Strong Solutions of the Matched Microstructure Model for Fluid Flow in Fractured Porous Media

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades

Doktorin der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation
von
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geboren am 05.12.1983 in Erfurt

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Datum des Prüfungskolloquiums: 28.10.2011

Acknowledgements

This thesis was written during a three year PhD programm at Leibniz University Hanover. As part of the DFG research training group 1463 “Analysis, Geometry and String Theory” I got the opportunity of a deep view into the connections of these fields.

I am especially grateful to my supervisor, Prof. Dr. J. Escher who strongly supported me over this time period. Also I want to express my gratitude to the members of my graduate school and my colleagues and thank them for all the fruitful discussions and the good working atmosphere. I thank the Deutsche Forschungsgemeinschaft (DFG) for the financial support of the program. Further I am much obliged to Prof. Dr. J.-F. Rodrigues from the University of Lisbon for hosting me in Lisbon during a three month visit and his interest in my research.

Last but not least I want to thank my family and my friends for their constant encouragement and support.

Hanover, June 2011 Daniela Treutler

Zusammenfassung

Die vorliegende Doktorarbeit beschäftigt sich mit einem Zweiskalenmodell zur Modellierung von Flüssigkeiten in rissigen porösen Medien, dem sogenannten Matched Microstructure Model. Es modelliert die Dichte einer Newtonschen Flüssigkeit in einem beschränkten und glatten Gebiet. Das homogenisierte mathematische Modell ist ein System von zwei gekoppelten, parabolischen Differentialgleichungen.

Zunächst beschäftigen wir uns mit den geometrischen Gegebenheiten und führen geeignete Funktionenräume ein, welche die Struktur des Materials widerspiegeln. Dann definieren wir einen Operator, so dass sich das Problem als abstraktes Anfangswertproblem formulieren lässt. Wir zeigen, dass der Operator eine analytische Halbgruppe erzeugt. Dies beweist die Wohlgestelltheit des Problems.

Es werden drei Fassung des Problems betrachtet: ein semilineares System mit linearen Randbedingungen, ein spezielles semilineares Modell mit nichtlinearen Randbedingungen und eine Verallgemeinerung zu einem quasilinearen System (mit linearen Randbedingungen). Im letzten Fall nutzen wir das Konzept der Maximalen Regularität zum Beweis der Lösbarkeit. Im semilinearen Fall mit linearen Randbedingungen untersuchen wir zudem das Langzeitverhalten der Lösung mit Hilfe der obigen Methoden. Generell ist die größere Regularität der Lösungen hervorzuheben im Vergleich mit Lösungen, die durch Standardmethoden der schwachen Theorie erhalten wurden.

Stichworte: poröse Struktur, Zweiskalenmodell, zweifach porös

Abstract

The subject of this thesis is a matched microstructure model for Newtonian fluid flows in fractured porous media. This is a homogenized model which takes the form of two coupled parabolic differential equations with boundary conditions in a given (two-scale) domain in Euclidean space, and the main objective is to establish the local well-posedness in the strong sense of the flow.

A first question is the geometry of the domain: the material structure reflects itself in the mathematical problem, and we introduce suitable function spaces adapted to this particular structure. These are used to define an operator such that the whole problem can be written as an abstract initial-value problem. We show that this operator generates an analytic semigroup, which, in turn, implies local well-posedness in the appropriate function spaces.

Three main settings are investigated: semi-linear systems with linear boundary conditions, semi-linear systems with nonlinear boundary conditions, and a generalization to quasi-linear systems (with linear boundary conditions). The last case requires an approach via maximal regularity. In the semilinear case the same methods as above can be used also to investigate the long-time behaviour of the solutions: we establish global existence and show that solutions converge to zero at an exponential rate. A general feature throughout this work is the higher regularity of the solutions found, as compared to solutions obtained via standard weak methods for the model under investigation.

key words: porous medium, two-scale model, double porosity

Contents

1 Introduction

Mineral oil is naturally found in porous stone layers deep under the ground. Often these oil reservoirs show a dense system of interconnected fractures. In the 1960's petroleum engineers observed that these fissures have a strong effect on the behaviour of a liquid in the porous rocks. This type of structure strongly facilitate fluid transport. The other part of the fluid remains in the pores. The exchange between the channels and the stone is rather small. This leads to the effect that storage of the liquid mostly happens in each block and the interaction is slow. So the processes in the system of the fractures and the porous material occur on very different length and time scales. This special structure has to be reflected by any model that tries to calculate the evolution of a fluid in this special circumstances.

The actual geometry inside the considered domain is discrete and unknown. All models therefore are derived in the spirit of an averaging process. There were many attempts to find a continuum model for the evolution of the oil inside such a double structure material. First the fissured medium equation and the double porosity parallel model were developed. Both descriptions do not reflect the different spatial scales. In contrast two-scale models emphasize this specification of the different magnitudes. Basic work on the mathematical concept was done by Barenblatt, Zheltov and Kochina [16] and later by Arbogast, Douglas and Hornung [10, 11, 13]. With the help of the ideas of homogenisation and a formal asymptotic expansion they derive an appropriate two-scale model. It is the basis of our ansatz. The dense fissering acts like a second porous structure in the domain. Of course the porosity may differ from the porosity of the stone. In this text we will refer to the fissure structure as the macro scale, the porous blocks constitute the micro scale.

We consider a domain $\Omega \subset \mathbb{R}^n$. To each point $x \in \Omega$, a cell Ω_x is associated as it can be seen in Figure 1. The sets Ω_x represent the blocks of porous material that are surrounded by the fissures. In this thesis the micro structure is composed of smooth domains that may vary smoothly with x. On Ω and on the family $\{\Omega_x, x \in \Omega\}$ we introduce density functions u and U respectively. Then a porous medium equation holds in each domain. The connection between the different scales is modeled through the amount of fluid $q(U)$ that enters the fissure system from inside the cells and a matching condition on the micro scale. In the discrete setting the porous cells are supposed to be very small. So the ambient density in the fissure is said to be constant. We assume that for all $x \in \Omega$ the boundary condition

$$U(x) = u(x), \qquad \text{on } \partial\Omega_x$$

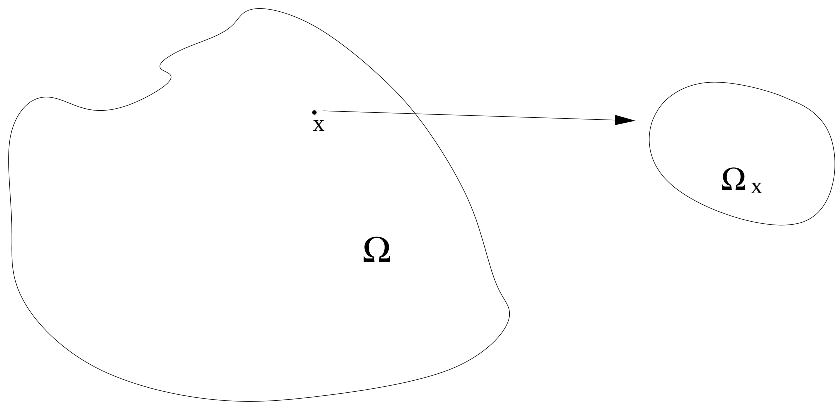

Figure 1: The domains on the macro and micro scale

holds. Assuming Dirichlet-zero-data for the boundary condition of the density u on Ω, we arrive at the following set of equations

$$(MM)\begin{cases} \partial_t u - \Delta_x u &= f + q(U), \quad \text{for } x \in \Omega, t \in (0,T] \\ \partial_t U - \Delta_z U &= g, \quad \text{for } z \in \Omega_x, x \in \Omega, t \in (0,T], \\ u &= 0, \quad \text{for } x \in \partial\Omega, t \in (0,T], \\ U(x) &= u(x), \quad \text{for } z \in \partial\Omega_x, x \in \Omega, t \in (0,T], \\ u(0) &= u_0, \quad U(0) = U_0. \end{cases}$$

Here u_0, U_0 are suitable initial data. It is the first goal of this work to show well-posedness in a Sobolev space setting of (MM). Weak solutions of this model have been of great interest in the 1990's. A good overview and seminal work is [61] of Showalter and Walkington. In their paper they give a good survey of the existing models and derive two different distributed microstructure models. The notion of the matched microstructure model goes back to this paper. So we will sometimes call the equations (MM) Showalter model. In addition the authors characterized the regularized microstructure model that mainly includes a different version of the coupling conditions. To prove well-posedness a variational formulation of the problem is given. Then the two mathematicians prove existence of solutions using technics from Hilbert space theory and holomorphic semigroups. The Trotter-Kato method of convergence of semigroups ensures continuous dependence on the data. With this ansatz the authors are able to treat quite general microstructure geometries.

It seems natural to consider a quasilinear version of the Showalter model. This generalisation allows to cover a broad range of applications. The most well-posedness results in this field are either based on homogenization theory

([36, 37] and the references in [38]) or on the theory of monotone operators (see [60, 22, 40]). The last method demands certain monotonicity characteristics for the right hand side. In [9] some results on the properties of the solutions are presented. Stationary solutions and the elliptic problem are for example treated in [62].

In this thesis we formulate the matched microstructure problem as an abstract evolution equation. This leads us to a rather flexible model. We prove that the involved operators generate an analytic semigroup even for varying cell geometry in an L_p-formulation. With this we show the existence of strong solutions. Many results from the abstract theory can be applied to the matched microstructure model. In particular the equation is satisfied pointwise in time and almost everywhere in space. The solutions exhibit a high regularity in the spatial and temporal variables. This is a big step towards the existence of classical solutions.

Some applications like ground water flow or concrete carbonation can also be discribed with a two-scale model. In particular for reaction processes the single pore geometry is of importance. So there are no fissures present but the microscopic and the pore scale are examined. The resulting equations show a big resemblance to the Showalter model. There are several papers [29, 30] on uniform pores. Friedman and Knabner [29] give a weak and classical formulation. Then they show that traveling wave solutions exist and that the propagation speed of a reaction front (in one dimension) is bounded. Recent publications in this field that worry about reactions with the solid structure and therefore consider an evolution of the pore geometry are [54, 55, 48, 49]. (See also the references therein.) Schemes for unsaturated porous media or models with several fluids [36] are another branch of this active research area.

The papers [12, 11, 47] also include numerical simulations. This problem has also attained much attention by applied scientists. We refer to [41] for more references.
Another important application of double porosity models is the simulation of contaminant transport and reaction in fissured porous media. The functions u and U then represent the concentration of a dissolved chemical. For an introduction see [56]. A specification of our ansatz to this topic is another goal of this work.

The concept of analytic operator semigroups and maximal regularity was developed in the past 30 years. It has proven to be an effective tool to treat linear and nonlinear evolution equations (see [24, 7] and the references inside). For quasilinear problems maximal regularity in the initial situation allows to prove well-posedness. Therefore the notion of $\mathcal{R}$-sectorial operators (see [23] for an overview) was introduced to show this property. It will be used in this work.

The outline for the thesis is the following: First we describe some aspects of the modeling and the derivation of the matched microstructure model. This has been done in several papers. Nevertheless we want to give the ideas because they justify some of the assumptions of Section 4.1. In Chapter 3 we prove a couple of auxiliary statements for uniform microscopic cells.

In the forth Chapter we first deal with the special geometry. The main ingredient here is to relate the domains Ω_x to a fixed reference domain B. So adequate function spaces are defined as images of well known Sobolev-Slobodetski spaces

$$L_p(\Omega, W_p^s(\Omega_x)) := \Phi_* L_p(\Omega, W_p^s(B)).$$

In connection with results from the previous part we use this to reformulate the problem as an abstract initial value problem

$$\partial_t(u,U) + \mathbf{A}(u,U) = f(u,U), \qquad (u,U)(0) = (u_0, U_0)$$

in a suitable product space. Then we show that the operator $\mathbf{A}$ generates an analytic semigroup. The interpolation theory for the new spaces is developed in Section 4.3. Now well-posedness of the intitial boundary value problem follows. The formulation enables us to apply our method to many semilinear versions of the Showalter model. At the end of this paragraph we present some results on the qualitative behaviour of the solutions. Using a Hilbert space setting we conclude that the solution of the Showalter model decays exponentially fast. In case of Neumann boundary conditions we prove that the total mass of the fluid is a preserved quantity.

In Chapter 5 we consider a variation of the model that allows to include the gravitational force into the description. Recall that u is the density of the fluid. Thus we arrive at a system with a nonlinear boundary condition

$$\partial_\nu u = -gu^2, \qquad \text{on } \Gamma_0.$$

Existence and uniqueness are first proved for weak solutions. The approach is funded on work of Escher [25] and Amann [4, 5]. Again the fact that the operator $\mathbf{A}$ is sectorial, is crutial for the proof. The abstract theory of evolution equations on interpolation-extrapolation scales ensures that also in this case the solution possesses additional regularity and the equations are fulfilled in a strong sense. The direct ansatz in the L_p-setting (compare with [28]) can not be applied since there is no trivial equilibrium known and the Fréchet derivative of the nonlinearity does not vanish.

The last part of this work is devoted to nonlinear generalisations of the model. With the help of Nemytskii operators we formulate a quasilinear version of the matched microstructure model. Based on results of Denk, Hieber and Prüss [23] we show that for suitable u the operator $\mathbf{A}(u)$ is $\mathcal{R}$-sectorial with an $\mathcal{R}$-angle less that $\frac{\pi}{2}$. This induces that the operator possessess maximal L_p-regularity. Now the well-posedness follows from results of Clément and Li [21]. The approach described above does not apply to the case of a nonlinearity depending on U. But from homogenisation theory it is known that such situations can appear. So in Section 6.3 we describe an ansatz to circumvent this problem. We vary the porosity of each porous cell with the amount of dissolved material inside of it. A fixed point argument demonstrates the existence and uniqueness of solutions.

In particular we collect some ideas for a possible continuation of this work.

2 A hydrodynamical model

In this chapter we present the background of the matched microstructure model. We try to motivate the equations which describe the evolution of a fluid in a densely fissured porous medium. For a detailed presentation of the foundations of porous medium equations we refer to the book of Bear and Bachmat [17]. The derivation of the two scale model can be found in [61] and [40]. The following assumptions are made for all porous systems that occur in this text. The first statements treat the solid structure of a porous material fillied with a single liquid. We suppose:

(A1) The (microscopic) fluid-solid interface is a material interface with respect to the fluid's mass, i.e. no mass of the considered fluid crosses it.

(A2) The solid phase preserves (microscopically) its volume. Furthermore it is macroscopically fixed in space.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that Ω is filled with a porous material of porosity k and a slightly compressible Newtonian fluid. The following equation of state is assumed to hold,

$$\rho = \rho_0 e^{cp}.$$

Here ρ denotes the density of the fluid, p is the pressure and $\rho_0, c > 0$ are material constants. The fluid's behaviour in the porous material is governed by Darcy's law

$$J = \rho v = -\rho k \nabla p = -\frac{k}{c}\nabla \rho,$$

where J denotes the flux and v is the velocity of the fluid. Then mass conservation yields

$$\partial_t \rho + div J = f.$$

Using Darcy's law this can be written as

$$\partial_t \rho - \frac{k}{c}\Delta \rho = f. \tag{1}$$

The function f collects all sources and sinks that contribute to the system. Later the density ρ will be denoted by u and U depending on the scale.

Now we assume that the material is burred by a dense system of fissures. So there are two geometric structures with significantly different length scales. Those systems are often treated via homogenization theory. To make an asymptotic expansion we consider a periodic structure as is shown in Figure 2. We ask for the following statements to hold true:

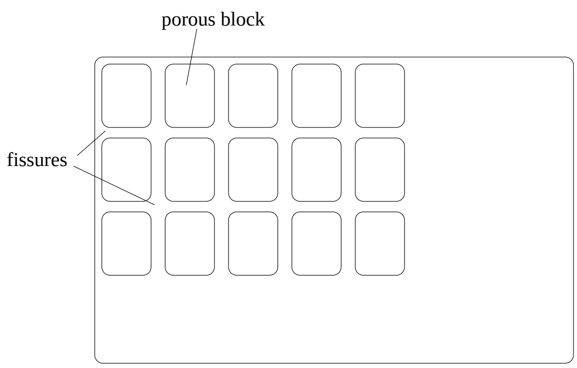

Figure 2: Periodic Geometry

(A3) The porous system of the material and also the porous system of the fractures are fully connected.

(A4) The porous blocks are so small that the density in the surrounding fissure can be considered constant.

(A5) Sources and sinks only exist in the fissure system.

(A6) The whole domain Ω is surrounded by impermeable rocks.

The next assumption has been used by engineers (see [13] and the references therein). It has proven to lead to good results when modeling real world situations.

(A7) The fissures are partially filled with rock debris. Thus the behaviour of the fluid inside is also described by equation 1.

The idea of the limit process is an appropriate scaling of the system. Let Ω_x denote the standard cell of size one. Then a scaling parameter $\varepsilon > 0$ is introduced. It enters the cell size and all constants and quantities in the framework

$$\Omega_x^\varepsilon := \varepsilon \cdot \Omega_x.$$

Condition (A6) and (A4) are interpreted as Dirichlet boundary conditions on Ω and Ω_x^ε. By a formal expansion in ε and a limit argument ($\varepsilon \to 0$) one gets the following two coupled equations:
The macro model for the density u living on Ω:

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) - \Delta_x u(t,x) &= f(t,x) + q(U)(t,x), && x \in \Omega, t \in (0,T], \\ u(t,x) &= 0, && x \in \Gamma, t \in (0,T], \\ u(t=0) &= u_0. \end{aligned}$$

The micro model for the density U living on all cells Ω_x:

$$\begin{aligned} \frac{\partial}{\partial t}U(t,x,z) - \Delta_z U(t,x,z) &= 0, && x \in \Omega, z \in \Omega_x, t \in (0,T], \\ U(t,x,z) &= u(x), && x \in \Omega, z \in \Gamma_x, t \in (0,T], \\ U(t=0) &= U_0. \end{aligned}$$

For more details of the derivation we refer to [13]. Of course the functions u_0, U_0 connect to the initial conditions of the discrete system. Therefore we will refer to them by the same name. The coupling between the macro scale and the microcopic one is reflected by two terms. The boundary condition in the cells Ω_x,

$$U(x) = u(x) \qquad \text{on } \partial\Omega_x, \text{ for all } x \in \Omega$$

models the matching of the densities on the material interface (A4). For this reason the model was introduced in [61] as matched microstructure model. The term

$$q(U)(t,x) = -\int_{\Gamma_x} \frac{\partial U(t,x,s)}{\partial \nu}\, ds = -\frac{\partial}{\partial t}\int_{\Omega_x} U(t,x,z)\, dz. \tag{2}$$

represents the amount of fluid that is exchanged between the two structures. It acts as a source or sink term in the macroscopic system.

The condition (A6) can also be interpreted as no-flux boundary condition on Ω. Some changes are considered in Chapter 4.5 and 5. The formal derivation is still possible and can be justified by homogenization results. The above derivation is done for cells of uniform shape. There is no other approach known to the author that does not rely on this. Nevertheless in [61], Showalter and Walkington treat a matched microstructure model with nearly arbitrarily varying cells Ω_x. There main assumption is that

$$Q = \bigcup_{x\in\Omega} \{x\} \times \Omega_x$$

is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. This can not be justified easily. Our ansatz requires some more regularity. For this reason we consider uniform cells first. In Section 4.1 we allow smooth variations of the cell shape.

3 Some Aspects for Uniform Cells

Our model is based on the derivation of the coupled equations in case of a uniform cell at each point x in the considered domain Ω. So we prove some properties for shifted operators that act on functions that live on all the cells in such a system. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Together with the usual Lebesque measure it is a measure space. For the rest of the chapter we denote with X, Y two Banach spaces. Later this will be the function spaces on a cell. If A is a closed linear operator from X to Y we denote with

$$D(A) = (dom(A), \|\cdot\|_A)$$

the domain of definition of A equipped with the graph norm. Further we will write $\hat{U}$ if we mean a representativ in $\mathcal{L}_p$ of a give function $U \in L_p$. With $[\cdot]$ we indicate the equivalence class again. The shifted operators will always be denoted with bold letters.

Lemma 1. *Assume that* $(x \mapsto A(x)) \in C(\overline{\Omega}, \mathcal{L}(X, Y))$. *Let*

$$\begin{aligned} dom(\boldsymbol{A}) &= L_p(\Omega, X), \\ \boldsymbol{A}U &= \left[A(x)\hat{U}(x)\right], \qquad \textit{for } U \in L_p(\Omega, X), \hat{U} \in U. \end{aligned}$$

Then $\boldsymbol{A}$ *is a well defined, bounded linear operator from* $L_p(\Omega, X)$ *to* $L_p(\Omega, Y)$. *If further* $A(x) = A$ *independent of* x *and* A *is a retraction, then* $\boldsymbol{A}$ *is a retraction as well.*

Proof. First we show that $\mathbf{A}$ is well defined. Let $U \in L_p(\Omega, X)$ and let $\hat{U} \in \mathcal{L}_p(\Omega, X)$ be a representative of U. Since $x \mapsto A(x)$ is continuous on Ω and $\hat{U}$ is measurable we know that $x \mapsto A(x)\hat{U}(x)$ is measurable on Ω.
We easily get that $\mathbf{A}U \in L_p(\Omega, Y)$ and $\mathbf{A}$ is bounded because

$$\|\mathbf{A}U\|^p_{L_p(\Omega,X)} = \int_\Omega \|A(x)\hat{U}(x)\|^p_Y \, d\mu(x) \le \max_{x\in\overline{\Omega}} \|A(x)\|^p_{\mathcal{L}(X,Y)} \cdot \|U\|^p_{L_p(\Omega,X)}.$$

Now assume that $A(x) = A$ is a retraction and let $R \in \mathcal{L}(Y, X)$ be a continuous right inverse of A, this means that

$$A \circ R = id_Y.$$

Let $V \in L_p(\Omega, Y)$ and $\hat{V} \in \mathcal{L}_p(\Omega, Y)$ a representative of V. As before we can define

$$\mathbf{R} \in \mathcal{L}(L_p(\Omega, Y), L_p(\Omega, X)), \quad \mathbf{R}V = [R\hat{V}(x)].$$

Then it holds

$$(\mathbf{A} \circ \mathbf{R})V = \mathbf{A}[R\hat{V}(x)] = \Big[(A \circ R)\hat{V}(x)\Big] = V.$$

This shows that $\mathbf{A}$ is a retraction. $\square$

Lemma 2. *For $x \in \Omega$, let $A(x) \in \mathcal{A}(X, Y)$ be a closed linear operator. Assume that there is $A_0 \in \mathcal{A}(X, Y)$, such that*

$$dom(A(x)) = dom(A_0), \qquad \text{for all } x \in \Omega.$$

Furthermore let $(x \mapsto A(x)) \in C(\overline{\Omega}, \mathcal{L}(D(A_0), Y))$. Then the operator

$$\begin{aligned} dom(\boldsymbol{A}) &= L_p(\Omega, dom(A_0)), \\ \boldsymbol{A}U &= \Big[A(x)\hat{U}(x)\Big], \qquad\qquad \text{for } U \in dom(\boldsymbol{A}), \hat{U} \in U. \end{aligned}$$

is a well defined, closed linear operator from $L_p(\Omega, X)$ to $L_p(\Omega, Y)$. If further $dom(A_0)$ is dense in X, then $\boldsymbol{A}$ is densely defined.

Proof. It is well known that $D(A_0)$ is a Banach space. Thus Lemma 1 can be applied and shows that the operator $\mathbf{A}$ is well defined. Next we prove that $\mathbf{A}$ is closed. Take $(U_n)_{n\in\mathbb{N}} \subset dom(\mathbf{A})$ such that there are $U \in L_p(\Omega, X)$, $V \in L_p(\Omega, Y)$ with

$$U_n \to U, \quad \mathbf{A}U_n \to V \qquad (n \to \infty)$$

in the respective spaces. Let $\hat{U}_n, \hat{U}, \hat{V}$ be representatives. Then $\hat{U}_n(x) \to \hat{U}(x)$ a.e. on Ω. From the fact that $A(x)$ is closed and $\hat{U}(x) \in dom(A(x))$, we conclude that

$$A(x)\hat{U}_n(x) \to A(x)\hat{U}(x) = \hat{V}(x), \text{ for a.e. } x \in \Omega.$$

Taking the equivalence class the last equation implies that $U \in dom(\mathbf{A})$ and $\mathbf{A}U = V$.
For the last statement we assume that $dom(A_0)$ is dense in X. For $\phi \in L_p(\Omega)$, $u \in dom(A_0)$ we write $\phi \otimes u := \phi u$. Moreover, we set

$$L_p(\Omega) \otimes dom(A_0) = \left\{ \sum_{j=0}^{m} \phi_j \otimes u_j; \phi_j \in L_p(\Omega), u_j \in dom(A_0), m \in \mathbb{N} \right\}.$$

This set contains the simple functions on Ω with values in $dom(A_0)$. Thus it follows that $L_p(\Omega) \otimes X$ is a dense subset of $L_p(\Omega, dom(A_0))$. It is also dense in $L_p(\Omega) \otimes X$ since $dom(A_0)$ is dense in X by assumption. So by the same argumentation as before we conclude that $L_p(\Omega) \otimes X \overset{d}{\subset} L_p(\Omega, X)$ and the assertion follows. $\square$

The last statement is due to Amann [7]. The next lemma is devoted to the shift of sectorial operators. Let $\omega \in \mathbb{R}, \theta \in (0,\pi)$. We define

$$S_{\theta,\omega} = \{\lambda \in \mathbb{C}; \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}.$$

For the rest of the chapter let $X = Y$.

Lemma 3. *Let the assumptions of the previous Lemma be fulfilled. Assume further that there exist constants $\omega \in \mathbb{R}, \theta \in (0,\pi)$, $M \geq 1$ such that for every $x \in \Omega$*

$$S_{\theta,\omega} \subset \rho(-A(x)),$$
$$\|(\lambda + A(x))^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \qquad \text{for } \lambda \in S_{\theta,\omega}.$$

Then $\boldsymbol{A}$ is sectorial in $L_p(\Omega, X)$.

Note that the assumption means that for every $x \in \Omega$, $A(x)$ is sectorial and $-A(x)$ is the generator of an analytic semigroup in X.

Proof. Let $\lambda \in S_{\theta,\omega}$. With Lemma 1 we define $\mathbf{R}_\lambda \in \mathcal{L}(L_p(\Omega, X))$ by

$$\mathbf{R}_\lambda U := [(\lambda + A(x))^{-1}\hat{U}(x)]$$

for $U \in L_p(\Omega, X)$. Then one easily calculates that this is the inverse of $\lambda + \mathbf{A}$. Thus $\lambda \in \rho(-\mathbf{A})$. Furthermore it holds

$$\begin{aligned}\|(\lambda + \mathbf{A})^{-1}U\|^p_{L_p(\Omega,X)} &\leq \int_\Omega \|(\lambda + A(x))^{-1}\|^p_{\mathcal{L}(X)} \|U(x)\|^p_X \, d\mu(x) \\ &\leq \left(\frac{M}{|\lambda - \omega|}\right)^p \int_\Omega \|U(x)\|^p_X \, d\mu(x).\end{aligned}$$

So we conclude that $\mathbf{A}$ is sectorial. □

The described method can be used to write the matched microstructure problem for uniform cells in another way. We only give some parts here. Let $B = B(0,1)$ denote the unit ball in $\mathbb{R}^n$ and S its boundary. With $tr_S : W^1_p(B) \to W_p^{1-\frac{1}{p}}(S)$ we mean the trace operator w.r.t. the unit sphere. Let y denote the points in B and Δ_y the Laplacian on B. We set

$$\begin{aligned}&dom(A_0) = \{U \in W^2_p(B); tr_S U = 0\},\\ &A_0 U = -\Delta_y U, \qquad U \in dom(A_0).\end{aligned}$$

Then it follows from the standard theory for PDE that A_0 is a closed, densly defined, sectorial operator. Let $X = Y = L_p(B)$. Then due to the previous lemma we can shift the operator to the space $L_p(\Omega, L_p(B))$. It follows that $\mathbf{A}_0$ defined as in the previous lemmata, i.e.

$$
\begin{aligned}
dom(\mathbf{A}_0) &= \{U \in L_p(\Omega, W_p^2(B)); tr_S U = 0\}, \\
\mathbf{A}_0 U(x) &= [-\Delta_y \hat{U}(x)], \qquad\qquad U \in dom(\mathbf{A}_0),
\end{aligned}
$$

is a well defined, sectorial operator in $L_p(\Omega, L_p(B))$. This will be used to write the whole problem as an abstract evolution equation.

4 Existence of solutions for the semilinear problem

4.1 Geometry

Here we present our main assumptions for a more general geometric configuration. We will still assume that the cell's shapes are related to one standard cell. In our case this is the unit ball $B = B(0,1) \subset \mathbb{R}^n$. With $S = \partial B$ we denote the $(n-1)$-dimensional boundary. Showalter and Walkington considered a more general geometry in their article [61]. We focus on the following configuration: Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $\Gamma = \partial\Omega$ be its boundary. We assume that there are two mappings Ψ, Φ in the way that

$$\begin{aligned}\Psi &: \Omega \times B \to \mathbb{R}^n,\\ \Phi &: \Omega \times B \to \mathbb{R}^n \times \mathbb{R}^n,\\ &\quad (x,y) \mapsto (x, \Psi(x,y)).\end{aligned}$$

These maps describe the geometry of our problem. This means the cell at a point $x \in \Omega$ is the image of B at x, i.e.

$$\Omega_x := \Psi(x, B).$$

Of course we want to ensure that Ω_x is a bounded smooth domain as well. Therefore we need some more assumptions on the properties of Φ, Ψ. First we define

$$Q := \bigcup_{x\in\Omega} \{x\} \times \Omega_x.$$

Then it holds that $Q = \Phi(\Omega \times B)$:

Proof. Let $(x,y) \in \Omega \times B$, then $\Phi(x,y) = (x, \Psi(x,y)) = (x,z)$ for some $z \in \Omega_x$, so $\Phi(x,y) \in Q$. Conversely if $(x,z) \in Q$ then $z \in \Omega_x = \Psi(x,B)$. Therefore $(x,z) \in (x, \Psi(x,B)) = \Phi(x,B)$. □

We summarize our conditions on Φ. We assume that

$$\Phi \in \mathrm{Lip}(\Omega \times B, Q), \tag{3}$$

$$\Phi^{-1} \in \mathrm{Lip}(Q, \Omega \times B), \tag{4}$$

$$\Phi(x,\cdot) \in \mathrm{Diff}(\overline{B}, \overline{\Omega}_x), \qquad \text{for all } x \in \Omega, \tag{5}$$

$$\sup_{x\in\Omega, |\alpha|\le 2} \|\partial_y^\alpha \Phi(x)\|_p, \|\partial_z^\alpha \Phi^{-1}(x)\|_p < \infty. \tag{6}$$

Here $\|\cdot\|_p$ denotes the usual L_p-norm. In this work we generally assume that the regularity conditions (3) - (6) are satisfied. Some examples are given at the end of this Section. Now it follows from [8], Th. IX 5.12, that Q is measurable. Further for every $x \in \Omega$, the set Ω_x is a bounded domain with smooth boundary $\Gamma_x := \partial\Omega_x$. Note that the construction of Φ implies that it is injectiv. The set $\mathrm{Diff}(\overline{B}, \overline{\Omega_x})$ shall denote all diffeomorphism from $\overline{B}$ to $\overline{\Omega_x}$ such that the restriction to the boundary S gives a diffeomorphism to Γ_x. Thus we will be able to work with the trace operator on B and later pull back to Ω_x. We also assume that $\Phi(x, \cdot)$ is sufficiently smooth. In Chapter 4.2 we require $\Phi(x, \cdot)$ to be at least twice differentiable. The conditions ensure that the following maps are well defined isomorphisms. Given $1 < p < \infty$, we define pull back and push forward operators

$$\begin{aligned} \Phi_* : L_p(\Omega \times B) &\to L_p(Q), \\ U &\mapsto U \circ \Phi^{-1}, \\ \Phi^* : L_p(Q) &\to L_p(\Omega \times B), \\ V &\mapsto V \circ \Phi. \end{aligned}$$

Proposition 4. *Φ_* and Φ^* are isomorphic maps and it holds*

$$(\Phi_*)^{-1} = \Phi^*.$$

Proof. Let $U \in L_p(\Omega \times B)$. We show that $U \circ \Phi^{-1} \in L_p(Q)$. Clearly $U \circ \Phi^{-1}$ is measurable since U is measurable and Φ is Lipschitz continuous. Due to the regularity condition (6) and the transformation theorem there exists a constant $C > 0$ such that

$$\|U \circ \Phi^{-1}\|^p_{L_p(Q)} \leq C\|U\|^p_{L_p(\Omega\times B)}.$$

It follows by construction that Φ^* and Φ_* are homeomorphisms. Additionally it holds

$$\begin{aligned} &\Phi^*\Phi_*U = \Phi^*(U \circ \Phi^{-1}) = (U \circ \Phi^{-1}) \circ \Phi = U, &&\text{for } U \in L_p(Q), \\ &\Phi_*\Phi^*V = \Phi_*(V \circ \Phi) = (V \circ \Phi) \circ \Phi^{-1} = V, &&\text{for } V \in L_p(\Omega \times B). \end{aligned}$$

This proves the statement. □

We collect some facts to motivate our definition of a class of function spaces. Assume that the diffeomorphisms $\Phi(x, \cdot)$ are of class C^m for some $m \geq 1$. In [2] (3.34, 3.35), it is proven that $W^s_p(\overline{B})$ is mapped onto $W^s_p(\overline{\Omega}_x)$ for $0 \leq s \leq m$. For $s = 0$ we identify $W^0_p(\overline{B}) = L_p(B)$ while W^2_p means the usual Sobolev-Slobodetski spaces. Furthermore we know from Fubini's Theorem that

$$L_p(\Omega \times B) \cong L_p(\Omega, L_p(B)).$$

The space $L_p(\Omega, W^s_p(B))$ is now defined by means of the Bochner integration theory. We summarize some well known properties:

Lemma 5. *(a) If $f : \Omega \times B$ is measurable and $f(x) \in L_p(B)$ for a.e. $x \in \Omega$ then $f : \Omega \to L_p(B)$ is Bochner measurable.*

(b) $L_p(\Omega, W_p^s(B))$ is a closed subspace of $L_p(\Omega, L_p(B))$. So it is a Banach space.

Proof. This follows directly from the theory of Bochner integration. □

We define

$$L_p(\Omega, W_p^s(\Omega_x)) := \Phi_*(L_p(\Omega, W_p^s(B))). \tag{7}$$

Equipped with the induced norm

$$\|f\|_{x,s} := \|\Phi^* f\|_{L_p(\Omega, W_p^s(B))}, \qquad f \in L_p(\Omega, W_p^s(\Omega_x)),$$

this is a Banach space:

Proof. Recall that Φ^*, Φ_* are linear maps. Fix $0 < s < \infty$. Let $\|\cdot\|$ denote the norm in $L_p(\Omega, W_p^s(B))$. For $f \in L_p(\Omega, W_p^s(\Omega_x))$ we calculate

(i) $\|f\|_{x,s} = 0 \Leftrightarrow \|\Phi^* f\| = 0 \Leftrightarrow \Phi^* f = 0 \Leftrightarrow f = 0,$

(ii) $\|\lambda f\|_{x,s} = \|\Phi^*(\lambda f)\| = |\lambda| \|\Phi^* f\| = |\lambda| \|f\|_{x,s},$

(iii) $\|f + g\|_{x,s} = \|\Phi^*(f + g)\| = \|\Phi^* f + \Phi^* g\| \le \|\Phi^* f\| + \|\Phi^* g\|.$

Now let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $L_p(\Omega, W_p^s(\Omega_x))$. Then also $g_n = (\Phi^* f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L_p(\Omega, W_p^s(B))$. Since this is a Banach space, g_n converges to an element $g \in L_p(\Omega, W_p^s(B))$. By definition $\Phi_* g \in L_p(\Omega, W_p^s(\Omega_x))$. It holds

$$\|f_n - \Phi_* g\|_{x,s} = \|\Phi^* f_n - g\| \to 0 \qquad (n \to \infty).$$

This completes the proof. □

[1] For the formulation of the boundary conditions on the cells we need to shift the trace operators. Remember that the geometry fullfills the following conditon:

$$\Phi(x, \cdot)|_S \in \mathrm{Diff}(S, \Gamma_x), \tag{8}$$

fo every $x \in \Omega$. This implies that all boundaries Γ_x are at least of class C^2. Additionally

$$\Phi(\Omega \times S) = \bigcup_{x \in \Omega} \{x\} \times \Gamma_x =: R_M.$$

[1]Note that hypothesis (6) ensures that different Φ's within the class (3) to (6) lead to equivalent norms.

This can be proven in the same way as for Q. We set as above

$$\begin{aligned} \Phi_* U &:= U \circ \Phi^{-1}, &&\text{for } U \in L_p(\Omega \times S), \\ \Phi^* V &:= V \circ \Phi, &&\text{for } V \in L_p(R_M). \end{aligned}$$

We use the same notation for the pullback and push forward as on $\Omega \times B$. It should be clear from the context which version has to be applied. We define

$$\begin{aligned} &L_p(\Omega, L_p(\Gamma_x)) := \Phi_*(L_p(\Omega, L_p(S))), \\ &\|U\|_{L_p(\Omega, L_p(\Gamma_x))} = \|\Phi^* U\|_{L_p(\Omega, L_p(S))}, \qquad U \in L_p(\Omega, L_p(\Gamma_x)). \end{aligned}$$

As before this is a Banach space. Let $s > 0$. If $\Phi(x, \cdot)|_S$ is a C^m-diffeomorphism with $m \geq s$ then we define in a consistent way the Sobolev space version

$$L_p(\Omega, W_p^s(\Gamma_x)) := \Phi_* \left(L_p(\Omega, W_p^s(S)) \right).$$

With the induced norm this is also a Banach space. From Lemma 1 we deduce that the shifted trace

$$\operatorname{tr}_S : L_p(\Omega, W_p^1(B)) \to L_p(\Omega, W_p^{1-\frac{1}{p}}(S)) : \operatorname{tr}_S U = [tr_S \hat{U}],$$

is a well defined linear operator. The last trace in the brackets is the usual trace on B. Next we transport this operator to Q. We set

$$\begin{aligned} &\operatorname{tr} : L_p(\Omega, W_p^1(\Omega_x)) \to L_p(\Omega, W_p^{1-\frac{1}{p}}(\Gamma_x)), \\ &\operatorname{tr} := \Phi_* \operatorname{tr}_S \Phi^*. \end{aligned}$$

The continuity of $\Phi_*, \Phi^*, \operatorname{tr}_S$ ensures that tr is a continuous operator. Finally $\operatorname{tr} U = 0$ implies $\operatorname{tr}_S(\Phi^* U) = 0$. This means that Dirichlet boundary conditions are preserved. From Lemma 1 we conclude that tr_S is a retraction. Let R_S be the right inverse of tr_S in the way that it maps constant functions on the boundary to constant functions on B. Hence for $U \in L_p(\Omega, W_p^{1-\frac{1}{p}}(S))$ with $U(x) = const.$ for a.e. $x \in \Omega$ we know that

$$\partial_{z_i}(R_S f(x)) = 0 \qquad \text{for } i \in \{1, \ldots, n\}.$$

In other words for a.e. $x \in \Omega$, $R_S f(x) \in W_p^1(B)$ and all derivatives vanish. We define

$$R := \Phi_* R_S \Phi^*.$$

Then it holds

$$\operatorname{tr} \circ R = id|_{L_p(\Omega, W_p^{1-\frac{1}{p}}(\Gamma_x))}.$$

Proof. Let $U \in L_p(\Omega, W_p^{1-\frac{1}{p}}(\Gamma_x))$. Then

$$\begin{aligned}(\operatorname{tr} \circ R)U &= \Phi_* \operatorname{tr}_S \Phi^* \Phi_* R_S \Phi^* U, \\ &= \Phi_* (\operatorname{tr}_S R_S) \Phi^* U, \\ &= \Phi_* \Phi^* U = U.\end{aligned}$$

This completes the proof. □

So $R : L_p(\Omega, W_p^{1-\frac{1}{p}}(\Gamma_x)) \to L_p(\Omega, W_p^1(\Omega_x))$ is a continuous right inverse to tr. It has the same properties as R_S. Let $u \in L_p(\Omega)$. The function $\tilde{u} = \Phi_*(u \cdot \mathbb{1}_S)$ is in $L_p(\Omega, W_p^{1-\frac{1}{p}}(\Gamma_x)$. Further

$$\|\tilde{u}\|_{L_p(\Omega, W_p^{1-\frac{1}{p}}(\Gamma_x))} = vol_S \|u\|_{L_p(\Omega)}.$$

In this way we will identify functions on Ω with functions on R_M or on $\Omega \times S$. We just write Ru if we mean $R\tilde{u}$ or $R_S u$ respectively. Let Δ_z denote the Laplace operator in the coordinates $z \in \Omega_x$. Similarly we write Δ_y and Δ_x for the Laplace acting on functions over B or Ω. The definitions above ensure that

$$-\Delta_z Ru(x) = 0, \qquad \text{for a.e. } x \in \Omega.$$

This will be used in later calculations.

Examples To justify our approach we consider some examples that fit into the scheme of our assumptions on the geometry.

(a) Let $\Phi : \Omega \times B \to \mathbb{R}^n \times \mathbb{R}^n$ be the identity mapping. Then Φ, Φ^{-1} fulfill the regularity conditions (3) - (6) with

$$\sup_{x \in \Omega} \left\{ \|\Phi(x)\|, \|\Phi(x)^{-1}\| \right\} = 1.$$

This is the trivial example of uniform spherical cells. The definitions are consistent in this case. If the mapping $\Phi(x, \cdot)$ is the same for all $x \in \Omega$ (and sufficiently smooth) we can treat the case of uniform cells with any smooth shape different from the unit ball.

(b) Another easy geometry are blocks that have the form of ellipsoids. For $i \in 1, \dots, n$ let $p_i : \mathbb{R}_+ \to \mathbb{R}_+$ be given polynomials. Suppose $p_i(\|x\|)$ is bounded away from zero for all $x \in \Omega$. Set

$$\begin{aligned}\Phi : \Omega \times B &\to \mathbb{R}^n \times \mathbb{R}^n, \\ (x, y) &\mapsto (x, (p_1(\|x\|)y_1, \dots p_n(\|x\|)y_n)).\end{aligned}$$

It is true that p_i is Lipschitz continuous on any closed intervall. So, since Ω is bounded, for any $x, \tilde{x} \in \overline{\Omega}$, there exists $L > 0$, such that

$$\begin{aligned} |p_i(\|x\|) - p_i(\|\tilde{x}\|)| &\leq L(\|x\| - \|\tilde{x}\|), \\ &\leq L\|x - \tilde{x}\|. \end{aligned}$$

The constant L can be chosen indepent of i. For any $(x, y), (\tilde{x}, \tilde{y}) \in \Omega \times B$ we calculate

$$\begin{aligned} |p_i(\|x\|)y_i - p_i(\|\tilde{x}\|)\tilde{y}_i| &= |p_i(\|x\|)y_i - p_i(\|\tilde{x}\|)y_i + p_i(\|\tilde{x}\|)y_i - p_i(\|\tilde{x}\|)\tilde{y}_i|, \\ &\leq |y_i|\,|p_i(\|x\|) - p_i(\|\tilde{x}\|)| + p_i(\|\tilde{x}\|)|y_i - \tilde{y}_i|. \end{aligned}$$

This implies that there is a constant $C > 0$ such that

$$\begin{aligned} &\|\Phi(x, y) - \Phi(\tilde{x}, \tilde{y})\|_{\mathbb{R}^n \times \mathbb{R}^n} \\ &\leq \|x - \tilde{x}\|(1 + L\sum_{i=1}^{n}|y_i|) + \sup_{x \in \Omega, i=1,\dots n} p_i(\|\tilde{x}\|)\|y - \tilde{y}\| \\ &\leq C\|(x - \tilde{x}, y - \tilde{y})\|_{\mathbb{R}^n \times \mathbb{R}^n}. \end{aligned}$$

So $\Phi \in \mathrm{Lip}(\Omega \times B, Q)$. The inverse exists and is also Lipschitz continuous because for all $i \in \{1, \dots n\}$ it holds

$$\frac{1}{p_i(\|x\|)}y_i - \frac{1}{p_i(\|\tilde{x}\|)}\tilde{y}_i = \frac{1}{p_i(\|x\|)p_i(\|\tilde{x}\|)}\big(y_i p_i(\|\tilde{x}_i\|) - \tilde{y}_i p_i(\|x\|)\big).$$

The boundedness of the first factor on the right hand side follows from the boundedness of the p_i. The second term can be estimated as before. For every $x \in \Omega$ the map $\Phi(x) : \overline{B} \to \overline{\Omega}_x := \Phi(\overline{B})$ is a diffeomorphism because it is linear in $y \in B$ and every p_i is independent of y. Now the boundedness of p_i and p_i^{-1} shows that the regularity conditions are satisfied.

(c) This example lives in $\mathbb{R}^2$. Let $\Omega \subseteq B(0, \frac{1}{2})$ be smooth enough. Again we parametrize the deformation using the euclidean norm of $x \in \Omega$. Let $t \in (0, \frac{1}{2})$, $y_1 \in [-1, 1]$. Set

$$R(y_1, t) := \begin{cases} \sqrt{(1-t)^2 - (y_1 + t)^2}, & -1 \leq y_1 \leq -t, \\ (1-t) - \frac{t^2}{(1-t)\pi^2}(1 - \cos(\frac{\pi}{t}y_1 - \pi)), & -t < y_1 < t, \\ \sqrt{(1-t)^2 - (y_1 - t)^2}, & t \leq y_1 \leq 1. \end{cases}$$

For each $t \in (0, \frac{1}{2}]$ we have $R(t) : (y_1 \mapsto R(y_1, t)) \in C^2[-1, 1]$. Set $R(0) = id_{[-1,1]}$. Then for each $t \in [0, \frac{1}{2}]$ the mappings

$$\Phi_t : (x, y) \mapsto \left(x, \frac{y}{\sqrt{1 - x^2}} R(x, t) \right),$$
$$\Phi_t^{-1} : (x, y) \mapsto \left(x, \sqrt{1 - x^2} \frac{y}{R(x, t)} \right)$$

are C^2-diffeomorphisms between $\overline{B}$ and $\overline{\Omega}_t := \Phi_t(\overline{B})$. Since Φ is continuous on a bounded closed domain, it is Lipschitz continuous from $\Omega \times B \to Q$. By an easy calculation one sees that (3) - (6) are satisfied. Figure 3 shows the shape of the cells for different values of the parameter t. So this is a nontrivial example that even includes nonconvex domains.

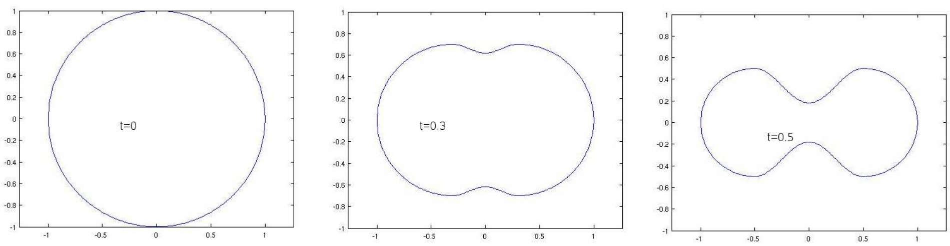

Figure 3: Barbell geometry

Function spaces for the matched microstructure problem have been considered by several authors before. In [49], S. Meier and M. Böhm present a different ansatz. They assume that there exists a bounded set $Y \subset \mathbb{R}^n$ such that $\Omega_x \subseteq Y$ for all $x \in \Omega$. Then they define

$$L_p(\Omega, W_p^s(\Omega_x))_{MB} := \big\{ f \in L_p(\Omega, W_p^s(Y)); \quad f(x, \cdot) = 0 \text{ on } Y \setminus \Omega_x \text{ for a.e. } x \in \Omega \big\},$$
$$\|f\|_{MB} = \|f\|_{L_p(\Omega, W_p^s(Y))}.$$

Let R_M be as in our notation. For $2 \leq q < \infty$ the two mathematicians define

$$L_{p,q}(R_M) := \{ f : R_M \to \mathbb{R} \text{ measurable such that } f(x) \in L_q(\Gamma_x) \text{ for a.e. } x \in \Omega \text{ and } \|f\|_{L_{p,q}(R_M)} < \infty \big\},$$
$$\|f\|_{L_{p,q}(R_M)} = \left(\int_\Omega \|f(x)\|_{L_q(\Gamma_x)}^p \, dx \right)^{\frac{1}{p}}.$$

We show that these two definitions are equivalent to our definition if $p = q$. For the manifold $\Omega \times S$ we get by Fubini's Theorem

$$\int_{\Omega\times S} f\,dz = \int_\Omega \int_S f\,d\sigma_S\,dx.$$

The integration over S uses local coordinates. At the moment let $f \in L_p(\Omega, L_p(S))$. Then $\Phi^* f$ is measurable on R_M because Φ_* is continuous and for a.e. $x \in \Omega$ the relation $\Phi^* f(x) \in L_p(\Gamma_x)$ holds. Hence by the transformation theorem for submanifolds it holds

$$\begin{aligned}
\|\Phi^* f\|_{L_{p,p}(R_M)} &= \left(\int_\Omega \|(\Phi^* f)(x)\|^p_{L_p(\Gamma_x)}\,dx\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \|(f\circ\Phi(x))(x)\|^p_{L_p(\Gamma_x)}\,dx\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |\det \partial\Phi^{-1}(x)| \|f(x)\|^p_{L_p(S)}\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

For functions on the domains Ω_x a similar calculation proves the equivalences. Nevertheless note that the spaces are not isometric isomorph. As an example we take $\Omega \subset \mathbb{R}^2$ the unit sphere and $\Omega_x = B(0, \|x\|+1)$ the sphere with radius $\|x\| + 1$. Let $f : R_M \to \mathbb{R}$ be the constant function $f = 1$. Then

$$\begin{aligned}
\|f\|^p_{L_{p,p}(R_M)} &= \int_\Omega \int_{\Gamma_x} 1^p\,d\sigma_y\,dx = \int_\Omega 2\pi(\|x\|+1)\,dx \\
&= \int_0^1 \int_0^{2\pi} 2\pi r(r+1)\,d\phi\,dr = \frac{7}{3}\pi^2,
\end{aligned}$$

while

$$\begin{aligned}
\|f\|^p_{L_p(\Omega, L_p(\Gamma_x))} &= \int_\Omega \int_S |f(\Phi(x)y)|^p\,d\sigma_y\,dx = \int_\Omega \int_S 1\,d\sigma_y\,dx \\
&= \int_0^1 \int_0^{2\pi} 2\pi r\,d\phi\,dr = 2\pi^2.
\end{aligned}$$

In fact in $L_p(R_M)$ the norm of the function has even a different value,

$$\begin{aligned}
\|f\|^p_{L_p(R_M)} &= \int_{R_M} 1^p\,d\sigma_{x,y} = \int_\Omega \int_{\Gamma_x} |G| 1\,d\sigma_x\,dx \\
&= \sqrt{2}\,\frac{4}{3}\pi^2.
\end{aligned}$$

Here G denotes the Gramschian determinant. The ansatz of Meier and Böhm was used by Meier his PhD-thesis to model an evolving microstructure that stays inside Y. Our ansatz focuses on the properties of the map

$$x \mapsto \Omega_x,$$

via the strucutre function Φ. In contrast to Meier we assume that Φ is autonomous.

4.2 Operators

This chapter is devoted to the definition of an operator $\mathbf{A}$ on the product space $L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x))$ that includes the coupling terms. The domain of definition of this operator will reflect the boundary condition on both scales. Then we prove that $-\mathbf{A}$ generates an analytic semigroup which finally implies well-posedness of the matched microstructure problem. To use existing results for strongly elliptic operators we first consider some auxiliary operators. Let A_1 be the Dirichlet-Laplace operator on Ω,

$$\begin{aligned} dom(A_1) &= W_p^2(\Omega) \cap W_p^{1,0}(\Omega), \\ A_1 u &= -\Delta_x u, \qquad\qquad \text{for } u \in dom(A_1). \end{aligned}$$

It is well known that A_1 is sectorial. Next we present some more definitions with respect to the cell geometry. For each $x \in \Omega$ we define a Riemannian metric $g(x)$ on the unit ball B. We write

$$\begin{aligned} g_{ij}(x) &:= (\partial_{z_i}\Phi(x)|\partial_{z_j}\Phi(x)), \\ \sqrt{|g(x)|} &:= \sqrt{\det g_{ij}(x)}, \\ g^{ij}(x) &:= (g_{ij}(x))^{-1}. \end{aligned}$$

Then the regularity assumptions on Φ imply that this metric is well defined, positive definite and

$$C_1|\xi|^2 \leq \sum_{i,j} g^{ij}\xi_i\xi_j \leq C_2|\xi|^2. \tag{9}$$

Let $U \in L_p(\Omega, W_p^1(\Omega_x))$, $V = \Phi^* U$. We set

$$q(U)(x) := -\int_S \sqrt{|g(x)|}g^{ij}(x)\partial_{y_i}\hat{V}(x) \cdot \nu_j \, ds. \tag{10}$$

Here $\nu = (\nu_1, \ldots, \nu_n)$ denotes the outer normal vector of B. Then $q(U)$ is a function in $L_p(\Omega)$. Using the transformation rule for integrals one sees that

this definition is consistent with (2).
We define the operator $\mathbf{A}_2$ using the transformed setting

$$dom(\mathbf{A}_2) = \{U \in L_p(\Omega, W_p^2(\Omega_x)); \operatorname{tr} U = 0\}, \tag{11}$$

$$\mathbf{A}_2 U = \Phi_*[\mathcal{A}_x \hat{V}(x)]. \tag{12}$$

The brackets $[\cdot]$ again indicate taking the equivalence class and $\hat{V}$ is a representative of V. Given $x \in \Omega$, the operator $\mathcal{A}_x$ acts in the following way on $v \in W_p^2(B)$

$$\mathcal{A}_x v = -\frac{1}{\sqrt{|g(x)|}} \sum_{i,j} \partial_{y_i} \left(\sqrt{|g(x)|} g^{ij}(x) \partial_{y_j} \right) v.$$

Note that $\mathcal{A}_x$ is the Laplace-Beltrami-operator with respect to the Riemannian metric g. It holds

Lemma 6. *The operator $\boldsymbol{A}_2$ is well defined.*

Proof. The coefficients of $\mathcal{A}_x$ depend continuously on x. Moreover the domain of definition is independent of $x \in \Omega$. Since Φ is defined up to the boundary of Ω the definition can be extended to the closure. So the hypothesis follows from Lemma 2 and the properties of Φ_*. $\square$

In the later we will use the letter y to denote coordinates in B. To points in Ω_x we refer by z. The following lemma collects some properties of the defined operators. Let $R(\lambda, A) = (\lambda + A)^{-1}$ denote the resolvent operator of $-A$ for $\lambda \in \rho(-A)$.

Lemma 7. *Assume that for any $x \in \Omega$, $\Phi_x := \Phi(x, \cdot)$ is orientation preserving. Further assume that the Riemannian metric g^{ij} induced from Φ is well defined. For each cell we define the transformation B_x of the Dirichlet-Laplace operator ,*

$$dom(B_x) = W_p^2(B) \cap W_p^{1,0}(B),$$
$$B_x v = \mathcal{A}_x v, \qquad \textit{for } v \in dom(B_x).$$

It holds

(a) The operator B_x can be written as

$$-B_x = \sum_{i,j=1}^{n} b_{ij}(x) \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial y^i}$$

where for $i,j \in \{1,\dots,n\}$

$$b_{ij}(x) = \sum_{k=1}^{n} \frac{(\partial\Phi_x^{-1})_i}{\partial z_k}\frac{(\partial\Phi_x^{-1})_j}{\partial z_k} \qquad \text{and} \qquad b_i(x) = \sum_{k=1}^{n} \frac{\partial^2(\Phi_x^{-1})_i}{\partial z_k^2}.$$

The operators B_x *are strongly elliptic in* $L_p(B)$.

(b) *Given* $x \in \Omega$, *the operator* B_x *is sectorial. In addition there exists a sector*

$$S_{\theta,\omega} = \{\lambda \in \mathbb{C}; \lambda \neq \omega, |arg(\lambda-\omega)| < \theta\},$$

and a constant $M_2 > 0$, *both independent of* x, *such that*

$$\rho(-B_x) \supseteq S_{\theta,\omega}, \tag{13}$$

$$\|R(\lambda, B_x)\|_{\mathcal{L}(L_p(B))} \leq \frac{M_2}{|\lambda-\omega|}, \qquad \text{for all } \lambda \in S_{\theta,\omega}. \tag{14}$$

(c) *The operator* $\boldsymbol{B}$ *in* $L_p(\Omega \times B)$, *given by*

$$dom(\boldsymbol{B}) = L_p(\Omega, W_p^2(B) \cap W_p^{1,0}(B)),$$
$$\boldsymbol{B}V = [B_x\hat{V}(x)], \qquad\qquad V \in dom(\boldsymbol{B}), \hat{V} \in V,$$

is well defined and sectorial.

(d) *Set* $\tilde{f} := \Phi^* f \in L_p(\Omega\times B)$. *If the function* $V \in L_p(\Omega, W_p^2(B) \cap W_p^{1,0}(B))$ *is a solution of* $\boldsymbol{B}V = \tilde{f}$, *then* $U := \Phi_* V$ *fulfills*

$$-\Delta_z V(x,\cdot) = f(x,\cdot), \quad \text{for a.e. } x \in \Omega.$$

Moreover

$$U(x,z) = 0, \quad \text{for a.e. } x \in \Omega, z \in \Gamma_x.$$

Proof. (a) Recall that $\Phi(x)$ is a C^2-diffeomorphism between $\overline{B}$ and $\overline{\Omega}_x$ for each $x \in \Omega$. So using the chain rule, we can calculate $b_{ij}(x)$ and $b_i(x)$ ($i,j \in \{1,\dots,n\}$). For $\zeta \in \mathbb{R}^n$ the fact that (9) holds, implies

$$\sum_{i,j} b_{ij}\zeta_i\zeta_j = \sum_{k=1}^{n}\sum_{i,j} \frac{(\partial\Phi_x^{-1})_i}{\partial z_k}\frac{(\partial\Phi_x^{-1})_j}{\partial z_k}\zeta_i\zeta_j \geq \kappa|\zeta|^2.$$

(b) With (a) we conclude that $b_{ij} \in C^0(B)$ and $b_i \in L^\infty(B)$ for a.e. $x \in \Omega$ and $i,j \in \{1,\dots,n\}$. Since Φ_x is orientation preserving and Φ is Lipschitz continuous on $\Omega \times B$ we deduce

$$|b_{ij}|, |b_i| \leq \Lambda \qquad \text{for some constant } \Lambda > 0.$$

In particular Λ can be chosen independent of $(x, y) \in \Omega \times B$. From Lunardi [46], Theorem 3.1.3, we know that all B_x are sectorial. Theorem 9.14 in [32] states that the sector $S_{\theta,\omega}$ and the occuring constant $M > 0$ depend only on the space dimension n, p, the constants κ, Λ, the domain B and the moduli of continuity of b_{ij} on B. The parameters n, p, κ, Λ and B are independent of $x \in \Omega$. It remains to consider the moduli of continuity of b_{ij}. Here the definition of $\Phi(x)$ up to the boundary of B ensures that they do not depend on x: Every $b_{ij}(x)$ is continuous on $\overline{B}$ and so uniformly continuous on B. Since Φ is Lipschitz continuous on $\Omega \times B$ we conclude that the moduli of continuity of b_{ij} $(i, j \in \{1, \dots, n\})$ are uniformly bounded.
(c) By definition it holds

$$\mathbf{B} = \Phi^* \mathbf{A}_2 \Phi_*.$$

Lemma 6 tells us that $\mathbf{B}$ is well defined. Let $S_{\theta,\omega} \subset \rho(-B_x)$ be the sector contained in the resolvent of all $-B_x$. Take $\lambda \in S_{\theta,\omega}$. Let $R(\lambda, B_x)$ be the resolvent of $-B_x$. We define $R_\lambda \in \mathcal{L}(L_p(\Omega, L_p(B)))$ by

$$R_\lambda V = [R(\lambda, B_x)\hat{V}], \qquad \hat{V} \text{ representative of } V \in L_p(\Omega \times B).$$

Now we show that $\lambda \in \rho(-\mathbf{B})$ and that $R_\lambda = R(\lambda, \mathbf{B})$. For $x, y \in \Omega$ it holds:

$$\begin{aligned} \|R(\lambda, B_x) - R(\lambda, B_y)\|_{\mathcal{L}(L_p(B))} &= \|(B_y - B_x)(\lambda + B_x)^{-1}(\lambda + B_y)^{-1}\|_{\mathcal{L}(L_p(B))} \\ &\leq C\|B_y - B_x\|_{\mathcal{L}(W_p^2(B) \cap W_p^{1,0}(B), L_p(B))}. \end{aligned}$$

The map $x \mapsto B_x$ is continuous on $\overline{\Omega}$ considered as a function in $\mathcal{L}(W_p^2(B) \cap W_p^{1,0}(B), L_p(B))$ since the coefficients b_{ij}, b_i depend continuously on x. So by the above considerations the map $(x \mapsto R(\lambda, B_x))$ is continuous from $\overline{\Omega}$ into $\mathcal{L}(L_p(B))$. Hence Lemma 2 can be applied and R_λ is well defined. It is the resolvent operator for $-\mathbf{B}$ which can be checked by a straight forward calculation. This also proves that $\lambda \in \rho(-\mathbf{B})$. Let $U \in L_p(\Omega, L_p(B))$. From (14) it follows by integration over Ω that

$$\begin{aligned} \|R_\lambda U\|^p_{L_p(\Omega, L_p(B))} &\leq \int_\Omega \|(\lambda + B_x)^{-1}U(x)\|^p_{L_p(B)}\, dx \\ &\leq \left(\frac{M_2}{|\lambda - \omega|}\right)^p \|U\|^p. \end{aligned}$$

(d) This part follows immediately from the sectoriality of $\mathbf{B}$ and the connection to the shifted Laplacian $\mathbf{A}_2$. □

The metric g^{ij} is automatically sufficiently regular because of the regularity we imposed on the geometry in the previous chapter. Now we are ready to treat the coupled problem. Given $u \in L_p(\Omega)$, we set

$$D_0(u) := \left\{U \in L_p\left(\Omega, W_p^2(\Omega_x)\right); \operatorname{tr} U = u\right\}.$$

This is a closed linear subspace of $L_p(\Omega, W_p^2(\Omega_x))$. We use this to define the operator $\mathbf{A}$ by

$$dom(\mathbf{A}) = \bigcup_{u \in W_p^2(\Omega) \cap W_p^{1,0}(\Omega)} \{u\} \times D_0(u),$$

$$\mathbf{A}(u, U) = \left(-\Delta_x u, [\Phi_* \mathcal{A}_x \Phi^* \hat{U}(x)]\right), \qquad \text{for } (u, U) \in dom(\mathbf{A}).$$

Observe that this operator contains the matching condition. The exchange term $q(U)$ will appear as a term on the right hand side of the abstract problem (18). Let $(f, g) \in L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x))$, $\lambda \in S_{\theta,\omega}$. We consider the system

$$(\lambda + \mathbf{A})(u, U) = (f, g), \qquad \text{for } (u, U) \in dom(\mathbf{A}).$$

This formally corresponds to

$$\lambda u - \Delta_x u = f, \qquad u \in W_p^2(\Omega) \cap W_p^{1,0}(\Omega), \tag{15}$$

$$\lambda U - \Delta_z U = g, \qquad U \in D_0(u). \tag{16}$$

Proposition 8. *The operator* $-\boldsymbol{A}$ *generates an analytic semigroup on the product space* $L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x))$.

Proof. Let ω_i, θ_i such that $S_{\theta_1,\omega_1} \subset \rho(-A_1)$, $S_{\theta_2,\omega_2} \subset \rho(-\mathbf{A}_2)$. Set

$$\omega = \max\{\omega_1, \omega_2\}, \qquad \theta = \min\{\theta_1, \theta_2\}.$$

Then $S_{\theta,\omega} \subset \rho(-A_1) \cap \rho(-\mathbf{A}_2)$. Take $\lambda \in S_{\theta,\omega}$. Without restriction we take $\omega = 0$. Since A_1 is sectorial the function $u = R(\lambda, A_1)f$ solves (15). Furthermore there is $M_1 \geq 1$ such that

$$|\lambda|\|u\| \leq \||\lambda| R(\lambda, A_1) f\| \leq M_1 \|f\|.$$

For $U \in D_0(u)$, it holds

$$U - Ru \in L_p(\Omega, W_p^2(\Omega_x)) \cap \ker \operatorname{tr} = dom(\mathbf{A}_2).$$

Here R is the extension operator defined in the previous chapter. So

$$\partial_{z_i} Ru = 0, \qquad i \in \{1, \ldots, n\}$$

and (16) is equivalent to

$$\lambda(U - Ru) + \mathbf{A}_2(U - Ru) = g - \lambda Ru. \tag{17}$$

Now we use the sectoriality of $\mathbf{B}$. It implies that (17) has the unique solution

$$U = \Phi_* R(\lambda, \mathbf{B}) \Phi^* (g - \lambda R u) + R u.$$

So we have shown that

$$(\lambda - \mathbf{A})(u, U) = (f, g)$$

has a unique solution for $\lambda \in S_{\theta,\omega}$. Hence we conclude that $\lambda \in \rho(-\mathbf{A})$. To shorten the notation we write $X_0 = L_p(\Omega, L_p(\Omega_x))$. It holds

$$\begin{aligned} |\lambda| \|U\| \leq |\lambda| & \|\Phi_* R(\lambda, \mathbf{B}) \Phi^* g\|_{X_0} \\ & + |\lambda| \|\Phi_* R(\lambda, \mathbf{B}) \Phi^* \lambda R \; R(\lambda, A_1) f\|_{X_0} \\ & + |\lambda| \|R \; R(\lambda, A_1) f\|_{X_0} \\ \leq \; & M_2 \|\Phi_*\| \|\Phi^*\| \|g\|_{X_0} + M_2 \|\Phi_*\| \|\Phi^*\| \|R\| M_1 \|f\|_{L_p(\Omega)} \\ & + M_1 \|R\| \|f\|_{L_p(\Omega)}. \end{aligned}$$

With this we estimate the norm of the resolvent of $-\mathbf{A}$

$$\begin{aligned} |\lambda| \|R(\lambda, \mathbf{A})\| = \sup\{ & |\lambda| \|U\| + |\lambda| \|u\|; u = R(\lambda, A_1) f, \\ & U = R(\lambda, \mathbf{A}_2)(g - \lambda R u) + R u, \|f\| + \|g\| \leq 1 \}, \\ \leq \; & M_2 \|\Phi_*\| \|\Phi^*\| + M_1 M_2 \|\Phi_*\| \|\Phi^*\| \|R\| + M_1 \|R\| + M_1 =: M. \end{aligned}$$

Hence the sector is contained in the resolvent set $\rho(-\mathbf{A})$ and the inequality

$$|\lambda| \|R(\lambda, \mathbf{A})\| \leq M$$

holds for some constant $M \geq 1$ independent of λ. So $\mathbf{A}$ is sectorial and $-\mathbf{A}$ generates a holomorphic semigroup. □

We are now prepared to write the matched microstructure problem as an abstract evolution equation. Set $w = (u, U)$, $w_0 = (u_0, U_0)$. We look for w satisfying

$$\begin{cases} \partial_t w + \mathbf{A} w = f(w), & t \in (0, T) \\ w(0) = w_0. \end{cases} \tag{18}$$

To solve this semilinear problem we take $\frac{1}{2} < \Theta < 1$. Our goal then is to show that

$$f : (0, T) \times [Y_0, D(\mathbf{A})]_\Theta \to Y_0 := L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x))$$

is locally Hölder continuous in t and locally Lipschitz continuous in w and that $w_0 \in [Y_0, D(\mathbf{A})]_\Theta$. Then results from Amann [6] imply local existence and uniqueness. So we have to investigate the interpolation spaces. The domains of the fractional powers of $\mathbf{A}$ are strongly connected to these spaces. In [35] it is proven an existence theorem on this bases. Let $[\cdot,\cdot]_\Theta$ denote complex interpolation for $0 \le \Theta \le 1$. Then

$$D(\mathbf{A}^\Theta) \subset [Y_0, D(\mathbf{A})]_\Theta.$$

Equality holds if the operator has bounded imaginary powers.

4.3 Interpolation and Existence Results

4.3.1 Complex interpolation

Let X_0, X_1 be two Banach spaces. We say that they form an interpolation couple $\{X_0, X_1\}$, if there is a locally convex space X such that

$$X_j \hookrightarrow X, \qquad j = 0, 1.$$

For the definition we refer to [7]. Now let $\{X_0, X_1\}$ be an interpolation couple. In this paragraph let $0 \le \Theta \le 1$. We denote with $[X_0, X_1]_\Theta$ the complex interpolation spaces of order Θ. The following theorem is well known (see [19], [63]):

Theorem 9. *Let $\{X_0, X_1\}$ be an interpolation couple, let $1 \le p_0 < \infty$, $1 \le p_1 < \infty$ and $0 < \Theta < 1$. Then for any bounded domain $\Omega \subset \mathbb{R}^n$ it holds*

$$[L_{p_0}(\Omega, X_0), L_{p_1}(\Omega, X_1)]_\Theta = L_p(\Omega, [X_0, X_1]_\Theta),$$

where

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}.$$

The spaces are isometric isomorph.

For the proof we refer to Calderon [20]. We need some additional statements.

Proposition 10. *Let $\{X_0, X_1\}$, $\{Y_0, Y_1\}$ be two interpolation couples of Banach spaces such that $X_1 \hookrightarrow X_0$, $Y_1 \hookrightarrow Y_0$. Let $\mathcal{F}$ be an arbitrary interpolation functor. Then $\{X_0 \times Y_0, X_1 \times Y_1\}$ is an interpolation couple as well and*

$$X_1 \times Y_1 \hookrightarrow \mathcal{F}(X_0, X_1) \times \mathcal{F}(Y_0, Y_1) \doteq \mathcal{F}(X_0 \times Y_0, X_1 \times Y_1) \hookrightarrow X_0 \times Y_0.$$

With $\doteq$ we mean that the sets coincide and that they are topologically equivalent w.r.t. to equivalent norms. This theorem has been proven in [7]. The next proposition states that retractions on interpolation couples are also retractions on the interpolation spaces. This is Prop. I 2.3.2 in [7]:

Proposition 11. *Let $R : \{X_0, X_1\} \to \{Y_0, Y_1\}$ be a retraction in the category of interpolation couples, and let $S : \{Y_0, Y_1\} \to \{X_0, X_1\}$ be a coretraction for R. If $\mathcal{F}$ is an arbitrary interpolation functor then*

$$R \in \mathcal{L}(\mathcal{F}(X_0, X_1), \mathcal{F}(Y_0, Y_1)),$$

is a retraction and S is a coretraction for $R = \mathcal{F}(R)$.

This has been used to prove the following statement ([63], 4.3.1). Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞-domain. Then it holds

$$\left[L_p(\Omega), W_p^2(\Omega)\right]_\Theta = W_p^{2\Theta}(\Omega) \qquad \text{for } 0 < \Theta < 1,\ 1 < p < \infty. \tag{19}$$

Let $\mathrm{tr}_{\partial\Omega} : W_p^1(\Omega) \to W_p^{1-\frac{1}{p}}(\partial\Omega)$ denote the trace operator. R. Seeley showed in [58] that

$$\left[L_p(\Omega), W_p^2(\Omega) \cap \ker \mathrm{tr}_{\partial\Omega}\right]_\Theta = \begin{cases} W_p^{2\Theta}, & \text{if } 2\Theta < \frac{1}{p}, \\ W_p^{2\Theta}(\Omega) \cap \ker \mathrm{tr}_{\partial\Omega}, & \text{if } 2\Theta > \frac{1}{p}. \end{cases} \tag{20}$$

He actually gives a proof for any normal boundary system (defined in the sense of [58], §3). To determine $[Y_0, D(\mathbf{A})]_\Theta$ we start with the case of uniform spherical small cells B. Let again $0 < \Theta < 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. We set

$$\begin{aligned} X_0 &= L_p(\Omega) \times L_p(\Omega, L_p(B)), \\ X_1 &= \left(W_p^2(\Omega) \cap \ker \mathrm{tr}_\Gamma\right) \times L_p(\Omega, W_p^2(B) \cap \ker \mathrm{tr}_S). \end{aligned}$$

Then $\{X_0, X_1\}$ is an interpolation couple. With Proposition 10 applied to the complex interpolation functor it suffices to interpolate both factors separately. We calculate

$$\left[L_p(\Omega), W_p^2(\Omega) \cap \ker \mathrm{tr}_\Gamma\right]_\Theta \text{ and } \left[L_p(\Omega, L_p(B)), L_p(\Omega, W_p^2(B) \cap \ker \mathrm{tr}_S)\right]_\Theta.$$

Let $2\Theta > \frac{1}{p}$. We know from (20)

$$\left[L_p(\Omega), W_p^2(\Omega) \cap \ker \mathrm{tr}_\Gamma\right]_\Theta = \ker \mathrm{tr}_\Gamma \cap W_p^{2\Theta}(\Omega).$$

Further Theorem 9 and (20) show that

$$\begin{aligned}
&\big[L_p(\Omega, L_p(B)), L_p(\Omega, W_p^2(B) \cap \ker \operatorname{tr}_S)\big]_\Theta \\
&= L_p(\Omega, \big[L_p(B), W_p^2(B) \cap \ker \operatorname{tr}_S\big]_\Theta), \\
&= L_p(\Omega, \ker \operatorname{tr}_S \cap W_p^{2\Theta}(B)), \\
&= L_p(\Omega, W_p^{2\Theta}(B)) \cap \ker \operatorname{tr}_S .
\end{aligned}$$

The last trace tr_S is the lifted trace operator. For $0 < \Theta < \frac{1}{2p}$ there is no more restriction on the boundary. So the complex interpolation space $[X_0, X_1]_\Theta$ is known explicitely for $0 < \Theta < 1$. By definition we have

$$\begin{aligned}
L_p(\Omega, L_p(\Omega_x)) &= \Phi_* \left(L_p(\Omega, L_p(B))\right), \\
L_p(\Omega, W_p^2(\Omega_x)) &= \Phi_* \left(L_p(\Omega, W_p^2(B))\right).
\end{aligned}$$

It also holds $\Phi_*(\ker \operatorname{tr}_S) = \ker \operatorname{tr}$ because

$$\operatorname{tr} = \Phi_* \operatorname{tr}_S \Phi^*$$

is the lifted trace on Q and Φ_*, Φ^* are linear isomorphisms. The mapping $(I, \Phi_*) : (u, U) \mapsto (u, \Phi_* U)$ is an isomorphism. It maps

$$X_0 = L_p(\Omega) \times L_p(\Omega, L_p(B)) \to L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x)) =: Y_0$$

as well as

$$X_1 \to (W_p^2(\Omega) \cap \ker \operatorname{tr}_\Gamma) \times L_p(\Omega, W_p^2(\Omega_x)) \cap \ker \operatorname{tr} =: Y_1.$$

The pair $\{Y_0, Y_1\}$ is again an interpolation couple. Using Proposition 11 we get for $2\Theta > \frac{1}{p}$,

$$\begin{aligned}
[Y_0, Y_1]_\Theta &= (I, \Phi_*) \, [X_0, X_1]_\Theta \\
&= \left(W_p^{2\Theta}(\Omega) \cap \ker \operatorname{tr}_\Gamma\right) \times \left(L_p(\Omega, W_p^{2\Theta}(\Omega_x)) \cap \ker \operatorname{tr}\right).
\end{aligned}$$

Finally we define the isomorphism

$$\begin{aligned}
&J : Y_0 \to Y_0, \\
&(u, U) \mapsto (u, U + Ru).
\end{aligned}$$

Here R is the retraction of the lifted trace. Then J maps Y_1 onto $D(\mathbf{A})$. This was already used in Section 4.1. Clearly

$$T : Y_0 \to Y_0 : (u, U) \mapsto (u, U - Ru)$$

is the inverse of J. So J fulfills the conditions of Proposition 11 for the interpolation couples $\{Y_0, Y_1\}$ and $\{Y_0, D(\mathbf{A})\}$. Hence it maps $[Y_0, Y_1]_\Theta$ onto $[Y_0, D(\mathbf{A})]_\Theta$. So given $2\Theta > \frac{1}{p}$, we get explicitely

$$\begin{aligned}[Y_0, D(\mathbf{A})]_\Theta &= J\left([Y_0, Y_1]_\Theta\right), \\ &= \bigcup_{\substack{u \in W_p^{2\Theta}(\Omega) \\ \cap \ker \operatorname{tr}_\Gamma}} \{u\} \times \{U \in L_p(\Omega, W_p^{2\Theta}(\Omega_x)); \operatorname{tr} U = u\}.\end{aligned}$$

If $2\Theta < \frac{1}{p}$ the boundary condition drops in both scales. Hence we conclude

$$[Y_0, D(\mathbf{A})]_\Theta = W_p^{2\Theta}(\Omega) \times L_p(\Omega, W_p^{2\Theta}(\Omega_x)).$$

As $\mathbf{A}$ is sectorial it follows that the fractional powers of $\mathbf{A}$ exist. Further in [63], Thm. 1.15.2 it is proven that

$$D(\mathbf{A}^\Theta) \subseteq [Y_0, D(\mathbf{A})]_\Theta .$$

4.3.2 Real Interpolation

A similar analysis can be done for real interpolation functors. Let the notation for the spaces be as before. Let $(X_0, X_1)_{\Theta,q}$ denote the real interpolation space for a Banach couple $\{X_0, X_1\}$ with $0 < \Theta < 1$ and $1 \leq q < \infty$. Then similar statements as in Theorem 9 hold, in case that the value of q is maintained constant. For this and the following results we refer to [63]. For Besov spaces $B^s_{p,q}(\Omega)$ interpolation results are known. For $s - 0, s_1 > 0$, $1 < q_0, q_1, q, p < \infty$, it holds

$$\left(B^{s_0}_{p,q_0}(\Omega), B^{s_1}_{p,q_1}(\Omega)\right)_{\Theta,q} = B^s_{p,q}(\Omega) \qquad s = (1-\Theta)s_0 + \Theta s_1.$$

Finally for $q_0 = q_1 = q = p$ one gets

$$(L_p(\Omega), W_p^2(\Omega))_{\Theta,p} = B^{2\Theta}_{p,p}(\Omega) = W_p^{2\Theta}(\Omega), \qquad (2\Theta \notin \mathbb{Z}).$$

The real interpolation of Sobolev spaces with boundary conditions is due to Grisvard [33]. Let $\{B_j\}_{j=1}^k$ be a normal system of boundary value operators with grades $\{m_j\}$. For $s > 0$, $1 < p < \infty$ set

$$W^s_{p,\{B_j\}}(\Omega) = \left\{ f \in W^s_p(\Omega); B_j f|_{\partial\Omega} = 0 \text{ for } m_j < s - \frac{1}{p} \right\}.$$

Then for $m \in \mathbb{N}$,

$$(L_p(\Omega), W^m_{p,\{B_j\}}(\Omega))_{\Theta,q} = B^{\Theta m}_{p,q,\{B_j\}}(\Omega) \qquad \text{if } m\Theta - \frac{1}{p} \neq m_j, j = 1, .., k.$$

So especially

$$(L_p(\Omega), W_p^2(\Omega) \cap \ker \operatorname{tr}_\Gamma)_{1-\frac{1}{p},p} = B_{p,p}^{2-\frac{2}{p}}(\Omega) \cap \ker \operatorname{tr}_\Gamma$$
$$= W_p^{2-\frac{2}{p}}(\Omega) \cap \ker \operatorname{tr}_\Gamma, \quad \text{for } 2 - \frac{2}{p} \notin \mathbb{Z}.$$

For $p = 2$ one can use complex interpolation instead. Propositions 10 and 11 hold for arbitrary interpolation functors. So the same considerations as in the previous chapter can be done here. Finally for $q = p$ we get

$$(Y_0, D(\mathbf{A}))_{\theta,p} = [Y_0, D(\mathbf{A})]_\theta \qquad \text{if } p \neq 2, \Theta \neq \frac{1}{2p}. \tag{21}$$

In particular

$$(Y_0, D(\mathbf{A}))_{1-\frac{1}{p},p} = \bigcup_{\substack{u \in W_p^{2-\frac{2}{p}}(\Omega) \\ \cap \ker \operatorname{tr}_\Gamma}} \{u\} \times \{U \in L_p(\Omega, W_p^{2-\frac{2}{p}}(\Omega_x)); \operatorname{tr} U = u\}. \tag{22}$$

Note that Seeley and Grisvard suppose C^∞-regularity for the boundary and the occuring coefficients. This can be reduced to $C^{2+\varepsilon}$ for second order operators (see [34]) but we will not focus on this point.

4.3.3 Well-posedness

At last we put everything together. Let $0 < \Theta < 1$ and let

$$X^\Theta = [Y_0, D(\mathbf{A})]_\Theta.$$

We suppose that

$$\underline{f} = (f, g) : [0, \infty) \times X^\Theta \to Y_0 \tag{23}$$

is sufficiently smooth. We get

Theorem 12. *Let $\underline{f} = (f, g)$ be as in (23). Assume*

$$\underline{f} \in C^{1-}([0, \infty) \times X^\Theta, Y_0),$$

is locally Lipschitz continuous for some $0 < \Theta < 1$. Then for any $(u_0, U_0) \in X^\Theta$ there exists $T = T(u_0, U_0, \Theta) > 0$ such that (18) has a unique strong solution $w = (u, U)$ on $(0, T)$ which satisfies

$$u(t = 0) = u_0 \qquad \textit{and} \qquad U(t = 0) = U_0.$$

In particular it holds

$$w \in C^1\left(([0, T), Y_0\right) \cap C\left([0, T), X^\Theta\right).$$

Proof. With the above considerations, Theorem 12.1 and Remark 12.2. (b) from [6] can be applied to the abstract equation (18). The regularity results are proved in [3]. □

Corollary 13. *For* $(u_0, U_0) \in X^{\frac{1}{2}+\frac{1}{2p}}$ *there exists* $T > 0$ *such that the matched microstructure problem has a unique strong solution on* $(0, T)$.

Proof. Let $\Theta = \frac{1}{2} + \frac{1}{2p}$. Let $U \in L_p(\Omega, W_p^{1+\frac{1}{p}}(\Omega_x))$, $V = \Phi_* U$. We have to check the properties of $U \mapsto q(U)$. Recall that

$$q(U)(x) = \int_{\Gamma_x} \frac{\partial U(x)}{\partial \nu}\, ds := \int_S \sqrt{|g(x)|} g^{ij}(x) \partial_{y_i} \hat{V}(x) \nu_j(x)\, ds.$$

Since $q(U)$ is explicitely time independent it remains to show the Lipschitz continuity in U. It holds

$$\begin{aligned}
&\|q(U)\|^p_{L_p(\Omega)} \\
&\leq \int_\Omega \left| \int_S \sqrt{|g(x,s)|} g^{ij}(x,s) \partial_{y_i} \hat{V}(x,s) \nu_j \, ds \right|^p dx \\
&\leq c_p^p \int_\Omega \sum_{j=1}^n \int_S \left| \sqrt{|g(x)|} g^{ij}(x) \partial_{y_i} \hat{V}(x) \right|^p ds\, dx \\
&\leq c_p^p \max_{\substack{(x,y) \in \overline{\Omega} \times \overline{B}, \\ i,j}} \left\{ \sqrt{|g|(x,y)}\, \left| g^{ij}(x,y) \right| \right\}^p \int_\Omega \sum_{k=1}^n \int_S |\partial_{y_k} \hat{V}(x,s)|^p \, ds\, dx \\
&\leq C \int_\Omega \|\hat{V}(x)\|^p_{W_p^1(S)}\, dx \leq C \int_\Omega \|\hat{V}(x)\|^p_{W_p^{1+\frac{1}{p}}(B)}\, dx \\
&= C \|U\|^p_{L_p(\Omega, W_p^{1+\frac{1}{p}}(\Omega_x))}.
\end{aligned}$$

The constant c_p is the embedding constant of $L_p(B)$ into $L_1(B)$. The value of $C > 0$ is adapted to each term. Together with the linearity of q this shows that q is locally Lipschitz continuous on $W_p^{1+\frac{1}{2}}(\Omega) \times L_p(\Omega, W_p^{1+\frac{1}{2}}(\Omega_x)) \supset X^\Theta$. Thus the assumption follows from Theorem 12. □

4.4 Exponential Decay under Dirichlet Boundary Conditions

With the help of the strong approach it is possible to investigate the asymptotic behaviour of the solutions to the matched microstructure model. Assume that there are no external sources in the system, i.e. $f = 0$. We will

show that the corresponding solutions decay exponentially fast in this case. We try to give an interpretation in terms of pollution:
Assume that a part Ω of the ground has been polluted with a chemical. In this area the rock shows a typical porous structure. In addition there is a dense system of fissures. Suppose that it is possible to ensure that there is no pollution outside of Ω. This could be done with the help of neutralisation and cleaning. The Dirichlet boundary condition $u = 0$ on Γ, means that we are able to guarantee a clean environment. Then the polluting particles will leave the rock matrix and diffuse to the boundary of Ω. The density in the inside decays exponentially. This means that the solute gets somehow washed out quickly. So an effective removal process should be possible.
To obtain this result we investigate the spectrum of $\mathbf{A}$. In the rest of the Chapter let $p = 2$. The space $L_2(\Omega) \times L_2(\Omega, L_2(B))$ is a Hilbert space. This implies that Y_0 is a Hilbert space with the inner product

$$((u, U), (w, W))_{Y_0} = (u, w)_{L_2(\Omega)} + (\Phi_* U, \Phi_* W)_{L_2(\Omega \times B)}$$

for $(u, U), (w, W) \in Y_0$. We introduce an extended operator $\mathbf{A}_q$ by

$$\begin{aligned} &dom(\mathbf{A}_q) = dom(\mathbf{A}), \\ &\mathbf{A}_q(u, U) = \mathbf{A}(u, U) - (q(U), 0), \qquad \text{for all } (u, U) \in dom(\mathbf{A}_q). \end{aligned}$$

We first investigate the spectrum of $\mathbf{A}_q$. It is convenient to introduce a weighted space Y_g. We set

$$\begin{aligned} Y_g &= L_2(\Omega) \times L_2(\Omega, L_2(\Omega_x, \sqrt{|g|})), \\ \|(u, U)\|_{Y_g}^2 &= \|u\|_{L_2(\Omega)}^2 + \|\Phi_* U\|_{L_2(\Omega, L_2(B, \sqrt{|g|}))}^2. \end{aligned}$$

This is a well defined Hilbert space with respect to the inner product

$$\begin{aligned} ((u, U), (w, W))_{Y_g} &= (u, w)_{L_2(\Omega)} + (U, W)_{L_2(\Omega, L_2(\Omega_x, \sqrt{|g|}))} \\ &= \int_\Omega uw + \int_{\Omega \times B} \sqrt{|g|} \Phi^* U \Phi^* V. \end{aligned}$$

for $(u, U), (w, W) \in Y_g$.

Lemma 14. *The operator $\mathbf{A}_q$ is self adjoint in Y_g.*

Proof. Take $(u, U), (w, W) \in dom(\mathbf{A}_q)$. Let $V = \Phi^* U$, $Z = \Phi^* W$. Then

$$((u, U), (w, W))_{Y_g} = (u, w)_{L_2(\Omega)} + (V, Z)_{L_2(\Omega, L_2(B, \sqrt{|g|}))}$$

and

$$
\begin{aligned}
&(\mathbf{A}_q(u,U),(w,W))_{Y_q} = \\
&\quad \int_\Omega (-\Delta_x u(x)w(x) - q(U)(x)w(x))\; dx \qquad (24)\\
&\qquad - \int_\Omega \int_B \frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_{y_i} \left(g^{ij}\sqrt{|g|}\partial_{y_j} V(x,y)\right) Z(x,y)\sqrt{|g|}\, dy\, dx.
\end{aligned}
$$

Since $u = w = 0$ on Γ, by partial integration we conclude

$$\int_\Omega \Delta_x u(x)w(x)\, dx = \int_\Omega u(x)\Delta_x w(x)\, dx.$$

For a.e. $x \in \Omega$ by definition of $\mathbf{A}$ it holds $V(x,y) = u(x)$, $Z(x,y) = w(x)$ if $y \in S$. Manipulating the last integral in (24) by partial integration we get

$$
\begin{aligned}
&- \int_\Omega \int_B \frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_{y_i} \left(g^{ij}\sqrt{|g|}\partial_{y_j} V(x,y)\right) Z(x,y)\sqrt{|g|}\, dy\, dx \\
&= \int_\Omega \int_B \sum_{i,j} g^{ij}\sqrt{|g|}\partial_{y_j} V(x,y)\partial_{y_i} Z(x,y)\, dy\, dx \\
&\qquad - \int_\Omega \int_S \sum_{i,j} g^{ij}\sqrt{|g|}\partial_{y_j} V(x,y)\cdot \nu_j \cdot Z(x,y)\, dy\, dx \\
&= + \int_\Omega \int_B \sum_{i,j} g^{ij}\sqrt{|g|}\partial_{y_j} V(x,y)\partial_{y_i} Z(x,y)\, dy\, dx + \int_\Omega q(U)w \\
&= - \int_\Omega \int_B V(x,y) \sum_{i,j} \partial_{y_j} \left(g^{ij}\sqrt{|g|}\partial_{y_i} Z(x,y)\right)\, dy\, dx \\
&\qquad + \int_\Omega q(U)w - \int_\Omega u\; q(W).
\end{aligned}
$$

This implies that

$$(\mathbf{A}_q(u,U),(w,W))_{Y_g} = ((u,U),\mathbf{A}_q(w,W))_{Y_g}.$$

Hence $\mathbf{A}_q$ is symmetric. To show that it is self adjoint we need to prove $im(\mathbf{A}_q) = Y_g \doteq Y$. From the theory of elliptic operators and the representation of $\mathbf{A}$ we know that $\mathbf{A}$ is invertible and hence $im\mathbf{A} = Y$. Let $(v,V) \in Y$. We show that there exist $(z,Z) \in dom(A_q)$ with $\mathbf{A}_q(z,Z) = (v,V)$. First we know that there are $(u,U) \in dom(\mathbf{A})$ such that $\mathbf{A}(u,U) = (v,V)$. Then

$$\mathbf{A}_q(u,U) = (v - q(U), V).$$

Clearly $(q(U),0) \in Y$. So there exits functions $(w,W) \in dom(\mathbf{A})$ with $\mathbf{A}(w,W) = (q(U),0)$. This implies that $W(x) = const.$ for a.e. $x \in \Omega$. Thus $q(W) = 0$. Now from the linearity of $\mathbf{A}_q$ it follows that

$$\mathbf{A}_q\{(w,W) + (u,U)\} = (q(U),0) + (v - q(U), V) = (v,V).$$

Thus $\mathbf{A}_q$ is symmetric and $im(\mathbf{A}_q) = Y$. It follows from the fact that

$$\ker(\mathbf{A}_q^*) = im(\mathbf{A}_q)^\perp = \{0\},$$

that the dual operator is injektiv. Thus $\mathbf{A}_q \subset \mathbf{A}_q^*$ implies the assertion. $\square$

Lemma 15. *There exists a constant* $s(\boldsymbol{A}_q) > 0$ *such that*

$$(-\boldsymbol{A}_q(u,U),(u,U))_{Y_g} \leq -s(\boldsymbol{A}_q)((u,U),(u,U))_{Y_g},$$

for all $(u,U) \in dom(\boldsymbol{A}_q)$.

Proof. As before we calculate the inner product. Let $(u,U) \in dom(\mathbf{A}_q)$. Set $V = \Phi^* U$. We make use of equivalent norms in W_2^1, Sobolev embedding results and that the metric g^{ij} is positive definit with a uniform constant on $\Omega \times B$. Let $C > 0$ denote an appropriate constant that may vary from line to line. It holds

$$\begin{aligned}
&(-\mathbf{A}_q(u,U),(u,U))_{Y_g} \\
&\quad = -\int_\Omega |\nabla_x u|^2 + \int_\Omega q(U)u - \int_\Omega \int_B \sum_{i,j} g^{ij}\sqrt{|g|}(\partial_{y_i} V)(\partial_{y_j} V) \\
&\qquad + \underbrace{\int_\Omega \int_S \sum_{i,j} g^{ij}\sqrt{|g|}\partial_{y_i} V \cdot \nu_j}_{=-q(U)} \overbrace{V(x)}^{=u(x)} \\
&\quad \leq -\int_\Omega |\nabla_x u|^2 - C\int_\Omega \int_B |\nabla_y V|^2 \\
&\quad \leq -C\left(\|u\|^2_{W_2^1(\Omega)} + \|U\|^2_{L_2(\Omega,W_2^1(\Omega_x))}\right) \\
&\quad \leq -C\left(\|u\|^2_{L_2(\Omega)} + \|U\|^2_{L_2(\Omega,L_2(\Omega_x))}\right) = -s(\mathbf{A}_q)\left((u,U),(u,U)\right).
\end{aligned}$$

So we get that $s(\mathbf{A}_q)$ is the last constant in the assertion. $\square$

Hence we have obtained a bound for the numerical range of $-\mathbf{A}_q$ in the weighted space. The spectrum of a self adjoint operator is contained in the closure of its numerical range (see [42], Section V, §3). Hence the spectrum of

$-\mathbf{A}_q$ lies totally on the right hand side of $-s(\mathbf{A}_q)$. Since the weighted norm and the usual norm on Y are equivalent, we also get a spectral bound for $-\mathbf{A}_q$ in the unweighted space. The last step is to show that the operator generates an analytic semigroup. We show this using perturbation arguments. Let

$$Q : D(\mathbf{A}^{\frac{1}{2}}) \to Y_0 : (u, U) \mapsto (-q(U), 0).$$

Then $Q \in \mathcal{L}(Y_{\frac{1}{2}}, Y_0)$. Obviously

$$-\mathbf{A}_q = -\mathbf{A} + Q$$

and the conditions of Proposition 2.4.1 in [46] are satisfied. So $\mathbf{A}_q$ is sectorial. The matched microstructure problem is equivalent to

$$\begin{cases} \partial_t(u, U) + \mathbf{A}_q(u, U) = 0, & t \in (0, T), \\ (u, U)(0) = (u_0, U_0). \end{cases} \tag{25}$$

Proposition 16. *Let (u, U) be a solution of (25).*
Then $(u, U) \searrow (0, 0)$ exponentially fast.

Proof. This follows from the the fact that for analytic semigroups the growth bound and the spectral bound coincide as it is e.g. shown in [24] Corollary 3.12. □

Proposition 16 allows also to apply the principle of linearized stability to the semilinear version of (MM), provided f is of class C^1, cf. [46].
Comment The Hilbert space setting allows to get more results on the domains of fractual powers of the operator. The following statements are shown in [7]. Since $\mathbf{A}_q$ is self adjoint and of positive type in Y_g, the fractual powers can be defined using its spectral resolution $\{E_\lambda, \lambda \in \mathbb{R}\}$,

$$\mathbf{A}_q^z = \int_0^\infty \lambda^z \, d\, E_\lambda, \qquad z \in \mathbb{C}.$$

This especially impies that the purely imaginary powers of $\mathbf{A}_q$ are bounded. In this case Theorem I 2.9. in [7] says that

$$D(\mathbf{A}_q^\Theta) = [Y_g, D(\mathbf{A}_q)]_\Theta, \qquad \text{for } 0 < \Theta < 1.$$

The question remains open whether this can be transfered to $\mathbf{A}$ in Y_0 and to $p > 2$. We would expect that $\mathbf{A}$ has bounded imaginary powers in $L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x))$.

4.5 Neumann Boundary Condtions

The main part of this thesis deals with the matched microstructure problem with Dirichlet boundary conditions. It is also natural to consider no-flux condition on the boundary Γ. Therefore we want to treat a modified model. We impose

$$\partial_\nu u = 0 \qquad \text{on } \Gamma.$$

Now we set

$$\begin{aligned} dom(A_1^N) &= \left\{u \in W_p^2(\Omega); \partial_\nu u = 0 \text{ on } \Gamma\right\}, \\ A_1^N u &= -\Delta_x u, \qquad\qquad \text{for all } u \in dom(A_1^N). \end{aligned}$$

The boundary conditions in the cells are not changed. In fact Hornung and Jäger showed in [39] that this model results from the formal expansion with no flux through the boundary of Ω as it was described in Section 2. Of course this boundary condition fulfills the so called nontangentiallity condition. So by using Agmon-Douglas-Nierenberg estimates Lunardi proved in [46] that $-A_1^N$ is the generator of a strongly continuous, analytic semigroup in $L_p(\Omega)$ for $1 < p < \infty$. We set

$$\begin{aligned} dom(\mathbf{A}^N) &= \bigcup_{u \in dom(A_1^N)} \{u\} \times D_0(u), \\ \mathbf{A}^N(u, U) &= (A_1^N u, [\Phi_* \mathcal{A}_x \hat{V}(x)]), \qquad \text{for } (u, U) \in dom(\mathbf{A}^N), V = \Phi^* U. \end{aligned}$$

The modified model can be formulated as an evolution equation

$$\begin{cases} \partial_t(u, U) + \mathbf{A}^N(u, U) = (q(U), 0), & t \in (0, T), \\ (u, U)(0) = (u_0, U_0). \end{cases} \tag{26}$$

The changes in the operator occur only on the macro scale. Thus this part can be treated with well known results for elliptic operators on bounded domains. So the same considerations as for $\mathbf{A}$ can be done for $\mathbf{A}^N$. Existence and uniqueness can be proved similarly as in Chapter 4.3. But the qualitative behaviour is different.

Proposition 17. *Let $(u_0, U_0) \in W_p^1(\Omega) \times L_p(\Omega, W_p^1(\Omega_x))$ and let (u, U) be the solution of the matched microstructure problem with Neumann boundary conditions (26) on some time interval $[0, T)$. Then the material value*

$$S(u, U)(t) := \int_\Omega u(t) + \int_\Omega \int_B \sqrt{|g|} \Phi^* U(t), \qquad t \in (0, T),$$

is preserved.

Proof. Let $V = \Phi^* U$. We use Gauss' Theorem. It holds

$$\begin{aligned}
\partial_t \left(\int_\Omega u + \int_\Omega \int_B \sqrt{|g|} V \right) &= \int_\Omega \partial_t u + \int_\Omega \int_B \sqrt{|g|} \partial_t V \\
&= \int_\Omega \Delta_x u + \int_\Omega q(U) + \int_\Omega \int_B \sum_{i,j} \partial_{y_i} \sqrt{|g|} g^{ij} \partial_{y_j} V \\
&= \int_\Gamma \nabla_x u \cdot \nu + \int_\Omega q(U) + \int_\Omega \int_S \sum_{i,j} \nu_i \sqrt{|g|} g^{ij} \partial_{y_j} V \\
&= \int_\Omega (q(U) - q(U)) = 0.
\end{aligned}$$

The second equality is due to the differential equation. □

The conserved quantity $S(u, U)$ is exactly the mass of the fluid in the whole system - micro and macro scale. The factor $\sqrt{|g|}$ is the correct adjustment for the geometry. So we proved mass conservation. Hence this model may be appropriate for closed reservoirs surrounded by impermeable rocks.

5 Including Gravity - an ansatz with nonlinear boundary conditions

The model we present here is based on the sharp interface model for ground water flow. We consider a two dimensional model with uniform cells $\Omega_x = B$ for all $x \in \Omega$. This model shall describe a slightly compressible, Newtonian fluid in an unsaturated fractured porous medium. One more assumption is needed in addition to (A1) - (A7) from Chapter 2:

(A8) The solid structure is either fully saturated with the fluid or filled with air. Thus we have a sharp interface that separates the saturated and unsaturated part.

The following state equation holds:

$$\rho = \rho_0 e^{cp}, \tag{27}$$

where ρ is the density of the fluid, p is the pressure and $\rho_0, c > 0$ are given material constants of the liquid. For sake of simplicity we assume that the atmospherical pressure p_0 outside the fluid is constant and equals zero. This will lead to a description for the fluid surface. Assume gravity acts in negative x_2- respectively negative z_2-direction. The gravity constant is denoted by g. Then from [17], p.176 we know that Darcy's law holds in the form

$$v = -\frac{k}{\mu} \nabla \overline{\phi}^*.$$

Here k is the porosity of the material and μ is the effective viscosity of the fluid. The function $\overline{\phi}^*$ is Hubbert's potential defined by

$$\overline{\phi}^* = x_2 + \int_{p_0}^{p} \frac{d\tilde{p}}{g\rho(\tilde{p})}.$$

Plugging in the state equation and $p_0 = 0$, we get

$$\overline{\phi}^* = x_2 + \frac{1}{cg} \left(\frac{1}{\rho_0} - \frac{1}{\rho} \right).$$

These considerations have to be done on both scales as it was suggested in engineeral literature. This is due to the fact, that the fractures are assumed to be partially filled with rock debris. Conservation of mass then leads to the following equation ([17], p.293)

$$\begin{aligned} \nabla \left(\rho \frac{k}{\mu} \nabla \overline{\phi}^* \right) &= nc\rho^2 g \frac{\partial \overline{\phi}^*}{\partial t} \\ &= n \frac{\partial \rho}{\partial t}. \end{aligned}$$

Altogether we get

$$\nabla \left(\frac{k}{\mu} \nabla \rho + \rho^2 g \nabla x_2 \right) = n \frac{\partial \rho}{\partial t}.$$

In the two scale model let again u, U denote the density on the macroscopic domain and the microscopic ones respectively. For the cells the matching condition $U(x) = u(x)$ on $\partial B = S$ has to be fulfilled. For the macroscopic domain we assume a fixed boundary layer at $\Gamma_0 \subseteq \{x_1 = 0\}$ with no-flux boundary condition

$$\begin{aligned} \partial_2 \overline{\phi}^* &= 0 && \text{on } \Gamma_0, \\ &\Leftrightarrow \\ \partial_2 u &= -u^2 cg && \text{on } \Gamma_0. \end{aligned}$$

We also assume that the domain is periodic in x_1-direction. So consider $x_1 \in \mathbb{S}^1$. Further the pressure is assumed to be continuous and zero outside the domain. So $p = 0$ defines the position of the upper moving boundary. In terms of density this is equivalent to $u = \rho_0$ on the surface. Assume that the boundary can be described by

$$x_2 = f(t, x_1).$$

If we set $F(t, x) = f(t, x_1) - x_2$, then obviously

$$\frac{d}{dt} F = 0.$$

Written differently, the following equation determines the boundary

$$\begin{aligned} 0 &= \partial_t f + \nabla F \cdot \dot{x} \\ &= \partial_t f + (\partial_1 f, -1) \cdot v \\ &= \partial_t f - \frac{k}{\mu} \partial_1 f \partial_1 \overline{\phi}^* + \frac{k}{\mu} \partial_2 \overline{\phi}^* \\ &= \partial_t f - \frac{k}{\mu u^2} \partial_1 f \partial_1 u + \frac{k}{\mu} + \frac{k}{\mu g c u^2} \partial_2 u. \end{aligned}$$

Here we made use of Darcy's law and the special form of the Hubbert's potential. Since $u = \rho_0$ on the boundary we finally get an evolution equation for f

$$\partial_t f = \frac{k}{\mu \rho_0^2} \partial_1 f \partial_1 u - \frac{k}{\mu g c \rho_0^2} \partial_2 u - \frac{k}{\mu}. \tag{28}$$

Together with the initial conditions this results in the following system (P_f):

$$\begin{aligned}
\partial_t u(t) - div_x \left(\frac{k}{\mu} \nabla_x u(t) \right) &= -\partial_2 (u(t)^2 g) + q(U(t)), && \text{on } \Omega_{f(t)}, \\
\partial_2 u(t) &= -u(t)^2 cg, && \text{on } \Gamma_0, \\
u(t) &= \rho_0, && \text{on } \Gamma_{f(t)}, \\
\partial_t f(t) - \frac{k}{\mu \rho_0^2} \partial_1 f(t) \partial_1 u(t) &= -\frac{k}{\mu g \rho_0^2 c} \partial_2 u(t) - \frac{k}{\mu}, && \text{on } \mathbb{S}^1, \\
f(0) &= f_0,
\end{aligned}$$

and for every $x \in \Omega_{f(t)}$ holds

$$\begin{aligned}
\partial_t U(t) - div_z \left(\frac{K}{\mu} \nabla_z U(t) \right) &= -\partial_{z_2} (U(t)^2 g), && \text{on } \Omega_x, \\
U(t, x) &= u(t, x), && \text{on } \Gamma_x, \\
U(0) &= U_0.
\end{aligned}$$

In our L_p-setting the term $\partial_2(U(t)^2)$ is not in the space $L_p(\Omega, L_p(B))$ anymore. That is the reason why we modify our model and neglect the term on both scales. Also we restrict ourselves to a fixed domain. The following version of problem (P_f) allows us to include gravity into the scheme. Assume $f : (0, 2\pi) \to (0, \infty)$ i a given periodic and sufficiently smooth function. We consider the fixed domain

$$\Omega_f = \left\{ (x, y) \in \mathbb{S}^1 \times \mathbb{R}; 0 < y < f(x) \right\}.$$

It is shown in Figure 4. The gravitational force points into the $-y$ direction. The almost cylindrical domain Ω_f can be treated with the same methods as before. For a work considering the torus see [27]. Remember that we restrict ourselves to uniform cells $\Omega_x = B$ on the micro scale. Assume that $k = \mu = c = g = K = 1$. Let $h : \Omega_f \times (0, T) \to \mathbb{R}$ describe the sources and sinks in the macro system. Now a solution of the matched microstructure model with gravity is a pair of functions (u, U) that satisfies

$$(P) \quad \left\{ \begin{array}{rcll}
\partial_t u - \Delta_x u &=& h + q(U), & \text{on } \Omega_f, t \in (0, T), \\
\partial_2 u &=& -u^2, & \text{on } \Gamma_0, t \in (0, T), \\
u &=& \rho_0, & \text{on } \Gamma_f, t \in (0, T), \\
\partial_t U - \Delta_y U &=& 0, & \text{in } \Omega_f \times B, t \in (0, T), \\
U &=& u, & \text{on } \Omega_f \times S, t \in (0, T), \\
(u, U)(0) &=& (u_0, U_0), & \text{on } \Omega_f \times (\Omega_f \times B).
\end{array} \right.$$

Let $v = u - \rho_0 \cdot \mathbb{1}_{\Omega_f}$, $V = U - \rho_0 \mathbb{1}_{\Omega_f \times B}$. By definition (2) it holds

$$q(U) = q(V + \rho_0) = q(V).$$

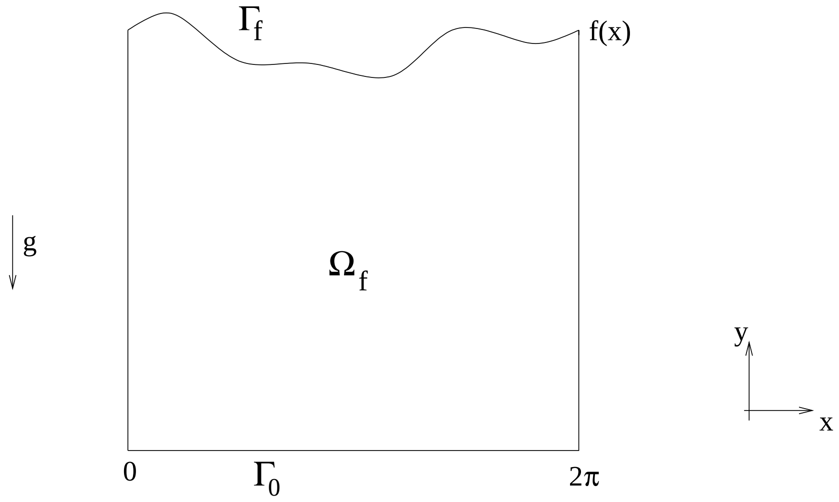

Figure 4: The periodic domain Ω_f

So (v, V) solves

$$
\text{(P')} \quad \left\{ \begin{array}{rcll} \partial_t v - \Delta_x v &=& h + q(V), & \text{on } \Omega_f, t \in (0,T), \\ \partial_2 v &=& -(v+\rho_0)^2, & \text{on } \Gamma_0, t \in (0,T), \\ v &=& 0, & \text{on } \Gamma_f, t \in (0,T), \\ \partial_t V - \Delta_y V &=& 0, & \text{in } \Omega_f \times B, t \in (0,T), \\ V &=& v, & \text{on } \Omega_f \times S, t \in (0,T), \\ (v,V)(0) &=& (u_0 - \rho_0, U_0 - \rho_0), & \text{on } \Omega_f \times (\Omega_f \times B). \end{array} \right.
$$

To treat the nonlinear boundary condition in the macroscopic scale is the main challenge. As before we define operators A_1 and $\mathbf{A}_q$ with linear zero boundary conditions ($\partial_2 v = 0$ on Γ_0, $v = 0$ on Γ_f). The mixed Dirichlet-Neumann conditions on Ω_f does not effect the properties of the operators. In particular $\mathbf{A}_q$ is selfadjoint and $-\mathbf{A}_q$ is the generator of an analytic semigroup in the Hilbert space

$$
Y_0 = L_2(\Omega_f) \times L_2(\Omega_f \times B).
$$

The operator A_1 is a well known form of the Laplace operator and so it is invertible. For the problem (P') we will use a weak ansatz to show well-posedness. The method is due to Amann [4] and Escher [25]. The main idea is to move the nonlinearity from the boundary to the right hand side h. Therefore we need to construct an appropriate inverse operator from the trace space on Γ_0 to the domain Ω_f. The resulting semilinear evolution equation can be treated with the help of the theory of analytic semigroups.

For $u \in L_2(\Omega_f)$ we set

$$D_0^s(u) = \left\{U \in L_2(\Omega_f, W_2^{2s}(B)), \operatorname{tr}_S U = u\right\}, \qquad \frac{1}{4} < s < \infty.$$

In the rest of the chapter we drop the index f of Ω_f. Let $\operatorname{tr}_0, \operatorname{tr}_f$ denote the trace operators onto Γ_0 and Γ_f. With the boundary operator on Ω we mean an operator $\mathcal{B}$ acting as

$$\mathcal{B}u = \operatorname{tr}_0 \partial_\nu u + \operatorname{tr}_f u.$$

We define

$$Y_s = \begin{cases} \{(u,U), u \in W_2^{2s}(\Omega), U \in D_0^s(u), \mathcal{B}u = 0\}, & \text{for } \frac{3}{2} < 2s \leq \infty, \\ \{(u,U), u \in W_2^{2s}(\Omega), U \in D_0^s(u), \operatorname{tr}_f u = 0\}, & \text{for } \frac{1}{2} < 2s \leq \frac{3}{2}, \\ W_2^{2s}(\Omega) \times L_2(\Omega, W_2^{2s}(B)), & \text{for } 0 \leq 2s \leq \frac{1}{2}. \end{cases} \qquad (29)$$

To construct a suitable retract, we first restrict ourselves to the macro scale. Let $\mathcal{A}_1 = -\Delta_x$. Considered as an unbounded operator in $L_2(\Omega)$ it is closable. Together with $\mathcal{B}$ it fits into the scheme of [25], Chapter 3. We will use the same notation. Let $\overline{\mathcal{A}}_1$ be the closure of $\mathcal{A}_1$. Then $W_2^2(\Omega) \overset{d}{\hookrightarrow} D(\overline{\mathcal{A}}_1)$. In addition we set

$$\mathcal{C}u = \operatorname{tr}_0 u + \operatorname{tr}_f \partial_\nu u,$$

and

$$\partial W_2^{2s} = W_2^{2s-\frac{3}{2}}(\Gamma_0) \times W_2^{2s-\frac{1}{2}}(\Gamma_f),$$
$$\partial_1 W_1^{2s} = W_2^{2s-\frac{1}{2}}(\Gamma_0) \times W_2^{2s-\frac{3}{2}}(\Gamma_f),$$

for $0 \leq s \leq 1$. The trace theorem implies that the operator $(\mathcal{B}, \mathcal{C}) \in \mathcal{L}(W_2^2(\Omega), \partial W_2^2 \times \partial_1 W_2^2)$ is a retraction. This means that it has a continuous right inverse. So we can apply Theorem 4.1. from [4]:

Proposition 18. *There exists a unique extension $\left(\overline{\mathcal{B}}, \overline{\mathcal{C}}\right) \in \mathcal{L}(D\left(\overline{\mathcal{A}}_1\right), \partial W_2^0 \times \partial_1 W_2^0)$ of $(\mathcal{B}, \mathcal{C})$ such that for $u \in D(\overline{\mathcal{A}}_1)$, $v \in W_2^2(\Omega)$ the generalized Green's formula*

$$\langle v, \overline{\mathcal{A}}_1 u\rangle_{Y_0} + \langle \mathcal{C}v, \overline{\mathcal{B}}u\rangle_{\partial W_2^0} = \langle \mathcal{A}_1 v, u\rangle_{Y_0} + \langle \mathcal{B}v, \overline{\mathcal{C}}u\rangle_{\partial_1 W_2^0}$$

is valid.

Then from interpolation theory (Prop. 11) and well known a priori estimates for $\mathcal{A}_1$, it follows that

$$\left(\overline{\mathcal{A}}_1, \overline{\mathcal{B}}\right) \in Isom((D(\overline{\mathcal{A}}_1), W_2^2(\Omega))_\theta, L_2(\Omega) \times \partial W_2^{2\theta}), \qquad \theta \in [0,1].$$

Therefore we can define the right inverse

$$R_\theta = \left(\overline{\mathcal{A}}_1, \overline{\mathcal{B}}\right)^{-1} |\{0\} \times \partial W_2^{2\theta}, \qquad \theta \in [0,1].$$

Then $R_\theta \in \mathcal{L}(\partial W_2^{2\theta}, W_2^{2\theta}(\Omega))$. We now add the microscopic scale. With $\mathcal{Y}^s$ we mean

$$\mathcal{Y}^s = \begin{cases} \{(u,U) \in W_2^{2s}(\Omega) \times L_2(\Omega, W_2^{2s}(B)); U \in D_0^{2s}(u)\}, & \frac{1}{2} < 2s \leq 2, \\ W_2^{2s}(\Omega) \times L_2(\Omega, W_2^{2s}(B)), & 0 \leq 2s \leq \frac{1}{2}. \end{cases}$$

So if $s < \frac{3}{4}$ the two sets Y_s and $\mathcal{Y}^s$ coincide. Define $\mathbf{R}_\theta \in \mathcal{L}(\partial W_2^{2\theta}, \mathcal{Y}^{2\theta})$ by

$$\mathbf{R}_\theta u = (R_\theta u, R_\theta u \cdot \mathbb{1}_B), \qquad u \in \partial W_2^{2\theta}.$$

We set

$$\partial_0 W_2^{2\theta} = \{u \in \partial W_2^{2\theta}; \operatorname{tr}_f u = 0\}.$$

Obviously it is a closed linear subspace of $\partial W_2^{2\theta}$. It can be identified with $W_2^{2\theta - \frac{3}{2}}(\Gamma_0)$. So it holds

$$\mathbf{R}_\theta(\partial_0 W_2^{2\theta}) \subset Y_{2\theta}, \qquad \text{if } 2\theta \leq \frac{3}{2}.$$

Again we denote with $\vec{u}$ the pair (u,U). For the formulation of the abstract evolution problem we use the scale of interpolation and extrapolation spcaes $\{(Y_\alpha, (\mathbf{A}_q)_\alpha), \alpha \in \mathbb{R}\}$ as it is defined by Amann in [4]. For $0 \leq \alpha \leq 1$ this corresponds to the interpolation spaces in Section 4.3 and Definition (29). We set in analogy to [25]

$$\mathbb{A} = (\mathbf{A}_q)_{-\frac{1}{2}}, \qquad H = Y_{-\frac{1}{2}}, \qquad D = Y_{\frac{1}{2}} = D(\mathbb{A}).$$

Then the duality theory tells us that $D = H'$ and the duality pairings satisfy

$$\langle \vec{u}, \vec{v} \rangle_H = \langle \vec{u}, \vec{v} \rangle_{Y_0} = (\vec{u}|\vec{v})_{Y_0}, \quad \text{for } \vec{u} \in D, \vec{v} \in Y_0.$$

Let $a : D \times D \to \mathbb{R}$ be the coercive bilinear form

$$a(\vec{u}, \vec{v}) = \int_\Omega \nabla_x u \cdot \nabla_x v \, dx + \int_{\Omega \times B} \nabla_z U \cdot \nabla_z V \, d(x,z), \quad \vec{u}, \vec{v} \in D.$$

Please note that we refer with u to a function living on Ω and with $\vec{u}$ to a pair $\vec{u} = (u,U)$. For $\vec{u}, \vec{v} \in D$ we get

$$\begin{aligned} \langle \vec{v}, \mathbb{A}\vec{u} \rangle_H &= \int_\Omega v(-\Delta_x u - q(U)) + \int_{\Omega \times B} V(-\Delta_z U) \\ &= \int_\Omega \nabla_x v \cdot \nabla_x u - \int_\Omega v q(U) + \int_{\Omega \times B} \nabla_z V \cdot \nabla_z U \\ &\quad - \int_{\Omega \times S} V \nabla_z U \cdot \nu - \int_{\Gamma_0} v \partial_\nu u - \int_{\Gamma_f} v \nabla_x u \cdot \nu = a(\vec{u}, \vec{v}). \end{aligned}$$

The possible approximation of u by functions in Y_1 and the continuity of the left and right hand side justify this formal calculation. To treat the nonlinear boundary condition we define the map $G : D \to L_2(\Gamma_0)$,

$$G(u) = -\operatorname{tr}_0(u + \rho_0)^2.$$

We want to show that (h, G) satisfies the assumption (3.6) of the abstract theory in [25]. For h we just assume that $h \in Y_0$. For G the properties are summarized in the following Lemma.

Lemma 19.

$$G \in C^1(D, W_2^{2\beta+\frac{1}{2}}(\Gamma_0))$$

for any fixed $\beta \in (-\frac{1}{2}, -\frac{1}{4})$*, and the Lipschitz continuity is uniform on bounded sets.*

Proof. Fix $\beta \in \left(-\frac{1}{2}, -\frac{1}{4}\right)$. Let $\vec{u} \in D$. Then also $u + \rho_0 \in W_2^1(\Omega)$. From [5], Theorem 4.1 and the fact that the Besov space $B_{22}^s(\Omega) = W_2^s(\Omega)$, we know that the multiplication

$$W_2^1(\Omega) \cdot W_2^1(\Omega) \to W_2^{1-\varepsilon}(\Omega)$$

is continuous for $0 < \varepsilon < 1$. We conclude that for fixed $\varepsilon < \frac{1}{2}$,

$$(u + \rho_0)^2 \in W_2^{1-\varepsilon}(\Omega) \qquad \text{and} \qquad \operatorname{tr}_0(u + \rho_0)^2 \in W_2^{\frac{1}{2}-\varepsilon}(\Gamma_0).$$

Then by Sobolev embedding it holds

$$W_2^{\frac{1}{2}-\varepsilon}(\Gamma_0) \overset{\mathrm{d}}{\hookrightarrow} L_2(\Gamma_0) \overset{\mathrm{d}}{\hookrightarrow} W_2^{2\beta+\frac{1}{2}}(\Gamma_0).$$

The second inclusion follows from the definition of W_2^{-s} as a dual spaces for $s > 0$. So finally

$$-\operatorname{tr}_0(u + \rho_0)^2 \in W_2^{2\beta+\frac{1}{2}}(\Gamma_0).$$

The Fréchet derivative of G is the linear operator $\partial G(u)v = -2\operatorname{tr}_0(u + \rho_0)v$. Thus it holds $G \in C^1(D, W_2^{2\beta+\frac{1}{2}}(\Gamma_0))$. It remains to show that the map is uniformly Lipschitz continuous on bounded sets. Let $W \subset D$ be bounded. Take $\vec{u}, \vec{v} \in W$. Then

$$\operatorname{tr}_0(u + \rho_0)^2 - \operatorname{tr}_0(v + \rho_0)^2 = \operatorname{tr}_0 u^2 - \operatorname{tr}_0 v^2 + 2\rho_0(\operatorname{tr}_0 u - \operatorname{tr}_0 v).$$

Clearly the last term is uniformly Lipschitz on W. Further $W_2^1(\Omega) \hookrightarrow C(\overline{\Omega})$. So a bounded set in D is bounded in $C(\overline{\Omega})$. Thus there exists a constant

$c_1 > 0$, such that $\|u\|_\infty \le c_1$ for all $\vec{u} \in W$. It follows from this and Sobolev embeddings that

$$\begin{aligned} \|\operatorname{tr}_0 u^2 - \operatorname{tr}_0 v^2\|_{W_2^{2\beta+\frac{1}{2}}(\Gamma_0)} &\le C\|\operatorname{tr}_0(u^2 - v^2)\|_{L_2(\Gamma_0)} \\ &\le 2c_1 C\|\operatorname{tr}_0(u-v)\|_{L_2(\Gamma_0)} \\ &\le L\|u-v\|_{W_2^1(\Omega)}. \end{aligned}$$

The last constant L is independent of $\vec{u}, \vec{v} \in W$. This completes the proof. □

Now we define the right hand side to write (P') as an abstract evolution equation. Let $\mathbf{R} := \mathbf{R}_{\frac{1}{2}}$. Then it holds for $u \in W_2^{\frac{1}{2}}(\Gamma_0)$ and $\vec{v} \in D$

$$\langle \vec{v}, \mathbb{A}\mathbf{R}u\rangle_H = \langle v, u\rangle_{W_2^{\frac{1}{2}}(\Gamma_0)} =: \langle v, u\rangle_{\Gamma_0}, \tag{30}$$

in the sense of trace. We set

$$F(\vec{u}) = (h, 0) + \mathbb{A}\mathbf{R}G(\vec{u}), \qquad \vec{u} \in D.$$

Note that the second component of $F(\vec{u})$ vanishes since $R_{\frac{1}{2}}u \cdot \mathbb{1}_B$ is constant on each cell. By assumption $h \in Y_0 \hookrightarrow Y_\beta$. It was shown in [25], p.301, that under these circumstances F is well defined and the previous lemma ensures that

$F \in C^1(D, Y_\beta)$ is uniformly Lipschitz continuous on bounded sets.

So all assumptions for the abstract theory are satisfied. It holds:

Proposition 20. *For each $\vec{u}_0 \in D$ there is a unique maximal solution $\vec{u}(\cdot, \vec{u}_0) \in C([0, T_1), D)$ of the semilinear Cauchy problem*

$$\dot{\vec{u}} + \mathbb{A}\vec{u} = F(\vec{u}), \qquad \vec{u}(0) = \vec{u}_0 \tag{31}$$

with $0 < T_1 \le \infty$. In addition

$$\vec{u} \in C((0, T_1), Y_{\varepsilon+\frac{1}{2}}) \cap C^1((0, T_1), Y_{\varepsilon-\frac{1}{2}})$$

for any $\varepsilon \in (0, \frac{1}{4})$.

Proof. Take $\beta = -\frac{1}{2} + \varepsilon$. Then the assertion follow from [4], Sect. 12 and the previous lemma. □

By a *weak solution* of (P') we mean a function $\vec{u} \in C^1([0,T_1),D)$ such that $\vec{u}(0) = \vec{u}_0 - \rho$, and

$$-\int_0^T \langle \dot{\varphi}, \vec{u} \rangle_H + a(\varphi, \vec{u})\, dt = \int_0^T \left(\langle \varphi, (h,0) \rangle_H + \int_{\Gamma_0} \varphi G(\vec{u}) \right) dt \\ + \langle \varphi(0), \vec{u}_0 - \rho_0 \rangle_H$$

for all $0 < T < T_1$, $\varphi \in C([0,T],D) \cap C^1([0,T],H)$ with $\varphi(T) = 0$. So the above considerations show that

Corollary 21.
For each $\vec{u}_0 \in Y_{\frac{1}{2}}$ there exists a unique maximal weak solution of (P').

Proof. This follows from the representation of $\mathbb{A}$ and (30). □

Remarks:
(i) The construction in [25] allows to consider a more general $h \in C^1(D, Y_\beta)$. This means a full semilinear version of (P') can be treated. Consider especially the term in (P_f):

$$-\partial_2(u^2 g) = -2u\partial_2 u =: H(u).$$

In the same way as in Lemma 19 we can show that

$$H(u) \in W_2^{2\beta+\frac{1}{2}}(\Omega).$$

Since $\partial H(u)v = -2v\partial_2 u - 2u\partial_2 v$, it holds

$$H(u) \in C^1(D, W_2^{2\beta+\frac{1}{2}}(\Omega)).$$

So

$$\tilde{F}(\vec{u}) = (H(u), 0) + \mathbb{A}\mathbf{R}G(\vec{u}), \qquad \vec{u} \in D$$

satisfies the assumptions in [25]. Hence Proposition 20 holds for $\tilde{F}$ instead of F. Unfortunately the square term in the cell system can not be handled in a similar way.
(ii) Abstract results on evolution equations in interpolation-extrapolation scales ensure that the solutions possesses additional regularity, i.e.

$$(u,U) \in C((0,T_1),Y_1) \cap C^1(0,T_1),Y_0).$$

So the system (P') is satisfied pointwise in time.

6 A Quasilinear Generalisation

6.1 Modeling

Naturally fissured rocks appear in many applications. In the 90's a large number of research projects from engineering, physics and mathematics investigated the flow of oil in fractured material (e.g. [18]). This is of great interest to open up new wells and locate new reservoirs. Another aim is to boost the extraction. In connection with this the behaviour of two or three inmiscible fluids in a fractured system is investigated. Here the goal is to replace oil by water in the reservoir to get a higher recovery rate. The work from Auriault, Lebaique and Bonnet [14] consider this case together with a deformable porous structure. In contrast to the substitution method sometimes the pressure has to be aumented using another miscible liquid. The behavior of several fluids like oil, water and gas, in the same fractured domain has been studied for example in [50].

The oldest application of a two-scale model was for a pipe system (see [59]). Consider a complex network of pipes surrounded by concrete. Now water circulates in the tubes that make up for the fractures. The whole framework now serves as an energy storrage. Heat diffusion and absorption is the principal process in consideration. The thermal energy of the fluid is transferred to the walls and stored there. When cold water enters the pipes it can be recovered and brought into use. In this task the functions u, U on the macro and micro system have to be interpreted differently. They stand for the heat.

Similar equations can be used to model the propagation of a dissolved species in a saturated double porous material. Then the fluid is often considered stationary. This has been used to examine concrete carbonation. Reinforcing steel bars in concrete are normally protected from corrosion by the high pH value of the surrounding material. But in pratice unexpected deterioration of the stabilizing rods has been observed. When atmospherical carbon dyoxid enters the concrete in a humid environment, it dissolves in the pore water. There it can react with dissolved calcium hydroxid of the structure. The chemical reaction is

$$CO_2(aq) + Ca(OH)_2(aq) \longrightarrow CaCO_3(aq) + H_2O.$$

When dried, the produced calcium carbonate becomes part of the solid structure again. But the important aspect of this process is not the change in porosity but in pH. The protection of the steel bars due to a hight pH value is destroyed and therefore corrosion can take place. The durability and stability of buildings is reduced significantly. A two scale approach treating this topic and further references can be found in [48]. Their model shows a great resemblance to the Showalter (see [61]) model. But the utilized functions live on the micro scale and the pore scale. The reaction takes place in each individual pore of the concrete and changes the pore geometry. Meier [48] introduces a model with an evolving pore structure and proves existence of weak solutions. For chemical aspects we refer to [53] and the reference within. Another approach to this phenomenon was presented by Muntean and Böhm [51]. They observed that the alteration starts at the surface of the components. A reaction front then evolves into the interior of the concrete structure. Its formation is described using a moving boundary framework. A work on the effect of fissures in the concrete on the carbonation effect is not known to the author.

Usually many different types of contamination are investigated. The trans-

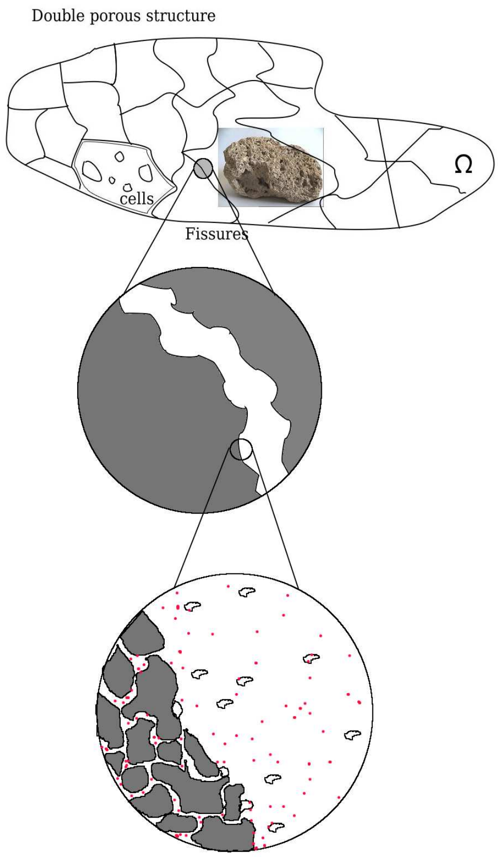

Figure 5: Colloid facilitated contaminant transport

port of a contaminant in a carrier fluid in porous media is well understood. For more details and references see [18], [15]. But the existence of fissures magnifies the pollution radius significantly. Other aspects like reactions of the solute with the solid matrix come into play. They can have positive consequences and bind hazardous particles or change the chemical composition of the rock which may influence the porosity or stability. This is not only a topic in soil sciences. It is a political one. A final repository for nuclear waste will not be safe for all times. To model the probable spreading of radioactive particles in the groundwater is therefore one of the challenges that have to be solved. This may result somehow in a better control of the dispersion of the pathogenic nuclids.

The opposite process is the removal of partical waste in rocks. A liquid flow, mostly water, is used to wash out the contamination. In fissured porous material a retardation of this process has been noticed. The fluid mainly moves in the fissure system. In this way it reduces quickly the pollution in the macro system. In contrast the rate of exchange between the porous blocks and the fissure determines the advances in the micro scale. Particles that have entered the rock matrix are removed with a severe delay.

Another aspect that comes into play is the presence of colloids in the fluid. The so-called colloid facilitated transport of pollutants has recently been a project of the DFG (for further references see [31]). Colloids are small organic or anorganic particles suspended in the liquid. Scientists consider three different types of colloidal particles: intrinsic colloidal particles, carrier colloids and biocolloids. In the first class the contaminant is an important part of the solid particle. Bacteria or viruses are examples for the last type. Carrier colloids can be considered as small former parts of the solid matrix. Their size prevents them from entering the porous matrix. Hence their contribution is limited to the fissure system. A schematic description of the situation is shown in Figure 5. The colloids have a great influence on the dispersion of a solute. The dissolved species can react with the colloids. The restriction of their range to the fratures leads to the observed effect. The particles are transported a lot further than expected in a single porous material. The dispersion of the colloids in the fluid has to be modeled separately. For more details of the modeling aspects consult Abdel-Salam, Chrysikopoulos [1] or Neretnieks [52] and the references within. An introduction to reaction-diffusion equations can be found in the book of Logan [45].

In this paragraph we generalize our ansatz to quasilinear equations. Afterwards this is linked to contaminant transport. In Part 6.3 we allow a modification of the porosity inside the cells due to reactive processes. Finally different sorption isotherms are discussed.

6.2 General Approach

Applications in general do not show a linear structure. Usually they include certain nonlinear terms. So we improve our ansatz and treat quasilinear problems. The special geometry and spaces of the problem nevertheless give a certain limit to the usual approach. We use the notation of Nemytskii operators to define a more general version of the operator $\mathbf{A}$. Let $2 \leq p < \infty$. We will prove maximal L_p-regularity in case that the initial data is sufficiently regular. We consider the problem again in an abstract framework. The proof of local existence is based on work of Clément and Li [21]. It uses the maximal regularity and Banach's Fixed Point Theorem. Results for more general operators may be obtained from perturbation theory.

First we define a Nemytskii operator a_1. Let $\tilde{a}_1 \in C^\infty(\overline{\Omega} \times \mathbb{R})$. Then

$$a_1(v)(x) := \tilde{a}_1(x, v(x)),$$

$$a_1(v) \in C(\overline{\Omega}), \qquad \text{for } v \in C(\overline{\Omega}).$$

So for $v \in C^1(\overline{\Omega})$ we define A_1 in the following way

$$\begin{aligned} dom(A_1(v)) &= W_p^2(\Omega) \cap W_p^{1,0}(\Omega), \\ A_1(v)u &= -div_x(a_1(v)\nabla_x u), \qquad \text{for all } u \in \ dom(A_1(v)). \end{aligned}$$

Assume that there is $\eta > 0$, such that

$$\tilde{a}_1(x, r) \geq \eta, \qquad \text{for all } (x, r) \in \overline{\Omega} \times \mathbb{R}.$$

Then $A_1(v)$ is a strongly elliptic operator for all $v \in C^1(\Omega)$. This is a standard definition for Nemytskii operators. More details can be found in [57]. To define the operator acting in the small cells we need to be more careful. Let $\tilde{a}_2 \in C^\infty(Q \times \mathbb{R})$, $r \in \mathbb{R}$. We set

$$\begin{aligned} \tilde{b}_2(r) &:= \Phi^* \tilde{a}_2(r), \\ b_2(v)(x, y) &:= \tilde{b}_2((x, y), v(x)), \qquad \text{for all } (x, y) \in \Omega \times B. \end{aligned}$$

It holds that $\tilde{b}_2 \in C^{2+\alpha}(\Omega \times B \times \mathbb{R})$ and thus $b_2 \in C^1(\Omega \times B)$. For any $(x, y) \in \Omega \times B$, $r \in \mathbb{R}_+$ assume that

$$\tilde{b}_2((x, y), r) \geq \eta.$$

This leads to a strongly elliptic operator for each fixed $x \in \Omega$. For $u \in W_p^2(B)$ we set

$$\mathcal{A}_x(v(x))u = -\frac{1}{\sqrt{|g(x)|}} \sum_{i,j} \partial_{y_i} \left(b_2(v)(x)\sqrt{|g(x)|} g^{ij}(x) \partial_{y_j} u \right).$$

We define $\mathbf{A}_2(v)$ for $v \in C^1(\overline{\Omega})$ as

$$\begin{aligned} dom(\mathbf{A}_2(v)) &= \left\{ U \in L_p(\Omega, W_p^2(\Omega_x)); \operatorname{tr} U = 0 \right\}, \\ \mathbf{A}_2(v)U &= \Phi_* \left[\mathcal{A}_x(v(x))\hat{V} \right] \qquad U \in dom(\mathbf{A}_2(v)). \end{aligned}$$

As before $V = \Phi^* U$ and $\hat{V}$ is a representative of V such that $\hat{V}(x) \in W_p^2(B)$ for all $x \in \Omega$.

Lemma 22. *Let $v \in C^1(\Omega)$.*
Then $\boldsymbol{A}_2(v)$ is well defined. It is a closed, densly defined, sectorial operator on $L_p(\Omega, L_p(\Omega_x))$.

Proof. Let $v \in C^1(\overline{\Omega})$, $x \in \overline{\Omega}$. First we consider the following operator

$$\begin{aligned} dom(B_x) &= W_p^2(B) \cap W_p^{1,0}(B) =: W_B, \\ B_x u &= \mathcal{A}_x(v(x))u \qquad \text{for } u \in W_B. \end{aligned}$$

Then B_x is a closed operator in $L_p(B)$. The function v as well as $\Phi(x)$ is in $C(\overline{\Omega})$. Hence the map $x \mapsto B_x$ is continuous from $\overline{\Omega}$ into $\mathcal{L}(W_B, L_p(B))$. Further the domain of definition of B_x is the same for all x. So Lemma 2 can be applied. It shows that the operator $\mathbf{B}$ defined by

$$\begin{aligned} dom(\mathbf{B}) &= \{ V \in L_p(\Omega, W_p^2(B)); \operatorname{tr}_S V = 0 \}, \\ \mathbf{B}V &= \left[\mathcal{A}_x(v(x))\hat{V}(x) \right], \end{aligned}$$

is well defined and closed. It is also a densely defined operator as it was shown in Chapter 4.2. Again it holds

$$\mathbf{B} := \Phi^* \mathbf{A}_2(v) \Phi_*. \tag{32}$$

Note that we can write as before

$$B_x u = -\sum_{i,j} b_{ij}(x) \partial_i \partial_j u + \sum_j b_j(x) \partial_j u, \qquad \text{for } u \in W_B.$$

The coefficients are of course different as in Chapter 4.2. But they still have the property that $b_{ij} = b_{ji}$, b_{ij} and the b_j are uniformly bounded by some $\Lambda > 0$. Further

$$\sum_{i,j} b_{ij}(x) \xi^i \xi^j \geq \eta |\xi|^2.$$

Since $b_2(v) \in C^{2+\alpha}(\overline{\Omega} \times B)$ and Φ^*, Φ_* are bounded isomorphisms on the bounded domain Ω the moduli of continuity of $b_{ij}(x)$ are uniformly bounded. So in the way as before a priori estimates (see [32]) work for all B_x with the same constant C. It only depends on the dimension n, $p, \eta_2, \|\Phi_*\|, \|\Phi^*\|, \Lambda$, the domain B and the moduli of continuity of $b_{ij}(x)$ on B. So for all $x \in \overline{\Omega}$, there is a common sector $S_{\Theta,\omega} \subset \rho(-B_x)$ and there exists $M \geq 1$ such that

$$\|(\lambda + B_x)^{-1}\|_{\mathcal{L}(L_p(\mathbf{B}))} \leq \frac{M}{|\lambda - \omega|}, \qquad \text{for } \lambda \in S_{\Theta,\omega}, x \in \overline{\Omega}.$$

Since M is just a multiple of C (see Lunardi [46], Ch. 3.1.1), it is independent of x. Thus the assumptions of Lemma 3 are satisfied and so $\mathbf{B}$ is sectorial. Because of (32) and the properties of the geometry, we can conclude that $\mathbf{A}_2(v)$ is a well and densely defined, closed, sectorial operator in $L_p(\Omega, L_p(\Omega_x))$. $\square$

Remark that $A_1(v)$ is a sectorial operator in $L_p(\Omega)$. This follows from standard elliptic theory.

6.2.1 $\mathcal{R}$-Sectoriality and Maximal Regularity

To prove maximal regularity we use a generalisation of the definition of sectorial operators. The concept of $\mathcal{R}$-boundedness and $\mathcal{R}$-sectorial operators was invented in the last 15 years and is e.g. introduced by Denk, Hieber and Prüss in [23] or by Kunstmann and Weis in [44]. We recall the most important definitions. Let E, F be Banach spaces.

Definition 23.

(a) A family of operators $\{S_t, t \in \mathbb{R}_+\} \subset \mathcal{L}(E, F)$ is $\mathcal{R}$-bounded if there is a constant $C > 0$, an exponent $1 < q < \infty$ such that for each $N \in \mathbb{N}$, any combination of N values t_i, N elements y_i of E and all N independent, symmetric, $\{1, -1\}$-valued random variables ε_i on a probability space (Σ, M, μ) the inequality

$$\|\sum_{i=1}^{N} \varepsilon_i S_{t_i} y_i\|_{L_q(\Sigma,F)} \leq C\|\sum_{i=1}^{N} \varepsilon_i y_i\|_{L_q(\Sigma,E)}$$

holds. The smallest such constant C is called the $\mathcal{R}$-bound of the set written as

$$\mathcal{R}\{S_t; t > 0\} = \min C.$$

(b) *An operator A from a Banach space X to a Banach space Y is called $\mathcal{R}$-sectorial if it is sectorial and the set*

$$\{t(t+A)^{-1}; t>0\}$$

is $\mathcal{R}$-bounded. The $\mathcal{R}$-angle $\Phi^{\mathcal{R}}(A)$ is defined as

$$\Phi^{\mathcal{R}}(A) = \inf\left\{\alpha \in (0,\pi); \mathcal{R}\{\lambda(\lambda+A)^{-1}; |\arg\lambda| \le \alpha\} < \infty\right\}.$$

Denk, Hieber and Prüss showed that for operators on Banach spaces the properties to possess maximal L_p-regularity and $\mathcal{R}$-sectoriality with an $\mathcal{R}$-angle smaller than $\frac{\pi}{2}$ are equivalent. They also show maximal regularity for a certain class of elliptic operators on domains. So they impose several conditions. First we have to varify these constraints for $A_1(v)$ and B_x ($x \in \Omega$). Let E be a Banach space and $\mathcal{A}$ act on E. In the following let

$$D = (D_1, \dots, D_n) =: (D', D_n)$$

with

$$D_j := -i\frac{\partial}{\partial x_j}, \quad j = 1, \dots, n.$$

Definition 24.

- *The **principal symbol** of a differential operator*

$$\mathcal{A}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$$

  is the polynomial

$$\mathcal{A}_\#(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha, \qquad \xi \in \mathbb{R}^n$$

- *A $\mathcal{L}(E)$-valued polynomial $\mathcal{A}(\xi)$ is called **parameter-elliptic** if there is an angle $\varphi \in [0,\pi)$ such that the spectrum $\sigma(\mathcal{A}(\xi))$ in $\mathcal{L}(E)$ is included in the sector of opening angle φ*

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\varphi, \qquad \text{for all } \xi \in \mathbb{R}^n, |\xi| = 1.$$

  *The value $\Phi_{\mathcal{A}} := \inf\{\varphi;$ condition holds$\}$ is called **angle of ellipicity of $\mathcal{A}$**.*

The next definition is Part 8.1 in [23].

Definition 25. *(Smoothness and Ellipticity conditions)*

(a) Smoothness conditions (SC)
A partial differential operator $\mathcal{A}(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ *and boundary operators*

$$B_j(x,D) = \sum_{|\beta|\leq m_j} b_{j\beta}(x)D^\beta$$

with $\mathcal{L}(E)$ *valued coeffients, shall satisfy (SC) if*

(i) $a_\alpha \in C_c(\overline{\Omega}, \mathcal{L}(E))$ *for each* $|\alpha| = 2m$.

(ii) $a_\alpha \in [L_\infty + L_{r_k}](\Omega, \mathcal{L}(E))$ *for each* $|\alpha| = k < 2m$ *with* $r_k \geq p$ *and* $2m - k > \frac{n}{r_k}$.

(iii) $b_{j\beta} \in C^{2m-2m_j}(\partial\Omega, \mathcal{L}(E))$ *for each* j, β.

(b) Ellipticity conditions
$\mathcal{A}$ *satisfies (EC) if there exists* $\varphi_{\mathcal{A}} \in [0,\pi)$ *such that the following assertions hold.*

(i) The principal symbol $\mathcal{A}_\#(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$ *is parameter-elliptic with angle of ellipticity* $< \varphi_{\mathcal{A}}$ *for each* $x \in \overline{\Omega} \cup \{\infty\}$.

(ii) (Lopatinskii-Shapiro condition)
Let $B_{j\#}$ *be the principal part of* B_j *and* $B_\# := (B_{1\#}, \ldots, B_{m\#})$. *For each* $x_0 \in \partial\Omega$ *we write the boundary value problem*

$$(\mathcal{A}_\#(x_0,D), B_\#(x_0,D))$$

in local coordinates corresponding to x_0. *Then the ODE problem in* $\mathbb{R}_+$

$$\begin{aligned} (\lambda + \mathcal{A}_\#(x_0,\xi',D_n)v(y), &= 0 \qquad && y > 0, \\ B_{j\#}(x_0,\xi',D_n)v(0) &= h_j && j = 1,\ldots,m \end{aligned}$$

has a unique solution $v \in C_0(\mathbb{R}_+, E)$ *for each* $(h_1,\ldots,h_m) \in E^m$ *and each* $\lambda \in \overline{S}_{\pi-\Phi_{\mathcal{A}},0}$ *and* $\xi' \in \mathbb{R}^{n-1}$ *with* $|\xi'| + |\lambda| \neq 0$.

We show that indeed the operators defined above satisfy these conditions.

Lemma 26. *Let* $v \in C^1(\overline{\Omega})$. *Take* $0 < \varepsilon < \frac{\pi}{2}$. *Then for each* $x \in \Omega$ *the operator* B_x *is* $\mathcal{R}$*-sectorial of* $\mathcal{R}$*-angle* $\Phi_{\mathcal{A}} < \varepsilon$. *This means that for each* $\Phi > \Phi_{\mathcal{A}}$ *there is* $\mu_\Phi \geq 0$ *such that the parabolic initial-boundary value problem*

$$\begin{aligned} \partial_t u + \mu_\Phi u + B_x u &= f, \qquad && t > 0, \\ u(0) &= u_0 \end{aligned}$$

has a unique solution in $L_p(\mathbb{R}_+, L_p(B))$.

Proof. Let $v \in C^1(\overline{\Omega})$. Fix $x \in \Omega$. By the assumptions B_x fulfills (SC) and the ellipticity condition. So it remains to check whether the condition of Lopatinskii-Shapiro is satisfied. We write B_x in spherical coordinates. Let J be the coordinate transform on B such that

$$(dy_1, \dots, dy_n) = J \cdot (dr, d\theta, d\varphi_1, \dots, d\varphi_{n-1}),$$
$$(d\tilde{y}_1, \dots, d\tilde{y}_n) := (dr, d\theta, d\varphi_1, \dots, d\varphi_{n-1}).$$

Define

$$h_{ij} = (J \circ \Phi(x))^T (J \circ \Phi(x)).$$

This is a metric on B in spherical coordinates. Let h^{ij} be the inverse metric, $\sqrt{|h|(x)}$ the square root of the determinant and $\tilde{b}_2(v)$ the version of the function b_2 in spherical coordinates. Set

$$\tilde{\mathcal{A}}_x(v(x)) = -\frac{1}{\sqrt{|h(x)|}} \sum_{i,j} \partial_{\tilde{y}_i} \left(\tilde{b}_2(v)(x) \sqrt{|h(x)|} h^{ij}(x) \partial_{\tilde{y}_j} u \right).$$

Then

$$B_x u = \tilde{\mathcal{A}}_x(v(x)) u$$

in these coordinates. Further

$$\tilde{\mathcal{A}}_x^{\#}(\xi) = \sum_{i,j} \tilde{b}_2(v)(x) h^{ij}(x) \xi_i \xi_j$$

is the principal symbol of B_x. Since h_{ij} is a metric and $\tilde{b}_2 \geq \eta > 0$, $\tilde{\mathcal{A}}_x^{\#}$ is parameter-elliptic and has only one negativ real eigenvalue. Hence $\Phi_{\tilde{\mathcal{A}}} = 0$. Take $(1, y_0') = y_0 \in \partial B$. The spherical coordinates give us a local coordinate chart around y_0. We just set $r = \frac{1}{\tilde{y}_1}$. Now we show that for any $g \in \mathbb{R}$ the ODE problem

$$\left(\lambda + \tilde{\mathcal{A}}_x(v(x), y_0, r, \xi') \right) u(r) = 0, \qquad \text{in } (1, \infty),$$
$$u(1) = g$$

has a unique solution in $C_0([1, \infty))$ if $|\lambda| + |\xi'|^2 \neq 0$, $\lambda \notin (-\infty, 0)$, $\xi' \in \mathbb{R}^{n-1}$. Keep λ and ξ' fixed. Let

$$c_0 = \sum_{i,j=2}^{n} \tilde{b}_2(v)(x, y_0) h^{ij}(x) \xi_i' \xi_j' > 0,$$
$$c_1 = \sum_{i=2}^{n} 2\xi_i' \tilde{b}_2(v)(x, y_0) h^{1i}(x, y_0),$$
$$c_2 = \tilde{b}_2(v)(x, y_0) h^{11}(x, y_0) > 0.$$

Then the equation can be written as

$$0 = \left(\lambda + \tilde{\mathcal{A}}_x(v(x), y_0, r, \xi')\right) u(r) = (\lambda + c_0)u(r) - ic_1 u'(r) - c_2 u''(r).$$

The solutions are of the form

$$u(r) = D_1 e^{\kappa_1 r} + D_2 e^{\kappa_2 r}$$

where $\kappa_{1/2} \in \mathbb{C}$ are the solutions of $\lambda + c_0 - ic_1\kappa - c_2\kappa^2 = 0$. It holds

$$\kappa_{1/2} = \frac{-ic_1}{2c_2} \pm \sqrt{\frac{-c_1^2}{4c_2^2} + \frac{\lambda + c_0}{c_2}}.$$

The first term is purely imaginary. Thus $Re\ \kappa_1 = -Re\ \kappa_2$. If one of them is positive then $D_j = 0$, because we need to ensure that u is bounded. Then $u(1) = g$ determines the other constant. It follows that the solution is unique. This only fails if $Re\ \kappa_{1/2} = 0$. This is equivalent to $\lambda \in \mathbb{R}$ and

$$-\frac{c_1^2}{4c_2} + \lambda + c_0 < 0 \quad \Leftrightarrow \quad \lambda < \frac{c_1^2}{4c_2} - c_0 < 0.$$

The last inequality follows by a short calculation from the fact that the metric is positiv definit,

$$\frac{c_1^2}{4c_2} - c_0 = \tilde{b}_2(v)\left(\frac{h^{1i}\xi_i' h^{1j}\xi_j'}{h^{11}} - h^{ij}\xi_i'\xi_j'\right) < 0.$$

Thus for $\kappa + B_x$ the smoothness and ellipticity conditions are fulfilled. So Theorem 8.2 of [23] proves the lemma. □

The uniform bounds for the transformation and b_2 make sure that μ_Φ can be found independent of x. So we can take the term to the right hand side $f_2(u, U)$. Thus w.l.o.g. we assume that for fixed $\Phi < \frac{\pi}{2}$, it holds $\mu_\Phi = 0$.

Proposition 27. *Let $v \in C^1(\overline{\Omega})$. Then $\boldsymbol{A}_2(v)$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle less or equal Φ.*

Proof. (i) Let $v \in C^1(\overline{\Omega})$ and $\{B_x, x \in \Omega\}$ as before. We start to consider $\mathbf{B}$. From the proof of Lemma 22 we know that $\mathbf{B}$ is sectorial. So it remains to show that

$$\mathcal{R}(\{t(t + \mathbf{B})^{-1}; t > 0\}) < \infty$$

and that the $\mathcal{R}$-angle of $\mathbf{B}$ is less or equal Φ. We will need the following estimate. Let $\lambda \in S_{\pi-\Phi,0} \subset \rho(-B_x)$ for arbitrary $x \in \Omega$. For any $x, y \in \Omega$ it holds

$$\begin{aligned}
&\|\lambda(\lambda + B_x)^{-1} - \lambda(\lambda + B_y)^{-1}\|_{\mathcal{L}(L_p(B))} \\
&= \|(B_y - B_x)\lambda(\lambda + B_x)^{-1}(\lambda + B_y)^{-1}\|_{\mathcal{L}(L_p(B))} \\
&\leq \|B_y - B_x\|_{\mathcal{L}(W_B, L_p(B))} \frac{M^2}{|\lambda|} \\
&\leq C\|y - x\| \frac{M^2}{|\lambda|}.
\end{aligned}$$

Here C and M are independent of x, y. The last inequality holds because $b_2(v) \in C^1(\Omega)$.

Next let (Σ, M, μ) be a probability space and let $N \in \mathbb{N}$. For $j = 1, \dots, N$ let ε_j be a random, $\{-1, 1\}$-valued variable, let $U_j \in L_p(\Omega, L_p(B)) =: X$ and let $\lambda_j \in S_{\pi-\Phi,0} \subset \rho(-\mathbf{B})$. Take further $q = p$. There exists a constant $C > 0$ such that

$$\begin{aligned}
&\Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + \mathbf{B})^{-1} U_j \Big\|_{L_p(\Sigma, X)} \\
&\leq \sum_j \|\varepsilon_j \lambda_j (\lambda_j + \mathbf{B})^{-1} U_j\|_{L_p(\Sigma, X)} \\
&\leq C \sum_j \|\lambda_j (\lambda_j + \mathbf{B})^{-1} U_j\| \leq C \sum_j M \|U_j\| < \infty.
\end{aligned}$$

So the intergals exist and we can apply Fubini's Theorem. It holds

$$\begin{aligned}
&\Big\| \sum_{j=1}^{N} \varepsilon_j \lambda_j (\lambda_j + \mathbf{B})^{-1} U_j \Big\|^p_{L_p(\Sigma, X)} \\
&= \int_\Sigma \Big\| \sum_j \varepsilon_j(s) \lambda_j (\lambda_j + \mathbf{B})^{-1} U_j \Big\|^p_X \, d\mu(s) \\
&= \int_\Sigma \left(\int_\Omega \Big\| \sum_j \varepsilon_j(s) \lambda_j (\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|^p_{L_p(B)} dx \right) d\mu(s) \\
&= \int_\Omega \left(\int_\Sigma \Big\| \sum_j \varepsilon_j(s) \lambda_j (\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|^p_{L_p(B)} d\mu(s) \right) dx.
\end{aligned}$$

This expression helps us to shift results for B_x to $\mathbf{B}$. With $\hat{U}(x)$ in the integrals we mean an adequate representative.

(ii) From the previous Lemma and [23] we know that B_x is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle smaller that Φ for any $x \in \Omega$. So there is a constant $C(x) > 0$ such that

$$\Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \leq C(x) \Big\| \sum_j \varepsilon_j \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))}. \quad (33)$$

We assume that $C(x)$ is choosen optimal, this is as small as possible. Fix $x_0 \in \Omega$. We estimate

$$\begin{aligned} &\Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \\ &\leq \Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + B_{x_0})^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \\ &\quad + \Big\| \sum_j \varepsilon_j \left(\lambda_j (\lambda_j + B_x)^{-1} - \lambda_j (\lambda_j + B_{x_0})^{-1} \right) \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))}. \end{aligned}$$

The first term can be assessed with the help of (33),

$$\Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + B_{x_0})^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \leq C(x_0) \Big\| \sum_j \varepsilon_j \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))}.$$

The second term is more complicated. With (33) for B_x and B_{x_0} we get

$$\begin{aligned} &\Big\| \sum_j \varepsilon_j \left(\lambda_j (\lambda_j + B_x)^{-1} - \lambda_j (\lambda_j + B_{x_0})^{-1} \right) \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \\ &= \Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + B_{x_0})^{-1} (B_{x_0} - B_x)(\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \\ &= \Big\| \sum_j \varepsilon_j (B_{x_0} - B_x) \lambda_j (\lambda_j + B_{x_0})^{-1} B_x^{-1} (-\lambda_j + B_x + \lambda_j)(\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\| \\ &\leq \Big\| (B_{x_0} - B_x) B_x^{-1} \sum_j \varepsilon_j \lambda_j (\lambda_j + B_{x_0})^{-1} \lambda_j (\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \\ &\quad + \Big\| (B_{x_0} - B_x) B_x^{-1} \sum_j \varepsilon_j \lambda_j (\lambda_j + B_{x_0})^{-1} \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))} \\ &\leq \| (B_{x_0} - B_x) B_x^{-1} \|_{\mathcal{L}(L_p(B))} (C(x_0) C(x) + C(x_0)) \Big\| \sum_j \varepsilon_j \hat{U}_j(x) \Big\|_{L_p(\Sigma, L_p(B))}. \end{aligned}$$

Furthermore we can calculate

$$\| (B_{x_0} - B_x) B_x^{-1} \|_{\mathcal{L}(L_p(B))} \leq \| (B_{x_0} - B_x) \|_{\mathcal{L}(W_B, L_p(B))} \| B_x^{-1} \|_{\mathcal{L}(L_p(B), W_p^2(B))}.$$

The assumptions on the geometry and $b_2(v)$ ensure that we have uniform a priori estimates for B_x. Further the map $x \mapsto B_x$ is Lipschitz continuous with some constant $\tilde{L} > 0$. For $u \in L_p(B)$ then

$$\|B_x^{-1}u\|_{W_p^2} \leq C_p M_p \|B_x B_x^{-1} u\|_{L_p(\Omega)} = C_p M_p \|u\|_{L_p(\Omega)}.$$

Since $C(x)$ was choosen minimal we can conclude that

$$C(x) \leq C(x_0)(1 + L|x - x_0|) + L|x - x_0| C(x) C(x_0), \tag{34}$$

for $L := \tilde{L} C_p M_p$. The constant C_p occurs if we consider the equivalent norm on $W_p^2(B)$ that only includes highest derivatives. With this we prove that the constants $\{C(x), x \in \Omega\}$ are unifomly bounded. Suppose instead,

$$\sup_{x \in \Omega} C(x) = \infty.$$

Then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \Omega$ such that $C(x_n) \to \infty$. Of course there is a subsequence of (x_n) that converges to some $x \in \overline{\Omega}$. We call it again (x_n). Two cases may occur: If $x \in \Omega$ then we know that $C(x) < \infty$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$. By (34) with the identification of x_0 with x and x_n instead of x it holds

$$C(x_n) \leq C(x)(1 + L\varepsilon) + L\varepsilon C(x) C(x_n).$$

The term $C(x)(1 + L\varepsilon)$ is bounded by some value S if ε is small enough. So

$$C(x_n) \leq L\varepsilon C(x) C(x_n) + S.$$

Since $C(x_n) \neq 0$ this is equivalent to

$$1 \leq L\varepsilon C(x) + \frac{S}{C(x_n)}.$$

If we let $\varepsilon \to 0$ then the right hand side vanishes since also $n \to \infty$. So the boundedness of $(C(x_n))_{n \in \mathbb{N}}$ follows by contradiction. If the limit $x \in \Gamma = \partial\Omega$ we can use the same argumentation. The regularity assumptions make sure that the map $x \mapsto B_x$ can be extended continuously to $\overline{\Omega}$. It is even Lipschitz continuous on $\overline{\Omega}$. The operator B_x, for $x \in \Gamma$, is defined with the help of the limits of b and g. It also satisfies (EC) and (SC). So the same approximations as in (34) and part (i) lead to a contradiction.
(iii) Let $C > 0$ be the uniform bound from step (ii). This means $C(x) \leq C$

for any $x \in \Omega$. Then with the considerations of part (i) we conclude

$$
\begin{aligned}
&\Big\| \sum_{j=1}^{N} \varepsilon_j \lambda_j (\lambda_j + \mathbf{B})^{-1} U_j \Big\|^p_{L_p(\Sigma, X)} \\
&= \int_\Omega \left(\int_\Sigma \Big\| \sum_j \varepsilon_j(s) \lambda_j (\lambda_j + B_x)^{-1} \hat{U}_j(x) \Big\|^p_{L_p(B)} d\mu(s) \right) dx \\
&\leq C^p \Big\| \sum_j \varepsilon_j U_j \Big\|^p_{L_p(\Sigma, X)}.
\end{aligned}
$$

So $\mathbf{B}$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle less or equal than Φ. By the permanence properties for $\mathcal{R}$-sectorial operators ([23], 4.1) we conclude that $\mathbf{A}_2(v)$ is $\mathcal{R}$-sectorial with the same $\mathcal{R}$-angle. □

Now Theorem 4.4 in [23] tells us that $\mathbf{A}_2(v)$ has maximal L_p-regularity. For fixed $v \in C^1(\overline{\Omega})$ it is true that $A_1(v)$ possesses maximal L_p-regularity.

6.2.2 Main Results

The results from the last section are now transfered to the coupled operator. In Chapter 4.2 we defined $D_0(u) = \{u \in L_p(\Omega, W^2_p(\Omega_x)); \operatorname{tr} U = u\}$. For $v \in C^1(\Omega)$ we identify

$$
\begin{aligned}
dom(\mathbf{A}(v)) &= \bigcup_{u \in W^2_p(\Omega) \cap W^{1,0}_p(\Omega)} \{u\} \times D_0(u), \\
\mathbf{A}(v)(u, U) &= \Big(-div_x(a_1(v) \nabla_x u), \Phi_*[\mathcal{A}_x(v(x)) \hat{V}] \Big).
\end{aligned}
$$

Here $(u, U) \in dom(\mathbf{A}(v)), V = \Phi^* U$. The domain of definition does not change with v. It is the same as for the linear operator. From the considerations before we know that there is a sector $S_{\pi-\Phi,0} \subset \rho(-\mathbf{A}_2(v)) \cap \rho(-A_1(v))$.

Theorem 28. *For any $v \in C^1(\overline{\Omega})$, the operator $\boldsymbol{A}(v)$ is $\mathcal{R}$-sectorial and possesses maximal L_p-regularity.*

Proof. Let $v \in C^1(\overline{\Omega})$. Take $\lambda \in S_{\pi-\Phi,0} \subset \rho(-\mathbf{B}) \cap \rho(-A_1(v))$. Let

$$(f, g) \in Y_0 = L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x)).$$

We set

$$
\begin{aligned}
u &= (\lambda + A_1(v))^{-1} f, \\
U &= \Phi_* (\lambda + \mathbf{B})^{-1} \Phi^* (g - \lambda R u) + R u,
\end{aligned}
$$

Here R is the retract from Chapter 4.1. Then $(u,U) \in dom(\mathbf{A}(v))$ and it holds

$$(f,g) = (\lambda + \mathbf{A}(v))(u,U).$$

In analogy with the considerations in the proof of Proposition 8 it holds

$$\begin{aligned}
&|\lambda| \|(\lambda + \mathbf{A}(v))^{-1}\|_{\mathcal{L}(L_p(Y_0))} \\
&\leq |\lambda| \|(\lambda + A_1(v))^{-1}\| + |\lambda| \|\Phi_*(\lambda + \mathbf{B})^{-1}\Phi^*\|(1 + |\lambda| \|R\| \|(\lambda + A_1(v))^{-1}\|) \\
&\qquad + |\lambda| \|R\| \|(\lambda + A_1(v))^{-1}\| \\
&\leq (1 + \|R\|)\|(\lambda + A_1(v))^{-1}\| \\
&\qquad + (1 + |\lambda| \|R\| \|(\lambda + A_1(v))^{-1}\|)|\lambda| \|\Phi_*(\lambda + \mathbf{B})^{-1}\Phi^*\| \\
&\leq M.
\end{aligned}$$

Hence $\mathbf{A}(v)$ is sectorial. Take (Σ, M, μ), $N \in \mathbb{N}$ and ε_j, λ_j as in Proposition 27. For $j = 1, \dots, n$ let $u_j \in L_p(\Omega)$, $U_j \in L_p(\Omega, L_p(\Omega_x)) = X$. We use again Fubini's Theorem and the methods of the proof for $\mathbf{B}$. Let C_1, C_2 be the bounds from the $\mathcal{R}$-calcul for $A_1(v)$ and $\mathbf{B}$. We calculate

$$\begin{aligned}
&\Big\| \sum_{j=1}^N \varepsilon_j \lambda_j (\lambda_j + \mathbf{A}(v))^{-1}(u_j, U_j) \Big\|_{L_p(\Sigma, Y_0)} \\
&= \Big\| \sum_j \varepsilon_j \big(\lambda_j (\lambda_j + A_1(v))^{-1} u_j, \Phi_*(\lambda_j + \mathbf{B})^{-1}\Phi^*(U_j - \lambda_j R(\lambda_j + A_1(v))^{-1} u_j) \\
&\qquad + R\lambda_j(\lambda_j + A_1(v))^{-1} u_j\big) \Big\|_{L_p(\Sigma, Y_0)} \\
&\leq (1 + \|R\|) \Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + A_1(v))^{-1} u_j \Big\|_{L_p(\Sigma, L_p(\Omega))} \\
&\qquad + \|\Phi_*\| \Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + \mathbf{B})^{-1}\Phi^* U_j \Big\|_{L_p(\Sigma, X)} \\
&\qquad + \|\Phi_*\| \Big\| \sum_j \varepsilon_j \lambda_j (\lambda_j + \mathbf{B})^{-1}\Phi^* R(\lambda_j + A_1(v))^{-1} u_j \Big\|_{L_p(\Sigma, X)} \\
&\leq C_1(1 + \|R\|) \Big\| \sum_j \varepsilon_j u_j \Big\|_{L_p(\Sigma, L_p(\Omega))} + C_2 \|\Phi_*\| \Big\| \sum_j \varepsilon_j \Phi^* U_j \Big\|_{L_p(\Sigma, X)} \\
&\qquad + (C_2 + 1)\, C_1 \|\Phi_*\| \|\Phi^*\| \|\mathbf{B}^{-1}\| \|R\| \Big\| \sum_j \varepsilon_j u_j \Big\|_{L_p(\Sigma, L_p(\Omega))} \\
&\leq C \Big\| \sum_j \varepsilon_j (u_j, U_j) \Big\|_{L_p(\Sigma, Y_0)}.
\end{aligned}$$

The rest follows from [23]. □

We will write $\mathbf{A}(v) \in MR(p, Y_0)$ meaning that $\mathbf{A}(v)$ possesses maximal L_p-regularity over Y_0. Take $n+2 < p < \infty$. Then due to Sobolev embeddings it holds

$$W_p^{2-\frac{2}{p}}(\Omega) \hookrightarrow C^1(\overline{\Omega}).$$

The special definition of $\mathbf{A}(v)$ and the form of the interpolation space $Y_{1-\frac{1}{p},p}$ ensures that

$$\big(v \mapsto \mathbf{A}(v)\big) \in C^{1-}\left(Y_{1-\frac{1}{p},p}, \mathcal{L}(D(\mathbf{A}), Y_0)\right).$$

Local existence of the solution for the quasilinear initial-boundary value problem now follows from well known abstract theory.

Corollary 29. *Let $p > n+2$, let $\boldsymbol{A}(\cdot)$ be defined as above, $T_0 > 0$ and*

$$f \in C^{1-,1-}\left([0, T_0] \times Y_{1-\frac{1}{p},p}, Y_0\right),$$
$$g \in L_p([0, T_0], Y_0).$$

Let $(u_0, U_0) \in Y_{1-\frac{1}{p},p}$. Then there exists $T_1 \in (0, T_0]$ and unique functions

$$(u, U) \in L_p((0, T_1), D(\boldsymbol{A})) \cap W_p^1((0, T_1), Y_0) \cap C([0, T_1], Y_{1-\frac{1}{p},p})$$

that satisfy

$$\begin{cases} \left(\dot{u}, \dot{U}\right) + \boldsymbol{A}(u(t))(u(t), U(t)) = f\big(t, u(t), U(t)\big) + g(t), & \text{on } (0, T_1), \\ (u(0), U(0)) = (u_0, U_0). \end{cases}$$

Proof. By Theorem 28 it holds

$$\mathbf{A}(u_0) \in MR(p, Y_0).$$

So the assumptions of Theorem 2.4 in [21] are fulfilled and the assertion follows. □

6.3 Contaminant Transport in Fissured Rocks

In the previous chapter the operator $\mathbf{A}$ was allowed to depend on the macroscopic density u. But usually the homogenisation process produces a dependence on the micro density U. We try to circumvent this problem and give a possible way of handling it here. As an application we think of contaminant transport in fissured porous material. Let u, U denote the concentration of a contaminant in the macro and micro scale of a double porosity system.

With m, M we denote the two porosities. Assume that the diffusivity in both scales depends on the concentration u. This can be modeled by using the Nemytskii operators a_1, b_2 introduced in the last chapter. For $u \in C^1(\overline{\Omega})$ set $a_2(u) = \Phi^* b_2(u) \in C^1(Q)$. Then the matched microstructure model reads as follows:

$$\begin{cases} \partial_t(mu) - div_x(ma_1(u)\nabla_x u) = & f_1(x,t,u,U), & \text{on } \Omega, t \in (0,T], \\ u(x,t) = & 0, & \text{on } \Gamma, t \in (0,T], \\ u(t=0) = & u_0, & \\ \partial_t(MU) - div_z(Ma_2(u)\nabla_z U) = & f_2(x,t,u,U), & \text{on } Q, t \in (0,T], \\ U(x,z,t) = & u(x,t), & \text{on } R_M, t \in (0,T], \\ U(t=0) = & U_0. & \end{cases}$$

Let us assume that the material is homogenious in Ω and in each cell. So the porosities are scalar functions on the respective domains. Up to now we had $m = M = 1$. In general the solute can interact with the solid structure. Assume that it gets attached to the walls with a certain rate depending on u. All different effects like reaction, van-der-Waals forces, electric forces etc. which bind the solute, are summed in one sorption term. The concentration of the sorped contaminant shall be given by a function u^*. Here we suppose that u^* depends linearily on the concentration u in the fluid. Thus

$$\partial_t u^* = K \partial_t u.$$

We neglect any dependence of the sorption on the changing porosity. The interaction results in a prefactor in our equations. So for $t \in (0,T]$ we get

$$\begin{cases} (1+K)\partial_t(mu) - div_x(ma_1(u)\nabla_x u) = & f_1(x,t,u,U), & \text{on } \Omega, \\ u(x,t) = & 0, & \text{on } \Gamma, \\ u(t=0) = & u_0, & \\ (1+K)\partial_t(MU) - div_z(Ma_2(u)\nabla_z U) = & f_2(x,t,u,U), & \text{on } Q, \\ U(x,z,t) = & u(x,t), & \text{on } R_M, \\ U(t=0) = & U_0. & \end{cases}$$

Assume that the porosities m, M are constant. Then we can still define $\mathbf{A}(u)$ as before. Thus Corollary 29 can be applied and gives well-posedness. Now we drop the assumption that the reaction has no effect on the solid structure of the porous blocks. Since the fissures are of another, bigger length scale we do not consider a change of m and set it to one. As before we do not model individuell pores. But we define an average and possibly evolving porosity of each block. Let $M : \Omega \times [0,T] \to (0,\infty]$. We assume that M has values in a bounded interval not including zero. This excludes the case of total pore closure in the cells and also that large parts of the solid structure dissolve in the fluid. So $M(x)$ is bounded between some minimal $M_{min} > 0$ and a maximal value $M_{min} < M_{max} < \infty$. Clearly this also has to hold for the initial value M_0. Let $U \in L_p(\Omega, L_p(\Omega_x)), V = \Phi_* U$. The total amount of reactant in the cell is approximated by

$$|\Phi_x(B)| \int_B \hat{V}(x, y)\, dy.$$

We suppose that there is a function G_0,

$$G_0 : [0, \infty) \times [M_{min}, M_{max}] \to [0, C_G],$$

which describes the change of the porosity due to the reaction. It shall depend on the amount of contaminant in the cell and the porosity. We assume that G_0 is Lipschitz continuous in both variables.

In the sense of continuum mechanics G_0 has to have the same form in all cells. Thus the same Lipschitz constant and bounds are valid for arbitrary $x \in \Omega$. We define a kind of sorption velocity

$$\begin{aligned} G : L_p(\Omega, L_p(\Omega_x)) \times L_p(\Omega) &\to L_p(\Omega), \\ (U, M) &\mapsto \left(x \mapsto G_0 \left(|\Phi_x(B)| \int_B \hat{V}(x, y)\, dy, M(x) \right) \right). \end{aligned}$$

Since G_0 is continuous and the arguments of G are measurable on Ω this is a measurable function on Ω. Usually in applications the reaction velocity is bounded. The amount of particles gives a restriction to the sorption speed. This implies that the change of the porosity can not exceed a certain value. For the complexity of the process this value can not be measured or calculated easily. Nevertheless we assume that there exists $c_g > 0$ such that

$$\sup_{U,M} |G(U, M)| < c_g. \tag{35}$$

The supremum is taken over all admissible U and M. Actually G should also depend on the concentration of the contaminant that is combined with the solid. But this goes linearly with U. We assume that this effect is already included in G_0. In particular let $M_0 \in L_p(\Omega)$ be admissible. We suppose that for fixed $U \in L_p(\Omega, W_p^2(\Omega_x))$ the evolution equation

$$\begin{cases} \partial_t M(t) &= -G(U, M), \\ M(0) &= M_0 \end{cases} \tag{36}$$

has an unique solution in $L_p((0, T_0), L_p(\Omega)) \cap W_p^1((0, T_0), L_p(\Omega))$. We add this to our system of equations. Then we transform the problem and write all terms which include M or its derivative on the right hand side. Thus for some $0 < T_1 < T_0$, the functions (u, U) have to fulfill

$$\begin{cases} \Big(\dot{u}(t), \dot{U}(t))\Big) + \mathbf{A}(u(t))(u(t), U(t)) &= \tilde{f}\big(t, u(t), U(t), M(t)\big), \quad \text{on } (0, T_1), \\ (u(0), U(0)) &= (u_0, U_0), \\ \dot{M}(t) &= -G(U, M), \qquad\qquad \text{on } (0, T_1), \\ M(0) &= M_0. \end{cases} \tag{37}$$

Here $\tilde{f}$ is defined by

$$\begin{aligned} \tilde{f}(t, u, U, M) &= \left(f_1(t, u, U), \frac{1}{M} f_2(t, u, U) - \partial_t M \frac{U}{M} \right) \\ &= \left(f_1(t, u, U), \frac{1}{M} f_2(t, u, U) + G(U, M) \frac{U}{M} \right). \end{aligned}$$

Let us first summerize our notation. We use

$$\begin{aligned} Y_0 &= L_p(\Omega) \times L_p(\Omega, L_p(\Omega_x)), \\ Y_1 &= dom(\mathbf{A}), \\ Y_{1-\frac{1}{p}} &= (Y_0, Y_1)_{1-\frac{1}{p},p}. \\ E_1 &= L_p(\Omega), \qquad E_2 = L_p(\Omega, L_p(\Omega_x)). \end{aligned}$$

Further we write

$$\begin{aligned} X^T &= L_p(0, T; Y_0), & E_1^T &= L_p(0, T; E_1), \\ Y^T &= W_p^1(0, T; Y_0) \cap L_p(0, T; Y_1), & E_2^T &= L_p(0, T; E_2), \\ Z^T &= \{\vec{u} \in Y^T; \vec{u}(0) = 0\}. \end{aligned}$$

With $\vec{u}$ we denote a pair of functions (u, U) from Y_0 or Y^T. In the setting of Chapter 6.2 we write down the well-posedness result.

Theorem 30. *Suppose $f = (f_1, f_2) \in C^{1-,1-}([0, T_0) \times Y_{1-\frac{1}{p},p}, Y_0)$ for some $T_0 > 0$ and G_0, G are given functions as above. For all $(v, V) \in Y_{1-\frac{1}{p},p}$ let $\boldsymbol{A}(v)$ be defined as in Section 6.2. Let $(u_0, U_0) \in Y_{1-\frac{1}{p},p}$, $M_0 \in C(\bar{\Omega})$, $M_{min} < M_0 < M_{max}$. Assume that*

$$\begin{aligned} \|f_2(t, \vec{u}_0)\|_{E_2} &\le \frac{M_{min}^2}{4}, & &\textit{for all } t \in [0, T_0], \\ \|U_0\|_{E_2} &\le \frac{M_{min}^2}{4c_g}. \end{aligned}$$

Then there exists $T_1(\vec{u}_0) \in (0, T_0]$ and unique functions

$$(u, U) \in L_p(0, T_1; Y_1) \cap W_p^1(0, T_1; Y_0) \cap C([0, T_1], Y_{1-\frac{1}{p},p})$$

and $M \in W_p^1(0, T_1; E_0)$ that satisfy (37).

Proof. In this proof we write v' instead of $\dot{v}$ to denote the time derivative. Let $\vec{w}, \tilde{M}$ be the solutions of the following linear problems. Let $\tilde{M} \in W_p^1(0, T_0; E_1)$ be the solution of

$$\begin{aligned} &\tilde{M}'(t) = -G(U_0, \tilde{M}), && t \in (0, T_0), \\ &M(0) = M_0. \end{aligned}$$

The solution exists due to our assumptions (36). Let $\vec{w}$ satisfy

$$\begin{aligned} \vec{w}'(t) + \mathbf{A}(u_0)\vec{w}(t) &= \tilde{f}(t, \vec{u}_0, \tilde{M}), && t \in (0, T_0), \\ \vec{w}(0) &= \vec{u}_0. \end{aligned}$$

Because of the properties of $\mathbf{A}(u_0)$ this is well defined. For the rest of the proof we write f instead of $\tilde{f}$. For $0 < T < T_0$, $\rho > 0$, let

$$\Sigma_{\rho,T} = \Big\{ (\vec{u}, M) \in Y^T \times E_1^T; \vec{u}(0) = \vec{u}_0,\ M(0) = M_0, \\ \|\vec{u} - \vec{w}\|_{Y^T} \le \rho,\ \|M - \tilde{M}\|_{E_1^T} \le \rho \Big\}.$$

We follow the steps of the proof of Theorem 2.1 in [21]. There exists $T_2 > 0$ small enough, such that

$$\{\vec{u}(t); t \in (0, T], \vec{u} \in \Sigma_{\rho,T}\} \subset Y_{1-\frac{1}{p}}$$

if $T \in (0, T_2]$. So for those $\vec{u}$ maximal regularity of $\mathbf{A}(u(t))$ is preserved in this time interval. We deduce that there is a constant $L > 0$, such that for $\vec{u}_1, \vec{u}_2 \in \Sigma_{\rho,T}$, $t \in (0, T)$, it holds

$$\|\mathbf{A}(u_1(t)) - \mathbf{A}(u_2(t))\|_{\mathcal{L}(Y_1, Y_0)} \le L\|\vec{u}_1(t) - \vec{u}_2(t)\|_{Y_{1-\frac{1}{p}}} \tag{38}$$

$$\|f_i(t, \vec{u}_1(t)) - f_i(t, \vec{u}_2(t))\|_{E_i} \le L\|\vec{u}_1(t) - \vec{u}_2(t)\|_{Y_{1-\frac{1}{p}}}, \qquad i = 1, 2. \tag{39}$$

We define the mapping

$$\gamma : Y^T \times E_1^T \to Y^T \times E_1^T : \gamma(\vec{u}, M) = (\vec{v}, N),$$

where $(\vec{v}, N)$ is the unique solution of the linear problem

$$\begin{aligned}
\vec{v}'(t) + \mathbf{A}(u_0)\vec{v}(t) &= \mathbf{A}(u_0)\vec{u}(t) - \mathbf{A}(u(t))\vec{u}(t) + f(t, \vec{u}(t), M(t)), \quad t \in (0,T),\\
\vec{v}(0) &= \vec{u}_0,\\
N'(t) &= -G(U(t), M(t)), \qquad t \in (0,T),\\
N(0) &= M_0.
\end{aligned}$$

To use Banach's Fixed Point Theorem we show that γ is a contraction on some Σ_{ρ_1,T_1}. So we estimate $\|\vec{v} - \vec{w}\|_{Z^T}$, $\|M - \tilde{M}\|_{E_1^T}$ and $\|\gamma(\vec{u}_1, M_1) - \gamma(\vec{u}_2, M_2)\|$. Let $(\vec{u}, M) \in \Sigma_{\rho,T}$ and $(\vec{v}, N) = \gamma(\vec{u}, M)$. Then it holds for $t \in (0,T)$,

$$\begin{aligned}
(\vec{v} - \vec{w})'(t) + \mathbf{A}(u_0)(\vec{v} - \vec{w})(t) = \mathbf{A}(u_0)\vec{u}(t) &- \mathbf{A}(u(t))\vec{u}(t)\\
&+ f(t, \vec{u}(t), M(t)) - f(t, \vec{u}_0, \tilde{M}(t)),
\end{aligned}$$

and
$$(\vec{v} - \vec{w})(0) = 0.$$

Now the maximal regularity of $\mathbf{A}(u_0)$ allows us to apply Corollary 2.3 from [21]. Thus there is a constant $\mathcal{M} > 0$, such that

$$\left\| \left(\frac{d}{dt} + \mathbf{A}(u_0) \right)^{-1} \right\|_{\mathcal{L}(X^T, Z^T)} \leq \mathcal{M}, \tag{40}$$

$$\|\vec{v}\|_{C([0,T], Y_{1-\frac{1}{p}})} \leq \mathcal{M}\|\vec{v}\|_{Z^T} \qquad \text{for } \vec{v} \in Z^T. \tag{41}$$

So with (38) we get

$$\begin{aligned}
&\|\vec{v} - \vec{w}\|_{Z^T}\\
&\leq \mathcal{M}\|(\mathbf{A}(u_0) - \mathbf{A}(u))\vec{u}\|_{X^T} + \mathcal{M}\|f(t, \vec{u}, M) - f(t, \vec{u}_0, \tilde{M})\|_{X^T}\\
&\leq \mathcal{M}\|(\mathbf{A}(u_0) - \mathbf{A}(u))(\vec{u} - \vec{w})\|_{X^T} + \mathcal{M}\|(\mathbf{A}(u_0) - \mathbf{A}(u))\vec{w}\|_{X^T}\\
&\quad + \mathcal{M}\|f(t, \vec{u}, M) - f(t, \vec{u}_0, \tilde{M})\|_{X^T}\\
&\leq \mathcal{M}\|\mathbf{A}(u_0) - \mathbf{A}(u)\|_{C(0,T;\mathcal{L}(Y_1,Y_0))}\|\vec{u} - \vec{w}\|_{Z^T}\\
&\quad + \mathcal{M}\|\mathbf{A}(u_0) - \mathbf{A}(u)\|_{C(0,T;\mathcal{L}(Y_1,Y_0))}\|\vec{w}\|_{Y^T}\\
&\quad + \mathcal{M}\|f(t, \vec{u}, M) - f(t, \vec{u}_0, \tilde{M})\|_{X^T}\\
&\leq \mathcal{M}L\|\vec{u}_0 - \vec{u}\|_{C([0,T], Y_{1-\frac{1}{p}})}(\rho + \|\vec{w}\|_{Y^T}) + \mathcal{M}\|f(t, \vec{u}, M) - f(t, \vec{u}_0, \tilde{M})\|_{X^T}\\
&\leq \mathcal{M}L(\mathcal{M}\rho + \|\vec{w} - \vec{u}_0\|_{C([0,T], Y_{1-\frac{1}{p}})})(\rho + \|\vec{w}\|_{Y^T})\\
&\quad + \mathcal{M}\|f(t, \vec{u}, M) - f(t, \vec{u}_0, \tilde{M})\|_{X^T}.
\end{aligned}$$

With (39) and the assumptions on G we estimate the last term

$$\begin{aligned}&\|f(t,\vec{u},M)-f(t,\vec{u}_0,\tilde{M})\|_{X^T}\\&\le \|f_1(t,\vec{u})-f_1(t,\vec{u}_0)\|_{E_1^T}+\left\|\frac{1}{M}f_2(t,\vec{u})-\frac{1}{\tilde{M}}f_2(t,\vec{u}_0)\right\|_{E_2^T}\\&\quad+\left\|\frac{U}{M}G(U,M)-\frac{U_0}{\tilde{M}}G(U_0,\tilde{M})\right\|_{E_2^T}.\end{aligned}$$

It holds

$$\begin{aligned}\|f_1(t,\vec{u})-f_1(t,\vec{u}_0)\|_{E_1^T}&\le L\|\vec{u}-\vec{u}_0\|_{L_p((0,T),Y_{1-\frac{1}{p}})}\\&\le LT^{1/p}(\mathcal{M}\rho+\|\vec{w}-\vec{u}_0\|_{C([0,T],Y_{1-\frac{1}{p}})}).\end{aligned}$$

The second part is calculated in a similar way

$$\begin{aligned}&\left\|\frac{1}{M}f_2(t,\vec{u})-\frac{1}{\tilde{M}}f_2(t,\vec{u}_0)\right\|_{E_2^T}+\left\|\frac{U}{M}G(U,M)-\frac{U_0}{\tilde{M}}G(U_0,\tilde{M})\right\|_{E_2^T}\\&\le\frac{1}{M_{min}^2}\Big[\|\tilde{M}f_2(t,\vec{u})-Mf_2(t,\vec{u}_0)\|_{E_2^T}\\&\qquad+\|\tilde{M}UG(U,M)-MU_0G(U_0,\tilde{M})\|_{E_2^T}\Big]\\&\le\frac{1}{M_{min}^2}\Big[\|\tilde{M}(f_2(t,\vec{u})-f_2(t,\vec{u}_0))\|_{E_2^T}+\|(\tilde{M}-M)f_2(t,\vec{u}_0)\|_{E_2^T}\\&\quad+\|\tilde{M}(UG(U,M)-U_0G(U_0,\tilde{M}))\|_{E_2^T}+\|(\tilde{M}-M)U_0G(U_0,\tilde{M})\|_{E_2^T}\Big]\\&\le\frac{1}{M_{min}^2}\Big[\|\tilde{M}\|_{E_1^T}\|f_2(t,\vec{u})-f_2(t,\vec{u}_0)\|_{C([0,T],E_2)}\\&\quad+\|\tilde{M}-M\|_{E_1^T}\|f_2(t,\vec{u}_0)\|_{C([0,T],E_2)}+\|\tilde{M}\|_{E_1^T}2c_g\|\vec{u}-\vec{u}_0\|_{C([0,T],Y_{1-\frac{1}{p}})}\\&\quad+\|\tilde{M}-M\|_{E_1^T}\|U_0G(U_0,\tilde{M})\|_{C([0,T],E_2)}\Big].\end{aligned}$$

Observe that $\|f_2(t,\vec{u}_0)\|\le\frac{M_{min}^2}{4}$ and $\|U_0G(\vec{u}_0,\tilde{M}(t))\|\le\frac{M_{min}^2}{4}$. The values $\|\tilde{M}\|_{E_1^T}$ and $\Psi(T):=\|\vec{w}-\vec{u}_0\|_{C([0,T],Y_{1-\frac{1}{p}})}$ vanish if $T\to0$. Further

$$\|f_2(t,\vec{u})-f_2(t,\vec{u}_0)\|_{C([0,T],E_2)}\le L\|\vec{u}-\vec{u}_0\|_{C([0,T],Y_{1-\frac{1}{p}})}\le L(\mathcal{M}\rho+\Psi(T)).$$

We summarize using $\|\tilde{M}-M\|_{E_1^T}\le\rho$,

$$\begin{aligned}\|\vec{v}-\vec{w}\|_{Z^T}&\le\mathcal{M}L(\mathcal{M}\rho+\Psi(T))(\rho+\|\vec{w}\|_{Y^T})+\mathcal{M}LT^{1/p}(\mathcal{M}\rho+\Psi(T))\\&+\frac{1}{M_{min}^2}\Big(\mathcal{M}\|\tilde{M}\|_{E_1^T}L(\mathcal{M}\rho+\Psi(T))+2c_g\|\tilde{M}\|_{E_1^T}(\mathcal{M}\rho+\Psi(T))\Big)+\frac{1}{2}\rho.\end{aligned}\tag{42}$$

The values T and ρ can be choosen in the way, such that the right hand side is smaller than ρ. Now the difference between N and $\tilde{M}$ remains to be considered. For $t \in (0,T]$ we have

$$(N-\tilde{M})(t) = \int_0^t (N'(s) - \tilde{M}'(s))\,ds = \int_0^t (G(U_0,\tilde{M}) - G(U,M))(s)\,ds.$$

Hence using Corollary 2.3 from [21] and (35) we get

$$\begin{aligned}\|N-\tilde{M}\|_{E_1^T} &\le \left(\int_0^T \left(\int_0^t \|G(U_0(s),\tilde{M}(s)) - G(U(s),M(s))\|_{E_1}\,ds\right)^p dt\right)^{\frac{1}{p}}\\ &\le 2c_g(1+p)^{-\frac{1}{p}} T^{\frac{p+1}{p}}.\end{aligned}$$

So for

$$T < \left(\frac{1+p}{(2c_g)^p}\right)^{\frac{1}{p+1}} \rho^{\frac{p}{p+1}} \tag{43}$$

this is smaller than ρ. In the next step we show that γ is a contraction. Let

$$(\vec{u}_i, M_i) \in \Sigma_{\rho,T}, \qquad \gamma(\vec{u}_i, M_i) = (\vec{v}_i, N_i) \quad \text{for } i = 1,2.$$

It holds for $t \in (0,T)$,

$$\begin{aligned}(\vec{v}_1 - \vec{v}_2)'(t) + \mathbf{A}(u_0)(\vec{v}_1(t) - \vec{v}_2(t)) &= \mathbf{A}(u_0)(\vec{u}_1(t) - \vec{u}_2(t)) - \mathbf{A}(u_1(t))\vec{u}_1(t)\\ &\quad + \mathbf{A}(u_2(t))\vec{u}_2(t) + f(t,\vec{u}_1,M_1)\\ &\quad - f(t,\vec{u}_2,M_2),\\ (\vec{v}_1 - \vec{v}_2)(0) &= 0.\end{aligned}$$

Similar arguments as above show that

$$\begin{aligned}&\|\vec{v}_1 - \vec{v}_2\|_{Z^T}\\ &\le \mathcal{M}\|(\mathbf{A}(u_0) - \mathbf{A}(u_1))(\vec{u}_1 - \vec{u}_2)\|_{X^T}\\ &\quad + \mathcal{M}\|(\mathbf{A}(u_2) - \mathbf{A}(u_1))(\vec{u}_2 - \vec{w})\|_{X^T}\\ &\quad + \mathcal{M}\|(\mathbf{A}(u_2) - \mathbf{A}(u_1))\vec{w}\|_{X^T} + \mathcal{M}\|f(t,\vec{u}_1,M) - f(t,\vec{u}_2,\tilde{M})\|_{X^T}\\ &\le \mathcal{M}L\|\vec{u}_1 - \vec{u}_2\|_{C([0,T],Y_{1-\frac{1}{p}})}\|\vec{u}_1 - \vec{u}_2\|_{Z^T}\\ &\quad + \mathcal{M}L\|\vec{u}_1 - \vec{u}_2\|_{C([0,T],Y_{1-\frac{1}{p}})}(\rho + \|\vec{w}\|_{Y^T})\\ &\quad + \mathcal{M}\|f(t,\vec{u}_1,M) - f(t,\vec{u}_2,\tilde{M})\|_{X^T}\\ &\le \mathcal{M}L\|\vec{u}_1 - \vec{u}_2\|_{Z^T}(3\mathcal{M}\rho + \|\vec{w}\|_{Y^T}) + \mathcal{M}\|f(t,\vec{u}_1,M) - f(t,\vec{u}_2,\tilde{M})\|_{X^T}.\end{aligned}$$

Again we treat the last term separately. It holds

$$\begin{aligned}
&\|f(t,\vec{u}_1,M_1)-f(t,\vec{u}_2,M_2)\|_{X^T}\\
&\le \|f_1(t,\vec{u}_1)-f_1(t,\vec{u}_2)\|_{E_1^T}+\left\|\frac{1}{M_1}f_2(t,\vec{u}_1)-\frac{1}{M_2}f_2(t,\vec{u}_2)\right\|_{E_2^T}\\
&\quad+\left\|\frac{U_1}{M_1}G(U_1,M_1)-\frac{U_2}{M_2}G(U_2,M_2)\right\|_{E_2^T}.
\end{aligned}$$

We have

$$\|f_1(t,\vec{u}_1)-f_1(t,\vec{u}_2)\|_{E_1^T}\le L\|\vec{u}_1-\vec{u}_2\|_{L_p((0,T),Y_{1-\frac{1}{p}})}\le \mathcal{M}LT^{1/p}\|\vec{u}_1-\vec{u}_2\|_{Z^T}.$$

With the use of $\vec{w}$ and the assumptions on G we conclude (all norms here are in E_2^T)

$$\begin{aligned}
&\left\|\frac{1}{M_1}f_2(t,\vec{u}_1)-\frac{1}{M_2}f_2(t,\vec{u}_2)\right\|+\left\|\frac{U_1}{M_1}G(U_1,M_1)-\frac{U_2}{M_2}G(U_2,M_2)\right\|\\
&\le \frac{1}{M_{min}^2}\Big[\|M_2f_2(t,\vec{u}_1)-M_1f_2(t,\vec{u}_2)\|\\
&\quad+\|M_2U_1G(U_1,M_1)-M_1U_2G(U_2,M_2)\|\Big]\\
&\le \frac{1}{M_{min}^2}\Big[\|(M_2-M_1)(f_2(t,\vec{u}_2)-f_2(t,\vec{w}))\|\\
&\quad+\|(M_2-\tilde{M})(f_2(t,\vec{u}_1)-f_2(t,\vec{u}_2))\|\\
&\quad+\|\tilde{M}(f_2(t,\vec{u}_1)-f_2(t,\vec{u}_2))\|\\
&\quad+\|(M_2-M_1)(f_2(t,\vec{w})-f_2(t,\vec{u}_0)+f_2(t,\vec{u}_0))\|\\
&\quad+\|M_2U_1G(U_1,M_1)-M_1U_2G(U_2,M_2)\|\Big]\\
&\le \frac{1}{M_{min}^2}\Big[\|M_2-M_1\|_{E_1^T}(\mathcal{M}L\rho+\frac{M_{min}^2}{4}+L\Psi(T))\\
&\quad+\|\vec{u}_1-\vec{u}_2\|_{Z^T}(\rho\mathcal{M}L+\mathcal{M}L\|\tilde{M}\|_{E_1^T})\\
&\quad+\|M_2U_1G(U_1,M_1)-M_1U_2G(U_2,M_2)\|\Big].
\end{aligned}$$

Further

$$\begin{aligned}
&\|M_2U_1G(U_1,M_1)-M_1U_2G(U_2,M_2)\|_{E_2^T}\\
&\le \|M_2-\tilde{M}\|_{E_1^T}\|U_1G(U_1,M_1)-U_2G(U_2,M_2)\|_{C([0,T],E_2)}\\
&\quad+\|\tilde{M}\|_{E_1^T}\|U_1G(U_1,M_1)-U_2G(U_2,M_2)\|_{C([0,T],E_2)}\\
&\quad+\|(M_2-M_1)(U_2G(U_2,M_2)-U_0G(U_0,\tilde{M}))\|_{E_2^T}\\
&\quad+\|M_2-M_1\|_{E_1^T}\|U_0G(U_0,\tilde{M})\|_{C([0,T],E_2)}\\
&\le \|M_2-\tilde{M}\|_{E_1^T}2c_g\|\vec{u}_1-\vec{u}_2\|_{C([0,T],Y_{1-\frac{1}{p}})}+\|\tilde{M}\|_{E_1^T}2c_g\|\vec{u}_1-\vec{u}_2\|_{C([0,T],Y_{1-\frac{1}{p}})}\\
&\quad+\|M_2-M_1\|_{E_1^T}2c_g\|\vec{u}_2-\vec{u}_0\|_{C([0,T],Y_{1-\frac{1}{p}})}\\
&\quad+\|M_2-M_1\|_{E_1^T}\|U_0G(U_0,\tilde{M})\|_{C([0,T],E_2)}\\
&\le \mathcal{M}2c_g\|\vec{u}_1-\vec{u}_2\|_{Z^T}(\rho+\|\tilde{M}\|_{E_1^T})+\|M_2-M_1\|_{E_1^T}(2c_g(\mathcal{M}\rho+\Psi(T))\\
&\quad+\|U_0G(U_0,\tilde{M})\|_{C([0,T],E_2)}).
\end{aligned}$$

Last but not least the continuity of G implies

$$\begin{aligned}
\|N_1-N_2\|_{E_1^T}&=\left(\int_0^T\|\int_0^t(G(U_2(s),M_2(s))-G(U_1(s),M_1(s)))\,ds\|_{E_1}^p\,dt\right)^{\frac{1}{p}}\\
&\le T^{\frac{1}{p}}\int_0^T\|G(U_2(s),M_2(s))-G(U_1(s),M_1(s))\|_{E_1}\,ds\\
&\le T^{\frac{1}{p}}c_p\|G(U_2,M_2)-G(U_1,M_1)\|_{L_p([0,T],E_1)}\\
&\le T^{\frac{1}{p}}c_pc(\|\vec{u}_1-\vec{u}_2\|_{Z^T}+\|M_1-M_2\|_{E_1^T}).
\end{aligned}$$

Because of the assumptions there exist (ρ_1,T_1), such that (42) is smaller than ρ_1, (43) is satisfied and it holds

$$\|\gamma(\vec{u}_1,M_1)-\gamma(\vec{u}_2,M_2)\|_{\Sigma_{\rho_1T_1}}\le\frac{3}{4}\|(\vec{u}_1,M_1)-(\vec{u}_2,M_2)\|_{\Sigma_{\rho_1T_1}}.$$

Now Banach's Fixed Point Theorem proves the assertion. $\square$

7 Discussion and Outlook

Within this work we presented a new flexible tool to solve variations of the matched microstructure model. The cell geometry is allowed to vary smoothly over the domain Ω. For the semilinear system we proved that $-\mathbf{A}$ generates an analytic semigroup. So results from the theory of operator semigroups lead to local existence and uniqueness of solutions. In 4.4 and 4.5 it was shown that the solution either vanishes exponentially (Dirichlet case) or the total mass is preserved. In Section 5 existence of a weak solution for a Showalter model with nonlinear boundary conditions was established. In the last part we demonstrated well-posedness for a class of quasilinear versions of the matched microstructure model. Nevertheless we were not able to prove well-posedness for more general quasilinear systems. Some attempts to compass the problems were presented in 6.3. All solutions possess a high regularity and satisfy the equations pointwise. This is an important improvement in comparison with former results.

There are some points that remain for future consideration:

In Chapter 4.1 a relaxation of the regularity assumptions on Φ, Ψ may be possible. In particular for second order operators it suffices if the domains Ω_x or the transformations are of grade $C^{2+\alpha}$ (for some small $\alpha > 0$). It could also be examined which assumptions are sharp. Some results from semigroup theory (long term existence etc.) may be applied directly to the semilinear case. The question about general Robin type boundary conditions on $\partial\Omega$ remains open, too.

The special case with a variable fluid-air interface from Chapter 5 might be a challeging future project. A transformation to a fixed reference domain leads to fully nonlinear equations. The treatment for one scale can be found in [26]. There a nearly incompressable fluid in a deformable porous material is under consideration. For two scales several problems occur already in the modelation. One needs to ensure that cells Ω_x that belong to some x under the surface at a certain time $t > 0$ immediately clean out if $x \notin \Omega_{f(t')}$ for some $t' > t$. Since the fluid transport mostly takes place in the fissure system a quasistationary ansatz that neglects the time derivative on the macro scale does not seem appropriate to us. If we drop the time derivative in the micro scale this leads to trivial solutions in the cells and in this way reduces to a one scale model.

In Chapter 6 the main challenge is to allow a quasilinear structure that includes U. Regrettably the L_p-approach does not support this. Here an ansatz with Hölder spaces could be effective. An additional question is what kinds of modifications of the matching condition are possible. This can help to find the answer what types of sorption isotherms can be included in our model. Sorption isotherms summarize all effects like reactions, van-der-Waals forces, electric forces etc. which bind solutes to colloids or fissure walls. Let s denote the concentration of the sorped material and u the original concentration in the fluid. We used the linear form

$$s = Ku,$$

where $K > 0$ is a given constant usually said to be one. Well known nonlinear relations are the Freundlich isotherm

$$s = Ku^N,$$

for $N > 0$, and the Langmuir isotherm

$$s = \frac{s_{max}u}{\frac{s_{max}}{K} + u},$$

for a given maximal sorption s_{max}. The different equations are valid in different chemical situations. We would like to incorporate these situations into our approach. If we collect all time derivatives in one term the equation reads as

$$g(u)_t - div_x(D\nabla_x u) = \tilde{f}(u, \nabla_x u, U, \nabla_z U, n, n^*).$$

Assume that $g(u) = u + h(u)$ is a continuous, invertible and differentiable function with $g(0) = 0$. Then there are functions h', f' such that the equation is equivalent to

$$\begin{aligned} \partial_t w - div_x(h'(w)D\nabla_x w)) &= f'(w, \nabla_x w, U, \nabla_z U, n, n^*), \quad \text{on } \Omega \times (0, T], \\ w(x,t) &= 0, \qquad\qquad\qquad\qquad \text{for } x \in \Gamma, t \in (0, T], \\ w(t=0) &= w_0 = u_0 + h(u_0). \end{aligned}$$

Here $w = g(u)$. Of course the boundary condition in the micro problem transforms as well:

$$U(x, z, t) = g^{-1}(w)(x, t) \qquad \text{for } x \in \Omega, z \in \Gamma_x, t \in (0, T].$$

This fact makes it complicated to generalize our ansatz into this direction. The properties of g that allow a two scale treatement are not clear. For example the effect of colloidal particals on contaminant transport could be investigated with this theory.

There are many more applications that could be discussed: Of special interest at the moment is the behaviour of non-Newtonian fluids in a two scale system. Some classes of non-Newtonian fluids were described by Kondic et.all. in [43]. They consider a nonlinear Darcy's law in the form

$$\mathbf{v} = -\frac{1}{\overline{\mu}(|\nabla u|^2)} \nabla u.$$

There is no approach known to the author to treat such fluids in a two scale system. The presented concept can not be applied directly.

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