Finite elements/boundary elements for electromagnetic interface problems, especially the skin effect

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To Antonia, Valentina, Jesus David and Emmanuel.

Abstract

This thesis deals with the coupling of finite elements and boundary elements for electromagnetic interface problems, especially the skin effect in $\mathbb{R}^3$.
The first part (Chapter 1) is dedicated to the study of transmission problems of electromagnetic waves in materials with strong contrast. We report the ideas which were developed by MacCamy and Stephan [30, 31], who consider the scattering of time-periodic electromagnetic fields by metallic obstacles, the eddy current problem. In this interface problem different sets of Maxwell equations must be solved in the obstacle and outside, while the tangential components of both electric and magnetic fields are continuous across the obstacle surface. We present two solution procedures.One is an asymptotic procedure which applies for large conductivity and reflects the skin effect in metals. This asymptotic procedure gives for the computation of the solution of the transmission problem a great reduction in complexity since it involves solving only the exterior boundary value problem (perfect conductor problem). The latter is solved numerically by the boundary element method. We give numerical experiments which show the efficiency of this procedure. The other solution procedure is a new coupling method with finite elements and boundary elements which allows the use of standard, conforming test and trial functions which are easy to implement.
In the second part (Chapters 2, 3, 4) we consider two different problems in the whole space $\mathbb{R}^3$, the scalar and the electromagnetic transmission problems. For both problems we prove a priori estimates. We calculate the terms of an asymptotic expansion of the electrical field and study its convergence. The ideas of this part are based on those of Peron [42], who considered a bounded exterior domain, while we extend his results to the case of an unbounded exterior domain. For this extension we use Beppo-Levi spaces with weights at infinity.
The third part (Chapter 5) is concerned with a non-conforming fem/bem coupling to solve the two-dimensional eddy current problem for the time harmonic Maxwell's equations. We use Crouzeix-Raviart elements in the interior domain and piecewise linear and piecewise constant boundary elements on the interface boundary.

Keywords. Skin effect, scalar and electromagnetic transmission problems, asymptotic expansion, non-conforming FEM/BEM coupling.

Zusammenfassung

Diese Arbeit behandelt die Kopplung von finiten Elementen und Randelementen für elektromagnetische Transmissionsprobleme, insbesondere den Skin-Effekt im $\mathbb{R}^3$.
Der erste Teil (Kapitel 1) ist der Analyse von Transmissionsproblemen von elektromagnetischen Wellen in Materialien mit starkem Kontrast gewidmet. Wir wiederholen die Ideen, die von MacCamy und Stephan entwickelt wurden [30, 31]. Sie betrachten die Streuung der zeitperiodischen elektromagnetischen Felder verursacht durch metallische Hindernisse, das sogenannte Wirbelstromproblem. In diesem Interface-Problem müssen verschiedene Maxwell-Gleichungen einmal im Hindernis und einmal außerhalb gelöst werden, wobei die Tangentialkomponenten der beiden elektrischen und magnetischen Felder stetig über die Oberfläche des Hindernisses sind. Wir betrachten ein asymptotisches Verfahren, das für große Leitfähigkeit gültig ist und den Skin-Effekt im Metall berücksichtigt. Das asymptotische Verfahren reduziert die Komplexität des Ausgangsproblems, da jetzt nur noch das äußere Randwertproblem gelöst werden muss. Dieses lösen wir numerisch mit der Randelementmethode. Unsere numerischen Experimente zeigen die Effizienz des Verfahrens. Des weiteren leiten wir eine neue Finite Elemente/Randelement-Kopplungsmethode für das Transmissionsproblems her, die erlaubt stückweise lineare sowie stückweise konstante Ansatzfunktion im Innengebiet und auf dem Rand zu benutzen.
Im zweiten Teil (Kapitel 2, 3, 4) betrachten wir zwei verschiedene Probleme über dem ganzen Raum $\mathbb{R}^3$, das skalare und das elektromagnetische Übertragungsproblem. Für beide Probleme beweisen wir jeweils eine a priori Abschätzung. Wir berechnen die Terme einer asymptotischen Entwicklung des elektrischen Feldes und untersuchen ihre Konvergenz. Die Ideen aus diesem Teil basieren auf der Arbeit von Peron [42], der ein beschränktes Außengebiet betrachtet, während wir seine Ergebnisse für den Fall eines unbeschränkten Außengebiets erweitern. Für diese Erweiterung benutzen wir Beppo-Levi-Räume mit Gewicht im Unendlichen.
Im dritten Teil (Kapitel 5) wird das zweidimensionale Wirbelstromproblem für die zeitharmonischen Maxwell-Gleichungen mit einer Kopplung von nicht-konformen Finiten Elementen und Randelementmethoden gelöst. Wir nehmen Crouzeix-Raviart-Elemente im Innengebiet und stw. lineare sowie stw. konstante Randelemente auf dem Übergangsrand. Unsere numerischen Experimente zeigen die Effizienz dieser FEM/BEM Kopplung.

Schlagwörter. Skin-Effekt, skalare und elektromagnetische Übertragungsprobleme, asymptotische Entwicklung, nicht-konforme FEM/BEM Kopplung.

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Jorge Eliécer Ospino Portillo

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Introduction

This thesis deals with finite element and boundary element procedures for electromagnetic transmission problems in $\mathbb{R}^3$. Special emphasis is on investigation of the behaviour of the electrical and magnetical fields for material with higher conductivity. We analyze the phenomenon of the skin effect with the aid of a multi-scale analysis and numerical simulation.

In Chapter 1 we present asymptotic expansions with respect to inverse powers of conductivity for the electrical and magnetical fields and report the algorithm of MacCamy and Stephan [31] which allows to compute the expansion terms of the electrical field in the exterior domain by solving sucessively only exterior problems (so-called perfect conductor problems) with different data on the interface between conductor (metal) and insulator (air). We solve these exterior problems numerically by applying the Galerkin boundary element method to first kind boundary integral equations which were originally introduced by MacCamy and Stephan in [30]. This system of integral equations on the interface Σ results from a single layer potential ansatz for the electrical field and has unknown densities namely a vector field and a scalar function on Σ which we approximate with lower order Raviart Thomas elements and continous piecewise linear functions on a regular, triangular mesh on Σ. As in the two dimensional case, investigated by Hariharan [22, 23] and MacCamy and Stephan [32], the asymptotic procedure gives for the computation of the solution of the transmission problem a great reduction in complexity since it involves solving only the exterior problem and furthermore only a few expansion terms must be computed. This is due to the fast convergence of the expansion for large conductivity which we obtain in Chapters 3, 4 by extending Peron's results [42] - valid for a bounded domain - to our transmission problem with unbounded exterior domain. We describe in detail how to implement the boundary element method for the perfect conductor problem. As an alternative to the asymptotic expansions for the solution of the transmission problem we introduce a new finite element/boundary element Galerkin coupling procedure which converges quasi-optimally in the energy norm (Theorem 2).

In Chapter 2 as in Peron [42] we investigate a scalar transmission problem for the Laplacian with parameter. But we use a setting in Beppo-Levi spaces (Sobolev spaces with weight) to incorporate in a weak sense the decay condition at infinity; in this way we extend Peron's results [42, 7] to an unbounded exterior domain.

In Chapter 3 we analyze electromagnetic transmission problems (Maxwell's equations) in $\mathbb{R}^3$ for a large parameter in weighted spaces (vectorial Beppo-Levi spaces). Again we follow Peron but consider unbounded exterior domains. Therefore we must consider appropriate weighted spaces and use a Helmholtz decomposition for the electrical field in weighted spaces obtained by Girault [18], and compactness results for the embedding in weighted Sobolev spaces by Avantaggiati and Troisi [2]. These ingredients allow us to derive an a priori estimate for the solution of the regularised Maxwell's interface problem which holds uniformly with respect to the conductivity parameter. In deriving this result we follow step by step Peron's approach [42] and modify it appropriately for the unbounded exterior domain. Our a priori estimate (Theorem 6) implies uniqueness and existence of the solution of the electromagnetic transmission problem in weighted spaces (Theorem 5).

In Chapter 4 we mainly report on Peron's results for an asymptotic expansion of the electrical field for large conductivity [42, 15]. We show that his results (for a bounded exterior domain) remain valid for an unbounded exterior domain. Since as we have shown in Theorem 5 the solution of Maxwell's interface problem is unique, it can be obtained on the other hand by the boundary integral equation procedure by MacCamy and Stephan [31] considered in Chapter 1. This on the other hand shows that the formal asymptotic expansion in Chapter 1 converges, too, and the effectiveness of the procedure in Chapter 1 (computing only a couple terms in the expansion via solving only perfect conductor problems) is guaranteed.

In Chapter 5 we present a non-conforming finite element/boundary element coupling method to solve the two-dimensional eddy current problem for the time harmonic Maxwell's equations. Here we combine the approach by Brenner et al. [4, 5, 6] for the fem part with the approach by Carstensen and Funken [8] for the fem/bem coupling. We present numerical simulations which show the effectiveness of our non-conforming fem/bem coupling method.

1 Asymptotic expansion for large conductivity, skin effect and boundary element computations

We consider the scattering of time periodic electro-magnetic fields by metallic obstacles, the eddy current problem. In this interface problem different sets of Maxwell equations must be solved in the obstacle and outside, while the tangential components of both electric and magnetic fields are continuous across the interface. In Subsection 1.1 we describe an asymptotic procedure from [31] which applies for large conductivity and reflects the skin effect in metals. The key to our method is to introduce a special integral equation procedure (derived in [31]) for the exterior boundary value problem corresponding to perfect conductors (see Subsection 1.2). The asymptotic procedure leads to a great reduction in complexity for the numerical solution since it involves solving only the exterior boundary value problem. In this chapter we extend the procedure from the two-dimensional case in [32] to three dimensions. Furthermore we introduce in Subsection 1.3 a new fem/bem coupling procedure for the transmission problem. Finally, in Subsection 1.4 we consider the implementation of the Galerkin elements for the perfect conductor problem and present numerical experiments in Subsection 1.5.

1.1 Asymptotic expansion for large conductivity and skin effect

Let Ω_- be a bounded region in $\mathbb{R}^3$ representing a metallic conductor and $\Omega_+ := \mathbb{R}^3 \backslash \overline{\Omega_-}$ representing air. Throughout Chapter 1 we assume that the boundary Σ of Ω_- is a regular analytic surface. The parameters ε, μ, σ denote permittivity, permeability and conductivity. We assume zero conductivity in Ω_+. Let the incident electric and magnetic fields, $\mathbf{E}^0$ and $\mathbf{H}^0$, satisfy Maxwell's equations in air. The total fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the same Maxwell's equations as $\mathbf{E}^0$ and $\mathbf{H}^0$ in Ω_+ but different equations in Ω_-. Across the interface $\Sigma := \partial\Omega_- = \partial\Omega_+$, the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous. $\mathbf{E} - \mathbf{E}^0$ and $\mathbf{H} - \mathbf{H}^0$ represent the scattered fields. All fields are time-harmonic with frequency ω. As in [31] we neglect conduction (displacement) currents in air (metal).

Then, with appropriate scaling, the eddy current problem is (see [31], [44])

Problem ($\boldsymbol{P}_{\alpha\beta}$): Given parameters α and $\beta > 0$, find $\mathbf{E}$ and $\mathbf{H}$ such that

$$\begin{aligned}
&\operatorname{curl}\mathbf{E} = \mathbf{H}, \quad \operatorname{curl}\mathbf{H} = \alpha^2\mathbf{E} \quad \text{in} \quad \Omega_+ \quad \text{(air)} \\
&\operatorname{curl}\mathbf{E} = \mathbf{H}, \quad \operatorname{curl}\mathbf{H} = i\beta^2\mathbf{E} \quad \text{in} \quad \Omega_- \quad \text{(metal)} \\
&\mathbf{E}_T^+ = \mathbf{E}_T^-, \qquad \mathbf{H}_T^+ = \mathbf{H}_T^-, \qquad \text{on} \quad \Sigma. \\
&\frac{\partial}{\partial r}\mathbf{E}(\mathbf{x}) - i\alpha\mathbf{E}(\mathbf{x}) = O\left(\frac{1}{r^2}\right) \quad \text{with} \quad r = |\mathbf{x}|, \quad \text{as} \quad |\mathbf{x}| \to \infty.
\end{aligned} \tag{1.1}$$

Here $\alpha^2 = \omega^2\mu_0\varepsilon_0$ and $\beta^2 = \omega\mu\sigma - i\omega^2\mu\varepsilon$ are dimensionless parameters, and $\beta^2 = \omega\mu\sigma > 0$ if displacement currents are neglected in metal ($\varepsilon = 0$). Here e.g. $\mathbf{E}_T^+$ denotes the limit from Ω_+ of the tangential component on Σ.
At higher frequencies the constant β is usually large leading to the *perfect conductor approximation*. Then in (1.1) one only solves the equation in Ω_+ and requires $\mathbf{E}_T = 0$ on Σ, that is

Problem ($\boldsymbol{P}_{\alpha\infty}$):For given $\mathbf{E}_T^0$ and $\alpha > 0$, find the scattered fields $\mathbf{E}$ and $\mathbf{H}$ such that

$$\begin{aligned}
&\operatorname{curl}\mathbf{E} = \mathbf{H}, \quad \operatorname{curl}\mathbf{H} = \alpha^2\mathbf{E} \quad \text{in} \quad \Omega_+ \\
&\mathbf{E}_T = -\mathbf{E}_T^0, \quad \text{on} \quad \Sigma.
\end{aligned} \tag{1.2}$$

Remark 1. *There exists at most one solution of problem* ($\boldsymbol{P}_{\alpha\beta}$) *for any* $\alpha > 0$ *and* $0 < \beta \leq \infty$ *(see [39]).*

We are interested in an asymptotic expansion of the solution of problem ($\mathbf{P}_{\alpha\beta}$) with respect to inverse powers of conductivity. With τ denoting the distance from Σ measured into Ω_- along the normal to Σ the expansions read (see [31]):

$$\mathbf{E} \sim \mathbf{E}^0 + \sum_{n=0}^{\infty} \mathbf{E}_n \beta^{-n} \quad \text{in} \quad \Omega_+ \tag{1.3}$$

$$\mathbf{H} \sim \mathbf{H}^0 + \sum_{n=0}^{\infty} \mathbf{H}_n \beta^{-n} \quad \text{in} \quad \Omega_+ \tag{1.4}$$

$$\mathbf{E} \sim e^{-\sqrt{-i}\beta\tau} \sum_{n=0}^{\infty} \mathbf{E}_n \beta^{-n} \quad \text{in} \quad \Omega_- \tag{1.5}$$

$$\mathbf{H} \sim e^{-\sqrt{-i}\beta\tau} \sum_{n=0}^{\infty} \mathbf{H}_n \beta^{-n} \quad \text{in} \quad \Omega_- \tag{1.6}$$

Here $\mathbf{E}_n$ and $\mathbf{H}_n$ are independent of β which is proportional to $\sqrt{\sigma}$. The exponential-factor in (1.5) and (1.6) represents *the skin effect.* Next we present from [31] these expansions for the half-space case where the various coefficients can be computed recursively. Note $\mathbf{E}_0$ and $\mathbf{H}_0$ in (1.3) and (1.4) is simply the perfect conductor approximation, that is, the solution of $(\mathbf{P}_{\alpha\infty})$. As observed in [31] $\mathbf{E}_n$ and $\mathbf{H}_n$ in (1.3) and (1.4) can be calculated successively by solving a sequence of problems of the same form as $(\mathbf{P}_{\alpha\infty})$ but with boundary values determined from earlier coefficients. The $\mathbf{E}_n$ and $\mathbf{H}_n$ in (1.5) and (1.6) are obtained by solving ordinary differential equations in the variable x_3.
In the half-space case $\Omega_+ = \mathbb{R}^3_+$ i.e. $x_3 > 0$ and $\Omega_- = \mathbb{R}^3_-$ i.e. $x_3 < 0$ a formal procedure to compute $\mathbf{E}_n$, $\mathbf{H}_n$ was given by MacCamy and Stephan [31]. They substitute in (1.3)-(1.6) into $(\mathbf{P}_{\alpha\beta})$ for $\Sigma = \mathbb{R}^2$ and equate coefficients of β^{-n}. Here we give a short description of their approach.
Let $\chi = e^{\sqrt{-i}\beta x_3}$ and decompose fields $\mathbf{F}$ into tangential and normal components

$$\mathbf{F} = \mathfrak{F} + f\mathbf{e}_3, \quad \mathfrak{F} = \mathcal{F}^1\mathbf{e}_1 + \mathcal{F}^2\mathbf{e}_2, \tag{1.7}$$

with orthogonal component $\mathfrak{F}^\perp = \mathbf{e}_3 \times \mathfrak{F}$, and unit vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$.
Then one computes with the surface gradient $grad_T$ for the rotation

$$\operatorname{curl} \mathbf{F} = \mathfrak{F}^\perp_{x_3} - (\operatorname{grad}_T f)^\perp - (\operatorname{div} \mathfrak{F}^\perp)\mathbf{e}_3 \tag{1.8}$$

and

$$\operatorname{curl}(\chi\mathbf{F}) = \chi[\sqrt{-i}\beta\mathfrak{F}^\perp + \mathfrak{F}^\perp_{x_3} - (\operatorname{grad}_T f)^\perp - (\operatorname{div} \mathfrak{F}^\perp)\mathbf{e}_3]. \tag{1.9}$$

Now setting $\mathbf{E}_n = \mathcal{E}_n + \ell_n\mathbf{e}_3$ one obtains for $x_3 < 0$

$$\operatorname{curl} \mathbf{E} \sim \chi\{\sqrt{-i}\beta\mathcal{E}_0^\perp + \sum_{n=0}^{\infty}[\sqrt{-i}\mathcal{E}^\perp_{n+1} + \mathcal{E}^\perp_{n,x_3} - (\operatorname{grad}_T \ell_n)^\perp - (\operatorname{div} \mathcal{E}^\perp_n)\mathbf{e}_3]\beta^{-n}\}, \tag{1.10}$$

and

$$\begin{aligned} \operatorname{curl}\operatorname{curl} \mathbf{E} \sim \chi \Bigg\{ & i\beta^2\mathcal{E}_0 - \sqrt{-i}\beta\mathcal{E}_{0,x_3} + \sqrt{-i}\beta\operatorname{div} \mathcal{E}_0\mathbf{e}_3 + \sum_{n=0}^{\infty} \Big[i\beta\mathcal{E}_{n+1} - \sqrt{-i}\mathcal{E}_{n+1,x_3} \\ & - \sqrt{-i}\operatorname{div} \mathcal{E}_{n+1}\mathbf{e}_3 - \sqrt{-i}\beta\mathcal{E}_{n,x_3} - \mathcal{E}_{n,x_3,x_3} + \operatorname{div} \mathcal{E}_{n,x_3}\mathbf{e}_3 \\ & + \sqrt{-i}\beta\operatorname{grad} \ell_n + (\operatorname{grad}_T \ell_n)_{x_3} + \operatorname{div}\operatorname{grad} \ell_n\mathbf{e}_3 \Big] \beta^{-n} + \operatorname{grad}\operatorname{div} \beta^{-n}\mathbf{e}_3 \Bigg\} \\ & = \chi[i\beta^2\mathcal{E}_0 + i\beta^2\ell_0\mathbf{e}_3 + i\beta\mathcal{E}_1 + i\beta\ell_1\mathbf{e}_3 + \sum_{n=0}^{\infty}(i\mathcal{E}_{n+2} + i\ell_{n+2}\mathbf{e}_3)\beta^{-n}] \sim i\beta^2\mathbf{E}. \end{aligned} \tag{1.11}$$

Hence, equating coefficients of β^2 and β, respectively yields $\ell_0 \equiv 0$, $i\ell_1 = \sqrt{-i}\operatorname{div} \mathcal{E}_0$ and $\mathcal{E}_{0,x_3} = 0$ implying $\mathcal{E}_0(x_1, x_2, x_3) = \mathcal{E}_0(x_1, x_2, 0)$.

As coefficients of β^0 one obtains

$$-\sqrt{-i}\mathcal{E}_{1,x_3} + \sqrt{-i}\text{grad}\,\ell_1 = 0,$$

$$\sqrt{-i}\text{div}\,\mathcal{E}_1 + \text{div}\,\mathcal{E}_{0,x_3} - \text{grad div}\,\mathcal{E}_0 = i\ell_2.$$

Now the gauge condition div $\mathcal{E}_0 = 0$ implies $\ell_1 \equiv 0$ and div $\mathcal{E}_{0,x_3} = 0$, hence $\mathcal{E}_{1,x_3} = 0$ and $\sqrt{-i}\text{div}\,\mathcal{E}_1 = i\ell_2$.

Thus $\mathcal{E}_1(x_1, x_2, x_3) = \mathcal{E}_1(x_1, x_2, 0)$.

Equating coefficients of β^{-1} in (1.11) gives

$$-\sqrt{-i}\mathcal{E}_{2,x_3} - \sqrt{-i}\mathcal{E}_{2,x_3} + \sqrt{-i}\text{grad}\,\ell_2 = 0,$$

$$\sqrt{-i}\text{div}\,\mathcal{E}_2 - \text{grad div}\,\mathcal{E}_1 = i\ell_3.$$

Setting

$$\mathbf{H} = \chi\sum_{n=0}^{\infty}(\mathcal{H}_n + h_n\mathbf{e}_3)\beta^{-n} \tag{1.12}$$

MacCamy and Stephan obtain in [31] with $\ell_1 = 0$, $h_0 = 0$ $\mathcal{E}_0 = 0$:

$$\sqrt{-i}\mathcal{E}_1^{\perp} + \mathcal{E}_{0,x_3}^{\perp} = \mathcal{H}_0,\;\; \sqrt{-i}\mathcal{H}_0^{\perp} = i\mathcal{E}_1,\;\; h_0 = \text{div}\,\mathcal{E}_0^{\perp} = 0. \tag{1.13}$$

and

$$\sqrt{-i}\mathcal{E}_2^{\perp} + \mathcal{E}_{1,x_3}^{\perp} = \mathcal{H}_1,\quad \sqrt{-i}\mathcal{H}_1^{\perp} + \mathcal{H}_{0,x_3}^{\perp} = i\mathcal{E}_2 \tag{1.14}$$

$$h_1 = -\text{div}\,\mathcal{E}_1^{\perp},\quad -\text{div}\,\mathcal{H}_0^{\perp} = i\ell_2. \tag{1.15}$$

and

$$\begin{aligned}&\mathcal{H}_{0,x_3} \equiv \mathcal{E}_{1,x_3} \equiv 0\\ &\mathcal{H}_0 \equiv \sqrt{-i}\mathcal{E}_1^{\perp} \qquad \text{in} \quad x_3 < 0\end{aligned} \tag{1.16}$$

For $x_3 > 0$, we have with curl $\mathbf{E} = \mathbf{H}$ yields

$$\text{curl}\,\mathbf{E}^0 + \sum_{n=0}^{\infty}\text{curl}\,\mathbf{E}_n\beta^{-n} = \mathbf{H}^0 + \sum_{n=0}^{\infty}\mathbf{H}_n\beta^{-n}$$

Equating coefficients of β^{-n} one finds in $x_3 > 0$

$$\text{curl}\,\mathbf{E}^0 = \mathbf{H}^0,\quad \text{curl}\,\mathbf{E}_n = \mathbf{H}_n,\;\; n \geq 0,$$

(and corresponding due to curl $\mathbf{H} = \alpha^2\mathbf{E}$)

$$\text{curl}\,\mathbf{H}^0 = \alpha^2\mathbf{E}^0,\quad \text{curl}\,\mathbf{H}_n = \alpha^2\mathbf{E}_n,\;\; n \geq 0.$$

With the above relations the recursion process goes as follows. First one use (6.10) for $n = 0$ and (6.13), in [31], to conclude that

$$\text{curl}\,\mathbf{E}_0 = \mathbf{H}_0,\quad \text{curl}\,\mathbf{H}_0 = \alpha^2\mathbf{E}_0 \qquad \text{in} \qquad x_3 > 0$$

$$\mathbf{E}_0^+ = -(\mathbf{E}_T^0)^-, \qquad\qquad \text{on} \qquad\qquad x_3 = 0.$$

Now $(\mathbf{E}_0, \mathbf{H}_0)$ is just the solution of $(\mathbf{P}_{\alpha\infty})$ which we can solve by the boundary integral equation procedure (1.29), (1.30) introduced by MacCamy and Stephan in [31] and revisited below in Section 1.2. But from $(1.1)_3$ we obtain

$$\mathcal{H}_0^- = \mathcal{H}_0^+ = (\mathbf{H}_0)_T^+ \quad \text{on} \quad x_3 = 0. \tag{1.17}$$

Now the right side of (1.17) is known and easily computed. Then $(1.1)_3$ and (1.17) yield

$$(\mathbf{E}_1)_T^+ = (\mathbf{E}_1)_T^- = \mathcal{E}_1^- = -\sqrt{i}(\mathcal{H}_0^\perp)^- = -\sqrt{i}((\mathbf{H}_0)_T^+)^\perp. \tag{1.18}$$

Therefore by (6.10), in [31], we have a new, again solvable problem for $(\mathbf{E}_1, \mathbf{H}_1)$ which is just like $(\mathbf{P}_{\alpha\infty})$, that is

$$\operatorname{curl} \mathbf{E}_1 = \mathbf{H}_1, \quad \operatorname{curl} \mathbf{H}_1 = \alpha^2 \mathbf{E}_1 \quad \text{in} \quad x_3 > 0,$$

but with new boundary values for $\mathbf{E}_T$ as given by (1.18).
For the complete algorithm see [31]. Note, with $\lambda = \sqrt{-i}$ we have $\mathcal{E}_1^-(x_1, x_2, 0) = -\frac{1}{\lambda}(\mathbf{n} \times \operatorname{curl} \mathbf{E}_0)$ yielding in $x_3 < 0$

$$\mathbf{E}_1(x_1, x_2, x_3) = \int_0^{-\tau} e^{\lambda\beta\widetilde{x}_3} \mathcal{E}_1^{-1}(x_1, x_2, 0) d\widetilde{x}_3 = -\frac{1}{\lambda^2\beta}(\mathbf{n} \times \operatorname{curl} \mathbf{E}_0)[e^{-\lambda\beta\tau} - 1]$$

A comparison with Peron's results (see Chapter 5 in [42]) shows that $\mathbf{W}_j^{cd}(y_\alpha, h_\rho) = e^{-\sqrt{-i}\beta\tau}\mathbf{E}_j$, $j \geq 0$, in Ω^{cd}, $\lambda Y_3 = \sqrt{-i}\beta\tau$ and $w_j = \ell_j$. Furthermore we see that the first terms in the asymptotic expansion of the electrical field for a smooth surface Σ derived by Peron [42] coincide with those for the half-space $x_3 = 0$ investigated by MacCamy and Stephan [31], namely $\ell_0 = w_0 = 0$, $\ell_1 = w_1 = 0$, $\mathcal{E}_0 = \mathbf{W}_0^{cd} = 0$.

Remark 2. *Since due to Theorem 5 in Chapter 3 there exists only one solution of the electromagnetic transmission problem for a smooth interface this solution can be computed by the boundary integral equation procedure below, when we assume that (1.22) holds. Then for the electrical field $\boldsymbol{E}$ obtained via the boundary integral equation system we have that in the tubular region $\Omega_\pm(\delta) = \{x \in \Omega_\pm, dist(x, \Sigma) < \delta\}$ there holds for the remainders $\boldsymbol{E}_m^{is(cd)}$ obtained by truncating (1.3) and (1.5) at $n = m$*

$$\|\boldsymbol{E}_{m,\rho}^{is}\|_{\boldsymbol{W}(curl,\Omega^{is})} \leq C_1\rho^{-m-1} \quad \text{and} \quad \|\boldsymbol{E}_{m,\rho}^{cd}\|_{L^2(\Omega_\pm(\delta))} \leq C_2 e^{C_3\tau}$$

for constants $C_1, C_2, C_3 > 0$, independent of ρ.

We set

$$\underline{\mathbf{E}}_m := \begin{cases} \mathbf{E}^0 + \sum_{k=0}^{m} \mathbf{E}_k \beta^{-k} & \text{in} \quad x_3 > 0 \\ \\ \chi \sum_{k=0}^{m} \mathbf{E}_k \beta^{-k} & \text{in} \quad x_3 < 0 \end{cases} \tag{1.19}$$

$$\underline{\mathbf{H}}_m := \begin{cases} \mathbf{H}^0 + \sum_{k=0}^{m} \mathbf{H}_k \beta^{-k} & \text{in} \quad x_3 > 0 \\ \\ \chi \sum_{k=0}^{m} \mathbf{H}_k \beta^{-k} & \text{in} \quad x_3 < 0 \end{cases} \tag{1.20}$$

where $\chi = e^{\sqrt{-i}\beta x_3}$, for $x_3 < 0$ and $m \geq 0$.
We call these the m^{th} order asymptotic approximations.
Now we have the following result which follows readily from the definition of the $\mathbf{E}_k$ and $\mathbf{H}_k$.

Theorem 1. *For each $m \geq 0$,*

$$\begin{aligned} &curl\, curl\, \underline{\boldsymbol{E}}_m - \alpha^2 \underline{\boldsymbol{E}}_m = 0, \quad in \;\; x_3 > 0, \\ &curl\, curl\, \underline{\boldsymbol{E}}_m - i\beta^2 \underline{\boldsymbol{E}}_m = \boldsymbol{F}_m + curl\, \mathcal{G}_m \equiv \widehat{\boldsymbol{F}}_m, \quad in \;\; x_3 < 0, \\ &(\underline{\boldsymbol{E}}_m^-)_T - (\underline{\boldsymbol{E}}_m^+)_T = 0, \quad on \;\; x_3 = 0, \\ &(\underline{\boldsymbol{H}}_m^-)_T - (\underline{\boldsymbol{H}}_m^+)_T = (curl\, \underline{\boldsymbol{E}}_m^-)_T - (curl\, \underline{\boldsymbol{E}}_m^+)_T - (\mathcal{G}_m)_T \equiv g_m, \quad on \;\; x_3 = 0, \end{aligned} \tag{1.21}$$

Where

$$\begin{aligned} \chi = &\quad e^{\sqrt{-i}\beta x_3} \\ \mathcal{G}_m = &\quad \chi[\mathcal{E}^\perp_{m,x_3} - (grad_T\, \ell_m)^\perp - \mathcal{H}_m]\beta^{-m} \\ \boldsymbol{F}_m = &\quad \chi[(\sqrt{-i}\mathcal{H}^\perp_m + \mathcal{H}^\perp_{m-1,x_3} - (grad_T\, h_{m-1})^\perp - (div\, \mathcal{H}^\perp_{m-1})\boldsymbol{e}_3)\beta^{-m+1} \\ &\quad +(\mathcal{H}^\perp_{m,x_3} - (grad_T\, h_m)^\perp - (div\, \mathcal{H}^\perp_m)\boldsymbol{e}_3)\beta^{-m}]. \end{aligned}$$

Proof. In $x_3 > 0$, from (1.19), (1.1) and (1.8)

$$\begin{aligned} \text{curl}\, \underline{\mathbf{E}}_m &= \text{curl}\, \mathbf{E}^0 + \textstyle\sum_{k=0}^m (\text{curl}\, \mathbf{E}_k)\beta^{-k} \\ &= \mathbf{H}^0 + \textstyle\sum_{k=0}^m \mathbf{H}_k \beta^{-k}, \end{aligned}$$

then

$$\begin{aligned} \text{curl curl}\, \underline{\mathbf{E}}_m &= \text{curl}\, \mathbf{H}^0 + \textstyle\sum_{k=0}^m (\text{curl}\, \mathbf{H}_k)\beta^{-k} \\ &= \alpha^2 \mathbf{E}^0 + \textstyle\sum_{k=0}^m \alpha^2 \mathbf{E}_k \beta^{-k} \\ &= \alpha^2 \underline{\mathbf{E}}_m, \end{aligned}$$

then

$$\text{curl curl}\, \underline{\mathbf{E}}_m - \alpha^2 \underline{\mathbf{E}}_m = 0.$$

In $x_3 < 0$, from (1.19), (1.1) and (1.9)

$$\begin{aligned}
\operatorname{curl} \underline{\mathbf{E}}_m &= \textstyle\sum_{k=0}^{m} (\operatorname{curl}(\chi \mathbf{E}_k))\beta^{-k} \\
&= \textstyle\sum_{k=0}^{m} \chi \left[\sqrt{-i}\beta \mathcal{E}_k^{\perp} + \mathcal{E}_{k,x_3}^{\perp} - (\operatorname{grad}_T \ell_k)^{\perp} - (\operatorname{div} \mathcal{E}_k^{\perp})\mathbf{e}_3\right] \beta^{-k} \\
&= \chi[\sqrt{-i}\beta\mathcal{E}_0^{\perp} + \textstyle\sum_{k=0}^{m-1} \sqrt{-i}\mathcal{E}_{k+1}^{\perp}\beta^{-k} \\
&+ \textstyle\sum_{k=0}^{m} (\mathcal{E}_{k,x_3}^{\perp} - (\operatorname{grad}_T \ell_k)^{\perp} - (\operatorname{div} \mathcal{E}_k^{\perp})\mathbf{e}_3)\beta^{-k}] \\
&= \underline{\mathbf{H}}_m = \chi \textstyle\sum_{k=0}^{m} (\mathcal{H}_k + h_k \mathbf{e}_3)\beta^{-k},
\end{aligned}$$

but from (I_n) and (I_n') in [31]

$$\sqrt{-i}\mathcal{E}_{k+1}^{\perp} + \mathcal{E}_{k,x_3}^{\perp} - (\operatorname{grad}_T \ell_k)^{\perp} = \mathcal{H}_k, \quad k = 0,1,2...$$

$$h_k = -\operatorname{div} \mathcal{E}_k^{\perp}, \quad k = 0,1,2...$$

and from (6.13) in [31] $\mathcal{E}_o^{\perp} = 0$, then

$$\operatorname{curl} \underline{\mathbf{E}}_m - \underline{\mathbf{H}}_m = \chi[\mathcal{E}_{m,x_3}^{\perp} - (\operatorname{grad}_T \ell_m)^{\perp} - \mathcal{H}_m]\beta^{-m} =: \mathcal{G}_m,$$

and

$$\begin{aligned}
\operatorname{curl} \underline{\mathbf{H}}_m &= \textstyle\sum_{k=0}^{m} (\operatorname{curl}(\chi \mathbf{H}_k))\beta^{-k} \\
&= \textstyle\sum_{k=0}^{m} \chi \left[\sqrt{-i}\beta \mathcal{H}_k^{\perp} + \mathcal{H}_{k,x_3}^{\perp} - (\operatorname{grad}_T h_k)^{\perp} - (\operatorname{div} \mathcal{H}_k^{\perp})\mathbf{e}_3\right] \beta^{-k} \\
&= \chi[\sqrt{-i}\beta\mathcal{H}_0^{\perp} + \textstyle\sum_{k=0}^{m-1} \sqrt{-i}\mathcal{H}_{k+1}^{\perp}\beta^{-k} \\
&+ \textstyle\sum_{k=0}^{m} (\mathcal{H}_{k,x_3}^{\perp} - (\operatorname{grad}_T h_k)^{\perp} - (\operatorname{div} \mathcal{H}_k^{\perp})\mathbf{e}_3)\beta^{-k}] \\
&= i\beta^2 \underline{\mathbf{E}}_m = i\beta^2 \chi \textstyle\sum_{k=0}^{m} (\mathcal{E}_k + \ell_k \mathbf{e}_3)\beta^{-k} \\
&= \chi[i\beta^2\mathcal{E}_0 + i\beta^2\ell_0\mathbf{e}_3 + i\beta\mathcal{E}_1 + i\beta\ell_1\mathbf{e}_3 + \textstyle\sum_{k=0}^{m-2} (i\mathcal{E}_{k+2} + i\ell_{k+2}\mathbf{e}_3)\beta^{-k}].
\end{aligned}$$

But from (II_n) and (I_n') in [31]

$$\sqrt{-i}\mathcal{H}_{k+1}^{\perp} + \mathcal{H}_{k,x_3}^{\perp} - (\operatorname{grad}_T h_k)^{\perp} = i\mathcal{E}_{k+2}, \quad k = 0,1,2...$$

$$\ell_{k+2} = i\operatorname{div} \mathcal{H}_k^{\perp}, \quad k = 0,1,2...$$

and from (6.13) and (6.14) in [31] $\mathcal{E}_0 = 0$, $\ell_0 = 0$, $\ell_1 = 0$ and $i\mathcal{E}_1 = \sqrt{-i}\mathcal{H}_0^{\perp}$, then

$$\begin{aligned}
\operatorname{curl} \underline{\mathbf{H}}_m - i\beta^2\underline{\mathbf{E}}_m &= \chi[(\sqrt{-i}\mathcal{H}_m^{\perp}) + \mathcal{H}_{m-1,x_3} - (\operatorname{grad}_T h_{m-1})^{\perp} - (\operatorname{div} \mathcal{H}_{m-1}^{\perp})\mathbf{e}_3)\beta^{-m+1} \\
&+ (\mathcal{H}_{m,x_3} - (\operatorname{grad}_T h_m)^{\perp} - (\operatorname{div} \mathcal{H}_m^{\perp})\mathbf{e}_3)\beta^{-m}] \\
&= \mathbf{F}_m
\end{aligned}$$

then

$$\mathrm{curl}\,\mathrm{curl}\,\underline{\mathbf{E}}_m - \mathrm{curl}\,\underline{\mathbf{H}}_m = \mathrm{curl}\,\mathcal{G}_m,$$

then

$$\mathrm{curl}\,\mathrm{curl}\,\underline{\mathbf{E}}_m - i\beta^2\underline{\mathbf{E}}_m - \mathbf{F}_m = \mathrm{curl}\,\mathcal{G}_m,$$

then

$$\mathrm{curl}\,\mathrm{curl}\,\underline{\mathbf{E}}_m - i\beta^2\underline{\mathbf{E}}_m = \mathbf{F}_m + \mathrm{curl}\,\mathcal{G}_m \equiv \widehat{\mathbf{F}}_m.$$

In $x_3 = 0$, $\chi \equiv 1$, from (1.19) and (1.1)

$$\begin{aligned}
(\underline{\mathbf{E}}_m^+)_T &= \mathbf{E}_T^0 + \left(\textstyle\sum_{k=0}^m \mathbf{E}_k\beta^{-k}\right)_T^+ \\
&= \mathbf{E}_T^0 + \textstyle\sum_{k=0}^m (\mathbf{E}_k^+)_T\beta^{-k} \\
&= \mathbf{E}_T^0 + \textstyle\sum_{k=1}^m (\mathbf{E}_k^-)_T\beta^{-k} + (\mathbf{E}_0)_T^+ \\
&= \mathbf{E}_T^0 + (\underline{\mathbf{E}}_m^-)_T - \mathbf{E}_T^0 \\
&= (\underline{\mathbf{E}}_m^-)_T,
\end{aligned}$$

then $(\underline{\mathbf{E}}_m^-)_T - (\underline{\mathbf{E}}_m^+)_T = 0$.
On the other hand

$$\begin{aligned}
(\underline{\mathbf{H}}_m^+)_T &= \mathbf{H}_T^0 + \left(\textstyle\sum_{k=0}^m \mathbf{H}_k\beta^{-k}\right)_T^+ \\
&= \mathbf{H}_T^0 + \textstyle\sum_{k=0}^m (\mathbf{H}_k^+)_T\beta^{-k} \\
&= \mathbf{H}_T^0 + \textstyle\sum_{k=0}^m (\mathbf{H}_k^-)_T\beta^{-k} \\
&= \mathbf{H}_T^0 + (\underline{\mathbf{H}}_m^-)_T,
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{n} \times (\mathrm{curl}\,\underline{\mathbf{E}}_m \times \mathbf{n}))^+ &= (\mathrm{curl}\,\mathbf{E}^0)_T^+ + \textstyle\sum_{k=0}^m (\mathrm{curl}\,\mathbf{E}_k)_T^+\beta^{-k} \\
&= \mathbf{H}_T^0 + \textstyle\sum_{k=0}^m (\mathbf{H}_k)_T^+\beta^{-k} \\
&= (\underline{\mathbf{H}}_m)_T^+,
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{n}\times(\operatorname{curl}\underline{\mathbf{E}}_m\times\mathbf{n}))^- &= \chi[\textstyle\sum_{k=0}^m \sqrt{-i}\beta(\mathcal{E}_k^\perp)_T\beta^{-k} + \sum_{k=0}^m((\mathcal{E}_{k,x_3}^\perp)_T - (\operatorname{grad}_T \ell_k)_T^\perp)\beta^{-k}] \\
&= \chi[\sqrt{-i}\beta(\mathcal{E}_0^\perp)_T + \textstyle\sum_{k=0}^{m-1}\sqrt{-i}(\mathcal{E}_k^\perp)_T\beta^{-k} \\
&+ \textstyle\sum_{k=0}^m((\mathcal{E}_{k,x_3}^\perp)_T - (\operatorname{grad}_T \ell_k)_T^\perp)\beta^{-k}] \\
&= \chi\textstyle\sum_{k=0}^{m-1}(\mathbf{H}_k^-)_T\beta^{-k} + \chi((\mathcal{E}_{m,x_3}^\perp)_T - (\operatorname{grad}_T \ell_m)_T^\perp)\beta^{-m} \\
&= (\underline{\mathbf{H}}_m)_T^- - \chi(\mathcal{H}_m^-)_T\beta^{-m} + \chi((\mathcal{E}_{m,x_3}^\perp)_T - (\operatorname{grad}_T \ell_m)_T^\perp)\beta^{-m} \\
&= (\underline{\mathbf{H}}_m)_T^- + (\mathcal{G}_m)_T,
\end{aligned}
$$

then $(\underline{\mathbf{H}}_m^-)_T - (\underline{\mathbf{H}}_m^+)_T = (\operatorname{curl}\underline{\mathbf{E}}_m^-)_T - (\operatorname{curl}\underline{\mathbf{E}}_m^+)_T - (\mathcal{G}_m)_T$. $\square$

The convergence of the asymptotic expansion can be derived from the results of Peron [42], modified in Chapter 4 where the case of an unbounded exterior domain is treated whereas Peron considered the case of a bounded exterior domain. Since the solution of problem $(\mathbf{P}_{\alpha\beta})$ is unique, the results of Chapter 4 apply to the solution of (1.1).

1.2 Boundary integral equation method of the first kind

Next we describe the integral equation procedure for $(\mathbf{P}_{\alpha\infty})$ from [31].
We note the following well-know result:

Remark 3. *There exists a sequence $\{\alpha_k\}_{k=1}^\infty$, such that if $\alpha \neq \alpha_k$ then curl $\boldsymbol{E} = \boldsymbol{H}$, curl $\boldsymbol{H} = \alpha^2\boldsymbol{E}$ in Ω_+, $\boldsymbol{E}_T \equiv 0$ on Σ implies $\boldsymbol{E}\equiv\boldsymbol{H}\equiv 0$ in Ω_+.*

Now we require that

$$\alpha \neq \alpha_k, \quad k = 1, 2, \dots \tag{1.22}$$

This integral equation procedure is based on the Stratton-Chu formula [44]. Let $\mathbf{n}$ denote the exterior normal to Σ. Any vector field $\mathbf{v}$ on Σ can be written as

$$\mathbf{v} = \mathbf{v}_T + v_N\mathbf{n}, \quad \mathbf{v}_T = \mathbf{n}\times(\mathbf{v}\times\mathbf{n}) \tag{1.23}$$

with tangential component $\mathbf{v}_T$ of $\mathbf{v}$.
We define the single layer potential $\mathcal{V}_\kappa$ for density ψ (correspondingly for a vector field) for the surface Σ by

$$\mathcal{V}_\kappa(\psi)(x) = \int_\Sigma \psi(\mathbf{y})G_\kappa(|\mathbf{x}-\mathbf{y}|)ds_y, \quad G_\kappa(r) = \frac{e^{i\kappa r}}{4\pi r}. \tag{1.24}$$

For a vector field $\mathbf{v}$ on Σ we define $\mathcal{V}_\kappa(\mathbf{v})$ by (1.24) with $\mathbf{v}$ replacing ψ.
Next, we collect some well-known results about the single layer potential $\mathcal{V}_\kappa$.

Remark 4. *[31, Lemma 2.1,Lemma 2.2] For $\kappa \in \mathbb{C}$, $0 \leq arg\kappa \leq \dfrac{\pi}{2}$ and any $\psi \in C^0(\Sigma)$ there holds:*

(i) $\mathcal{V}_\kappa(\psi)$ *is continuous in* $\mathbb{R}^3$,

(ii) $\Delta \mathcal{V}_\kappa(\psi) = -\kappa^2 \mathcal{V}_\kappa(\psi)$ *in* $\Omega_- \cup \Omega_+$,

(iii) $\mathcal{V}_\kappa(\psi)(\boldsymbol{x}) = O\left(\dfrac{e^{i\kappa|\boldsymbol{x}|}}{|\boldsymbol{x}|}\right)$ *as* $|\boldsymbol{x}| \to \infty$,

(iv)

$$\left(\frac{\partial \mathcal{V}_\kappa(\psi)}{\partial \boldsymbol{n}}(\boldsymbol{x})\right)^{\pm} = \mp\frac{1}{2}\psi(\boldsymbol{x}) + \int_\Sigma K_\kappa(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{y}) ds_y, \quad on \ \ \Sigma,$$

where $K_\kappa(\boldsymbol{x}, \boldsymbol{y}) = O(|\boldsymbol{x} - \boldsymbol{y}|^{-1})$ *as* $\boldsymbol{y} \to \boldsymbol{x}$.

(v)

$$(\boldsymbol{n} \times curl\, \mathcal{V}_\kappa(\boldsymbol{v})(\boldsymbol{x}))^{\pm} = \pm\frac{1}{2}\boldsymbol{v}(\boldsymbol{x}) + \frac{1}{2}\int_\Sigma \boldsymbol{K}_\kappa(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{v}(\boldsymbol{y}) ds_y,$$

where the matrix function $\boldsymbol{K}_\kappa$ *satisfies* $\boldsymbol{K}_\kappa(\boldsymbol{x}, \boldsymbol{y}) = O(|\boldsymbol{x} - \boldsymbol{y}|^{-1})$ *as* $\boldsymbol{y} \to \boldsymbol{x}$.

For problem $(1.1)_1$, in Ω_+ the Stratton-Chu formula gives

$$\begin{aligned} \mathbf{E} &= \mathcal{V}_\alpha(\mathbf{n} \times \mathbf{H}) - \mathrm{curl}\, \mathcal{V}_\alpha(\mathbf{n} \times \mathbf{E}) + \mathrm{grad}\, \mathcal{V}_\alpha(\mathbf{n} \cdot \mathbf{E}), \\ \mathbf{H} &= \mathrm{curl}\, \mathcal{V}_\alpha(\mathbf{n} \times \mathbf{H}) - \mathrm{curl}\,\mathrm{curl}\, \mathcal{V}_\alpha(\mathbf{n} \times \mathbf{E}). \end{aligned} \tag{1.25}$$

Now for given $\mathbf{n}\times\mathbf{H}$, $\mathbf{n}\times\mathbf{E}$ and $\mathbf{n}\cdot\mathbf{E}$ on Σ, (1.25) yields a solution of $(\mathbf{P}_{\alpha\infty})$. Unfortunately we know only $\mathbf{n} \times \mathbf{E}$. The standard treatment of $(\mathbf{P}_{\alpha\infty})$, sets $\mathbf{n} \times \mathbf{H} = 0$ and $\mathbf{n} \cdot \mathbf{E} = 0$ in (1.25) and replaces $-\mathbf{n} \times \mathbf{E}$ by an unknown tangential field $\mathbf{L}$ yielding

$$\mathbf{E} = \mathrm{curl}\, \mathcal{V}_\alpha(\mathbf{L}), \quad \mathbf{H} = \mathrm{curl}\,\mathrm{curl}\, \mathcal{V}_\alpha(\mathbf{L}). \tag{1.26}$$

Then the boundary condition in (1.2) yields an integral equation of the second kind for $\mathbf{L}$ in the tangent space to Σ.
The method (1.26) corresponds to solving the Dirichlet problem for the scalar Helmholtz equation with a double layer potential ansatz. But having found $\mathbf{L}$ it is difficult to determine $\mathbf{H}_T$, on Σ, because one must compute a hypersingular integral operator which is still a challenge for numerical simulations since for calculating $\mathbf{n} \times \mathbf{H}$ on Σ one has to compute a second normal derivative of $\mathcal{V}_\alpha(\mathbf{L})$.
The method in [31] for $(\mathbf{P}_{\alpha\infty})$ is analogous to solving the scalar problems with a simple layer potential (see [26]). MacCamy and Stephan use (1.25) in [31] but this time they set $\mathbf{n} \times \mathbf{E} = 0$ and replace $\mathbf{n} \times \mathbf{H}$ and $\mathbf{n} \cdot \mathbf{E}$ by unknowns $\mathbf{J}$ and M. Thus they take

$$\mathbf{E} = \mathcal{V}_\alpha(\mathbf{J}) + \mathrm{grad}\, \mathcal{V}_\alpha(M), \quad \mathbf{H} = \mathrm{curl}\, \mathcal{V}_\alpha(\mathbf{J}). \tag{1.27}$$

After having determined $\mathbf{J}$ then they can use Remark 4 to determine $\mathbf{n} \times \mathbf{H}$, hence $\mathbf{H}_T$ on Σ.

With the surface gradient $\mathrm{grad}_T\psi = (\mathrm{grad}\,\psi)_T$ on Σ, the boundary condition in (1.2) and (1.27) imply, by continuity of $\mathcal{V}_\alpha$,

$$\mathbf{n}\times\mathbf{E} = \mathbf{n}\times\mathcal{V}_\alpha(\mathbf{J}) + \mathbf{n}\times\mathrm{grad}\,\mathcal{V}_\alpha(M) = -\mathbf{n}\times\mathbf{E}^0$$

or equivalently

$$\mathcal{V}_\alpha(\mathbf{J})_T + \mathrm{grad}_T\,\mathcal{V}_\alpha(M) = -\mathbf{E}^0_T. \tag{1.28}$$

We note that for any field $\mathbf{v}$ defined in a neighborhood of Σ one can define the surface divergence div_T by

$$\mathrm{div}\,\mathbf{v} = \mathrm{div}_T\,\mathbf{v} + \frac{\partial v}{\partial\mathbf{n}}\mathbf{n}.$$

As shown in [31, Lemma 2.3], there holds for any differentiable tangential field $\mathbf{v}$, $\mathrm{div}\,\mathcal{V}_\kappa(\mathbf{v}) = \mathcal{V}_\kappa(\mathrm{div}_T\,\mathbf{v})$ on Σ.

As derived in [31] setting $\mathrm{div}\mathbf{E} = 0$ on Σ yields therefore with (1.27)

$$0 = \mathrm{div}\,\mathbf{E} = \mathrm{div}\,\mathcal{V}_\alpha(\mathbf{J}) + \mathrm{div}\,\mathrm{grad}\,\mathcal{V}_\alpha(M)$$

and $\mathrm{div}\,\mathrm{grad}\mathcal{V}_\alpha(M) = -\alpha^2\mathcal{V}_\alpha(M)$ gives immediately

$$\mathcal{V}_\alpha(\mathrm{div}_T\,\mathbf{J}) - \alpha^2\mathcal{V}_\alpha(M) = 0. \tag{1.29}$$

In subsection 1.4 we will investigate a boundary element method for (1.28) and (1.29).

1.3 FEM/BEM coupling for the interface problem

Next we introduce a new coupling method for the interface problem $(P_{\alpha\beta})$. Integration by parts gives in Ω_- for the second equation in $(P_{\alpha\beta})$ with $\gamma_N\mathbf{E} = (\mathrm{curl}\,\mathbf{E})\times\mathbf{n}$, $\gamma_D\mathbf{E} = \mathbf{n}\times(\mathbf{E}\times\mathbf{n})$

$$\int_{\Omega_-}\mathrm{curl}\,\mathbf{E}\cdot\mathrm{curl}\,\overline{\mathbf{v}}d\mathbf{x} - \int_{\Omega_-} i\beta^2\mathbf{E}\cdot\overline{\mathbf{v}}d\mathbf{x} - \int_\Sigma \gamma_N^-\mathbf{E}\cdot\gamma_D^-\overline{\mathbf{v}}ds = 0. \tag{1.30}$$

Therefore with $\gamma_N^-\mathbf{E} = \gamma_N^+\mathbf{E} + \gamma_N\mathbf{E}^0$ and setting $\mathbf{E} = \mathcal{V}_\alpha(\mathbf{J}) + \mathrm{grad}\,\mathcal{V}_\alpha(M)$ in Ω_+ we obtain

$$\int_{\Omega_-}\mathrm{curl}\,\mathbf{E}\cdot\mathrm{curl}\,\overline{\mathbf{v}}d\mathbf{x} - \int_{\Omega_-} i\beta^2\mathbf{E}\cdot\overline{\mathbf{v}}d\mathbf{x} - \int_\Sigma \gamma_N^+(\mathcal{V}_\alpha(\mathbf{J}) + \mathrm{grad}\,\mathcal{V}_\alpha(M))\cdot\gamma_D^+\overline{\mathbf{v}}ds = \int_\Sigma\gamma_N\mathbf{E}^0\cdot\gamma_D^+\overline{\mathbf{v}}ds.$$

Note that $\gamma_N^+(\mathcal{V}_\alpha(\mathbf{J}) + \mathrm{grad}\,\mathcal{V}_\alpha(M)) = \frac{1}{2}\mathbf{J} + \frac{1}{2}\mathbf{K}_\alpha(\mathbf{J})$ where $\mathbf{K}_\alpha$ is a smoothing operator. As shown in [31, Lemma 4.5] there exists a continuous map $J_\alpha(\mathbf{J})_T$ from $\mathbf{H}^r(\Sigma)$ into $H^{r+1}(\Sigma)$, for any real number r with

$$\mathrm{div}_T\,\mathcal{V}_\alpha(\mathbf{J})_T = \mathcal{V}_\alpha(\mathrm{div}_T\,\mathbf{J}) + J_\alpha(\mathbf{J})_T. \tag{1.31}$$

As shown in [30] the following system of boundary operators on Σ (which is equivalent to (1.28) and (1.29))

$$\begin{aligned} \mathcal{V}_\alpha(\mathbf{J})_T + \mathrm{grad}_T\, \mathcal{V}_\alpha(M) \qquad &= -\mathbf{E}^0_T \\ -J_\alpha(\mathbf{J})_T - (\Delta_T + \alpha^2)\mathcal{V}_\alpha(M) \;\; &= \mathrm{div}_T\, \mathbf{E}^0_T. \end{aligned} \tag{1.32}$$

is strongly elliptic as a mapping from $\mathbf{H}^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$ into $\mathbf{H}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma)$, where $\mathrm{grad}_T(\mathrm{div}_T)$ denote the surface gradient (surface divergence) and Δ_T the Laplace-Beltrami operator on Σ.

Now, our fem/bem coupling method is based on the variational formulation: For given incident field $\mathbf{E}^0$ on Σ find $\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega_-)$, $\mathbf{J} \in \mathbf{H}^{-\frac{1}{2}}(\Sigma)$ and $M \in H^{\frac{1}{2}}(\Sigma)$ with

$$\int_{\Omega_-} \mathrm{curl}\, \mathbf{E} \cdot \mathrm{curl}\, \overline{\mathbf{v}} d\mathbf{x} - \int_{\Omega_-} i\beta^2 \mathbf{E} \cdot \overline{\mathbf{v}} d\mathbf{x} - \frac{1}{2}\int_\Sigma (\mathbf{J} + \mathbf{K}_\alpha(\mathbf{J})) \cdot \gamma_D^+ \overline{\mathbf{v}} ds = \int_\Sigma \gamma_N \mathbf{E}^0 \cdot \gamma_D^+ \overline{\mathbf{v}} ds,$$

$$\int_\Sigma \mathcal{V}_\alpha(\mathbf{J})_T \cdot \mathbf{j}\, dS + \int_\Sigma \mathrm{grad}_T \mathcal{V}_\alpha(M) \cdot \mathbf{j}\, dS = -\int_\Sigma \mathbf{E}^0_T \cdot \mathbf{j}\, dS,$$

$$-\int_\Sigma J_\alpha(\mathbf{J})_T m\, dS - \int_\Sigma (\Delta_T + \alpha^2)\mathcal{V}_\alpha(M) m\, dS = \int_\Sigma \left(\mathrm{div}_T\, \mathbf{E}^0_T\right) m dS, \tag{1.33}$$

$\forall \mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega_-)$, $\mathbf{j} \in \mathbf{H}^{-\frac{1}{2}}(\Sigma)$, $m \in H^{\frac{1}{2}}(\Sigma)$.

In order to formulate a conforming Galerkin scheme for (1.33) we take subspaces $\mathbf{H}^1_h \subset \mathbf{H}(\mathrm{curl}, \Omega_-)$, $\mathbf{H}_h^{-\frac{1}{2}} \subset \mathbf{H}^{-\frac{1}{2}}(\Sigma)$, $H_h^{\frac{1}{2}} \subset H^{\frac{1}{2}}(\Sigma)$ with mesh parameter h and look for $\mathbf{E}_h \in \mathbf{H}^1_h$, $\mathbf{J}_h \in \mathbf{H}_h^{-\frac{1}{2}}$, $M_h \in H_h^{\frac{1}{2}}$ such that for any $\mathbf{v}_h \in \mathbf{H}^1_h$, $\mathbf{j}_h \in \mathbf{H}_h^{-\frac{1}{2}}$, $m_h \in H_h^{\frac{1}{2}}$

$$\langle \mathcal{A}(\mathbf{E}_h, \mathbf{J}_h, M_h), (\mathbf{v}_h, \mathbf{j}_h, m_h)\rangle = \langle \mathcal{F}, (\mathbf{v}_h, \mathbf{j}_h, m_h)\rangle \tag{1.34}$$

where $\mathcal{A}$ is the operator given by the left hand side in (1.33), $\mathcal{F} = (\gamma_N \mathbf{E}^0, -\mathbf{E}^0_T, \mathrm{div}_T\, \mathbf{E}^0_T)$.

Theorem 2. *1. System (1.33) has a unique solution $(\boldsymbol{E}, \boldsymbol{J}, M)$ in $\boldsymbol{X} = \boldsymbol{H}(curl, \Omega_-) \times \boldsymbol{H}^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$.*

2. The Galerkin system (1.34) is uniquely solvable in $\boldsymbol{X}_h = \boldsymbol{H}^1_h \times \boldsymbol{H}_h^{-\frac{1}{2}} \times H_h^{\frac{1}{2}}$ and there exists $C > 0$, independent of h,

$$\begin{aligned} &\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(curl,\Omega_-)} + \|\boldsymbol{J} - \boldsymbol{J}_h\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Sigma)} + \|M - M_h\|_{H^{\frac{1}{2}}(\Sigma)} \\ &\le C \inf_{(\boldsymbol{v},\boldsymbol{j},m)\in \boldsymbol{X}_h} \left\{ \|\boldsymbol{E} - \boldsymbol{v}\|_{\boldsymbol{H}(curl,\Omega_-)} + \|\boldsymbol{J} - \boldsymbol{j}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Sigma)} + \|M - m\|_{H^{\frac{1}{2}}(\Sigma)} \right\} \end{aligned} \tag{1.35}$$

where $(\boldsymbol{E}, \boldsymbol{J}, M)$ and $(\boldsymbol{E}_h, \boldsymbol{J}_h, M_h)$ solve (1.33) and (1.34) respectively.

Proof. First we note that system (1.33) is strongly elliptic in $\mathbf{X}$ which follows by considering $\mathcal{A}$ as a system of pseudodifferential operators (cf. [30]). The only difference to [30] is that here we have additionally the first equation in (1.33). If we note $\Delta\mathbf{E} = \text{curl curl}\,\mathbf{E} - \text{grad div}\,\mathbf{E}$ and take $\text{div}\,\mathbf{E} = 0$ we have that the principal symbol of $\mathcal{A}$ has the form (with $|\xi|^2 = \xi_1^2 + \xi_2^2$)

$$\sigma(\mathcal{A})(\xi)(\mathbf{E}, \mathbf{J}, M)^t = \begin{pmatrix} |\xi|^2 + \xi_3^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & |\xi|^2 + \xi_3^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & |\xi|^2 + \xi_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{|\xi|} & 0 & i\xi_1 \frac{1}{|\xi|} \\ 0 & 0 & 0 & 0 & \frac{1}{|\xi|} & i\xi_2 \frac{1}{|\xi|} \\ 0 & 0 & 0 & 0 & 0 & |\xi| \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ J^1 \\ J^2 \\ M \end{pmatrix} \tag{1.36}$$

where $(E_1, E_2) = \mathbf{E}_T$ and E_3 is perpendicular to $x_3 = 0$. Here $\xi = (\xi_1, \xi_2)$ is the dual variable in the Fourier transform to (x_1, x_2) and ξ_3 is the dual variable to x_3.
Obviously the upper left and the lower right sub blocks are strongly elliptic (see [30] for the lower sub block). Assuming that $(\alpha, \sqrt{i}\beta)$ is not an eigenvalue of $(\mathbf{P}_{\alpha\beta})$ we have existence and uniqueness of the exact solution. Due to the strong ellipticity of $\mathcal{A}$ there exists a unique Galerkin solution and the a priori error estimate holds due to the abstract results by Stephan and Wendland [43]. $\square$

1.4 Galerkin procedure for the perfect conductor problem $(\mathbf{P}_{\alpha\infty})$

Next we consider the implementation of the Galerkin boundary element methods and present corresponding numerical experiments for the integral equations (1.28) and (1.29). These experiments are performed with the program package *Maiprogs*, cf. Maischak [34, 36], which is a Fortran-based program package used for finite element and boundary element simulations [35]. Initially developed by M. Maischak, *Maiprogs* has been extended for electromagnetic problems by Teltscher [45] and Leydecker [28].
We will investigate the exterior problem $(P_{\alpha\infty})$ by solving numerically the integral equations (1.28) and (1.29) with Galerkin's methods:
Testing with arbitrary functions $\mathbf{j} \in \mathbf{H}^{-\frac{1}{2}}(\Sigma)$ and $m \in H^{\frac{1}{2}}(\Sigma)$ in (1.28) and (1.29), we get

$$\begin{aligned} &\int_\Sigma \mathcal{V}_\alpha(\mathbf{J})_T \cdot \mathbf{j}\, dS + \int_\Sigma \text{grad}_T \mathcal{V}_\alpha(M) \cdot \mathbf{j}\, dS = -\int_\Sigma \mathbf{E}_T^0 \cdot \mathbf{j}\, dS, \\ -&\int_\Sigma \mathcal{V}_\alpha(\text{div}_T \mathbf{J}) \cdot m\, dS + \alpha^2 \int_\Sigma \mathcal{V}_\alpha(M) \cdot m\, dS = 0. \end{aligned} \tag{1.37}$$

Partial integration in the second term of $(1.37)_1$

$$\int_\Sigma \mathrm{grad}_T \mathcal{V}_\alpha(M) \cdot \mathbf{j}\, dS = -\int_\Sigma \mathcal{V}_\alpha(M) \cdot \mathrm{div}_T \mathbf{j}\, dS$$

shows that the formulation (1.37) is symmetric: By definition of symmetric bilinear forms a, c, of the bilinear form b and linear form ℓ through

$$\begin{aligned} a(\mathbf{J},\mathbf{j}) :&= \int_\Sigma \mathcal{V}_\alpha(\mathbf{J})_T \cdot \mathbf{j}\, dS, \quad c(M,m) = \alpha^2 \int_\Sigma \mathcal{V}_\alpha(M) \cdot m\, dS, \quad \ell(\mathbf{j}) = -\int_\Sigma \mathbf{E}_T^0 \cdot \mathbf{j}\, dS \\ b(\mathbf{J},m) :&= -\int_\Sigma \mathcal{V}_\alpha(\mathrm{div}_T \mathbf{J}) \cdot m\, dS = -\int_\Sigma \mathcal{V}_\alpha(m) \cdot \mathrm{div}_T \mathbf{J}\, dS. \end{aligned}$$

the variational formulation has the form: Find $(\mathbf{J}, M) \in \mathbf{H}^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$ such that

$$\begin{aligned} a(\mathbf{J},\mathbf{j}) + b(\mathbf{j},M) &= \ell(\mathbf{j}) \\ b(\mathbf{J},m) + c(M,m) &= 0 \end{aligned} \tag{1.38}$$

for all $(\mathbf{j}, m) \in \mathbf{H}^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$.
We now proceed to finite dimensional subspaces $\mathcal{R}_h \subset \mathbf{H}^{-\frac{1}{2}}(\Sigma)$ of dimension n and $\mathcal{M}_h \subset H^{\frac{1}{2}}(\Sigma)$ of dimension m, and seek approximations $\mathbf{J}_h \in \mathcal{R}_h$ and $M_h \in \mathcal{M}_h$ for $\mathbf{J}$ and M, such that

$$\begin{aligned} a(\mathbf{J}_h,\mathbf{j}) + b(\mathbf{j},M_h) &= \ell(\mathbf{j}), \\ b(\mathbf{J}_h,m) + c(M_h,m) &= 0 \end{aligned} \tag{1.39}$$

for all $\mathbf{j} \in \mathcal{R}_h$ and $m \in \mathcal{M}_h$.
Let $\{\boldsymbol{\psi}_i\}_{i=1}^n$ be a basis of $\mathcal{R}_h$ and $\{\varphi_j\}_{j=1}^m$ be a basis of $\mathcal{M}_h$. $\mathbf{J}_h$ and M_h are of the forms

$$\mathbf{J}_h := \sum_{i=1}^n \lambda_i \boldsymbol{\psi}_i \ \text{ and } \ M_h := \sum_{j=1}^m \mu_j \varphi_j. \tag{1.40}$$

Inserting (1.40) in (1.39) provides

$$\begin{aligned} \sum_{i=1}^n \lambda_i a(\boldsymbol{\psi}_i, \boldsymbol{\psi}_k) + \sum_{j=1}^m \mu_j b(\boldsymbol{\psi}_k, \varphi_j) &= \ell(\boldsymbol{\psi}_k) \\ \sum_{i=1}^n \lambda_i b(\boldsymbol{\psi}_i, \varphi_l) + \sum_{j=1}^m \mu_j c(\varphi_j, \varphi_l) &= 0 \end{aligned} \tag{1.41}$$

for all $\boldsymbol{\psi}_k$ and φ_l, $1 \leq k \leq n$, $1 \leq l \leq m$.
With matrices and vectors

$$\begin{aligned}
A &:= (a(\boldsymbol{\psi}_i, \boldsymbol{\psi}_k))_{i,k} \in \mathbb{C}^{n\times n},\\
B &:= (b(\boldsymbol{\psi}_i, \varphi_l))_{i,l} \in \mathbb{C}^{n\times m},\\
C &:= (c(\varphi_j, \varphi_l))_{j,l} \in \mathbb{C}^{m\times m},\\
\boldsymbol{\lambda} &:= (\lambda_i)_i \in \mathbb{C}^{n},\\
\boldsymbol{\mu} &:= (\mu_j)_j \in \mathbb{C}^{m},\\
\boldsymbol{\ell} &:= (\ell(\boldsymbol{\psi}_k))_k \in \mathbb{C}^{n}.
\end{aligned} \tag{1.42}$$

(1.41) has also the form

$$\begin{pmatrix} A & B^t \\ B & C \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\ell} \\ 0 \end{pmatrix}. \tag{1.43}$$

We have considered with $\{\boldsymbol{\psi}_i\}_{i=1}^{n}$ a basis of $\mathcal{R}_h$ and $\{\varphi_j\}_{j=1}^{m}$ a basis of $\mathcal{M}_h$. These functions, are chosen as piecewise polynomials. To win these bases, we consider suitable basis functions locally on the element of a grid, i.e. on each component grid.
If we start from a grid

$$\{\Sigma_k\}_{k=1}^{N} \quad \text{with} \quad \bigcup_{1\leq k\leq N} \Sigma_k = \Sigma$$

with N elements, and let $\{\widehat{\boldsymbol{\psi}}_i\}_{i=1}^{\widehat{n}}$ and $\{\widehat{\varphi}_j\}_{j=1}^{\widehat{m}}$ respectively bases on a square reference element $\widehat{\Sigma}$. The local basis functions on an element Σ_k are each $\{\boldsymbol{\psi}_i\}_{i=1}^{n_k}$ or $\{\varphi_j\}_{j=1}^{m_k}$.
First we calculate

$$A := (a(\boldsymbol{\psi}_{j_s}, \boldsymbol{\psi}_{i_z}))_{i_z,j_s} \in \mathbb{C}^{n\times n},$$

where $\boldsymbol{\psi}_{j_s}$ or $\boldsymbol{\psi}_{i_z}$ are the basics function of $\mathcal{R}_h$ and

$$a(\boldsymbol{\psi}_{j_s}, \boldsymbol{\psi}_{i_z}) = \int_{\Sigma} \mathcal{V}_\alpha(\boldsymbol{\psi}_{j_s})_T \cdot \boldsymbol{\psi}_{i_z}\, dS = \sum_{k=1}^{N} \int_{\Sigma_k} \mathcal{V}_\alpha(\boldsymbol{\psi}_{j_s})_T \cdot \boldsymbol{\psi}_{i_z}\, dS,$$

We test each local basis function against any other local basis function and sum the result to the test value of the global basis functions, which include these local basis functions.
Let $I_N = \{1, \dots, N\}$ the index set for the grid elements, $I_{\widehat{n}} = \{1, \dots, \widehat{n}\}$ the index set for the basic functions on the reference element and $I_n = \{1, \dots, n\}$ the index set for the global basis functions.
Let $\boldsymbol{\zeta} : I_N \times I_{\widehat{n}} \to I_n$ the mapping from local to global basis functions such that $\boldsymbol{\zeta}(k, i) = j$, if the local basis function $\boldsymbol{\psi}_{k,i}$ component of the global basis function is $\boldsymbol{\psi}_j$.
Let $\boldsymbol{\zeta}^{-1}$ the set of all pairs of (k, j) with $\boldsymbol{\zeta}(k, j) = i$, then

$$\int_{\Sigma} \mathcal{V}_\alpha(\boldsymbol{\psi}_{j_s})_T \cdot \boldsymbol{\psi}_{i_z}\, dS = \sum_{\substack{(k,i)\in\\ \boldsymbol{\zeta}^{-1}(i_z)}} \sum_{\substack{(l,j)\in\\ \boldsymbol{\zeta}^{-1}(j_s)}} \int_{\Sigma_k} \mathcal{V}_\alpha(\boldsymbol{\psi}_{l,j})_T \cdot \boldsymbol{\psi}_{k,i}\, dS$$

$$= \sum_{\substack{(k,i)\in\\ \boldsymbol{\zeta}^{-1}(i_z)}} \sum_{\substack{(l,j)\in\\ \boldsymbol{\zeta}^{-1}(j_s)}} \int_{\Sigma_k}\int_{\Sigma_l} G_\alpha(|\mathbf{x}-\mathbf{y}|)(\boldsymbol{\psi}_{l,j}(\mathbf{y}))^t \cdot \boldsymbol{\psi}_{k,i}(\mathbf{x})\, dS_{\mathbf{y}}\, dS_{\mathbf{x}}.$$

We are dealing in this implementation with Raviart-Thomas basis functions. The transformation of these functions requires a Peano transformation $\boldsymbol{\psi}_{k,i} = \dfrac{1}{|\det A_k|} A_k \widehat{\boldsymbol{\psi}}_i$. Thus, if $A_k = (\mathbf{a}_1, \mathbf{a}_2)$, $\det A_k$ is calculated by $\det A_k = (\mathbf{a}_1 \times \mathbf{a}_2)\cdot \dfrac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$. The Peano-transformation of the local basis functions to the basic functions on the reference element then gives

$$\begin{aligned} I &= \sum_{\substack{(k,i)\in\\ \boldsymbol{\zeta}^{-1}(i_z)}} \sum_{\substack{(l,j)\in\\ \boldsymbol{\zeta}^{-1}(j_s)}} \int_{\Sigma_k}\int_{\Sigma_l} G_\alpha(|\mathbf{x}-\mathbf{y}|)(\boldsymbol{\psi}_{l,j}(\mathbf{y}))^t \cdot \boldsymbol{\psi}_{k,i}(\mathbf{x})\, dS_{\mathbf{y}}\, dS_{\mathbf{x}} \\ &= \sum_{\substack{(k,i)\in\\ \boldsymbol{\zeta}^{-1}(i_z)}} \sum_{\substack{(l,j)\in\\ \boldsymbol{\zeta}^{-1}(j_s)}} \int_{\widehat{\Sigma}}\int_{\widehat{\Sigma}} \frac{G_\alpha(|\mathbf{x}-\mathbf{y}|)}{|\det A_k \cdot \det A_l|} (\widehat{\boldsymbol{\psi}}_i(\widehat{\mathbf{x}}))^t (A_k)^t \cdot A_l \widehat{\boldsymbol{\psi}}_j(\widehat{\mathbf{y}})\, dS_{\widehat{\mathbf{y}}}\, dS_{\widehat{\mathbf{x}}} \end{aligned} \tag{1.44}$$

with $\mathbf{x} = \mathbf{a}_k + A_k\widehat{\mathbf{x}}$ and $\mathbf{y} = \mathbf{a}_l + A_l\widehat{\mathbf{y}}$, and referent element $\widehat{\Sigma}$.
The calculation of the integrals with Helmholtz kernel G_α is not exact. We consider the expansion of the Helmholtz kernel in a Taylor series. There holds

$$G_\alpha(|\mathbf{x}-\mathbf{y}|) = \frac{1}{4\pi}\frac{e^{\alpha i|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}\left[1 + \alpha i|\mathbf{x}-\mathbf{y}| + \frac{(\alpha i)^2}{2}|\mathbf{x}-\mathbf{y}|^2 + \ldots\right].$$

The first terms are singular for $\mathbf{x} = \mathbf{y}$ and the corresponding integral is treated by analytic evaluation in *Maiprogs*, cf. [33, 34, 36], but the integrals of all other summands can be calculated sufficiently well by Gaussian quadrature.
We compute

$$\begin{aligned} b(\boldsymbol{\psi}_{i_z}, \varphi_{j_s}) &= -\int_\Sigma \mathcal{V}_\alpha(\nabla_T \cdot \boldsymbol{\psi}_{i_z}) \cdot \varphi_{j_s}\, dS \\ &= -\sum_{\substack{(k,i)\in\\ \boldsymbol{\zeta}_\psi^{-1}(i_z)}} \sum_{\substack{(l,j)\in\\ \boldsymbol{\zeta}_\varphi^{-1}(j_s)}} \int_{\Sigma_l} \mathcal{V}_\alpha(\nabla \cdot \boldsymbol{\psi}_{k,i})_T \cdot \varphi_{l,j}\, dS \\ &= -\sum_{\substack{(k,i)\in\\ \boldsymbol{\zeta}_\psi^{-1}(i_z)}} \sum_{\substack{(l,j)\in\\ \boldsymbol{\zeta}_\varphi^{-1}(j_s)}} \int_{\Sigma_l}\int_{\Sigma_k} G_\alpha(|\mathbf{x}-\mathbf{y}|)\nabla_T \cdot \boldsymbol{\psi}_{k,i}(\mathbf{y}) \cdot \varphi_{l,j}(\mathbf{x})\, dS_{\mathbf{y}}\, \mathrm{d}S_{\mathbf{x}}. \end{aligned} \tag{1.45}$$

with $\boldsymbol{\zeta}_\psi^{-1} = \boldsymbol{\zeta}$ described above, and $\boldsymbol{\zeta}_\varphi^{-1}$, the analogously defined map for the basic functions of $\mathcal{M}_h$.
While a transformation of the scalar basis functions is not required, the transformation of the surface divergence of Raviart-Thomas elements is carried out by $\nabla_T \cdot \boldsymbol{\psi}_{k,i} = \frac{1}{|\det A_k|}\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}}_i$ and

we have

$$b(\boldsymbol{\psi}_{i_z}, \varphi_{j_s}) = - \sum_{\substack{(k,i)\in \\ \boldsymbol{\zeta}_{\psi}^{-1}(i_z)}} \sum_{\substack{(l,j)\in \\ \boldsymbol{\zeta}_{\varphi}^{-1}(j_s)}} \int_{\widehat{\Sigma}} \int_{\widehat{\Sigma}} \frac{G_\alpha(|\mathbf{x}-\mathbf{y}|)}{|\det A_k|} \widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}}_{k,i}(\widehat{\mathbf{y}}) \cdot \widehat{\varphi}_{l,j}(\widehat{\mathbf{x}}) \, dS_{\widehat{\mathbf{y}}} \, \mathrm{d}S_{\widehat{\mathbf{x}}} \tag{1.46}$$

with $\mathbf{y} = \mathbf{a}_k + A_k\widehat{\mathbf{y}}$ and $\mathbf{x} = \mathbf{a}_l + A_l\widehat{\mathbf{x}}$.
The calculation of $c(\varphi_i, \varphi_j)$ is similar to the above-mentioned case. Thus, for a possible p-version bem one would proceed with

$$c(\varphi_{i_z}, \varphi_{j_s}) = \sum_{\substack{(k,i)\in \\ \boldsymbol{\zeta}_{\varphi}^{-1}(i_z)}} \sum_{\substack{(l,j)\in \\ \boldsymbol{\zeta}_{\varphi}^{-1}(j_s)}} \sum_{\kappa=0}^{p}\sum_{\lambda=0}^{p}\sum_{\mu=0}^{p}\sum_{\nu=0}^{p} c_{\mathbf{v}} \mathcal{V}_\alpha^{l,k}(\kappa,\lambda,\mu,\nu) \tag{1.47}$$

with $c_{\mathbf{v}} := \alpha^2 k_{i\kappa\lambda} k_{j\mu\nu}$ to $\mathbf{v} = (i, j, \kappa, \lambda, \mu, \nu)$. For piecewise constant φ, it follows

$$c(\varphi_{i_z}, \varphi_{j_s}) = \alpha^2 \mathcal{V}_\alpha^{i_z,j_s}(0,0,0,0), \tag{1.48}$$

where

$$\mathcal{V}_\alpha^{l,k}(\kappa,\lambda,\mu,\nu) := \int_{\widehat{\Sigma}} \int_{\widehat{\Sigma}} G_\alpha(|\mathbf{x}-\mathbf{y}|) \widehat{y}_1^{\kappa} \widehat{y}_2^{\lambda} \widehat{x}_1^{\mu} \widehat{x}_2^{\nu} \, dS_{\widehat{\mathbf{y}}} \, dS_{\widehat{\mathbf{x}}}. \tag{1.49}$$

The calculation of the right-hand side in (1.38) looks simple, since there are no single layer potential terms. However the right hand side must be computed with quadrature.
The quadrature of an integral over $\mathbf{f}$ on the reference element is determined by the quadrature points $\widehat{\mathbf{x}}_{x,y}$, and the associated weights $w_{x,y} = w_x \cdot w_y$. We perform the two-dimensional quadrature as a combination of one-dimensional quadratures in each x and y direction, and use the weights from the one-dimensional quadrature formula. With $\widetilde{n}_x$ quadrature points in x-direction, and $\widetilde{n}_y$ quadrature points in y-direction, then the quadrature formula reads:

$$\mathcal{Q}_{\widehat{\Sigma}}(\mathbf{f}) = \sum_{i=1}^{\widetilde{n}_x} \sum_{j=1}^{\widetilde{n}_y} \mathbf{f}(\widehat{\mathbf{x}}_{i,j}) \cdot w_i w_j. \tag{1.50}$$

The quadrature points on the square reference element and the corresponding weights for Gaussian quadrature are already implemented in *Maiprogs*. For triangular elements, we use Duffy transformation.
Now we comment on the calculation of the right hand side in the Galerkin formulation, i.e. the linear form ℓ, applied to the bases functions $\boldsymbol{\psi}_i$, $i = 1, \ldots, n$. The quadrature takes place on the reference element. We decompose the global into local basis functions and then use the

Peano-transformation for the Raviart-Thomas functions. It is therefore

$$\begin{aligned}\ell(\boldsymbol{\psi}_{i_r}) &= -\int_{\Sigma} (\mathbf{E}_T^0(\mathbf{x}))^t \cdot \boldsymbol{\psi}_{i_r}(\mathbf{x})\, dS_{\mathbf{x}} \\ &= -\sum_{\substack{(k,i)\in \\ \zeta^{-1}(i_r)}} \int_{\Sigma_k} (\mathbf{E}_T^0(\mathbf{x}))^t \cdot \boldsymbol{\psi}_{k,i}(\mathbf{x})\, dS_{\mathbf{x}} \\ &= -\sum_{\substack{(k,i)\in \\ \zeta^{-1}(i_r)}} \int_{\Sigma_k} (\mathbf{E}_T^0(\mathbf{x}))^t \cdot \frac{A_k}{|\det A_k|} \boldsymbol{\psi}_{k,i}(\widehat{\mathbf{x}})\, dS_{\widehat{\mathbf{x}}} \\ &= -\sum_{\substack{(k,i)\in \\ \zeta^{-1}(i_r)}} \int_{\widehat{\Sigma}} (\mathbf{E}_T^0(\mathbf{x}))^t \cdot \frac{A_k}{|\det A_k|} \widehat{\boldsymbol{\psi}}_{k,i}(\widehat{\mathbf{x}}) |\det A_k|\, dS_{\widehat{\mathbf{x}}}\end{aligned}$$

then

$$\ell(\boldsymbol{\psi}_{i_r}) = -\sum_{\substack{(k,i)\in \\ \zeta^{-1}(i_r)}} \int_{\widehat{\Sigma}} (\mathbf{E}_T^0(\mathbf{x}))^t \cdot A_k \cdot \widehat{\boldsymbol{\psi}}_{k,i}(\widehat{\mathbf{x}})\, dS_{\widehat{\mathbf{x}}} \tag{1.51}$$

with $\mathbf{x} = \mathbf{a}_k + A_k\widehat{\mathbf{x}}$. Applying (1.50) leads with $\widetilde{n}_x = \widetilde{n}_y := \widetilde{n}$ to

$$\mathcal{Q}(\ell(\boldsymbol{\psi}_i)) = -\sum_{\substack{(k,i)\in \\ \zeta^{-1}(i_r)}} \sum_{i_1=1}^{\widetilde{n}} \sum_{i_2=1}^{\widetilde{n}} (\mathbf{E}_T^0(\mathbf{x}_{i_1,i_2}))^t \cdot A_k \cdot \widehat{\boldsymbol{\psi}}_{k,i}(\widehat{\mathbf{x}}_{i_1,i_2}) \cdot w_{i_1} w_{i_2} \tag{1.52}$$

with $\mathbf{x}_{i,j} = \mathbf{a}_k + A_k\widehat{\mathbf{x}}_{i,j}$. As before, the task is carried out by looping through all grid components, and the values are added to the entries for each of its base function.
The electrical field can be calculated by post-processing.
The subroutine

```
subroutine electricfield(x,nx,sp1,sp2,ckom1,ckom2,electric)
```

compute the electrical field

$$\mathbf{E}_h = \mathcal{V}_\alpha(\mathbf{J}_h) + \text{grad}\ \mathcal{V}_\alpha(M_h) \tag{1.53}$$

with the help of subroutines

```
subroutine vpsi232(x,nx,sp1,ckom1,vp)
```

and

```
subroutine grdvpsi2(x,nx,sp2,ckom2,grvp)
```

Where *vpsi232* calculates the first term on the right side of equation (1.53) and *grdvpsi2* the second term respectively.
We proceed as follows:
We have for the first term in (1.53) with $(1.40)_1$

$$\mathcal{V}_\alpha(\mathbf{J}_h)(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \int_\Sigma G_\alpha(|\mathbf{x}-\mathbf{y}|)\boldsymbol{\psi}_i(\mathbf{y})dS_{\mathbf{y}}. \tag{1.54}$$

Then using Peano-transformation we have

$$\begin{aligned}\mathcal{V}_\alpha(\boldsymbol{\psi}_{i_s})(\mathbf{x}) &= \int_\Sigma G_\alpha(|\mathbf{x}-\mathbf{y}|)\boldsymbol{\psi}_{i_s}(\mathbf{y})dS_{\mathbf{y}} \\ &= \sum_{\substack{(l,i)\in \\ \boldsymbol{\zeta}^{-1}(i_s)}} \int_{\Sigma_l} G_\alpha(|\mathbf{x}-\mathbf{y}|)\boldsymbol{\psi}_{l,i}(\mathbf{y})\, dS_{\mathbf{y}} \\ &= \sum_{\substack{(l,i)\in \\ \boldsymbol{\zeta}^{-1}(i_s)}} \int_{\widehat{\Sigma}} \frac{G_\alpha(|\mathbf{x}-\mathbf{y}|)}{|\det A_l|} A_l \widehat{\boldsymbol{\psi}}_i(\widehat{\mathbf{y}})\, dS_{\widehat{\mathbf{y}}}\end{aligned} \tag{1.55}$$

with $\mathbf{y} = \mathbf{a}_k + A_k\widehat{\mathbf{y}}$.
For the second term in (1.53) we have with

$$\operatorname{grad} \mathcal{V}_\alpha(\varphi_{j_z})(\mathbf{x}) = \sum_{\substack{(l,j)\in \\ \boldsymbol{\zeta}^{-1}(j_z)}} \int_{\widehat{\Sigma}} \operatorname{grad}_{\mathbf{x}} G_\alpha(|\mathbf{x}-\mathbf{y}|)\widehat{\varphi}_j(\widehat{\mathbf{y}})\, dS_{\widehat{\mathbf{y}}} \tag{1.56}$$

The calculation of $\mathbf{H}_T^{\pm}$ is done as follows (compare Remark 4)

$$\mathbf{H}_T^{\pm} = [\mathbf{n} \times \operatorname{curl} \mathcal{V}_\alpha(\mathbf{J})]^{\pm} = \pm\frac{1}{2}\mathbf{J}(\mathbf{x}) + \frac{1}{2}\mathbf{n}(\mathbf{x}) \times \int_\Sigma \operatorname{grad}_{\mathbf{x}} G_\alpha(|\mathbf{x}-\mathbf{y}|) \times \mathbf{J}(\mathbf{y})dS_{\mathbf{y}}. \tag{1.57}$$

1.5 Numerical experiments

Example 1. *As domain we take the cube $\Omega_- = [-2,2]^3$, and we now want to test the Galerkin method in (1.39). We choose the wave numbers $\alpha = 0.1, 0.5, 1.5$ and the exact solution*

$$\boldsymbol{J} = \frac{1}{8}\boldsymbol{n} \times \begin{pmatrix} (1-x_1)(1-x_2) \\ 0 \\ 0 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 \\ (1-x_1)(1-x_2)\cdot n_3 \\ -(1-x_1)(1-x_2)\cdot n_2 \end{pmatrix} \tag{1.58}$$

and

$$M = \frac{1}{8\alpha^2}\boldsymbol{n} \cdot \begin{pmatrix} 0 \\ 0 \\ (x_1-1) \end{pmatrix} = \frac{1}{8\alpha^2}(x_1-1)\cdot n_3 \tag{1.59}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ *denotes the outer normal vector at a point on the surface* $\Sigma = \cup_{k=1}^{6} \Sigma_k$. *We can write each term of equation (1.28) as:*

$$\mathcal{V}_\alpha(\boldsymbol{J})_T(\boldsymbol{x}) = \sum_{k=1}^{6} \int_{\Sigma_k} G_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{J}_k(\boldsymbol{y}))^t \, dS_{\boldsymbol{y}}, \tag{1.60}$$

and

$$grad_T \mathcal{V}_\alpha(M)_T(\boldsymbol{x}) = \sum_{k=1}^{6} grad_T \int_{\Sigma_k} G_\alpha(|\boldsymbol{x}-\boldsymbol{y}|) M_k(\boldsymbol{y}) \, dS_{\boldsymbol{y}}. \tag{1.61}$$

Then, from (1.28), (1.60) and (1.61)there holds

$$\boldsymbol{E}_T = \sum_{k=1}^{6} \left(\int_{\Sigma_k} G_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{J}_k(\boldsymbol{y}))^t \, dS_{\boldsymbol{y}} + grad_T \int_{\Sigma_k} G_\alpha(|\boldsymbol{x}-\boldsymbol{y}|) M_k(\boldsymbol{y}) \, dS_{\boldsymbol{y}} \right). \tag{1.62}$$

This last expression is calculated with the following subroutines:

```
subroutine osilv2(x,nx,osilvx2)
```

and

```
subroutine osilgrdv2(x,nx,osilgrx2)
```

which compute (1.60) and (1.61) respectively. With the help of the subroutines

```
subroutine helmvpotll(x,nx,by,dy1,dy2,ny,ty,py,vkl)
```

and

```
subroutine helmgrdvpotll(x,nx,by,dy1,dy2,ny,ty,py,grvkl)
```

Now, we write the fields $\boldsymbol{J}$ *and* M *in each component of* $\Sigma = \cup_{k=1}^{6} \Sigma_k$:

- $\Sigma_1 = \{(x_1, x_2, 2) \mid -2 \le x_1, x_2 \le 2\}$, $\boldsymbol{n}_1 = (0,0,1)^t$,

  $\boldsymbol{J}_1 = \frac{1}{8}(0, (1-x_1)(1-x_2), 0)^t$ *and* $M_1 = \frac{1}{8\alpha^2}(x_1 - 1)$.
- $\Sigma_2 = \{(x_1, x_2, -2) \mid -2 \le x_1, x_2 \le 2\}$, $\boldsymbol{n}_2 = (0,0,-1)^t$,

  $\boldsymbol{J}_2 = \frac{1}{8}(0, -(1-x_1)(1-x_2), 0)^t$ *and* $M_2 = -\frac{1}{8\alpha^2}(x_1 - 1)$.
- $\Sigma_3 = \{(x_1, 2, x_3) \mid -2 \le x_1, x_3 \le 2\}$, $\boldsymbol{n}_3 = (0,1,0)^t$,

  $\boldsymbol{J}_3 = \frac{1}{8}(0, 0, 1-x_1)^t$ *and* $M_3 = 0$.

- $\Sigma_4 = \{(x_1, -2, x_3) \mid -2 \leq x_1, x_3 \leq 2\}$, $\boldsymbol{n}_4 = (0, -1, 0)^t$,

  $\boldsymbol{J}_4 = \dfrac{1}{8}(0, 0, 3 \cdot (1 - x_1))^t$ *and* $M_4 = 0$.
- $\Sigma_5 = \{(2, x_2, x_3) \mid -2 \leq x_2, x_3 \leq 2\}$, $\boldsymbol{n}_5 = (1, 0, 0)^t$,

  $\boldsymbol{J}_5 = (0, 0, 0)^t$ *and* $M_5 = 0$.
- $\Sigma_6 = \{(-2, x_2, x_3) \mid -2 \leq x_2, x_3 \leq 2\}$, $\boldsymbol{n}_6 = (-1, 0, 0)^t$,

  $\boldsymbol{J}_6 = (0, 0, 0)^t$ *and* $M_6 = 0$.

We use different values of α for our investigation. In Table 1.1 we present the results of the errors in energy norm for $\alpha = 0.1, 0.5, 1.5$ for the uniform h-version with polynomial degree $p = 1$. In Figure 1.2 we compare the h-version with different α. The exact norm is known by extrapolation for $\alpha = 0.1$ is $|C| = 8.580798$, for $\alpha = 0.5$ is $|C| = 1.6171534$, and for $\alpha = 1.5$ is $|C| = 1.8042380$. Here $C = Re\langle \mathbf{E}_T^0, \mathbf{J}\rangle$, $C_h = Re\langle \mathbf{E}_T^0, \mathbf{J}_h\rangle$ (see Holm and al. [25]).
In Table 1.1 we present the results of the errors in $L2$-norm for $\alpha = 0.1, 0.5, 1.5$ for the uniform h-version with polynomial degree $p = 1$. In Figures 1.1 and 1.2 we compare the h-version with $\alpha = 0.1, 0.5, 1.5$ respectively. The exact L2-norm is known by extrapolation for $\alpha = 0.1$ are $\|\mathbf{J}\|_{L^2} = 2.1066356$ and $\|M\|_{L^2} = 81.9249906$, for $\alpha = 0.5$ are $\|\mathbf{J}\|_{L^2} = 2.1977966$ and $\|M\|_{L^2} = 3.9588037$ and for $\alpha = 1.5$ are $\|\mathbf{J}\|_{L^2} = 2.3826646$ and $\|M\|_{L^2} = 0.7763804$.
The convergence rate η for $\alpha = 0.1$ are for the "energy norm" $\eta_C = 1.325363$, for L^2-norm $\eta_{\mathbf{J}} = 1.617988$ and $\eta_M = 1.184964$. For $\alpha = 0.5$ are for the "energy norm" $\eta_C = 1.165255$, for L^2-norm $\eta_{\mathbf{J}} = 0.976440$ and $\eta_M = 1.211619$ and for $\alpha = 1.5$ are for the "energy norm" $\eta_C = 1.552163$, for L^2-norm $\eta_{\mathbf{J}} = 0.174124$ and $\eta_M = 0.295586$.

Let as compare our numerical convergence rates above for the boundary element methods obtained in the above example with the theoretical convergence rates predicted by Theorem 2. Note that we have implemented the boundary integral equation system (1.28), (1.29) and note the strongly elliptic system (1.32), where convergence is guaranteed due to Theorem 2. Nevertheless our experiments show convergence for the boundary element solution, but with suboptimal convergence rates. Theorem 2 predicts (when Raviart-Thomas elements are used to approximate $\mathbf{J}$ and piecewise linear elements to approximate M) a convergence rate of order $\eta = \frac{3}{2}$ in the energy norm for smooth solutions $\mathbf{J}$ and M. Our computations depend on the parameter α which is a well-known effect with boundary integral equations where it may come to spurious eigenvalues diminishing the orders of the Galerkin approximations. Due to the cube $\Omega_- = [-2, 2]^3$ the numerical solution might become singular near the edges and corners of Ω_-; hence the Galerkin scheme converges suboptimally.

Now we present the implementation of the matrices in system (1.43), all matrices involve computation of integrals over the boundary Σ. The programming is done within the program package *Maiprogs* using Fortran 90/95, cf. Maischak [34, 36]. The executable program is called *maicoup3*. Within the folder structure of *Maiprogs*, we mainly work in ../fo3c and use for BEM

the folder ../fo23. The executable program *maicoup3* has to be called in connection with an appropriate bcl-file. The so called batch control language bcl is a special script language, developed as a part of *Maiprogs* and calls the different subroutines in the right context and sets the specific structure of the problem regarded.
Implementation of the matrix A: The subroutine *rthelmgv2* computes the Galerkin element of matrix A for the single layer potential with Raviart-Thomas functions. The important parts of the source code for the calculation of the integral in (1.44) are given below.

The subroutine

```
subroutine rthelmgv2(vkl,bx,dx1,dx2,nx,tx,by,dy1,dy2,ny,ty,px,py,ptyp1,ptyp2),
! Table of the element matrix
real(kind=dp), intent(inout) :: vkl(0:,0:,0:)
! Polynomial degree
integer, intent(in) :: px(0:1),py(0:1)
! Element typ: 3=triangle, 4=rectangle
integer, intent(in) :: tx,ty
! Type of base function: 10=Raviart-Thomas, 0=Monomials
integer, intent(in) :: ptyp1,ptyp2
! Vector of the origin, boundaries, normal direction
real(kind=dp), dimension(0:2), intent(in) :: bx,dx1,dx2,nx ! Test element
real(kind=dp), dimension(0:2), intent(in) :: by,dy1,dy2,ny ! Trial element
```

computes the matrix for the single layer potential.

The calculation of the number of Raviart-Thomas basis functions in x and y are with the subroutine

```
call basenum2(px,tx,ptyp1,dofx) and call basenum2(py,ty,ptyp2,dofy).
```

The computation of the Galerkin elements for the Helmholtz kernel is done with

```
call helminteg3(...,dy1,dy2,ny,ty,px,py,pdu,pdu,pdu,pdu,vklmn,iklmn,iklmn).
```

The two subroutine

```
call trafo_koeff_rt(bx,dx1,dx2,tx,a1x,a2x,detx)
```

and

```
call trafo_koeff_rt(by,dy1,dy2,ty,a1y,a2y,dety)
```

calculate the transformation matrix A_k, A_l and their determinant for element x and element y.

The subroutine

```
call trafo_iklmn(vklmn,a1x,a2x,tx,a1y,a2y,ty,px,py,ptyp1,ptyp2,trafo,vkl)
```

transformed vklmn into a field vkl that is composed with Raviart-Thomas coefficient. trafo is the transformation routine trafo-rt-helm, which represents the terms in (1.44).
Implementation of the matrix B: The subroutine *rthelmgv4* computes the Galerkin matrix elements for the single layer potential with Raviart-Thomas functions and monomials. The important parts of the source code for the calculation of the integral in (1.46) are given below.

The subroutine

```
subroutine rthelmgv4(vkl,bx,dx1,dx2,nx,tx,by,dy1,dy2,ny,ty,px,py,ptyp1,ptyp2),
```

computes the matrix for the single layer potential.

The calculation of the number of Raviart-Thomas basis functions in x and y are with

```
call basenum2(px,tx,ptyp1,dofx) and call basenum2(py,ty,ptyp2,dofy).
```

The computation of the Galerkin elements for the Helmholtz kernel is with

```
call helminteg3(...,dy1,dy2,ny,ty,px,py,pdu,pdu,pdu,pdu,vklmn,iklmn,iklmn).
```

The subroutine

```
call trafo_koeff_rt(by,dy1,dy2,ty,a1y,a2y,dety)
```

calculates the transformation matrix A_k and its determinant for element y.

The following subroutine calculates the div-term in (1.46)

```
call trafo_divrt_helm(hmn,py,ty,ptyp2,dety,c2)
```

Implementation of the matrix C: The subroutine *helmgv* computes the Galerkin matrix elements for Helmholtz single layer potential of the integral in (1.47).

```
subroutine helmgv(ikl,bx,dx1,dx2,nx,tx,by,dy1,dy2,ny,ty,px,py,ptyp1,ptyp2).
```

Next, we apply the boundary element method above to compute the first terms in the asymptotic expansion of the electrical field considered in subsection 1.1 (Remark 1). In this way we obtain good results for the electrical field at some point away from the transmission surface Σ by only computing a few terms in the expansion.

Algorithm for the asymptotic of the eddy current problem:

1. First solve the exterior Problem $(\mathbf{P}_{\alpha\infty})$ by integral equations (1.28) and (1.29) i.e. (1.37) with given incident field $-\mathbf{E}_T^0$.
2. Compute $\mathbf{H}_T^+$ from (1.57).
3. Go back to 1: Solve the exterior problem $(\mathbf{P}_{\alpha\infty})$ with new right hand side from (1.18).
4. Go back to 2.
5. $\mathbf{E} = \mathbf{E}^0 + \beta^{-1}\mathbf{E}_1 + \beta^{-2}\mathbf{E}_2 + \mathbf{R}_m$.

We have $\widetilde{\mathbf{E}} = \mathbf{E}^0 + \beta^{-1}\mathbf{E}_1 + \beta^{-2}\mathbf{E}_2$ and calculate the error $|\widetilde{\mathbf{E}} - \mathbf{E}_{\text{exact}}(\mathbf{x}_i)|$, $i = 1, 2, 3$, where $\mathbf{x}_1 = (3, 0, 0)$, $\mathbf{x}_2 = (6, 0, 0)$ and $\mathbf{x}_3 = (9, 0, 0)$, $\beta = 10^3$. We present the results in Table 1.2 and in Figure 1.3.

N	DOF	$\|C\|$	$\|C - C_h\|$	$\\|\mathbf{J}\\|_{\mathbf{L}^2}$	$\\|M\\|_{\mathbf{L}^2}$	$\\|\mathbf{J} - \mathbf{J}_h\\|_{\mathbf{L}^2}$	$\\|M - M_h\\|_{\mathbf{L}^2}$
				$\alpha = 0.1$			
1	144	8.502965	1.153119	2.085189	80.704374	0.299829	14.08929
2	576	8.568451	0.460150	2.104369	81.690279	0.097681	6.196968
3	2304	8.578833	0.033717	2.106395	81.879637	0.031823	2.725645
4	9216	8.654072	0.073274	2.117002	83.123825	0.010367	1.198835
				$\alpha = 0.5$			
1	144	1.603519	0.209552	2.149511	3.8937090	0.458159	0.714952
2	576	1.614451	0.093436	2.185426	3.9467491	0.232851	0.308704
3	2304	1.616616	0.041661	2.194608	3.9565591	0.118342	0.133293
4	9216	1.617260	0.018576	2.198619	3.9592220	0.060145	0.057554
				$\alpha = 1.5$			
1	144	1.774450	0.326497	2.350909	0.7243729	0.387707	0.279375
2	576	1.800799	0.111334	2.365011	0.7422644	0.343627	0.227618
3	2304	1.803838	0.037965	2.382843	0.7539064	0.304558	0.185450
4	9216	1.804284	0.012946	2.397906	0.7909461	0.269932	0.151093

Table 1.1: Errors in L^2-norm and energy norm with respect to the degrees of freedom for $\alpha = 0.1, 0.5, 1.5$.

DOF	$\lvert\widetilde{\mathbf{E}} - \mathbf{E}_{\text{exact}}(\mathbf{x}_1)\rvert$	$\lvert\widetilde{\mathbf{E}} - \mathbf{E}_{\text{exact}}(\mathbf{x}_2)\rvert$	$\lvert\widetilde{\mathbf{E}} - \mathbf{E}_{\text{exact}}(\mathbf{x}_3)\rvert$
144	0.4959	0.6499	0.8049
576	0.1043	0.0910	0.0347
2304	0.0998	0.0067	0.0378

Table 1.2: Errors for electrical field in $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

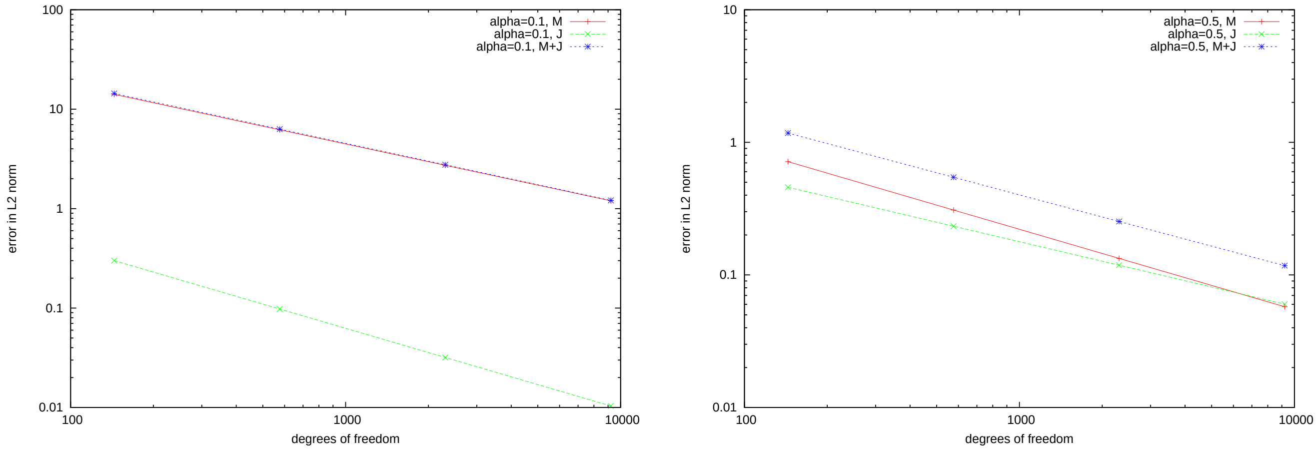

Figure 1.1: Errors in L2-norm for $\alpha = 0.1, 0.5$.

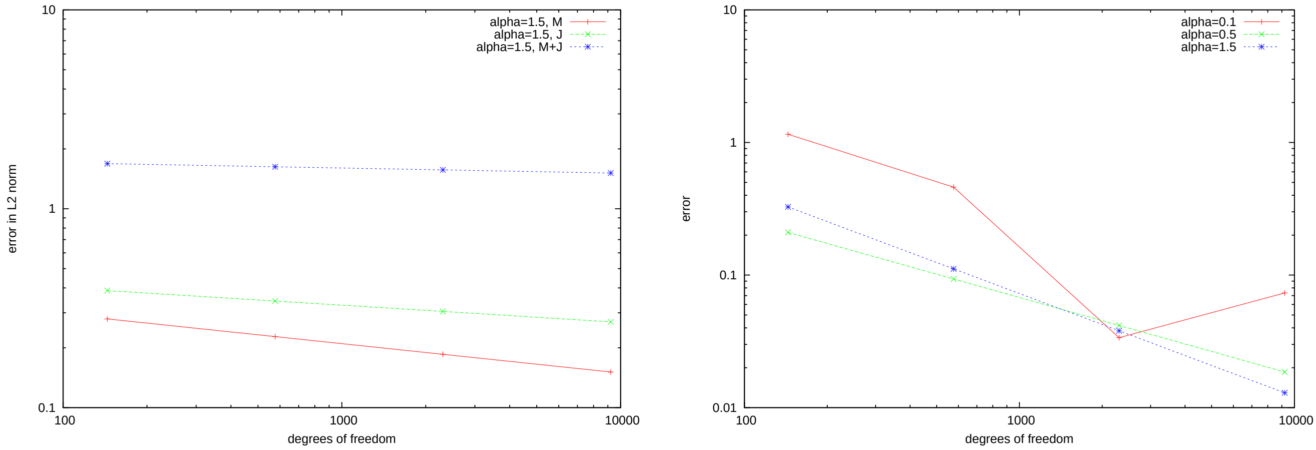

Figure 1.2: Errors in L2-norm for $\alpha = 1.5$ and in energy norm $|C - C_N| = O(h^\eta)$.

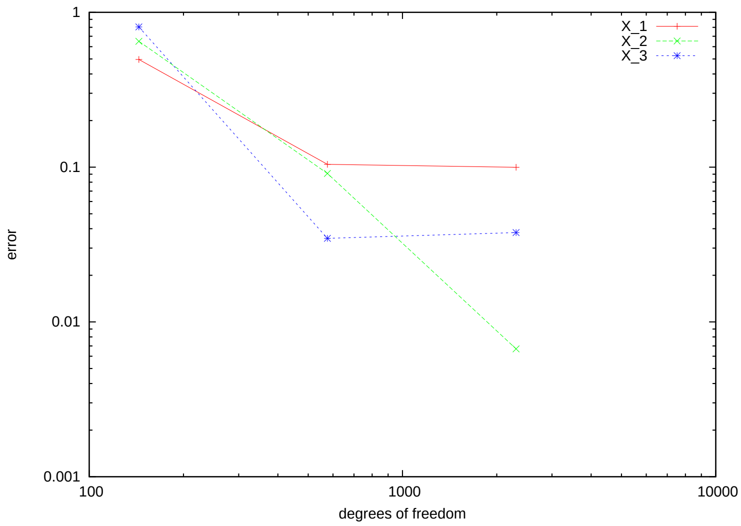

Figure 1.3: Errors for electrical field with respect to the degrees of freedom for $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

2 Transmission problem for the Laplacian with a parameter in $\mathbb{R}^3$

In this chapter, we present a priori estimates for a scalar transmission problem of the Laplacian with parameter in $\mathbb{R}^3$. The behavior of the solution at infinity is described by means of a family of weighted Sobolev spaces, so-called Beppo-Levi spaces. The main result is Theorem 3. To prove this we have to extend Peron's work who considered a bounded exterior domain in [42, 7], while we analyze the case of an unbounded exterior domain.

2.1 A scalar transmission problem in weighted spaces

In this section we analyze a scalar transmission problem (2.6)-(2.7) in an unbounded setting. Let Ω_- be a bounded region in $\mathbb{R}^3$ and $\Omega_+ = \mathbb{R}^3 \setminus \overline{\Omega_-}$. Let $\Sigma = \partial\Omega_- = \partial\Omega_+$ the interface be of class C^∞, see figure 2.1. Throughout this chapter, $\mathfrak{D}$ denotes the space consisting of all C^∞-functions with compact support and $\mathfrak{D}'$ is the topological dual space of $\mathfrak{D}$ (space of distributions).

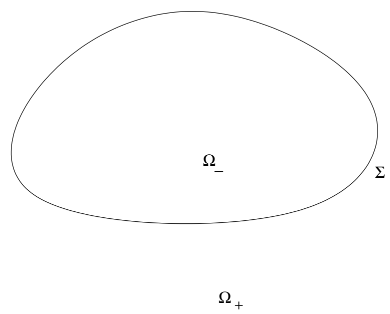

Figure 2.1: Region of the problem.

Consider the basic weight

$$\ell(r) = \sqrt{1 + r^2}, \tag{2.1}$$

with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, for $\mathbf{x} = (x_1, x_2, x_3)$, is the distance of the origin. For any scalar function $u = u(x_1, x_2, x_3)$, we define the Laplace and grad operator of u by

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2},$$

and

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right).$$

Due to the unboundedness of the exterior domain $A = \Omega_+$, the transmission problem is based on the weighted Sobolev spaces, also known as the Beppo-Levi spaces (see [27]), these spaces were introduced and studied by Hanouzet in [21] and a wide range of basic elliptic problems were solved in these spaces by Giroire in [20], defined by

$$\mathbb{W}_0^1(A) = \left\{ u \in \mathfrak{D}'(A) \ | \ (\ell(r))^{-1} u \in L^2(A), \nabla u \in \mathbf{L}^2(A) \right\} \tag{2.2}$$

and

$$\mathbb{W}_1^2(A) = \left\{ u \in \mathfrak{D}'(A) \ \middle| \ \frac{u}{\ell(r)} \in L^2(A), \nabla u \in \mathbf{L}^2(A), \ell(r) \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(A), 1 \leq i, j \leq 3 \right\}. \tag{2.3}$$

They are reflexive Banach spaces equipped, respectively, with natural norms:

$$\|u\|_{\mathbb{W}_0^1(A)} = \left(\|(\ell(r))^{-1} u\|^2_{L^2(A)} + \|\nabla u\|^2_{\mathbf{L}^2(A)} \right)^{\frac{1}{2}}, \tag{2.4}$$

and

$$\|u\|_{\mathbb{W}_1^2(A)} = \left(\left\| \frac{u}{\ell(r)} \right\|^2_{L^2(A)} + \|\nabla u\|^2_{\mathbf{L}^2(A)} + \sum_{1 \leq i,j \leq 3} \left\| \ell(r) \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(A)} \right)^{\frac{1}{2}} \tag{2.5}$$

We also define semi-norms

$$|u|_{\mathbb{W}_0^1(A)} = \|\nabla u\|_{\mathbf{L}^2(A)},$$

and

$$|u|_{\mathbb{W}_1^2(A)} = \left(\sum_{1 \leq i,j \leq 3} \left\| \ell(r) \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(A)} \right)^{\frac{1}{2}}.$$

Here $\mathbf{L}^2(A) = (L^2(A))^3$, and also we define for all m in $\mathbb{N} \cup \{0\}$ and all k in $\mathbb{Z}$

$$L^2_{m,k}(\mathbb{R}^3) := \left\{ u \in \mathbb{R} \ | \ \forall \alpha \in \mathbb{N}^3, \ \ 0 \leq |\alpha| \leq m, \ \ \ell(r)^{|\alpha| - m + k} u \in L^2(\mathbb{R}^3) \right\},$$

with the norm

$$\|u\|_{L^2_{m,k}(\mathbb{R}^3)} = \|\ell(r)^{|\alpha| - m + k} u\|_{L^2(\mathbb{R}^3)}.$$

We set the following spaces:

$$\mathring{\mathbb{W}}_0^1(A) = \overline{\mathfrak{D}(A)}^{\|\cdot\|_{\mathbb{W}_0^1(A)}} \ \text{ and } \ \mathring{\mathbb{W}}_1^2(A) = \overline{\mathfrak{D}(A)}^{\|\cdot\|_{\mathbb{W}_1^2(A)}}.$$

We denote by $\mathbb{W}_0^{-1}(A)$ (respectively $\mathbb{W}_1^0(A)$) the dual space of $\mathring{\mathbb{W}}_0^1(A)$ (respectively of $\mathring{\mathbb{W}}_1^2(A)$). They are spaces of distributions.
With $a(\mathbf{x}) = a_- \in \Omega_-$, $a(\mathbf{x}) = a_+ \in \Omega_+$ for constants $a\pm$, its jump $[a]_\Sigma = a_+ - a_-$, across Σ and the restriction $\varphi^+(\varphi^-)$ of a function φ to $\Omega_+(\Omega_-)$ we consider the problem:
For given

$$f \in L^2(\Omega_-) \cup \mathbb{W}_1^0(\Omega_+) \ \text{ and } \ g \in H^{\frac{1}{2}}(\Sigma), \tag{2.6}$$

find $\varphi \in \mathcal{V}$, such that

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi^+} dx + a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi^-} dx = -\int_{\Omega_+ \cup \Omega_-} f \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds, \ \ \forall \psi \in \mathcal{V} \quad (2.7)$$

with

$$\varphi \in \mathcal{V} = H^1_0(\Omega_-) \cup \mathbb{W}^1_0(\Omega_+), \ \ H^1_0(\Omega_-) = \left\{ \varphi \in H^1(\Omega_-) \ \middle| \ \int_{\Omega_-} \varphi dx = 0 \right\}. \quad (2.8)$$

The transmission problem (2.6)-(2.7) is elliptic (see [20]) . By elliptic regularity, φ has more regularity on sub-domains when the data are more regular.
We introduce

$$PH^2(\mathbb{R}^3) = \{\varphi = (\varphi^+, \varphi^-) \ | \ \varphi^+ \in \mathbb{W}^2_1(\Omega_+) \text{ and } \varphi^- \in H^2(\Omega_-)\}, \quad (2.9)$$

with norm

$$\|\varphi\|^2_{PH^2(\mathbb{R}^3)} = \|\varphi^-\|^2_{H^2(\Omega_-)} + \|\varphi^+\|^2_{\mathbb{W}^2_1(\Omega_+)}. \quad (2.10)$$

The following result is an extension of Peron's results [42] (for a bounded exterior domain) to an unbounded exterior domain Ω_+.

Proposition 1. *For f and g satisfying (2.6) and (2.13) we have*

$$\varphi \in PH^2(\mathbb{R}^3), \quad (2.11)$$

where $\varphi \in \mathcal{V}$ is the solution of (2.7). Furthermore φ solves(in the sense of distributions)

$$\begin{aligned} &a_+\Delta\varphi^+ = f^+ \ \ in \ \Omega_+, \ \ a_-\Delta\varphi^- = f^- \ \ in \ \ \Omega_-, \\ &\varphi^+ = \varphi^-, \ \ a_+\partial_{\boldsymbol{n}}\varphi^+ - a_-\partial_{\boldsymbol{n}}\varphi^- = [a]_\Sigma \cdot g \ \ on \ \ \Sigma, \\ &\varphi = O\left(\frac{1}{|\boldsymbol{x}|}\right), \ \ \partial_{\boldsymbol{n}}\varphi = o\left(\frac{1}{|\boldsymbol{x}|^2}\right) \ \ as \ \ |\boldsymbol{x}| \longrightarrow \infty, \end{aligned} \quad (2.12)$$

where $\partial_{\boldsymbol{n}}$ denotes the normal derivative where $\boldsymbol{n}$ is the normal pointing from Ω^+ in Ω^-.

Proof. The proof of Proposition 1 is given in several steps and follows Peron's original proof. We only modify it for the unbounded exterior domain Ω^+ by looking for the solution in weighted spaces. We show that φ satisfies an a priori estimate (Theorem 3) yielding (the assertion (2.11)). First we prove (2.12).
We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_-. Let $\Omega_R = B_R \cap \Omega_+$, with $\partial\Omega_R = \partial B_R \cup \Sigma$, see figure 2.2.

Then, the first term in (2.7) is

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = \lim_{R\to\infty} a_+ \int_{\Omega_R} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx,$$

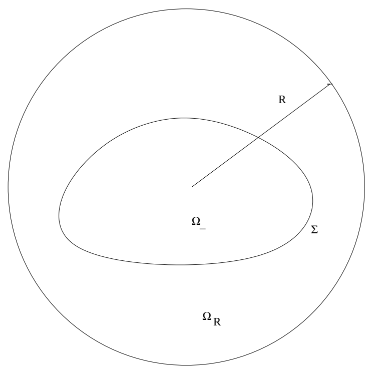

Figure 2.2: The domain $\Omega_R = B_R \cap \Omega_+$.

and by integration by parts in Ω_R

$$a_+ \int_{\Omega_R} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = -a_+ \int_{\Omega_R} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\partial\Omega_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds$$

$$= -a_+ \int_{\Omega_R} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds + a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds,$$

then, when $R \to \infty$, comes

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = -a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds + \lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds.$$

The second term in (2.7) by integration by parts, yields

$$a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi} dx = -a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx - a_- \int_{\Sigma} \partial_{\mathbf{n}}\varphi^- \cdot \overline{\psi} ds.$$

Then

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx + a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi} dx =$$

$$= -a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx - a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx +$$

$$+ \int_{\Sigma} (a_+\partial_{\mathbf{n}}\varphi^+ - a_-\partial_{\mathbf{n}}\varphi^-) \cdot \overline{\psi} ds + \lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds.$$

The right part in (2.7) is

$$-\int_{\Omega_+\cup\Omega_-} f \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds = -\int_{\Omega_+} f^+ \cdot \overline{\psi} dx - \int_{\Omega_-} f^- \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds,$$

then, we have

$$a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx = \int_{\Omega_+} f^+ \cdot \overline{\psi} dx,$$

$$a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx = \int_{\Omega_-} f^- \cdot \overline{\psi} dx,$$

$$\int_\Sigma (a_+ \partial_{\mathbf{n}} \varphi^+ - a_- \partial_{\mathbf{n}} \varphi^-) \cdot \overline{\psi} ds = [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds,$$

and

$$\lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}} \varphi^+ \cdot \overline{\psi} ds. = 0.$$

This implies (2.12).

Next we set $a_+ = 1$, $a_- = \rho \in \mathbb{C}$, and consider:
Find $\varphi_\rho \in \mathcal{V}$, such that, for all $\psi \in \mathcal{V}$,

$$\int_{\Omega_+} \nabla\varphi_\rho^+ \cdot \overline{\nabla\psi^+} dx + \rho \int_{\Omega_-} \nabla\varphi_\rho^- \cdot \overline{\nabla\psi^-} dx = -\int_{\Omega_+\cup\Omega_-} f \cdot \overline{\psi} dx + (1-\rho) \int_\Sigma g \cdot \overline{\psi} ds, \quad (\mathbf{P}_\rho)$$

with f and g satisfying (2.6) independent of ρ.
Following Peron [42] we construct a mapping $\rho \longmapsto \varphi_\rho$ where φ_ρ solves $(\mathbf{P}_\rho)$ and consider its behavior when $|\rho| \to \infty$.
We assume

$$\int_{\Omega_+\cup\Omega_-} f dx = 0 \quad \text{and} \quad \int_\Sigma g ds = 0. \tag{2.13}$$

and show an a priori estimate for φ_ρ uniformly in ρ.
We observe now that $\varphi_\rho \in \mathcal{V}$. By construction, φ_ρ is a solution of problem (2.12), with $a_- = \rho$, $a_+ = 1$. Especially $\varphi_\rho \in H^1(\Omega_-) \cup \mathbb{W}_0^1(\Omega_+)$. Finally $\int_{\Omega_-} \varphi_\rho^- dx = 0$ because every φ_n^- in (2.15) has integral mean zero. To complete the proof of Proposition 1 we prove now the following a priori estimate. Its application gives the assertion of Proposition 1. □

2.2 A priori estimate in weighted spaces

The main result for this chapter is to show an a priori estimate in PH^2 uniformly in ρ for a solution $\varphi_\rho \in \mathcal{V}$ of $(\mathbf{P}_\rho)$; that is the following theorem which in case of a bounded exterior domain was originally derived by Peron in his thesis [42]

Theorem 3. *Assuming (2.6) and (2.13), there exists a constant $\rho_0 > 0$ such that for all $\rho \in \{\vec{z} \in \mathbb{C} | |\vec{z}| \geq \rho_0\}$, problem $(\boldsymbol{P}_\rho)$ has a solution $\varphi_\rho \in PH^2(\mathbb{R}^3)$ with*

$$\|\varphi_\rho\|_{PH^2(\mathbb{R}^3)} \leq C_{\rho_0}(\|f^-\|_{L^2(\Omega_-)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}), \tag{2.14}$$

where $C_{\rho_0} > 0$ is independent of ρ, f and g.

Proof. Our proof modifies Peron's approach in [42] and is given via the following steps.

First we expand φ_ρ in a power series in ρ^{-1}.

$$\varphi_\rho = \begin{cases} \sum_{n=0}^{\infty} \varphi_n^+ \rho^{-n}, & \text{in} \quad \Omega_+, \\ \sum_{n=0}^{\infty} \varphi_n^- \rho^{-n}, & \text{in} \quad \Omega_-. \end{cases} \tag{2.15}$$

We show that these series converge in the norm in the space PH^2 to a solution of problem $(\mathbf{P}_\rho)$.

Inserting (2.15) in (2.12) and identifying terms of like powers of ρ^{-1} we obtain a family of problems independent of ρ, coupled by their conditions on Σ.
$(2.15)_2$ in $(2.12)_1$

$$\sum_{n=0}^{\infty} \Delta\varphi_n^- \rho^{-n+1} = f^-$$

or

$$\rho\Delta\varphi_0^- + \Delta\varphi_1^- + \rho^{-1}\Delta\varphi_2^- + \cdots = f^-$$

then

$$\Delta\varphi_0^- = 0, \;\; \Delta\varphi_1^- = f^-, \;\; \Delta\varphi_k^- = 0, \;\; k \geq 2,$$

or

$$\Delta\varphi_0^- = 0, \;\; \Delta\varphi_k^- = \delta_{k,1} f^-, \;\; k \geq 1, \;\; \text{in} \;\; \Omega_-.$$

$(2.15)_1$ in $(2.12)_2$

$$\sum_{n=0}^{\infty} \Delta\varphi_n^+ \rho^{-n} = f^+$$

or

$$\Delta\varphi_0^+ + \rho^{-1}\Delta\varphi_1^- + \cdots = f^+$$

then

$$\Delta\varphi_0^- = f^+, \;\; \Delta\varphi_k^+ = 0, \;\; k \geq 1, \;\; \text{in} \;\; \Omega_+.$$

(2.15) in $(2.12)_3$

$$\sum_{n=0}^{\infty} \Delta\varphi_n^- \rho^{-n} = \sum_{n=0}^{\infty} \Delta\varphi_n^+ \rho^{-n}$$

then

$$\varphi_k^- = \varphi_k^+, \;\; k \geq 0, \;\; \text{on} \;\; \Sigma.$$

(2.15) in $(2.12)_4$

$$\sum_{k=0}^{\infty} \partial_{\mathbf{n}}\varphi_k^+ \rho^{-k} - \sum_{k=0}^{\infty} \partial_{\mathbf{n}}\varphi_k^- \rho^{-k+1} = (1-\rho)g$$

or

$$\sum_{k=0}^{\infty} \partial_{\mathbf{n}}\varphi_k^+ \rho^{-k} - \sum_{k=-1}^{\infty} \partial_{\mathbf{n}}\varphi_{k+1}^- \rho^{-k} = (1-\rho)g$$

or

$$-\rho\partial_{\mathbf{n}}\varphi_0^- + \sum_{k=0}^{\infty} (\partial_{\mathbf{n}}\varphi_k^+ - \partial_{\mathbf{n}}\varphi_{k+1}^-)\rho^{-k} = (1-\rho)g$$

then

$$\partial_{\mathbf{n}}\varphi_0^- = g,\ \ \partial_{\mathbf{n}}\varphi_0^+ - \partial_{\mathbf{n}}\varphi_1^- = g,\ \ \partial_{\mathbf{n}}\varphi_k^+ - \partial_{\mathbf{n}}\varphi_{k+1}^- = 0,\ \ k \geq 1$$

or

$$\partial_{\mathbf{n}}\varphi_0^- = g,\ \ \partial_{\mathbf{n}}\varphi_k^- = -\delta_{k,1}g + \partial_{\mathbf{n}}\varphi_{k-1}^+,\ \ k \geq 1,\ \text{ on } \Sigma.$$

This implies

$$\begin{aligned} \Delta\varphi_0^- &= 0, \quad \text{in} \quad \Omega_-, \\ \partial_{\mathbf{n}}\varphi_0^- &= g, \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \Delta\varphi_0^+ &= f^+, \quad \text{in} \quad \Omega_+, \\ \varphi_0^+ &= \varphi_0^-, \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.17}$$

and for $k \in \mathbb{N}$ with the Kronecker symbol $\delta_{k,1}$

$$\begin{aligned} \Delta\varphi_k^- &= \delta_k^1 f^-, \quad \text{in} \quad \Omega_-, \\ \partial_{\mathbf{n}}\varphi_k^- &= -\delta_{k,1}g + \partial_{\mathbf{n}}\varphi_{k-1}^+, \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \Delta\varphi_k^+ &= 0, \quad \text{in} \quad \Omega_+, \\ \varphi_k^+ &= \varphi_k^-, \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.19}$$

and the condition at infinity

$$\varphi_\rho = O\left(\frac{1}{|\mathbf{x}|}\right),\quad \partial_{\mathbf{n}}\varphi_\rho = o\left(\frac{1}{|\mathbf{x}|^2}\right) \quad \text{as} \quad |\mathbf{x}| \longrightarrow \infty, \tag{2.20}$$

In addition we assume $\int_{\Omega_-} \varphi_0^- = 0$ and $\int_{\Omega_-} \varphi_k^- = 0$ (*) We construct successively every term φ_n^- and φ_n^+, by beginning in φ_0^- and φ_0^+.
Let us assume that $\{\varphi_k^-\}_{k=0}^{n-1}$ and $\{\varphi_k^+\}_{k=0}^{n-1}$ are known. Then, problem (2.18) defines a unique φ_n^-. Its trace on Σ is inserted in (2.19) as Dirichlet data to determine the external part φ_n^+.

The Neumann problem (2.16) has a unique solution $\varphi_0^- \in H^1(\Omega_-)$ if $\int_{\Omega_-} \varphi_0^- dx = 0$. But this holds since $\int_\Sigma g ds = 0$. Also, by elliptic regularity, $\varphi_0^- \in H^2(\Omega_-)$ and there is a constant $C_N > 0$, independent of ρ, such that (see [40, Theorem 2.5.2])

$$\|\varphi_0^-\|_{H^2(\Omega_-)} \leq C_N \|g\|_{H^{\frac{1}{2}}(\Sigma)}. \tag{2.21}$$

We are interested in φ_0^+ in (2.17). Problem (2.17) has a unique solution (see [20, Chapter 2]), $\varphi_0^+ \in \mathbb{W}_0^1(\Omega_+)$. Also, by elliptic regularity and since $\varphi_0^- \in H^2(\Omega_-)$, $\varphi_0^+ \in \mathbb{W}_1^2(\Omega_+)$ and there is a constant $C_{DN} > 0$ independent of ρ, such that (see [3, Theorem 6])

$$\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}(\|\varphi_0^-\|_{H^2(\Omega_-)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)}). \tag{2.22}$$

Now that (2.20) guarantees that $\varphi_0^+ \in \mathbb{W}_0^1(\Omega_+)$ and not only in $\mathbb{W}_0^1(\Omega_+)/\mathbb{R}$. Similarly we can deal with (2.18) and (2.19). Since φ_ρ satisfies the decay condition at infinity, φ_ρ can not

behave like a constant. Therefore the constraints (*) are not necessary.
Next we study the Neumann problem (2.18).
For $k = 1$, we show that there holds

$$\int_{\Omega_-} f^- dx + \int_\Sigma (-g + \partial_{\mathbf{n}} \varphi_0^+) ds = 0. \tag{2.23}$$

According to (2.17) and (2.20)

$$\begin{aligned} &\Delta \varphi_0^+ = f^+, && \text{in } \ \Omega_+, \\ &\varphi_0^+ = \varphi_0^-, && \text{on } \ \Sigma, \\ &\partial_{\mathbf{n}} \varphi_0^+ = o\left(\frac{1}{|\mathbf{x}|^2}\right), && \text{as } \ |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{2.24}$$

We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_- (see figure 2.2). Then for the bounded domain $\Omega_+ \cap B_R$, integrating by part in $(2.24)_1$ gives

$$\int_{\Omega_+ \cap B_R} f^+ \overline{\psi^+} dx = \int_{\Omega_+ \cap B_R} \Delta \varphi_0^+ \overline{\psi^+} dx$$

$$= -\int_{\Omega_+ \cap B_R} \nabla \varphi_0^+ \cdot \overline{\nabla \psi^+} dx + \int_{\partial(\Omega_+ \cap B_R)} \overline{\psi^+} \cdot \partial_{\mathbf{n}} \varphi_0^+ ds,$$

for $\psi \equiv 1$ yields

$$\int_{\Omega_+ \cap B_R} f^+ dx = \int_{\partial(\Omega_+ \cap B_R)} \partial_{\mathbf{n}} \varphi_0^+ ds$$

and $\partial(\Omega_+ \cap B_R) = \partial B_R \cup \Sigma$, then

$$\int_{\Omega_+ \cap B_R} f^+ dx = \int_{\partial B_R} \partial_{\mathbf{n}} \varphi_0^+ ds + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds$$

$$= \int_{\partial B_R} o\left(\frac{1}{R^2}\right) ds + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds$$

$$= o\left(\frac{1}{R^2}\right) R^2 + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds,$$

then

$$\int_{\Omega_+} f^+ dx = o(1) + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds, \quad \text{as} \quad R \longrightarrow \infty,$$

then

$$\int_{\Omega_+} f^+ dx = \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds.$$

Under the hypothesis (2.13)

$$\int_\Sigma g ds = 0, \quad \text{and} \quad \int_{\mathbb{R}^3} f dx = 0,$$

then

$$\int_{\Omega_+} f^+ dx = -\int_{\Omega_-} f^- dx,$$

the compatibility condition (2.23) is deducted.
For $k \geq 2$, we assume that the term φ^+_{k-1} is constructed.and we show that

$$\int_{\Sigma} \partial_{\mathbf{n}} \varphi^+_{k-1} ds = 0. \tag{2.25}$$

According to (2.19) and (2.20)

$$\begin{aligned} &\Delta\varphi^+_{k-1} = 0, && \text{in} \quad \Omega_+, \\ &\varphi^+_{k-1} = \varphi^-_{k-1}, && \text{on} \quad \Sigma, \\ &\partial_{\mathbf{n}}\varphi^+_{k-1} = o\left(\frac{1}{|\mathbf{x}|^2}\right), && \text{as} \quad |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{2.26}$$

Again we choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_-. Then for the bounded domain $\Omega_+ \cap B_R$, integrating by part in $(2.26)_1$ gives

$$0 = \int_{\Omega_+\cap B_R} \Delta\varphi^+_{k-1}\overline{\psi^+}dx = -\int_{\Omega_+\cap B_R} \nabla\varphi^+_{k-1}\cdot\overline{\nabla\psi^+}dx + \int_{\partial(\Omega_+\cap B_R)} \overline{\psi^+}\cdot\partial_{\mathbf{n}}\varphi^+_{k-1}ds,$$

for $\psi \equiv 1$ yields

$$0 = \int_{\partial(\Omega_+\cap B_R)} \partial_{\mathbf{n}}\varphi^+_{k-1}ds$$

and $\partial(\Omega_+ \cap B_R) = \partial B_R \cup \Sigma$, then

$$\begin{aligned} 0 &= \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+_{k-1}ds + \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+_{k-1}ds \\ &= \int_{\partial B_R} o\left(\frac{1}{R^2}\right) ds + \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+_{k-1}ds \\ &= o\left(\frac{1}{R^2}\right) R^2 + \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+_{k-1}ds, \end{aligned}$$

then

$$0 = o(1) + \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+_{k-1}ds, \quad \text{as} \quad R \longrightarrow \infty,$$

and

$$0 = \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+_{k-1}ds.$$

Consequently, the Neumann problem (2.18) admits a solution $\varphi^-_k \in H^1(\Omega_-)$, which is unique under condition $\int_{\Omega_-} \varphi^-_k dx = 0$ (see [40, Theorem 2.5.10]). Also, $\varphi^-_k \in H^2(\Omega_-)$ and

$$\|\varphi^-_k\|_{H^2(\Omega_-)} \leq C_N[\delta^1_k(\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}) + \|\partial_{\mathbf{n}}\varphi^+_{k-1}\|_{H^{\frac{1}{2}}(\Sigma)}]. \tag{2.27}$$

Finally, problem (2.19) has a unique solution $\varphi_k^+ \in \mathbb{W}_0^1(\Omega_+)$ (see [20, Chapter 2] and there holds the estimate (see [40, Theorem 2.5.14]) which the constant $C_{DN} > 0$

$$\|\varphi_k^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_k^-\|_{H^2(\Omega_-)}. \tag{2.28}$$

Next, we demonstrate the convergence in $PH^2(\mathbb{R}^3)$ of the series (2.15) for large $|\rho|$.

For the Neumann trace

$$\begin{array}{rccc} \gamma_{1,\Sigma}: & \mathbb{W}_1^2(\Omega_+) & \longrightarrow & H^{\frac{1}{2}}(\Sigma), \\ & \varphi & \longmapsto & \partial_{\mathbf{n}}\varphi \end{array}$$

we have with a constant $C_1 > 0$,

$$\|\gamma_{1,\Sigma}(\varphi)\|_{H^{\frac{1}{2}}(\Sigma)} \leq C_1\|\varphi\|_{\mathbb{W}_1^2(\Omega_+)}. \tag{2.29}$$

We pose $\alpha = C_N C_1 C_{DN}$, where C_N and C_{DN} are the respective constants of estimates (2.21) and (2.22). With (2.27), (2.28) and (2.29) we show by induction

$$\begin{aligned} \|\varphi_n^-\|_{H^2(\Omega_-)} &\leq \alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)}, \\ \|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN}\cdot\alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)}. \end{aligned} \tag{2.30}$$

$(2.30)_1$ can be see as follows: For $n = 1$,

$$\|\varphi_1^-\|_{H^2(\Omega_-)} = \alpha^0\|\varphi_1^-\|_{H^2(\Omega_-)}.$$

With (2.27) we have for $k = 2$

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_1^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and with (2.29)

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)};$$

hence by (2.28) we have for $k = 1$

$$\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

and therefore

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1 C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)} = \alpha\|\varphi_1^-\|_{H^2(\Omega_-)}.$$

We assume that $(2.30)_1$ is true for $k = n-1$, this is

$$\|\varphi_{n-1}^-\|_{H^2(\Omega_-)} \leq \alpha^{n-2}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

then, according to (2.27), for $k = n$

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_{n-1}^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (2.29)

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)};$$

according to (2.28) for $k = n-1$

$$\|\varphi_{n-1}^{+}\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_{n-1}^{-}\|_{H^2(\Omega_-)},$$

then

$$\begin{aligned} \|\varphi_n^{-}\|_{H^2(\Omega_-)} &\leq C_N C_1 C_{DN}\|\varphi_{n-1}^{-}\|_{H^2(\Omega_-)} \\ &\leq \alpha \cdot \alpha^{n-2}\|\varphi_1^{-}\|_{H^2(\Omega_-)} \\ &= \alpha^{n-1}\|\varphi_1^{-}\|_{H^2(\Omega_-)}, \end{aligned}$$

then $(2.30)_1$ is true for all n.
$(2.30)_2$ can be see as follows:
According to (2.28) for $k = 1$

$$\|\varphi_1^{+}\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_1^{-}\|_{H^2(\Omega_-)},$$

and for $k = 2$

$$\|\varphi_2^{+}\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_2^{-}\|_{H^2(\Omega_-)}.$$

According to (2.27) for $k = 2$

$$\|\varphi_2^{-}\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_1^{+}\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (2.29)

$$\|\varphi_2^{-}\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_1^{+}\|_{\mathbb{W}_1^2(\Omega_+)},$$

then

$$\begin{aligned} \|\varphi_2^{+}\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN} C_N C_1\|\varphi_1^{+}\|_{\mathbb{W}_1^2(\Omega_+)} \\ &\leq C_{DN} \cdot \alpha\|\varphi_1^{-}\|_{H^2(\Omega_-)}. \end{aligned}$$

We assume that $(2.30)_2$ is true for $k = n-1$, this is

$$\|\varphi_{n-1}^{+}\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \cdot \alpha^{n-2}\|\varphi_1^{-}\|_{H^2(\Omega_-)}$$

then, according to (2.28), for $k = n$

$$\|\varphi_n^{+}\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_n^{-}\|_{H^2(\Omega_-)},$$

and according to (2.27) for $k = n$

$$\|\varphi_n^{-}\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_{n-1}^{+}\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (2.29)

$$\|\varphi_n^{-}\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_{n-1}^{+}\|_{\mathbb{W}_1^2(\Omega_+)},$$

then

$$\begin{aligned} \|\varphi_n^{-}\|_{H^2(\Omega_-)} &\leq C_N C_1 C_{DN} \cdot \alpha^{n-2}\|\varphi_1^{-}\|_{H^2(\Omega_-)} \\ &= \alpha^{n-1}\|\varphi_1^{-}\|_{H^2(\Omega_-)}, \end{aligned}$$

then

$$\|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \cdot \alpha^{n-1} \|\varphi_1^-\|_{H^2(\Omega_-)},$$

then $(2.30)_2$ is true for all n.
Hence for all $\rho \in \mathbb{C}$, with $|\rho|^{-1}\alpha < 1$, the series (2.15) converges in $\mathbb{W}_1^2(\Omega_+)$ and $H^2(\Omega_-)$, respectively. Now we are in the position to prove Theorem 3.
We show first the estimate (2.14) for φ_ρ in (2.15). Let $\rho_0 > 0$, such that $\rho_0^{-1}\alpha < 1$, where $\alpha = C_N C_1 C_{DN}$.
Let $\rho \in \{z \in \mathbb{C} | |z| \geq \rho_0\}$. According to (2.30) φ_ρ converges geometrically in $PH^2(\mathbb{R}^3)$ with $|\rho^{-1}|\alpha$, bounded by $\rho_0^{-1}\alpha$. Hence,

$$\begin{aligned} \|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN} \frac{1}{1-\rho_0^{-1}\alpha} \rho_0^{-1} \|\varphi_1^-\|_{H^2(\Omega_-)} + \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}, \\ \|\varphi_\rho^-\|_{H^2(\Omega_-)} &\leq \rho_0^{-1} \frac{1}{1-\rho_0^{-1}\alpha} \|\varphi_1^-\|_{H^2(\Omega_-)} + \|\varphi_0^-\|_{H^2(\Omega_-)} \end{aligned} \tag{2.31}$$

This can be seen as follows.
From $(2.15)_1$, $(2.30)_2$ and the triangular inequality, we have

$$\begin{aligned} \|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &= \|\textstyle\sum_{n=0}^\infty \varphi_n^+ \rho^{-n}\|_{\mathbb{W}_1^2(\Omega_+)} \\ &\leq \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} + \textstyle\sum_{n=1}^\infty \|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} |\rho^{-n}| \\ &\leq \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} + C_{DN} \cdot \alpha^{-1} \|\varphi_1^-\|_{H^2(\Omega_-)} \textstyle\sum_{n=1}^\infty |\rho^{-n}|\alpha^n, \end{aligned}$$

and

$$\sum_{n=1}^\infty |\rho^{-n}|\alpha^n = \sum_{n=1}^\infty (\rho^{-1}\alpha)^n = \frac{1}{1-\rho^{-1}\alpha} \leq \frac{1}{1-\rho_0^{-1}\alpha}, \tag{2.32}$$

then

$$\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \frac{1}{1-\rho_0^{-1}\alpha} \rho_0^{-1} \|\varphi_1^-\|_{H^2(\Omega_-)} + \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}.$$

Using the triangle inequality, $(2.15)_2$ and $(2.30)_1$, we have

$$\begin{aligned} \|\varphi_\rho^-\|_{H^2(\Omega_-)} &= \|\textstyle\sum_{n=0}^\infty \varphi_n^- \rho^{-n}\|_{H^2(\Omega_-)} \\ &\leq \|\varphi_0^-\|_{H^2(\Omega_-)} + \textstyle\sum_{n=1}^\infty \|\varphi_n^-\|_{H^2(\Omega_-)} |\rho^{-n}| \\ &\leq \|\varphi_0^-\|_{H^2(\Omega_-)} + \alpha^{-1} \|\varphi_1^-\|_{H^2(\Omega_-)} \textstyle\sum_{n=1}^\infty |\rho^{-n}|\alpha^n, \end{aligned}$$

this and (2.32) implies $(2.31)_2$.
Now, from (2.27), for $k = 1$

$$\|\varphi_1^-\|_{H^2(\Omega_-)} \leq C_N [\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|\partial_{\mathbf{n}} \varphi_0^+\|_{H^{\frac{1}{2}}(\Sigma)}], \tag{2.33}$$

according to (2.33) and (2.29), get

$$\|\varphi_1^-\|_{H^2(\Omega_-)} \leq C_N [\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)} + C_1 \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}]. \tag{2.34}$$

From (2.34), (2.31), (2.21) and (2.22), we have

$$\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \frac{1}{1-\rho_0^{-1}\alpha} \rho_0^{-1} C_N [\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}$$

$$+C_1 C_{DN}(C_N \|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)})] + C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)}),$$

and

$$\|\varphi_\rho^-\|_{H^2(\Omega_-)} \leq \rho_0^{-1} \frac{1}{1-\rho_0^{-1}\alpha} C_N [\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}$$

$$+C_1 C_{DN}(C_N \|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)})] + C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)}.$$

This yields the desired estimate (2.14)and the proof of Theorem 3 is complete.

□

The a priori estimate of Theorem 3 is a crucial tool in the proof of Theorem 6 in Chapter 3 where it is taken for one part in the Helmholtz decomposition.

3 Electromagnetic transmission problem for large conductivity - Analysis in weighted Sobolev spaces

In this chapter, we present an a priori estimate for an electromagnetic transmission problem with unbounded exterior domain in $\mathbb{R}^3$. We consider Maxwell's equations in two sub-domains, the bounded interior one $\Omega^{cd} = \Omega_-$ representing a conducting material (metal) and the unbounded exterior one $\Omega^{is} = \Omega_+$ an insulating material (air). The behavior of the solution at infinity is described by means of families of weighted Sobolev spaces, so-called Beppo-Levi spaces (see [27]). Existence and uniqueness of the solution are obtained. The results of this chapter are modifications of Peron's results [42], which he derived for a bounded exterior domain Ω^{is}. We follow closely his thesis, present some of his results but make modifications for the unbounded exterior domain Ω^{is}. This leads us to use weighted spaces and to apply an embedding result for weighted spaces. The compactness of such an embedding allows us to perform a contradiction argument implying an a priori estimate for the electrical field (Theorem 6). From this a priori estimate then follows existence and uniqueness of the solution of the electromagnetic transmission problem (3.2).

3.1 The electromagnetic transmission problem in $\mathbb{R}^3$

We consider Maxwell's equations in a bounded domain Ω^{cd} and in an unbounded domain $\Omega^{is} = \mathbb{R}^3 \diagdown \bar{\Omega}^{cd}$

$$
\begin{aligned}
&\operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0, \quad \text{in} \quad \Omega^{cd} \cup \Omega^{is} \quad , \\
&\operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \sigma)\mathbf{E} = \mathbf{J}, \quad \text{in} \quad \Omega^{cd} \cup \Omega^{is}
\end{aligned}
\tag{3.1}
$$

for electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ with real-valued constants ω, ε_0, μ_0. Across the smooth interface surface Σ, the boundary of Ω^{cd}, the tangential components of both $\mathbf{E}$ and $\mathbf{H}$ must be continuous, i.e. $\mathbf{E}_T^{is} = \mathbf{E}_T^{cd}$, $\mathbf{H}_T^{is} = \mathbf{H}_T^{cd}$.
Furthermore the Silver-Müller radiation condition must hold at infinity.
Setting $\rho := \sqrt{\dfrac{\sigma}{\omega\varepsilon_0}} > 0$, $\mu := \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$, $\varepsilon(\rho) := \dfrac{1}{\mu}(1_{\Omega^{is}} + (1 + i\rho^2)1_{\Omega^{cd}})$, and $\mathbf{F} = i\kappa\mathbf{J}$, then (3.1)

writes with $\widehat{\mathbf{x}} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$ and $\kappa = \omega(\mu\varepsilon)^{\frac{1}{2}}$ as

$$\begin{aligned}
&\operatorname{curl} \mathbf{E}_\rho - i\kappa\mu\mathbf{H}_\rho = 0, && \text{in} \quad \Omega^{cd} \cup \Omega^{is} \quad ,\\
&\operatorname{curl} \mathbf{H}_\rho + i\kappa\varepsilon(\rho)\mathbf{E}_\rho = \frac{1}{i\kappa}\mathbf{F}, && \text{in} \quad \Omega^{cd} \cup \Omega^{is} \quad ,\\
&|\mathbf{E}_\rho|, |\mathbf{H}_\rho| = o\left(\frac{1}{|\mathbf{x}|}\right), \ |\mathbf{H}_\rho \times \widehat{\mathbf{x}} - i\kappa\mathbf{E}_\rho| = o\left(\frac{1}{|\mathbf{x}|}\right), && \text{as} \quad |\mathbf{x}| \to \infty,
\end{aligned} \tag{3.2}$$

Note (3.2) can be reduced to

$$\frac{1}{\mu}\operatorname{curl}\operatorname{curl} \mathbf{E}_\rho - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho = \mathbf{F}, \quad \text{in} \quad \Omega^{cd} \cup \Omega^{is},$$

setting $\mathbf{H}_\rho = \dfrac{1}{i\kappa\mu}\operatorname{curl} \mathbf{E}_\rho$; the Silver-Müller radiation condition at infinity becomes

$$|\operatorname{curl} \mathbf{E}_\rho \times \widehat{\mathbf{x}} - i\kappa\mathbf{E}_\rho| = o\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as} \quad |\mathbf{x}| \longrightarrow \infty.$$

We are interested in the behavior of the electric and magnetic fields as the conductivity σ becomes large. i.e. $\sigma \to \infty$, thus $\rho \to \infty$.
Peron considers in [42] problem (3.2) in a bounded domain $\Omega = \Omega^{cd} \cup \Omega is$ with bounded domains Ω^{cd} and Ω^{is}. Peron considers on the boundary $\partial\Omega$ of Ω either Dirichlet or Neumann conditions. In our case Ω is unbounded (since in our case Ω^{is} is unbounded) and the boundary conditions on $\partial\Omega$ are replaced by the Silver-Müller decay condition at infinity. There are two ways to analyze problem (3.2) - either one uses $H^s_{loc}(\mathbb{R}^3)$ or Beppo-Levi spaces (as done in Costabel and Stephan [12] or in Giroire [20], respectively, for boundary value problems of the Laplacian). For the use of H^s_{loc} spaces to electromagnetic problems see the book by Nedelec [40]. Here we investigate problem (3.2) in Beppo-Levi spaces.
Before we introduce those weighted Sobolev spaces for an unbounded domain let us remember the definition of the spaces associated with Maxwell's system in a bounded domain $\Omega^{cd} \subset \mathbb{R}^3$ (see [38]), which are based on the space

$$\mathbf{L}^2(\Omega^{cd}) = (L^2(\Omega^{cd}))^3 := \left\{\mathbf{u} \in \mathbb{R}^3 \ \middle| \ \int_{\Omega^{cd}} |\mathbf{u}|^2 dx < \infty\right\},$$

with norm

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega^{cd})} = \left(\int_{\Omega^{cd}} |\mathbf{u}|^2 dx\right)^{\frac{1}{2}},$$

namely

$$\mathbf{H}(\operatorname{curl}, \Omega^{cd}) = \{\mathbf{u} \in \mathbf{L}^2(\Omega^{cd}) \ | \ \operatorname{curl} \mathbf{u} \in \mathbf{L}^2(\Omega^{cd})\},$$

$$\mathbf{H}(\operatorname{div}, \Omega^{cd}) = \{\mathbf{u} \in \mathbf{L}^2(\Omega^{cd}) \ | \ \operatorname{div} \mathbf{u} \in L^2(\Omega^{cd})\},$$

with norms

$$\|\mathbf{u}\|^2_{\mathbf{H}(\operatorname{curl}, \Omega^{cd})} = \|\operatorname{curl} \mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})},$$

$$\|\mathbf{u}\|^2_{\mathbf{H}(\mathrm{div},\Omega^{cd})} = \|\mathrm{div}\ \mathbf{u}\|^2_{L^2(\Omega^{cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})}.$$

From Peron [42] we collect the following variational spaces

$$\mathbf{X}(\Omega^{cd}) = \mathbf{H}(\mathrm{curl}, \Omega^{cd}) \cap \mathbf{H}(\mathrm{div}, \Omega^{cd}),$$

with norm

$$\|\mathbf{u}\|^2_{\mathbf{X}(\Omega^{cd})} = \|\mathrm{curl}\ \mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})} + \|\mathrm{div}\ \mathbf{u}\|^2_{L^2(\Omega^{cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})},$$

and

$$\mathbf{X}_{\mathtt{T}}(\Omega^{cd}) = \{\mathbf{u} \in \mathbf{X}(\Omega^{cd}) \ \mid \ \mathbf{n} \cdot \mathbf{u} = 0, \ \text{ on } \ \partial\Omega^{cd}\},$$

$$\mathbf{X}_{\mathtt{N}}(\Omega^{cd}) = \{\mathbf{u} \in \mathbf{X}(\Omega^{cd}) \ \mid \ \mathbf{n} \times \mathbf{u} = 0, \ \text{ on } \ \partial\Omega^{cd}\},$$

$$\mathbf{X}_{\mathtt{T}}(\Omega^{cd}, \rho) = \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \Omega^{cd}) \ \mid \ \varepsilon(\rho)\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega^{cd}), \ \mathbf{n} \cdot \mathbf{u} = 0, \ \text{ on } \ \partial\Omega^{cd}\},$$

$$\mathbf{X}_{\mathtt{N}}(\Omega^{cd}, \rho) = \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \Omega^{cd}) \ \mid \ \varepsilon(\rho)\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega^{cd}), \ \mathbf{n} \times \mathbf{u} = 0, \ \text{ on } \ \partial\Omega^{cd}\}.$$

with norm

$$\|\mathbf{u}\|^2_{\mathbf{X}(\Omega^{cd},\rho)} = \|\mathrm{curl}\ \mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})} + \|\mathrm{div}(\varepsilon(\rho)\mathbf{u})\|^2_{L^2(\Omega^{cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{cd})},$$

Note, $\mathbf{X}_{\mathtt{T}}(\Omega^{cd})$, $\mathbf{X}_{\mathtt{N}}(\Omega^{cd})$, $\mathbf{X}_{\mathtt{T}}(\Omega^{cd}, \rho)$ and $\mathbf{X}_{\mathtt{N}}(\Omega^{cd}, \rho)$ are Hilbert spaces.

Furthermore let $\mathfrak{D}$ denote the space consisting of all C^∞-functions with compact support and $\mathfrak{D}'$ is the topological dual space of $\mathfrak{D}$ (space of distributions).
Consider the basic weight $\ell(r) = \sqrt{1 + r^2}$, with distance $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, for $\mathbf{x} = (x_1, x_2, x_3)$, from the origin. Due to the unboundedness of the domain Ω^{is}, the problem is based on the vectorial weighted Sobolev space (also known as the vectorial Beppo-Levi space), a fairly complete treatment of these spaces is given in [3], [18], [27] and [40, Section 2.5.4]:

$$\mathbf{W}(\mathrm{curl}, \mathbb{R}^3) = \{\mathbf{u} \in \mathfrak{D}'(\mathbb{R}^3) \ \mid \ \ell(r)^{-1}\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3), \ \mathrm{curl}\ \mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3)\},$$

$$\mathbf{W}(\mathrm{div}, \mathbb{R}^3) = \{\mathbf{u} \in \mathfrak{D}'(\mathbb{R}^3) \ \mid \ \ell(r)^{-1}\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3), \ \mathrm{div}\ \mathbf{u} \in L^2(\mathbb{R}^3)\},$$

$\mathbf{W}(\mathrm{curl}, \mathbb{R}^3)$ and $\mathbf{W}(\mathrm{div}, \mathbb{R}^3)$ are Hilbert spaces equipped with the norms

$$\|\mathbf{u}\|^2_{\mathbf{W}(\mathrm{curl},\mathbb{R}^3)} = \|\mathrm{curl}\ \mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)} + \|\ell(r)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)},$$

and

$$\|\mathbf{u}\|^2_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)} = \|\mathrm{div}\ \mathbf{u}\|^2_{L^2(\mathbb{R}^3)} + \|\ell(r)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}.$$

Furthermore we need the following spaces

$$\mathbb{X}(\mathbb{R}^3) = \mathbf{W}(\mathrm{curl}, \mathbb{R}^3) \cap \mathbf{W}(\mathrm{div}, \mathbb{R}^3),$$

with norm

$$\|\mathbf{u}\|^2_{\mathbb{X}(\mathbb{R}^3)} = \|\text{curl }\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)} + \|\text{div }\mathbf{u}\|^2_{L^2(\mathbb{R}^3)} + \|\ell(r)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)},$$

and with $\Sigma = \bar{\Omega}^{cd} \cap \bar{\Omega}^{is}$ and $\Omega^{is} = \mathbb{R}^3 \setminus \bar{\Omega}^{cd}$

$$\mathbb{X}_{\mathtt{T}}(\mathbb{R}^3) = \{\mathbf{u} \in \mathbb{X}(\mathbb{R}^3) \ | \ [\mathbf{n}\cdot\mathbf{u}] = 0, \ \text{ on } \ \Sigma\},$$

$$\mathbb{X}_{\mathtt{N}}(\mathbb{R}^3) = \{\mathbf{u} \in \mathbb{X}(\mathbb{R}^3) \ | \ [\mathbf{n}\times\mathbf{u}] = 0, \ \text{ on } \ \Sigma\},$$

$$\mathbb{X}_{\mathtt{T}}(\mathbb{R}^3,\rho) = \{\mathbf{u} \in \mathbf{W}(\text{curl},\mathbb{R}^3) \ | \ \varepsilon(\rho)\mathbf{u} \in \mathbf{W}(\text{div},\mathbb{R}^3), \ [\mathbf{n}\cdot\mathbf{u}] = 0, \ \text{ on } \ \Sigma\},$$

$$\mathbb{X}_{\mathtt{N}}(\mathbb{R}^3,\rho) = \{\mathbf{u} \in \mathbf{W}(\text{curl},\mathbb{R}^3) \ | \ \varepsilon(\rho)\mathbf{u} \in \mathbf{W}(\text{div},\mathbb{R}^3), \ [\mathbf{n}\times\mathbf{u}] = 0, \ \text{ on } \ \Sigma\},$$

$$\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3,\rho) = \{\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3), \text{curl }\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3), \ell(r)\text{div}\varepsilon(\rho)\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3) \ \ , \ \ [\mathbf{n}\times\mathbf{u}] = [\mathbf{n}\cdot\mathbf{u}] = 0, \ \text{ on } \ \Sigma\},$$

with norm

$$\|\mathbf{u}\|^2_{\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3,\rho)} = \|\text{curl }\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)} + \|\ell(r)\text{div}(\varepsilon(\rho)\mathbf{u})\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)},$$

Note, $\mathbb{X}_{\mathtt{T}}(\mathbb{R}^3)$, $\mathbb{X}_{\mathtt{N}}(\mathbb{R}^3)$, $\mathbb{X}_{\mathtt{T}}(\mathbb{R}^3,\rho)$ and $\mathbb{X}_{\mathtt{N}}(\mathbb{R}^3,\rho)$ are Hilbert spaces.
And also we define for all m in $\mathbb{N}\cup\{0\}$ and all k in $\mathbb{Z}$

$$\mathbf{L}^2_{m,k}(\mathbb{R}^3) := \left\{\mathbf{u} \in \mathbb{R}^3 \ | \ \forall\alpha \in \mathbb{N}^3, \ 0 \le |\alpha| \le m, \ \ \ell(r)^{|\alpha|-m+k}\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3)\right\},$$

with the norm

$$\|\mathbf{u}\|_{\mathbf{L}^2_{m,k}(\mathbb{R}^3)} = \|\ell(r)^{|\alpha|-m+k}\mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^3)},$$

where $\mathbf{L}^2_{m,k}(\mathbb{R}^3) = \left(L^2_{m,k}(\mathbb{R}^3)\right)^3$. Note $\mathbf{L}^2_{0,1}(\mathbb{R}^3) \subset \mathbf{L}^2(\mathbb{R}^3) \subset \mathbf{L}^2_{0,-1}(\mathbb{R}^3)$.
First, we address the setting of Maxwell's transmission problem (3.1) in $\mathbb{R}^3$ in standard Sobolev spaces in the bounded conductor and weighted spaces in the unbounded insulator. Modifying [29, Lemma 1.3.1,Lemma 1.3.2], [42, Lemma 2.7] for the unbounded exterior domain we have the followings results. Louér [29] derives her results in $H_{loc}(\text{curl},\mathbb{R}^3) = \{\mathbf{u} \in \mathbf{L}^2_{loc}(\mathbb{R}^3), \text{curl}\mathbf{u} \in \mathbf{L}^2_{loc}(\mathbb{R}^3)\}$ where $\mathbf{u} \in \mathbf{L}^2_{loc}(\mathbb{R}^3)$ means $\mathbf{u} \in \mathbf{L}^2(\Omega)$ for any bounded domain Ω.

Lemma 1. *Let $\boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3)$. Let $\boldsymbol{E}_\rho$ and $\boldsymbol{H}_\rho$ in $\boldsymbol{L}^2(\mathbb{R}^3)$ be a solution of (3.2). Then, $\boldsymbol{E}_\rho, \boldsymbol{H}_\rho \in \boldsymbol{W}(curl,\mathbb{R}^3)$ if and only if $\boldsymbol{E}^{cd}_\rho, \boldsymbol{H}^{cd}_\rho \in \boldsymbol{H}(curl,\Omega^{cd})$ and $\boldsymbol{E}^{is}_\rho, \boldsymbol{H}^{is}_\rho \in \boldsymbol{W}(curl,\Omega^{is})$ and there holds $[\boldsymbol{n}\times\boldsymbol{E}_\rho]_\Sigma = 0$, $[\boldsymbol{n}\times\boldsymbol{H}_\rho]_\Sigma = 0$, where $[\boldsymbol{u}]_\Sigma = \boldsymbol{u}^{is} - \boldsymbol{u}^{cd}$. denotes the jump across Σ.*

Proof. If $\mathbf{E}_\rho, \mathbf{H}_\rho \in \mathbf{W}(\text{curl},\mathbb{R}^3)$, then by definition $\mathbf{E}^{cd}_\rho, \mathbf{H}^{cd}_\rho \in \mathbf{H}(\text{curl},\Omega^{cd})$ and $\mathbf{E}^{is}_\rho, \mathbf{H}^{is}_\rho \in \mathbf{W}(\text{curl},\Omega^{is})$. Thus for $\mathbf{u}_\rho = \text{curl}\,\mathbf{E}_\rho$ or $\text{curl}\,\mathbf{H}_\rho$, we have for all $\mathbf{v} \in \mathbf{C}^\infty(\mathbb{R}^3)$ satisfying the radiation condition in (3.2) and assuming $\ell(r)\text{curl}\mathbf{v}$, $\ell(r)\text{curl}\mathbf{u}_\rho \in L^2(\mathbb{R}^3)$

$$\int_{\Omega^{cd}\cup\Omega^{is}} \mathbf{v}\cdot\text{curl }\mathbf{u}_\rho dx = \int_{\Omega^{cd}} \mathbf{v}\cdot\text{curl }\mathbf{u}^{cd}_\rho dx + \int_{\Omega^{is}} \mathbf{v}\cdot\text{curl }\mathbf{u}^{is}_\rho dx,$$

and

$$\int_{\Omega^{cd}\cup\Omega^{is}} \mathbf{u}_\rho\cdot\text{curl }\mathbf{v}dx = \int_{\Omega^{cd}} \mathbf{u}^{cd}_\rho\cdot\text{curl }\mathbf{v}dx + \int_{\Omega^{is}} \mathbf{u}^{is}_\rho\cdot\text{curl }\mathbf{v}dx,$$

in Ω^{cd} integration by parts (see [29, Lemma 1.3.1]) gives

$$\int_{\Omega^{cd}} [\mathbf{v} \cdot \text{curl } \mathbf{u}_\rho^{cd} dx - \mathbf{u}_\rho^{cd} \cdot \text{curl } \mathbf{v}] dx = \int_\Sigma [\mathbf{n} \times (\mathbf{n} \times \mathbf{u}_\rho^{cd})] \cdot (\mathbf{n} \times \mathbf{v}) ds.$$

where

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{u}_\rho^{cd}) = \mathbf{n}(\mathbf{n} \cdot \mathbf{u}_\rho^{cd}) - \mathbf{u}_\rho^{cd}(\mathbf{n} \cdot \mathbf{n}) = \mathbf{n}(\mathbf{n} \cdot \mathbf{u}_\rho^{cd}) - \mathbf{u}_\rho^{cd}.$$

Then

$$[\mathbf{n} \times (\mathbf{n} \times \mathbf{u}_\rho^{cd})] \cdot (\mathbf{n} \times \mathbf{v}) = [\mathbf{n}(\mathbf{n} \cdot \mathbf{u}_\rho^{cd}) - \mathbf{u}_\rho^{cd}] \cdot (\mathbf{n} \times \mathbf{v}) = -\mathbf{u}_\rho^{cd} \cdot (\mathbf{n} \times \mathbf{v}),$$

and

$$-\mathbf{u}_\rho^{cd} \cdot (\mathbf{n} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{cd}),$$

yield

$$\int_{\Omega^{cd}} \mathbf{v} \cdot \text{curl } \mathbf{u}_\rho^{cd} dx = \int_{\Omega^{cd}} \mathbf{u}_\rho^{cd} \cdot \text{curl } \mathbf{v} dx + \int_\Sigma \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{cd}) ds.$$

We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω^{cd}.
Let $\Omega^{is} = \lim_{R\to\infty} \Omega_R$ where $\Omega_R = B_R \cap \Omega^{is}$, with $\partial\Omega_R = \partial B_R \cup \Sigma$.
In the domain Ω_R, we have, by integration by parts, (see [29, Theorem 1.2.17]),

$$\int_{\Omega_R} \mathbf{v} \cdot \text{curl } \mathbf{u}_\rho^{is} dx = \int_{\Omega_R} \mathbf{u}_\rho^{is} \cdot \text{curl } \mathbf{v} dx - \int_\Sigma \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{is}) ds + \int_{\partial B_R} \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{is}) ds.$$

The Silver-Müller radiation conditions yield

$$\lim_{R\to\infty} \int_{\partial B_R} |\text{curl}\mathbf{E}_\rho^{is} \times n_R - i\kappa \mathbf{E}_\tau^{is}|^2 = 0$$

Hence choosing $\mathbf{v} = \mathbf{E}^{is}, \mathbf{u}_\rho = \text{curl}\mathbf{E}^{is}$ gives with a generic constant C

$$\lim_{R\to\infty} |\int_{\partial B_R} \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{is}) ds| \leq C \lim_{R\to\infty} \int_{\partial B_R} |\mathbf{E}^{is}|^2 ds = 0.$$

Hence

$$\int_{\Omega^{is}} \mathbf{v} \cdot \text{curl } \mathbf{u}_\rho^{is} dx = \int_{\Omega^{is}} \mathbf{u}_\rho^{is} \cdot \text{curl } \mathbf{v} dx - \int_\Sigma \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{is}) ds.$$

Thus

$$\int_{\Omega^{cd}\cup\Omega^{is}} \mathbf{v} \cdot \text{curl } \mathbf{u}_\rho dx = \int_{\Omega^{cd}\cup\Omega^{is}} \mathbf{u}_\rho \cdot \text{curl } \mathbf{v} dx + \int_\Sigma \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{cd} - \mathbf{n} \times \mathbf{u}_\rho^{is}) ds,$$

yielding

$$\int_\Sigma \mathbf{v} \cdot (\mathbf{n} \times \mathbf{u}_\rho^{cd} - \mathbf{n} \times \mathbf{u}_\rho^{is}) ds = 0,$$

and therefore $[\mathbf{n} \times \mathbf{u}_\rho]_\Sigma = 0$. By density for $\mathbf{u}_\rho \in \mathbf{W}(\text{curl}, \mathbb{R}^3)$ this follows because integration by parts holds in $\mathbb{R}^3$ in this Beppo-Levi space. The reverse statement follows again by integration by parts. □

Our next results is a modification of Lemma 2.8 in [42] according to our unbounded exterior domain. Note that despite of a similar formulation, our lemma is for weighted Sobolev spaces.

Lemma 2. *Let $\boldsymbol{F} \in \boldsymbol{W}(div, \mathbb{R}^3)$. Let $\boldsymbol{E}_\rho$ and $\boldsymbol{H}_\rho$ in $\boldsymbol{L}^2(\mathbb{R}^3)$ be solutions of (3.2). If $\boldsymbol{E}_\rho, \boldsymbol{H}_\rho \in \boldsymbol{W}(curl, \mathbb{R}^3)$, then $\varepsilon(\rho)\boldsymbol{E}_\rho, \boldsymbol{H}_\rho \in \boldsymbol{W}(div, \mathbb{R}^3)$ and $[\boldsymbol{n}\cdot(\varepsilon(\rho)\boldsymbol{E}_\rho)]_\Sigma = 0$, $[\boldsymbol{n}\cdot \boldsymbol{H}_\rho]_\Sigma = 0$. Furthermore,*

$$div(\varepsilon(\rho)\boldsymbol{E}_\rho) = -\frac{1}{\kappa^2} div\boldsymbol{F}, \quad and \quad div\boldsymbol{H}_\rho = 0 \quad in \quad \boldsymbol{L}^2(\mathbb{R}^3).$$

Proof. If $\mathbf{F} \in \mathbf{W}(\text{div}, \mathbb{R}^3)$ and $\mathbf{E}_\rho$ and $\mathbf{H}_\rho$ in $\mathbf{L}^2(\mathbb{R}^3)$ are solutions of (3.2), then applying divergence operator in (3.2), we have

$$\text{div}(\varepsilon(\rho)\mathbf{E}_\rho) = -\frac{1}{\kappa^2}\text{div}\,\mathbf{F}, \quad \text{div}\,\mathbf{H}_\rho = 0 \;\text{ in }\; \mathbf{L}^2(\mathbb{R}^3),$$

and $\varepsilon(\rho)\mathbf{E}_\rho \in \mathbf{W}(\text{div}, \mathbb{R}^3)$, $\mathbf{H}_\rho \in \mathbf{W}(\text{div}, \mathbb{R}^3)$.
Now, for $\mathbf{u}_\rho = \varepsilon(\rho)\mathbf{E}_\rho$ or $\mathbf{H}_\rho$, we have for all $\phi \in \mathbf{V} = H^1_0(\Omega^{cd}) \cup \mathbb{W}^1_0(\Omega^{is})$,

$$\int_{\Omega^{cd}\cup\Omega^{is}} \phi \text{ div } \mathbf{u}_\rho dx = \int_{\Omega^{cd}} \phi \text{ div } \mathbf{u}^{cd}_\rho dx + \int_{\Omega^{is}} \phi \text{ div } \mathbf{u}^{is}_\rho dx,$$

and

$$\int_{\Omega^{cd}\cup\Omega^{is}} \mathbf{u}_\rho \cdot \nabla\phi dx = \int_{\Omega^{cd}} \mathbf{u}^{cd}_\rho \cdot \nabla\phi dx + \int_{\Omega^{is}} \mathbf{u}^{is}_\rho \cdot \nabla\phi dx.$$

In Ω^{cd}, by integration by parts (see [29, Theorem 1.2.16]) there holds

$$\int_{\Omega^{cd}} \phi \text{ div } \mathbf{u}^{cd}_\rho dx = \int_{\Omega^{cd}} \mathbf{u}^{cd}_\rho \cdot \nabla\phi dx - \int_\Sigma (\mathbf{n}\cdot\mathbf{u}^{cd}_\rho)\phi ds.$$

We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω^{cd}.
Let $\Omega^{is} = \lim_{R\to\infty} \Omega_R$ with $\Omega_R = B_R \cap \Omega^{is}$, $\partial\Omega_R = \partial B_R \cup \Sigma$.
In the domain Ω_R, we have, by integration by parts, (see [29, Lemma 1.3.2]),

$$\begin{aligned}\int_{\Omega_R} \phi \text{ div } \mathbf{u}^{is}_\rho dx &= \int_{\Omega_R} \mathbf{u}^{is}_\rho \cdot \nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{u}^{is}_\rho)\phi ds - \int_{\partial B_R} (\mathbf{n}\cdot\mathbf{u}^{is}_\rho)\phi ds \\ &= \int_{\Omega_R} \mathbf{u}^{is}_\rho \cdot \nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{u}^{is}_\rho)\phi ds - \int_{\partial B_R} (\mathbf{n}\cdot(\mathbf{u}^{is}_\rho + \mathbf{U}^{is}_\rho \times \mathbf{n}))\phi ds,\end{aligned}$$

where, if $\mathbf{u}^{is}_\rho = \varepsilon(\rho)\mathbf{E}_\rho$, $\mathbf{U}^{is}_\rho = -\frac{\varepsilon(\rho)}{i\kappa}\mathbf{H}_\rho$ or if $\mathbf{u}^{is}_\rho = \mathbf{H}_\rho$, $\mathbf{U}^{is}_\rho = -i\kappa\mathbf{E}_\rho$ and $\mathbf{n}\cdot(\mathbf{U}^{is}_\rho \times \mathbf{n}) = 0$.
Then, applying Silver-Müller conditions, yields

$$\int_{\partial B_R} \mathbf{n}\cdot(\mathbf{u}^{is}_\rho + \mathbf{U}^{is}_\rho \times \mathbf{n})\cdot\phi ds \to 0 \;\text{ as }\; R\to\infty.$$

Hence

$$\int_{\Omega^{is}} \phi \text{ div } \mathbf{u}^{is}_\rho dx = \int_{\Omega^{is}} \mathbf{u}^{is}_\rho \cdot \nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{u}^{is}_\rho)\phi ds.$$

Altogether we have

$$\int_{\Omega^{cd}\cup\Omega^{is}} \phi \text{ div } \mathbf{u}_\rho dx = \int_{\Omega^{cd}\cup\Omega^{is}} \mathbf{u}_\rho \cdot \nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{u}^{is}_\rho - \mathbf{n}\cdot\mathbf{u}^{cd}_\rho)\cdot\phi ds$$

then

$$\int_\Sigma (\mathbf{n}\cdot\mathbf{u}^{is}_\rho - \mathbf{n}\cdot\mathbf{u}^{cd}_\rho)\cdot\phi ds = 0,$$

for all ϕ implies $[\mathbf{n}\cdot\mathbf{u}_\rho]_\Sigma = 0$. This follows because integration by parts holds in $\mathbb{R}^3$ in this Beppo-Levi space. $\square$

Next we introduce the following notation:
For $\mathbf{E}_\rho \in \mathbf{W}(\mathrm{curl}, \mathbb{R}^3)$, $\mathbf{E}' \in \widetilde{\mathbf{W}}(\mathrm{curl}, \mathbb{R}^3) = \{\ell(r)\mathbf{E}' \in \mathbf{L}^2(\mathbb{R}^3), \mathrm{curl}\mathbf{E}' \in \mathbf{L}^2(\mathbb{R}^3)\}$, set

$$b_\rho(\mathbf{E}_\rho, \mathbf{E}') := \int_{\Omega^{cd}\cup\Omega^{is}} \left(\frac{1}{\mu} \mathrm{curl}\, \mathbf{E}_\rho \cdot \mathrm{curl}\, \overline{\mathbf{E}'} - \kappa^2 \varepsilon(\rho) \mathbf{E}_\rho \cdot \overline{\mathbf{E}'} \right) dx. \tag{3.3}$$

Proposition 2. *Let $\boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3)$. Let $\boldsymbol{E}_\rho$ and $\boldsymbol{H}_\rho$ in $\boldsymbol{L}^2(\mathbb{R}^3)$ solve (3.2). Then, $\boldsymbol{E}_\rho \in \boldsymbol{W}(curl, \mathbb{R}^3)$ and for all $\boldsymbol{E}' \in \widetilde{\boldsymbol{W}}(curl, \mathbb{R}^3)$,*

$$b_\rho(\boldsymbol{E}_\rho, \boldsymbol{E}') = \int_{\Omega^{cd}\cup\Omega^{is}} \boldsymbol{F} \cdot \overline{\boldsymbol{E}'} dx. \tag{3.4}$$

Proof. The proof of Proposition 2 is an extension of Proposition 3 in Peron [42] from a bounded to unbounded exterior domain Ω^{is} . First $(3.2)_1$ gives curl $\mathbf{E}_\rho \in \mathbf{L}^2(\mathbb{R}^3)$ for $\mathbf{H}_\rho$ in $\mathbf{L}^2(\mathbb{R}^3)$. Thus $\mathbf{E}_\rho \in \mathbf{W}(\mathrm{curl}, \mathbb{R}^3)$. Next we test $(3.2)_1$ by curl $\overline{\mathbf{E}'}$ and then we test $(3.2)_2$ by $\mathbf{E}' \in \widetilde{\mathbf{W}}(\mathrm{curl}, \mathbb{R}^3)$. Adding these equations yields (3.4). Note (3.4) also holds for $\mathbf{E}_\rho \in \widetilde{\widetilde{\mathbf{W}}}(\mathrm{curl}, \mathbb{R}^3) := \{\ell(r)^{-1}\mathbf{E} \in \mathbf{L}^2(\mathbb{R}^3), \mathrm{curl}\mathbf{E} \in \mathbf{L}^2(\mathbb{R}^3), \ell(r)\mathrm{curlcurl}\mathbf{E} \in \mathbf{L}^2(\mathbb{R}^3)\}$ and $\mathbf{E}' \in \widetilde{\mathbf{W}}(\mathrm{curl}, \mathbb{R}^3)$. This follows directly by integration by parts from (3.2) applying the arguments in [37, Section 3] where the Laplacian in weighed spaces is considered and a weighted Poincare inequality is used. □

The strong formulation of (3.4) is our next result, which corresponds to Peron's Proposition 2.15 in [42]. Again our formulation follows Peron but replaces the boundary conditions on $\partial\Omega$ by the decay conditions at infinity. In our proof we modify Peron's proof in the exterior domain. For the bounded interior domain his arguments remain unchanged, and only those parts are repeated here from [42] which are necessary for better reading.

Proposition 3. *If $\boldsymbol{E}_\rho \in \boldsymbol{W}(curl, \mathbb{R}^3)$ solves (3.4), then $\boldsymbol{E}_\rho$ solves (in the distributional sense):*

$$\begin{aligned}
&curl\; curl\, \boldsymbol{E}_\rho - \kappa^2 \boldsymbol{E}_\rho &&= \mu \boldsymbol{F}^{is}, &&in \;\; \Omega^{is},\\
&curl\; curl\, \boldsymbol{E}_\rho - \kappa^2 (1 + i\rho^2) \boldsymbol{E}_\rho &&= \mu \boldsymbol{F}^{cd}, &&in \;\; \Omega^{cd},\\
&[\boldsymbol{n} \times \boldsymbol{E}_\rho]_\Sigma = 0, \;\; [\boldsymbol{n} \times curl\, \boldsymbol{E}_\rho]_\Sigma &&= 0 &&on \;\; \Sigma,\\
&\textit{with Silver-Müller condition} && && \\
&|curl\, \boldsymbol{E}_\rho \times \widehat{\boldsymbol{x}} - i\kappa \boldsymbol{E}_\rho| &&= o\left(\frac{1}{|\boldsymbol{x}|}\right), &&as \;\; |\boldsymbol{x}| \longrightarrow \infty,
\end{aligned} \tag{3.5}$$

On the other hand, if $\boldsymbol{E}_\rho$ solves (3.5) then

$$\frac{1}{\mu} curl\; curl\, \boldsymbol{E}_\rho - \kappa^2 \varepsilon(\rho) \boldsymbol{E}_\rho = \boldsymbol{F}, \quad in \;\; \Omega^{cd} \cup \Omega^{is}, \tag{3.6}$$

and

$$div(\varepsilon(\rho)\boldsymbol{E}_\rho) = -\frac{1}{\kappa^2} div\, \boldsymbol{F}, \quad in \;\; \Omega^{cd} \cup \Omega^{is}. \tag{3.7}$$

Proof. For proving the first two equations in (3.5) we follow Peron [42].
Taking $\mathbf{E}' \in \mathfrak{D}'(\mathbb{R}^3)$ with support in Ω^{is} as test function in (3.4) and using

$$\int_{\Omega^{cd}\cup\Omega^{is}} \text{curl } \mathbf{E}_\rho \cdot \text{curl } \overline{\mathbf{E}'} dx = \langle \text{curl curl } \mathbf{E}_\rho, \mathbf{E}' \rangle_{\Omega^{is}},$$

we deduce the first equation of (3.5).
Next, we take $\mathbf{E}' \in \mathfrak{D}'(\mathbb{R}^3)$ with support in Ω^{cd} as test function in (3.4) and using

$$\int_{\Omega^{cd}\cup\Omega^{is}} \text{curl } \mathbf{E}_\rho \cdot \text{curl } \overline{\mathbf{E}'} dx = \langle \text{curl curl } \mathbf{E}_\rho, \mathbf{E}' \rangle_{\Omega^{cd}},$$

we deduce the second equation of (3.5).
The third relation (3.5) holds due to Lemma 1.
To continue with the proof, we choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω^{cd}. we apply Stokes formula for the bounded domain $\Omega = \Omega^{cd} \cup (\Omega^{is} \cap B_R)$, with $\partial(\Omega^{is} \cap B_R) = \Sigma \cup \partial B_R$ and $\partial\Omega = \partial B_R$, for all $\mathbf{E}, \mathbf{H} \in \mathbf{H}(\text{curl}, \Omega)$,

$$\int_\Omega \text{curl } \mathbf{E} \cdot \overline{\mathbf{H}} - \mathbf{E} \cdot \text{curl } \overline{\mathbf{H}})dx = \langle \mathbf{n} \times \mathbf{E}, \mathbf{H}_\tau \rangle_{\partial\Omega}, \tag{3.8}$$

where $\mathbf{H}_\tau = (\mathbf{n} \times \mathbf{H}) \times \mathbf{n}$.
According to $(3.5)_1$ we have curl $\mathbf{E}^{is} \in \mathbf{H}(\text{curl}, \Omega^{is} \cap B_R)$. So, applying formula (3.8) in $\Omega^{is} \cap B_R$ to $\mathbf{E} = \text{curl } \mathbf{E}_\rho^{is}$ and $\mathbf{H} = \mathbf{E}' \in \mathbf{H}(\text{curl}, \Omega)$, we have

$$\int_{\Omega^{is}\cap B_R} \text{curl } \mathbf{E}_\rho^{is} \cdot \text{curl } \overline{(\mathbf{E}')^{is}} dx =$$

$$\int_{\Omega^{is}\cap B_R} \text{curl curl } \mathbf{E}_\rho^{is} \cdot \overline{(\mathbf{E}')^{is}} dx + \langle \text{curl } \mathbf{E}_\rho^{is} \times \mathbf{n}, (\mathbf{E}')_\tau^{is} \rangle_{\partial(\Omega^{is}\cap B_R)}. \tag{$*$}$$

Applying formula (3.8) in Ω^{cd} to $\mathbf{E} = \text{curl } \mathbf{E}_\rho^{cd}$ and $\mathbf{H} = \mathbf{E}' \in \mathbf{H}(\text{curl}, \Omega)$, we have

$$\int_{\Omega^{cd}} \text{curl } \mathbf{E}_\rho^{cd} \cdot \text{curl } \overline{(\mathbf{E}')^{cd}} dx =$$

$$\int_{\Omega^{cd}} \text{curl curl } \mathbf{E}_\rho^{cd} \cdot \overline{(\mathbf{E}')^{cd}} dx + \langle \text{curl } \mathbf{E}_\rho^{cd} \times \mathbf{n}, (\mathbf{E}')_\tau^{cd} \rangle_\Sigma. \tag{$**$}$$

In $(*)$

$$\langle \text{curl } \mathbf{E}_\rho^{is} \times \mathbf{n}, (\mathbf{E}')_\tau^{is} \rangle_{\partial(\Omega^{is}\cap B_R)} = -\langle \text{curl } \mathbf{E}_\rho^{is} \times \mathbf{n}, (\mathbf{E}')_\tau^{is} \rangle_\Sigma + \langle \text{curl } \mathbf{E}_\rho^{is} \times \mathbf{n}, (\mathbf{E}')_\tau^{is} \rangle_{\partial B_R}.$$

Applying Silver-Müller radiation conditions as in Lemma 1 yields

$$\lim_{R\to\infty} |\langle \text{curl } \mathbf{E}_\rho^{is} \times \mathbf{n}, (\mathbf{E}')_\tau^{is} \rangle_{\partial B_R}| = 0$$

Then, letting $R \longrightarrow \infty$ in $(*)$ yields

$$\int_{\Omega^{is}} \text{curl } \mathbf{E}_\rho^{is} \cdot \text{curl } \overline{(\mathbf{E}')^{is}} dx =$$

$$\int_{\Omega^{is}} \text{curl curl } \mathbf{E}_\rho^{is} \cdot \overline{(\mathbf{E}')^{is}} dx - \langle \text{curl } \mathbf{E}_\rho^{is} \times \mathbf{n}, (\mathbf{E}')_\tau^{is} \rangle_\Sigma. \tag{$***$}$$

Adding $(**)$ and $(***)$ gives for all $\mathbf{E}' \in \mathbf{W}(\mathrm{curl}, \mathbb{R}^3)$

$$\int_{\Omega^{cd} \cup \Omega^{is}} \mathrm{curl}\, \mathbf{E}_\rho \cdot \mathrm{curl}\, \overline{\mathbf{E}'} dx =$$

$$\int_{\Omega^{cd} \cup \Omega^{is}} \mathrm{curl}\, \mathrm{curl}\, \mathbf{E}_\rho \cdot \overline{\mathbf{E}'} dx + \langle [\mathrm{curl}\, \mathbf{E}_\rho \times \mathbf{n}]_\Sigma, \mathbf{E}'_\tau \rangle_\Sigma.$$

Then (3.4), $(3.5)_1$ and $(3.5)_2$, yield

$$\langle [\mathrm{curl}\, \mathbf{E}_\rho \times \mathbf{n}]_\Sigma, \mathbf{E}'_\tau \rangle_\Sigma = 0,$$

which gives $(3.5)_3$. This follows because integration by parts holds in $\mathbb{R}^3$ in this Beppo-Levi space. □

Our next results are similar to those of Peron [42] whose results are for a bounded exterior domain, but here we investigate the case of an unbounded exterior domain.
The regularized form of problem (3.4) is: Find $\mathbf{E}_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$, such that, for all $\mathbf{E}'_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$,

$$\int_{\Omega^{cd} \cup \Omega^{is}} \left(\frac{1}{\mu} \mathrm{curl}\, \mathbf{E}_\rho \cdot \mathrm{curl}\, \overline{\mathbf{E}'_\rho} + \alpha\, \mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho) \cdot \mathrm{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho}) - \kappa^2 \varepsilon(\rho) \mathbf{E}_\rho \cdot \overline{\mathbf{E}'_\rho} \right) dx = \langle f, \mathbf{E}'_\rho \rangle, \tag{3.9}$$

where

$$\langle f, \mathbf{E}'_\rho \rangle = \int_{\Omega^{cd} \cup \Omega^{is}} \left(\mathbf{F} \cdot \overline{\mathbf{E}'_\rho} - \frac{\alpha}{\kappa^2} \mathrm{div}\, \mathbf{F} \cdot \mathrm{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho}) \right) dx, \tag{3.10}$$

and where $\alpha > 0$ is a parameter that will be needed next. We will use the following Theorem, (compare Peron [42, Theorem 2.21], see also Costabel et al. [11]), which corresponds to Peron's Theorem 2.21 in [42] and is its modification for an unbounded exterior domain and weighted spaces. Again for the bounded interior domain we repeat Peron's proofs almost verbatim.

Theorem 4. *There exists a real $\alpha > 0$, independent of ρ, such that if $\boldsymbol{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ is a solution of (3.9)-(3.10) for $\boldsymbol{F} \in \widetilde{\boldsymbol{W}}(div, \mathbb{R}^3) = \{\boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3), \ell(r) div \boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3)\}$, then*

$$div(\varepsilon(\rho)\boldsymbol{E}_\rho) + \frac{1}{\kappa^2} div\, \boldsymbol{F} = 0, \quad in \quad \Omega^{cd} \cup \Omega^{is}. \tag{3.11}$$

Furthermore $\boldsymbol{E}_\rho$ and $\boldsymbol{H}_\rho = \dfrac{1}{i\omega\varepsilon_0} curl\, \boldsymbol{E}_\rho$ solve Maxwell's equations (3.2).

Proof. Let us define the operator $\Delta_{\varepsilon(\rho)}$ from $\mathbb{W}^1_0(\mathbb{R}^3)$ to $\mathbb{W}^1_0(\mathbb{R}^3)'$ mapping φ to $\mathrm{div}(\varepsilon(\rho)\nabla\varphi)$, where $\mathrm{div}(\varepsilon(\rho)\nabla\varphi) \in \mathbb{W}^1_0(\mathbb{R}^3)'$ is defined for any $\psi \in \mathbb{W}^1_0(\mathbb{R}^3)$ by

$$\int_{\Omega^{cd} \cup \Omega^{is}} \varepsilon(\rho) \nabla\varphi \cdot \overline{\nabla\psi} dx.$$

If we define the domain of this operator by

$$\mathbf{D}(\Delta_{\varepsilon(\rho)}) = \{\varphi \in \mathbb{W}^1_0(\mathbb{R}^3) \mid \ell(r)\mathrm{div}(\varepsilon(\rho)\nabla\varphi) \in L^2(\mathbb{R}^3)\}.$$

Then $\nabla\varphi \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3,\rho)$ for $\varphi \in \mathbf{D}(\Delta_{\varepsilon(\rho)})$.
Let $\mathbf{E}_\rho$ solves (3.9). Choosing $\mathbf{E}' = \nabla\varphi$ which $\varphi \in \mathbf{D}(\Delta_{\varepsilon(\rho)})$. (3.9) gives

$$\begin{aligned}\int_{\Omega^{cd}\cup\Omega^{is}} (\alpha\,\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot \mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi}) - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \\ = \int_{\Omega^{cd}\cup\Omega^{is}} \left(\mathbf{F}\cdot\overline{\nabla\varphi} - \frac{\alpha}{\kappa^2}\mathrm{div}\,\mathbf{F}\cdot\mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi})\right)dx.\end{aligned} \tag{3.12}$$

Now, since $\varepsilon(\rho)\mathbf{E}_\rho, \mathbf{F} \in \mathbf{W}(\mathrm{div},\mathbb{R}^3)$ and $\varphi \in \mathbb{W}^1_0(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega^{cd}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx + \int_{\Omega^{is}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx.$$

Green's formula in Ω^{cd}, yields

$$\int_{\Omega^{cd}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega^{cd}} \kappa^2\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi})dx + \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho\overline{\nabla\varphi}))dS.$$

Next we choose a ball B_R with radius $R>0$ and boundary ∂B_R containing Ω^{cd}.
Let $\Omega^{is} = \lim_{R\to\infty}\Omega_R$ and $\Omega_R = B_R\cap\Omega^{is}$, with $\partial\Omega_R = \partial B_R\cup\Sigma$.
In the domain Ω_R, we have, by integration by parts,

$$\int_{\Omega_R} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega_R} \kappa^2\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi})dx + \int_{\partial\Omega_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds,$$

and

$$\int_{\partial\Omega_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds = -\int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds + \int_{\partial B_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds.$$

As in the proof of Lemma 2, applying Silver-Müller condition there holds

$$\int_{\partial B_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi}ds \longrightarrow 0, \quad \text{as} \quad R\longrightarrow\infty,$$

Hence

$$\int_{\Omega^{is}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega^{is}} \kappa^2\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi}dx - \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho\overline{\nabla\varphi}))ds - \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi}ds.$$

for all $\varphi \in \mathbf{D}(\Delta_{\varepsilon(\rho)})$ yielding $[\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)] = 0$ on Σ. Now

$$\int_{\mathbb{R}^3} \mathbf{F}\cdot\overline{\nabla\varphi}dx = \int_{\Omega^{cd}} \mathbf{F}\cdot\overline{\nabla\varphi}dx + \int_{\Omega^{is}} \mathbf{F}\cdot\overline{\nabla\varphi}dx,$$

Green's formula in Ω^{cd}, yields

$$\int_{\Omega^{cd}} \mathbf{F}\cdot\overline{\nabla\varphi}dx = -\int_{\Omega^{cd}} \mathrm{div}\,\mathbf{F}\cdot\overline{\varphi}dx - \int_\Sigma (\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds.$$

Again, we choose a ball B_R with radius $R>0$ and boundary ∂B_R containing Ω^{cd}. Setting $\Omega^{is} = \lim_{R\to\infty}\Omega_R$ with $\Omega_R = B_R\cap\Omega^{is}$, and $\partial\Omega_R = \partial B_R\cup\Sigma$.
In the domain Ω_R, we have, by integration by parts,

$$\int_{\Omega_R} \mathbf{F}\cdot\overline{\nabla\varphi}dx = -\int_{\Omega_R} \mathrm{div}\,\mathbf{F}\cdot\overline{\varphi}dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds + \int_{\partial B_R} (\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds,$$

from (3.2) note that there holds $\mathbf{F} = i\kappa \mathrm{curl}\mathbf{H} - \kappa^2\varepsilon(\rho)\mathbf{E}$, therefore applying Silver-Müller condition gives

$$\int_{\partial B_R} (\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty,$$

hence

$$\int_{\Omega^{is}} \mathbf{F}\cdot\overline{\nabla\varphi}dx = -\int_{\Omega^{is}} \mathrm{div}\,\mathbf{F}\cdot\overline{\varphi}dx + \int_{\Sigma}(\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds.$$

Then, we have altogether

$$\int_{\mathbb{R}^3} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\mathbb{R}^3} \kappa^2 \mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi}dx + \int_{\Sigma}\kappa^2\mathbf{n}\cdot[\varepsilon(\rho)\mathbf{E}_\rho]\overline{\varphi}ds = \int_{\mathbb{R}^3}\kappa^2\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi}dx$$

and

$$\int_{\mathbb{R}^3} \mathbf{F}\cdot\overline{\nabla\varphi}dx = -\int_{\mathbb{R}^3}\mathrm{div}\,\mathbf{F}\cdot\overline{\varphi}dx.$$

Similarly, according to (3.12) there holds

$$\int_{\mathbb{R}^3}(\alpha\,\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi}) + \frac{\alpha}{\kappa^2}\mathrm{div}\,\mathbf{F}\cdot\mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi})$$

$$-\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi} - \mathbf{F}\cdot\overline{\nabla\varphi}\big)\,dx = 0.$$

Then

$$\int_{\mathbb{R}^3}(\alpha\,\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi}) + \frac{\alpha}{\kappa^2}\mathrm{div}\,\mathbf{F}\cdot\mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi}) +$$

$$+\kappa^2\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi} + \mathrm{div}\,\mathbf{F}\cdot\overline{\varphi}\big)\,dx = 0.$$

Therefore, for all $\varphi \in \mathbf{D}(\Delta_{\varepsilon(\rho)})$

$$\int_{\mathbb{R}^3}\left(\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho) + \frac{1}{\kappa^2}\mathrm{div}\,\mathbf{F}\right)\cdot(\alpha\,\mathrm{div}(\overline{\varepsilon(\rho)\nabla\varphi}) + \kappa^2\overline{\varphi})dx = 0. \tag{3.13}$$

The sesquilinear form associated with the operator $-\Delta_{\varepsilon(\rho)}$ is uniformly coercive on $\mathbb{W}_0^1(\mathbb{R}^3)$, (see Giroire [20]), i. e. $\exists C > 0$

$$\mathrm{Re}\left(\int_{\mathbb{R}^3}\varepsilon(\rho)\nabla\varphi\cdot\overline{\nabla\varphi}dx\right) = \frac{1}{\mu}|\varphi|^2_{\mathbb{W}_0^1(\mathbb{R}^3)} \geq C\|\varphi\|^2_{\mathbb{W}_0^1(\mathbb{R}^3)}. \tag{3.14}$$

Next, we follow again Peron and examine the real non-zero eigenvalues λ of $-\Delta_{\varepsilon(\rho)}$, i.e :

$$-\Delta_{\varepsilon(\rho)}\varphi = \lambda\varphi \quad \text{in} \quad \mathbb{R}^3, \tag{3.15}$$

which gives after integration by parts

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\cdot\overline{\nabla\varphi}dx = \lambda\int_{\mathbb{R}^3}\varphi\cdot\overline{\varphi}dx.$$

Now (3.14) gives $\lambda \geq C$ and we take $\alpha > 0$ large enough such that $\dfrac{\kappa^2}{\alpha} < C$. Then $\dfrac{\kappa^2}{\alpha}$ is not an eigenvalue of $-\Delta_{\varepsilon(\rho)}$. Consequently to (3.13) implies

$$\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho) + \frac{1}{\kappa^2}\mathrm{div}\,\mathbf{F} = 0, \quad \text{in} \quad \mathbb{R}^3.$$

This way, according to (3.9) and (3.10) for all $\mathbf{E}'_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$,

$$\int_{\mathbb{R}^3} \left(\frac{1}{\mu}\mathrm{curl}\,\mathbf{E}_\rho \cdot \mathrm{curl}\,\overline{\mathbf{E}'_\rho} - \kappa^2 \varepsilon(\rho)\mathbf{E}_\rho \cdot \overline{\mathbf{E}'_\rho} \right) dx = \int_{\mathbb{R}^3} \mathbf{F} \cdot \overline{\mathbf{E}'_\rho} dx.$$

We set $\mathbf{H}_\rho = \dfrac{1}{i\omega\varepsilon_0}\mathrm{curl}\,\mathbf{E}_\rho$ in $\mathbb{R}^3$. Then from Proposition 2 follows, for all $\mathbf{E}'_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$,

$$\int_{\mathbb{R}^3} \left(\frac{i\omega\varepsilon_0}{\mu}\mathrm{curl}\,\mathbf{H}_\rho \cdot \overline{\mathbf{E}'_\rho} - \kappa^2 \varepsilon(\rho)\mathbf{E}_\rho \cdot \overline{\mathbf{E}'_\rho} \right) dx = \int_{\mathbb{R}^3} \mathbf{F} \cdot \overline{\mathbf{E}'_\rho} dx,$$

implying

$$\mathrm{curl}\,\mathbf{H}_\rho + i\kappa\varepsilon(\rho)\mathbf{E}_\rho = \frac{1}{i\kappa}\mathbf{F}, \quad \text{in} \quad \mathbb{R}^3.$$

□

In this section, we give a variational formulation for the term $\varphi_\rho \in \mathcal{V}$ (see (2.8)), which appears in the decomposition of the electrical field, compare Theorem 8. Again, we extend the ideas of Peron [42] to prove Lemma 3 for the unbounded exterior domain. Our Lemma 3 corresponds to Peron's Lemma 2.33 in [42] and gives the appropriate setting for an unbounded exterior domain

Lemma 3. *Let $\boldsymbol{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ solve of (3.9)-(3.10) for $\boldsymbol{F} \in \widetilde{\boldsymbol{W}}(div, \mathbb{R}^3)$, and let $(\boldsymbol{w}_\rho, \varphi_\rho) \in \boldsymbol{W}^1_0(\mathbb{R}^3) \times \mathcal{V}$ with $div\boldsymbol{w}_\rho = 0$ given by Theorem 8. Then, φ_ρ solves the variational problem: Find $\varphi_\rho \in \mathcal{V}$, such that for all $\psi \in \mathcal{V}$,*

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\nabla\varphi_\rho \cdot \overline{\nabla\psi} dx = \frac{1}{\kappa^2}\int_{\mathbb{R}^3} div\,\boldsymbol{F} \cdot \overline{\psi} dx + \frac{1}{\mu} i\rho^2 \int_\Sigma \boldsymbol{w}_\rho \cdot \boldsymbol{n}|_\Sigma \overline{\psi} ds. \tag{3.16}$$

Proof. Due to Theorem 8 there exists an unique couple $(\mathbf{w}_\rho, \varphi_\rho) \in \mathbb{W}^1_0(\mathbb{R}^3) \times \mathcal{V}$ such that $\mathbf{E}_\rho = \mathbf{w}_\rho + \nabla\varphi_\rho$. Thus we have

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\nabla\varphi_\rho \cdot \overline{\nabla\psi} dx = \int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{E}_\rho \cdot \overline{\nabla\psi} dx - \int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx, \forall\psi \in \mathcal{V}.$$

Then since $\ell(r)\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho) \in \mathbf{L}^2(\mathbb{R}^3)$and $\varepsilon(\rho)\mathbf{E}_\rho \in \mathbf{L}^2(\mathbb{R}^3)$ there holds

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{E}_\rho \cdot \overline{\nabla\psi} dx = -\int_{\mathbb{R}^3} \mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho) \cdot \overline{\psi} dx,$$

so, due to Theorem 4,

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{E}_\rho \cdot \overline{\nabla\psi} dx = \frac{1}{\kappa^2}\int_{\mathbb{R}^3} \mathrm{div}\,\mathbf{F} \cdot \overline{\psi} dx.$$

Next, we have

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx = \int_{\Omega^{is}} \varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx + \int_{\Omega^{cd}} \varepsilon(\rho)^{cd}\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx,$$

and, by integration by parts,

$$\int_{\Omega^{cd}} \varepsilon(\rho)^{cd}\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx = \int_{\Omega^{cd}} \mathrm{div}(\varepsilon(\rho)^{cd}\mathbf{w}_\rho)\overline{\psi} dx - \int_{\Sigma} (\varepsilon(\rho)^{cd}\mathbf{w}_\rho \cdot \mathbf{n})\overline{\psi} ds.$$

Next, we choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω^{cd}. Setting $\Omega^{is} = \lim_{R\to\infty} \Omega_R$ with $\Omega_R = B_R \cap \Omega^{is}$, with $\partial\Omega_R = \partial B_R \cup \Sigma$ we have

$$\int_{\Omega_R} \varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx = \int_{\Omega_R} \mathrm{div}(\varepsilon(\rho)^{is}\mathbf{w}_\rho)\overline{\psi} dx + \int_{\Sigma} (\varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \mathbf{n})\overline{\psi} ds + \int_{\partial B_R} (\varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \mathbf{n})\overline{\psi} ds.$$

Note Silver-Müller conditions yield

$$\int_{\partial B_R} (\varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \mathbf{n})\overline{\psi} ds = \int_{\partial B_R} (\varepsilon(\rho)^{is}[\mathbf{w}_\rho - \mathbf{w}_\rho \times \mathbf{n}] \cdot \mathbf{n})\overline{\psi} ds \to 0 \text{ as } R \to \infty,$$

Hence

$$\int_{\Omega^{is}} \varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx = \int_{\Omega^{is}} \mathrm{div}(\varepsilon(\rho)^{is}\mathbf{w}_\rho)\overline{\psi} dx + \int_{\Sigma} (\varepsilon(\rho)^{is}\mathbf{w}_\rho \cdot \mathbf{n})\overline{\psi} ds.$$

Thus

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{w}_\rho \cdot \overline{\nabla\psi} dx = \int_{\mathbb{R}^3} \mathrm{div}(\varepsilon(\rho)\mathbf{w}_\rho)\overline{\psi} dx + \int_{\Sigma} (\varepsilon(\rho)^{is} - \varepsilon(\rho)^{cd})\mathbf{w}_\rho \cdot \mathbf{n}|_\Sigma \overline{\psi} ds.$$

Since div $\mathbf{w}_\rho = 0$ in $\mathbb{R}^3$, we get (3.16) because (see before (3.2)) $\varepsilon(\rho)^{is} - \varepsilon(\rho)^{cd} = -\frac{1}{\mu} i\rho^2$. □

3.2 Uniform a priori estimate of the electrical field in $\mathbb{R}^3$

Next we give an existence and uniqueness result for the solution of (3.9)-(3.10). The proof uses an a priori estimate for the electrical field for large conductivity, hence for large ρ with a constant C independent of ρ. The ideas of this section are based on those of Peron [42], but using compactness results for the embedding of weighted spaces with unbounded domains. This is a crucial difference of our proof compared to Peron's proof.

In the following we assume there holds the

Spectral hypothesis (**SH**): κ^2 is not an eigenvalue of the limit problem: Find $\mathbf{E}_0 \in \mathbf{W}(\mathrm{curl}, \Omega^{is})$ with $\mathbf{E}_0 \times n = 0$ on Σ such that, for all $\mathbf{E}' \in \mathbf{W}(\mathrm{curl}, \Omega^{is})$, with $\mathbf{E}' \times n = 0$ on Σ

$$\int_{\Omega^{is}} (\mathrm{curl}\, \mathbf{E}_0 \cdot \mathrm{curl}\, \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_0 \cdot \overline{\mathbf{E}'}) dx = 0 \text{ and } \mathbf{n} \times \mathbf{E} = 0 \text{ on } \Sigma. \tag{3.17}$$

Now, we can formulate our main theorem of this chapter which is the weighted version of Peron's Theorem 2.27.

Theorem 5. *Under the spectral hypothesis (**SH**), there exists a constant $\rho_0 > 0$, such that for all $\rho > \rho_0$, problem (3.9)-(3.10) admits an unique solution $\boldsymbol{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ for $\boldsymbol{F} \in \widetilde{\boldsymbol{W}}(div, \mathbb{R}^3)$, satisfying*

$$\begin{aligned} \|curl\, \boldsymbol{E}_\rho\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} + \|div(\varepsilon(\rho)\boldsymbol{E}_\rho)\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} + \|\boldsymbol{E}_\rho\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} + \rho\|\boldsymbol{E}_\rho\|_{\boldsymbol{L}^2(\Omega^{cd})} \\ \leq C\|\boldsymbol{F}\|_{\boldsymbol{W}(div,\mathbb{R}^3)}, \end{aligned} \tag{3.18}$$

with a constant $C > 0$, independent of ρ.

The proof of Theorem 5 is given in various steps, below. The estimate (3.18) is based on the a priori estimate (3.19).

Theorem 6. *Let (**SH**) hold, then there exists a constant $\rho_0 > 0$, such that, for all $\rho > \rho_0$, if $\boldsymbol{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ solves (3.9)-(3.10) for $\boldsymbol{F} \in \widetilde{\boldsymbol{W}}(div, \mathbb{R}^3)$, then*

$$\|\boldsymbol{E}_\rho\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} \leq C\|\boldsymbol{F}\|_{\boldsymbol{W}(div,\mathbb{R}^3)}, \tag{3.19}$$

where $C > 0$ is a constant independent of ρ.

Proof. The proof is similar to the one by Peron [42] but here we use a compact embedding of $\mathbf{PH}^1(\mathbb{R}^3)$ into $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$ where

$$\mathbf{PH}^1(\mathbb{R}^3) = \{\varphi \;\mid\; \varphi^{is} \in (\mathbb{W}^1_0(\Omega^{is}))^3 \;\text{ and }\; \varphi^{cd} \in (H^1(\Omega^{cd}))^3\}.$$

Taking $\mathbf{E}_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$, solution of (3.9)-(3.10), we have $\forall \Phi \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$,

$$\begin{aligned} \int_{\mathbb{R}^3} \left(\frac{1}{\mu}\text{curl } \mathbf{E}_\rho \cdot \text{curl } \overline{\Phi} + \alpha \text{ div}(\varepsilon(\rho)\mathbf{E}_\rho) \cdot \text{div}(\overline{\varepsilon(\rho)\Phi}) - \frac{\kappa^2}{\mu}\mathbf{E}_\rho \overline{\Phi} \right) dx- \\ -\frac{1}{\mu} i\rho^2 \int_{\Omega^{cd}} \mathbf{E}_\rho \overline{\Phi} dx = \int_{\mathbb{R}^3} \left(\mathbf{F} \cdot \overline{\Phi} - \frac{\alpha}{\kappa^2}\text{div } \mathbf{F} \cdot \text{div}(\overline{\varepsilon(\rho)\Phi}) \right) dx. \end{aligned} \tag{3.20}$$

Then, due to Theorem 4 there holds,

$$\text{div}(\varepsilon(\rho)\mathbf{E}_\rho) + \frac{1}{\kappa^2}\text{div } \mathbf{F} = 0, \quad \text{in} \quad \mathbb{R}^3.$$

$\ell(r)^{-2}\Phi \in (C_0^\infty(\mathbb{R}^3)^3) \subset \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ implies with $\ell(r) \equiv 1$ on Ω^{cd}

$$\int_{\mathbb{R}^3} (\text{curl } \mathbf{E}_\rho \cdot \text{curl } (\ell(r)^{-2}\overline{\Phi}) - \kappa^2 \mathbf{E}_\rho \overline{\Phi}\, \ell(r)^{-2}) dx - i\rho^2 \int_{\Omega^{cd}} \mathbf{E}_\rho \overline{\Phi} dx = \mu \int_{\mathbb{R}^3} \mathbf{F} \cdot \overline{\Phi}\, \ell(r)^{-2} dx, \forall \Phi \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho). \tag{3.21}$$

Just like Peron we prove the theorem by contradiction argument, but we crucially apply a compactness result for the embedding in weighted Sobolev spaces by Avantaggiati and Troisi. Since Peron considers only bounded domains, he can, in contrary, apply standard embedding arguments (Rellich's theorem).
Suppose that exists a sequence $\{\mathbf{F}_{\rho_n}\}_{n \geq 1}$ in $\widetilde{\mathbf{W}}(\text{div}, \mathbb{R}^3)$ with $\rho_n \longrightarrow \infty$, and $\|\mathbf{F}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = 1$

and $\mathbf{F}_{\rho_n} \cdot \mathbf{n} = 0$ in Σ, and such that for the corresponding solution $\mathbf{E}_{\rho_n} \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho_n)$ there holds

$$\lim_{n\to\infty} \|\mathbf{E}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = \infty.$$

Writing $\;\widetilde{}\;$ for dividing by $\|\mathbf{E}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}$, i.e. $\widetilde{\mathbf{E}}_{\rho_n} = (\|\mathbf{E}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)})^{-1}\mathbf{E}_{\rho_n}$ we have

$$\|\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = 1, \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = 0. \tag{3.22}$$

We will show that $\{\widetilde{\mathbf{E}}_{\rho_n}\}_{n\geq 1}$ is bounded in $\mathbf{X}_{\mathtt{TN}}(\mathbb{R}^3)$.
With $\Phi = \widetilde{\mathbf{E}}_{\rho_n}$, (3.21) becomes with l.o.t$=\int_{\mathbb{R}^3} \mathrm{curl}\widetilde{\mathbf{E}}_\rho . \widetilde{\mathbf{E}}_\rho \mathrm{curl}(\ell(r)^{-2})dx$

$$\|\mathrm{curl}\, \widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \text{l.o.t} - \kappa^2 \|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} - i\rho_n^2 \|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\Omega^{cd})} = \mu(\widetilde{\mathbf{F}}_{\rho_n}, \widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \tag{3.23}$$

Taking imaginary parts we have

$$\rho_n^2 \|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\Omega^{cd})} = -\mu \mathrm{Im}(\widetilde{\mathbf{F}}_{\rho_n}, \widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}, \tag{3.24}$$

and with Cauchy-Schwarz inequality we obtain

$$|\mathrm{Im}(\widetilde{\mathbf{F}}_{\rho_n}, \widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}| \leq \|\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \|\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}.$$

Hence (3.22) yields

$$\lim_{n\to\infty} \|\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{L}^2(\Omega^{cd})} = 0. \tag{3.25}$$

Also, taking real parts in (3.23), we get

$$\|\mathrm{curl}\, \widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \text{l.o.t} - \kappa^2 \|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = \mu \mathrm{Re}(\widetilde{\mathbf{F}}_{\rho_n}, \widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \tag{3.26}$$

Hence due to Cauchy-Schwarz inequality and (3.22), there are constants C_1 and C_2 independent of n, such that

$$\|\mathrm{curl}\, \widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \leq C_1 + C_2 \|\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \tag{3.27}$$

Therefore, $\{\mathrm{curl}\, \widetilde{\mathbf{E}}_{\rho_n}\}_{n\geq 1}$ is bounded in $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$.
Let $(\mathbf{w}_{\rho_n}, \varphi_{\rho_n}) \in \mathbb{W}^1_0(\mathbb{R}^3) \times \mathcal{V}$, (for definition of $\mathcal{V}$ see (2.8)) be given by Theorems 7 and 8 by Girault [18], such that

$$\widetilde{\mathbf{E}}_{\rho_n} = \widetilde{\mathbf{w}}_{\rho_n} + \nabla \widetilde{\varphi}_{\rho_n}, \quad \text{and} \quad \mathrm{div}\, \widetilde{\mathbf{w}}_{\rho_n} = 0, \quad \text{in} \quad \mathbb{R}^3,$$

and

$$\|\widetilde{\mathbf{w}}_{\rho_n}\|_{\mathbb{W}^1_0(\mathbb{R}^3)} \leq C \|\mathrm{curl}\, \widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}, \tag{3.28}$$

where $C > 0$ is a constant independent of ρ_n. Therefore, $\{\widetilde{\mathbf{w}}_{\rho_n}\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{W}^1_0(\mathbb{R}^3)$.
According to Lemma 3 and (3.11), $\widetilde{\varphi}_{\rho_n}$ satisfies for all $\psi \in \mathcal{V}$,

$$\int_{\mathbb{R}^3} \varepsilon(\rho) \nabla \widetilde{\varphi}_{\rho_n} \cdot \overline{\nabla \psi} dx = \frac{1}{\kappa^2} \int_{\mathbb{R}^3} \mathrm{div}\, \widetilde{\mathbf{F}}_{\rho_n} \cdot \overline{\psi} dx + \frac{1}{\mu} i\rho^2 \int_\Sigma \widetilde{\mathbf{w}}_{\rho_n} \cdot \mathbf{n}|_\Sigma \overline{\psi} ds. \tag{3.29}$$

Let $\rho_0 > 0$ and the constant $C_{\rho_0} > 0$ be given by Theorem 3. We set $\delta_n = 1 + i\rho_n^2$. Then there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ we have $|\delta_n| \geq \rho_0$. Note div $\widetilde{\mathbf{F}}_{\rho_n}$ and $\widetilde{\mathbf{w}}_{\rho_n} \cdot \mathbf{n}$ verify the hypotheses of Theorem 3. Also, problem (3.29) is coercive on $\mathcal{V}$. Hence the solution of (3.29) belongs in $PH^2(\mathbb{R}^3)$ and there holds for any $n \geq n_0$,

$$\|\widetilde{\varphi}^{cd}_{\rho_n}\|_{H^2(\Omega^{cd})} + \|\widetilde{\varphi}^{is}_{\rho_n}\|_{\mathbb{W}^2_1(\Omega^{is})} \leq C_{\delta_0}\left(\|\mathrm{div}\, \widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\widetilde{\mathbf{w}}_{\rho_n}\cdot \mathbf{n}\|_{H^{\frac{1}{2}}(\Sigma)}\right).$$

Thus $\{\nabla\widetilde{\varphi}_{\rho_n}\}_{n\geq 1}$ is bounded in $\mathbf{PH}^1(\mathbb{R}^3)$, and $\{\widetilde{\mathbf{E}}_{\rho_n}\}_{n\geq 1}$ is bounded in $\mathbf{H}^1(\Omega^{cd}) \cup \left(\mathbb{W}^1_0(\Omega^{is})\right)^3$. According to Lemma 5, the embedding of $\mathbf{PH}^1(\mathbb{R}^3)$ in $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$ is compact. This implies that there exists a subsequence $\{\widetilde{\mathbf{E}}_{\rho_n}\}_{n\geq 1}$ and $\widetilde{\mathbf{E}} \in \mathbf{L}^2_{0,-1}(\mathbb{R}^3)$, such that

$$\widetilde{\mathbf{E}}_{\rho_n} \rightharpoonup \widetilde{\mathbf{E}} \quad \text{in} \quad \left(\mathbf{PH}^1(\mathbb{R}^3)\right)^3, \widetilde{\mathbf{E}}_{\rho_n} \to \widetilde{\mathbf{E}} \quad \text{in} \quad \mathbf{L}^2_{0,-1}(\mathbb{R}^3), \tag{3.30}$$

with (3.22) we have therefore

$$\|\widetilde{\mathbf{E}}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = 1. \tag{3.31}$$

To get a contradiction, we show that $\widetilde{\mathbf{E}} = 0$ in $\Omega^{is} \cup \Omega^{cd}$.
Note $\|\widetilde{\mathbf{E}}\|_{\mathbf{L}^2(\Omega^{cd})} = 0$ due to (3.25) yielding

$$\widetilde{\mathbf{E}} = 0, \quad \text{in} \quad \Omega^{cd}. \tag{3.32}$$

Next, we take $\Phi \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ with support in Ω^{is}. Then $\mathbf{n}\cdot\Phi = 0$, $\mathbf{n}\times\Phi = 0$ on Σ and due to (3.21), we have

$$(\mathrm{curl}\, \widetilde{\mathbf{E}}_{\rho_n}, \mathrm{curl}\, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{is})} + \text{l.o.t} - \kappa^2(\widetilde{\mathbf{E}}_{\rho_n}, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{is})} = \mu(\widetilde{\mathbf{F}}_{\rho_n}, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{is})}. \tag{3.33}$$

According to (3.30) letting $n \longrightarrow \infty$ we obtain

$$(\mathrm{curl}\, \widetilde{\mathbf{E}}, \mathrm{curl}\, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{is})} + \text{l.o.t.} - \kappa^2(\widetilde{\mathbf{E}}, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{is})} = 0. \tag{3.34}$$

Now (**SH**) gives $\widetilde{\mathbf{E}} = 0$, in Ω^{is}, and therefore altogether $\widetilde{\mathbf{E}} = 0$, in $\mathbb{R}^3$, which is a contradiction to (3.31) and therefore (3.19) holds. □

Now with the help of Theorem 6 we can prove Theorem 5. The proof follows directly Peron's proof of his Theorem 2.27 in [42]. But we must analyze in weighted spaces and therefore present in our proof the necessary modifications.

Proof of Theorem 5: Let $\rho_0 > 0$ be given by Theorem 6. Let us assume $\mathbf{E}_\rho$ solves (3.9)-(3.10). Then, $\mathbf{E}_\rho$ solves (3.21) and taking $\Phi = \mathbf{E}_\rho$ we get

$$\|\mathrm{curl}\, \mathbf{E}_\rho\|^2_{\mathbf{L}^2(\mathbb{R}^3)} - \kappa^2\|\mathbf{E}_\rho\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} - i\rho^2\|\mathbf{E}_\rho\|^2_{\mathbf{L}^2(\Omega^{cd})} = \mu(\mathbf{F}, \mathbf{E}_\rho)_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \tag{3.35}$$

Taking successively again imaginary and real parts as in the proof of Theorem 6 we obtain the a priori estimate (3.21) from (3.11) and

$$\rho\|\mathbf{E}_\rho\|_{\mathbf{L}^2(\Omega^{cd})} \leq C_1\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}, \tag{3.36}$$

and

$$\|\mathrm{curl}\, \mathbf{E}_\rho\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C_2\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}. \tag{3.37}$$

Next we derive existence and uniqueness of the solution of (3.9)-(3.10). The proof is almost the same as Peron's proof, only that we have to argue differently with the compact embedding due to our unbounded exterior domain. Next, note that the a priori estimate (3.18) implies the injectivity of the solution operator of the variational problem (3.9). Therefore to show existence of the solution it suffices to demonstrate that this operator is surjective. We introduce the sesquilinear form c_ρ defined for all $\mathbf{E}_\rho, \mathbf{E}'_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ as

$$c_\rho(\mathbf{E}_\rho, \mathbf{E}'_\rho) = \int_{\mathbb{R}^3} \left(\frac{1}{\mu} \text{curl } \mathbf{E}_\rho \cdot \text{curl } \overline{\mathbf{E}'_\rho} + \alpha \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) \cdot \operatorname{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho}) \right) dx. \tag{3.38}$$

We can demonstrate that c_ρ is coercive on $\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$, by a suitable modification of Corollary 3.16 in [1]. As another possibility we can apply the arguments in Hiptmair's proof of his theorem 2.1 [24] which is listed as Theorem 10 in the next subsection for convenience. According to the generalization of the Lax-Milgram Theorem (see [27, Lemma 13.6]), we deduce that the operator C_ρ from $\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ into $\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)'$ is a isomorphism and thus C_ρ is a Fredholm operator. Now the embedding $I_\rho(\mathbf{E}_\rho) = \varepsilon(\rho)\mathbf{E}_\rho$ for $\mathbf{E}_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ from $\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ into $\mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)'$ is compact. Hence $C_\rho - \kappa^2 I_\rho$ is a Fredholm operator. In particular, it is surjective if and only if his adjoint $C^*_\rho - \kappa^2 I^*_\rho$ is injective where $I^*_\rho = \overline{\varepsilon(\rho)} I_\rho$. Let c^*_ρ be the sesquilinear form associated with the operator C^*_ρ, i.e.

$$c^*_\rho(\mathbf{E}_\rho, \mathbf{E}'_\rho) = \int_{\mathbb{R}^3} \left(\frac{1}{\mu} \text{curl } \mathbf{E}_\rho \cdot \text{curl } \overline{\mathbf{E}'_\rho} + \alpha \operatorname{div}(\overline{\varepsilon(\rho)}\mathbf{E}_\rho) \cdot \operatorname{div}(\varepsilon(\rho)\overline{\mathbf{E}'_\rho}) \right) dx, \forall \mathbf{E}_\rho, \mathbf{E}'_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho). \tag{3.39}$$

Now similar to Theorem 5, we can prove an a priori estimate for $C^*_\rho - \kappa^2 I^*_\rho$ yielding its injectivity and therefore the desired surjectivity of the operator $C_\rho - \kappa^2 I_\rho$.

3.3 Mathematical tools: Decomposition of vector fields and compact embeddings in weighted spaces

In this subsection we collect the mathematical tools we have needed in the proof of our a priori estimate (Theorem 5), namely a vector Helmholtz decomposition in $\mathbb{R}^3$ and a compactness results (Lemma 6) for the embedding in weighted spaces.
First we consider the vector potential of divergence-free vector fields and present results for a Helmholtz decomposition by Girault [18]. The weighted Sobolev spaces used here were introduced and studied by Hanouzet in [21].
For any multi-index α in $\mathbb{N}^3$, we denote by ∂^α the differential operator of order α:

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

Then, for all m in $\mathbb{N}$ and all k in $\mathbb{Z}$, we define the weighted Sobolev space:

$$\mathbb{W}^m_k(\Omega^{is}) := \left\{ v \in \mathfrak{D}'(\Omega^{is}) \mid \forall \alpha \in \mathbb{N}^3,\ 0 \le |\alpha| \le m,\ \ell(r)^{|\alpha|-m+k} \partial^\alpha v \in L^2(\Omega^{is}) \right\},$$

which is a Hilbert space for the norm:

$$\|v\|_{\mathbb{W}_k^m(\Omega^{is})} = \left\{ \sum_{|\alpha|=0}^{m} \|\ell(r)^{|\alpha|-m+k}\partial^\alpha v\|^2_{L^2(\Omega^{is})} \right\}^{\frac{1}{2}}.$$

Observe that, as a particular case,

$$\mathbb{W}_0^0(\Omega^{is}) = L^2(\Omega^{is}) \quad \text{and} \quad \mathbb{W}_{-1}^0(\mathbb{R}^3) = L^2_{0,-1}(\mathbb{R}^3).$$

For all $n \in \mathbb{Z}$, $\mathbf{P}_n$ denotes the space of all polynomials (of three variables) of degree at most n, with the convention that the space is reduced to zero when n is negative.
$\mathcal{P}_n$ is the subspace of all harmonic polynomials of $\mathbf{P}_n$, again with the convention that the space is reduced to zero when n is negative.
For all integers $k \geq 0$, we define the following subspace of $(\mathcal{P}_k)^3$,

$$\mathcal{G}_k := \{\nabla q \mid q \in \mathcal{P}_{k+1}\}.$$

Note that $\mathcal{G}_0 = \mathbb{R}^3$.
The following theorem by Girault [18] characterizes the vector potentials of some divergence-free vector fields.

Theorem 7. *(V. Girault [18, Theorem 3.2]) Let m belong to $\mathbb{Z}$ and k belong to $\mathbb{N} \cup \{-1, -2\}$ and let $\boldsymbol{u}$ be a vector field in $\mathbb{W}_{m-k}^m(\mathbb{R}^3)^3$ such that*

$$\mathit{div}\, \boldsymbol{u} = 0. \tag{3.40}$$

Then $\boldsymbol{u}$ has a unique vector potential Ψ in $\mathbb{W}_{m-k}^{m+1}(\mathbb{R}^3)^3/\mathcal{G}_{k-1}$ such that

$$\boldsymbol{u} = \mathit{curl}\,\Psi, \quad \mathit{div}\,\Psi = 0, \tag{3.41}$$

and

$$\|\Psi\|_{\mathbb{W}_{m-k}^{m+1}(\mathbb{R}^3)^3/\mathcal{G}_{k-1}} \leq C\|\boldsymbol{u}\|_{\mathbb{W}_{m-k}^m(\mathbb{R}^3)^3}. \tag{3.42}$$

When $k = 0, -1, -2$, the vector potential is unique in $\mathbb{W}_{m-k}^{m+1}(\mathbb{R}^3)^3$ and (3.42) can be slightly refined:

$$\|\Psi\|_{\mathbb{W}_{m-k}^{m+1}(\mathbb{R}^3)^3/\mathcal{G}_{k-1}} \leq C\|\mathit{curl}\,\boldsymbol{u}\|_{\mathbb{W}_{m-k}^{m-1}(\mathbb{R}^3)^3}. \tag{3.43}$$

The following result is based on the paper by Girault [18]. In the case of a bounded domain, there are two classical orthogonal decompositions of vector fields: a decomposition in $\mathbf{L}^2$ and a decomposition in H_0^1 (cf. for example [19]). The following theorem establishes the analogue of the decomposition in $\mathbf{L}^2$ for vector fields in $\mathbb{R}^3$. Beforehand, we introduce space

$$\mathbf{V}_k^m(\mathbb{R}^3) := \left\{\mathbf{v} \in \mathbb{W}_k^m(\mathbb{R}^3)^3 \mid \operatorname{div} \mathbf{v} = 0\right\},$$

and the following subspace of $(\mathcal{P}_k)^3$, which is analogue of $\mathcal{G}_k$:

$$\mathcal{C}_k := \{\operatorname{curl} \mathbf{q} \mid \mathbf{q} \in (\mathcal{P}_{k+1})^3\},$$

with the usual convention that $\mathcal{C}_k = \{0\}$, when $k < 0$, observe that $\mathcal{C}_0 = \mathbb{R}^3 = \mathcal{G}_0$.
In addition, for all $k \geq 1$, $\mathcal{G}_k \subset \mathcal{C}_k$, but the inverse inclusion is false.

Theorem 8. *(V. Girault [18, Theorem 5.1]) Let the integers m and k belong to $\mathbb{Z}$ and let $\boldsymbol{u}$ be a vector field in $\mathbb{W}^m_{m+k}(\mathbb{R}^3)^3$.*

1. *If $k \leq 1$, $\boldsymbol{u}$ has the decomposition*

$$\boldsymbol{u} = \nabla p + curl\,\Phi, \tag{3.44}$$

where Φ is unique in $\boldsymbol{V}^{m+1}_{m+k}(\mathbb{R}^3)/\mathcal{C}_{-k-1}$ and p is uniquely determined by $\boldsymbol{u}$ and Φ in $\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)/\mathbb{R}$, or $\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)$ if $k = 0$ or 1. They satisfy the bounds:

$$\|\Phi\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)^3/\mathcal{C}_{-k-1}} + \|p\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)/\mathbb{R}} \leq C\|\boldsymbol{u}\|_{\mathbb{W}^m_{m+k}(\mathbb{R}^3)^3}, \tag{3.45}$$

with the convention that the quotient norm of p is replaced by $\|p\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)}$ when $k = 0$ or 1.

2. *If $k \geq 2$ has the decomposition (3.44) with a unique p in $\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)$ and a unique Φ in $\boldsymbol{V}^{m+1}_{m+k}(\mathbb{R}^3)$ if and only if $\boldsymbol{u}$ is orthogonal to $\mathcal{C}_{k-2}$ (for the duality paring). The analogue of (3.45) holds:*

$$\|\Phi\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)^3} + \|p\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)} \leq C\|\boldsymbol{u}\|_{\mathbb{W}^m_{m+k}(\mathbb{R}^3)^3}, \tag{3.46}$$

3. *When both m and k belong to $\mathbb{N}$, the decomposition is orthogonal for the scalar product of $\boldsymbol{L}^2(\mathbb{R}^3)$.*

Now, this part is concerned with compact embedding in weighted Sobolev spaces for unbounded domains, and is based on the paper by Avantaggiati and Troisi [2].
Let Ω be an unbounded set of $\mathbb{R}^n$, provided with the cone property, and $\delta \in C^0(\overline{\Omega})$, a positive continuous function divergent for $|\mathbf{x}| \to \infty$, satisfying the following conditions:

1. There exist two open and separated subsets Ω_1 and Ω_2 of $\mathbb{R}^n$, such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ and

$$\delta(\mathbf{x}) \leq 1, \ \ \forall \mathbf{x} \in \Omega_1, \ \ \delta(\mathbf{x}) \geq 1, \ \ \forall \mathbf{x} \in \Omega_2.$$

We will put also, $\Omega_0 = \Omega$.

2. We put, for each $\mathbf{x}_0 \in \Omega_i$, $i = 0, 1, 2$,

$$A_i(\mathbf{x}_0) = \Omega_i \cap \{\mathbf{x} \ | \ |\mathbf{x} - \mathbf{x}_0| < \delta(\mathbf{x}_0)\},$$

it is able

$$c_1\delta(\mathbf{x}_0) \leq \delta(\mathbf{x}) \leq c_2\delta(\mathbf{x}_0), \ \ \forall \mathbf{x} \in A_i(\mathbf{x}_0),$$

where c_1 and c_2 are two positive constants independent of $\mathbf{x}_0$ and $\mathbf{x}$.

3. If $\varphi_i(\mathbf{x}, \mathbf{x}_0)$ is the characteristic function of the set $A_i(\mathbf{x}_0)$, then the inequalities

$$c_3\delta^n(\mathbf{x}) \leq \int_{\Omega_i} \varphi_i(\mathbf{x}, \mathbf{x}_0)d\mathbf{x}_0 \leq c_4\delta^n(\mathbf{x}), \ \ \forall \mathbf{x} \in A_i(\mathbf{x}_0),$$

hold, where c_3 and c_4 are two positive constants independent of $\mathbf{x}$.

If $s, \lambda \in \mathbb{R}$ and $0 < p \leq \infty$, we will indicate with $\widetilde{L}^p_{s,\lambda}(\Omega)$ the space of the functions $u(\mathbf{x})$, such that $\delta^s \left(\dfrac{\delta}{1+\delta^2}\right)^\lambda u \in L^p(\Omega)$, with the norm

$$\|u\|_{\widetilde{L}^p_{s,\lambda}(\Omega)} := \left\| \delta^s \left(\frac{\delta}{1+\delta^2}\right)^\lambda u \right\|_{L^p(\Omega)}. \tag{3.47}$$

If $s, \lambda \in \mathbb{R}$, $r \in \mathbb{N}_0$ and $p \in (1, \infty)$, we will indicate with $W^{r,p}_{s,\lambda}(\Omega)$ the space of the distributions u on Ω, such that $\partial^\alpha u \in \widetilde{L}^p_{s+|\alpha|-r,\lambda}(\Omega)$ for $|\alpha| \leq r$, with the norm

$$\|u\|_{W^{r,p}_{s,\lambda}(\Omega)} := \left[\sum_{k=0}^{r} \|\partial^k u\|^p_{\widetilde{L}^p_{s+|\alpha|-r,\lambda}(\Omega)} \right]^{\frac{1}{p}}. \tag{3.48}$$

We observe that the algebraic and topology inclusion

$$W^{r,p}_{s,\lambda}(\Omega) \subseteq W^{k,p}_{s+k-r+t,\lambda+\tau}(\Omega), \quad \text{for} \quad k \leq r, \tau \geq 0, \quad \text{and} \quad t \in [-\tau, \tau], \tag{3.49}$$

exists. Observe that, as a particular case,

$$W^{0,p}_{s,\lambda}(\Omega) = \widetilde{L}^p_{s,\lambda}(\Omega).$$

We have also, $L^2_{0,-1}(\Omega) = \widetilde{L}^2_{-1,1}(\Omega)$.

Definition 1. *Let X, Y be two normed linear spaces. We say that the space X is embedded into the space Y, and denote this fact by the symbol*

$$X \hookrightarrow Y,$$

if

(i) X is a subspace of the space Y.

(ii) There is a constant $C > 0$ such that

$$\|u\|_Y \leq C \|u\|_X,$$

for all elements $u \in X$.

Lemma 4. *[2] Let s, λ, p, r real numbers with $p \geq 1$ and $r > 0$ entire. For each distribution u on Ω such that $u \in \widetilde{L}^p_{-r,\lambda}(\Omega)$ and $\partial^\alpha u \in \widetilde{L}^p_{s,\lambda}(\Omega)$ for $|\alpha| = r$, for each not negative entire $k < r$ and for each $a \in [k/r, 1[\cap [k/r, k/r + n/pr]$ there exists the limitation*

$$\|\partial^\alpha u\|_{\widetilde{L}^{pn/(pk+n-apr)}_{s-(1-a)r,\lambda}(\Omega)} \leq c(\|\partial^r u\|^a_{\widetilde{L}^p_{s,\lambda}(\Omega)} \cdot \|u\|^{1-a}_{\widetilde{L}^p_{s-r,\lambda}(\Omega)} + \|u\|_{\widetilde{L}^p_{s-r,\lambda}(\Omega)}), \tag{3.50}$$

where c is a constant independent of u.

Theorem 9. *(Avantaggiati and Troisi [2, Theorem 6.1]) There are s, λ, r, p real numbers, where $r \in \mathbb{Z}_+$ and $p > 1$. For each not negative integer $k < r$, for each real number $\tau > 0$ and for each $t \in (-\tau, \tau)$ the injection*

$$W^{r,p}_{s,\lambda}(\Omega) \hookrightarrow W^{k,p}_{s+k-r+t,\lambda+\tau}(\Omega) \tag{3.51}$$

is compact.

As a consequence of the forgoing results we have the following lemma

Lemma 5. *The embedding of $\boldsymbol{PH}^1(\mathbb{R}^3)$ into $\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)$ is compact.*

Proof. First, we observe that by definition $\widetilde{L}^2_{-1,1}(\Omega) = L^2_{0,-1}(\Omega) = W^{0,2}_{-1,1}(\Omega)$. On the other hand choosing $t = s = \lambda = k = 0$, $\tau = r = 1$, $p = 2$ in (3.51) gives the compact embedding $W^{1,2}_{0,0}(\Omega) \subset\subset W^{0,2}_{-1,1}(\Omega)$. Altogether $W^{1,2}_{0,0}(\Omega) \subset\subset L^2_{0,-1}(\Omega)$ where we can set $\Omega = \mathbb{R}^3$.
Furthermore $\varphi \in PH^1(\mathbb{R}^3) := \{\varphi = (\varphi^{is}, \varphi^{cd}) : \varphi^{is} \in \mathbb{W}^1_0(\Omega^{is}), \varphi^{cd} \in H^1(\Omega^{cd})\}$ gives due to the definition of $\mathbb{W}^1_0$ that $\nabla\varphi \in L^2$ and hence $\nabla\varphi \in \widetilde{L}^2_{0,0}$ with $s = \lambda = 0$ in (3.47). Therefore $\varphi \in W^{1,2}_{0,0}(\Omega)$ with $r = 1$, $p = 2$, $s = \lambda = 0$ in (3.48) because with $s = \lambda = 0 = |\alpha|$, $r = 1$ there holds

$$\|\varphi\|_{\widetilde{L}^2_{-1,0}}(\Omega) = \|\delta^{-1}\varphi\|_{L^2(\Omega)} \le c \left\| \frac{\varphi}{\sqrt{1+x^2}} \right\|_{L^2(\Omega)} < \infty$$

by taking δ proportional to $\sqrt{1+x^2}$. □

A variational formulation of the eddy current problem is given by Hiptmair [24] who shows existence and uniqueness of the weak solution. He consider the problem

$$\begin{aligned}
&\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} + i\kappa\omega\sigma\mathbf{E} = -i\omega\mathbf{J}_0, \quad \text{in } \mathbb{R}^3,\\
&\operatorname{div}(\varepsilon_0 \mathbf{E}) = 0, \quad \text{in } \Omega^{is},\\
&\int_{\Sigma_i} \mathbf{E} \cdot \mathbf{n} ds = 0, \quad i = 1, \cdots N_{cd},\\
&\mathbf{E}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as } |\mathbf{x}| \to \infty,\\
&\operatorname{curl} \mathbf{E}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as } |\mathbf{x}| \to \infty.
\end{aligned} \tag{3.52}$$

Here $\{\Sigma_i\}_{i=1}^{N_{cd}}$ stands for the finitely many connected components of $\Sigma = \partial\Omega^{cd}$, $\varepsilon_0 > 0$ is a constant in Ω^{is}, $\operatorname{Supp}(\mathbf{J}_0) \subset \overline{\Omega^{cd}}$, $\mathbf{J}_0$ is divergence-free source current.
With the constrained space

$$\mathbb{X}_{\mathtt{C}}(\mathbb{R}^3) = \left\{ \mathbf{u} \in \mathbf{W}(\operatorname{curl}, \mathbb{R}^3) \;\middle|\; \operatorname{div} \mathbf{u} = 0, \text{ in } \Omega^{is}, \int_{\Sigma_i} \mathbf{E}|_{\Omega^{is}} \cdot \mathbf{n} ds = 0, \; i = 1, \cdots N_{cd} \right\},$$

the weak eddy current problem reads:
Seek $\mathbf{E} \in \mathbb{X}_{\mathtt{C}}(\mathbb{R}^3)$, such that for all $\mathbf{v} \in \mathbb{X}_{\mathtt{C}}(\mathbb{R}^3)$,

$$a(\mathbf{E}, \mathbf{v}) := (\mu^{-1}\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{v})_{\mathbf{L}^2(\mathbb{R}^3)} + i\omega(\sigma\mathbf{E}, \mathbf{v})_{\mathbf{L}^2(\Omega^{cd})} = -i\omega(\mathbf{J}_0, \mathbf{v})_{\mathbf{L}^2(\Omega^{cd})}. \tag{3.53}$$

Theorem 10. *(R. Hiptmair [24, Theorem 2.1]) A solution of the variational problem (3.53) exists and is unique.*

The above theorem could be used alternatively in the proof of Theorem 5.

4 Asymptotic expansion for large conductivity - Revisited

In this chapter we summarize the results from Peron's thesis [42] on the terms in the asymptotic expansion of the electric field w.r.t. large conductivity. We show that his results on the form of the terms and the convergence of the expansion which he derived for bounded domains remain valid for a unbounded exterior domain. A key issue in [42] is an appropriate scaling in normal direction near the interface (see [17, 32, 31, 47, 42]). We follow Peron's analysis but use a weighted space setting for Ω^{is}. For the bounded interior domain we just quote or repeat the respective results from Peron [42, 15]. For better reading we use Peron's notation

4.1 Asymptotic expansion - Revisited

In the following we assume that the interface Σ is compact orientable C^∞ surface and denote by n the normal to Σ pointing into Ω^{cd}. We can locally define coordinates such that $y = (y_\alpha, y_3)$ in a tubular neighborhood of Σ. We consider

$$\begin{aligned} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0, \quad \text{in} \quad \Omega^{cd} \cup \Omega^{is}, \\ \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \sigma)\mathbf{E} = \mathbf{J}, \quad \text{in} \quad \Omega^{cd} \cup \Omega^{is}. \end{aligned} \tag{4.1}$$

Let $\mathbf{J} \in \mathbf{H}(div, \Omega)$ with $\mathbf{J} = 0$ in Ω^{cd}. Under the assumption that ω is not an eigenvalue there exists $\rho_0 > 0$ such that for all $\rho \geq \rho_0$ the problem (4.1) admits a unique solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2_{0,-1}$. Furthermore, for Ω^{is} bounded Peron derives an asymptotic expansion in powers of the ρ^{-1} [42, 15]

$$\begin{aligned} \mathbf{E}^{is}_\rho(\mathbf{x}) \sim \sum_{j\geq 0} \mathbf{E}^{is}_j(\mathbf{x})\rho^{-j}, \quad \text{for} \quad \mathbf{x} \in \Omega^{is} \\ \mathbf{E}^{cd}_\rho(\mathbf{x}) \sim \sum_{j\geq 0} \mathbf{E}^{cd}_j(\mathbf{x};\rho)\rho^{-j}, \quad \text{for} \quad \mathbf{x} \in \Omega^{cd}, \end{aligned} \tag{4.2}$$

where

$$\mathbf{E}^{cd}_j(\mathbf{x};\rho) = \mathbf{W}^{cd}_j(y_\alpha, h\rho), \ \alpha = 1, 2 \quad , \tag{4.3}$$

and

$$\mathbf{W}^{cd}_j(y_\alpha, Y_3) \to 0 \quad \text{when} \quad Y_3 = h\rho \to \infty. \tag{4.4}$$

Let us write $J_{\alpha,k}(y_\beta) := \lambda^{-1}(\mathrm{curl} E_k^+ \times n)_\alpha(y_\beta, 0)$ for $k = 0, 1$ and $\lambda = \kappa e^{-i\pi/4}$ ($\kappa = \omega\sqrt{\varepsilon_0\mu_0}$ is a wave number). There holds (see Dauge et al. [15] and Peron [42]) with $\mathbf{W}_j = (W_{\alpha,j}, w_j)$:

$$\mathbf{W}_0(y_\alpha, Y_3) = 0$$

$$W_{\alpha,1}(y_\alpha, Y_3) = J_{\alpha,0}(y_\beta)e^{-\lambda Y_3} \text{ and } w_1 = 0$$

$$W_{\alpha,2}(y_\alpha, Y_3) = [-J_{\alpha,1} + (\lambda^{-1} + Y_3)(b_\alpha^\sigma)J_{\sigma,0} - \mathcal{H}J_{\alpha,0}](y_\beta)e^{-\lambda Y_3}$$

$$\text{and } w_2(y_\alpha, Y_3) = -\lambda^{-1}D_\alpha J_0^\alpha(y_\beta)e^{-\lambda Y_3}$$

where $\mathcal{H} = \frac{1}{2}b_\alpha^\alpha$ is a mean curvature of the interface Σ and D_α is the covariant derivative on Σ. In order to show that the above statements hold also for an unbounded exterior domain we only have to modify Peron's proofs as follows:
We adopt his arguments in the bounded interior domain and in the unbounded exterior domain we use Beppo-Levi spaces. In the unbounded domain Ω^{is} we use these weighted spaces to extend his argument as follows where elliptic regularity plays a key role.

In the insulating domain we investigate equation (3.4) where we insert our expansion ansatz (4.2) so that we get further equations for the coefficients.
Let's consider now the functional space

$$\mathbb{X}(\Omega^{is}) = \mathbf{W}(\mathrm{curl}, \Omega^{is}) \cap \mathbf{W}(\mathrm{div}, \Omega^{is}) \ .$$

We introduce an operator Φ_Σ^{is} extending the tangential traces on Σ into the insulating domain Ω^{is}

$$\Phi_\Sigma^{is} : \mathbf{H}^{s-\frac{1}{2}}(\Sigma) \longrightarrow \left(\mathbb{W}_1^s(\Omega^{is})\right)^3$$

with

$$\left(\mathbb{W}_1^s(\Omega^{is})\right)^3 = \{\mathbf{u} \in \mathcal{D}'(\Omega^{is}) \ | \ \ell(r)^{-1}\mathbf{u}\ldots, \ell(r)^{s-1}\frac{\partial^s \mathbf{u}}{\partial x_i^{s_i}\partial x_j^{s_j}} \in \mathbf{L}^2(\Omega^{is}), s_i + s_j = s\}$$

where s is a real number fixed large enough. So, for all $j \in \mathbb{N}$, if $\mathbf{E}_j^{cd} \times \mathbf{n} \in \mathbf{H}^{s-\frac{1}{2}}(\Sigma)$, then $\mathbf{u}_j^{is} := \Phi_\Sigma^{is}(\mathbf{E}_j^{cd} \times \mathbf{n}) \in \left(\mathbb{W}_1^s(\Omega^{is})\right)^3$, verifies $\mathbf{u}_j^{is} \times \mathbf{n} = \mathbf{E}_j^{cd} \times \mathbf{n}$ on Σ. Φ_Σ^{is} is defined as the inverse of the trace operator

$$\gamma_0 : \left(\mathbb{W}_1^s(\Omega^{is})\right)^3 \longrightarrow \mathbf{H}^{s-\frac{1}{2}}(\Sigma).$$

We assume now that the data $\mathbf{F}$ of problem (3.4) is independent of ρ. By substituting ansatz (4.2) in equation (3.4), we have

$$\begin{aligned}&\int_{\Omega^{is}} \frac{1}{\mu}\left(\mathrm{curl}\ \mathbf{E}_0^{is} \cdot \mathrm{curl}\ \overline{\mathbf{E}'} - \kappa^2\mathbf{E}_0^{is} \cdot \overline{\mathbf{E}'}\right) dx + \\ &+\sum_{j\geq 1}\rho^{-j}\int_{\Omega^{is}} \frac{1}{\mu}\left(\mathrm{curl}\ \mathbf{E}_j^{is} \cdot \mathrm{curl}\ \overline{\mathbf{E}'} - \kappa^2\mathbf{E}_j^{is} \cdot \overline{\mathbf{E}'}\right) dx = \int_{\Omega^{is}} \mathbf{F}^{is} \cdot \overline{\mathbf{E}'} dx.\end{aligned} \tag{4.5}$$

Identifying terms according to powers of ρ^{-1}, we get the following equations

$$\begin{aligned} &\int_{\Omega^{is}} \frac{1}{\mu}\left(\operatorname{curl} \mathbf{E}_0^{is} \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_0^{is} \cdot \overline{\mathbf{E}'}\right) dx = \int_{\Omega^{is}} \mathbf{F}^{is} \cdot \overline{\mathbf{E}'} dx \\ &\int_{\Omega^{is}} \left(\operatorname{curl} \mathbf{E}_j^{is} \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_j^{is} \cdot \overline{\mathbf{E}'}\right) dx = 0 \quad \text{for all} \quad j \geq 1. \end{aligned} \tag{4.6}$$

Next we solve like Peron [42, (5.16)] the following problems in the insulating domain: Find $\mathbf{E}_0^{is} \in \mathbb{X}(\Omega^{is})$, with $\mathbf{E}_0^{is} - \Phi_\Sigma^{is}(\mathbf{E}_0^{cd} \times \mathbf{n}) \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$, such that for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is}) = \{\mathbf{u} \in \mathbb{X}(\Omega^{is}) : \mathbf{n} \times \mathbf{u} = 0 \text{ on } \Sigma\}$

$$\int_{\Omega^{is}} \frac{1}{\mu}\left(\operatorname{curl} \mathbf{E}_0^{is} \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_0^{is} \cdot \overline{\mathbf{E}'}\right) dx = \int_{\Omega^{is}} \mathbf{F}^{is} \cdot \overline{\mathbf{E}'} dx, \tag{4.7}$$

and for all $j \in \mathbb{N}^*$, find $\mathbf{E}_j^{is} \in \mathbb{X}(\Omega^{is})$, with $\mathbf{E}_j^{is} - \Phi_\Sigma^{is}(\mathbf{E}_j^{cd} \times \mathbf{n}) \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$, such that for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$

$$\int_{\Omega^{is}} \left(\operatorname{curl} \mathbf{E}_j^{is} \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_j^{is} \cdot \overline{\mathbf{E}'}\right) dx = 0. \tag{4.8}$$

Now, the terms of the asymptotic expansion of the electrical field can be successively constructed. The terms $\mathbf{W}_j^{cd}$ are exponentially decreasing in the variable Y_3. We set $I = (0, \infty)$ and $\Omega_0 = \Sigma \times I$.

First we present from Peron [42] terms of order 0 in the conductor Ω^{cd} and repeat his derivation. Following his argument we can also for unbounded Ω^{is} compute the expansion terms in (4.2):

According to (5.14) in [42], we have

$$-i\kappa^2 w_0^{cd} = 0$$

then $w_0^{cd} = 0$ in Ω_0.
Due to (5.13) and (5.14) in [42], there holds the second order ODE

$$\begin{aligned} -\partial_3^2 W_{\alpha,0}^{cd} - i\kappa^2 W_{\alpha,0}^{cd} = 0 \quad &\text{in} \quad \Omega_0 \\ \partial_3 W_{\alpha,0}^{cd} = 0 \quad &\text{on} \quad \Sigma, \text{i.e.} Y_3 = 0 \end{aligned} \tag{4.9}$$

The boundary condition in system (4.9) and the hypothesis (4.4) assure the uniqueness of the solution of the ordinary differential equation. Since $w_0^{cd} = 0$ and $W_{\alpha,0}^{cd} = 0$, then $\mathbf{W}_0^{cd} = 0$ in Ω_0. But due to (4.3), we have $\mathbf{E}_0^{cd}(\mathbf{x}, \rho) = \mathbf{W}_0^{cd}(y_\alpha, h\rho)$ yielding $\mathbf{E}_0^{cd} = 0$ in Ω^{cd}. This derivation was originally applied by Peron [42].

Next we consider the terms of order 0 in insulating domain Ω^{is}:

We know that $\mathbf{E}_0^{cd} = 0$ in Ω^{cd}, then the term $\mathbf{E}_0^{is}$ verifies the perfect conductor condition

$$\mathbf{E}_0^{is} \times \mathbf{n} = 0 \quad \text{on} \quad \Sigma \tag{4.10}$$

Due to (4.7) and (4.10), the problem to be solved for the term $\mathbf{E}_0^{is}$ is then the following: Let $\mathbf{E}_0^{is} \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$, such that for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$

$$\int_{\Omega^{is}} \frac{1}{\mu} \left(\text{curl } \mathbf{E}_0^{is} \cdot \text{curl } \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_0^{is} \cdot \overline{\mathbf{E}'}\right) dx = \int_{\Omega^{is}} \mathbf{F}^{is} \cdot \overline{\mathbf{E}'} dx. \tag{4.11}$$

And, according to spectral hypothesis (**SH**) κ^2 is not eigenvalue of problem: Let $\mathbf{E}_0 \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$, such that for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$

$$\int_{\Omega^{is}} \left(\text{curl } \mathbf{E}_0 \cdot \text{curl } \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_0 \cdot \overline{\mathbf{E}'}\right) dx = 0. \tag{4.12}$$

Therefore due to Fredholm's alternative (4.11) has an unique solution $\mathbf{E}_0^{is} \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$. Due to elliptic regularity, $\mathbf{E}_0^{is} \in \left(\mathbb{W}_1^s(\Omega^{is})\right)^3$ for $\mathbf{F} \in \left(\mathbb{W}_1^{s-2}(\mathbb{R}^3)\right)^3$.

Next we present from Peron [42] the terms of order 1 in conducting domain Ω^{cd}:

According to (5.14) in [42], we have

$$-i\kappa^2 w_1^{cd} + \gamma_\alpha^\alpha(\partial_3(\mathbf{W}_0^{cd})) + b_\alpha^\alpha \partial_3 w_0^{cd} = 0$$

and $\mathbf{W}_0^{cd} = 0$ in Ω_0, implying $w_1^{cd} = 0$ in Ω_0.
According to (5.13) and (5.14) in [42], and since $W_{\alpha,0}^{cd} = 0$ and $w_0^{cd} = 0$, there holds

$$\begin{aligned} -\partial_3^2 W_{\alpha,1}^{cd} - i\kappa^2 W_{\alpha,1}^{cd} &= 0 \text{ in } \Omega_0 \\ \partial_3 W_{\alpha,1}^{cd} &= (\text{curl } \widetilde{\mathbf{E}}_0^{is} \times \mathbf{n})_\alpha \text{ on } \Sigma. \end{aligned} \tag{4.13}$$

The following results by Peron give an expression of the term $W_{\alpha,1}^{cd}$.

Proposition 4. *[42, Propositions 5.3 and 5.4] Suppose (**SH**) holds. Then the ordinary differential equations (4.13) admits a unique solution $W_{\alpha,1}^{cd}$. Furthermore, for all $(y_\beta, Y_3) \in \Omega_0$, we have with $\lambda := \kappa e^{-i\frac{\pi}{4}}$, $Re(\lambda) > 0$*

$$W_{\alpha,1}^{cd}(y_\beta, Y_3) = -\frac{1}{\lambda}(\mathit{curl}\, \widetilde{\boldsymbol{E}}_0^{is} \times \boldsymbol{n})_\alpha(y_\beta, 0)e^{-\lambda Y_3} \tag{4.14}$$

and for all $(y_\beta, Y_3) \in \Omega_0$, we have

$$\boldsymbol{W}_1^{cd}(y_\beta, Y_3) = -\frac{1}{\lambda}(\mathit{curl}\, \widetilde{\boldsymbol{E}}_0^{is} \times \boldsymbol{n})(y_\beta, 0)e^{-\lambda Y_3} \tag{4.15}$$

Next we present the terms of order 1 in insulating domain Ω^{is}:

According to (4.3) we have in a tubular neighbourhood of $\Sigma \in \Omega^{cd}$.

$$\mathbf{E}_j^{cd}(\mathbf{x}; \rho) = \mathbf{W}_j^{cd}(y_\alpha, h\rho) \quad \text{if} \quad \mathbf{x} \in \mathcal{O},$$

in particular for $j=1$ there holds

$$\mathbf{E}_1^{cd}(\mathbf{x};\rho) = \mathbf{W}_1^{cd}(y_\alpha, h\rho).$$

Now (4.15) implies

$$\mathbf{E}_1^{cd} = -\frac{1}{\lambda}(\text{curl}\, \mathbf{E}_0^{is} \times \mathbf{n}) \quad \text{on} \quad \Sigma$$

since

$$\mathbf{E}_0^{is}(\mathbf{x}) = \widetilde{\mathbf{E}}_0^{is}(y_\alpha, 0) \quad \text{on} \quad \Sigma.$$

Consequently

$$\mathbf{E}_1^{is} \times \mathbf{n} = \mathbf{E}_1^{cd} \times \mathbf{n} = -\frac{1}{\lambda}(\text{curl}\, \mathbf{E}_0^{is} \times \mathbf{n}) \times \mathbf{n} \quad \text{on} \quad \Sigma,$$

We know that $\mathbf{E}_0^{is} \in \left(\mathbb{W}_1^s(\Omega^{is})\right)^3$, hence curl $\mathbf{E}_0^{is} \times \mathbf{n} \in \mathbf{H}^{s-\frac{3}{2}}(\Sigma)$ for smooth Σ and therefore

$$\mathbf{E}_1^{cd} \times \mathbf{n}|_\Sigma \in \mathbf{H}^{s-\frac{3}{2}}(\Sigma).$$

Now due to (4.8), the problem for $\mathbf{E}_1^{is}$ reads: Find $\mathbf{E}_1^{is} \in \mathbb{X}(\Omega^{is})$, with $\mathbf{E}_1^{is} - \Phi_\Sigma^{is}(\mathbf{E}_1^{cd} \times \mathbf{n}) \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$ such that, for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$,

$$\int_{\Omega^{is}} \left(\text{curl}\, \mathbf{E}_1^{is} \cdot \text{curl}\, \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_1^{is} \cdot \overline{\mathbf{E}'}\right) dx = 0. \tag{4.16}$$

Due to the spectral hypothesis (**SH**), the variational problem (4.16) admits a unique solution $\mathbf{E}_1^{is} \in \mathbb{X}(\Omega^{is})$ and again elliptic regularity implies, $\mathbf{E}_1^{is} \in \left(\mathbb{W}_1^{s-1}(\Omega^{is})\right)^3$.
Higher order terms can be written in a similar manner (see [42]). In the same way one can see that Peron's expansion procedure remains valid for unbounded Ω^{is}.

4.2 Convergence of the asymptotic expansion of the electrical field

Next, we study the convergence of the asymptotic expansion of the electrical field $\mathbf{E}_\rho$, (4.2). Therefore we consider the remainder of order m, $\mathbf{R}_{m,\rho}$ which consists of the difference between $\mathbf{E}_\rho$ and the first m terms in the asymptotic expansion of $\mathbf{E}_\rho$. Theorem 11 shows the convergence of the remainder. Important tools are the uniform a priori estimates in ρ derived in Chapter 3. Again, we extend the results of Peron [42] to the case of the unbounded exterior domain using Beppo-Levi spaces.
Now, we consider data $\mathbf{F} \in \mathbf{PH}^s(\mathbb{R}^3) = \{\mathbf{F} = (\mathbf{F}^{cd}, \mathbf{F}^{is}) \mid \mathbf{F}^{cd} \in \mathbf{H}^s(\Omega^{cd}), \mathbf{F}^{is} \in \left(\mathbb{W}_1^s(\Omega^{is})\right)^3\}$, $s \geq 2$, with support in Ω^{is} such that div$\mathbf{F} = 0$ in $\mathbb{R}^3$. Note $\mathbf{E}_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3, \rho)$ is a solution of the following equations in the sense of distributions

$$\begin{aligned} &\frac{1}{\mu}\text{curl}\,\text{curl}\, \mathbf{E}_\rho - \kappa^2 \varepsilon(\rho) \mathbf{E}_\rho = \mathbf{F}, && \text{in } \mathbb{R}^3 \\ &|\text{curl}\, \mathbf{E}_\rho \times \widehat{\mathbf{x}} - i\kappa \mathbf{E}_\rho| = o\left(\frac{1}{|\mathbf{x}|}\right), && \text{as } |\mathbf{x}| \to \infty, \end{aligned} \tag{4.17}$$

(see Proposition 3). Since div $\mathbf{F} = 0$ and $\kappa \neq 0$, we deduced that $\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) = 0$ and thus $\mathbf{E}_\rho$ verifies

$$-\Delta \mathbf{E}_\rho - \kappa^2 \mu \varepsilon(\rho) \mathbf{E}_\rho = \mu \mathbf{F}, \text{ in } \Omega^{cd} \cup \Omega^{is},$$

as well as the transmission conditions (see Lemmas 1 and 2)

$$[\varepsilon(\rho)\mathbf{E}_\rho \cdot \mathbf{n}]_\Sigma = 0, \ [\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)]_\Sigma = 0, \ [\mathbf{E}_\rho \times \mathbf{n}]_\Sigma = 0, \ [\operatorname{curl} \mathbf{E}_\rho \times \mathbf{n}]_\Sigma = 0. \tag{4.18}$$

Under the spectral hypothesis (**SH**) on κ, for all $j \in [|0, [s] - 2|]$, with $s \geq 2$,for the terms $\mathbf{E}_j^{is}$ and $\mathbf{W}_j^{cd}$ of the asymptotic expansion (4.2) there holds

$$\mathbf{E}_j^{is} \in \left(\mathbb{W}_1^{s-j}(\Omega^{is})\right)^3 \quad \text{and} \quad \mathbf{W}_j^{cd} \in \mathbf{H}^{s-j-\frac{1}{2}}(\Sigma, C^\infty(I)), \tag{4.19}$$

where $I = (0, \infty)$. This can be seen as done for $j = 0, 1$ in Subsection 4.1.

From the asymptotic expansion (4.2), one defines the associated partial sums for all $N \in [|0, [s] - 2|]$

$$\begin{aligned} \mathbf{E}_{[N],\rho}^{is}(\mathbf{x}) &= \sum_{j=0}^{N} \mathbf{E}_j^{is}(\mathbf{x})\rho^{-j}, \qquad &&\text{if } \ \mathbf{x} \in \Omega^{is}, \\ \mathbf{E}_{[N],\rho}^{cd}(\mathbf{x}) &= \sum_{j=0}^{N} \mathbf{W}_j^{cd}(y_\alpha, Y_3)\rho^{-j}, &&\text{if } \ \mathbf{x} \in \mathcal{O} \end{aligned} \tag{4.20}$$

and sets with a cut-off-function $\chi \in C^\infty(\Omega^{cd})$ such that $\chi = 1$ in a tubular neighborhood $\mathcal{O}'$ of Σ, where $\mathcal{O}' \subset \mathcal{O}$, and $\chi = 0$ in $\Omega^{cd} \setminus \mathcal{O}$, see figure 4.1,

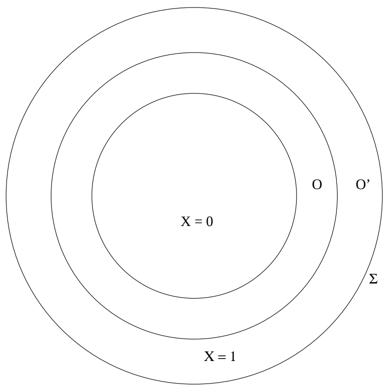

Figure 4.1: A tubular neighborhood of Σ.

$$\widetilde{\mathbf{E}}_{[N],\rho} = \begin{cases} \mathbf{E}_{[N],\rho}^{is}, & \text{in } \ \Omega^{is}, \\ \chi \mathbf{E}_{[N],\rho}^{cd}, & \text{in } \ \mathcal{O}, \\ 0, & \text{in } \ \Omega^{cd} \setminus \mathcal{O}. \end{cases} \tag{4.21}$$

According to (4.19), we have

$$\widetilde{\mathbf{E}}^{is}_{[N],\rho} \in \left(\mathbb{W}^{s-N}_1(\Omega^{is})\right)^3 \quad \text{and} \quad \widetilde{\mathbf{E}}^{cd}_{[N],\rho} \in \mathbf{H}^{s-N-\frac{1}{2}}(\Omega^{cd}). \tag{4.22}$$

We remember that $\mathbf{E}_\rho \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3,\rho) \cap \mathbf{PH}^s(\mathbb{R}^3)$ is a strong solution of (4.17) for $\mathbf{F} \in \mathbf{PH}^{s-2}(\mathbb{R}^3)$ where $s \geq 2$. Almost verbatim there holds the following modification of Peron's Proposition 7.4 in [42] for the remainder $\mathbf{R}_{m,\rho} = \mathbf{E}_\rho - \widetilde{\mathbf{E}}_{[m],\rho}$. But of course we consider an unbounded exterior domain Ω^{is}. For better reading we adopt Peron's notation and repeat some essential parts of his proof, but modified to our situation, i.e. weighted spaces in Ω^{is}.

Proposition 5. *Under the spectral hypothesis (**SH**), for all $m \in [|0,[s]-2|]$, we have*

$$\begin{aligned} &\mathit{curl}\ \mathit{curl}\ \boldsymbol{R}^{is}_{m,\rho} - \kappa^2 \boldsymbol{R}^{is}_{m,\rho} = 0, \quad \mathit{in}\ \Omega^{is}, \\ &\boldsymbol{R}^{cd}_{m,\rho} \times \boldsymbol{n} = \boldsymbol{R}^{is}_{m,\rho} \times \boldsymbol{n}, \quad \mathit{on}\ \Sigma, \\ &|\mathit{curl}\ \boldsymbol{R}_{m,\rho} \times \widehat{\boldsymbol{x}} - i\kappa \boldsymbol{R}_{m,\rho}| = o\left(\frac{1}{|\boldsymbol{x}|}\right), \quad \mathit{as}\quad |\boldsymbol{x}| \to \infty. \end{aligned} \tag{4.23}$$

Moreover,

$$\mathit{div}[\rho]\boldsymbol{R}^{cd}_{m,\rho} = -\rho^{-m-1} \sum_{k=0}^{m} \mathit{div}^{m+1-k}\, \boldsymbol{W}^{cd}_k + O(\rho^{-m-2}), \quad \mathit{in}\ \mathcal{O}', \tag{4.24}$$

and on Σ, we have

$$\varepsilon(\rho)^{cd} \boldsymbol{R}^{cd}_{m,\rho} \cdot \boldsymbol{n} = \varepsilon(\rho)^{is} \boldsymbol{R}^{is}_{m,\rho} \cdot \boldsymbol{n} - \varepsilon(\rho)^{is} \rho^{-m+1} \sum_{k=0}^{1} (w^{cd}_{m-k} - \widetilde{\boldsymbol{E}}^{is}_{m-k} \cdot \boldsymbol{n}) \rho^{k-1}. \tag{4.25}$$

Proof. We have that $\mathbf{E}^{is}_0 \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$ satisfies

$$\int_{\Omega^{is}} \frac{1}{\mu} (\operatorname{curl} \mathbf{E}^{is}_0 \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}^{is}_0 \cdot \overline{\mathbf{E}'}) dx = \int_{\Omega^{is}} \mathbf{F}^{is} \cdot \overline{\mathbf{E}'} dx, \forall \mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is}) \tag{4.26}$$

and for all $j \geq 1$, $\mathbf{E}^{is}_j \in \mathbb{X}(\Omega^{is})$ satisfies $\mathbf{E}^{is}_j - \Phi^{is}_\Sigma(\mathbf{E}^{cd}_j \times \mathbf{n}) \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$, with extension operator Φ^{is}_Σ (see Section 4.1). Furthermore for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$,

$$\int_{\Omega^{is}} (\operatorname{curl} \mathbf{E}^{is}_j \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{E}^{is}_j \cdot \overline{\mathbf{E}'}) dx = 0. \tag{4.27}$$

Thus, $\mathbf{R}^{is}_{m,\rho} \in \mathbb{X}_{\mathtt{T}}(\Omega^{is})$ satisfies

$$\mathbf{R}^{is}_{m,\rho} - \Phi^{is}_\Sigma(\mathbf{R}^{cd}_{m,\rho} \times \mathbf{n}) \in \mathbb{X}_{\mathtt{N}}(\Omega^{is}),$$

and for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{N}}(\Omega^{is})$,

$$\int_{\Omega^{is}} (\operatorname{curl} \mathbf{R}^{is}_{m,\rho} \cdot \operatorname{curl} \overline{\mathbf{E}'} - \kappa^2 \mathbf{R}^{is}_{m,\rho} \cdot \overline{\mathbf{E}'}) dx = 0. \tag{4.28}$$

Next, integration by parts in (4.28), gives

$$\begin{aligned} &\operatorname{curl}\operatorname{curl}\mathbf{R}^{is}_{m,\rho} - \kappa^2\mathbf{R}^{is}_{m,\rho} = 0, \quad \text{in } \Omega^{is}, \\ &\mathbf{R}^{cd}_{m,\rho}\times\mathbf{n} = \mathbf{R}^{is}_{m,\rho}\times\mathbf{n}, \qquad\qquad \text{on } \Sigma. \end{aligned} \tag{4.29}$$

The remainder of the proof goes verbatim as the proof of Proposition 7.4 in Peron [42].

□

Next we give for $\mathbf{R}_{m,\rho}$ an asymptotic expansion in powers of ρ^{-1}, and we present estimates of the remainder. The crucial point in the proof is the a priori estimate (3.18).
As observed by Peron we cannot apply directly the a priori estimate (3.18) to $\mathbf{R}_{m,\rho}$, because of the lack of continuity of the terms curl $\mathbf{R}_{m,\rho}\times\mathbf{n}$ and $\varepsilon(\rho)\mathbf{R}_{m,\rho}\cdot\mathbf{n}$ on the interface Σ, see the relations $(4.23)_3$ and (4.25). Following Peron we construct a correction term $\mathbf{C}_{m,\rho}$ with support in the domain Ω^{is} such that we can apply estimate (3.18) to $\mathbf{u}_{m,\rho} := \mathbf{R}_{m,\rho} - \mathbf{C}_{m,\rho}$. The following result corresponds to Proposition 7.7 in Peron's thesis [42]. Again we give here the modification for an unbounded exterior domain Ω^{is} and weighted spaces. For better reading and completeness we give the details of the proof which repeats to some extend Peron's proof.

Proposition 6. *Let $s \geq 2$ and $\boldsymbol{F}_{m,\rho} \in \boldsymbol{PH}^2(\mathbb{R}^3)$ with support in Ω^{is}, such that div $\boldsymbol{F} = 0$ in $\mathbb{R}^3$. Under the spectral hypothesis (**SH**), for all $m \in [|0,[s]-2|]$ there exists $\boldsymbol{C}_{m,\rho} \in \boldsymbol{PH}^2(\mathbb{R}^3)$ with support in Ω^{is}, such that $\boldsymbol{u}_{m,\rho} \in \mathbb{X}_{TN}(\mathbb{R}^3,\rho)$ satisfies for all $\boldsymbol{E}' \in \mathbb{X}_{TN}(\mathbb{R}^3,\rho)\cap\widetilde{\boldsymbol{W}}(curl,\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3}\frac{1}{\mu}(curl\,\boldsymbol{u}_{m,\rho}\cdot curl\,\overline{\boldsymbol{E}'} - \kappa^2\varepsilon(\rho)\boldsymbol{u}_{m,\rho}\cdot\overline{\boldsymbol{E}'})dx = \int_{\mathbb{R}^3}\boldsymbol{F}_{m,\rho}\cdot\overline{\boldsymbol{E}'}\,dx, \tag{4.30}$$

and

$$div(\varepsilon(\rho)\boldsymbol{u}_{m,\rho}) = -\frac{1}{\kappa^2}div\,\boldsymbol{F}_{m,\rho}, \quad in \quad \boldsymbol{L}^2_{0,-1}(\mathbb{R}^3),$$

where $\boldsymbol{F}_{m,\rho} \in \boldsymbol{W}(div,\mathbb{R}^3)$ is defined by

$$\mu\boldsymbol{F}^{is}_{m,\rho} = -curl\,curl\,\boldsymbol{C}^{is}_{m,\rho} + \kappa^2\boldsymbol{C}^{is}_{m,\rho} \quad in\ \Omega^{is}, \tag{4.31}$$

$$\mu\boldsymbol{F}^{cd}_{m,\rho} = curl\,curl\,\boldsymbol{R}^{cd}_{m,\rho} - \kappa^2(1+i\rho^2)\boldsymbol{R}^{cd}_{m,\rho} \quad in\,\Omega^{cd}. \tag{4.32}$$

Moreover, there exists a constant $C_m > 0$ independent of ρ such that

$$\|\boldsymbol{C}_{m,\rho}\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} \leq C_m\rho^{-m+1}. \tag{4.33}$$

Proof. Let $\mathbf{C}_{m,\rho} \in \mathbf{PH}^2(\mathbb{R}^3)$ with support in $\Omega^{is} = \Omega_-$ and $\mathbf{u}_{m,\rho} := \mathbf{R}_{m,\rho} - \mathbf{C}_{m,\rho}$. According to Proposition 5, there holds

$$\begin{aligned} \mathbf{u}^{cd}_{m,\rho}\times\mathbf{n} &= \mathbf{u}^{is}_{m,\rho}\times\mathbf{n} + \mathbf{C}^{is}_{m,\rho}\times\mathbf{n}, \ \text{ on } \Sigma, \\ (1+i\rho^2)\mathbf{u}^{is}_{m,\rho}\cdot\mathbf{n} &= \mathbf{u}^{is}_{m,\rho}\cdot\mathbf{n} + \sum_{k=0}^{1}(w^{cd}_{m-k} - \widetilde{\mathbf{E}}^{is}_{m-k}\cdot\mathbf{n})\rho^{-m+k} - \mathbf{C}^{is}_{m,\rho}\cdot\mathbf{n}, \ \text{ on } \Sigma, \\ |\text{curl}\,\mathbf{u}_{m,\rho}\times\widehat{\mathbf{x}} - i\kappa\mathbf{u}_{m,\rho}| &= o\left(\frac{1}{|\mathbf{x}|}\right), \ \text{ as } |\mathbf{x}|\to\infty. \end{aligned} \tag{4.34}$$

According to (7.38) in [42], we have

$$\partial_3 \mathcal{W}^{cd}_{m+1}|_\Sigma \in \mathbf{H}^{\frac{1}{2}}(\Sigma) \tag{4.35}$$

and define on Σ

$$\mathcal{H}_{m,\rho} := -\rho^{-m}\partial_3 \mathcal{W}^{cd}_{m+1} \quad \text{and} \quad g_{m,\rho} := \sum_{k=0}^{1}(w^{cd}_{m-k} - \widetilde{\mathbf{E}}^{is}_{m-k}\cdot \mathbf{n})\rho^{-m+k}.$$

According to (4.35) and (4.19), we have

$$\mathcal{H}_{m,\rho} \in \mathbf{H}^{\frac{1}{2}}(\Sigma) \quad \text{and} \quad g_{m,\rho} \in H^{\frac{3}{2}}(\Sigma).$$

As in [42] we construct the correction terms $\mathbf{C}_{m,\rho}$ via the traces $\mathcal{H}_{m,\rho}$ and $g_{m,\rho}$ on Σ: There exists $\mathbf{C}_{m,\rho} \in \mathbf{PH}^2(\mathbb{R}^3)$ with support in Ω^{is}, such that

$$\begin{gathered}\mathbf{C}^{is}_{m,\rho} \times \mathbf{n} = 0, \quad \mathbf{C}^{is}_{m,\rho}\cdot \mathbf{n} = g_{m,\rho}, \quad \partial_3 \mathbf{C}^{is}_{m,\rho} = \mathcal{H}_{m,\rho}, \quad \text{on} \quad \Sigma,\\ |\text{curl}\, \mathbf{C}_{m,\rho} \times \widehat{\mathbf{x}} - i\kappa \mathbf{C}_{m,\rho}| = o\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as} \quad |\mathbf{x}| \to \infty,\end{gathered} \tag{4.36}$$

and there exists $C_m > 0$ independent of ρ such that

$$\|\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \leq C_m(\|\mathcal{H}_{m,\rho}\|_{\mathbf{H}^{\frac{1}{2}}(\Sigma)} + \|g_{m,\rho}\|_{H^{\frac{3}{2}}(\Sigma)}). \tag{4.37}$$

With definition of $\mathcal{H}_{m,\rho}$ and $g_{m,\rho}$ this proves the estimate (4.33). Now the regularity for $\mathbf{R}_{m,\rho}$ yields

$$\mathbf{u}^{is}_{m,\rho} \in \left(\mathbb{W}^2_1(\Omega^{is})\right)^3 \quad \text{and} \quad \mathbf{u}^{cd}_{m,\rho} \in \mathbf{H}^{3/2}(\Omega^{cd}). \tag{4.38}$$

Due to Proposition 5 and (4.34) there holds

$$\begin{gathered}\text{curl curl}\, \mathbf{u}^{is}_{m,\rho} - \kappa^2 \mathbf{u}^{is}_{m,\rho} = \mu \mathbf{F}^{is}_{m,\rho}, \quad \text{in} \quad \Omega^{is},\\ \text{curl curl}\, \mathbf{u}^{cd}_{m,\rho} - \kappa^2(1+i\rho^2)\mathbf{u}^{cd}_{m,\rho} = \mu \mathbf{F}^{cd}_{m,\rho}, \quad \text{in} \quad \Omega^{cd},\\ \mathbf{u}^{cd}_{m,\rho} \times \mathbf{n} = \mathbf{u}^{is}_{m,\rho} \times \mathbf{n}, \quad (1+i\rho^2)\mathbf{u}^{cd}_{m,\rho}\cdot \mathbf{n} = \mathbf{u}^{is}_{m,\rho}\cdot \mathbf{n}, \quad \text{on} \quad \Sigma,\\ |\text{curl}\, \mathbf{u}_{m,\rho} \times \widehat{\mathbf{x}} - i\kappa \mathbf{u}_{m,\rho}| = o\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as} \quad |\mathbf{x}| \to \infty,\end{gathered} \tag{4.39}$$

with $\mathbf{F}^{is}_{m,\rho}$, $\mathbf{F}^{cd}_{m,\rho}$ as in (4.31), (4.32). Proposition 3 implies that $\mathbf{u}_{m,\rho}$ solves

$$\frac{1}{\mu}\text{curl curl}\, \mathbf{u}_{m,\rho} - \kappa^2 \varepsilon(\rho)\mathbf{u}_{m,\rho} = \mathbf{F}_{m,\rho}, \quad \text{in} \quad \mathbb{R}^3 \tag{4.40}$$

in the sense of distributions. Now (4.40) yields for all $\mathbf{E}' \in \mathbb{X}_{\mathtt{TN}}(\mathbb{R}^3,\rho) \cap \widetilde{\mathbf{W}}(\text{curl}, \mathbb{R}^3)$ in Ω^{cd}

$$\int_{\Omega^{cd}} \left(\frac{1}{\mu}\text{curl curl}\, \mathbf{u}^{cd}_{m,\rho} - \kappa^2\varepsilon(\rho)\mathbf{u}^{cd}_{m,\rho}\right)\cdot \overline{\mathbf{E}}' dx = \int_{\Omega^{cd}} \mathbf{F}^{cd}_{m,\rho}\cdot \overline{\mathbf{E}}' dx,$$

and gives by integration by parts

$$\int_{\Omega^{cd}} \left(\frac{1}{\mu} \text{curl } \mathbf{u}^{cd}_{m,\rho} \cdot \text{curl } \overline{\mathbf{E}}' - \kappa^2 \varepsilon(\rho) \mathbf{u}^{cd}_{m,\rho} \cdot \overline{\mathbf{E}}' \right) dx = \int_{\Omega^{cd}} \mathbf{F}^{cd}_{m,\rho} \cdot \overline{\mathbf{E}}' dx + \int_{\Sigma} (\mathbf{n} \times \mathbf{u}^{cd}_{m,\rho}) \cdot \overline{\mathbf{E}}' ds, \tag{4.41}$$

Next as in Chapter 3 applying integration by parts in Ω_R and letting $R \to \infty$ we end up at

$$\int_{\Omega^{is}} \left(\frac{1}{\mu} \text{curl } \mathbf{u}^{is}_{m,\rho} \cdot \text{curl } \overline{\mathbf{E}}' - \kappa^2 \varepsilon(\rho) \mathbf{u}^{is}_{m,\rho} \cdot \overline{\mathbf{E}}' \right) dx = \int_{\Omega^{is}} \mathbf{F}^{is}_{m,\rho} \cdot \overline{\mathbf{E}}' dx - \int_{\Sigma} (\mathbf{n} \times \mathbf{u}^{is}_{m,\rho}) \cdot \overline{\mathbf{E}}' ds, \tag{4.42}$$

By adding (4.41) and (4.42) and incorporating $(4.39)_4$ gives

$$\int_{\Omega^{is} \cup \Omega^{cd}} \left(\frac{1}{\mu} \text{curl } \mathbf{u}_{m,\rho} \cdot \text{curl } \overline{\mathbf{E}}' - \kappa^2 \varepsilon(\rho) \mathbf{u}_{m,\rho} \cdot \overline{\mathbf{E}}' \right) dx = \int_{\Omega^{is} \cup \Omega^{cd}} \mathbf{F}_{m,\rho} \cdot \overline{\mathbf{E}}' dx. \tag{4.43}$$

Moreover

$$\text{div}(\varepsilon(\rho) \mathbf{u}_{m,\rho}) = -\frac{1}{\kappa^2} \text{div } \mathbf{F}_{m,\rho}, \quad \text{in} \quad \mathbf{L}^2_{0,-1}(\mathbb{R}^3). \tag{4.44}$$

□

The next result corresponds to Peron's Theorem 7.9 in [42] but here our result covers now the case of an unbounded exterior domain. Here we show in detail in the proof where the weighted spaces do appear. Again we follow Peron's proof of Theorem 7.9 in [42].

Theorem 11. *Let $s \geq 2$ and $\boldsymbol{F} \in \boldsymbol{PH}^2(\mathbb{R}^3)$ with support in Ω^{is}, such that div $\boldsymbol{F} = 0$ in $\mathbb{R}^3$. Under the spectral hypothesis (**SH**), for all $m \in [|0, [s] - 2|]$, the solution $\boldsymbol{E}_\rho$ of problem (4.17) admits the asymptotic expansion*

$$\boldsymbol{E}_\rho = \sum_{j=0}^{m} \boldsymbol{E}_j \rho^{-j} + \boldsymbol{R}_{m,\rho}, \quad \text{where} \quad \boldsymbol{E}_j|_{\Omega^{is}} = \boldsymbol{E}^{is}_j \quad \text{and} \quad \boldsymbol{E}_j|_{\mathcal{O}}(\boldsymbol{x}, \rho) = \boldsymbol{W}^{cd}_j(y_\alpha, h\rho),$$

and $\boldsymbol{R}_{m,\rho}$ satisfies for $\rho > 0$

$$\begin{aligned} &\|\mathit{curl}\, \boldsymbol{R}_{m,\rho}\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathit{div}(\varepsilon(\rho) \boldsymbol{R}_{m,\rho})\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} + \|\boldsymbol{R}_{m,\rho}\|_{\boldsymbol{L}^2_{0,-1}(\mathbb{R}^3)} + \rho \|\boldsymbol{R}^{cd}_{m,\rho}\|_{\boldsymbol{L}^2(\Omega^{cd})} \\ &\qquad \leq C_m \rho^{-m+1}, \end{aligned} \tag{4.45}$$

where $C_m > 0$ is a constant independent of ρ.

Proof. We can apply the a priori estimation (3.18) of the Theorem 5 to the term $\mathbf{u}_{m,\rho}$ defined in the Proposition 6. We have for all $\rho > 0$

$$\begin{aligned} &\|\text{curl } \mathbf{u}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\text{div}(\varepsilon(\rho) \mathbf{u}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathbf{u}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \rho \|\mathbf{u}^{cd}_{m,\rho}\|_{\mathbf{L}^2(\Omega^{cd})} \\ &\qquad \leq C_m \|\mathbf{F}_{m,\rho}\|_{\mathbf{W}(\text{div}, \mathbb{R}^3)}, \end{aligned} \tag{4.46}$$

where $C_m > 0$ is a constant independent of ρ and $\mathbf{F}_{m,\rho}$ is defined by (4.31) and (4.32) . According to (4.22)

$$\mathbf{R}^{is}_{m,\rho} \in \left(\mathbb{W}^2_1(\Omega^{is}) \right)^3 \quad \text{and} \quad \mathbf{R}^{cd}_{m,\rho} \in \mathbf{H}^{\frac{3}{2}}(\Omega^{cd}), \tag{4.47}$$

because $s - m \geq 2$. Therefore we have

$$
\begin{aligned}
&\|\mathrm{curl}\,\mathbf{R}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \leq \|\mathrm{curl}\,\mathbf{u}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathrm{curl}\,\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)},\\
&\|\mathrm{div}(\varepsilon(\rho)\mathbf{R}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \leq \|\mathrm{div}(\varepsilon(\rho)\mathbf{u}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathrm{div}(\varepsilon(\rho)\mathbf{C}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)},\\
&\|\mathbf{R}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \leq \|\mathbf{u}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)},\\
&\|\mathbf{R}^{cd}_{m,\rho}\|_{\mathbf{L}^2(\Omega^{cd})} \leq \|\mathbf{u}^{cd}_{m,\rho}\|_{\mathbf{L}^2(\Omega^{cd})}.
\end{aligned}
\tag{4.48}
$$

According to (4.46) and (4.48)

$$
\begin{aligned}
&\|\mathrm{curl}\,\mathbf{R}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathrm{div}(\varepsilon(\rho)\mathbf{R}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathbf{R}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \rho\|\mathbf{R}^{cd}_{m,\rho}\|_{\mathbf{L}^2(\Omega^{cd})}\\
&\leq C_m\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)} + \|\mathrm{curl}\,\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathrm{div}(\varepsilon(\rho)\mathbf{C}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}.
\end{aligned}
\tag{4.49}
$$

We have with $\mathrm{div}(\mathbf{F}) = 0 \in \mathbb{R}^3$ and due to (4.31), (4.32) and (4.33)

$$
\|\mathbf{F}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \lesssim \rho^{-m+1}. \tag{4.50}
$$

Finally, according to estimate (4.33) we have

$$
\|\mathrm{curl}\,\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathrm{div}(\varepsilon(\rho)\mathbf{C}^{is}_{m,\rho})\|_{\mathbf{L}^2_{0,-1}(\Omega^{is})} + \|\mathbf{C}_{m,\rho}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \lesssim \rho^{-m+1}. \tag{4.51}
$$

Hence from (4.49), (4.50) and (4.51)the estimate (4.45) is derived. □

Finally as in Peron's Thesis [42, Lemma 7.11 and Corollary 7.12] (describing the case of a bounded insulator domain) we can also (using suitable modifications) derive optimal estimates for unbounded insulating domains.

Lemma 6. *Let $s \geq 2$ and $\boldsymbol{F} \in \boldsymbol{PH}^{s-2}(\mathbb{R}^3)$ with support in Ω^{is}, such that $div\boldsymbol{F} = 0$ in $\mathbb{R}^3$. There exists a constant $C > 0$ independent of ρ such that for all $m \in [|0, [s] - 2|]$, we have*

$$
\|\boldsymbol{E}^{is}_m\|_{\boldsymbol{W}(curl,\Omega^{is})} \leq C, \tag{4.52}
$$

and

$$
\|\chi\boldsymbol{E}^{cd}_m\|_{\boldsymbol{L}^2(\Omega^{cd})} \leq C\rho^{-\frac{1}{2}}, \ \ \textit{and} \ \ \|\chi\boldsymbol{E}^{cd}_m\|_{\boldsymbol{H}(curl,\Omega^{cd})} \leq C\rho^{\frac{1}{2}}. \tag{4.53}
$$

Proof. From (4.19), the function $\mathbf{W}^{cd}_j$ are profiles defined on $\Sigma \times I$. Moreover, for any $j \in \mathbb{N}$

$$
\mathbf{E}^{is}_j \in \mathbf{W}(\mathrm{curl}, \Omega^{is}), \ \text{ and } \ \mathbf{W}^d_j \in \mathbf{H}(\mathrm{curl}, \Sigma \times I)
$$

then, for any $j \in \mathbb{N}$, we have

$$
\|\mathbf{E}^{is}_j\|_{\mathbf{W}(\mathrm{curl},\Omega^{is})} + \rho^{\frac{1}{2}}\|\mathbf{E}^{cd}_j\|_{\mathbf{L}^2(\Omega^{cd})} + \rho^{-\frac{1}{2}}\|\mathrm{curl}\,\mathbf{E}^{cd}_j\|_{\mathbf{L}^2(\Omega^{cd})} \leq C
$$

where $C > 0$ is a constant independent of ρ. Therefore the assertion follows from

$$
\|\chi\mathbf{E}^{cd}_m\|_{\mathbf{L}^2(\Omega^{cd})} \leq C\rho^{-\frac{1}{2}}, \quad \|\chi\mathrm{curl}\,\mathbf{E}^{cd}_m\|_{\mathbf{L}^2(\Omega^{cd})} \leq C\rho^{\frac{1}{2}}
$$

and

$$\|\mathbf{E}_m^{is}\|_{\mathbf{W}(\mathrm{curl},\Omega^{is})} \leq C.$$

□

Application of Lemma 6 and Thoerem 11 gives the following result which corresponds to Corollary 7.12 in [42].

Corollary 1. *Let $s \geq 4$ and $\boldsymbol{F} \in \boldsymbol{PH}^{s-2}(\mathbb{R}^3)$ with support in Ω^{is}, such that $div\boldsymbol{F} = 0$ in $\mathbb{R}^3$. Under the spectral hypothesis (**SH**), for all $\rho > 0$, for all $m \in [|0, [s] - 2|]$, the remainder $\boldsymbol{R}_{m,\rho}$ of the asymptotic development satisfies the followings estimates:*

$$\|\boldsymbol{R}_{m,\rho}^{cd}\|_{\boldsymbol{H}(curl,\Omega^{cd})} \leq C_{m+2}\rho^{-m-\frac{1}{2}}, \quad and \quad \|curl\,\boldsymbol{R}_{m,\rho}^{is}\|_{\boldsymbol{L}^2_{0,-1}(\Omega^{is})} \leq C_{m+2}\rho^{-m-1}, \tag{4.54}$$

where $C_{m+2} > 0$ is a constant independent of ρ.

Proof. The fist estimate in (4.54) is shown in Corollary 7.12 in [42]. The second estimate in (4.54) follows by writing

$$\mathbf{R}_{m,\rho}^{is} = \mathbf{R}_{m+2,\rho}^{is} + \rho^{-m-1}\mathbf{E}_{m+1}^{is} + \rho^{-m-2}\mathbf{E}_{m+2}^{is}, \quad \text{in} \quad \Omega^{is}.$$

By Lemma 6 we have

$$\|\mathbf{E}_m^{is}\|_{\mathbf{W}(\mathrm{curl},\Omega^{is})} \leq C$$

and due to Theorem 11 for $\rho > 0$ we have

$$\|\mathrm{curl}\,\mathbf{R}_{m+2,\rho}^{is}\|_{\mathbf{L}^2_{0,-1}(\Omega^{is})} \leq C'_{m+2}\rho^{-m-1},$$

implying the second estimate in (4.54). □

Let us comment that altogether we have shown that Peron's results for the asymptotic expansion remain valid for an unbounded exterior domain. Thus we have convergence of the asymptotic expansion with respect to inverse powers of conductivity for the electrical field and also analogously for the magnetic field for the transmission problem ($\mathbf{P}_{\alpha\beta}$) in $\mathbb{R}^3$ in Chapter 1. Comparison of the asymptotic expansion found in Chapter 1 for the halfspace case and of the asymptotic expansion in Chapter 4 for a smooth interface Σ shows that the first terms of the expansions coincide. These terms can be efficiently computed numerically by the boundary element procedure discussed in Chapter 1.

5 Non-conforming FE/BE coupling for a two-dimensional eddy current problem

Let Ω_- be a simply connected bounded region in $\mathbb{R}^2$ and $\Omega_+ := \mathbb{R}^2 \setminus \overline{\Omega_-}$ its complement. Here, Ω_+ represents the air and Ω_- the inter-section of a metallic obstacle in the x_1, x_2 plane. The obstacle is assumed to be parallel to the x_3-axis (see figure 5.1).
The classical macroscopic electromagnetic field is described by four vector functions of position $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and time $t \geq 0$, denoted by $\widehat{\mathbf{E}}$, $\widehat{\mathbf{D}}$, $\widehat{\mathbf{H}}$ and $\widehat{\mathbf{B}}$ (see [10, 39, 40, 41, 44]). Then we obtain the first order system

$$\begin{aligned}
\nabla \times \mathbf{E} &= i\omega\mu\mathbf{H} && \text{in } \mathbb{R}^2, \\
\nabla \times \mathbf{H} &= (\sigma - i\omega\varepsilon)\mathbf{E} + \mathbf{J}_0 && \text{in } \mathbb{R}^2, \\
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon} && \text{in } \mathbb{R}^2, \\
\nabla \cdot \mathbf{H} &= 0 && \text{in } \mathbb{R}^2.
\end{aligned} \tag{5.1}$$

The coefficients ε, μ and σ are bounded real valued scalar functions satisfying almost everywhere in Ω

$$\varepsilon_0 \leq \varepsilon(\mathbf{x}) \leq \varepsilon_1, \;\; \mu_0 \leq \mu(\mathbf{x}) \leq \mu_1 \;\text{ and }\; 0 \leq \sigma(\mathbf{x}) \leq \sigma_1,$$

where ε_0, ε_1, μ_0, μ_1 and σ_1 are positive constants and ε_0 and μ_0 are electric permittivity and magnetic permeability, respectively, of the free space.
Moreover, since the medium is dielectric and homogeneous, outside the obstacle there holds

$$\varepsilon(\mathbf{x}) = \varepsilon_0, \;\; \mu(\mathbf{x}) = \mu_0, \;\text{ and }\; \sigma(\mathbf{x}) = 0 \;\text{ in }\; \Omega_+.$$

Eliminating $\mathbf{E}$ from (5.1) yields in $\mathbb{R}^2$ the second order system with $\mathbf{j}_0 = 0$ in Ω_+

$$\nabla \times (\nabla \times \mathbf{H}) - (\sigma - i\omega\varepsilon)i\omega\mu\mathbf{H} = \mathbf{j}_0 \;\text{ in }\; \mathbb{R}^2. \tag{5.2}$$

This choice is arbitrary. We can also eliminate $\mathbf{H}$. We suppose the incident electric and magnetic fields $\mathbf{E}^0$, $\mathbf{H}^0$, and the fields $\mathbf{E}$, $\mathbf{H}$ are transverse magnetic and time harmonic. This

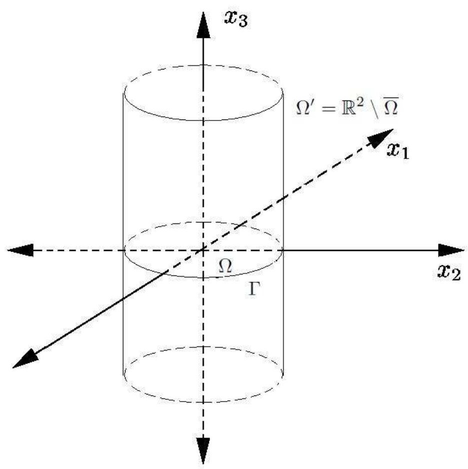

Figure 5.1: Region of the problem.

means that, with a proper choice of x_1, x_2, x_3 axes.
The amplitudes $\mathbf{E}$ and $\mathbf{H}$, which are independent of x_3, are given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\sqrt{\varepsilon_0}} \begin{pmatrix} 0 \\ 0 \\ u(x_1, x_2) \end{pmatrix}$$

and

$$\mathbf{H}(\mathbf{x}) = \frac{1}{\sqrt{\mu_0}} \begin{pmatrix} h_1(x_1, x_2) \\ h_2(x_1, x_2) \\ 0 \end{pmatrix}$$

where the complex-valued functions u and $\mathbf{h} = (h_1, h_2)^T$ are now the unknowns of the problem.
By calculations

$$\nabla \times \mathbf{E} = \frac{1}{\sqrt{\varepsilon_0}} \begin{pmatrix} \dfrac{\partial u}{\partial x_2} \\ -\dfrac{\partial u}{\partial x_1} \\ 0 \end{pmatrix}, \qquad \nabla \times \mathbf{H} = \frac{1}{\sqrt{\mu_0}} \begin{pmatrix} 0 \\ 0 \\ \dfrac{\partial h_2}{\partial x_1} - \dfrac{\partial h_1}{\partial x_2} \end{pmatrix}, \tag{5.3}$$

hence $\nabla \times u = (\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1})^T$ and $\nabla \times \mathbf{h} = \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2}$.
From (5.1) and (5.3) we have the first order system

$$\begin{aligned} \nabla \times \mathbf{h} + ikau &= \sqrt{\mu_0}\mathbf{J}_0 \quad \text{in} \quad \mathbb{R}^2, \\ \nabla \times u - ikb\mathbf{h} &= 0 \qquad\quad \text{in} \quad \mathbb{R}^2, \end{aligned} \tag{5.4}$$

with $\mathbf{j}_0 = 0$ in Ω_+ and $\forall \mathbf{x} \in \mathbb{R}^2$

$$k = \omega\sqrt{\varepsilon_0 \mu_0}, \quad b(\mathbf{x}) = \frac{\mu(\mathbf{x})}{\mu_0}, \quad a(\mathbf{x}) = \frac{\varepsilon(\mathbf{x})}{\varepsilon_0} + i\frac{\sigma(\mathbf{x})}{\omega\varepsilon_0},$$

(wave number, relative permeability, relative permittivity or index of refraction, respectively). Eliminating u from (5.4) yields with $\mathbf{j}_0 = a^{-1}\sqrt{\mu_0}\nabla \times \mathbf{J}_0$ the second order system

$$\begin{aligned} &\nabla \times (a^{-1}\nabla \times \mathbf{h}) - k^2 b\mathbf{h} = \mathbf{j}_0 && \text{in} \quad \mathbb{R}^2, \\ &\mathbf{h} = \mathbf{h}^i + \mathbf{h}^s && \text{in} \quad \mathbb{R}^2. \end{aligned} \tag{5.5}$$

where $\mathbf{h}^i$ and $\mathbf{h}^s$ belong to the incident and scattered waves, respectively.
Furthermore the behavior of the scattered electromagnetic field at infinity (known as the Sylver-Müller condition) implies the Sommerfeld radiation condition

$$\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) \longrightarrow 0, \quad \text{when} \quad r = |\mathbf{x}| \longrightarrow \infty. \tag{5.6}$$

We have

$$1 \le b(\mathbf{x}) \le \frac{\mu_1}{\mu_0}, \quad \text{for a.e.} \quad \mathbf{x} \in \mathbb{R}^2, \tag{5.7}$$

and $a(\mathbf{x}) := a_R(\mathbf{x}) + i a_I(\mathbf{x})$

$$1 \le a_R(\mathbf{x}) \le \frac{\varepsilon_1}{\varepsilon_0}, \quad \text{and} \quad 0 \le a_I(\mathbf{x}) \le \frac{\sigma_1}{\omega\varepsilon_0} \quad \text{for a.e.} \quad \mathbf{x} \in \mathbb{R}^2, \tag{5.8}$$

and

$$b(\mathbf{x}) = 1 \quad \text{and} \quad a(\mathbf{x}) = 1 \quad \text{and} \quad \Omega_+.$$

Then, testing (5.5) with the vector-valued function $\mathbf{q} \in \mathbf{H}_0(\mathrm{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, 0, \Omega_-)$ and integrating by parts yields: Find $\mathbf{h} \in \mathbf{H}_0(\mathrm{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, 0, \Omega_-)$ such that

$$(\nabla \times \mathbf{h}, \nabla \times \mathbf{q}) + \alpha(\mathbf{h}, \mathbf{q}) = (\mathbf{f}, \mathbf{q}) \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathrm{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, 0, \Omega_-), \tag{5.9}$$

where $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega_-)$ (or $[L_2(\Omega_-)]^2$), $\alpha \in \mathbb{R}$ is a constant, $\mathbf{f} \in [L_2(\Omega_-)]^2$, and besides the usual Hilbert space $H^s(\Omega_-)$ we define the following spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{curl}, \Omega_-) \quad &:= \left\{ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in [L_2(\Omega_-)]^2 \ : \ \nabla \times \mathbf{v} \in L_2(\Omega_-) \right\}, \\
\mathbf{H}_0(\mathrm{curl}, \Omega_-) \quad &:= \{ \mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega_-) \ : \ \mathbf{n} \times \mathbf{v} = 0 \ \text{on} \ \Gamma = \partial\Omega_- \}, \\
\mathbf{H}(\mathbf{div}, \Omega_-) \quad &:= \left\{ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in [L_2(\Omega_-)]^2 \ : \ \nabla \cdot \mathbf{v} \in L_2(\Omega_-) \right\}, \\
\mathbf{H}(\mathbf{div}, 0, \Omega_-) \quad &:= \{ \mathbf{v} \in \mathbf{H}(\mathbf{div}, \Omega_-) \ : \ \nabla \cdot \mathbf{v} = 0 \ \text{in} \ \Omega_- \},
\end{aligned}$$

$\mathbf{H}(\mathrm{curl}, \Omega_-)$ and $\mathbf{H}(\mathbf{div}, \Omega_-)$ also form Hilbert spaces (see [16]) with respect to the following associated norms

$$\|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{curl},\Omega_-)} := \|\mathbf{v}\|^2_{\mathbf{L}_2(\Omega_-)} + \|\nabla \times \mathbf{v}\|^2_{L_2(\Omega_-)},$$

$$\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{div},\Omega_-)} := \|\mathbf{v}\|^2_{\mathbf{L}_2(\Omega_-)} + \|\nabla\cdot\mathbf{v}\|^2_{L_2(\Omega_-)}.$$

The strong form associated with (5.9) is given by (see [5, 6]): Find $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl},\Omega_-) \cap \mathbf{H}(\mathbf{div},0,\Omega_-)$ such that

$$\nabla\times(\nabla\times\mathbf{u}) + \alpha\mathbf{u} = Q\mathbf{f} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl},\Omega_-) \cap \mathbf{H}(\mathbf{div},0,\Omega_-).$$

where $Q : [L_2(\Omega_-)]^2 \longrightarrow \mathbf{H}(\mathbf{div},0,\Omega_-)$ is the orthogonal projection onto the divergence-free functions. It is necessary to consider the projection of $\mathbf{f}$ onto $\mathbf{H}(\mathbf{div},0,\Omega_-)$ (see [4, 5, 6]).
We will also consider the more general formulation following Brenner et al. [4]: Find $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl},\Omega_-) \cap \mathbf{H}(\mathbf{div},\Omega_-)$ such that

$$(\nabla\times\mathbf{u},\nabla\times\mathbf{v}) + \beta(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}) + \alpha(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl},\Omega_-) \cap \mathbf{H}(\mathbf{div},\Omega_-), \tag{5.10}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$ are constants, $\mathbf{f} \in [L_2(\Omega_-)]^2$. The associated boundary value problem is:

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{u}) - \beta\nabla(\nabla\cdot\mathbf{u}) + \alpha\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega_-,\\
\mathbf{n}\times\mathbf{u} &= 0 \quad \text{on } \Gamma,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{on } \Gamma.
\end{aligned}$$

For the latter formulation we consider a non-conforming finite element procedure in the next subsection. Finally, in subsection 5.2 we consider a 2D transmission problem (5.18) where the solution in the exterior domain is represented by boundary integral operators on the interface. This leads to the non-conforming fe/be coupling method (5.25). Numerical experiments in subsection 5.3 show the efficiency of this coupling method.

5.1 Non-conforming finite element method

In this section we present the implementation for a non-conforming finite element method for Maxwell's equations introduced by Brenner et al. [4, 5, 6]. We consider two Galerkin schemes (one with divergence free elements and one without divergence free finite elements, but with an additional term $\beta(\nabla_h\cdot\mathbf{u}_h,\nabla_h\cdot\mathbf{v})$ in the formulation).
First, non-conforming divergence free finite element method is: Find $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{div}}$ such that

$$\begin{aligned}
&(\nabla_h\times\mathbf{u}_h,\nabla_h\times\mathbf{v}) + \alpha(\mathbf{u}_h,\mathbf{v})\\
&+\sum_{e\in\mathcal{E}_h}\frac{[\Phi_\mu(e)]^2}{|e|}\int_e [|\mathbf{n}\times\mathbf{u}_h|][|\mathbf{n}\times\mathbf{v}|]\,ds\\
&+\sum_{e\in\mathcal{E}_h^i}\frac{[\Phi_\mu(e)]^2}{|e|}\int_e [|\mathbf{n}\cdot\mathbf{u}_h|][|\mathbf{n}\cdot\mathbf{v}|]\,ds = (\mathbf{f},\mathbf{v}),
\end{aligned} \tag{5.11}$$

for all $\mathbf{v} \in \mathbf{V}_h^{\text{div}} \subset \mathbf{H}_0(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, 0, \Omega_-)$, where

$$\begin{aligned}\mathbf{V}_h^{\text{div}} := \quad & \{\mathbf{v} \in [L_2(\Omega_-)]^2 \ : \ \mathbf{v}_{\mathtt{T}} = \mathbf{v}|_{\mathtt{T}} \in [\mathbf{P}_1(\mathtt{T})]^2, \forall \mathtt{T} \in \mathbf{T}_h, \\ & \nabla \cdot \mathbf{v}|_{\mathtt{T}} = 0, \ \text{for all } \ \mathtt{T} \in \mathbf{T}_h, \\ & \mathbf{v} \ \text{ is continuous at the midpoint of any } \ e \in \mathcal{E}_h^i, \\ & \mathbf{n} \times \mathbf{v} \ \text{ vanishes at the midpoint of any } \ e \in \mathcal{E}_h^b\}.\end{aligned}$$

with the set of interior edges $\mathcal{E}_h^i$ and boundary edges $\mathcal{E}_h^b$ and

$$(\mathbf{u}_h, \mathbf{v}) = \int_\Omega \mathbf{u}_h(x) \cdot \mathbf{v}(x) dx.$$

The edge weight $\Phi_\mu(e)$ is defined by

$$\Phi_\mu(e) = \prod_{l=1}^{L} |c_l - m_e|^{1-\mu_l}. \tag{5.12}$$

where μ_l, $1 \leq l \leq L$ is the grading parameter at the corner c_l, m_e and $|e|$ denote the midpoint and length of the edge e.
We will measure the discretization error in the L_2 norm and the mesh-dependent energy norm $\|\cdot\|_{h,\text{div}}$ defined by

$$\begin{aligned}\|\mathbf{u}\|_{h,\text{div}}^2 = & \|\nabla_h \times \mathbf{u}\|_{\mathbf{L}_2(\Omega_-)}^2 + \|\mathbf{u}\|_{\mathbf{L}_2(\Omega_-)}^2 \\ & + \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[|\mathbf{n} \times \mathbf{u}|]\|_{L_2(e)}^2 + \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \|[|\mathbf{n} \cdot \mathbf{u}|]\|_{L_2(e)}^2.\end{aligned} \tag{5.13}$$

Secondly, we consider the non-conforming (non-divergence free) finite element method: Find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\begin{aligned}& (\nabla_h \times \mathbf{u}_h, \nabla_h \times \mathbf{v}) + \beta(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \mathbf{v}) + \alpha(\mathbf{u}_h, \mathbf{v}) \\ & + \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [|\mathbf{n} \times \mathbf{u}_h|][|\mathbf{n} \times \mathbf{v}|] \, ds + \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [|\mathbf{n} \cdot \mathbf{u}_h|][|\mathbf{n} \cdot \mathbf{v}|] \, ds = (\mathbf{f}, \mathbf{v}),\end{aligned} \tag{5.14}$$

for all $\mathbf{v} \in \mathbf{V}_h \subset \mathbf{H}_0(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, \Omega_-)$, where

$$\begin{aligned}\mathbf{V}_h := \quad & \{\mathbf{v} \in [L_2(\Omega_-)]^2 \ : \ \mathbf{v}_{\mathtt{T}} = \mathbf{v}|_{\mathtt{T}} \in [\mathbf{P}_1(\mathtt{T})]^2, \forall \mathtt{T} \in \mathbf{T}_h, \\ & \mathbf{v} \ \text{ is continuous at the midpoint of any } \ e \in \mathcal{E}_h, \\ & \mathbf{n} \times \mathbf{v} \ \text{ vanishes at the midpoint of any } \ e \in \mathcal{E}_h^b\}.\end{aligned}$$

We note that this space is related to the classical Crouzeix-Raviart space (see [14]).
In both formulations $[|\mathbf{n} \times \mathbf{u}|]$ and $[|\mathbf{n} \cdot \mathbf{u}|]$ denote the jumps of the tangential and normal

components across the triangle sides respectively (see Brenner et al. [4, 5, 6]). Here we use the following notation:
Let $e \in \mathcal{E}_h^i$ be shared by the two triangles $\mathtt{T}_{e,1}, \mathtt{T}_{e,2} \in \mathbf{T}_h$ and $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be the unit normal of e pointing towards the outside of $\mathtt{T}_{e,1}$ (resp. $\mathtt{T}_{e,2}$). We define, on e,

$$[|\mathbf{n} \times \mathbf{v}|] := \mathbf{n}_1 \times (\mathbf{v}_{\mathtt{T}_{e,1}}|_e) + \mathbf{n}_2 \times (\mathbf{v}_{\mathtt{T}_{e,2}}|_e), \tag{5.15}$$

$$[|\mathbf{n} \cdot \mathbf{v}|] := \mathbf{n}_1 \cdot (\mathbf{v}_{\mathtt{T}_{e,1}}|_e) + \mathbf{n}_2 \cdot (\mathbf{v}_{\mathtt{T}_{e,2}}|_e). \tag{5.16}$$

For an edge $e \in \mathcal{E}_h^b$, we take $\mathbf{n}_e$ to be the unit normal of e pointing towards the outside of Ω and define

$$[|\mathbf{n} \times \mathbf{v}|] := \mathbf{n}_e \times (\mathbf{v}|_e). \tag{5.17}$$

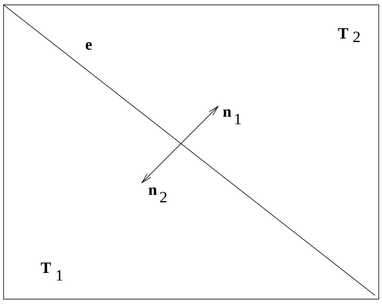

Figure 5.2: Triangles and normals in the definition of $[|\mathbf{n} \times \mathbf{v}|]$ and $[|\mathbf{n} \cdot \mathbf{v}|]$ [6].

Here the edge weight $\Phi_\mu(e)$ is defined by (5.12) (see Brenner et al. [4, 5, 6]).
Based on the Jan Thiedau's finite element program, (see [46]), which gives the numerical results for divergence free case (5.11), we have extended his programm to the non-divergence free case (5.14) which is used in Example 2.
The corresponding convergence analysis of both schemes (5.11) and (5.14) is given in the works by Brenner et al. [4, 5, 6].

5.2 The coupling of non-conforming finite element and boundary element methods

The coupling of non-conforming finite element and boundary element methods was established by Carstensen and Funken [8], where quasi-optimal a priori error estimates are provided for a (nonlinear) interface problem for the Laplacian and also a posteriori error estimates was established in Carstensen and Funken [9]. Here we present a different derivation for a two-dimensional electromagnetic transmission problem.
We consider the following transmission problem: In a bounded two-dimensional Lipschitz domain $\Omega_- \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega_-$ and an unbounded exterior domain $\Omega_+ := \mathbb{R}^2 \setminus \overline{\Omega_-}$

we are given a right-hand side $\mathbf{f} \in \mathbf{L}^2(\Omega_-)$, constants $\alpha \in \mathbb{R}$, $\beta > 0$ and seek functions $\mathbf{h} \in \mathbf{H}(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, \Omega_-)$, $u \in H^1_{\text{loc}}(\Omega_+)$ and a real constant b satisfying

$$\begin{aligned}
&\nabla \times \nabla \times \mathbf{h} - \beta \nabla (\nabla \cdot \mathbf{h}) + \alpha \mathbf{h} = \mathbf{f} && \text{in} \quad \Omega_-, \\
&\nabla \times \nabla \times u = 0, \quad \nabla \cdot u = 0 && \text{in} \quad \Omega_+, \\
&\gamma_\tau(\mathbf{h}) = \frac{\partial u}{\partial \mathbf{n}}, \;\; \gamma(\nabla \times \mathbf{h}) = u, \;\; \nabla \cdot \mathbf{h} = 0 && \text{on} \quad \Gamma, \\
&u(\mathbf{x}) - b \log |\mathbf{x}| \to 0 && \text{as} \quad |\mathbf{x}| \to \infty,
\end{aligned} \tag{5.18}$$

where $u|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ and $\frac{\partial u}{\partial \mathbf{n}}|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$.
Then, testing $(5.18)_1$ with the vector-valued function $\mathbf{q} \in \mathbf{H}(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, \Omega_-)$ and integrating by parts yields: Find $\mathbf{h} \in \mathbf{H}(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, \Omega_-)$, such that

$$(\nabla \times \mathbf{h}, \nabla \times \mathbf{q}) + \beta(\nabla \cdot \mathbf{h}, \nabla \cdot \mathbf{q}) + \alpha(\mathbf{h}, \mathbf{q}) - \langle \gamma_\tau(\mathbf{q}), \gamma(\nabla \times \mathbf{h}) \rangle = (\mathbf{f}, \mathbf{q}), \tag{5.19}$$

for all $\mathbf{q} \in \mathbf{H}(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, \Omega_-)$, where $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega_-)$ (or $[L_2(\Omega_-)]^2$), $\langle \cdot, \cdot \rangle$ the duality pairing on Γ. Note that $\gamma_\tau(\mathbf{q}) = \mathbf{q} \cdot \mathbf{t}$, where $\mathbf{t}$ is the unit tangential vector.
We have from equation $(5.18)_2$ with the constraints $\nabla \cdot u = 0$ and $\nabla \times \nabla \times u = 0$ in Ω_+ that $\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \Delta u$ implies $\Delta u = 0$ in Ω_+.
Next, we employ a boundary integral equation method (see Costabel and Stephan [13] and Carstensen and Funken [8]) to complete (5.19) with the corresponding variational formulation in the unbounded exterior domain Ω_+. Here

$$\begin{aligned}
\xi(\mathbf{x})|_\Gamma &= -\frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial n_{\mathbf{y}}} \log(|\mathbf{x} - \mathbf{y}|) \xi(\mathbf{y}) ds_{\mathbf{y}} + \frac{1}{2\pi} \int_\Gamma \log(|\mathbf{x} - \mathbf{y}|) \phi(\mathbf{y}) ds_{\mathbf{y}} \\
&= K\xi(\mathbf{x}) - V\phi(\mathbf{x})
\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
\phi(\mathbf{x})|_\Gamma &= -\frac{1}{2\pi} \int_\Gamma \frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}} \log(|\mathbf{x} - \mathbf{y}|) \xi(\mathbf{y}) ds_{\mathbf{y}} + \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial n_{\mathbf{x}}} \log(|\mathbf{x} - \mathbf{y}|) \phi(\mathbf{y}) ds_{\mathbf{y}} \\
&= -W\xi(\mathbf{x}) - K'\phi(\mathbf{x})
\end{aligned} \tag{5.21}$$

with $(\xi, \phi) = \left(u, \frac{\partial u}{\partial \mathbf{n}}\right) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$.
Now the non-conforming coupling method reads: Find $(\mathbf{h}, \xi, \phi) \in \mathbf{X} := \mathbf{H}(\text{curl}, \Omega_-) \cap \mathbf{H}(\mathbf{div}, \Omega_-) \times H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ such that $\forall (\mathbf{q}, \eta, \mu) \in \mathbf{X}$

$$\begin{aligned}
(\nabla \times \mathbf{h}, \nabla \times \mathbf{q}) + \beta(\nabla \cdot \mathbf{h}, \nabla \cdot \mathbf{q}) + \alpha(\mathbf{h}, \mathbf{q}) - \langle \gamma_\tau(\mathbf{q}), \xi \rangle &= (\mathbf{f}, \mathbf{q}) \\
-\langle \gamma_\tau(\mathbf{h}), \eta \rangle - \langle W\xi, \eta \rangle + \langle (\tfrac{1}{2} - K')\phi, \eta \rangle &= 0 \\
\langle (\tfrac{1}{2} - K)\xi, \mu \rangle + \langle V\phi, \mu \rangle &= 0
\end{aligned} \tag{5.22}$$

Let as abbreviate the above equations by

$$\mathbf{A}((\mathbf{h},\xi,\phi),(\mathbf{q},\eta,\mu)) = (\mathbf{f},\mathbf{q}). \tag{5.23}$$

Combining the results of Brenner et al. [4] for the interior problem in Ω_- and the mapping properties of the boundary integral operators (see for example Costabel and Stephan [12] and Carstensen and Funken [8]) we obtain there holds the Gårding inequality, i.e. $\exists\gamma_1,\gamma_2 > 0$ such that $\forall(\mathbf{h},\xi,\phi) \in \mathbf{X}$

$$\mathrm{Re}\{\mathbf{A}((\mathbf{h},\xi,\phi),(\mathbf{h},\xi,\phi))\} \geq \gamma_1(\|\mathbf{h}\|^2_{\mathbf{H}(\mathrm{curl},\Omega_-)} + \|\xi\|^2_{H^{1/2}(\Gamma)} + \|\phi\|^2_{H^{-1/2}(\Gamma)}) - \gamma_2\|\mathbf{h}\|^2_{\mathbf{L}^2(\Omega_-)}. \tag{5.24}$$

In order to formulate a finite element/boundary element coupling methods we introduce the following discrete spaces: As finite elements we use the Crouzeix-Raviart elements to span:

$$\begin{aligned} \mathbf{S}_h := \; & \big\{ \mathbf{v} \in [L_2(\Omega)]^2 \;\; : \;\; \mathbf{v}_{\mathrm{T}} = \mathbf{v}|_{\mathrm{T}} \in [\mathbf{P}_1(\mathrm{T})]^2, \forall \mathrm{T} \in \mathbf{T}_h, \\ & \mathbf{v} \text{ is continuous at the midpoint of any } e \in \mathcal{E}_h^i, \big\}. \end{aligned}$$

As boundary elements we use the space of piecewise linear, continuous functions S_h^1, and the space of piecewise constants S_h^0.
Now our non-conforming fem/bem coupling method reads: Find $(\mathbf{h}_h,\xi_h,\phi_h) \in \mathbf{X}_h := \mathbf{S}_h \times S_h^1 \times S_h^0$ such that $\forall(\mathbf{q},\eta,\mu) \in \mathbf{X}_h$

$$\begin{aligned} &(\nabla\times\mathbf{h}_h,\nabla\times\mathbf{q}) + \beta(\nabla\cdot\mathbf{h}_h,\nabla\cdot\mathbf{q}) + \alpha(\mathbf{h}_h,\mathbf{q}) + b_h(\mathbf{h}_h,\mathbf{q}) - \langle\gamma_\tau(\mathbf{q}),\xi_h\rangle &&= (\mathbf{f},\mathbf{q}) \\ &-\langle\gamma_\tau(\mathbf{h}_h),\eta\rangle - \langle W\xi_h,\eta\rangle + \langle(\tfrac{1}{2}-K')\phi_h,\eta\rangle &&= 0 \\ &\langle(\tfrac{1}{2}-K)\xi_h,\mu\rangle + \langle V\phi_h,\mu\rangle &&= 0 \end{aligned} \tag{5.25}$$

where

$$b_h(\mathbf{h}_h,\mathbf{q}) := \sum_{e\in\mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|}\int_e [|\mathbf{n}\times\mathbf{h}_h|][|\mathbf{n}\times\mathbf{q}|]ds + \sum_{e\in\mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|}\int_e [|\mathbf{n}\cdot\mathbf{h}_h|][|\mathbf{n}\cdot\mathbf{q}|]ds, \tag{5.26}$$

is the penalty term.

5.3 Numerical examples

In this part we report the results of a series of numerical experiments for the non-conforming finite element method (5.14) and for the non-conforming FE/BE coupling method (5.25). All computations where done with *Matlab*.Our finite element simulations are obtained with the software developed by Thiedau in [46]. We have developed new Matlab codes for the boundary element part and FE/BE coupling. Our computations show that the methods (5.14) and (5.25) converge and give good results.

We take $\beta = 1$ and $\alpha = \pm k^2$ in the experiment for the scheme (5.14). Besides the errors in the $\mathbf{L}_2$-norm $\|\cdot\|_{\mathbf{L}_2(\Omega)}$ and the mesh-dependent energy norm $\|\cdot\|_{h,\text{div}}$ defined by

$$\begin{aligned}\|\mathbf{u}\|_h^2 &= \|\nabla_h \times \mathbf{u}\|^2_{\mathbf{L}_2(\Omega)} + \|\nabla_h \cdot \mathbf{u}\|^2_{L_2(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{L}_2(\Omega)} \\ &+ \sum_{e\in\mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[|\mathbf{n}\times\mathbf{u}|]\|^2_{L_2(e)} + \sum_{e\in\mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \|[|\mathbf{n}\cdot\mathbf{u}|]\|^2_{L_2(e)}\end{aligned} \tag{5.27}$$

we also include the errors in the semi-norms $|\cdot|_{\text{curl}}$ and $|\cdot|_{\text{div}}$ defined by $|\mathbf{u}|_{\text{curl}} = \|\nabla_h \times \mathbf{u}\|_{\mathbf{L}_2(\Omega)}$ and $|\mathbf{u}|_{\text{div}} = \|\nabla_h \cdot \mathbf{u}\|_{L_2(\Omega)}$.

Example 2. *We examine the convergence for the general scheme (5.14) on the square domain $[0, 0.5]^2$ for a family of uniform meshes, where the exact solution is given by (see Brenner et al. [4])*

$$\boldsymbol{u}(x, y) = [(\frac{x^3}{3} - \frac{x^2}{4})(y^2 - 0.5y)\sin(ky), (\frac{y^3}{3} - \frac{y^2}{4})(x^2 - 0.5x)\cos(kx)]. \tag{5.28}$$

Hence for the right-hand side we get $\boldsymbol{f}(x, y) = \nabla\times\nabla\times\boldsymbol{u}(x, y) - \beta\nabla(\nabla\cdot\boldsymbol{u}(x, y)) + \alpha\boldsymbol{u}(x, y)$. The results are tabulated in Table 5.1 for $\alpha = k^2$ and $k = 0, 1, 5, 10$ and in Table 5.2 for $\alpha = -k^2$ and $k = 1, 5, 10$ and are plotted in Figure 5.3 for $\alpha = -1, 1$. They show that the scheme (5.14) is second order accurate in the $\boldsymbol{L}_2$-norm and first order accurate in the energy norm and in the semi-norms, which agrees with the error estimates in [4, Theorem 13] and [5, Theorem 14]. The results in Table 5.3 confirm that the scheme (5.14) does not converge without the consistency terms. Our numerical simulations agree with those in [4, 5, 6] and [46].

Next we present numerical experiments for the non-conforming fe/be coupling (5.25) where we had to develop a matlab implementation for the boundary integral operators in 5.25 and a coupling with finite element method in 5.14.

Example 3. *We consider the interface problem (5.18) on the square domain $[0, 0.5]^2$ with a family of uniform meshes, where*

$$\boldsymbol{h}(x, y) = [(\frac{x^3}{3} - \frac{x^2}{4})(y^2 - 0.5y)\sin(ky), (\frac{y^3}{3} - \frac{y^2}{4})(x^2 - 0.5x)\cos(kx)], \tag{5.29}$$

the right-hand side we take $\boldsymbol{f}(x, y) = \nabla\times\nabla\times\boldsymbol{h}(x, y) - \beta\nabla(\nabla\cdot\boldsymbol{h}(x, y)) + \alpha\boldsymbol{h}(x, y)$. The errors and convergence rates for the discrete solutions $\boldsymbol{h}_h$, ξ_h and ϕ_h of (5.25) are tabulated in Tables 5.4, 5.5 and 5.6 for $\beta = 1$, $\alpha = \pm k^2$ and $k = 1$, and are plotted in Figures 5.4, 5.5 and 5.6. The error in energy norm of $\boldsymbol{h}$ is calculate by (5.27). The energy norm $\|\xi\|_V$ is approximated by the energy norm $\|\xi_{h=\frac{1}{200}}\|_V = 2.723\cdot 10^{-6}$ of the solution to the finest mesh for $\alpha = 1$ and $\|\xi_{h=\frac{1}{200}}\|_V = 2.727\cdot 10^{-6}$ for $\alpha = -1$. Analogously for $\|\phi\|_W$, we have $\|\phi_{h=\frac{1}{200}}\|_W = 2.647\cdot 10^{-6}$ for $\alpha = 1$ and $\|\phi_{h=\frac{1}{200}}\|_W = 2.6516\cdot 10^{-6}$ for $\alpha = -1$. The asymptotic convergence rate of $\boldsymbol{h}$ is 0.25 with respect to the degrees of freedom which is suboptimal to the best possible rate of 0.5. The convergence of $\|\phi_h\|_W$ to $\|\phi\|_W$ and $\|\xi_h\|_V$ to $\|\xi\|_V$ at rate 1.2 to 1.3 is also suboptimal. Here $\|\xi\|_V = \xi^T\boldsymbol{V}\xi$ where $\boldsymbol{V}$ is the matrix representation of V and ξ are the coefficients of the solution vector.

Example 4. *Let we consider the interface problem (5.18) on the square domain* $[0, 0.5]^2$ *with a family of uniform meshes, where*

$$\boldsymbol{h}(x,y) = [x^2(y^2 - 0.5y), y^2(x^2 - 0.5x)], \tag{5.30}$$

the right-hand side we take $\boldsymbol{f}(x,y) = \nabla \times \nabla \times \boldsymbol{h}(x,y) - \beta\nabla(\nabla \cdot \boldsymbol{h}(x,y)) + \alpha\boldsymbol{h}(x,y)$.
The results for $\boldsymbol{h}$, ξ *and* ϕ *solutions of (5.25) are tabulated in Tables 5.7, 5.8 and 5.9 for* $\beta = 1$, $\alpha = \pm 1$, *and are plotted in Figures 5.7, 5.8 and 5.9. The exact energy norm of* $\boldsymbol{h}$ *is known by extrapolation for* $\alpha = 1$ *is* $\|\boldsymbol{h}\|_h = 0.0339734151$ *and for* $\alpha = -1$ *is* $\|\boldsymbol{h}\|_h = 0.0350692138$. *The energy norm* $\|\xi\|_V$ *is approximated by the energy norm* $\|\xi_{h=\frac{1}{200}}\|_V = 5.54937 \cdot 10^{-5}$ *of the solution to the finest mesh for* $\alpha = 1$ *and* $\|\xi_{h=\frac{1}{200}}\|_V = 5.774 \cdot 10^{-5}$ *for* $\alpha = -1$. *Analogously for* $\|\phi\|_W$, *we have* $\|\phi_{h=\frac{1}{200}}\|_W = 7.89382 \cdot 10^{-6}$ *for* $\alpha = 1$ *and* $\|\phi_{h=\frac{1}{200}}\|_W = 8.20526 \cdot 10^{-5}$ *for* $\alpha = -1$. *The convergence of* $\|\boldsymbol{h}_h\|_h$ *to* $\|\boldsymbol{h}\|_h$ *at rate* 0.21 *to* 0.22 *with respect to the degrees of freedom which is suboptimal to the best possible rate of* 0.24. *The convergence of* $\|\phi_h\|_W$ *to* $\|\phi\|_W$ *and* $\|\xi_h\|_V$ *to* $\|\xi\|_V$ *at rate* 1.1 *to* 1.2 *is also suboptimal.*

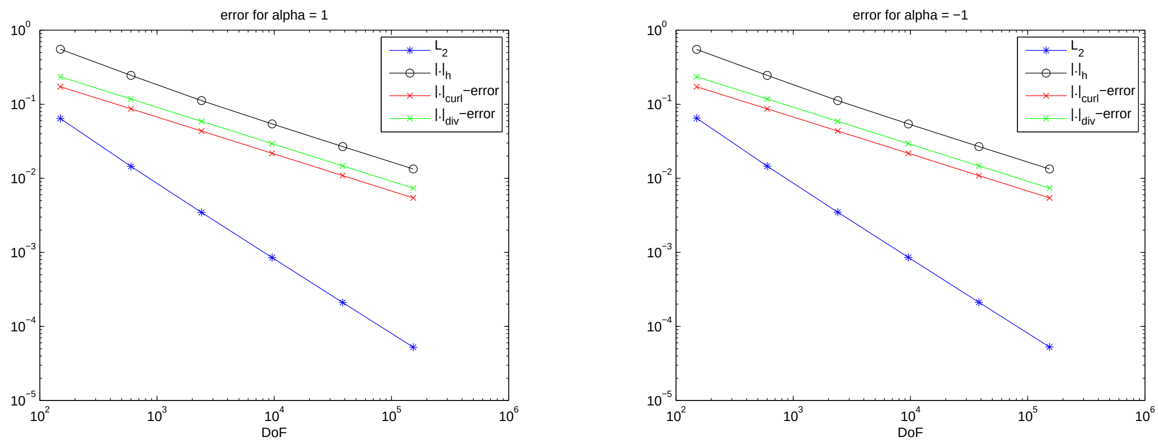

Figure 5.3: Errors of the finite element scheme (5.14) for $\alpha = 1$ and $\alpha = -1$.

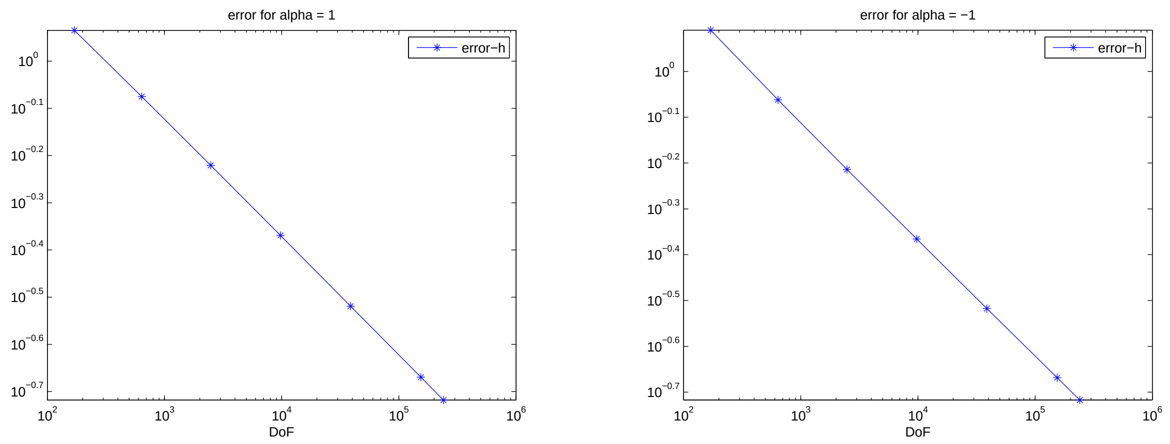

Figure 5.4: Errors in energy norm of $\mathbf{h}$ for $\alpha = 1$ and $\alpha = -1$ in Example 3 (fe/be coupling)

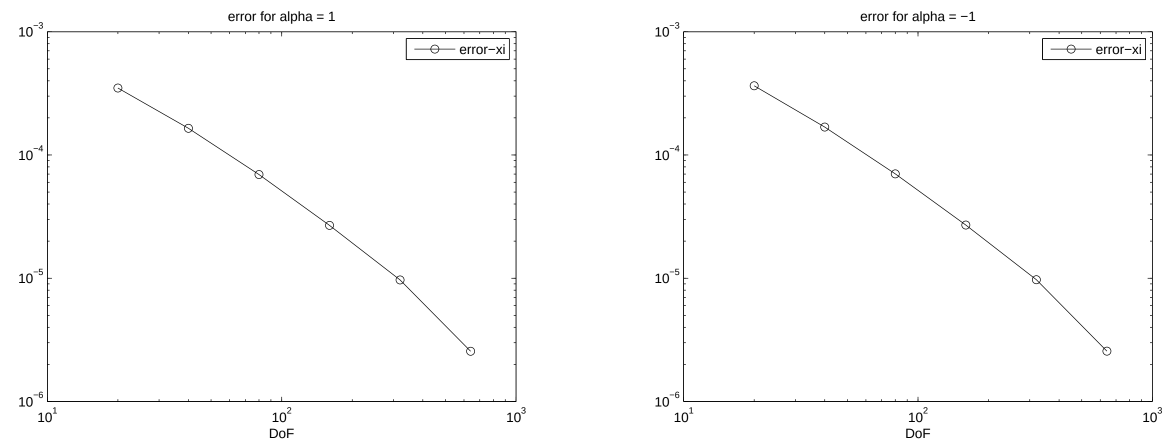

Figure 5.5: Errors in energy norm of ξ for $\alpha = 1$ and $\alpha = -1$ in Example 3 (fe/be coupling)

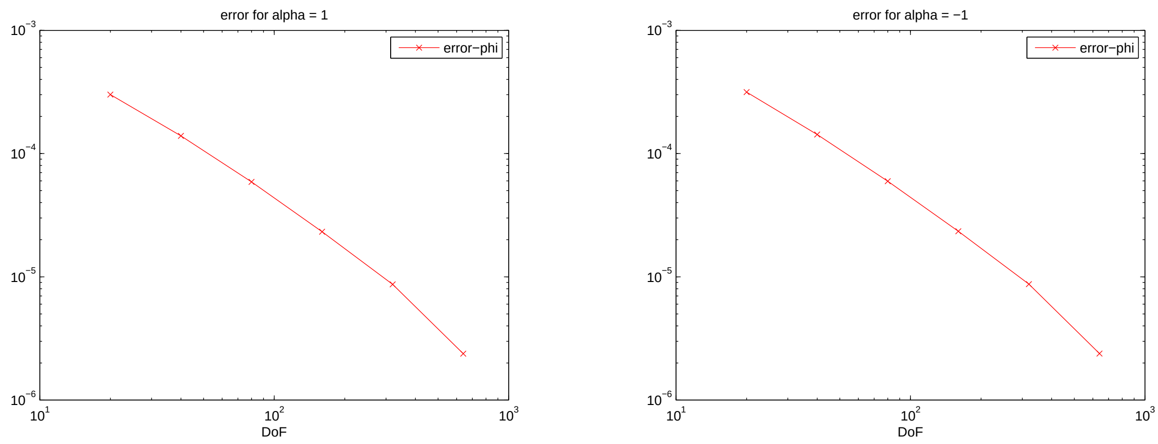

Figure 5.6: Errors in energy norm of ϕ for $\alpha = 1$ and $\alpha = -1$ in Example 3(fe/be coupling)

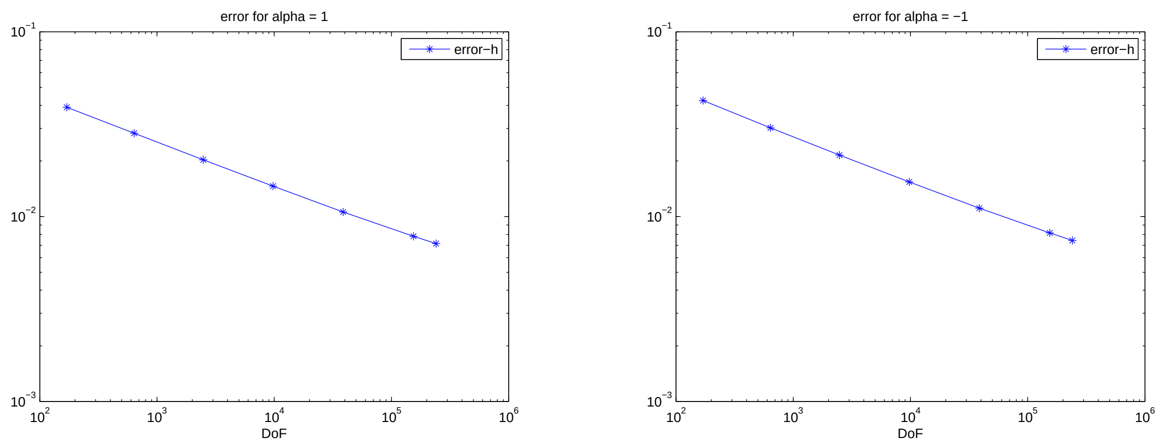

Figure 5.7: Errors in energy norm of $\mathbf{h}$ for $\alpha = 1$ and $\alpha = -1$ in Example 4(fe/be coupling)

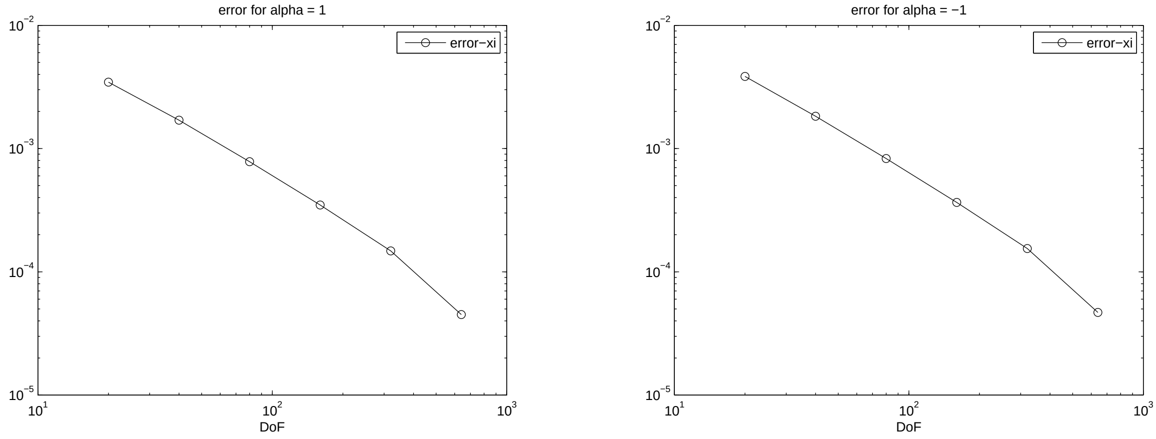

Figure 5.8: Errors in energy norm of ξ for $\alpha = 1$ and $\alpha = -1$ in Example 4(fe/be coupling)

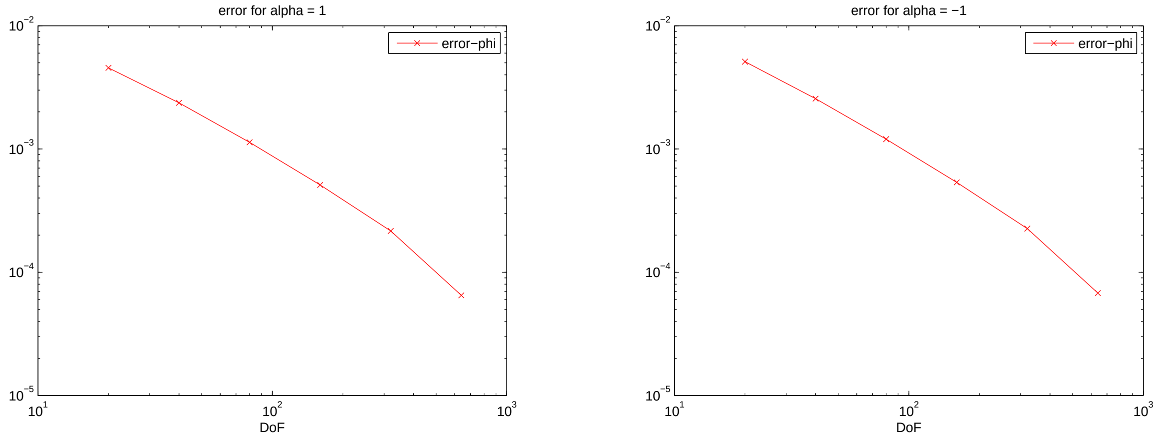

Figure 5.9: Errors in energy norm of ϕ for $\alpha = 1$ and $\alpha = -1$ in Example 4(fe/be coupling)

h	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2}}{\|\mathbf{u}\|_{\mathbf{L}^2}}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_h}{\|\mathbf{u}\|_h}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{curl}}}{\|\mathbf{u}\|_{\mathrm{curl}}}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{div}}}{\|\mathbf{u}\|_{\mathrm{div}}}$	order
				$k=0$				
1/5	0.0745		0.6271		0.1894		0.3342	
1/10	0.0165	2.1785	0.2514	1.3187	0.0950	0.9953	0.1672	0.9993
1/20	0.0039	2.0824	0.1119	1.1678	0.0475	0.9991	0.0836	0.9995
1/40	0.0009	2.0391	0.0539	1.0536	0.0238	1.0000	0.0418	1.0000
1/80	0.0002	2.0190	0.0267	1.0163	0.0119	1.0001	0.0209	1.0001
1/160	0.0001	2.0093	0.0133	1.0052	0.0059	1.0001	0.0104	1.0001
				$k=1$				
1/5	0.0642		0.5544		0.1728		0.2354	
1/10	0.0145	2.1445	0.2458	1.1730	0.0870	0.9902	0.1177	0.9993
1/20	0.0034	2.0652	0.1120	1.1339	0.0435	0.9976	0.0589	0.9985
1/40	0.0008	2.0301	0.0542	1.0451	0.0217	0.9994	0.0294	0.9995
1/80	0.0002	2.0143	0.0268	1.0141	0.0109	0.9998	0.0147	0.9998
1/160	0.0001	2.0070	0.0133	1.0046	0.0054	0.9999	0.0073	0.9999
				$k=5$				
1/5	0.1252		0.4128		0.3113		0.5464	
1/10	0.0281	2.1548	0.2508	0.7188	0.1565	0.9915	0.2748	0.9917
1/20	0.0066	2.0837	0.1284	0.9662	0.0783	0.9988	0.1375	0.9988
1/40	0.0016	2.0415	0.0643	0.9973	0.0392	1.0001	0.0688	1.0001
1/80	0.0004	2.0205	0.0321	1.0012	0.0196	1.0002	0.0344	1.0002
1/160	0.0001	2.0102	0.0160	1.0013	0.0098	1.0002	0.0172	1.0002
				$k=10$				
1/5	0.1995		0.4324		0.3903		1.6516	
1/10	0.0494	2.0153	0.3194	0.4371	0.1979	0.9794	0.8373	0.9801
1/20	0.0118	2.0683	0.1754	0.8646	0.0992	0.9973	0.4194	0.9974
1/40	0.0029	2.0421	0.0893	0.9733	0.0496	0.9997	0.2097	0.9997
1/80	0.0007	2.0223	0.0448	0.9956	0.0248	1.0001	0.1049	1.0001
1/160	0.0002	2.0114	0.0224	0.9999	0.0124	1.0001	0.0524	1.0001

Table 5.1: Convergence of the finite element scheme (5.14) for $\alpha = k^2$.

h	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2}}{\|\mathbf{u}\|_{\mathbf{L}^2}}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_h}{\|\mathbf{u}\|_h}$	order	$\frac{\lvert\mathbf{u}-\mathbf{u}_h\rvert_{\text{curl}}}{\lvert\mathbf{u}\rvert_{\text{curl}}}$	order	$\frac{\lvert\mathbf{u}-\mathbf{u}_h\rvert_{\text{div}}}{\lvert\mathbf{u}\rvert_{\text{div}}}$	order
				$k=1$				
1/5	0.0649		0.5529		0.1729		0.2354	
1/10	0.0146	2.1492	0.2458	1.1697	0.0870	0.9904	0.1178	0.9995
1/20	0.0035	2.0662	0.1120	1.1335	0.0436	0.9976	0.0589	0.9985
1/40	0.0009	2.0304	0.0543	1.0450	0.0218	0.9994	0.0295	0.9995
1/80	0.0002	2.0145	0.0269	1.0141	0.0109	0.9999	0.0147	0.9999
1/160	0.0001	2.0070	0.0134	1.0046	0.0055	1.0000	0.0074	1.0000
				$k=5$				
1/5	0.1607		0.3867		0.3115		0.5469	
1/10	0.0326	2.3019	0.2480	0.6409	0.1565	0.9929	0.2748	0.9931
1/20	0.0075	2.1165	0.1281	0.9536	0.0783	0.9987	0.1375	0.9987
1/40	0.0018	2.0502	0.0643	0.9946	0.0392	1.0001	0.0687	1.0001
1/80	0.0004	2.0231	0.0321	1.0006	0.0196	1.0002	0.0344	1.0002
1/160	0.0001	2.0110	0.0160	1.0011	0.0098	1.0002	0.0172	1.0002
				$k=10$				
1/5	0.4483		0.3429		0.4099		1.7347	
1/10	0.0672	2.7374	0.3137	0.1284	0.1987	1.0447	0.8405	1.0454
1/20	0.0148	2.1866	0.1749	0.8428	0.0992	1.0019	0.4197	1.0020
1/40	0.0035	2.0664	0.0893	0.9703	0.0496	1.0004	0.2098	1.0004
1/80	0.0009	2.0273	0.0448	0.9949	0.0248	1.0002	0.1049	1.0002
1/160	0.0002	2.0123	0.0224	0.9998	0.0124	1.0001	0.0524	1.0001

Table 5.2: Convergence of the finite element scheme (5.14) for $\alpha = -k^2$.

h	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2}}{\|\mathbf{u}\|_{\mathbf{L}^2}}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_h}{\|\mathbf{u}\|_h}$	order	$\frac{\lvert\mathbf{u}-\mathbf{u}_h\rvert_{\text{curl}}}{\lvert\mathbf{u}\rvert_{\text{curl}}}$	order	$\frac{\lvert\mathbf{u}-\mathbf{u}_h\rvert_{\text{div}}}{\lvert\mathbf{u}\rvert_{\text{div}}}$	order
				$k=1$				
1/5	41.183		0.0015		0.4540		0.6146	
1/10	41.698	-0.017	0.0005	1.6966	0.4398	0.0459	0.5947	0.0476
1/20	41.828	-0.004	0.0002	1.5043	0.4361	0.0119	0.5898	0.0120
1/40	41.861	-0.001	0.0001	1.4234	0.4352	0.0030	0.5885	0.0030
1/80	41.869	-0.000	0.0000	1.3895	0.4350	0.0008	0.5882	0.0008
1/160	41.871	-0.000	0.0000	1.3747	0.4349	0.0002	0.5881	0.0002

Table 5.3: Non-convergence of the finite element scheme (5.14).

h	dof-$\mathbf{h}$	$\|\mathbf{h}-\mathbf{h}_h\|_h$	$\frac{\|\mathbf{h}-\mathbf{h}_h\|_h}{\|\mathbf{h}\|_h}$	order w.r.t. dof-$\mathbf{h}$
		$\alpha = 1$		
1/5	170	0.0020170812	1.1624543206	
1/10	640	0.0014596589	0.8412087631	0.2439891112
1/20	2480	0.0010439860	0.6016543757	0.2474310882
1/40	9760	0.0007415150	0.4273388386	0.2497061840
1/80	38720	0.0005250301	0.3025775152	0.2505252099
1/160	154240	0.0003713494	0.2140105165	0.2505599558
1/200	240800	0.0003321473	0.1914181694	0.2504501719
		$\alpha = -1$		
1/5	170	0.0021333017	1.2294328114	
1/10	640	0.0015034630	0.8664532872	0.2639420700
1/20	2480	0.0010599082	0.6108303967	0.2580856962
1/40	9760	0.0007472096	0.4306206281	0.2551702286
1/80	38720	0.0005270525	0.3037429856	0.2532869465
1/160	154240	0.0003720655	0.2144232458	0.2519474417
1/200	240800	0.0003326599	0.1917135701	0.2513136839

Table 5.4: Errors in energy norm and convergence rate for **h** with respect to the degrees of freedom, h-version in Example 3(fe/be coupling 5.25).

h	dof-ξ	$\|\xi\|_V$	$(\|\xi\|_V^2 - \|\xi_h\|_V^2)^{\frac{1}{2}}$	order w.r.t. dof-ξ
		$\alpha = 1$		
1/5	20	0.0003489617	0.0003489511	
1/10	40	0.0001646595	0.0001646370	1.0837361987
1/20	80	0.0000694849	0.0000694316	1.2456251635
1/40	160	0.0000269819	0.0000268441	1.3709842442
1/80	320	0.0000100613	0.0000096858	1.4706607015
1/160	640	0.0000037343	0.0000025555	1.9222788474
1/200	800	0.0000027230		
		$\alpha = -1$		
1/5	20	0.0003640640	0.0003640448	
1/10	40	0.0001683813	0.0001683398	1.1127401096
1/20	80	0.0000703053	0.0000702057	1.2617158325
1/40	160	0.0000271498	0.0000268908	1.3844759487
1/80	320	0.0000100947	0.0000093759	1.5200865467
1/160	640	0.0000037410	0.0000025610	1.9241725093
1/200	800	0.0000027270		

Table 5.5: Errors in energy norm and convergence rate for ξ with respect to the degrees of freedom, h-version in Example 3(fe/be coupling 5.25).

h	dof-ϕ	$\|\phi\|_W$	$(\|\phi\|_W^2 - \|\phi_h\|_W^2)^{\frac{1}{2}}$	order w.r.t. dof-ϕ
		$\alpha = 1$		
1/5	20	0.0003007662	0.0003007546	
1/10	40	0.0001392414	0.0001392162	1.1112591635
1/20	80	0.0000589137	0.0000588542	1.2421110217
1/40	160	0.0000233855	0.0000232353	1.3408289386
1/80	320	0.0000090799	0.0000086855	1.4196329953
1/160	640	0.0000035594	0.0000023797	1.8678494785
1/200	800	0.0000026470		
		$\alpha = -1$		
1/5	20	0.0003150112	0.0003149910	
1/10	40	0.0001426874	0.0001426428	1.1429039400
1/20	80	0.0000596844	0.0000595778	1.2595613093
1/40	160	0.0000235504	0.0000232787	1.3557626398
1/80	320	0.0000091149	0.0000083880	1.4726168404
1/160	640	0.0000035669	0.0000023858	1.8699730295
1/200	800	0.0000026516		

Table 5.6: Errors in energy norm and convergence rate for ϕ with respect to the degrees of freedom, h-version in Example 3(fe/be coupling 5.25).

h	dof-$\mathbf{h}$	$\|\mathbf{h}\|_h$	$(\|\mathbf{h}\|_h^2 - \|\mathbf{h}_h\|_h^2)^{\frac{1}{2}}$	order w.r.t. dof-$\mathbf{h}$
		$\alpha = 1$		
1/5	170	0.0517514059	0.0390386357	
1/10	640	0.0441725639	0.0282315863	0.2444880286
1/20	2480	0.0395776551	0.0203026562	0.2433950506
1/40	9760	0.0369811976	0.0146087659	0.2402349191
1/80	38720	0.0355876850	0.0105967158	0.2329918388
1/160	154240	0.0348638630	0.0078291765	0.2189964978
1/200	240800	0.0347168123	0.0071459161	0.2049947195
		$\alpha = -1$		
1/5	170	0.0550492986	0.0424331888	
1/10	640	0.0462864284	0.0302090003	0.2563164096
1/20	2480	0.0411522141	0.0215326490	0.2499506465
1/40	9760	0.0383032688	0.0154042410	0.2444665714
1/80	38720	0.0367886139	0.0111154109	0.2367889920
1/160	154240	0.0360056484	0.0081582448	0.2237837145
1/200	240800	0.0358468907	0.0074262922	0.2110245629

Table 5.7: Errors in energy norm and convergence rate for $\mathbf{h}$ with respect to the degrees of freedom, h-version in Example 4(fe/be coupling 5.25).

h	dof-ξ	$\|\xi\|_V$	$(\|\xi\|_V^2 - \|\xi_h\|_V^2)^{\frac{1}{2}}$	order w.r.t. dof-ξ
		$\alpha = 1$		
1/5	20	0.0034582495	0.0034578042	
1/10	40	0.0016986709	0.0016977642	1.0262200580
1/20	80	0.0007846831	0.0007827183	1.1170709903
1/40	160	0.0003524066	0.0003480099	1.1693648943
1/80	320	0.0001578402	0.0001477633	1.2358407851
1/160	640	0.0000714102	0.0000449430	1.7171188027
1/200	800	0.0000554937		
		$\alpha = -1$		
1/5	20	0.0038444560	0.0038440223	
1/10	40	0.0018304028	0.0018294919	1.0711737064
1/20	80	0.0008318333	0.0008298270	1.1405605609
1/40	160	0.0003702129	0.0003656825	1.1822189543
1/80	320	0.0001649231	0.0001544854	1.2431214191
1/160	640	0.0000743616	0.0000468587	1.7210820019
1/200	800	0.0000577400		

Table 5.8: Errors in energy norm and convergence rate for ξ with respect to the degrees of freedom, h-version in Example 4(fe/be coupling 5.25).

h	dof-ϕ	$\|\phi\|_W$	$(\|\phi\|_W^2 - \|\phi_h\|_W^2)^{\frac{1}{2}}$	order w.r.t. dof-ϕ
		$\alpha = 1$		
1/5	20	0.0004555453	0.0004554769	
1/10	40	0.0002366844	0.0002365527	0.9452159835
1/20	80	0.0001134675	0.0001131926	1.0633824574
1/40	160	0.0000516756	0.0000510692	1.4825405220
1/80	320	0.0000230079	0.0000216114	1.2406604906
1/160	640	0.0000102277	0.0000065034	1.7325113097
1/200	800	0.0000078938		
		$\alpha = -1$		
1/5	20	0.0051133846	0.0051127262	
1/10	40	0.0025600687	0.0025587534	0.9986516559
1/20	80	0.0012038727	0.0012010732	1.0911169870
1/40	160	0.0005426186	0.0005363789	1.1629996273
1/80	320	0.0002401819	0.0002257315	1.2486448657
1/160	640	0.0001063980	0.0000677340	1.7366548937
1/200	800	0.0000820526		

Table 5.9: Errors in energy norm and convergence rate for ϕ with respect to the degrees of freedom, h-version in Example 4(fe/be coupling 5.25).

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