of Lemma 4.28. The other banana classes that exist on $\overline{R}_{1,2}$ resp. $\overline{R}_{1,3}$, can not cause problems, since they are all of dimension 0. So by the rationality of $\overline{R}_{1,2}$ and $\overline{R}_{1,3}$ they are equivalent to a rational multiple of any other point on $\overline{R}_{1,2}$ resp. $\overline{R}_{1,3}$.

(ii): The banana classes of $\overline{R}_{1,4}$ of dimension > 0 lie in $A^2(\overline{R}_{1,4})$ and $A^3(\overline{R}_{1,4})$. All the banana classes in $A^3(\overline{R}_{1,4})$ are shown to lie inside $\tau_4^* A^*(\overline{M}_{1,4})$ in Lemma 4.28 (v).

So $A^*(\overline{R}_{1,4})$ is spanned by $\tau_n^* A^*(\overline{M}_{1,4})$, and the banana classes in $A^2(\overline{R}_{1,4})$, i.e. the classes of the form $d''_{\alpha,\{i\}}$, $d^r_{\alpha,\{i\}}$, $d''_{\beta,\{ij\}}$ or $d^r_{\beta,\{ij\}}$. Using the equations (4.2) and (4.3) in Lemma 4.28 (iv) we get

$$\begin{aligned}
&\forall\, i,j \in \underline{4} \quad d''_{\alpha,\{i\}} - 2d^r_{\alpha,\{i\}} = d''_{\alpha,\{j\}} - 2d^r_{\alpha,\{j\}} \\
\Rightarrow\ &\forall\, i,j \in \underline{4} \quad 2d''_{\alpha,\{i\}} - \tau_4^*\delta_{\alpha,\{i\}} = 2d''_{\alpha,\{j\}} - \tau_4^*\delta_{\alpha,\{j\}} \\
\Rightarrow\ &\forall\, i,j \in \underline{4} \quad (d''_{\alpha,\{i\}} - d''_{\alpha,\{j\}}) \in \tau_4^* A^*(\overline{M}_{1,4})
\end{aligned}$$

Analogously one shows

$$\forall\, i,j \in \underline{4} \quad (d^r_{\alpha,\{i\}} - d^r_{\alpha,\{j\}}),\ (d''_{\beta,\{1i\}} - d''_{\beta,\{1j\}}),\ (d^r_{\beta,\{1i\}} - d^r_{\beta,\{1j\}}) \in \tau_4^* A^*(\overline{M}_{1,4})$$

This, together with $d''_{\alpha,\{i\}} + 2d^r_{\alpha,\{i\}} = \tau_4^*\, \delta_{\alpha,\{i\}} \in \tau_4^* A^*(\overline{M}_{1,4})$, and $d''_{\beta,\{1i\}} + 2d^r_{\beta,\{1i\}} \in \tau_4^* A^*(\overline{M}_{1,4})$, implies that $A^*(\overline{R}_{1,4})$ is spanned by $\tau_4^* A^*(\overline{M}_{1,4})$ and say the two banana cycle classes $d''_{\alpha,\{1\}}$, $d''_{\beta,\{12\}}$. But if we choose $(i,j,k,l) = (1,2,3,4)$ in equation (4.4) from Lemma 4.28 (iv) , it can be rewritten as

$$2d''_{\alpha,\{1\}} + 2d''_{\beta,\{12\}} = (d''_{\alpha,\{1\}} - d''_{\alpha,\{3\}}) + (d''_{\beta,\{12\}} - d''_{\beta,\{14\}}) + d''_{0,\{13\}} + d''_{0,\{123\}} + d''_{0,\{124\}} + d''_{0,4}.$$

Every summand on the right hand side lies in $\tau_4^* A^*(\overline{M}_{1,4})$, either by what we have just shown or by Lemma 4.4. So $d''_{\alpha,\{1\}} + d''_{\beta,\{12\}} \in \tau_4^* A^*(\overline{M}_{1,4})$, and claim (ii) of our Lemma follows. □

We cite the following Lemma from [Bel98]:

Lemma 4.30 (3.4.8. in [Bel98]) *The following* 23 *linearly independent classes span the* $\mathbb{Q}$*-vector space* $A^2(\overline{M}_{1,4})$*:*

$$\delta_{0,\{ij\}} \text{ (6 classes)}, \quad \delta_{0,\{ijk\}} \text{ (4 classes)}, \quad \delta_{0,4}, \quad \delta_{\{ij\},\{kl\}} \text{ (3 classes)},$$

$$\delta_{\{1,\{23\}\}}, \quad \delta_{\{1,\{24\}\}}, \quad \delta_{\{1,\{34\}\}}, \quad \delta_{\{2,\{34\}\}},$$

$$\delta_{\{jk,\{1i\}\}} \text{ (3 classes)}, \quad \delta_{\{1,\{234\}\}}, \quad \delta_{\{2,\{134\}\}}.$$

The 23×23 *matrix of the intersection numbers of these* 23 *classes has full rank. (In [Bel98] this matrix is not written down, but it is stated that one can compute it by Fabers algorithm [Fab99].)*

Lemma 4.31 *The class $d''_{\beta,\{12\}} \in A^2(\overline{R}_{1,4})$ is not contained in $\tau_4^* A^*(\overline{M}_{1,4})$.*

Proof: We choose $(i,j,k,l) = (1,3,2,4)$ in equation (4.4) from Lemma 4.28 (iv), and multiply it by $d''_{\beta,\{12\}}$. Since every boundary cycle class on the right hand side of (4.4) can be expressed as a product of d''_0 with some other boundary divisor class, we can apply Lemma 4.8 (ii) and obtain:

$$d''_{\alpha,\{1\}}d''_{\beta,\{12\}} + d''_{\alpha,\{2\}}d''_{\beta,\{12\}} + d''_{\beta,\{13\}}d''_{\beta,\{12\}} + d''_{\beta,\{14\}}d''_{\beta,\{12\}} = 0.$$

The intersections $d''_{\beta,\{13\}}d''_{\beta,\{12\}}$, $d''_{\beta_{14}}d''_{\beta,\{12\}}$ are proper, and each of $D''_{\beta,\{13\}} \cap D''_{\beta,\{12\}}$ and $D''_{\beta,\{14\}} \cap D''_{\beta,\{12\}}$ is a point which parametrises a prym curve without non-trivial automorphisms. Thus $d''_{\beta,\{13\}}d''_{\beta,\{12\}} = d''_{\beta,\{14\}}d''_{\beta,\{12\}} = 1$. Note that the intersection numbers $d''_{\alpha,\{1\}}d''_{\beta,\{12\}}$ and $d''_{\alpha,\{2\}}d''_{\beta,\{12\}}$ must be the same since we can replace one by the other by exchanging the names of the indices 1 and 2. Thus $d''_{\alpha,\{1\}}d''_{\beta,\{12\}} = d''_{\alpha,\{2\}}d''_{\beta,\{12\}} = -1$. Using such "swapping of indices" arguments, one can show that for all $\{i,j,i',j'\} = \underline{4}$:

$$d''_{\alpha,\{i\}}d''_{\alpha,\{j\}} = d''_{\alpha,\{i'\}}d''_{\alpha,\{j'\}}, \quad d''_{\alpha,\{i\}}d''_{\beta,\{1j\}} = d''_{\alpha,\{i'\}}d''_{\beta,\{1j'\}},$$

$$\text{and} \quad d''_{\beta,\{1i\}}d''_{\beta,\{1j\}} = d''_{\beta,\{1i'\}}d''_{\beta,\{1j'\}}$$

Multiplying equation (4.4) by $d''_{\beta,\{14\}}$ we get

$$d''_{\alpha,\{1\}}d''_{\beta,\{14\}} + d''_{\alpha,\{2\}}d''_{\beta,\{14\}} + d''_{\beta,\{13\}}d''_{\beta,\{14\}} + (d''_{\beta,\{14\}})^2 = 0$$

Inserting $d''_{\alpha,\{1\}}d''_{\beta,\{14\}} = d''_{\alpha,\{2\}}d''_{\beta,\{14\}} = -1$ and $d''_{\beta,\{13\}}d''_{\beta,\{14\}} = 1$, yields $(d''_{\beta,\{14\}})^2 = 1$. We obtain $(d^r_{\beta,\{14\}})^2 = \frac{1}{8}$ by an analogous argument, using equation (4.5) instead of (4.4). Here we have to take into account, that the curve parametrised by the point $D^r_{\beta,\{13\}} \cap D^r_{\beta,\{12\}}$ or $D^r_{\beta,\{14\}} \cap D^r_{\beta,\{12\}}$ has 8 automorphisms.

Lemma 4.30 gives 23 classes which generate $A^*(\overline{M}_{1,4})$. We pull back these classes via τ_4. Let M be the 23×23 matrix of intersection numbers of the pulled back classes. M is just the intersection matrix of Lemma 4.30, multiplied by $\deg \tau_4 = 3$, and thus has full rank.

We want to determine the intersections of $d''_{\beta,\{12\}}$ and $d^r_{\beta,\{12\}}$ with these 23 classes generating $\tau_4^* A^*(\overline{M}_{1,4})$. By Lemma 4.8 (ii) the intersections with all of the first 11 classes are 0. The class $\tau_4^* \delta_{\{12\},\{34\}} = d_{\{12\},\{34\}}$ intersects both $d''_{\beta,\{12\}}$ and $d^r_{\beta,\{12\}}$ properly. The points $D''_{\beta,\{12\}} \cap D_{\{12\},\{34\}}$ resp. $D^r_{\beta,\{12\}} \cap D_{\{12\},\{34\}}$ parametrise prym curves with 2 resp. 4 automorphisms. (The first prym curve has a non-trivial automorphism swapping the two non-disconnecting nodes, the second prym curve caries a lifting of this automorphism, and furthermore its number of inessential automorphisms is 2.) Thus $d''_{\beta,\{12\}}d_{\{12\}\{34\}} = \frac{1}{2}$ and $d^r_{\beta,\{12\}}d_{\{12\}\{34\}} = \frac{1}{4}$. It is easy to check that the components of all the other 11 pulled back classes, do meet neither $D''_{\beta,\{12\}}$ nor $D^r_{\beta,\{12\}}$, so the intersections with these classes are 0. From our calculations above we know $(d''_{\beta,\{12\}})^2 = 1$ and $(d^r_{\beta,\{12\}})^2 = \frac{1}{8}$. We get $d''_{\beta,\{12\}}d^r_{\beta,\{12\}} = 0$, since $D''_{\beta,\{12\}} \cap D^r_{\beta,\{12\}} = \emptyset$. Putting together this information we see that the 25×25 matrix of intersection numbers of the 23 pulled back classes together with the classes $d''_{\beta,\{12\}}$ and $d^r_{\beta,\{12\}}$ is of the form:

$$\left(\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & M & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \frac{1}{4} \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & \frac{1}{2} & & 1 & \\
 & & & \frac{1}{4} & & & \frac{1}{8}
\end{array}\right)$$

The empty spaces in the matrix are meant to be filled by zeros. Since the 23×23 matrix M sitting in the upper left corner has full rank, it is easy to see that the whole matrix has at least rank 24. So $\dim_{\mathbb{Q}} A^2(\overline{R}_{1,4}) \geq 24$. Together with Corollary 4.29 (ii) this implies that $\dim_{\mathbb{Q}} A^2(\overline{R}_{1,4}) = 24$ and $d''_{\beta,\{12\}} \notin \tau_4^* A^*(\overline{M}_{1,4})$. □

Theorem 4.32 *(i) The Chow ring $A^*(\overline{R}_{1,4})$ is given by*

$$\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]/I$$

where $D_1, ..., D_{12}$ are the 12 divisor classes obtained by pulling back the 12 boundary divisor classes of $\overline{M}_{1,4}$, and where I is an ideal described below. The dimensions of the homogeneous pieces of the Chow ring are $1, 12, 24, 12, 1$. The pairing

$$A^k(\overline{R}_{1,4}) \times A^{4-k}(\overline{R}_{1,4}) \to \mathbb{Q}$$

is perfect.

(ii) The ideal I is generated by the union of:

1. All the 56 relations one obtains by pulling back the generating relations of the ideal K, described in Summary 4.10 (v). (55 are in codimension 2, one in codimension 3.)

2. The following relations involving $d''_{\beta,\{12\}}$. (12 in codimension 3, one in codimension 4.) Here we denote the pullback of a boundary divisor $\delta_{...}$ of $\overline{M}_{1,4}$ via τ_4, by the same symbol $\delta_{...}$ again:

$$\delta_{\{13\}}d''_{\beta,\{12\}} = 0, \quad \delta_{\{14\}}d''_{\beta,\{12\}} = 0, \quad \delta_{\{23\}}d''_{\beta,\{12\}} = 0, \quad \delta_{\{24\}}d''_{\beta,\{12\}} = 0$$

$$\forall \{i,j,k\} \subset \{1, ..., 4\}: \quad \delta_{\{ijk\}}d''_{\beta,\{12\}} = 0 \quad \textit{(4 relations)},$$

$$\delta_4 d''_{\beta,\{12\}} = 0, \quad \delta_0 d''_{\beta,\{12\}} = 0,$$

$$2\delta_{\{12\}}d''_{\beta,\{12\}} = d''_0\delta_{\{12\}}(\delta_{\{123\}} + \delta_{\{124\}} - \delta_{\{34\}} + \delta_4),$$

$$2\delta_{\{34\}}d''_{\beta,\{12\}} = d''_0\delta_{\{34\}}(\delta_{\{134\}} + \delta_{\{234\}} - \delta_{\{12\}} + \delta_4)$$

$$(d''_{\beta,\{12\}})^2 = 2\delta_4\delta_{\{234\}}\delta_{\{34\}}$$

Proof: (i): The $\mathbb{Q}$-algebra $\tau_4^* A^*(\overline{M}_{1,4})$ is generated by $D_1, ..., D_{12}$, since these are the pullbacks of the generators of $A^*(\overline{M}_{1,4})$ (cf. Summary 4.10 (iv)). So together with the class $d''_{\beta,\{12\}}$ they generate $A^*(\overline{R}_{1,4})$ by Corollary 4.29 (ii). This Corollary also implies together with Lemma 4.31 and Summary 4.10 (iv) that the dimensions of the homogeneous pieces are $1, 12, 24, 12, 1$.

The perfect pairing claim in (i) follows for $k \neq 2$ from the analogous statement in Summary 4.10 (iv), since the only graded piece of the Chow ring which is not contained in $\tau_4^* A^*(\overline{M}_{1,4})$ is $A^2(\overline{R}_{1,4})$. For $k = 2$ it follows from the fact that the intersection matrix in the proof of Lemma 4.31 has rank 24.

(ii): We say that a relation between elements of $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]$ is *true* if it holds in $A^*(\overline{R}_{1,4})$. It is clear (since τ_4^* is injective) that the pullbacks of the 56 generating relations of K yield 56 independent true relations. We know that $A^3(\overline{R}_{1,4})$ and $A^4(\overline{R}_{1,4})$ are generated by products of the $D_1, ..., D_{12}$. The degree 3 part of the polynomial ring $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]$ with adjusted grading $\deg d''_{\beta,\{12\}} := 2$, is spanned over the degree 3 part of $\mathbb{Q}[D_1, ..., D_{12}]$ by the 12 elements of the form $D_i d''_{\beta,\{12\}}$. So if I' is an ideal generated by the 56 old relations and 12 independent true codimension 3 relations involving the elements $D_i d''_{\beta,\{12\}}$, we get that the degree ≤ 3 pieces of $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]/I'$ and $A^*(\overline{R}_{1,4})$ coincide. Furthermore the degree 4 component of $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]/I'$ is then spanned by products of the D_i together with the class $(d''_{\beta,\{12\}})^2$. So if I is an ideal generated by I' and one true codimension 4 relation expressing $(d''_{\beta,\{12\}})^2$ in terms of products of the D_i, then $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]/I = A^*(\overline{R}_{1,4})$. This includes that the degree ≥ 5 homogeneous parts of $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]/I$ are 0. To check this, note that every element of such a homogeneous part can be generated by the $D_1, ..., D_{12}$, and that the sub-algebra of $\mathbb{Q}[D_1, ..., D_{12}, d''_{\beta,\{12\}}]/I$ generated by these divisor classes is isomorphic to $A^*(M_{1,n})$ by definition of I and Summary 4.10 (iv).

Now the ideal I defined in (ii) is of the form just described, provided that the relations we used to define it are true: That the new relations are independent as required, is clear with Lemma 4.30. We will check that they are true:

The relation $\delta_0 d''_{\beta,\{12\}} = 0$ is true by Lemma 4.8 (ii). All the other relations of the form $D_i d''_{\beta,\{12\}} = 0$ are obtained by observing that the divisors D_i involved do not even meet $D''_{\beta,\{12\}}$ as sets. We calculated in the proof of Lemma 4.31 that $(d''_{\beta,\{12\}})^2 = 1$. The intersection $\delta_4 \delta_{\{234\}} \delta_{\{34\}}$ is proper, and the point $\Delta_4 \cap \Delta_{\{234\}} \cap \Delta_{\{34\}}$ parametrises a prym curve with 2 automorphisms. (There is an elliptic involution on the genus 1 component). Thus $(d''_{\beta,\{12\}})^2 = 1 = 2\delta_4 \delta_{\{234\}} \delta_{\{34\}}$.

The remaining two relations are just the equations one gets from Lemma 4.28 (vi), if one chooses $\{i, j\} = \{1, 2\}$ resp. $\{i, j\} = \{3, 4\}$. All one has to do is to expresses the boundary cycles whose classes appear in the equations in the natural way as intersections of boundary divisors (cf. Remark 4.6 (i), Proposition 4.2). Here one also uses that $D''_{\beta,\{34\}} = D''_{\beta,\{12\}}$ as subvarieties of $\overline{R}_{1,4}$ and thus $d''_{\beta,\{34\}} = d''_{\beta,\{12\}}$.) □

Chapter 5

Orbifold cohomology of $\overline{R}_{1,n}$

Following Nicola Paganis article [Pag08] where the Chen-Ruan cohomology $H^*_{CR}(\overline{M}_{1,n})$ of $\overline{M}_{1,n}$ is computed as an algebra over the usual cohomology ring of $\overline{M}_{1,n}$, we do (nearly) the same for $\overline{R}_{1,n}$. For any $n \in \mathbb{N}$, the two moduli spaces $\overline{R}_{1,n}$ and $\overline{S}^+_{1,n}$ are isomorphic as coarse moduli spaces, but differ slightly as stacks or orbifolds, since some of the singular objects in $\overline{S}^+_{1,n}$ have more exceptional components than their counterparts in $\overline{R}_{1,n}$ which leads to additional inessential automorphisms. Very similarly $\overline{S}^-_{1,n}$ is isomorphic to $\overline{M}_{1,n}$ as variety but differs slightly as a stack. Accordingly $H^*_{CR}(\overline{S}^+_{1,n})$ is not isomorphic to $H^*_{CR}(\overline{R}_{1,n})$. After examining $H^*_{CR}(\overline{R}_{1,n})$, we will (in section 5.5.6) remark on how $H^*_{CR}(\overline{S}^+_{1,n})$ differs from $H^*_{CR}(\overline{R}_{1,n})$.

After providing the necessary general background in Chen-Ruan cohomology in the first section, the second and fourth section of this chapter deal with the additive structure of $H^*_{CR}(\overline{R}_{1,n})$. The main results there will be the description of the inertia stack $I_1(\overline{R}_{1,n})$ by giving a decomposition into 1-sectors (Thm. 5.32), and Thm. 5.40 expressing the graded $\mathbb{Q}$ vector space $H^*_{CR}(\overline{R}_{1,n})$ explicitly as a direct sum of $H^*(\overline{R}_{1,n})$, and known other cohomology spaces. Section 3 in between provides information about the simple banana cycles of $\overline{R}_{1,n}$, many of which appear as supports of 1-sectors. These are 1-sectors belonging to inessential automorphisms, and they are responsible for the main differences between $H^*(\overline{R}_{1,n})$ and $H^*(\overline{M}_{1,n})$.

The fifth section is concerned with the multiplicative structure of $H^*_{CR}(\overline{R}_{1,n})$. Of course here one would like to determine this ring as a $\mathbb{Q}$-algebra, in terms of generators and relations. But unfortunately since even the ring structure of the usual cohomology $H^*(\overline{R}_{1,n})$ is far from known, this seems out of reach. ($H^*(\overline{R}_{1,n})$ is a part of $H^*_{CR}(\overline{R}_{1,n})$.) What is possible, is to (mostly) determine the structure of $H^*_{CR}(\overline{R}_{1,n})$ as an $H^*(\overline{R}_{1,n})$-algebra, in terms of generators and relations. We determine independent generators of this algebra, and many relations involving these generators (Thm. 5.58). For each $n \in \mathbb{N}$, these relations are all that exist, if and only if $H^*_{BCl}(\overline{R}_{1,n})$, the subalgebra of $H^*(\overline{R}_{1,n})$ generated by boundary cycle classes of $\overline{R}_{1,n}$, is already the whole even part $H^{2*}(\overline{R}_{1,n})$ of the cohomology. For $\overline{M}_{1,n}$ the analogue is an old but still not proven claim by Ezra Getzler, but I do not know whether one should expect the same for $\overline{R}_{1,n}$:

Claim 5.1 (E. Getzler, [Get97], page 1) (i) For all $n \in \mathbb{N}$, $H^*_{BCl}(\overline{M}_{1,n}) = H^{2*}(\overline{M}_{1,n})$.

(ii) The space of relations between the boundary cycle classes of $\overline{M}_{1,n}$ in $H^*(\overline{M}_{1,n})$ is generated by the pushforwards of (Keel-)relations from the spaces $\overline{M}_{0,n}$ via the gluing morphisms to the boundary cycles, together with the relations obtained on $H^*(\overline{M}_{1,n})$ from the new relation in $H^*(\overline{M}_{1,4})$ computed in [Get97].

In a sixth section we will use the information gathered in [Pag08] and the earlier parts of this chapter about the automorphisms of objects in $\overline{M}_{1,n}$ resp. $\overline{R}_{1,n}$ to determine the singular locus and the locus of canonical singularities of $\overline{M}_{1,n}$ and $\overline{R}_{1,n}$. This will be done in the style of [Lud07], and also implies a result about lifting of pluricanonical forms, that is necessary to make computations of the Kodaira dimension of these spaces rigorous. In the last part of this section we compute the Kodaira dimension of $\overline{R}_{1,11}$, which seems to be the only $\overline{R}_{1,n}$ for which the Kodaira dimension was not known before.

The results on $H^*_{CR}(\overline{R}_{1,n})$ obtained are mainly relative to $H^*(\overline{R}_{1,n})$. Unlike to the case of $\overline{M}_{1,n}$, whose cohomology was investigated in work of E. Getzler, not much is known about $H^*(\overline{R}_{1,n})$. In an appendix to this chapter (section 5.7) we show that the Chow-Rings $A^*(\overline{R}_{1,n})$ computed for $n \leq 4$ in section 4.4 coincide with $H^*(\overline{R}_{1,n})$ via the cycle map. So for $n \leq 4$ our results determine the structure of $H^*_{CR}(\overline{R}_{1,n})$ as a $\mathbb{Q}$-algebra.

5.1 Orbifolds and the Chen-Ruan orbifold cohomology

We give a short summary of the basic definitions and results of Chen-Ruan orbifold cohomology, mainly from [CR04], [Pag06] and [Pag08].

Definition 5.2 Let X be a paracompact Hausdorff space.

(i) Let $U \subseteq X$ be open. Then a *complex uniformising system* of dimension n for U is a triple (V, G, ρ, π) such that: V is a connected open subset of $\mathbb{C}^n$, G is a finite group, $\rho : G \to \mathrm{Aut}(V)$ a group homomorphism (not necessarily injective), where $\mathrm{Aut}(V)$ is the group of holomorphic automorphisms of V. And $\pi : V \to U$ is a continuous map that factors through the quotient $V/G := V/\rho(G)$ and induces a homeomorphisms $V/G \to U$.

(ii) An *embedding of complex uniformising systems* $(V, G, \rho, \pi) \hookrightarrow (V', G', \rho', \pi')$ is a pair (φ, λ), where $\varphi : V \to V'$ is a holomorphic embedding, $\pi = \pi' \circ \varphi$, while $\lambda : G \to G'$ is a group homomorphism such that $\varphi \circ \rho(g) = \rho'(\lambda(g)) \circ \varphi$.

We will usually suppress the ρ in our notation of uniformising systems.

(iii) A *complex orbifold atlas* on X is a family $\mathcal{V}$ of complex uniformising systems (V, G, π) such that: The family of the $\pi(V)$ covers X. Let $(V, G, \pi), (V', G', \pi') \in \mathcal{V}$. Then for every point $x \in \pi(V) \cap \pi(V')$, there is a $(V'', G'', \pi'') \in \mathcal{V}$ such that $x \in \pi(V'') \subseteq \pi(V) \cap \pi(V')$. Furthermore, if $\pi(V) \subseteq \pi(V')$ then there exists an embedding of uniformising systems $(V, G, \pi) \hookrightarrow (V', G', \pi')$.

(iv) Two orbifold atlases are called *equivalent* if they have a common refinement with respect to embeddings of uniformising systems. A *complex orbifold* $[X]$ is a paracompact

Hausdorff space X together with an equivalence class of complex orbifold atlases on X.

(v) For a complex orbifold $[X]$, it makes sense to say that a given uniformising system (V, G, π) belongs to the orbifold. For each point $x \in X$, there is a uniformising system (V_x, G_x, π_x), belonging to $[X]$, such that: V_x is the complex n-ball centred at o, $\pi^{-1}(x) = o$, i.e. G_x fixes o. One calls G_x the *local group* at x, and calls $[X]$ a *reduced orbifold* if G_x acts effectively on V_x for every $x \in X$.

Definition 5.3 (i) For a complex orbifold $[X]$ we define the *k-th inertia orbifold (or inertia stack)* $I_k([X])$ to be the set of all tuples

$$I_k([X]) := \{(x, \mathbf{g}) | x \in X,\ \mathbf{g} = (g_1, ..., g_k),\ g_1, ..., g_k \in G_x\} / \sim,$$

where $\sim$ is defined by: $(x, (g_1, ..., g_k)) \sim (x', (g'_1, ..., g'_k))$ if $x = x'$ and there is a $g \in G_x$, such that $g g_j g^{-1} = g'_j$ for each $j \in \underline{k}$. Note that $\sim$ is trivial if G_x is abelian. In case $k = 1$, $I_1([X]) = \{(x, g)\} / \sim$ is endowed with an orbifold structure by charts

$$\pi_{(x,g)} : (V_x^g, C(g)) \to V_x^g / C(g)$$

around each point $(x, g) \in I_1([X])$, where $V_x^g = \mathrm{Fix}(g)$ is the subset of V_x fixed by g, and $C(g)$ is the centraliser of g in G_x. For general k this is generalised to charts

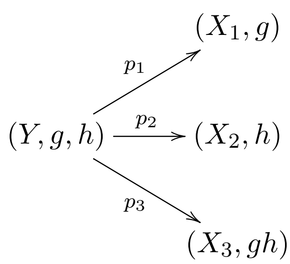
$$\pi_{(x,\mathbf{g})} : (V_x^{\mathbf{g}}, C(\mathbf{g})) \to V_x^{\mathbf{g}} / C(\mathbf{g})$$

around $(x, \mathbf{g}) \in I_k([X])$, where $V_x^{\mathbf{g}} := V_x^{g_1} \cap V_x^{g_1} \cap ... \cap V_x^{g_k}$ and $C(\mathbf{g}) = C(g_1) \cap C(g_2) \cap ... \cap C(g_k)$.

(ii) For any k there is a forgetful morphism $\chi_k : I_k([X]) \to [X]$, sending $(x, (g_1, ..., g_k))$ to x. The connected components of $I_k([X])$ are called the *sectors* of $I_k([X])$ or the *k-sectors* of $[X]$. If $[S] \subseteq I_k([X])$ is such a sector, we usually denote it by $(Y, g_1, ..., g_k)$, where $Y := \chi_k([S])$ is called the *support* of $[S]$ and $g_1, ..., g_k$ are the group elements belonging to some point $(x, (g_1, ..., g_k)) \in [S]$. Note that Y and $(g_1, ..., g_k)$ determine $[S]$. Also note that one sector of $I_1([X])$ is $([X], 1)$ where 1 stands for the unit in G_x for any $x \in X$.

The term "k-sectors" is not really standard. Usually the 1-sectors except $([X], 1)$ are called the *twisted sectors*, while $([X], 1)$ is called the *untwisted sector*. The 2-sectors are sometimes called double-twisted sectors. But we will not use this terminology often.

(iii) For every 2-sector $(Y, g, h) \subseteq I_2([X])$, there are forgetful morphisms:



By the same symbols we denote the forgetful morphisms $p_i : I_2([X]) \to I_1([X])$, where p_i (for $i \in \underline{3}$) is the morphism obtained as the union, over all 2-sectors $(Y, g, h) \subseteq I_2([X])$, of the p_i introduced before.

Definition 5.4 Let V be an n-dimensional $\mathbb{C}$ vector space, φ an automorphism of finite order m on V.

(i) Then one can choose a basis of V relative to which φ is represented by a diagonal matrix $M(\varphi)$. If ζ is any primitive m-th root of unity, then

$$M(\varphi) = \begin{pmatrix} \zeta^{b_1} & & \\ & \ddots & \\ & & \zeta^{b_n} \end{pmatrix}$$

for appropriate $0 \leq b_i < m$. We define the *age of φ with respect to ζ* to be

$$\operatorname{age}(\varphi, \zeta) := \frac{1}{m} \sum_{i=1}^{n} b_i.$$

This is also called the Reid-Tai sum of φ with respect to ζ. Note that this sum depends on ζ but not on the chosen basis of V.

(ii) For $\zeta_1 := e^{2\pi i \frac{1}{m}}$ we denote $a(\varphi) := \operatorname{age}(\varphi, \zeta_1)$ and call it the age of φ.

(iii) For a point $(x, \varphi) \in I_1([X])$ for some orbifold $[X]$, φ acts on V_x, fixing the origin, and the action on this complex n-ball can be linearised and extended to $\mathbb{C}^n$. Then we define $a(x, \varphi) := a(\varphi)$ for this action of φ on $\mathbb{C}^n$. For a sector (Y, g) of $I_1([X])$, $a(x, \varphi)$ is the same for all $(x, \varphi) \in (Y, g)$. We define $a(Y, g) := a(x, \varphi)$ for any $(x, \varphi) \in (Y, g)$, and call this the age of (Y, g)

Definition 5.5 Now we define the Chen-Ruan cohomology ring $H^*_{CR}([X])$ (with rational coefficients) for an orbifold $[X]$.

(i) Denote by $H^*(...)$ the usual singular cohomology with coefficients in $\mathbb{Q}$. By $H^*([Y])$ of a orbifold $[Y]$ we mean $H^*(Y)$ of the underlying topological space. On $H^*(Y)$ the usual cup-product $\cup$ is defined. (We denote it by " $\cup$ " here, but after this section will return to our usual convention and write it as " $\cdot$ ".)

(ii) As a $\mathbb{Q}$ vector space:

$$H^*_{CR}([X]) := H^*(I_1([X])) = \bigoplus_{(Y,g) \text{ sector of } I_1([X])} H^*((Y, g))$$

(iii) $H^*_{CR}([X])$ is made into a graded vector space by setting for $d \in \mathbb{Q}$

$$H^d_{CR}([X]) := \bigoplus_{(Y,g) \text{ sector of } I_1([X])} H^{d-2a(Y,g)}((Y, g)).$$

This grading is sometimes called the *age grading*, in general $H^d_{CR}([X])$ is non-zero also for some $d \in \mathbb{Q} \setminus \mathbb{Z}$.

If we write $H^*((Y, g))$ for a 1-sector (Y, g) in the following, we usually interpret it as a subspace of $H^*_{CR}([X])$.

(iv) On $H^*_{CR}([X])$ a product $*$ is defined as follows: If p_1, p_2, p_3 are the forgetful morphisms $I_2([X]) \to I_1([X])$ as defined in Def. 5.3 (iii). Then for two classes $\alpha, \beta \in H^*_{CR}([X])$:

$$\alpha * \beta = (p_3)_* \left(p_1^*(\alpha) \cup p_2^*(\beta) \cup c_{top}(E)\right),$$

where $\cup$ is the usual cup-product, and E is the *Chen-Ruan excess intersection bundle* on $I_2([X])$ as defined below.

If there are 1-sectors (X_1, g) and (X_2, h), such that $\alpha \in H^*(X_1, g)$ and $\beta \in H^*(X_2, h)$ then $p_1^*(\alpha), p_2^*(\beta) \in H^*(Y, g, h)$, and it suffices to know $E_{(Y,g,h)} := E_{|(Y,g,h)}$ to compute

$$p_1^*(\alpha) \cup p_2^*(\beta) \cup c_{top}(E) = p_1^*(\alpha) \cup p_2^*(\beta) \cup c_{top}(E_{(Y,g,h)})$$

In this case $\alpha * \beta \in H^*(X_3, gh)$.

(v) The CR-excess intersection bundle $E_{(Y,g,h)}$ on a 2-sector (Y, g, h) is defined as follows: Let G be the group generated by g and h. The fundamental group $\pi_1(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})$ is generated by tree small loops $\gamma_0, \gamma_1, \gamma_\infty$ arround the points $0, 1, \infty \in \mathbb{P}^1$, and we have $\gamma_0 \cdot \gamma_1 = \gamma_\infty^{-1}$. Any group homomorphism $\pi_1(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}) \to G$ corresponds to a G-principal bundle on $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. Let $\tau^0 : C^0 \to \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ be the G-principal bundle corresponding to the morphism defined by $\gamma_0 \mapsto g$, $\gamma_1 \mapsto h$, $\gamma_\infty \mapsto (gh)^{-1}$. This bundle can be uniquely extended to a ramified G-Galois-cover $\tau : C \to \mathbb{P}^1$, with C a smooth curve. G acts on C and so also on $H^1(C, \mathcal{O}_C)$. Let $f : (Y, g, h) \to [X]$ be the restriction of $\chi_2 : I_2([X]) \to [X]$. Then one defines:

$$E_{(Y,g,h)} := \left(H^1(C, \mathcal{O}_C) \otimes_{\mathbb{C}} f^*(T_{[X]})\right)^G,$$

where $T_{[X]}$ denotes the orbifold tangent bundle of $[X]$ (cf. Example 2.4 of [CR04]), and where $(...)^G$ denotes the subspace of G-invariants. Since $H^1(C, \mathcal{O}_C)^G = 0$, and G acts trivially on T_Y, we also have

$$E_{(Y,g,h)} = \left(H^1(C, \mathcal{O}_C) \otimes_{\mathbb{C}} N_Y[X]\right)^G,$$

where $N_Y[X]$ is the normal bundle, i.e. the cokernel of $T_Y \to f^*(T_{[X]})$.

Remark 5.6 The pullbacks in (iv) of the above definition are pullbacks via morphisms of orbifolds, i.e. behave like pullbacks via morphism of stacks and can be calculated locally on uniformising systems. Later when we compute such pullbacks for moduli spaces of spin/prym curves and their sectors, and are interested in how they act on cycle classes coming from the Chow ring, we thus have to compute pullbacks over the corresponding morphisms of stacks, or equivalently compute the adjusted pullbacks as introduced in Summary 2.6 (iv). (Also cf. Summary 2.4 (iii).)

Summary 5.7 *(i) The CR-Product $*$ is associative, and its restriction to $H^*(([X], 1))$, i.e. to the "untwisted sector", coincides with the usual cup product on X. Also $*$ respects the age-grading.*

(ii) For (Y, g) a 1-sector of $[X]$, and (Y, g^{-1}) the sector "inverse to it", we have:

$$a(Y, g) + a(Y, g^{-1}) = \operatorname{codim}(Y, [X]) \text{ [1]}$$

(iii) For (Y, g, h), (X_1, g), (X_2, h), (X_3, gh) and $E_{(Y,g,h)}$ as above:

$$rk(E_{(Y,g,h)}) = a(X_1, g) + a(X_2, h) + a(X_3, (gh)^{-1}) - \operatorname{codim}(Y, [X])$$

[1] The codimension of a sub-orbifold is defined as the codimension on the V's of the uniformising systems.

(iv) If all local groups of $[X]$ are abelian, then for each $k \in \mathbb{N}$, the forgetful morphism $\chi_k : I_k([X]) \to [X]$ restricts to a closed embedding $(Y, \mathbf{g}) \to [X]$ (with image Y), on every sector $(Y, \mathbf{g})$ of $I_k([X])$.

5.1.1 $\overline{M}_{g,n}$, $\overline{R}_{g,n}$, $\overline{S}_{g,n}$ as complex orbifolds

By the results from section 1.5, $\overline{M}_{g,n}$, $\overline{R}_{g,n}$ and $\overline{S}_{g,n}$ are endowed with a complex orbifold atlas in a natural way: For example for every point $[\mathfrak{C}] \in \overline{M}_{g,n}$ locally around $[\mathfrak{C}]$, $\overline{M}_{g,n}$ is isomorphic to the quotient $(B, b_0)/\operatorname{Aut}(\mathfrak{C})$, where (B, b_0) is the local universal deformation space of $\mathfrak{C}$, by Summary 1.30 (v). Analogous results hold for $\overline{R}_{g,n}$ and $\overline{S}_{g,n}$ by Summary 1.31.

We will for the rest of this chapter always consider our moduli spaces as orbifolds with this structure defined by the deformation spaces. We stick with our definition of automorphisms of spin or prym curves from Def. 1.11 (ii). So our automorphism groups are smaller as if we would have included an isomorphism of the spin resp. prym sheaves in the data of our automorphisms, as done for example in [Cor91] and [Lud10]. To be more precise, for any spin curve $\mathfrak{X}$ of $\overline{S}_{g,n}$ (or a prym curve) denote by $\operatorname{Aut}(\mathfrak{X})$ the automorphism group according to our definition, and $\operatorname{Aut}'(\mathfrak{X})$ the group for the alternative definition. Then $|\operatorname{Aut}'(\mathfrak{X})| = 2|\operatorname{Aut}(\mathfrak{X})|$, and $\operatorname{Aut}'(\mathfrak{X})$ is an extension of $\operatorname{Aut}(\mathfrak{X})$ by the inessential automorphism ι, which acts trivially on the support X but acts as multiplication by -1 on all fibres of the spin resp. prym sheaf. Since ι extends to every object of $\overline{S}_{g,n}$ it acts trivially on the deformation space. How would the presence of ι change the Chen-Ruan cohomology of $\overline{S}_{g,n}$?

Denote the Chen-Ruan cohomology ring of $\overline{S}_{g,n}$ defined using the alternative definition of automorphisms by $H^*_{CR}(\overline{S}_{g,n})'$, the one defined by our definition by $H^*_{CR}(\overline{S}_{g,n})$. Firstly to each 1 sector (Y, g) of $\overline{S}_{g,n}$ for our definition, there correspond two 1-sectors (X, φ) and $(X, \iota\varphi)$ for the alternative definition, such that both of them are isomorphic to (Y, g) as orbifolds (ι as above). So $\dim_{\mathbb{Q}} H^*_{CR}(\overline{S}_{g,n})' = 2 \dim_{\mathbb{Q}} H^*_{CR}(\overline{S}_{g,n})$. Furthermore we denote by $[(\overline{S}_{g,n}, \iota)] \in H^*((\overline{S}_{g,n}, \iota)) \subseteq H^*_{CR}(\overline{S}_{g,n}))'$ the fundamental class of the 1-sector $(\overline{S}_{g,n}, \iota)$. Then it is not difficult to show that the multiplication (using the Chen-Ruan product $*$) by $[(\overline{S}_{g,n}, \iota)]$ induces, for each 1-sector (X, φ) of $\overline{S}_{g,n}$ an isomorphism between the subspaces $H^*((X, \varphi))$ and $H^*((X, \iota\varphi))$ of $H^*_{CR}(\overline{S}_{g,n})'$. The ring $H^*_{CR}(\overline{S}_{g,n})'$ is a $H^*_{CR}(\overline{S}_{g,n})$-algebra, and as such a algebra generated by the class $[(\overline{S}_{g,n}, \iota)]$, with the single relation $[(\overline{S}_{g,n}, \iota)] * [(\overline{S}_{g,n}, \iota)] = [(\overline{S}_{g,n}, 1)]$ $(= 1)$, where $[(\overline{S}_{g,n}, 1)]$ is the fundamental class of the untwisted 1-sector. Put differently $H^*_{CR}(\overline{S}_{g,n})'$ is isomorphic to the quotient ring $H^*_{CR}(\overline{S}_{g,n})[T]/(T^2-1)$, by the isomorphism sending the variable T to the class $[(\overline{S}_{g,n}, \iota)]$. The relation between $H^*_{CR}(\overline{R}_{g,n})'$ and $H^*_{CR}(\overline{R}_{g,n})$ is completely analogous.

5.2 First steps towards determining $I_1(\overline{R}_{1,n})$

5.2.1 The 1-sectors of $R_{1,n} = S_{1,n}$

We summarize the following definitions and results about twisted sectors of $M_{1,n}$ from [Pag08]. We will use all of these definitions too.

Summary 5.8 (N. Pagani) *(i) If $(C; p_1)$ is an elliptic curve, G its automorphism group, then G acts effectively on the cotangent space $T^\vee_{p_1}(C)$, which is canonically isomorphic to $\mathbb{C}$. We use this action to identify G with the group of N-th roots of unity μ_N, where $N = |G|$. Use the notation $\epsilon := e^{\frac{2\pi i}{6}}$ to fix one generator of μ_6. Using the Weierstrass representation for elliptic curves one can determine which automorphisms exist on which curves. One obtains that only $N = 1, 2, 3, 4, 6$ are possible. More specifically:*

(ii) In $M_{1,1}$ there is one isomorphism class of curves $C_4 = [(\mathscr{C}_4; p_1)]$, such that $G = \mu_4$ (where $G = \mathrm{Aut}((\mathscr{C}_4; p_1))$) and one isomorphism class of curves $C_6 := [(\mathscr{C}_6; p_1)]$ such that $G = \mu_6$. For all other curves $G = \mu_2$.

(iii) In $M_{1,2}$, with the same curves $\mathscr{C}_4$ and $\mathscr{C}_6$ (with p_1, p_2 in special position), there is one isomorphism class of curves $C'_4 = [(\mathscr{C}_4; p_1, p_2)]$, such that $G = \mu_4$ and one isomorphism class of curves $C'_6 := [(\mathscr{C}_6; p_1, p_2)]$ such that $G = \mu_3$, where μ_3 is the subgroup of the automorphism group μ_6 of $\mathscr{C}_6$ generated by ϵ^2. For every smooth elliptic curve $(C; p_1)$ there are three position on which p_2 can be put such that $(C; p_1, p_2)$ has $G \supseteq \mu_2$, we call the locus of these pointed curves A_2. If we form the closure $\overline{A}_2$ of A_2 in $\overline{M}_{1,2}$ then $A_2 \cong \mathbb{P}^1$ as varieties. For all other curves $(C; p_1, p_2)$, $G = \{id\}$.

(iv) In $M_{1,3}$ there is still one isomorphism class $C''_6 = [(\mathscr{C}_6; p_1, p_2, p_3)]$ with $G = \mu_3$. The locus of curves $(C; p_1, p_2, p_3)$ with $G \supseteq \mu_2$ is called A_3. Again $\overline{A}_3 \cong \mathbb{P}^1$.

(v) For $M_{1,4}$ a curve can have at most 2 automorphisms. The locus of curves $(C, p_1, ..., p_4)$ with points in the special position admitting two automorphisms is called A_4. Again $\overline{A}_4 \cong \mathbb{P}^1$.

(vi) For $n \geq 5$ any curve of $M_{1,n}$ has $G = \{id\}$.

(vii) Since all the groups are abelian, if (X, g) is a 1-sector of $M_{1,n}$ then the restriction of the forgetful map $I_1(M_{1,n}) \to M_{1,n}$ to (X, g), which has image X, is a closed embedding.

(viii) From the above one can conclude that the inertia stacks $I_1(R_{1,n})$ decompose into sectors as follows:

- $I_1(M_{1,1}) = (M_{1,1}, 1) \biguplus (M_{1,1}, -1) \biguplus (C_4, i/-i) \biguplus (C_6, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5)$
- $I_1(M_{1,2}) = (M_{1,2}, 1) \biguplus (A_2, -1) \biguplus (C'_4, i/-i) \biguplus (C'_6, \epsilon^2/\epsilon^4)$
- $I_1(M_{1,3}) = (M_{1,3}, 1) \biguplus (A_3, -1) \biguplus (C''_6, \epsilon^2/\epsilon^4)$
- $I_1(M_{1,4}) = (M_{1,4}, 1) \biguplus (A_4, -1)$
- $I_1(M_{1,n}) = (M_{1,n}, 1)$ *if* $n \geq 5$

Here and in the rest of the chapter, if we write something like $(C_4, i/-i)$ *this is an abbreviation, to make lists of 1-sectors shorter. Here, and also several times later, it has to be interpreted as* $(C_4, i) \uplus (C_4, -i)$*. At some points it will mean "*(C_4, i) and $(C_4, -i)$*" or "*(C_4, i) or $(C_4, -i)$*" instead, but that should be clear from the context.*

Since $\omega_C = \mathcal{O}_C$ on an elliptic curve C, the stacks $R_{1,n}$ and $S^+_{1,n}$ are isomorphic (while $S^-_{1,n} \cong M_{1,n}$). Furthermore, for smooth prym curves with marked points, $(C; p_1, .., p_n; \mathcal{L}, b)$ all automorphisms come from automorphisms of the underlying curve with marked points $(C; p_1, ..., p_n)$. Thus there is a forgetful morphism $\pi : I_1(R_{1,n}) \to I_1(M_{1,n})$. We now describe the preimages of all sectors of $I_1(M_{1,n})$ under π.

Lemma & Definition 5.9 *(i) For the following sectors* X*, we have* $\pi^{-1}(X) = \emptyset$*:*

$$(C_6, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5), \quad (C_6', \epsilon^2/\epsilon^4), \quad (C_6'', \epsilon^2/\epsilon^4)$$

I.e. for all 1-sectors of automorphisms of order divisible by 3*.*

(ii) For the following sectors X *the preimage* $\pi^{-1}X$ *has exactly one component, and* $(\pi_1)_{|\pi_1^{-1}X}$ *is an isomorphism:*

$$(C_4, i/-i), \quad (C_4', i/-i)$$

we denote the preimages of these sectors in $I_1(R_{1,n})$ *by the same symbols again.*

(iii) We describe $\pi^{-1}X$ *for the remaining sectors* X*, as the union of their connected components:*

$$\pi_1^{-1}(M_{1,n}, 1) = (R_{1,n}, 1), \quad \pi_1^{-1}(M_{1,1}, -1) = (R_{1,1}, -1)$$

$$\pi_1^{-1}(A_2, -1) = (A_{2,a}, -1) \uplus (A_{2,b}, -1), \quad \pi_1^{-1}(A_3, -1) = (A_{3,a}, -1) \uplus (A_{3,b}, -1) \uplus (A_{3,c}, -1),$$

$$\pi_1^{-1}(A_4, -1) = (A_{4,a}, -1) \uplus (A_{4,b}, -1) \uplus (A_{4,c}, -1)$$

The new symbols for 1*-sectors occurring in the second and third line we define by explaining which kind of objects* $((C; p_1, ..., p_n; \mathcal{L}; b), \varphi)$ *these sectors parametrise. The* $(C, p_1, .., p_n)$ *and automorphism* φ *appearing are the same as in the* $(A_n, -1)$ *for* $n \in \{2, 3, 4\}$*, independent of the indices* a, b, c*. The indices* a, b, c *correspond to ways* $\mathcal{L}$ *is related to the marked points* $p_1, ...p_n$*:*

- *For* $A_{2,a}$*,* $\mathcal{L} \cong \mathcal{O}_C(p_1 - p_2)$*, while for* $A_{2,b}$*,* $\mathcal{L}$ *is one of the other two possible prym sheaves.*
- *For* $A_{3,a}$ *and* $A_{4,a}$*,* $\mathcal{L} \cong \mathcal{O}_C(p_1 - p_2)$*.*
- *For* $A_{3,b}$ *and* $A_{4,b}$*,* $\mathcal{L} \cong \mathcal{O}_C(p_1 - p_3)$*.*
- *For* $A_{3,c}$ *and* $A_{4,c}$*,* $\mathcal{L} \cong \mathcal{O}_C(p_2 - p_3)$*.*

Proof: (i) If $(\mathscr{C}_6, p_1)$ is an elliptic curve parametrised by the point C_6, then the elliptic involution fixes p_1 and three other points q_1, q_2, q_3. We know that there are three isomorphism classes of prym sheaves on $\mathscr{C}_6$, like on any elliptic curve, which are represented by

$\mathcal{O}_{\mathscr{C}_6}(p_1 - q_1)$, $\mathcal{O}_{\mathscr{C}_6}(p_1 - q_2)$, $\mathcal{O}_{\mathscr{C}_6}(p_1 - q_3)$. Using for example the Weierstrass representation in Theorem 3.8. in [Pag08] it is easy to see that any automorphism g in μ_6 with order ≥ 3, fixes none of the q_i but cyclically permutes them. Hence for any prym sheaf $\mathcal{L}$ on $\mathscr{C}_6$, $g^*\mathcal{L}$ is not isomorphic to $\mathcal{L}$. Hence there are no automorphisms of order divisible by 3 on prym curves of $R_{1,1}$. This of course implies the same for $R_{1,n}$ for all $n \geq 1$.

(ii) Analogously one has three points q_1, q_2, q_3 on $\mathscr{C}_4$, fixed by the elliptic involution of $\mathscr{C}_4$. Again it is easy to see that the automorphisms i and $-i$, both fix the same of the points q_i, say q_1, and transpose the two other points q_2, q_3. Hence the prym sheaf $\mathcal{O}_{\mathscr{C}_4}(p_1 - q_1)$ is fixed by i and $-i$, while $\mathcal{O}_{\mathscr{C}_4}(p_1 - q_2)$ and $\mathcal{O}_{\mathscr{C}_4}(p_1 - q_3)$ are swapped. Hence there is only one class of prym curves of $\overline{R}_{1,n}$ which carries an automorphism of order 4, namely the one represented by $(\mathscr{C}_4, \mathcal{O}_{\mathscr{C}_4}(p_1 - q_2))$. If we denote the point in $\overline{R}_{1,n}$ corresponding to this class by C_4 again, part (ii) of our Lemma follows. (The case of C_4' is analogous.)

(iii): Recall Definition 2.1 and Summary 2.14. It is clear that $A_n = HM_{1,n}$ and $\pi^{-1}(A_n) = HR_{1,n}$. By Summary 2.14, $HR_{1,2}$ has two irreducible components, namely the images of $a'_{R_{1,2},2,\{p_1,p_2\}}$ and $a'_{R_{1,2},2,\{p_1\}}$, which are just $A_{2,a}$ and $A_{2,b}$. Similarly $HR_{1,3}$ has the three components $A_{3,a}$, $A_{3,b}$, $A_{3,c}$, which are the images of $a'_{R_{1,3},2,\{p_1,p_2\}}$, $a'_{R_{1,3},2,\{p_1,p_3\}}$, $a'_{R_{1,3},2,\{p_2,p_3\}}$ (cf. Example 2.20). The argument for A_4 is analogous. □

Corollary 5.10 *The inertia stacks $I_1(R_{1,n})$ decompose into sectors as follows:*

- $I_1(R_{1,1}) = (R_{1,1}, 1) \uplus (R_{1,1}, -1) \uplus (C_4, i/-i)$
- $I_1(R_{1,2}) = (R_{1,2}, 1) \uplus (A_{2,a}, -1) \uplus (A_{2,b}, -1) \uplus (C_4', i/-i)$
- $I_1(R_{1,3}) = (R_{1,3}, 1) \uplus (A_{3,a}, -1) \uplus (A_{3,b}, -1) \uplus (A_{3,c}, -1)$
- $I_1(R_{1,4}) = (R_{1,4}, 1) \uplus (A_{4,a}, -1) \uplus (A_{4,b}, -1) \uplus (A_{4,c}, -1)$
- $I_1(R_{1,n}) = (R_{1,n}, 1)$ *if* $n \geq 5$

Warning: Since all the automorphism groups are abelian, for all the sectors (Z, g) of $I_1(R_{1,n})$, the restriction to (Z, g) of the forgetful morphism $\chi_1 : I_1(R_{1,n}) \to R_{1,n}$, which would in general by a finite cover, is a closed embedding (cf. Summary 5.7 (v)). The same is true, as will be shown later, for all sectors of $I_1(\overline{R}_{1,n})$ (and $I_1(\overline{S}^+_{1,n})$). We call the locus Z the support of the sector, but since the forgetful morphism is an embedding, we will sometimes abuse notation and call the Z's sectors.

5.2.2 Constructing sectors of $I_1(\overline{R}_{1,n})$

Definition 5.11 For $k \in \underline{4}$ and $x \in \{a, b, c\}$, let $\overline{A}_{k,x}$ be the closure of $A_{k,x}$ in $\overline{R}_{1,k}$. The automorphism -1 that exists on a $A_{k,x}$ extends to $\overline{A}_{k,x}$.

We call *basic* 1*-sectors* all the $\overline{A}_{k,x}$ as well as the points $C_4 \subset \overline{R}_{1,1}$ and $C_4' \subset \overline{R}_{1,2}$, for reasons explained below. We will see that these are all the twisted sectors of $I_1(\overline{R}_{1,n})$ for any n, that have a non-empty intersection with the interior $R_{1,n}$.

Definition 5.12 (i) If $P = (I_1, ..., I_k)$ is an (ordered) partition of $\underline{n}$ with all $I_i \neq \emptyset$, denote by $\Delta_{I_1,...,I_k}$ the boundary cycle of $\overline{M}_{1,n}$, which generally parametrises curves with one smooth elliptic component, to which k rational tails are attached, such that the k-th tail carries exactly the marked points with indices in I_k, i.e. $\Delta_{I_1,....,I_k} = \Delta_{I_1} \cap .... \cap \Delta_{I_k}$. We define the morphism

$$\xi_P := \xi_{\Delta_{I_1,...,I_k}} : \overline{M}_{1,\{\bullet_1,...,\bullet_k\}} \times \overline{M}_{0,I_1 \uplus \circ_1} \times ... \times \overline{M}_{0,I_k \uplus \circ_k} \to \overline{M}_{1,n}$$

to be the gluing morphism surjecting to $\Delta_{I_1,...,I_k}$, defined in Proposition 1.26 (i). By

$$f_P : \overline{M}_{1,\{\bullet_1,...,\bullet_k\}} \times \overline{M}_{0,I_1 \uplus \circ_1} \times ... \times \overline{M}_{0,I_k \uplus \circ_k} \to \overline{M}_{1,k}$$

we denote the projection to the first factor.

(ii) For $\overline{R}_{1,n}$ we set $D_{I_1,...,I_k} := \tau_n^{-1}\Delta_{I_1,...,I_k}$, where $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ is the forgetful morphism. We define a morphism

$$\zeta_P := \zeta_{D_{I_1,...,I_k}} : \overline{R}_{1,\{\bullet_1,...,\bullet_k\}} \times \overline{M}_{0,I_1 \uplus \circ_1} \times ... \times \overline{M}_{0,I_k \uplus \circ_k} \to \overline{R}_{1,k}.$$

to be the gluing morphisms surjecting to $D_{I_1,...,I_k}$ (cf. Remark 4.6 (iii)). Here we call the projection to the first factor F_P.

In case $|I_j| = 1$ delete the factor $\overline{M}_{0,I_j \cup \{\circ_j\}}$ in the product and just replace the index $\bullet_j$ by the index in I_j.

If we have a (prym) curve from $\overline{M}_{1,k}$ resp. $\overline{R}_{1,k}$ one can use these morphisms to glue a rational tree with some marked points on it to each of the k marked points of the curve, producing a curve in $\overline{M}_{1,n}$ resp. $\overline{R}_{1,n}$. It is clear that all automorphism of the old curve lift to the new curve obtained by this procedure, and that this curve has no new automorphisms. Applying this operation one can construct sectors of $I_1(\overline{M}_{1,n})$ resp. $I_1(\overline{R}_{1,n})$ out of sectors of $I_1(M_{1,n})$ resp. $I_1(R_{1,n})$:

Lemma & Definition 5.13 *Let $(\overline{Z}, g)$ be a basic sector in $I_1(\overline{R}_{1,k})$ for $k \in \underline{4}$ (cf. Def. 5.11).*

(i) For a partition $P = (I_1, ..., I_k)$ of $\underline{n}$ we set

$$\overline{Z}^P := \zeta_P(F_P^{-1}(\overline{Z})).$$

We call $\overline{Z}$ the basic sector associated to $\overline{Z}^P$.

(ii) As we will show in the Proof of Theorem 5.32 below, the automorphism g lifts to $\overline{Z}^P \subset \overline{R}_{1,n}$, and does not extend to a larger locus in $\overline{R}_{1,n}$, hence $(\overline{Z}^P, g)$ is a sector of $I_1(\overline{R}_{1,n})$.

(iii) For $k = 1$, i.e. $I_1 = \underline{n}$, we will denote the resulting sector by $\overline{Z}^{\underline{n}}$.

For $k = 2$, for all possible $\overline{Z}$, we have $\overline{Z}^{(I_1,I_2)} = \overline{Z}^{(I_2,I_1)}$. To symbolise this invariance, we will write the possible sectors obtained for $k = 2$ as $C_4^{\{I_1,I_2\}}$, $\overline{A}_{2,a}^{\{I_1,I_2\}}$ and $\overline{A}_{2,b}^{\{I_1,I_2\}}$.

For $k = 3$ the only possible $\overline{Z}$ are the $\overline{A}_{3,x}$ for $x \in \{a, b, c\}$. Here we have

$$\overline{A}_{3,a}^{(I_1,I_2,I_3)} = \overline{A}_{3,b}^{(I_1,I_3,I_2)} = \overline{A}_{3,c}^{(I_2,I_3,I_1)}.$$

Also $\overline{A}_{3,a}^{(I_1,I_2,I_3)} = \overline{A}_{3,a}^{(I_2,I_1,I_3)}$. *Hence all possible cases for* $\overline{Z}^{(I_1,I_2,I_3)}$ *are covered by the* $\overline{A}_{3,a}^{(I_1,I_2,I_3)}$, *when considering all possible partitions* (I_1, I_2, I_3) *of* $\underline{n}$. *To express the invariance under transposing the first two entries, and to get rid of the index* x, *we write these sectors as*

$$\overline{A}_3^{\{I_1,I_2\},I_3} := \overline{A}_{3,a}^{(I_1,I_2,I_3)}$$

For $k = 4$ *the possible* $\overline{Z}$ *are the* $\overline{A}_{4,x}$ *for* $x \in \{a, b, c\}$. *Here we have*

$$\overline{A}_{4,a}^{(I_1,I_2,I_3,I_4)} = \overline{A}_{4,b}^{(I_1,I_3,I_2,I_4)} = \overline{A}_{4,c}^{(I_2,I_3,I_1,I_4)}.$$

Also $\overline{A}_{4,a}^{(I_1,I_2,I_3,I_4)}$ *is invariant under transposing the first two or the last two entries of* (I_1, I_2, I_3, I_4), *and under replacing by* (I_3, I_4, I_1, I_2). *Hence we get all possible* $\overline{Z}^{(I_1,I_2,I_3,I_4)}$ *by taking the sectors*

$$\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}} := \overline{A}_{4,a}^{(I_1,I_2,I_3,I_4)}$$

Proof: Only (iii) is more than a definition, and all claimed there is clear, considering the definition of the $\overline{A}_{k,x}$, and the fact that if q_1, q_2, q_3, q_4 are the points fixed by the elliptic involution of a curve C, then $\mathcal{O}_C(q_1 - q_2) = \mathcal{O}_C(q_2 - q_1) = \mathcal{O}_C(q_3 - q_4)$. □

In the case of $\overline{M}_{1,n}$ the analogous construction of forming $\overline{Z}^P = \xi_P(f_P^{-1}(\overline{Z}))$ starting from the basic sectors $\overline{Z} = C_4, C_4', C_6, C_6', C_6'', \overline{A}_n$ yields all sectors of $I_1(\overline{M}_{1,n})$ (cf. [Pag08] Theorem 3.24). For our case of $\overline{R}_{1,n}$ this is not quite true, since stable prym curve can have exceptional components and inessential automorphisms acting non-trivially only on these exceptional components. Since there are no such inessential automorphisms on smooth prym curves, the sectors of $I_1(\overline{R}_{1,n})$ corresponding to such inessential automorphisms lie entirely inside the boundary of $\overline{R}_{1,n}$. They do not originate from the basic sectors.

Definition 5.14 (i) For $(I_1, ..., I_m)$ a partition of $\underline{n}$, recall the definition of the simple banana cycles $B_{I_1,...,I_m}$ and $B^r_{I_1,...,I_m}$ from Def. 4.1 (ii) resp. 4.5 (i). $B^r_{I_1,...,I_m}$ is the closure of the locus of $\overline{R}_{1,n}$ parametrising prym curves $(X; p_1, ..., p_n; \mathcal{L}, b)$ of the following type: Consider the indices $1, ..., m$ as elements of $\mathbb{Z}/m$. Then the stable model C of X is a "circuit" of rational curves: It consist of smooth rational components $C_1, ..., C_m$, such that each C_i meets C_{i-1} and C_{i+1} in one simple node each, and meets on other component. The component C_i carries all marked points with indices in I_i. Now X is obtained by blowing up all nodes in C. The prym sheaf $\mathcal{L}$ restricts to $\mathcal{O}_{C_i}(-1)$ on each of the C_i and to $\mathcal{O}(1)$ on each exceptional component.

(ii) In case m is even, let ι_m be the inessential automorphism of $(X; p_1, ..., p_n; \mathcal{L}, b)$, that corresponds to multiplying by -1 on the fibres of the prym sheaf over the components C_i with i even, and acting as identity on the fibres over all C_j with j odd. (Note that with our definition of automorphisms this is the same inessential automorphism as the one multiplying by -1 on the fibres over components with i odd, and by 1 on those with i even.) We will later often denote partitions of $\underline{n}$ by P, and sometimes more precisely denote the automorphism ι_m on B^r_P by ι_P.

We will see later that each $(B^r_{I_1,\ldots,I_m}, \iota_m)$ for m even, is a sector of $I_1(\overline{R}_{1,n})$, and that together with the sectors obtained from basic sectors in Lemma & Definition 5.13, they are the only sectors of $I_1(\overline{R}_{1,n})$. These "banana-sectors" are the ones that are really new, compared to the sectors that appear in $I_1(\overline{M}_{1,n})$. We will need more information about the simple banana cycles, then provided in Chapter 4, to compute $H^*_{CR}(\overline{R}_{1,n})$ later. The next section will provide this information, also parts of it that will be used only much later. But before that we state the following

Remark 5.15 (i) We have seen in Summary 5.8 that all objects of any $M_{1,n}$ have abelian automorphism groups. The same holds for all objects of $\overline{M}_{1,n}$, since (as shown in [Pag08]) the 1-sectors of $\overline{M}_{1,n}$ all stem from basic 1-sectors via the procedure explained in Lemma & Definition 5.13. It is easy to see that an automorphism of a stable genus 1 curve $\mathfrak{C}$ with marked points can not exchange two components of $\mathfrak{C}$. Hence for each object $\mathfrak{X}$ of $\overline{R}_{1,n}$ an automorphism can not exchange components of the non exceptional subcurve. From this it follows by the description of the inessential automorphism $\mathrm{Aut}_0(\mathfrak{X})$ in Remark 1.12, that $\mathrm{Aut}_0(\mathfrak{X})$ is contained in the centre of $\mathrm{Aut}(\mathfrak{X})$. Since $\mathrm{Aut}(\mathfrak{X})$ is an extension of a subgroup of $\mathrm{Aut}(\mathfrak{C})$ by $\mathrm{Aut}_0(\mathfrak{X})$ for $\mathfrak{C}$ the stable modle of $\mathfrak{X}$ (cf. Def. 1.11 (v)), it follows that each $\mathrm{Aut}(\mathfrak{X})$ is abelian.

(ii) Hence for every 1-sector (X, g) of $\overline{M}_{1,n}$ resp. $\overline{R}_{1,n}$, (X, g) is isomorphic to its image $X \subset \overline{M}_{1,n}$ resp. $X \subset \overline{R}_{1,n}$ as variety as well as as orbifold. (The same holds for the k-sectors for $k > 1$. Cf. Summary 5.7 (iii).)

5.3 Simple banana cycles

5.3.1 Circular partitions and set-theoretic intersections of banana cycles

The following combinatorial notions are closely related to simple banana cycles on $\overline{M}_{1,n}$ (and $\overline{R}_{1,n}$), and their intersection behaviour. To begin with there is quite obviously a $1:1$ relation between the simple banana cycles of $\overline{M}_{1,n}$, and the (non-trivial) circular partitions of $\underline{n}$:

Definition 5.16 (and first remarks) (i) Let M be a finite set. An *arrangement* of M is a map $e : M \times M \to \mathbb{N}_0$ such that for $i_1, i_2 \in M$, $e(i_1, i_2) = e(i_2, i_1)$. To an arrangement we define a graph $\Lambda(M, e)$, by interpreting the elements of M as the vertices of $\Lambda(M, e)$, and by connecting each pair $i_1, i_2 \in M$ by $e(i_1, i_2)$ many edges. [2]

(ii) An *arrangement as string* of a finite set M, is an arrangement e, such that the graph $\Lambda(M, e)$ is a connected graph, and no vertex meets more than two edges. (A self edge at a vertex counts as meeting the vertex twice.)

If $|M| \geq 2$, this implies: For all $i \in M$, $1 \leq \sum_{i' \in M} e(i, i') \leq 2$.

[2] The arrangement e and the graph $\Lambda(M, e)$ determine each other uniquely, so everything in this section could also be done using only graphs instead of arrangements.

If M is a finite set, together with a fixed such arrangement, we call M a *string*. We then denote by e_M this fixed arrangement. For $h^1(\Lambda(M, e_M))$ the first Betti number, we call the string M *open* if $h^1(\Lambda(M, e_M)) = 0$ and *closed* if $h^1(\Lambda(M, e_M)) = 1$. [3]

If $|M| \geq 2$, this is equivalent to saying: A string is called closed if for all $i \in M$, $\sum_{i' \in M} e_M(i, i') = 2$, and is called open otherwise.

We sometimes also call an arrangement as a closed string a *circular arrangement*.

(iii) If (M, e) is a string, we define a reflexive, symmetric (but usually not transitive) relation $\|$ between elements of M, called *neighbouring*, by saying $i_1 \in M$ neighbours $i_2 \in M$, written $i_1 \| i_2$, if $e(i_1, i_2) \geq 1$ or $i_1 = i_2$.

For $|M| \geq 3$, the relation $\|$ fixes e. For $|M| \leq 2$ it does this only after declaring whether M should be an open or closed string.

We often write a closed string as $\langle i_1, ..., i_m \rangle$, by which we mean the set $\{i_1, ..., i_m\}$ with neighbouring relations $i_1 \| i_2 \| ... \| i_m \| i_1$.

Choosing a circular arrangement of M is furthermore the same as choosing an equivalence class from the set

$$\{f : M \to \mathbb{Z}/|M| \cdot \mathbb{Z} \mid f \text{ bijective}\}/\sim,$$

where $\sim$ is the equivalence relation generated by the relations $f \sim f + a$ for all constant maps $a : M \to \mathbb{Z}/|M| \cdot \mathbb{Z}$, and $f \sim -f$. [4]

(iv) An *end-point* of a string M is a point $i \in M$ such that $\sum_{i' \in M} e(i, i') = 1$. A closed string has no end-points, an open string has one end-point if $|M| = 1$ and two end-points otherwise.

Let i_1, i_2 be the two end points of an open string M ($i_1 = i_2$ iff $|M| = 1$). One can make the open string M into a closed string by increasing the value of $e_M(i_1, i_2)$ by 1, we call this procedure *closing the string* M. In the opposite direction one can *cut open* a closed string M by choosing any pair $i_1, i_2 \in M$ with $e_M(i_1, i_2) \geq 1$ and decreasing this value by 1.

(v) A subset $S \subseteq M$ of a string M is called a *set of neighbours* in M, if the elements of S are the vertices of a connected subgraph of $\Lambda(M, e_M)$. This is equivalent to saying that for $e_{M|S}$ the restriction of e_M to $S \times S \subseteq M \times M$, $(S, e_{M|S})$ is a string. Instead of saying that S is a set of neighbours, we often say it is a *substring* (of M), since we can always consider a set of neighbours S as string, using this induced arrangement.

We say that two substrings S_1 and S_2 in M are neighbours, written $S_1 \| S_2$, if firstly $S_1 \cup S_2$ is a substring and secondly, if the string $S_1 \cup S_2$ is open then S_1 and S_2 each contain at least one of its end-points.

If we write $S_1 \| S_2$ for sets $S_1, S_2 \subseteq M$ this always is also meant to imply that S_1 and S_2 both are substrings of M.

[3] $h^1(\Lambda(M, e_M)) > 1$ is not possible for a string. For an open string $e_M(i_1, i_2) = 2$ never occurs. For a closed string it occurs iff $\{i_1, i_2\} = M$.

[4] The relation between this and the former definitions is: We say $i_1 \| i_2$ iff $f(i_1) = f(i_2)$ or $f(i_1) = f(i_2) + 1$ or $f(i_1) = f(i_2) - 1$.

By $S_1 \parallel S_2 \parallel .... \parallel S_n$ we mean that for every choice of $1 \leq q \leq r \leq n$, firstly $\bigcup_{i=q}^{r} S_i$ is a substring, and secondly if this substring is open, then its end-points are contained in $S_q \cup S_r$, and each of S_q and S_r contains at least one of the end-points.

(vi) If M is an open resp. closed string with $|M| \geq 2$, we can define a new open resp. closed string N by *deleting an element* $i \in M$. For this set $N := M \setminus \{i\}$, and if i meets two edges in $\Lambda(M, e_M)$ connecting i to points i_- and i_+, then replace these two edges by one new edge between i_- and i_+ to obtain $\Lambda(N, e_N)$ ($i_- = i_+$ possible if $|M| = 2$). [5]

(vii) A *refinement* of a string is a pair (N, ρ) of a string N and a surjective *refinement map* $\rho : N \to M$. I.e for ρ there is a contraction $c : \Lambda(N, e_N) \rightsquigarrow \Lambda(M, e_M)$ (cf. Definition 1.18 (ii)), such that c restricted to the sets of vertices N and M, acts as ρ. For $|M| \geq 2$, (N, ρ) is a refinement if and only if for every $i \in M$, $\rho^{-1}(i)$ is a substring of M, and for all $i_1 \neq i_2 \in M$,

$$e_M(i_1, i_2) = \sum_{j_1 \in \rho^{-1}(i_1), j_2 \in \rho^{-1}(i_2)} e_N(j_1, j_2).$$

If $\rho : N \to M$ is a refinement map, then either N and M are both closed or both open.

For a given M, the relationship between refinements $\rho : N \to M$ and contractions $c : \Lambda(N, e_N) \rightsquigarrow \Lambda(M, e_M)$ is a $1 : 1$ correspondence except if $|M| \leq 2$. In the letter case one refinement will always be induced by two different contractions, since a contraction chooses for each of the two edges connecting the two vertices of $\Lambda(M, e_M)$, a preimage-edge in $\Lambda(N, e_N)$. (This will be discussed in more detail at the beginning of the proof of Lemma 5.28.)

(viii) A circularly arranged set P which is a partition of $\underline{n}$ (cf. Notation1.1), we will call a *circular partition* of $\underline{n}$. We often write such partitions as $P = \langle I_1, ..., I_m \rangle$, and then usually carry over the circular arrangement to the set of indices $\underline{m}$, i.e. arrange as $\underline{m} = \langle 1, ..., m \rangle$, without further mentioning it. *To simplify notation in later applications, we require in addition that* $|P| \geq 2$ *for a circular partition.*

(ix) If we have a tuple $(a_1, ..., a_m)$ this of course defines a closed string $\langle a_1, ..., a_m \rangle$. Hence to any ordered partition $(I_1, ..., I_m)$ we can associate a circular partition $\langle I_1, ..., I_m \rangle$.

(x) A circular partition $P' := \langle J_1, ..., J_r \rangle$ is called a *refinement* of $P := \langle I_1, ..., I_m \rangle$, if it can be obtained by replacing each I_i in $\langle I_1, ..., I_m \rangle$ by an (ordered) partition $J_{j_1}, ..., J_{j_s}$ of I_i. More precisely this means that each J_j is contained in one of the I_i, and that the surjective map $\rho : \langle J_1, ..., J_r \rangle \to \langle I_1, ..., I_m \rangle$ sending each J_j to the I_j containing it, is a refinement map in the sense of (v).

With respect to the induced arrangement on the set of indices (cf. (iv)), ρ induces a refinement map $i : \underline{r} \to \underline{m}$, such that $J_j \subseteq I_{i(j)}$.

(xi) As one would expect we call a circular partition $\bar{P}$ a common refinement for a collection of circular partitions P_1, ..., P_s, if $\bar{P}$ is a refinement of each P_k, $k \in \underline{s}$. In this case, there are s *refinement maps* $\rho_k : \bar{P} \to P_k$.

[5] I.e. set $e_N := e_{M|N}$ if $\sum_{i' \in M} e_M(i, i') = 1$, or obtain e_N by increasing the value of $e_{M|N}(i_-, i_+)$ by 1 if $\sum_{i' \in M} e_M(i, i') = 2$.

We call this $\bar{P}$, a *coarsest common refinement* of $P_1, ..., P_s$ if it has the property that there is no common refinement $\widetilde{P} \neq \bar{P}$ of $P_1, ..., P_s$, such that $\bar{P}$ is a refinement of $\widetilde{P}$. As we see later, there may be more than one coarsest common refinement for given $P_1, ..., P_s$.

(xii) To a circular partition $P = \langle I_1, ..., I_m \rangle$ of $\underline{n}$, with $m \geq 2$, we assign a stable $(1, n)$-graph

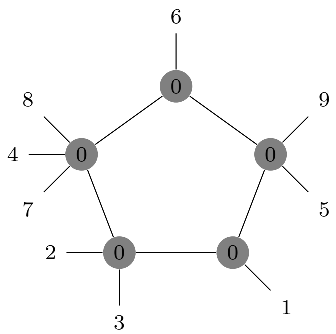
$$\Gamma(P) = (V, H, a : H \to V, i : H \to H, g : V \to \mathbb{Z}_{\geq 0}, p : \underline{n} \to H)$$

(cf. Definition 1.16) as follows: V is the set P. Define H', a' and i', such that $\Lambda(P, e_P) = (V, H', a', i')$. Let g be constant 0, and $b : \underline{n} \to V$ be the map sending each element of a set I_i to $I_i \in V$. Then set $H := H' \cup \underline{n}$, $a := a' \cup b$ and $i := i' \cup id_{\underline{n}}$. The marking p is just the inclusion $\underline{n} \hookrightarrow H = H' \cup \underline{n}$. I.e. $\Gamma(P)$ can be visualised as the graph one obtains by attaching to each vertex I_i of $\Lambda(P, e_P)$, for each $k \in I_i$, a leg labelled by k.

Obviously the $\Gamma(P)$ defined such is the graph of a simple banana cycle (cf. Def. 4.1 (ii)). On the other hand we can assign to each graph Γ of a simple banana cycle in $\overline{M}_{1,n}$ a circularly ordered partition $P(\Gamma) = \langle I_1, ..., I_m \rangle$ with $m \geq 2$: Let $e_V(v, v')$ be the number of edges connecting a pair $v, v' \in V$ of vertices. Then carry over this circular arrangement e_V of $V = \langle v_1, ..., v_m \rangle$ to the set $\langle I_1, ..., I_m \rangle$, where $I_i := b^{-1}(v_i)$. It is clear that $P(\Gamma(P)) = P$ and $\Gamma(P(\Gamma)) = \Gamma$. By this there is a $1 : 1$ correspondance between the simple banana cycles in $\overline{M}_{1,n}$ and the circular partitions $P = \langle I_1, ..., I_m \rangle$ of $\underline{n}$, with $m \geq 2$.

The following example for the relationship between P and $\Gamma(P)$ depicts a stable graph in the way introduced in Example 1.24

$$P = \langle \{2, 3\}, \{1\}, \{5, 9\}, \{6\}, \{4, 7, 8\} \rangle \quad \longleftrightarrow \quad \Gamma(P) =$$



Lemma 5.17 *Let N, M be strings, either both closed or both open. Then:*

(i) For a surjective map $\rho : M \to N$ the following are equivalent:

(1) *ρ is a refinement map*

(2) *For all $S_1, S_2 \subseteq N$: $S_1 \parallel S_2 \Rightarrow \rho^{-1}(S_1) \parallel \rho^{-1}(S_2)$.* [6]

(3) *For all substrings $S \subseteq N$, $\rho^{-1}(S)$ is a substring of M.*

(ii) Let $r : 2^N \to 2^M$ be a map between the power sets, such that $r(S_1 \cup S_2) = r(S_1) \cup r(S_2)$ for each $S_1, S_2 \in 2^N$. If r has the property that for $S_1, S_2 \in 2^N$, $S_1 \parallel S_2$ implies $r(S_1) \parallel r(S_2)$, then even

$$S_1 \parallel S_2 \parallel ... \parallel S_n \Rightarrow r(S_1) \parallel r(S_2) \parallel ... \parallel r(S_n),$$

[6] Recall that in our notation $A \parallel B$ includes the assertion that A and B are substrings.

for any $S_1, ..., S_n \in 2^N$.

(iii) If $\rho : N \to M$ *is a refinement map, then also for all sets* $S_1, S_2 \subseteq N$,

$$S_1 \parallel S_2 \Rightarrow \rho(S_1) \parallel \rho(S_2).$$

(But this condition is strictly weaker than being a refinement map.)

(iv) If M *is a finite set with* $|M| = m \geq 3$ *then it can be circularly arranged in* $\frac{m!}{2m}$ *different ways.*

(v) If M *is a string, and if* $S_1, S_2 \subseteq M$ *are substrings, then* $S_1 \cap S_2$ *is either a string, or consists of exactly two substrings* T *and* T' *of* M *with* $T \nparallel T'$*. In the second case* T *and* T' *each contain one end point of the (then open) string* S_1 *and one end-point of the (then open) string* S_2*. The second case is only possible if* M *is closed and* $S_1 \cup S_2 = M$*.*

Proof: (i): "(1) ⇒ (2)": WLOG assume that we do not have $S_1 = S_2$. For $S_1 \parallel S_2$, by definition of refinement maps, $\rho^{-1}(S_1) \cup \rho^{-1}(S_2) = \rho^{-1}(S_1 \cup S_2)$ has to be a substring of M. Furthermore if $S_1 \cup S_2$ is an open string then it is clear that $\rho^{-1}(S_1) \cup \rho^{-1}(S_2)$ is open too. Let $i_1 \in S_1$ and $i_2 \in S_2$ be the end-points of $S_1 \cup S_2$. Then

$$\sum_{i' \in S_1 \cup S_2} e_N(i_1, i') = \sum_{i' \in S_1 \cup S_2} e_N(i_1, i') = 1.$$

But by (1) this implies

$$\sum_{\substack{j_1 \in \rho^{-1}(i_1),\\ j' \in \rho^{-1}(S_1 \cup S_2)}} e_N(j_1, j') = \sum_{\substack{j_1 \in \rho^{-1}(i_1),\, j' \in \rho^{-1}(i'),\\ i' \in S_1 \cup S_2}} e_N(j_1, j') = \sum_{i' \in S_1 \cup S_2} e_N(i_1, i') = 1,$$

and the same for i_2 instead of i_1. This means that the open strings $\rho^{-1}(i_1)$ resp. $\rho^{-1}(i_2)$ are connected to the rest of the string $\rho^{-1}(S_1 \cup S_2)$ by only one edge. But this implies that $\rho^{-1}(i_1) \subseteq \rho^{-1}(S_1)$ and $\rho^{-1}(i_2) \subseteq \rho^{-1}(S_2)$ each contain an end-point of $\rho^{-1}(S_1 \cup S_2)$.

"(2) ⇒ (3)", is clear by Def. 5.16 (v).

"(3) ⇒ (1)": By (3) for $i \in N$, $\rho^{-1}(i)$ is a string. One gets that

$$e_N(i_1, i_2) \leq \sum_{j_1 \in \rho^{-1}(i_1),\, j_2 \in \rho^{-1}(i_2)} e_M(j_1, j_2),$$

since if $\{i_1, i_2\}$ is a substring then also $\rho^{-1}(\{i_1, i_2\})$ is a substring, and since N is closed iff M is. So from (3) we can conclude that M is as set the disjoint unions of the substrings $\rho^{-1}(i)$ for $i \in M$, and that two such substrings $\rho^{-1}(i_1)$ and $\rho^{-1}(i_2)$ are connected (in the graph $\Lambda(M, e_M)$) by at least as many edges, as i_1 and i_2 in the graph $\Lambda(N, e_N)$. But now it is easy to see that $\sum_{j_1 \in \rho^{-1}(i_1), j_2 \in \rho^{-1}(i_2)} e_M(j_1, j_2) > e_N(i_1, i_2)$ would either imply that M is closed while N is not, or that M is not even a string.

(ii): The definition of $S_1 \parallel S_2 \parallel ... \parallel S_n$ in Def. 5.16 (v), can obviously be reformulated as: For all $1 \leq q \leq r \leq n$, $S_q \parallel \bigcup_{i=q+1}^{r} S_i$ and $\bigcup_{i=q}^{r-1} S_i \parallel S_r$. Under the conditions of the Lemma this implies $r(S_q) \parallel \bigcup_{i=q+1}^{r} r(S_i)$ and $\bigcup_{i=q}^{r-1} r(S_i) \parallel r(S_r)$, and hence $r(S_1) \parallel r(S_2) \parallel ... \parallel r(S_n)$.

(iii): This is quite obvious if we look at one of the contractions $c : \Lambda(N, e_N) \rightsquigarrow \Lambda(M, e_M)$ corresponding to ρ. (Cf. Def. 5.17 (vii).)

(iv): There are $m!$ different bijective maps $M \to \mathbb{Z}/|M| \cdot \mathbb{Z}$, and we formed equivalence classes of $2m$ elements each (cf. Def 5.16 (iii)).

(v): For M open the statement is obvious. If either S_1 or S_2 is M, (ii) is again obvious. Otherwise the substrings S_1 and S_2 are open strings. Let a and b be the two end-points of the string S_2. If S_1 contains none of these points either $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$. In both cases $S_1 \cap S_2$ is a string. If exactly one of these points, say a, is contained in S_1 then $S_1 \cap S_2$ contains all elements of S_1 that lie on one side of a and none of the elements lying on the other side, hence $S_1 \cap S_2$ is a string. If a and b are both in S_1 then either S_2 consists of the elements in M_1 lying between a and b, in which case $S_1 \cap S_2 = S_2$ is a string, or S_2 is the complement in M of the elements of S_1 lying strictly between a_1 and a_2. In this last case $S_1 \cap S_2$ is not a string but a union of two strings, and $S_1 \cup S_2 = M$. □

Definition 5.18 For a string M we let $\mathrm{EnP}(M)$ be the end-points of M if M is open, and we set $\mathrm{EnP}(M) = M$ if M is closed.

Lemma 5.19 *Let M_1 and M_2 be two closed strings, N a set, $\rho_1 : N \to M_1$, $\rho_2 : N \to M_2$ two surjective maps. For all sets $S \subseteq M_1$ and $T \subseteq M_2$ use the notation $r_{12}(S) := \rho_2\rho_1^{-1}(S)$, $r_{21}(T) := \rho_1\rho_2^{-1}(T)$. Then the following two conditions (1) and (2) are equivalent:*

(1) *The following two conditions hold:*

(a) For all sets $S, S' \subseteq M_1$ resp. $T, T' \subseteq M_2$, we have

$$S \parallel S' \Rightarrow r_{12}(S) \parallel r_{12}(S'), \quad \text{and} \quad T \parallel T' \Rightarrow r_{21}(T) \parallel r_{21}(T').$$ [7]

(b) For all $i \in M_1$ set $C_1(i) := M_1 \smallsetminus i$. Then $|r_{12}(i) \cap r_{12}(C_1(i))| \le 2$, and furthermore $r_{12}(i) \cap r_{12}(C_1(i)) \subseteq \mathrm{EnP}(r_{12}(i)) \cap \mathrm{EnP}(r_{12}(C_1(i)))$. The analogous statement holds for, $j \in M_2$, $C_2(j) := M_2 \smallsetminus \{j\}$ and $r_{21}(j) \cap r_{21}(C_2(j))$.

(2) *N can be arranged as a closed string in such a way, that with this arrangement (N, ρ_1) is a refinement of M_1 and (N, ρ_2) is a refinement of M_2.*

Proof: "(2) $\Rightarrow$ (1)": It is clear that (2) implies (1) (a), by Lemma 5.17 (i) and (iii). To show (1) (b), define for all $i \in M_1$, $j \in M_2$: $R_1(i) := \rho_1^{-1}(i)$, $R_2(j) := \rho_2^{-1}(j)$. Set $K_1(i) := \rho_1^{-1}(C_1(i))$ and $K_2(j) := \rho_2^{-1}(C_2(j))$. We have $N = R_1(i) \uplus K_1(i)$, and by (2) the $R_1(i)$, $R_2(j)$, $K_1(i)$, $K_2(j)$ are substrings of N. Now if $j \in r_{12}(i) \cap r_{12}(C_1(i))$ this means $R_1(i) \cap R_2(j) \neq \emptyset \neq K_1(i) \cap R_2(j)$. But then $R_2(j)$ has to contain one end-point of $R_1(i)$ and one end-point of $K_1(i)$. Hence, since the $R_2(j)$ are disjoint there can be at most one other $j' \in M_2$, such that $j' \in r_{12}(i) \cap r_{12}(C_1(i))$. This shows the first claim of (1) (b). For the second claim, note that $j \in r_{12}(i) \cap r_{12}(C_1(i))$ implies $R_1(i) \parallel R_2(j)$ and $K_1(i) \parallel R_2(j)$. Then, using Lemma 5.17 (iii):

$$r_{12}(i) = \rho_2(R_1(i)) \parallel \rho_2(R_2(j)) = \{j\}, \quad \text{and} \quad r_{12}(C_1(i)) = \rho_2(K_1(i)) \parallel \rho_2(R_2(j)) = \{j\}.$$

[7] Again, recall that in our notation $A \parallel B$ includes the assertion that A and B are substrings.

Since also $j \in r_{12}(i)$ and $j \in r_{12}(C_1(i))$, we can conclude that $j \in \mathrm{EnP}(r_{12}(i)) \cap \mathrm{EnP}(r_{12}(C_1(i)))$. [8]

To show "(1) $\Rightarrow$ (2)", we first restrict WLOG to the case that N is "as coarse as possible": We define a set $\bar{N}$ by replacing each *non-empty* set $R_1(i) \cap R_2(j) \subseteq N$ for $i \in M_1$ and $j \in M_2$, by one element denoted $[i,j]$, and define a surjective map $\pi : N \to \bar{N}$ sending all elements of each non-empty $R_1(i) \cap R_2(j)$ to $[i,j]$. Then it is easy to check that $\rho_1 = \bar{\rho}_1 \circ \pi$ and $\rho_2 = \bar{\rho}_2 \circ \pi$ for the surjections

$$\bar{\rho}_1 : \bar{N} \to M_1,\ [i,j] \mapsto i, \quad \text{and} \quad \bar{\rho}_2 : \bar{N} \to M_2,\ [i,j] \mapsto j.$$

It is true that condition (1) resp. (2) holds for $(N, M_1, M_2, \rho_1, \rho_2)$ if and only if condition (1) resp. (2) holds for $(\bar{N}, M_1, M_2, \bar{\rho}_1, \bar{\rho}_2)$, but we will only show the implications we need in our proof: For condition (1) just note that r_{12} and r_{21} are not changed when replacing (N, ρ_1, ρ_2) by $(\bar{N}, \bar{\rho}_1, \bar{\rho}_2)$. For condition (2), if we have an arrangement $e_{\bar{N}}$ as string on $\bar{N}$, which makes the $\bar{\rho}_i$ into refinement maps, we arrange N as a string such that the surjective map $\pi : N \to \bar{N}$ becomes a refinement map. For this just arrange the preimage $\pi^{-1}([i,j])$ of each point $[i,j] \in \bar{N}$ in an arbitrary way as open string by an arrangement $e_{\pi^{-1}([i,j])}$ and replace in the graph $\Lambda(\bar{N}, e_{\bar{N}})$ each vertex $[i,j]$ by the graph $\Lambda(\pi^{-1}([i,j]), e_{\pi^{-1}([i,j])})$. [9] Then π will be a refinement map by Lemma 5.17 (i) (3). Hence, as compositions of two refinement maps, the $\rho_i = \bar{\rho}_i \circ \pi$ are refinement maps with this arrangement of N.

Hence we can WLOG assume that $(N, \rho_1, \rho_2) = (\bar{N}, \bar{\rho}_1, \bar{\rho}_2)$. Note that under this assumption, $R_1(i) = \rho_1^{-1}(i) = \{[i,j] \mid j \in r_{12}(i)\}$, $R_2(j) = \rho_2^{-1}(j) = \{[i,j] \mid i \in r_{21}(j)\}$.

For $|M_1| = 1$ or $|M_2| = 1$ it is clear that conditions (1) and (2) are always satisfied. So assume WLOG that $|M_1| \geq 2$ and $|M_2| \geq 2$.

Assuming (1) we now construct an arrangement e_N of $N = \bar{N}$ fulfilling (2), as follows: The restricted maps $\rho_{2|R_1(i)} : R_1(i) \to r_{12}(i) \subseteq M_2$ and $\rho_{1|R_2(j)} : R_2(j) \to r_{21}(j) \subseteq M_1$, are bijections, and the images $r_{12}(i)$ resp. $r_{21}(j)$ are strings. We use this to arrange the $R_1(i)$ and $R_2(j)$ as open strings: First carry over the arrangement of the string $r_{12}(i)$ to $R_1(j)$, and call this arrangement $\widetilde{e}_{R_1(i)}$. If $(R_1(i), \widetilde{e}_{R_1(i)})$ is an open string, set $e'_{R_1(i)} := \widetilde{e}_{R_1(i)}$. Otherwise, we have to cut open (cf. Def. 5.16 (iv)) this closed string: By (1) (b), $1 \leq |r_{12}(C_1(i))| = |r_{12}(i) \cap r_{12}(C_1(i))| \leq 2$. If $|r_{12}(i) \cap r_{12}(C_1(i))| = 2$, let j_1, j_2 be its elements, if $|r_{12}(i) \cap r_{12}(C_1(i))| = 1$ call its only element j. Now in the first case, cut $(R_1(i), \widetilde{e}_{R_1(i)})$ open between $[i, j_1]$ and $[i, j_2]$. In the second case choose one neighbour j^* of j in M_2 and cut $R_1(i)$ open between $[i,j]$ and $[i, j^*]$. Call the resulting arrangement as open string again $e'_{R_1(i)}$. Arrange the $R_2(j)$ as open strings in the same way.

We want e_N to restrict on the $R_1(i)$ and $R_2(j)$ to the arrangements $e'_{R_1(i)}$, $e'_{R_2(j)}$ just defined. So we start by defining a arrangement e'_N with this property, by for all $i_1, i_2 \in M_1$ and $j_1, j_2 \in M_2$ setting $e'_N([i_1, j_1], [i_2, j_2]) :=$

$$\max\{e'_{R_1(i)}([i_1, j_1], [i_2, j_2]),\ e'_{R_2(j)}([i_1, j_1], [i_2, j_2]) \mid i \in M_1,\ j \in M_2\}.$$

[8] Note that in general for a string M: $j \in \mathrm{EnP}(M) \Leftrightarrow (j \in M \ \wedge\ \{j\} \parallel M)$

[9] Each $\Lambda(\pi^{-1}([i,j]), e_{\pi^{-1}([i,j])})$ can be fitted in in two different ways but π will become a refinement map independent of this choice.

The arrangement e'_N defined by this is in general not an arrangement as string yet, since $\Lambda(N, e'_N)$ is in general not connected.

By this definition it is clear that always $e'_N([i_1, j_1], [i_2, j_2]) \leq 1$, with $= 1$ possible only if: Either $i_1 = i_2$ and $e_{M_2}(j_1, j_2) \geq 1$, or $j_1 = j_2$ and $e_{M_1}(i_1, i_2) \geq 1$. We refer to this remark by (†).

Next we show:

($*$) For all $i_1 \neq i_2 \in M_1$ we have $e_{M_1}(i_1, i_2) \geq \sum_{[i_1,j]\in R_1(i_1),[i_2,j']\in R_1(i_2)} e'_N([i_1, j], [i_2, j'])$. For all $j_1, j_2 \in M_2$ the analogous statement holds.

By (†), $e_{M_1}(i_1, i_2) = 0$ implies $\sum_{[i_1,j]\in R_1(i_1),[i_2,j']\in R_1(i_2)} e'_N([i_1, j], [i_2, j']) = 0$. So we now may assume $e_{M_1}(i_1, i_2) \geq 1$. If we have $e'_N([i, j], [i', j']) = 1$ for some $[i_1, j] \in R_1(i_1)$, $[i_2, j'] \in R_1(i_2)$, this means that $j = j'$. So $j \in r_{12}(i_1) \cap r_{12}(i_2)$, and then by definition of e'_N:

$$\sum_{[i_1,j]\in R_1(i_1),[i_2,j']\in R_1(i_2)} e'_N([i_1, j], [i_2, j']) = \sum_{j\in r_{12}(i_1)\cap r_{12}(i_2)} e'_{R_2(j)}([i_1, j], [i_2, j]).$$

The $e'_{R_2(j)}$ are ≤ 1 everywhere by their definition. So if $|r_{12}(i_1) \cap r_{12}(i_2)| = 1$ we are done. Otherwise by (1) (b), $|r_{12}(i_1) \cap r_{12}(i_2)| = 2$ and we call the two elements of the intersection j_a, j_b. Then we have $\{i_1, i_2\} \subseteq r_{21}(j_a) \cap r_{21}(j_b)$ By (1) (b) we must have $r_{21}(j_a) = r_{21}(C_2(j_b))$ and $r_{21}(j_a) = \{i_1, i_2\}$ or $r_{21}(j_b) = \{i_1, i_2\}$. WLOG $r_{21}(j_a) = \{i_1, i_2\}$ and hence $r_{21}(j_b) = M_1$. Recall from our construction of e'_N that then the open string $(R_2(j_b), e'_{R_2(j_b)})$ is obtained from the closed string $(R_2(j_b), \widetilde{e}_{R_2(j_b)})$ by cutting open between $[i_1, j_b]$ and $[i_2, j_b]$. So in this case $e'_{R_2(j_b)}([i_1, j_b], [i_2, j_b]) = 0$ and hence

$$\sum_{[i_1,j]\in R_1(i_1),[i_2,j']\in R_1(i_2)} e'_N([i, j], [i', j']) \leq 1 = e_{M_1}(i_1, i_2).$$

We have proven ($*$). A direct consequences of ($*$) is: For all $[i, j] \in N$,

$$\sum_{[i',j']\in N} e'_N([i, j], [i', j']) \leq 2.$$

This already implies:

(A) For some $1 \leq m \leq \min\{|M_1|, |M_2|\}$, e'_N arranges N as the disjoint union of strings $S_1, ..., S_m$, such that the different S_k are not connected to each other by edges in $\Lambda(N, e'_N)$.

It is clear by the definition of e'_N, that each $R_1(i)$ and each $R_2(j)$ is contained in exactly one of the S_k. Hence:

(B) $M_1 = \rho_1(S_1) \uplus .... \uplus \rho_1(S_m)$ and $M_2 = \rho_2(S_1) \uplus .... \uplus \rho_2(S_m)$. And $\rho_1^{-1}\rho_1(S_k) = S_k$, $\rho_2^{-1}\rho_2(S_k) = S_k$.

(C) The restrictions $\rho_{1|S_k} : S_k \to \rho_1(S_k)$ are refinement maps between strings, except in the possible case that $m = 1$, M_1 is a closed string and S_1 is an open string. In this case $\rho_{1|S_1} = \rho_1$ becomes a refinement map if one closes the string S_1. The same holds for ρ_2 instead of ρ_1. In particular the $\rho_1(S_k)$ and $\rho_2(S_k)$ are substrings of M_1 and M_2.

To show (C): N is the disjoint union of the $R_1(i)$ $(i \in M_1)$, and by (B), S_k is (as a set) the disjoint union of the $R_1(i)$ with $i \in \rho_1(S_k)$ and the $R_1(i)$ are substrings of the open string S_k by construction of e'_N. Furthermore by (†), two $R_1(i), R_1(i') \subset S_k$ can be connected by an edge only if $i \parallel i'$. Hence we can write the elements of $\rho_1(S_k)$ as $i_1, ..., i_r$ in such a way that $R_1(i_1) \parallel R_1(i_2) \parallel ... \parallel R_1(i_r)$, and then we must have $i_1 \parallel i_2 \parallel ... \parallel i_r$. If $\rho_1(S_k)$ is open, i.e. if $m \geq 2$, then this proves that $\rho_{1|S_k}$ is a refinement (using either the definition of refinement of strings or the criterion of Lemma 5.17 (i) (3)). If $m = 1$ then $\rho_1(S_k)$ is closed, but after closing S_k if it is not closed already, we also have $R_1(i_r) \parallel R_1(i_1)$, and $\rho_{1|S_k}$ is a refinement map by Lemma 5.17 (i) (3).

(D) If S_k is open, and $[i,j] \in \mathrm{EnP}(S_k)$, then either $R_1(i) = \{[i,j]\} \subseteq R_2(j)$ or $R_2(j) = \{[i,j]\} \subseteq R_1(i)$.

We get (D), by observing that $[i,j] \in \mathrm{EnP}(S_k)$, i.e. $e'_N([i,j]) := \sum_{[i',j'] \in N} e'_N([i,j],[i',j']) < 2$ is equivalent to

$$[i,j] \in \mathrm{EnP}(R_1(i)) \ \wedge \ [i,j] \in \mathrm{EnP}(R_2(j)) \ \wedge \ \big(|R_1(i)| = 1 \ \vee \ |R_2(j)| = 1\big),$$

which clearly implies (D). It is clear from the construction of e'_N, that we would have $e'_N([i,j]) = 2$ if one of the first two conditions was not satisfied. If the third condition was not satisfied there would have to be a $j_+ \parallel j$ and a $i_+ \parallel i$ such that in $R_1(i)$ resp. $R_2(j)$ we had $[i,j_+] \parallel [i,j]$ resp. $[i_+,j] \parallel [i_+,j]$. Hence again $e'_N([i,j]) = 2$.

Now we show by induction on m: If e'_N is an arrangement on N, such that (A), (B), (C), (D) are fulfilled by $((N,e'_N), M_1, M_2, \rho_1, \rho_2)$, then there is an arrangement e_N of N as a string, such that $e_N \geq e'_N$ [10] , and such that condition (2) is fulfilled. [11]

Now to the existence of e_N: For $m = 1$, if S_1, M_1, M_2 are all closed, (C) immediately implies (2) for $e_N := e'_N$. If $m = 1$, S_1 open, but M_1 and M_2 closed, close $(N, e'_N) = (S_1, e'_N)$ to obtain a string (N, e_N), fulfilling (2) by (C). For $m > 1$, the S_k are open, and we show below that there are end-points $[i_a, j_a] \in \mathrm{EnP}(S_1)$, and $[i_b, j_b] \in \mathrm{EnP}(S_k)$ for some $S_k \parallel S_1$, such that $i_a \parallel i_b \in M_1$ and $j_a \parallel j_b$ in M_2. Now define a new arrangement e''_N by keeping all edges of e'_N but additionally connect $[i_a, j_a]$ and $[i_b, j_b]$ by one edge. Then it not difficult to check that conditions (A)-(D) are still fulfilled with this new arrangement. But the number of not connected strings in (N, e''_N) is $m - 1$. Hence by induction hypothesis there is a $e_N \geq e''_N \geq e'_N$ fulfilling condition (2).

To finish the proof, it remains to show that the $[i_a, j_a] \in \mathrm{EnP}(S_1)$, $[i_b, j_b] \in \mathrm{EnP}(S_k)$ with $i_a \parallel i_b$ and $j_a \parallel j_b$ exist. By (D) we may WLOG assume that there is a $[i_a, j_a] \in \mathrm{EnP}(S_1)$ such that $R_1(i_a) = \{[i_a, j_a]\}$. By (A)-(C) there is some $S_k \neq S_1$ and a $i_b \in \mathrm{EnP}(\rho_1(S_k))$ such that $i_a \parallel i_b$. Hence $i_a \parallel i_b \parallel \rho_1(S_k)$. From this with (1) (a) we get $j_a = r_{12}(i_a) \parallel r_{12}(i_b) \parallel \rho_2(S_k)$. Since $r_{12}(i_b) \subseteq \rho_2(S_k)$, this implies that there is a $j_b \in r_{12}(i_b) \cap \mathrm{EnP}(\rho_2(S_k))$ with $j_a \parallel j_b$. Now $[i_b, j_b] \in S_k$ is contained in $\mathrm{EnP}(S_k)$, as can be concluded from the fact that $i_b \in \mathrm{EnP}(\rho_1(S_k))$ and $j_b \in \mathrm{EnP}(\rho_2(S_k))$ using (C) and (D). □

[10] By $e_N \geq e'_N$ we mean $e_N(\{[i_1,j_1],[i_2,j_2]\}) \geq e'_N(\{[i_1,j_1],[i_2,j_2]\})$for all $[i_1,j_1],[i_2,j_2] \in N$.

[11] We still assume that (1) holds, and that N is WLOG "as coarse as possible".

Lemma 5.20 *(i) For two circular partitions* $P_1 = \langle I_1, ..., I_m \rangle$, $P_2 = \langle I'_1, ..., I'_{m'} \rangle$ *of* $\underline{n}$ *define for* $k \in \underline{m}$ *and* $k' \in \underline{m'}$,

$$M_k := \{I'_{i'} \in P_2 \mid I_k \cap I'_{i'} \neq \emptyset\} \subseteq P_2, \quad \textit{resp.} \quad M'_{k'} := \{I_i \in P_1 \mid I'_{k'} \cap I_i \neq \emptyset\} \subseteq P_1.$$

Then P_1 *and* P_2 *have a common refinement if and only if, for all* $k_1, k_2 \in \underline{m}$ *and all* $k'_1, k'_2 \in \underline{m'}$, *we have:*

(a) $M_1 \parallel M_2 \parallel ... \parallel M_m \parallel M_1$ *and* $M'_1 \parallel M'_2 \parallel ... \parallel M'_{m'} \parallel M'_1$. [12]

(b) *For all but at most two elements of* M_k *we have* $I'_{i'} \subseteq I_k$, *and all* $I'_{i'} \in M_k$ *with* $I'_{i'} \nsubseteq I_k$ *are in* $\mathrm{EnP}(M_k) \cap \mathrm{EnP}(C_k)$, *where* C_k *is the string* $\bigcup_{i \in \underline{m} \setminus \{k\}} M_k$. *For* $M'_{k'}$ *the analogous claim holds.*

(ii) P_1 *and* P_2 *can only have more than one coarsest common refinement if there are* $k \in \underline{m}$, $k' \in \underline{m'}$ *such that* $I_k \cup I'_{k'} = \underline{n}$. *In this case, by cyclically permuting the indices if necessary, we can assume* $I_1 \cup I'_{m'} = \underline{n}$. *Then*

$$\bigcup_{i' \in \underline{m'} \setminus \{m'\}} I'_i \subseteq I_1 \quad \textit{and} \quad \bigcup_{i \in \underline{m} \setminus \{1\}} I_i \subseteq I'_{m'}.$$

If this condition is fulfilled, set

$$S := I_1 \cap I'_{m'} = I_1 \setminus \bigcup_{i' \in \underline{m'} \setminus \{m'\}} I'_i.$$

Then consider all ordered partitions (K_1, K_2) *of* S, *where, contrary to our usual convention, we allow that one of the* K_j *may be empty (or both if* $S = \emptyset$*). If we define for all these partitions* (K_1, K_2) *the partitions*

$$\langle K_1, I'_1, I'_2, ..., I'_{m'-1}, K_2, I_2, ..., I_m \rangle, \quad \textit{and} \quad \langle K_2, I'_{m'-1}, I'_{m'-2}, ..., I'_1, K_1, I_2, ..., I_m \rangle$$

then these are exactly all the coarsest common refinements of P_1 *and* P_2. *(If one of the* K_i *is empty, we have to delete it in the definition of these circular partitions, since such partitions do not contain empty sets as elements by our definition.)*

(iii) If $\bar{P}$ *and* $\bar{P}'$ *are two different coarsest common refinements of* P_1 *and* P_2, *then* $\bar{P}$ *and* $\bar{P}'$ *do not have a common refinement.*

Proof: (i): This is a special case of Lemma 5.19. For the "if" direction, set

$$M_1 := P_1, \quad M_2 := P_2, \quad N := \{I \cap I' \mid I \in P_1,\ I' \in P_2,\ I \cap I' \neq \emptyset\},$$

$$\text{and} \quad \rho_1(I \cap I') := I, \quad \rho_2(I \cap I') := I',$$

and note that conditions (a) resp. (b) of (i) are equivalent to (1) (a) resp. (1) (b) of Lemma 5.19.

(ii): In the following we denote by $(*)$ the condition that $I_k \cup I'_{k'} \neq \underline{n}$ for all $k \in \underline{m}$, $k' \in \underline{m'}$.

[12] Again, recall that in our notation $A \parallel B$ includes the assertion that A and B are substrings.

First we show that if condition $(*)$ holds, then there is at most one coarsest common refinement $\bar{P}$ of P_1 and P_2.

Note that if $P' = \langle K_1, ..., K_r \rangle$ is a common refinement of P_1 and P_2 and $\rho_1 : P' \to P_1$ resp. $\rho_2 : P' \to P_2$ are the refinement maps, then we get a coarsest common refinement $\bar{P}$ from P': Each substring $K_{k_1} \parallel ... \parallel K_{k_s}$ of P', which is of maximal length according to the property $\rho_1(K_{k_1}) = ... = \rho_1(K_{k_s})$ and $\rho_2(K_{k_1}) = ... = \rho_2(K_{k_r})$, is replaced by one element $K_{k_1} \cup ... \cup K_{k_r}$ when passing from P' to $\bar{P}$.

By what was just said it is clear that for any coarsest common refinement $\bar{P}$ of P_1, P_2, with refinement maps again called $\rho_1 : \bar{P} \to P_1$ and $\rho_2 : \bar{P} \to P_2$, we have: For all $I \in P_1$, $I' \in P_2$, if $\rho_1^{-1}(I) \cap \rho_2^{-1}(I')$ contains more than one element, then no two of them are neighbours. This is actually the property that distinguishes *coarsest* common refinements from others. So by Lemma 5.17 (v), $\rho_1^{-1}(I) \cap \rho_2^{-1}(I')$ can only contain more than two elements, if $\rho_1^{-1}(I) \cup \rho_2^{-1}(I') = \bar{P}$, i.e. if $I_{i_0} \cup I'_{i_0} = \underline{n}$. This means that $(*)$ is not fulfilled.

Hence condition $(*)$ implies for each $J \in \bar{P}$:

$$J = \rho_1(J) \cap \rho_2(J). \tag{\dagger}$$

We use this to show that under condition $(*)$ each neighbouring relation between two element J_a, J_b in $\bar{P}$ is already implied by the fact that $\bar{P}$ is a coarsest common refinement of P_1 and P_2.

Firstly Lemma 5.17 (iii) implies that for $I_a, I_b \in P_1$ and $I'_a, I'_b \in P_2$ [13] , $I_a \cap I'_a \parallel I_b \cap I'_b$ is only possible in $\bar{P}$ if $I_a \parallel I_b$ and $I'_a \parallel I'_a$. So it suffices to give criteria for $I_a \cap I'_a \parallel I_b \cap I'_b$ in this case.

Criterion (A) is for the case that $I_a = I_b$ and $I'_a \parallel I'_b$ with $I'_a \neq I'_b$ [14]. Then we have:

$$J_a := I_a \cap I'_a \parallel I_a \cap I'_b =: J_b$$

in $\bar{P}$, if and only if the condition that either $I_a \cup I'_a \cup I'_b \neq \underline{n}$ or $I'_a \cup I'_b = \underline{n}$ is fulfilled.

The "if" direction is true since

$$\{J_a, J_b\} = \rho_1^{-1}(I_a) \cap \rho_2^{-1}(\{I'_a, I'_b\})$$

is a set of neighbours by Lemma 5.17 (v), if $I_a \cup I'_a \cup I'_b \neq \underline{n}$. If $I'_a \cup I'_b = \underline{n}$ then $\{J_a, J_b\} = \rho_1^{-1}(I_a)$ is also a set of neighbours. For the "only if" direction, we use that if the condition does not hold then

$$S_1 := \rho_1^{-1}(P_1 \smallsetminus \{I_a\}), \quad S_2 := \rho_2^{-1}(P_2 \smallsetminus \{I'_a, I'_b\}), \quad T_1 := \rho_2^{-1}(I'_a), \quad T_2 := \rho_2^{-1}(I'_b)$$

all are non-empty substrings of $\bar{P}$, and $S_1 \uplus S_2 \uplus \{J_a, J_b\} = \bar{P}$. Furthermore S_1 is a substring of the open string $T_1 \cup T_2$, and T_1 consists only of J_a and some elements of S_1, while T_2

[13] We attach the a and b to the I and I' just to distinguish the two elements I_a and I_b and the two elements I'_a and I'_b. This should not indicate, that for example I_a and I'_a appear with the same index in $P_1 = \langle I_1, ..., I_m \rangle$ and in $P_2 = \langle I'_1, ...I'_{m'} \rangle$ respectively.

[14] Of course an analogous criterion holds if we switch the roles of P_1 and P_2, i.e. assume $I'_a = I'_b$ and $I_a \neq I_b$.

contains only J_b and some elements of S_1. So $\{J_a\} \parallel S_1 \parallel \{J_b\}$ and hence

$$\{J_a\} \parallel S_1 \parallel \{J_b\} \parallel S_2 \parallel \{J_a\}, \quad \text{and thus} \quad J_a \nparallel J_b.$$

Criterion (B) is for $I_a \neq I_b$, $I'_a \neq I'_b$, but $I_a \parallel I_b$ and $I'_a \parallel I'_b$. Then:

$$I_a \cap I'_a \parallel I_b \cap I'_b$$

in $\bar{P}$, if and only if

$$(I_a \subseteq I'_a \ \vee \ I_a \supseteq I'_{a'}) \ \wedge \ (I_b \subseteq I'_b \ \vee \ I_b \supseteq I'_b).$$

For the "if" direction note that under this condition, by Lemma 5.17 (v) and condition $(*)$,

$$T := \rho_1^{-1}(\{I_a, I_b\}) \cap \rho_2^{-1}(\{I'_a, I'_b\})$$

is a substring of $\bar{P}$. [15] Furthermore one can check that under the condition we have $T = \{I_a \cap I'_a,\, I_b \cap I'_b\}$, and hence $I_a \cap I'_a \parallel I_b \cap I'_b$.

To show the "only if" direction, assume for example that $I_a \nsubseteq I'_a$ and $I_a \nsupseteq I'_a$. Then by (ii), we have that

$$S_1 := \rho_1^{-1}(I_a) \cap \rho_2^{-1}(P_2 \setminus \{I'_a\}) \quad \text{and} \quad S_2 := \rho_2^{-1}(I'_a) \cap \rho_1^{-1}(P_1 \setminus \{I_a\})$$

are non-empty, disjoint substrings of $\bar{P}$. Since also $\rho_1^{-1}(I_a) = S_1 \cup \{I_a \cap I'_a\}$ and $\rho_2^{-1}(I'_a) = S_2 \cup \{I_a \cap I'_a\}$ are set of neighbours we get

$$S_1 \parallel \{I_a \cap I'_a\} \parallel S_2.$$

Since $I_b \cap I'_b$ is neither contained in S_1 nor in S_2 this implies $I_a \cap I'_a \nparallel I_b \cap I'_b$.

The two criteria we just have proven determine $\bar{P}$ completely, hence the coarsest common refinement of P_1 and P_2 is unique if condition $(*)$ holds.

If $(*)$ does not hold, and hence WLOG $I_1 \cup I'_{m'} = \underline{n}$, then it is easy to check that all the partitions that are claimed in (ii) to be coarsest common partitions of P_1 and P_2 are indeed such.

To show that these are all coarsest common partitions that exist, let $\bar{P}$ be any coarsest common partition of P_1 and P_2. Then $I_2, ..., I_m, I'_1, ..., I'_{m'-1}$ are pairwise different elements of $\bar{P}$, and by Lemma 5.17 (v), $T_1 := I_2 \parallel ... \parallel I_m$ and $T_2 = I'_1 \parallel ... \parallel I'_{m'-1}$ are two substrings of $\bar{P}$. If we view T_1 and T_2 as sets, then

$$\bar{P} \setminus (T_1 \cup T_2) = \rho_1^{-1}(I_1) \cap \rho_2^{-1}(I'_{m'}).$$

By Lemma 5.17 (v) and the distinguishing property of coarsest common refinements mentioned above, there can be at most two elements in $\rho_1^{-1}(I_1) \cap \rho_2^{-1}(I'_{m'})$, and if there are

[15] For this reduce the possible cases WLOG to $I_a \cup I_b \cup I'_a \cup I'_b = I_a \cup I_b$ and $I_a \cup I_b \cup I'_a \cup I'_b = I_a \cup I'_b$. In the second case apply Lemma 5.17 (v) and $(*)$ to get the claim. In the first case either $I_a \cup I_b \neq \underline{n}$, in which case we again apply (v), or $I_a \cup I_b = \underline{n}$, in which case the claim is also clear.

two, they are not neighbours. Also it is clear that the union over the sets that are elements of $\rho_1^{-1}(I_1) \cap \rho_2^{-1}(I'_{m'})$ is just the set $S \subset \underline{n}$ defined in (ii). Hence the elements of $\bar{P}$ not contained in the strings T and T' are either only S or two disjoint sets K_1, K_2 such that $K_1 \cup K_2 = S$. Putting together what we have seen in this paragraph we obtain that all coarsest common partitions of P_1 and P_2 are of the forms claimed in (ii).

(ii): Using the description given in (ii) of the coarsest common partitions in the case when there is more than one of them, one sees that for any two different of them condition (a) of (ii) fails to hold. □

Remark 5.21 Lemma 5.20 together with the proof gives us the following (cumbersome) procedure to determine all coarsest common refinements of two given circular partitions $P_1 = \langle I_1, ..., I_m \rangle$, $P_2 = \langle I'_1, ..., I'_{m'} \rangle$:

Write down the M_k and $M'_{k'}$ of Lemma 5.20. Then using (i)+(ii), one checks whether there is a coarsest common refinement, and whether there is more than one. If there is more than one we can write them all down by (ii). In the case that (i)+(ii) say that there is exactly one coarsest common refinement, each M_k is some substring $I'_{i_{k,1}} \parallel I'_{i_{k,2}} \parallel ... \parallel I'_{i_{k,r}}$ of P_2. If $M_k = P_2$ for some $k \in \underline{m}$, cut the closed string M_k open between its two unique elements I'_a and I'_b such that I'_a and I'_b also appear in some other M_l (i.e. $I'_a, I'_b \not\subseteq I_k$). Call the resulting open string $\widehat{M_k}$. If M_k is already open, set $\widehat{M_k} = M_k$. Write $\widehat{M_k}$ as $I'_{j_{k,1}} \parallel I'_{j_{k,2}} \parallel ... \parallel I'_{j_{k,s}}$, define $\widetilde{M_k}$ to be the string $I_k \cap I'_{j_{k,1}} \parallel I_k \cap I'_{j_{k,2}} \parallel ... \parallel I_k \cap I'_{j_{k,s}}$. Now the coarsest common refinement $\bar{P}$ of P_1 and P_2 is obtained from the $\widetilde{M_1}, \widetilde{M_2}, ..., \widetilde{M_m}$ as follows: For each $k \in \underline{m}$ [16] glue (i.e. declare to be neighbours) the unique pair of endpoints $I_k \cap I_a \in \mathrm{EnP}(\widetilde{M_k})$ and $I_{k+1} \cap I_b \in \mathrm{EnP}(\widetilde{M_{k+1}})$ such that the pair fulfils one of the two criteria (A) and (B) from the proof of Lemma 5.20. [17]

Notation: Until now we usually denoted simple banana cycles in the form $B_{I_1,...,I_m}$, for $(I_1, ..., I_m)$ an ordered partition of $\underline{n}$. But since it is clear that the banana cycle only depends on the associated circular partition $P := \langle I_1, ..., I_m \rangle$, we will for the rest of this chapter write these cycles as $B_{\langle I_1,...,I_m \rangle}$ or B_P. This should remind us not to count them too often when they appear as twisted sectors.

Now we determine the "set theoretic" intersections $B_{P_1} \cap B_{P_2}$ of simple banana cycles.

Lemma 5.22 *(i) The intersection $B_{P_1} \cap B_{P_2}$ is non-empty, if and only if the two circular partitions P_1 and P_2 of $\underline{n}$ have a common refinement.*

In this case, if we let $\bar{P}^{(1)}, ..., \bar{P}^{(\nu)}$ be all the coarsest common refinements P_1 and P_2 have,

$$B_{P_1} \cap B_{P_2} = \biguplus_{k=1}^{\nu} B_{\bar{P}^{(k)}}.$$

[16] View $\underline{m}$ as circularly ordered, i.e. $m + 1 = 1$.

[17] If $m = 2$ there will of course be two such pairs of end points which have to be glued.

(ii) This implies for the simple banana cycles of $\overline{R}_{1,n}$*:*

$$B''_{P_1} \cap B''_{P_2} = \biguplus_{k=1}^{\nu} B''_{\bar{P}^{(k)}} \quad \text{and} \quad B^r_{P_1} \cap B^r_{P_2} = \biguplus_{k=1}^{\nu} B^r_{\bar{P}^{(k)}}$$

while $B''_{P_1} \cap B^r_{P_2}$ *is always empty.*

Proof: (i): Recall from Def. 5.16, the definition of $\Gamma(P)$ and $P(\Gamma)$ and the discussion of their connection in (xiii), and the discussion of the connection between contractions and refinements in (vii). From this we see that for circularly ordered partitions P and P' the following are equivalent:

$$\exists \text{ refinement } \rho : P' \to P \;\Leftrightarrow\; \exists \text{ contraction } c : \Gamma(P') \to \Gamma(P) \;\Leftrightarrow\; B_{P'} \subseteq B_P. \qquad (*)$$

For the last equivalence cf. Def. 4.1 (ii) and Prop. 1.26 (iv).

Hence $B_{P_1} \cap B_{P_2} \supseteq B_{\bar{P}(k)}$, for each $k \in \underline{\nu}$. It is clear that every common refinement P' of P_1 and P_2 is a refinement of one of the $\bar{P}^{(k)}$, hence also $B_{P'} \subset B_{\bar{P}^{(k)}}$ for some $k \in \underline{\nu}$.

In the opposite direction each pointed curve parametrised by a point of $B_{P_1} \cap B_{P_2}$ must have a stable graph Γ' which is a specialisation of $\Gamma(P_1)$ as well as $\Gamma(P_2)$. It is clear that a stable graph of genus 1 is the graph of a simple banana cycle, (cf. Definition 4.1 (ii)), if and only if it has more than one vertex, and contains no rational trees.

Now we can contract all rational trees of Γ' and obtain a graph Γ'' which is still a specialisation of $\Gamma(P_1)$ as well of $\Gamma(P_2)$, since these two graphs do not contain rational trees. So there are contractions $\Gamma'' \rightsquigarrow \Gamma(P_1)$ and $\Gamma'' \rightsquigarrow \Gamma(P_2)$, and Γ'' is the graph of a simple banana cycle. Hence using $\Gamma(P(\Gamma'')) = \Gamma''$ from Def. 5.16 (xii), $(*)$ implies that $P(\Gamma'')$ is a common refinement of P_1 and P_2. So it is a refinement of some $\bar{P}^{(k)}$. Hence Γ'' and thus also Γ' are specialisations of $\Gamma(\bar{P}^{(k)})$. Therefore the class of every curve with dual graph Γ' is contained in $B_{\bar{P}^{(k)}}$. We have shown that every point of $B_{P_1} \cap B_{P_2}$ is contained in one of the $B_{\bar{P}^{(k)}}$. That the union over the $B_{\bar{P}^{(k)}}$ is disjoint follows from Lemma 5.20 (iii).

(ii) $B''_{P_1} \cap B^r_{P_2} = \emptyset$ is a direct consequence of Lemma 4.8 (i). But this together with (i) also implies the rest of (ii). This is because for $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ the forgetful morphism, we have $\tau_n^{-1}(B_{P_1}) = B''_{P_1} \cup B^r_{P_1}$, $\tau_n^{-1}(B_{P_2}) = B''_{P_2} \cup B^r_{P_2}$, $\tau_n^{-1}(B_{P^{(k)}}) = B''_{P^{(k)}} \cup B^r_{P^{(k)}}$. □

5.3.2 Cohomology of simple banana cycles

Since the usual rational cohomology of each sector of $I_1(\overline{R}_{1,n})$ appears as summand in $H^*_{CR}(\overline{R}_{1,n})$, and since all simple banana cycles $B^r_{\langle I_1,\ldots,I_m\rangle}$ with m even are the supports of such sectors, we will compute the cohomology of the simple banana cycles here.

First note that for all $m \in \mathbb{N}_{\geq 2}$, $n \in \mathbb{N}$, $\langle I_1, ..., I_m\rangle$ a circular partition of $\underline{n}$:

$$B^r_{\langle I_1,\ldots,I_m\rangle} \cong B''_{\langle I_1,\ldots,I_m\rangle} \cong B_{\langle I_1,\ldots,I_m\rangle} \qquad (5.1)$$

as varieties. This follows from Lemma 1.46, since the cycles are normal varieties by Lemma 4.3 and since the finite forgetful morphism $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$, restricted to $B^r_{\langle I_1,\ldots,I_m\rangle}$ or

$B''_{\langle I_1,\ldots,I_m\rangle}$ has degree 1 (cf. the proof of Lemma 4.4). Because of these isomorphisms, in this subsection, which is concerned with the "inner structure" of the simple banana cycles, it will suffice to speak about $B_{\langle I_1,\ldots,I_m\rangle}$ only.

Definition 5.23 Let f_{B_P} be the embedding $f_{B_P} : B_P \hookrightarrow \overline{M}_{1,n}$, then of course the gluing morphism ξ_{B_P} factors as $\xi_{B_P} = f_{B_P} \circ z_{B_P}$, where we denote by $z_{B_P} : \overline{M}_{\Gamma(P)} \to B_P$ the finite surjective morphism obtained by restricting the codomain of $\xi_{B_{P_1}}$ to the image B_{P_1}. Analogously we define $f_{B^r_P}$, $f_{B''_P}$, $z_{B^r_P}$ and $z_{B''_P}$ and note that $\zeta_{B^r_P} = f_{B^r_P} \circ z_{B^r_P}$ and $\zeta_{B''_P} = f_{B''_P} \circ z_{B''_P}$.

Lemma 5.24 *For a partition $I_1 \uplus I_2 = \underline{n}$, let $\mathbb{S}_2$ act on $\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}}$ by simultaneously permuting the indices $\bullet_1$ and $\bullet_2$ on the first component and the indices $\circ_1$ and $\circ_2$ on the second component, defining a quotient $(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})/\mathbb{S}_2$. Then:*

(i) $B_{\langle I_1,I_2\rangle} \cong (\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})/\mathbb{S}_2$, as varieties, and we may identify $z_{B_{\langle I_1,I_2\rangle}}$ with the quotient morphism.

(ii) Let $\mathbb{S}_2$ act on $H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}})$ resp. $H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})$ by interchanging indices $\bullet_1$ and $\bullet_2$, resp. $\circ_1$ and $\circ_2$. Denote by $(\ldots)^+$ the $\mathbb{S}_2$ invariant part of each algebra, and by $(\ldots)^-$ the part on which the non-trivial element of $\mathbb{S}_2$ acts as multiplication with -1. Then the algebra $H^*(B_{\langle I_1,I_2\rangle})$ is isomorphic to the following sub-algebra of $H^*_{\mathbb{Q}}(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})$:

$$(H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}}))^+ \otimes (H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}}))^+ \oplus (H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}}))^- \otimes (H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}}))^-$$

(iii) We denote by $D^{(1)}$ resp. $D^{(2)}$ boundary divisors of $\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}}$ resp. $\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}}$, and by $\widehat{D}^{(1)}$ resp. $\widehat{D}^{(2)}$ the divisors that are obtained from $D^{(1)}$ resp. $D^{(2)}$ by interchanging the indices $\bullet_1$ and $\bullet_2$ resp. $\circ_1$ and $\circ_2$. Then the sub-algebra just described is generated by elements of the form: $(D^{(1)} + \widehat{D}^{(1)}) \otimes 1$, $1 \otimes (D^{(2)} + \widehat{D}^{(2)})$ and $(D^{(1)} - \widehat{D}^{(1)}) \otimes (D^{(2)} - \widehat{D}^{(2)})$.

(iv) We use the notation $k_{|I_1|,|I_2|}(s) := \dim_{\mathbb{Q}} H^s(B_{\langle I_1,I_2\rangle})$, $g_{|I|}(s) := \dim_{\mathbb{Q}} H^s_{\mathbb{Q}}(\overline{M}_{0,I\cup\{\bullet_1,\bullet_2\}})$, $h_{|I|}(s) := \dim_{\mathbb{Q}} H^s(\overline{M}_{0,I\cup\{\bullet_1,\bullet_2\}}/\mathbb{S}_2) = (H^s(\overline{M}_{0,I\cup\{\bullet_1,\bullet_2\}}))^+$, where $\mathbb{S}_2$ acts as in (ii). With this notation:

$$k_{n_1,n_2}(s) = \sum_{s_1+s_2=s} h_{n_1}(s_1)h_{n_2}(s_2) + (g_{n_1}(s_1) - h_{n_1}(s_1))(g_{n_2}(s_2) - h_{n_2}(s_2))$$

The functions $g_n(s)$ and $h_n(s)$ are known by [Kee92] resp. by [Get98].

Proof: (i): The gluing morphism $\xi_{B_{\langle I_1,I_2\rangle}} : \overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}} \to B_{\langle I_1,I_2\rangle}$ (cf. Proposition 1.26 (i)) is finite, surjective and of degree 2, since the stable graph $\Gamma(B_{\langle I_1,I_2\rangle})$ has 2 automorphisms [18] . It is clear that $\xi_{B_{\langle I_1,I_2\rangle}}$ is invariant under the action of $\mathbb{S}_2$ defined in the Lemma. It thus factors through a degree 1 morphism

$$\xi' : (\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})/\mathbb{S}_2 \to B_{\langle I_1,I_2\rangle}.$$

Since $B_{\langle I_1,I_2\rangle}$ is normal by Lemma 4.3, ξ' has to be an isomorphism (of varieties) by Lemma 1.46.

[18] The non-trivial one exchanges the two edges connecting the two vertices

(ii): Denote by φ_1 resp. φ_2 the morphism as which the non-trivial element of $\mathbb{S}_2$ acts on $H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}})$ resp. $H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})$, for the actions defined in (ii). We can write

$$H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}} \times \overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}}) = H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}}) \otimes H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})$$

by the Künneth formula.

Now by (i) and Lemma 3.1, $H^*(B_{\langle I_1,I_2\rangle})$ can be seen as the sub-algebra of this product formed by the elements that are invariant under $\varphi_1 \otimes \varphi_2$. Denote this sub-algebra by A. Denote the algebra that is claimed to be isomorphic to $H^*(B_{\langle I_1,I_2\rangle})$ in (ii) by B. It is clear that $B \subseteq A$.

For the opposite direction, use for any $Z^{(1)} \in H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}})$, $Z^{(2)} \in H^*(\overline{M}_{0,I_1\cup\{\circ_1,\circ_2\}})$, the notation $\widehat{Z}^{(1)} := \varphi_1(Z^{(1)})$, $\widehat{Z}^{(2)} := \varphi_2(Z^{(2)})$. By a general fact about invariants of finite groups (cf. [FH91] Prop. 2.8.) we know that the homomorphism

$$H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\})\otimes H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})} \to A, \quad Z^{(1)} \otimes Z^{(2)} \mapsto \frac{1}{2}(Z^{(1)} \otimes Z^{(2)} + \widehat{Z}^{(1)} \otimes \widehat{Z}^{(2)}),$$

is surjective. Now $A = B$ follows from the fact that we can write each

$$Z^{(1)}\otimes Z^{(2)}+\widehat{Z}^{(1)}\otimes\widehat{Z}^{(2)} \quad \text{as} \quad \frac{1}{2}\big((Z^{(1)}+\widehat{Z}^{(1)})\otimes(Z^{(2)}+\widehat{Z}^{(2)})+(Z^{(1)}-\widehat{Z}^{(1)})\otimes(Z^{(2)}-\widehat{Z}^{(2)})\big) \in B.$$

(iii): For all $Z^{(1)}$, $Z^{(2)}$ as in the poof of (ii) define $\bar{Z}^{(i)} := Z^{(i)} + \widehat{Z}^{(i)}$, $\widetilde{Z}^{(i)} := Z^{(i)} - \widehat{Z}^{(i)}$ ($i \in \underline{2}$). Note that by Summary 1.48 (iv), $H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}})$ resp. $H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})$ is as $\mathbb{Q}$ vector space generated by $Z^{(i)}$'s ($i = 1$ resp. $i = 2$) that are of the form $D_1^{(i)} \cdot \ldots \cdot D_m^{(i)} \neq 0$ where the $D_k^{(i)}$ are pairwise different boundary divisor classes of $\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}}$ resp. $\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}}$. To prove (iii) it now suffices to show for all such $Z^{(i)}$ the following:

(1) $\bar{Z}^{(i)} = \alpha\bar{D}_1^{(i)} \cdot \ldots \cdot \bar{D}_m^{(i)}$, for some $\alpha \in \mathbb{Q}$.

(2) $\widetilde{Z}^{(i)} = \alpha\bar{D}_1^{(i)} \cdot \bar{D}_2^{(i)} \cdot \ldots \cdot \bar{D}_{m-1}^{(i)} \cdot \widetilde{D}_m^{(i)}$, for some $\alpha \in \mathbb{Q}$.

We will show these claims for $\bar{Z}^{(1)}$ and $\widetilde{Z}^{(1)}$, the cases of $\bar{Z}^{(2)}$ and $\widetilde{Z}^{(2)}$ being completely analogous. First we show (1), by induction on m. For $m = 1$, clearly $\bar{Z}^{(1)} = \bar{D}_1^{(1)}$. For $n > 1$, in general we have

$$\bar{Z}^{(1)} = D_1^{(1)} \cdot \ldots \cdot D_{m-1}^{(1)} \cdot D_m^{(1)} + \widehat{D}_1^{(1)} \cdot \ldots \cdot \widehat{D}_{m-1}^{(1)} \cdot \widehat{D}_m^{(1)}. \qquad (*)$$

Now every $D_k^{(1)}$ is either of the form $D_k^{(1)} = [\bullet_1, \bullet_2, J_k]$ or of the form $D_k^{(1)} = [\bullet_1, J_k]$, for some $J_k \subseteq I_1$. For divisors of the first kind $\widehat{D}_k^{(1)} = D_k^{(1)}$. We distinguish cases: Either at least one of the $D_k^{(1)}$, (WLOG it is $D_m^{(1)}$) is of the first kind (case (A)), or all the $D_k^{(1)}$ are of the second kind (case (B)). In case (A), equation $(*)$ can be continued by

$$= (D_1^{(1)} \cdot \ldots \cdot D_{m-1}^{(1)} + \widehat{D}_1^{(1)} \cdot \ldots \cdot \widehat{D}_{m-1}^{(1)}) \cdot \frac{1}{2}\bar{D}_m^{(1)} = \frac{1}{2}\alpha'\bar{D}_1^{(1)} \cdot \ldots \cdot \bar{D}_{m-1}^{(1)} \cdot \bar{D}_m^{(1)},$$

for some $\alpha' \in \mathbb{Q}$, where in the second step we applied the induction hypothesis. In case (B), since $D_k^{(1)} \cdot D_{k'}^{(1)} \neq 0$ for all $k \neq k' \in \underline{m}$, we have $J_k \subseteq J_{k'}$ or $J_k \supseteq J_{k'}$ by Summary

1.48 (iii). Since for all $k \in \underline{m}$, $\emptyset \neq J_k \neq \underline{n}$, this implies that $J_k \not\subseteq J^c_{k'}$ and $J_k \not\supseteq J^c_{k'}$, where $J^c_{k'} := \underline{n} \setminus J_{k'}$, hence $D^{(1)}_k \cdot \widehat{D}^{(1)}_{k'} = [\bullet_1, J_k] \cdot [\bullet_1, J^c_{k'}] = 0$. This implies for case (B):

$$\bar{D}^{(1)}_1 \cdot ... \cdot \bar{D}^{(1)}_{m-1} \cdot \bar{D}^{(1)}_m = \bar{Z}^{(1)}.$$

Now we show (2). For this first note that if all $D^{(1)}_k$ are of the form $[\bullet_1, \bullet_2, J_k]$, then $\widetilde{Z}^{(1)} = Z^{(1)} - \widehat{Z}^{(1)} = 0$. Hence WLOG $D^{(1)}_{r+1}, D^{(1)}_{r+2}, ..., D^{(1)}_m$ are of the form $[\bullet_1, J_k]$ for some $0 \leq r < m$ and $D^{(1)}_1, ..., D^{(1)}_r$ are of the form $[\bullet_1, \bullet_2, J_k]$. Then

$$\widetilde{Z}^{(1)} = D^{(1)}_1 \cdot ... \cdot D^{(1)}_{m-1} \cdot D^{(1)}_m - \widehat{D}^{(1)}_1 \cdot ... \cdot \widehat{D}^{(1)}_{m-1} \cdot \widehat{D}^{(1)}_m$$

$$= \frac{1}{2^r} \bar{D}^{(1)}_1 \cdot ... \cdot \bar{D}^{(1)}_r \cdot (D^{(1)}_{r+1} \cdot ... \cdot D^{(1)}_m - \widehat{D}^{(1)}_{r+1} \cdot ... \cdot \widehat{D}^{(1)}_m) = \frac{1}{2^r} \bar{D}^{(1)}_1 \cdot ... \cdot \bar{D}^{(1)}_r \cdot \bar{D}^{(1)}_{r+1} \cdot ... \cdot \bar{D}^{(1)}_{m-1} \cdot \widetilde{D}^{(1)}_m.$$

For the last equation one argues analogously to the proof of (1) in case (B).

(iv): From (ii) it follows that, for $n_1 := |I_1|$ and $n_2 := |I_2|$:

$$H^s(B_{\langle I_1, I_2 \rangle}) =$$

$$\bigoplus_{s_1+s_2=s} \big((H^{s_1}(\overline{M}_{0,n_1+2}))^+ \otimes (H^{s_2}(\overline{M}_{0,n_2+2}))^+\big) \oplus \big((H^{s_1}(\overline{M}_{0,n_1+2}))^- \otimes (H^{s_2}(\overline{M}_{0,n_2+2}))^-\big).$$

So, $\dim_{\mathbb{Q}}(B_{\langle I_1, I_2 \rangle}) = \sum_{s_1+s_2=s} h_{n_1}(s_1) h_{n_2}(s_2) + (g_{n_1}(s_1) - h_{n_1}(s_1))(g_{n_2}(s_2) - h_{n_2}(s_2))$. □

The simple banana cycles for $m \geq 3$ are easier to treat:

Lemma 5.25 *(i) For $P = \langle I_1, ..., I_m \rangle$ a circular partition of $\underline{n}$ with $m \geq 3$, the morphism*

$$z_{B_P} : \overline{M}_{0, I_1 \cup \{\circ_1, \bullet_2\}} \times \overline{M}_{I_2 \cup \{\circ_2, \bullet_3\}} \times ... \times \overline{M}_{I_{m-1} \cup \{\circ_{m-1}, \bullet_m\}} \times \overline{M}_{I_m \cup \{\circ_m, \bullet_1\}} \to B_P$$

is an isomorphism of varieties. The same holds for $z_{B''_P}$ and $z_{B^r_P}$.

(ii) Hence $z^*_{B_P}$ is an isomorphism of $\mathbb{Q}$-algebras:

$$H^*(B_P) \cong H^*(\overline{M}_{0, I_1 \cup \{\circ_1, \bullet_2\}}) \otimes H^*(\overline{M}_{0, I_2 \cup \{\circ_2, \bullet_3\}}) \otimes ... \otimes H^*(\overline{M}_{0, I_m \cup \{\circ_m, \bullet_1\}})$$

The same holds for B^r_P and B''_P.

Proof: Since the stable graph of $B_{\langle I_1, ..., I_m \rangle}$ for $m \geq 3$ has no non-trivial automorphism, $z_{B_{\langle I_1, ..., I_m \rangle}}$ has degree 1 (cf. Proposition 1.26 (i)). Hence $z_{B_{\langle I_1, ..., I_m \rangle}}$ is an isomorphism since $B_{\langle I_1, ..., I_m \rangle}$ is normal by Lemma 4.3. The rest is clear by equation (5.1), at the beginning of this section. □

Corollary 5.26 *Since z_{B_P}, $z_{B^r_P}$, $z_{B''_P}$ (cf. Def. 5.23) are finite surjective, the pullback along them is injective and by the previous lemmas the pullback is surjective for $|P| \geq 3$ and has image $H^*(\overline{M}_{\Gamma(P)})^{\mathbb{S}_2} \subset H^*(\overline{M}_{\Gamma(P)})$ if $|P| = 2$. Via these pullbacks* [19] *we usually identify $H^*(B_P)$, $H^*(B''_P)$ and $H^*(B^r_P)$ with $H^*(\overline{M}_{\Gamma(P)})$ if $|P| \geq 3$ and with $H^*(\overline{M}_{\Gamma(P)})^{\mathbb{S}_2}$ if $|P| = 2$. Assume that $n \geq 3$ then in case $|P| = 2$, with the chosen identification, the*

[19] Recall that, as always, we use the adjusted pullback, as introduced in Summary 1.34 (iv). This makes a difference especially for B^r_P for which the general object has $2^{|P|-1}$ automorphisms (if $n \geq 3$).

pushforwards $(z_{B_P})_*$ *and* $(z_{B''_P})_*$ *on the one hand, and* $(z_{B^r_P})_*$ *on the other hand, act on* $H^*(\overline{M}_{\Gamma(P)}) = H^*(\overline{M}_{0,I_1\cup\{\bullet_1,\bullet_2\}}) \otimes H^*(\overline{M}_{0,I_2\cup\{\circ_1,\circ_2\}})$ *by*

$$Z^{(1)}\otimes Z^{(2)} \mapsto Z^{(1)}\otimes Z^{(2)} + \widehat{Z}^{(1)}\otimes \widehat{Z}^{(2)}, \quad \textit{resp.} \quad Z^{(1)}\otimes Z^{(2)} \mapsto 2\cdot(Z^{(1)}\otimes Z^{(2)} + \widehat{Z}^{(1)}\otimes \widehat{Z}^{(2)}),$$

using the notation from the proof of Lemma 5.24. So restricted to the invariant part $H^*(\overline{M}_{\Gamma(P)})^{\mathbb{S}_2}$*), the pushforwards acts as multiplication by* 2 *resp.* 4*.*

For $|P| \geq 3$ *the pushforwards* $(z_{B_P})_*$ *and* $(z_{B''_P})_*$ *act as the identity, while* $(z_{B^r_P})_*$ *acts as multiplication by* $2^{|P|-1}$*.*

5.3.3 Intersection theory of simple banana cycles

Let $P_1 = \langle I_1, ...I_m\rangle$ and $P_2 = \langle I'_1, ...I'_{m'}\rangle$ have a common refinement $\widetilde{P} = \langle J_1, ..., J_\mu\rangle$, and let $\rho_1 : \widetilde{P} \to P_1$, $\rho_2 : \widetilde{P} \to P_2$ be the refinement maps. Then:

Definition 5.27 For any circular ordered partition P we set

$$\mathrm{N}(P) := \{\{I_1, I_2\} \subseteq P \mid I_1 \parallel I_2,\ I_1 \neq I_2\}$$

(i) If $\widetilde{P}$ as above is a refinement of P_1 define

$$\mathrm{ON}(P_1; \widetilde{P}) := \{\{J_1, J_2\} \in \mathrm{N}(\widetilde{P}) \mid \rho_1(J_1) \neq \rho_1(J_2)\}$$ [20]

It is clear that $\{J_1, J_2\} \mapsto \{\rho_1(J_1), \rho_1(J_2)\}$ defines a bijection $\mathrm{ON}(P_1; \widetilde{P}) \to \mathrm{N}(P_1)$.

(ii) For a common refinement $\widetilde{P}$, define $\mathrm{CN}(P_1, P_2; \widetilde{P}) \subseteq N(\widetilde{P})$ as

$$\mathrm{CN}(P_1, P_2; \widetilde{P}) := \{\{J_1, J_2\} \in \mathrm{N}(\widetilde{P}) \mid \rho_1(J_1) \neq \rho_1(J_2) \text{ and } \rho_2(J_1) \neq \rho_2(J_2)\}$$

$$= \mathrm{ON}(P_1; \widetilde{P}) \cap \mathrm{ON}(P_2; \widetilde{P}).$$

We also define subsets $\mathrm{CN}_1(P_1, P_2; \widetilde{P}) \subseteq \mathrm{N}(P_1)$, $\mathrm{CN}_2(P_1, P_2; \widetilde{P}) \subseteq \mathrm{N}(P_2)$ as

$$\mathrm{CN}_1(P_1, P_2; \widetilde{P}) := \{\{\rho_1(J_1), \rho_1(J_2)\} \mid \{J_1, J_2\} \in \mathrm{CN}(P_1, P_2; \widetilde{P})\},$$

$$\mathrm{CN}_2(P_1, P_2; \widetilde{P}) := \{\{\rho_2(J_1), \rho_2(J_2)\} \mid \{J_1, J_2\} \in \mathrm{CN}(P_1, P_2; \widetilde{P})\}.$$

(iii) We call the unordered pairs $\{I_1, I_2\}$ in $N(P)$ the *nodes* of P and the elements of $\mathrm{CN}(P_1, P_2; \widetilde{P})$ the *common nodes* of P_1 and P_2 on $\widetilde{P}$.

(iv) If P is a circular partition of $\underline{n}$, we set $d(P) := 2$ if $|P| = 2$ and $d(P) := 1$ otherwise. Then $d(P)|\mathrm{N}(P)|$ is the number of edges of the graph $\Gamma(P)$, or of the nodes of a general curve parametrised by B_P. So $\mathrm{codim}(B_P, \overline{M}_{1,n}) = d(P)|\mathrm{N}(P)|$.

(v) Denote by $\mathrm{CCR}(P_1, P_2)$ the set of all coarsest common refinements of P_1 and P_2.

[20] Let $c : \Lambda(\widetilde{P}, e_{\widetilde{P}}) \rightsquigarrow \Lambda(P_1, e_{P_1})$ be a contraction corresponding to the refinement map $\rho_1 : \widetilde{P} \to P_1$, and let $E(\widetilde{P})$ resp. $E(P_1)$ be the sets of edges of $\Lambda(\widetilde{P}, e_{\widetilde{P}})$ resp. $\Lambda(P_1, e_{P_1})$. Then $\mathrm{ON}(P_1; \widetilde{P})$ is the set of those pairs of vertices which are connected by edges in $c^{-1}(E(P_1)) \subseteq E(\widetilde{P})$.

One may think about $\mathrm{CN}(P_1, P_2; \widetilde{P})$ as determining which nodes of a general curve C parametrised by $B_{\widetilde{P}}$ come as well from a node of a general curve of B_{P_1} as from a node of a general curve parametrised by B_{P_2}. Therefore the name "common nodes". [21] In particular we have for any coarsest common refinement $\bar{P}$ of P_1, P_2,

$$|P_1| + |P_2| - |\bar{P}| = d(\bar{P})|\,\mathrm{CN}(P_1, P_2; \bar{P})|, \quad \text{and hence,} \tag{5.2}$$

$$\mathrm{codim}(B_{P_1}, \overline{M}_{1,n}) + \mathrm{codim}(B_{P_2}, \overline{M}_{1,n}) = \mathrm{codim}(B_{\bar{P}}, \overline{M}_{1,n}) + d(\bar{P})|\,\mathrm{CN}(P_1, P_2; \bar{P})|, \tag{5.3}$$

as can be checked using Lemma 5.20 (ii) and its proof. The "common nodes" will correspond to the common edges appearing in the excess intersection formula (cf. Proof of the next Lemma).

We want to determine the pullback of the class of one simple banana cycle B_{P_2} on $\overline{M}_{1,n}$ via the gluing map of another simple banana cycle B_{P_1}, i.e $\xi^*_{B_{P_1}}(b_{P_2})$. In a second step this will allow us to compute the intersection $b_{P_1} b_{P_2}$ on $\overline{M}_{1,n}$.

To be able to use our usual notation for the gluing morphism $\xi_{B_{P_1}}$, choose for the cyclically arranged set $P_1 = \langle I_1, ..., I_m \rangle$ a bijection $P_1 \overset{\cong}{\to} \mathbb{Z}/m\mathbb{Z}$, which is a representative of the cyclic arrangement, cf. Def. 5.16 (iii). (Of course writing the elements of P_1 with indices $1, ..., m$ makes it look like P_1 comes with such a representative anyway, but it is not meant like this.)

Now we will write the gluing map $\xi_{B_{P_1}}$ as

$$\xi_{B_{P_1}} : \overline{M}_{\Gamma(P_1)} = \prod_{i \in \underline{m}} \overline{M}_{0, I_i \cup \{\circ_i, \bullet_{i+1}\}} \to \overline{M}_{1,n}. \text{ [22]}$$

Lemma 5.28 *Assume $n \geq 3$ for the whole Lemma. For P' a refinement of a circular partition P of $\underline{n}$, let $\mathrm{Con}(P', P)$ be the set of all contractions of graphs $c : \Gamma(P') \rightsquigarrow \Gamma(P)$. We have $|\,\mathrm{Con}(P', P)| = d(P)$. (For the definition of $\Gamma(P)$ cf. Def. 5.16)*

(i) For a $c \in \mathrm{Con}(P', P)$ let $\xi_c : \overline{M}_{\Gamma(P')} \to \overline{M}_{\Gamma(P)}$ be the partial gluing morphism (cf. Proposition 1.26 (iii)). If $P = \langle I_1, ..., I_m \rangle$, and $\rho : P' \to P$ is the refinement map, we can write

$$P' = \big\langle J_{1,1}, J_{1,2}, ..., J_{1,\mu_1}, J_{2,1}, ..., J_{m,\mu_m} \big\rangle,$$

such that $\rho^{-1}(I_i) = \{J_{i,1}, ..., J_{i,\mu_i}\}$.

We determine the pushforward of the fundamental class $1_{P'} := [\overline{M}_{\Gamma(P')}]_Q$ of $\overline{M}_{\Gamma(P')}$,

$$(\xi_c)_* 1_{P'} \in H^*(\overline{M}_{\Gamma(P)}) \cong H^*(\prod_{i \in \underline{m}} \overline{M}_{0, I_i \cup \{\circ_i, \bullet_{i+1}\}}) :$$

In case $m \geq 3$, for the only $c \in \mathrm{Con}(P', P)$:

$$(\xi_c)_* 1_{P'} = D_1 \otimes D_2 \otimes ... \otimes D_m, \quad \textit{where}$$

[21] Note that this interpretation is not quite adequate if $|\widetilde{P}| = 2$ since then the two components of a general curve meet in two nodes, hence the number of "actual" common nodes on the general curve is $d(\widetilde{P}) \cdot |\,\mathrm{CN}(P_1, P_2; \widetilde{P})|$.

[22] The indices of the $\circ_i$ and $\bullet_{i+1}$ are elements of $\mathbb{Z}/m\mathbb{Z}$, so $\bullet_{m+1} = \bullet_1$.

$$D_i := \prod_{s=1}^{\mu_i} [\circ_i, \bigcup_{t=1}^{s} J_{i,t}] \ \in \ H^*(\overline{M}_{0,I_i \cup \{\circ_i, \bullet_{i+1}\}}).$$

Here the $[\circ_i, \bigcup_{t=1}^{s} J_{i,t}]$ *are boundary divisors of* $\overline{M}_{0,I_i \cup \{\circ_i, \bullet_{i+1}\}}$, *cf. 1.47 for the notation.*

In case $m = 2$ *there is a* $c \in \mathrm{Con}(P', P)$, *such that for* $(\xi_c)_* 1_{P'}$ *the same formula as in the case* $|P| \geq 3$ *holds. For the other element* $c' \in \mathrm{Con}(P', P)$, *the formula for* $(\xi_{c'})_* 1_{P'}$ *is obtained by replacing* $\circ_i$ *by* $\bullet_{i+1}$ *in the definition of the* D_i. [23]

Now define

$$\mathbb{B}(P', P) := \frac{1}{d(P')} \sum_{c \in \mathrm{Con}(P', P)} (\xi_c)_* 1_{P'}$$ [24]

(ii) We can express the pullback of the class of one simple banana cycle via the gluing map of another simple banana cycle as follows: Let P_1, P_2 *be two circular partitions of* $\underline{n}$. *For each* $\bar{P} \in \mathrm{CCR}(P_1, P_2)$ *denote by* $\Psi(P_1, P_2; \bar{P})$ *the set of all refinements* $\widehat{P}$ *of* $\bar{P}$, *such that for* $\rho : \widehat{P} \to \bar{P}$ *the refinement map, the following conditions are fulfilled:*

1. *There is a map* $r : \mathrm{CN}(P_1, P_2; \bar{P}) \to \bar{P}$, *such that for each* $\{J, J'\} \in \mathrm{CN}(P_1, P_2, \bar{P})$ *we have* $r(\{J, J'\}) \in \{J, J'\}$, *and with this map* r*:*
2. *If* $J \in \bar{P} \setminus r(\mathrm{CN}(P_1, P_2; \bar{P}))$ *then* $\rho^{-1}(J) = J$.
3. *If* $r^{-1}(J) = \{\{J, J'\}\}$ *then* $\rho^{-1}(J) = \{K_1, K_2\}$ *for some* $K_1 \uplus K_2 = J$ *such that* $\nu(J) \in K_2$ *and* $K_1 \parallel \rho^{-1}(J')$ *in* $\widehat{P}$. *Here* $\nu(J)$ *denotes the smallest number in* $J \subset \underline{n}$.
4. *If* $r^{-1}(J) = \{\{J, J'\}, \{J, J''\}\}$ *such that* $J' \parallel J \parallel J''$, *then* $\rho^{-1}(J) = \{K_1, K_2, K_3\}$ *for some* $K_1 \uplus K_2 \uplus K_3 = J$ *such that* $\nu(J) \in K_2$, $K_1 \neq \emptyset \neq K_2$ *and* $\rho^{-1}(J') \parallel K_1 \parallel K_2 \parallel K_3 \parallel \rho^{-1}(J'')$ *in* $\widehat{P}$. *Again* $\nu(J)$ *denotes the smallest number in* $J \subset \underline{n}$.

Note that if $\mathrm{CN}(P_1, P_2; \bar{P}) = \emptyset$ *then* $\Psi(P_1, P_2, \bar{P}) = \{\bar{P}\}$.

With this definitions we have:

$$\xi^*_{B_{P_1}}(b_{P_2}) = \sum_{\bar{P} \in \mathrm{CCR}(P_1, P_2)} (-1)^{|CN(P_1, P_2, \bar{P})|} \sum_{\widehat{P} \in \Psi(P_1, P_2, \bar{P})} \mathbb{B}(\widehat{P}, P_1)$$

The only exception from this formula is the case $P_1 = P_2$ *with* $|P_1| = |P_2| = 2$. *In that case* $\xi^*_{B_{P_1}}(b_{P_2}) = \sum_{\widehat{P} \in \Psi} \mathbb{B}(\widehat{P}, P)$ *where* Ψ *is the set of all refinements* $\widehat{P}$ *of* $P_1 = P_2$ *with* $|\widehat{P}| = 4$.

(iii) Hence:

$$\zeta^*_{B''_{P_1}}(b''_{P_2}) = \sum_{\bar{P} \in \mathrm{CCR}(P_1, P_2)} (-1)^{|CN(P_1, P_2, \bar{P})|} \sum_{\widehat{P} \in \Psi(P_1, P_2, \bar{P})} \mathbb{B}(\widehat{P}, P_1)$$

[23] When pushed forward further to $\overline{M}_{1,n}$, $(\xi_c)_* 1_{P'}$ and $(\xi_{c'})_* 1_{P'}$ are mapped to the same class. So for computing the intersection on $\overline{M}_{1,n}$ the difference between them is not relevant.

[24] Note that if $d(P') = 2$ i.e. if $|P'| = 2$, then $P' = P$ and the pushforward for both contractions is just 1_P. So all the factor $\frac{1}{d(P')}$ does is preventing to count this class twice in this special case.

$$\zeta^*_{B^r_{P_1}}(b^r_{P_2}) = \frac{1}{2^{|P_2|}} \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|CN(P_1,P_2,\bar{P})|} \sum_{\widehat{P}\in\Psi(P_1,P_2,\bar{P})} \mathbb{B}(\widehat{P},P_1)$$

(iv) Let $f_{B_{P_1}}$ be the embedding $f_{B_{P_1}} : B_{P_1} \hookrightarrow \overline{R}_{1,n}$, then of course the gluing morphism $\xi_{B_{P_1}}$ factors as $\xi_{B_{P_1}} = f_{B_{P_1}} \circ z_{B_{P_1}}$, as explained in Definition 5.23. By section 5.3.2, $z_{B_{P_1}}$ is an isomorphism if $|P_1| > 2$ and a quotient morphism of degree 2 if $|P_1| = 2$. For any refinement P' of P_1 let $B(P', P_1)$ denote the Q-class in $H^*(B_{P_1})$ of the subvariety $B_{P'} \subset B_{P_1}$. Then

$$(z_{B_{P_1}})_*\mathbb{B}(P',P_1) = d(P_1)B(P',P_1)$$

If we let $B''(P', P_1)$ resp. $B^r(P', P_1)$ be the Q-classes of $B''_{P'}$ resp. $B^r_{P'}$ in $H^*(B''_{P_1})$ resp. $H^*(B^r_{P_2})$. Then analogously

$$(z_{B''_{P_1}})_*\mathbb{B}(P',P_1) = d(P_1)B''(P',P_1) \quad \text{and} \quad (z_{B^r_{P_1}})_*\mathbb{B}(P',P_1) = d(P_1)2^{|P'|-1}B^r(P',P_1).$$

Hence, if we work with the identifications of cohomology rings defined in Corollary 5.26 then:

$$B(P',P_1) = \mathbb{B}(P',P_1), \quad B''(P',P_1) = \mathbb{B}(P',P_1), \quad B^r(P',P_1) = 2^{|P_1|-|P'|}\mathbb{B}(P',P_1).$$

(v) With this notation:

$$f^*_{B_{P_1}}(b_{P_2}) = \frac{1}{\deg z_{B_{P_1}}}(z_{B_{P_1}})_*\xi^*_{B_{P_1}}(b_{P_2}) = \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|\,\mathrm{CN}(P_1,P_2,\bar{P})|} \sum_{\widehat{P}\in\Psi(P_1,P_2,\bar{P})} B(\widehat{P},P_1)$$

$$f^*_{B''_{P_1}}(b''_{P_2}) = \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|\,\mathrm{CN}(P_1,P_2,\bar{P})|} \sum_{\widehat{P}\in\Psi(P_1,P_2,\bar{P})} B''(\widehat{P},P_1)$$

$$f^*_{B^r_{P_1}}(b^r_{P_2}) = \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|\,\mathrm{CN}(P_1,P_2,\bar{P})|} \sum_{\widehat{P}\in\Psi(P_1,P_2,\bar{P})} 2^{|\widehat{P}|-|P_1|-|P_2|}B^r(\widehat{P},P_1)$$

(vi) For a given circular partition P of $\underline{n}$, $|P| \geq 2$, let P_1 and P_2 be two refinements. For any $\bar{P} \in \mathrm{CCR}(P_1, P_2)$, set $\widehat{\mathrm{CN}}(P_1, P_2; \bar{P}; P) := \mathrm{CN}(P_1, P_2; \bar{P}) \smallsetminus (\mathrm{ON}(P; \bar{P}) \cap \mathrm{CN}(P_1, P_2; \bar{P}))$. Now define $\widehat{\Psi}(P_1, P_2; \bar{P}; P)$ by mimicking exactly the definition of $\Psi(P_1, P_2; \bar{P})$ from (ii), except of replacing each $\mathrm{CN}(P_1, P_2; \bar{P})$ appearing there by $\widehat{\mathrm{CN}}(P_1, P_2; \bar{P}; P)$. Then for $i_{P_1,P} : B_{P_1} \to B_P$, $i_{P_1,P,''} : B''_{P_1} \to B''_P$, $i_{P_1,P,r} : B^r_{P_1} \to B^r_P$ the inclusions, we have:

$$i^*_{P_1,P}(B(P_2,P)) = \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|\widehat{\mathrm{CN}}(P_1,P_2;\bar{P};P)|} \sum_{\widehat{P}\in\widehat{\Psi}(P_1,P_2;\bar{P};P)} B(\widehat{P},P_1)$$

$$i^*_{P_1,P,''}(B''(P_2,P)) = \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|\widehat{\mathrm{CN}}(P_1,P_2;\bar{P};P)|} \sum_{\widehat{P}\in\widehat{\Psi}(P_1,P_2;\bar{P};P)} B''(\widehat{P},P_1)$$

$$i^*_{P_1,P,r}(B^r(P_2,P)) = \sum_{\bar{P}\in \mathrm{CCR}(P_1,P_2)} (-1)^{|\widehat{\mathrm{CN}}(P_1,P_2;\bar{P};P)|} \sum_{\widehat{P}\in\widehat{\Psi}(P_1,P_2;\bar{P};P)} 2^{|P|+|\widehat{P}|-|P_1|-|P_2|}B^r(\widehat{P},P_1)$$

Proof: Here we use the formulas and notation from section 1.7 to calculate excess intersections.

There are exactly $d(P)$ contractions $c : \Gamma(P') \rightsquigarrow \Gamma(P)$: For $\rho : P' \to P$ the refinement map, it is clear that a vertex J of $\Gamma(P')$ for $J \in P'$ is contracted into the vertex $\rho(J)$ of $\Gamma(P_1)$ by c, since contractions respect the legs of a graph. All edges between two vertices J_1 and J_2 with $\rho(J_1) = \rho(J_2)$ are contracted by c since the graph of a banana cycle contains no self edges. If there are edges between J_1 and J_2 but $\rho(J_1) \neq \rho(J_2)$, i.e. if $\{J_1, J_2\} \in \mathrm{ON}(P, P')$, then the set of these edges is mapped invectively to the set of edges between $\rho(J_1)$ and $\rho(J_2)$ by c. The only case in which there can be more than two edges between $\rho(J_1)$ and $\rho(J_2)$, and hence in which c is not completely determined by ρ is the case $P = \langle \rho_1(J_1), \rho_1(J_2) \rangle$, i.e. $|P| = 2$. In this case there are exactly two edges between $\rho(J_1)$ and $\rho(J_2)$, whose preimages in $\Gamma(P')$ under c_1 must be specified. Hence in this case there are exactly two different possible contractions $\Gamma(P') \rightsquigarrow \Gamma(P)$.

(i): It is easy to see that the right hand side of the claimed equation

$$(\xi_c)_* 1_{P'} = D_1 \otimes D_2 \otimes \dots \otimes D_m$$

is the class $\left[\xi_c\big(\overline{M}_{\Gamma(P')}\big)\right]_Q$, where $\xi_c\big(\overline{M}_{\Gamma(P')}\big)$ is the image. Since ξ_c has degree 1 as a morphism of stacks [25] , this is the same as the pushforward of the fundamental class $1_{P'}$ of $\overline{M}_{\Gamma(P')}$ by ξ_c.

(ii): Since we want to use formula (1.5) from section 1.7, we first determine the set $G_{\Gamma(P_1)\Gamma(P_2)}$ appearing there. We claim that for $(\Lambda, c_1, c_2) \in G_{\Gamma(P_1)\Gamma(P_2)}$, Λ must be the graph $\Gamma(\bar{P})$ for $\bar{P}$ some coarsest common refinement of P_1 and P_2: Every edge of Λ must be either mapped to an edge of $\Gamma(P_1)$ by c_1 or of $\Gamma(P_2)$ by c_2, by definition of $G_{\Gamma(P_1)\Gamma(P_2)}$. These graphs do not have disconnecting edges, and it is easy to check that a contraction can not map a disconnecting edge to a non-disconnecting one. Hence Λ is a graph of a simple banana cycle by Definition 4.1 (ii) and hence by Definition 5.16 (xii) of the form $\Gamma(\bar{P})$ for some circular partition $\bar{P}$ of $\underline{n}$. Then the contractions c_1 and c_2 have to induce refinement maps $\rho_1 : \bar{P} \to P_1$ resp. $\rho_2 : \bar{P} \to P_2$ since contractions respect the marked legs of the graphs. Now the condition that no edge of Λ is contracted by both c_1 and c_2, defining $G_{\Gamma(P_1)\Gamma(P_2)}$ is equivalent to: $\bar{P}$ is a *coarsest* common refinement of P_1 and P_2.

So the set of all (Λ, c_1, c_2) allowed in $G_{\Gamma(P_1)\Gamma(P_2)}$ is the set of all $(\Gamma(\bar{P}), c_1, c_2)$ where $\bar{P} \in \mathrm{CCR}(P_1, P_2)$ and $c_1 \in \mathrm{Con}(\bar{P}, P_1)$ and $c_2 \in \mathrm{Con}(\bar{P}, P_2)$ are contractions. By the discussion above for a fixed $\bar{P}$ we have $d(P_1) \cdot d(P_2)$ pairs of contractions (c_1, c_2). Now a $G_{\Gamma(P_1)\Gamma(P_2)}$ is obtained by choosing a representative of every residue class in

$$\biguplus_{\bar{P} \in \mathrm{CCR}(P_1, P_2)} \{(c_1, c_2) \mid c_1 \in \mathrm{Con}(\bar{P}, P_1),\, c_2 \in \mathrm{Con}(\bar{P}, P_2)\} / \sim_{\bar{P}},$$

[25] One can see this as follows: In the proof of Proposition 1.26 (iii), the partial gluing morphisms ξ_c corresponding to a $c : \Gamma \rightsquigarrow \Gamma'$ is described as a product of the gluing morphisms corresponding to the subgraphs of Γ which are the preimages of the smooth cells of Γ'. In our case it is easy to see that these preimage-graphs have no automorphisms, and thus the factors of ξ_c all have degree 1 by Proposition 1.26 (i). Hence the product ξ_c has degree 1 too.

where $(c_1, c_2) \sim_{\bar{P}} (c_1', c_2')$ if there is a $\varphi \in \mathrm{Aut}(\Gamma(\bar{P}))$ such that $(c_1 \circ \varphi, c_2 \circ \varphi) = (c_1', c_2')$. A non-trivial automorphism on $\Gamma(\bar{P})$ exists if and only if $|\bar{P}| = 2$. In this case $|P_1| = |P_2| = 2$ too, and hence $\bar{P} = P_1 = P_2$. Check that we can choose as representatives of the two classes in $\{(c_1, c_2) \mid c_1 \in \mathrm{Con}(\bar{P}, P_1),\ c_2 \in \mathrm{Con}(\bar{P}, P_2)\}/\sim_{\bar{P}}$ in this case (c_1, c_2) and (c_1', c_2), where $c_1 \neq c_1'$ and c_2 is any of the two elements of $\mathrm{Con}(\bar{P}, P_2)$. So:

$$\overline{M}_{\Gamma(P_1)\Gamma(P_2)} \cong \coprod_{(\Lambda, c_1, c_2) \in G_{\Gamma(P_1)\Gamma(P_2)}} \overline{M}_\Lambda \cong \coprod_{\bar{P} \in \mathrm{CCR}(P_1, P_2)} \coprod_{h=1}^{d(P_1)d(P_2)/d(\bar{P})} \overline{M}_{\Gamma(\bar{P})} \qquad (*)$$

Let $\xi : \overline{M}_{\Gamma(P_1)\Gamma(P_2)} \to \overline{M}_{\Gamma(P_1)}$ be the forgetful morphism from Diagram 1.2 in Section 1.7, then using the isomorphism of $(*)$, we can make the identification

$$\xi = \coprod_{\bar{P} \in \mathrm{CCR}(P_1, P_2)} \coprod_{h=1}^{d(P_2)/d(\bar{P})} \coprod_{c \in \mathrm{Con}(\bar{P}, P_1)} \xi_c.$$

Since $\Gamma(\bar{P})$ has no self edges, an edge between J and J' is contracted by none of c_1 and c_2 for a pair (c_1, c_2) if and only if $\rho_1(J) \neq \rho_1(J')$ and $\rho_2(J) \neq \rho_2(J')$, i.e. if $\{J, J'\} \in \mathrm{CN}(P_1, P_2; \bar{P})$. The set CE appearing in the excess intersection formula (1.5) is the set of just these edges. So if we denote by $E(J, J')$ the set of edges $\{h, h'\}$ [26] between the vertices J and J', the excess intersection formula (1.5) yields, with $\eta_{\bar{P},J} := \eta_{\Gamma(\bar{P}),J}$ (cf. Def, 1.41 (i)) and using $|Aut(\Gamma(P_2))| = d(P_2)$:

$$\xi^*_{B_{P_1}}(b_{P_2}) =$$

$$\frac{1}{d(P_2)} \sum_{\bar{P} \in \mathrm{CCR}(P_1, P_2)} \frac{d(P_2)}{d(\bar{P})} \sum_{c \in \mathrm{Con}(\bar{P}, P_1)} (\xi_c)_* \left(\prod_{\substack{\{h,h'\} \in E(J,J'),\\ \{J,J'\} \in \mathrm{CN}(P_1,P_2;\bar{P})}} -\eta^*_{\bar{P},J}(\psi_h) - \eta^*_{\bar{P},J'}(\psi_{h'}) \right) \qquad (\dagger)$$

From now on assume that we are not in the case $P_1 = P_2$ with $|P_1| = |P_2| = 2$, and hence that $|\bar{P}| \geq 3$ for all $\bar{P} \in \mathrm{CCR}(P_1, P_2)$ [27]. Recall that $\eta_{\bar{P},J}$ is the projection from $\overline{M}_{\Gamma(\bar{P})}$ to the moduli space $\overline{M}_{0, J \cup \{h, h^*\}}$, where h denotes the point belonging to the half edge h, and where h^* belongs to a half edge h^* connecting J to its neighbour J'' different from J' (since $|\bar{P}| \geq 3$, $J'' \neq J'$). By Summary 1.42 (iv), we have:

$$\psi_h = \sum_{\emptyset \neq K \subseteq J \setminus \{\nu(J)\}} [h, K]. \qquad (\spadesuit)$$

where we choose $\nu(J) \in J$ to be the smallest number in $J \subseteq \underline{n}$. Using $(\spadesuit)$, we can check that for h, h^* as above

$$\eta^*_{\bar{P},J}(\psi_h) \cdot \eta^*_{\bar{P},J}(\psi_{h^*}) = \sum_{\substack{\emptyset \neq K \subseteq J \setminus \{\nu(J)\} \\ \emptyset \neq K^* \subseteq J \setminus \{\nu(J)\}}} [h, K] \cdot [h^*, K^*] = \sum_{\substack{K_1 \uplus K_2 \uplus K_3 = J \\ \nu(J) \in K_2,\ K_1 \neq \emptyset \neq K_3}} [h, K_1] \cdot [h, K_1 \cup K_2]. \qquad (\clubsuit)$$

[26] Where h is a half-edge attached to J, h' attached to J'.

[27] The formula of (ii) can be checked to hold also in this excluded case directly, using $(\dagger)$.

We can write the part of (†) in the large brackets as:

$$\sum_{r\in S}(-1)^{|\operatorname{CN}(P_1,P_2,\bar P)|}\prod_{\{J,J'\}\in\operatorname{CN}(P_1,P_2;\bar P)}\eta^*_{\bar P,r(\{J,J'\})}(\psi_{h_r(\{J,J'\})}) \tag{‡}$$

where S is the set of all maps $r:\operatorname{CN}(P_1,P_2,\bar P)\to\bar P$ such that for each $\{J,J'\}\in\operatorname{CN}(P_1,P_2,\bar P)$ we have $r(\{J,J'\})\in\{J,J'\}$ and where $h_r(\{J,J'\})$ is the unique half edge which is part of the edge joining J and J' and which is attached to $r(\{J,J'\})$. Now let $\mathbf{K}(r)$ be the set of all possible tuples $\mathscr{K}=\big(K_{r(\{J,J'\})}\big)_{\{J,J'\}\in\operatorname{CN}(P_1,P_2;\bar P)}$ such that $\emptyset\neq K_{r(\{J,J'\})}\subseteq r(\{J,J'\})\smallsetminus\nu(r(\{J,J'\})$, where again $\nu(r(\{J,J'\}))$ is the smallest number in $r(\{J,J'\})$. Then using (♠) we can rewrite the product in (‡) as

$$\sum_{\mathscr{K}\in\mathbf{K}(r)}\left(\prod_{\{J,J'\}\in\operatorname{CN}(P_1,P_2;\bar P)}\eta^*_{\bar P,r(\{J,J'\})}([h_r(\{J,J'\}),K_{r(\{J,J'\})}])\right) \tag{$\diamondsuit$}$$

where $[h_r(\{J,J'\}),K_{r(\{J,J'\})}]$ uses our standard notation for boundary divisors of spaces $\overline{M}_{0,M}$. Using (i) and (♣), we can check that the product in ($\diamondsuit$) is $(\xi_{c'})_*1_{\widehat P}$ for $\widehat P$ a certain refinement of $\bar P$ in $\widehat\Psi(P_1,P_2;\bar P)$ and for a certain $c'\in\operatorname{Con}(\widehat P,\bar P)$. Now substitute all this back into (†) and check that the result is the formula of (ii).

(iii): This is easy to show using the commutative diagrams

$$\begin{array}{ccc}\overline{M}_{\Gamma(P_1)} & \xrightarrow{\zeta_{B''_{P_1}}} & \overline{R}_{1,n}\\ \cong\downarrow & & \downarrow\tau_n\\ \overline{M}_{\Gamma(P_1)} & \xrightarrow[\xi_{B_{P_1}}]{} & \overline{M}_{1,n}\end{array}\qquad\text{and}\qquad\begin{array}{ccc}\overline{M}_{\Gamma(P_1)} & \xrightarrow{\zeta_{B^r_{P_1}}} & \overline{R}_{1,n}\\ \cong\downarrow & & \downarrow\tau_n\\ \overline{M}_{\Gamma(P_1)} & \xrightarrow[\xi_{B_{P_1}}]{} & \overline{M}_{1,n}\end{array}$$

together with $\tau_n^*\,b_{P_2}=b''_{P_2}+(2^{|P_2|})b^r_{P_2}$ and Lemma 4.8 (i).

(iv): We have

$$(z_{B_{P_1}})_*\mathbb{B}(P',P_1)=\frac{1}{d(P')}\sum_{c\in\operatorname{Con}(P',P_1)}(z_{B_{P_1}}\circ\xi_c)_*1_{P'}$$

$$=\frac{1}{d(P')}|\operatorname{Con}(P',P_1)|\cdot\deg'(z_{B_{P_1}}\circ\xi_c)\cdot B(P',P_1),$$

where $\deg'$ denotes the degree as a morphism of stacks, or equivalently the degree as morphism of varieties adjusted by the automorphism numbers, as in Remark 1.35 (ii). But $z_{B_{P_1}}\circ\xi_c$ is (for all $c\in\operatorname{Con}(P',P_1)$) the morphism one obtains from $\xi_{B_{P'}}$ by restricting its domain from $\overline{M}_{1,n}$ to B_{P_1}. Hence $\deg'(z_{B_{P_1}}\circ\xi_c)=\deg'(\xi_{B_{P'}})=d(P')$. Together with $|\operatorname{Con}(P',P_1)|=d(P_1)$, (iv) follows for the case of B_{P_1}. For B''_{P_1} everything goes analogously. For $B^r_{P_1}$ one has to take into account that the general object of $B^r_{P'}$ has $2^{|P'|-1}$ automorphisms.

(v): This is clear by projection formula, (iv), and (ii) resp. (iii).

(vi): We prove this for $i_{P_1,P,r}$, the other cases work analogously. Choose a $c \in \mathrm{Con}(P_1, P)$, then

$$\begin{array}{ccc} \overline{M}_{\Gamma(P_1)} & \xrightarrow{\xi_c} & \overline{M}_{\Gamma(P)} \\ {\scriptstyle z_{B^r_{P_1}}} \downarrow & & \downarrow {\scriptstyle z_{B^r_P}} \\ B^r_{P_1} & \xrightarrow[i_{P_1,P,r}]{} & B^r_P \end{array}$$

commutes, and by the projection formula and (iii) we get

$$i^*_{P_1,P,r}(B^r(P_2,P)) = \frac{1}{\deg' z_{B^r_{P_1}}}(z_{B^r_{P_1}})_*\xi^*_c z^*_{B^r_P}(B^r(P_2,P))$$

$$= \frac{1}{2^{|P_1|-1}d(P_1)}(z_{B^r_{P_1}})_*\xi^*_c(2^{|P|-|P_2|}\mathbb{B}(P_2,P)) = 2^{|P|-|P_1|-|P_2|+1}\frac{1}{d(P_1)}(z_{B^r_{P_1}})_*\xi^*_c(\mathbb{B}(P_2,P)).$$

Now for each $J \in P$, J is a vertex of $\Gamma(P)$ and we denote by $\Gamma(J)$ as usual the smooth cell of $\Gamma(P)$ containing J (cf. Def. 1.17 (iii)). Denote by $c_J : c^{-1}(\Gamma(J)) \rightsquigarrow \Gamma(J)$ the contraction induced by c and by ξ_{c_J} the corresponding gluing morphism. We have

$$\overline{M}_{\Gamma(P)} = \prod_{J\in P}\overline{M}_{\Gamma(J)}, \quad \overline{M}_{\Gamma(P_1)} = \prod_{J\in P}\overline{M}_{c^{-1}(\Gamma(J))}, \quad \text{and} \quad \xi_c = \prod_{J\in P}\xi_{c_J}$$

(cf. Remark 1.19). Now if $P = \langle J_1, ..., J_m\rangle$, write $\mathbb{B}(P_2, P)$ as in (i) in the from

$$D_1 \otimes ... \otimes D_m \in H^*(\prod_{j=1}^{m}\overline{M}_{\Gamma(J_j)}) = \bigotimes_{j=1}^{m} H^*(\overline{M}_{\Gamma(J_j)}),$$

where $m = |P|$, or as a sum of two such expressions if $|P| = 2$. For every appearing boundary cycle class $D_j \in H^*(\overline{M}_{\Gamma(J_j)})$ we can compute $\xi^*_{c_{J_j}}(D_j)$ by applying excess intersection formula (1.5). By putting the results together again in $H^*(\overline{M}_{\Gamma(P_1)}) = \bigotimes_{j=1}^{n} H^*(\overline{M}_{c^{-1}(\Gamma(J_j))})$ we obtain (as one can check)

$$\xi^*_c(\mathbb{B}(P_2,P)) = \sum_{\bar{P}\in\mathrm{CCR}(P_1,P_2)}(-1)^{|\widehat{\mathrm{CN}}(P_1,P_2;\bar{P};P)|}\sum_{\widehat{P}\in\widehat{\Psi}(P_1,P_2;\bar{P};P)}\mathbb{B}(\widehat{P},P_1)\ ^{28}.$$

Together with (iv) this yields (vi). [29] □

Corollary 5.29 *Let P_1, P_2 be as above. Using the notation of Lemma 5.28, we determine the intersection product $b_{P_1}b_{P_2}$:*

$$b_{P_1}b_{P_2} = \sum_{\bar{P}\in\mathrm{CCR}(P_1,P_2)}(-1)^{|CN(P_1,P_2,\bar{P})|}\sum_{\widehat{P}\in\Psi(P_1,P_2,\bar{P})}b_{\widehat{P}}$$

[28] Note that by condition of Lemma 5.20 (ii), here $\mathrm{CCR}(P_1, P_2)$ will only contain more than one element if $P = \langle J_1, J_2\rangle$ and $P_1 = \langle J_1, K_1, ..., K_{n_1}\rangle$, $P_2 = \langle L_1, ..., L_{n_2}, J_2\rangle$ for some partitions $K_1, .., K_{n_1}$ and $L_1, ..., L_{n_2}$ of J_2 resp. J_2. In this case $\mathrm{CCR}(P_1, P_2)$ consists exactly of $\langle L_1, L_2, ..., L_{n_2}, K_1, K_2, ..., K_{n_1}\rangle$ and $\langle L_{n_2}, L_{n_2-1}, ..., L_1, K_1, K_2, ..., K_{n_1}\rangle$.

[29] One also could proof (vi) analogously to (v) using a slightly generalised version of the excess intersection formula (1.5), which would determine the pullback of boundary cycle classes to boundary cycles from inside another boundary cycle. But I did not want to proof this generalised version and do not know a reference.

$$b''_{P_1} b''_{P_2} = \sum_{\bar{P} \in \mathrm{CCR}(P_1,P_2)} (-1)^{|CN(P_1,P_2,\bar{P})|} \sum_{\widehat{P} \in \Psi(P_1,P_2,\bar{P})} b''_{\widehat{P}}$$

$$b^r_{P_1} b^r_{P_2} = \sum_{\bar{P} \in \mathrm{CCR}(P_1,P_2)} (-1)^{|CN(P_1,P_2,\bar{P})|} \sum_{\widehat{P} \in \Psi(P_1,P_2,\bar{P})} 2^{|\widehat{P}|-|P_1|-|P_2|} b^r_{\widehat{P}}$$

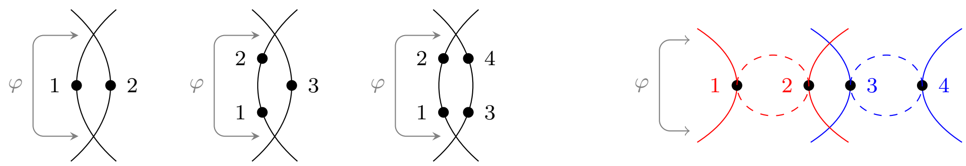
Proof: Just push forward the equations of Lemma 5.28 (v) via the inclusions $f_{B_{P_1}}$ resp. $f_{B''_{P_1}}$ resp. $f_{B^r_{P_1}}$. □

5.4 The additive structure of $H^*_{CR}(\overline{R}_{1,n})$

5.4.1 The sectors of $I_1(\overline{R}_{1,n})$

Definition 5.30 (i) In $\overline{M}_{1,k}$, $k \in \{2,3,4\}$, index the marked points by $\bullet_1, ..., \bullet_k$, and let $E_2 \subset \overline{M}_{1,\{\bullet_1,\bullet_2\}}$, $E_3 \subset \overline{M}_{1,\{\bullet_1,\bullet_2,\bullet_3\}}$, $E_4 \subset \overline{M}_{1,\{\bullet_1,\bullet_2,\bullet_3,\bullet_4\}}$ be the following points: $E_2 := B_{\langle\{\bullet_1\},\{\bullet_2\}\rangle}$ (i.e. the only banana cycle in $\overline{M}_{1,\{\bullet_1,\bullet_2\}}$). E_3 is the unique [30] point in $B_{\langle\{\bullet_1,\bullet_2\},\{\bullet_3\}\rangle}$ such that the parametrised curve has only two irreducible components and has an automorphism φ which swaps the two nodes, E_4 is the unique [31] point in $B_{\langle\{\bullet_1,\bullet_2\},\{\bullet_3,\bullet_4\}\rangle}$ with the same property.

As we shall see below these points lie inside the "hyperelliptic loci" $\overline{A}_k = \overline{HM}_{1,k}$ (cf. chapter 2). The non-trivial automorphism on them is the limit of the elliptic involution on the smooth curves of $\overline{A}_2$, $\overline{A}_3$, $\overline{A}_4$. Below there are symbolic pictures of the curves $\mathfrak{C}$ parametrised by the E_k. The picture on the right shows the curve parametrised by E_4 in a more "hyperelliptic" fashion (cf. Chapter 2).



$[\mathfrak{C}] \in E_2$ $[\mathfrak{C}] \in E_3$ $[\mathfrak{C}] \in E_4$ φ as hyperelliptic involution on $[\mathfrak{C}] \in E_4$

Denote the preimage points under $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ of these points E_n (two for each E_n) by E''_n and E^r_n. The prym curve parametrised by E''_n is supported on the stable curve C parametrised by E_n. For E^r_n it is supported on the curve X obtained from C by blowing up to the two nodes.

(ii) We define loci in $\overline{M}_{1,n}$, by, like in Lemma&Definition 5.13, attaching pointed rational tails to these E_k. We use the morphisms form Definition 5.12. Let (I_1, I_2), resp. (I_1, I_2, I_3),

[30] Each general point of $B_{\langle\{\bullet_1,\bullet_2\},\{\bullet_3\}\rangle}$ parametrises a curve consisting of two components $C_1 \cong \mathbb{P}^1$, $C_2 \cong \mathbb{P}^1$ meeting each other in two nodes q_1 and q_2. C_1 carries marked points $\bullet_1$, $\bullet_2$, while C_2 carries the marked point $\bullet_3$. It is easy to check (using the description of automorphisms of $\mathbb{P}^1$ as Möbius transformations) that on a $\mathbb{P}^1$ there is exactly one automorphism exchanging two given points (q_1 and q_2) and fixing one other given point (say $\bullet_1$ on $C_1 \cong \mathbb{P}^1$ resp. $\bullet_2$ on $C_2 \cong \mathbb{P}^1$). Also this automorphism has one unique additional fixed point, and on the component C_1, $\bullet_2$ has to be placed on this fixed point, for not to block the automorphism.

[31] The uniqueness of this point can be seen like for E_3.

resp. (I_1, I_2, I_3, I_4) be ordered partitions of $\underline{n}$. Then we define:

$$E_2^{\{I_1,I_2\}} := \xi_{I_1,I_2}(f^{-1}_{I_1,I_2}(E_2)), \quad E_3^{\{I_1,I_2\},I_3} := \xi_{I_1,I_2,I_3}(f^{-1}_{I_1,I_2,I_3}(E_3)),$$
$$E_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}} := \xi_{I_1,I_2,I_3,I_4}(f^{-1}_{I_1,I_2,I_3,I_4}(E_4)).$$

Like in 5.13, we write the exponents such that they express invariances of the loci $E_n^{\{\dots\}}$ under certain permutations of the sets in the partition $(I_1, ..., I_k)$.

Analogously define loci $E_k^{\prime\prime,\cdots}$ and $E_k^{r,\cdots}$, in $\overline{R}_{1,n}$ by attaching rational tails to the E_k'' and E_k^r

(iii) In $\overline{M}_{1,2}$, let F_2 be the unique point in the divisor Δ_0 parametrising a curve with a non-trivial automorphism. This point represents the following curve C: Mark as p_1 and p_2 on $\mathbb{P}^1$ the two points 0 and ∞. Glue the points 1 and -1 on the $\mathbb{P}^1$ together to obtain C. Then multiplication on $\mathbb{P}^1$ by -1 induces an automorphism on C fixing p_1 and p_2.

Denote by F_2^r and F_2'' the two points of $\overline{R}_{1,n}$ lying over F_2, like in (i).

Lemma 5.31 *Here we describe for each $\overline{A}_{k,x} \subset \overline{R}_{1,k}$ and $\overline{A}_k \subset \overline{M}_{1,k}$, for $k \in \{2, 3, 4\}$, $x \in \{a, b, c\}$, the boundary $\overline{A}_{k,x} \smallsetminus A_{k,x}$ resp. $\overline{A}_k \smallsetminus A_k$. We use the previous definition, but write for example $E_4^{\{\{1,3\}\{2,4\}\}}$ instead of the formally correct $E_4^{\{\{\{1\},\{3\}\}\{\{2\},\{4\}\}\}}$.*

(i) For the $\overline{A}_k \subset \overline{M}_{1,k}$:

$$\overline{A}_2 \smallsetminus A_2 = F_2 \cup E_2^{\{\{1\},\{2\}\}}$$
$$\overline{A}_3 \smallsetminus A_3 = E_3^{\{\{1,2\},\{3\}\}} \cup E_3^{\{\{1,3\},\{2\}\}} \cup E_3^{\{\{2,3\},\{1\}\}},$$
$$\overline{A}_4 \smallsetminus A_4 = E_4^{\{\{1,2\},\{3,4\}\}} \cup E_4^{\{\{1,3\},\{2,4\}\}} \cup E_4^{\{\{2,3\},\{1,4\}\}},$$

(ii) For the $\overline{A}_{k,x} \subset \overline{R}_{1,k}$

$$\overline{A}_{2,a} \smallsetminus A_{2,a} = F_2'' \cup E_2^{r,\{\{1\},\{2\}\}}, \qquad \overline{A}_{2,b} \smallsetminus A_{2,b} = F_2^r \cup E_2^{r,\{\{1\},\{2\}\}} \cup E_2^{\prime\prime,\{\{1\},\{2\}\}},$$
$$\overline{A}_{3,a} \smallsetminus A_{3,a} = E_3^{\prime\prime,\{1,2\},\{3\}} \cup E_3^{r,\{1,3\},\{2\}} \cup E_3^{r,\{2,3\},\{1\}},$$
$$\overline{A}_{3,b} \smallsetminus A_{3,b} = E_3^{\prime\prime,\{1,3\},\{2\}} \cup E_3^{r,\{1,2\},\{3\}} \cup E_3^{r,\{2,3\},\{1\}},$$
$$\overline{A}_{3,c} \smallsetminus A_{3,c} = E_3^{\prime\prime,\{2,3\},\{1\}} \cup E_3^{r,\{1,3\},\{2\}} \cup E_3^{r,\{1,2\},\{3\}},$$
$$\overline{A}_{4,a} \smallsetminus A_{4,a} = E_4^{\prime\prime,\{\{1,2\},\{3,4\}\}} \cup E_4^{r,\{\{1,3\},\{2,4\}\}} \cup E_4^{r,\{\{2,3\},\{1,4\}\}},$$
$$\overline{A}_{4,b} \smallsetminus A_{4,b} = E_4^{\prime\prime,\{\{1,3\},\{2,4\}\}} \cup E_4^{r,\{\{1,2\},\{3,4\}\}} \cup E_4^{r,\{\{2,3\},\{1,4\}\}},$$
$$\overline{A}_{4,c} \smallsetminus A_{4,c} = E_4^{\prime\prime,\{\{2,3\},\{1,4\}\}} \cup E_4^{r,\{\{1,3\},\{2,4\}\}} \cup E_4^{r,\{\{1,2\},\{3,4\}\}}$$

Proof: Part (i) is shown in [Pag08] section 3.b.1, but could like (ii) below also be shown using Propositions 2.14 and 2.19.

(ii): In the proof of Lemma 5.9, we saw that the $\overline{A}_{k,x}$ are the components of the "hyperelliptic locus" $\overline{HR}_{1,k}$, using the notation of Definition 2.1. But for $k = 2$ we determined the boundary divisors of these hyperelliptic loci in Example 2.20 as an application of Propositions 2.14 and 2.19. The decomposition of the boundaries for $k = 3$ and $k = 4$ can be determined analogously. □

Theorem 5.32 *The sectors produced from basic sectors by attaching rational tails (as described in Lemma & Definition 5.13), together with the sectors of the form* $(B^r_{I_1,...,I_m}, \iota_m)$ *(m even), are all the sectors that appear in* $I_1(\overline{R}_{1,n})$.

Thus the decomposition of $I_1(\overline{R}_{1,n})$ *into sectors is:*

$$(\overline{R}_{1,n}, 1) \biguplus (\overline{A}_1^{\underline{n}}, -1) \biguplus_{\{I_1,I_2\},\ I_2 \uplus I_2 = \underline{n}} (\overline{A}_{2,a}^{\{I_1,I_2\}}, -1) \biguplus_{\{I_1,I_2\},\ I_2 \uplus I_2 = \underline{n}} (\overline{A}_{2,b}^{\{I_1,I_2\}}, -1)$$

$$\biguplus_{\substack{(\{I_1,I_2\},I_3),\\ I_1 \uplus I_2 \uplus I_3 = \underline{n}}} (\overline{A}_3^{\{I_1,I_2\},I_3}, -1) \biguplus_{\substack{\{\{I_1,I_2\},\{I_3,I_4\}\},\\ I_1 \uplus I_2 \uplus I_3 \uplus I_4 = \underline{n}}} (\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}}, -1)$$

$$\biguplus (C_4^{\underline{n}}, i/-i) \biguplus_{\{I_1,I_2\},\ I_2 \uplus I_2 = \underline{n}} (C_4^{\{I_1,I_2\}}, i/-i) \biguplus_{\substack{m \leq n,\\ m \text{ even}}} \biguplus_{\substack{\langle I_1,...,I_m \rangle,\\ I_1 \uplus ... \uplus I_m = \underline{n}}} (B^r_{\langle I_1,...,I_m \rangle}, \iota_m)$$

Proof: If one only wants to show this Theorem, it is probably possible to give a shorter proof than the following. We will instead give a prove which provides additional information that will also be used to prove other statements later.

For a genus 1 prym curve with n-marked points $\mathfrak{X} := (X; p_1, ..., p_n; \mathcal{L}; b)$ denote the stable model of $\mathfrak{X}$ by $\mathfrak{C} := (C; p_1, ..., p_n)$. Then any $\varphi \in \mathrm{Aut}(\mathfrak{X})$ induces an automorphism $\varphi_{\mathfrak{C}} \in \mathrm{Aut}(\mathfrak{C})$ on the stable model $\mathfrak{C}$. In this way we can regard as a subgroup of $\mathrm{Aut}(\mathfrak{C})$ the group $\mathrm{Aut}(\mathfrak{X})/H$, where H is the subgroup of inessential automorphisms in $\mathrm{Aut}(\mathfrak{X})$ (cf. Definition 1.11 (v) and Remark 1.12). For any $\varphi \in \mathrm{Aut}(\mathfrak{X})$, $(\mathfrak{X}, \varphi)$ is an object parametrised by a point of a 1-sector (Y, g) of $\overline{R}_{1,n}$. We want to determine which (Y, g) exist.

As seen in the proof of Lemma 4.4, the quasi-stable curve X consists of one genus 1 component X' having only non-disconnecting nodes, to which several rational trees $X_1, .., X_k$ may be attached. These non-disconnecting nodes are either all exceptional (i.e. blown up) or all non-exceptional. On the rational trees any automorphism φ of the prym curve acts trivially [32]. Hence, for the prym curve $\mathfrak{X}' := (X'; p_{i_1}, ..., p_{i_\nu}, \bullet_1, ..., \bullet_k; \mathcal{L}_{|X'})$, where $p_{i_1}, ..., p_{i_\nu}$ is the (ordered) subset of the marked points $p_1, ..., p_n$ which lie on X', there is an isomorphism $\mathrm{Aut}(\mathfrak{X}) \cong \mathrm{Aut}(\mathfrak{X}')$. It correspnds to restricting each $\varphi \in \mathrm{Aut}(\mathfrak{X})$ to an automorphism $\varphi_{\mathfrak{X}'}$ of $\mathfrak{X}'$.

Let $\mathfrak{C}' := (C'; p_{i_1}, ..., p_{i_\nu}, \bullet_1, ..., \bullet_k)$ be the stable model of $\mathfrak{X}'$. Then $\mathrm{Aut}(\mathfrak{C}') \cong \mathrm{Aut}(\mathfrak{C})$ in the same way.

Now we will work with local universal deformations of our objects, so recall the notations and results summarized in section 1.5. Let $(X \hookrightarrow \mathcal{X} \to (S, s_0); \sigma_1, ..., \sigma_n; \mathbf{L}, \mathbf{b})$ be the local universal deformation of $\mathfrak{X}$. And let (B, b_0) be the local universal deformation space of $\mathfrak{C}$. Let $\pi : (S, s_0) \to (B, b_0)$ be the forgetful morphism of Summary 1.31 (ii). We denote the maps as which φ resp. $\varphi_{\mathfrak{C}}$ act on (S, s_0) resp. (B, b_0) again by φ resp. $\varphi_{\mathfrak{C}}$.

Summary 1.31 (iii) in particular implies that, if $\mathrm{Fix}(\varphi) \subseteq S$ is the subset fixed by φ, then $\pi(\mathrm{Fix}(\varphi)) \subseteq \mathrm{Fix}(\varphi_{\mathfrak{C}}) \subseteq B$. We want to determine $\mathrm{Fix}(\varphi)$, because if we extend φ to

[32] This is clear for the corresponding $\varphi_{\mathfrak{C}}$, and since by Summary 1.13 (i) none of the nodes on the rational trees is blown up, there are also no inessential automorphisms acting non-trivially on the rational trees.

$\mathcal{X} \to B$, then this extension restricts to an automorphism on a fibre $f^{-1}(p)$ for a $p \in S$ exactly if $p \in \mathrm{Fix}(\varphi)$. Hence $\mathrm{Fix}(\varphi) = \rho^{-1}(Y)$, where Y is the support of (Y, g) and $\rho : S \to \overline{R}_{1,n}$ is the map which describes $\overline{R}_{1,n}$ locally around $[\mathfrak{X}]$ as the quotient $S/\operatorname{Aut}(\mathfrak{X})$ (cf. Summary 1.30 (v) and 1.31 (i)). We will see in this way that for any possible 1-sector (Y, g), the support Y is one of the supports appearing in the theorem. This will allow us to show that the sectors in the theorem are all that exist, and also that they are indeed sectors, since the automorphisms on them do not extend to larger loci.

Let $E = E(\Gamma(\mathfrak{C}))$ be as in section 1.5. We can simultaneously interpret E as the set of nodes of C, and as set of nodes and blown up nodes (i.e. exceptional components) of X. We will always call the elements of E nodes, even if we interpret them on X. Set $\nu := |E|$.

As in 1.31 we identify (S, s_0) and (B, b_0) with the unit ball in $\mathbb{C}^n$, and endow them with standard bases

$$(\vec{y}_1, ..., \vec{y}_{n-\nu}, (\vec{y}_e)_{e\in E}) \quad \text{resp.} \quad (\vec{x}_1, ..., \vec{x}_{n-\nu}, (\vec{y}_e)_{e\in E}).$$

Now $\mathrm{Fix}(\varphi)$ resp. $\mathrm{Fix}(\varphi_{\mathfrak{C}})$ are linear sub spaces of S resp. B.

We divide E into three subsets: Let E_{nd} be the set of non-disconnecting nodes of C', $E_\bullet$ be the set of nodes connecting a rational tree to C', E_{tr} be the nodes in which two components of a rational tree of C meet. Then the permutation φ_E induced on E by φ, respects this partition of E, i.e. $\varphi_E(E_{nd}) = E_{nd}$, and so on. We also write $V = V' \uplus V_{tr}$ where V' contains the vertices corresponding to components of C', V_{tr} contains the vertices corresponding to components of the rational trees of C. Then of course

$$S = \bigoplus_{v\in V'} U_v \oplus \bigoplus_{v\in V_{tr}} U_v \oplus \operatorname{span}_S\big(\{\vec{y}_e\}_{e\in E_{nd}},\ \{\vec{y}_e\}_{e\in E_\bullet},\ \{\vec{y}_e\}_{e\in E_{tr}}\big)$$

where the U_v are as in Summary 1.31. Recall Lemma&Definition 1.32 (i), since we will use it from now on.

Since φ acts trivially on the rational trees, it is clear that φ and $\varphi_{\mathfrak{C}}$ extend in the directions $\vec{y}_e$ resp. $\vec{x}_e$ for $e \in E_{tr}$. For the same reason $\operatorname{span}_S(\{U_v\}_{v\in V_{tr}}) \subseteq \mathrm{Fix}(\varphi)$, where the U_v are as in Summary 1.31. We refer to these two remarks by: $(*)$

Now look at an $e \in E_\bullet$. If φ is not inessential, $\varphi_{\mathfrak{C}}$ fixes e, and acts non-trivially on the tangent space to the node e of the component X' but trivially on the tangent space to e of the rational tree meeting X' in e. Hence in these cases $\mathrm{Fix}(\varphi_{\mathfrak{C}}) \subseteq \bigcap_{e\in E_\bullet}\{x_e = 0\}$, hence $\mathrm{Fix}(\varphi) \subseteq \bigcap_{e\in E_\bullet}\{y_e = 0\}$. If φ is an inessential automorphism, then it acts trivially at each node $e \in E_\bullet$ on both branches, and $\varphi_{\mathfrak{C}}$ and φ extend in the direction $\vec{x}_e$ resp. $\vec{y}_e$ [33] . We refer to this paragraph by: $(\dagger)$

Let $(\overline{Z}, g)$ be one of the basic sectors of $I(\overline{R}_{1,k})$, for $k \in \underline{4}$, and look at a pair $(\mathfrak{X}, \varphi)$ such that $(\mathfrak{X}', \varphi_{\mathfrak{X}'}) \in (\overline{Z}, g)$. Then it is clear that $[\mathfrak{X}] \in \overline{Z}^{(I_1,\dots,I_k)}$ for some partition $(I_1, ..., I_k)$ of $\underline{n}$. Let $(\overline{Z}^{(I_1,\dots,I_k)}, g)$ be the corresponding sector obtained from $(\overline{Z}, g)$ (cf. Lemma&Definition 5.13). Now let $U_{\overline{Z}}$ be the preimage of $\overline{Z}$ on the local universal deformation space (S', s'_0)

[33]That $\varphi_{\mathfrak{C}}$ extends in direction $\vec{x}_e$ implies that φ extend in direction $\vec{y}_e$ in this case, since these nodes are not blown up on X.

of $\mathfrak{X}'$ ($U_{\overline{Z}} = \text{Fix}(\varphi_{\mathfrak{X}'})$). We can identify (S', s'_0) with the sub space

$$\bigoplus_{v\in V'} U_v \oplus \text{span}_S(\{\vec{y}_e\}_{e\in E_{nd}}) \subseteq S. \quad \text{Then:} \quad U_{\overline{Z}} = \text{Fix}(\varphi) \cap S' \text{ inside } S.$$

Since $\varphi_{\mathfrak{X}'}$ belongs to a basic sector $(\overline{Z}, g)$, $\varphi_{\mathfrak{X}'}$ and φ are not inessential. So together with $(*)$ and $(\dagger)$, $U_{\overline{Z}} = \text{Fix}(\varphi) \cap S'$ yields:

$$\text{Fix}(\varphi) = U_{\overline{Z}} \oplus \bigoplus_{v\in V_{tr}} U_v \oplus \text{span}_S(\{\vec{y}_e\}_{e\in E_{tr}}).$$

It is easy to check that this is just the preimage of $\overline{Z}^{(I_1,\dots,I_k)}$ on (S, s_0). This proves that all $(\overline{Z}^{(I_1,\dots,I_k)}, g)$ are indeed 1-sectors of $\overline{R}_{1,n}$. This covers all the sectors listed in the theorem except the sectors $(B^r_{\langle I_1,\dots I_m\rangle}, \iota_m)$. Furthermore we have seen that these sectors suffice to parametrise all pairs $(\mathfrak{X}, \varphi)$ such that $(\mathfrak{X}', \varphi_{\mathfrak{X}'})$ is parametrised by a basic sector.

So now look at any pair $(\mathfrak{X}, \varphi)$, where $\varphi \neq id$ is an inessential automorphism, i.e. $\varphi_{\mathfrak{C}} = id$. The existence of such a φ implies that the non-exceptional subcurve of X is disconnected, which is, using Summary 1.13 (i), only possible if the same holds for X'. So by our general description of the possible $\mathfrak{X}$ above, $[\mathfrak{X}]$ is contained in some $B^r_{\langle J_1,\dots J_{m'}\rangle}$, where $m' \leq n$ may be any number (possibly odd). Then it is clear, that $\bigcap_{e\in E_{nd}}\{y_e = 0\} \subseteq \text{Fix}(\varphi)$. Note that this is just the preimage of $B^r_{\langle J_1,\dots,J_{m'}\rangle}$ on (S, s_0). But φ may also extend in directions $\vec{y}_e$ for some $e \in E_{nd}$: Recall from Remark 1.12 and Lemma 1.32 (iii), that an inessential automorphism φ corresponds to choosing for each non-exceptional component X'_i of X' a number $a_i \in \{-1, 1\}$, up to multiplying all a_i simultaneously by -1, and that φ extends in direction $\vec{y}_e$ for $e \in E_{nd}$ iff e is a node between two components X_i, $X_{i'}$ with $a_i = a_{i'}$. Denote by E^*_{nd} the non-disconnecting nodes for which the two adjacent components have different a_i, $m := |E^*_{nd}|$. Note that m has to be even, since the components of X_i are circularly arranged. Let $\langle I_1, ..., I_m\rangle$ be the partition obtained by coarsening the partition $\langle J_1, ..., J_{m'}\rangle$, by replacing each sequence $J_{j_1} \parallel .... \parallel J_{j_s}$ of neighbouring sets, for which $a_{j_1} = a_{j_2} = ... = a_{j_s}$, by the union $J_{j_1} \cup .... \cup J_{j_s}$ [34] . Then φ fixes exactly the preimage of $B^r_{\langle I_1,\dots,I_m\rangle}$ on (S, s_0), and it is also clear that $[(\mathfrak{X}, \varphi)] \in (B^r_{\langle I_1,\dots,I_m\rangle}, \iota_m)$. This shows that all the $(B^r_{\langle I_1,\dots,I_m\rangle}, \iota_m)$ with $m \leq$ even are indeed 1-sectors of $\overline{R}_{1,n}$, and furthermore that every pair $(\mathfrak{X}, \varphi)$ with $\varphi \neq id$ inessential is parametrised by such a sector.

We have shown that all the loci of $I(\overline{R}_{1,n})$ listed in our theorem are indeed 1-sectors. It remains to show that every possible pair $(\mathfrak{X}, \varphi)$, is parametrised by one of them. By the above discussion we may already WLOG assume that φ is not inessential. Now we distinguish several cases:

We know that C' is either a smooth elliptic curve (case 1), a rational curve with one non-disconnecting node (case 2) or a curve parametrised by a general point of a simple banana cycle $B_{\langle I_1,\dots,I_m\rangle}$, where $m \geq 2$ is the number of (non-disconnecting) nodes of C' (case 3).

In case 1, we know by Corollary 5.10 that $(\mathfrak{X}', \varphi_{\mathfrak{X}'})$ is parametrised by a basic sector, so by the above discussion $(\mathfrak{X}, \varphi)$ is parametrised by one those sectors listed in our Theorem which are of the form $(\overline{Z}^{(I_1,\dots,I_k)}, g)$.

[34] This coarsening of the partition corresponds to smoothing all nodes in $E_{nd} \setminus E^*_{nd}$.

In case 2 we can argue as in the footnote to Definition 5.30 (i), to see that there is at most one non-trivial automorphism of $\mathfrak{C}'$, which is present if and only if $\mathfrak{C}'$ is parametrised by one of the points $\Delta_0 \subset \overline{M}_{1,1}$ or $F_2 \subset \overline{M}_{1,2}$ of Definition 5.30 (possibly after renaming the indices of marked points of $\mathfrak{C}'$). Hence $\mathfrak{X}'$ is parametrised by one of D_0'', D_0^r,F_2'', F_2^r (again cf. Def. 5.30). By Lemma 5.31 (ii), we conclude that $\mathfrak{X}'$ is contained in $\overline{A}_1 = \overline{R}_{1,1}$, $\overline{A}_{2,a}$ or $\overline{A}_{2,b}$. Since $\mathfrak{X}'$ has no inessential automorphisms $|\operatorname{Aut}(\mathfrak{X}')| = |\operatorname{Aut}(\mathfrak{C}')| = 2$, i.e. there is only one non-trivial automorphism. So $(\mathfrak{X}', \varphi)$ is parametrised by one of $(\overline{A}_1, -1)$, $(\overline{A}_{2,a}, -1)$ and $(\overline{A}_{2,b}, -1)$, which again are basic sectors.

In case 3 we distinguish two sub-cases: Either $\mathfrak{X}'$ is parametrised by a point of $B''_{\langle I_1,\ldots,I_m\rangle}$, (case 3.1), or of $B^r_{\langle I_1,\ldots,I_m\rangle}$, (case 3.2).

Regardless of the sub-case, there is again at most one non-trivial automorphism of $\mathfrak{C}'$, which is present if and only if $\mathfrak{C}'$ is parametrised, possibly after renaming its marked points, by E_2, E_3 or E_4 of Definition 5.30. To see this, first note that $\varphi_{\mathfrak{C}}$ can not interchange any components of C' since it has to fix the marked points, of which each component at least carries one. Since an automorphism fixing 3 points on $\mathbb{P}^1$ is trivial, this already suffices to show that $\varphi_{\mathfrak{C}}$ has to be trivial if C' has more than two components. If C' has two components we see (as in the footnote to Def. 5.30 (i)) that each component can carry at most two marked points, and that a non-trivial $\varphi_{\mathfrak{C}}$ has to interchange the two nodes of C'. By definition E_2, E_3 or E_4 are the loci of points parametrising such curves with such an automorphism. Again by Lemma 5.31 (ii) we conclude that $\mathfrak{X}'$ is contained in some $\overline{A}_{k,x}$ for $k \in \{2, 3, 4\}$ and $x \in \{a, b, c\}$. In sub-case 3.1 there are again no inessential automorphisms, so $\mathfrak{X}'$ has only one non-trivial automorphism, and thus $[(\mathfrak{X}', \varphi_{\mathfrak{X}'})] \in (\overline{A}_{k,x}, -1)$, which is a basic sector.

In sub-case 3.2., $[\mathfrak{X}']$ must be contained in one of the $E_k^{r,\cdots}$ for $r \in \{2, 3, 4\}$ appearing in Def. 5.30 (ii). There is one non-trivial inessential automorphism of $\mathfrak{X}'$, and hence $|\operatorname{Aut}(\mathfrak{X}')| = 4$, $|\operatorname{Aut}(\mathfrak{C}')| = 2$. By Lemma 5.31 (ii) we see that $E_k^{r,\cdots}$ is contained in three different supports of 1-sectors: Two are of the form $\overline{A}_{k,x}$ for $x \in \{a, b, c\}$ the third one is the unique $B^r_{\langle J_1,J_2\rangle}$ containing $E_k^{r,\cdots}$ ($\langle J_1, J_2\rangle$ a partition of $\underline{k}$). Hence $\operatorname{Aut}(\mathfrak{X}')$ must contain three non-trivial automorphisms $\varphi_1, \varphi_2, \varphi_3$, such that each of the two different sectors $(\overline{A}_{k,x}, -1)$ contains one of $[(\mathfrak{X}', \varphi_1)]$ and $[(\mathfrak{X}', \varphi_2)]$, and such that $[(\mathfrak{X}', \varphi_3)] \in (B^r_{\langle J_1,J_2\rangle}, \iota_2)$. Since $|\operatorname{Aut}(\mathfrak{X}')| = 4$, and since our $\varphi_{\mathfrak{X}'}$ is not inessential, we must have $\varphi_{\mathfrak{X}'} = \varphi_1$ or $\varphi_{\mathfrak{X}'} = \varphi_2$. So once again, $(\mathfrak{X}', \varphi_{\mathfrak{X}'})$ is parametrised by a basic sector.

Now we have shown that all 1-sectors (Y, φ) of $\overline{R}_{1,n}$ indeed appear in the list of our Lemma. □

Remark 5.33 In almost all cases it is clear how the automorphism group $\operatorname{Aut}(\mathfrak{X})$ for a prym curve $\mathfrak{X}$ with $[\mathfrak{X}] \in \overline{R}_{1,n}$ looks like, since either $\mathfrak{X}$ has only inessential automorphisms and $\operatorname{Aut}(\mathfrak{X})$ is then known by Remark 1.12 (ii), or $\mathfrak{X}$ has no inessential automorphisms, and then $\operatorname{Aut}(\mathfrak{X}) = \operatorname{Aut}(\mathfrak{C})$, for $\mathfrak{C}$ the stable model. The only exception appears if we are in the case 3.2. of the proof above, and $\operatorname{Aut}(\mathfrak{X})$ contains a non-inessential automorphism. I.e. for a prym curve $\mathfrak{X}$ such that $[\mathfrak{X}] \in E_k^{r,\cdots}$ ($k \in \{2, 3, 4\}$).

We know that in this case there are two nodes in E_{nd}, which we call e_1 and e_2, and that

$|\operatorname{Aut}(\mathfrak{X})| = 4$. More precisely $\operatorname{Aut}(\mathfrak{X}) = \{id, \iota, \varphi_1, \varphi_2\}$ where ι is inessential, $\iota^2 = id$ and φ_1, φ_2 are non-inessential automorphisms, such that $\varphi_1 = \iota\varphi_2$. Furthermore $\varphi_{\mathfrak{C}} := \varphi_{1,\mathfrak{C}} = \varphi_{2,\mathfrak{C}}$ is the automorphism swapping e_1 and e_2 on C', and we know by the discussion in Def. 5.30, that it restricts to the hyperelliptic involution on the stable hyperelliptic curve $\mathfrak{C}'$. So by Summary 2.8 (iii) we know that the liftings $\varphi_{1,\mathfrak{X}'}$ and $\varphi_{2,\mathfrak{X}'}$ of this hyperelliptic involution are of order 2. Hence the same holds for φ_1 and φ_2 (cf. the proof above) and thus we must have $\varphi_1^2 = \varphi_2^2 = \iota^2 = id$ (and thus, with the above, $\varphi_1\varphi_2 = \iota$). This determines the group $\operatorname{Aut}(\mathfrak{X})$ also in this case.

Lemma 5.34 *As varieties all of $\overline{R}_{1,1}$, and $\overline{A}_{k,x}$ for $k \in \{2,3,4\}$, $x \in \{a,b,c\}$, are isomorphic to $\mathbb{P}^1$.*

Proof: We know $\overline{R}_{1,1} \cong \mathbb{P}^1$ by Proposition 4.15 (i). If we restrict the finite forgetful morphisms $\tau_k : \overline{R}_{1,k} \to \overline{M}_{1,k}$ to one of the $\overline{A}_{k,x}$ we obtain a finite morphism $\overline{A}_{k,x} \to \overline{A}_k \subset \overline{M}_{1,k}$. It is easy to see by the description of the $\overline{A}_{k,x}$ in Lemma&Definition 5.9 (iii), that this morphism has degree 1 except in case $\overline{A}_{2,b}$ when it has degree 2. We know that the $A_k \subset \overline{M}_{1,k}$ are all isomorphic to $\mathbb{P}^1$ by [Pag08], and also by Proposition 2.14 (ii)+(iii). So in all cases but $\overline{A}_{2,b}$ the claim follows with Lemma 1.46. For $\overline{A}_{2,b}$ we know by Example 2.20 that there is a finite surjective morphism $a : \mathbb{P}^1 \cong \overline{M}_{0,4} \to \overline{A}_{2,b}$ of degree 1. By 1.46 again, a is an isomorphism if $\overline{A}_{2,b}$ is a normal variety. We can prove the normality by showing that for each $[\mathfrak{X}] \in \overline{A}_{2,b}$ the preimage of $\overline{A}_{2,b}$ on the local universal deformation space of $\mathfrak{X}$ is normal. This automatically holds for $\mathfrak{X}$ if its hyperelliptic local universal deformation space $\mathscr{S}$ has only one component, i.e. by the description of $\mathscr{S}$ from section 2.1.3, and by Lemma 5.31, for all cases except $[\mathfrak{X}] \in E_2^{r,\{\{1\},\{2\}\}}$. For $[\mathfrak{X}] \in E_2^{r,\{\{1\},\{2\}\}}$ the hyperelliptic deformation space has 2 irreducible components, but only one of these belongs to $\overline{A}_{2,b}$, the other one to $\overline{A}_{2,a}$. So again the preimage of $\overline{A}_{2,b}$ on the local universal deformation space is normal. □

Remark: It is possible to show the following more precise statement. If $\mathbb{P}(n,m)$ denotes the weighted projective 2-space with weights n and m (cf. [Man08], also cf. [Pag08] Lemma 3.17.), then we have the following isomorphisms of stacks/orbifolds:

- $\overline{R}_{1,1}$ and $\overline{A}_{2,a}$ are isomorphic to $\mathbb{P}(2,4)$.
- $\overline{A}_{2,b}$, $\overline{A}_{3,a}$, $\overline{A}_{3,b}$, $\overline{A}_{3,c}$, $\overline{A}_{4,a}$, $\overline{A}_{4,b}$ and $\overline{A}_{4,c}$ are all isomorphic to $\mathbb{P}(2,2)$.

Definition 5.35 By Theorem 5.32 and its proof, we know that all 1-sectors (X,g) of $\overline{R}_{1,n}$ are of one of the following two types:

(1) $X = \overline{Z}^{(I_1,\ldots,I_k)}$, where $\overline{Z}$ is a basic 1-sector from $\overline{R}_{1,k}$ (cf. Def. 5.11), $k \in \underline{4}$, and $\overline{Z}^{(I_1,\ldots,I_k)}$ is obtained by attaching rational tails as in Lemma&Definition 5.13. In this cases g is a non-inessential automorphism. We often call these sectors *essential* 1*-sectors.*

(2) $X = B_P^r$ with $|P| \geq 2$ even, and $g = \iota_P$ inessential. We call these sectors the *inessential* 1*-sectors.*

Corollary 5.36 *Each support X of a 1-sector (X,g) of $\overline{R}_{1,n}$ is as a variety isomorphic either to a product*

$$A \times \overline{M}_{0,n_1} \times \overline{M}_{0,n_2} \times \overline{M}_{0,n_3} \times \overline{M}_{0,n_4}$$

where $n_1, n_2, n_3, n_4 \geq 3$ are integers and A is either a point or $\mathbb{P}^1$, or X is a simple banana cycle $B^r_{I_1,\ldots,I_m}$ for m even. $B^r_{I_1,\ldots,I_m}$ is isomorphic to $\overline{M}_{0,|I_1|+2} \times \ldots \times \overline{M}_{0,|I_m|+2}$ if $m \geq 4$, or, for $m = 2$, is isomorphic to the quotient $(\overline{M}_{0,|I_1|+2} \times \overline{M}_{0,|I_2|+2})/\mathbb{S}_2$, where $\mathbb{S}_2$ acts as explained in Lemma 5.24.

Proof: If (X,g) is essential (cf. Def. 5.35), the corresponding basic sector $\overline{Z}$ is a point if $\overline{Z} \in \{C_4, C_4'\}$ and isomorphic to $\mathbb{P}^1$, by Lemma 5.34, if $\overline{Z} \in \{\overline{R}_{1,1}, \overline{A}_{k,x} \mid k \in \underline{4}, x \in \{a,b,c\}\}$. A support $\overline{Z}^{(I_1,\ldots,I_k)}$ obtained from $\overline{Z}$ is clearly isomorphic to $\overline{Z} \times \overline{M}_{0,|I_1|+1} \times \ldots \times \overline{M}_{0,|I_k|+1}$ if we define $\overline{M}_{0,2}$ to be a point. [35] The isomorphisms in the banana-cycle (i.e. inessential) case where shown in the Lemmas 5.24 and 5.25. □

5.4.2 Chen-Ruan cohomology of $\overline{R}_{1,n}$ as $\mathbb{Q}$ vector space

We use the notation

$$h_n := \dim_{\mathbb{Q}} H^*(\overline{M}_{0,n+1}), \quad k_{|I_1|,|I_2|} := \dim_{\mathbb{Q}} H^*(B^r_{\langle I_1,I_2\rangle}), \quad \binom{n}{i_1,\ldots,i_m} = \frac{n!}{i_1!\cdots i_m!}$$

for $i_1 + \ldots + i_m = n$. The values h_n are known from [Kee92], for $k_{|I_1|,|I_2|}$ cf. Lemma 5.24 (iv). We get

Corollary 5.37 *The vector space dimension of the Chen-Ruan cohomology of $\overline{R}_{1,n}$ is:*

$$\dim_{\mathbb{Q}} H^*_{CR}(\overline{R}_{1,n}) = \dim_{\mathbb{Q}} H^*(\overline{R}_{1,n}) + 4h_n + 3\sum_{i+j=n}\binom{n}{i,j}h_ih_j$$

$$+ \sum_{i+j+k=n}\binom{n}{i,j,k}h_ih_jh_k + \frac{1}{4}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}h_ih_jh_kh_l + \frac{1}{2}\sum\binom{n}{i,j}k_{i,j}$$

$$+ \sum_{\substack{4\leq m\leq n,\\ m\ \text{even}}}\frac{1}{2m}\sum_{i_1+\ldots+i_m=n}\binom{n}{i_1,\ldots,i_m}h_{i_1+1}h_{i_2+1}\cdots h_{i_m+1}$$

Proof: This is obtained by starting with the decomposition of $I_1(\overline{R}_{1,n})$ in Theorem 5.32, and applying to it (the proof of) Corollary 5.36, the Künneth formula, and Lemma 5.24 (iv) (and $\dim_{\mathbb{Q}} H^*(\mathbb{P}^1) = 2$):

$$\dim_{\mathbb{Q}} H^*_{CR}(\overline{R}_{1,n}) = \dim_{\mathbb{Q}} H^*(I_1(\overline{R}_{1,n})) = \dim_{\mathbb{Q}} H^*(\overline{R}_{1,n}) + 2h_n + \sum_{i+j=n}\frac{1}{2!}\binom{n}{i,j}2h_ih_j$$

$$+ \sum_{i+j=n}\frac{1}{2!}\binom{n}{i,j}2h_ih_j + \sum_{i+j+k=n}3\cdot\frac{1}{3!}\binom{n}{i,j,k}2h_ih_jh_k+$$

[35] Since $\overline{M}_{0,3}$ is a point too, we can replace all $\overline{M}_{0,2}$ by $\overline{M}_{0,3}$ and obtain the corollary.

$$\sum_{i+j+k+l=n} 3\cdot\frac{1}{4!}\binom{n}{i,j,k,l}2h_ih_jh_kh_l + 2\cdot h_n + 2\cdot\sum_{i+j=n}\frac{1}{2!}\binom{n}{i,j}h_ih_j$$

$$+\sum_{i+j=n}\frac{1}{2!}\binom{n}{i,j}k_{i,j} + \sum_{\substack{4\leq m\leq n,\\ m \text{ even}}}\frac{m!}{2m}\frac{1}{m!}\sum_{i_1+\ldots+i_m=n}\binom{n}{i_1,\ldots,i_m}h_{i_1+1}h_{i_2+1}\cdots h_{i_m+1}$$

For this, note that $\frac{1}{m!}\binom{n}{i_1,\ldots,i_m}$ is the number of possible unordered partitions $\{I_1,...,I_m\}$ of $\underline{n}$ into m sets of prescribed cardinalities $|I_1| = i_1,...,|I_m| = i_m$. In case of the sectors $\overline{A}_3^{\{I_1,I_2\},I}$ and $\overline{A}_4^{\{I_1,I_2\}\{I_3,I_4\}}$ this number has to be multiplied by 3 to account for the partial ordering of the partitions defining the rational tails. In the last term of this sum, the factor $\frac{m!}{2m}$ is the number of ways to put a circular arrangement on a set of m elements (cf. Lemma 5.17 (iv)).

The above formula simplifies to the formula in the Corollary. □

Remark: One may compare this Corollary to the corresponding result for $\overline{M}_{1,n}$ which is Corollary 3.26 in [Pag08]. Contrary to the case of $\dim_{\mathbb{Q}} H^*_{\mathbb{Q}}(\overline{M}_{1,n})$, the value $\dim_{\mathbb{Q}} H^*_{\mathbb{Q}}(\overline{R}_{1,n})$, which is part of our formula for $\dim_{\mathbb{Q}} H^*_{CR}(\overline{R}_{1,n})$, is not known for large n.

5.4.3 Age grading

Notation: If L is a line bundle on a variety or orbifold/stack X, especially for X a support of one of our 1-sectors, and we have the group μ_n with a fixed generator α acting on L respecting the fibres, then α has to act on all fibres as multiplication by the same power α^k for some $0 \leq k < n$. We denote L with this given group action by (α^k, L). Recall that we have identified the automorphism groups on our basic 1-sectors with μ_n for $n \in \{2, 4\}$ and fixed generators -1 resp. i (cf. Summary 5.8 (i)). We often use $\underline{\mathbb{C}}$ to denote the trivial line bundle. Also recall the definition of the bundles $\mathbb{L}_i$ on $\overline{M}_{g,n}$ and the classes $\psi_i = c_1(\mathbb{L}_i)$ from Definition 1.41.

Sectors (X, g) with an automorphism g of order 2 always have as age half their codimension, by the formula of Summary 5.7 (ii). The sectors coming from basic sectors C_4 and C_4' are the only ones carrying automorphisms g of order > 2. Their normal bundles in $\overline{R}_{1,n}$ are as g representations isomorphic to the normal bundles of the corresponding sectors of $\overline{M}_{1,n}$, which are computed in [Pag08]:

Lemma 5.38 *We know that $C_4^{\underline{n}} \cong C_4 \times \overline{M}_{0,\underline{n}\cup\{\circ_1\}}$ and $C_4^{I_1,I_2} \cong C_4' \times \overline{M}_{0,I_1\cup\{\circ_1\}} \times \overline{M}_{0,I_2\cup\{\circ_2\}}$ and we denote by p_1 resp. p_2 the projections to the first resp. second $\overline{M}_{0,\ldots}$ in these products. Then we have the following isomorphism of line bundles as representations of the group μ_4 generated by the automorphism i.*

(i) $N_{C_4^{\underline{n}}}\overline{R}_{1,n}$ is isomorphic to $(i^2, \underline{\mathbb{C}}) \oplus (i^3, p_1^*(\mathbb{L}_{\circ_1}^\vee))$.

(ii) $N_{C_4^{I_1,I_2}}\overline{R}_{1,n}$ is isomorphic to $(i^2, \underline{\mathbb{C}}) \oplus (i^3, \underline{\mathbb{C}}) \oplus (i^3, p_1^*(\mathbb{L}_{\circ_1}^\vee)) \oplus (i^3, p_2^*(\mathbb{L}_{\circ_2}^\vee))$.

If one of the I_i's cardinality is 1, $N_{C_4^{I_1,I_2}}\overline{R}_{1,n}$ has the same description after cancelling the corresponding component $(i^3, p_i^*(\mathbb{L}_{\circ_i}^\vee)$ in the direct sum.

Proof: For $[\mathfrak{X}] \in \overline{R}_{1,n}$ let $\mathfrak{C}$ be the stable model of $\mathfrak{X}$. The forgetful morphism $\pi : (S, s_0) \to (B, b_0)$ between the local universal defomation spaces of $\mathfrak{X}$ and $\mathfrak{C}$ is an isomorphism, unless $[\mathfrak{X}]$ is in the boundary divisor D_0^r (cf. Summary 1.31). Let Z be $C_4^{\underline{n}}$ or $C_4^{I_1,I_2}$, then Z does not meet D_0^r. Let Z' be the 1-sector of $\overline{M}_{1,n}$ to which Z is mapped (isomorphically) by $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ (this sector is also denoted by $C_4^{\underline{n}}$ resp. $C_4^{I_1,I_2}$). Since the normal bundle of Z resp. Z' in the orbifold sense is locally the normal bundle of the preimage of Z resp. Z' on the local uniformising systems of the orbifolds, and since our local uniformising systems are the deformation spaces, we obtain: The normal bundles of Z and Z' are isomorphic, and by Summary 1.31, the automorphisms also act on them in the same way. The normal bundles of the sectors Z' are computed as μ_4 representations in Prop. 4.7. of [Pag08]. □

The previous Lemma together with the fact that all sectors (X, g) not based on C_4 or C_4' have age $a(X, g) = \frac{1}{2}\operatorname{codim}(X, \overline{R}_{1,n})$ implies:

Corollary 5.39 *The following table lists for all* 1*-sectors* (X, g) *of* $\overline{R}_{1,n}$ *the codimension of* X *in* $\overline{R}_{1,n}$ *and the age* $a(X, g)$ *of the sector. The definition of each* X *involves a partition* $I_1, ..., I_m$ *of* $\underline{n}$*. Define the number* $\mu \le m$ *as* $\mu := |\{I_i \mid i \in \underline{m},\ |I_i| = 1\}|$*. (In case* $\underline{n} = 1$ *also set* $\mu = 1$ *for the sectors with* $\underline{n}$ *in the "exponent".)*

X	g	$\operatorname{codim}(X, \overline{R}_{1,n})$	$a(X, g)$
$\overline{R}_{1,n}$	1	0	0
$\overline{A}_1^{\underline{n}}$	-1	$1-\mu$	$\frac{1}{2}(1-\mu)$
$\overline{A}_{2,a}^{\{I_1,I_2\}}$	-1	$3-\mu$	$\frac{1}{2}(3-\mu)$
$\overline{A}_{2,b}^{\{I_1,I_2\}}$	-1	$3-\mu$	$\frac{1}{2}(3-\mu)$
$\overline{A}_3^{\{I_1,I_2\},I_3}$	-1	$5-\mu$	$\frac{1}{2}(5-\mu)$
$\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}}$	-1	$7-\mu$	$\frac{1}{2}(7-\mu)$
$C_4^{\underline{n}}$	i	$2-\mu$	$\frac{5}{4}-\frac{3}{4}\mu$
$C_4^{\underline{n}}$	$-i$	$2-\mu$	$\frac{3}{4}-\frac{1}{4}\mu$
$C_4^{\{I_1,I_2\}}$	i	$4-\mu$	$\frac{11}{4}-\frac{3}{4}\mu$
$C_4^{\{I_1,I_2\}}$	$-i$	$4-\mu$	$\frac{5}{4}-\frac{1}{4}\mu$
$B^r_{\langle I_1,\dots,I_m\rangle}$	ι_m	m	$\frac{m}{2}$

5.4.4 CR-cohomology of $\overline{R}_{1,n}$ as graded vector space

Like in [Pag08], we can encode the dimensions of the homogeneous components of the graded vector space $H^*_{CR}(\overline{R}_{1,n})$ for all $n \in \mathbb{N}$ in a compact way, by describing the *generating series of the Chen-Ruan Poincare polynomials*

$$P_1^{CR}(s,t) := \sum_{m\in\mathbb{Q},\, n\in\mathbb{Z}} \frac{1}{n!} \dim H^m_{CR}(\overline{R}_{1,n}) s^n t^m.$$

Obviously one can read out every value $\dim H^m_{CR}(\overline{R}_{1,n})$ from this series, and, if we view P_1^{CR} as a power series in s, then the coefficient of s^n is the Chen-Ruan Poincare polynomial of $\overline{R}_{1,n}$, divided by $n!$.

We set for $n, m \in \mathbb{N}_0$:

$$Q_0(n,m) := \dim H^{2m}(\overline{M}_{0,n+1}), \qquad Q_1(n,m) := \dim H^m(\overline{R}_{1,n}),$$

$$\widetilde{Q}_0(n,m) := \dim H^{2m}(\overline{M}_{0,n+2}/\mathbb{S}_2),$$

where $\mathbb{S}_2$ acts by permuting the indices $n+1$ and $n+2$, and

$$Q'_{1,\beta}(n,m) := \dim H^{(m+\beta)}_{CR}(\overline{R}_{1,n}), \quad \text{for } \beta \in \mathbb{Q}$$

If the right hand side in this definitions is not defined (i.e. for $n \leq 1$ for $Q_0(n,m)$, and for $n = 0$ in the other cases) we set the left hand side to be 0. The $Q_0(m,n)$ and $\widetilde{Q}_0(m,n)$ are known from [Kee92] resp. [Get98]. Define the power series:

$$P_0(s,t) := \sum_{n,m \in \mathbb{N}_0}^{\infty} \frac{Q_0(n,m)}{n!} s^n t^m \quad P_1(s,t) := \sum_{n,m \in \mathbb{N}_0}^{\infty} \frac{Q_1(n,m)}{n!} s^n t^m,$$

$$\widetilde{P}_0(s,t) := \sum_{n,m \in \mathbb{N}_0}^{\infty} \frac{\widetilde{Q}_0(n,m)}{n!} s^n t^m, \quad P'_{1,\beta}(s,t) := \sum_{n,m \in \mathbb{N}_0}^{\infty} \frac{Q'_{1,\beta}(n,m)}{n!} s^n t^m$$

Note that $H^m(\overline{M}_{0,n+1}) = H^m(\overline{M}_{0,n+2}/\mathbb{S}_2) = 0$ for m odd, so P_0 and $\widetilde{P}_0$ do not miss "interesting information". The rational numbers m for which $H^m_{CR}(\overline{R}_{1,n}) \neq 0$ all have *fractional part* $\langle m \rangle := m - \lfloor m \rfloor \in \{0, \frac{1}{2}\}$ (cf. Corollary 5.39). Thus we can decompose the Chen-Ruan Poincare series of $\overline{R}_{1,n}$ as

$$P_1^{CR}(s,t) = P'_{1,0}(s,t) + t^{\frac{1}{2}} P'_{1,\frac{1}{2}}(s,t),$$

such that $P'_{1,0}$ and $P'_{1,\frac{1}{2}}$ are power series with integer exponents. We want to make it more easy to compare our following Proposition to Thm. 4.13. of [Pag08]. In order to do this we define $H^*_{CR,\alpha}(\overline{R}_{1,n})$ as the subspace of the graded space $H^*_{CR}(\overline{R}_{1,n})$ coming from those *twisted* 1-sectors of $\overline{R}_{1,n}$ whose age a has fractional part $\langle a \rangle = \alpha$. Then for all $n \in \mathbb{N}$, $m \in \mathbb{Q}$,

$$\sum_{\alpha \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \dim H^m_{CR,\alpha}(\overline{R}_{1,n}) = h^m_{CR}(\overline{R}_{1,n}) - h^m(\overline{R}_{1,n}),$$

and we further decompose

$$P_1^{CR}(s,t) = P_1(s,t) + P^{CR}_{1,0}(s,t^2) + tP^{CR}_{1,\frac{1}{2}}(s,t^2) + t^{\frac{1}{2}} P^{CR}_{1,\frac{1}{4}}(s,t^2) + t^{\frac{3}{2}} P^{CR}_{1,\frac{3}{4}}(s,t^2)$$

where

$$P^{CR}_{1,\alpha}(s,t) := \sum_{n,m \in \mathbb{N}_0} \dim H^{2(m+\alpha)}_{CR,\alpha}(\overline{R}_{1,n}) s^n t^m.$$

$$\Big(P'_{1,0}(s,t) = P_1(s,t) + tP^{CR}_{1,0}(s,t^2) + P^{CR}_{1,\frac{1}{2}}(s,t^2), \qquad P'_{1,\frac{1}{2}} = P^{CR}_{1,\frac{1}{4}}(s,t^2) + tP^{CR}_{1,\frac{3}{4}}(s,t^2) \Big)$$

Our $P^{CR}_{1,\alpha}$ for $\overline{R}_{1,n}$ correspond roughly to what is called $P^{CR}_{1,\alpha}$ for $\overline{M}_{1,n}$ in Thm. 4.13. of [Pag08].

Theorem 5.40 *The $P^{CR}_{1,\alpha}$, belonging to the 4 possible values of α, can be expressed in terms of P_0 and $\widetilde{P}_0$ as follows*

$$\begin{aligned}
P^{CR}_{1,0} &= 2(t+t^2)sP_0 + \frac{3}{2}(t^2+t^3)sP_0^2 + \frac{1}{2}(t^2+t^3)s^3P_0 + \frac{1}{2}(t^3+t^4)sP_0^3 \\
&\quad + (t+t^2)P_0 + \frac{1}{2}t(2\widetilde{P}_0^2 + (\frac{\partial}{\partial s}P_0)^2 - (\frac{\partial}{\partial s}P_0)\widetilde{P}_0) + \sum_{2\leq\nu\in\mathbb{N}} \frac{1}{(2\nu)!}t^\nu(\frac{\partial}{\partial s}P_0)^{2\nu} + C_{1,0} \\
P^{CR}_{1,\frac{1}{4}} &= tP_0 + \frac{1}{2}t^2P_0^2 + C_{1,\frac{1}{4}} \\
P^{CR}_{1,\frac{1}{2}} &= (1+t)P_0 + (t+t^2)P_0^2 + \frac{3}{2}(t+t^2)s^2P_0 + \frac{1}{2}(t^2+t^3)P_0^3 \\
&\quad + \frac{3}{4}(t^2+t^3)s^2P_0^2 + \frac{1}{8}(t^3+t^4)P_0^4 + C_{1,\frac{1}{2}} \\
P^{CR}_{1,\frac{3}{4}} &= P_0 + \frac{1}{2}tP_0^2 + C_{1,\frac{3}{4}}
\end{aligned}$$

The terms $C_{1,\alpha}$ correspond to some twisted sectors (the basic sectors without any attached rational tails) that appear in $\overline{R}_{1,n}$ only for $n \leq 4$. So they can be ignored if one wants to read out cohomology dimensions for larger n. They are:

$$C_{1,0} = (3s^3+s)(1+t), \quad C_{1,\frac{1}{4}} = s^2t, \quad C_{1,\frac{1}{2}} = 2s^2(1+t) + 3s^4(1+t) + 2s, \quad C_{1,\frac{3}{4}} = s^2$$

Proof: If $P(s,t)$ is some power series in two variables t, s, we denote by $P[n,m]$ the coefficient of the monomial s^nt^m in $P(s,t)$ multiplied by $n!$. With this notation, for P_0 as above and $r \in \mathbb{N}$:

$$(P_0^r)[n,m] = \sum_{\substack{n_1+\ldots+n_r=n\\ m_1+\ldots+m_r=m}} \frac{\prod_{i=1}^r Q_0(n_i,m_i)}{\prod_{i=1}^r n_i!} n! = \sum_{\substack{n_1+\ldots+n_r=n\\ n_i\geq 2}} h^{2m}(\prod_{i=1}^r \overline{M}_{0,n_i+1})\binom{n}{n_1,\ldots,n_r}$$

$$\Rightarrow (\frac{1}{r!}P_0^r)[n,m] = \sum_{\substack{\{I_1,\ldots,I_r\},\,|I_i|\geq 2\\ I_1\uplus\ldots\uplus I_r=[n]}} h^{2m}(\prod_{i=1}^r \overline{M}_{0,I_i\uplus\{\circ_i\}})$$

Also we obtain for $r,l \in \mathbb{N}$:

$$(\frac{1}{r!l!}s^lP_0^r)[n,m] = \frac{n!}{(n-l)!l!}(\frac{1}{r!}P_0^r)[n-l,m] = \sum_{\substack{\{I_1,\ldots,I_r\},J\\ I_1\uplus\ldots\uplus I_r=[n]\smallsetminus J\\ |I_i|\geq 2,|J|=l}} h^{2m}(\prod_{i=1}^r \overline{M}_{0,I_i\uplus\{\circ_i\}})$$

If we multiply by $(1+t)$ we get, using the Künneth formula,

$$((1+t)\frac{1}{r!}P_0^r)[n,m] = \sum_{\substack{\{I_1,\ldots,I_r\},\,|I_i|\geq 2\\ I_1\uplus\ldots\uplus I_r=[n]}} h^{2m}(\mathbb{P}^1 \times \prod_{i=1}^r \overline{M}_{0,I_i\uplus\{\circ_i\}})$$

and an analogous expression for $((1+t)\frac{1}{r!l!}s^lP_0^r)[n,m]$.

Now we start computing $P^{CR}_{1,\frac{1}{2}}$. The other series can be determined analogously. We use the decomposition of $I_1(\overline{R}_{1,n})$ into sectors given in Theorem 5.32 and the table of Corollary

5.39, containing the age for each twisted sector. Also we need Corollary 5.36 and its proof. We work through the decomposition in Theorem 5.32 looking for components (X, g) such that the age $a := a(X, g)$ has fractional part $\langle a \rangle = \frac{1}{2}$. The first contribution comes from the sectors $(\overline{A}_1^{[n]}, -1)$ for $n \geq 2$, with age $a = \frac{1}{2}$. The $2m$-th cohomology dimension of these sectors is

$$h^{2m}(\overline{A}_1^{[n]}) = h^{2m}(\mathbb{P} \times \overline{M}_{0,n+1}) = ((1+t)P_0)[n, m]$$

The sectors of the form $(\overline{A}_1^{[n]}, -1)$ contribute $h^{2m}(\overline{A}_1^{[n]})$ to the number

$$\dim H^{2(m+a)}_{CR,\frac{1}{2}}(\overline{R}_{1,n}) = P^{CR}_{1,\frac{1}{2}}[n, m + (a - \frac{1}{2})].$$

In this case $a - \frac{1}{2} = 0$, so this means that the sectors $(\overline{A}_1^{[n]}, -1)$ for $n \geq 2$, contribute $(1+t)P_0$ to the series $P^{CR}_{1,\frac{1}{2}}$, otherwise we would have had to shift by multiplying with $t^{(a-\frac{1}{2})}$.

The next contribution comes from the two single sectors $(\overline{A}_{2,x}, -1) = (\overline{A}_{2,x}^{\{1\}\{2\}}, -1) \subset I_1(\overline{R}_{1,2})$, $x \in \{a, b\}$, with age $\frac{1}{2}$. The contribution is $2s^2(1+t)$, since $\overline{A}_2 \cong \mathbb{P}$, and belongs to $C_{1,\frac{1}{2}}$. Another term of $C_{1,\frac{1}{2}}$ comes from the sectors (C_4, i) and (C_4, i^2), with age $\frac{1}{2}$. It is $2s$, since C_4 is a point in $\overline{R}_{1,1}$.

The sectors of the form $(\overline{A}_{2,x}^{I_1,I_2}, -1)$ with $|I_i| \geq 2$, $x \in \{a, b\}$, have age $\frac{3}{2}$. They contribute $2t(1+t)\frac{1}{2!}P_0^2$. This is since $\overline{A}_{2,x}^{I_1,I_2} \cong \mathbb{P}^1 \times \overline{M}_{0,I_1 \uplus \{\circ_1\}} \times \overline{M}_{0,I_2 \uplus \{\circ_2\}}$, the coefficient 2 stems from the two possible choices of x, and we have to shift by $t = t^{(\frac{3}{2}-\frac{1}{2})}$, for age reasons.

Sectors of the form $(\overline{A}_{3,x}^{\{j\},\{k\},I_3})$, where $j \neq k \in [n]$ and $|I_3| \geq 2$, $x = a, b, c$, have age $\frac{3}{2}$ and contribute $3t(1+t)\frac{1}{2!}s^2 P_0$.

The other sectors contributing to $P^{CR}_{1,\frac{1}{2}}$ are those of the forms $(\overline{A}_{3,x}^{I_1,I_2,I_3}, -1)$, $(\overline{A}_{4,x}, -1)$, $(\overline{A}_{4,x}^{\{j\},\{k\},I_3,I_4}, -1)$ and $(\overline{A}_{4,x}^{I_1,I_2,I_3,I_4}, -1)$, and their contributions can be determined in the same way.

Some of the contributions to other series $P^{CR}_{1,\alpha}$ are of a somewhat different type as those encountered before. We will compute some of them as examples:

The sectors of the form $(C_4^{\{j\},I_2}, i)$, $j \in [n]$, $|I_2| \geq 2$ have age 2. Since $C_4^{\{j\},I_2} \cong \overline{M}_{0,|I_2|+1}$ these sectors contribute $t^2 s P_0$ to $P^{CR}_{1,0}$.

The sectors of the form $(B^r_{\langle I_1,\ldots,I_\nu \rangle}, \iota_\nu)$ for an even $\nu \geq 4$, and $|I_i| \geq 1$ have age $\frac{\nu}{2}$. We have $B^r_{\langle I_1,\ldots,I_\nu \rangle} \cong \prod_{i=1}^{\nu} \overline{M}_{0,|I_i|+2}$. Now we use that

$$(\frac{\partial}{\partial s}P_0)[n, m] = h^{2m}(\overline{M}_{0,n+2}),$$

to be able to describe the contribution of all sectors $(B^r_{\langle I_1,\ldots,I_\nu \rangle}, \iota_\nu)$ for a fixed ν as $\frac{1}{\nu!}t^{\frac{\nu}{2}}(\frac{\partial}{\partial s}P_0)^\nu$.

Using the formula for $h^m(B^r_{\langle I_1,I_2 \rangle}) = k_{|I_1|,|I_2|}(m)$ from Lemma 5.24 (iv), we get that the sectors of the form $B^r_{\langle I_1,I_2 \rangle}$ contribute $\frac{1}{2}t\left(2\widetilde{P}_0^2 + (\frac{\partial}{\partial s}P_0)^2 - (\frac{\partial}{\partial s}P_0)\widetilde{P}_0\right)$ to $P^{CR}_{1,0}$. □

5.5 Multiplicative structure of $H^*_{CR}(\overline{R}_{1,n})$

As one can see from Definition 5.5 (iv), to compute the product $*$ on $H^*_{CR}(\overline{R}_{1,n})$, we have to determine the second inertia stack $I_2(\overline{R}_{1,n})$, and we have to compute pullbacks and pushforwards along the forgetful morphisms $p_1, p_2, p_3 : I_2(\overline{R}_{1,n}) \to I_1(\overline{R}_{1,n})$.

In our case the support X of a 2-sector (X, g, h) is usually the set-theoretic intersection $X = X_1 \cap X_2$ of the supports of the 1-sectors (X_1, g) and (X_2, h). Therefore we will try to determine all set theoretic intersections of supports of 1-sectors. Then we will calculate the necessary pullbacks and pushforwards. These are the things the next few subsections will be concerned with. Several times the following notation will be used.

Notation 5.41 Let X be a 1-sector of $\overline{R}_{1,n}$, let $f : X \hookrightarrow \overline{R}_{1,n}$ be the inclusion of the subvariety X in $\overline{R}_{1,n}$.

(i) Suppose X is of the form $X = \overline{Z}^{(I_1,...,I_k)}$, $k \in \underline{4}$, i.e. X obtained from a basic sector $\overline{Z} \subseteq \overline{R}_{1,k}$ by attaching rational tails (cf. Lemma&Def. 5.13). Then we have that f is the restriction of the gluing morphisms $\zeta_{(I_1,...,I_k))}$ to

$$X \cong \overline{Z} \times \overline{M}_{0,I_1 \cup \{\circ_1\}} \times ... \times \overline{M}_{0,I_k \cup \{\circ_k\}}.$$

We denote by $\eta_{\overline{Z}} : X \to \overline{Z}$ the projection to the first factor, by $\eta_i : X \to \overline{M}_{0,I_i \cup \{\circ_i\}}$ the projection to the $i+1$-st factor.

(ii) Otherwise we have $X = B^r_P$ for some circular partition $P = \langle I_1, ..., I_m \rangle$ of $\underline{n}$ with m even. Then we write

$$\overline{M}_{\Gamma(P)} := \overline{M}_{0,I_1 \cup \{\circ_1,\bullet_2\}} \times \overline{M}_{0,I_2\{\circ_2,\bullet_3\}} \times ... \times \overline{M}_{0,I_m \cup \{\circ_m,\bullet_1\}}$$

and let η_i be the projection to the i-th factor. As seen in section 5.3.2, in this case $f = f_{B^r_P}$ can be identified with the gluing morphism $\zeta_{B^r_P}$ if $|P| > 2$ and in case $|P| = 2$ can be identified with the embedding $\overline{M}_{\Gamma(P)}/\mathbb{S}_2 \to \overline{R}_{1,n}$, through which $\zeta_{B^r_P}$ factors in this case.

5.5.1 Intersections of supports of 1-sectors and the second inertia stack

Lemma 5.42 *If $X \neq X'$ are the supports of sectors of $I_1(\overline{R}_{1,n})$, with $X \neq \overline{R}_{1,n} \neq X'$, then the set-theoretic intersection $X \cap X'$ is either empty, or specified in the following list:*

(1) $C_4^{\underline{n}} \cap \overline{A}_1^{\underline{n}} = C_4^{\underline{n}}$

(2) $C_4^{\{I_1,I_2\}} \cap \overline{A}_{2,a}^{\{I_1,I_2\}} = C_4^{\{I_1,I_2\}}$

(3) $\overline{A}_{2,a}^{\{I_1,I_2\}} \cap \overline{A}_{2,b}^{\{I_1,I_2\}} = \overline{A}_{2,a}^{\{I_1,I_2\}} \cap B^r_{\langle I_1,I_2\rangle} = \overline{A}_{2,b}^{\{I_1,I_2\}} \cap B^r_{\langle I_1,I_2\rangle} = E_2^{r,\{I_1,I_2\}}$

(4) $\overline{A}_3^{\{I_1,I_2\},I_3} \cap \overline{A}_3^{\{I_1,I_3\},I_2} = \overline{A}_3^{\{I_1,I_2\},I_3} \cap B^r_{\langle I_2 \cup I_3, I_1\rangle} = \overline{A}_3^{\{I_1,I_3\},I_2} \cap B^r_{\langle I_2 \cup I_3, I_1\rangle} = E_3^{r,\{I_2,I_3\},I_1}$

(5) $\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}} \cap \overline{A}_4^{\{\{I_1,I_3\},\{I_2,I_4\}\}} = \overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}} \cap B^r_{\langle I_2 \cup I_3, I_1 \cup I_4\rangle}$
$= \overline{A}_4^{\{\{I_1,I_3\},\{I_2,I_4\}\}} \cap B^r_{\langle I_2 \cup I_3, I_1 \cup I_4\rangle} = E_4^{r,\{\{I_2,I_3\},\{I_1,I_4\}\}}$

(6) *Intersections of the form* $B^r_{\langle I_1,...,I_m\rangle} \cap B^r_{\langle I'_1,...,I'_{m'}\rangle}$ *can be non-empty and are determined in Lemma 5.22.*

Proof: Recall the notation of Definitions 5.12 and 5.13. If $P = (I_1, ..., I_k)$ and $P' = (I'_1, ..., I'_{k'})$ for $k, k' \in \underline{4}$ are two ordered partitions of $\underline{n}$ and $\overline{Z}$, $\overline{Z}'$ are two basic 1-sectors of $\overline{R}_{1,k}$ resp. $\overline{R}_{1,k'}$, then $\overline{Z}^P, \overline{Z}'^{P'} \subset \overline{R}_{1,n}$ can only meet if $k = k'$ and $(I'_1, ..., I'_{k'})$ can be obtained from $(I_1, ..., I_k)$ by permuting the indices in $\underline{k}$. To meet without fulfilling this condition, WLOG $\overline{Z}^P$ would have to parametrise a curve with a rational tree carrying the marked points from at least two different sets I_{i_1} and I_{i_2}. But the curves in $\overline{Z}^P$ can not degenerate in this way: Otherwise by construction of $\overline{Z}^P$ there would have to be a curve in $\overline{Z}$ with a rational tail carrying the points i_1 and i_2. But we know that the basic sectors $\overline{Z}$ do not parametrise curves with rational tails. [36]

By Theorem 5.32 we know that each of X and X' is either of the form $\overline{Z}^P$ for some basic sector $\overline{Z} \in \{C_4, C'_4, \overline{A}_1 = \overline{R}_{1,1}, \overline{A}_{2,a}, \overline{A}_{2,b}, \overline{A}_{3,a}, \overline{A}_{4,a}\}$ [37] or of the form B^r_P.

Since C_4^P and $(C'_4)^P$ do not parametrise curves with non-disconnecting nodes, they do not meet any $B^r_{P'}$. Together with the discussion above this shows that the only intersections involving C_4^P and $(C'_4)^P$ which could be non-empty are those in (1) and (2) and possibly $A_{2,b}^{\{I_1,I_2\}} \cap C_4^{\{I_1,I_2\}}$. The equation in (1) is clear since $C_4 \subset \overline{A}_1 = \overline{R}_{1,1}$. We know that $C'_4 \subset A_{2,a} \cup A_{2,b}$. To get (2) and show $A_{2,b}^{\{I_1,I_2\}} \cap C_4^{\{I_1,I_2\}} = \emptyset$ it thus suffices to prove $C'_4 \cap A_{2,b} = \emptyset$. Let $(\mathscr{C}_4, p_1, p_2, \mathcal{L})$ be a prym curve parametrised by the point C'_4, then the elliptic involution $i^2 = -1$ fixes the two marked points p_1, p_2 and two other points q_1, q_2. We can see using the Weierstrass representation from Theorem 3.8 of [Pag08], that i interchanges q_1 and q_2. Hence we must have $\mathcal{L} \cong \mathcal{O}_{\mathscr{C}_4}(p_1 - p_2)$, i.e. $[(\mathscr{C}_4, p_1, p_2, \mathcal{L})] \notin A_{2,b}$.

If $X = (\overline{A}_{k,x})^P$, $X' = (\overline{A}_{k,x'})^{P'}$ for any $k \in \{2, 3, 4\}$, and $x, x' \in \{a, b\}$ for $k = 2$, and $x = x' = a$ for $k \in \{3, 4\}$, then by the discussion above, the intersection $X \cap X'$ can be non-empty only if it appears in (3), (4) or (5). To see that these intersections are described correctly in (3) – (5) note that, by Lemma&Definition 5.13 (iii), we can always rewrite them in the form $(\overline{A}_{k,y})^P \cap (\overline{A}_{k,y'})^P$ where now $y \neq y'$ and $y, y' \in \{a, b, c\}$ for any k. But

$$(\overline{A}_{k,y})^P \cap (\overline{A}_{k,y'})^P = \zeta_P(F_P^{-1}(\overline{A}_{k,y} \cap \overline{A}_{k,y'})),$$

and the intersections $\overline{A}_{k,y} \cap \overline{A}_{k,y'}$ can be found in Lemma 5.31.

To come to the last case (except (6)), $X = (\overline{A}_{k,x})^P$ and $X' = B^r_{P'}$ can only meet inside $\zeta_P(F_P^{-1}(\overline{A}_{k,x} \smallsetminus A_{k,x}))$, since the rest of $(\overline{A}_{k,x})^P$ parametrises no curves with non-disconnecting nodes. But the boundary $\overline{A}_{k,x} \smallsetminus A_{k,x}$ is described in Lemma 5.31, and using this together with Lemma&Definition 5.13 (iii) we can check that all remaining equations in (3) – (5) are correct. In this way we also see that all other intersection of the form $(\overline{A}_{k,x})^P \cap B^r_{P'}$ are empty. □

It is easy to check that the following lemma is true:

[36] The sectors C_4, C''_4 are points parametrising smooth curves, and we know the boundary points of the remaining basic sectors by Lemma 5.31.

[37] The basic sectors $\overline{A}_{3,b}$, $\overline{A}_{3,c}$, $\overline{A}_{4,b}$, $\overline{A}_{4,c}$ are not needed by Lemma&Definition 5.13 (iii)

Lemma 5.43 *Let P_1, P_2 be circular partitions of $\underline{n}$, and P a coarsest common refinement of P_1 and P_2. Let $\widetilde{P}$ be the circular partition obtained from $P = \langle I_1, I_2, ..., I_m \rangle$ by replacing in $\langle I_1, I_2, ..., I_m \rangle$ each pair I_i, I_{i+1} such that $\{I_i, I_{i+1}\} \in \mathrm{CN}(P_1, P_2; P)$ by $I_i \cup I_{i+1}$ (cf. Definition 5.27). (As usual $m + 1 = 1$ here.)*

Then P is also a coarsest *common refinement of the pairs P_1, $\widetilde{P}$ and P_2, $\widetilde{P}$.*

Proposition 5.44 *(i) If $(Y; g, h)$ is a 2-sector of $\overline{R}_{1,n}$, for any $(x; g, h) \in (Y; g, h)$, let (X_1, g) and (X_2, h) be the 1-sectors parametrising (x, g) resp. (x, h). Then $Y \subset \overline{R}_{1,n}$ is one of the connected components of $X_1 \cap X_2$. Furthermore $(Y; g, h) \cong Y$ as orbifolds.*

(ii) Often we denote a 2-sector $(Y; g, h)$ instead by $(Y, (g, h, (gh)^{-1}))$. This is a trick (from [Pag08]) to reduce the bookkeeping effort: We allow two actions on the labels $(g, h, (gh)^{-1})$. $\mathbb{S}_2$ acts by sending $(g, h, (gh)^{-1})$ to (g^{-1}, h^{-1}, gh), and $\mathbb{S}_3$ acts by permuting the three entries. For each of the 12 labels $(g', h', (g'h')^{-1})$ obtained from $(g, h, (gh)^{-1})$ in this way, there exists a 2-sector $(Y, (g', h', (g'h')^{-1}))$ of $\overline{R}_{1,n}$ (where Y is the same subvariety of $\overline{R}_{1,n}$ as before). [38]

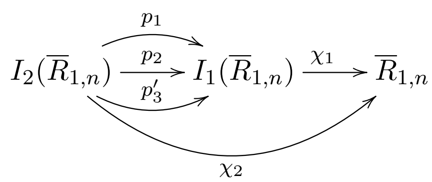
This also reduces the effort when dealing with the Chen-Ruan excess intersection bundles on the 2-sectors, since $E_{(Y,(g',h',(g'h')^{-1}))} \cong E_{(Y,(g,h,(gh)^{-1}))}$, for a label $(g', h', (g'h')^{-1})$ obtained from $(g, h, (gh)^{-1})$ by applying the $\mathbb{S}_3$ action. (This is not true for the $\mathbb{S}_2$ action).

(iii) The following table lists all 2-sectors $(Y, (g, h, (gh)^{-1}))$ of $I_2(\overline{R}_{1,n})$, up to the two operations on the labels allowed in (ii). We also list the corresponding 1-sectors (X_1, g), (X_2, h) and $(X_3, (gh)^{-1})$. For the rows of the table, listing 2-sectors supported on an $E_k^{r,\dots}$, note that by Remark 5.33, a general object $\mathfrak{X}$ of $E_k^{r,\dots}$ has one inessential automorphism ι_2 and two non-inessential automorphisms which act on the cotangent space at the first marked point of $\mathfrak{X}$ by -1, and which we call -1_a and -1_b here. Concerning which of -1_a and -1_b denotes which automorphism: Read this off from the listed (X_1, g), (X_2, h), $(X_3, (gh)^{-1})$. In the last row of the table let P, P_1, P_2 and $\widetilde{P}$ be as in Lemma 5.43.

[38] These are not 12 different 2-sectors, since transposition of the first two entries does not change the sector.

Support Y	$(g,h,(gh)^{-1})$	$\big((X_1,g),(X_2,h),(X_3,(gh)^{-1})\big)$
$\overline{R}_{1,n}$	$(1,1,1)$	$\big((\overline{R}_{1,n},1),(\overline{R}_{1,n},1),(\overline{R}_{1,n},1)\big)$
$\overline{A}_1^n$	$(1,-1,-1)$	$\big((\overline{R}_{1,n},1),(\overline{A}_1^n,-1),(\overline{A}_1^n,-1)\big)$
$\overline{A}_{2,x}^{\{I_1,I_2\}}$, $x\in\{a,b\}$	$(1,-1,-1)$	$\big((\overline{R}_{1,n},1),(\overline{A}_{2,x}^{\{I_1,I_2\}},-1),(\overline{A}_{2,x}^{\{I_1,I_2\}},-1)\big)$
$\overline{A}_3^{\{I_1,I_2\},I_3}$	$(1,-1,-1)$	$\big((\overline{R}_{1,n},1),(\overline{A}_3^{\{I_1,I_2\},I_3},-1),(\overline{A}_3^{\{I_1,I_2\},I_3},-1)\big)$
$\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}}$	$(1,-1,-1)$	$\big((\overline{R}_{1,n},1),(\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}},-1),(\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}},-1)\big)$
C_4^n	$(1,i,-i)$	$\big((\overline{R}_{1,n},1),(C_4^n,i),(C_4^n,-i)\big)$
C_4^n	$(i,i,-1)$	$\big((C_4^n,i),(C_4^n,i),(\overline{A}_1^n,-1)\big)$
$C_4^{I_1,I_2}$	$(1,i,-i)$	$\big((\overline{R}_{1,n},1),(C_4^{I_1,I_2},i),(C_4^{I_1,I_2},-i)\big)$
$C_4^{I_1,I_2}$	$(i,i,-1)$	$\big((C_4^{I_1,I_2},i),(C_4^{I_1,I_2},i),(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)\big)$
$E_2^{r,\{I_1,I_2\}}$	$(-1_a,-1_b,\iota_2)$	$\big((\overline{A}_{2,a}^{\{I_1,I_2\}},-1),(\overline{A}_{2,b}^{\{I_1,I_2\}},-1),(B^r_{\langle I_1,I_2\rangle},\iota_2)\big)$
$E_3^{r,\{I_2,I_3\},I_1}$	$(-1_a,-1_b,\iota_2)$	$\big((\overline{A}_3^{\{I_1,I_2\},I_3},-1),(\overline{A}_3^{\{I_1,I_3\},I_2},-1),(B^r_{\langle I_2\cup I_3,I_1\rangle},\iota_2)\big)$
$E_4^{r,\{\{I_2,I_3\},\{I_1,I_4\}\}}$	$(-1_a,-1_b,\iota_2)$	$\big((\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}},-1),(\overline{A}_4^{\{\{I_1,I_3\},\{I_2,I_4\}\}},-1),(B^r_{\langle I_2\cup I_3,I_1\cup I_4\rangle},\iota_2)\big)$
B_P^r	$(1,\iota_P,\iota_P)$	$\big((\overline{R}_{1,n},1),(B_P^r,\iota_P),(B_P^r,\iota_P)\big)$
B_P^r	$(\iota_{P_1},\iota_{P_2},\iota_{\widetilde{P}})$	$\big((B^r_{P_1},\iota_{P_1}),(B^r_{P_2},\iota_{P_2}),(B^r_{\widetilde{P}},\iota_{\widetilde{P}})\big)$

Proof: Let as in Definition 5.3, $p_1,p_2: I_2(\overline{R}_{1,n})\to I_1(\overline{R}_{1,n})$ be the forgetful morphisms corresponding on points $(x;g,h)$ of $I_2(\overline{R}_{1,n})$ to $(x;g,h)\mapsto(x;g)$, $(x;g,h)\mapsto(x;h)$, and let $p_3': I_2(\overline{R}_{1,n})\to I_1(\overline{R}_{1,n})$ be the forgetful morphism corresponding to $(x;g,h)\mapsto(x;(gh)^{-1})$. Let $\chi_2: I_2(\overline{R}_{1,n})\to\overline{R}_{1,n}$, $\chi_1: I_1(\overline{R}_{1,n})\to\overline{R}_{1,n}$ be the usual forgetful morphisms. Then the diagram



commutes. Furthermore by Summary 5.7 (iv) all the morphisms in it become closed embeddings of orbifolds when restricted to any sector of $I_2(\overline{R}_{1,n})$ resp. $I_1(\overline{R}_{1,n})$.

(i): If one looks at the definition of the structure of the orbifolds $I_1(\overline{R}_{1,n})$ and $I_2(\overline{R}_{1,n})$ locally around each point, it is clear that for every point $(x,g,h)\in(Y,g,h)$ the image $Y=\chi_2((Y,g,h))$ is locally around $\chi_2((x,g,h))=x$ equal to the intersection $X_1\cap X_2$. So Y is a connected component of $X_1\cap X_2$, and since χ_2 restricted to (Y,g,h) is a closed embedding, $(Y,g,h)\cong Y$.

(ii): Everything here should be clear but maybe the fact that for all labels $(g,h,(gh)^{-1})$ in the same orbit of the $\mathbb{S}_3$-action the CR-excess intersection bundles $E_{(Y;g,h)}$ are isomorphic. But this is easy to see by the definition of $E_{(Y;g,h)}$ (cf. Definition 5.5 (v)): The group G does not change under this action, and a permutation of $g,h,(gh)^{-1}$ corresponds for C and the G action on $H^1(C,\mathcal{O}_C)$ to a permutation of the marked points $0,1,\infty\in\mathbb{P}^1$ which clearly does not change the isomorphism class.

(iii): Which 2-sectors exist, follows from (i) together with Lemma 5.42. The third entry in the label, i.e. the corresponding 1-sector $(X_3,(gh)^{-1})$, is in most cases obvious. For the 2-sectors supported on a $E_k^{r,\dots}$, X_3 is determined by the information from Remark 5.33. For the last line of the table, let $\mathfrak{X}$ be a general object of B_P^r, $P = \langle I_1, ... I_m\rangle$. Then the inessential automorphisms ι_1 resp. ι_2 act on $\mathfrak{X}$ by multiplying the fibres of the prym sheaf over each non exceptional component X_i ($i \in \underline{m}$) by a number $a_{i,1} \in \{1, -1\}$ resp. $a_{i,2} \in \{1, -1\}$ (cf. Proof of Theorem 5.32). Now check that for two neighbouring $I_i \parallel I_{i'}$, $a_{i,1} = a_{i',1}$ and $a_{i,2} = a_{i',2}$ can never happen simultaneously since P is a *coarsest* common refinement of P_1 and P_2. Furthermore $a_{i,1} \neq a_{i',1}$ and $a_{i,2} \neq a_{i',2}$ happen simultaneously if and only if $\{I_i, I_{i'}\} \in \mathrm{CN}(P_1, P_2; P)$. Hence if $b_i \in \{1, -1\}$ are the numbers by which $(\iota_1\iota_2)^{-1} = \iota_1\iota_2$ acts, then $b_i = b_{i'}$ iff $\{I_i, I_{i'}\} \in \mathrm{CN}(P_1, P_2; P)$. So $(\iota_1\iota_2)^{-1}$ extends exactly to $B^r_{\tilde{P}}$ (cf. Proof of Theorem 5.32). □

Lemma 5.45 *(i) If $(Y, (g, h, (gh)^{-1}))$ is a 2-sector of $I_2(\overline{R}_{1,n})$ then the excess intersection bundle $E_{(Y,(g,h,(gh)^{-1}))}$ either has rank 0, or is listed in the following table. The table lists the 2-sectors up to permutation of the three entries of the label, as $E_{(Y,(g,h,(gh)^{-1}))}$ is invariant under such a permutation by Proposition 5.44 (ii). We identify $(Y, (g, h, (gh)^{-1}))$ with Y by the isomorphism of 5.44 (i) to be able to express the bundle $E_{(Y,(g,h,(gh)^{-1}))}$. Then:*

Support Y	*Label* $(g, h, (gh)^{-1})$	$E_{(Y,(g,h,(gh)^{-1}))}$
$C_4^{\underline{n}}$	$(i, i, -1)$	$\eta_1^*(\mathbb{L}_{\circ_1}^\vee)$
$C_4^{I_1,I_2}$	$(i, i, -1)$	$\underline{\mathbb{C}} \oplus \eta_1^*(\mathbb{L}_{\circ_1}^\vee) \oplus \eta_2^*(\mathbb{L}_{\circ_2}^\vee)$.

For the η_i, cf. Notation 5.41. For the $\mathbb{L}_{\dots}$ and $\psi_{\dots}$, cf. Def. 1.41.

(ii) The top Chern class of $E_{(Y,(g,h,(gh)^{-1}))}$ is either $1 = [Y]_Q$, or listed in the following table (again up to permutation of the three label-entries):

Support Y	*Label* $(g, h, (gh)^{-1})$	$c_{top}(E_{(Y,(g,h,(gh)^{-1}))})$
$C_4^{\underline{n}}$	$(i, i, -1)$	$-\eta_1^*(\psi_{\circ_1})$
$C_4^{I_1,I_2}$	$(i, i, -1)$	0

Proof: (i): Let $(Y, (g, h, (gh)^{-1}))$ be a 2-sector, and let (X_1, g), (X_2, h), $(X_3, (gh)^{-1})$ be corresponding 1-sectors. Recall from Summary 5.7 the two formulas

$$\mathrm{rk}(E_{(Y,(g,h,(gh)^{-1}))}) = a((X_1, g)) + a((X_2, h)) + a((X_3, (gh)^{-1})) - \mathrm{codim}(Y, \overline{R}_{1,n}) \quad (*)$$

$$\mathrm{codim}(X, \overline{R}_{1,n}) = a((X, \varphi)) + a((X, \varphi^{-1})) \quad (**)$$

where in the second formula (X, φ) is any 1-sector. These two formulas imply that if the label $(g, h, (gh)^{-1})$ contains an entry 1, then $\mathrm{rk}(E_{(Y,(g,h,(gh)^{-1}))}) = 0$, for then the other two automorphisms in the label are inverse to each other, and are supported exactly on Y. This already proves (i) for the most of the 2-sectors. The remaining 2-sectors for which we have to show that the rank of the excess intersection bundle is 0 are in the last 4 rows of the table of Prop. 5.44 (iii). In the labels of these sectors only automorphisms of order 2 appear. Thus inverting all entries does not change the label. From this we conclude by Prop. 5.44 (ii), that all 2-sectors we obtain by applying the $\mathbb{S}_2 \times \mathbb{S}_3$ action of 5.44 (ii) to the label have isomorphic excess intersection bundles. Now take for example the 2-sector

$(E_2^{r,\{I_1,I_2\}}, (-1_a, -1_b, \iota_2))$. The 1-sectors relevant in formula $(*)$ are then $(\overline{A}_{2,a}^{\{I_1,I_2\}}, -1_a)$, $(\overline{A}_{2,a}^{\{I_1,I_2\}}, -1_b)$ and $(B^r_{\langle I_1,I_2\rangle}, \iota_2)$. Using $(*)$ and the table of Corollary 5.39 we get:

$$\mathrm{rk}(E_{(E_2^{r,\{I_1,I_2\}},(-1_a,-1_b,\iota_2))}) = \frac{3}{2} + \frac{3}{2} + \frac{2}{2} - 4 = 0$$

The 2-sectors in the second and third last row of the table of Prop. 5.44 (iii) are shown to have excess intersection bundles of rank 0 in the same way. In the last row there appear sectors of the form $(Y, (g, h, (gh)^{-1})) = (B^r_P, (\iota_{P_1}, \iota_{P_2}, \iota_{\widetilde{P}}))$. In this case $(*)$ reads:

$$\begin{aligned}\mathrm{rk}(E_{(B^r_P,(\iota_{P_1},\iota_{P_2},\iota_{\widetilde{P}}))}) &= a((B^r_{P_1}, \iota_{P_1})) + a((B^r_{P_2}, \iota_{P_2})) + a((B^r_{\widetilde{P}}, \iota_{\widetilde{P}})) - \mathrm{codim}(B^r_{P_1}, \overline{R}_{1,n}) \\ &= \frac{1}{2}|P_1| + \frac{1}{2}|P_2| + \frac{1}{2}|\widetilde{P}| - |P|\end{aligned}$$

where we used Corollary 5.39 for the second line. With $\mathrm{CN} := d(P)\,\mathrm{CN}(P_1, P_2; P)$ and equation (5.2) from section 5.3.3 we can continue the equation by

$$= \frac{1}{2}|P_1| + \frac{1}{2}|P_2| + \frac{1}{2}(|P_1| + |P_2| - 2\,\mathrm{CN}) - (|P_1| + |P_2| - \mathrm{CN}) = 0.$$

It remains to compute the excess intersection bundle on the 2-sectors supported on $C_4^{\underline{n}}$ and $C_4^{I_1,I_2}$. By Prop. 5.44 (ii) it suffices to consider the sectors $(C_4^{\underline{n}}; i, i)$, $(C_4^{\underline{n}}; -i, -i)$, $(C_4^{\{I_1,I_2\}}; i, i)$ and $(C_4^{\{I_1,I_2\}}; -i, -i)$. For $(C_4^{\underline{n}}; -i, -i)$ and $(C_4^{\{I_1,I_2\}}; -i, -i)$, we see that the rank is 0 by $(*)$ and Corollary 5.39.

By definition (cf. Def. 5.5 (v))

$$E_{(Y,g,h)} = \left(H^1(C_{g,h}, \mathcal{O}_{C_{g,h}}) \otimes_{\mathbb{C}} N_Y \overline{R}_{1,n}\right)^{\mathrm{Grp}(g,h)}, \tag{†}$$

where $\mathrm{Grp}(g,h)$ is the group generated by the automorphisms g and h, and $C_{g,h}$ the curve C from Def. 5.5 (v). We have $\mathrm{Grp}(i,i) = \mathrm{Grp}(-i,-i) = \mu_4$. From Proposition 6.12. of [Pag08] we know that $H^1(C_{i,i}, \mathcal{O}_{C_{i,i}}) = (i, \mathbb{C})$ as a representation of μ_4 (Cf. Lemma 5.38, and the paragraph before, for the notation $(i, \mathbb{C})$) . Lemma 5.38 gives us the normal bundles $N_{C_4^{\underline{n}}}\overline{R}_{1,n}$ and $N_{C_4^{\{I_1,I_2\}}}\overline{R}_{1,n}$ as representations of μ_4. Plugging this into (†) yields:

$$E_{(C_4^{\underline{n}},i,i)} = \left[(i^3, \underline{\mathbb{C}}) \oplus (i^4, \eta_1^*(\mathbb{L}^\vee_{\circ_1}))\right]^{\mu_4}$$

$$E_{(C_4^{\{I_1,I_2\}},i,i)} = \left[(i^3, \underline{\mathbb{C}}) \oplus (i^4, \underline{\mathbb{C}}) \oplus (i^4, \eta_1^*(\mathbb{L}^\vee_{\circ_1})) \oplus (i^4, \eta_2^*(\mathbb{L}^\vee_{\circ_2}))\right]^{\mu_4}$$

Which gives the results in the table.

(ii): If $rk(E) = 0$ we have $c_{top}(E) = 1$. Since $E_{(C_4^{\{I_1,I_2\}},i,i)}$ contains a trivial sub-bundle, we have $c_{top}(E_{(C_4^{\{I_1,I_2\}},i,i)}) = 0$. □

5.5.2 The classes of supports of 2-sectors, expressed in the usual generators of $H^*_{BCl}(\overline{R}_{1,n})$

The class of a support of a 1-sector is a priori a class in $H^*(\overline{R}_{1,n})$. But in this section we will show that all these classes actually lie in $H^*_{BCl}(\overline{R}_{1,n})$ (cf. Def. 1.40). We are going to express each of these classes explicitly as a polynomial in the usual generators of the $\mathbb{Q}$-algebra $H^*_{BCl}(\overline{R}_{1,n})$, i.e. the boundary divisors and the simple banana cycles. We will calculate these expression following section 3.d of [Pag08].

First we will express the supports of the basic sectors as polynomials in the boundary divisor classes of $\overline{R}_{1,n}$, for $n \in \underline{4}$. Recall that $d_0'' = d_0^r \in H^*(\overline{R}_{1,n})$. We will use only d_0'' in our formulas.

Lemma 5.46 *(i) For the supports in $\overline{R}_{1,1}$:*

$$[C_4]_Q = \frac{1}{2}d_0'', \qquad [\overline{A}_1]_Q = [\overline{R}_{1,1}]_Q = 1.$$

(ii) For the supports in $\overline{R}_{1,2}$:

$$[C_4']_Q = \frac{1}{2}d_0''d_{\{12\}}, \qquad [\overline{A}_{2,a}]_Q = \frac{1}{4}d_0'' + d_{\{12\}}, \qquad [\overline{A}_{2,b}]_Q = \frac{1}{2}d_0'' + 2d_{\{12\}}.$$

(iii) For the supports in $\overline{R}_{1,3}$:

$$[\overline{A}_{3,a}]_Q = [\overline{A}_{3,b}]_Q = [\overline{A}_{3,c}]_Q = \frac{2}{3}\sum_{\{i,j\}\subset\underline{3}} d_3d_{\{ij\}} + \frac{1}{4}\sum_{\{i,j\}\subset\underline{3}} d_0''d_{\{ij\}} + \frac{1}{4}d_0''d_3.$$

(iv) For the supports in $\overline{R}_{1,4}$:

$$[\overline{A}_{4,a}]_Q = [\overline{A}_{4,b}]_Q = [\overline{A}_{4,c}]_Q =$$

$$\frac{1}{6}\sum_{\{ij\}\subset\{ijk\}\subset\underline{4}} d_4d_{\{ijk\}}d_{\{ij\}} + \frac{1}{12}\sum_{\{ij\}\subset\underline{4}} d_0''d_4d_{\{ij\}} + \frac{1}{12}\sum_{\{ij\}\subset\{ijk\}\subset\underline{4}} d_0''d_{\{ijk\}}d_{\{ij\}}.$$

Proof: Here I follow the proof of Theorem 3.33. of [Pag08].

(i): C_4 is a point parametrising a curve with 4 automorphisms, while D_0'' is a point with 2 automorphisms.

(ii): Also C_4' and $D_0'' \cap D_{\{12\}}$ are points. The latter is a transversal intersection.

Since the two classes on the right hand side form a basis of $A^1(\overline{R}_{1,2}) = H^2(\overline{R}_{1,2})$, it is clear that we can write

$$[\overline{A}_{2,b}]_Q = ad_0'' + bd_{\{12\}}, \qquad \text{for some } a, b \in \mathbb{Q} \qquad (*)$$

Let $\pi : \overline{R}_{1,2} \to \overline{R}_{1,1}$ be the morphism forgetting the last marked point. Since π is $2:1$ on $\overline{A}_{2,b}$ and $1:1$ on $D_{\{12\}}$, while the dimension of D_0'' drops by 1 under π, we obtain

$$\pi_*[\overline{A}_{2,b}]_Q = 2[\overline{R}_{1,1}]_Q = a0 + b[\overline{R}_{1,1}]_Q \quad \Rightarrow \quad b = 2$$

If we intersect any class $[\overline{A}_{n,x}]_Q$ ($n \in \underline{4}, x \in \{a,b,c\}$), with any boundary divisor of $\overline{R}_{1,n}$, except d_0'' or d_0^r, the result is 0 by Lemma 5.31. Intersect $(*)$ with $d_{\{12\}}$. We know $\delta^2_{\{12\}} = \frac{1}{24}$ (Example 1.43) from which we conclude $d^2_{\{12\}} = \frac{1}{8}$ using the projection formula. With this we get:

$$0 = ad_{\{12\}}d_0'' + 2d^2_{\{12\}} = a\frac{1}{2} + 2(-\frac{1}{8}) \quad \Rightarrow \quad a = \frac{1}{2}$$

The formula for $[\overline{A}_{2,a}]_Q$ can be obtained exactly the same way.

(iii): One can obtain these formulas by expressing $[\overline{A}_{3,x}]_Q$ as a polynomial in a basis of $A^2(\overline{R}_{1,3})$, and calculating pushforwards by morphisms forgetting one of the 3 marked points, and/or by intersecting with boundary divisors. To shorten the proof, we make the more special ansatz

$$[\overline{A}_{3,x}]_Q = a_1 v_1 + a_2 v_2 + a_3 v_3, \quad \text{where}$$

$$v_1 := d_0'' \left(\sum_{\{i,j\}\subset \underline{3}} d_{\{ij\}} \right), \quad v_2 := d_0'' d_3, \quad v_3 := d_3 \left(\sum_{\{i,j\}\subset \underline{3}} d_{\{ij\}} \right).$$

We intersect with all boundary divisor classes, and since the intersection pairing on $A^*(\overline{R}_{1,3})$ is perfect, this will not only determine the coefficients, but, if it does not impose contradictory conditions on the coefficients, will also ensure that our ansatz was correct. [39]

If we intersect both sides of our ansatz with all the boundary divisors of $\overline{R}_{1,3}$ we obtain the following equations of intersection numbers which are independent of $x \in \{a, b, c\}$:

$$\begin{aligned} d_0''[\overline{A}_{3,x}]_Q &= 1 = a_1 0 + a_2 0 + a_3 \frac{3}{2}, &\Rightarrow\quad a_3 = \frac{2}{3} \\ \forall\ \{i,j\} \subset \underline{3}:\ d_{\{ij\}}[\overline{A}_{3,x}]_Q &= 0 = a_1(-\frac{1}{2}) + a_2 \frac{1}{2} + a_3 0, &\Rightarrow\quad a_1 = a_2 \\ d_3[\overline{A}_{3,x}]_Q &= 0 = a_1 \frac{3}{2} + a_2(-\frac{1}{2}) + a_3(-\frac{3}{8}), &\Rightarrow\quad a_1 = \frac{1}{4} \end{aligned}$$

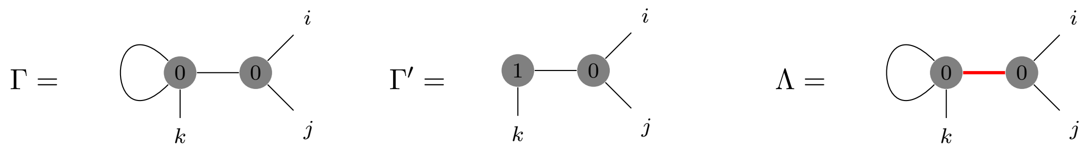
These equations obtained for the coefficients do not contradict each other and determine the coefficients.

The intersection numbers in these equations are determined as follows: The first equation in every row is obtained using the description of the boundary of the $\overline{A}_{3,x}$ from Lemma 5.31 (ii), which gives us $D_0'' \cap [\overline{A}_{3,x}]_Q = E_3^{\prime\prime,\cdots}$, $D_{\{ij\}} \cap [\overline{A}_{3,x}]_Q = D_3 \cap [\overline{A}_{3,x}]_Q = \emptyset$. By Summary 1.34 (v) we can locally calculate proper intersections of Q-classes on the deformation spaces. Now (compare to Remark 5.33) $E_3^{\prime\prime,\cdots}$ is a point, parametrising a prym curve $\mathfrak{X}$ with two (not blown up) nodes e_1 and e_2, and the automorphism -1 interchanges e_1 and e_2. We get by Lemma 1.32 (ii) that on the deformation space S of $\mathfrak{X}$ which coincides with the deformation space of the stable model $\mathfrak{C}$, since $\mathfrak{X}$ has no exceptional components, the fixed point set Fix(-1) which is the preimage of $\overline{A}_{3,x}$, can be written as Fix$(-1) = \text{span}_S(\vec{y}_{e_1} + \vec{y}_{e_2})$. The preimage of D_0'' on S is $\{y_{e_1} = 0\} \cup \{y_{e_2} = 0\}$. So we get $[D_0'']_Q \cdot [\overline{A}_{3,x}]_Q = 2[E_3^{\prime\prime,\cdots}]_Q$, and due to automorphisms $[E_3^{\prime\prime,\cdots}]_Q = \frac{1}{2}[p]$, where $[p]$ is the class of a general point.

The other intersection numbers in the first line are $d_0'' v_1 = d_0'' v_2 = 0$ since $(d_0'')^2 = 0$ by Lemma 4.8, and $d_0'' v_3 = \frac{3}{2}$, since for all three $\{ij\} \subset \underline{3}$, $D_0'' \cap D_3 \cap D_{\{i,j\}}$ is a point, hence the intersection is proper, and this point parametrises a prym curve with two automorphisms. The intersection numbers $d_{\{i,j\}} v_2$ and $d_3 v_1$ in the next lines are computed analogously.

[39] We could, as suggested by Nicola Pagani, also justify the ansatz beforehand, by showing that the classes $[\overline{A}_{3,x}]$ are invariant under the action of $\mathbb{S}_3$ permuting the indices of marked point, and by noting that Getzler's results on the equivariant cohomology of $H^*(\overline{M}_{1,m})$ ([Get98]) implie that v_1, v_2, v_3 is a basis of $A^2(\overline{R}_{1,3})^{\mathbb{S}_3}$. The same is true for the ansatz used in (iv).

The remaining $d_{\{ij\}}v_1$, $d_{\{ij\}}v_3$, d_3v_2, d_3v_3 are excess intersections which we compute using the projection formula and section 1.7: For example $d_{\{ij\}}v_1 = d_0'' d_{\{ij\}}^2 = d_0''\tau_3^*\delta_{\{ij\}}^2$, where $\tau_3 : \overline{R}_{1,3} \to \overline{M}_{1,3}$ is the forgetful morphism. By the projection formula this is the same number as $(\tau_3)_* d_0''\delta_{\{ij\}}^2 = \delta_0\delta_{\{ij\}}^2$. Now $\delta_0\delta_{\{ij\}}$ is a transversal intersection, and we use the excess intersection formula (1.6) of section 1.7 to compute $(\delta_0\delta_{\{ij\}})\delta_{\{ij\}}$. Here the relevant graphs are:



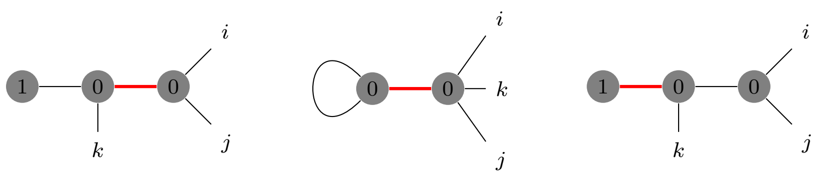
$G_{\Gamma\Gamma'}$ has only one element (Λ, c, c') and the edge in Λ which we have drawn thick and red is the only element of CE. With

$$\xi_\Lambda : \overline{M}_{0,\{k,\bullet_1,\circ_1,\bullet_2\}} \times \overline{M}_{0,\{i,j,\circ_2\}} \to \overline{M}_{1,3},$$

the gluing morphism, and p a point in $\overline{M}_{0,\{k,\bullet_1,\circ_1,\bullet_2\}} \cong \overline{M}_{0,4}$, q a point in $\overline{M}_{1,3}$ we have thus:

$$(\delta_0\delta_{\{ij\}})\delta_{\{ij\}} = \frac{1}{2}(\xi_\Lambda)_*(-\psi_{\bullet_2}\otimes 1 - 1\otimes\psi_{\circ_2}) = \frac{1}{2}(\xi_\Lambda)_*(-\psi_{\bullet_2}\otimes 1) = \frac{1}{2}(\xi_\Lambda)_*(-[p]\otimes 1) = -\frac{1}{2}[q]$$

To determine $d_{\{ij\}}v_3$, d_3v_2 resp. d_3v_3 [40] in the same way, we have to compute $(\delta_3\delta_{\{ij\}})\delta_{\{ij\}}$, $(\delta_0\delta_3)\delta_3$ resp. $(\delta_3\delta_{\{ij\}})\delta_3$. The corresponding graphs Λ, with elements of CE marked red are in this order



The excess intersection formula yields:

$$(\delta_3\delta_{\{ij\}})\delta_{\{ij\}} = 0, \quad (\delta_0\delta_3)\delta_3 = -\frac{1}{2}, \quad (\delta_3\delta_{\{ij\}})\delta_3 = -\frac{1}{24}.$$

(iv) Here we again make a special ansatz:

$$[\overline{A}_{4,x}]_Q = b_1w_1 + b_2w_2 + b_3w_3 + b_4w_4, \quad \text{where}$$

$$w_1 := \sum_{\{ij\}\subset\{ijk\}\subset\underline{4}} d_4 d_{\{ijk\}} d_{\{ij\}}, \quad w_2 := \sum_{\{ij\}\subset\underline{4}} d_0'' d_4 d_{\{ij\}},$$

[40] For the last case: $d_3v_3 = \sum_{\{i,j\}\subset\underline{3}} d_3^2 d_{\{ij\}}$ and $d_3^2 d_{\{ij\}} = \tau_3^*(\delta_3^2)d_{\{ij\}} = \delta_3^2(\tau_3)_* d_{\{ij\}} = 3\delta_3^2\delta_{\{ij\}}$.

$$w_3 := \sum_{\{ijk\}\subset \underline{4}} d_0'' d_4 d_{\{ijk\}}, \quad w_4 := \sum_{\{ij\}\subset\{ijk\}\subset \underline{4}} d_0'' d_{\{ijk\}} d_{\{ij\}}$$

Intersecting with all boundary divisor classes gives:

$$\begin{aligned} d_0''[\overline{A}_{4,x}]_Q = 1 &= b_1\frac{12}{2} + b_2 0 + b_3 0 + b_4 0, &\Rightarrow\quad b_1 = \frac{1}{6} \\ \forall\ \{i,j\}\subset \underline{4}:\ d_{\{ij\}}[\overline{A}_{4,x}]_Q = 0 &= b_1 0 + b_2 0 + b_3\frac{3}{2} + b_4 0, &\Rightarrow\quad b_3 = 0 \\ \forall\ \{i,j,k\}\subset \underline{4}:\ d_{\{ijk\}}[\overline{A}_{4,x}]_Q = 0 &= b_1 0 + b_2\frac{3}{2} + b_3(-\frac{1}{2}) + b_4(-\frac{2}{3}), &\Rightarrow\quad b_2 = b_4 \\ d_4[\overline{A}_{4,x}]_Q = 0 &= b_1(-\frac{12}{8}) + b_2(-\frac{6}{2}) + b_3 0 + b_4\frac{12}{2}, &\Rightarrow\quad b_2 = \frac{1}{12} \end{aligned}$$

To give another example of how such an intersection number is calculated: The number $d_{\{12\}}w_2 = 0$, appearing in the second line, is obtained by observing that the only two terms in the sum w_2 that meet $d_{\{12\}}$ as sets, are $d_0''d_4d_{\{12\}}$ and $d_0''d_4d_{\{34\}}$. With the first term, $d_{\{12\}}$ has excess intersection $-\frac{1}{2}$, calculated as above. With the second term the intersection is transversal and contributes $\frac{1}{2}$. □

Lemma 5.47 *Let $\overline{Z}\subseteq\overline{R}_{1,k}$, $k\in\underline{4}$, be a basic 1-sector, let $(I_1,...,I_k)$ be a partition of $\underline{n}$, and let $\overline{Z}^{(I_1,...,I_k)}$ be defined as in Definition 5.13.*

(i) In Lemma 5.46 we expressed $[\overline{Z}]_Q \in H^*(\overline{R}_{1,k})$ as a polynomial in the classes of the form d_0'' and d_J for $J\subseteq\underline{k}$. Let $\widehat{Z}\in H^*(\overline{R}_{1,n})$ be the class one obtains by replacing in this formula each d_0'' by the class of the same name in $H^*(\overline{R}_{1,n})$ and by replacing each d_J by $d_{\widehat{J}}\in H^*(\overline{R}_{1,n})$, where $\widehat{J}:=\bigcup_{i\in J} I_i$. Then the class $[\overline{Z}^{(I_1,...,I_k)}]_Q\in H^*(\overline{R}_{1,n})$ can be expressed as:

$$[\overline{Z}^{(I_1,...,I_k)}]_Q = \widehat{Z}\cdot d_{I_1}\cdot ...\cdot d_{I_k}.$$

(ii)For P a circular partition of $\underline{n}$ with $|P|\geq 2$, by definition $[B_P^r]_Q = b_P^r$, which already is one of the generators of $H^*_{BCl}(\overline{R}_{1,n})$.

(iii) In particular the classes of all supports of 1-sectors of $\overline{R}_{1,n}$ lie inside $H^*_{BCl}(\overline{R}_{1,n})$.

Proof: (i): With Definition 5.13 it is easy to show, using the projection formula and the fact that $\zeta_{(I_1,...,I_k)}$ is a closed embedding, that

$$\begin{aligned}[\overline{Z}^{(I_1,...,I_k)}]_Q &= (\zeta_{(I_1,...,I_k)})_*([\overline{Z}]_Q\otimes 1\otimes ...\otimes 1) \\ &= (\zeta_{(I_1,...I_k)})_*(\zeta^*_{(I_1,...,I_k)}(\widehat{Z})) = \widehat{Z}\cdot d_{I_1}\cdot ...\cdot d_{I_k}.\end{aligned}$$

□

5.5.3 Pullbacks from $H^*(\overline{R}_{1,n})$ to the 1-sectors.

Let (X,g) be a 1-sector of $I_1(\overline{R}_{1,n})$, let $f: X\hookrightarrow\overline{R}_{1,n}$ be the inclusion of the subvariety X in $\overline{R}_{1,n}$. In this section we study the pull-back homomorphism

$$f^*: H^*(\overline{R}_{1,n})\to H^*(X).$$

This is a part of our attempt to determine the structure of $H^*_{CR}(\overline{R}_{1,n})$ as an $H^*(\overline{R}_{1,n})$-algebra.

For this recall Notation 5.41. Furthermore in the case $X = B^r_P$, we use the identification of $H^*(B^r_P)$ with $H^*(\overline{M}_{\Gamma(P)})$ resp. with $H^*(\overline{M}_{\Gamma(P)})^{\mathbb{S}_2} \subset H^*(\overline{M}_{\Gamma(P)})$ introduced in Corollary 5.26, to express the pullback f^*.

First we determine f^* on the subalgebra $H^*_{BCl}(\overline{R}_{1,n})$ (cf. Def. 1.40). As we know from Remark 4.6 (i), the $\mathbb{Q}$-algebra $H^*_{BCl}(\overline{R}_{1,n})$ is generated by the boundary divisors together with the classes of the simple banana cycles. Hence the pullbacks $f^* : H^*_{BCl}(\overline{R}_{1,n}) \to H^*(X)$ are for all 1-sectors X determined by the following two Propositions.

Lemma 5.48 *In case (X, α) is a sector $(\overline{Z}^{I_1,\ldots,I_k}, \alpha)$, $k \in \underline{4}$, obtained from a basic sector $\overline{Z}$ by attaching rational tails (cf. Definition 5.13), we get the following results, analogous to those in section 7.a. of [Pag08].*

(i) If β is the class of any banana cycle, then $f^*(\beta) = 0$

(ii) Since $d''_0 = d^r_0$, always $f^*(d''_0) = f^*(d^r_0)$. We have $f^*(d''_0) = 0$ if $\overline{Z}$ is C_4 or C'_4. For the other possible X's, the pullback $f^*(d''_0)$ is given in the following table. There $\eta_{\overline{Z}} : X \to \overline{Z}$ is the projection as defined in Notation 5.41 (i), and p is the cycle class of a point in the cohomology of the basic sector $H^*(\overline{Z})$:

Sector X	$f^*(d''_0)$ $(=f^*(d^r_0))$
$\overline{A}^{n}_1$	$\frac{1}{2}\eta^*_{\overline{Z}}(p)$
$\overline{A}^{I_1,I_2}_{2,a}$	$\frac{1}{2}\eta^*_{\overline{Z}}(p)$
$\overline{A}^{I_1,I_2}_{2,b}$	$\eta^*_{\overline{Z}}(p)$
$\overline{A}^{\{I_1,I_2\},I_3}_3$	$\eta^*_{\overline{Z}}(p)$
$\overline{A}^{\{\{I_1,I_2\},\{I_3,I_4\}\}}_4$	$\eta^*_{\overline{Z}}(p)$

(iii) For $J \subseteq \underline{n}$ we have that $f^*(d_J) = 0$ if J is not contained in any of the $I_1, \ldots, I_k$. If $J \subsetneq I_i$ then, with η_i as in Notation 5.41,

$$f^*(d_J) = \eta^*_i([J]),$$

where $[J]$ denotes a divisor in $\overline{M}_{0,I_i \cup \{\circ_i\}}$ (cf. Notation 1.47). If $J = I_i$ then

$$f^*(d_J) = -\frac{1}{4} f^*(d''_0) - \eta^*_i(\psi_{\circ_i}).$$

(For the definition of $\psi_{\circ_i}$ cf. Def. 1.41 (iii))

(iv) From these results together we can conclude that $f^* : H^*(\overline{R}_{1,n}) \to H^*(X)$ is surjective for all X of the form $\overline{Z}^{(I_1,\ldots,I_k)}$.

Proof: (i): By Lemma 5.42 we see that (i) could only possibly be wrong for $\overline{Z} = \overline{A}_{k,x}$ for $k \in \{2, 3, 4\}$, and $\beta = B^r_{J_1,J_2}$ for a certain partition (J_1, J_2) of $\underline{n}$ depending on X. If

$\pi : \overline{R}_{1,n} \to \overline{R}_{1,k}$ is a morphism forgetting all marked points except one from each set I_k, there is a commutating diagram

$$\begin{array}{ccccc} \overline{A}_{k,x}^{(I_1,\dots,I_k)} & \xrightarrow{f} & \overline{R}_{1,n} & \longleftarrow & B^r_{\langle J_1,J_2\rangle} \\ \downarrow \eta_{\overline{Z}} & & \downarrow \pi & & \downarrow \\ \overline{A}_{k,x} & \xrightarrow{g} & \overline{R}_{1,k} & \longleftarrow & B^r_{\langle K_1,K_2\rangle} \end{array}$$

where (K_1, K_2) is the partition of $\underline{k}$ obtained from (J_1, J_2) by forgetting the mentioned points. Then

$$\eta^*_{\overline{Z}} g^* b^r_{\langle K_1,K_2\rangle} = f^*\pi^* b^r_{\langle K_1,K_2\rangle} = f^* b^r_{\langle J_1,J_2\rangle},$$

where the second equality is obtained by checking with Lemma 5.42, that $B^r_{\langle J_1,J_2\rangle}$ is the only component of $\pi^{-1}(B^r_{\langle K_1,K_2\rangle})$ meeting $\overline{A}_{k,x}^{(I_1,\dots,I_k)}$ as a set. But $\eta^*_{\overline{Z}} g^* b^r_{\langle K_1,K_2\rangle} = 0$, since $\dim \overline{A}_{k,x} = 1 < 2 = \operatorname{codim} B^r_{\langle K_1,K_2\rangle}$ (cf. the table of Corollary 5.39).

(ii): That $f^*(d''_0) = 0$ if $\overline{Z}$ is C_4 or C'_4, can be shown using a similar argument, by forgetting all but 1 resp. 2 marked points, and then arguing by dimension of the intersected classes on $\overline{R}_{1,1}$ resp. $\overline{R}_{1,2}$. In case $\overline{Z} \in \{\overline{A}^n_1, \overline{A}_{k,x} \mid k \in \{1,2,3\}, x \in \{a,b,c\}\}$, again define morphisms of the same names as in the proof of (i), forgetting all but k marked points, and obtain that $f^*d''_0 = \eta^*_{\overline{Z}} g^* d''_0$ where on the right hand side d''_0 is the divisor class d''_0 on $\overline{R}_{1,k}$. We compute $g^*d''_0$ by determining how the preimages of $\overline{Z}$ and D''_0 meet on the deformation spaces of their finitely many common objects, like in the computation of $d''_0[\overline{A}_{3,x}]_Q$ in the proof of Lemma 5.46 (iii).

(iii): That $f^*(d_J) = \eta^*_i([J])$ for $J \subsetneq I_i$ is clear. Now consider the commutative diagram

$$\begin{array}{ccc} \overline{R}_{1,\{\bullet_1,\dots,\bullet_k\}} \times \overline{M}_{0,I_1\cup\{\circ_1\}} \times \dots \times \overline{M}_{0,I_k\cup\{\circ_k\}} & \longrightarrow & \overline{R}_{1,n} \\ \downarrow \varphi & & \downarrow \tau \\ \overline{M}_{1,\{\bullet_1,\dots,\bullet_k\}} \times \overline{M}_{0,I_1\cup\{\circ_1\}} \times \dots \times \overline{M}_{0,I_k\cup\{\circ_k\}} & \longrightarrow & \overline{M}_{1,n} \end{array}$$

in the notation of Definition 5.12, the horizontal arrows are $\zeta_{(I_1,\dots,I_1)}$ respectively $\xi_{(I_1,\dots,I_n)}$. Now use formula 1.5 from section 1.7 together with Summary 1.42 to compute that

$$\begin{aligned} \xi^*_{(I_1,\dots,I_n)}(d_{I_i}) &= -(\psi_{\bullet_1} \otimes 1 \otimes \dots \otimes 1) - (1 \otimes \dots \otimes 1 \otimes \psi_{\circ_i} \otimes 1 \otimes \dots \otimes 1) \\ &= -(\frac{1}{12}\delta_0 \otimes 1 \otimes \dots \otimes 1) - (1 \otimes \dots \otimes 1 \otimes \psi_{\circ_i} \otimes 1 \otimes \dots \otimes 1). \end{aligned}$$

Pulling this back via φ gives:

$$-(\frac{1}{4}d''_0 \otimes 1 \otimes \dots \otimes 1) - (1 \otimes \dots \otimes 1 \otimes \psi_{\circ_i} \otimes 1 \otimes \dots \otimes 1) = -\frac{1}{4}f^*(d''_0) - \eta^*_i(\psi_{\circ_i}.)$$

□

Lemma 5.49 *Here we look at the remaining* 1*-sectors* (X, g)*, whose supports are of the form* $X = B^r_P$ *for some circular partition* $P = \langle I_1, \dots, I_m\rangle$ *of* $\underline{n}$*. For them:*

(i) $f^*(d''_0) = f^*(d^r_0) = 0$.

(ii) For $J \subseteq \underline{n}$ we have that $f^*(d_J) = 0$ if I is not contained in any of the $I_1, ..., I_k$. If $J \subseteq I_i$ then

$$f^*(d_J) = \eta_i^*([J]),$$

where η_i is as defined in Notation 5.41 (ii), and $[J]$ denotes a divisor in $\overline{M}_{0,I_i \cup \{\circ_i, \bullet_{i+1}\}}$ (cf. Notation 1.47).

(iii) It remains to determine $f^*(\beta)$ if β is a simple banana cycle class. Then β is either of the form $\beta = [B''_{P_2}]_Q = b''_{P_2}$ or $\beta = [B^r_{P_2}]_Q = b^r_{P_2}$ for some circular partition P_2 of $\underline{n}$. In the first case $f^*(\beta) = 0$. For the second case note that $f = f_{B^r_P}$ and that $f^*_{B^r_P}(\beta) = f^*_{B^r_P}(b^r_{P_2})$ is computed in Lemma 5.28 (v).

Proof: From Lemma 4.8, we get (i) as well as $f^*(b''_{P_2}) = 0$ from (iii).

(ii): This can be easily seen using formula 1.5 from section 1.7, since the graphs Γ of B^r_P and Γ' of D_J allow only one common specialisation (Λ, c, c'), and the corresponding CE is empty. In case of $m = 2$ this argument only computes the pullback $\zeta^*_{B^r_P}(d_J)$, but we obtain with the projection formula

$$f^*(d_J) = \frac{1}{4}(z_{B^r_P})_* \zeta^*_{B^r_P}(d_J) = \frac{1}{4} \cdot 4 \cdot \eta_i^*([J]).$$

□

Lemma 5.50 *For any twisted 1-sector X of $\overline{R}_{1,n}$, i.e. $X \neq \overline{R}_{1,n}$, the pullback f^* maps the whole odd part of $H^*(\overline{R}_{1,n})$ to 0. I.e. $f^*(H^{2*+1}(\overline{R}_{1,n})) = \{0\} \subset H^*(X)$.*

Proof: Using the description of the twisted sectors in Corollary 5.36 as products of spaces whose cohomology is well known, and applying the Künneth formula, one gets that $H^{2*+1}(X) = 0$ for any twisted 1-sector X. □

Summing up the results of this section, for any 1-sector X we know the pullback $f^* : H^*(\overline{R}_{1,n}) \to H^*(X)$ on the subspaces $H^{2*+1}(\overline{R}_{1,n})$ and $H^*_{BCl}(\overline{R}_{1,n}) \subseteq H^{2*}(\overline{R}_{1,n})$ of $H^*(\overline{R}_{1,n})$. It seems possible, but it is not known, that $H^*_{BCl}(\overline{R}_{1,n}) = H^{2*}(\overline{R}_{1,n})$ for all n, in which case the results of this section would determine f^* completely. For $\overline{M}_{1,n}$, $H^*_{BCl}(\overline{M}_{1,n}) = H^{2*}(\overline{M}_{1,n})$ is an old claim of Getzler (cf. Claim 5.1) for which no proof has appeared so far.

5.5.4 The $H^*(\overline{R}_{1,n})$-module $H^*_{CR}(\overline{R}_{1,n})$

Definition 5.51 (i) For R a ring and S a set, denote by $R[(S)]$ the polynomial ring over R in all the elements of S and by $R^{(S)}$ the free R-algebra generated by the elements of S.

(ii) If M is an R-module which is generated by a finite subset $\mathcal{G} \subseteq M$, let $q : R^{(\mathcal{G})} \to M$ be the surjective homomorphism of R-modules defined by sending each element of $\mathcal{G}$ to the element of the same name in M. We call q the *$\mathcal{G}$-evaluation*, and we call the R-module $\ker q$ the *module of relations* (with respect to the set of generators $\mathcal{G}$).

(iii) Similarly for M an R-algebra generated by $\mathcal{G}$ we again call the surjective homomorphism of R-algebras $q : R[(\mathcal{G})] \to M$, sending $\mathcal{G}$ to $\mathcal{G}$, the $\mathcal{G}$-evaluation, and we call $\ker q$ the *ideal of relations*.

(iv) We say that a set of generators $\mathcal{G}$ of an R-module or algebra M is *minimal* if no proper subset of $\mathcal{G}$ generates M.

We regard $H^*_{CR}(\overline{R}_{1,n})$ as an $H^*(\overline{R}_{1,n})$-module as follows: Let $*$ be the product on $H^*_{CR}(\overline{R}_{1,n})$. For each 1-sector (X,g), $H^*((X,g)) \subset H^*_{CR}(\overline{R}_{1,n})$ is, as $\mathbb{Q}$ vector space identified with $H^*(X)$ by the definition of H^*_{CR}, Summary 5.7 (iv), and Remark 5.15. For any 1-sector (X,g), and $f : X \to \overline{R}_{1,n}$ the inclusion as in the previous section, we know by Summary 5.7 (iv) that f can also be identified with the restricted forgetful morphism $(\chi_2)_{|(X,g,1)}$: $(X,g,1) \to \overline{R}_{1,n}$, form the 2-sector $(X,g,1)$. Then for $\alpha \in H^*((\overline{R}_{1,n},1)) \subset H^*_{CR}(\overline{R}_{1,n})$ and $\beta \in H^*((X,g))$, we have by definition of the CR-product $*$ (Def. 5.5 (iv)):

$$\alpha * \beta = f^*(\alpha) \cdot \beta \in H^*((X,g)), \tag{5.4}$$

where on the right hand side $\cdot$ is the usual (cup) product on $H^*(X)$, which space we identified with $H^*((X,g))$ as above for this purpose. [41] If also $\beta \in H^*((\overline{R}_{1,n},1))$, then $\alpha*\beta = \alpha\cdot\beta \in H^*(\overline{R}_{1,n})$, so on the untwisted sector $*$ restricts to $\cdot$. So by (5.4), $H^*(\overline{R}_{1,n}) = H^*((\overline{R}_{1,n},1))$ is a subring of $H^*_{CR}(\overline{R}_{1,n})$, and $H^*_{CR}(\overline{R}_{1,n})$ as well as every $H^*((X,g)) \subset H^*_{CR}(\overline{R}_{1,n})$ is a $H^*(\overline{R}_{1,n})$-module via $*$.

Of course, this makes all the $H^*((X,g))$ and $H^*_{CR}(\overline{R}_{1,n})$ also into modules over the subring $H^*_{BCl}(\overline{R}_{1,n}) \subseteq H^*(\overline{R}_{1,n})$ (cf. Definition 1.40).

Notation 5.52 For a 1-sector (X,g) we will in the following often consider its *fundamental class* $[(X,g)] \in H^*_{CR}(\overline{R}_{1,n})$. By this we mean the fundamental class of the orbifold (X,g) in $H^*((X,g)) \subset H^*_{CR}(\overline{R}_{1,n})$. Under the identification of $H^*((X,g))$ with $H^*(X)$ it coincides with the Q-class $[X]_Q \in H^*(X)$.

Lemma 5.53 *Let (X,g) be a 1-sector of $\overline{R}_{1,n}$ which is of the form $\overline{Z}^{(I_1,\dots,I_k)}$ as in Lemma 5.48, then:*

(i) As $H^*_{BCl}(\overline{R}_{1,n})$-module, the submodule $H^*((X,g)) \subset H^*_{CR}(\overline{R}_{1,n})$ is generated by the fundamental class $[(X,g)]$.

(ii) Let $\mathrm{FB}((X,g))$ *be the free $H^*_{BCl}(\overline{R}_{1,n})$-module in the generator $[(X,g)]$, and denote the scalar multiplication in this module by the same symbol $*$ as in $H^*((X,g))$. Let q :* $\mathrm{FB}((X,g)) \to H^*((X,g))$ *be the evaluation sending $[(X,g)] \in$* $\mathrm{FB}((X,g))$ *to $[(X,g)] \in H^*((X,g))$* [42]*. The module of relations* $\mathrm{RB}((X,g)) := \ker q$ *is generated by the elements in the following list:*

(1) *For all simple banana cycles $\beta \in H^*_{BCl}(\overline{R}_{1,n})$: $\beta * [(X,g)]$.*

[41] Note that $\cdot$ is not part of the structure of $H^*_{CR}(\overline{R}_{1,n})$ so it is defined on $H^*((X,g)) \subset H^*_{CR}(\overline{R}_{1,n})$ only by making this identification.

[42] We have of course $\mathrm{FB}((X,g)) \cong H^*_{BCl}((X,g))$ and q corresponds (using the notation of 5.48) to the pullback homomorphism f^* and $\mathrm{RB}((X,g)) \cong \ker f^*$.

(2) *If* $\overline{Z} \in \{C_4, C_4'\}\}$*:* $d_0'' * [(X,g)]$.

(3) *For each* $J \subseteq \underline{n}$ *which is not contained in any of the* $I_1, ..., I_k$*:* $d_J * [(X,g)]$

(4) *For each set* I_i *of the partition* $(I_1, ..., I_k)$ *and* a_i, b_i *the two smallest numbers in* I_i*:*

$$(d_{I_i} + \frac{1}{4}d_0'' + \sum_{\{a_i,b_i\}\subseteq J \subsetneq I_i} d_J) * [(X,g)]$$

(5) *From the Keel relations on* $H^*(\overline{M}_{0,I_i\cup\{\circ_i\}})$*: For all* $i \in \underline{k}$ *and* $q,r,s \in I_i$ *with* $|\{q,r,s\}| = 3$*:*

$$(\sum_{\substack{J\subset I_i,\\ q,r\in J,\ s\notin J}} d_J - \sum_{\substack{J\subset I_i,\\ r,s\in J,\ q\notin J}} d_J) * [(X,g)]$$

Proof: (i): Follows from equation (5.4) above and Lemma 5.48 (iv).

(ii): It is clear that in the free module FB$((X,g))$ we have RB$((X,g)) = (\ker f^*) * [(X,g)]$, where $f^* : H^*_{BCl}(\overline{R}_{1,n}) \to H^*((X,g))$ is as in Lemma 5.48. (1)-(5) list elements of the form $\gamma * [(X,g)]$ with $\gamma \in H^*_{BCl}(\overline{R}_{1,n})$, and proving (ii) is equivalent to showing that the collection of the γ's generates the ideal $\ker f^* \subset H^*_{BCl}(\overline{R}_{1,n})$. Denote by G the ideal in $H^*_{BCl}(\overline{R}_{1,n})$ generated by these γ.

That the γ's in (1), (2), (3) are contained in $\ker f^*$ follows from (i)-(iii) of Lemma 5.48. For the γ's from (4) the same follows from the equation

$$f^*(d_J) = -\frac{1}{4}f^*(d_0'') - \eta_i^*(\psi_{\circ_i}),$$

of Lemma 5.48, if one applies Summary 1.42 and $f^*(d_J) = \eta_i^*([J])$ of 5.48 (iii).

In general an $\alpha \in H^*_{BCl}(\overline{R}_{1,n})$ can be expressed as a $\mathbb{Q}$-polynomial in the classes d_0'', d_J for $J \subseteq \underline{n}$ and simple banana cycle classes (cf. Remark 4.6 (i)). By now we know that $\ker f^*$ as well as G contain the γ's listed in (1)-(4). So if we want to check whether $\alpha \in G \Leftrightarrow \alpha \in \ker f^*$, by (1) and (3) we can WLOG assume that α is a polynomial in only the class d_0'' and classes d_J with J contained in some I_i. Furthermore by adding to our α suitable multiples of γ's from (4) we may WLOG assume that only classes d_J with $J \subsetneq I_i$ for some $i \in \underline{k}$ appear in the polynomial α. Let S be the set containing as elements d_0'' and those d_J we did not WLOG exclude yet. We continue our proof for the case that $\overline{Z} \notin \{C_4, C_4'\}$. Otherwise (2) would furthermore allow us to assume that α is a polynomial only in classes d_J. This would make the rest of the proof only easier than in the cases we will treat.

Let $H(S) \subset H^*_{BCl}(\overline{R}_{1,n})$ be the sub-$\mathbb{Q}$-algebra of $H^*_{BCl}(\overline{R}_{1,n})$ generated by the classes in S. It is clear that $G \cap H(S)$ is generated by the γ's in (5). With this notation our WLOG assumptions above tell us that it suffices to show that $\ker f^* \cap H(S)$ is also generated by these elements.

Since $\overline{Z} \notin \{C_4, C_4'\}$ we have $\overline{Z} \cong \mathbb{P}^1$ by the proof of Corollary 5.36. Then $H^*((X,g)) \cong H^*(\overline{Z}) \otimes H^*(\overline{M}_{0,I_1\cup\{\circ_1\}}) \otimes ... \otimes H^*(\overline{M}_{0,I_k\cup\{\circ_k\}})$ is generated as $\mathbb{Q}$-algebra [43] by the class

[43] $H^*((X,g))$ as a subspace of $H^*_{CR}(\overline{R}_{1,n})$ has of course no $\mathbb{Q}$-algebra structure in general, but we identified it with $H^*(X)$ which has.

$\eta^*_{\overline{Z}}(p)$ for p the class of a point in $\overline{Z}$, together with the classes of the form $\eta^*_i([J])$ where $i \in \underline{k}$ and $J \subsetneqq I_i$, with $|J| \geq 2$. Call the set of these generators S', and let $\pi : \mathbb{Q}[(S')] \to H^*((X,g))$ be the evaluation. The ideal of relations $\ker \pi$ is generated by the relations one obtains by pulling back the Keel-relations via η_i from each $\overline{M}_{0,I_i \cup \{\circ_i\}}$, i.e. (cf. Summary 1.48 (iii)) $\ker \pi$ is generated by the following collection of elements:

(a) For $i \in \underline{k}$ and all $q, r, s \in I_i$ with $|\{q,r,s\}| = 3$:

$$\sum_{\substack{J \subset I_i,\\ q,r \in J,\ s \notin J}} \eta^*_i([J]) - \sum_{\substack{J \subset I_i,\\ r,s \in J,\ q \notin J}} \eta^*_i([J]).$$

(b) For all $i \in \underline{k}$ and all $J, J' \subseteq I_i$ such that neither $J \subseteq J'$ nor $J' \subseteq J$: $\eta^*_i([J]) \cdot \eta^*_i([J'])$.

Define a bijection $\rho : S' \to S$ by $\rho(\eta^*_{\overline{Z}}([p])) := d''_0$ and $\rho(\eta^*_i([J])) := d_J$. Let $\varphi : \mathbb{Q}[(S')] \to H(S)$ be the morphism of $\mathbb{Q}$-algebras, induced by extending ρ to polynomials in elements of S'. Now 5.48 tells us that $\pi = f^* \circ \varphi$. Hence $\ker f^* \cap H(S) = \varphi(\ker \pi)$. So $\ker f^* \cap H(S)$ is generated by the images of the classes from (a) and (b) under φ. From (a) one obtains exactly the γ's of (5). From (b) one obtains that $d_J \cdot d_{J'} \in \ker f^* \cap H(S)$ for certain J, J'. But it is easy to see that for these pairs J, J', one has $d_J \cdot d_{J'} = 0 \in H^*_{BCl}(\overline{R}_{1,n})$, so $\ker q \cap H(S)$ is already generated by (5) alone. □

The twisted 1-sectors (X,g) which are not of the form assumed in the previous Lemma, are of the form $(X,g) = (B^r_P, \iota_m)$ and are treated in the next Lemma.

Lemma 5.54 *Let $P = \langle I_1, ..., I_m \rangle$, with $m \geq 2$ even, be a circular partition of $\underline{n}$, recall the definition of the classes $B^r(P',P) \in H^*(B^r_P) = H^*((B^r_P, \iota_m))$ from Lemma 5.28 (v).*

(i) The $H^*_{BCl}(\overline{R}_{1,n})$-module $H^*((B^r_P, \iota_m)) \subset H^*_{CR}(\overline{R}_{1,n})$ is generated by the following (larger than necessary) collection of classes: All classes $B^r(P',P) \in H^*((B^r_P, \iota_m))$ for refinements P' of P. Here $P' = P$ is allowed and defines the fundamental class $B^r(P,P) = [(B^r_P, \iota_m)]$.

(ii) Set $\mathrm{FB}((B^r_P, \iota_m)) := H^*_{BCl}(\overline{R}_{1,n})^{(\mathcal{G})}$ for $\mathcal{G}$ the set of generators listed in (i). Let $q : \mathrm{FB}((B^r_P, \iota_m)) \to H^*((B^r_P, \iota_m))$ be the $\mathcal{G}$-evaluation. Let $\mathrm{RB}((B^r_P, \iota_m)) := \ker q$ be the module of relations. Then $\mathrm{RB}((B^r_P, \iota_m))$ is generated by the set A containing for each refinement $P' = \langle J_1, ..., J_{m'} \rangle$ of P:

(1) $d''_0 * B^r(P',P) \in A$

(2) *For all circular partitions P_2 of n: $b''_{P_2} * B^r(P',P) \in A$.*

(3) *For every $K \subseteq \underline{n}$ which is not contained in any of the sets $J_1, ..., J'_m$: $d_K * B^r(P',P) \in A$.*

(4) *For P_2 a circular partition of $\underline{n}$, using the notation of Lemma 5.28:*

$$b^r_{P_2} * B^r(P',P) - \sum_{\bar{P} \in \mathrm{CCR}(P_2,P')} (-1)^{|CN(P',P_2,\bar{P})|} \sum_{\widehat{P} \in \Psi(P',P_2,\bar{P})} 2^{|\widehat{P}|-|P'|-|P_2|} B(\widehat{P},P) \in A$$

Before we describe the remaining elements of A, note that we can write each refinement P' of P with refinement map $\rho : P' \to P$, in the form

$$P' = \left\langle J_{1,1}, J_{1,2}, ..., J_{1,\mu_1}, J_{2,1}, ..., J_{m,\mu_m} \right\rangle, \qquad (\dagger)$$

such that $\rho^{-1}(I_i) = \{J_{i,1}, ..., J_{i,\mu_i}\}$. For any P' as above denote for a ordered partition (L_1, L_2) of $J_{i,j}$ by $P'(L_1, L_2)$ the refinement one obtains by replacing in ($\dagger$) the symbol $J_{i,j}$ by L_1, L_2 (in this order). With this notation, include in our set A for every P' the classes:

(5) *For each $J_{i,j}$ and for each two distinct elements $x, y \in J_{i,j}$:*

$$\sum_{\{x,y\}\subseteq L\subseteq J_{i,j}} d_L * B^r(P', P) - 4 \cdot \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ x\in L_1, y \in L_2}} B^r(P'(L_1, L_2), P) \in A,$$

(6) *For each $J_{i,j}$ and pairwise different $x, y, z \in J_{i,j}$, denote for each set $L \subset J_{i,j}$ by $\widehat{L}$ the set one obtains by replacing y by z and vice versa. Then A contains:*

$$\Bigg(\sum_{\{x,y\}\subseteq L\subseteq J_{i,j}\smallsetminus\{z\}} (d_L - d_{\widehat{L}})\alpha * B^r(P', P) + 4 \cdot \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ \{x,y\}\subseteq L_1,\, z \in L_2}} B^r(P'(L_1, L_2), P) - B^r(P'(\widehat{L}_1, \widehat{L}_2), P) \Bigg).$$

(7) *For each $i \in \underline{m}$ and each $1 \le j < \mu_i$, and for any $x \in J_{i,j}$ and $y \in J_{i,j+1}$, A contains:*

$$\Bigg(\sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ x\in L_1,\, L_2 \neq\emptyset}} B^r(P'(L_1, L_2), P) + \sum_{\substack{L_1 \uplus L_2 = J_{i,j+1} \\ y\in L_2,\, L_1 \neq\emptyset}} B^r(P'(L_1, L_2), P) - \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ \nu(J_{i,j})\in L_1}} B^r(P'(L_1, L_2), P) - \sum_{\substack{L_1 \uplus L_2 = J_{i,j+1} \\ \nu(J_{i,j+1})\in L_2}} B^r(P'(L_1, L_2), P) \Bigg),$$

where $\nu(J)$ stands for the smallest number in J.

Proof: We again identify $H^*((B^r_P, \iota_m))$ with $H^*(B^r_P)$ to be able to use in our proof the multiplication $\cdot$ from this ring. Beside $B^r(P', P)$ we also use the cycles $\mathbb{B}(P', P)$ as defined in Lemma 5.28 (i).

As $\mathbb{Q}$-algebra $H^*(\overline{M}_{\Gamma(P)}) = \bigotimes_{i=1}^m H^*(\overline{M}_{0,I_i\cup\{\circ_i,\bullet_{i+1}\}})$ is generated by the elements of the form $\eta_i^*([\circ_i, J])$ and $\eta_i^*([K])$ for all $i \in \underline{m}$, and $\emptyset \neq J \subsetneqq I_i$, $K \subseteq I_i$, $|K| \ge 2$, since each $H^*(\overline{M}_{0,I_i\cup\{\circ_i,\bullet_{i+1}\}})$ is generated by boundary divisor classes of the form $[\circ_i, J]$ and $[K]$. Call $\mathcal{K}$ the set of all classes of the form $\eta_i^*([K])$, and $\mathcal{J}$ the set of all classes of the form $\eta_i^*([\circ_i, J])$.

Let $\lambda : \mathbb{Q}[(\mathcal{K} \cup \mathcal{J})] \to \bigotimes_{i=1}^m H^*(\overline{M}_{0,I_i\cup\{\circ_i,\bullet_{i+1}\}})$ be the evaluation. The ideal of relations between these generators, $\ker\lambda$, is generated by pulled back (Keel) relations from each $H^*(\overline{M}_{0,I_i\cup\{\circ_i,\bullet_{i+1}\}})$. I.e., for all $i \in \underline{m}$:

(a) For all $x, y \in I_i$:

$$\sum_{L \subseteq I_i \smallsetminus \{x,y\}} \eta_i^*([\circ_1, x, L]) - \eta_i^*([x, y, L])$$

(b) For all $x, y, z \in I_i$:

$$\sum_{L \subseteq I_i \smallsetminus \{x,y,z\}} \eta_i^*([x, y, L]) + \eta_i^*([\circ_i, x, y, L]) - \eta_i^*([x, z, L]) - \eta_i^*([\circ_i, x, z, L])$$

(c) For all $J, K \subseteq I_i$, unless $K \subseteq J$ or $K \subseteq I_i \smallsetminus J$:

$$\eta_i^*([K]) \cdot \eta_i^*([\circ_i, J])$$

(d) For all $K, K' \subseteq I_i$, unless $K \subseteq K'$ or $K \subseteq I_i \smallsetminus K'$ or $K' \subseteq K$ or $K' \subseteq I_i \smallsetminus K$:

$$\eta_i^*([K]) \cdot \eta_i^*([K'])$$

(e) For all $J, J' \subsetneqq I_i$, unless $J \subseteq J'$ or $J' \subseteq J$:

$$\eta_i^*([\circ_i, J]) \cdot \eta_i^*([\circ_i, J'])$$

For $z^*_{B^r_P} : H^*(\overline{M}_{\Gamma(P)}) = \bigotimes_{i=1}^m H^*(\overline{M}_{0, I_i \cup \{\circ_i, \bullet_{i+1}\}}) \to H^*(B^r_P)$ the surjective pushforward via $z_{B^r_P}$. In Corollary 5.26 we identified $H^*(B^r_P)$ with $H^*(\overline{M}_{\Gamma(P)})$ if $|P| \geq 4$ and with $H^*(\overline{M}_{\Gamma(P)})^{\mathbb{S}_2}$ if $|P| = 2$, and have seen that then $z^*_{B^r_P}$ acts on the part of $H^*(\overline{M}_{\Gamma(P)})$ identified with $H^*(B^r_P)$ as multiplication by $d(P)2^{|P|-1}$. Let z be the homomorphism obtained from $z^*_{B^r_P}$ by dividing through $d(P)2^{|P|-1}$, i.e. z acts as identity on the part identified with $H^*(B^r_P)$. Then z acts as identity on $\mathcal{K}$ always, and acts as identity on $\mathcal{J}$ in case $|P| \geq 4$. For $|P| = 2$ we have

$$z(\eta_i^*([\circ_i, J])) = \frac{1}{2}(\eta_i^*([\circ_i, J]) + \eta_i^*([\bullet_{i+1}, J])) = \frac{1}{2}(\eta_i^*([\circ_i, J]) + \eta_i^*([\circ_i, J^c])),$$

where $J^c := I_i \smallsetminus J$. So if we set $\pi' := z \circ \lambda$, then $\ker(z \circ \lambda) = \ker \pi'$ is generated by $(a) - (e)$ together with

(g) If $|P| = 2$, for all $i \in \underline{m}$ and all $J \subseteq I_i$: $\eta_i^*([\circ_i, J]) - \eta_i^*([\circ_i, J^c])$.

Let $\mathcal{MJ}$ be the set of all monomials in elements of $\mathcal{J}$. As $\mathbb{Q}[(\mathcal{K})]$-modules, we can naturally identify $\mathbb{Q}[(\mathcal{J} \cup \mathcal{K})]$ with $\mathbb{Q}[(\mathcal{K})]^{(\mathcal{MJ})}$. Then we can regard π' as a homomorphism of $\mathbb{Q}[(\mathcal{K})]$-modules, $\pi' : \mathbb{Q}[(\mathcal{K})]^{(\mathcal{MJ})} \to H^*(B^r_P)$. We obtain a set of generators of $\ker \pi'$ as a $\mathbb{Q}[(\mathcal{K})]$ module by multiplying each relation from a (a)-(g) by each element of $\mathcal{MJ}$. We refer to the resulting new list of generators by (a')-(g').

Let $\mathcal{G}'$ be the subset of $\mathcal{MJ}$ consisting of all monomials of the form

$$\mathscr{D} = \prod_{i=1}^m \prod_{j=1}^{\mu_i - 1} \eta_i^*([\circ_i, \widetilde{J}_{i,j}]), \qquad (\diamondsuit)$$

for some numbers $\mu_i \geq 1$, with the empty product considered as 1, fulfilling the condition that for all $i \in \underline{m}$,

$$\emptyset \subsetneqq \widetilde{J}_{i,1} \subsetneqq \widetilde{J}_{i,2} \subsetneqq ... \subsetneqq \widetilde{J}_{i,\mu_i-1} \subsetneqq I_i. \qquad (\spadesuit)$$

For each $\mathscr{D}$ let $\widehat{\mathscr{D}}$ be the monomial obtained by replacing in the product each $[\circ_i, \widetilde{J}_{i,j}]$ by $[\circ_i, \widetilde{J}^c_{i,j}]$. We also denote $\mathscr{D}$ resp. $\widehat{\mathscr{D}}$ by $\mathscr{D}(P', P)$ resp. $\widehat{\mathscr{D}}(P', P)$, where P' the refinement of P defined as follows: Set $\widetilde{J}_{i,0} := \emptyset$, $\widetilde{J}_{i,\mu_i} := I_i$ and for each $j \in \{1, ..., \mu_i\}$ set $J_{i,j} := \widetilde{J}_{i,j} \smallsetminus \widetilde{J}_{i,j-1}$, and define

$$P' = \big\langle J_{1,1}, J_{1,2}, ..., J_{1,\mu_1}, J_{2,1}, ..., J_{2,\mu_2}, ..., J_{m,\mu_m} \big\rangle. \qquad (\clubsuit)$$

In $\mathcal{MJ}$ we formally write $\mathbb{B}(P', P) := \mathscr{D}(P', P)$ if $|P| \geq 4$ and $\mathbb{B}(P', P) := \mathscr{D}(P', P) + \widehat{\mathscr{D}}(P', P)$ if $|P| = 2$. This is justified by Lemma 5.28 (i), which implies that each so defined $\mathbb{B}(P', P) \in \mathcal{MJ}$ is mapped by λ to the class $\mathbb{B}(P', P)$ of the same name in $H^*(\overline{M}_{\Gamma(P)}) = \bigotimes_{i=1}^m H^*(\overline{M}_{0,I_i \cup \{\circ_i, \bullet_{i+1}\}})$.

Let $\mathcal{G}^*$ be the subset of $\mathcal{MJ}$ consisting of all those classes $\mathbb{B}(P', P)$. (For $|P| \geq 4$, $G^* = G'$.)

We can check using $(a')-(g')$, or more easily by excess intersection theory (also cf. Lemma 5.28 (vi) + proof), that $\ker \pi'$ contains for any refinement P' of P as above, and each $i \in \underline{m}$ and each class $\eta_i^*([\circ_i, \widetilde{J}_{i,j}])$ for $1 \leq j \leq \mu_i - 1$ as above, the relation [44]:

$$\mathscr{D}(P', P) \cdot (\eta_i^*[\circ_i, \widetilde{J}_{i,j}]) = - \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ \nu(J_{i,j}) \in L_1}} \mathscr{D}(P'(L_1, L_2), P) - \sum_{\substack{L_1 \uplus L_2 = J_{i,j+1} \\ \nu(J_{i,j+1}) \in L_2}} \mathscr{D}(P'(L_1, L_2), P), \quad (\ddagger)$$

where for a set $J \subset \underline{n}$, $\nu(J)$ is the smallest number in J.

Using e' we can see that every element of $\mathcal{MJ}$ is in the same fibre of π' as an element of the form

$$\prod_{i=1}^{m} \prod_{j=1}^{\mu_i - 1} (\eta_i^*([\circ_i, \widetilde{J}_{i,j}]))^{\epsilon_{i,j}},$$

where the $\widetilde{J}_{i,j}$ fulfil $(\spadesuit)$ with $\epsilon_{i,j} \in \mathbb{Z}_{\geq 1}$. Then using $(\ddagger)$ on finds inductively that each element of $\mathcal{MJ}$ is even in the same fibre of π' as a linear combination of elements $\mathscr{D} \in \mathcal{G}'$. By (g') there is even a linear combination of elements $\mathbb{B}(P', P) \in \mathcal{G}^*$ in the same fibre.

This shows firstly that $\pi : \mathbb{Q}[(\mathcal{K})]^{(\mathcal{G}^*)} \to H^*(B_P^r)$, which we define as the restriction of π', is still surjective. Secondly it shows that we can obtain a different list of generators of $\ker \pi'$ as follows: Let (a^*)-(d^*) be the relations obtained by multiplying each relation from (a)-(d) by each element $\mathbb{B}(P', P) \in \mathcal{G}^*$. Then (a^*)-(d^*) together with $(\ddagger)$, (e'), and (g') generate $\ker \pi'$ as a $\mathbb{Q}[(\mathcal{K})]$-module.

We claim that $\ker \pi = \ker \pi' \cap \mathbb{Q}[(\mathcal{K})]^{(\mathcal{G}^*)}$ is generated by (a^*)-(d^*) alone: Indeed the elements from (a^*)-(b^*) are obviously in $\mathbb{Q}[\mathcal{K}]^{(\mathcal{G}^*)}$, and it is also easy to check that there is no $\mathbb{Q}[(\mathcal{K})]$-linear combination of the relations described in $(\ddagger)$ and (e') and (g') which lies in $\mathbb{Q}[(\mathcal{K})]^{(\mathcal{G}^*)} \smallsetminus \{0\}$.

We write down the relations from (a^*) and (b^*) explicitly, we used also (c^*) to simplify them: For all $\mathbb{B}(P', P)$:

[44] Written here in form of an equation $a = b$ not as the corresponding element $a - b \in \ker \pi'$

(a^*) For all $J_{i,j} \in P'$ and all $x, y \in J_{i,j}$ with $x \neq y$:

$$\sum_{L \subseteq J_{i,j} \smallsetminus \{x,y\}} \mathbb{B}(P'(L \cup \{x\}, L^c \cup \{y\}), P) - \eta_i^*([x,y,L]) \cdot \mathbb{B}(P', P),$$

where $L^c := J_{i,j} \smallsetminus L$. And for all $1 \leq j < j' \leq \mu_i$ and all $x \in J_{i,j}$ and $y \in J_{i,j'}$:

$$\sum_{x \in L \subseteq J_{i,j}} \mathbb{B}(P'(L, L^c), P) + \sum_{L \subseteq J_{i,j'} \smallsetminus \{y\}} \mathbb{B}(P'(L, L^c \cup \{y\}), P) + \sum_{j<r<j'} \sum_{L \subseteq J_{i,r}} \mathbb{B}(P'(L, L^c), P) +$$

$$+ \sum_{j<r<j'} \big(- \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ \nu(J_{i,j}) \in L_1}} \mathbb{B}(P'(L_1, L_2), P) - \sum_{\substack{L_1 \uplus L_2 = J_{i,j+1} \\ \nu(J_{i,j+1}) \in L_2}} \mathbb{B}(P'(L_1, L_2), P) \big).$$

(b^*) For all $J_{i,j} \in P'$ and all $x, y, z \in J_{i,j}$:

$$\sum_{L \subseteq J_{i,j} \smallsetminus \{x,y,z\}} \big((\eta_i^*([x,y,L]) - \eta_i^*([x,z,L]) \big) \cdot \mathbb{B}(P', P) + \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ x,y \in L_1,\, z \in L_2}} \mathbb{B}(P'(L_1, L_2), P)$$

$$- \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ x,z \in L_1,\, y \in L_2}} \mathbb{B}(P'(L_1, L_2), P).$$

And for all $1 \leq j < j' \leq \mu_i$ and all $x, z \in J_{i,j}$ and $z \in J_{i,j'}$:

$$\sum_{L \subseteq J_{i,j} \smallsetminus \{x,y,z\} \eta_i^*([x,y,L]) \cdot \mathbb{B}(P',P)} + \sum_{\substack{L_1 \uplus L_2 = J_{i,j} \\ x,y \in L_1}} \mathbb{B}(P'(L_1, L_2), P)$$

$$+ \sum_{\substack{L_1 \uplus L_2 = J_{i,j'} \\ z \in L_2}} \mathbb{B}(P'(L_1, L_2), P) + \sum_{j<r<j'} \sum_{L_1 \uplus L_2 = J_{i,r}} \mathbb{B}(P'(L_1, L_2), P) +$$

$$+ \sum_{j \leq r<j'} \big(- \sum_{\substack{L_1 \uplus L_2 = J_{i,r} \\ \nu(J_{i,r}) \in L_1}} \mathbb{B}(P'(L_1, L_2), P) - \sum_{\substack{L_1 \uplus L_2 = J_{i,r+1} \\ \nu(J_{i,j+1}) \in L_2}} \mathbb{B}(P'(L_1, L_2), P) \big).$$

And for all $1 \leq j < j' \leq \mu_i$ and all $x \in J_{i,j}$ and $y, z \in J_{i,j'}$:

$$\sum_{\substack{L_1 \uplus L_2 = J_{i,j'} \\ x \in L_1,\, y \in L_2}} \mathbb{B}(P'(L_1, L_2), P) - \sum_{\substack{L_1 \uplus L_2 = J_{i,j'} \\ y \in L_1,\, x \in L_2}} \mathbb{B}(P'(L_1, L_2), P).$$

For x, y, z, lying in three different sets $J_{i,j} \in P'$ one obtains again the relations of the second type listed in (a^*).

Now let H be the $\mathbb{Q}$-sub algebra of $H^*_{BCl}(\overline{R}_{1,n})$ generated by the boundary divisor classes d_K for all $K \subseteq \underline{n}$ ($|K| \geq 2$), and let $\rho' : \mathbb{Q}[(\mathcal{K})] \to H$ be the homomorphism of $\mathbb{Q}$-algebras induced by sending each $\eta_i^*([K]) \in \mathcal{K}$ to $d_K \in H$. Recall that $\mathcal{G}$ is the set of generators listed in (i), i.e. the set of all classes $B^r(P', P)$. Let $\rho : \mathbb{Q}[(\mathcal{K})]^{(\mathcal{G}^*)} \to H^{(\mathcal{G})}$ be the homomorphism of $\mathbb{Q}[(\mathcal{K})]$-modules induced by ρ' and by sending each $\mathbb{B}(P', P) \in \mathcal{G}^*$

to $2^{|P'|-|P|}B^r(P',P)$. Now by definition of π and $\pi' = z \circ \lambda$, by Lemma 5.28 (iv), and the discussion of the properties of z at the beginning of the proof, we see that π factors as

$$\underbrace{\mathbb{Q}[(\mathcal{K})]^{(\mathcal{G}^*)} \xrightarrow{\ \rho\ } H^{(\mathcal{G})} \xrightarrow{\ q'\ } H^*(B^r_P)}_{\pi}\ , \tag{$\heartsuit$}$$

where q' is the restriction of q to $H^{(\mathcal{G})} \subset H^*_{BCl}(\overline{R}_{1,n})^{(\mathcal{G})} = \mathrm{FB}((B^r_P, \iota_m))$. The fact that π is surjective so implies that q is surjective, i.e. part (i) of our Lemma.

(ii): Let $M \subset \mathrm{FB}((B^r_P, \iota_m))$ be the submodule generated by the collection of relations listed in (1)-(7). [45] We use the shorthand $\mathrm{RB} := \mathrm{RB}((B^r_P, \iota_m))$ and have to show that $M = \mathrm{RB}$. The classes listed in (1)-(3) are contained in RB by Lemma 5.49. Concerning (4): Let $i : B^r_{P'} \hookrightarrow B^r_P$ the inclusion. Then using the notation of Lemma 5.28:

$$b^r_{P_2} * B^r(P',P) = f^*_{B^r_P}(b^r_{P_2}) \cdot B^r(P',P) = i_*i^*f^*_{B^r_P}(b^r_{P_2}) = i_*f^*_{B^r_{P'}}(b^r_{P_2}).$$

Now (4) follows from Lemma 5.28 (v) together with the obvious fact that for every refinement $\widehat{P}$ of P' we have $i_*(B^r(\widehat{P},P')) = B^r(\widehat{P},P)$.

We know by now that the classes from (1)-(4) are contained in RB and M. Similar to the proof of Lemma 5.53 (ii) this allows to reduce to showing that $\mathrm{RB}' := \mathrm{RB} \cap H^{(\mathcal{G})} = M \cap H^{(\mathcal{G})} =: M'$. Note that $\mathrm{RB}' = \ker q'$ for the q' appearing in ($\heartsuit$). Hence $\mathrm{RB}' = \rho(\ker \pi)$. So RB' is generated by the relations one obtains by applying ρ to (a^*)-(d^*), i.e. formally by replacing in them each $\eta_i^*([K])$ by d_K, and each $\mathbb{B}(P',P)$ by $2^{|P'|-|P|}B^r(P',P)$.

Now one can check that the relations from (5), (6), (7) are direct translations of some of the relations from (a^*) and (b^*). Furthermore the other relations coming from (a^*) and (b^*) are all H-linear combinations of those in (5)-(7). (3) is the translation of (c^*). Finally (d^*) translates to relations which hold in $H \subset H^*_{BCl}(\overline{R}_{1,n})$ anyway [46], so they are contained in RB' trivially. Hence $\mathrm{RB}' = M'$. □

By the Lemmas 5.53 and 5.54 we know for every *twisted* 1-sector (X,g) of $\overline{R}_{1,n}$ the structure of $H^*((X,g)) \subset H^*_{CR}(\overline{R}_{1,n})$ as an $H^*_{BCl}(\overline{R}_{1,n})$-module. (Since we have explicitly described the two modules $\mathrm{FB}((X,g))$ and $\mathrm{RB}((X,g)) \subset \mathrm{FB}((X,g))$, and obviously $H^*((X,g)) \cong \mathrm{FB}((X,g))/\,\mathrm{RB}((X,g))$ as an $H^*_{BCl}(\overline{R}_{1,n})$ module.) So the only information missing to describe $H^*_{CR}(\overline{R}_{1,n})$ as an $H^*_{BCl}(\overline{R}_{1,n})$-module is a description of $H^*(\overline{R}_{1,n})$ as an $H^*_{BCl}(\overline{R}_{1,n})$-module, to account for the untwisted sector $(\overline{R}_{1,n}, 1)$. Unfortunately this module structure is not known, we do not even know generators of the module. [47] To avoid this problem we only attempt to give the coarser description of $H^*_{CR}(\overline{R}_{1,n})$ as an $H^*(\overline{R}_{1,n})$-module:

[45] Contrary to what we did in the proof of Lemma 5.53, here M really contains the elements $\gamma * B^r(P',P)$ listed in (1)-(5) and not only the γ's. This is since we do not have $\mathrm{FB}((B^r_P, \iota_m)) \cong H^*_{BCl}(\overline{R}_{1,n})$ in this case.

[46] Cf. the end of the proof of Lemma 5.53 (ii)

[47] The same problem exist in case of $\overline{M}_{1,n}$ instead of $\overline{R}_{1,n}$. Such a description seems to be difficult to obtain. For example on the way one would obviously either have to proof or falsify Claim 5.1 (i) (by Getzler), and would need additional information about the odd part of $H^*(\overline{M}_{1,n})$.

For every 1-sector $(X,g) \neq (\overline{R}_{1,n})$ of $\overline{R}_{1,n}$, the generators of $H^*((X,g))$ as $H^*_{BCl}(\overline{R}_{1,n})$-module listed in Lemma 5.53 resp. 5.54 of course also generate it as $H^*(\overline{R}_{1,n})$-module. So denote by $\mathrm{F}((X,g))$ the free $H^*(\overline{R}_{1,n})$-module in the same generators as $\mathrm{FB}((X,g))$. (So $\mathrm{F}((X,g)) = \mathrm{FB}((X,g)) \otimes_{H^*_{BCl}(\overline{R}_{1,n})} H^*(\overline{R}_{1,n})$.) Let $Q : \mathrm{F}((X,g)) \to H^*((X,g))$ be the evaluation. Let $\mathrm{R}((X,g)) := \ker Q$ be the $H^*(\overline{R}_{1,n})$-module of relations. For the untwisted sector let $\mathrm{F}((\overline{R}_1,1))$ be the free $H^*(\overline{R}_{1,n})$ module generated by $[(\overline{R}_{1,n},1)]$, then $\mathrm{F}((\overline{R}_1,1)) \cong H^*(\overline{R}_{1,n}) \cong H^*((\overline{R}_{1,n},1))$, and $\mathrm{R}((\overline{R}_{1,n},1)) = \{0\}$. Since for any $\gamma \in H^*((\overline{R}_{1,n},1))$ and $\beta \in H^*((X,g))$ for any 1-sector (X,g), we have $\alpha * \beta \in H^*((X,g))$ by definition of the Chen-Ruan product $*$, it is clear that as $H^*(\overline{R}_{1,n})$-modules:

$$H^*_{CR}(\overline{R}_{1,n}) \cong \bigoplus_{(X,g)\ \text{1-sector of}\ \overline{R}_{1,n}} \mathrm{F}((X,g))/\,\mathrm{R}((X,g))$$

Let $\widehat{\mathrm{RB}}((X,g))$ be the submodule of $\mathrm{F}((X,g))$ generated by the same list of relations as the $H^*_{BCl}(\overline{R}_{1,n})$-module $\mathrm{RB}((X,g))$. It is clear that $\widehat{\mathrm{RB}}((X,g)) \subseteq \mathrm{R}((X,g))$. Let $H^{2*}(\overline{R}_{1,n}) \oplus H^{2*+1}(\overline{R}_{1,n}) = H^*(\overline{R}_{1,n})$ be the decomposition of $H^*(\overline{R}_{1,n})$ into the even and odd part. Denote by $\mathrm{RB}^+((X,g)) \subseteq \mathrm{F}((X,g))$ the $H^*(\overline{R}_{1,n})$-module generated by the set

$$\widehat{\mathrm{RB}}((X,g)) \cup \{\gamma * \beta \mid \gamma \in H^{2*+1}(\overline{R}_{1,n}),\ \beta \in H^*((X,g))\}.$$

Using Lemma 5.50, we see that also $\mathrm{RB}^+((X,g)) \subseteq \mathrm{R}((X,g))$.

Now we claim that for all $(X,g) \neq (\overline{R}_{1,n},1)$: $\mathrm{RB}^+((X,g)) = \mathrm{R}((X,g))$ if $H^*_{BCl}(\overline{R}_{1,n}) = H^{2*}(\overline{R}_{1,n})$. If $\mathscr{G}_1,...,\mathscr{G}_r$ are our generators of $\mathrm{F}((X,g))$, then we have to show that for all $\gamma_1,...,\gamma_r \in H^*(\overline{R}_{1,n})$, $\sum_{i=1}^r \gamma_i * \mathscr{G}_i = 0 \in H^*((X,g))$ implies $\sum_{i=1}^r \gamma_i * \mathscr{G}_i \in \mathrm{RB}^+((X,g))$ under the condition $H^*_{BCl}(\overline{R}_{1,n}) = H^{2*}(\overline{R}_{1,n})$. Let $\widetilde{\gamma}_i$ be the part of γ_i lying in H^{2*}, i.e. by our assumption $\widetilde{\gamma}_i \in H^*_{BCl}(\overline{R}_{1,n})$. Then using in this order Lemma 5.50, the definition of $\mathrm{RB}((X,g))$ and the definition of $\mathrm{RB}^+((X,g))$:

$$\sum_{i=1}^r \gamma_i * \mathscr{G}_i = 0 \ \Rightarrow\ \sum_{i=1}^r \widetilde{\gamma}_i * \mathscr{G}_i = 0 \ \Rightarrow\ \sum_{i=1}^r \widetilde{\gamma}_i * \mathscr{G}_i \in \mathrm{RB}((X,g)) \ \Rightarrow\ \sum_{i=1}^r \gamma_i * \mathscr{G}_i \in \mathrm{RB}^+((X,g))$$

From this discussion our next proposition follows quite directly:

Proposition 5.55 *(i) The $H^*(\overline{R}_{1,n})$-module $H^*_{CR}(\overline{R}_{1,n})$ is generated by the set $\mathcal{G}$ consisting of the following classes:*

(1) *For every 1-sector (X,g), the fundamental class $[(X,g)]$. (Cf. Theorem 5.32 for a complete list of the 1-sectors.)*

(2) *For all circular partitions P of $\underline{n}$ with $|P| \geq 2$ even, and all refinements P' of P, the classes $B^r(P',P)$.*

(ii) Set $H' := H^*(\overline{R}_{1,n})^{(\mathcal{G})}$, and let $\pi : H' \to H^*_{CR}(\overline{R}_{1,n})$ be the evaluation. Then the module of relations $\ker \pi$ contains the following relations:

(1) *For every 1-sector (X,g) with $X = \overline{Z}^{(I_1,\ldots,I_k)}$ for some basic sector $\overline{Z}$, all relations listed for this sector in Lemma 5.53 (ii).*

(2) *For every 1-sector (X,g) with X a banana cycle, all relations listed for this sector in Lemma 5.54 (ii).*

(3) *For all classes $\gamma \in H^{2*+1}(\overline{R}_{1,n})$ and all generators $\mathscr{G}$ listed in (i), $\gamma * \mathscr{G} = 0$.* [48]

For a given $n \in \mathbb{N}$ the module of relations of $\ker q$ *is generated by the listed relations as an $H^*(\overline{R}_{1,n})$-module, if $H^{2*}(\overline{R}_{1,n}) = H^*_{BCl}(\overline{R}_{1,n})$.*

5.5.5 The $H^*(\overline{R}_{1,n})$-algebra $H^*_{CR}(\overline{R}_{1,n})$

In this section we try to determine the ring structure of $H^*_{CR}(\overline{R}_{1,n})$. Since not even $H^*(\overline{R}_{1,n})$ is known as a $\mathbb{Q}$-algebra for larger n [49], we have no chance of determining $H^*_{CR}(\overline{R}_{1,n})$ as a $\mathbb{Q}$-algebra. What we can do is to give a set of independent generators of the $H^*(\overline{R}_{1,n})$-algebra $H^*_{CR}(\overline{R}_{1,n})$, and many relations in these generators holding on $H^*_{CR}(\overline{R}_{1,n})$. I was not able to prove that these relations span the whole ideal of relations. One can say that they span the ideal if the whole even part of $H^*(\overline{R}_{1,n})$, i.e. $H^{2*}(\overline{R}_{1,n})$ is generated by boundary cycle classes, for all n [50] . In this regard we are in an analogous situation for $\overline{R}_{1,n}$ as for $\overline{M}_{1,n}$ in [Pag08]. But the analogy is broken by the fact that it is an old, but yet unproven, claim of Getzler that $H^{2*}(\overline{M}_{1,n})$ is generated by boundary cycle classes (cf. Claim 5.1). In the case of $\overline{R}_{1,n}$, I do not know whether one should expect the same.

First we compute the products of the fundamental classes of 1-sectors of $\overline{R}_{1,n}$.

Proposition 5.56 *(i) If (X_1,g), (X_2,h) are* 1*-sectors of $I_1(\overline{R}_{1,n})$, such that not both of X_1, X_2 are banana cycles, then there is a sector (X_3,gh) such that $[(X_1,g)] * [(X_2,g)] \in H^d((X_3,gh))$, for some $d \in \mathbb{N}_0$. Furthermore in our case we always have*

$$[(X_1,g)] * [(X_2,g)] = \gamma * \mathcal{D}$$

for some $\gamma \in H^*_{BCl}(\overline{R}_{1,n})$, and a $\mathcal{D} \in H^d((X_3,gh))$, such that $\mathcal{D}$ is one of the generators of the $H^*(\overline{R}_{1,n})$-module $H^*_{CR}(\overline{R}_{1,n})$ listed in Proposition 5.55 (i). [51]

For each pair $((X_1,g),(X_2,h))$, either $[(X_1,g)] * [(X_2,h)] = 0$, or the pair appears (up to swapping $[(X_1,g)]$ and $[(X_2,h)]$) in the table on the next page. (Or $X_1 = B^r_{P_1}$ and $X_2 = B^r_{P_2}$, which case is treated in (ii)) Then this table lists the corresponding sector (X_3,gh), and a geometric class $cl = [(X_1,g)]*[(X_2,h)] \in H^*(X_3,gh)$. If we write $cl = [V]_Q$ for some subvariety V of X_3 we mean by this the Q-class taken inside X_3, not in $\overline{R}_{1,n}$. (This explanation will be continued after the table on the next page.)

[48]Actually these are infinitely many relations, but of course, as soon as one knows a finite generating system S of $H^{2*+1}(\overline{R}_{1,n})$, one can replace these by the finitely many relations $\gamma * \mathscr{G}$ for $\gamma \in S$.

[49]Here, contrary to the case of $H^*(\overline{M}_{1,n})$, not even the Betti number are known.

[50]More precisely our relations span the ideal of relations if and only if $H^*(\overline{R}_{1,n})$ is generated as $\mathbb{Q}$-algebra by $H^*_{BCl}(\overline{R}_{1,n}) \oplus H^{2*+1}(\overline{R}_{1,n})$, which is formally a weaker condition.

[51]So if X_3 is an essential 1-sector, the only possibility is $\mathcal{D} = [(X_3,gh)]$. Otherwise $X_3 = B^r_P$ and $\mathcal{D} = B^r(P',P)$ for some refinement P' of P.

Nr.	(X_1,g)	(X_2,h)	(X_3,gh)	cl	d	$\gamma * \mathcal{D}$
(1)	$(\overline{A}_1^{\underline{n}},-1)$	$(C_4^{\underline{n}},i)$	$(C_4^{\underline{n}},-i)$	$\eta_1^*(-\psi_{\circ_1})$	2	$-(\sum_{\{1,2\}\subseteq I\subsetneq \underline{n}} d_I) * [(X_3,gh)]$
(2)	$(\overline{A}_1^{\underline{n}},-1)$	$(C_4^{\underline{n}},-i)$	$(C_4^{\underline{n}},i)$	$[(C_4^{\underline{n}})]_Q$	0	$[(X_3,gh)]$
(3)	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	$(\overline{A}_{2,b}^{\{I_1,I_2\}},-1)$	$(B^r_{\langle I_1,I_2\rangle},\iota_2)$	$[E_2^{r,\{I_1,I_2\}}]_Q$	$2(2-\mu)$	$d_{I_1}d_{I_2} * [(X_3,gh)]$
(4)	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	$(C_4^{\{I_1,I_2\}},i)$	$(C_4^{\{I_1,I_2\}},-i)$	0	0	0
(5)	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	$(C_4^{\{I_1,I_2\}},-i)$	$(C_4^{\{I_1,I_2\}},i)$	$[C_4^{\{I_1,I_2\}}]_Q$	0	$[(X_3,gh)]$
(6)	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	$(B^r_{\langle I_1,I_2\rangle},\iota_2)$	$(\overline{A}_{2,b}^{\{I_1,I_2\}},-1)$	$[E_2^{r,\{I_1,I_2\}}]_Q$	2	$\frac{1}{4}d_0'' * [(X_3,gh)]$
(7)	$(\overline{A}_{2,b}^{\{I_1,I_2\}},-1)$	$(B^r_{\langle I_1,I_2\rangle},\iota_2)$	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	$[E_2^{r,\{I_1,I_2\}}]_Q$	2	$\frac{1}{2}d_0'' * [(X_3,gh)]$
(8)	$(\overline{A}_3^{\{I_1,I_2\},I_3},-1)$	$(\overline{A}_3^{\{I_1,I_3\},I_2},-1)$	$(B^r_{\langle I_2\cup I_3,I_1\rangle},\iota_2)$	$[E_3^{r,\{I_2,I_3\},I_1}]_Q$	$2(4-\mu)$	$d_{I_1}d_{I_2}d_{I_3} * B^r(\langle I_1,I_2,I_3\rangle,\langle I_2\cup I_3,I_1\rangle)$
(9)	$(\overline{A}_3^{\{I_1,I_2\},I_3},-1)$	$(B^r_{\langle I_2\cup I_3,I_1\rangle},\iota_2)$	$(\overline{A}_3^{\{I_1,I_3\},I_2},-1)$	$[E_3^{r,\{I_2,I_3\},I_1}]_Q$	2	$\frac{1}{4}d_0'' * [(X_3,gh)]$
(10)	$(\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}},-1)$	$(\overline{A}_4^{\{\{I_1,I_3\},\{I_2,I_4\}\}},-1)$	$(B^r_{\langle I_2\cup I_3,I_1\cup I_4\rangle},\iota_2)$	$[E_4^{r,\{\{I_2,I_3\},\{I_1,I_4\}\}}]_Q$	$2(6-\mu)$	$2d_{I_1}d_{I_2}d_{I_3}d_{I_4}b^r_{\langle I_1\cup I_2,I_3\cup I_4\rangle} * [(X_3,gh)]$
(11)	$(\overline{A}_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}},-1)$	$(B^r_{\langle I_2\cup I_3,I_1\cup I_4\rangle},\iota_2)$	$(\overline{A}_4^{\{\{I_1,I_3\},\{I_2,I_4\}\}},-1)$	$[E_4^{r,\{\{I_2,I_3\},\{I_1,I_4\}\}}]_Q$	2	$\frac{1}{4}d_0'' * [(X_3,gh)]$
(12)	$(C_4^{\underline{n}},i)$	$(C_4^{\underline{n}},i)$	$(\overline{A}_1^{\underline{n}},-1)$	$[C_4^{\underline{n}}]_Q\cdot\eta_1^*(-\psi_{\circ_1})$	$2(2-\mu)$	$-d_0''(\sum_{\{1,2\}\subseteq I\subsetneq \underline{n}} d_I) * [(X_3,gh)]$
(13)	$(C_4^{\underline{n}},-i)$	$(C_4^{\underline{n}},-i)$	$(\overline{A}_1^{\underline{n}},-1)$	$[C_4^{\underline{n}}]_Q$	2	$d_0'' * [(X_3,gh)]$
(14)	$(C_4^{\{I_1,I_2\}},i)$	$(C_4^{\{I_1,I_2\}},i)$	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	0		0
(15)	$(C_4^{\{I_1,I_2\}},-i)$	$(C_4^{\{I_1,I_2\}},-i)$	$(\overline{A}_{2,a}^{\{I_1,I_2\}},-1)$	$[C_4^{\{I_1,I_2\}}]_Q$	2	$d_0'' * [(X_3,gh)]$
For any 1-sector (X,g) of $\overline{R}_{1,n}$, with $c(X) := \mathrm{codim}(X,\overline{R}_{1,n})$:						
(16)	(X,g)	$(\overline{R}_{1,n},1)$	(X,g)	$[X]_Q$	0	$[(X,g)]$
(17)	(X,g)	(X,g^{-1})	$(\overline{R}_{1,n},1)$	$[X]_Q$	$c(X)$	$[X]_Q * [(\overline{R}_{1,n},1)]$

Furthermore the table lists the degree d such that $[(X_1,g)]*[(X_2,h)] \in H^d(X_3,gh)$, and an expression of the form $\gamma * \mathcal{D}$ as above, such that $\gamma * \mathcal{D} = [(X_1,g)]*[(X_2,h)]$. In row (17) also in $\gamma * \mathcal{D}$ a class of the form $[X]_Q$ appears, and here this means the Q-class of X taken on $\overline{R}_{1,n}$, so here $[X]_Q \in H^*(\overline{R}_{1,n})$. Note that the classes $[X]_Q$ appearing there are described explicitly as polynomials in the usual generators of $H^*_{BCl}(\overline{R}_{1,n})$ in section 5.5.2, so we always know γ explicitly as such a polynomial. If there appears a μ in the expression of d it means the number of sets containing only one element among the sets $I_1,...,I_k$ of a the partition defining the sectors. In the expression for $\gamma * \mathcal{D}$ the symbol d_I has to be interpreted as 1, if $|I| = 1$. (Otherwise it denotes the boundary divisor class d_I, as usual.) For the class $\eta_1^*(-\psi_{\circ_1})$ appearing, η_1 is as defined in Notation 5.41.

(ii) Let P_1, P_2 be two circular partitions of $\underline{n}$ with $|P_1|$, $|P_2|$ even, P_2' a refinement of P_2. Recall the notation of Lemma 5.28 and Definition 5.27, and the definition of $\widetilde{P}$ from Lemma 5.43. For any $P \in \mathrm{CCR}(P_1,P_2)$ and any $P' \in \mathrm{CCR}(P_2',P)$, define $\widehat{CN}(P,P_2';P';P_2)$ and $\widehat{\Psi}(P,P_2';P';P_2)$ like in Lemma 5.28 (vi). Then:

$$[(B^r_{P_1},\iota_{P_1})]*B^r(P_2',P_2)$$

$$= \sum_{P\in \mathrm{CCR}(P_1,P_2)} \sum_{P'\in \mathrm{CCR}(P,P_2')} (-1)^{|\widehat{\mathrm{CN}}(P,P_2';P';P_2)|} \sum_{\widehat{P}\in\widehat{\Psi}(P,P_2';P';P_2)} 2^{|P_2|+|\widehat{P}|-|P|-|P_2'|} B^r(\widehat{P},\widetilde{P})$$ [52]

Proof: (i): Recall that

$$[(X_1,g)]*[(X_2,h)] = (p_3)_*\Big(p_1^*([(X_1,g)]) \cdot p_2^*([X_2,h]) \cdot c_{top}(E)\Big) \qquad (\dagger)$$

where E is the Chen-Ruan excess intersection bundle. Since we exclude in (i) the case that X_1 and X_2 are both banana cycles, we know $X_1 \cap X_2$ explicitly from Lemma 5.42 and see that it is irreducible and possibly empty. From Proposition 5.44 we then know that the only 2-sector of $\overline{R}_{1,n}$ whose images under both p_1 and p_2 meet (X_1,g) resp. (X_2,h) is $(X_1 \cap X_2; g,h)$, if $X_1 \cap X_2 \neq \emptyset$. If $X_1 \cap X_2 = \emptyset$, then there is no such 2-sector and hence $[(X_1,g)]*[(X_2,h)] = 0$. For this reason, by 5.42, all products $[(X_1,g)]*[(X_2,h)]$ not listed in the table are 0. In the remaining cases, in (†) we can restrict the domains of p_1, p_2 and p_3 to $(X_1\cap X_2;g,h)$. Furthermore $p_1 : (X_1\cap X_2;g,h) \to (X_1,g)$ and $p_2 : (X_1\cap X_2;g,h) \to (X_2,h)$ are closed embeddings, and so

$$p_1^*([(X_1,g)]) = p_2^*([X_2,h]) = [(X_1\cap X_2;g,h)] = p_1^*([(X_1,g)]) \cdot p_2^*([X_2,h]).$$

$$\text{Hence:} \qquad [(X_1,g)]*[(X_2,h)] = (p_3)_*(c_{top}(E_{(X_1\cap X_2;g,h)})). \qquad (\ddagger)$$

The 1-sector (X_3,gh) into which $(X_1 \cap X_2)$ is mapped by p_3 is known from Proposition 5.44 (iii). One only has to take into account that one finds $(X_3,(gh)^{-1})$ instead of (X_3,gh) in that table. Furthermore for all 2-sectors $(Y;g,h)$ of $\overline{R}_{1,n}$, the class $c_{top}(E_{(X\cap X';g,h)})$ is determined in Lemma 5.45. With this and (‡) one directly computes the entries cl and d

[52]Note that in the "generic case" already the first sum is empty, and in the generic nonempty case, $|\mathrm{CCR}(P_1,P_2)| = |\mathrm{CCR}(P,P_2')| = 1$, and $|\widehat{\mathrm{CN}}(P')| = 0$, and then this long sum consists only of one term of the form $B^r(P',P)$.

in the table of (ii). To write cl in the form $\gamma * \mathcal{D}$ in the last column of the table one also has to use information from section 5.5.3. We explicitly compute some examples:

In row (1): $(X_1 \cap X_2; g, h) = (C_4^{\underline{n}}; -1, i)$, or alternatively $(C_4^{\underline{n}}; (-1, i, i))$ in the notation of Proposition 5.44 with automorphism label $(g, h, (gh)^{-1})$. So $X_3 = (C_4^{\underline{n}}, -i)$, and $p_3 : (C_4^{\underline{n}}; -1, i) \to (C_4^{\underline{n}}, -i)$ is an isomorphism in this case. With $c_{top}(E_{(C_4^{\underline{n}};(-1,i,i))}) = c_{top}(E_{(C_4^{\underline{n}};(i,i,-1))}) = -\eta_1^*(\psi_{\circ_1})$ from Lemma 5.45 we thus already know all entries in row (1), except $\gamma * \mathcal{D}$. To obtain this last entry use Summary 1.42 (iv), Lemma 5.48 and equation (5.4), to write, for $f : (C_4^{\underline{n}}, -i) \to \overline{R}_{1,n}$ the closed embedding:

$$-\eta_1^*(\psi_{\circ_1}) = -\eta_1^*(\sum_{\{1,2\}\subseteq I \subsetneq \underline{n}} [I]) = -\sum_{\{1,2\}\subseteq I \subsetneq \underline{n}} f^*(d_I) = -\sum_{\{1,2\}\subseteq I \subsetneq \underline{n}} d_I * [(C_4^{\underline{n}}, -i)].$$

In row (2) everything is very similar, except that $c_{top}(E_{(C_4^{\underline{n}};(-1,-i,-i))}) = 1 = [(C_4^{\underline{n}}; -1, -i)]$, by Lemma 5.45. In row (4) we get $cl = 0$ since $c_{top}(E_{(C_4^{\{I_1,I_2\}};(-1,i,i))}) = 0$. For row (6) we obtain $[(X_1, g)] * [(X_2, h)] = [E_2^{r,\{I_1,I_2\}}]_Q \in H^*((\overline{A}_{2,b}^{\{I_1,I_2\}}, -1))$ as before. Now from the definitions of $\overline{A}_{2,b}^{\{I_1,I_2\}}$ and $E_2^{r,\{I_1,I_2\}}$ it is clear that $[E_2^{r,\{I_1,I_2\}}]_Q = \eta^*_{\overline{A}_{2,b}}([E_2^r]_Q)$ (using Notation 5.41). We know that E_2^r is a point in $\overline{A}_{2,b}$ parametrising an object with 4 automorphisms (cf. section 5.4.1). Hence with Lemma 5.48 (ii), and equation (5.4)

$$\eta^*_{\overline{A}_{2,b}}([E_2^r]_Q) = \frac{1}{4} f^*(d_0'') = \frac{1}{4} d_0'' * [(\overline{A}_{2,b}^{\{I_1,I_2\}}, -1)], \quad \text{where} \quad f : (\overline{A}_{2,b}^{\{I_1,I_2\}}, -1) \to \overline{R}_{1,n},$$

is the closed embedding. We remark that all the d_0'' appearing in the last column arise in a way similar to this.

As a last example, in row (8), we have $cl = [E_3^{r,\{I_2,I_3\},I_1}]_Q \in H^*((B^r_{\langle I_2 \cup I_3, I_1\rangle}))$. Use

$$z_{B^r_{\langle I_2\cup I_3,I_1\rangle}} : \overline{M}_{0,I_2\cup I_3\cup\{\circ_1,\bullet_2\}} \times \overline{M}_{0,I_1\cup\{\circ_2,\bullet_1\}} \to B^r_{\langle I_2\cup I_3,I_1\rangle}, \quad \text{and}$$

$$h : \overline{M}_{0,I_2\cup\{\triangle_2\}} \times \overline{M}_{0,I_3\cup\{\triangle_3\}} \times \overline{M}_{0,\{\blacktriangle_2,\blacktriangle_3,\circ_1,\bullet_2\}} \times \overline{M}_{0,\{\blacktriangle_1,\circ_2,\bullet_1\}} \times \overline{M}_{0,I_1\cup\{\triangle_1\}}$$
$$\to \overline{M}_{0,I_2\cup I_3\cup\{\circ_1,\bullet_2\}} \times \overline{M}_{0,I_1\cup\{\circ_2,\bullet_1\}}$$

the morphism gluing each $\triangle_i$ to $\blacktriangle_i$. Then

$$z^*_{B^r_{\langle I_2\cup I_3,I_1\rangle}}([E_3^{r,\{I_2,I_3\},I_1}]_Q) = h_*(1 \otimes 1 \otimes q \otimes 1 \otimes 1)$$

where q is the class of the special point on $\overline{M}_{0,\{\blacktriangle_2,\blacktriangle_3,\circ_1,\bullet_2\}}$ which parametrises $\mathbb{P}^1$ with points $\blacktriangle_2, \blacktriangle_3, \circ_1, \bullet_2$ in such a position, that there is an automorphism of $\mathbb{P}^1$ fixing $\blacktriangle_2$, and $\blacktriangle_3$ and swapping $\circ_1$ and $\bullet_2$. Now on $\overline{M}_{0,\{\blacktriangle_2,\blacktriangle_3,\circ_1,\bullet_2\}} \cong \mathbb{P}^1$, q is equivalent to the divisor class $[\circ_1, \blacktriangle_2]$. So

$$[E_3^{r,\{I_2,I_3\},I_1}]_Q = \frac{1}{4}(z_{B^r_{\langle I_2\cup I_3,I_1\rangle}})_* h_*(1 \otimes 1 \otimes [\circ_1, \blacktriangle_2] \otimes 1 \otimes 1)$$

$$= \frac{1}{4}(z_{B^r_{\langle I_2\cup I_3,I_1\rangle}})_* \Big(([I_2] \cdot [I_3] \cdot [\circ_1, I_2]) \otimes ([I_1])\Big)$$

$$= \frac{1}{4}\frac{1}{2}\Big(([I_2] \cdot [I_3] \cdot [\circ_1, I_2]) \otimes ([I_1]) + ([I_2] \cdot [I_3] \cdot [\bullet_2, I_2]) \otimes ([I_1])\Big)$$

$$= \frac{1}{8} f^*(d_{I_1} d_{I_2} d_{I_3}) \cdot \mathbb{B}(\langle I_1, I_2, I_3 \rangle, \langle I_2 \cup I_3, I_1 \rangle) = d_{I_1} d_{I_2} d_{I_3} * B^r(\langle I_1, I_2, I_3 \rangle, \langle I_2 \cup I_3, I_1 \rangle)$$

using projection formula, Lemma 5.48 (ii), equation (5.4) and Lemma 5.28 (iv). For (10), additionally use that

$$b^r_{\langle I_1 \cup I_2, I_3 \cup I_4 \rangle} * [(B^r_{\langle I_2 \cup I_3, I_1 \cup I_4 \rangle}, \iota_2)] = B^r(\langle I_2, I_3, I_1, I_4 \rangle, \langle I_2 \cup I_3, I_1 \cup I_4 \rangle)$$

by Lemma 5.28 (v).

(ii): Here $B^r_{P_1} \cap B^r_{P_2}$, may have several components, namely all B^r_P where P are all the elements of $\mathrm{CCR}(P_1, P_2)$. The 2-sectors to which the pull back of both of our classes might be nonzero are all the corresponding $(B^r_P; \iota_{P_1}, \iota_{P_2})$. So with $i_{P,P_1}, i_{P,P_2}, i_{P,\widetilde{P}}$ the closed embeddings of B^r_P into $B^r_{P_1}$, $B^r_{P_2}$ and $B^r_{\widetilde{P}}$ we have, by Proposition 5.44, Lemma 5.45 and (†) :

$$[(B^r_{P_1}, \iota_{P_1})] * B^r(P_2', P_2) = \sum_{P \in \mathrm{CCR}(P_1, P_2)} (i_{P,\widetilde{P}})_* \big((i^*_{P,P_1}(B^r(P_1, P_1))) \cdot (i^*_{P,P_2}(B^r(P_2', P_2))) \big)$$

$$= \sum_{P \in \mathrm{CCR}(P_1, P_2)} (i_{P,\widetilde{P}})_* (i^*_{P,P_2}(B^r(P_2', P_2))).$$

Now (ii) follows from Lemma 5.28 (vi) together with $(i_{P,\widetilde{P}})_*(B^r(\widehat{P}, P)) = B^r(\widehat{P}, \widetilde{P})$, which is clear. □

Lemma 5.57 *Here we use the shorthand $\mathscr{B}^r_Q := [(B^r_Q, \iota_Q)]$ for any circular partition Q of $\underline{n}$, with $|Q| \geq 2$ even.*

(i) Let $P = \langle I_1, ..., I_m \rangle$ be a circular partition of $\underline{n}$ with $m \geq 2$ even, let P' be a refinement of P with refinement map $\rho : P' \to P$. Write

$$P' = \langle J_{1,1}, J_{1,2}, ..., J_{1,\nu_1}, J_{2,1}, ..., J_{2,\nu_2}, ..., J_{m,1}, ..., J_{m,\nu_m} \rangle,$$

such that for each $i \in \underline{m}$, $\rho^{-1}(I_i) = \{J_{i,1}, ..., J_{i,\nu_i}\}$. Also for $m' := |P'|$ set $J_1 := J_{1,1}$, $J_2 := J_{1,2}$,...,$J_{m'} := J_{m,\nu_m}$.

In the following, regard the indices of the I_i resp. J_j in $\underline{m}$ and $\underline{m'}$ as elements of $\mathbb{Z}/m\mathbb{Z}$ resp. $\mathbb{Z}/m'\mathbb{Z}$, when adding numbers to them. We distinguish two cases:

(a) If $|P'| = m'$ is even, set for $l = 1, 2, ..., \frac{m'}{2}$:

$$\widehat{P}'_l := \langle J_l \cup J_{l+1} \cup ... \cup J_{l+\frac{m'}{2}-1},\ J_{l+\frac{m'}{2}} \cup J_{l+\frac{m'}{2}+1} \cup ... \cup J_{l+m'-1} \rangle.$$

Furthermore if $P' \neq P$, let

$$P^* = \langle K_1, K_2, ..., K_\mu \rangle$$

be the partition obtained from $P' = \langle J_1, J_2, ..., J_{m'} \rangle$ by contracting each edge between each two $\{J_{j_1}, J_{j,2}\} \in \mathrm{ON}(P, P')$ (cf. Def. 5.27), i.e. by replacing in $P' = \langle J_1, J_2, ..., J_{m'} \rangle$ the " , " between J_j and J_{j+1} by a " ∪ " if $\{J_j, J_{j+1}\} \in \mathrm{ON}(P, P')$ [53]. Note that $\mu = m' - m$ is then even. For $s = 1, 2, ..., \frac{m}{2}$ set

$$\widehat{P}^*_s := \langle K_s \cup K_{s+1} \cup ... \cup K_{s+\frac{\mu}{2}-1},\ K_{s+\frac{\mu}{2}} \cup K_{s+\frac{\mu}{2}+1} \cup ... \cup K_{s+\mu-1} \rangle.$$

[53] Here WLOG assume that $\{J_{m'}, J_1\} \notin \mathrm{ON}(P, P')$

Then we have

$$B^r(P',P) = \mathscr{B}^r_{\widehat{P}'_1} * \mathscr{B}^r_{\widehat{P}'_2} * ... * \mathscr{B}^r_{\widehat{P}'_{\frac{m'}{2}}} * \mathscr{B}^r_{\widehat{P}^*_1} * \mathscr{B}^r_{\widehat{P}^*_2} * ... * \mathscr{B}^r_{\widehat{P}^*_{\frac{\mu}{2}}} \tag{5.5}$$

(b) If $|P'| = m'$ is odd, there is at least one $\nu_i \geq 2$, WLOG $\nu_1 \geq 2$, i.e. $\{J_1, J_2\} \notin \mathrm{ON}(P,P')$. Then set $\widetilde{J}_1 := J_1 \cup J_2$, and $\widetilde{J}_j := J_{j+1}$ for $j = 2, 3, ..., m'-1$. Set $Q' := \langle \widetilde{J}_1, \widetilde{J}_2, ..., \widetilde{J}_{m'-1}\rangle$. Then Q' is still a refinement of P. Define for $l = 1, 2, ..., \frac{m'-1}{2}$, treating the indices of the $\widetilde{J}_j$ as elements of $\mathbb{Z}/(m'-1)\mathbb{Z}$:

$$\widehat{Q}'_l := \langle \widetilde{J}_l \cup \widetilde{J}_{l+1} \cup ... \cup \widetilde{J}_{l+\frac{m'-1}{2}-1},\ \widetilde{J}_{l+\frac{m'-1}{2}} \cup \widetilde{J}_{l+\frac{m'-1}{2}+1} \cup ... \cup \widetilde{J}_{l+m'-2}\rangle.$$

Let $Q^* = \langle \widetilde{K}_1, \widetilde{K}_2 ...., \widetilde{K}_\kappa\rangle$ be the partition obtained from Q' by contracting all edges belonging to $\mathrm{ON}(P,Q')$ like in the definition of P^*. Note that $\kappa = m' - m - 1$ is even. For $s = 1, 2, ..., \frac{\kappa}{2}$ set

$$\widehat{Q}^*_s := \langle \widetilde{K}_s \cup \widetilde{K}_{s+1} \cup ... \cup \widetilde{K}_{s+\frac{\kappa}{2}-1},\ \widetilde{K}_{s+\frac{\kappa}{2}} \cup \widetilde{K}_{s+\frac{\kappa}{2}+1} \cup ... \cup \widetilde{K}_{s+\kappa-1}\rangle.$$

With $\check{Q}' := \langle J_1,\ J_2 \cup J_3 \cup ... \cup J_{\frac{m'-1}{2}+1},\ J_{\frac{m'-1}{2}+2} \cup .... \cup J_{m'}\rangle$ [54] , we have

$$B^r(P',P) = B^r(\check{Q}', \widehat{Q}'_1) * \mathscr{B}^r_{\widehat{Q}'_2} * ... * \mathscr{B}_{\widehat{Q}'_{\frac{m'-1}{2}}} * \mathscr{B}^r_{\widehat{Q}^*_1} * ... * \mathscr{B}^r_{\widehat{Q}^*_{\frac{\kappa}{2}}} \tag{5.6}$$

(ii) If $P = \langle I_1, I_2\rangle$ is a circular partition of $\underline{n}$, and $P' = \langle J_1, J_2, I_2\rangle$ is a refinement of P, i.e. $J_1 \uplus J_2 = I_1$, [55] then:

$$B^r(P',P) = \mathscr{B}^r_P * \mathscr{B}^r_{\langle J_1,\ J_2 \cup I_2\rangle} - \sum_{\substack{K_a \uplus K_b = J_2 \\ K_1 \neq \emptyset \neq K_2}} \mathscr{B}^r_{\langle K_a \cup J_1,\ K_b \cup I_2\rangle} * \mathscr{B}^r_{\langle J_1 \cup K_b,\ I_2 \cup K_a\rangle}. \tag{5.7}$$

Proof: (i): One shows this by induction, using Proposition 5.56 (ii) and Lemma 5.20. Note that the $\widehat{P}'_l$, $\widehat{P}^*_s$, and $\widehat{Q}'_l$, $\widehat{Q}^*_s$, have been chosen such that for all steps of the multiplication the right hand side of the formula of 5.56 (ii) reduces to something of the form $B^r(P, \widetilde{P})$.

(ii): This follows from Proposition 5.56 (ii) together with Lemma 5.20 (ii). □

Theorem 5.58 *(i) The following collection of classes forms a minimal system of generators of the $H^*(\overline{R}_{1,n})$-algebra $H^*_{CR}(\overline{R}_{1,n})$:*

(1) *Include the fundamental class $[(X,g)]$ for every essential 1-sector (X,g) (cf. Def 5.35), except the class $[(C_4^{\underline{n}}, i)]$ and classes of the form $[(C_4^{\{I_1,I_2\}}, i)]$, which we exclude from our set of generators.*

(2) *For each circular partition $\langle I_1, I_2\rangle$ of $\underline{n}$ into two sets, include the fundamental class $\mathscr{B}^r_{\langle I_1,I_2\rangle} := [(B^r_{\langle I_1,I_2\rangle}, \iota_2)]$.*

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[54] Note that $\check{Q}'$ is a refinement of $\widehat{Q}'_1$.

[55] Note that the pair of partitions $\widehat{Q}'_1$ and $\check{Q}'$ from (i) is of this form.

(ii) In the $H^*(\overline{R}_{1,n})$-algebra $H^*_{CR}(\overline{R}_{1,n})$ the following relations hold, and so the elements $a-b$ corresponding to these equations $a=b$ lie in the ideal of relations $\mathfrak{I}$ [56] *:*

(1) *For each pair $[(X_1,g)]$, $[(X_2,h)]$ of fundamental classes of 1-sectors, such that not both of X_1 and X_2 are banana cycles, the equation of the form $[(X_1,g)]*[(X_2,h)] = \gamma * \mathcal{D}$, which can be read out of the table of Proposition 5.56 (i), or, if the pair is not to be found in the table, then $[(X_1,g)]*[(X_2,h)]=0$.*

(2) *From Proposition 5.56 (ii) for each pair P_1, P_2 of circular partitions of $\underline{n}$, such that $|P_1|=2$ and $|P_2|$ even, and each refinement P_2' of P_2, the equation*

$$[(B^r_{P_1},\iota_{P_1})]*B^r(P_2',P_2)$$

$$=\sum_{P\in\mathrm{CCR}(P_1,P_2)}\ \sum_{P'\in\mathrm{CCR}(P,P_2')}(-1)^{|\widehat{\mathrm{CN}}(P,P_2';P';P_2)|}\sum_{\widehat{P}\in\widehat{\Psi}(P,P_2';P';P_2)}2^{|P_2|+|\widehat{P}|-|P|-|P_2'|}B^r(\widehat{P},\widetilde{P}).$$

(3) *All relations between the generators of the $H^*(\overline{R}_{1,n})$-*module *$H^*_{CR}(\overline{R}_{1,n})$ from Proposition 5.55 (ii) are also included in the list.*

Now many of these relations contain terms that are not written as polynomials over $H^*(\overline{R}_{1,n})$ in the generators listed in (i). Firstly these are relations containing classes of the form $[(C_4^{\frac{n}{2}},i)]$ or $[(C_4^{\{I_1,I_2\}},i)]$. This is remedied by substituting via $[(A_1^{\frac{n}{2}})]*[(C_4^{\frac{n}{2}},-i)] = [(C_4^{\frac{n}{2}},i)]$ resp. $[(A_{2,a}^{\{I_1,I_2\}},-1]*[(C_4^{\{I_1,I_2\}},-i)] = [(C_4^{\{I_1,I_2\}},i)]$. Secondly there appear classes of the form $B^r(P',P)$ with $|P'|\geq 3$. Substitute each $B^r(P',P)$ by a polynomial in classes of the form $\mathscr{B}^r_{\langle I_1,I_2\rangle}$, using equation (5.5) from Lemma 5.57, if $|P'|$ is even, or using equations (5.6) and (5.7) from 5.57, if $|P'|$ is odd. After this procedure all relations in the list are explicit relations between polynomials in the generators listed in (i).

(iii) The relations described in (ii) generate the complete ideal of relations $\mathfrak{I}$ between the generators given in (i), if $H^*_{BCl}(\overline{R}_{1,n}) = H^{2*}(\overline{R}_{1,n})$.

Proof: First note that (ii) is only a collection of relations we have already proven to hold on $H^*_{CR}(\overline{R}_{1,n})$ earlier.

Let $\mathcal{G}$ be the set of the generators listed in (i), let $\mathcal{G}'$ be the larger set of generators of the $H^*(\overline{R}_{1,n})$-module $H^*_{CR}(\overline{R}_{1,n})$ listed in 5.55 (i). Let I the ideal in $H^*(\overline{R}_{1,n})[(\mathcal{G})]$ generated by the relations listed in (ii) (using the notation from the proof of Lemma 5.54).

Let $q' : H^*(\overline{R}_{1,n})^{(\mathcal{G}')} \to H^*_{CR}(\overline{R}_{1,n})$, $q : H^*(\overline{R}_{1,n})[(\mathcal{G})] \to H^*_{CR}(\overline{R}_{1,n})$ be the evaluations. Let $\pi : H^*(\overline{R}_{1,n})^{(\mathcal{G}')} \to H^*(\overline{R}_{1,n})[(\mathcal{G})]$ be the surjective homomorphism defined by sending every $B^r(P',P)\in\mathcal{G}'$ to the unique polynomial in classes of the form $\mathscr{B}^r_{\langle I_1,I_2\rangle}$ as which it can be expressed using the formulas (5.5), (5.6) and (5.7) from Lemma 5.57 [57] , and sending

[56] where $\mathfrak{I}=\ker q$ for $q : H^*(\overline{R}_{1,n})[(\mathcal{G})] \to H^+_{CR}(\overline{R}_{1,n})$ the evaluation with respect to the set of generators $\mathcal{G}$ as specified in (i).

[57] A $B^r(P',P)$ can in $H^*_{CR}(\overline{R}_{1,n})$ usually be expressed as a polynomial in classes $\mathscr{B}^r_{\langle I_1,I_2\rangle}$ in several ways, but there is only one way to do it using only the mentioned formulas.

$[(C_4^{\frac{n}{4}}, i)]$ to $[(A_1^{\frac{n}{1}})] * [(C_4^{\frac{n}{4}}, -i)]$ and each $[(C_4^{\{I_1,I_2\}}, i)]$ to $[(A_{2,a}^{\{I_1,I_2\}}, -1] * [(C_4^{\{I_1,I_2\}}, -i)]$. Since these formulas hold on $H^*_{CR}(\overline{R}_{1,n})$ we have a commutating diagram:

$$H^*(\overline{R}_{1,n})^{(\mathcal{G}')} \xrightarrow{\;\pi\;} H^*(\overline{R}_{1,n})[(\mathcal{G})] \xrightarrow{\;q\;} H^*_{CR}(\overline{R}_{1,n})\ .$$
$$H^*(\overline{R}_{1,n})^{(\mathcal{G}')} \xrightarrow{\;q'\;} H^*_{CR}(\overline{R}_{1,n})$$

By Proposition 5.55, q' is surjective, so q is too, which proves (i), except the claim that $\mathcal{G}$ is minimal. For (iii): The equations of (ii) (1) and (2), after the substitution via Lemma 5.57 suffice to express each element of $H^*(\overline{R}_{1,n})[(\mathcal{G})]$ as an element in $H := \pi(H^*(\overline{R}_{1,n})^{(\mathcal{G}')}) \subseteq H^*(\overline{R}_{1,n})[(\mathcal{G})]$. Since these equations are contained in I as well as $\mathfrak{I} = \ker q$, for (iii) it suffices to show that $I \cap H = \ker q \cap H$ if $H^*_{BCl}(\overline{R}_{1,n}) = H^{2*}(\overline{R}_{1,n})$. By (ii) we already know that $I \subseteq \ker q$. The diagram also tells us that $\ker q \cap H = \pi(\ker q')$. Proposition 5.55 (ii) lists relations which generate $\ker q'$ if $H^*_{BCl}(\overline{R}_{1,n}) = H^{2*}(\overline{R}_{1,n})$. But we included the images under π of these relations into I as part (3) of our list in (ii). So $\pi(\ker q) \subseteq I$, which concludes the proof of (iii).

It remains to show that $\mathcal{G}$ is a *minimal* set of generators: First note that $\mathcal{G}$ consists only of fundamental classes of 1-sector. Suppose there was a class $[(X,g)] \in \mathcal{G}$ which could be expressed over $H^*(\overline{R}_{1,n})$ as a polynomial in the other classes from $\mathcal{G}$. Like every element of $H^*_{CR}(\overline{R}_{1,n})$ we can express such a polynomial in the form

$$\sum_{[(X',h)] \text{ a 1-sector}} \ \sum_{i=1}^{\nu((X',h))} \alpha_{(X',h),i}$$

for some $\nu((X',h)) \in \mathbb{Z}_{\geq 0}$ and homogeneous classes $a_{(X',h),i} \in H^*((X',h))$. Here all summands have to cancel except $\sum_{i=1}^{\nu((X,g))} \alpha_{(X,g),i}$, and in this sum at least one of the $\alpha_{(X,g),i}$ has to have degree 0 in $H^*((X,g))$ [58] , while all summands of higher degree cancel out.

Now let $[(X',h')] \neq [(X,g)]$ be the fundamental class of a 1-sector, and look at a product $\beta * [(X',h')]$, where β is either also the class of a fundamental 1-sector, or $\beta \in H^*(\overline{R}_{1,n})$: As we check with Proposition 5.56 the product is a $\mathbb{Q}$-linear combination of classes of the following 4 types:

type 1: $[(X',h')]$ itself.

type 2: Classes of form $[(C_4^{\frac{n}{4}}, i)]$ or $[(C_4^{\{I_1,I_2\}}, i)]$.

type 3: Classes of form $[(B_P^r, \iota_P)]$ with $|P| \geq 4$.

type 4: Classes $\alpha \in H^*((X'',h''))$ with (X'',h'') a 1-sector, α homogeneous of degree ≥ 1.

One can subsume types 1, 2 and 3 under: Fundamental classes of form $[(X'',h'')] \neq [(X,g)]$ (type A), since $\mathcal{G}$ does not contain classes of type 2 and 3. Now by multiplying a class of type A or of type 4 again with a fundamental class of a 1-sector or a class from $H^*(\overline{R}_{1,n})$, we again obtain a $\mathbb{Q}$-linear combination of classes of these types. Starting with a class of type A this is just repeating the same step as before, to see this for classes of type 4

[58] Here and in the rest of the proof we always mean by degree in a $H^*((X',h'))$ the degree without adjustment by the age number.

is suffices to check, using the definition of the product $*$, that for $\alpha \in H^{d_1}((X,h))$ and $\beta \in H^{d_2}((X',h'))$ the product $\alpha * \beta$ is a sum of classes of the form $\gamma \in H^{d_3}((X'',h''))$ for $d_3 \geq d_1 + d_2$ [59] . Obviously classes of type A or type 4 are not homogeneous of degree 0 in $H^*((X,g))$. □

How "explicit" are the relations in Theorem 5.58 (ii) ? What would one have to do to write down all the relations from (ii) for a given $n \in \mathbb{N}$ "really explicitly"? After gathering the relations together along the various backward references and before doing the substitutions mentioned in (ii), one has to plug in all the various possible partitions of $\underline{n}$ which enter into the relations. Then one has to deal with the many relations involving sums over the coarsest common refinements of certain given circular partitions, i.e. one has to determine all these coarsest common refinements. This is no problem in principle, because we have given a recipe of how to do this in Remark 5.21. Then substitute via Lemma 5.57, as indicated in Theorem 5.58 (ii). After that the relations are as explicit as one could wish for. The only problem is that for all but very small n there are so many of them that one would not want to do all this work.

Here we also remark that the generating set of relations we gave in Theorem 5.58 (iii) is far from minimal. The main reason why it is "too large" is that in determining the relations we worked with sectors (B_P^r, ι_P) for arbitrary large $|P|$, and only later, by substitution, adapted the obtained relations to our smaller set of generators of the algebra $H^*_{CR}(\overline{R}_{1,n})$ which contains only sectors (B_P^r, ι_P) for $|P| = 2$. (It seems to me that, using the results of Theorem 5.58, one can work out a simpler set of relations, which only contains polynomials of small degree in the (B_P^r, ι_P) with $|P| = 2$, and which can be determined for each given $\overline{R}_{1,n}$ without calculating coarsest common refinements for large circularly arranged partitions P. This is something which I would like to finish after handing in this thesis.)

Comparison of $H^*_{CR}(\overline{R}_{1,n})$ and $H^*_{CR}(\overline{M}_{1,n})$

We conclude our examination of $H^*_{CR}(\overline{R}_{1,n})$ by a short discussion of the main differences between our results on this ring, and the results on $H^*_{CR}(\overline{M}_{1,n})$ in [Pag08].

- It is clear that the main differences between $H^*_{CR}(\overline{R}_{1,n})$ and $H^*_{CR}(\overline{M}_{1,n})$ arise from the existence of inessential automorphisms on $\overline{R}_{1,n}$. Since a prym curve with m disjoint non-exceptional components has 2^{m-1} inessential automorphisms, and since more marked points allow a curve to acquire more non-exceptional rational components, the maximal size of $\mathrm{Aut}(\mathfrak{X})$ for $[\mathfrak{X}] \in \overline{R}_{1,n}$ tends towards infinity with growing n. This is a phenomenon which can not occur for $\overline{M}_{g,n}$ for any g: On the contrary, for $n' \leq n$, $\mathfrak{C} \in \overline{M}_{g,n}$ and $\mathfrak{C}'$ obtained from $\mathfrak{C}$ by forgetting all but n' marked points, one always has $|\mathrm{Aut}(\mathfrak{C})| \leq |\mathrm{Aut}(\mathfrak{C}')|$. For larger g the 1-sectors of $\overline{R}_{g,n}$ parametrising inessential automorphisms will become more diverse, since then the underlying curves may contain several "loops" of rational components connected by blown up

[59] Note that it is not the degree in $H^*_{CR}(\overline{R}_{1,n})$ (adjusted by age) we are talking about here, which of course behaves additively under $*$.

nodes, not only one single loop as for the banana cycle sectors of $\overline{R}_{1,n}$. Since a product of two inessential automorphisms is inessential, like for $\overline{R}_{1,n}$, the subspace of $H^*_{CR}(\overline{R}_{g,n})$ coming from inessential automorphisms, will always form a subring.

- For $\overline{R}_{1,n}$ this subring has compared to $H^*_{CR}(\overline{M}_{1,n})$ a relatively "rich" multiplicative structure. One respect in which this shows up is the following: While $H^*_{CR}(\overline{M}_{1,n})$ is generated by the fundamental classes of 1-sectors as a $H^*(\overline{M}_{1,n})$-module, the $H^*(\overline{R}_{1,n})$ module $H^*_{CR}(\overline{R}_{1,n})$ is not. On the other hand, $H^*_{CR}(\overline{R}_{1,n})$ is generated by a collection of fundamental classes of 1-sectors as $H^*(\overline{R}_{1,n})$-algebra, which is considerably smaller than the set of all fundamental classes of 1-sectors, while for the algebra $H^*_{CR}(\overline{M}_{1,n})$ most of the fundamental classes are needed as generators.
- Since much less is known about $H^*(\overline{R}_{1,n})$ then about $H^*(\overline{M}_{1,n})$ (Betti-numbers, Getzler's claims), our results on $H^*_{CR}(\overline{R}_{1,n})$ depending on $H^*(\overline{R}_{1,n})$ give less concrete information than the analogous results in [Pag08].

5.5.6 Remarks on $H^*_{CR}(\overline{S}^+_{1,n})$

The isomorphism $\psi : \overline{S}^+_{1,n} \to \overline{R}_{1,n}$ which holds on the level of varieties (as seen in the introduction of Chapter 4) is not induced by an isomorphism of the stacks of the two moduli problems, and accordingly the natural orbifold structures on $\overline{R}_{1,n}$ and $\overline{S}^+_{1,n}$ as defined in section 5.1.1 are not isomorphic. A smooth pointed spin curve of genus 1 is the same as a smooth pointed prym curve of genus 1 since $\omega_C = \mathcal{O}_C$ for an elliptic curve. But this does not hold for the singular spin and prym curves $\mathfrak{X} = (X; p_1, ..., p_n; \mathcal{L}, b)$. On a non-exceptional component $X_i \cong \mathbb{P}^1$ of X (i.e. X_i carries at last 3 special points), $\mathcal{L}_{|X_i} \cong \mathcal{O}_{X_i}$ in the prym case but $\mathcal{L}_{|X_i} \cong \mathcal{O}(-1)$ in the spin case. More important for us, every disconnecting node of X is exceptional in the spin case, while in the prym case no such node can be exceptional (cf. Summary 1.13 (iii)). Hence the objects of $\overline{S}_{1,n}$ have more inessential automorphisms then the objects of $\overline{R}_{1,n}$. To make this more precise: If $[\mathfrak{X}] \in \overline{S}^+_{1,n}$ and $[\mathfrak{X}'] = \psi([\mathfrak{X}]) \in \overline{R}_{1,n}$ then for (S, s_0) resp. (S', s'_0) the local universal deformation spaces of $\mathfrak{X}$ resp. $\mathfrak{X}'$ there are morphism:

$$S \xrightarrow{f} S' \xrightarrow{\pi'} \overline{R}_{1,n} \xrightarrow{\psi^{-1}} \overline{S}^+_{1,n}, \quad \text{with} \quad f(s_0) = s'_0, \quad \pi(s'_0) = [\mathfrak{X}'],$$

such that f is a ramified cover of complex balls and π' and $\pi := \pi' \circ f \circ \psi^{-1}$ are the usual quotient maps from deformation space to moduli space. Choose a standard basis $\vec{x}_1, ..., \vec{x}_n$ on (S, s_0) in the sense of Summary 1.31, such that with r the number of disconnecting nodes of the stable model C of $\mathfrak{X}$ and $\mathfrak{X}'$, $\vec{x}_1, ..., \vec{x}_r$ are the basis vectors corresponding to these nodes. Set $\vec{x}'_i := f(\vec{x}_i)$. Then f is the map $f(\sum_{i=1}^n \alpha_i \vec{x}_i) = \sum_{i=1}^r \alpha_i^2 \vec{x}'_i + \sum_{i=r+1}^n \alpha_i \vec{x}'_i$. So the orbifold $\overline{S}^+_{1,n}$ is in a sense a cover of $\overline{R}_{1,n}$. There are inessential automorphisms $\varepsilon_1, ..., \varepsilon_r$ on $\mathfrak{X}$ such that ε_i acts non-trivial only on the exceptional component corresponding to the i-th disconnecting node of C. They generate a subgroup $\operatorname{Aut}_0(\mathfrak{X})^+ \subseteq \operatorname{Aut}_0(\mathfrak{X}) \subseteq \operatorname{Aut}(\mathfrak{X})$, such that $\operatorname{Aut}(\mathfrak{X}) / \operatorname{Aut}_0(\mathfrak{X})^+ \cong \operatorname{Aut}(\mathfrak{X}')$, and such that $f : S \to S'$ is the quotient morphism $S \to S / \operatorname{Aut}_0(\mathfrak{X}')^+$.

Accordingly there are more 1-sectors of $\overline{S}^+_{1,n}$ then of $\overline{R}_{1,n}$: The banana cycle sectors (B^r_P, ι_P) lift to isomorphic sectors of $\overline{S}^+_{1,n}$. Furthermore there are the following new inessential 1-sectors: Let, for each $I \subseteq \underline{n}$, $D_I \subset \overline{S}_{g,n}$ be the divisor defined analogously to the divisor of the same name on $\overline{R}_{1,n}$ (cf. section 4.1.1 and Def. 4.5). For an object $\mathfrak{X} = (X'; p_1, ..., p_n; \mathcal{L}', b')$ parametrised by a point of D_I, X contains a rational tree X_I which carries exactly the marked points with index in I. Let ε_I be the inessential automorphism of $\mathfrak{X}$ which acts nontrivially only on the exceptional component connecting X_I to the rest of X. Then (D_I, ε_I) is a 1-sector, and so is every $(D_{I_1,...,I_m}, \varepsilon_{I_1,...,I_m})$ for each $D_{I_1,...,I_m} := D_{I_1} \cap ... \cap D_{I_m} \neq \emptyset$ with $\varepsilon_{I_1,...,I_m} := \varepsilon_{I_1} \cdot ... \cdot \varepsilon_{I_m}$. It is not difficult to show that together these are all inessential 1-sectors of $\overline{S}^+_{1,n}$ and that $[(D_{I_1,...,I_m}, \varepsilon_{I_1,...,I_m})] = [(D_{I_1}, \varepsilon_{I_1})] * ... * [(D_{I_m}, \varepsilon_{I_m})]$. The non inessential sectors (Z^P, g) of $\overline{R}_{1,n}$ all lift to $\overline{S}^+_{1,n}$, but not 1 : 1. As one can check, for example using the summaries of section 1.5, the automorphisms on such sectors Z^P which we denoted by -1 resp. i and $-i$ lift to automorphism of order 4 resp. 8. The second resp. fourth power of these lifings $-\widehat{1}$ and $\widehat{i}$ is inessential. More precisely for the second resp. forth power of classes in $H^*_{CR}(\overline{S}_{1,n})$: $[(A^{I_1,...,I_k}_{k,x}, -\widehat{1})]^2 \in H^*((D_{I_1,...,I_k}, \varepsilon_{I_1,...,I_k}))$ and $[(C^{I_1,...,I_k}_4, \widehat{i})]^4 \in H^*((D_{I_1,...,I_k}, \varepsilon_{I_1,...,I_k}))$. It does not seem to be a problem to determine the additive Chen-Ruan cohomology of $\overline{S}^+_{1,n}$ applying the same methods as for $\overline{R}_{1,n}$ and also to produce a multiplication table for the fundamental classes of 1-sectors like the one in Proposition 5.56. But determining $H^*_{CR}(\overline{S}^+_{1,n})$ as a $H^*(\overline{S}_{1,n})$-algebra will probably be much more difficult: Since for a given n, $D_I \cong \overline{S}^+_{1,n-|I|+1} \times \overline{M}_{0,|I|+1}$, for $n - |I| + 1 \geq 11$ the cohomology $H^*((D_I, \varepsilon_I))$ will have a non-vanishing odd part. (Because $H^{11}(\overline{M}_{1,11}) \neq 0$.) So one can not expect the odd cohomology of $\overline{S}^+_{1,n}$ to pull back to 0 on every twisted 1-sector. Thus one will probably need much more information about the odd part of $H^*(\overline{S}^+_{1,n})$ than we have now to be able to obtain a generating set of relations of the $H^*(\overline{S}^+_{1,n})$-algebra $H^*_{CR}(\overline{S}^+_{1,n})$.

5.6 Singularities and Kodaira dimension of $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$

Thematically this section would have fitted better into the previous chapter 4, but it uses information from this chapter and was therefore put here.

5.6.1 Singularities of $\overline{M}_{1,n}$ and $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$

In order to compute the Chen-Ruan cohomology, N. Pagani determined all automorphisms that exist on $\overline{M}_{1,n}$, the loci on which they exist, and the way they act on the tangent space of the stack $\overline{M}_{1,n}$, i.e. on the local deformation spaces. We adapted his result for $\overline{R}_{1,n}$. But with this information at hand it is quite easy to determine the singular locus of these moduli spaces (as varieties), and the locus of (non-) canonical singularities, using the generalised Reid-Tai-Criterion. The idea how to do this basically comes from [HM82]. The method was refined by taking into account so called quasi-reflections and applied to $\overline{S}_g$ and $\overline{R}_g$, for $g \geq 4$, by Katharina Ludwig in [Lud07], [Lud10] and [FL10].

We will cite some definitions and theorems for which we take section 4.1. of [Lud07] as a reference.

Definition 5.59 (i) For a normal quasiprojective variety, let K_X be the Weil divisor such that $\omega_X = \mathcal{O}_X(K_X)$ (for its existence cf. [Rei87]). X is said to have *canonical singularities* if:

(1) For some integer $r \geq 1$, rK_X is a Cartier divisor, and

(2) if $f : \widetilde{X} \to X$ is a desingularisation of X and $\{E_i\}$ is the family of all exceptional prime divisors of f, then for K_X and $K_{\widetilde{X}}$ the canonical divisors:

$$rK_{\widetilde{X}} = f^*(rK_X) + \sum a_i E_i$$

where all $a_i \geq 0$.

Now let V be an m-dimensional $\mathbb{C}$ vector space, φ an automorphism of finite order n on V. Then:

(ii) φ is called a *quasi-reflection* if 1 is a eigenvalue of φ of order exactly $m-1$.

(iii) One can choose a basis of V relative to which φ is represented by a diagonal matrix $M(\varphi)$. If ζ is any primitive n-th root of unity, then

$$M(\varphi) = \begin{pmatrix} \zeta^{b_1} & & \\ & \ddots & \\ & & \zeta^{b_m} \end{pmatrix}$$

for appropriate $0 \leq b_i < n$. We define the *age of φ with respect to ζ* to be

$$\text{age}(\varphi, \zeta) := \frac{1}{n} \sum_{i=1}^{m} b_i.$$

This is also called the Reid-Tai sum of φ with respect to ζ. Note that this sum depends on ζ but not on the chosen basis of V.

We will apply the following criteria:

Theorem 5.60 *Let V be a finite dimensional $\mathbb{C}$ vector space, and let $G \subset GL(V)$ be a finite subgroup. Let V/G be the quotient. Then:*

(i) V/G is non-singular if and only if G is generated by quasi-reflections (or by the identity).

(ii) V/G has only canonical singularities, if for every $\varphi \in G$, and for every primitive n-th root of unity ζ we have

$$\text{age}(\varphi, \zeta) \geq 1.$$

This is called the Reid-Tai criterion.

(iii) If G contains no quasi-reflections, the "if" in (ii) can be replaced by "if and only if".

Theorem 5.61 *(i) The singular locus of $\overline{M}_{1,n}$ for $n \geq 1$ is*

$$\bigcup_{\substack{\{I_1,I_2\},\\ I_1 \uplus I_2 = \underline{n}}} A_2^{I_1,I_2} \cup \bigcup_{\substack{\{I_1,I_2,I_3\},\\ I_1 \uplus I_2 \uplus I_3 = \underline{n}}} A_3^{I_1,I_2,I_3} \cup \bigcup_{\substack{\{I_1,\dots,I_4\},\\ I_1 \uplus \dots \uplus I_4 = \underline{n}}} A_4^{I_1,I_2,I_3,I_4}$$

$$\cup C_4^{\underline{n}} \cup \bigcup_{\substack{\{I_1,I_2\},\\ I_1 \uplus I_2 = \underline{n}}} C_4^{I_1,I_2} \cup C_6^{\underline{n}} \cup \bigcup_{\substack{\{I_1,I_2\},\\ I_1 \uplus I_2 = \underline{n}}} C_6^{I_1,I_2} \cup \bigcup_{\substack{\{I_1,I_2,I_3\},\\ I_1 \uplus I_2 \uplus I_3 = \underline{n}}} C_6^{I_1,I_2,I_3}$$

(In all the unions all the I_i are required to be non-empty.)

(ii) The singular locus of $\overline{R}_{1,n}$ for $n \geq 1$ is

$$\bigcup_{\substack{\{I_1,I_2\},\\ I_1 \uplus I_2 = \underline{n}}} A_2^{\{I_1,I_2\}} \cup \bigcup_{\substack{\{\{I_1,I_2\},I_3\},\\ I_1 \uplus I_2 \uplus I_3 = \underline{n}}} A_3^{\{I_1,I_2\},I_3} \cup \bigcup_{\substack{\{\{I_1,I_2\},\{I_3,I_4\}\},\\ I_1 \uplus \dots \uplus I_4 = \underline{n}}} A_4^{\{\{I_1,I_2\},\{I_3,I_4\}\}}$$

$$\cup C_4^{\underline{n}} \cup \bigcup_{\substack{\{I_1,I_2\},\\ I_1 \uplus I_2 = \underline{n}}} C_4^{\{I_1,I_2\}} \cup \bigcup_{\substack{\{I_1,I_2\},\\ I_1 \uplus I_2 = \underline{n}}} B^r_{\langle I_1,I_2\rangle}$$

(Again the I_i are all required to be nonempty.)

(iii) $\overline{M}_{1,n}$ has non-canonical singularities for all $n \geq 2$. For these n, the locus of non-canonical singularities on $\overline{M}_{1,n}$ is $C_6^{\underline{n}}$.

(iv) $\overline{R}_{1,n}$ has only canonical singularities.

Proof: First we note that describing the action of a automorphisms on a deformation space of a pointed stable curve $\mathfrak{C}$ or prym curve $\mathfrak{X}$, is the same as describing the action on the tangent space of the moduli stack at the point $[\mathfrak{C}]$ resp. $[\mathfrak{X}]$. So we can use the description of the action of automorphisms on this tangent space in [Pag08] (for $\mathfrak{C}$) and in this chapter (for $\mathfrak{X}$) to prove the theorem.

(i): By Theorem 3.24. of [Pag08], the locus of curves with nontrivial automorphisms in $\overline{M}_{1,n}$ consists of the locus we claim to be the singular locus, and of $A_1^{[n]}$. First note that an automorphism of order m acts as a quasireflection if an only if it acts with age $\frac{1}{m}$. So by the table in Corollary 4.8. of [Pag08] the only 1-sector of $\overline{M}_{1,n}$ belonging to a quasi-reflection is $(A_1^{[n]}, -1)$. For a general object $\mathfrak{C}$ of $A_1^{[n]}$, -1 is the only nontrivial automorphism, and hence generates $\mathrm{Aut}(\mathfrak{C})$. So $\overline{M}_{1,n}$ is nonsingular at a general point of $A_1^{[n]}$, while at every point outside $A_1^{[n]}$ parametrising objects with non-trivial automorphisms, $\overline{M}_{1,n}$ is singular.

(ii): Here one argues analogously to (i), using instead of the results of [Pag08] our results Thm. 5.32 and Corollary 5.39.

(iii): For an automorphism of order 2 there is only one possible choice of the root of unity appearing in the Reid-Tai sum, and the Reid-Tai sum equals the age by which the automorphism acts. So for all objects $\mathfrak{X}$ of $\overline{M}_{1,n}$ for which $\mathrm{Aut}(\mathfrak{X})$ is generated by automorphisms of order 2 one sees by the table in Corollary 4.8. of [Pag08] that they fulfill the Reid-Tai criterion, except in the case $[\mathfrak{X}] \in A_1^{\underline{n}}$, in which $\mathrm{Aut}(\mathfrak{X})$ is generated by a quasi-reflection. So, using the list of the 1-sectors of $\overline{M}_{1,n}$ in Theorem 3. 24 of [Pag08], the only candidates for non-canonical singularities are the points $[\mathfrak{C}]$ in $C_4^{[n]}$, $C_4^{I_1,I_2}$, $C_6^{[n]}$,

$C_6^{I_1,I_2}$, $C_6^{I_1,I_2,I_3}$. For these we know the action of $\mathrm{Aut}(\mathfrak{C})$ on the deformation space explicitly by [Pag08] Prop. 4.7. Using this one can check that for objects in $C_6^{I_1,I_2}$ and $C_6^{I_1,I_2,I_3}$ the automorphisms all have Reid-Tai sums which are ≥ 1, so there are no non-canonical singularities in these loci, by the Reid-Tai criterion. For $C_4^{[n]}$, $C_6^{[n]}$, and in the special case of $C_4^{I_1,I_2}$ with $|I_1| = 1$ and $|I_2| = 1$, there are automorphisms for which not all Reid-Tai sums are ≥ 1. But for a $\mathfrak{C}$ in one of these three loci, the automorphism $-1 = i^2 \in \mathrm{Aut}(\mathfrak{C})$ resp. $-1 = \epsilon^3 \in \mathrm{Aut}(\mathfrak{C})$ acts as a quasireflection. Thus we can not directly conclude by the Reid-Tai criterion that these loci are non-canonical singularities. Instead we have to quotient the deformation space by the quasi-reflection first, and then have to consider the action of $\mathrm{Aut}(\mathfrak{C})$ on the resulting smooth quotient:

First consider the case $[\mathfrak{C}] \in C_4^{[n]}$ $(n \geq 2)$. On the deformation space B we can (by [Pag08] Prop. 4.7) choose a basis[60] $\vec{x}_1, ..., \vec{x}_n$ such that the automorphisms i, i^3 and $-1 = i^2$ of $\mathfrak{C}$ act by diagonal matrices of the form

$$M(i) = \begin{pmatrix} i^2 & & \\ & i^3 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}, \quad M(i^3) = \begin{pmatrix} i^2 & & \\ & i & \\ & & \mathbb{1}_{n-2} \end{pmatrix}, \quad M(-1) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \mathbb{1}_{n-2} \end{pmatrix},$$

where $\mathbb{1}_{n-2}$ denotes a identity matrix of size $(n-2) \times (n-2)$. Now if $\pi : B \to B/M(-1)$ is the quotient-morphism, $B/M(-1)$ is again isomorphic to a open complex n-ball, and $(\vec{z}_1, ..., \vec{z}_n) := (\pi(\vec{x}_1), ..., \pi(\vec{x}_n))$ is a basis of $B/M(-1)$. The map π can be described with respect to these bases by

$$\pi(\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \alpha_3\vec{x}_3 ... + \alpha_n\vec{x}_n) = \alpha_1\vec{z}_1 + \alpha_2^2\vec{z}_2 + \alpha_3\vec{z}_3 ... + \alpha_n\vec{z}_n, \quad \text{for all} \quad (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n.$$

Now it is clear that the actions of i and i^3 descend to actions on the quotient $B/M(-1)$ which are relative to the basis $\vec{z}_1, ..., \vec{z}_n$ represented by the matrices:

$$\bar{M}(i) = \begin{pmatrix} i^2 & & \\ & i^6 = i^2 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}, \quad \bar{M}(i^3) = \begin{pmatrix} i^2 & & \\ & i^2 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}.$$

So both automorphism act by the same matrix on $B/M(-1)$ and one can check that the Reid-Tai sums of this Matrix are 1 for both primitive second roots of unity i and i^3. Hence the quotient $B/\mathrm{Aut}(\mathfrak{C}) \cong (B/M(-1))/\bar{M}(i)$ has canonical singularities.

The case of $[\mathfrak{C}] \in C_4^{I_1,I_2}$ with $|I_1| = 1$ and $|I_2| = 1$ can analogously be shown to yield only canonical singularities.

If $[\mathfrak{C}] \in C_6^{[n]}$ we have $\mathrm{Aut}(\mathfrak{C}) = \mu_6 = \langle \epsilon \rangle$. Here the automorphisms ϵ, ϵ^2 and $\epsilon^3 = -1$ act relative to a suitably chosen basis by

$$M(\epsilon) = \begin{pmatrix} \epsilon^4 & & \\ & \epsilon^5 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}, \quad M(\epsilon^2) = \begin{pmatrix} \epsilon^2 & & \\ & \epsilon^4 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}, \quad M(-1) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}.$$

[60] Cf. Notation 1.29 (i)

We do not have to consider the actions of ϵ^4 and ϵ^5, since an automorphism and its inverse yield the same sets of Reid-Tai sums. On $B/M(-1)$, ϵ and ϵ^2 act by

$$\bar{M}(\epsilon) = \begin{pmatrix} \epsilon^4 & & \\ & \epsilon^4 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}, \quad \bar{M}(\epsilon^2) = \begin{pmatrix} \epsilon^2 & & \\ & \epsilon^2 & \\ & & \mathbb{1}_{n-2} \end{pmatrix}.$$

If we choose the primitive 6-th root of unity ϵ then $\bar{M}(\epsilon)$ yields the Reid-Tai sum $\frac{8}{6} \geq 1$, but if we choose the primitive root ϵ^5 instead, the Reid-Tai sum of $\bar{M}(\epsilon)$ is $\frac{4}{6} < 1$, since $(\epsilon^5)^2 = \epsilon^4$. So all points of $C_6^{[n]}$ are non-canonical singularities of $\overline{M}_{1,n}$ (for $n \geq 2$).

(Note that $C_4^{[n]}$, $C_6^{[n]}$ can be said to parametrise objects with special elliptic tails, and $C_4^{\{I_1,I_2\}}$ to parametrise objects with special elliptic bridges. In case of $\overline{S}_g$ with $g \geq 4$ such loci are investigated in [Lud07] section 4.3, and shown to contain non-canonical singularities exactly in the case analogous to $C_6^{[n]}$. Our proof above is probably the same as the proof given for the analogous cases there.)

(iv): Here we argue analogously to (iii) but use Lemma 5.38 instead of [Pag08] Prop. 4.7. □

Part (iv) of Theorem 5.61 directly implies:

Corollary 5.62 *Let $\widetilde{R}_{1,n}$ be a desingularisation of the variety $\overline{R}_{1,n}$, let $\overline{R}_{1,n}^{reg} \subseteq \overline{R}_{1,n}$ be the open subvariety of nonsingular points. Then:*

(i) Every pluricanonical form on $\overline{R}_{1,n}^{reg}$ extends to $\widetilde{R}_{1,n}$, i.e.

$$H^0(\overline{R}_{1,n}^{reg}, \mathcal{O}_{\overline{R}_{1,n}}(mK_{\overline{R}_{1,n}})) = H^0(\widetilde{R}_{1,n}, \mathcal{O}_{\overline{R}_{1,n}}(mK_{\widetilde{R}_{1,n}}))$$

for all m and n.

(ii) Thus for the Kodaira dimension $\kappa(\overline{R}_{1,n})$ we have:

$$\kappa(\overline{R}_{1,n}) = \kappa(\widetilde{R}_{1,n}, K_{\widetilde{R}_{1,n}}) = \kappa(\overline{R}_{1,n}, K_{\overline{R}_{1,n}})$$ [61]

Remark: It should be possible to prove a complete analogue of Corollary 5.62 for $\overline{M}_{1,n}$, by applying the method on page 40-44 of [HM82] to the non-canonical singularities in C_6^n (like in [Lud07], section 5.2). But we will not attempt this here. Furthermore, Corollary 5.62 (ii) and its analogue for $\overline{M}_{1,n}$ seem to be implicitly applied in [BF06].

5.6.2 The Kodaira Dimension

The Kodaira dimension (cf. Def. 1.51) of $\overline{M}_{1,n}$ is computed in [BF06] for all $n \in \mathbb{N}$. It is

$$\kappa(\overline{M}_{1,n}) = \begin{cases} -\infty, & 1 \leq n \leq 10 \\ 0, & n = 11 \\ 1, & n \geq 12 \end{cases}$$

[61] The direction $\kappa(R_{1,n}) \leq \kappa(\overline{R}_{1,n}, K_{\overline{R}_{1,n}})$ follows from the fact that for a normal variety X of dimension n with $j : X^{reg} \to X$ the embedding, $\omega_X = j_*(\Omega^n_{X^{reg}})$.

By Thm. 3 of [BF06] and Belorousski's result that $\overline{M}_{1,n}$ is rational for $n \leq 10$.

For $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$ the Kodaira dimension $\kappa(\overline{R}_{1,n})$ is computed for all $n \neq 11$ in [BF06], and turns out to be equal to $\kappa(\overline{M}_{1,n})$ in these cases. For $n = 11$ it is shown that $0 \leq \kappa(\overline{R}_{1,11}) \leq 1$. (Lemma 2, Proposition 4 and Proposition 5 of [BF06].)

We will show $\kappa(\overline{R}_{1,11}) = 1$, and therefore $\kappa(\overline{R}_{1,11}) \neq \kappa(\overline{M}_{1,11})$. This answers Question 1 asked in [BF06].

In order to compute $\kappa(\overline{M}_{1,n})$ the following Proposition was shown:

Proposition 5.63 (Prop. 3 in [BF06]) *For any integer $n \geq 3$, and for $K_{\overline{M}_{1,n}}$ the canonical divisor of $\overline{M}_{1,n}$:*

$$K_{\overline{M}_{1,n}} = (n-11)\lambda + (n-3)\delta_n + \sum_{\substack{I \subset \underline{n}, \\ |I| \geq 2,\ I \neq \underline{n}}} (|I|-2)\delta_I$$

where λ as usual denotes is the first Chern class of the Hodge bundle on $\overline{M}_{1,n}$.

(In [BF06] a different notation for the boundary divisors is used.)

Lemma 5.64 *The ramification divisor of the forgetful morphism $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$, viewed as a morphism of varieties, is the boundary divisor D^r_0.*

Proof: By Summary 1.31 (vi) and Summary 1.13 (i) we know that for every $[\mathfrak{X}] \in \overline{R}_{1,n}$ the forgetful morphism $\pi : (S, s_0) \to (B, b_0)$ between the local universal deformation spaces of $\mathfrak{X}$ and of its stable model $\mathfrak{C}$ is an isomorphisms if $[\mathfrak{X}] \notin D^r_0$ (case 1). For a general $[\mathfrak{X}] \in D^r_0$, one can choose standard bases $(\vec{y}_i)_{i \in \underline{n}}$, $(\vec{x}_i)_{i \in \underline{n}}$ of (S, s_0) and (B, b_0) (cf. Summary 1.31) such that for the coordinate y_1 corresponding to $\vec{y}_1$, $\{y_1 = 0\}$ is the subspace of (S, s_0) parametrising objects of D^r_0. Then for all $z = \alpha_1\vec{y}_1 + \alpha_2\vec{y}_2 + ... + \alpha_n\vec{y}_n \in S$, $\pi(z) = \alpha_1^2\vec{x}_1 + \alpha_2\vec{x}_2 + ... + \alpha_n\vec{x}_n$. Since a general point of D^r_0 has no non-trivial automorphisms by Theorem 5.32, we can conclude with Summary 1.31 (iii) that locally analytically around general points of D^r_0, τ_n can be identified with π and hence the ramification divisor of τ_n indeed contains D^r_0 with multiplicity 1.

We again use Summary 1.31 (iii), to see that in case 1, $[\mathfrak{X}] \in \overline{R}_{1,n}$ can only lie on a component of the ramification divisor if there is a $g \in \mathrm{Aut}(\mathfrak{C})$ such that firstly g does not lift to $\mathfrak{X}$, and secondly the set of fixed points $\mathrm{Fix}(g) \subset (B, b_0)$ is of codimension 1. But we know that all pairs $(\mathfrak{C}, g) \in I_1(\overline{M}_{1,n})$ fulfilling the second part of this condition are parametrised by $(\overline{A}^{\underline{n}}_1, -1)$. [62] But the automorphism -1 lifts to all objects $\mathfrak{X}$ in $\tau_n^{-1}(\overline{A}^{\underline{n}}_1)$ [63] so the first part of the condition can not be fulfilled if the second part is. Hence the ramification divisor of τ_n is supported entirely on D^r_0. □

[62] This is for example clear by the fact that among the supports of 1-sectors of $\overline{M}_{1,n}$, $\overline{A}^{\underline{n}}_1$ is the only one of codimension 1.

[63] $\tau_n^{-1}(\overline{A}^{\underline{n}}_1) \subset \overline{R}_{1,n}$ is the locus we denoted again by $\overline{A}^{\underline{n}}_1$. This is just the boundary divisor $D_{\underline{n}}$.

Corollary 5.65 *For any integer $n \geq 3$:*

$$K_{\overline{R}_{1,n}} = d_0^r + (n-11)\lambda + (n-3)d_n + \sum_{\substack{I \subset \underline{n}, \\ |I| \geq 2,\ I \neq \underline{n}}} (|I|-2)d_I$$

where λ denotes the first Chern class of the Hodge bundle on $\overline{R}_{1,n}$.

Proposition 5.66 *The Kodaira dimension of $\overline{R}_{1,11} \cong \overline{S}^+_{1,11}$ is $\kappa(\overline{R}_{1,11}) = 1$.*

Proof: We already know that $\kappa(\overline{R}_{1,11}) \leq 1$ from [BF06]. Thus it suffices to show $\kappa(\overline{R}_{1,11}) \geq 1$. This works similar to the proof that $\kappa(\overline{M}_{1,n}) \geq 1$ for $n \geq 12$ in [BF06].

By Corollary 5.65 we have

$$K_{\overline{R}_{1,11}} = d_0^r + 8d_{11} + \sum_{\substack{I \subset \underline{11}, \\ |I| \geq 2,\ I \neq \underline{11}}} (|I|-2)d_I$$

Thus $K_{\overline{R}_{1,11}}$ is the sum of $d_0^r = [D_0^r]_Q = [D_0^r]$ and an effective divisor. Hence we have an inequality of Iitaka dimensions $\kappa(\overline{R}_{1,11},\, K_{\overline{R}_{1,11}}) \geq \kappa(\overline{R}_{1,11},\, D_0^r)$, and together with Corollary 5.62 (ii) this yields $\kappa(\overline{R}_{1,11}) \geq \kappa(\overline{R}_{1,11},\, D_0^r)$.

Let $\pi : \overline{R}_{1,11} \to \overline{R}_{1,1}$ be the morphism forgetting the last 10 marked points. Denote by $D_0^{r,1}$ the boundary divisor D_0^r of $\overline{R}_{1,1}$ to distinguish it from the boundary divisor D_0^r of $\overline{R}_{1,11}$. Then $d_0^r = \pi^* d_0^{r,1}$. But $d_0^{r,1} = [D_0^{r,1}]_Q = \frac{1}{2}[D_0^{r,1}]$, and $D_0^{1,r}$ is a point on $\overline{R}_{1,1} \cong \mathbb{P}^1$. Hence a multiple of $d_0^{r,1}$ is ample. (For $\overline{R}_{1,1} \cong \mathbb{P}^1$, cf. Prop. 4.15.) Thus the Iitaka dimension $\kappa(\overline{R}_{1,1},\, d_0^{r,1})$ is 1. Since π is surjective, $\kappa(\overline{R}_{1,1}, d_0^{r,1}) = \kappa(\overline{R}_{1,11}, \pi^* d_0^{r,1})$ by Theorem 5.13 of [Uen75]. Hence we have $\kappa(\overline{R}_{1,11}) \geq \kappa(\overline{R}_{1,11},\, d_0^r) = \kappa(\overline{R}_{1,1},\, d_0^{r,1}) = 1$. □

5.7 Euler characteristic and Cohomology of $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$ for small n

In this section we use previous results of this chapter for some simple observations about the Euler characteristic of $\overline{R}_{1,n} \cong \overline{S}^+_{1,n}$. Using them we compute the Euler characteristic for $n \leq 5$. This result implies that for $n \leq 4$, the Chow-Rings $A^*(\overline{R}_{1,n})$ we computed in section 4.4 are isomorphic to the cohomology rings $H^*(\overline{R}_{1,n})$.

We denote the Euler characteristic of a space X by $\chi(X)$. Recall that χ behaves multiplicative under cartesian products, and that for $f : X \to Y$ a unramified finite morphism of degree m (i.e. covering of degree m), $\chi(X) = m\chi(Y)$. Furthermore for subvarieties $X_1, ..., X_n$ of a complex algebraic variety, but not in general, χ fulfils the inclusion-exclusion principle, i.e. $\chi(X_1 \cup ... \cup X_n) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{1 \geq i_1 \geq ... \geq i_k \geq n} \chi(X_{i_1} \cap ... \cap X_{i_k})$ (cf. the exercise on page 95 of [Ful93] and the corresponding endnote 13 on page 141).

Summary 5.67 *Let $M'_{0,n} := M_{0,n}/\mathbb{S}_2$ be the quotient of $M_{0,n}$ by the $\mathbb{S}_2$-action transposing the indices n and $n-1$ of marked points. For $(M_{0,m} \times M_{0,n})' := (M_{0,m} \times M_{0,n})/\mathbb{S}_2$ let $\mathbb{S}_2$ act by simultaneously transposing m with $m-1$ and n with $n-1$. Then:*

(i) For all $n \geq 3$*:* $\chi(M_{0,n}) = (-1)^{n-3}(n-3)!$ *(with* $0! := 1$*).*

(ii) $\chi(M'_{0,3}) = 1$*,* $\chi(M'_{0,4}) = 0$*, for all* $n \geq 5$*:* $\chi(M'_{0,n}) = \frac{1}{2}\chi(M_{0,n}) = \frac{1}{2}(-1)^{n-3}(n-3)!$*.*

(iii) $\chi((M_{0,3} \times M_{0,4})') = 0$*,* $\chi((M_{0,4} \times M_{0,4})') = 1$*, and for all* $m \geq 3$*,* $n \geq 5$*:*

$$\chi((M_{0,m} \times M_{0,n})') = \frac{1}{2}(-1)^{m+n}(m-3)!(n-3)!.$$

(iv) $\chi(M_{1,1}) = \chi(M_{1,2}) = 1$*,* $\chi(M_{1,3}) = \chi(M_{1,4}) = 0$*,* $\chi(M_{1,5}) = -2$*, and for all* $n \geq 5$*:* $\chi(M_{1,n}) = \frac{1}{12}(-1)^n(n-1)!$*.*

(v) For all $n \in \mathbb{N}$*,* $H^1(\overline{R}_{1,n}) = H^3(\overline{R}_{1,n}) = 0$*.*

Proof: For (i) cf. [AC98] page 121, for (iv) cf. [Get99] Proposition 5.7., for (v) cf. [BF09b]. Also (ii) is more or less from [AC98]: For (ii) and (iii) note that the quotient maps $M_{0,n} \to M'_{0,n}$ and $M_{0,m} \times M_{0,n} \to (M_{0,m} \times M_{0,n})'$ are $2:1$ covers which are ramified exactly at the fixed points of the $\mathbb{S}_2$ action. If we denote by $\mathrm{Fix}_{\mathbb{S}_2}(M_{0,n})$ the set of fixed points on $M_{0,n}$ then $\mathrm{Fix}_{\mathbb{S}_2}(M_{0,m} \times M_{0,n}) = \mathrm{Fix}_{\mathbb{S}_2}(M_{0,m}) \times \mathrm{Fix}_{\mathbb{S}_2}(M_{0,n})$. Since $Fix_{\mathbb{S}_2}(M_{0,n}) = \emptyset$ for $n \geq 5$ (there is no automorphism of $\mathbb{P}^1$ fixing three points and exchanging two), the quotient maps are unramified in this case and $\chi(M_{0,n}) = 2\chi(M'_{0,n})$, $\chi(M_{0,m})\chi(M_{0,n}) = \chi(M_{0,m} \times M_{0,n}) = 2\chi((M_{0,m} \times M_{0,n})')$. There is one isomorphism class of configurations of 4 points on $\mathbb{P}^1$ allowing an automorphism which fixes two and exchanges two, so $\mathrm{Fix}_{\mathbb{S}_2}(M_{0,n})$ is a point p. Hence $\chi(M'_{0,4}) = 0$:

$$2\chi(M'_{0,4}) - 2 = 2\chi(M'_{0,4} \setminus p) = \chi(M_{0,4} \setminus p) = \chi(M_{0,4}) - 1 = -2.$$

The rest of (iii) is proven analogously. □

We remark that for $n \leq 3$ the results of the next Proposition where already computed in [BF09b].

Proposition 5.68 *(i)* $\chi(R_{1,1}) = \chi(R_{1,2}) = 0$*,* $\chi(R_{1,3}) = -2$*,* $\chi(R_{1,4}) = 0$*, and for all* $n \geq 5$*:*

$$\chi(R_{1,n}) = 3\chi(M_{1,n}) = \frac{1}{4}(-1)^n(n-1)!.$$

(ii) $\chi(\overline{R}_{1,1}) = 2$*,* $\chi(\overline{R}_{1,2}) = 4$*,* $\chi(\overline{R}_{1,3}) = 12$*,* $\chi(\overline{R}_{1,4}) = 50$*,* $\chi(\overline{R}_{1,5}) = 270$*.*

(iii) Define $\Delta_{I_1,\dots,I_k} := \Delta_{I_1} \cap \dots \cap \Delta_{I_k}$ *and* $D_{I_1,\dots,I_k} := D_{I_1} \cap \dots \cap D_{I_k}$*. For* $n \geq 5$*:*

$$\chi(\overline{M}_{1,n}) = \frac{1}{12}(-1)^n(n-1)! + n! \sum_{m=1}^{n} \frac{(-1)^{n-m}}{2m} \sum_{r_1+r_2+\dots r_m=n} \frac{1}{r_1 \cdot r_2 \cdot \dots \cdot r_m}$$

$$+ \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{\{I_1,\dots,I_k\} \\ I_i \subset \underline{n},\, |I_i| \geq 2}} \chi(\Delta_{I_1,\dots,I_k}) \ ^{64}.$$

$$\chi(\overline{R}_{1,n}) = \frac{1}{4}(-1)^n(n-1)! + n! \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m} \sum_{r_1+r_2+\dots r_m=n} \frac{1}{r_1 \cdot r_2 \cdot \dots \cdot r_m}$$

[64] Note that $\chi(\overline{M}_{1,n})$ is calculated in [Get98], so the formula given here is only needed for comparison with the next formula for $\chi(\overline{R}_{1,n})$.

$$+\sum_{k=1}^{n}(-1)^{k+1}\sum_{\substack{\{I_1,\ldots,I_k\}\\ I_i\subset \underline{n},\,|I_i|\geq 2}}\chi(D_{I_1,\ldots,I_k})\ ^{65}.$$

Proof: (i): The forgetful morphism $\tau_n : \overline{R}_{1,n} \to \overline{M}_{1,n}$ is of degree 3 and for a point $[\mathfrak{C}] \in M_{1,n}$ we have $|\tau_n^{-1}([\mathfrak{C}])| < 3$ if and only if $\mathfrak{C} = (C; p_1, .., p_n)$ has an automorphism which does not fix all three isomorphism classes of prym sheaves on C. By the proof of Lemma 5.9 the only such points are $C_4, C_6 \in \overline{M}_{1,1}$, $C_4', C_6' \in \overline{M}_{1,2}$ and $C_6'' \in \overline{M}_{1,3}$. Furthermore we have seen there that the automorphisms of C_4 and C_4' transpose two classes of prym sheaves and fix one, while the automorphism ϵ^2 of C_6, C_6', C_6'' cyclically permutes all three isomorphism classes. Hence $|\tau_1^{-1}(C_4)| = |\tau_2^{-1}(C_4')| = 2$, and $|\tau_1^{-1}(C_6)| = |\tau_2^{-1}(C_6')| = |\tau_3^{-1}(C_6'')| = 1$. With this:

$$\chi(R_{1,1}) = 3\chi(M_{1,1} \smallsetminus \{C_4, C_6\}) + \chi(\tau_1^{-1}(C_4)) + \chi(\tau_1^{-1}(C_6)) = 3\chi(M_{1,1}) - 3$$

similarly: $\chi(R_{1,2}) = 3\chi(M_{1,2}) - 3$, $\chi(R_{1,3}) = 3\chi(M_{1,3}) - 2$, $\forall n \geq 4\ \chi(R_{1,n}) = 3\chi(M_{1,n})$.

This together with Summary 5.67 (iv) yields (i).

For each n, set inside $\overline{M}_{1,n}$ resp. $\overline{R}_{1,n}$

$$T_{1,n} := \bigcup_{I\subset \underline{n},\,|I|\geq 2}\Delta_I,\quad \mathscr{T}_{1,n} := \bigcup_{I\subset \underline{n},\,|I|\geq 2} D_I,\quad \widetilde{\Delta}_0 := \Delta_0 \smallsetminus (T_{1,n}\cap \Delta_0),$$

$$\widetilde{D}_0'' := D_0'' \smallsetminus (\mathscr{T}_{1,n}\cap D_0''),\quad \widetilde{D}_0^r := D_0^r \smallsetminus (\mathscr{T}_{1,n}\cap D_0^r).$$

Then we have

$$\overline{M}_{1,n} = M_{1,n} \uplus \widetilde{\Delta}_0 \uplus T_{1,n},\quad \overline{R}_{1,n} = R_{1,n} \uplus \widetilde{D}_0'' \uplus \widetilde{D}_0^r \uplus \mathscr{T}_{1,n}.$$

Let S be the set of all circular partitions P of n with $|P| \geq 2$, denote by U_P the boundary stratum of $\overline{M}_{1,n}$ parametrising curves with dual graph $\Gamma(P)$, i.e. the interior of the banana cycle B_P, denote by U_0 the interior of Δ_0. Then with U_0'', U_P'', U_0^r, U_P^r defined analogously,

$$\widetilde{\Delta}_0 = U_0 \uplus \biguplus_{P\in S} U_P,\quad \widetilde{D}_0'' = U_0'' \uplus \biguplus_{P\in S} U_P'',\quad \widetilde{D}_0^r = U_0^r \uplus \biguplus_{P\in S} U_P^r.$$

By the proof of Lemma 4.4 there are bijective morphisms $\widetilde{D}_0'' \to \widetilde{\Delta}_0$ and $\widetilde{D}_0^r \to \widetilde{\Delta}_0$. So with Lemma 4.11, $\widetilde{\Delta}_0 \cong \widetilde{D}_0'' \cong \widetilde{D}_0^r$. Hence, using notation of Summary 5.67, and results from section 5.3.2 in the second line, and notation and arguments similar to the proof of Corollary 5.37:

$$\chi(\overline{M}_{1,n}) = \chi(M_{1,n}) + \chi(\widetilde{\Delta}_0) + \chi(T_{1,n}),\quad \chi(\overline{R}_{1,n}) = \chi(\overline{R}_{1,n}) + 2\chi(\widetilde{\Delta}_0) + \chi(\mathscr{T}_{1,n}) \qquad (\dagger)$$

$$\chi(\widetilde{\Delta}_0) = \chi(U_0) + \sum_{P\in S}\chi(U_P) = \chi(M_{0,n+2}') + \sum_{r_1+r_2=n}\frac{1}{2!}\binom{n}{r_1, r_2}\chi((M_{0,r_1+2}\times M_{0,r_2+2})')$$

[65] If one wants to compute numbers $\chi(\overline{R}_{1,n})$ for larger n, it would not be difficult to write a computer program which does this recursively using this formula (although this program might be quite slow). For this note that every non-empty $D_{I_1,\ldots,I_k}$ is isomorphic to a certain $\overline{R}_{1,q}\times\overline{M}_{0,l_1}\times\ldots\times\overline{M}_{0,l_k}$ for a $q < n$, and that $\chi(\overline{M}_{0,n})$ is known by [Kee92]

$$+\sum_{m=3}^{n} \sum_{r_1+\ldots+r_m=n} \frac{1}{m!}\binom{n}{r_1,\ldots,r_m}\frac{m!}{2m}\prod_{i=1}^{m}\chi(M_{0,r_i+2}). \qquad (\ddagger)$$

The last term can for all n be rewritten with 5.67 as:

$$\sum_{m=3}^{n} \sum_{r_1+\ldots+r_m=n} (-1)^{n-m}\frac{1}{2m}\frac{n!}{r_1\cdot\ldots\cdot r_m}$$

$$\text{and for } n \geq 5: \quad \chi(\widetilde{\Delta}_0) = n!\sum_{m=1}^{n}\frac{(-1)^{n-m}}{2m}\sum_{r_1+r_2+\ldots r_m=n}\frac{1}{r_1\cdot r_2\cdot\ldots\cdot r_m} \qquad (\clubsuit)$$

By the inclusion-exclusion principle we get:

$$\chi(T_{1,n}) = \sum_{k=1}^{n}(-1)^{k+1}\sum_{\substack{\{I_1,\ldots,I_k\}\\ I_i\subset \underline{n},\,|I_i|\geq 2}}\chi(\Delta_{I_1,\ldots,I_k}), \qquad \chi(\mathscr{T}_{1,n}) = \sum_{k=1}^{n}(-1)^{k+1}\sum_{\substack{\{I_1,\ldots,I_k\}\\ I_i\subset \underline{n},\,|I_i|\geq 2}}\chi(D_{I_1,\ldots,I_k}). \qquad (\diamondsuit)$$

With Summary 5.67, we obtain (iii) from our previous observations.

Now we compute $\chi(\overline{R}_{1,n})$ for $n \leq 5$. One may do this directly using (†), (‡) and Summary 5.67 and [Kee92], but it is easier to compute the difference $d(n) := \chi(\overline{R}_{1,n}) - \chi(\overline{M}_{1,n})$ and then add it to the value of $\chi(\overline{M}_{1,n})$ known by [Get98] (page 8). By ($\diamondsuit$), $\chi(\mathscr{T}_{1,n}) - \chi(T_{1,n})$ is a sum over terms $\chi(D_{I_1,\ldots,I_k}) - \chi(\Delta_{I_1,\ldots,I_k})$. But, if non-empty, $D_{I_1,\ldots,I_k} \cong \overline{R}_{1,q} \times \overline{M}_{rest}$ and $\Delta_{I_1,\ldots,I_k} \cong \overline{M}_{1,q} \times \overline{M}_{rest}$, where $q < n$ and $\overline{M}_{rest}$ is a product of some $\overline{M}_{0,l_i}$. Hence

$$\chi(D_{I_1,\ldots,I_k}) - \chi(\Delta_{I_1,\ldots,I_k}) = \big(\chi(\overline{R}_{1,q} - \chi(\overline{M}_{1,q})\big)\chi(\overline{M}_{rest}) = d(q)\chi(\overline{M}_{rest}). \qquad (\spadesuit)$$

So if for an $n \in \mathbb{N}$, $d(q) = 0$ for all $q < n$, then by (†)

$$d(n) = e(n) := \chi(R_{1,n}) - \chi(M_{1,n}) + \chi(\widetilde{\Delta}_0).$$

We compute $e(n)$ for $n \leq 4$ using (‡) and 5.67, and obtain $e(1) = e(2) = e(3) = 0$ and $e(4) = 1$, hence these are also the values for $d(n)$. For $n = 5$ for the first time there may be a contribution from $\chi(\mathscr{T}_{1,n}) - \chi(T_{1,n})$, coming from those terms $\chi(D_{I_1,\ldots,I_k}) - \chi(\Delta_{I_1,\ldots I_k})$ for which $q = 4$ in ($\spadesuit$). It is easy to check that this is only the case for $k = 1$ and $|I_1| = 2$. There are $\binom{5}{2} = 10$ such sets I_1, and in these cases, $\overline{M}_{rest} \cong \overline{M}_{0,3}$. Hence $\chi(\mathscr{T}_{1,5}) - \chi(T_{1,5}) = 10$. Since, using ($\clubsuit$), $e(5) = -4 + 12 - 25 + 35 - 30 + 12 = 0$, we have $d(5) = 10$. □

Corollary 5.69 *For $n \leq 4$, $H^*(\overline{R}_{1,n}) = A^*(\overline{R}_{1,n})$ via the cycle map. So in particular the Betti numbers $h^i(\overline{R}_{1,n}) = h^i(\overline{S}^+_{1,n})$ for $n \leq 4$ are:*

	h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8
$\overline{R}_{1,1}$	*1*	*0*	*1*						
$\overline{R}_{1,2}$	*1*	*0*	*2*	*0*	*1*				
$\overline{R}_{1,3}$	*1*	*0*	*5*	*0*	*5*	*0*	*1*		
$\overline{R}_{1,4}$	*1*	*0*	*12*	*0*	*24*	*0*	*12*	*0*	*1*

Proof: By Summary 5.67 (v), $\overline{R}_{1,n}$ has no odd cohomology for $n \leq 4$, hence in this range $\dim H^*(\overline{R}_{1,n}) = \chi(\overline{R}_{1,n})$. [66] In our computation of the Chow rings we bounded the dimension of each homogeneous part $A^d(\overline{R}_{1,n})$ from below by means of computing an intersection matrix [67]. Hence the $\mathbb{Q}$ vector space $A^d(\overline{R}_{1,n})$ for every d has a basis of $\dim A^d(\overline{R}_{1,n})$-many *numerically* independent elements. Since numerical equivalence is weaker then homological equivalence, the cycle map $A^*(\overline{R}_{1,n}) \to H^*(\overline{R}_{1,n})$ is thus injective (cf. Chapter 19 of [Ful98]). Since the dimensions of $A^*(\overline{R}_{1,n})$ computed in section 4.4 agree with $\chi(\overline{R}_{1,n})$ as computed in Proposition 5.68 (ii) for all $n \leq 4$ the cycle map is then also surjective. □

Remark 5.70 If we assume that for $\overline{R}_{1,5}$ the cycle map surjects on the even cohomology $H^{2*}(\overline{R}_{1,5})$ (which is in this case equivalent to $H^{2*}(\overline{R}_{1,n}) = H^*_{BCl}(\overline{R}_{1,n})$), then it is not difficult to show that the Betti numbers of $\overline{R}_{1,5}$ are $1, 0, 27, 0, 105, 0, 105, 0, 27, 0, 1$. This would use the above results, and the knowledge of the Betti numbers of $\overline{M}_{1,5}$ from [Get98]. But since I do not know how to proof $H^{2*}(\overline{R}_{1,5}) = H^*_{BCl}(\overline{R}_{1,5})$, I will not give any details here. (One can check that $\dim A^2(\overline{R}_{1,5}) \leq 105$ in the style of section 4.4, then with the assumption everything follows quickly. Also one obtains the mentioned Betti numbers quite directly if one assumes instead that the even cohomology vanishes. [68])

[66] From this, together with Proposition 5.68 (i), and the knowledge of all Betti numbers $h^i(\overline{M}_{1,n})$ for $n \leq 4$ ([Get98], page 10), one can compute the Betti numbers $h^i(\overline{R}_{1,n})$ without knowing the Chow ring: It is clear that $h^i(\overline{R}_{1,n}) \geq h^i(\overline{M}_{1,n})$ always. For $n \leq 3$, $\chi(\overline{R}_{1,n}) = \chi(\overline{M}_{1,n})$, so $h^i(\overline{R}_{1,n}) = h^i(\overline{M}_{1,n})$ for all i here. For $n = 4$, $\chi(\overline{R}_{1,4}) = \chi(\overline{M}_{1,4}) + 1$, and hence by Poincare duality we must have $h^4(\overline{R}_{1,4}) = h^4(\overline{M}_{1,4}) + 1$ and $h^i(\overline{R}_{1,4}) = h^i(\overline{M}_{1,4})$ for all $i \neq 4$. This would though not determine the ring structure of $H^*(\overline{R}_{1,4})$, so the work in section 4.4 was not completely gratuitous.

[67] Or, in many cases we showed that $A^d(\overline{R}_{1,n}) = \tau_n^*(A^d(\overline{M}_{1,n}))$ and so can use, that the dimension of $A^d(\overline{M}_{1,n})$ is bounded from below in [Bel98] by computing an intersection matrix.

[68] To me both assumptions seem very plausible, since $\overline{R}_{1,5}$ is a rational variety, and since for $\overline{M}_{1,n}$ the analogous assumptions hold for all $n < 11$ which is also the range in which $\overline{M}_{1,n}$ is rational. If these Betti numbers are correct $H^5(\overline{R}_{1,5}) = 0$, and with the same inductive arguments as used in [BF09b] to show $H^1(\overline{R}_{1,n}) = H^3(\overline{R}_{1,n}) = 0$, it would follow that $H^5(\overline{R}_{1,n}) = 0$ for all n.

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