A geometric approach to the μ-variant of the periodic b-equation and some two-component extensions

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Abstract

In the present thesis we discuss some integrable equations and systems of equations suitable for the modelling of 1D water waves, using methods coming from geometric analysis. A novel variant of the periodic b-equation is defined by the operator $\mu - \partial_x^2$; here, μ gives the mean of a periodic function. For $b = 2$ and $b = 3$ we obtain the μ-Camassa-Holm equation and the μ-Degasperis-Procesi equation, respectively. A two-component generalization of the Camassa-Holm equation and its μ-variant is obtained by including the continuity equation for the fluid velocity and density. Analogously, it is possible to define a two-component system for the Degasperis-Procesi equation, its μ-variant or related equations like the Hunter-Saxton equation. We show that the equations under consideration reexpress a geodesic flow on the group of orientation-preserving diffeomorphisms of the circle $\mathbb{S}$ (or a suitable semidirect product, respectively); in particular, they can be treated within Arnold's geometric approach. The geometric picture yields some local well-posedness theorems, in particular for the smooth category, as well as stability results. The thesis also shows ways to generalize the obtained results to non-periodic equations and other modified variants, e.g., coming up in the study of water waves under the influence of weak energy dissipation.

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Key words and phrases. (μ-)b-equation, geodesic flow, circle diffeomorphism group

Zusammenfassung

In der vorliegenden Arbeit werden integrable Gleichungen und Gleichungssysteme zur Modellierung von 1D Wasserwellen mit Methoden der geometrischen Analysis studiert. Eine Variante der periodischen b-Gleichung wird durch den linearen Operator $\mu - \partial_x^2$ realisiert; hierbei liefert μ den Mittelwert einer periodischen Funktion. Für $b = 2$ und $b = 3$ erhält man die μ-Camassa-Holm Gleichung bzw. die μ-Degasperis-Procesi Gleichung. Eine Verallgemeinerung der Camassa-Holm Gleichung und ihrer μ-Variante wird durch Hinzunahme der Kontinuitätsgleichung für die Geschwindigkeit und die Fluiddichte erhalten. Analog definiert man ein Zwei-Komponenten-System für die Degasperis-Procesi Gleichung, ihre μ-Variante und verwandte Gleichungen wie die Hunter-Saxton Gleichung. Wir zeigen auf, daß die genannten Modellgleichungen äquivalent sind zu Geodätengleichungen auf der Diffeomorphismengruppe des Einheitskreises (bzw. einem geeigneten semidirekten Produkt); insbesondere lassen sie sich im Rahmen der Arnold'schen Theorie beschreiben. Aus der geometrischen Betrachtung resultieren Theoreme zur lokalen Wohlgestelltheit, insbesondere in Fréchet-Räumen, sowie Aussagen zur Stabilität von Lösungen. Die Arbeit beinhaltet auch Ansätze zur Verallgemeinerung der Resultate auf nicht-periodische Gleichungen und diskutiert Modifizierungen der untersuchten Gleichungen, etwa zur Modellierung von schwacher Energiedissipation.

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Schlagwörter. (μ-)b-Gleichung, geodätischer Fluß, Diffeomorphismengruppe des Einheitskreises

Preface

Mathematical studies of fluid motion have been carried out for more than 300 years and there is a number of famous mathematicians, physicists and engineers who contributed important results to the mathematical theory of fluids: Daniel Bernoulli (Hydrodynamica, 1738), Georges Gabriel Stokes (Mathematical and Physical Papers, 1880-1905) or the universal genius Leonhard Euler (Principes généraux du mouvement des fluides, 1757), to name only a few. Isaac Newton (Principia II, 1687) was the first to attempt a mathematical theory of water waves. Much later, after the derivation of Euler's equations of hydrodynamics, Pierre-Simon Laplace (1776) reexamined wave motion (although his work remained disregarded). Joseph-Louis Lagrange (Méchanique analytique, 1788), perhaps independently, derived the linearized governing equations for small amplitude waves and obtained a solution in the limiting case of long plain waves in shallow water, [29].

Laplace was the first who posed the general initial value problem: Given any localized initial disturbance of the liquid surface what is the subsequent motion? Even nowadays, problems like that are of great importance; the reader might think of the prediction of tsunamis and huge cresting waves which motivate the study of water wave problems from the mathematical point of view, within the framework of a modern area of research. The well-posedness problem for the general Navier-Stokes system in three dimensions is only one prominent example among many other attractive open problems where we probably need some deep new ideas.

Until the second half of the 20th century, the study of wave motion was confined almost exclusively to linear theory, [27]. Nevertheless, linear water wave theory cannot capture effects like wave breaking or solitary waves. On account of that, nonlinear equations have been introduced and a pioneering candidate has been proposed by Boussinesq (1877),

$$u_t - 6uu_x + u_{xxx} = 0,$$

later named after Diederik Korteweg and Gustav de Vries (1895). The Korteweg-de Vries (KdV) equation is a paradigmatic example of an integrable nonlinear PDE, i.e., the solutions can be exactly and precisely specified. In addition, KdV is bi-Hamiltonian and can be solved by means of the inverse scattering transform; hence there is an infinite number of conservation laws and a corresponding Lax pair representation, [78, 110]. Eventually, it has been shown that KdV allows for soliton solutions but not for wave breaking, cf. [10] where global existence for $L_2(\mathbb{S})$ initial data is proved.

Not more than 20 years ago, Roberto Camassa and Darryl D. Holm derived a novel nonlinear equation for the motion of shallow water waves, applying Hamiltonian methods:

$$u_t + 3uu_x = 2u_x u_{xx} + uu_{xxx} + u_{txx}.$$

Camassa and Holm proved the existence of solitary waves and studied the associated Lax pair, showing in that way the integrability of the Camassa-Holm (CH) equation, [11]. In the subsequent years, the CH became subject of a wide range of papers, e.g., dealing with blow-up solutions and hence wave breaking, cf., e.g., [19, 21, 22, 111]. In 1999, the research for integrable nonlinear PDEs in form similar to the CH led to the Degasperis-Procesi (DP) equation

$$u_t + 4uu_x = 3u_x u_{xx} + uu_{xxx} + u_{txx},$$

see [30, 32], and later to a general family of nonlinear equations,

$$m_t = -(m_x u + b u_x m), \quad m = u - u_{xx}, \quad b \in \mathbb{R},$$

which is also called b-equation, [36]. For $b = 2$ and $b = 3$, the b-equation becomes the CH equation and the DP equation, respectively, and only for these choices of b, the resulting equation is integrable, [30, 69]. Interestingly, although discovered solely because of its mathematical properties, it turned out later that the DP equation plays a role in the water wave theory, quite similar to the CH, [27]. Further recent papers establish the bi-Hamiltonian formulation and the scattering approach, [30, 31], and until today, researchers try to obtain various types of solutions of DP by using numerical methods as well as powerful analytical tools, see, e.g., [44, 60, 83].

The present thesis is devoted to some variants of the periodic b-equation. If we replace the momentum variable $m = u - u_{xx}$ by $m = \mu(u) - u_{xx}$ where $\mu(u)$ is the mean of the function $u(t,x)$, i.e., the real valued time-dependent function $\int_0^1 u(t,x)\,\mathrm{d}x$, we obtain the novel family of equations

$$m_t = -(m_x u + b m u_x), \quad m = \mu(u) - u_{xx}, \quad b \in \mathbb{R},$$

which first appeared in a paper by Lenells, Misiołek and Tiglăy in 2009, [99]. The motivation for considering this partially averaged variant comes from geometry: In the pioneering work [5], Arnold explains that the motion of inertia rigid bodies and the motion of an ideal fluid can be described within the same mathematical approach: Euler's equations of motion for the body and the fluid can both be obtained as the geodesic equations of a one-sided invariant Riemannian metric on a Lie group. In each case the metric corresponds to the kinetic energy and is given by an inner product on the Lie algebra of the group. The inertia matrix for the rigid body corresponds to an inertia operator for the fluid motion which maps the fluid velocity u to the momentum variable m. For the b-equation, the inertia operator is $1 - \partial_x^2$ and choosing $-\partial_x^2$, the b-equation for $b = 2$ becomes the Hunter-Saxton (HS) equation which appears in the study of nematic liquid crystals, [65]. In some intuitive sense, choosing $\mu - \partial_x^2$, we obtain an equation which might inherit properties of the b-equation and the HS equation. In this thesis, we mainly discuss the μ-b-equation for $b = 2$ and $b = 3$ where we obtain the μCH and the μDP equation (which are two integrable members of the μ-family, [99]).

In physical experiments with real water waves, it is not possible to omit the effect of energy dissipation. For the CH and the DP, some recent studies show that by adding a term proportional to m on the right-hand side one obtains a suitable model for water waves with weak energy dissipation, [46, 124]. This motivates the study of a weakly dissipative μ-b-equation

$$m_t = -(m_x u + b u_x m + \lambda m), \quad m = \mu(u) - u_{xx}, \quad (b, \lambda) \in \mathbb{R} \times (0, \infty),$$

which we discuss only for $b = 3$.

The periodic CH equation possesses an integrable two-component extension, denoted as 2CH, which includes the continuity equation for fluid velocity and fluid density in the second component:

$$\begin{cases} m_t = -u m_x - 2 m u_x - \rho \rho_x, \\ \rho_t = -(\rho u)_x, \end{cases}$$

where $m = u - u_{xx}$, cf., e.g., [13, 49]. A two-component variant of DP has been suggested by Popowicz [115],

$$\begin{cases} m_t = -3 m u_x - m_x u - \rho u_x + 2 \rho \rho_x, \\ \rho_t = -2 \rho u_x - \rho_x u. \end{cases}$$

That 2DP is in fact integrable — which manifests itself in the existence of a Lax pair and a bi-Hamiltonian structure — is not proved but conjectured in [115]; Popowicz only generalizes a Hamiltonian operator for the DP to a suitable matrix Hamiltonian operator for the extended equation. A two-component variant of HS has been suggested by Lenells and Lechtenfeld [90]; it is of the same form as the 2CH but with $m = -u_{xx}$. The 2HS can be regarded as a supersymmetric extension of the Camassa-Holm equation. In [90], the authors also work out the bi-Hamiltonian formulation and a Lax pair representation for the 2HS equation and present some explicit solutions like bounded travelling waves.

Very often the main step to obtaining a solution of a mathematical problem is to find an adequate representation for it. After that finding the solution becomes easy or is at least possible. For the b-equation, there is a beautiful generalization of Arnold's powerful geometric approach, see, e.g., [41]. In the geometric picture, the b-equation reexpresses a geodesic flow and, concerning local well-posedness, it is much easier to discuss the geodesic equation than the equation in its initial form. It will be the general concept of the present work to rewrite the equations under consideration in a suitable geometric picture. The geometric viewpoint on our equations and families is not only aesthetically appealing but will also be useful in the study of well-posedness and stability issues.

Up to now, there are only a few results about the μ-variant of the b-equation and two-component systems, related to the fact that these equations have only begun lately to appear in the literature. In particular, geometric interpretations of nonlinear PDEs for the water wave theory are a current area of research. Let us summarize in detail the main results collected in this work which is organized as follows:

The first chapter introduces the basic concepts of fluid mechanics to the reader. We explain the derivation of Euler's equations of motion and give a precise description of the classical water wave problem. We also make clear how the families of model equations mentioned above come up within the mathematical theory of ideal fluids. In addition, we present Arnold's geometric approach to fluid dynamics and recall some elementary concepts from Riemannian geometry like geodesics, curvature and the geometric aspects of some Lie groups. This introductory chapter does not contain any new results and can be skipped by the experienced reader.

In Chap. 2, we consider a more general family of CH equations obtained from the inertia operator $1 - \lambda \partial_x^2$ for $\lambda \in [0, 1]$; this is motivated by the variational principle. As for the CH, it is possible to show that the so obtained generalization is integrable since it possesses a bi-Hamiltonian structure and a Lax par. Similar to Lenells' approach [94] for the CH equation, the generalized family is a reexpression of the geodesic flow for a canonically defined affine connection on the group Diff($\mathbb{S}$) of orientation-preserving diffeomorphisms of the circle $\mathbb{S}$. We specify the Christoffel map and derive a convenient formula for the sectional curvature of the circle diffeomorphism group associated with the generalized CH. Next, we present an infinite-dimensional subspace of positive sectional curvature and compute explicit formulas for the variation of the Christoffel map and the sectional curvature with respect to the parameter λ.

The CH and the DP share many similarities: They allow for breaking waves, solitary waves, peakon solutions, have an integrable structure and are both obtained from the b-equation. Nevertheless, in the geometric picture we find the following main difference: While the affine connection defined for the CH is compatible with a Riemannian metric, this is not the case for the DP. We thus call the CH a metric Euler equation; the DP belongs to the class of non-metric Euler equations. In [45], Escher and Seiler prove that only for $b = 2$ (that is, the Camassa-Holm) the b-equation is a metric Euler equation: For any $b \neq 2$, it is impossible to find a regular inertia operator A such that the corresponding b-equation reexpresses geodesic motion with respect to the right-invariant metric induced by A. For non-metric Euler equations, the geometric theory only works on account of the affine connection defined in terms of the Christoffel map.

For the DP equation (which is a prototypical example for the general case $b \neq 2$), Escher and Kolev established a meaningful local well-posedness result for the smooth category in 2009, cf. [41]. One goal of Chap. 3 is to point out that the arguments in [41, 45] also work well for the μ-b-equation. Precisely, we show that the μCH is the only equation obtained from the μ-b-equation which is compatible with a Riemannian structure; the corresponding inertia operator is $\mu - \partial_x^2$. For any $b \neq 2$ we prove that the μ-b-equation is of non-metric type. We then consider the μDP equation and prove that it is locally well-posed in the smooth category, i.e., for any smooth initial value $u_0 \in C^\infty(\mathbb{S})$, there exists a unique smooth short-time solution which depends smoothly on time and on the initial data. The strategy of our proof is to make consequently use of the geometric reformulation: On the diffeomorphism group of the circle (in the $C^n(\mathbb{S})$-category with $n \geq 3$) the μDP becomes a geodesic equation which is an ODE with smooth right-hand side. Applying standard Banach space theory, we immediately get a local well-posedness result for the geodesic flow. But concluding that the geodesic flow for smooth initial data is smooth, is not trivial for several technical reasons: First $C^\infty(\mathbb{S})$ is a Fréchet space in which we cannot apply the local existence and uniqueness theorems for Banach spaces. On the other hand, in the $C^n(\mathbb{S})$-category the diffeomorphism group of the circle is only a topological group and not a Lie group. And letting $n \to \infty$ we have to make sure that the existence intervals for the C^n-flows corresponding to smooth initial values do not converge to zero.

Finally, Chap. 3 presents some well-posedness and blow-up results for a weakly dissipative μDP equation and continues the discussion of one-parameter families of Riemannian metrics, as explained in Chap. 2: We define a one-parameter family of μCH equations which is obtained from the inertia operator $\mu - \lambda\partial_x^2$ for $\lambda \in [0, 1]$. Again, we find the Christoffel map for the novel family, compute the sectional curvature and establish a positivity result. The chapter ends with a computation of the λ-derivatives of some geometric quantities.

Chap. 3 also contains a detailed overview about the different inertia operators coming into the play, as well as a short introduction to Sobolev spaces on the circle, which we will need in this and in our next chapter.

Chap. 4 is about two-component variants of CH, DP and HS as well as the associated μ-equations. After a short summary of well-known facts about semidirect product groups we show that 2CH and 2DP can be regarded as geodesic equations on the semidirect product $\mathrm{Diff}(\mathbb{S}) \circledS \mathcal{F}(\mathbb{S})$, where $\mathcal{F}(\mathbb{S})$ denotes a space of sufficiently smooth real-valued functions on the circle. For 2CH, the geodesic equation derives from a natural right-invariant Riemannian metric, whereas for 2DP the affine connection is not compatible with any such metric. The geometric construction will give immediate proofs of local well-posedness for both systems in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ or $C^n(\mathbb{S}) \times C^{n-1}(\mathbb{S})$ for sufficiently smooth initial data. Moreover, we will show that the local well-posedness can be extended to the Fréchet space $C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$. For 2CH, we perform some explicit calculations of the sectional curvature and we prove the existence of a large subspace of positive sectional curvature. Finally, we point out that our approach to the 2CH is analogous to Euler's formalism for the rigid body motion which already proved to be successful for the one-component CH, [81]. Our treatment of the 2HS and its μ-variant is quite similar: We find a suitable semidirect product configuration space for the 2HS equation, prove that 2HS reexpresses a geodesic flow and show that the sectional curvature associated with the 2HS equation is constant and positive. This generalizes a result for the one-component HS established by Lenells in [95]. Our discussion of 2μHS begins with the presentation of a Lax pair and the geometric setting. We also perform some curvature computations for the 2μHS.

In Chap. 5 we are concerned with the non-periodic b-equation, i.e., the family $m_t = -(m_x u + b u_x m)$ with $b \in \mathbb{R}$, $m = u - u_{xx}$ and x a real variable. Some recent studies show that local well-posedness in the smooth category can be achieved from the geometric picture, quite similarly to the periodic case, but using a different Lie group setting. For $b = 2$ and the group of H^∞-diffeomorphisms, a proof is written down in [34] and we generalize the approach to an arbitrary b and diffeomorphism groups of general Sobolev class. The main problem is to establish that the groups under consideration have the structure of a regular Fréchet Lie group in the sense of Milnor, cf. [105, 108].

There are three appendices which summarize some key results of the analysis in Banach and Fréchet spaces, Kato's semigroup approach to abstract evolution equations and the theory of integrable infinite-dimensional systems.

To sum it up, the thesis shows how analytical methods coming from physics, differential geometry and analysis lead to new interesting results in the mathematical theory of water waves. Some unanswered questions and further tasks can be found in the open problem chapter: there is still a lot of work to do! Some of the results mentioned above have already been published by the author, see the reference list, and further preprints will follow. I hope to have succeeded in writing a text accessible for mathematicians, engineers and physicists working in different fluid mechanics research communities and I am thankful for any kind of feedback.

Hannover, January 18, 2011 *Martin Kohlmann*

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List of Symbols

Sets and numbers

$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers with zero
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{S}$	unit circle
i	imaginary unit

Operators

∇	nabla operator
grad	gradient
div	divergence
rot	curl operator
$Df, D_p f$	derivative of f (at p)
∂_x	partial derivative with respect to x
$-\partial_x^2, -\Delta$	Laplacian
$\frac{\mathrm{D}}{\mathrm{D}t}$	material derivative (Sect. 1.1.1), covariant derivative (Sect. 1.2.3)
μ	mean of a function on $\mathbb{S}$
$\mathscr{F}$	Fourier transform (also: $\mathscr{F}f = \hat{f}$ and $\mathscr{F}^{-1}f = \check{f}$)

Function spaces

$\langle \cdot, \cdot \rangle_X$	scalar product in X
$\\|\cdot\\|_X$	norm in X
$B(x_0, r), B_r(x_0)$	open ball with radius r around x_0
$\overline{B}_r(x_0)$	closed ball with radius r around x_0
X^*	dual space of X, adjoint operator of X
$(\cdot, \cdot)$	dual pairing
$\mathcal{L}(X, Y)$	bounded linear operators $X \to Y$
$\mathcal{L}(X)$	$\mathcal{L}(X, X)$
$\mathcal{L}^k(X, Y)$	$\mathcal{L}(X, \mathcal{L}(X, \cdots, \mathcal{L}(X, Y), \cdots))$
$\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(X)$	symmetric topological isomorphisms of X
$\mathcal{L}_{\mathrm{sym}}^2(X, Y)$	symmetric bilinear maps $X \to Y$
$C^n(X; Y)$	space of C^n-maps $X \to Y$

$C^n(X)$	space of C^n-maps on X
$C^\infty(X;Y)$	space of smooth maps $X \to Y$
$C^\infty(X)$	space of smooth maps on X
$\lVert\cdot\rVert_\infty$	supremum norm
$L_p(X)$	(Banach) space of measurable functions f defined on X with $\lVert f\rVert_{L_p}^p = \int_X \lvert f(x)\rvert^p \,\mathrm{d}x < \infty$ for $p \in [1,\infty)$, $\lVert f\rVert_{L_\infty} = \text{ess sup} f < \infty$ for $p = \infty$
$\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R})$	Schwartz space on the real axis, tempered distributions
$\mathscr{S}(\mathbb{Z})$	Schwartz space of rapidly decreasing sequences
$\lvert x \wedge y\rvert^2$	$\lVert x\rVert^2 \lVert y\rVert^2 - \langle x, y\rangle^2$

Manifolds

T_pM	tangent space of M at $p \in M$
T_p^*M	cotangent space of M at $p \in M$
TM	tangent bundle of M
T^*M	cotangent bundle of M

Other general notations

$\forall$	for all
$\exists$	there exists
$[\,\cdot\,]$	equivalence class
Re, Im	real part, imaginary part
$:=, \equiv$	definition ($\equiv$ is also used to reinforce $=$)
$\simeq$	identification with
$\hookrightarrow$	embedding
l.i.m.	limit in the L_2-norm
$f\vert_U$	restriction of f to U
$\dot{x}(t), x'(t)$	time derivative
$u_x, u^{(k)}$	spatial derivative of u (similarly for higher order and mixed derivatives), derivative of order k
χ_M	characteristic function of M
$*$	convolution
$\otimes$	tensor product
$\mathcal{O}$	Landau symbol
∂M	boundary of M
tr	trace
diag$(\cdot,\cdot)$	2×2-diagonal matrix
$D(A)$	domain of the operator A
ker(A)	kernel of A
δ	Dirac distribution, Kronecker delta, variation
$\mathcal{F}(\mathbb{S})$	some space of scalar functions on $\mathbb{S}$
Diff$(\mathbb{S})$	some group of diffeomorphisms $\mathbb{S} \to \mathbb{S}$
Rot$(\mathbb{S})$	set of all maps $\mathbb{S} \to \mathbb{S}$: $x \mapsto x + d$ for $d \in \mathbb{R}$
End, Aut, Der, Isom	endomorphisms, automorphisms, derivations, isomorphisms

Variables describing a fluid flow

Sobolev spaces

Diffeomorphism groups

Lie groups and geometric notions

Γ_{ij}^k	Christoffel symbols	20
Γ_p, B	Christoffel operator	21
$\nabla_X Y$	affine connection	20
$\exp_p$	exponential map at p	23
$R(X,Y)Z$	curvature tensor	24
$S(x,y)$	sectional curvature in the direction spanned by x, y	24

Geometric theory of rigid bodies and the motion of an ideal fluid

$SO(3), \mathfrak{so}(3)$	group of orthogonal 3×3-matrices and its Lie algebra	14
$\hat{\cdot}, \check{\cdot}$	isomorphisms $\mathbb{R}^3 \to \mathfrak{so}(3)$ and $\mathbb{R}^3 \to \mathfrak{so}(3)^*$ respectively	15
$\mathcal{B}$	rigid body in its initial configuration	14
$\omega, \omega_c, \omega_s$	angular velocities	17
ω, Ω	angular velocities for a rigid body	104
m, m_c, m_s	momentum variables	17
π, Π	momentum variables for a rigid body	104
$I = (I_1, I_2, I_3)$	inertia matrix	103
A, ρ_A	inertia operator, metric associated with A	22
u	Eulerian velocity for a fluid flow	21
(u, ρ)	Eulerian velocity for two-component equations	85
u_0, m_0	initial values of velocity and momentum	9
$(u_0, \rho_0), (m_0, \rho_0)$	initial values for two-component equations	90
$\pi, m_0, (m_0, \rho_0)$	conserved momentum variables	104
U	analogue of the body velocity	14
(U_1, U_2)	analogue of the body velocity for two-component equations	105
φ	geodesic flow on $\mathrm{Diff}(\mathbb{S})$	34
(φ, f)	geodesic flow on $\mathrm{Diff}(\mathbb{S})\circledS\mathcal{F}(\mathbb{S})$	90
S_1, S_2	one-component, two-component sectional curvature	99

Equations

KdV	Korteweg-de Vries (equation)	8
CH	Camassa-Holm (equation)	8
DP	Degasperis-Procesi (equation)	8
HS	Hunter-Saxton (equation)	29
2CH	two-component Camassa-Holm (equation)	84
2DP	two-component Degasperis-Procesi (equation)	84
2HS	two-component Hunter-Saxton (equation)	109
μB	μ-Burgers (equation)	76
μCH	μ-Camassa-Holm (equation)	50
μDP	μ-Degasperis-Procesi (equation)	53
μHS	μ-Hunter-Saxton (equation)	50
2μCH	two-component μ-Camassa-Holm (equation)	109
2μDP	two-component μ-Degasperis-Procesi (equation)	118
2μHS	two-component μ-Hunter-Saxton (equation)	109

Chapter 1
Preliminaries

"What we know is a drop, what we don't know is an ocean." (Sir Isaac Newton, 1643-1727)

Fig. 1.1 The free surface of a water wave.
(`http://www.how-to-purify-water.com/images/waterwave.jpg`, cited 13 May 2010)

The mathematical theory of water waves is a modern area of research which is based on the classical analysis of partial differential equations but also uses methods coming from geometry, harmonic analysis or the theory of infinite-dimensional Hamiltonian systems. This introductory chapter aims to explain the fundamental aspects of the mathematical modelling of fluids, in particular the governing equations of fluid motion. We show that one-dimensional (1D) water waves can be described by a novel family of evolution equations, the so called b-equation. The b-equation includes the Camassa-Holm equation as well as the Degasperis-Procesi equation. We also recall some basic concepts from Riemannian geometry and the group structure of fluid dynamics since this is absolutely necessary for all the following considerations. Finally, we introduce the Hunter-Saxton equation by physical arguments and explain its connection to the water wave problem.

1.1 The mathematical theory of ideal fluids and water waves

The motion of a perfect fluid is described by a system of partial differential equations named after Leonhard Euler who first published them in 1757 in his famous article *Principes généraux du mouvement des fluides*. We explain the intuitive and mathematical ideas which lead to the notion of an ideal fluid and derive a complete set of conservation laws for the motion of such a fluid. Most importantly, we discuss a general water wave problem in which we are interested in the water's free surface over a flat bottom, moving under the influence of gravity. We discuss different approximations to the Euler equations for this model leading to the famous Korteweg-de Vries equation, the Camassa-Holm equation or the Degasperis-Procesi equation which was derived recently in [30, 32]. Our summary mainly follows [4, 14] for the general theory and [27, 36, 69] for the modelling of water waves.

1.1.1 Euler's equations for the flow of an ideal fluid

Let Ω be a region in two- or three-dimensional space filled with a fluid. Our aim is to describe the fluid's motion. The basic mathematical idea of a fluid motion is that it can be regarded as a point transformation. We imagine the fluid to consist of small moving particles: A fluid particle which is at a position ξ at time $t = 0$ is at position x at a later time so that

$$x = x(\xi, t) \quad \text{or} \quad x_i = x_i(\xi_1, \xi_2, \xi_3, t). \tag{1.1}$$

Clearly, this model violates the concepts of the kinetic theory of fluids saying that the fluid particles are the molecules which are in random motion. Our treatment of fluid motion is based on a *continuum model*, which turned out to be suitable for macroscopic phenomena, and we assume that the velocity at any point is the average velocity of the molecules in a suitable neighborhood of this point. The initial coordinates ξ of a particle are called *material coordinates* (or *convected coordinates*, *Lagrangian coordinates*) of the particle and the particle itself may be called the *fluid particle* ξ. The *spatial coordinates* x of the particle may be referred to as its *position* or *place*. The transformation (1.1) can be regarded as a curve with parameter t and we call this curve the *particle path* of the particle ξ. Let us assume that the motion is continuous, single valued and that Eq. (1.1) can be inverted, i.e.,

$$\xi = \xi(x, t) \quad \text{or} \quad \xi_i = \xi_i(x_1, x_2, x_3, t)$$

and that ξ is continuous and single valued. Physically, these assumptions mean that the particle paths are continuous functions and that fluid particles do not split up or that two distinct particles occupy the same place. If we consider a physical quantity Q of the fluid, we thus have two ways of interpreting the values of Q: On the one hand, $Q(\xi, t)$ is obtained by an observer riding on the particle ξ through the fluid whereas $Q(x, t)$ is obtained by an observer who is fixed at the spatial position x and watches the fluid motion through a small neighborhood of the point x. We call the first picture the *Lagrangian description* of fluid motion and the second picture the *Eulerian description*. For each fixed time t, we define the *Eulerian velocity* of the fluid by

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i(x, t)$$

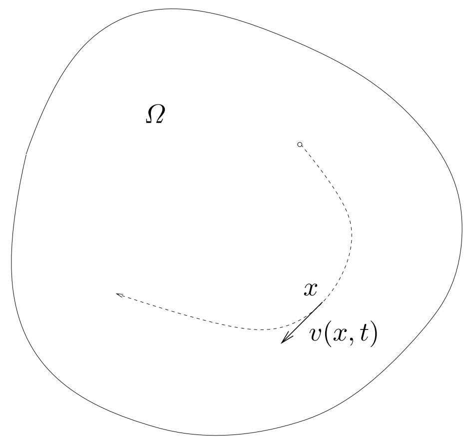


Fig. 1.2 Fluid particle flowing in a region Ω.

and observe that $v = (v_1, v_2, v_3)$ is a time-dependent vector field on Ω. We assume that for each time t the fluid has a well-defined *mass density* $\rho(x,t)$, i.e., if $\Omega' \subset \Omega$ is a subregion of Ω, then the *mass* of fluid inside Ω' at time t is

$$m(\Omega', t) = \int_{\Omega'} \rho(x,t)\,\mathrm{d}V,$$

where $\mathrm{d}V$ denotes the volume element in plane or in space. Again, the existence of ρ follows from our continuum assumption which ignores somehow the molecular structure of matter. In the following, we assume that ρ and v inherit appropriate smoothness so that standard operations of calculus can be applied to them. To obtain the governing equations for the fluid motion we stick to the following three basic principles:

1. Mass is neither created nor destroyed.
2. Newton's second law: The rate of change of momentum of a portion of the fluid equals the force applied to it.
3. Energy is neither created nor destroyed.

Let $\Omega' \subset \Omega$ be a subregion of the fluid domain Ω. Conservation of mass means that the rate of change of mass in Ω' equals the volume flow across $\partial\Omega'$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega'} \rho(x,t)\,\mathrm{d}V = -\int_{\partial\Omega'} \rho v \cdot n\,\mathrm{d}A, \tag{1.2}$$

where n denotes the outward normal at points of $\partial\Omega'$ and $\mathrm{d}A$ the area element on $\partial\Omega'$. Applying the divergence theorem, we find the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0. \tag{1.3}$$

Second, the total *force* $F_{\partial\Omega'}$ on $\partial\Omega'$ is given by the surface stress and hence the *pressure* $p = p(x,t)$ on $\partial\Omega'$ and external forces, i.e.,

$$F_{\partial\Omega'} = -\int_{\partial\Omega'} pn\,\mathrm{d}A + \int_{\Omega'} \rho b\,\mathrm{d}V,$$

where $b = b(x,t)$ is the given *body force per unit mass*. If we fix some vector e in space, it follows from the divergence theorem that

$$e \cdot F_{\partial\Omega'} = \int_{\Omega'} (-\operatorname{grad} p + \rho b) \cdot e \, \mathrm{d}V$$

and if we write

$$\frac{\mathrm{D}v}{\mathrm{D}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} v(x(t),t) = \frac{\partial v}{\partial t} + (v \cdot \nabla) v$$

for the *material derivative* of v, we see that Newton's law reads as

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = -\operatorname{grad} p + \rho b. \tag{1.4}$$

In three-dimensional space, (1.3) and (1.4) are four equations for the five unknown quantities $v = (v_1, v_2, v_3)$ and the scalar variables p and ρ. To describe the fluid motion completely, we need a third equation which is obtained from conservation of energy. The fluid's *energy* E is given by the sum of the *kinetic energy*

$$E_{\mathrm{kin}} = \frac{1}{2} \int_{\Omega'} \rho |v|^2 \, \mathrm{d}V, \qquad |v| = (v_1^2 + v_2^2 + v_3^2)^{1/2},$$

and an *internal energy* which we cannot see on a macroscopic scale and which comes from intermolecular potentials and the thermodynamics of the fluid. A straightforward computation shows that, for a moving fluid portion $\Omega'(t)$, the rate of change of the kinetic energy is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{kin}} = \int_{\Omega'(t)} \rho \left(v \cdot \left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) \right) \mathrm{d}V. \tag{1.5}$$

If we assume that $E = E_{\mathrm{kin}}$, then the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{kin}} = -\int_{\partial\Omega'(t)} p v \cdot n \, \mathrm{d}A + \int_{\Omega'(t)} \rho v \cdot b \, \mathrm{d}V. \tag{1.6}$$

Using (1.5) together with the divergence theorem we get from (1.6) under the assumption $\operatorname{div} v = 0$ the identity

$$\int_{\Omega'(t)} \rho v \cdot \frac{\mathrm{D}v}{\mathrm{D}t} \, \mathrm{d}V = -\int_{\Omega'(t)} (v \cdot \operatorname{grad} p - \rho v \cdot b) \, \mathrm{d}V$$

which is also a consequence of Eq. (1.4). We say that a fluid with divergence free velocity field is *incompressible*[1]. If the density ρ is only depending on time, i.e., $\operatorname{grad} \rho = 0$, we call the fluid *homogeneous*. An incompressible, homogeneous and non-viscous fluid is called *ideal fluid*. For an ideal fluid, the governing equations of motion are

---

[1] Letting J be the Jacobian determinant of the coordinate transformation $\xi \to x$, an easy computation shows that $J' = J \cdot \operatorname{div} v$. Since J can be regarded as the ratio of an elementary material volume to its initial volume, it follows that $\operatorname{div} v = 0$ implies that the fluid does neither expand nor squeeze.

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = -\mathrm{grad}\, p + \rho b, \tag{1.7}$$

$$\rho = \text{const.}, \tag{1.8}$$

$$\mathrm{div}\, v = 0. \tag{1.9}$$

We call the system (1.7)–(1.9) the system of *Euler equations* for the ideal fluid flow. Usually, one adds the boundary condition $v \cdot n = 0$ on $\partial\Omega'$.

1.1.2 The classical problem of 1D water waves

We consider the the unidirectional irrotational motion of water waves on a free surface under the influence of gravity. The water layer is regarded as an ideal fluid in $\mathbb{R}^3$ with Euclidean coordinates x, y, z over a flat bad which is assumed to be at $z = 0$. For simplicity, we assume that the wave propagates in x-direction and that all physical variables do not depend on y. We write $v = (u, 0, w)$ and $b = (0, 0, -g)$ for the constant acceleration due to gravity of earth. Let h be the mean level of water and $\eta(x, t)$ the shape of the water surface, i.e., the deviation from the average level. The total pressure follows from *Bernoulli's equation*

$$P = P_A + \rho g(h - z) + p$$

where P_A is the constant atmospheric pressure and p measures the deviation from the hydrostatic pressure distribution. On the surface $z = h + \eta$, $P = P_A$ and hence $p = \eta\rho g$. We also have the boundary conditions

$$w = \tfrac{\partial \eta}{\partial t} + u \tfrac{\partial \eta}{\partial x}, \quad z = h + \eta, \tag{1.10}$$

$$w = 0, \quad z = 0, \tag{1.11}$$

which are explained in [69]. The Euler equations (1.7)–(1.9) together with (1.10) and (1.11) yield

$$\left\{ \begin{array}{cclcc} u_t + uu_x + wu_z & = & -\frac{1}{\rho} p_x, & & \\ w_t + uw_x + ww_z & = & -\frac{1}{\rho} p_z, & & \\ u_x + w_z & = & 0, & & \\ w & = & \eta_t + u\eta_x & \text{on} & z = h + \eta, \\ p & = & \eta\rho g & \text{on} & z = h + \eta, \\ w & = & 0 & \text{on} & z = 0. \end{array} \right. \tag{1.12}$$

In the next step, one introduces the dimensionless parameters

$$\varepsilon = \frac{a}{h}, \quad \delta = \frac{h}{\lambda}, \quad \mu = \delta^2,$$

where a denotes the typical amplitude and λ the typical wavelength of waves under consideration, and scales the variables

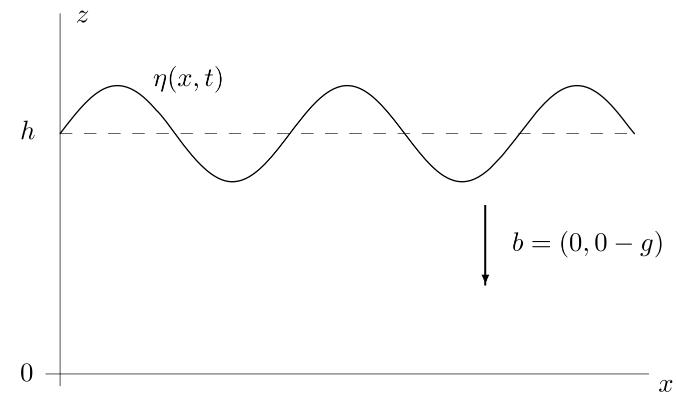


Fig. 1.3 The classical water wave problem.

$$\begin{pmatrix} x \\ z \\ t \\ \eta \\ u \\ w \\ p \end{pmatrix} \to \begin{pmatrix} \lambda x \\ zh \\ \frac{\lambda}{\sqrt{gh}}t \\ a\eta \\ \varepsilon\sqrt{gh}u \\ \varepsilon\delta\sqrt{gh}w \\ \varepsilon\rho g h \end{pmatrix}.$$

The idea behind this is that making assumptions on the respective size of ε and δ one is led to derive simpler asymptotic models for (1.12). Substituting the new dimensionless variables in the system (1.12) gives

$$\begin{cases} u_t + \varepsilon(uu_x + wu_z) &= -p_x, \\ \mu(w_t + \varepsilon(uw_x + ww_z)) &= -p_z, \\ u_x + w_z &= 0, \\ w &= \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta, \\ p &= \eta \quad \text{on} \quad z = 1 + \varepsilon\eta, \\ w &= 0 \quad \text{on} \quad z = 0. \end{cases}$$

Finally, for right-moving waves, one introduces the *far-field quantities*

$$\zeta = \sqrt{\varepsilon}(x - t), \quad \tau = \varepsilon^{3/2}t, \quad w = \sqrt{\varepsilon}W,$$

and obtains the system

$$\begin{cases} \varepsilon u_\tau - u_\zeta + \varepsilon(uu_\zeta + Wu_z) &= -p_\zeta, \\ \varepsilon\mu(\varepsilon W_\tau - W_\zeta + \varepsilon(uW_\zeta + WW_z)) &= -p_z, \\ u_\zeta + W_z &= 0, \\ W &= \varepsilon\eta_\tau - \eta_\zeta + \varepsilon u\eta_\zeta \quad \text{on} \quad z = 1 + \varepsilon\eta, \\ p &= \eta \quad \text{on} \quad z = 1 + \varepsilon\eta, \\ W &= 0 \quad \text{on} \quad z = 0. \end{cases}$$

Now the heuristical strategy is to assume that the variables u, W and p can be expressed as double expansions in ε and δ with terms depending only on $\eta(x,t)$ and explicitly on z. As a result, one obtains a single nonlinear equation for η and all the variables can be expressed in terms of the solution of this equation. In the so-called *long wave regime* we have

$$\mu \ll 1, \quad \varepsilon = \mathcal{O}(\mu),$$

and explicit calculations show that a right-going wave should satisfy the *KdV equation*

$$u_t + u_x + \varepsilon \frac{3}{2} u u_x + \mu \frac{1}{6} u_{xxx} = 0$$

which becomes the transport equation with speed 1 if $\varepsilon, \mu \to 0$. Benjamin, Bona and Mahoney [9] found out that the KdV equation belongs to a wider class of equations, the so-called *BBM equations*, which provide an approximation of exact water waves equations of the same accuracy as the KdV equation:

$$u_t + u_x + \frac{3}{2} \varepsilon u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = 0, \quad \alpha - \beta = \frac{1}{6}. \tag{1.13}$$

Observe that Eq. (1.13) contains both non-linear effects, described by the uu_x-term, and dispersive effects, modelled by the u_{xxx}-term and the u_{xxt}-term.

For medium or large amplitude waves, it was observed that the behavior is more nonlinear than dispersive and thus one uses the scaling

$$\mu \ll 1, \quad \varepsilon = \mathcal{O}(\delta), \tag{1.14}$$

which characterizes the *medium amplitude shallow water regime*. Observe that we still have $\varepsilon \ll 1$ and thus the same reduction to a simple wave equation at leading order, but since the dimensionless parameter is larger than in the long wave regime, we capture stronger nonlinear effects. Observe that stronger nonlinearity could allow the appearance of breaking waves which are *not* modelled by the BBM equations. It is shown in [27], that the correct generalization of the family (1.13) under the scaling (1.14) is provided by the class

$$u_t + u_x + \frac{3}{2} \varepsilon u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon \mu (\gamma u u_{xxx} + \delta u_x u_{xx}), \tag{1.15}$$

with appropriate conditions on the parameters α, β, γ and δ. It can be shown that Eq. (1.15) is not well-posed if β is positive. Second, among all $\beta \leq 0$-members of the family (1.15) only two have a bi-Hamiltonian structure: the *Camassa-Holm equation* and the *Degasperis-Procesi equation*. In general, bi-Hamiltonian equations are of particular interest since they may form completely integrable Hamiltonian systems (see Appendix C). Notice that the KdV equation is the only bi-Hamiltonian member of the family (1.13). In addition, the Camassa-Holm and Degasperis-Procesi equations allow for solitons, i.e., wave packets which do not change their shape while travelling at constant speed.

The Camassa-Holm (CH) equation is usually written in the form

$$U_t + \kappa U_x + 3UU_x - U_{txx} = 2U_x U_{xx} + UU_{xxx}, \quad \kappa \in \mathbb{R}. \tag{1.16}$$

For $\kappa \neq 0$, we can transform (1.16) to (1.15) by setting

$$u(t,x) = aU(b(x - vt), ct), \tag{1.17}$$

with

$$a = \frac{2}{\varepsilon\kappa}(1 - v), \quad b^2 = -\frac{1}{\beta\mu}, \quad v = \frac{\alpha}{\beta} \neq 1, \quad c = \frac{b}{\kappa}(1 - v)$$

which requires $\beta < 0$ and yields $\beta = -2\gamma$ and $\delta = 2\gamma$. Similarly, the Degasperis-Procesi (DP) equation is usually written as

$$U_t + \kappa U_x + 4UU_x - U_{txx} = 3U_x U_{xx} + UU_{xxx}, \quad \kappa \in \mathbb{R},$$

and using the transformation (1.17) with

$$a = \frac{8}{3\varepsilon\kappa}(1-v), \quad b^2 = -\frac{1}{\beta\mu}, \quad v = \frac{\alpha}{\beta}, \quad c = \frac{b}{\kappa}(1-v)$$

and $\beta < 0$, $\alpha \neq \beta$, $\beta = -8\gamma/3$ and $\delta = 3\gamma$, one sees that the DP equation is of the form (1.15).

The Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.18}$$

is an asymptotic equation for the unidirectional motion of water waves in the long wave regime. The function $u(t,x)$ depends on a time variable t and a space variable x and represents the wave profile over the flat bottom. In the shallow water medium amplitude regime, the Camassa-Holm (CH) equation

$$u_t + 3uu_x = u_{txx} + 2u_x u_{xx} + uu_{xxx} \tag{1.19}$$

and the Degasperis-Procesi (DP) equation

$$u_t + 4uu_x = u_{txx} + 3u_x u_{xx} + uu_{xxx} \tag{1.20}$$

are approximations to the governing equations of wave motion which capture stronger nonlinear effects.

The KdV equation first appeared in Boussinesq's article *Essai sur la théorie des eaux courantes* (1877) and is named for Diederik Korteweg and Gustav de Vries who studied the equation in 1895. The KdV equation is the prototypical example of an exactly solvable non-linear partial differential equation and its solutions in turn are paradigmatic examples for soliton solutions. The mathematical theory behind the KdV equation is rich and interesting: The method of inverse scattering is applicable to KdV and the equation possesses a bi-Hamiltonian structure, a Lax pair and hence an infinite number of conserved quantities (see Appendix C). Furthermore, KdV is obtained from a variational principle and defines a symplectic structure in the theory of infinite-dimensional Hamiltonian systems, cf. [104].

The CH equation was introduced by Roberto Camassa and Darryl D. Holm as a bi-Hamiltonian model for water waves in shallow water, see [11] where the authors also specify a Lax pair and so-called peakon solutions for (1.19); that are solitons with a sharp peak and hence a discontinuity at the peak in the wave slope. In [18, 23] it is shown that the CH equation is solvable via the inverse scattering transform.

The DP equation was discovered by A. Degasperis and M. Procesi [30, 32] in the search for a bi-Hamiltonian equation in form similar to the CH equation. In [30], the authors present a Lax pair and show that the DP equation has peaked solitons.

In the general theory of 1D water wave equations, one distinguishes between two types of models: in the periodic case, we assume that the space variable x is defined on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Formally, $\mathbb{S}$ consists of equivalence classes of real numbers such that x and y are equivalent if and only if $x - y$ is an integer. Since for any real x there is an integer n satisfying $n \leq x < n + 1$, each equivalence class can be represented by an element of $[0, 1)$. We write $u \colon \mathbb{S} \to \mathbb{R}$ if $u \colon \mathbb{R} \to \mathbb{R}$ is periodic with period 1. Physically, $x \in \mathbb{S}$ means that the wave has a periodic profile. For non-periodic equations, we write $x \in \mathbb{R}$; in this case the equation is considered on the real line.

For applications, one is often led to study the following question: Let u be a function depending on time and space. Given the initial data u_0 in some function space and the evolution equation $u_t = Au$ with the unknown u and some (in general nonlinear) operator A, can we solve the initial value problem

$$\begin{cases} u_t = Au, \\ u(0,x) = u_0(x) \end{cases} \tag{1.21}$$

in the sense that

1. the problem in fact has a solution in the underlying function space, at least for some open time interval containing zero,
2. this solution is unique,
3. the solution depends continuously on the initial data u_0?

We call the problem (1.21) a *Cauchy problem* (or *initial value problem*) and say that it is *well-posed* if it has a solution u as specified above. If the problem (1.21) is well-posed and the solution u exists for all $t \in \mathbb{R}$, we say that (1.21) is *globally* well-posed, otherwise, the problem is *locally* well-posed. The third condition in the above definition is of great practical importance since we would prefer that our (unique) solution changes only a little when the conditions specifying the problem only change a little.

Note also that we have not carefully defined what we mean by a solution; presumably, we would demand that our solution possesses as much as regularity as necessary to plug it into the evolution equation. However, apart from so called *classical solutions* or *strong solutions*, there might be *weak solutions* which are obtained if we multiply the equation $u_t = Au$ with some smooth test function, integrate and perform integration by parts. Note that *peakon solutions*

$$u(t,x) = \sum_{i=1}^{n} m_i(t) e^{-|x - x_i(t)|}$$

of a water wave equation are weak solutions since they fail to be differentiable at the cusps.

For (1.18), (1.19) and (1.20), various well-posedness results and properties of strong and weak solutions have been established. Here, we only mention some examples for the periodic case since we will mainly discuss periodic equations in the following. For the KdV equation (1.18), Bourgain [10] proved global well-posedness for square integrable initial data, see also [72, 78] for further results. The Cauchy problem for the periodic CH equation (1.19) in spaces of classical solutions has been studied extensively (see, e.g., [109]); in [33] the authors explain that this equation is also well-posed in spaces which include peakons, showing in this way that peakons are indeed meaningful solutions of CH. The precise blow-up setting, the blow-up rate and examples for finite time solutions of the CH equation are presented in [19, 21, 22, 111]. Well-posedness for the periodic DP

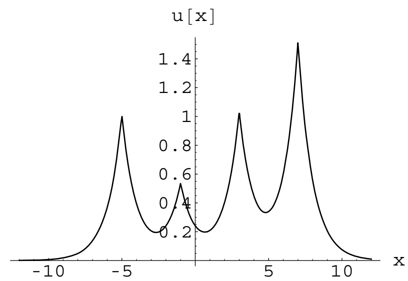


Fig. 1.4 Four-peakon wave profile formed by adding peakons at $x_1 = -5$, $x_2 = -1$, $x_3 = 3$, $x_4 = 7$ and with $m_1 = m_3 = 1$, $m_2 = 1/2$ and $m_4 = 3/2$.

equation (1.20) and various features of solutions of the DP on the circle are discussed in [44]. Just a small selection for further reading is [15, 16, 47, 126] and [59], and for readers with a particular interest in travelling waves [93, 119] and in peakons, (multi)solitons and shock waves [100, 103].

1.2 Euler's equations as reexpression of a geodesic flow on the circle diffeomorphisms

Euler found out that the motion of a rigid three-dimensional body can be described along geodesics in the group of rotations of three-dimensional Euclidean space equipped with a left-invariant Riemannian metric. A significant part of Euler's theory depends only upon this invariance so that it can be extended to other groups. Most interestingly, Euler's formalism can be applied to the hydrodynamics of an ideal fluid where the relevant group is the diffeomorphism group of smooth and volume-preserving diffeomorphisms. Basically, the kinetic energy defines a right-invariant metric and the key result is that the fluid motion is described by the geodesics with respect to this metric. Of course, we have to pay attention when generalizing results from a finite-dimensional Lie algebra to an infinite-dimensional one. This section summarizes the most important facts about the circle diffeomorphism group (in the smooth category) and the geometric approach to fluid dynamics. In addition, we also recall some elementary facts from differential geometry which we will need in the following.

1.2.1 The diffeomorphism group of the circle as manifold configuration space for the motion of an ideal fluid

To describe the dynamics of a physical system, one first needs a configuration space, i.e., a Lie group such that the motion of the system is given by a smooth path in this Lie group. For clarity, let us first recall the following definition.

Definition 1.1. A *group* G is a non-empty set G together with a map $G \times G \to G$, $(g,h) \to gh$, such that

1. for all $g_1, g_2, g_3 \in G$ we have that $(g_1g_2)g_3 = g_1(g_2g_3)$,
2. there is an element $e \in G$ satisfying $eg = ge = g$ for all $g \in G$,
3. for any $g \in G$ there is a unique element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

If, in addition, G is a smooth manifold and the group product $G \times G \to G$, $(g,h) \mapsto gh$, and the inversion $G \to G$ sending any g to g^{-1} are smooth, we say that G is a *Lie group*.

Let $C^\infty(M;N)$ denote the set of all smooth maps between smooth manifolds M and N. For an ideal fluid, filling a domain M, we choose the configuration space

$$\begin{aligned}\mathrm{Diff}^\infty(M) := \big\{\varphi \in C^\infty(M;M);\ &\varphi \text{ bijective, volume-preserving and} \\ &\varphi^{-1} \in C^\infty(M;M)\big\}\end{aligned}$$

of smooth and volume-preserving diffeomorphisms of M. The group product is just the composition of two diffeomorphisms and the neutral element is the identity map id. Indeed, the fluid flow determines for every time t a map $t \mapsto \varphi(t)$ in $\mathrm{Diff}^\infty(M)$ such that the initial position of every fluid particle is mapped to its position at time t. To model periodic 1D waves, we will consider diffeomorphisms on the circle $\mathbb{S}$.

Definition 1.2. Let $C^\infty(\mathbb{S})$ denote the set of all functions $\mathbb{S} \to \mathbb{R}$ which have continuous derivatives of order n for any $n \in \mathbb{N}$. We write $C^n(\mathbb{S})$ for the space of n-times continuously differentiable functions $\mathbb{S} \to \mathbb{R}$. By $C^0(\mathbb{S}) \equiv C(\mathbb{S})$, we denote the continuous functions on $\mathbb{S}$.

Obviously,

$$C^\infty(\mathbb{S}) = \bigcap_{n \in \mathbb{N} \cup \{0\}} C^n(\mathbb{S}).$$

Clearly, the spaces $C^n(\mathbb{S})$ are Banach spaces, where

$$\|u\|_{C^n} := \sum_{j=0}^{n} \left\|u^{(j)}\right\|_\infty, \quad u^{(j)}(x) := \frac{\partial^j u}{\partial x^j}(x), \quad u^{(0)} := u, \quad \|v\|_\infty := \max_{x \in \mathbb{S}} |v(x)|.$$

The space $C^\infty(\mathbb{S})$ is a Fréchet space (see Appendix A.3); more precisely, its topology is induced by the countable family $\{\|\cdot\|_{C^n}\,;\ n \geq 0\}$. A sequence $(u_k)_{k\in\mathbb{N}}$ converges to u in $C^\infty(\mathbb{S})$ if and only if

$$\|u_k - u\|_{C^n(\mathbb{S})} \to 0, \quad k \to \infty, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

In a Fréchet space, only *directional derivatives* (*Gateaux derivatives*) are meaningful.

Definition 1.3. Let X, Y be Fréchet spaces. A function $f\colon X \to Y$ is called continuously differentiable (or C^1) on an open subset $U \subset X$, if the directional derivative

$$[Df(x)]u = \lim_{h\to 0} \frac{1}{h}\left(f(x+hu) - f(x)\right)$$

exists for all $x \in U$ and all u in X and the map $(x,u) \mapsto [Df(x)]u$ is continuous.

Higher order derivatives and C^n-classes in Fréchet spaces are defined inductively. Note that for Banach spaces X, Y our definition of continuous differentiability is weaker than the usual one, cf. Appendix A. Now we introduce some diffeomorphism groups.

Definition 1.4. We write $\mathrm{Diff}^n(\mathbb{S})$ for the set of all diffeomorphisms $\varphi\colon \mathbb{S} \to \mathbb{S}$ which are C^n-functions with strictly positive derivative. Similarly, we let $\mathrm{Diff}^\infty(\mathbb{S})$ denote the set of all smooth and orientation-preserving diffeomorphisms of the circle $\mathbb{S}$.

Given a diffeomorphism $\varphi \in \mathrm{Diff}^\infty(\mathbb{S})$, we define its derivative $\varphi_x \in C^\infty(\mathbb{S})$ by the following construction, cf. [56]. We denote by $p\colon \mathbb{R} \to \mathbb{S}$, $x \mapsto e^{2\pi i x}$ the universal cover of the circle. A *lift* of φ is a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $p \circ f = \varphi \circ p$, i.e.,

$$\varphi\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} f(x)}, \quad \forall x \in \mathbb{R}.$$

By definition, a lift f for φ is unique only up to some integer constant. The fact that φ is orientation-preserving implies $f(x+1) = f(x) + 1$. The map f' is smooth and periodic and we set

$$\varphi_x := f'$$

to obtain a well-defined derivative of φ. Next, we observe that $\mathrm{Diff}^\infty(\mathbb{S})$ is naturally equipped with a *Fréchet manifold structure*[2] modelled on the Fréchet vector space $C^\infty(\mathbb{S})$. We briefly sketch how to obtain a smooth atlas with only two charts, cf. [56]. Given a $\varphi \in \mathrm{Diff}^\infty(\mathbb{S})$ it is always possible to find a lift $f\colon \mathbb{R} \to \mathbb{R}$ of φ such that

$$-1/2 < f(0) < 1/2 \quad \text{or} \quad 0 < f(0) < 1;$$

these conditions being not exclusive. We now let

$$\begin{aligned} V_1 &:= \{\varphi \in \mathrm{Diff}^\infty(\mathbb{S});\ \varphi \text{ has a lift } f \text{ satisfying } -1/2 < f(0) < 1/2\}, \\ V_2 &:= \{\varphi \in \mathrm{Diff}^\infty(\mathbb{S});\ \varphi \text{ has a lift } f \text{ satisfying } 0 < f(0) < 1\}, \end{aligned}$$

and obtain open subsets of $\mathrm{Diff}^\infty(\mathbb{S})$ with $V_1 \cup V_2 = \mathrm{Diff}^\infty(\mathbb{S})$. For any $\varphi \in \mathrm{Diff}^\infty(\mathbb{S})$ let

$$u = f - \mathrm{id}.$$

Then u has period 1 and hence lies in $C^\infty(\mathbb{S})$. In addition $u'(x) > -1$ and $u(0) = f(0)$. Thus, defining the open sets

$$\begin{aligned} U_1 &:= \{u \in C^\infty(\mathbb{S});\ -1/2 < u(0) < 1/2 \text{ and } u' > -1\}, \\ U_2 &:= \{u \in C^\infty(\mathbb{S});\ 0 < u(0) < 1 \text{ and } u' > -1\}, \end{aligned}$$

and the maps

$$\Phi_j\colon U_j \to V_j, \quad u \mapsto f = \mathrm{id} + u, \quad j = 1, 2,$$

we get charts of $\mathrm{Diff}^\infty(\mathbb{S})$ with values in $C^\infty(\mathbb{S})$. The change of charts corresponds to a change of lift and is just translation in $C^\infty(\mathbb{S})$ by ± 1.

Since the composition and the inversion are smooth maps $\mathrm{Diff}^\infty(\mathbb{S}) \times \mathrm{Diff}^\infty(\mathbb{S}) \to \mathrm{Diff}^\infty(\mathbb{S})$ and $\mathrm{Diff}^\infty(\mathbb{S}) \to \mathrm{Diff}^\infty(\mathbb{S})$ respectively, $\mathrm{Diff}^\infty(\mathbb{S})$ is a *Fréchet Lie group*, [58]. The tangent space at the identity of $\mathrm{Diff}^\infty(\mathbb{S})$ is naturally identified with the smooth vector fields on the circle. To see this, one may choose a smooth path $t \mapsto \varphi(t) \subset \mathrm{Diff}^\infty(\mathbb{S})$ with $\varphi(0) = \mathrm{id}$ so that, on the one hand, $\varphi_t(0) \in T_{\mathrm{id}}\mathrm{Diff}^\infty(\mathbb{S})$, and on the other hand, $\varphi_t(0, x) \in T_x\mathbb{S}$ for any $x \in \mathbb{S}$. Since $T\mathbb{S} \simeq \mathbb{S} \times \mathbb{R}$, we also have $T_{\mathrm{id}}\mathrm{Diff}^\infty(\mathbb{S}) \simeq C^\infty(\mathbb{S})$. Later on, we will explain that $T_{\mathrm{id}}\mathrm{Diff}^\infty(\mathbb{S})$ is the Lie algebra $\mathfrak{g}$ of $\mathrm{Diff}^\infty(\mathbb{S})$, equipped with the Lie bracket $[u, v] = u_x v - v_x u$. Observe that $\mathrm{Diff}^\infty(\mathbb{S})$ is itself parallelizable, i.e.,

$$T\mathrm{Diff}^\infty(\mathbb{S}) \simeq \mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}).$$

[2] Fréchet manifolds are defined as sets which can be covered by charts taking values in a given Fréchet space and such that the transition maps are smooth. The notions *Banach manifold*, *Hilbert manifold* etc. are defined analogously.

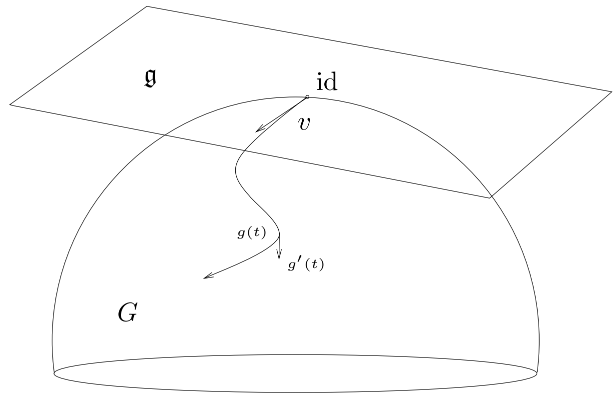


Fig. 1.5 Lie group G with Lie algebra $\mathfrak{g}$. The curve $g(t)$ starts at id with velocity v.

For $n \in \mathbb{N}$, the diffeomorphism groups $\mathrm{Diff}^n(\mathbb{S})$ are equipped with a smooth Banach manifold structure modelled on the Banach space $C^n(\mathbb{S})$. However, $\mathrm{Diff}^n(\mathbb{S})$ is only a topological group an not a Banach Lie group; the composition and the inversion map are continuous but not differentiable, cf. [37].

1.2.2 A geometric approach to Euler's equations of motion

Arnold [5, 6, 7], Ebin and Marsden [37] found out that the motion of inertia rigid objects in Classical Mechanics and the incompressible flow of some ideal fluid can be described by the same mathematical approach. In this section, our aim is to introduce this powerful geometric formalism and to explain the results using the example of a rigid body in $\mathbb{R}^3$ and some ideal fluid inside a domain $M \subset \mathbb{R}^3$ (see also [62, 81, 83]). The configuration space for a rigid three-dimensional body is the Lie group $SO(3)$. Recall that an ideal fluid inside a domain M is modelled on the manifold configuration space $\mathrm{Diff}^\infty(M)$. Let $\varphi(t) \subset \mathrm{Diff}^\infty(M)$ be a smooth path. The velocity field of the fluid motion described by $\varphi(t)$ is given by $v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)$ and hence $v(t)$ is an element of the tangent space $T_{\varphi(t)}\mathrm{Diff}^\infty(M)$. The kinetic energy $E_{\mathrm{kin}} = \frac{1}{2}\int_M \rho v^2 \,\mathrm{d}x$ is a quadratic form on $T_{\varphi(t)}\mathrm{Diff}^\infty(M)$. Observe that, since our fluid is incompressible, the integration can be carried out with the volume element occupied by an initial fluid particle or with the volume element $\mathrm{d}x$ occupied at time t. Moreover, the kinetic energy is right-invariant in the sense that it is invariant under right translations on the diffeomorphism group.

Definition 1.5. Let G be a Lie group and let $g \in G$. The maps

$$R_g : G \to G, \quad h \mapsto hg, \qquad L_g : G \to G, \quad h \mapsto gh, \tag{1.22}$$

are called the *right translation* and *left translation* given by g. The map $I_g \coloneqq R_{g^{-1}} L_g : G \to G$ sending h to ghg^{-1} is called the *inner automorphism* of G.

Observe that the operations L_g and R_g commute and that I_g is indeed an automorphism because $I_g(h_1h_2) = I_g(h_1)I_g(h_2)$. Since $I_{gh} = I_g I_h$ the map sending any $g \in G$ to the inner automorphism I_g is a group homomorphism. Note that the differential of I_g at the identity is a map $T_eG \to T_eG$.

Definition 1.6. Let G be a Lie group. The tangent space at the unity T_eG is called the *Lie algebra* $\mathfrak{g}$ of the Lie group G.

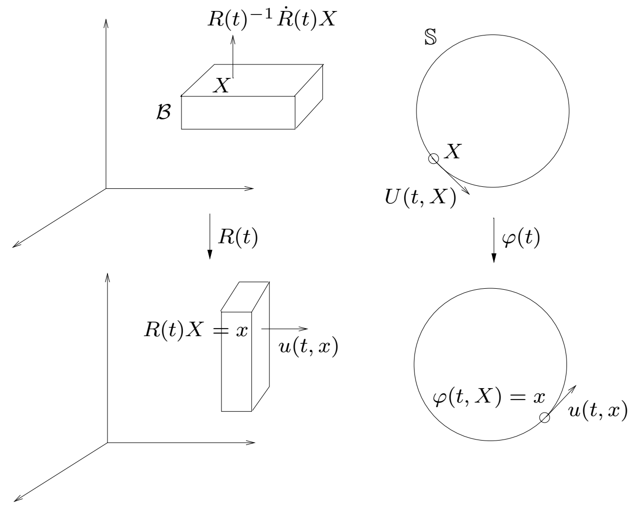


Fig. 1.6 Angular velocities for a rigid body $\mathcal{B}$ and a circle diffeomorphism. We denote by X a point in the body and describe the body's motion by a curve $R(t) \subset SO(3)$; similarly for the ideal fluid with $\varphi(t) \subset \mathrm{Diff}^\infty(\mathbb{S})$. For the body, the Eulerian velocity satisfies $u(t,x) = \dot{R}(t)X = \dot{R}(t)R^{-1}(t)x$ and similarly for the fluid $u(t,x) = \varphi_t(t,X) = (\varphi_t \circ \varphi^{-1})(t,x)$.

From the invariance of the kinetic energy under right translations (left translations for the rigid body, respectively), we get the crucial idea that it will often be enough to define geometric objects on the Lie algebra so that the values on all the other tangent spaces follow from right invariance (left invariance, respectively).

The Lie algebra of $SO(3)$ is denoted by $\mathfrak{so}(3)$ and consists of all real antisymmetric 3×3-matrices. The space $\mathfrak{so}(3)$ is three-dimensional and its elements are called *angular velocities*. More precisely, for a smooth path $R(t) \subset SO(3)$, we call $\dot{R}(t)$ the *material angular velocity*,

$$(D_{R(t)} R_{R^{-1}(t)})\dot{R}(t) = \dot{R}(t)R^{-1}(t) \in \mathfrak{g}$$

the *spatial angular velocity* and

$$(D_{R(t)} L_{R^{-1}(t)})\dot{R}(t) = R^{-1}(t)\dot{R}(t) \in \mathfrak{g}$$

the *body angular velocity* since these velocities correspond to the spatial reference frame and the body's reference frame respectively. Observe that L_R and R_R are linear maps so that $DL_R = L_R$ and $DR_R = R_R$. The Lie algebra of $\mathrm{Diff}^\infty(M)$ consists of the smooth divergence-free vector fields on M and is denoted as $\mathrm{Vect}^\infty(M)$. For $M = \mathbb{S}$, applying $D_\varphi R_{\varphi^{-1}}$ and $D_\varphi L_{\varphi^{-1}}$ to the velocity φ_t, we obtain the velocities $u = \varphi_t \circ \varphi^{-1}$ (from the linearity of $R_{\varphi^{-1}}$) and

$$U = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi^{-1} \circ (\varphi + \varepsilon\varphi_t)\bigg|_{\varepsilon=0} = \frac{\varphi_t}{\varphi_x} = \frac{u \circ \varphi}{\varphi_x}.$$

Definition 1.7. We define the *adjoint action of* G *on* $\mathfrak{g}$ by

$$\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}_g\, \xi := (D_e I_g)\, \xi, \quad \xi \in \mathfrak{g}.$$

For the rigid body, the map Ad_R sends body angular velocities Ω to spatial angular velocities ω, i.e., $\omega = \mathrm{Ad}_R \Omega = R\Omega R^{-1}$. The adjoint action of $\mathrm{Diff}^\infty(\mathbb{S})$ on $C^\infty(\mathbb{S})$ is given by $u = \mathrm{Ad}_\varphi U = (U\varphi_x) \circ \varphi^{-1}$.

Definition 1.8. Let $\mathrm{End}(\mathfrak{g})$ be the space of linear operators taking $\mathfrak{g}$ to itself and let

$$\mathrm{Ad} \colon G \to \mathrm{End}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g.$$

We define the *adjoint representation of the Lie algebra* $\mathfrak{g}$ as the map

$$\mathrm{ad} := D_e \mathrm{Ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}_\eta = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{g(t)} \Big|_{t=0},$$

where $t \mapsto g(t)$ is a curve in G which starts at $g(0) = e$ with velocity $g'(0) = \eta$.

If $G = SO(3)$, then by direct computation $\mathrm{ad}_a b = ab - ba = [a, b]$ and $[\cdot, \cdot]$ is the commutator of 3×3-matrices. Using that $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ via the map

$$\hat{\cdot} \colon \mathbb{R}^3 \to \mathfrak{so}(3), \quad x = (x_1, x_2, x_3) \mapsto \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

we have $\mathrm{ad}_a b = a \times b$, where $\times$ denotes the vector product in $\mathbb{R}^3$. For $G = \mathrm{Diff}^\infty(\mathbb{S})$ and a curve $\varphi(t)$ with $\varphi(0) = \mathrm{id}$ and $\varphi_t(0) = u$, we compute

$$\begin{aligned} \mathrm{ad}_u v &= \frac{\mathrm{d}}{\mathrm{d}t} (v \varphi_x) \circ \varphi^{-1} \Big|_{t=0} \\ &= \left[v \varphi_{tx} - (v \varphi_x)_x \frac{\varphi_t}{\varphi_x} \right] \circ \varphi^{-1} \Big|_{t=0} \\ &= u_x v - v_x u, \end{aligned}$$

where we have used that $\varphi(t) \circ \varphi^{-1}(t) = \mathrm{id}$ and hence $\varphi_t \circ \varphi^{-1} + (\varphi_x \circ \varphi^{-1}) \frac{\mathrm{d}}{\mathrm{d}t} \varphi^{-1} = 0$, $\varphi_{tx}(0) = u_x$ and $\varphi_{xx}(0) = 0$.

Definition 1.9. We define the *commutator* in the Lie algebra $\mathfrak{g}$ as the map

$$[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (\xi, \eta) \mapsto \mathrm{ad}_\xi \eta = [\xi, \eta].$$

The pair $(\mathfrak{g}, [\cdot, \cdot])$ is called *(abstract) Lie algebra* of the Lie group G.

It is easy to see that the operation $[\cdot, \cdot]$ is bilinear, skew-symmetric and satisfies the Jacobi identity. A vector space V equipped with a bilinear, skew-symmetric operation $[\cdot, \cdot] \colon V \times V \to V$ satisfying the Jacobi identity is called *abstract Lie algebra*. Every finite-dimensional abstract Lie algebra is the Lie algebra of some Lie group G. However, this correspondence fails in the infinite-dimensional case.

The adjoint operators $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ form a representation of the Lie group G by the automorphisms of its Lie algebra $\mathfrak{g}$:

$$[\mathrm{Ad}_g \, \xi, \mathrm{Ad}_g \, \eta] = \mathrm{Ad}_g \, [\xi, \eta], \quad \mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h.$$

Definition 1.10. The orbit of $\xi \in \mathfrak{g}$ under the action of Ad_g for all $g \in G$ is called the *adjoint (group) orbit* of ξ.

The adjoint orbits of $SO(3)$ are spheres centered at the origin and the origin itself. If $v \in \mathrm{Vect}^\infty(\mathbb{S})$, the associated adjoint orbit under the action of $\mathrm{Diff}^\infty(\mathbb{S})$ is the set $\big\{(v\varphi_x) \circ \varphi^{-1};\ \varphi \in \mathrm{Diff}^\infty(\mathbb{S})\big\}$. Note that the vectors $\mathrm{ad}_u v$, $u \in \mathrm{Vect}^\infty(\mathbb{S})$, form the tangent space to the adjoint orbit of v.

We denote by $\mathfrak{g}^*$ the vector space dual to the Lie algebra $\mathfrak{g}$. The space $\mathfrak{g}^*$ consists of continuous linear functionals on $\mathfrak{g}$. To every linear operator $A \colon X \to Y$, mapping a vector space X to a vector space Y, one can associate an adjoint operator A^* acting in the reverse direction, between the corresponding dual spaces, by

$$A^*\colon Y^* \to X^*, \quad (A^*y)(x) = y(Ax), \quad \forall y \in Y^*,\ x \in X.$$

For the differentials of the translation maps (1.22) on a Lie group G we have

$$D_h L_g\colon T_h G \to T_{gh} G, \quad D_h R_g\colon T_h G \to T_{hg} G,$$

and thus

$$(D_h L_g)^*\colon T^*_{gh} G \to T^*_h G, \quad (D_h R_g)^*\colon T^*_{hg} G \to T^*_h G.$$

This motivates why several authors use the notation $DR_g = (R_g)_*$ and $(DR_g)^* = (R_g)^*$ (and $DL_g = (L_g)_*$ and $(DL_g)^* = (L_g)^*$ respectively).

Definition 1.11. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The map Ad^* which associates to any group element $g \in G$ the linear transformation

$$\mathrm{Ad}^*_g\colon \mathfrak{g}^* \to \mathfrak{g}^*$$

is called the *coadjoint (anti)representation of* G. The orbit of a point $\omega \in \mathfrak{g}^*$ under the action of the coadjoint representation of G is the set $\left\{\mathrm{Ad}^*_g \omega;\ g \in G\right\} \subset \mathfrak{g}^*$ which is called the *coadjoint orbit of* ω.

Recall that the dual transformation $\mathrm{Ad}^*_g\colon \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by

$$(\mathrm{Ad}^*_g \omega)(\xi) = \omega(\mathrm{Ad}_g \xi), \quad g \in G,\ \xi \in \mathfrak{g},\ \omega \in \mathfrak{g}^*.$$

The operators Ad^*_g form an antirepresentation since

$$\mathrm{Ad}^*_{gh} = \mathrm{Ad}^*_h \mathrm{Ad}^*_g.$$

Definition 1.12. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the *coadjoint representation* of an element $\eta \in \mathfrak{g}$ is the rate of change of the operator $\mathrm{Ad}^*_{g(t)}$ of the coadjoint group representation as the group element $g(t)$ leaves the unity $g(0) = e$ with velocity $\dot{g}(0) = \eta$. We denote the operator of the coadjoint representation of the algebra element $\eta \in \mathfrak{g}$ by

$$\mathrm{ad}^*_\eta\colon \mathfrak{g}^* \to \mathfrak{g}^*.$$

The operator ad^*_η is dual to the adjoint representation, i.e.,

$$\mathrm{ad}^*_\eta(\omega)(\xi) = \omega(\mathrm{ad}_\eta \xi) = \omega([\eta, \xi]),$$

for all $\eta \in \mathfrak{g}$, $\xi \in \mathfrak{g}$ and $\omega \in \mathfrak{g}^*$. For any $\omega \in \mathfrak{g}^*$, the vectors $\mathrm{ad}^*_\eta \omega$, $\eta \in \mathfrak{g}$, form the tangent space to the coadjoint orbit of ω.

Assume now that we are given a Riemannian metric on a Lie group G which is invariant under left translations L_g, i.e., for any $g \in G$, there is a bilinear map $\langle \cdot, \cdot \rangle_g : T_g G \times T_g G \to \mathbb{R}$, depending smoothly on g, such that

$$\langle \xi, \eta \rangle_e = \langle (D_e L_g)\xi, (D_e L_g)\eta \rangle_g$$

for all $\xi, \eta \in \mathfrak{g}$ and for any $g \in G$. Clearly, such a metric is defined uniquely by its restriction to the tangent space at the group unity, i.e., by a quadratic form on the Lie algebra $\mathfrak{g}$ of the group. We are dealing with left-invariant metrics to describe the motion of a rigid body. To model the motion of an ideal fluid, one uses right-invariant metrics and the theory is similar.

Definition 1.13. Let $A\colon \mathfrak{g} \to \mathfrak{g}^*$ be the linear, symmetric and positive definite operator which defines the inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_e$, i.e.,

$$\langle \xi, \eta \rangle = (A\xi, \eta) = (A\eta, \xi).$$

Here, $(\cdot, \cdot)$ denotes the dual pairing of elements belonging to $\mathfrak{g}$ and $\mathfrak{g}^*$. Then A is called the *inertia operator* for $\langle \cdot, \cdot \rangle$. For any $g \in G$, let

$$A_g\colon T_gG \to T_g^*G, \quad A_g\xi = [(D_eL_g)^*]^{-1}A[(D_gL_{g^{-1}})\xi],$$

i.e., $\langle \xi, \eta \rangle_g = (A_g\xi, \eta) = (A_g\eta, \xi) = \langle \eta, \xi \rangle_g$ for all $\xi, \eta \in T_gG$.

The dual space $\mathfrak{so}(3)^*$ has a vector representation given by the map

$$\check{\cdot}\colon \mathbb{R}^3 \mapsto \mathfrak{so}(3)^*, \quad y = (y_1, y_2, y_3) \mapsto \check{y},$$

where

$$\langle \check{y}, \hat{x} \rangle = y_1x_1 + y_2x_2 + y_3x_3.$$

In the following, we omit the hat and check notation and identify elements of $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$ directly with $\mathbb{R}^3$-vectors. The dual space of $\mathrm{Vect}^\infty(\mathbb{S})$ is given by the *distributions* $\mathrm{Vect}'(\mathbb{S})$ on $\mathbb{S}$. The subspace of *regular distributions* which can be represented by smooth densities is denoted by $\mathrm{Vect}^*(\mathbb{S})$, i.e., $T \in \mathrm{Vect}^*(\mathbb{S})$ if and only if there is a $C^\infty(\mathbb{S})$-function ρ such that

$$T(\varphi) = \int_{\mathbb{S}} \varphi(x)\rho(x)\,\mathrm{d}x, \quad \forall \varphi \in C^\infty(\mathbb{S}).$$

Given a curve $g(t) \subset G$, the velocity $\dot{g}(t)$ is an element of the tangent space of G at the point $g(t)$. Recall that we can apply left and right translations to transport $\dot{g}$ to the Lie algebra $\mathfrak{g}$ to obtain

$$\omega_c := (D_gL_{g^{-1}})\dot{g} \in \mathfrak{g} \quad \text{and} \quad \omega_s := (D_gR_{g^{-1}})\dot{g} \in \mathfrak{g},$$

the angular velocity in the body frame and the spatial angular velocity, related by $\omega_s = \mathrm{Ad}_g\omega_c$. The kinetic energy E_{kin} is left-invariant and thus completely determined by ω_c, i.e.,

$$E_{\mathrm{kin}} = \frac{1}{2}\langle \dot{g}, \dot{g} \rangle_g = \frac{1}{2}\langle \omega_c, \omega_c \rangle_e = \frac{1}{2}(A\omega_c, \omega_c) = \frac{1}{2}(A_g\dot{g}, \dot{g}).$$

We now apply left and right translations to

$$m := A_g\dot{g} \in T_g^*G$$

to obtain two elements of $\mathfrak{g}^*$.

Definition 1.14. The dual space $\mathfrak{g}^*$ is called the *space of angular momenta*. The vectors

$$m_c := (D_eL_g)^*m \in \mathfrak{g}^* \quad \text{and} \quad m_s := (D_eR_g)^*m \in \mathfrak{g}^*$$

are called the vector of the *angular momentum relative to the body* and the *spatial angular momentum*.

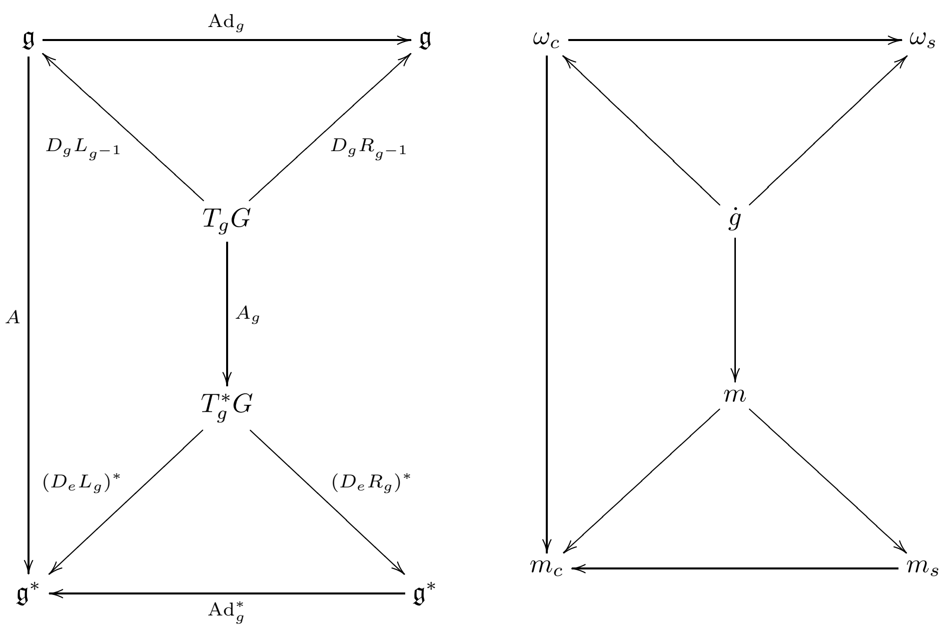
We have $m_c = \mathrm{Ad}_g^*m_s$ and

$$E_{\mathrm{kin}} = \frac{1}{2}(m_c, \omega_c) = \frac{1}{2}(m, \dot{g}).$$

Altogether, we consider four vectors moving in the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$: the vectors of angular velocity and momentum in the body and in space, i.e.,

$$\omega_c(t), \omega_s(t) \in \mathfrak{g} \quad \text{and} \quad m_c(t), m_s(t) \in \mathfrak{g}^*. \tag{1.23}$$

We furthermore obtain the following commutative diagram.



The following system of differential equations for the four moving vectors (1.23) in the rigid body problem was established by Leonhard Euler.

Theorem 1.15 (Euler). *The vector of spatial angular momentum is preserved under motion, i.e.,*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s = 0. \tag{1.24}$$

The vector of angular momentum relative to the body satisfies the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t} m_c = \mathrm{ad}^*_{\omega_c} m_c. \tag{1.25}$$

The first statement is a consequence of the symmetry of E_{kin} with respect to left translations. The Euler equation follows from this conservation law and $m_c(t) = \mathrm{Ad}^*_{g(t)} m_s$ by differentiating at $t = 0$ with $g(0) = e$. If we replace $\omega_c = A^{-1}m_c$, we see that the Euler equation defines a quadratic vector field on $\mathfrak{g}^*$ and its flow determines the motion of m_c. Using the isomorphism $A^{-1}\colon \mathfrak{g}^* \to \mathfrak{g}$, we can also obtain an Euler equation on the Lie algebra $\mathfrak{g}$ which is an an evolution equation for the vector $\omega_c = A^{-1}m_c$.

Theorem 1.16. *The vector of angular velocity in the body evolves according to the following equation with quadratic right-hand side:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega_c = B(\omega_c, \omega_c), \tag{1.26}$$

where the bilinear (nonsymmetric) map $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\langle [a,b], c\rangle = \langle B(c,a), b\rangle\,, \quad \forall a,b,c \in \mathfrak{g}, \tag{1.27}$$

i.e., the operator B is the image of the operator of the algebra coadjoint representation under the isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$ defined by the operator A.

For a fixed first argument, B is skew-symmetric with respect to the second argument, i.e.,

$$\langle B(c,a), b\rangle + \langle B(c,b), a\rangle = 0.$$

For a proof of Theorem 1.16, we refer to [7].

For the group $G = SO(3)$ Euler's equation takes the following form: The angular momentum $m = A\omega$ evolves according to $\dot{m} = m \times \omega$. With the inertia operator $A = \operatorname{diag}(I_1, I_2, I_3)$, one has

$$\begin{cases} \dot{m}_1 = \gamma_{23} m_2 m_3, \\ \dot{m}_2 = \gamma_{31} m_3 m_1, \\ \dot{m}_3 = \gamma_{12} m_1 m_2, \end{cases}$$

with $\gamma_{ij} = I_j^{-1} - I_i^{-1}$. The I_i are called the *principal moments of inertia* and they satisfy the triangle inequality $|I_i - I_j| \leq I_k$. The Euler equation (1.25) describes the evolution of the momentum vector in the three-dimensional space $\mathfrak{so}(3)^*$. Any solution $m_c(t)$ of Euler's equation lies in the intersection of the coadjoint orbits (which are spheres centered at the origin) with the energy levels. Note that the kinetic energy is a quadratic first integral on the dual space and the energy level surfaces are given by the ellipsoids $\left\langle A^{-1} m_c, m_c \right\rangle = \text{const.}$

Remark 1.17. The inertia operator for a rigid body is usually defined as the integral

$$A = \int_{\mathcal{B}} \rho(X) \left(|X|^2 \operatorname{id}_{\mathbb{R}^3} - X X^T\right) \mathrm{d}^3 X$$

where $\mathcal{B} \subset \mathbb{R}^3$ is the region of space occupied by the body in its reference configuration, X is the spatial position of a particle in the body and ρ denotes the mass density. Since A is symmetric it is diagonalizable by a rotation matrix and transforming to the system of principal axes, we can assume that A is a diagonal matrix.

We now apply Euler's theorem to ideal hydrodynamics where we have the infinite-dimensional group of smooth and volume-preserving diffeomorphisms. Among all the parallels in the formalism, let us recall one crucial difference: Motions of an ideal (= incompressible, homogeneous, inviscid) fluid filling a domain M are modelled by a *right*-invariant metric on the Lie group $G = \operatorname{Diff}^\infty(M)$. To transfer our results about left-invariant metrics to the right-invariant case, it suffices to change the sign of the commutator $[\cdot,\cdot]$ as well as of all operators depending linearly on it, i.e., ad_v, ad_v^* and B. Generalizing Euler's results for the motion of a rigid body to the group $\operatorname{Diff}^\infty(M)$, we obtain Euler's equations of fluid motion as well as the conservation laws for them. In particular, the right invariance of the metric results in the following form of the Euler equation:

$$\dot{v} = -B(v,v),$$

with B according to Theorem 1.16. Arnold also showed that the bilinear operator B on the Lie algebra $\mathfrak{g} = \operatorname{Vect}^\infty(M)$ for the Euler equation has the form

$$B(c,a) = \operatorname{rot} c \times a + \operatorname{grad} p,$$

where p is a function on M which represents the pressure of the fluid. Hence the Euler equation for three-dimensional ideal hydrodynamics is the evolution

$$\frac{\partial v}{\partial t} = v \times \operatorname{rot} v - \operatorname{grad} p$$

of a divergence-free vector field v in $M \subset \mathbb{R}^3$ tangent to ∂M. To finish this section and to lead over to the next one, we consider the following theorem which is proved in [7].

Theorem 1.18. *The operation $B(v, v)$ for a divergence-free vector field v on a Riemannian manifold M of any dimension is*

$$B(v, v) = \nabla_v v + \operatorname{grad} p.$$

Here $\nabla_v v$ is the vector field on M which is the covariant derivative of v along itself in the Riemannian connection on M given by the chosen Riemannian metric and p is determined modulo a constant by the same conditions as above.

1.2.3 Affine connections, Riemannian structures and geodesics on Lie groups

Let us assume that M is a smooth manifold of finite dimension $n \in \mathbb{N}$. Local coordinates are denoted as $x^1, \ldots, x^n$, the coordinate derivatives as $\partial_1, \ldots, \partial_n$ and the set of smooth vector fields on M as $\mathrm{Vect}^\infty(M)$. Let us assume that M is equipped with an *affine connection*, i.e., an $\mathbb{R}$-bilinear map

$$\nabla \colon \mathrm{Vect}^\infty(M) \times \mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M), \quad (X, Y) \mapsto \nabla_X Y$$

satisfying $\nabla_{fX} Y = f \nabla_X Y$ for all $f \in C^\infty(M)$ and $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$. Given a local chart, an affine connection ∇ is completely determined by the *Christoffel symbols*

$$\Gamma^k_{ij} = (\nabla_{\partial_i} \partial_j)^k. \tag{1.28}$$

Let $X(t)$ be a vector field along the curve $t \mapsto x(t) \subset M$, i.e., $X(t) = X(x(t))$. Then the *covariant derivative of X along the path $x(t)$* is

$$\frac{\mathrm{D}X}{\mathrm{D}t}(t) = (\nabla_{\dot{x}} X)(x(t)).$$

In local coordinates, we have

$$\left(\frac{\mathrm{D}X}{\mathrm{D}t}\right)^k = \dot{X}^k + \Gamma^k_{ij} \dot{x}^i X^j,$$

where we use Einstein summation convention, cf. [35].

Definition 1.19. Let M be a finite-dimensional smooth manifold, equipped with an affine connection ∇. A *geodesic* on M is a smooth curve $x(t)$ in M such that

$$\frac{\mathrm{D}\dot{x}}{\mathrm{D}t} = 0. \tag{1.29}$$

The geodesic equation in local coordinates is

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0.$$

For affine connections, we next define the notion of invariance under diffeomorphisms.

Definition 1.20. Let φ be a diffeomorphism of M and $X \in \mathrm{Vect}^\infty(M)$. We define

$$(\varphi^* X)(x) = (D_{\varphi(x)}\varphi^{-1})X(\varphi(x))$$

and say that an affine connection ∇ on M is *invariant under* φ if

$$\varphi^*(\nabla_X Y) = \nabla_{\varphi^* X}\varphi^* Y, \quad \forall X, Y \in \mathrm{Vect}^\infty(M).$$

If $M = G$ is a Lie group, one usually considers right and left translations $G \times G \to G$ as in (1.22) and says that a connection ∇ is *right-(left-)invariant*, if it is invariant under R_g (L_g) for any $g \in G$. A connection which is both right- and left-invariant is called *bi-invariant*.

On any Lie group G with affine connection ∇, a canonical bi-invariant connection is defined by

$$\nabla^0_{\xi_u}\xi_v \coloneqq \frac{1}{2}[\xi_u, \xi_v],$$

where $[\cdot,\cdot]$ denotes the Lie bracket on the Lie algebra T_eG of G and ξ_u and ξ_v are the right-invariant vector fields on G with values u and v at the identity. (Observe that *right*-invariant vector fields are of great importance for the study of the motion of incompressible fluids.) If ∇ is right-invariant,

$$B(X,Y) \coloneqq \nabla_X Y - \nabla^0_X Y \tag{1.30}$$

defines a right-invariant tensor field on G which is uniquely determined by its value at the identity, i.e., a bilinear operator $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Conversely, any bilinear operator B on $\mathfrak{g}$ defines uniquely a right-invariant affine connection on G via (1.30), i.e.,

$$\nabla_{\xi_u}\xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v), \tag{1.31}$$

where we use the same notation for B and the right-invariant tensor field it generates on G. The operator B is called *Christoffel operator*, since it generalizes the Christoffel symbols defined in (1.28).

Finally, let us choose a basis $(e_k)_{1\le k\le n}$ for the Lie algebra $\mathfrak{g}$. Let $(\xi_k)_{1\le k\le n}$ be the global right-invariant frame on G which equals $(e_k)_{1\le k\le n}$ at the identity and let $(\omega^k)_{1\le k\le n}$ be its dual co-frame. For a smooth path $t \mapsto g(t) \subset G$, let

$$u(t) = (u^k(t))_{1\le k\le n} = (D_{g(t)}R_{g^{-1}(t)})\dot{g}(t)$$

be the *Eulerian velocity*; the components u^k are given by

$$u^k = \omega^k_e(u) = \omega^k_g((D_eR_g)u) = \omega^k_g(\dot{g}).$$

The covariant derivative along $g(t)$ is obtained from

$$\left(\frac{\mathrm{D}X}{\mathrm{D}t}\right)^k = \dot{X}^k + \left(\frac{1}{2}c^k_{ij} + b^k_{ij}\right)u^iX^j,$$

where c_{ij}^k are the structure constants of $\mathfrak{g}$ and b_{ij}^k are the tensor components of B, and since $c_{ij}^k = -c_{ji}^k$, Eq. (1.29) in terms of u reads as

$$\dot{u}^k + b_{ij}^k u^i u^j = 0.$$

We have the following theorem.

Theorem 1.21. *A smooth curve $g(t)$ on a Lie group G with right-invariant affine connection ∇ is a geodesic if and only if its Eulerian velocity $u = (D_g R_{g^{-1}})\dot{g}$ is a solution of the Euler equation*

$$u_t = -B(u, u). \tag{1.32}$$

In general, for a bilinear operator B which defines an Euler equation of the type (1.32), the associated affine connection ∇ given by (1.31) is not necessarily Riemannian in the sense that it is compatible with a Riemannian metric[3] on G.

Let A be an inertia operator on G, i.e., $A\colon \mathfrak{g} \to \mathfrak{g}^*$ is a symmetric isomorphism. The corresponding right-invariant metric on G is denoted by ρ_A. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot,\cdot]$ and write $(\mathrm{ad}_v)^*$ for the adjoint with respect to ρ_A of the natural action of $\mathfrak{g}$ on itself given by $\mathrm{ad}_v\colon \mathfrak{g} \to \mathfrak{g}$, $u \mapsto [v, u]$. Formally, the bilinear operator B in (1.32) is $B(u, v) = \mathrm{ad}_v^* u$, cf. Theorem 1.16, but since we prefer to have a symmetric operator, we will work with

$$B(u, v) = \frac{1}{2}\left[(\mathrm{ad}_u)^* v + (\mathrm{ad}_v)^* u\right].$$

Fix now $G = \mathrm{Diff}^\infty(\mathbb{S})$ and recall that the topological dual space of $\mathrm{Vect}^\infty(\mathbb{S}) \simeq C^\infty(\mathbb{S})$ is given by the distributions $\mathrm{Vect}'(\mathbb{S})$ on $\mathbb{S}$. In order to get a convenient representation of the Christoffel operator B we restrict ourselves to $\mathrm{Vect}^*(\mathbb{S})$, the set of all regular distributions as introduced in Sect. 1.2.2. By Riesz' representation theorem we may identify $\mathrm{Vect}^*(\mathbb{S}) \simeq C^\infty(\mathbb{S})$. This motivates the following definition.

Definition 1.22. Let $\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ denote the set of all continuous isomorphisms on $C^\infty(\mathbb{S})$ which are symmetric with respect to the L_2 inner product. Each $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ is called a *regular inertia operator on* $\mathrm{Diff}^\infty(\mathbb{S})$.

We now come to one of the most important definitions for all the following considerations.

Definition 1.23. Let $G = \mathrm{Diff}^\infty(\mathbb{S})$ and let $u_t = -B(u, u)$ be an Euler equation on the Lie algebra $\mathfrak{g}$. We call this Euler equation *metric* if there exists a regular inertia operator A on G such that $B(u, v) = \mathrm{ad}_v^* u$ where ad^* is the adjoint of $\mathrm{ad}_v u = [v, u]$ with respect to the right-invariant metric ρ_A on G induced by A. Otherwise, we say that the Euler equation is *non-metric*.

While the periodic Camassa-Holm equation is of metric type with $A = 1 - \partial_x^2$ this does not hold true for the Degasperis-Procesi equation (see [45, 83]). In the metric case, it is important to establish that the metric ρ_A and the connection (1.31) are compatible in the following sense.

Definition 1.24. Let M be a Banach manifold endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$ and let $\mathrm{Vect}^\infty(M)$ denote the space of smooth vector fields on M. An $\mathbb{R}$-bilinear operator $(X, Y) \mapsto \nabla_X Y\colon \mathrm{Vect}^\infty(M) \times \mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M)$ is a *Riemannian covariant derivative* if the following properties are satisfied:

[3] Observe that, if G is a finite-dimensional Riemannian manifold, the Levi-Civita Theorem guarantees the existence and uniqueness of a symmetric affine connection ∇ on G compatible with the Riemannian metric. This does not hold true in the infinite-dimensional case in general.

1. *punctual dependence on X:*

$$X(m) = 0 \quad \Longrightarrow \quad (\nabla_X Y)(m) = 0$$

   for $m \in M$ and $X, Y \in \mathrm{Vect}^\infty(M)$,
2. *torsion-freeness:*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

   for $X, Y \in \mathrm{Vect}^\infty(M)$,
3. *derivation in Y:*

$$\nabla_X (fY) = (Xf)Y + f\nabla_X Y$$

   for $f \in C^\infty(M)$ and $X, Y \in \mathrm{Vect}^\infty(M)$,
4. *compatibility with the metric:*

$$X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

   for X, Y and Z in $\mathrm{Vect}^\infty(M)$.

It is important to recall that in the case of an infinite-dimensional Riemannian manifold, the Levi-Civita Theorem does in general not hold true. Let $\langle\cdot,\cdot\rangle$ be a Riemannian metric on a finite-dimensional manifold M and denote by ∇ the *Levi-Civita connection*. For vector fields X, Y and Z on M, one obtains $\nabla_X Y$ from the formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = &- \langle [Y, X], Z\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle \\ &+ X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle, \end{aligned} \tag{1.33}$$

see [35]. The bracket $\langle\cdot,\cdot\rangle$ establishes an isomorphism $T_m M \to T_m^* M$ for each $m \in M$ which guarantees the existence of $\nabla_X Y(m)$ for all m. In general, this approach fails if one does not have a finite number of local coordinates. In our setting, the crucial point is that the natural topology on any $T_\varphi G \simeq C^\infty(\mathbb{S})$ is stronger than the topology given by the right-invariant metric ρ_A—we have defined a *weak Riemannian metric* on G and there are elements in $T_\varphi^* G$ which cannot be written as $\rho_A(\cdot, \xi)$ for some $\xi \in T_\varphi G$. In other words: Any open set in the topology induced by ρ_A on $T_\varphi G$ is open in $C^\infty(\mathbb{S})$, but the converse is not true. Nevertheless, uniqueness of ∇ can be deduced from formula (1.33): Since ∇ satisfies the properties in Definition 1.24, writing down the compatibility relation for the cyclic permutations of $X, Y, Z \in \mathrm{Vect}^\infty(M)$ yields

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Adding the first two and subtracting the third of these equations shows that (1.33) holds with necessity. The non-degeneracy of $\langle\cdot,\cdot\rangle$ now implies that ∇ is unique.

If M is a Banach manifold with a (Riemannian) covariant derivative ∇ in the above sense, the theory established in [88, 89] guarantees the existence of geodesics and a curvature tensor on M. In addition, we have a well-defined *exponential map* which is defined as the geodesic flow at time one, i.e., if $t \mapsto \gamma(t)$ is the (unique) geodesic in M starting at $p = \gamma(0)$ with velocity $\gamma_t(0) = u \in T_p M$ then $\exp_p(u) = \gamma(1)$, cf. [35]. Roughly speaking, the map $\exp_p(\cdot)$ is a projection from $T_p M$ to the manifold M. Moreover, geodesics are *homogeneous* in the sense that $\exp_p(tu) = \gamma(t)$ for any $t > 0$. Since the derivative of $\exp_p$ at zero is the identity, the exponential map is a local diffeomorphism from a neighbourhood of zero of T_pM to a neighbourhood of $p \in M$. However, this fails for Fréchet manifolds like $\mathrm{Diff}^\infty(\mathbb{S})$ in general. Nevertheless, it could be shown that, for the Camassa-Holm equation, the exponential map is in fact a smooth local diffeomorphism, [25, 26]. Recently, this result was generalized to the Degasperis-Procesi equation, [41, 82].

1.2.4 Curvature in a two-dimensional direction

Given a manifold M with affine connection ∇, one defines the *curvature tensor*

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.34}$$

where X, Y and Z are vector fields on M. Since $R(X,Y)Z \equiv 0$ for $M = \mathbb{R}^n$, we are able to think of R as a way of measuring how much M deviates from being Euclidean. Closely related to the curvature operator is the sectional curvature S that we are now going to define. Given a vector space V with inner product $\langle\cdot,\cdot\rangle$, we denote

$$|x \wedge y|^2 = \|x\|^2\,\|y\|^2 - \langle x, y\rangle^2\,,$$

which represents the area of a two-dimensional parallelogram determined by the pair of vectors $x, y \in V$. Let $\sigma \subset T_pM$ be a two-dimensional subspace of the tangent space T_pM and assume that $\langle\cdot,\cdot\rangle$ is an inner product on T_pM. Let $x, y \in \sigma$ be two linearly independent vectors for which we define the *sectional curvature*

$$S(x,y) := \frac{\langle R(x,y)y, x\rangle}{|x \wedge y|^2}.$$

Indeed, S only depends on the two-dimensional space σ and not on the particular basis $\{x, y\}$ for σ. Another important motivation for studying the sectional curvature is that knowledge of $S(\sigma)$ for all σ determines the curvature R completely, see [35].

The sectional curvature of a manifold is closely connected to the question of stability of its geodesics. Consider a geodesic on M, starting at $p \in M$ with velocity $v \in T_pM$, and then alter the initial conditions p and v slightly to obtain a second geodesic, which at first only differs very little from the geodesic with initial data (p, v). In order to describe the divergence of both geodesics one linearizes the geodesic equation close to the original geodesic and obtains a so-called variational equation which is also known as the *Jacobi equation*. Let $x(t)$ be a point moving along a geodesic in M with velocity $v(t) \in T_{x(t)}M$. If the initial conditions of the curve $x(t)$ depend smoothly on an additional parameter s, then the geodesic also depends smoothly on s. For fixed t, we now consider the motion $s \mapsto x(t,s)$ with $x(t,0) = x(t)$ and define the *vector field of geodesic variation*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}x(t,s)\right|_{s=0} = \xi(t) \in T_{x(t)}M.$$

Then the Jacobi equation reads as

$$\frac{\mathrm{D}^2\xi}{\mathrm{D}t^2} = -R(\xi, v)v. \tag{1.35}$$

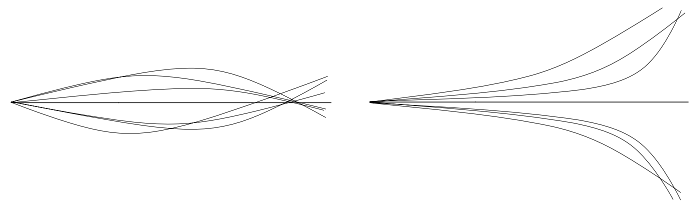


Fig. 1.7 Nearby geodesics on manifolds with positive and negative sectional curvature.

Conversely, every solution of Eq. (1.35) is a field of variation of the original geodesic, cf. [6]. Now we decompose the variation vector ξ into components parallel and perpendicular to the velocity vector v. Then, since $R(v, v) = 0$, the second covariant derivative of the parallel component vanishes and for the normal component we get again the Jacobi equation. More precisely, for the normal component, we find that

$$\frac{\mathrm{D}^2\xi}{\mathrm{D}t^2} = -\operatorname{grad} U, \quad U(\xi) = \frac{1}{2}\left\langle R(v,\xi)\xi, v\right\rangle = \frac{1}{2} S\left\langle \xi,\xi\right\rangle\left\langle v,v\right\rangle,$$

and if we assume $\|v\| = 1$ then the normal component of the variation vector is described by the equation of a non-autonomous linear oscillator with potential energy U equal to the product of the curvature in the direction of the plane of velocity vectors and variations with the square of length of the normal component of the variation.

Assume now that the sectional curvature S is negative in all two-dimensional directions containing the velocity vector v. In this case the divergence of nearby geodesics in the normal direction is described by the equation of an oscillator with negative potential energy. In the stability theory of dynamical systems, this suggests that the geodesics near the given will diverge exponentially from it—we might compare this behavior to an unstable equilibrium of some rigid object. The exponential instability of geodesics can be concluded rigorously if we assume that the curvature in the different two-dimensional directions containing v has values in the interval $[-a^2, -b^2]$, where $0 < b < a$. Then the solutions of the Jacobi equation (1.35) for normal divergence are linear combinations of exponential functions with exponents $\pm\lambda_i$ and $b < \lambda_i < a$ and hence every solution of the Jacobi equation grows at least as fast as $e^{b|t|}$ as $t \to \pm\infty$; most solutions grow even faster, with rate $e^{a|t|}$, cf. [6]. On the contrary, $S > 0$ implies that the perturbed geodesics might converge as depicted in Fig. 1.7.

1.3 A one-parameter family of evolution equations on spaces of tensor densities

Recall that the configuration space for the motion of periodic 1D waves is the diffeomorphism group $\mathrm{Diff}^\infty(\mathbb{S})$; the Lie algebra $\mathfrak{g}$ of $\mathrm{Diff}^\infty(\mathbb{S})$ coincides with the space of smooth vector fields $\mathrm{Vect}^\infty(\mathbb{S})$. The dual space $\mathrm{Vect}'(\mathbb{S})$ is the space of distributions on $\mathbb{S}$. Again, we are only interested in the regular part $\mathrm{Vect}^*(\mathbb{S})$ of $\mathfrak{g}^*$ which can be identified with the space of quadratic differentials $\left\{m(x)\,\mathrm{d}x^2;\ m \in C^\infty(\mathbb{S})\right\}$ with the pairing

$$\left(m\,\mathrm{d}x^2, v\partial_x\right) = \int_0^1 m(x)v(x)\,\mathrm{d}x, \tag{1.36}$$

where $\mathrm{d}x^2 \coloneqq (\mathrm{d}x)^2 = \mathrm{d}x \otimes \mathrm{d}x$. As usual, vector fields $X \in \mathrm{Vect}^\infty(\mathbb{S})$ are directional derivatives, i.e., if $\gamma(t)$ is a smooth curve in $\mathrm{Diff}^\infty(\mathbb{S})$ with $\gamma(0) = \mathrm{id}$ and $\gamma'(0) = v$, we associate $X = X_v$ via

$$Xf = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} = f_x v$$

and have $X = v(x)\partial_x$.

Proposition 1.25. *The Euler equation on $\mathfrak{g}^*$ reads as*

$$m_t = -\mathrm{ad}^*_{A^{-1}m} m = -um_x - 2u_x m, \quad m = Au. \tag{1.37}$$

Proof. We have $\mathrm{ad}^*_{u\partial_x} m\,\mathrm{d}x^2 = (um_x + 2u_x m)\,\mathrm{d}x^2$ since

$$\begin{aligned}
\left(\mathrm{ad}^*_{u\partial_x} m\,\mathrm{d}x^2, v\partial_x\right) &= \left(m\,\mathrm{d}x^2, \mathrm{ad}_{u\partial_x} v\partial_x\right) \\
&= -\left(m\,\mathrm{d}x^2, [u\partial_x, v\partial_x]\right) \\
&= -\left(m\,\mathrm{d}x^2, (uv_x - u_x v)\partial_x\right) \\
&= \int_0^1 m(u_x v - v_x u)\,\mathrm{d}x \\
&= \int_0^1 (m_x u + 2u_x m) v\,\mathrm{d}x,
\end{aligned}$$

where we have used (1.36) and the identity $[u\partial_x, v\partial_x] = (uv_x - u_x v)\partial_x$. □

Writing down Eq. (1.37) on $\mathfrak{g}$ we find that

$$Au_t + 2u_x Au + u(Au)_x = 0$$

which is equivalent to the Camassa-Holm equation (1.19) if $A = 1 - \partial_x^2$. We now extend this formalism to include the Degasperis-Procesi equation (1.20) and replace the space of quadratic differentials by the space of all tensor densities of weight b on the circle, cf. [56, 99].

Definition 1.26. Let $b \in \mathbb{Z}$. A *tensor density of weight* $b \geq 0$ ($b < 0$) on the circle $\mathbb{S}$ is a section of the bundle $\bigotimes^b T^*\mathbb{S}$ ($\bigotimes^{-b} T\mathbb{S}$, respectively).

Choosing a parameter x on the circle, a tensor density α of weight b can be written as $\alpha = m(x)\,\mathrm{d}x^b$ where m is a smooth function on the circle and

$$\mathrm{d}x^b \coloneqq \begin{cases} \mathrm{d}x \otimes \cdots \otimes \mathrm{d}x \;\; (b \text{ factors}), & b \geq 0, \\ \frac{\mathrm{d}}{\mathrm{d}x} \otimes \cdots \otimes \frac{\mathrm{d}}{\mathrm{d}x} \;\; (-b \text{ factors}), & b < 0. \end{cases}$$

In order to generalize the concept of tensor densities $\alpha = m(x)\,\mathrm{d}x^b$ for $b \in \mathbb{R}$, we define

$$\alpha_x \colon T_x\mathbb{S} \to \mathbb{R} \qquad \delta x \mapsto m(x)(\mathrm{d}x(\delta x))^b$$

where m is a smooth periodic function. We write $\mathcal{F}_b = \left\{m(x)\,\mathrm{d}x^b;\ m \in C^\infty(\mathbb{S})\right\}$ for the set of tensor densities of weight b on $\mathbb{S}$. Clearly, $\mathcal{F}_b$ is a vector space isomorphic to $C^\infty(\mathbb{S})$. We have $\mathcal{F}_{-1} = \mathrm{Vect}^\infty(\mathbb{S})$, $\mathcal{F}_0 = C^\infty(\mathbb{S})$, $\mathcal{F}_1 = \Omega^1(\mathbb{S})$ (the space of 1-forms on $\mathbb{S}$) and $\mathcal{F}_2$ coincides with the space of quadratic differentials. In order to generalize the coadjoint action $\mathrm{Ad}^* \colon \mathrm{Diff}^\infty(\mathbb{S}) \to \mathrm{End}(\mathcal{F}_2)$ on the space of quadratic differentials we define, for any $\varphi \in \mathrm{Diff}^\infty(\mathbb{S})$, the action

$$\mathcal{F}_b \to \mathcal{F}_b\colon \quad m\,\mathrm{d}x^b \mapsto (m\circ\varphi)\varphi_x^b\,\mathrm{d}x^b.$$

The infinitesimal generator of this action is given by

$$\begin{aligned} L^b_{u\partial_x}(m\,\mathrm{d}x^b) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\circ\varphi(t))\varphi_x(t)^b\,\mathrm{d}x^b\Big|_{t=0} \\ &= \big([m_x\circ\varphi(t)]\varphi_t(t)\varphi_x(t)^b\,\mathrm{d}x^b + b[m\circ\varphi(t)]\varphi_x(t)^{b-1}\varphi_{tx}(t)\,\mathrm{d}x^b\big)\big|_{t=0} \\ &= (um_x + bu_x m)\,\mathrm{d}x^b \end{aligned} \tag{1.38}$$

where $\varphi(t)$ is a curve with $\varphi(0) = \mathrm{id}$ and $\varphi_t(0) = u \in C^\infty(\mathbb{S})$; in particular, we have $\varphi_x(0) = 1$ and $\varphi_{tx}(0) = u_x$. The operator L can be thought of as the *Lie derivative* of tensor densities. Furthermore it represents the action of $\mathrm{Vect}^\infty(\mathbb{S})$ on $\mathcal{F}_b$ which coincides with the (algebra) coadjoint action on $\mathcal{F}_2$ for $b = 2$, i.e., $L^2_{u\partial_x} = \mathrm{ad}^*_{u\partial_x}$. Using (1.38), we generalize the Euler equation in Proposition 1.25 to

$$m_t = -um_x - bu_x m \tag{1.39}$$

and substituting $m = Au$ we finally arrive at

$$Au_t + bu_x Au + u(Au)_x = 0. \tag{1.40}$$

If $A = 1 - \partial_x^2$, we will call the family (1.40) the *b-equation*. Observe that, if $b = 3$, we obtain the DP equation (1.20).

The *periodic b-equation* is the 1-parameter family of evolution equations

$$u_t = -(1-\partial_x^2)^{-1}\left(bu_x(1-\partial_x^2)u + u(1-\partial_x^2)u_x\right), \quad b \in \mathbb{R}, \tag{1.41}$$

where $u(t,x)$ is a function depending on time $t \in \mathbb{R}$ and a space variable $x \in \mathbb{S}$.

The b-equation (1.41) attracted a considerable amount of attention in the fluid dynamics research community in recent years, see [48]. Each of these equations models the unidirectional irrotational free surface flow of a shallow layer of an inviscid fluid moving under the influence of gravity over a flat bed, cf. Sect. 1.1.2, where $u(t,x)$ represents the wave's height at time t and position x above the flat bottom. For $b = 2$, the single terms in Eq. (1.39) model convection, stretching and expansion of the fluid, cf. [62]; observe that in the one-dimensional case the stretching term equals the expansion term.

For further details concerning the hydrodynamical relevance of Eq. (1.41) we refer to [27, 36, 69, 70, 71]. As shown in [32, 36, 64, 68, 107], the b-equation is asymptotically integrable which is a necessary condition for complete integrability, but only for $b = 2$ and $b = 3$ for which it becomes the Camassa-Holm equation (1.19) and the Degasperis-Procesi equation (1.20) respectively.

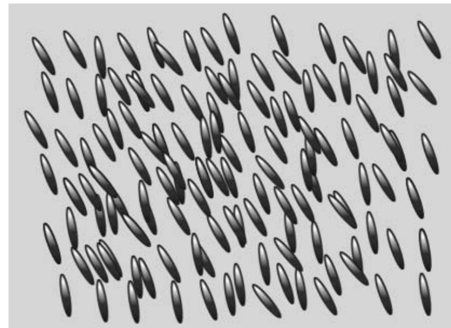


Fig. 1.8 Nematic phase liquid crystals. (`http://birgeneau.berkeley.edu/lxtaltest.php`, cited 15 Mai 2010)

1.4 The Hunter-Saxton equation and nematic liquid crystals

The Hunter-Saxton (HS) equation is an integrable PDE that arises in the theoretical study of nematic liquid crystals. Liquid crystals are a state of matter that has properties between those of a conventional liquid and those of a solid crystal. For instance, a liquid crystal may flow like a liquid, but its molecules may be oriented in a crystal-like way. The various phases of a liquid crystal can be characterized by the type of ordering. Due to the high viscosity of liquid crystals, it is assumed in many models that there is no fluid flow, i.e., no kinetic energy, so that only the orientation of the molecules is of interest. Within the elastic continuum theory, the orientation is described by a field of unit vectors $n(x,y,z,t)$. For nematic liquid crystals, there is no difference between orienting a molecule in the n direction or in the $-n$ direction, and the vector field n is then called a *director field*. The potential energy density of a director field is usually assumed to be given by the *Oseen-Frank energy functional*

$$W(n,\nabla n)=\frac{1}{2}\left(\alpha(\nabla\cdot n)^2+\beta(n\cdot(\nabla\times n))^2+\gamma|n\times(\nabla\times n)|^2\right),$$

where the positive coefficients α,β,γ are known as the elastic coefficients of splay, twist, and bend, respectively. Hunter and Saxton investigated the case when viscous damping is ignored and a kinetic energy term is included in the model, [65]. The governing equations follow from minimizing the action defined by the Lagrangian

$$\mathcal{L}=\frac{1}{2}|n_t|^2-W(n,\nabla n)-\frac{\lambda}{2}(1-|n|^2);$$

λ is a Lagrange multiplier enforcing $|n|=1$. For splay waves, the director field is of the form

$$n(x,y,z,t)=(\cos\varphi(x,t),\sin\varphi(x,t),0)$$

and the Lagrangian reduces to

$$\mathcal{L}=\frac{1}{2}\left(\varphi_t^2-a(\varphi)^2\varphi_x^2\right),\quad a(\varphi)^2=\alpha\sin^2\varphi+\gamma\cos^2\varphi.$$

The Euler equation for the angle φ is

$$\varphi_{tt}=a(\varphi)[a(\varphi)\varphi_x]_x.$$

Apart form trivial constant solutions $\varphi = \varphi_0$ where the molecules are perfectly aligned, the linearization

$$\varphi(x,t;\varepsilon) = \varphi_0 + \varepsilon\varphi_1(\theta,\tau) + \mathcal{O}(\varepsilon^2), \quad \theta \coloneqq x - a(\varphi_0)t, \quad \tau \coloneqq \varepsilon t$$

around such an equilibrium yields in order ε^2 the equation

$$(\varphi_{1\tau} + a'(\varphi_0)\varphi_1\varphi_{1\theta})_\theta = \frac{1}{2}a'(\varphi_0)\varphi_{1\theta}^2.$$

Under the assumption $a'(\varphi_0) \neq 0$, this equation is equivalent to $(u_t + uu_x)_x = \frac{1}{2}u_x^2$, after renaming and scaling the variables. Taking the x-derivative yields the *Hunter-Saxton equation*

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0. \tag{1.42}$$

The HS equation is closely related to the CH equation since we can rewrite HS in the form

$$m_t = -(m_x u + 2mu_x), \quad m = -u_{xx}. \tag{1.43}$$

Replacing the inertia operator $1 - \partial_x^2$ for the CH equation by $-\partial_x^2$, the b-equation for $b = 2$ becomes the HS equation. That is why some authors call the HS equation the $\lambda \to \infty$-limit of the Camassa-Holm equation

$$m_t = -(m_x u + 2mu_x), \quad m = (1 - \lambda\partial_x^2)u = u - \lambda u_{xx}.$$

Equation (1.43) possesses a bi-Hamiltonian structure and has an associated Lax pair, cf. [66]. Local existence of strong solutions to the periodic HS equation is established in [127], using semi-group methods. Formulas for the classical periodic solutions are presented in [97], proving existence up to breaking time.

Chapter 2

A one-parameter family of Camassa-Holm equations on the diffeomorphism group of the circle

The Camassa-Holm equation can be obtained from Lagrange's variational principle, i.e., defining an appropriate Lagrangian $\mathcal{L}$, the CH equation is the Euler-Lagrange equation obtained from

$$\delta \int \mathcal{L} \, \mathrm{d}t = 0.$$

In this chapter we generalize the Lagrangian $\mathcal{L}$ for CH and obtain a one-parameter family of integrable equations similar to the CH equation (1.19) and lying in some sense between the CH equation and the Burgers equation. We construct a family of Riemannian metrics on $\mathrm{Diff}^n(\mathbb{S})$, $n \geq 2$, such that the general CH equation is the geodesic equation on $\mathrm{Diff}^n(\mathbb{S})$ for a covariant derivative compatible with the Riemannian structure and obtained via (1.31). While this is a little bit reminiscent of the results of Kouranbaeva and Lenells [84, 94] for the CH equation we then perform an explicit calculation of the sectional curvature S for all two dimensional directions and find a large subspace of $C^\infty(\mathbb{S})$ for which $S > 0$. Finally, we derive formulae to describe the variation of geometric quantities like the Christoffel map or the sectional curvature. This chapter is also important due to some lengthy computations which will be needed in the following.

2.1 A variational approach to variants of the Camassa-Holm equation

We consider the infinite-dimensional Lie group $G = \mathrm{Diff}^\infty(\mathbb{S})$ of smooth and orientation-preserving diffeomorphisms of $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, equipped with the L_2 and the H^1 right-invariant metric respectively, given at the identity by the positive definite and symmetric inner products

$$\langle f, g \rangle_{L_2} := \int_{\mathbb{S}} f(x) g(x) \, \mathrm{d}x$$

and

$$\langle f, g \rangle_{H^1} := \langle f, g \rangle_{L_2} + \langle f_x, g_x \rangle_{L_2}$$

for $f, g \in \mathfrak{g} \simeq C^\infty(\mathbb{S})$. The corresponding norms are denoted by $\|\cdot\|_{L_2}$ and $\|\cdot\|_{H^1}$. It is well-known that the triple $(G, \mathfrak{g}, \langle \cdot, \cdot \rangle_{L_2})$ corresponds to the Burgers equation

$$u_t + 3uu_x = 0$$

in the sense that the Burgers equation is the Euler-Lagrange equation obtained by minimizing the functional $\gamma \mapsto \frac{1}{2}\int_a^b \left\|\gamma_t \circ \gamma^{-1}\right\|_{L_2}^2 \mathrm{d}t$, where $\gamma\colon [a,b] \to G$ is a smooth path. Similarly, the triple $(G, \mathfrak{g}, \langle\cdot,\cdot\rangle_{H^1})$ yields the Camassa-Holm equation (1.19) which is derived by minimizing $\gamma \mapsto \frac{1}{2}\int_a^b \left\|\gamma_t \circ \gamma^{-1}\right\|_{H^1}^2 \mathrm{d}t$, cf. [25, 67].

Here, our aim is to study a one-parameter family of Riemannian metrics which are "between" the L_2- and the H^1-metric in the following sense: Let $\lambda \in [0,1]$ and let

$$\langle f, g\rangle_\lambda := (1-\lambda)\langle f, g\rangle_{L_2} + \lambda \langle f, g\rangle_{H^1}.$$

Then $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_{L_2}$ and $\langle\cdot,\cdot\rangle_1 = \langle\cdot,\cdot\rangle_{H^1}$ and for $\lambda \in (0,1)$, one obtains a one-parameter family of metrics which are convexly combined of the L_2-metric and the H^1-metric. Applying the least action principle to

$$\gamma \mapsto \frac{1}{2}\int_a^b \left\langle \gamma_t \circ \gamma^{-1}, \gamma_t \circ \gamma^{-1}\right\rangle_\lambda \mathrm{d}t,$$

we obtain the Euler equation

$$u_t + 3uu_x - \lambda(u_{txx} + 2u_x u_{xx} + uu_{xxx}) = 0. \tag{2.1}$$

As expected, this is the Burgers equation for $\lambda = 0$ and the Camassa-Holm equation if we set $\lambda = 1$. Introducing the operator $A_\lambda := 1 - \lambda\partial_x^2\colon C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$, we can rewrite (2.1) in the form

$$m_t = -(m_x u + 2u_x m), \quad m = A_\lambda u; \tag{2.2}$$

here, $A = A_\lambda$ is the inertia operator which induces the λ-metric $\langle\cdot,\cdot\rangle_\lambda$ in the sense that

$$\langle f, g\rangle_\lambda = \int_{\mathbb{S}} f A g \,\mathrm{d}x = \int_{\mathbb{S}} g A f \,\mathrm{d}x, \quad f, g \in \mathfrak{g}.$$

It is an interesting question and the goal of this chapter to find out which geometric properties coming from the CH equation and the Burgers equation are inherited to the λ-equation (2.1) which lies between both of these equations, cf. [82]. Note that

$$\langle f, g\rangle_\lambda = \langle f, g\rangle_{L_2} + \lambda\langle f, g\rangle_*, \quad \langle f, g\rangle_* = \int_{\mathbb{S}} f_x g_x \,\mathrm{d}x,$$

so that we can regard $\langle\cdot,\cdot\rangle_\lambda$ as the L_2-metric plus a perturbation controlled by the parameter λ. We will explain in Sect. 3.1 that $(G, \mathfrak{g}, \langle\cdot,\cdot\rangle_*)$ leads to the Hunter-Saxton equation (1.42). If one is interested in the effect of the $\langle\cdot,\cdot\rangle_*$-metric compared to the effect of the L_2-metric, it might be useful to study the family

$$[0,1] \ni \lambda \mapsto \lambda\langle f, g\rangle_* + (1-\lambda)\langle f, g\rangle_{L_2}. \tag{2.3}$$

Note that this modified family gives the Burgers equation for $\lambda = 0$ and the CH equation if $\lambda = 1/2$. Choosing $\lambda = 1$, one obtains the HS equation. For $\lambda \neq 1$, the metric (2.3) equals (up to a scalar factor) the metric $\langle f, g\rangle_{L_2} + \beta\langle f, g\rangle_*$, where $\beta = \frac{\lambda}{1-\lambda} \in [0,\infty)$, and reduces for $\lambda \leq 1/2$ to the case studied in the following section.

First, we show that Eq. (2.1) is a geodesic equation on $(\mathrm{Diff}^n(\mathbb{S}), \langle\cdot,\cdot\rangle_\lambda)$, $n \geq 2$. Then we compute the Christoffel operator and the sectional curvature of $\mathrm{Diff}^n(\mathbb{S})$ associated with Eq. (2.1) and derive formulas for the λ-derivatives of these geometric quantities.

2.2 The geometric setting

The operator $A = A_\lambda = 1 - \lambda\partial_x^2$ is a textbook example for a regular inertia operator on $C^\infty(\mathbb{S})$ in the sense of Definition 1.22; for $\lambda = 1$, we will show this in detail in Sect. 3.1. We write

$$\langle u, v\rangle_{\mathrm{id}} = \int_{\mathbb{S}} uAv \,\mathrm{d}x = \int_{\mathbb{S}} (uv + \lambda u_x v_x)\,\mathrm{d}x$$

for the metric $\langle\cdot,\cdot\rangle_\lambda$ on $C^n(\mathbb{S})$, $n \geq 2$, and denote the corresponding right-invariant inner product on $\mathrm{Diff}^n(\mathbb{S})$ by

$$\langle U, V\rangle_\varphi = \left\langle U \circ \varphi^{-1}, V \circ \varphi^{-1}\right\rangle_{\mathrm{id}} = \int_{\mathbb{S}} (U \circ \varphi^{-1}) A (V \circ \varphi^{-1})\,\mathrm{d}x,$$

for all $U, V \in T_\varphi \mathrm{Diff}^n(\mathbb{S}) \simeq C^n(\mathbb{S})$. Recall that $(D_\varphi R_{\varphi^{-1}})U = U \circ \varphi^{-1}$ since the map $R_{\varphi^{-1}} \colon \mathrm{Diff}^n(\mathbb{S}) \to \mathrm{Diff}^n(\mathbb{S})$ is linear. Then $\langle\cdot,\cdot\rangle_\lambda$ is indeed a Riemannian metric on $\mathrm{Diff}^n(\mathbb{S})$ which is compatible with the connection defined locally by

$$\nabla_X Y(\varphi) = \frac{1}{2}[X(\varphi), Y(\varphi)] + B'(X(\varphi), Y(\varphi)). \tag{2.4}$$

Here, B' is the symmetric operator given by $2B'(u, v) = B(u, v) + B(v, u)$ and

$$B(u, v) = A^{-1}\left((Au_x)v + 2(Au)v_x\right). \tag{2.5}$$

Observe that the CH equation now reads as $u_t = -B(u, u)$ and hence is rewritten as a metric Euler equation in the sense of Definition 1.23. As explained in Theorem 1.16 the operator B defined in (2.5) can also be regarded as the $\langle\cdot,\cdot\rangle_\lambda$-adjoint of the natural action $\mathrm{ad}_u v = u_x v - u v_x$, i.e.,

$$\int_{\mathbb{S}} (u_x v - u v_x) A w \,\mathrm{d}x = \int_{\mathbb{S}} B(w, u) A v \,\mathrm{d}x.$$

Recall that, for finite n, $\mathrm{Diff}^n(\mathbb{S})$ is not a Lie group; nevertheless, we now regard the Camassa-Holm equation $u_t = -B(u, u)$ as an evolution equation on the tangent space at the identity of $\mathrm{Diff}^n(\mathbb{S})$, $n \geq 2$. For technical purposes it is useful to introduce another bilinear symmetric operator $\Gamma_\varphi(X(\varphi), Y(\varphi)) = \Gamma(\varphi; X(\varphi), Y(\varphi))$ such that

$$\nabla_X Y(\varphi) = DY(\varphi) \cdot X(\varphi) - \Gamma_\varphi(Y(\varphi), X(\varphi)). \tag{2.6}$$

Observe that the commutator of vector fields is locally given by

$$[X, Y](\varphi) = DY(\varphi) \cdot X(\varphi) - DX(\varphi) \cdot Y(\varphi) \tag{2.7}$$

so that the representation (2.6) is a direct consequence of (2.4). Precisely, we have

$$\Gamma_{\mathrm{id}}(u, v) = \frac{1}{2}(uv)_x - B'(u, v) = -A^{-1}\partial_x\left(uv + \frac{\lambda}{2}u_x v_x\right) \tag{2.8}$$

and

$$\Gamma_\varphi(X(\varphi), Y(\varphi)) = \Gamma_{\mathrm{id}}(X(\varphi) \circ \varphi^{-1}, Y(\varphi) \circ \varphi^{-1}) \circ \varphi.$$

We will use the notion Christoffel operator optionally for the maps Γ and B'.

Proposition 2.1. *The pair* $(\mathrm{Diff}^n(\mathbb{S}), \langle\cdot,\cdot\rangle_\lambda)$*,* $n \geq 2$*, with the right-invariant metric* $\langle\cdot,\cdot\rangle_\lambda$ *is a Riemannian manifold. The bilinear map* ∇ *on* $\mathrm{Vect}^\infty(\mathrm{Diff}^n(\mathbb{S}))$ *defined in* (2.6) *depends smoothly on* φ *and is a Riemannian covariant derivative on* $\mathrm{Diff}^n(\mathbb{S})$*; in particular,* ∇ *is compatible with the right-invariant metric* $\langle\cdot,\cdot\rangle_\lambda$*.*

Proof. The proof is similar to the proofs of Theorem 3.4, Theorem 4.1 and Theorem 5.3 in [94]. □

As a key result in [84, 94], we now obtain a unique *geodesic flow* $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S})$ for the connection (2.6) standing in a one to one correspondence with the solution u of the Camassa-Holm equation (2.1). Observe that $\varphi_t(t) \in T_{\varphi(t)}\mathrm{Diff}^n(\mathbb{S})$ and hence for any t the Eulerian velocity $u(t) \coloneqq \varphi_t(t)\circ\varphi^{-1}(t)$ lies in $T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S})$; more interestingly, the function u is a solution of the CH equation. Contrariwise, a solution u of the CH equation can be interpreted as a time-dependent vector field on $\mathbb{S}$ whose flow is the geodesic flow for the connection (2.6).

Theorem 2.2. *For* $n \geq 2$*, let* $\varphi\colon J \to \mathrm{Diff}^n(\mathbb{S})$ *be a* C^2*-curve and define* $u\colon J \to C^n(\mathbb{S})$ *by* $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ *so that* $u \in C(J, C^n(\mathbb{S})) \cap C^1(J, C^{n-1}(\mathbb{S}))$*. Then* φ *is a geodesic for the connection* ∇ *defined in* (2.6) *if and only if* u *solves the Camassa-Holm equation* (2.1)*.*

Proof. This is a direct consequence of the arguments in the proof of Theorem 6.5 in [94]. □

Remark 2.3. (i) The geodesic equation in Lagrangian coordinates reads as $\Gamma_\varphi(\varphi_t, \varphi_t) = \varphi_{tt}$ which follows from differentiating $\varphi_t = u \circ \varphi$ with respect to t, Eq. (2.2) and the definition of the Christoffel map, cf. [94]. Writing the CH as $u_t + uu_x = \Gamma_{\mathrm{id}}(u,u)$ we also see that $n \geq 2$ is sufficient for our purposes.

(ii) Recently it could be shown that the periodic b-equation is an Euler equation in the sense of Theorem 1.21 for any real b but it is compatible with a Riemannian structure if and only if $b = 2$. Whenever $b \neq 2$, geometric information is obtained only by using the connection ∇ defined via the Christoffel operator for the corresponding equation. Only for $b = 2$ there is a unique regular inertia operator, namely $1 - \partial_x^2$, cf. [45].

We will now establish a formula which shows that the sectional curvature of a plane spanned by two vectors $u, v \in T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S})$ can be expressed explicitly in terms of the Christoffel map Γ. We denote by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

the curvature tensor for the family (2.1), in accordance to definition (1.34). (Observe that we use the same sign convention for R as in [79, 94] so that we will be able to compare our results for μ-equations and two-component systems in the following.) In particular, the sectional curvature for the CH equation is given by

$$S(u,v) = \langle R(u,v)v, u\rangle\,,$$

for orthonormal functions u and v.

Theorem 2.4. *Let* $S(u,v) = \langle R(u,v)v,u\rangle$ *be the sectional curvature of* $\mathrm{Diff}^n(\mathbb{S})$ *endowed with the right-invariant metric given at the identity by* $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\lambda$*. Then*

$$S(u,v) = \langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle\,, \quad u, v \in T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S}).$$

Proof. By the definition of R and the local formula (2.6) for the affine connection ∇ we have

$$\begin{aligned} R(X,Y)Z &= \nabla_X\left[DZ\cdot Y - \Gamma_\varphi(Z,Y)\right] - \nabla_Y\left[DZ\cdot X - \Gamma_\varphi(Z,X)\right] \\ &\quad - DZ\cdot[X,Y] + \Gamma_\varphi(Z,[X,Y]) \\ &= D(DZ\cdot Y)\cdot X - D_1\Gamma_\varphi(Z,Y)X - \Gamma_\varphi(DZ\cdot X,Y) \\ &\quad -\Gamma_\varphi(Z,DY\cdot X) - \Gamma_\varphi(DZ\cdot Y,X) + \Gamma_\varphi(\Gamma_\varphi(Z,Y),X) \\ &\quad -D(DZ\cdot X)\cdot Y + D_1\Gamma_\varphi(Z,X)Y + \Gamma_\varphi(DZ\cdot Y,X) \\ &\quad +\Gamma_\varphi(Z,DX\cdot Y) + \Gamma_\varphi(DZ\cdot X,Y) - \Gamma_\varphi(\Gamma_\varphi(Z,X),Y) \\ &\quad -DZ\cdot[X,Y] + \Gamma_\varphi(Z,[X,Y]) \\ &= D_1\Gamma_\varphi(Z,X)Y - D_1\Gamma_\varphi(Z,Y)X + \Gamma_\varphi(\Gamma_\varphi(Z,Y),X) \\ &\quad -\Gamma_\varphi(\Gamma_\varphi(Z,X),Y), \end{aligned}$$

for all $X,Y,Z \in T_\varphi \mathrm{Diff}^n(\mathbb{S}) \simeq C^n(\mathbb{S})$. Here

$$\begin{aligned} D_1\Gamma_\varphi(Z,X)Y &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Gamma(\varphi+\varepsilon Y; Z,X)\right|_{\varepsilon=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Gamma(\mathrm{id}; Z\circ(\varphi+\varepsilon Y)^{-1}, X\circ(\varphi+\varepsilon Y)^{-1})\circ(\varphi+\varepsilon Y)\right|_{\varepsilon=0}. \end{aligned}$$

Writing $u = X\circ\varphi^{-1}$, $v = Y\circ\varphi^{-1}$ and $w = Z\circ\varphi^{-1}$ we find that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} Z\circ(\varphi+\varepsilon Y)^{-1}\right|_{\varepsilon=0} = -(Z\circ\varphi^{-1})_x(Y\circ\varphi^{-1}) = -w_x v; \tag{2.9}$$

recall that the derivative of $(\varphi+\varepsilon Y)^{-1}$ is obtained from the identity

$$(\varphi+\varepsilon Y)\circ(\varphi+\varepsilon Y)^{-1} = \mathrm{id}$$

by differentiating with respect to ε at $\varepsilon = 0$. Together with

$$(Z\circ\varphi^{-1})_x = \frac{Z_x\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}}$$

we get (2.9). Writing $\Gamma(\cdot,\cdot)$ for $\Gamma_{\mathrm{id}}(\cdot,\cdot)$ we have

$$\begin{aligned} D_1\Gamma(w,u)v &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Gamma_{\mathrm{id}+\varepsilon v}(w,u) \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left.\left\{\left[-A^{-1}\left(\left(wu + \frac{\lambda w_x u_x}{2(1+\varepsilon v_x)^2}\right)\circ(\mathrm{id}+\varepsilon v)^{-1}\right)_x\right]\circ(\mathrm{id}+\varepsilon v)\right\}\right|_{\varepsilon=0} \\ &= A^{-1}\partial_x\left((wu)_x v + \frac{\lambda}{2}(w_x u_x)_x v + \lambda w_x u_x v_x\right) \\ &\qquad -v\partial_x A^{-1}\partial_x\left(wu + \frac{\lambda}{2}w_x u_x\right) \\ &= -\Gamma(w_x v, u) - \Gamma(u_x v, w) + \Gamma(w,u)_x v. \end{aligned}$$

It follows that

$$\begin{aligned} S(u,v) = &\langle \Gamma(\Gamma(v,v),u),u\rangle - \langle \Gamma(\Gamma(v,u),v),u\rangle + \langle \Gamma(v,u)_x v - \Gamma(v,v)_x u, u\rangle \\ &+ \langle -\Gamma(v_x v,u) - \Gamma(v,u_x v) + 2\Gamma(v_x u,v),u\rangle \,. \end{aligned}$$

Recall the definition of Γ in Eq. (2.8) and that

$$-\langle B(u,v),w\rangle = \langle u,[v,w]\rangle\,, \quad u,v,w \in C^n(\mathbb{S}),$$

where $[v,w] = vw_x - v_x w$ is the Lie bracket induced by right-invariant vector fields. (Indeed, if $X(\varphi)$ and $Y(\varphi)$ denote the right-invariant vector fields with values u,v at id we conclude from

$$DY(\varphi)\cdot X(\varphi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} Y(\varphi + \varepsilon X(\varphi))\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} v\circ(\varphi + \varepsilon(u\circ\varphi))\Big|_{\varepsilon=0} = (v_x u)\circ\varphi$$

and Eq. (2.7) that $[X,Y](\varphi)\circ\varphi^{-1} = [u,v] = v_x u - u_x v$. Observe that this bracket differs from the Lie bracket introduced in Sect. 1.2.2 by a sign.) We thus can rewrite

$$\begin{aligned} &\langle \Gamma(v,u)_x v - \Gamma(v,v)_x u,u\rangle + \langle \Gamma(\Gamma(v,v),u),u\rangle - \langle \Gamma(\Gamma(v,u),v),u\rangle \\ &= \langle \Gamma(v,u)_x v - \Gamma(v,v)_x u,u\rangle + \frac{1}{2}\langle (\Gamma(v,v)u)_x - B(\Gamma(v,v),u) - B(u,\Gamma(v,v)),u\rangle \\ &\quad -\frac{1}{2}\langle (\Gamma(v,u)v)_x - B(\Gamma(v,u),v) - B(v,\Gamma(v,u)),u\rangle \\ &= \frac{1}{2}\langle [v,\Gamma(v,u)],u\rangle + \frac{1}{2}\langle [\Gamma(v,v),u],u\rangle + \frac{1}{2}\langle u,[\Gamma(v,v),u]\rangle \\ &\quad -\frac{1}{2}\langle \Gamma(v,u),[v,u]\rangle - \frac{1}{2}\langle v,[\Gamma(v,u),u]\rangle \\ &= -\frac{1}{2}\langle B(u,v),\Gamma(v,u)\rangle + \langle B(u,u),\Gamma(v,v)\rangle - \frac{1}{2}\langle \Gamma(v,u),[v,u]\rangle - \frac{1}{2}\langle B(v,u),\Gamma(v,u)\rangle \\ &= \langle \Gamma(u,v),\Gamma(u,v)\rangle - \langle \Gamma(u,u),\Gamma(v,v)\rangle \\ &\quad -\frac{1}{2}\langle (uv)_x,\Gamma(v,u)\rangle + \langle uu_x,\Gamma(v,v)\rangle - \frac{1}{2}\langle \Gamma(v,u),[v,u]\rangle \\ &= \langle \Gamma(u,v),\Gamma(u,v)\rangle - \langle \Gamma(u,u),\Gamma(v,v)\rangle - \langle u_x v,\Gamma(v,u)\rangle + \langle uu_x,\Gamma(v,v)\rangle \,. \end{aligned}$$

Therefore

$$\begin{aligned} S(u,v) = &\langle \Gamma(u,v),\Gamma(u,v)\rangle - \langle \Gamma(u,u),\Gamma(v,v)\rangle \\ &+ \langle -\Gamma(v_x v,u) - \Gamma(v,u_x v) + 2\Gamma(v_x u,v),u\rangle \\ &- \langle u_x v,\Gamma(v,u)\rangle + \langle uu_x,\Gamma(v,v)\rangle \,. \end{aligned} \tag{2.10}$$

Using (2.8) we find that all but the first two terms on the right-hand side of Eq. (2.10) cancel. This proves the theorem. □

As explained in Sect. 1.2.4, subspaces of positive curvature are of particular interest since the positivity of $S(u,v)$ is related to meaningful results from stability theory. For the family (2.1) it turns out that $S(u,v)$ is strictly positive on all planes spanned by two trigonometric functions[1] of the form $\cos kx$ and $\sin lx$ for $k,l \in 2\pi\mathbb{N}$. Recall that, by standard results from Fourier theory, periodic functions on $\mathbb{S}$ can be written as a Fourier series with respect to the trigonometric functions which we discuss here.

[1] The calculations also appear in a preprint of J. Lenells, G. Misiołek and S.C. Preston, 2009.

Theorem 2.5. *Let* $S(u,v) = \langle R(u,v)v, u\rangle$ *denote the sectional curvature of* $\mathrm{Diff}^n(\mathbb{S})$ *endowed with the right-invariant metric given at the identity by* $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\lambda$. *Then, for* $k \neq l \in 2\pi\mathbb{N}$,

$$S(\cos kx, \cos lx) = S(\cos kx, \sin lx) = S(\sin kx, \sin lx) = \mathrm{Sec}(k,l) > 0 \tag{2.11}$$

where

$$\mathrm{Sec}(k,l) = \frac{1}{8}\left(\frac{(1+\frac{\lambda}{2}kl)^2}{1+\lambda(k-l)^2}(k-l)^2 + \frac{(1-\frac{\lambda}{2}kl)^2}{1+\lambda(k+l)^2}(k+l)^2\right).$$

Moreover, for $k \in 2\pi\mathbb{N}$,

$$S(\cos kx, \sin kx) = 2\,\mathrm{Sec}(k,k) = \frac{(1-\frac{\lambda}{2}k^2)^2}{1+4\lambda k^2}k^2 \tag{2.12}$$

and

$$S(\cos kx, 1) = S(\sin kx, 1) = 2\,\mathrm{Sec}(k,0) = \frac{k^2}{2(1+\lambda k^2)} > 0. \tag{2.13}$$

Proof. By the previous theorem,

$$S(u,1) = \int_{\mathbb{S}} u_x A^{-1} u_x \,\mathrm{d}x.$$

Since $A^{-1}\sin kx = \frac{1}{1+\lambda k^2}\sin kx$ and $\int_{\mathbb{S}} \sin^2 kx\,\mathrm{d}x = \frac{1}{2}$ for $k \in 2\pi\mathbb{N}$ the expression for $S(\cos kx, 1)$ follows. A similar computation gives the same value for $S(\sin kx, 1)$ and (2.13) follows. To obtain the equalities (2.11) and (2.12) we use that, for $k, l \in 2\pi\mathbb{N}$,

$$A\Gamma(\cos kx, \cos lx) = \partial_x\left[-\frac{1}{2}\left(1-\frac{\lambda}{2}kl\right)\cos(k+l)x - \frac{1}{2}\left(1+\frac{\lambda}{2}kl\right)\cos(k-l)x\right], \tag{2.14}$$

$$A\Gamma(\cos kx, \sin lx) = \partial_x\left[-\frac{1}{2}\left(1-\frac{\lambda}{2}kl\right)\sin(k+l)x + \frac{1}{2}\left(1+\frac{\lambda}{2}kl\right)\sin(k-l)x\right], \tag{2.15}$$

$$A\Gamma(\sin kx, \sin lx) = \partial_x\left[\frac{1}{2}\left(1-\frac{\lambda}{2}kl\right)\cos(k+l)x - \frac{1}{2}\left(1+\frac{\lambda}{2}kl\right)\cos(k-l)x\right], \tag{2.16}$$

$$\Gamma(\cos kx, \cos lx) = \partial_x\left[-\frac{\frac{1}{2}(1-\frac{\lambda}{2}kl)}{1+\lambda(k+l)^2}\cos(k+l)x - \frac{\frac{1}{2}(1+\frac{\lambda}{2}kl)}{1+\lambda(k-l)^2}\cos(k-l)x\right], \tag{2.17}$$

$$\Gamma(\cos kx, \sin lx) = \partial_x\left[-\frac{\frac{1}{2}(1-\frac{\lambda}{2}kl)}{1+\lambda(k+l)^2}\sin(k+l)x + \frac{\frac{1}{2}(1+\frac{\lambda}{2}kl)}{1+\lambda(k-l)^2}\sin(k-l)x\right], \tag{2.18}$$

$$\Gamma(\sin kx, \sin lx) = \partial_x\left[\frac{\frac{1}{2}(1-\frac{\lambda}{2}kl)}{1+\lambda(k+l)^2}\cos(k+l)x - \frac{\frac{1}{2}(1+\frac{\lambda}{2}kl)}{1+\lambda(k-l)^2}\cos(k-l)x\right]. \tag{2.19}$$

We only give a proof for $u = \cos kx$ and $v = \cos lx$ with $k \neq l \in 2\pi\mathbb{N}$. The other computations are similar and we leave them to the reader. Again, by the previous theorem,

$$S(\cos kx, \cos lx) = \int_{\mathbb{S}} \Gamma(\cos kx, \cos lx) A\Gamma(\cos kx, \cos lx)\mathrm{d}x - \int_{\mathbb{S}} \Gamma(\cos kx, \cos kx) A\Gamma(\cos lx, \cos lx)\mathrm{d}x.$$

Using Eqs. (2.14) and (2.17), we can rewrite the right-hand side terms as

$$-\int_{\mathbb{S}} \left[-\frac{\frac{1}{2}(1-\frac{\lambda}{2}kl)}{1+\lambda(k+l)^2}\cos(k+l)x - \frac{\frac{1}{2}(1+\frac{\lambda}{2}kl)}{1+\lambda(k-l)^2}\cos(k-l)x \right] \times \partial_x^2 \left[-\frac{1}{2}\left(1-\frac{\lambda}{2}kl\right)\cos(k+l)x - \frac{1}{2}\left(1+\frac{\lambda}{2}kl\right)\cos(k-l)x \right] \mathrm{d}x$$
$$-\int_{\mathbb{S}} \partial_x \left[-\frac{\frac{1}{2}(1-\frac{\lambda}{2}k^2)}{1+4\lambda k^2}\cos 2kx \right] \partial_x \left[-\frac{1}{2}\left(1-\frac{\lambda l^2}{2}\right)\cos 2lx \right] \mathrm{d}x.$$

Using the orthogonality relations for trigonometric functions, we find that

$$\begin{aligned} S(\cos kx, \cos lx) &= \frac{1}{4}\frac{(1+\frac{\lambda}{2}kl)^2}{1+\lambda(k-l)^2}(k-l)^2 \int_{\mathbb{S}} \cos^2(k-l)x\,\mathrm{d}x \\ &+ \frac{1}{4}\frac{(1-\frac{\lambda}{2}kl)^2}{1+\lambda(k+l)^2}(k+l)^2 \int_{\mathbb{S}} \cos^2(k+l)x\,\mathrm{d}x \\ &= \mathrm{Sec}(k,l). \end{aligned}$$

□

2.3 Variation of geometric quantities

In classical differential geometry, one studies one-parameter families of Riemannian metrics, i.e., if M is a manifold and I is a non-empty interval, then for any $t \in I$, there is a Riemannian metric

$$g_t \colon M \ni p \mapsto \langle \cdot, \cdot \rangle_{t,p} : T_pM \times T_pM \to \mathbb{R}$$

such that $t \mapsto g_t$ is smooth. A trivial example is a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ where a one-parameter family of metrics is obtained simply by scaling the given metric, i.e., for $t \in (0, \infty)$, one defines $\langle \cdot, \cdot \rangle_t = t \cdot \langle \cdot, \cdot \rangle$. A more interesting example which is closely related to our problem is the following: Given two Riemannian metrics g_0 and g_1, then

$$g_t := (1-t)g_0 + tg_1, \quad t \in [0,1],$$

defines a one-parameter family of Riemannian metrics between g_0 and g_1. If the manifold M has finite dimension, then any $p \in M$ has a neighborhood V such that $p = \varphi(x)$, $x = (x_1, \ldots, x_n) \in U$, $U \subset \mathbb{R}^n$ open and $\varphi \colon U \to V$, and T_pM is spanned by $\{\partial_{x_i}\}_{i=1,\ldots,n}$. One defines covariant components of g_t via

$$g_{ij}(t,x) := \left\langle \frac{\partial \varphi}{\partial x_i}(x), \frac{\partial \varphi}{\partial x_j}(x) \right\rangle_{t,\varphi(x)}$$

and the Christoffel symbols $\Gamma_{ij}^k = (\nabla_{\partial_i}\partial_j)^k$; ∇ denoting the Levi-Civita connection. Contravariant components are defined by inverting the matrix $g = (g_{ij})_{i,j=1,\dots,n}$. A standard question is how quantities of the inner geometry of M change when the metric varies, controlled by the parameter t. In [8], the author discusses this problem for regular surfaces in $\mathbb{R}^3$ and comes to the following results: First, the derivative of g_{ij} follows from the Taylor expansion

$$g_{ij}(t,x) = g_{ij}(t_0,x) + (t-t_0)\dot{g}_{ij}(t_0,x) + \mathcal{O}((t-t_0)^2)$$

and with $\dot{g}^{jk} = -\sum_{il} g^{ij}\dot{g}_{il}g^{lk}$ one checks that

$$\dot{\Gamma}_{ij}^k = \frac{1}{2}\sum_\alpha g^{k\alpha}\left(\frac{\partial \dot{g}_{j\alpha}}{\partial x_i} + \frac{\partial \dot{g}_{i\alpha}}{\partial x_j} - \frac{\partial \dot{g}_{ij}}{\partial x_\alpha}\right) - \sum_{\beta,l}\Gamma_{ij}^\beta g^{lk}\dot{g}_{\beta l}$$

and

$$2\dot{\mathcal{G}} = \operatorname{div}(\operatorname{div}\dot{g}) - \Delta(\operatorname{tr}\dot{g}) - \mathcal{G}\operatorname{tr}\dot{g}$$

where $\mathcal{G} = \frac{1}{2}\sum_{ijk} g^{jk}R_{ijk}^i$ denotes the Gaussian curvature.

In this section, we discuss a similar question for the family $\{\langle\cdot,\cdot\rangle_\lambda\,;\ \lambda\in[0,1]\}$ of Riemannian metrics on the circle diffeomorphisms. Basically, we compute the λ-derivatives of the Christoffel map $\Gamma(u,v)$ in (2.8) and the sectional curvature $S(u,v)$ obtained in Theorem 2.4. Therefore, all we need is to compute the λ-derivative of A^{-1}. In the following lemma we find the Green's function for A^{-1}.

Lemma 2.6. *The operator* $A_\lambda = 1 - \lambda\partial_x^2\colon C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$, $\lambda\in[0,1]$, *is invertible and for* $\lambda\neq 0$ *its inverse is*

$$A_\lambda^{-1}f = G_\lambda * f, \quad G_\lambda(x) := \frac{1}{2\lambda^{1/2}}\sum_{k\in\mathbb{Z}} e^{-\lambda^{-1/2}|x+k|}.$$

Furthermore, the map $\lambda\mapsto\lambda^j G_\lambda$ *is differentiable for all* $j\in\mathbb{N}_0$ *and* $[\partial_x, A_\lambda^{-1}] = 0$.

Proof. For $\lambda\in(0,1]$, let $f_1 := \exp(\frac{x}{\sqrt{\lambda}})\chi_{\{x<0\}}$ be the function which equals $\exp(\frac{x}{\sqrt{\lambda}})$ on the negative half-axis and which is zero for positive x. Similarly, let $f_2 := \exp(-\frac{x}{\sqrt{\lambda}})\chi_{\{x>0\}}$. Note that $f_1, f_2 \in L_2(\mathbb{R})\backslash\mathscr{S}(\mathbb{R})$ so that

$$(\mathscr{F}f_1)(k) = \text{l.i.m.}_{R\to\infty}\frac{1}{\sqrt{2\pi}}\int_{|x|<R} f_1(x)e^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda^{-1/2}-\mathrm{i}k}$$

where $\mathscr{F}$ is the Fourier transform $L_2(\mathbb{R})\to L_2(\mathbb{R})$. Similarly,

$$(\mathscr{F}f_2)(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda^{-1/2}+\mathrm{i}k}$$

and

$$\frac{\lambda}{1+\lambda k^2} = \frac{1}{\lambda^{-1/2}+\mathrm{i}k}\frac{1}{\lambda^{-1/2}-\mathrm{i}k} = 2\pi(\mathscr{F}f_1)(k)(\mathscr{F}f_2)(k).$$

Given $f\in C^\infty(\mathbb{S})$, we consider the equation $u - \lambda u_{xx} = f$ in $\mathscr{S}'$. Applying $\mathscr{F}$, we conclude that $(1+\lambda k^2)\mathscr{F}u = \mathscr{F}f$. If we apply $\mathscr{F}^{-1}$ to the equation $\mathscr{F}u = (1+\lambda k^2)^{-1}\mathscr{F}f$ and use the identity $\mathscr{F}(u*v) = \sqrt{2\pi}\mathscr{F}u\mathscr{F}v$, we get

$$u = \frac{1}{\sqrt{2\pi}}\left(f * \mathscr{F}^{-1}(1+\lambda k^2)^{-1}\right) = \frac{1}{\lambda}(f*(f_1*f_2)).$$

Note that

$$\begin{aligned}
f_1 * f_2 &= \int_{\mathbb{R}} \exp\left(\frac{x-y}{\sqrt{\lambda}}\right) \chi_{\{x<y\}} \exp\left(-\frac{y}{\sqrt{\lambda}}\right) \chi_{\{y>0\}} \, \mathrm{d}y \\
&= \exp\left(\frac{x}{\sqrt{\lambda}}\right) \int_{\max\{0,x\}}^{\infty} \exp\left(-\frac{2y}{\sqrt{\lambda}}\right) \mathrm{d}y \\
&= \begin{cases} \frac{\sqrt{\lambda}}{2} \exp(\frac{x}{\sqrt{\lambda}})[-\exp(-\frac{2y}{\sqrt{\lambda}})]_0^{\infty}, & x \le 0, \\ \frac{\sqrt{\lambda}}{2} \exp(\frac{x}{\sqrt{\lambda}})[-\exp(-\frac{2y}{\sqrt{\lambda}})]_x^{\infty}, & x > 0, \end{cases} \\
&= \frac{\sqrt{\lambda}}{2} \exp\left(-\frac{|x|}{\sqrt{\lambda}}\right).
\end{aligned}$$

Since f has period 1, it follows that

$$\begin{aligned}
u &= (1-\lambda\partial_x^2)^{-1} f \\
&= \frac{1}{2\sqrt{\lambda}} \int_{\mathbb{R}} f(x-y) \exp\left(-\frac{|y|}{\sqrt{\lambda}}\right) \mathrm{d}y \\
&= \frac{1}{2\sqrt{\lambda}} \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x-y) \exp\left(-\frac{|y|}{\sqrt{\lambda}}\right) \mathrm{d}y \\
&= \frac{1}{2\sqrt{\lambda}} \sum_{k\in\mathbb{Z}} \int_0^1 f(x-y-k) \exp\left(-\frac{|y+k|}{\sqrt{\lambda}}\right) \mathrm{d}y \\
&= \int_{\mathbb{S}} f(x-y) \left(\frac{1}{2\sqrt{\lambda}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{|y+k|}{\sqrt{\lambda}}\right)\right) \mathrm{d}y.
\end{aligned}$$

Thus G_λ is as desired. □

Remark 2.7. For fixed $x \in [0,1]$ we can rewrite the integral kernel of $(1-\lambda\partial_x^2)^{-1}$ as

$$\begin{aligned}
2\sqrt{\lambda} G_\lambda(x) &= \sum_{k=-\infty}^{-1} \exp\left(\frac{x+k}{\sqrt{\lambda}}\right) + \sum_{k=0}^{\infty} \exp\left(-\frac{x+k}{\sqrt{\lambda}}\right) \\
&= \exp\left(\frac{x}{\sqrt{\lambda}}\right) \sum_{k=1}^{\infty} \left[\exp\left(-\frac{1}{\sqrt{\lambda}}\right)\right]^k + \exp\left(-\frac{x}{\sqrt{\lambda}}\right) \sum_{k=0}^{\infty} \left[\exp\left(-\frac{1}{\sqrt{\lambda}}\right)\right]^k \\
&= \exp\left(\frac{x}{\sqrt{\lambda}}\right) \left(\frac{1}{1-\exp\left(-\frac{1}{\sqrt{\lambda}}\right)} - 1\right) + \exp\left(-\frac{x}{\sqrt{\lambda}}\right) \frac{1}{1-\exp\left(-\frac{1}{\sqrt{\lambda}}\right)} \\
&= \frac{\exp\left(\frac{x}{\sqrt{\lambda}}\right) + \exp\left(\frac{1-x}{\sqrt{\lambda}}\right)}{\exp\left(\frac{1}{\sqrt{\lambda}}\right) - 1} \\
&= \frac{\cosh\left(\frac{1}{\sqrt{\lambda}}\left(x-\frac{1}{2}\right)\right)}{\sinh\left(\frac{1}{2\sqrt{\lambda}}\right)}.
\end{aligned}$$

Since G_λ has period 1, we obtain

$$G_\lambda(x) = \frac{\cosh\left(\frac{1}{\sqrt{\lambda}}\left(x-[x]-\frac{1}{2}\right)\right)}{2\sqrt{\lambda}\sinh\left(\frac{1}{2\sqrt{\lambda}}\right)}$$

for any $x \in \mathbb{R}$ and $\lambda \in (0,1]$, cf. [19] for the case $\lambda = 1$.

From Remark 2.7 we get that

$$\partial_\lambda G_\lambda(x) = -\frac{\sinh\left(\frac{1}{\sqrt{\lambda}}\left(x-[x]-\frac{1}{2}\right)\right)}{4\lambda^2\sinh\left(\frac{1}{2\sqrt{\lambda}}\right)}\left(x-[x]-\frac{1}{2}\right) - \frac{G_\lambda(x)}{2\lambda}\left(1-\frac{\coth\left(\frac{1}{2\sqrt{\lambda}}\right)}{2\sqrt{\lambda}}\right). \tag{2.20}$$

Next we derive the formula

$$\partial_\lambda \Gamma(u,v) = -(\partial_\lambda G_\lambda) * \partial_x\left(uv + \frac{\lambda}{2}u_x v_x\right) - \frac{1}{2}G_\lambda * (u_x v_x)_x.$$

Using Theorem 2.4 and Eq. (2.20) we have

$$\begin{aligned}
\partial_\lambda S(u,v) = &\left\langle (\partial_\lambda G_\lambda) * \partial_x\left(uv+\frac{\lambda}{2}u_x v_x\right), \partial_x\left(uv+\frac{\lambda}{2}u_x v_x\right)\right\rangle_{L_2} \\
&+\frac{1}{2}\left\langle G_\lambda * (u_x v_x)_x, \partial_x\left(uv+\frac{\lambda}{2}u_x v_x\right)\right\rangle_{L_2} \\
&+\frac{1}{2}\left\langle G_\lambda * \partial_x\left(uv+\frac{\lambda}{2}u_x v_x\right), (u_x v_x)_x\right\rangle_{L_2} \\
&-\left\langle (\partial_\lambda G_\lambda) * \partial_x\left(u^2+\frac{\lambda}{2}u_x^2\right), \partial_x\left(v^2+\frac{\lambda}{2}v_x^2\right)\right\rangle_{L_2} \\
&-\left\langle G_\lambda * (u_x u_{xx}), \partial_x\left(v^2+\frac{\lambda}{2}v_x^2\right)\right\rangle_{L_2} \\
&-\left\langle G_\lambda * \partial_x\left(u^2+\frac{\lambda}{2}u_x^2\right), v_x v_{xx}\right\rangle_{L_2}.
\end{aligned}$$

Letting u and v be equal to a trigonometric function, we computed the sectional curvature in Theorem 2.5 and found that it equals, up to a factor, the quantity $\mathrm{Sec}(k,l)$. Note that

$$\partial_\lambda \mathrm{Sec}(k,l) = A(\lambda)(k-l)^2 + B(\lambda)(k+l)^2$$

where

$$A(\lambda) = \frac{kl-(k-l)^2+\frac{\lambda}{2}k^2l^2+\frac{\lambda^2}{4}(k-l)^2k^2l^2}{8[1+\lambda(k-l)^2]^2}$$

and

$$B(\lambda) = \frac{-kl-(k+l)^2+\frac{\lambda}{2}k^2l^2+\frac{\lambda^2}{4}(k+l)^2k^2l^2}{8[1+\lambda(k+l)^2]^2}.$$

Chapter 3
A partially averaged version of the periodic *b*-equation

A novel family of equations related to the b-equation is proposed in [99]. The key idea is to replace the inertia operator $1-\partial_x^2$ for the b-equation by the operator $\mu-\partial_x^2$ where $\mu(u)$ is the average value of the periodic function u. We will call this new family the μ-b-equation. In this chapter we come to the following results:

First, we comment on the fact that the b-equation, the μ-b-equation and the HS equation have the same form and only differ in the particular choice of the inertia operator and the parameter b respectively. Then we show that a novel method proving local well-posedness in the smooth category for the b-equation with smooth initial data (see [41]) can be generalized to the μDP equation. In particular, we obtain a geodesic flow and an exponential map for μDP which are smooth objects if the initial data are smooth.

Escher and Seiler showed in [45] that only for $b=2$ the b-equation is a metric Euler equation. We generalize this result to the μ-b-equation and prove that the μCH equation is the only member of the novel family which possesses a regular inertia operator. In addition, we extend our discussion of a one-parameter family of CH equations in Chap. 2 and compute the Christoffel operator and the sectional curvature for a general μCH equation.

As a corollary, our results show that the μDP equation is a non-metric Euler equation; the same has been established for the DP equation for which some recent studies dealing with a dissipative term have been done, [46, 122]. This motivates our study of a weakly dissipative μDP equation for which we prove local well-posedness —using our geometric methods— and specify the precise blow-up scenario. We finally discuss blow-up solutions as well as criteria for the global existence of strong solutions.

3.1 A couple of regular inertia operators and some preliminary remarks

In this introductory section our aim is to give a brief overview about the different inertia operators with which we will deal in the following. We make clear how the operators have to be defined so that they are topological isomorphisms, we compute the Green's functions (as far as they exist) and explain how our theory can be extended to Sobolev spaces which play an important role for various applications.

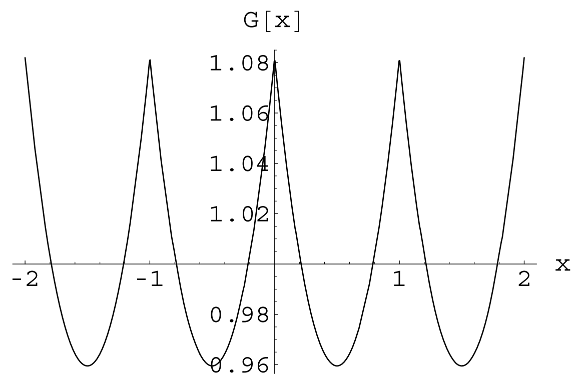


Fig. 3.1 Green's function for the operator $A = 1 - \partial_x^2$, periodically extended to the real axis.

3.1.1 The b-equation

The inertia operator for the b-equation is $A = 1 - \partial_x^2$. We have proved in Lemma 2.6 that the operator A as a map $C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ has the inverse

$$A^{-1}u = G * u, \quad G(x) = \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{-|x+k|} = \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}. \tag{3.1}$$

Note that $-\partial_x^2 A^{-1} f = f - A^{-1} f$ and that the existence of a Green's function implies that A^{-1} commutes with ∂_x (which is a consequence of carrying out the differentiation under the integral sign). Since for any $n \geq 2$ we have $\|Au\|_{C^{n-2}} \leq 2\,\|u\|_{C^n}$ it follows that A is continuous and since

$$\begin{aligned}
\left\|A^{-1}u\right\|_{C^{n+2}} &= \left\|A^{-1}u\right\|_\infty + \left\|A^{-1}u_x\right\|_\infty + \left\|u - A^{-1}u\right\|_{C^n} \\
&\leq \|G\|_\infty \left(\|u\|_\infty + \|u_x\|_\infty\right) + \|u\|_{C^n} + \sum_{k=0}^{n} \left\|A^{-1}u^{(k)}\right\|_\infty \\
&\leq \left(\|G\|_\infty + 1\right)\|u\|_{C^n} + \|G\|_\infty \sum_{k=0}^{n} \left\|u^{(k)}\right\|_\infty \\
&\leq \left(2\,\|G\|_\infty + 1\right)\|u\|_{C^n}
\end{aligned}$$

for any $n \geq 2$ it follows that A^{-1} is continuous. Observe that the inner product on $C^\infty(\mathbb{S})$ generated by A is the H^1 inner product. Hence A is a regular inertia operator in the sense of Definition 1.22.

3.1.2 The μ-b-equation

If we replace the momentum variable $m = u - u_{xx}$ by the partially averaged momentum $m = \mu(u) - u_{xx}$ in (1.39) we obtain the *periodic μ-b-equation*. Note that if u is a $C^n(\mathbb{S})$-function then $\mu(\partial_x^k u) = 0$ for $k \in \{1, \ldots, n\}$. Furthermore, it is important to mention that $\mu(u(t,\cdot))$ does not depend on x but is still a function of the time variable t.

The *periodic μ-b-equation* is the 1-parameter family of evolution equations

$$u_t = -(\mu - \partial_x^2)^{-1} \left(b\mu(u)u_x - bu_x u_{xx} - uu_{xxx} \right), \quad b \in \mathbb{R}, \tag{3.2}$$

where $u(t,x)$ is a function depending on time $t \in \mathbb{R}$ and a space variable $x \in \mathbb{S}$.

The inertia operator for the μ-variant (3.2) is $A = \mu - \partial_x^2$. We first establish that $A\colon C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ defines indeed an inner product on $C^\infty(\mathbb{S})$.

Lemma 3.1. *The bilinear map*

$$\langle \cdot,\cdot \rangle_\mu : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to \mathbb{R}, \quad \langle u,v \rangle_\mu = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x(x)v_x(x)\,\mathrm{d}x$$

defines an inner product on $C^\infty(\mathbb{S})$.

Proof. Clearly, $\langle \cdot,\cdot \rangle_\mu$ is a symmetric bilinear form and $\langle u,u \rangle_\mu \geq 0$ for all u. If $u \in C^\infty(\mathbb{S})$ satisfies $\langle u,u \rangle_\mu = 0$, then $u_x = 0$ on $\mathbb{S}$ and hence u is constant. The fact that $\mu(u) = 0$ implies $u = 0$ and hence $\langle \cdot,\cdot \rangle_\mu$ is positive definite. □

Next we show that A is invertible and that A and A^{-1} are continuous maps on $C^\infty(\mathbb{S})$.

Lemma 3.2. *The operator $A = \mu - \partial_x^2$ maps $C^n(\mathbb{S})$ to $C^{n-2}(\mathbb{S})$, $n \geq 2$, and has the inverse*

$$\begin{aligned}(A^{-1}f)(x) = \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right)\int_0^1 f(a)\,\mathrm{d}a + \left(x - \frac{1}{2}\right)\int_0^1\int_0^a f(b)\,\mathrm{d}b\,\mathrm{d}a \\ - \int_0^x\int_0^a f(b)\,\mathrm{d}b\,\mathrm{d}a + \int_0^1\int_0^a\int_0^b f(c)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a.\end{aligned} \tag{3.3}$$

In particular, A and A^{-1} are continuous maps on $C^\infty(\mathbb{S})$.

Proof. Clearly, $\mu(A^{-1}f) = \mu(f)$ and $(A^{-1}f)_{xx} = \mu(f) - f$ so that $A(A^{-1}f) = f$ for any $f \in C^{n-2}(\mathbb{S})$. To conclude that A is surjective, we observe that $(\partial_x^k A^{-1}f)(0) = (\partial_x^k A^{-1}f)(1)$ for all $k \in \{0,\ldots,n\}$. To see that A is injective, assume that $Au = 0$ for $u \in C^n(\mathbb{S})$ and $n \geq 2$. Then there are constants $c,d \in \mathbb{R}$ such that $u = \frac{1}{2}\mu(u)x^2 + cx + d$. Since u must be periodic, $c = 0$ and $\mu(u) = 0$ from which we get $d = 0$ and hence $u = 0$. □

Remark 3.3. We get from Lemma 3.2 that $\partial_x^2 A^{-1} = \mu - 1$.

In fact, the operator A^{-1} is an integral operator: To obtain Green's function for A^{-1} we first look for elements in the kernel of A. As explained in the above proof, $Au = 0$ implies that $u = \frac{1}{2}\mu(u)x^2 + cx + d$. Applying μ to this equation shows that

$$u = \frac{1}{2}\mu(u)x^2 + \left(\frac{5}{3}\mu(u) - 2d\right)x + d.$$

Using that $u(0) = u(1)$ yields

$$u = \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right)\mu(u).$$

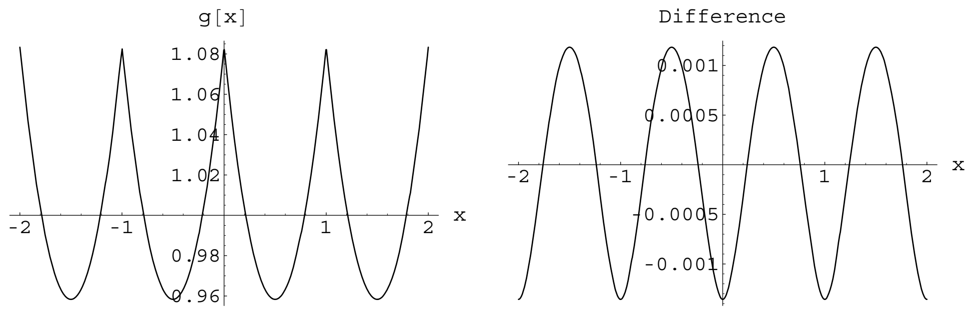


Fig. 3.2 Green's function g for the operator $A = \mu - \partial_x^2$, periodically extended to the real axis, and the difference to the function G in Fig. 3.1. The fact that the difference is small suggests that results for the b-equation might also be valid for its μ-variant.

Clearly, since $2u_x(0) = -\mu(u)$ and $2u_x(1) = \mu(u)$ we have $\mu(u) = 0$, i.e., $u = 0$, but nevertheless, we find that

$$(\mu - \partial_x^2)\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right) = 0.$$

This motivates the identity

$$((\mu - \partial_x^2)^{-1}u)(x) = \int_0^1 g(x-y)u(y)\,\mathrm{d}y, \quad g(x) = \frac{1}{2}x^2 - \frac{1}{2}|x| + \frac{13}{12}. \tag{3.4}$$

Observe that[1] $-\partial_x^2|\cdot| = -2\delta$. In conclusion, we have shown that $\mu - \partial_x^2$ is a regular inertia operator in the sense of Definition 1.22.

Remark 3.4. The μ-b-equation can be rewritten as

$$u_t + uu_x + A^{-1}\partial_x\left(b\mu(u)u + \frac{1}{2}(3-b)u_x^2\right) = 0. \tag{3.5}$$

Applying μ to this equation shows that $\mu(u_t) = 0$. Thus the μ-b-equation reduces to

$$-u_{txx} + b\mu(u)u_x - bu_xu_{xx} - uu_{xxx} = 0. \tag{3.6}$$

3.1.3 The Hunter-Saxton equation

As explained in Chap. 1, the Hunter-Saxton equation is of the same form as the CH equation since it can be written as $m_t = -m_xu - 2u_xm$, but with $m = -u_{xx}$. That is why we introduce the inertia operator $A = -\partial_x^2$ for the Hunter-Saxton equation. It is not a priori clear how to chose the domain of A so that it is an isomorphism.

Lemma 3.5. *Let A be the operator $-\partial_x^2$ with domain*

$$D(A) = \{f \in C^n(\mathbb{S});\ f(0) = 0\}, \quad n \geq 2.$$

[1] Using integration by parts we see that $\langle|\cdot|, -\varphi''\rangle = \int_{-\infty}^0 x\varphi''(x)\,\mathrm{d}x - \int_0^\infty x\varphi''(x)\,\mathrm{d}x = -2\varphi(0)$ for any test function φ.

For any $n \geq 2$, A is a topological isomorphism

$$D(A) \to \left\{ f \in C^{n-2}(\mathbb{S});\ \int_{\mathbb{S}} f(x)\,\mathrm{d}x = 0 \right\}$$

with the inverse

$$(A^{-1}f)(x) = -\int_0^x \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y + x \int_{\mathbb{S}} \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y. \tag{3.7}$$

Proof. Clearly, for all $f \in C^{n-2}(\mathbb{S})$ with zero mean, $-\partial_x^2(A^{-1}f) = f$ and $A^{-1}f \in D(A)$ since $(\partial_x^k A^{-1}f)(0) = (\partial_x^k A^{-1}f)(1)$ for all $k \in \{0, \ldots, n\}$. To see that A is injective, we assume that $u \in \ker(A)$ and have $u = ax+b$ with real constants a and b. Since $u(1) = u(0)$ we see that $a = 0$ and since $u(0) = 0$, we also get $b = 0$. □

Remark 3.6. Note that we have $-A^{-1}f_{xx} = f - f(0)$ and $-\partial_x^2 A^{-1} f = f$ for any C^2-function f and that $\partial_x A^{-1} f_x = -f + \mu(f)$. Note also that ∂_x and A^{-1} do not commute since

$$(A^{-1}f_x)(x) = -\int_0^x f(y)\,\mathrm{d}y + x\mu(f) \neq -\int_0^x f(y)\,\mathrm{d}y + \int_{\mathbb{S}} \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y = (A^{-1}f)_x(x).$$

Lemma 3.7. *The bilinear form on $\{u \in C^\infty(\mathbb{S});\ u(0) = 0\}$ defined by*

$$(u, v) \mapsto \int_{\mathbb{S}} uAv\,\mathrm{d}x$$

is a positive definite inner product.

Proof. It is clear that the map under consideration is bilinear and symmetric and that $\int_{\mathbb{S}} uAu\,\mathrm{d}x = \int_{\mathbb{S}} u_x^2\,\mathrm{d}x \geq 0$. If $\int_{\mathbb{S}} uAu\,\mathrm{d}x = 0$ we get $u_x = 0$; then u is constant and $u(0) = 0$ enforces $u = 0$. □

Definition 3.8. We call the metric induced on $\{u \in C^\infty(\mathbb{S}); u(0) = 0\}$ by the symmetric operator $A = -\partial_x^2$ the *$\dot{H}^1$-metric* and write

$$\langle u, v\rangle_{\dot{H}^1} = \int_{\mathbb{S}} uAv\,\mathrm{d}x = \int_{\mathbb{S}} u_x v_x\,\mathrm{d}x. \tag{3.8}$$

Remark 3.9. More general, one considers the *homogeneous Sobolev spaces*

$$\dot{H}^s(\mathbb{S}) = \left\{ f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x};\ f(0) = 0,\ \|f\|^2_{\dot{H}^s} = \sum_{n\in\mathbb{Z}} (4\pi^2 n^2)^s \left|\hat{f}(n)\right|^2 < \infty \right\}$$

where the $\hat{f}(n)$ are the Fourier coefficients of f and s is non-negative[2]. The reader can compare this definition to the one for the ordinary Sobolev spaces on the circle in the following section. Note that the operator $-\partial_x^2$ maps a function in $\dot{H}^s(\mathbb{S})$ to a function with zero mean. On the real axis, the spaces $\dot{H}^s(\mathbb{R})$ are defined by completing the Schwartz space $\mathscr{S}(\mathbb{R})$ with respect to

$$\|u\|^2_{\dot{H}^s} = \int_{\mathbb{R}} |\xi|^{2s} |\hat{u}(\xi)|^2\,\mathrm{d}\xi;$$

[2] Observe that $\{u \in H^s(\mathbb{S});\ u(0) = 0\} \simeq \{v \in H^s(\mathbb{S});\ \mu(v) = 0\}$ via the isomorphism $u \mapsto u - \mu(u)$ with the inverse $v \mapsto v - v(0)$.

again this is similar to the definition of $H^s(\mathbb{R})$ which is the completion of $\mathscr{S}(\mathbb{R})$ with respect to the Sobolev norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

We see that $A = -\partial_x^2$ is a regular inertia operator in the sense of Definition 1.22. Let us summarize our results.

For

$$A = \begin{cases} 1 - \partial_x^2 & \text{and} \quad b \in \mathbb{R}, \\ \mu - \partial_x^2 & \text{and} \quad b \in \mathbb{R}, \\ -\partial_x^2 & \text{and} \quad b = 2, \end{cases} \tag{3.9}$$

the equation $m_t = -(m_x u + b u_x m)$, $m = Au$, becomes the b-equation, the μ-b-equation and the Hunter-Saxton equation respectively. In any case, the operator A defines a regular inertia operator.

The fact that the Green's functions for $(1 - \partial_x^2)^{-1}$ and $(\mu - \partial_x^2)^{-1}$ resemble each other motivates to study the μ-b-equation under similar aspects as it has been done for the b-equation recently. We can also motivate the study of μ-variants of Eq. (1.41) by perturbing the inertia operators for the b-equation or the HS equation respectively, either by adding the operator μ to the inertia operator for the HS equation or by replacing id by μ in the inertia operator for the b-equation. In the following we will always write A for the inertia operator of the respective equation and we do not introduce different notations for the three types of operators in (3.9).

3.1.4 A short introduction to Sobolev spaces

In this section, we briefly recall elementary facts about Sobolev spaces, in particular on the circle $\mathbb{S}$, and explain how the inertia operators introduced in the previous section can be defined on Sobolev spaces.

We denote by $H^s = H^s(\mathbb{S})$, $s \geq 0$, the Sobolev space of periodic functions. If $s \in \mathbb{N}_0$, H^s is the space of all $L_2(\mathbb{S})$-functions f with square integrable distributional derivatives up to the order s, $\partial_x^i f \in L_2(\mathbb{S})$, $i \in \{0, \ldots, s\}$. Endowed with the norm

$$\|f\|_{H^s}^2 = \sum_{i=0}^{s} \int_{\mathbb{S}} (\partial_x^i f)^2(x) \, \mathrm{d}x = \sum_{i=0}^{s} \left\langle \partial_x^i f, \partial_x^i f \right\rangle_{L_2(\mathbb{S})} = \sum_{i=0}^{s} \left\| \partial_x^i f \right\|_{L_2(\mathbb{S})}^2,$$

the spaces H^s become Hilbert spaces. Note that we have $H^0 = L_2(\mathbb{S})$. For general $s \geq 0$, we define the Sobolev spaces $H^s(\mathbb{S})$ by using the Fourier transform on $L_2(\mathbb{S})$ which maps a periodic function f to its Fourier series $(\hat{f}(n))_{n \in \mathbb{Z}}$, see [129]. Let $Q^s = (1 - \partial_x^2)^{s/2}$ be the elliptic pseudo-differential operator with the symbol $(1 + 4\pi^2 n^2)^{s/2}$, i.e., for any $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$ we have

$$\widehat{Q^s f}(n) = (1 + 4\pi^2 n^2)^{s/2} \hat{f}(n).$$

The Sobolev space $H^s(\mathbb{S})$ is the function space

$$H^s(\mathbb{S}) := \left\{ f \in L_2(\mathbb{S});\ \|f\|_{H^s}^2 = \sum_{n\in\mathbb{Z}} \left|\widehat{Q^s f}(n)\right|^2 < \infty \right\}.$$

The operator $Q^2 = 1 - \partial_x^2 \colon H^s(\mathbb{S}) \to H^{s-2}(\mathbb{S})$ is an isomorphism for all $s \geq 2$. That Q^2 and Q^{-2} are continuous follows from

$$\left\|Q^2 u\right\|_{H^{s-2}}^2 = \left\langle Q^{2(s-2)} Q^2 u, Q^2 u \right\rangle_{L_2(\mathbb{S})} = \left\langle Q^{2s} u, u\right\rangle_{L_2(\mathbb{S})} = \|u\|_{H^s}^2 .$$

It is easy to check that the operator $\mu - \partial_x^2 \colon H^s(\mathbb{S}) \to H^{s-2}(\mathbb{S})$ has the inverse (3.3) and the Green's function (3.4), see [99]. We compute for any $f \in H^s$

$$\begin{aligned}
\left\|(\mu - \partial_x^2) f\right\|_{H^{s-2}}^2 &= \left\| \hat{f}(0) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{f}(n) 4\pi^2 n^2 e^{2\pi i n x} \right\|_{H^{s-2}}^2 \\
&\leq 2\left|\hat{f}(0)\right|^2 + 2 \sum_{n\in\mathbb{Z}\setminus\{0\}} (4\pi^2 n^2)^2 (1+4\pi^2 n^2)^{s-2} \left|\hat{f}(n)\right|^2 \\
&\leq 2 \sum_{n\in\mathbb{Z}} (1+4\pi^2 n^2)^s \left|\hat{f}(n)\right|^2 \\
&= 2\,\|f\|_{H^s}^2
\end{aligned}$$

and together with the open mapping theorem this achieves the continuity of both, $\mu - \partial_x^2$ and its inverse; i.e., $\mu - \partial_x^2$ is a topological isomorphism. For the HS equation, it is explained in [97] that $-\partial_x^2$ is a topological isomorphism

$$E^s := \{f \in H^s(\mathbb{S});\ f(0) = 0\} \to \left\{f \in H^{s-2}(\mathbb{S});\ \mu(f) = 0\right\}, \quad s \geq 3.$$

We now introduce the $H^s(\mathbb{S})$-diffeomorphisms

$$H^s\mathrm{Diff}(\mathbb{S}) = \left\{\varphi \in H^s(\mathbb{S});\ \varphi \text{ bijective, orientation-preserving and } \varphi^{-1} \in H^s(\mathbb{S})\right\}.$$

The H^s-diffeomorphisms form a topological group for any $s > 3/2$, as we will see in Lemma 3.35. Furthermore, $H^s\mathrm{Diff}(\mathbb{S})$ is a Hilbert manifold modelled on $H^s(\mathbb{S})$ and $T_\varphi H^s\mathrm{Diff}(\mathbb{S}) \simeq H^s(\mathbb{S})$ for all $\varphi \in H^s\mathrm{Diff}(\mathbb{S})$. We will implicitly use the natural identification

$$TH^s\mathrm{Diff}(\mathbb{S}) \simeq H^s\mathrm{Diff}(\mathbb{S}) \times H^s(\mathbb{S}) \tag{3.10}$$

and a vector field X on $H^s\mathrm{Diff}(\mathbb{S})$ is viewed as a map $H^s\mathrm{Diff}(\mathbb{S}) \to H^s(\mathbb{S})$; the evaluation of X at $\varphi \in H^s\mathrm{Diff}(\mathbb{S})$ is viewed as a map $\mathbb{S} \to T\mathbb{S}$ covering φ with value $(X(\varphi))(x) \in \mathbb{R} \simeq T_{\varphi(x)}\mathbb{S}$ at the point $\varphi(x)$ for $x \in \mathbb{S}$. The identification (3.10) is given explicitly as follows. The map $\varphi \mapsto (\varphi(0), \varphi(x) - x - \varphi(0))$ is a diffeomorphism $H^s\mathrm{Diff}(\mathbb{S}) \to \mathbb{S} \times U^s$, where

$$U^s := \{f \in H^s(\mathbb{S});\ f(0) = 0,\ f_x > -1\}.$$

Since U^s is an open subset of the closed linear subspace $E^s \subset H^s$, this map provides a local chart on $H^s\mathrm{Diff}(\mathbb{S})$ with values in $I \times U^s \subset \mathbb{R} \times E^s$ for any open subinterval $I \subset \mathbb{S}$. Moreover, using that $T\mathbb{S} \simeq \mathbb{S} \times \mathbb{R}$, we find

$$TH^s\mathrm{Diff}(\mathbb{S}) \simeq T(\mathbb{S} \times U^s) \simeq \mathbb{S} \times U^s \times \mathbb{R} \times E^s \simeq H^s\mathrm{Diff}(\mathbb{S}) \times H^s(\mathbb{S}).$$

Remark 3.10. Note that, according to Sobolev's embedding theorem, $H^s(\mathbb{S}) \subset C^1(\mathbb{S})$ for all $s > 3/2$.

3.2 The special role of the case $b = 2$

In [45], the authors explain that for the b-equation the case $b = 2$ is of particular interest since only for $b = 2$ one obtains a metric Euler equation. More precisely, for all $b \neq 2$, Eq. (1.41) is a family of non-metric Euler equations. In this section, we generalize this result to the periodic μ-b-equation.

Proposition 3.11. *Let $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$. Then the associated bilinear symmetric operator $B(u,v) = \frac{1}{2}[\mathrm{ad}_u^* v + \mathrm{ad}_v^* u]$ has the form*

$$B(u,v) = \frac{1}{2} A^{-1} \left[2(Au)v_x + 2(Av)u_x + u(Av)_x + v(Au)_x \right],$$

for all $u, v \in C^\infty(\mathbb{S})$.

Proof. Let ρ_A be the metric induced by A, i.e., $\rho_A(u,v) = \langle Au, v\rangle_{L_2}$. By direct computation we find that

$$\rho_A(\mathrm{ad}_u^* v, w) = \rho_A(v, \mathrm{ad}_u w) = \int_{\mathbb{S}} Av(u_x w - w_x u)\,\mathrm{d}x = \int_{\mathbb{S}} [(Av)u_x + ((Av)u)_x] w\,\mathrm{d}x$$

for all $u, v, w \in C^\infty(\mathbb{S})$. Hence

$$\mathrm{ad}_u^* v = A^{-1}(2(Av)u_x + (Av)_x u)$$

and symmetrization achieves the proof. □

It may be instructive to recall the following paradigmatic examples.

Example 3.12. Let $\lambda \in [0,1]$ and let A be the inertia operator for the equation $m_t = -(m_x u + 2u_x m)$.

1. The choice $A = -\partial_x^2$ yields $B(u,u) = -A^{-1}(2u_x u_{xx} + u u_{xxx})$ and $u_t = -B(u,u)$ is the Hunter-Saxton equation
$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0.$$
2. We choose $A = 1 - \lambda\partial_x^2$. If $\lambda = 0$, the equation $m_t = -(m_x u + 2u_x m)$ becomes the periodic inviscid Burgers equation $u_t + B(u,u) = u_t + 3uu_x = 0$. For $\lambda \neq 0$, we obtain
$$u_t + B(u,u) = u_t + 3uu_x - \lambda(2u_x u_{xx} + u u_{xxx} + u_{txx}) = 0,$$
a version of the Camassa-Holm equation which we discussed in the previous chapter.
3. Choosing $A = \mu - \partial_x^2$, we arrive at the μCH equation
$$\mu(u_t) - u_{txx} + 2\mu(u)u_x = 2u_x u_{xx} + u u_{xxx}$$
which coincides with the μHS equation which we will study later.

Each regular inertia operator induces an Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$. We now consider the question for which $b \in \mathbb{R}$ there is a regular inertia operator such that the μ-b-equation is the corresponding Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$. Example 3.12 shows that, for $b = 2$, the operator $\mu - \partial_x^2 \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ induces the μCH. Our goal is to show that this works only for $b = 2$.

Theorem 3.13. *Let $b \in \mathbb{R}$ be given and suppose that there is a regular inertia operator $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ such that the μ-b-equation*

$$m_t = -(m_x u + bmu_x), \quad m = \mu(u) - u_{xx},$$

is the Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ with respect to ρ_A. Then $b = 2$ and $A = \mu - \partial_x^2$.

Proof. We write $L = \mu - \partial_x^2$. Let us assume that, for given $b \in \mathbb{R}$ and $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$, the μ-b-equation is the Euler equation on the circle diffeomorphisms with respect to ρ_A. By Proposition 3.11

$$u_t = -A^{-1}((Au)_x u + 2(Au)u_x), \quad u \in C^\infty(\mathbb{S}),$$

and the μ-b-equation can be written as

$$(Lu)_t = -((Lu)_x u + b(Lu)u_x), \quad u \in C^\infty(\mathbb{S}).$$

Using that $(Lu)_t = Lu_t$ and resolving the second equation with respect to u_t we get that

$$A^{-1}\left(2(Au)u_x + u(Au)_x\right) = L^{-1}\left(b(Lu)u_x + u(Lu)_x\right), \tag{3.11}$$

for all $u \in C^\infty(\mathbb{S})$. Denote by $\mathbf{1}$ the constant function with value 1. If we set $u = \mathbf{1}$ in (3.11), then $A^{-1}(\mathbf{1}(A\mathbf{1})_x) = 0$ and hence $(A\mathbf{1})_x = 0$, i.e., $A\mathbf{1} = c\mathbf{1}$. Scaling (3.11) shows that we may assume $c = 1$. Replacing u by $u + \lambda$ in (3.11) and scaling with λ^{-1}, we get on the left-hand side

$$\begin{aligned}
&\frac{1}{\lambda}A^{-1}\big(2(A(u+\lambda))(u+\lambda)_x + (u+\lambda)(A(u+\lambda))_x\big) \\
&\quad = \frac{1}{\lambda}A^{-1}\big(2((Au)+\lambda)u_x + (u+\lambda)(Au)_x\big) \\
&\quad = A^{-1}\left(\frac{2(Au)u_x + u(Au)_x}{\lambda} + 2u_x + (Au)_x\right) \\
&\quad \to A^{-1}(2u_x + (Au)_x),\ \lambda \to \infty,
\end{aligned}$$

and a similar computation for the right-hand side gives

$$\begin{aligned}
&\frac{1}{\lambda}L^{-1}\big(b(L(u+\lambda))(u+\lambda)_x + (u+\lambda)(L(u+\lambda))_x\big) \\
&\quad \to L^{-1}(bu_x + (Lu)_x),\ \lambda \to \infty.
\end{aligned}$$

We obtain

$$A^{-1}\left(2u_x + (Au)_x\right) = L^{-1}(bu_x + (Lu)_x), \tag{3.12}$$

for all $u \in C^\infty(\mathbb{S})$. Let us consider the functions $u_n = e^{inx}$, $n \in 2\pi\mathbb{Z}\backslash\{0\}$, for which we have $Lu_n = n^2 u_n$ and

$$L^{-1}(b(u_n)_x + (Lu_n)_x) = \mathrm{i}\alpha_n u_n, \quad \alpha_n = \frac{b}{n} + n.$$

We now apply A to (3.12) with $u = u_n$ and see that

$$2\mathrm{i}nu_n + (Au_n)_x = \mathrm{i}\alpha_n(Au_n).$$

Therefore $v_n \coloneqq Au_n$ solves the ordinary differential equation

$$v' - \mathrm{i}\alpha_n v = -2\mathrm{i}nu_n. \tag{3.13}$$

If $b = 0$, then $\alpha_n = n$ and hence the general solution of (3.13) is

$$v(x) = (c - 2\mathrm{i}nx)u_n, \quad c \in \mathbb{R},$$

which is not periodic for any $c \in \mathbb{R}$. Hence $b \neq 0$ and there are numbers γ_n so that

$$v_n = Au_n = \gamma_n e^{\mathrm{i}\alpha_n x} + \beta_n u_n, \quad \beta_n = \frac{2}{b}n^2.$$

We first discuss the case $\gamma_n = 0$ for all n, secondly we show that $\gamma_p \neq 0$ for some $p \in 2\pi\mathbb{Z}\backslash\{0\}$ is not possible. If all γ_n vanish, then $Au_n = \beta_n u_n$ and A is a Fourier multiplication operator; in particular A commutes with L. Therefore (3.11) with $u = u_n$ is equivalent to

$$L(2(Au_n)(u_n)_x + u_n(Au_n)_x) = A(b(Lu_n)(u_n)_x + u_n(Lu_n)_x)$$

and by direct computation

$$12\mathrm{i}n^3\beta_n u_{2n} = \mathrm{i}(b+1)n^3\beta_{2n}u_{2n}.$$

Inserting $\beta_n = 2n^2/b$ we see that $b = 2$ and $\beta_n = n^2$. Therefore $A = L$. Assume that there is $p \in 2\pi\mathbb{Z}\backslash\{0\}$ with $\gamma_p \neq 0$. Since $v_p = Au_p$ is periodic, $\alpha_p \in 2\pi\mathbb{Z}$ and hence $b = kp$ for some $k \in 2\pi\mathbb{Z}\backslash\{0\}$. Let $\alpha_p = m$. If $m = p$, then $b = 0$ which is impossible. We thus have $\langle u_m, u_p\rangle = 0$ and

$$\langle Au_p, u_m\rangle = \left\langle \gamma_p e^{\mathrm{i}mx}, u_m\right\rangle = \gamma_p.$$

The symmetry of A yields

$$\gamma_p = \langle Au_p, u_m\rangle = \langle u_p, Au_m\rangle = \overline{\gamma_m}\left\langle u_p, e^{\mathrm{i}\alpha_m x}\right\rangle.$$

Since $\gamma_p \neq 0$, γ_m is non-zero and periodicity implies $\alpha_m \in 2\pi\mathbb{Z}$. More precisely, $\alpha_m = p$ since otherwise $\left\langle u_p, e^{\mathrm{i}\alpha_m x}\right\rangle = 0 = \gamma_p$. Using $b = kp$ and the definition of α_p, we see that $m = \alpha_p = k + p$. Furthermore,

$$p(k+p) = \alpha_m(k+p) = \alpha_{k+p}(k+p) = kp + (k+p)^2$$

and hence $0 = k^2 + 2pk$. Since $k \neq 0$, it follows that $k = -2p$ and hence $b = -2p^2$. We get $\alpha_p = -p$ and observe that $\gamma_n = 0$ for all $n \neq \pm p$, since otherwise repeating the above calculations would yield $b = -2n^2$ contradicting $b = -2p^2$. Inserting $u = u_p$ in (3.11) shows that

$$\mathrm{i}p\gamma_p\mathbf{1} - \frac{3\mathrm{i}p}{\beta_{2p}}u_{2p} = \mathrm{i}p^3(b+1)\frac{u_{2p}}{4p^2};$$

here we have used that $Au_p = \gamma_p/u_p + \beta_p u_p$, $\beta_p = -1$ and $A^{-1}u_{2p} = u_{2p}/\beta_{2p}$, since $2p$ does not coincide with $\pm p$ and hence $\gamma_{2p} = 0$. It follows that $p\gamma_p = 0$ in contradiction to $p, \gamma_p \neq 0$. □

From the above theorem we immediately get the following result for the μDP equation.

Corollary 3.14. *The μDP equation on the circle*

$$m_t = -(m_x u + 3mu_x), \quad m = \mu(u) - u_{xx},$$

cannot be realized as a metric Euler equation in the sense of Definition 1.22 for any regular inertia operator $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$.

3.3 Local well-posedness in the smooth category and a smooth exponential map for μDP

This section is about a local well-posedness result for the periodic μDP equation

$$\mu(u_t) - u_{txx} + 3\mu(u)u_x - 3u_x u_{xx} - uu_{xxx} = 0, \tag{3.14}$$

which belongs to the family (3.2) and is obtained for $b = 3$. Throughout the following considerations, we will assume that

$$u \in C((-T,T), C^n(\mathbb{S})) \cap C^1((-T,T), C^{n-1}(\mathbb{S}))$$

for some $n \geq 3$ (so that all derivatives exist in the classical sense), where T denotes a positive real number. We will reformulate the μDP equation in terms of a geodesic flow on $\mathrm{Diff}^\infty(\mathbb{S})$ to obtain the following main result: For smooth initial data $u_0(x)$ for which $\|u_0\|_{C^3(\mathbb{S})}$ is small, we prove the short-time existence of a smooth solution $u(t,x)$ of (3.14) which depends smoothly on (t, u_0).

Theorem 3.15. *There exists an open interval J centered at zero and $\delta > 0$ such that for each $u_0 \in C^\infty(\mathbb{S})$ with $\|u_0\|_{C^3(\mathbb{S})} < \delta$, there exists a unique solution $u \in C^\infty(J, C^\infty(\mathbb{S}))$ of the μDP equation such that $u(0) = u_0$. Moreover, the solution u depends smoothly on $(t, u_0) \in J \times C^\infty(\mathbb{S})$.*

Furthermore, we show that the exponential map for the μDP equation is a smooth local diffeomorphism from a neighbourhood of $0 \in C^\infty(\mathbb{S})$ to a neighbourhood of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{S})$.

Theorem 3.16. *The exponential map* exp *at the unity element for the μDP equation on* $\mathrm{Diff}^\infty(\mathbb{S})$ *is a smooth local diffeomorphism from a neighbourhood of zero in* $\mathrm{Vect}^\infty(\mathbb{S})$ *to a neighbourhood of* id *on* $\mathrm{Diff}^\infty(\mathbb{S})$.

Our results will extend the work done in [41] to the periodic μDP equation and presumably our analysis will also work for general b, cf. Chap. 6. For concreteness, we restrict ourselves to the case $b = 3$ where we deal with a non-metric Euler equation. The results have been published by the author in [39].

This section is organized as follows: First, we explain how to rewrite (3.14) in terms of a local flow $\varphi \in \mathrm{Diff}^n(\mathbb{S})$, $n \geq 3$, and briefly comment on the geometric setting. The resulting equation is an ordinary differential equation and we can apply the Cauchy-Lipschitz Theorem to obtain a solution of class $C^n(\mathbb{S})$ with smooth dependence on t and

$u_0(x)$. In addition, we show that the solution (φ, φ_t) in $\mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ does neither lose nor gain spatial regularity as t varies through the existence interval. We then approximate the Fréchet Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ by the topological groups $\mathrm{Diff}^n(\mathbb{S})$ and the Fréchet space $C^\infty(\mathbb{S})$ by the Banach spaces $C^n(\mathbb{S})$ to obtain an analogous existence result for the geodesic equation on $\mathrm{Diff}^\infty(\mathbb{S})$. From this, we directly conclude Theorem 3.15 and Theorem 3.16 which is more or less a corollary to our theorem which shows the existence of a smooth geodesic flow.

The crucial point here is that $\mathrm{Diff}^\infty(\mathbb{S})$ is a Lie group so that if $\varphi(t)$ is the smooth geodesic flow for the μDP with smooth initial data, the solution u of (3.14) is given by $u(t) = \varphi_t(t) \circ \varphi^{-1}(t)$ and is again smooth. Nevertheless, to obtain a smooth flow $\varphi(t)$ is rather difficult since our standard local existence theorem for ODEs only applies in Banach spaces and not in Fréchet spaces like $C^\infty(\mathbb{S})$. Hence the strategy is to obtain first a geodesic flow $\varphi(t)$ in some $\mathrm{Diff}^n(\mathbb{S})$ and then to check that, for smooth initial data, $\varphi(t)$ is in fact smooth. A further technical difficulty is that $\mathrm{Diff}^n(\mathbb{S})$ only possesses the structure of a topological group and not a Banach Lie group. Our approach will point out how we obtain a smooth solution within this difficult scenario.

3.3.1 The periodic b-equation with smooth initial values

In this subsection, we recast some important results from [41], where the authors discuss the periodic b-equation in a geometric framework and prove local well-posedness for smooth initial data as well as smoothness of the corresponding exponential map as a diffeomorphism $C^\infty(\mathbb{S}) \to \mathrm{Diff}^\infty(\mathbb{S})$. The b-equation can be written in the form

$$u_t = -A^{-1}\left[u(Au)_x + b(Au)u_x\right],$$

where A is the operator $1 - \partial_x^2$. Hence

$$u_t + uu_x = -A^{-1}\left[3u_x u_{xx} + b(Au)u_x\right]. \tag{3.15}$$

Let $J \subset \mathbb{R}$ be an open interval. If we regard $u(t,x)$ as a time-dependent vector field on $J \times \mathbb{S}$ of class C^n, then u has a (unique) local flow $\varphi(t,x)$ of class C^n such that

$$u = \varphi_t \circ \varphi^{-1}.$$

Conversely, for any pair (φ, ξ), where $\varphi \in \mathrm{Diff}^n(\mathbb{S})$ is a $C^n(\mathbb{S})$-diffeomorphism and ξ is a $C^n(\mathbb{S})$-function, $\xi \circ \varphi^{-1}$ is of class C^n. It is easy to check that (3.15) is equivalent to

$$\begin{cases} \varphi_t = \quad \xi, \\ \xi_t = -P_\varphi(\xi), \end{cases} \tag{3.16}$$

where

$$P_\varphi(\xi) = P(\xi \circ \varphi^{-1}) \circ \varphi, \quad P = A^{-1}Q, \quad Q(u) = 3u_x u_{xx} + b(Au)u_x.$$

The fact that A and Q are polynomial differential operators with constant coefficients and the inverse mapping theorem for Banach spaces show that the second order vector field $F(\varphi, \xi) := (\xi, -P_\varphi(\xi))\colon \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}) \to C^n(\mathbb{S}) \times C^n(\mathbb{S})$ is smooth so that the short-time existence of a solution of (3.16) in $\mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ with smooth dependence on t

and $u_0 \in C^n(\mathbb{S})$ follows immediately from the Cauchy-Lipschitz Theorem. Furthermore, it can be shown that the solution $(\varphi(t), \xi(t))$ does neither lose nor gain spatial regularity as t increases or decreases from zero. This follows from the equations

$$\varphi_{xx}(t) = \varphi_x(t)\left[\int_0^t \xi(s)\varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \varphi_x(s)^{1-b}\,\mathrm{d}s\right] \tag{3.17}$$

and

$$\begin{aligned}\xi_{xx}(t) = \xi_x(t)&\left[\int_0^t \xi(s)\varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \varphi_x(s)^{1-b}\,\mathrm{d}s\right]\\ &+\varphi_x(t)\left[\xi(t)\varphi_x(t) - m_0\varphi_x(t)^{1-b}\right],\end{aligned} \tag{3.18}$$

where $m_0 = Au_0 = u_0 - (u_0)_{xx}$. To obtain Eqs. (3.17) and (3.18), one uses the conservation of the quantity $[(u - u_{xx}) \circ \varphi]\varphi_x^b$, i.e.,

$$[(u - u_{xx}) \circ \varphi]\varphi_x^b = m_0. \tag{3.19}$$

Lemma 3.17. *Let $(\varphi(t), \xi(t)) \in \mathrm{Diff}^3(\mathbb{S}) \times C^3(\mathbb{S})$, $t \in J$, be a short-time solution of (3.16). If $u_0 \in C^n(\mathbb{S})$, $n \geq 3$, then we have $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ for all $t \in J$.*

Lemma 3.18. *Let $(\varphi(t), \xi(t)) \in \mathrm{Diff}^3(\mathbb{S}) \times C^3(\mathbb{S})$, $t \in J$, be a short-time solution of (3.16). If there exists a nonzero $t \in J$ such that $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S})$ or $\xi(t) \in C^n(\mathbb{S})$, $n \geq 3$, then $\xi(0) = u_0 \in C^n(\mathbb{S})$.*

The main theorem (formulated in the geometric picture) reads as follows.

Theorem 3.19. *There exists an open interval J centered at zero and $\delta > 0$ such that for all $u_0 \in C^\infty(\mathbb{S})$ with $\|u_0\|_{C^3(\mathbb{S})} < \delta$, there exists a unique solution $(\varphi, \xi) \in C^\infty(J, \mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}))$ of (3.16) such that $\varphi(0) = \mathrm{id}$ and $\xi(0) = u_0$. Moreover, the flow (φ, ξ) depends smoothly on $(t, u_0) \in J \times C^\infty(\mathbb{S})$.*

In the smooth category, the map

$$\mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S}), \quad (\varphi, \xi) \mapsto \xi \circ \varphi^{-1} = u$$

is smooth. We thus have the following result.

Corollary 3.20. *There exists an open interval J centered at zero and $\delta > 0$ such that for each $u_0 \in C^\infty(\mathbb{S})$ with $\|u_0\|_{C^3(\mathbb{S})} < \delta$, there exists a unique solution $u \in C^\infty(J, C^\infty(\mathbb{S}))$ of the b-equation such that $u(0) = u_0$. Moreover, the solution u depends smoothly on $(t, u_0) \in J \times C^\infty(\mathbb{S})$.*

The flow $\varphi(t) \subset \mathrm{Diff}^\infty(\mathbb{S})$ can be interpreted as geodesic flow with respect to a right-invariant affine connection ∇ which is defined on the Lie algebra of $\mathrm{Diff}^\infty(\mathbb{S})$ by the sum of the Lie bracket and a bilinear operator, see Eq. (1.31). Hence it makes sense to study the exponential map defined by ∇ which is just evaluation of the geodesic flow at time $t = 1$. For finite n, the exponential map $\exp_\varphi(\cdot)$ is a map from $T_\varphi\mathrm{Diff}^n(\mathbb{S}) \simeq C^n(\mathbb{S})$ to the manifold $\mathrm{Diff}^n(\mathbb{S})$. Moreover, $\exp_\varphi(\cdot)$ is a local diffeomorphism. In general, this fails to hold true for Fréchet manifolds and taking the example of the right-invariant L_2-metric (Burgers equation) on $\mathrm{Diff}^\infty(\mathbb{S})$ we see that we do not get a local C^1-diffeomorphism near the origin, cf. [25, 26]. For the Camassa-Holm equation and more general for the H^k-metrics, $k \geq 1$, the Riemannian exponential map in fact is a smooth local diffeomorphism. This result has been extended to the general (non-metric) b-equation in [41].

Theorem 3.21. *The exponential map* exp *at the unity element for the b-equation on* $\mathrm{Diff}^{\infty}(\mathbb{S})$ *is a smooth local diffeomorphism from a neighbourhood of zero in $C^{\infty}(\mathbb{S})$ to a neighbourhood of* id *on* $\mathrm{Diff}^{\infty}(\mathbb{S})$.

3.3.2 A generalization to the μDP equation

Recall that the μDP equation (3.14) can be written as

$$u_t = -A^{-1}(u(Au)_x + 3(Au)u_x) \tag{3.20}$$

where $A = \mu - \partial_x^2$. As for the b-equation, the vector field $u(t,x)$ possesses a unique local flow φ of class $C^n(\mathbb{S})$, i.e., $\varphi_t(t,x) = u(t,\varphi(t,x))$ for all $x \in \mathbb{S}$ and all t in some open interval $J \subset \mathbb{R}$. Again, we use the short-hand notation $\varphi_t = u \circ \varphi$ for $\varphi_t(t,x) = u(t,\varphi(t,x))$; i.e., $\circ$ denotes the composition with respect to the space variable. Hence $u = \varphi_t \circ \varphi^{-1}$. The other way round, if $(\varphi,\xi) \in C^1(J, \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}))$ is given, then $\varphi^{-1}(t)$ is a $C^n(\mathbb{S})$-diffeomorphism and $(\xi \circ \varphi^{-1})(t)$ is a continuously differentiable curve in $C^n(\mathbb{S})$. The Christoffel operator for the μDP equation is

$$B(u,v) = \frac{1}{2}A^{-1}(u(Av)_x + v(Au)_x + 3(Au)v_x + 3(Av)u_x) \tag{3.21}$$

since we have

$$B(u,u) = A^{-1}(u(Au)_x + 3(Au)u_x) = -u_t,$$

which is the μDP equation written as an Euler equation on the tangent space at the identity of the C^n-diffeomorphisms of $\mathbb{S}$. We also know that the μDP belongs to the class of non-metric Euler equations and hence we cannot expect to obtain geometric information by defining some right-invariant metric on the diffeomorphism group of the circle. Instead of that we will work with the affine connection

$$\nabla_{\xi_u}\xi_v = \frac{1}{2}[\xi_u,\xi_v] + B(\xi_u,\xi_v), \tag{3.22}$$

where ξ_u, ξ_v are the right-invariant vector fields on the circle diffeomorphism group with values u, v at the identity. Let $X(t) = (\varphi(t),\xi(t))$ be a vector field along the curve $\varphi(t) \in \mathrm{Diff}^{\infty}(\mathbb{S})$ or $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S})$ respectively. The covariant derivative of $X(t)$ in the present case is defined as

$$\frac{\mathrm{D}X}{\mathrm{D}t}(t) = \left(\varphi(t), \xi_t + \frac{1}{2}[u(t),\xi(t)] + B(u(t),\xi(t))\right),$$

where $u = \varphi_t \circ \varphi^{-1}$. In particular, we see that u is a solution of the μDP if and only if its local flow φ is a geodesic for the connection ∇ defined by B in (3.21) via (3.22). That is why we will call $\varphi(t)$ the *geodesic flow* for the μDP equation in the following.

Although our goal is to handle the $\mathrm{Diff}^{\infty}(\mathbb{S})$-case, we will first discuss flows $\varphi(t)$ in $\mathrm{Diff}^n(\mathbb{S})$ for technical purposes[3]. Regarding $\mathrm{Diff}^n(\mathbb{S})$ as a smooth Banach manifold modelled on $C^n(\mathbb{S})$, the following result has to be understood locally, i.e., in any local

[3] Observe that Eq. (3.20) is not an ODE on $C^n(\mathbb{S})$ since the term $(Au)_x$ is not regularized by the operator A^{-1} of order -2. In particular, if u is in $C((-T,T);C^n(\mathbb{S}))$ then Eq. (3.20) implies that $u_t \in C((-T,T);C^{n-1}(\mathbb{S}))$.

chart of $\mathrm{Diff}^n(\mathbb{S})$.

Proposition 3.22. *The function $u \in C(J, C^n(\mathbb{S})) \cap C^1(J, C^{n-1}(\mathbb{S}))$, for $n \geq 3$, is a solution of (3.14) if and only if $(\varphi, \xi) \in C^1(J, \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}))$ is a solution of*

$$\begin{cases} \varphi_t = \quad \xi, \\ \xi_t = -P_\varphi(\xi), \end{cases} \tag{3.23}$$

where $P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}}$ and $P(f) := 3A^{-1}(f_x f_{xx} + (Af)f_x)$.

Proof. The function u and the corresponding flow $\varphi \in \mathrm{Diff}^n(\mathbb{S})$ satisfy the relation $\varphi_t = u \circ \varphi$. If we set $\varphi_t = \xi$, then, by the chain rule,

$$\xi_t = (u_t + uu_x) \circ \varphi.$$

Using (3.20), we see that u is a solution of the μDP equation (3.14) if and only if

$$\begin{aligned} u_t + uu_x &= -A^{-1}(u(Au)_x - A(uu_x) + 3(Au)u_x) \\ &= -A^{-1}(-uu_{xxx} + u_{xx}u_x + uu_{xxx} + 2u_x u_{xx} + 3(Au)u_x) \\ &= -3A^{-1}(u_x u_{xx} + (Au)u_x) \\ &= -P(u). \end{aligned}$$

Recall that

$$\mu(uu_x) = \int_0^1 uu_x \,\mathrm{d}x = \frac{1}{2}\int_0^1 \partial_x(u^2)\,\mathrm{d}x = \frac{1}{2}(u^2(1) - u^2(0)) = 0,$$

since u is continuous on $\mathbb{S}$. With $u = \xi \circ \varphi^{-1}$ the desired result follows. □

We now define the vector field

$$F(\varphi, \xi) := (\xi, -P_\varphi(\xi))$$

so that $(\varphi_t, \xi_t) = F(\varphi, \xi)$. We know that

$$F : \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}) \to C^n(\mathbb{S}) \times C^n(\mathbb{S}),$$

since P has order zero. Note that the second order vector field F is *equivariant* by the action of $\mathrm{Diff}^n(\mathbb{S})$ on $\mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$, i.e., $F(X \circ \psi) = F(X) \circ \psi$, for any $X \in \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ and $\psi \in \mathrm{Diff}^n(\mathbb{S})$. We aim to prove smoothness of the map F. It is important to mention that this will not follow from the smoothness of P since neither the composition nor the inversion are smooth maps on $\mathrm{Diff}^n(\mathbb{S})$. The following lemma will be crucial for our purposes.

Lemma 3.23. *Assume that p is a polynomial differential operator of order r with coefficients depending only on μ, i.e.,*

$$p(u) = \sum_{\substack{I=(\alpha_0,\dots,\alpha_r),\\ \alpha_i \in \mathbb{N}\cup\{0\},\, |I|\leq K}} a_I(\mu(u))\, u^{\alpha_0}(u')^{\alpha_1} \cdots (u^{(r)})^{\alpha_r}.$$

Then the action of $p_\varphi := R_\varphi \circ p \circ R_{\varphi^{-1}}$ is

$$p_\varphi(u) = \sum_I a_I \left(\int_0^1 u(y)\varphi_x(y)\,\mathrm{d}y \right) q_I(u; \varphi_x, \dots, \varphi^{(r)}),$$

where q_I are polynomial differential operators of order r with coefficients being rational functions of the derivatives of φ up to the order r. Moreover, the denominator terms only depend on φ_x.

Proof. It is sufficient to consider a monomial

$$m(u) = a(\mu(u))u^{\alpha_0}(u')^{\alpha_1} \cdots (u^{(r)})^{\alpha_r}.$$

We have

$$m_\varphi(u) = a(\mu(u \circ \varphi^{-1}))u^{\alpha_0}[(u \circ \varphi^{-1})' \circ \varphi]^{\alpha_1} \cdots [(u \circ \varphi^{-1})^{(r)} \circ \varphi]^{\alpha_r}.$$

First, we observe that

$$\mu(u \circ \varphi^{-1}) = \int_{\mathbb{S}} u(\varphi^{-1}(x))\,\mathrm{d}x = \int_0^1 u(y)\varphi_x(y)\,\mathrm{d}y,$$

where we have omitted the time dependence of u and φ. Recall that $\varphi(\mathbb{S}) = \mathbb{S}$, $\varphi_x > 0$ and that $\mu(u \circ \varphi^{-1})$ is a constant with respect to the space variable $x \in \mathbb{S}$. Let us introduce the notation

$$a_k = (u \circ \varphi^{-1})^{(k)} \circ \varphi, \quad k = 1, 2, \dots, r.$$

Then, by the chain rule,

$$a_1 = (\partial_x(u \circ \varphi^{-1})) \circ \varphi = \frac{u_x \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \circ \varphi = \frac{u_x}{\varphi_x}$$

and

$$\begin{aligned} a_{k+1} &= (\partial_x(u \circ \varphi^{-1})^{(k)}) \circ \varphi \\ &= (\partial_x(a_k \circ \varphi^{-1})) \circ \varphi \\ &= \frac{\partial_x a_k}{\varphi_x}, \end{aligned}$$

so that our theorem follows by induction. □

In the Banach algebras $C^n(\mathbb{S})$, $n \geq 1$, addition and multiplication as well as the mean value operation μ and the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ are smooth maps. We see that if the coefficients a_I are smooth functions for any multi-index I and u and φ are at least r times continuously differentiable, then $p_\varphi(u)$ depends smoothly on (φ, u).

Proposition 3.24. *The vector field*

$$F\colon \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}) \to C^n(\mathbb{S}) \times C^n(\mathbb{S})$$

is smooth for any $n \geq 3$.

Proof. We write $F = (F_1, F_2)$. Since $F_1\colon (\varphi, \xi) \mapsto \xi$ is smooth, it remains to check that $F_2\colon (\varphi, \xi) \mapsto -P_\varphi(\xi)$ is smooth. For this purpose, we consider the map

$$\tilde{P}\colon \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}) \to \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$$

defined by

$$\tilde{P}(\varphi, \xi) = (\varphi, R_\varphi \circ P \circ R_{\varphi^{-1}}(\xi)).$$

Observe that we have the decomposition $\tilde{P} = \tilde{A}^{-1} \circ \tilde{Q}$ with

$$\tilde{A}(\varphi, \xi) = (\varphi, R_\varphi \circ A \circ R_{\varphi^{-1}}(\xi))$$

and

$$\tilde{Q}(\varphi, \xi) = (\varphi, R_\varphi \circ Q \circ R_{\varphi^{-1}}(\xi)),$$

where $Q(f) \coloneqq 3(f_x f_{xx} + (Af)f_x)$. We now apply Lemma 3.23 to deduce that

$$\tilde{A}, \tilde{Q} \colon \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}) \to \mathrm{Diff}^n(\mathbb{S}) \times C^{n-2}(\mathbb{S})$$

are smooth. To show that $\tilde{A}^{-1} \colon \mathrm{Diff}^n(\mathbb{S}) \times C^{n-2}(\mathbb{S}) \to \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ is smooth, we compute the derivative $D\tilde{A}$ at an arbitrary point (φ, ξ). We have the following directional derivatives of the components $\tilde{A}_1$ and $\tilde{A}_2$:

$$D_\varphi \tilde{A}_1 = \mathrm{id}, \quad D_\xi \tilde{A}_1 = 0, \quad D_\xi \tilde{A}_2 = R_\varphi \circ A \circ R_{\varphi^{-1}},$$

and it remains to compute $(D_\varphi \tilde{A}_2(\varphi, \xi))(\psi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \tilde{A}_2(\varphi + \varepsilon\psi, \xi)\big|_{\varepsilon=0}$. In a first step, we calculate

$$\begin{aligned} \partial_x^2(\xi \circ (\varphi + \varepsilon\psi)^{-1}) &= \partial_x \left[\left(\frac{\xi_x}{\varphi_x + \varepsilon\psi_x} \right) \circ (\varphi + \varepsilon\psi)^{-1} \right] \\ &= \left(\frac{\xi_{xx}}{(\varphi_x + \varepsilon\psi_x)^2} - \xi_x \frac{\varphi_{xx} + \varepsilon\psi_{xx}}{(\varphi_x + \varepsilon\psi_x)^3} \right) \circ (\varphi + \varepsilon\psi)^{-1}, \end{aligned}$$

from which we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\partial_x^2(\xi \circ (\varphi + \varepsilon\psi)^{-1}) \circ (\varphi + \varepsilon\psi) \right) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\frac{\xi_{xx}}{(\varphi_x + \varepsilon\psi_x)^2} - \xi_x \frac{\varphi_{xx} + \varepsilon\psi_{xx}}{(\varphi_x + \varepsilon\psi_x)^3} \right) \\ &= -2 \frac{\xi_{xx}\psi_x}{(\varphi_x + \varepsilon\psi_x)^3} - \frac{\xi_x \psi_{xx}}{(\varphi_x + \varepsilon\psi_x)^3} \\ &\quad + 3 \frac{\xi_x \psi_x}{(\varphi_x + \varepsilon\psi_x)^4} (\varphi_{xx} + \varepsilon\psi_{xx}) \end{aligned}$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\partial_x^2(\xi \circ (\varphi + \varepsilon\psi)^{-1}) \circ (\varphi + \varepsilon\psi) \right) \Big|_{\varepsilon=0} = -2 \frac{\xi_{xx}\psi_x}{\varphi_x^3} - \frac{\xi_x \psi_{xx}}{\varphi_x^3} + 3 \frac{\varphi_{xx}\xi_x\psi_x}{\varphi_x^4}.$$

Secondly, we observe that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mu(\xi \circ (\varphi + \varepsilon\psi)^{-1}) \Big|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_{\mathbb{S}} \xi(y)(\varphi_x + \varepsilon\psi_x)(y) \, \mathrm{d}y \Big|_{\varepsilon=0} \\ &= \int_{\mathbb{S}} \xi(y)\psi_x(y) \, \mathrm{d}y, \end{aligned}$$

since $\varphi + \varepsilon\psi \in \mathrm{Diff}^n(\mathbb{S})$ for small $\varepsilon > 0$. Hence

$$(D_\varphi \tilde{A}_2(\varphi, \xi))(\psi) = \int_{\mathbb{S}} \xi(y)\psi_x(y) \, \mathrm{d}y + 2 \frac{\xi_{xx}\psi_x}{\varphi_x^3} + \frac{\xi_x \psi_{xx}}{\varphi_x^3} - 3 \frac{\varphi_{xx}\xi_x\psi_x}{\varphi_x^4}$$

and

$$D\tilde{A}(\varphi,\xi) = \begin{pmatrix} \mathrm{id} & 0 \\ D_\varphi \tilde{A}_2(\varphi,\xi) & R_\varphi \circ A \circ R_{\varphi^{-1}} \end{pmatrix}.$$

It is easy to check that $D\tilde{A}(\varphi,\xi)$ is an invertible bounded linear operator $C^n(\mathbb{S})\times C^n(\mathbb{S}) \to C^n(\mathbb{S}) \times C^{n-2}(\mathbb{S})$. By the open mapping theorem, $D\tilde{A}$ is a topological isomorphism and, by the inverse mapping theorem, $\tilde{A}^{-1}$ is smooth. □

Remark 3.25. In fact, Proposition 3.24 shows that the Christoffel map $\Gamma_\varphi(\varphi_t,\varphi_t) = \varphi_{tt}$ for μDP is smooth, cf. Remark 2.3.

Since F is smooth, we can apply the Banach space version of the Picard-Lindelöf Theorem (also known as *Cauchy-Lipschitz Theorem*) as explained in Appendix A. This yields the following theorem about the existence and uniqueness of integral curves for the vector field F.

Theorem 3.26. *Let $n \geq 3$. Then there is an open interval J_n centered at zero and an open ball $B(0,\delta_n) \subset C^n(\mathbb{S})$ such that for any $u_0 \in B(0,\delta_n)$ there exists a unique solution $(\varphi,\xi) \in C^\infty(J_n, \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S}))$ of (3.23) with initial conditions $\varphi(0) = \mathrm{id}$ and $\xi(0) = u_0$. Moreover, the flow (φ,ξ) depends smoothly on (t,u_0).*

From Theorem 3.26 we get a unique short-time solution $u = \xi\circ\varphi^{-1}$ of μDP in $C^n(\mathbb{S})$ with continuous dependence on (t,u_0). Note that, to obtain an analogous result for smooth initial data u_0, we cannot send $n \to \infty$ in Theorem 3.26 since the δ_n or the J_n might converge to zero. On the other hand, since $C^\infty(\mathbb{S})$ is a Fréchet space, classical results like the Picard-Lindelöf Theorem or the local inverse theorem for Banach spaces are no longer valid. In the proof of our main theorem, we will make use of a Banach space approximation of the Fréchet space $C^\infty(\mathbb{S})$. First, we observe that any solution (φ,ξ) of the μDP equation (3.23) does not lose nor gain spatial regularity as t increases or decreases from zero. For this purpose, we are in need of a conservation law.

Lemma 3.27. *Let u be a $C^3(\mathbb{S})$-solution of the μDP equation on $(-T,T)$ and let φ be the corresponding flow. Then*

$$(m\circ\varphi)\varphi_x^3 = m_0,$$

for all $t \in (-T,T)$, where $m = Au = (\mu - \partial_x^2)u$ and $m_0 = Au_0$.

Proof. We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(m\circ\varphi)\varphi_x^3 &= [(m_t + m_x u)\circ\varphi]\,\varphi_x^3 + 3\varphi_x^2\varphi_{tx}(m\circ\varphi) \\ &= [(-3u_x m)\circ\varphi]\varphi_x^3 + 3\varphi_x^3(mu_x)\circ\varphi \\ &= 0.\end{aligned}$$

Since $\varphi(0) = \mathrm{id}$ and $\varphi_x(0) = 1$, we are done. □

Lemma 3.28. *Let $(\varphi,\xi) \in C^\infty(J_3, \mathrm{Diff}^3(\mathbb{S}) \times C^3(\mathbb{S}))$ be a solution of (3.23) with initial data (id, u_0) according to Theorem 3.26. Then, for all $t \in J_3$,*

$$\varphi_{xx}(t) = \varphi_x(t)\left(\int_0^t \mu(u)\varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \frac{1}{\varphi_x(s)^2}\,\mathrm{d}s\right) \tag{3.24}$$

and

$$\xi_{xx}(t) = \xi_x(t)\frac{\varphi_{xx}(t)}{\varphi_x(t)} + \varphi_x(t)\left[\mu(u)\varphi_x(t) - \frac{m_0}{\varphi_x(t)^2}\right]. \tag{3.25}$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = \frac{\varphi_{xxt}\varphi_x - \varphi_{xt}\varphi_{xx}}{\varphi_x^2}.$$

Here

$$\varphi_{xt} = \dot{\varphi}_x = \partial_x(u \circ \varphi) = (u_x \circ \varphi)\varphi_x$$

and

$$\begin{aligned}\varphi_{xxt} &= \dot{\varphi}_{xx} \\ &= \partial_x^2(u \circ \varphi) \\ &= \partial_x[(u_x \circ \varphi)\varphi_x] \\ &= (u_{xx} \circ \varphi)\varphi_x^2 + (u_x \circ \varphi)\varphi_{xx}.\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = (u_{xx} \circ \varphi)\varphi_x.$$

According to the previous lemma, we can replace

$$u_{xx} \circ \varphi = \mu(u) - \frac{m_0}{\varphi_x^3}.$$

Integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = \mu(u)\varphi_x - \frac{m_0}{\varphi_x^2}$$

over $[0,t]$ leads to equation (3.24) and taking the time derivative of (3.24) yields (3.25). □

From Remark 3.4 we know that $\mu(u_t) = 0$ and hence $\mu(u) = \mu(u_0)$ so that $\mu(u)$ can in fact be written in front of the first integral sign in equation (3.24). As in the discussion of the periodic b-equation, we obtain the following corollaries which guarantee that the geodesic flow for the μDP equation does not lose its spatial regularity as t increases or decreases from zero.

Corollary 3.29. *Let (φ, ξ) be as in Lemma 3.28. If $u_0 \in C^n(\mathbb{S})$ then we have $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ for all $t \in J_3$.*

Proof. We proceed by induction on n. For $n = 3$ the corollary follows immediately from our assumption on $(\varphi(t), \xi(t))$. Let us assume that $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ for some $n \geq 3$. Then Lemma 3.28 shows that, if $u_0 \in C^{n+1}(\mathbb{S})$, then $(\varphi(t), \xi(t)) \in \mathrm{Diff}^{n+1}(\mathbb{S}) \times C^{n+1}(\mathbb{S})$, finishing the proof. □

Corollary 3.30. *Let (φ, ξ) be as in Lemma 3.28. If there exists a nonzero $t \in J_3$ such that $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S})$ or $\xi(t) \in C^n(\mathbb{S})$ then $\xi(0) = u_0 \in C^n(\mathbb{S})$.*

Proof. Again, we use a recursive argument. For $n = 3$, there is nothing to do. For some $n \geq 3$, suppose that $u_0 \in C^n(\mathbb{S})$. By the previous corollary, $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$ for all $t \in J_3$. Assume that there is $0 \neq t_0 \in J_3$ such that $\varphi(t_0) \in \mathrm{Diff}^{n+1}(\mathbb{S})$ or $\xi(t_0) \in C^{n+1}(\mathbb{S})$. In the first case, we can resolve (3.24) with respect to m_0 and see that $m_0 \in C^{n-1}(\mathbb{S})$, which of course implies $u_0 \in C^{n+1}(\mathbb{S})$. In the second case, we use Eqs. (3.24) and (3.25) to obtain the identity

$$\xi_{xx}(t_0) = \mu(u_0)\xi_x(t_0)\int_0^{t_0}\varphi_x(s)\,\mathrm{d}s + \mu(u_0)\varphi_x^2(t_0) + m_0\left[-\xi_x(t_0)\int_0^{t_0}\frac{\mathrm{d}s}{\varphi_x(s)^2} - \frac{1}{\varphi_x(t_0)}\right].$$

To resolve this identity with respect to m_0 we have to guarantee that the expression in brackets does not vanish. To see that this is true we claim that there is a nontrivial interval $I \subset \mathbb{R}$ containing zero such that for all $x \in \mathbb{S}$ and any $t \in I$

$$f(t,x) \coloneqq -\xi_x(t,x)\int_0^t \frac{\mathrm{d}s}{\varphi_x(s,x)^2} - \frac{1}{\varphi_x(t,x)} \neq 0.$$

Replacing J_3 by $J_3 \cap I$ this will achieve the proof. Let us show that f converges to -1, uniformly in x, as $t \to 0$ and $t \geq 0$ (w.l.o.g.):

$$|f(t,x)+1| \leq t\,\|\xi_x(t)\|_\infty \max_{0\leq s\leq t}\left\|\frac{1}{\varphi_x(s)}\right\|_\infty^2 + \left|1 - \frac{1}{\varphi_x(t,x)}\right|.$$

Note that φ_x is the unique solution of

$$\begin{cases} v_t &= (u_x \circ \varphi)v, \\ v(0) &= 1, \end{cases}$$

and thus

$$\varphi_x(t,x) = \exp\left(\int_0^t (u_x\circ\varphi)(s,x)\,\mathrm{d}s\right).$$

We now estimate

$$\begin{aligned}\left|1 - \frac{1}{\varphi_x(t,x)}\right| &= \left|\exp\left(-\int_0^t (u_x\circ\varphi)(s,x)\,\mathrm{d}s\right) - 1\right| \\ &= \left|\sum_{k=1}^\infty \frac{(-1)^k}{k!}\left[\int_0^t (u_x\circ\varphi)(s,x)\,\mathrm{d}s\right]^k\right| \\ &\leq \sum_{k=1}^\infty \frac{1}{k!}\left(t\max_{0\leq s\leq t}\|u_x(s)\|_\infty\right)^k \\ &= \exp\left(t\max_{0\leq s\leq t}\|u_x(s)\|_\infty\right) - 1\end{aligned}$$

and obtain

$$\|f(t)+1\|_\infty \leq t\,\|\xi_x(t)\|_\infty \max_{0\leq s\leq t}\left\|\frac{1}{\varphi_x(s)}\right\|_\infty^2 + \exp\left(t\max_{0\leq s\leq t}\|u_x(s)\|_\infty\right) - 1 \to 0$$

as $t \to 0$; recall that $\xi_x(t) \to u_{0x}$, $\varphi_x(t) \to 1$ and $u_x(t) \to u_{0x}$ as $t \to 0$. We conclude that, for any $\varepsilon > 0$, there is an interval $I_\varepsilon \subset \mathbb{R}$ with positive measure and containing zero, such that

$$|f(t,x)+1| < \varepsilon, \quad \forall x \in \mathbb{S},\, \forall t \in I_\varepsilon,$$

and in particular $|f(t,x)| > 1 - \varepsilon$. Finally, we choose $\varepsilon \in (0,1)$ arbitrarily and are done. □

Now we discuss Banach space approximations of Fréchet spaces.

Definition 3.31. Let X be a Fréchet space. A *Banach space approximation* of X is a sequence $\{(X_n, \|\cdot\|_n);\ n \in \mathbb{N}_0\}$ of Banach spaces such that

$$X_0 \supset X_1 \supset X_2 \supset \cdots \supset X, \quad X = \bigcap_{n=0}^{\infty} X_n$$

and $\{\|\cdot\|_n;\ n \in \mathbb{N}_0\}$ is a sequence of norms inducing the topology on X with

$$\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \dots$$

for any $x \in X$.

We have the following result. For a proof, we refer to [41].

Lemma 3.32. *Let X and Y be Fréchet spaces with the Banach space approximations $\{(X_n, \|\cdot\|_n);\ n \in \mathbb{N}_0\}$ and $\{(Y_n, \|\cdot\|_n);\ n \in \mathbb{N}_0\}$. Let $\Phi_0 \colon U_0 \to V_0$ be a smooth map between the open subsets $U_0 \subset X_0$ and $V_0 \subset Y_0$. Let*

$$U := U_0 \cap X \quad \text{and} \quad V := V_0 \cap Y,$$

as well as

$$U_n := U_0 \cap X_n \quad \text{and} \quad V_n := V_0 \cap Y_n,$$

for any $n \geq 0$. Furthermore, we assume that, for each $n \geq 0$, the following properties are satisfied:

1. $\Phi_0(U_n) \subset V_n$,
2. the restriction $\Phi_n := \Phi_0|_{U_n} : U_n \to V_n$ is a smooth map.

Then $\Phi_0(U) \subset V$ and the map $\Phi := \Phi_0|_U : U \to V$ is smooth.

Now we come to our main theorem which we first formulate in the geometric picture.

Theorem 3.33. *There exists an open interval J centered at zero and $\delta > 0$ such that for all $u_0 \in C^\infty(\mathbb{S})$ with $\|u_0\|_{C^3(\mathbb{S})} < \delta$, there exists a unique solution $(\varphi, \xi) \in C^\infty(J, \mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}))$ of (3.23) such that $\varphi(0) = \mathrm{id}$ and $\xi(0) = u_0$. Moreover, the flow (φ, ξ) depends smoothly on $(t, u_0) \in J \times C^\infty(\mathbb{S})$.*

Proof. Theorem 3.26 for $n = 3$ shows that there is an open interval J centered at zero and an open ball $U_3 := B(0, \delta) \subset C^3(\mathbb{S})$ such that for any $u_0 \in U_3$ there exists a unique solution $(\varphi, \xi) \in C^\infty(J, \mathrm{Diff}^3(\mathbb{S}) \times C^3(\mathbb{S}))$ of (3.23) with initial data (id, u_0) and a smooth flow

$$\Phi_3 \colon J \times U_3 \to \mathrm{Diff}^3(\mathbb{S}) \times C^3(\mathbb{S}).$$

Let

$$U_n := U_3 \cap C^n(\mathbb{S}) \quad \text{and} \quad U_\infty := U_3 \cap C^\infty(\mathbb{S}).$$

By Corollary 3.29, we have

$$\Phi_3(J \times U_n) \subset \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$$

for any $n \geq 3$ and the map

$$\Phi_n := \Phi_3|_{J \times U_n} : J \times U_n \to \mathrm{Diff}^n(\mathbb{S}) \times C^n(\mathbb{S})$$

is smooth. Lemma 3.32 yields that

$$\Phi_3(J \times U_\infty) \subset \mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}),$$

proving the short-time existence in the smooth category, and that the map

$$\Phi_\infty := \Phi_3|_{J\times U_\infty} : J \times U_\infty \to \mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$$

is smooth, proving the smooth dependence on time and the initial condition. □

In the smooth category, the map

$$\mathrm{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S}), \quad (\varphi, \xi) \mapsto \xi \circ \varphi^{-1} = u$$

is smooth. Thus we obtain the result stated in Theorem 3.15.

3.3.3 The exponential map for the μDP equation

The basic idea of the proof of Theorem 3.16 is to consider a perturbed problem: Let $(\varphi^\varepsilon, \xi^\varepsilon)$ denote the local expression of an integral curve of (3.23) in $T\mathrm{Diff}^n(\mathbb{S})$ with initial data $(\mathrm{id}, u + \varepsilon w)$, where $u, w \in C^n(\mathbb{S})$. We have

$$(\varphi^\varepsilon, \xi^\varepsilon) \to (\varphi, \xi), \quad \varepsilon \to 0,$$

where (φ, ξ) is the solution with initial values (id, u). Let

$$\psi(t) := \left.\frac{\partial \varphi^\varepsilon(t)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Then

$$\psi(t) = L_n(t, u)w,$$

where $L_n(t,u)$ is a bounded linear operator $C^n(\mathbb{S}) \to C^n(\mathbb{S})$, for any t in the existence interval of (φ, ξ).

In our next lemma we show that, for $u \in C^{n+1}(\mathbb{S})$ and any $t \neq 0$, we have

$$L_n(t, u)w \in C^{n+1}(\mathbb{S}) \quad \Longrightarrow \quad w \in C^{n+1}(\mathbb{S}).$$

In the proof, we proceed in a similar manner as Escher and Kolev in their proof of Lemma 15 in [41].

Lemma 3.34. *Suppose that $u \in C^{n+1}(\mathbb{S})$. Then, for $t \neq 0$,*

$$L_n(t, u)(C^n(\mathbb{S}) \backslash C^{n+1}(\mathbb{S})) \subset C^n(\mathbb{S}) \backslash C^{n+1}(\mathbb{S}).$$

Proof. First, we write down Eq. (3.24) for $\varphi^\varepsilon(t)$,

$$\varphi^\varepsilon_{xx}(t) = \varphi^\varepsilon_x(t)\left[\mu(u + \varepsilon w)\int_0^t \varphi^\varepsilon_x(s)\,\mathrm{d}s - m_0^\varepsilon \int_0^t \frac{1}{\varphi^\varepsilon_x(s)^2}\,\mathrm{d}s\right],$$

and take the derivative with respect to ε,

$$\begin{aligned}\frac{\partial \varphi^\varepsilon_{xx}}{\partial \varepsilon}(t) = \frac{\partial \varphi^\varepsilon_{x}}{\partial \varepsilon}(t)\left[\mu(u+\varepsilon w)\int_0^t \varphi^\varepsilon_x(s)\,\mathrm{d}s - m^\varepsilon_0 \int_0^t \frac{1}{\varphi^\varepsilon_x(s)^2}\,\mathrm{d}s\right] \\ + \varphi^\varepsilon_x(t)\left[\mu(w)\int_0^t \varphi^\varepsilon_x(s)\,\mathrm{d}s + \mu(u+\varepsilon w)\int_0^t \frac{\partial \varphi^\varepsilon_x}{\partial \varepsilon}(s)\,\mathrm{d}s\right] \\ - \varphi^\varepsilon_x(t)\left[\frac{\partial m^\varepsilon_0}{\partial \varepsilon}\int_0^t \frac{1}{\varphi^\varepsilon_x(s)^2}\,\mathrm{d}s + m^\varepsilon_0 \int_0^t \frac{\partial}{\partial \varepsilon}\frac{1}{\varphi^\varepsilon_x(s)^2}\,\mathrm{d}s\right].\end{aligned}$$

Notice that

$$\frac{\partial m^\varepsilon_0}{\partial \varepsilon} = \mu(w) - w_{xx} = Aw$$

and that $m^\varepsilon_0 \to m_0 = Au$ as $\varepsilon \to 0$. Hence

$$\begin{aligned}\psi_{xx}(t) &= \psi_x(t)\left[\mu(u)\int_0^t \varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \frac{1}{\varphi_x(s)^2}\,\mathrm{d}s\right] \\ &\quad + \varphi_x(t)\left[\mu(w)\int_0^t \varphi_x(s)\,\mathrm{d}s + \mu(u)\int_0^t \psi_x(s)\,\mathrm{d}s\right] \\ &\quad - \varphi_x(t)\left[(\mu(w)-w_{xx})\int_0^t \frac{1}{\varphi_x(s)^2}\,\mathrm{d}s - 2m_0\int_0^t \frac{\psi_x(s)}{\varphi_x(s)^3}\,\mathrm{d}s\right] \\ &= a(t)\psi_x(t) + b(t)\int_0^t c(s)\psi_x(s)\,\mathrm{d}s + d(t) + e(t)w_{xx}\end{aligned}$$

with $a(t), b(t), c(t), d(t), e(t) \in C^{n-1}(\mathbb{S})$ and $e(t) \neq 0$ for $t \neq 0$. Finally, if

$$w \in C^n(\mathbb{S})\backslash C^{n+1}(\mathbb{S}),$$

then

$$\psi(t) = L_n(t,u)w \in C^n(\mathbb{S})\backslash C^{n+1}(\mathbb{S}).$$

□

Let us now come to the proof of Theorem 3.16. Since $C^3(\mathbb{S})$ is a Banach space and $\mathrm{Diff}^3(\mathbb{S})$ is a Banach manifold modelled on $C^3(\mathbb{S})$, we know that the μDP exponential map $C^3(\mathbb{S}) \to \mathrm{Diff}^3(\mathbb{S})$ is a smooth diffeomorphism near zero, i.e., there are neighbourhoods U_3 of zero in $C^3(\mathbb{S})$ and V_3 of id in $\mathrm{Diff}^3(\mathbb{S})$ such that

$$\exp_3 := \exp|_{U_3} \colon U_3 \to V_3$$

is a smooth diffeomorphism. For $n \geq 3$, we now define

$$U_n := U_3 \cap C^n(\mathbb{S}) \quad \text{and} \quad V_n = V_3 \cap \mathrm{Diff}^n(\mathbb{S}).$$

Let $\exp_n := \exp_3|_{U_n}$. Since $\exp_n$ is a restriction of $\exp_3$, it is clearly injective. We now use Corollary 3.29 and Corollary 3.30 to deduce that $\exp_n$ is also surjective, more precisely, $\exp_n(U_n) = V_n$. If the geodesic φ with $\varphi(1) = \exp(u)$ starts at id $\in \mathrm{Diff}^n(\mathbb{S})$ with velocity vector u belonging to $C^n(\mathbb{S})$, then $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S})$ for any t and hence $\exp_n(U_n) \subset V_n$. Conversely, if $v \in V_n$ is given, then there is $u \in U_3$ with $\exp_3(u) = v$. Corollary 3.30 immediately implies that $u \in C^n(\mathbb{S})$; hence $u \in U_n$ and $\exp_n(u) = v$. We conclude that $\exp_n$ is a bijection from U_n to V_n. Furthermore, $\exp_n$ is a smooth map and diffeomorphic as a map $U_n \to V_n$ (since it is a restriction of $\exp_3$). We now show that $\exp_n$ is a smooth diffeomorphism; precisely we show that $\exp_n^{-1} \colon V_n \to U_n$ is smooth by virtue of the inverse mapping theorem. For each $u \in C^n(\mathbb{S})$, $D\exp_n(u)$ is a bounded linear operator $C^n(\mathbb{S}) \to C^n(\mathbb{S})$. Notice that

$$D\exp_n(u) = D\exp_3(u)|_{C^n(\mathbb{S})},$$

from which we conclude that $D\exp_n(u)$ is injective. Let us prove the surjectivity of $D\exp_n(u)$, $n \geq 3$, by induction. For $n = 3$, this follows from the fact that $\exp_3 \colon U_3 \to V_3$ is diffeomorphic and hence a submersion. Assume that $D\exp_n(u)$ is surjective for some $n \geq 3$ and that $u \in C^{n+1}(\mathbb{S})$. We have to show that this implies the surjectivity of $D\exp_{n+1}(u)$. But this is a direct consequence of $D\exp_n(u) = L_n(1,u)$ and the previous lemma: Let $f \in C^{n+1}(\mathbb{S})$. We have to find $g \in C^{n+1}(\mathbb{S})$ with the property $D\exp_{n+1}(u)g = f$. By our assumption, there is $g \in C^n(\mathbb{S})$ such that $D\exp_n(u)g = f$. It remains to check that $g \in C^{n+1}(\mathbb{S})$. To see this, we assume $g \in C^n(\mathbb{S})\backslash C^{n+1}(\mathbb{S})$. But then $f = L_n(1,u)g \notin C^{n+1}(\mathbb{S})$ in contradiction to the choice of f. Thus $g \in C^{n+1}(\mathbb{S})$ and $D\exp_{n+1}(u)g = f$. Now we can apply the open mapping theorem to deduce that for any $n \geq 3$ and any $u \in C^n(\mathbb{S})$ the map

$$D\exp_n(u) \colon C^n(\mathbb{S}) \to C^n(\mathbb{S})$$

is a topological isomorphism. By the inverse function theorem, $\exp_n \colon U_n \to V_n$ is a smooth diffeomorphism. If we define

$$U_\infty \coloneqq U_3 \cap C^\infty(\mathbb{S}) \quad \text{and} \quad V_\infty \coloneqq V_3 \cap \mathrm{Diff}^\infty(\mathbb{S}),$$

Lemma 3.32 yields that

$$\exp_\infty \coloneqq \exp_3|_{U_\infty} \colon U_\infty \to V_\infty$$

as well as

$$\exp_\infty^{-1} \colon V_\infty \to U_\infty$$

are smooth maps. Thus $\exp_\infty$ is a smooth diffeomorphism between U_∞ and V_∞.

3.4 The μDP equation with weak dissipation

In general, it is difficult to avoid energy dissipation mechanisms in real experiments with water waves. On account of that, Ott and Sudan [114] investigated how the KdV equation has to be modified to include the effect of energy dissipation. Ghidaglia [53] studied the long-time behavior of solutions of the weakly dissipative KdV equation as a finite-dimensional dynamical system. Some results for a weakly dissipative CH equation are proved in [124] and recently, [46, 123] discussed blow-up and global existence for a weakly dissipative DP equation.

The goal of the present section is to study the Cauchy problem for the periodic weakly dissipative μDP equation

$$\begin{cases} m_t + u m_x + 3u_x m + \lambda m = 0, \\ m = \mu(u) - u_{xx}, \\ u(0,x) = u_0(x), \end{cases} \tag{3.26}$$

which has not been discussed up to now. Again, the function $u(t,x)$ is depending on time $t \geq 0$ and a space variable $x \in \mathbb{S}$ and μ is the projection $\mu(u) = \int_0^1 u(t,x)\,dx$. The constant λ is assumed to be positive and the term $\lambda(\mu(u) - u_{xx})$ models energy dissipation. By the replacement $\mu(u) \to u$ in (3.26), we obtain the weakly dissipative DP equation discussed in [46, 123]. Note that the quantity $E_1(u) = \int_{\mathbb{S}} m \, \mathrm{d}x$ is conserved for the DP equation and that E_1 can be interpreted as an energy, since it equals (up to a factor) a Hamiltonian function for the DP as explained in [30, 31]. However, for the weakly dissipative DP equation, $\frac{\mathrm{d}}{\mathrm{d}t}E_1(u) = -\lambda\mu(u)$, so that $\mu(u_0) > 0$ implies that the quantity E_1 decreases as t increases. The weak dissipation also breaks other conservation laws of the DP equation like $E_2(u) = \int_{\mathbb{S}} mv \, \mathrm{d}x$ or $E_3(u) = \int_{\mathbb{S}} u^3 \, \mathrm{d}x$, where $v = (4 - \partial_x^2)^{-1}u$, cf. [46].

The general framework in which we discuss Eq. (3.26) is based on a geometric technique since we will regard Eq. (3.26) as an evolution equation on the group $H^s\mathrm{Diff}(\mathbb{S})$ of orientation-preserving diffeomorphisms of the circle $\mathbb{S}$ of class H^s, for $s > 3/2$, cf. Sect. 3.1.4: The vector field $u(t, \cdot) \in H^s(\mathbb{S})$ has a unique local flow $\varphi(t, \cdot) \in H^s\mathrm{Diff}(\mathbb{S})$ such that $\varphi_t \circ \varphi^{-1} = u$, $\varphi(0) = \mathrm{id}$ and $\varphi_{tt} = -F(\varphi, \varphi_t)$ with some map F defined on $H^s\mathrm{Diff}(\mathbb{S}) \times H^s(\mathbb{S})$. The latter equation can be handled with standard ODE methods for Banach spaces. Altogether, it will turn out that the weakly dissipative μDP equation behaves quite similarly to the μDP equation (for which $\lambda = 0$) or the weakly dissipative DP equation. That we work with the Sobolev classes H^s has to do with the spadework in [99].

This section is organized as follows: We first prove local well-posedness for the initial value problem (3.26) with $u_0 \in H^s(\mathbb{S})$ for $s > 3/2$. Secondly, we establish the precise blow-up scenario for $s = 3$. Then we give an example that for smooth initial data with zero-mean, the solution $u(t, \cdot)$ of (3.26) can blow-up in finite time. If $\mu(u_0) \neq 0$ and $\mu(u_0) - (u_0)_{xx}$ is non-negative or non-positive, the corresponding solution $u(t, \cdot)$ will exist globally in time.

In this section, we write $A = \mu - \partial_x^2$ for the inertia operator and $Au = m$, $Au_0 = m_0$. We begin with some preliminary remarks about the manifold configuration space $H^s\mathrm{Diff}(\mathbb{S})$. Recall that $T_\varphi H^s\mathrm{Diff}(\mathbb{S}) \simeq H^s(\mathbb{S})$ for any $\varphi \in H^s\mathrm{Diff}(\mathbb{S})$. Our first lemma establishes that $H^s\mathrm{Diff}(\mathbb{S})$ is a topological group for any $s > 3/2$. The reader can find a proof in [109].

Lemma 3.35. *For $s > 3/2$, the composition map $\varphi \mapsto \omega \circ \varphi$ with an H^s-function ω and the inversion map $\varphi \mapsto \varphi^{-1}$ are continuous maps $H^s\mathrm{Diff}(\mathbb{S}) \to H^s(\mathbb{S})$ and $H^s\mathrm{Diff}(\mathbb{S}) \to H^s\mathrm{Diff}(\mathbb{S})$ respectively and*

$$\|\omega \circ \varphi\|_{H^s} \leq C(1 + \|\varphi\|_{H^s}^s) \, \|\omega\|_{H^s} \, ;$$

C only depending on $\sup_{x \in \mathbb{S}} |\varphi_x(x)|$ and $\inf_{x \in \mathbb{S}} |\varphi_x(x)|$.

Before we proceed, we prepare the following lemma ensuring the existence and uniqueness of a local flow for the weakly dissipative μDP on the Hilbert manifold $H^s\mathrm{Diff}(\mathbb{S})$.

Lemma 3.36. *Let $u(t, x)$ be a time-dependent H^s-vector field on $\mathbb{S}$ for $s > 3/2$. Then the problem*

$$\begin{cases} \varphi_t(t, x) = u(t, \varphi(t, x)), \\ \varphi(0, x) = x, \end{cases}$$

has a unique solution $\varphi \in C^1([0, T_{\max}), H^s\mathrm{Diff}(\mathbb{S}))$ and $T_{\max} > 0$ is maximal.

Local flows on diffeomorphism groups of Sobolev class have approved to be powerful tools for the analysis of model equations for water waves, see, e.g., [94].

In many texts, local well-posedness results for Cauchy problems similar to (3.26) are obtained by applying Kato's theory for abstract quasi-linear evolution equations, cf. Appendix B. We now present a method of proof which is based on a geometric argument, most importantly using local flows as introduced in Lemma 3.36. A technical disadvantage of this method is that it does not yield a priori a maximal existence time for our solution which we will obtain inductively. The key idea is to rewrite the weakly dissipative μDP equation in the form

$$u_t + uu_x + 3\mu(u)\partial_x A^{-1}u + \lambda u = 0. \tag{3.27}$$

Equation (3.27) is suitable for a reformulation of (3.26) in the geometric picture, i.e., in terms of a local flow on the group $H^s\mathrm{Diff}(\mathbb{S})$. To improve the structure of the subsequent well-posedness proof, we begin with the following lemma.

Lemma 3.37. *Let R_φ denote the right translation map on $H^s\mathrm{Diff}(\mathbb{S})$ and let $A_\varphi^{-1} = R_\varphi \circ A^{-1} \circ R_{\varphi^{-1}}$ and $\partial_{x,\varphi} = R_\varphi \circ \partial_x \circ R_{\varphi^{-1}}$. Then*

$$3\mu(\xi\circ\varphi^{-1})\left(A^{-1}\partial_x\left(\xi\circ\varphi^{-1}\right)\right)\circ\varphi = A_\varphi^{-1}\partial_{x,\varphi}h(\varphi,\xi) \tag{3.28}$$

with $h(\varphi,\xi) = 3\xi\int_{\mathbb{S}}\xi\circ\varphi^{-1}\,\mathrm{d}x$. Furthermore, we have the identities

$$\partial_\varphi A_\varphi^{-1}(v) = -A_\varphi^{-1}\left[(v\circ\varphi^{-1})\partial_x, A\right]_\varphi A_\varphi^{-1}, \tag{3.29}$$

$$\partial_\varphi \partial_{x,\varphi}(v) = \left[(v\circ\varphi^{-1})\partial_x, \partial_x\right]_\varphi, \tag{3.30}$$

$$\partial_\varphi h(\varphi,\xi)(v) = 3\xi\int_{\mathbb{S}}\xi\circ\varphi^{-1}\partial_x(v\circ\varphi^{-1})\,\mathrm{d}x. \tag{3.31}$$

Proof. Equation (3.28) follows directly from our definitions. We have

$$\partial_\varphi(f\circ\varphi^{-1})(v) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} f\circ(\varphi+\varepsilon v)^{-1} = (f_x\circ\varphi^{-1})\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}(\varphi+\varepsilon v)^{-1}$$

and differentiating $(\varphi+\varepsilon v)\circ(\varphi+\varepsilon v)^{-1} = \mathrm{id}$ with respect to ε at $\varepsilon = 0$, we obtain that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}(\varphi+\varepsilon v)^{-1} = -\frac{v\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}}$$

so that $\partial_\varphi(f\circ\varphi^{-1})(v) = -(f\circ\varphi^{-1})_x(v\circ\varphi^{-1})$. Using this, we directly get

$$\begin{aligned}
[\partial_\varphi(A_\varphi^{-1}w)](v) &= \left(A^{-1}\left(\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} w\circ(\varphi+\varepsilon v)^{-1}\right)\right)\circ\varphi + \left((A^{-1}(w\circ\varphi^{-1})_x)\circ\varphi\right)v \\
&= -A^{-1}\left((w\circ\varphi^{-1})_x(v\circ\varphi^{-1})\right)\circ\varphi + \left((v\circ\varphi^{-1})A^{-1}(w\circ\varphi^{-1})_x\right)\circ\varphi \\
&= -A_\varphi^{-1}\left[(v\circ\varphi^{-1})\partial_x, A\right]_\varphi A_\varphi^{-1}w,
\end{aligned}$$

$$\begin{aligned}
[\partial_\varphi(\partial_{x,\varphi}w)](v) &= \left(\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{t=0} w\circ(\varphi+\varepsilon v)^{-1}\right)_x\circ\varphi + \left((w\circ\varphi^{-1})_{xx}\circ\varphi\right)v \\
&= -\left((w\circ\varphi^{-1})_x(v\circ\varphi^{-1})\right)_x\circ\varphi + \left((v\circ\varphi^{-1})(w\circ\varphi^{-1})_{xx}\right)\circ\varphi \\
&= \left[(v\circ\varphi^{-1})\partial_x, \partial_x\right]_\varphi w
\end{aligned}$$

and (3.31) after integration by parts. □

Let us now come to our local well-posedness result. Observe that we establish the existence of a solution of Eq. (3.26) on an interval $[0,T)$ for physical reasons, in contrast to our discussion of the μDP in Sect. 3.3 where we also allowed negative values of t.

Theorem 3.38. *Let $s>3/2$ and $u_0 \in H^s(\mathbb{S})$. Then there is a maximal time $T=T(u_0)\in(0,\infty]$ and a unique solution*

$$u \in C([0,T);H^s(\mathbb{S})) \cap C^1([0,T);H^{s-1}(\mathbb{S}))$$

of the Cauchy problem (3.26) *which depends continuously on the initial data u_0, i.e., the mapping*

$$H^s(\mathbb{S}) \to C([0,T);H^s(\mathbb{S})) \cap C^1([0,T);H^{s-1}(\mathbb{S})), \quad u_0 \mapsto u(\cdot,u_0)$$

is continuous.

Proof. Writing

$$Tu = 3\mu(u)\partial_x A^{-1}u + \lambda u = T_0 u + \lambda u,$$

(3.27) shows that (3.26) is equivalent to $u_t + uu_x = -Tu$. Let $\varphi(t) \in H^s\mathrm{Diff}(\mathbb{S})$ denote the local flow for the vector field $u(t,\cdot)$ according to Lemma 3.36, i.e., $\varphi_t = u\circ\varphi$ and $\varphi(0)=\mathrm{id}$ on $[0,T_{\max})$. We then have

$$\varphi_{tt} = (u_t + uu_x)\circ\varphi = -T(\varphi_t\circ\varphi^{-1})\circ\varphi.$$

Let $F(\varphi,\xi) \coloneqq R_\varphi \circ T \circ R_{\varphi^{-1}}\xi$ so that

$$\varphi_{tt} = -F(\varphi,\varphi_t), \quad \varphi_t(0)=u_0, \quad \varphi(0)=\mathrm{id}, \tag{3.32}$$

which is an ordinary second order equation. By Lemma 3.35, $F\colon H^s\mathrm{Diff}(\mathbb{S})\times H^s(\mathbb{S}) \to H^s(\mathbb{S})$. We next show that F is continuously differentiable in a neighborhood of any $(\varphi,\xi)\in TH^s\mathrm{Diff}(\mathbb{S}) \simeq H^s\mathrm{Diff}(\mathbb{S})\times H^s(\mathbb{S})$ and therefore decompose $F=F_1+F_2$ with $F_1 = R_\varphi\circ T_0\circ R_{\varphi^{-1}}$ and F_2 just being multiplication with λ. By (3.28), $F_1(\varphi,\xi) = A_\varphi^{-1}\partial_{x,\varphi}h(\varphi,\xi)$, and using (3.29)–(3.31) and the relation $\partial_x^2A^{-1} = A^{-1}\partial_x^2 = \mu - 1$, cf. Sect. 3.1.2, we get

$$\begin{aligned}
\partial_\varphi F_1(\varphi,\xi)v &= [(\partial_\varphi A_\varphi^{-1})(\partial_{x,\varphi}h(\varphi,\xi))]v + A_\varphi^{-1}[(\partial_\varphi\partial_{x,\varphi})h(\varphi,\xi)]v + A_\varphi^{-1}\partial_{x,\varphi}[\partial_\varphi h(\varphi,\xi)]v\\
&= -3\mu(\xi\circ\varphi^{-1})\left(A^{-1}(v\circ\varphi^{-1})\partial_x^2(\xi\circ\varphi^{-1})\right)\circ\varphi\\
&\quad + 3\mu(\xi\circ\varphi^{-1})\left((v\circ\varphi^{-1})\partial_x^2A^{-1}(\xi\circ\varphi^{-1})\right)\circ\varphi\\
&\quad + 3\mu(\xi\circ\varphi^{-1})\left(A^{-1}(v\circ\varphi^{-1})\partial_x^2(\xi\circ\varphi^{-1})\right)\circ\varphi\\
&\quad - 3\mu(\xi\circ\varphi^{-1})\left(A^{-1}\partial_x((v\circ\varphi^{-1})\partial_x(\xi\circ\varphi^{-1}))\right)\circ\varphi\\
&\quad + 3\left(\int_{\mathbb{S}}(\xi\circ\varphi^{-1})\partial_x(v\circ\varphi^{-1})\,\mathrm{d}x\right)\left(A^{-1}\partial_x(\xi\circ\varphi^{-1})\right)\circ\varphi\\
&= 3\Bigg(-v\xi\int_{\mathbb{S}}\xi\circ\varphi^{-1}\,\mathrm{d}x + v\left(\int_{\mathbb{S}}\xi\circ\varphi^{-1}\,\mathrm{d}x\right)^2\\
&\quad - A_\varphi^{-1}\partial_{x,\varphi}(v\partial_{x,\varphi}\xi)\int_{\mathbb{S}}\xi\circ\varphi^{-1}\,\mathrm{d}x + A_\varphi^{-1}\partial_{x,\varphi}\xi\int_{\mathbb{S}}\xi\circ\varphi^{-1}\partial_x(v\circ\varphi^{-1})\,\mathrm{d}x\Bigg)\\
&= 3\Big(-\mu(\xi\varphi_x)v\xi + \mu(\xi\varphi_x)^2v - \mu(\xi\varphi_x)A_\varphi^{-1}\partial_{x,\varphi}(v\partial_{x,\varphi}\xi) + \mu(\xi v_x)A_\varphi^{-1}\partial_{x,\varphi}\xi\Big)
\end{aligned}$$

and

$$\begin{aligned}\partial_\xi F_1(\varphi,\xi)v &= 3A_\varphi^{-1}\partial_{x,\varphi}v\int_{\mathbb{S}}\xi\circ\varphi^{-1}\,\mathrm{d}x + 3A_\varphi^{-1}\partial_{x,\varphi}\xi\int_{\mathbb{S}}v\circ\varphi^{-1}\,\mathrm{d}x\\ &= 3\mu(\xi\varphi_x)A_\varphi^{-1}\partial_{x,\varphi}v + 3\mu(v\varphi_x)A_\varphi^{-1}\partial_{x,\varphi}\xi,\end{aligned}$$

for arbitrary $(\varphi,\xi)\in TH^s\mathrm{Diff}(\mathbb{S})$. Since we have the relation

$$\partial_x A^{-1}w = \left(x-\frac{1}{2}\right)\int_0^1 w(x)\,\mathrm{d}x - \int_0^x w(y)\,\mathrm{d}y + \int_0^1\int_0^x w(y)\,\mathrm{d}y\,\mathrm{d}x$$

we get

$$\begin{aligned}A_\varphi^{-1}\partial_{x,\varphi}(v\partial_{x,\varphi}\xi) &= \Bigg[\left(x-\frac{1}{2}\right)\mu\left((v\circ\varphi^{-1})(\xi\circ\varphi^{-1})_x\right) - \int_0^x (v\circ\varphi^{-1})(\xi\circ\varphi^{-1})_x\,\mathrm{d}y\\ &\quad + \int_0^1\int_0^x (v\circ\varphi^{-1})(\xi\circ\varphi^{-1})_x\,\mathrm{d}y\,\mathrm{d}x\Bigg]\circ\varphi\\ &= \left(\varphi(x)-\frac{1}{2}\right)\mu(v\xi_x) - \int_{\varphi^{-1}(0)}^x v\xi_x\,\mathrm{d}y + \int_0^1\int_{\varphi^{-1}(0)}^{\varphi^{-1}(x)} v\xi_x\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}A_\varphi^{-1}\partial_{x,\varphi}\xi &= \left[\left(x-\frac{1}{2}\right)\mu(\xi\circ\varphi^{-1}) - \int_0^x \xi\circ\varphi^{-1}\,\mathrm{d}y + \int_0^1\int_0^x \xi\circ\varphi^{-1}\,\mathrm{d}y\,\mathrm{d}x\right]\circ\varphi\\ &= \left(\varphi(x)-\frac{1}{2}\right)\mu(\xi\varphi_x) - \int_{\varphi^{-1}(0)}^x \xi\varphi_x\,\mathrm{d}y + \int_0^1\int_{\varphi^{-1}(0)}^{\varphi^{-1}(x)} \xi\varphi_x\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

To prove that $v\mapsto\partial_\varphi F_1(\varphi,\xi)v$ and $v\mapsto\partial_\xi F_1(\varphi,\xi)v$ are bounded operators on $H^s(\mathbb{S})$, it suffices to estimate the sum

$$\|\partial_\varphi F_1(\varphi,\xi)v\|_{L_2} + \|\partial_x(\partial_\varphi F_1(\varphi,\xi)v)\|_{H^{s-1}}$$

by a constant independent of v times $\|v\|_{H^s}$, and similarly for the other partial derivative. Therefore, we differentiate the equations for $\partial_\varphi F_1(\varphi,\xi)v$ and $\partial_\xi F_1(\varphi,\xi)v$ with respect to x and use Sobolev's embedding theorem, the fact that H^s is an algebra and some standard estimates to obtain

$$\|\partial_\varphi F_1(\varphi,\xi)v\|_{H^s} \le C\,\|v\|_{H^s}\,\|\xi\|_{H^s}^2$$

and

$$\|\partial_\xi F_1(\varphi,\xi)v\|_{H^s} \le C\,\|v\|_{H^s}\,\|\xi\|_{H^s}\,,$$

with C only depending on φ. Now it remains to establish the continuity of both partials in φ and ξ. Continuity in the ξ-variable follows from the fact that the dependence of $\partial_\varphi F_1$ and $\partial_\xi F_1$ on ξ is polynomial. To see that $\partial_\varphi F_1(\varphi,\xi)$ depends continuously on φ, we perform a tedious but straightforward computation of a bound for the sum

$$\|\partial_\varphi F_1(\varphi,\xi)v - \partial_\varphi F_1(\tilde\varphi,\xi)v\|_{L_2} + \|\partial_x[\partial_\varphi F_1(\varphi,\xi)v - \partial_\varphi F_1(\tilde\varphi,\xi)v]\|_{H^{s-1}}$$

which tends to zero as $\varphi\to\tilde\varphi$ in H^s; therefor, we use again the algebra property of H^s, Sobolev's embedding theorem and Lemma 3.35. That $\varphi\mapsto\partial_\xi F_1(\varphi,\xi)v$ is continuous can be proved very similarly. Clearly, $F_2(\varphi,\xi)=\lambda\xi$ is differentiable; the directional

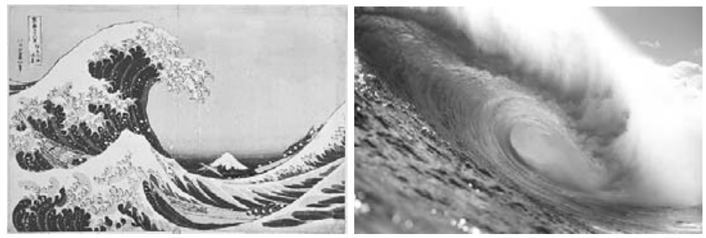


Fig. 3.3 Wave breaking. (Hokusai, "The breaking wave of Kanagawa" and snapshot of a wave crest, `www.heartsandminds.org/global/actnow.htm`, cited 15 August 2010)

derivatives $\partial_\varphi F_2 = 0$ and $\partial_\xi F_2 = \lambda$ are bounded linear operators on $H^s(\mathbb{S})$ with continuous dependence on (φ, ξ). Since F is continuously differentiable near $(\mathrm{id}, 0)$, our local existence theorem for Banach spaces (cf. Appendix A) establishes the well-posedness of (3.32), i.e., there is a time $0 < T_1 < T_{\max}$ and a unique solution (φ, φ_t) of (3.32) on $[0, T_1]$ with continuous dependence on t and u_0. To show that there is a maximal interval of existence, we apply the local existence theorem once more to the problem $\varphi_{tt} = -F(\varphi, \varphi_t)$ with initial data $(\varphi(T_1), \varphi_t(T_1))$ to continue the solution (φ, φ_t) to a solution on a time interval $[0, T_2]$ with $T_1 < T_2 < T_{\max}$. Iterating this procedure, the local well-posedness of (3.26) is now a simple consequence of $u = \varphi_t \circ \varphi^{-1}$ and the fact that $H^s\mathrm{Diff}(\mathbb{S})$ is a topological group whenever $s > 3/2$. □

Remark 3.39. As explained in the proof of Theorem 3.38, we obtain a strictly increasing sequence $(T_n)_{n\in\mathbb{N}}$ describing the continuation of our solution $u(t, x)$ in $H^s(\mathbb{S})$. Note that $T_n \to T$ with either $T < \infty$ or $T = \infty$. In the first case, we say that the solution has a *finite* existence time, whereas $T = \infty$ means that the solution exists *globally* in time. It is an interesting problem and the aim of the following considerations to describe the behavior of finite time solutions as $t \to T$ from below and to find criteria for the global existence of strong solutions as well as so called *finite time blow-up*.

In physics, a breaking wave is a wave whose amplitude reaches a critical level at which some process suddenly starts to occur that causes large amounts of wave energy to be transformed in turbulent kinetic energy. At this point, simple physical models describing the dynamics of the wave will often become invalid, particularly those which assume linear behavior. Wave breaking has been studied for various classes of non-linear 1D wave equations, [21, 22, 38, 101, 102, 130], and a reasonable way is to show that there is a finite-time solution u satisfying an L_∞-bound for all $t \in [0, T)$ so that the norm of u is unbounded as $t \to T$ if and only if the first order derivative u_x approaches $-\infty$ as $t \to T$ from below (cf., e.g., [46] for a discussion of the DP equation with a dissipative term). The physical interpretation of this is that the wave steepens, while the height of its crests stays bounded, until wave breaking occurs in the sense that u ceases to be a classical solution (see Figs. 3.3 and 3.4).

Here, we first describe the blow-up of finite time-solutions of (3.26) in terms of the first order derivative and then discuss examples in which blow-up occurs or where one gets global solutions, respectively. Henceforth, we will restrict ourselves to $s = 3$.

Theorem 3.40. *Given $u_0 \in H^3(\mathbb{S})$, the solution u of* (3.26) *obtained in Theorem 3.38 blows up in finite time $T > 0$ if and only if*

$$\liminf_{t\to T} \min_{x\in\mathbb{S}} u_x(t, x) = -\infty.$$

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (3.26) with initial data u_0. Since $H^3(\mathbb{S}) \subset C^2(\mathbb{S})$ we find that

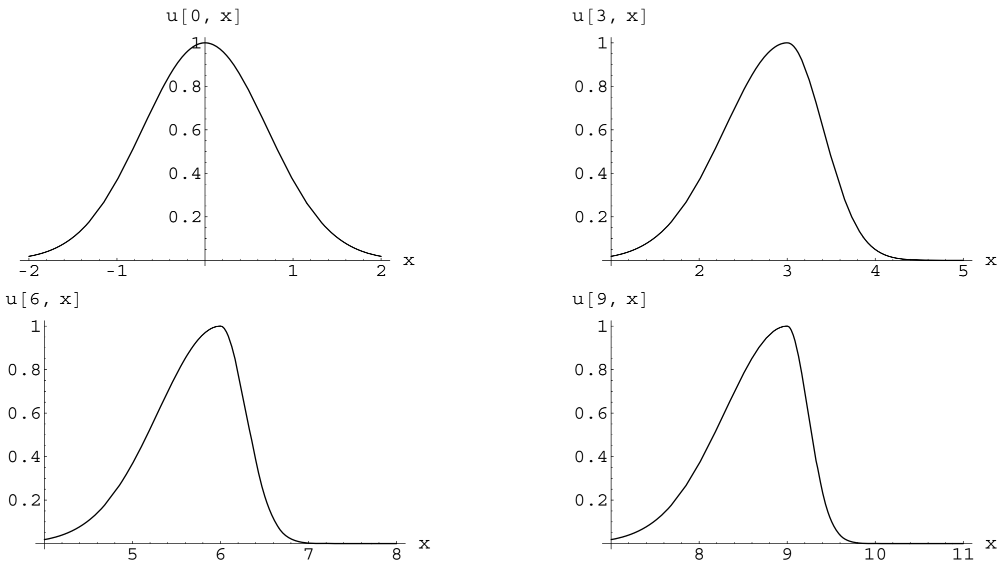


Fig. 3.4 A breaking wave profile $u(t,x)$. The wave propagates in the positive x direction with constant speed 1 and steepens while its height does not change. We say that the wave breaks at $t = T < \infty$ if its slope u_x becomes unbounded from below as $t \to T$ and hence u ceases to be a classical solution of the governing wave equation for $t \geq T$. We also say that the solution u blows up in finite time T.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} m^2\,\mathrm{d}x &= 2\int_{\mathbb{S}} m m_t\,\mathrm{d}x\\ &= -2\int_{\mathbb{S}} u m_x m\,\mathrm{d}x - 6\int_{\mathbb{S}} u_x m^2\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} m^2\,\mathrm{d}x\\ &= -5\int_{\mathbb{S}} u_x m^2\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} m^2\,\mathrm{d}x. \end{aligned} \tag{3.33}$$

If we assume $u_0 \in H^4(\mathbb{S})$ and use that $H^4(\mathbb{S}) \subset C^3(\mathbb{S})$, we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} m_x^2\,\mathrm{d}x &= 2\int_{\mathbb{S}} m_x m_{tx}\,\mathrm{d}x\\ &= -2\int_{\mathbb{S}} m_x m_{xx} u\,\mathrm{d}x - 8\int_{\mathbb{S}} m_x^2 u_x\,\mathrm{d}x - 6\int_{\mathbb{S}} m m_x u_{xx}\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} m_x^2\,\mathrm{d}x\\ &= -7\int_{\mathbb{S}} m_x^2 u_x\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} m_x^2\,\mathrm{d}x. \end{aligned} \tag{3.34}$$

Adding (3.33) and (3.34) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m\|_{H^1}^2 = -7\int_{\mathbb{S}} m_x^2 u_x\,\mathrm{d}x - 5\int_{\mathbb{S}} u_x m^2\,\mathrm{d}x - 2\lambda\|m\|_{H^1}^2\,. \tag{3.35}$$

Next we observe that (3.35) also holds true for $u_0 \in H^3(\mathbb{S})$: We approximate u_0 in $H^3(\mathbb{S})$ by functions $u_0^n \in H^4(\mathbb{S})$, $n \geq 1$. Let $u^n = u^n(\cdot, u_0^n)$ be the solution of (3.26) with initial data u_0^n. By Theorem 3.38 we know that

$$u^n \in C([0,T_n); H^4(\mathbb{S})) \cap C^1([0,T_n); H^3(\mathbb{S})), \quad n \geq 1,$$

$$m^n = \mu(u^n) - u^n_{xx} \in C([0,T_n); H^2(\mathbb{S})) \cap C^1([0,T_n); H^1(\mathbb{S})), \quad n \geq 1,$$

$u^n \to u$ in $H^3(\mathbb{S})$ and $T_n \to T$ as $n \to \infty$. Since $u^n_0 \in H^4(\mathbb{S})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (m^n_x)^2\,\mathrm{d}x = -7\int_{\mathbb{S}} (m^n_x)^2 u^n_x\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} (m^n_x)^2\,\mathrm{d}x.$$

Since $u_n \to u$ in $H^3(\mathbb{S})$ it follows that $u^n_x \to u_x$ in $L_\infty(\mathbb{S})$ as $n \to \infty$. Note also that $m^n \to m$ in $H^1(\mathbb{S})$ and $m^n_x \to m_x$ in $L_2(\mathbb{S})$ as $n \to \infty$. We deduce that, as $n \to \infty$, (3.34) also holds for $u_0 \in H^3(\mathbb{S})$. If u_x is bounded from below on $[0,T)$, i.e., $u_x \geq -c$, where c is a positive constant, then we can apply Gronwall's inequality to Eq. (3.35) and have

$$\|m\|^2_{H^1} \leq \|m_0\|^2_{H^1}\exp((7c - 2\lambda)t).$$

This shows that $\|u\|_{H^3}$ does not blow up in finite time. The converse direction follows from Sobolev's embedding theorem. This completes the proof of our assertion. □

Remark 3.41. The previous theorem shows that if u_x stays bounded from below, then u also persists in H^3. Thus Theorem 3.40 provides us with a sufficient criterion for global existence: The boundedness of u_x from below implies $T = \infty$ in Theorem 3.38.

We already know that the mean $\mu(u)$ of a solution $u(t,\cdot)$ of the μDP equation is conserved, i.e., $\mu(u_0) = \mu(u)$, cf. Remark 3.4. We now show that the mean $\mu(u)$ of a solution of the weakly dissipative μDP equation decreases exponentially as t increases from zero. More precisely, we prove that the damping constant is equal to the dissipation parameter λ.

Lemma 3.42. *Let $u_0 \in H^3(\mathbb{S})$ and let $u(t,x)$ be the solution of* (3.26) *obtained in Theorem 3.38. Then the mean of u satisfies*

$$\mu(u) = \mu(u_0)e^{-\lambda t}$$

for $t \geq 0$ in the existence interval of u. In particular, if $\mu(u_0) = 0$, then the mean of the solution u is conserved.

Proof. We apply μ to (3.27) and change the order of time derivative and integration to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu(u) &= \mu\left(-uu_x - 3\mu(u)\partial_x A^{-1}u - \lambda u\right)\\ &= -\mu\left(\frac{1}{2}\partial_x\left(u^2\right)\right) - 3\mu(u)\mu\left(\partial_x A^{-1}u\right) - \lambda\mu(u),\end{aligned}$$

as long as the solution $u(t,\cdot) \in H^3(\mathbb{S})$ exists. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(u) = -\lambda\mu(u)$$

from which the lemma follows. □

With the help of Lemma 3.42, we are able to establish the following blow-up scenario. It is important to notice that our result shows the blow-up of smooth initial data. A similar blow-up setting for the μDP equation is discussed in [99].

Theorem 3.43. *Assume that $0 \neq u_0 \in C^\infty(\mathbb{S})$ has zero mean and that there is $y \in \mathbb{S}$ satisfying*

$$0 < 1 + \frac{\lambda}{u_{0x}(y)} < 1. \tag{3.36}$$

Let u be the corresponding solution of (3.26). Then there is $0 < \tau < \infty$ such that $\|u_x(t)\|_\infty$ blows up as $t \to \tau$ from below. In particular, the solution u blows up in the H^3-norm in finite time.

Proof. Differentiating Eq. (3.27) with respect to x and the identity $\partial_x^2 A^{-1} = \mu - 1$ yield

$$u_{tx} + uu_{xx} + u_x^2 + \lambda u_x = 3\mu(u)(u - \mu(u)).$$

By Lemma 3.42, it follows that the right-hand side equals zero. Again, we denote by φ the local flow of the time-dependent vector field $u(t,\cdot)$, i.e., $\varphi_t = u \circ \varphi$. We set

$$w := \frac{\varphi_{tx}}{\varphi_x} = u_x \circ \varphi$$

and with

$$\varphi_{ttx} = [(u_{tx} + uu_{xx} + u_x^2) \circ \varphi]\varphi_x$$

we obtain

$$w_t = \frac{\varphi_{ttx}\varphi_x - (\varphi_{tx})^2}{\varphi_x^2} = (u_{tx} + uu_{xx}) \circ \varphi$$

and hence

$$w_t + w^2 + \lambda w = 0.$$

With $\Lambda := -\lambda < 0$, we finally arrive at the logistic equation

$$w_t = w(\Lambda - w)$$

and standard ODE techniques show that the solution is given by

$$w(t) = \frac{\Lambda}{1 + \left(\frac{\Lambda}{w(0)} - 1\right) e^{-\Lambda t}}.$$

Recall that $w(0) = u_{0x}(x)$. By our assumption on u_0, we can find a point $y \in \mathbb{S}$ satisfying (3.36). Setting

$$\tau = -\frac{1}{\lambda} \ln\left(1 + \frac{\lambda}{u_{0x}(y)}\right),$$

it follows that the solution must blow up in the H^3-norm. □

Remark 3.44. Condition (3.36) means that we can find $y \in \mathbb{S}$ such that

1. $u_{0x}(y) < 0$ and
2. $|u_{0x}(y)| > \lambda$.

Since we assume $\mu(u_0) = 0$, it follows that u_0 must change sign. Since $u_0 \in C^\infty(\mathbb{S})$, u_0 has to change sign at least twice and so it is always possible to find $y \in \mathbb{S}$ satisfying the first condition. Our second condition says that the slope of u_0 must exceed λ in order to obtain blow-up: The larger the dissipation given by λ, the larger must $|u_{0x}|$ be locally in order to obtain a blow-up solution. So (3.36) is a non-trivial condition for u_0 in our blow-up setting.

The following lemma is similar to Lemma 2.2. in [46]. Furthermore, we see that as $\lambda \to 0$, we obtain conservation of the quantity $(m \circ \varphi)\varphi_x^3$, which is explained for the DP and the μDP equation in [41, 99].

Lemma 3.45. *Let $u_0 \in H^3(\mathbb{S})$ and let $T > 0$ be the maximal existence time of the corresponding solution $u(t,x)$ according to Theorem 3.38. Let φ be the associated local flow according to Lemma 3.36. Then we have*

$$m(t,\varphi(t,x))\varphi_x^3(t,x) = m_0(x)e^{-\lambda t}.$$

Proof. An easy calculation shows that the function

$$[0,T) \mapsto \mathbb{R}, \quad t \mapsto e^{\lambda t} m(t,\varphi(t,x))\varphi_x^3(t,x)$$

is constant. Using $\varphi(0) = \mathrm{id}$ and $\varphi_x(0) = 1$, we are done. □

Finally, we come to the following global well-posedness result. Note that our assumptions on the initial condition u_0 are quite similar to the ones in Theorem 5.4. in [99].

Theorem 3.46. *Assume that $u_0 \in H^3(\mathbb{S})$ has positive mean and satisfies the condition $Au_0 \geq 0$. Then the Cauchy problem* (3.26) *has a unique global solution u in $C([0,\infty), H^3(\mathbb{S})) \cap C^1([0,\infty), H^2(\mathbb{S}))$.*

Proof. Let $u(t,\cdot) \in H^3(\mathbb{S})$, $t \in [0,T)$, denote the solution of (3.26) obtained in Theorem 3.38. According to Theorem 3.40, we only have to show that $\|u_x(t,\cdot)\|_\infty$ stays bounded as t approaches T from below. Note that, for any periodic function w, differentiating formula (3.3) yields

$$\|\partial_x w\|_\infty \leq C \|Aw\|_{L_1},$$

where C is a positive constant. Now Lemma 3.45 and the assumption $Au_0 \geq 0$ imply that

$$\|Au\|_{L_1} = \mu(Au).$$

Using Lemma 3.42, we obtain the estimate

$$\|\partial_x u(t,\cdot)\|_\infty \leq C \int_0^1 Au \,\mathrm{d}x = C\mu(u) \leq C\mu(u_0) < \infty,$$

from which the indefinite persistence of the solution u follows. □

Remark 3.47. It is easy to see that Theorem 3.46 also holds if $\mu(u_0) < 0$ and $Au_0 \leq 0$.

3.5 A one-parameter family of μCH equations

In Chap. 2 we discussed a one-parameter family of CH equations coming up from the variational principle for the inner product induced by the operator $1 - \lambda\partial_x^2$ which is convexly combined of the canonical inner products on L_2 and H^1 respectively. We computed the Christoffel map and sectional curvatures and determined the variation of geometric quantities with respect to the parameter $\lambda \in [0,1]$. Here, we want to extend this discussion to the novel family of equations which is obtained from the CH equation if we replace the inertia operator by $A = \mu - \lambda\partial_x^2$; precisely, we discuss the family

$$\mu(u_t) + 2u_x\mu(u) = \lambda(u_{txx} + uu_{xxx} + 2u_xu_{xx}), \quad \lambda \in (0,1]. \tag{3.37}$$

The inner product defined by A is the bilinear form

$$\langle f, g\rangle_{\mu,\lambda} = \mu(f)\mu(g) + \lambda \int_{\mathbb{S}} f_x(x)g_x(x)\,\mathrm{d}x$$

which can be defined on any tangent space of the circle diffeomorphisms by right invariance. For $\lambda = 1$ we obtain the μCH equation which is mentioned in [99]. Observe that, for $\lambda = 0$, Eq. (3.37) does *not* become the so-called μ-Burgers (μB) equation, which one might expect in analogy to what we get from the family (2.1) for $\lambda = 0$; as explained in [99], the μB reads as $u_{txx} + 3u_xu_{xx} + uu_{xxx} = 0$. Note that, for any $0 < \lambda \leq 1$ we have that $\mu(u_t) = 0$ if u is sufficiently regular since Eq. (3.37) is equivalent to the evolution equation

$$u_t + uu_x + A^{-1}\partial_x\left(2\mu(u)u + \frac{\lambda}{2}u_x^2\right) = 0,$$

cf. Remark 3.4. The Christoffel operator $\Gamma = \Gamma_{\mathrm{id}}$ for Eq. (3.37) is

$$\Gamma(u,v) = -A^{-1}\left(\mu(u)v + \mu(v)u + \frac{\lambda}{2}u_xv_x\right)_x, \tag{3.38}$$

since

$$\begin{aligned} u_t + uu_x &= -A^{-1}(um_x + 2mu_x - A(uu_x)) \\ &= -A^{-1}(-\lambda uu_{xxx} + 2(\mu(u) - \lambda u_{xx})u_x + 3\lambda u_xu_{xx} + \lambda uu_{xxx}) \\ &= -A^{-1}\left(2\mu(u)u + \frac{\lambda}{2}u_x^2\right)_x \\ &= \Gamma(u,u). \end{aligned}$$

Let Γ_φ be the associated right-invariant Christoffel map on $\mathrm{Diff}^n(\mathbb{S})$. In the following proposition, we show that μCH possesses a unique geodesic flow $\varphi \in \mathrm{Diff}^n(\mathbb{S})$ for $n \geq 2$. The μCH equation thus reads as $\Gamma_\varphi(\varphi_t, \varphi_t) = \varphi_{tt}$ in local coordinates (see Remark 2.3).

Proposition 3.48. *The pair* $(\mathrm{Diff}^n(\mathbb{S}), \langle\cdot,\cdot\rangle_{\mu,\lambda})$*,* $n \geq 2$*, is a Riemannian manifold. The bilinear map* ∇ *defined on* $\mathrm{Vect}^\infty(\mathrm{Diff}^n(\mathbb{S}))$ *via* (2.6) *with the Christoffel operator* (3.38) *depends smoothly on* φ *and is a Riemannian covariant derivative on* $\mathrm{Diff}^n(\mathbb{S})$*; in particular it is compatible with the right-invariant metric* $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mu,\lambda}$.

Proof. Clearly, $\langle\cdot,\cdot\rangle_{\mu,\lambda}$ is a positive definite and symmetric bilinear from on $C^n(\mathbb{S})$. That the map

$$g(\varphi)(X,Y) = \int_{\mathbb{S}} (X \circ \varphi^{-1})\left(\mu - \lambda\partial_x^2\right)(Y \circ \varphi^{-1})\,\mathrm{d}x$$

is smooth for any $X, Y \in T_\varphi\mathrm{Diff}^n(\mathbb{S}) \simeq C^n(\mathbb{S})$ follows from Proposition 2.1 and

$$\int_{\mathbb{S}} (X \circ \varphi^{-1})(\mu - 1)(Y \circ \varphi^{-1})\,\mathrm{d}x = \mu(X\varphi_x)\mu(Y\varphi_x) - \mu(XY\varphi_x).$$

Obviously, ∇ satisfies the properties 1.–3. in Definition 1.24. That ∇ depends smoothly on φ follows from the smoothness of $\varphi \mapsto \Gamma_\varphi$; this can be proved as explained in Proposition 3.24, with the aid of Lemma 3.23, cf. also Remark 3.25. We finally show the compatibility of ∇ with the right-invariant metric on $\mathrm{Diff}^n(\mathbb{S})$ induced by the operator

A. Let X, Y, Z be vector fields on $\mathrm{Diff}^n(\mathbb{S})$ and define the functions u, v and w by

$$X(\varphi) \circ \varphi^{-1} = u, \quad Y(\varphi) \circ \varphi^{-1} = v, \quad Z(\varphi) \circ \varphi^{-1} = w,$$

for $\varphi \in \mathrm{Diff}^n(\mathbb{S})$. Using Eq. (2.9), we have

$$\begin{aligned}(X \langle Y, Z \rangle)(\varphi) = &\int_{\mathbb{S}} \left[(DY(\varphi) \cdot X(\varphi)) \circ \varphi^{-1} - v_x u \right] Aw \, \mathrm{d}x \\ &+ \int_{\mathbb{S}} \left[(DZ(\varphi) \cdot X(\varphi)) \circ \varphi^{-1} - w_x u \right] Av \, \mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\langle \nabla_X Y, Z \rangle_{\varphi} &= \left\langle (DY(\varphi) \cdot X(\varphi)) \circ \varphi^{-1} - \Gamma(v, u), Aw \right\rangle_{L_2} \\ &= \int_{\mathbb{S}} \left[(DY(\varphi) \cdot X(\varphi)) \circ \varphi^{-1} \right] Aw \, \mathrm{d}x \\ &\quad + \int_{\mathbb{S}} \left(\mu(u) v_x + \mu(v) u_x + \frac{\lambda}{2} (u_x v_x)_x \right) w \, \mathrm{d}x.\end{aligned}$$

Using integration by parts, it is now easy to see that

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle \nabla_X Z, Y \rangle .$$

□

Note that we also have a well-defined curvature tensor R for μCH which is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Expressing the sectional curvature in terms of the Christoffel map Γ we obtain an additional term compared to the result in Theorem 2.4, cf. [79] for the case $\lambda = 1$.

Theorem 3.49. *The sectional curvature* $S(u, v) = \langle R(u, v)v, u \rangle$ *for the family* (3.37) *is given by*

$$\begin{aligned}S(u, v) &= \langle \Gamma(u, v), \Gamma(u, v) \rangle - \langle \Gamma(u, u), \Gamma(v, v) \rangle - 3\mu(u_x v)^2 \\ &= \mu(u)^2 \left(\frac{1}{\lambda} \mu(v^2) + \mu(v_x^2) \right) + \mu(v)^2 \left(\frac{1}{\lambda} \mu(u^2) + \mu(u_x^2) \right) \\ &\quad + \mu(u) \mu((v u_x - u v_x) v_x) + \mu(v) \mu((u v_x - v u_x) u_x) \\ &\quad - 2\mu(u)\mu(v) \left(\frac{1}{\lambda} \mu(uv) + \mu(u_x v_x) \right) \\ &\quad - \frac{\lambda}{4} \mu(u_x v_x)^2 + \frac{\lambda}{4} \mu(u_x^2) \mu(v_x^2) - 3\mu(u_x v)^2,\end{aligned}$$

for any $u, v \in T_{\mathrm{id}} \mathrm{Diff}^n(\mathbb{S})$.

Proof. Replacing $1 - \lambda \partial_x^2$ by $\mu - \lambda \partial_x^2$ the same calculations as in the proof of Theorem 2.4 show that

$$\begin{aligned}\langle R(u, v)v, u \rangle = &\langle \Gamma(u, v), \Gamma(u, v) \rangle - \langle \Gamma(u, u), \Gamma(v, v) \rangle \\ &+ \langle -\Gamma(v_x v, u) - \Gamma(v, u_x v) + 2\Gamma(v_x u, v), u \rangle \\ &- \langle u_x v, \Gamma(v, u) \rangle + \langle u u_x, \Gamma(v, v) \rangle .\end{aligned}$$

Now, using (3.38), it is easy to derive that the second and third row terms are equal to $-3\mu(u_x v)^2$. Substituting the Christoffel symbol in $\langle\Gamma(u,v),\Gamma(u,v)\rangle - \langle\Gamma(u,u),\Gamma(v,v)\rangle$ we see that $S(u,v)$ equals

$$-\int_{\mathbb{S}}\left(\mu(u)v+\mu(v)u+\frac{\lambda}{2}u_xv_x\right)\partial_x A^{-1}\partial_x\left(\mu(u)v+\mu(v)u+\frac{\lambda}{2}u_xv_x\right)\mathrm{d}x$$
$$+\int_{\mathbb{S}}\left(2\mu(u)u+\frac{\lambda}{2}u_x^2\right)\partial_x A^{-1}\partial_x\left(2\mu(v)v+\frac{\lambda}{2}v_x^2\right)\mathrm{d}x-3\mu(u_xv)^2.$$

Since $\partial_x A^{-1}\partial_x = \partial_x^2 A^{-1} = \frac{1}{\lambda}(-1+\mu)$ our theorem follows by simplifying the above expression. $\square$

To deduce an expression for the sectional curvature of a plane spanned by two vectors u and v, we may assume, after taking linear combinations, that u and v are orthonormal with respect to $\langle\cdot,\cdot\rangle_{\mu,\lambda}$ and that v has zero mean, i.e.,

$$\mu(u)^2+\lambda\mu(u_x^2)=1,\quad \mu(v)=0,\quad \lambda\mu(v_x^2)=1,\quad \mu(u_xv_x)=0.$$

With these assumptions, the previous theorem yields

$$S(u,v)=\frac{1}{\lambda}\mu(u)^2(\mu(v^2)+1)+\mu(u)\mu((vu_x-uv_x)v_x)+\frac{1}{4}\mu(u_x^2)-3\mu(u_xv)^2. \qquad (3.39)$$

If u also has zero mean and λ is large enough, we obtain the following positivity result for the sectional curvature.

Theorem 3.50. *For any orthonormal vectors $u,v\in T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S})$ with $\mu(u)=\mu(v)=0$, the sectional curvature $S(u,v)$ of the plane spanned by u and v satisfies*

$$S(u,v)=\frac{1}{4\lambda}-3\mu(u_xv)^2\geq\frac{1}{4\lambda}\left(1-\frac{3}{\lambda\pi^2}\right).$$

In particular, the sectional curvature $S(u,v)$ is strictly positive for all $\lambda\in(3/\pi^2,1]$.

Proof. Letting $\mu(u)=0$ in Eq. (3.39) we find the expression stated in the theorem. Since v has zero mean, we further deduce that

$$\lambda^2\mu(u_xv)^2\leq\lambda^2\mu(u_x^2)\mu(v^2)=\lambda\mu(v^2)\leq\frac{\lambda}{4\pi^2}\mu(v_x^2)=\frac{1}{4\pi^2}.$$

For the latter estimate, we have used that v can be written as a Fourier series $v=\sum_{k\in\mathbb{Z}\setminus\{0\}}v_ke^{2\pi\mathrm{i}kx}$. This achieves

$$\mu(v_x^2)=\langle v_x,v_x\rangle_{L_2(\mathbb{S})}=\sum_{k\neq0}|v_k|^24\pi^2k^2\geq4\pi^2\sum_{k\neq0}|v_k|^2=4\pi^2\langle v,v\rangle_{L_2(\mathbb{S})}=4\pi^2\mu(v^2).$$

$\square$

In [79], the authors discuss the μHS equation under geometric aspects and obtain that the sectional curvature, for u and v satisfying the assumptions of the above theorem, is always positive. Furthermore, a result comparable to the following one is established.

Theorem 3.51. *Let $v\in T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S})$ be orthonormal to the constant function* 1*. Then*

$$S(1,v) = \frac{1}{\lambda}\mu(v^2) > 0.$$

Proof. This follows from Eq. (3.39) by inserting $u = 1$. □

Note that the functions

$$v_k := \frac{\sqrt{2}}{k}\sin kx, \quad k \in 2\pi\mathbb{Z}\backslash\{0\},$$

satisfy the assumptions of Theorem 3.51 and

$$S(1, v_k) = \frac{1}{\lambda k^2} \to 0, \quad k \to \infty.$$

In conclusion, let us decompose the tangent space at the identity $T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S}) = U \oplus V$ in such a way that U consists of the zero mean functions on $\mathbb{S}$ and $V \simeq \mathbb{R}$ are the constants so that $u = \tilde{u} + \mu(u)$ and $\mu(\tilde{u}) = 0$ for any $u \in T_{\mathrm{id}}\mathrm{Diff}^n(\mathbb{S})$. Theorem 3.50 shows that, for $\lambda > 3/\pi^2$, the sectional curvature for any plane contained in (i.e., parallel to) the subspace U is strictly positive. Theorem 3.51 establishes that the sectional curvature is also positive on the planes perpendicular to U, i.e., the planes containing constant functions, which constitute V.

To obtain formulas describing the variation with respect to λ of the Christoffel map and the sectional curvature for the μCH we will need the λ-derivative of A^{-1}.

Lemma 3.52. *The operator* $A = \mu - \lambda\partial_x^2\colon C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$, $\lambda \in (0, 1]$, *is invertible and its inverse is*

$$(\mu - \lambda\partial_x^2)^{-1} f = G * f, \quad G(x) = \frac{1}{2\lambda}\left(x^2 - |x| + \frac{1}{6}\right) + 1.$$

In particular, the map $\lambda \mapsto \lambda^j G$ *is differentiable for all* $j \in \mathbb{N}_0$ *and we have the relation* $[\partial_x, (\mu - \lambda\partial_x^2)^{-1}] = 0$.

Proof. This follows from $AG = \delta$. □

Remark 3.53. The kernel G can be computed as in the case $\lambda = 1$ (see Sect. 3.1). For $\lambda = 1$ we obtain the formula presented in (3.4) for the Green's function of $\mu - \partial_x^2$.

It is now easy to derive the formula

$$\partial_\lambda \Gamma(u, v) = -(\partial_\lambda G) * \partial_x \left(\mu(u)v + \mu(v)u + \frac{\lambda}{2}u_x v_x\right) - \frac{1}{2}G * (u_x v_x)_x.$$

From this, we get immediately a result for $\partial_\lambda S(u, v)$ by applying Theorem 3.49. We leave it to the reader to write down the explicit formulae which follow from elementary calculations.

Example 3.54. For the family (3.37) we computed the sectional curvature $S(u, v)$ for orthonormal vectors u and v with zero mean in Theorem 3.50. We have

$$\partial_\lambda S(u, v) = -\frac{1}{4\lambda^2};$$

i.e., the λ-derivative of $S(u, v)$ is strictly negative.

Example 3.55. For the orthonormal vectors 1 and v we found the sectional curvature $S(u,v)$ in Theorem 3.51. We have

$$\partial_\lambda S(1,v) = -\frac{1}{\lambda^2}\mu(v^2);$$

in particular, the λ-derivative of the strictly positive quantity $S(1,v)$ is strictly negative.

Chapter 4
Two-component generalizations of the periodic b-equation and its μ-variant

The CH equation (1.19) possesses an integrable two-component extension, denoted as 2CH, [13, 42, 49, 86], which involves both fluid density and momentum. What makes the 2CH particularly interesting is that it possesses peakon and multi-kink solutions as well as a bi-Hamiltonian structure and a Lax pair formulation. The basic idea of generalizing the CH equation was to include an additional function in the Lax pair and to derive some properties of the new equation from the generalized Lax pair representation, [115]. A first geometric approach to two-component variants of CH and DP is shown in [54, 55].

In [90, 116] the authors show that the HS equation has a supersymmetric two-component generalization, called 2HS, and discuss the geometric interpretation of the 2HS as an Euler equation on the superconformal algebra of contact vector fields on a certain supercircle. Simple examples of explicit solutions and a description of the bounded travelling wave solutions of 2HS are given in [90].

In this chapter it is our goal to extend the results from Sect. 3.3 to the two-component CH and HS as well as a two-component version of the DP introduced in [115] where the author generalizes a Hamiltonian operator of the DP to a suitable 2×2-matrix operator. We also consider the corresponding μ-variants as introduced in Chap. 3. To establish the geometric setting we first need a brief introduction to semidirect products of Lie groups, which proved to be suitable configuration spaces. To obtain the existence of a geodesic flow and a sectional curvature for 2CH we show that 2CH allows for a smooth Riemannian structure compatible with a smooth affine connection. From this, we will conclude local well-posedness for the geodesic equation (and later for the original equation) in different function spaces: the H^s-category, the C^n-category and finally the smooth category (which again requires additional technical effort). Finally, we present some explicit calculations of the curvature of 2CH and obtain subspaces of positive sectional curvature. To round everything off, we compare the 2CH equation to a rotating rigid body and the one-component CH in the context of Arnold's geometric framework. After that we discuss two-component extensions of the DP and the HS and some μ-equations under similar aspects.

Some of the results presented in this chapter have been published by the author, cf. [40].

4.1 Generalities on semidirect products

As explained in Chap. 1, the geometric analysis for rigid bodies and fluids is based on the same mathematical principles and uses the same analytical tools like Lagrangian or Hamiltonian formulations or Lie group-techniques.

As a motivation for the issues of this chapter, let us consider a three-dimensional rigid body having three translational degrees of freedom modelled by $v \in \mathbb{R}^3$ and with rotations parametrized by $R \in SO(3)$. The configuration space is the Lie group $SE(3) \simeq SO(3) \times \mathbb{R}^3$, the *special Euclidean group*, of 4×4-matrices of the form

$$E(R, v) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}.$$

For any $w \in \mathbb{R}^3$,

$$\begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix} = \begin{pmatrix} Rw + v \\ 1 \end{pmatrix}$$

so that $E(R, v)$ corresponds to rotation by R followed by translation by v, cf. [62]. The group operation

$$\star\colon SE(3) \times SE(3) \to SE(3), \quad (R_1, v_1) \star (R_2, v_2) = (R_1 R_2, v_1 + R_1 v_2)$$

can be generalized naturally to arbitrary Lie groups, leading to the notion of a *semidirect product* of a Lie group G with a vector space V. Assuming that G acts on the left on V with the left-action denoted by $(g, v) \mapsto gv$, the operation

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2)$$

defines a Lie group structure on $G \times V$; we denote this Lie group by $G \circledS V$. If G acts on the right on V, one defines

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + v_1 g_2)$$

similarly. It is easy to see that $(e, 0)$ is the neutral element, where e denotes the neutral element of G, and that (g, v) has the inverse $(g^{-1}, -vg^{-1})$. While for rigid bodies, left-invariant formulations lead to the correct equations of motion, the mathematical analysis of fluid motion is always based on right actions and right invariance. That is why we will use the second definition stated above henceforth in this chapter. To obtain the Lie bracket on the Lie algebra $\mathfrak{g} \circledS V$, we consider the inner automorphism

$$I_{(g,v)}(h, w) = (g, v)(h, w)(g, v)^{-1} = (ghg^{-1}, -vg^{-1} + (w + vh)g^{-1}).$$

Writing $v\xi$ for the induced infinitesimal action of $\mathfrak{g}$ on V, i.e., the map

$$V \times \mathfrak{g} \mapsto V, \quad (v, \xi) \mapsto v\xi := \frac{\mathrm{d}}{\mathrm{d}t} vg(t)\bigg|_{t=0},$$

$g(t)$ being a curve in G starting from e in the direction of ξ, we obtain

$$\mathrm{Ad}_{(g,v)}(\xi, w) = (\mathrm{Ad}_g \xi, (w + v\xi)g^{-1}),$$

$$\mathrm{ad}_{(\eta,v)}(\xi, w) = (\mathrm{ad}_\eta \xi, v\xi - w\eta)$$

and hence

$$[(\xi_1, v_1), (\xi_2, v_2)] = \mathrm{ad}_{(\xi_2,v_2)}(\xi_1, v_1) = ([\xi_1, \xi_2], v_2\xi_1 - v_1\xi_2).$$

In [61], the authors explain the main differences when working with semidirect products in case of right and left actions.

From now on, we consider the semidirect product of the orientation-preserving diffeomorphisms $\mathrm{Diff}(\mathbb{S})$ with a space of scalar functions $\mathcal{F}(\mathbb{S})$; the exact regularity assumptions will be made precise in the following. We will use the notation G and $\mathfrak{g}$ for the Lie group $\mathrm{Diff}(\mathbb{S})Ⓢ\mathcal{F}(\mathbb{S})$ and its Lie algebra $\mathrm{Vect}(\mathbb{S})Ⓢ\mathcal{F}(\mathbb{S})$. The group product in G is defined by

$$(\varphi_1, f_1)(\varphi_2, f_2) \coloneqq (\varphi_1 \circ \varphi_2, f_2 + f_1\varphi_2)$$

where $\circ$ denotes the group product in $\mathrm{Diff}(\mathbb{S})$ (i.e., composition) and $f\varphi \coloneqq f \circ \varphi$ is a right action of $\mathrm{Diff}(\mathbb{S})$ on $\mathcal{F}(\mathbb{S})$. The neutral element of G is $(\mathrm{id}, 0)$ and (φ, f) has the inverse $(\varphi^{-1}, -f \circ \varphi^{-1})$. The above calculations show that

$$\mathrm{Ad}_{(\varphi,f)}(u, \rho) = (\mathrm{Ad}_\varphi u, (f_x u + \rho) \circ \varphi^{-1}),$$

$$\mathrm{ad}_{(v,f)}(u, \rho) = (\mathrm{ad}_v u, f_x u - \rho_x v)$$

and

$$[(u_1, u_2), (v_1, v_2)] = ([u_1, v_1], v_{2x}u_1 - u_{2x}v_1),$$

where $[u_1, v_1] = v_{1x}u_1 - u_{1x}v_1$ is the Lie bracket induced by right-invariant vector fields on $\mathrm{Diff}(\mathbb{S})$.

Several different regularity assumptions can be imposed on the elements of G. The structure of the two-component equations under consideration suggests that the density variable ρ should be allowed to have one spatial derivative less than the velocity u. This suggests the following choice for G:

$$H^sG \coloneqq H^s\mathrm{Diff}(\mathbb{S})ⓈH^{s-1}(\mathbb{S}), \tag{4.1}$$

where $H^s\mathrm{Diff}(\mathbb{S})$ denotes the group of orientation-preserving diffeomorphisms of $\mathbb{S}$ of Sobolev class H^s. We will assume that $s > 5/2$. In this case, $H^s\mathrm{Diff}(\mathbb{S})$ is a Hilbert manifold and a topological group and the composition map

$$(\varphi, f) \mapsto f \circ \varphi \colon H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to H^{s-1}(\mathbb{S})$$

is continuous, cf. [37]. Thus, H^sG is a topological group and a smooth manifold modelled on the Hilbert space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$.

Another natural choice for G is

$$C^nG \coloneqq \mathrm{Diff}^n(\mathbb{S})ⓈC^{n-1}(\mathbb{S}); \tag{4.2}$$

recall that $\mathrm{Diff}^n(\mathbb{S})$ is the set of orientation-preserving diffeomorphisms of $\mathbb{S}$ of class C^n. We will assume that $n \geq 2$. In this case, C^nG is a topological group and a smooth manifold modelled on the Banach space $C^n(\mathbb{S}) \times C^{n-1}(\mathbb{S})$. Note that H^sG and C^nG are *not* Lie groups, since left multiplication is only continuous and not smooth.

Finally, we may choose G as

$$C^\infty G \coloneqq \mathrm{Diff}^\infty(\mathbb{S})ⓈC^\infty(\mathbb{S}), \tag{4.3}$$

with $\mathrm{Diff}^\infty(\mathbb{S})$ the smooth orientation-preserving diffeomorphisms of $\mathbb{S}$. This is a Lie group (the multiplication and inverse maps are smooth) and a Fréchet manifold modelled on $C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$. In contrast to H^sG and C^nG, it is *not* a Banach manifold.

The three choices (4.1)–(4.3) for G are all of interest due to their different advantages. We will first develop the theory for H^sG and then consider C^nG and $C^\infty G$. We refer to [61, 63] for further information on geodesic flows on semidirect products.

4.2 The 2CH equation as a metric Euler equation

We now introduce the generalizations of the CH equation and the DP equation which we want to study in this and in the following section; see [115] where the author also considers an interacting system of equations and works out Hamiltonian structures.

Let $t \geq 0$ and $x \in \mathbb{S}$. By a solution of the periodic 2-component Camassa-Holm equation with initial data (u_0, ρ_0) we mean a function $(u(t,x), \rho(t,x))$ which satisfies

$$\begin{cases} m_t = -m_x u - 2mu_x - \rho\rho_x, \\ \rho_t \;= -(\rho u)_x, \end{cases} \tag{4.4}$$

for $t > 0$, and $(u(0,x), \rho(0,x)) = (u_0, \rho_0)$, where $m = u - u_{xx}$. Similarly, we say that (u, ρ) solves the 2DP equation with initial data (u_0, ρ_0) if

$$\begin{cases} m_t = -m_x u - 3mu_x - \rho u_x + 2\rho\rho_x, \\ \rho_t \;= -2\rho u_x - \rho_x u, \end{cases} \tag{4.5}$$

for $t > 0$, and $(u(0,x), \rho(0,x)) = (u_0, \rho_0)$.

Clearly, (4.4) and (4.5) reduce to (1.19) and (1.20) for $\rho = 0$. It is our first aim to study Eq. (4.4) under geometric aspects as explained in Sect. 2.2: We show that Eq. (4.4) is a reexpression of a geodesic flow on H^sG. In a preliminary step we find the Christoffel operator $\Gamma_{(\varphi,f)}$ for (4.4); this is a smooth bilinear map $\Gamma_{(\varphi,f)}$ which defines a smooth connection ∇ on H^sG. In addition we consider the right-invariant metric $\langle\cdot,\cdot\rangle_{(\varphi,f)}$ on H^sG equal to the H^1-metric for the first plus the L_2-metric for the second component at $(\mathrm{id}, 0)$. Then we show that the connection ∇ is compatible with the metric $\langle\cdot,\cdot\rangle_{(\varphi,f)}$ and obtain the existence and uniqueness of a geodesic flow. The scenario is similar for C^nG and in both cases, we establish local well-posedness for the original equation from the geometric theory.

The next step is to prove that, for smooth initial data in the geometric picture, the 2CH equation possesses a smooth short-time solution. As a corollary, we see that Eq. (4.4) is well-posed in the smooth category.

Throughout the whole discussion, we deal with several geometric quantities which we relate to the corresponding quantities for a rotating rigid body in the end (cf. Sect. 1.2 and [81] for a discussion of the CH equation in this context). This section also deals with the sectional curvature of H^sG associated with the 2CH equation.

4.2.1 Geometric aspects of the 2CH equation

The CH equation is the Euler-Lagrange equation for the Lagrangian $\mathcal{L}\colon T\mathrm{Diff}^\infty(\mathbb{S}) \to \mathbb{R}$ defined by $\mathcal{L}(g,\dot g) = \frac{1}{2}\|\dot g(t)\|^2_{g(t)}$, where $\|\cdot\|_g$ denotes the H^1 right-invariant metric on $\mathrm{Diff}^\infty(\mathbb{S})$ and $g(t) \subset \mathrm{Diff}^\infty(\mathbb{S})$ is a smooth curve. Precisely, the CH is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\mathcal{L}}{\delta u} = -\mathrm{ad}_u^*\frac{\delta\mathcal{L}}{\delta u}, \quad u(t) = D_{g(t)}R_{g(t)^{-1}}\dot g(t),$$

where ad^* is the adjoint of ad with respect to the H^1 inner product and u is the Eulerian velocity of the curve $g(t)$. Similarly, as explained in [62, 63], Eq. (4.4) comes up from the variational principle

$$\delta\int_a^b \mathcal{L}(u,\rho)\,\mathrm{d}t = 0, \quad \mathcal{L}(u,\rho) = \frac{1}{2}\int u(1-\partial_x^2)u\,\mathrm{d}x + \frac{1}{2}\int \rho^2\,\mathrm{d}x,$$

in the sense that the 2CH is equivalent to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\frac{\delta L}{\delta u}\\ \frac{\delta L}{\delta \rho}\end{pmatrix} = \begin{pmatrix}-\mathrm{ad}_u^* & \cdot\diamond\rho\\ 0 & -u\end{pmatrix}\begin{pmatrix}\frac{\delta L}{\delta u}\\ \frac{\delta L}{\delta \rho}\end{pmatrix}.$$

In terms of the Eulerian velocity (u,ρ), Eq. (4.4) can be regarded as an equation on the Lie algebra $T_{(\mathrm{id},0)}C^\infty G$. In this section, it is our aim to write (4.4) as an evolution equation on the semidirect product $C^\infty G$ and furthermore to show that the resulting equation reexpresses a geodesic flow. We begin to develop the geometric theory for the configuration space H^sG and come to the following key observation which gets us started:

To any tangent vector $v \in T_pH^sG$, $p \in H^sG$, we associate an element of the Lie algebra $\mathfrak{g} \simeq H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ by applying the differential of the right shift $R_{p^{-1}}\colon H^sG \to H^sG$ sending any $q \in H^sG$ to qp^{-1}. Let us write $p = (\varphi, f)$ and $v = (v_1, v_2)$. To compute $D_pR_{p^{-1}}v$ explicitly, we choose a curve $\gamma(t) = (\gamma_1(t),\gamma_2(t)) \subset H^sG$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. Then

$$\begin{aligned} D_pR_{p^{-1}}v &= \frac{\mathrm{d}}{\mathrm{d}t}R_{p^{-1}}\gamma(t)\Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t),\gamma_2(t))(\varphi^{-1}, -f\circ\varphi^{-1})\Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t)\circ\varphi^{-1}, -f\circ\varphi^{-1}+\gamma_2(t)\circ\varphi^{-1})\Big|_{t=0}\\ &= (v_1\circ\varphi^{-1}, v_2\circ\varphi^{-1})\\ &= v\circ\varphi^{-1}. \end{aligned}$$

Note that this result is similar to what we obtained in the one-component case, where $R_\psi\colon \varphi \mapsto \varphi\circ\psi$ is a linear map. For a curve $(\varphi(t), f(t))$ in H^sG, we write

$$(u,\rho) = (\varphi_t\circ\varphi^{-1}, f_t\circ\varphi^{-1}) \tag{4.6}$$

for the Eulerian velocity, i.e., we have $\varphi_t = u\circ\varphi$ and $f_t = \rho\circ\varphi$. Next, we define a bilinear operator $\Gamma_{(\mathrm{id},0)}\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ by

$$\Gamma_{(\mathrm{id},0)}(X,Y) \coloneqq \begin{pmatrix} \Gamma^0_{\mathrm{id}}(X_1,Y_1) - \frac{1}{2}A^{-1}(X_2Y_2)_x \\ -\frac{1}{2}(X_{1x}Y_2 + Y_{1x}X_2) \end{pmatrix}, \tag{4.7}$$

for all $X=(X_1,X_2), Y=(Y_1,Y_2) \in \mathfrak{g}$. Here, A is the operator $1-\partial_x^2$ and

$$\Gamma^0_{\mathrm{id}}(u,v) = -A^{-1}\partial_x\left(uv + \frac{1}{2}u_xv_x\right)$$

is the Christoffel operator for the CH equation (see Sect. 2.2). For vector fields X, Y on H^sG, we define

$$\Gamma_{(\varphi,f)}(X,Y) = \Gamma_{(\mathrm{id},0)}(X(\varphi,f)\circ\varphi^{-1}, Y(\varphi,f)\circ\varphi^{-1})\circ\varphi.$$

Differentiating Eq. (4.6) with respect to t and using (4.4) and (4.6) shows that

$$\begin{aligned}\begin{pmatrix}\varphi_{tt}\\ f_{tt}\end{pmatrix} &= \begin{pmatrix}(u_t+uu_x)\circ\varphi \\ (\rho_t+u\rho_x)\circ\varphi\end{pmatrix}\\
&= \begin{pmatrix} -[A^{-1}(u(Au)_x + 2(Au)u_x - A(uu_x))]\circ\varphi - [A^{-1}(\rho\rho_x)]\circ\varphi \\ -(\rho u_x)\circ\varphi\end{pmatrix}\\
&= \begin{pmatrix} -[A^{-1}(u^2+\frac{1}{2}u_x^2)_x]\circ\varphi - [A^{-1}(\rho\rho_x)]\circ\varphi \\ -(\rho u_x)\circ\varphi\end{pmatrix}\\
&= \Gamma_{(\varphi,f)}((\varphi_t,f_t),(\varphi_t,f_t)).\end{aligned} \tag{4.8}$$

Let us define locally an affine connection on H^sG by setting

$$\nabla_X Y(\varphi,f) \coloneqq DY(\varphi,f)\cdot X(\varphi,f) - \Gamma_{(\varphi,f)}(Y(\varphi,f),X(\varphi,f)). \tag{4.9}$$

We also define an inner product on $\mathfrak{g}$,

$$\langle X,Y\rangle_{(\mathrm{id},0)} \coloneqq \langle X_1,Y_1\rangle_{H^1(\mathbb{S})} + \langle X_2,Y_2\rangle_{L_2(\mathbb{S})},$$

and obtain a right-invariant inner product on H^sG by setting

$$\langle X,Y\rangle_{(\varphi,f)} \coloneqq \left\langle X(\varphi,f)\circ\varphi^{-1}, Y(\varphi,f)\circ\varphi^{-1}\right\rangle_{(\mathrm{id},0)}, \tag{4.10}$$

where X, Y are vector fields on H^sG. In the following, we will use the short hand notation $\langle\cdot,\cdot\rangle_{(\mathrm{id},0)} = \langle\cdot,\cdot\rangle$. First we prove that the right-invariant metric (4.10) defines indeed a Riemannian metric on H^sG, $s>5/2$. Since H^sG is only a topological group, it is not a priori clear that $p\mapsto\langle\cdot,\cdot\rangle_p$ is smooth.

Proposition 4.1. *Let $s>5/2$. Let $H^sG = H^s\mathrm{Diff}(\mathbb{S})\circledS H^{s-1}(\mathbb{S})$ and let Γ be the Christoffel map defined in* (4.7). *Then Γ defines a smooth spray on H^sG, i.e., the map*

$$(\varphi,f)\mapsto\Gamma_{(\varphi,f)}\colon H^sG\to\mathcal{L}^2_{\mathrm{sym}}\left(H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}); H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})\right) \tag{4.11}$$

is smooth. Moreover, the metric $\langle\cdot,\cdot\rangle$ defined by (4.10) *is a smooth (weak) Riemannian metric on H^sG, i.e., the map*

$$(\varphi,f)\mapsto\langle\cdot,\cdot\rangle_{(\varphi,f)}: H^sG\to\mathcal{L}^2_{\mathrm{sym}}\left(T_{(\varphi,f)}H^sG;\mathbb{R}\right) \tag{4.12}$$

is a smooth section of the bundle $\mathcal{L}^2_{\mathrm{sym}}(TH^sG;\mathbb{R})$. Finally, the connection ∇ in (4.9) *is a Riemannian covariant derivative in the sense of Definition 1.24.*

Proof. In order to establish smoothness of (4.11), it is sufficient to show that the following map is smooth:

$$((\varphi, f), w) \mapsto \Gamma_{(\varphi,f)}(w, w) \colon H^sG \times [H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})] \to H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}),$$

where $w = (w_1, w_2) \in T_{(\varphi,f)}H^sG \simeq H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ and

$$\Gamma_{(\varphi,f)}(w, w) = \begin{pmatrix} \Gamma^0_{\mathrm{id}}(w_1 \circ \varphi^{-1}, w_1 \circ \varphi^{-1}) - \frac{1}{2}A^{-1}\partial_x(w_2^2 \circ \varphi^{-1}) \\ -(w_1 \circ \varphi^{-1})_x w_2 \circ \varphi^{-1} \end{pmatrix} \circ \varphi.$$

We will show that the term $-\frac{1}{2}\big(A^{-1}\partial_x(w_2^2 \circ \varphi^{-1})\big) \circ \varphi$ makes a smooth contribution to Γ; the other terms can be treated by similar arguments[1]. Consider the map

$$P \colon H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to H^s\mathrm{Diff}(\mathbb{S}) \times H^s(\mathbb{S})$$

defined by

$$P(\varphi, w) = \big(\varphi, \big(A^{-1}\partial_x(w^2 \circ \varphi^{-1})\big) \circ \varphi\big)\,.$$

We write P as the composition $P = \tilde{A}^{-1} \circ P_2 \circ P_1$, where the maps

$$\begin{aligned} P_1 &\colon\ H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-1}(\mathbb{S}), \\ P_2 &\colon\ H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-2}(\mathbb{S}), \\ \tilde{A} &\colon\ \ H^s\mathrm{Diff}(\mathbb{S}) \times H^s(\mathbb{S}) \ \ \to H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-2}(\mathbb{S}) \end{aligned}$$

are defined by

$$\begin{aligned} P_1(\varphi, w) &= (\varphi, w^2), \\ P_2(\varphi, w) &= \big(\varphi, (w \circ \varphi^{-1})_x \circ \varphi\big) = \left(\varphi, \frac{w_x}{\varphi_x}\right), \\ \tilde{A}(\varphi, w) &= \big(\varphi, \big(A(w \circ \varphi^{-1})\big) \circ \varphi\big) = \left(\varphi, w - \frac{w_{xx}}{\varphi_x^2} + \frac{w_x \varphi_{xx}}{\varphi_x^3}\right). \end{aligned}$$

The maps P_1, P_2, and $\tilde{A}$ are smooth since $H^s(\mathbb{S})$ is a Banach algebra under pointwise multiplication for $s > 1/2$. To show that $\tilde{A}^{-1}$ is smooth, we compute

$$D\tilde{A}(\varphi, w) = \begin{pmatrix} \mathrm{id} & 0 \\ * & \mathrm{id} - \frac{1}{\varphi_x^2}\partial_x^2 + \frac{\varphi_{xx}}{\varphi_x^3}\partial_x \end{pmatrix}.$$

This is, for each $(\varphi, w) \in H^s\mathrm{Diff}(\mathbb{S}) \times H^s(\mathbb{S})$, a bijective bounded linear map $H^s(\mathbb{S}) \times H^s(\mathbb{S}) \to H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$. The open mapping theorem implies that its inverse is also bounded. The inverse mapping theorem now implies that $\tilde{A}^{-1}$, and hence also P, is a smooth map.

We next establish the smoothness of (4.12). It is sufficient to show that the map

$$g \colon H^sG \times \big[H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})\big] \to \mathbb{R},$$

defined by

[1] The smoothness of $\varphi \mapsto \Gamma^0_\varphi$ has already been established for the one-componet CH in [94].

$$g((\varphi,f),w)=\int_{\mathbb{S}}(w_1\circ\varphi^{-1})A(w_1\circ\varphi^{-1})\,\mathrm{d}x+\int_{\mathbb{S}}(w_2\circ\varphi^{-1})^2\,\mathrm{d}x$$

is smooth. The change of variables $y=\varphi^{-1}(x)$ yields

$$g((\varphi,f),w)=\int_{\mathbb{S}}\left(w_1^2\varphi_x+\frac{w_{1x}^2}{\varphi_x}+w_2^2\varphi_x\right)\,\mathrm{d}y,$$

and written in this form the smoothness of g is clear. Let us check the properties 1.–4. in Definition 1.24 for (4.9). While 1.–3. are almost trivial, the check of 4. is a lengthy but straightforward computation. By our local definition (4.9), $X(\varphi,f)=0$ implies $(\nabla_X Y)(\varphi,f)=0$. That ∇ is torsion-free is an immediate consequence of the symmetry of Γ and the fact that the commutator of two vector fields is defined locally by

$$[X,Y](\varphi,f)=DY(\varphi,f)\cdot X(\varphi,f)-DX(\varphi,f)\cdot Y(\varphi,f).$$

Another direct consequence of our definition is $\nabla_X(hY)=X(h)Y+h\nabla_X Y$ for all vector fields X,Y and functions h on H^sG. It remains to check, that ∇ is compatible with the right-invariant metric $\langle\cdot,\cdot\rangle$ defined in (4.10), i.e.,

$$X\langle Y,Z\rangle=\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle$$

for all vector fields X,Y,Z on H^sG. Let us write $u_i=X_i(\varphi,f)\circ\varphi^{-1}$ for $i=1,2$ and $v_i=Y_i(\varphi,f)\circ\varphi^{-1}$, $w_i=Z_i(\varphi,f)\circ\varphi^{-1}$ analogously. Let $\gamma(t)\subset H^sG$ be a curve with $\gamma(0)=(\varphi,f)$ and $\gamma'(0)=X(\varphi,f)$. On the one hand,

$$\begin{aligned}(X\langle Y,Z\rangle)(\varphi,f)&=\frac{\mathrm{d}}{\mathrm{d}t}\langle Y(\gamma(t)),Z(\gamma(t))\rangle_{\gamma(t)}\Big|_{t=0}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Y_1(\gamma(t))\circ\gamma_1^{-1},Z_1(\gamma(t))\circ\gamma_1^{-1}\right\rangle_{H^1}\Big|_{t=0}\\&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Y_2(\gamma(t))\circ\gamma_1^{-1},Z_2(\gamma(t))\circ\gamma_1^{-1}\right\rangle_{L_2}\Big|_{t=0},\end{aligned}$$

and a straightforward computation yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Y_2(\gamma(t))\circ\gamma_1^{-1},Z_2(\gamma(t))\circ\gamma_1^{-1}\right\rangle_{L_2}\Big|_{t=0}\\&\quad=\left\langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}-v_{2x}u_1,w_2\right\rangle_{L_2}\\&\qquad+\left\langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}-w_{2x}u_1,v_2\right\rangle_{L_2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\nabla_X Y,Z\rangle_{(\varphi,f)}&=\left\langle DY_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}-\Gamma^0_\varphi(Y_1,X_1)\circ\varphi^{-1},w_1\right\rangle_{H^1}\\&\quad+\left\langle\frac{1}{2}(v_{2x}u_2+u_{2x}v_2),w_1\right\rangle_{L_2}\\&\quad+\left\langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1}+\frac{1}{2}(v_{1x}u_2+u_{1x}v_2),w_2\right\rangle_{L_2}\end{aligned}$$

and similarly

$$\begin{aligned}\langle Y, \nabla_X Z\rangle_{(\varphi,f)} &= \left\langle DZ_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - \Gamma^0_\varphi(Z_1,X_1)\circ\varphi^{-1}, v_1\right\rangle_{H^1} \\ &\quad + \left\langle \frac{1}{2}(w_{2x}u_2 + u_{2x}w_2), v_1\right\rangle_{L_2} \\ &\quad + \left\langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} + \frac{1}{2}(w_{1x}u_2 + u_{1x}w_2), v_2\right\rangle_{L_2}.\end{aligned}$$

The calculations in [94] for the CH equation show that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Y_1(\gamma(t))\circ\gamma_1^{-1}, Z_1(\gamma(t))\circ\gamma_1^{-1}\right\rangle_{H^1}\Big|_{t=0} = \\ &\quad + \left\langle DY_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - \Gamma^0_\varphi(Y_1,X_1)\circ\varphi^{-1}, w_1\right\rangle_{H^1} \\ &\quad + \left\langle DZ_1(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - \Gamma^0_\varphi(Z_1,X_1)\circ\varphi^{-1}, v_1\right\rangle_{H^1},\end{aligned}$$

so that it remains to check that

$$\begin{aligned}&\left\langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - v_{2x}u_1, w_2\right\rangle_{L_2} \\ &\quad + \left\langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} - w_{2x}u_1, v_2\right\rangle_{L_2} \\ &\quad\quad = \left\langle \frac{1}{2}(v_{2x}u_2 + u_{2x}v_2), w_1\right\rangle_{L_2} \\ &\quad\quad + \left\langle DY_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} + \frac{1}{2}(v_{1x}u_2 + u_{1x}v_2), w_2\right\rangle_{L_2} \\ &\quad\quad + \left\langle \frac{1}{2}(w_{2x}u_2 + u_{2x}w_2), v_1\right\rangle_{L_2} \\ &\quad\quad + \left\langle DZ_2(\varphi,f)\cdot X(\varphi,f)\circ\varphi^{-1} + \frac{1}{2}(w_{1x}u_2 + u_{1x}w_2), v_2\right\rangle_{L_2}\end{aligned}$$

which is equivalent to

$$\begin{aligned}\int_{\mathbb{S}}\Big(&u_1v_{2x}w_2 + u_1v_2w_{2x} + \frac{1}{2}u_2v_{2x}w_1 + \frac{1}{2}u_{2x}v_2w_1 + \frac{1}{2}u_2v_1w_{2x} + \\ &+\frac{1}{2}u_{2x}v_1w_2 + \frac{1}{2}u_2v_{1x}w_2 + \frac{1}{2}u_{1x}v_2w_2 + \frac{1}{2}u_2v_2w_{1x} + \frac{1}{2}u_{1x}v_2w_2\Big)\,\mathrm{d}x = 0.\end{aligned}$$

Since the left-hand side is equal to

$$\int_{\mathbb{S}}\left(\frac{1}{2}\partial_x(u_2v_1w_2) + \frac{1}{2}\partial_x(u_2v_2w_1) + \partial_x(u_1v_2w_2)\right)\,\mathrm{d}x = 0$$

we are done. □

Remark 4.2. In general, the Christoffel map is only defined locally. In Proposition 4.1, we implicitly use the natural smooth identification

$$TH^sG \simeq H^sG \times \left(H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})\right) \tag{4.13}$$

and view Γ as a map from H^sG to the space of bilinear symmetric maps from $H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ to itself. Similarly, a vector field X on H^sG is viewed as a map $H^sG \to H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$. The identification (4.13), for the non-trivial part, is given in Sect. 3.1.4.

Since the existence of a smooth connection on a Banach manifold immediately yields the local existence and uniqueness of a geodesic flow (see [88]), Proposition 4.1 implies the following result.

Theorem 4.3. *Let $s > 5/2$. Then there exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, H^s G)$ of (4.8) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi\colon J \times U \to H^s G$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

We write the Cauchy problem for 2CH in the form

$$\begin{cases} u_t + uu_x &= -A^{-1}\partial_x\big(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\big), \\ \rho_t + u\rho_x &= \qquad\qquad -\rho u_x, \\ (u(0), \rho(0)) &= \qquad\qquad (u_0, \rho_0). \end{cases} \tag{4.14}$$

This formulation of 2CH is suitable for the formulation of weak solutions. It follows from Theorem 4.3 that the 2CH equation is locally well-posed in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ for $s > 5/2$.

Corollary 4.4. *Suppose $s > 5/2$. Then for any $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ there exists an open interval J centered at 0 and a unique solution*

$$(u, \rho) \in C(J, H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1(J, H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})) \tag{4.15}$$

of the Cauchy problem (4.14) which depends continuously on the initial data (u_0, ρ_0).

Proof. Theorem 4.3 yields the existence of a smooth curve $(\varphi(t), f(t)) \in H^s G$ such that $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Define $(u(t), \rho(t))$ by equation (4.6). Then, (u, ρ) has the regularity specified in (4.15) and depends continuously on (u_0, ρ_0). By right-invariance of Γ, the geodesic equation (4.8) can be written as

$$\begin{pmatrix} u_t + uu_x \\ \rho_t + u\rho_x \end{pmatrix} = \Gamma_{(\mathrm{id},0)}((u, \rho), (u, \rho)).$$

This is equation (4.14). □

Remark 4.5. The well-posedness result of Corollary 4.4 can also be proved using Kato's semigroup approach (see [42] for the case on the line).

The results of the previous discussion hold with the obvious changes also in the C^n-category. Assuming $n \geq 2$, the proofs are the same with $H^s G$ replaced with $C^n G$. In particular, Γ defines a smooth spray on $C^n G = \mathrm{Diff}^n(\mathbb{S}) \circledS C^{n-1}(\mathbb{S})$ compatible with the metric defined in (4.10). For the sake of brevity, we only state the analog of Theorem 4.3.

Theorem 4.6. *Let $n \geq 2$. Then there exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in C^n(\mathbb{S}) \times C^{n-1}(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, C^n G)$ of (4.8) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi\colon J \times U \to C^n G$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

4.2.2 Local well-posedness for smooth initial data

We now want to extend the results of the previous subsection to the space $C^\infty G = C^\infty \mathrm{Diff}(\mathbb{S}) \circledS C^\infty(\mathbb{S})$. Since $C^\infty G$ is not a Banach manifold, the local existence and uniqueness theorems for differential equations fail. We will therefore take an indirect approach and start with the local geodesic flows on $H^s G$, $s > 5/2$. We will first show that the domains of definition of these flows do not shrink to zero as $s \to \infty$. By considering the limit as $s \to \infty$, the existence of a smooth local geodesic flow on $C^\infty G$ will then be established. We will use a blow-up result for the 2CH equation which is proved in [42] for the 2CH on the real axis; observe that for the non-periodic 2CH, the term $\rho\rho_x$ has to be replaced by the term $-\rho\rho_x$ in (4.4). The following conservation law for the two-component CH equation will be essential for our purposes.

Lemma 4.7. *Let (u, ρ) be a solution of (4.4) with the geodesic flow (φ, f). Then, for any time t in the existence interval of (u, ρ), we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(m \circ \varphi)\varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x\right] = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\rho \circ \varphi)\varphi_x] = 0.$$

Proof. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[(m \circ \varphi)\varphi_x^2\right] &= [m_t \circ \varphi + (m_x \circ \varphi)\varphi_t]\varphi_x^2 + (m \circ \varphi) 2\varphi_x \varphi_{tx} \\ &= [(m_t + m_x u + 2m u_x) \circ \varphi]\varphi_x^2 \\ &= [-(\rho\rho_x) \circ \varphi]\varphi_x^2\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\rho \circ \varphi)\varphi_x] = [(\rho_t + u\rho_x) \circ \varphi]\varphi_x + [(\rho u_x) \circ \varphi]\varphi_x = 0.$$

Since

$$f_{tx} = \partial_x(\rho \circ \varphi) = (\rho_x \circ \varphi)\varphi_x$$

the lemma follows. □

Remark 4.8. Since $\varphi_{tx} = (u_x \circ \varphi)\varphi_x$ and $\varphi_x(0) = 1$ we have

$$\varphi_x(t) = \exp\left(\int_0^t (u_x \circ \varphi)(s)\,\mathrm{d}s\right).$$

If there exists $M > 0$ such that $u_x(t, x) \geq -M$ for all $(t, x) \in [0, T) \times \mathbb{S}$, then $\|1/\varphi_x\|_\infty \leq e^{MT}$. Hence we get from Lemma 4.7 that

$$\|\rho(t)\|_\infty = \|(\rho \circ \varphi)(t)\|_\infty = \left\|\frac{\rho_0}{\varphi_x(t)}\right\|_\infty \leq \|\rho_0\|_\infty e^{MT}, \quad \forall t \in [0, T). \tag{4.16}$$

Proposition 4.9. *Let $s > 5/2$. Let $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ and let $T > 0$ be the maximal time of existence of the solution*

$$(u, \rho) \in C([0, T), H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$$

of the Cauchy problem (4.14). *Then the solution* (u, ρ) *blows up in finite time if and only if*

$$\lim_{t\to T}\inf_{x\in\mathbb{S}} u_x(t,x) = -\infty \quad \textit{or} \quad \limsup_{t\to T}\|\rho_x(t)\|_{L^\infty} = \infty. \tag{4.17}$$

Proof. We multiply the first equation in (4.4) by $2m$ and integrate over $\mathbb{S}$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|m\|_{L_2}^2 &= -2\int_{\mathbb{S}} m m_x u \,\mathrm{d}x - 4\int_{\mathbb{S}} u_x m^2 \,\mathrm{d}x - 2\int_{\mathbb{S}} m\rho\rho_x \,\mathrm{d}x \\ &= -3\int_{\mathbb{S}} u_x m^2 \,\mathrm{d}x + \int_{\mathbb{S}} u_x \rho^2 \,\mathrm{d}x - \int_{\mathbb{S}} u_{xxx}\rho^2 \,\mathrm{d}x.\end{aligned}$$

Differentiating the first equation in (4.4) with respect to x, multiplying the obtained equation by $2m_x$ and integrating over $\mathbb{S}$ we next find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|m_x\|_{L_2}^2 &= -2\int_{\mathbb{S}} m_{xx} m_x u \,\mathrm{d}x - 6\int_{\mathbb{S}} m_x^2 u_x \,\mathrm{d}x - 4\int_{\mathbb{S}} u_{xx} m m_x \,\mathrm{d}x \\ &\quad -2\int_{\mathbb{S}} \rho_x^2 m_x \,\mathrm{d}x - 2\int_{\mathbb{S}} \rho\rho_{xx} m_x \,\mathrm{d}x \\ &= -5\int_{\mathbb{S}} m_x^2 u_x \,\mathrm{d}x + 2\int_{\mathbb{S}} u_{xxx} m^2 \,\mathrm{d}x + \int_{\mathbb{S}} u_{xxx}(2\rho_x^2 + 2\rho\rho_{xx} - \rho^2) \,\mathrm{d}x.\end{aligned}$$

Combining both equations we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (m^2 + m_x^2) \,\mathrm{d}x = &-\int_{\mathbb{S}} u_x m^2 \,\mathrm{d}x - 5\int_{\mathbb{S}} m_x^2 u_x \,\mathrm{d}x + \int_{\mathbb{S}} u_x \rho^2 \,\mathrm{d}x \\ &+ \int_{\mathbb{S}} u_{xxx}(2\rho_x^2 + 2\rho\rho_{xx} - 2\rho^2) \,\mathrm{d}x.\end{aligned}$$

Since the second equation in (4.4) is the same as for the 2CH on the real line, we refer to [42] for the derivation of the equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (\rho^2 + \rho_x^2 + \rho_{xx}^2) \,\mathrm{d}x = &-\int_{\mathbb{S}} u_x \rho^2 \,\mathrm{d}x - 3\int_{\mathbb{S}} u_x \rho_x^2 \,\mathrm{d}x - 5\int_{\mathbb{S}} u_x \rho_{xx}^2 \,\mathrm{d}x \\ &+ \int_{\mathbb{S}} u_{xxx}(\rho^2 + 3\rho_x^2 - 2\rho\rho_{xx}) \,\mathrm{d}x.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) \,\mathrm{d}x = &-\int_{\mathbb{S}} u_x m^2 \,\mathrm{d}x - 5\int_{\mathbb{S}} m_x^2 u_x \,\mathrm{d}x - 3\int_{\mathbb{S}} u_x \rho_x^2 \,\mathrm{d}x \\ &-5\int_{\mathbb{S}} u_x \rho_{xx}^2 \,\mathrm{d}x + \int_{\mathbb{S}} u_{xxx}(5\rho_x^2 - \rho^2) \,\mathrm{d}x.\end{aligned}$$

Assume that there exist $M_1, M_2 > 0$ such that

$$u_x(t,x) \geq -M_1, \quad \forall (t,x) \in [0,T) \times \mathbb{S}, \tag{4.18}$$

and

$$\|\rho_x(t)\|_\infty \leq M_2, \quad \forall t \in [0,T). \tag{4.19}$$

In view of (4.16), (4.18), (4.19), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}(m^2+m_x^2+\rho^2+\rho_x^2+\rho_{xx}^2)\,\mathrm{d}x \le C(M_1,M_2,T,\rho_0)\int_{\mathbb{S}}(m^2+m_x^2+\rho^2+\rho_x^2+\rho_{xx}^2)\,\mathrm{d}x$$

with a positive constant C. By means of Gronwall's inequality,

$$\|u(t)\|_{H^3}^2+\|\rho(t)\|_{H^2}^2 \le \|m(t)\|_{H^1}^2+\|\rho(t)\|_{H^2}^2 \le (\|m_0\|_{H^1}^2+\|\rho_0\|_{H^2}^2)e^{Ct},$$

for all $t\in[0,T)$. By the above inequality, Sobolev's imbedding theorem and the fact that the solution does not blow up in $H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ on $[0,T)$ if

$$\exists M>0\,\forall t\in[0,T):\quad \|u_x(t)\|_\infty+\|\rho(t)\|_\infty+\|\rho_x(t)\|_\infty \le M, \tag{4.20}$$

see [42], we obtain that the solution (u,ρ) does not blow up in finite time. On the other hand, by Sobolev's imbedding theorem, if

$$\lim_{t\to T}\inf_{x\in\mathbb{S}} u_x(t,x)=-\infty \quad \text{or} \quad \limsup_{t\to T}\|\rho_x(t)\|_{L^\infty}=\infty$$

then the solution will blow up in finite time. This completes the proof of our theorem. □

Let

$$\Phi_3\colon [0,T_3)\times U_3\to H^3G,$$

where $T_3>0$ and $U_3\subset H^3(\mathbb{S})\times H^2(\mathbb{S})$, be the local geodesic flow on H^3G whose existence is guaranteed by Theorem 4.3. In the next proposition, we show that the restriction of Φ_3 to $H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\ge 3$, defines a smooth flow on H^sG for $t\in[0,T_3)$. Thus, the flow on H^sG exists for all $t\in[0,T_3)$ for any $s\ge 3$.

Proposition 4.10. *Suppose $s>3$ and let Φ_s denote the restriction of Φ_3 to $[0,T_3)\times U_s$, where $U_s=U_3\cap(H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}))$. Then Φ_s is a smooth local flow of the geodesic equation (4.8) on H^sG, that is:*

a. Φ_s is a smooth map from $[0,T_3)\times U_s$ to H^sG.
b. For each $(u_0,\rho_0)\in U_s$, $\Phi_s(\cdot,u_0,\rho_0)$ is a smooth solution of Eq. (4.8) on $[0,T_3)$ satisfying $\Phi_s(0,u_0,\rho_0)=(\mathrm{id},0)$ and $\partial_t\Phi_s(0,u_0,\rho_0)=(u_0,\rho_0)$.

Proof. Fix $(u_0,\rho_0)\in U_3$ and let $(u(t;u_0,\rho_0),\rho(t;u_0,\rho_0))$ be the corresponding solution in $H^3(\mathbb{S})\times H^2(\mathbb{S})$ of the Cauchy problem (4.14). This solution is defined at least on $[0,T_3)$. Since the criterion (4.17) is independent of $s\ge 3$, it follows from Proposition 4.9 that if $(u_0,\rho_0)\in U_s$ for some $s\ge 3$, then the curve $t\mapsto(u(t;u_0,\rho_0),\rho(t;u_0,\rho_0))$ belongs to the space

$$C([0,T_3),H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}))\cap C^1([0,T_3),H^{s-1}(\mathbb{S})\times H^{s-2}(\mathbb{S})).$$

Let $(\varphi,f)\in C^\infty([0,T_3),H^3G)$ be the geodesic flow defined on $[0,T_3)$. Let $s>3$. Suppose $(u_0,\rho_0)\in U_s$ and $\varphi\in C^1([0,T_3),H^r\mathrm{Diff}(\mathbb{S}))$ for some r with $3\le r\le s-1$. We will show that $\varphi\in C^1([0,T_3),H^{r+1}\mathrm{Diff}(\mathbb{S}))$. Since

$$\varphi_{tx}=(u_x\circ\varphi)\varphi_x,\qquad \varphi_{txx}=(u_{xx}\circ\varphi)\varphi_x^2+(u_x\circ\varphi)\varphi_{xx},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right)=(u_{xx}\circ\varphi)\varphi_x.$$

Thus,

$$\varphi_{xx}(t)=\varphi_x(t)\int_0^t(u_{xx}\circ\varphi)(s)\varphi_x(s)\,\mathrm{d}s. \tag{4.21}$$

Since $\varphi_x \in C^1([0,T_3), H^{r-1}(\mathbb{S}))$ and $u_{xx} \in C([0,T_3), H^{s-2}(\mathbb{S}))$, Eq. (4.21) implies that

$$\varphi_{xx} \in C^1([0,T_3), H^{r-1}(\mathbb{S})). \tag{4.22}$$

This implies that $\varphi \in C^1([0,T_3), H^{r+1}\mathrm{Diff}(\mathbb{S}))$. Indeed,

$$\left\| \frac{\varphi(t)-\varphi(s)}{t-s} - u\circ\varphi \right\|_{H^{r+1}}^2 = \left\| \frac{\varphi(t)-\varphi(s)}{t-s} - u\circ\varphi \right\|_{H^1}^2 + \left\| \frac{\varphi_{xx}(t)-\varphi_{xx}(s)}{t-s} - (u\circ\varphi)_{xx} \right\|_{H^{r-1}}^2 .$$

As $t \to s$, the first term on the right-hand side vanishes since $\varphi \in C^\infty([0,T_3), H^3\mathrm{Diff}(\mathbb{S}))$ and the second vanishes in view of (4.22). Induction shows that

$$\varphi \in C^1([0,T_3), H^s\mathrm{Diff}(\mathbb{S})). \tag{4.23}$$

We now show that in fact $(\varphi, f) \in C^\infty([0,T_3), H^sG)$. By Lemma 4.7, $f_t\varphi_x = (\rho\circ\varphi)\varphi_x = \rho_0$ and we infer that

$$f(t) = \rho_0 \int_0^t \frac{\mathrm{d}s}{\varphi_x(s)}. \tag{4.24}$$

It follows that

$$f \in C^2([0,T_3), H^{s-1}(\mathbb{S})). \tag{4.25}$$

Moreover, by Theorem 4.3, (φ, f) is a smooth solution of (4.8) in $H^s\mathrm{Diff}(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ for sufficiently small $t \geq 0$. Standard ODE results show that the only way this solution can cease to exist (Corollary IV.1.8 in [88]) is either that the condition $\varphi_x > 0$ ceases to hold or that one of the norms

$$\|(\varphi_t, f_t)\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}, \quad \left\|\Gamma_{(\varphi,f)}\big((\varphi_t, f_t), (\varphi_t, f_t)\big)\right\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})} \tag{4.26}$$

blows up. But we know that $\varphi_x > 0$ on $[0,T_3)$ and Eqs. (4.23) and (4.25) together with the smoothness of Γ imply that the norms in (4.26) remain bounded on $[0,T_3)$. This proves (b). The standard ODE theorems on smooth dependence on initial data (Theorem IV.1.16 in [88]) imply (a). □

The Sobolev spaces $H^s(\mathbb{S})$ provide a Banach space approximation of the Fréchet space $C^\infty(\mathbb{S})$ in the sense of Definition 3.31. Proposition 4.10 together with Lemma 3.32 imply local well-posedness of the geodesic flow on $C^\infty G$.

Theorem 4.11. *There exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, C^\infty G)$ of (4.8) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi\colon J \times U \to C^\infty G$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

Since $C^\infty G$ is a Lie group with smooth multiplication and $(u, \rho) = (\varphi_t \circ \varphi^{-1}, f_t \circ \varphi^{-1})$, we immediately get the following result.

Corollary 4.12. *There exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution*

$$(u, \rho) \in C^\infty(J, C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}))$$

of (4.4) *with* $(u(0), \rho(0)) = (u_0, \rho_0)$. *Furthermore, the solution depends smoothly on the initial data in the sense that the local flow* $\Phi\colon J \times U \to C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ *defined by* $\Phi(t, u_0, \rho_0) = (u(t; u_0, \rho_0), \rho(t; u_0, \rho_0))$ *is a smooth map.*

Remark 4.13. In [120, 121], we find the general form of the geodesic equation on a semidirect product of two Lie groups G and H. In terms of the right logarithmic derivative (which corresponds to the Eulerian velocity in our terminology), the geodesic equation on $G \circledS H$ with a right-invariant metric given by the sum of positive definite inner products on the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is

$$\begin{cases} u_t = -\mathrm{ad}_u^* u + h(\rho, \rho), \\ \rho_t = -\mathrm{ad}_\rho^* \rho - b(u)^* \rho. \end{cases} \tag{4.27}$$

(If the smooth map $B\colon G \times H \to H$ denotes a left action of G on H and if we define the map $\beta\colon G \to \mathrm{Aut}(\mathfrak{h})$, $\beta(g) = D_e B(g)$ then $b\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is the derivative of β at the identity; $\mathrm{Aut}(\mathfrak{h})$ denotes the automorphism group of $\mathfrak{h}$ and $\mathrm{Der}(\mathfrak{h})$ the set of derivations of $\mathfrak{h}$. The map $h\colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ is defined by the relation $\langle b(X)Y_1, Y_2\rangle_{\mathfrak{h}} = \langle h(Y_1, Y_2), X\rangle_{\mathfrak{g}}$.) If V is a vector space with inner product and B is a linear action of G on V, the geodesic equation (4.27) on $G \circledS V$ becomes

$$\begin{cases} u_t = -\mathrm{ad}_u^* u + h(\rho, \rho), \\ \rho_t = \qquad -b(u)^* \rho. \end{cases}$$

In [120, 121] it is worked out that this general system reduces to the classical equations modelling ideal hydrodynamical flow for $G = \mathrm{Diff}^\infty(\mathbb{S})$ and $V = C^\infty(\mathbb{S})$ with the (left) action $\varphi f = f \circ \varphi^{-1}$.

4.2.3 Subspaces of positive sectional curvature

We have shown that the 2CH equation is a geodesic equation on the semidirect product $H^s G = H^s \mathrm{Diff}(\mathbb{S}) \circledS H^{s-1}(\mathbb{S})$ with respect to a smooth affine connection. The existence of a smooth connection ∇ on a Banach manifold immediately implies the existence of a smooth curvature tensor R defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields on $H^s G$. Since there exists a metric $\langle \cdot, \cdot \rangle$ for 2CH, we can also define an (unnormalized) sectional curvature S by[2]

$$S(X, Y) := \langle R(X, Y)Y, X\rangle.$$

[2] Recall that the sectional curvature $S(\sigma)$ of a subspace σ spanned by two tangent vectors u and v is defined by

$$S(\sigma) = \frac{\langle R(u, v)v, u\rangle}{|u \wedge v|^2}.$$

In this section, we will derive a convenient formula for S and use it to determine large subspaces of positive curvature for the 2CH equation. We will work in the H^s-category; similar results are valid with H^sG replaced with C^nG. In view of the right-invariance of ∇, it is enough to consider the curvature at the identity $(\mathrm{id}, 0)$. We will write Γ for $\Gamma_{(\mathrm{id},0)}$.

In a first step, we rewrite the Christoffel operator (4.7) as

$$\Gamma(u,v) = \frac{1}{2}\left[\begin{pmatrix} (u_1v_1)_x \\ u_{2x}v_1 + v_{2x}u_1 \end{pmatrix} + B(u,v) + B(v,u)\right] \tag{4.28}$$

with the bilinear operator $B = (B_1, B_2)$ on the Lie algebra $\mathfrak{g} \simeq H^s(\mathbb{S}) \circledS H^{s-1}(\mathbb{S})$ satisfying

$$\langle B(u,v), w\rangle = \langle u, [v,w]\rangle\,,$$

cf. Theorem 1.16. Writing $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$, we obtain from

$$\begin{aligned} \langle u, [v,w]\rangle &= \int_{\mathbb{S}} u_1 A(w_{1x}v_1 - v_{1x}w_1)\,\mathrm{d}x + \int_{\mathbb{S}} u_2(w_{2x}v_1 - v_{2x}w_1)\,\mathrm{d}x \\ &= -\int_{\mathbb{S}} A^{-1}(2v_{1x}Au_1 + v_1Au_{1x})Aw_1\,\mathrm{d}x \\ &\quad + \int_{\mathbb{S}} (-(u_2v_1)_x w_2 - u_2v_{2x}w_1)\,\mathrm{d}x \end{aligned}$$

that

$$\begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix} = \begin{pmatrix} -A^{-1}(2v_{1x}Au_1 + v_1Au_{1x} + u_2v_{2x}) \\ -(u_2v_1)_x \end{pmatrix}$$

and hence

$$\begin{aligned} B(u,u) + \begin{pmatrix} u_{1x}u_1 \\ u_{2x}u_1 \end{pmatrix} &= \begin{pmatrix} -A^{-1}(2u_{1x}Au_1 + u_1Au_{1x} + u_2u_{2x} - A(u_{1x}u_1)) \\ -(u_2u_1)_x + u_{2x}u_1 \end{pmatrix} \\ &= \begin{pmatrix} -A^{-1}(u_1^2 + \frac{1}{2}u_{1x}^2)_x - \frac{1}{2}A^{-1}(u_2^2)_x \\ -u_2u_{1x} \end{pmatrix} \\ &= \Gamma_{(\mathrm{id},0)}(u,u). \end{aligned}$$

Let us write $\Gamma_p(\cdot,\cdot) = \Gamma(p;\cdot,\cdot)$ and denote by D_1 differentiation of Γ_p with respect to p. The resulting formula for the sectional curvature for the 2CH equation is analogous to the formula obtained in Theorem 2.4—we only have to replace the Christoffel operator Γ^0 by Γ and the metric by its two-component extension in Theorem 2.4.

Theorem 4.14. *Let $s > 5/2$. Let R be the curvature tensor on H^sG associated with the 2CH equation. Then $S(u,v) := \langle R(u,v)v, u\rangle$ is given at the identity by*

$$S(u,v) = \langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle\,, \qquad u, v \in T_{(\mathrm{id},0)}H^sG.$$

Proof. Let $(\varphi, f) = p \in H^sG$, $X, Y, Z \in T_pH^sG$ and $(X,Y,Z)\circ\varphi^{-1} = (u,v,w)$. By the local formula for the curvature,

$$R(X,Y)Z = D_1\Gamma_p(Z,X)Y - D_1\Gamma_p(Z,Y)X + \Gamma_p(\Gamma_p(Z,Y),X) - \Gamma_p(\Gamma_p(Z,X),Y),$$

see the proof of Theorem 2.4. For the CH equation we found that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Gamma^0_{\mathrm{id}+\varepsilon v_1}(w_1,u_1) = -\Gamma^0_{\mathrm{id}}(w_{1x}v_1,u_1) - \Gamma^0_{\mathrm{id}}(u_{1x}v_1,w_1) + \Gamma^0_{\mathrm{id}}(w_1,u_1)_x v_1.$$

Since

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left[-\frac{1}{2}(u_2w_2)\circ(\mathrm{id}+\varepsilon v_1)^{-1}\right]_x = \frac{1}{2}((w_{2x}v_1)u_2)_x + \frac{1}{2}((u_{2x}v_1)w_2)_x$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left\{\left[-\frac{1}{2}(w_{1x}u_2+u_{1x}w_2)\right]\circ R_{(\mathrm{id}+\varepsilon v_1)^{-1}}\right\} = \frac{1}{2}((w_{1x}v_1)u_2)_x + \frac{1}{2}((u_{1x}v_1)w_2)_x,$$

we get

$$D_1\Gamma(w,u)v = -\Gamma(w_xv_1,u) - \Gamma(u_xv_1,w) + \Gamma(w,u)_xv_1$$

and hence

$$\begin{aligned} S(u,v) = &\left\langle \Gamma(\Gamma(v,v),u),u\right\rangle - \left\langle \Gamma(\Gamma(v,u),v),u\right\rangle + \left\langle \Gamma(v,u)_xv_1 - \Gamma(v,v)_xu_1,u\right\rangle \\ &+ \left\langle -\Gamma(v_xv_1,u) - \Gamma(v,u_xv_1) + 2\Gamma(v_xu_1,v),u\right\rangle . \end{aligned}$$

Using that $\Gamma = (\Gamma_1,\Gamma_2)$ is given by (4.28) we now compute

$$\begin{aligned} &\left\langle \Gamma(v,u)_xv_1 - \Gamma(v,v)_xu_1,u\right\rangle + \left\langle \Gamma(\Gamma(v,v),u),u\right\rangle - \left\langle \Gamma(\Gamma(v,u),v),u\right\rangle \\ &= \left\langle \Gamma(v,u)_xv_1 - \Gamma(v,v)_xu_1,u\right\rangle + \\ &\quad + \frac{1}{2}\left\langle \begin{pmatrix} (\Gamma_1(v,v)u_1)_x \\ \Gamma_2(v,v)_xu_1 + u_{2x}\Gamma_1(v,v)\end{pmatrix} + B(\Gamma(v,v),u) + B(u,\Gamma(v,v)),u\right\rangle \\ &\quad - \frac{1}{2}\left\langle \begin{pmatrix} (\Gamma_1(v,u)v_1)_x \\ \Gamma_2(v,u)_xv_1 + v_{2x}\Gamma_1(v,u)\end{pmatrix} + B(\Gamma(v,u),v) + B(v,\Gamma(v,u)),u\right\rangle \\ &= \frac{1}{2}\left\langle \begin{pmatrix} \Gamma_1(v,u)_xv_1 - \Gamma_1(v,u)v_{1x} \\ \Gamma_2(v,u)_xv_1 - \Gamma_1(v,u)v_{2x}\end{pmatrix},u\right\rangle + \frac{1}{2}\left\langle u,[\Gamma(v,v),u]\right\rangle \\ &\quad + \frac{1}{2}\left\langle \begin{pmatrix} \Gamma_1(v,v)u_{1x} - \Gamma_1(v,v)_xu_1 \\ \Gamma_1(v,v)u_{2x} - \Gamma_2(v,v)_xu_1\end{pmatrix},u\right\rangle - \frac{1}{2}\left\langle \Gamma(v,u),[v,u]\right\rangle \\ &\quad - \frac{1}{2}\left\langle v,[\Gamma(v,u),u]\right\rangle \\ &= \frac{1}{2}\left\langle [v,\Gamma(v,u)],u\right\rangle + \left\langle u,[\Gamma(v,v),u]\right\rangle - \frac{1}{2}\left\langle \Gamma(v,u),[v,u]\right\rangle - \frac{1}{2}\left\langle v,[\Gamma(v,u),u]\right\rangle \\ &= \frac{1}{2}\left\langle B(u,v),\Gamma(v,u)\right\rangle - \left\langle B(u,u),\Gamma(v,v)\right\rangle - \frac{1}{2}\left\langle \Gamma(v,u),[v,u]\right\rangle \\ &\quad + \frac{1}{2}\left\langle B(v,u),\Gamma(v,u)\right\rangle \\ &= \left\langle \Gamma(u,v),\Gamma(u,v)\right\rangle - \frac{1}{2}\left\langle \begin{pmatrix} (u_1v_1)_x \\ u_{2x}v_1 + v_{2x}u_1\end{pmatrix},\Gamma(u,v)\right\rangle - \frac{1}{2}\left\langle \Gamma(v,u),[v,u]\right\rangle \\ &\quad - \left\langle \Gamma(u,u),\Gamma(v,v)\right\rangle + \frac{1}{2}\left\langle \begin{pmatrix} (u_1^2)_x \\ u_{2x}u_1 + u_{2x}u_1\end{pmatrix},\Gamma(v,v)\right\rangle \\ &= \left\langle \Gamma(u,v),\Gamma(u,v)\right\rangle - \left\langle \Gamma(u,u),\Gamma(v,v)\right\rangle + \left\langle \begin{pmatrix} u_1u_{1x} \\ u_1u_{2x}\end{pmatrix},\Gamma(v,v)\right\rangle \\ &\quad - \frac{1}{2}\left\langle \begin{pmatrix} (u_1v_1)_x + u_{1x}v_1 - u_1v_{1x} \\ u_{2x}v_1 + v_{2x}u_1 + u_{2x}v_1 - v_{2x}u_1\end{pmatrix},\Gamma(u,v)\right\rangle \end{aligned}$$

which is equal to

$$\langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle + \left\langle \begin{pmatrix} u_1 u_{1x} \\ u_1 u_{2x} \end{pmatrix}, \Gamma(v,v) \right\rangle - \left\langle \begin{pmatrix} u_{1x} v_1 \\ u_{2x} v_1 \end{pmatrix}, \Gamma(u,v) \right\rangle .$$

Hence

$$\begin{aligned} S(u,v) = &\langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle \\ &+ \langle -\Gamma(v_x v_1, u) - \Gamma(v, u_x v_1) + 2\Gamma(v_x u_1, v), u\rangle \\ &- \left\langle \begin{pmatrix} u_{1x} v_1 \\ u_{2x} v_1 \end{pmatrix}, \Gamma(u,v) \right\rangle + \left\langle \begin{pmatrix} u_{1x} u_1 \\ u_{2x} u_1 \end{pmatrix}, \Gamma(v,v) \right\rangle . \end{aligned}$$

We now claim that that the sum of the last three terms is zero. To see this, we use that

$$\Gamma(u,v) = \begin{pmatrix} \Gamma^0(u_1, v_1) - \frac{1}{2} A^{-1}(u_2 v_2)_x \\ -\frac{1}{2}(u_{1x} v_2 + v_{1x} u_2) \end{pmatrix} \tag{4.29}$$

and that the terms involving Γ^0 cancel out as explained in the proof of Theorem 2.4. The remaining terms are

$$\begin{aligned} &\left\langle \begin{pmatrix} \frac{1}{2} A^{-1}(v_{2x} v_1 u_2)_x \\ \frac{1}{2}((v_{1x} v_1)_x u_2 + u_{1x}(v_{2x} v_1)) \end{pmatrix}, u \right\rangle + \left\langle \begin{pmatrix} \frac{1}{2} A^{-1}(v_2 u_{2x} v_1)_x \\ \frac{1}{2}((v_{1x} u_{2x} v_1 + (u_{1x} v_1)_x v_2) \end{pmatrix}, u \right\rangle \\ &- \left\langle \begin{pmatrix} A^{-1}(v_{2x} u_1 v_2)_x \\ ((v_{1x} u_1)_x v_2 + v_{1x} v_{2x} u_1) \end{pmatrix}, u \right\rangle + \left\langle \begin{pmatrix} u_{1x} v_1 \\ u_{2x} v_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} A^{-1}(u_2 v_2)_x \\ \frac{1}{2}(u_{1x} v_2 + v_{1x} u_2) \end{pmatrix} \right\rangle \\ &- \left\langle \begin{pmatrix} u_{1x} u_1 \\ u_{2x} u_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} A^{-1}(v_2^2)_x \\ v_{1x} v_2 \end{pmatrix} \right\rangle . \end{aligned}$$

We first consider the H^1-terms, i.e., the first row terms:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{S}} (v_{2x} v_1 u_2)_x u_1 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} (v_2 u_{2x} v_1)_x u_1 \,\mathrm{d}x - \int_{\mathbb{S}} (v_{2x} u_1 v_2)_x u_1 \,\mathrm{d}x \\ &+ \frac{1}{2}\int_{\mathbb{S}} u_{1x} v_1 (u_2 v_2)_x \,\mathrm{d}x - \int_{\mathbb{S}} u_{1x} u_1 v_2 v_{2x} \,\mathrm{d}x. \end{aligned}$$

The L_2-terms can be found in the second row:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{S}} (v_{1x} v_1)_x u_2^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} u_{1x} v_{2x} v_1 u_2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} v_{1x} u_{2x} v_1 u_2 \,\mathrm{d}x \\ &+ \frac{1}{2}\int_{\mathbb{S}} (u_{1x} v_1)_x v_2 u_2 \,\mathrm{d}x - \int_{\mathbb{S}} (v_{1x} u_1)_x v_2 u_2 \,\mathrm{d}x - \int_{\mathbb{S}} v_{1x} v_{2x} u_1 u_2 \,\mathrm{d}x + \\ &+ \frac{1}{2}\int_{\mathbb{S}} u_{2x} v_1 u_{1x} v_2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} u_{2x} v_1 v_{1x} u_2 \,\mathrm{d}x - \int_{\mathbb{S}} u_{2x} u_1 v_{1x} v_2 \,\mathrm{d}x. \end{aligned}$$

The terms quadratic in u_2 cancel out since

$$\frac{1}{2}\int_{\mathbb{S}} (v_{1x} v_1)_x u_2^2 \,\mathrm{d}x + \int_{\mathbb{S}} v_{1x} v_1 u_{2x} u_2 \,\mathrm{d}x = 0.$$

Similarly, the terms quadratic in u_1 cancel out:

$$- \int_{\mathbb{S}} (v_{2x} u_1 v_2)_x u_1 \, \mathrm{d}x - \int_{\mathbb{S}} u_1 u_{1x} v_2 v_{2x} \, \mathrm{d}x = 0.$$

A careful observation shows that the other terms also give zero:

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{S}} u_{1x} v_{2x} v_1 u_2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{S}} (u_{1x} v_1)_x v_2 u_2 \, \mathrm{d}x - \int_{\mathbb{S}} (v_{1x} u_1)_x v_2 u_2 \, \mathrm{d}x \\
&- \int_{\mathbb{S}} v_{1x} v_{2x} u_1 u_2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{S}} u_{2x} v_1 u_{1x} v_2 \, \mathrm{d}x - \int_{\mathbb{S}} u_{2x} u_1 v_{1x} v_2 \, \mathrm{d}x \\
&+ \frac{1}{2} \int_{\mathbb{S}} (v_{2x} v_1 u_2)_x u_1 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{S}} (v_2 u_{2x} v_1)_x u_1 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 (u_2 v_2)_x \, \mathrm{d}x \\
= &\frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_2 v_{2x} \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_{2x} v_2 \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_2 v_{2x} \, \mathrm{d}x \\
&- \int_{\mathbb{S}} (u_1 v_{1x})_x u_2 v_2 \, \mathrm{d}x - \int_{\mathbb{S}} u_1 v_{1x} u_2 v_{2x} \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_{2x} v_2 \, \mathrm{d}x \\
&- \int_{\mathbb{S}} u_1 v_{1x} u_{2x} v_2 \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_2 v_{2x} \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_{2x} v_2 \, \mathrm{d}x \\
&+ \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_{2x} v_2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{S}} u_{1x} v_1 u_2 v_{2x} \, \mathrm{d}x \\
= &\int_{\mathbb{S}} u_1 v_{1x} u_{2x} v_2 \, \mathrm{d}x + \int_{\mathbb{S}} u_1 v_{1x} u_2 v_{2x} \, \mathrm{d}x - \int_{\mathbb{S}} u_1 v_{1x} u_2 v_{2x} \, \mathrm{d}x - \int_{\mathbb{S}} u_1 v_{1x} u_{2x} v_2 \, \mathrm{d}x \\
= &\, 0.
\end{aligned}$$

This finishes our proof. □

Let us write S_2 for the sectional curvature of 2CH and S_1 for the CH sectional curvature. We are now interested in two-dimensional subspaces for which S_2 is positive. As explained in Chap. 1, this has various interesting geometric interpretations, e.g., concerning stability of the geodesics. Since

$$S_2 \left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \right) = S_1(u_1, v_1), \tag{4.30}$$

we directly conclude that the same examples of subspaces of positive curvature for the CH equation found in Sect. 2.2 work for the 2CH: Recall Theorem 2.5 where we showed for the CH equation that $S_1(u_1, v_1)$ is positive whenever u_1, v_1 are trigonometric functions of the form $\cos kx, \sin lx$ with $k \neq l \in 2\pi\mathbb{N}$. We now investigate the curvature of H^sG in directions which are non-trivial along the second component.

Proposition 4.15. *Let $s > 5/2$. Let $S(u,v) = S_2(u,v) := \langle R(u,v)v, u \rangle$ be the unnormalized sectional curvature on H^sG associated with the 2CH equation. Then*

$$S(u,v) > 0$$

for all vectors $u, v \in T_{(\mathrm{id},0)} H^sG$, $u \neq v$, of the form

$$u = \begin{pmatrix} \cos k_1 x \\ \cos k_2 x \end{pmatrix}, \quad v = \begin{pmatrix} \cos l_1 x \\ \cos l_2 x \end{pmatrix}, \qquad k_1, k_2, l_1, l_2 \in \{2\pi, 4\pi, \dots\}.$$

Moreover, the normalized sectional curvature satisfies

$$\frac{S(u,v)}{|u \wedge v|^2} \geq \frac{1}{8} \tag{4.31}$$

for all vectors $u, v \in T_{(\mathrm{id},0)}H^sG$, $u \neq v$, *of the form*

$$u = \begin{pmatrix} 0 \\ \cos k_2 x \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ \cos l_2 x \end{pmatrix}, \qquad k_2, l_2 \in \{2\pi, 4\pi, \dots\}.$$

Proof. Let us denote the components of u and v by u_1, u_2 and v_1, v_2. In the following computations, we use the relation

$$A^{-1}\cos\alpha x = \frac{1}{1+\alpha^2}\cos\alpha x,$$

Eq. (2.17) for $\Gamma^0(\cos k_1 x, \cos l_1 x)$,

$$\Gamma^0(\cos k_1 x, \cos l_1 x) = \partial_x\left[-\frac{\frac{1}{2}(1-\frac{1}{2}k_1 l_1)}{1+(k_1+l_1)^2}\cos(k_1+l_1)x - \frac{\frac{1}{2}(1+\frac{1}{2}k_1 l_1)}{1+(k_1-l_1)^2}\cos(k_1-l_1)x\right],$$

the trigonometric identities

$$\begin{aligned}
\cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)),\\
\sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),\\
\sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)),
\end{aligned}$$

as well as the orthogonality relations

$$\int_0^1 \cos(\alpha x)\cos(\beta x)\,\mathrm{d}x = \int_0^1 \sin(\alpha x)\sin(\beta x)\,\mathrm{d}x = \frac{1}{2}\left(\delta_{\alpha,\beta} \pm \delta_{\alpha,-\beta}\right) \tag{4.32}$$

and

$$\int_0^1 \cos(\alpha x)\sin(\beta x)\,\mathrm{d}x = 0 \tag{4.33}$$

for $\alpha, \beta \in 2\pi\mathbb{Z}$. According to Theorem 4.14 and (4.29),

$$\begin{aligned}
S_2(u,v) &= \langle\Gamma(u,v),\Gamma(u,v)\rangle - \langle\Gamma(u,u),\Gamma(v,v)\rangle\\
&= \int_{\mathbb{S}}\Gamma_1(u,v)A\Gamma_1(u,v)\,\mathrm{d}x + \int_{\mathbb{S}}\Gamma_2(u,v)^2\,\mathrm{d}x\\
&\quad - \int_{\mathbb{S}}\Gamma_1(u,u)A\Gamma_1(v,v)\,\mathrm{d}x - \int_{\mathbb{S}}\Gamma_2(u,u)\Gamma_2(v,v)\,\mathrm{d}x\\
&= \int_{\mathbb{S}}\left(\Gamma^0(u_1,v_1) - \frac{1}{2}A^{-1}(u_2v_2)_x\right)A\left(\Gamma^0(u_1,v_1) - \frac{1}{2}A^{-1}(u_2v_2)_x\right)\mathrm{d}x\\
&\quad - \int_{\mathbb{S}}\left(\Gamma^0(u_1,u_1) - \frac{1}{2}A^{-1}(u_2^2)_x\right)A\left(\Gamma^0(v_1,v_1) - \frac{1}{2}A^{-1}(v_2^2)_x\right)\mathrm{d}x\\
&\quad + \frac{1}{4}\int_{\mathbb{S}}(u_{1x}v_2 + v_{1x}u_2)^2\,\mathrm{d}x - \int_{\mathbb{S}}u_{1x}u_2v_{1x}v_2\,\mathrm{d}x,
\end{aligned}$$

and thus

$$S_2(u,v) = S_1(u_1,v_1) + \sum_{j=1}^{4} I_j$$

where

$$\begin{aligned}
I_1 &= \frac{1}{4}\int_{\mathbb{S}} (u_2 v_2)_x A^{-1} (u_2 v_2)_x \,\mathrm{d}x,\\
I_2 &= -\frac{1}{4}\int_{\mathbb{S}} (u_2^2)_x A^{-1} (v_2^2)_x \,\mathrm{d}x,\\
I_3 &= \frac{1}{2}\int_{\mathbb{S}} \left[\Gamma^0(u_1,u_1)(v_2^2)_x + \Gamma^0(v_1,v_1)(u_2^2)_x - 2\Gamma^0(u_1,v_1)(u_2v_2)_x\right] \mathrm{d}x,\\
I_4 &= \frac{1}{4}\int_{\mathbb{S}} (u_{1x}^2 v_2^2 + v_{1x}^2 u_2^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}} u_{1x} u_2 v_{1x} v_2 \,\mathrm{d}x
\end{aligned}$$

so that $S_2(u,v)$ equals the CH-curvature of u_1 and v_1 plus additional terms. Now using integration by parts we obtain

$$\begin{aligned}
I_1 &= -\frac{1}{4}\int_{\mathbb{S}} (u_2v_2)_{xx} A^{-1}(u_2v_2)\,\mathrm{d}x\\
&= -\frac{1}{16}\int_{\mathbb{S}} (\cos(k_2-l_2)x + \cos(k_2+l_2)x)_{xx}\, A^{-1}\,(\cos(k_2-l_2)x + \cos(k_2+l_2)x)\,\mathrm{d}x\\
&= \frac{1}{16}\int_{\mathbb{S}} \big((k_2-l_2)^2\cos(k_2-l_2)x + (k_2+l_2)^2\cos(k_2+l_2)x\big)\\
&\qquad \times A^{-1}\,(\cos(k_2-l_2)x + \cos(k_2+l_2)x)\,\mathrm{d}x\\
&= \frac{1}{16}\int_{\mathbb{S}} \big((k_2-l_2)^2\cos(k_2-l_2)x + (k_2+l_2)^2\cos(k_2+l_2)x\big)\\
&\qquad \times \left(\frac{\cos(k_2-l_2)x}{1+(k_2-l_2)^2} + \frac{\cos(k_2+l_2)x}{1+(k_2+l_2)^2}\right)\mathrm{d}x\\
&= \frac{1}{32}\left(\frac{(k_2-l_2)^2}{1+(k_2-l_2)^2} + \frac{(k_2+l_2)^2}{1+(k_2+l_2)^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= -\int_{\mathbb{S}} u_2 u_{2x} A^{-1} v_2 v_{2x}\,\mathrm{d}x\\
&= -\frac{k_2 l_2}{4}\int_{\mathbb{S}} (\sin 2k_2x) A^{-1} (\sin 2l_2 x)\,\mathrm{d}x\\
&= -\frac{1}{8}\frac{k_2^2}{1+(2k_2)^2}\delta_{k_2,l_2}
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{\mathbb{S}} \Gamma^0(u_1,v_1)_x(u_2v_2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(u_1,u_1)_x v_2^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(v_1,v_1)(u_2^2)_x\,\mathrm{d}x\\
&= \int_{\mathbb{S}} \left(\frac{\frac{1}{2}(1-\frac{1}{2}k_1l_1)(k_1+l_1)^2}{1+(k_1+l_1)^2}\cos(k_1+l_1)x + \frac{\frac{1}{2}(1+\frac{1}{2}k_1l_1)(k_1-l_1)^2}{1+(k_1-l_1)^2}\cos(k_1-l_1)x\right)\\
&\qquad \times \frac{1}{2}\,(\cos(k_2-l_2)x + \cos(k_2+l_2)x)\,\mathrm{d}x\\
&\quad -\frac{1}{4}\int_{\mathbb{S}} \frac{\frac{1}{2}(1-\frac{1}{2}k_1^2)}{1+(2k_1)^2}(2k_1)^2\cos(2k_1x)(1+\cos 2l_2x)\,\mathrm{d}x\\
&\quad +\frac{1}{2}\int_{\mathbb{S}} \Gamma^0(v_1,v_1)(u_2^2)_x\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\frac{(1-\frac{1}{2}k_1l_1)(k_1+l_1)^2}{1+(k_1+l_1)^2}\left(\delta_{k_1+l_1,k_2-l_2}+\delta_{k_1+l_1,l_2-k_2}+\delta_{k_1+l_1,k_2+l_2}\right)\\
&\quad+\frac{1}{8}\frac{(1+\frac{1}{2}k_1l_1)(k_1-l_1)^2}{1+(k_1-l_1)^2}(\delta_{k_1-l_1,k_2-l_2}+\delta_{k_1-l_1,l_2-k_2}+\delta_{k_1-l_1,k_2+l_2}+\delta_{l_1-k_1,k_2+l_2})\\
&\quad-\frac{k_1^2}{4}\frac{1-\frac{1}{2}k_1^2}{1+(2k_1)^2}\delta_{k_1,l_2}-\frac{l_1^2}{4}\frac{1-\frac{1}{2}l_1^2}{1+(2l_1)^2}\delta_{k_2,l_1}
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \frac{1}{4}k_1^2\int_{\mathbb{S}}\sin^2 k_1x\cos^2 l_2x\,\mathrm{d}x+\frac{1}{4}l_1^2\int_{\mathbb{S}}\sin^2 l_1x\cos^2 k_2x\,\mathrm{d}x\\
&\quad-\frac{1}{2}k_1l_1\int_{\mathbb{S}}\sin k_1x\cos k_2x\sin l_1x\cos l_2x\,\mathrm{d}x\\
&= \frac{1}{16}k_1^2\int_{\mathbb{S}}(1-\cos 2k_1x)\,(1+\cos 2l_2x)\,\mathrm{d}x+\frac{1}{16}l_1^2\int_{\mathbb{S}}(1-\cos 2l_1x)\,(1+\cos 2k_2x)\,\mathrm{d}x\\
&\qquad-\frac{1}{8}k_1l_1\int_{\mathbb{S}}(\cos(k_1-l_1)x-\cos(k_1+l_1)x)\,(\cos(k_2-l_2)x+\cos(k_2+l_2)x)\,\mathrm{d}x\\
&= \frac{1}{16}k_1^2\left(1-\frac{1}{2}\delta_{k_1,l_2}\right)+\frac{1}{16}l_1^2\left(1-\frac{1}{2}\delta_{l_1,k_2}\right)\\
&\quad-\frac{1}{16}k_1l_1(\delta_{k_1-l_1,k_2-l_2}+\delta_{k_1-l_1,l_2-k_2}+\delta_{k_1-l_1,k_2+l_2}+\delta_{l_1-k_1,k_2+l_2}\\
&\qquad-\delta_{k_1+l_1,k_2-l_2}-\delta_{k_1+l_1,l_2-k_2}-\delta_{k_1+l_1,k_2+l_2}).
\end{aligned}
$$

The sum of the negative terms occurring in the above computations can be estimated as follows:

$$
\begin{aligned}
&-\frac{1}{8}\frac{k_2^2}{1+(2k_2)^2}\delta_{k_2,l_2}\\
&-\frac{1}{16}k_1l_1\frac{(k_1+l_1)^2}{1+(k_1+l_1)^2}\left(\delta_{k_1+l_1,k_2-l_2}+\delta_{k_1+l_1,l_2-k_2}+\delta_{k_1+l_1,k_2+l_2}\right)\\
&-\frac{1}{16}k_1l_1\big(\delta_{k_1-l_1,k_2-l_2}+\delta_{k_1-l_1,l_2-k_2}+\delta_{k_1-l_1,k_2+l_2}+\delta_{l_1-k_1,k_2+l_2}\big)\\
\geq&-\frac{1}{32}-\frac{k_1l_1}{16}-\frac{k_1l_1}{16},
\end{aligned}
\tag{4.34}
$$

because at most one delta function within each bracket can give a nonzero contribution for a given set of values of $k_1,k_2,l_1,l_2\in\{2\pi,4\pi,\dots\}$. On the other hand, the term $S_1(u_1,v_1)$ contributes to $S(u,v)$ the positive term

$$
\frac{1}{8}\frac{(1-\frac{1}{2}k_1l_1)^2}{1+(k_1+l_1)^2}(k_1+l_1)^2, \tag{4.35}
$$

and the sum of the right-hand side of (4.34) and (4.35) is positive:

$$
\begin{aligned}
\frac{1}{8}\frac{(1-\frac{1}{2}k_1l_1)^2}{1+(k_1+l_1)^2}(k_1+l_1)^2-\frac{1}{32}-\frac{k_1l_1}{8} &\geq \frac{1}{16}\left(1-\frac{1}{2}k_1l_1\right)^2-\frac{1}{32}-\frac{k_1l_1}{8}\\
&= \frac{k_1^2l_1^2}{16}\left[\frac{1}{k_1^2l_1^2}-\frac{1}{k_1l_1}+\frac{1}{4}-\frac{1}{2k_1^2l_1^2}-\frac{2}{k_1l_1}\right]\\
&> 0,
\end{aligned}
$$

where we used that $k_1, l_1 \geq 2\pi$. This shows that $S_2(u,v) > 0$. In remains to prove (4.31). Suppose $u_1 = v_1 = 0$ and $u_2 \neq v_2$. Then

$$\begin{aligned} S\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right) &= I_1 + I_2 \\ &= \frac{1}{32}\left(\frac{(k_2 - l_2)^2}{1 + (k_2 - l_2)^2} + \frac{(k_2 + l_2)^2}{1 + (k_2 + l_2)^2}\right) - \frac{1}{8}\frac{k_2^2}{1 + (2k_2)^2}\delta_{k_2, l_2} \\ &\geq \frac{1}{64} + \frac{1}{64}, \end{aligned}$$

where we used that $k_2 \neq l_2$. On the other hand, for this choice of u and v,

$$\langle u, v \rangle = 0,$$

and hence

$$\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2 = \frac{1}{4}.$$

This yields (4.31). □

Remark 4.16. Although Proposition 4.15 establishes the existence of a large subspace of positive curvature, there are also directions for 2CH of strictly negative curvature. Indeed, it can be shown that there exist directions of strictly negative sectional curvature for the CH equation, [40]. In view of (4.30), this implies that 2CH also admits directions of negative curvature.

4.2.4 The 2CH equation and the motion of a rigid body

In this section, we make clear that the 2CH fits into the geometric approach introduced in [5, 37] to describe the motion of an ideal fluid in analogy to the motion of a rigid body (which already proved to be successful for the CH equation, cf. [62]). For the motion of a rigid body, we now recall the results presented in Sect. 1.2.2 for clarity. In the next step, we discuss the CH equation under the same aspects and include our results for 2CH in the final subsection.

4.2.4.1 The rotating rigid body

The configuration space of a rigid body in $\mathbb{R}^3$ rotating around its center of mass is the Lie group $SO(3)$. The corresponding Lie algebra is $\mathfrak{so}(3)$, the space of antisymmetric 3×3-matrices, which is canonically identified with $\mathbb{R}^3$. We can also identify the dual space $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$. Let $I \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the inertia matrix of the body. A *left*-invariant metric $\langle \cdot, \cdot \rangle$ on $SO(3)$ is given at the identity by

$$\langle a, b \rangle = a \cdot Ib, \quad \forall a, b \in \mathfrak{so}(3) \simeq \mathbb{R}^3.$$

The fact that the body's motion is described by the classical Euler equation can then be reformulated in the geometric picture: $R(t)$ is a geodesic on $(SO(3), \langle \cdot, \cdot \rangle)$ if and only if $\Omega(t) := R(t)^{-1}\dot{R}(t)$ solves the Euler equation

$$I\dot{\Omega} = (I\Omega) \times \Omega.$$

Physically, $\Omega(t)$ represents the angular velocity in the reference frame of the rotating body. The angular velocity in the spatially fixed frame of reference is given by $\dot{R}(t)R^{-1}(t)$. In other words: Applying left and right translations to the material angular velocity $\dot{R}(t)$, one obtains the body and the spatial angular velocity which both are elements of the Lie algebra $\mathfrak{so}(3)$. The body and spatial angular momenta, which are elements of the dual $\mathfrak{so}(3)^*$, are given by $\Pi(t) = I\Omega(t)$ and $\pi(t) = R(t)\Pi(t)$, respectively. The body and spatial quantities are related by the adjoint and coadjoint actions

$$\omega(t) = \mathrm{Ad}_{R(t)}\Omega(t) = R(t)\Omega(t)R^{-1}(t), \quad \Pi(t) = \mathrm{Ad}^*_{R(t)}\pi(t). \tag{4.36}$$

Conservation of (spatial) angular momentum means that π is constant in time, i.e.,

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} = 0. \tag{4.37}$$

4.2.4.2 The CH equation

For the CH equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{S}, \quad t > 0,$$

the configuration space is Diff($\mathbb{S}$) with multiplication $(\varphi, \psi) \mapsto \varphi \circ \psi$. Elements of the Lie algebra $\mathfrak{g}$ are identified with functions $\mathbb{S} \to \mathbb{R}$. A *right*-invariant metric is defined at the identity by

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{S}} uAv \,\mathrm{d}x = \int_{\mathbb{S}} (uv + u_x v_x)\,\mathrm{d}x;$$

here, $A = 1 - \partial_x^2 \colon \mathfrak{g} \to \mathfrak{g}^*$ is the inertia operator. In this picture, the CH equation is the Euler equation on the diffeomorphism group Diff($\mathbb{S}$) in the sense that $\varphi(t)$ is a geodesic in $(\mathrm{Diff}(\mathbb{S}), \langle\cdot,\cdot\rangle_{H^1})$ if and only if its Eulerian velocity $u(t) = D_{\varphi(t)}R_{\varphi^{-1}(t)}\varphi_t(t) = \varphi_t(t) \circ \varphi^{-1}(t)$ solves the CH equation. Letting $U = D_\varphi L_{\varphi^{-1}}\varphi_t = (u \circ \varphi)/\varphi_x$, U and u are the analogs of the body and spatial angular velocities: they are obtained by left resp. right translation of the material velocity φ_t to the Lie algebra. The momentum in the spatial frame is $m = Au$. The analog of Eq. (4.36) is

$$u(t) = \mathrm{Ad}_{\varphi(t)}U(t), \qquad m_0(t) = \mathrm{Ad}^*_{\varphi(t)}m(t),$$

where $m_0 = (m \circ \varphi)\varphi_x^2$ is the momentum in the body frame. Since the metric is now right- instead of left-invariant, the analog of the conservation law (4.37) is that the momentum m_0 in the body frame is conserved,

$$\frac{\mathrm{d}m_0}{\mathrm{d}t} = 0, \qquad \text{i.e.,} \qquad (m \circ \varphi)\varphi_x^2 = m_0.$$

4.2.4.3 The 2CH equation

For the 2CH equation (4.4) the configuration space is the semidirect product $G = \mathrm{Diff}(\mathbb{S})\circledS\mathcal{F}(\mathbb{S})$ introduced in Section 4.1. The Lie algebra $\mathfrak{g}$ is identified with $\mathcal{F}(\mathbb{S})\times\mathcal{F}(\mathbb{S})$. The inertia operator is $\mathrm{diag}(A, \mathrm{id})$ and the metric is the right-invariant metric $\langle\cdot,\cdot\rangle$ defined in (4.10). The basic observation is that $(\varphi(t), f(t))$ is a geodesic in $(\mathrm{Diff}(\mathbb{S})\circledS\mathcal{F}(\mathbb{S}), \langle\cdot,\cdot\rangle)$ if and only if

$$(u(t), \rho(t)) = D_{(\varphi(t),f(t))}R_{(\varphi(t),f(t))^{-1}}(\varphi_t(t), f_t(t))$$

satisfies (4.4). The analog of the body angular velocity is $(U_1, U_2) = D_{(\varphi,f)}L_{(\varphi,f)^{-1}}(\varphi_t, f_t)$ and is obtained from

$$\begin{aligned} D_{(\varphi,f)}L_{(\varphi,f)^{-1}}v &= \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}, -f\circ\varphi^{-1})(\gamma_1(t),\gamma_2(t))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\varphi^{-1}\circ\gamma_1(t), \gamma_2(t) - (f\circ\varphi^{-1})\circ\gamma_1(t))\Big|_{t=0} \\ &= \left(\frac{v_1}{\varphi_x}, v_2 - \frac{f_x}{\varphi_x}v_1\right), \end{aligned}$$

where $\gamma(t)$ is a curve starting at (φ, f) with velocity v. Thus

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \frac{u\circ\varphi}{\varphi_x} \\ \rho\circ\varphi - \frac{f_x}{\varphi_x}u\circ\varphi \end{pmatrix}.$$

The spatial momentum is $(m, \rho) = (Au, \rho)$. The analog of equation (4.36) is

$$(u(t), \rho(t)) = \mathrm{Ad}_{(\varphi(t),f(t))}(U_1(t), U_2(t)), \qquad (m_0(t), \rho_0(t)) = \mathrm{Ad}^*_{(\varphi(t),f(t))}(m(t), \rho(t)),$$

where (m_0, ρ_0) is the momentum in the body frame. In order to find an explicit expression for (m_0, ρ_0), we need to compute the adjoint and coadjoint actions. We have

$$\mathrm{Ad}_{(\varphi,f)}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} (v\varphi_x)\circ\varphi^{-1} \\ (\eta + f_x v)\circ\varphi^{-1} \end{pmatrix}.$$

The L_2-pairing is used to identify the (regular part of the) dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathcal{F}(\mathbb{S})\times\mathcal{F}(\mathbb{S})$. Since

$$\begin{aligned} \left\langle \begin{pmatrix} m \\ \rho \end{pmatrix}, \mathrm{Ad}_{(\varphi,f)}\begin{pmatrix} v \\ \eta \end{pmatrix} \right\rangle &= \int_{\mathbb{S}} m[(v\varphi_x)\circ\varphi^{-1}]\,\mathrm{d}x + \int_{\mathbb{S}} \rho[(f_x v + \eta)\circ\varphi^{-1}]\,\mathrm{d}x \\ &= \int_{\mathbb{S}} m(\varphi(y))\varphi_x^2(y)v(y)\,\mathrm{d}y + \int_{\mathbb{S}} \rho(\varphi(y))(f_x v + \eta)(y)\varphi_x(y)\,\mathrm{d}y \\ &= \left\langle \begin{pmatrix} (m\circ\varphi)\varphi_x^2 + (\rho\circ\varphi)f_x\varphi_x \\ (\rho\circ\varphi)\varphi_x \end{pmatrix}, \begin{pmatrix} v \\ \eta \end{pmatrix} \right\rangle \end{aligned}$$

we find

$$\mathrm{Ad}^*_{(\varphi,f)}\begin{pmatrix} m \\ \rho \end{pmatrix} = \begin{pmatrix} (m\circ\varphi)\varphi_x^2 + (\rho\circ\varphi)f_x\varphi_x \\ (\rho\circ\varphi)\varphi_x \end{pmatrix}, \qquad \begin{pmatrix} m \\ \rho \end{pmatrix} \in \mathfrak{g}^*.$$

The analog of the conservation law (4.37) is that the momentum (m_0, ρ_0) in the body frame is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix} = 0, \qquad \text{i.e.,} \qquad \begin{pmatrix} (m\circ\varphi)\varphi_x^2 + (\rho\circ\varphi)f_x\varphi_x \\ (\rho\circ\varphi)\varphi_x \end{pmatrix} = \begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix}.$$

This explains the origin of the conservation law established in Lemma 4.7.

Table 4.1 A rigid body, the CH equation and the 2CH equation: Geometric aspects.

	Rigid body	CH	2CH
configuration space	$SO(3)$	$\mathrm{Diff}(\mathbb{S})$	$\mathrm{Diff}(\mathbb{S}) \circledS \mathcal{F}(\mathbb{S})$
Lie algebra	$\mathfrak{so}(3)$	$\mathcal{F}(\mathbb{S})$	$\mathcal{F}(\mathbb{S}) \times \mathcal{F}(\mathbb{S})$
material velocity	$\dot{R}$	φ_t	(φ_t, f_t)
spatial velocity	$\omega = \dot{R}R^{-1}$	$u = \varphi_t \circ \varphi^{-1}$	$(u,\rho) = (\varphi_t, f_t)\circ\varphi^{-1}$
body velocity	$\Omega = R^{-1}\dot{R}$	$U = \frac{u\circ\varphi}{\varphi_x}$	$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \frac{u\circ\varphi}{\varphi_x} \\ \rho\circ\varphi - \frac{f_x}{\varphi_x}(u\circ\varphi) \end{pmatrix}$
inertia operator	I	$A = 1 - \partial_x^2$	$\begin{pmatrix} A & 0 \\ 0 & \mathrm{id} \end{pmatrix}$
spatial momentum	$\pi = RI\Omega$	$m = Au = u - u_{xx}$	$(m,\rho) = (Au,\rho)$
body momentum	$\Pi = I\Omega$	$m_0 = (m\circ\varphi)\varphi_x^2$	$\begin{pmatrix} m_0 \\ \rho_0 \end{pmatrix} = \begin{pmatrix} (m\circ\varphi)\varphi_x^2 + (\rho\circ\varphi)f_x\varphi_x \\ (\rho\circ\varphi)\varphi_x \end{pmatrix}$
spatial velocity (Ad)	$\omega = \mathrm{Ad}_R\Omega$	$u = \mathrm{Ad}_\varphi U$	$(u,\rho) = \mathrm{Ad}_{(\varphi,f)}(U_1,U_2)$
body momentum (Ad*)	$\Pi = \mathrm{Ad}_R^*\pi$	$m_0 = \mathrm{Ad}_\varphi^* m$	$(m_0,\rho_0) = \mathrm{Ad}^*_{(\varphi,f)}(m,\rho)$
momentum conservation	$\pi = \mathrm{const}$	$m_0 = \mathrm{const}$	$(m_0,\rho_0) = \mathrm{const}$

4.3 A generalization to the 2DP equation

Here we study the two-component generalization (4.5) of the DP equation as proposed in [115] on account of an appropriate Hamiltonian structure. Most of the results for 2CH presented in the previous section have direct counterparts in the case of 2DP; the main exception being that the geodesic flow associated with 2DP is not induced by any right-invariant metric. Therefore, we can apply the technology developed for the 2CH, working for the configuration spaces H^sG for $s > 5/2$, C^nG for $n \geq 2$ and finally $C^\infty G$. We introduce a bilinear operator on $\mathfrak{g} = T_{(\mathrm{id},0)}H^sG \simeq H^s \times H^{s-1}$ by setting

$$\Gamma_{(\mathrm{id},0)}(X,Y) = \begin{pmatrix} \Gamma^0_{\mathrm{id}}(X_1,Y_1) - \frac{1}{2}A^{-1}(X_2Y_{1x} + X_{1x}Y_2) + A^{-1}(X_2Y_2)_x \\ -(X_2Y_{1x} + X_{1x}Y_2) \end{pmatrix} \tag{4.38}$$

for $X = (X_1,X_2), Y = (Y_1,Y_2) \in \mathfrak{g}$, where

$$\Gamma^0_{\mathrm{id}}(u,v) = -\frac{3}{2}A^{-1}(uv)_x$$

denotes the Christoffel operator for the DP equation. It is important to recall that the DP equation belongs to the family of non-metric Euler equations, i.e., there exists no Riemannian metric $\langle\cdot,\cdot\rangle$ on $H^s\mathrm{Diff}(\mathbb{S})$ such that the DP equation can be written in the form

$$u_t = -\mathrm{ad}_u^* u,$$

where u is the Eulerian velocity of some smooth path in $H^s\mathrm{Diff}(\mathbb{S})$ and ad^* denotes the adjoint of ad with respect to $\langle\cdot,\cdot\rangle$, cf. [45]. We will use the bilinear operator introduced in (4.38) to define a right-invariant affine connection ∇ on H^sG by the local formula (4.9) and it will turn out that the 2DP equation is a geodesic equation on on H^sG with respect to this connection (although ∇ is not compatible with any Riemannian metric). The proof of the following proposition is similar to that of Proposition 4.1.

Proposition 4.17. *Let $s > 5/2$. Let $H^sG = H^s\mathrm{Diff}(\mathbb{S})\circledS H^{s-1}(\mathbb{S})$ and let Γ be the 2DP Christoffel map defined in* (4.38). *Then Γ defines a smooth spray on H^sG, i.e., the map*

$$(\varphi, f) \mapsto \Gamma_{(\varphi,f)} \colon H^sG \to \mathcal{L}^2_{\mathrm{sym}}(H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}))$$

is smooth.

Proof. The first component terms in (4.38) are of the form A^{-1} applied to a polynomial expression in X_i, Y_i and X_{ix}, Y_{ix}. The second component is a polynomial term in X_{1x}, X_2, Y_{1x}, Y_2. Hence the same arguments as in the proof of Proposition 4.1 can be applied. □

The existence of a smooth spray implies local existence and uniqueness of the geodesic flow. We obtain, by our definition of $\Gamma_{(\varphi,f)}$, that

$$\begin{aligned}
\begin{pmatrix} \varphi_{tt} \\ f_{tt} \end{pmatrix} &= \begin{pmatrix} (u_t + uu_x) \circ \varphi \\ (\rho_t + u\rho_x) \circ \varphi \end{pmatrix} \\
&= \begin{pmatrix} -[A^{-1}((Au)_x u + 3(Au)u_x - A(uu_x))] \circ \varphi - [A^{-1}(\rho u_x - 2\rho\rho_x)] \circ \varphi \\ -2(\rho u_x) \circ \varphi \end{pmatrix} \\
&= \begin{pmatrix} -3[A^{-1}(uu_x)] \circ \varphi - [A^{-1}(\rho u_x)] \circ \varphi + 2[A^{-1}(\rho\rho_x)] \circ \varphi \\ -2(\rho u_x) \circ \varphi \end{pmatrix} \\
&= \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t))
\end{aligned} \tag{4.39}$$

so that the 2DP equation is in fact the geodesic equation for the connection ∇, cf. Remark 2.3 and [88].

Theorem 4.18. *Let $s > 5/2$. Let Γ be the 2DP Christoffel map defined in* (4.38). *Then there exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, H^sG)$ of the geodesic equation* (4.39) *satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi\colon J \times U \to H^sG$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

We write the Cauchy problem for 2DP in the form

$$\begin{cases} u_t + uu_x &= -A^{-1}\left(\left(\frac{3}{2}u^2 - \rho^2\right)_x + \rho u_x\right), \\ \rho_t + u\rho_x &= -2\rho u_x, \\ (u(0), \rho(0)) &= (u_0, \rho_0). \end{cases} \tag{4.40}$$

It follows from Theorem 4.18 that 2DP is locally well-posed in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ for $s > 5/2$. According to a referee's suggestion it might be useful to repeat the arguments used in the proof of the well-posedness of the original 2DP equation (although everything works as for the 2CH equation).

Corollary 4.19. *Suppose $s > 5/2$. Then for any $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ there exists an open interval J centered at 0 and a unique solution*

$$(u,\rho) \in C(J, H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1(J, H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$$

of the Cauchy problem (4.40) *which depends continuously on the initial data* (u_0, ρ_0).

Proof. Let $(\varphi(t), f(t)) \in H^s G$ be the smooth curve with $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$ obtained in Theorem 4.18 and define $(u(t), \rho(t)) := (\varphi_t(t), f_t(t)) \circ \varphi^{-1}(t)$. Then, (u,ρ) has the regularity specified in the corollary and depends continuously on (u_0, ρ_0). By right-invariance of the 2DP Christoffel map Γ, the geodesic equation $(\varphi_{tt}, f_{tt}) = \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t))$ can be written as

$$\begin{pmatrix} u_t + uu_x \\ \rho_t + u\rho_x \end{pmatrix} = \Gamma_{(\mathrm{id},0)}((u,\rho),(u,\rho)).$$

This is equation (4.40).

The results of the above discussion hold with the obvious changes also in the C^n-category, $n \geq 2$. The following conservation laws are also only slightly different to the conservation laws presented in Lemma 4.7 for 2CH.

Lemma 4.20. *Let* (u,ρ) *be a solution of* (4.5) *with geodesic flow* (φ, f). *Then for any time* t *in the existence interval of* (u,ρ) *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}[(m \circ \varphi)\varphi_x^3 - (\rho \circ \varphi)\varphi_x^2(-\varphi_x + 2f_x)] = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\rho \circ \varphi)\varphi_x^2] = 0.$$

Proof. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[(m \circ \varphi)\varphi_x^3\right] &= [(m_t + um_x) \circ \varphi]\varphi_x^3 + 3\varphi_x^3(mu_x) \circ \varphi \\ &= [(-\rho u_x + 2\rho\rho_x) \circ \varphi]\varphi_x^3 \end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\rho \circ \varphi)\varphi_x^2\right] = [(\rho_t + u\rho_x) \circ \varphi]\varphi_x^2 + 2[(\rho u_x) \circ \varphi]\varphi_x^2 = 0.$$

With

$$-\varphi_{tx} + 2f_{tx} = -(u \circ \varphi)_x + 2(\rho \circ \varphi)_x = [(-u_x + 2\rho_x) \circ \varphi]\varphi_x$$

we are done. □

We have the following blow-up result for 2DP; the proof is similar to that of Proposition 4.9, cf. [40].

Proposition 4.21. *Let* $s > 5/2$. *Let* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ *and let* $T > 0$ *be the maximal time of existence of the solution*

$$(u,\rho) \in C([0,T), H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0,T), H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$$

of the Cauchy problem (4.40). *Then the solution* (u,ρ) *blows up in finite time if and only if*

$$\liminf_{t \to T} \inf_{x \in \mathbb{S}} u_x(t,x) = -\infty \quad or \quad \limsup_{t \to T} \|\rho_x(t)\|_{L^\infty} = \infty.$$

Let

$$\Phi_3 \colon [0, T_3) \times U_3 \to H^3 G,$$

where $T_3 > 0$ and $U_3 \subset H^3(\mathbb{S}) \times H^2(\mathbb{S})$, be the local geodesic flow on $H^3 G$ whose existence is guaranteed by Theorem 4.18.

Proposition 4.22. *Suppose $s > 3$ and let Φ_s denote the restriction of Φ_3 to $[0, T_3) \times U_s$, where $U_s = U_3 \cap (H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}))$. Let Γ be the 2DP Christoffel map defined in* (4.38)*. Then Φ_s is a smooth local flow of the geodesic equation* (4.39) *on $H^s G$, that is,*

a. Φ_s is a smooth map from $[0, T_3) \times U_s$ to $H^s G$.
b. For each $(u_0, \rho_0) \in U_s$, $\Phi_s(\cdot, u_0, \rho_0)$ is a smooth solution of equation (4.39) *on $[0, T_3)$ satisfying $\Phi_s(0, u_0, \rho_0) = (\mathrm{id}, 0)$ and $\partial_t \Phi_s(0, u_0, \rho_0) = (u_0, \rho_0)$.*

Proof. The proof is identical to that of Proposition 4.10 except that equation (4.24) must be replaced with

$$f(t) = \rho_0 \int_0^t \frac{\mathrm{d}s}{\varphi_x^2(s)},$$

cf. Lemma 4.20. □

We thus obtain the following well-posedness results.

Theorem 4.23. *Let Γ be the 2DP Christoffel map. There exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, C^\infty G)$ of the geodesic equation* (4.39) *satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi \colon J \times U \to C^\infty G$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

Corollary 4.24. *There exists an open interval J centered at 0 and an open neighborhood U of $(0,0) \in C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution*

$$(u, \rho) \in C^\infty(J, C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}))$$

of (4.5) *with $(u(0), \rho(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi \colon J \times U \to C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ defined by $\Phi(t, u_0, \rho_0) = (u(t; u_0, \rho_0), \rho(t; u_0, \rho_0))$ is a smooth map.*

4.4 The Euler formalism for the 2HS equation and the 2μHS equation

We now consider a two-component variant of the HS equation which we introduced in Sect. 1.4. The idea is to start with Eq. (4.4) and to replace $m = -u_{xx}$, cf. [90]. From the 2HS equation we also obtain the corresponding μ-version by setting $m = \mu(u) - u_{xx}$.

Let $t \geq 0$ and $x \in \mathbb{S}$. By a solution of the periodic 2-component Hunter-Saxton equation with initial data (u_0, ρ_0) we mean a function $(u(t,x), \rho(t,x))$ which satisfies

$$\begin{cases} m_t &= -um_x - 2mu_x - \rho\rho_x, \; t > 0, \\ \rho_t &= -(\rho u)_x, \qquad\qquad\quad\;\; t > 0, \\ m &= -u_{xx}, \qquad\qquad\qquad t \geq 0, \\ u &= u_0, \qquad\qquad\qquad\quad\; t = 0, \\ \rho &= \rho_0, \qquad\qquad\qquad\quad\; t = 0. \end{cases} \tag{4.41}$$

Similarly, we say that (u, ρ) solves the 2μHS equation if (4.41) holds true with $m = \mu(u) - u_{xx}$.

Remark 4.25. Eq. (4.41) reduces to the HS (the μHS, respectively) for $\rho = 0$. The 2μHS equation is also called 2μCH equation since the μ-variant of HS equals the μ-variant of CH. Observe that we obtain the same operator if we add μ to the inertia operator for the HS or replace the identity by μ in the inertia operator for the CH.

In [90] the authors derive the 2HS equation as the $N = 2$ supersymmetric extension of the Camassa-Holm equation. They also work out the bi-Hamiltonian formulation and a Lax pair representation for the 2HS equation. Concerning geometry, the 2HS can be regarded as an Euler equation on the superconformal algebra of contact vector fields on the 1|2-dimensional supercircle. Finally, the paper [90] presents some explicit solutions of Eq. (4.41), like bounded travelling waves.

In this section we are concerned with some geometric aspects of the 2HS equation. We prove that the 2HS can be regarded as an evolution equation on a semidirect product obtained from the H^s-diffeomorphisms on $\mathbb{S}$ for s sufficiently large[3]. Most importantly, we show that Eq. (4.41) is compatible with the Riemannian structure induced by the $\dot{H}^1$ inner product for the first component plus the L_2 inner product for the second one at the identity $(\mathrm{id}, 0)$. Defining an affine connection in terms of the Christoffel operator for the 2HS equation, we therefrom establish that 2HS is related to a geodesic flow on the underlying semidirect product configuration space. We are mainly concerned with the sectional curvature for the 2HS equation and show that it has the constant value $1/4$ for any two-dimensional subspace. An analogous result has been obtained in [95] for the one-component Hunter-Saxton equation.

Finally, we discuss the 2μHS equation for which we first construct a Lax pair. Second, we explain that the 2μHS is an Euler equation on $H^s\mathrm{Diff}(\mathbb{S}) ⓢ H^{s-1}(\mathbb{S})$ and a reexpression of a geodesic flow on the semidirect product; therefore we specify the Christoffel operator for 2μHS. We then obtain an infinite-dimensional subspace of positive sectional curvature for 2μHS.

The reader can easily see how the geometric point of view on the 2HS equation and the 2μHS equation corresponds to Arnold's powerful geometric picture which proved to be successful not only for the motion of inertia rigid objects in Classical Mechanics but also for water wave equations like the CH and its two-component version, as explained in Sect. 4.2.4. Since many of the arguments for HS and 2μHS are very similar to the corresponding arguments for 2CH we will often refer to Sect. 4.2.

[3] More precisely, we will work with the diffeomorphisms modulo rotations in the first component to enforce that any $u \in T_{\mathrm{id}} H^s\mathrm{Diff}(\mathbb{S})$ satisfies $u(0) = 0$. This will be explained in the following subsection.

4.4.1 Geometry associated with the HS equation

In Sect. 3.3.1, we explained taking the example of the periodic b-equation how to obtain a local well-posedness result in case of smooth initial data: The solution $u(t,x)$ of (1.41) can be regarded as a vector field on $\mathbb{S}$ so that, if $u(t,\cdot) \in C^n(\mathbb{S})$ for some finite $n \in \mathbb{N}$, there exists a local flow $\varphi(t,\cdot) \in \mathrm{Diff}^n(\mathbb{S})$ such that $u = \varphi_t \circ \varphi^{-1}$ and $\varphi(0) = \mathrm{id}$. Then some elementary calculations show that Eq. (1.41) is equivalent to a first order differential equation for the flow $X(t) = (\varphi(t), \varphi_t(t))$ where the right-hand side depends smoothly on X. The Cauchy-Lipschitz Theorem proves the existence of a solution, uniqueness and smooth dependence on time and the initial data, for some time interval containing zero. Some further arguments show that this solution is in $C^\infty(\mathbb{S})$, provided $u_0 \in C^\infty(\mathbb{S})$.

Let us try to adopt this technique for the HS equation, i.e., let us rewrite the HS as an autonomous system in terms of the local flow $X(t) \in \mathrm{Diff}(\mathbb{S}) \times \mathcal{F}(\mathbb{S})$ for the time-dependent vector field $u(t,\cdot)$ on $\mathbb{S}$. The usual starting point is to compute φ_{tt}. By the chain rule and from the relation $\varphi_t = u \circ \varphi$, we obtain

$$\varphi_{tt} = (u_t + uu_x) \circ \varphi.$$

We now have to replace u_t by using Eq. (1.42). But since Eq. (1.42) only includes u_{txx}, we differentiate twice with respect to x to obtain

$$\begin{aligned}\partial_x^2(u_t + uu_x) &= u_{txx} + 3u_x u_{xx} + uu_{xxx} \\ &= u_x u_{xx} \\ &= \frac{1}{2}\partial_x u_x^2. \end{aligned} \tag{4.42}$$

Let A^{-1} be the inverse of the operator $A = -\partial_x^2$ studied in Lemma 3.5. If $u_x \in C(\mathbb{S})$, the right-hand side of (4.42) is a function with zero mean; hence it is in the domain of A^{-1} and we conclude

$$\varphi_{tt} = -\frac{1}{2}\left[A^{-1}\partial_x(\varphi_t \circ \varphi^{-1})_x^2\right] \circ \varphi.$$

Note also that $u_t + uu_x$ must belong to the domain of A which suggests that we will need the assumption $u(0) = 0$. Setting

$$\Gamma(u,v) = -\frac{1}{2}A^{-1}(u_x v_x)_x, \tag{4.43}$$

we obtain a symmetric bilinear operator $\mathcal{F}(\mathbb{S}) \times \mathcal{F}(\mathbb{S}) \to \mathcal{F}(\mathbb{S})$. In fact, this Christoffel map is smooth (in the categories under consideration) which enables the following geometric approach, established in [97]:

For $s \geq 3$ we consider the the Banach manifold $H^s\mathrm{Diff}(\mathbb{S})$ of orientation-preserving diffeomorphisms $\mathbb{S} \to \mathbb{S}$ of Sobolev class H^s. Let $\mathrm{Rot}(\mathbb{S}) \subset H^s\mathrm{Diff}(\mathbb{S})$ be the subgroup of rotations $x \mapsto x + d$ for some $d \in \mathbb{R}$. We denote by $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ the space of right cosets $\mathrm{Rot}(\mathbb{S}) \circ \varphi = \{\varphi(\cdot) + d;\ d \in \mathbb{R}\}$, for $\varphi \in H^s\mathrm{Diff}(\mathbb{S})$, and set $M^s = \{\varphi \in H^s\mathrm{Diff}(\mathbb{S});\ \varphi(0) = 0\}$. We have

$$M^s = \{\mathrm{id} + u;\ u \in H^s,\ u_x > -1,\ u(0) = 0\}$$

and thus M^s is an open subset of the closed hyperplane

$$\mathrm{id} + E^s = \mathrm{id} + \{u \in H^s;\ u(0) = 0\} \subset H^s.$$

Writing the elements of $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ as $[\varphi]$, the map $[\varphi] \mapsto \varphi - \varphi(0)$ establishes a diffeomorphism $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S}) \to M^s$, showing in this way that M^s is a global chart for $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$. Furthermore, all tangent spaces $T_\varphi M^s$ can be identified with E^s. Next, we define a right-invariant metric on $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ by setting

$$\langle U, V\rangle_\varphi = \frac{1}{4}\int_{\mathbb{S}} (U\circ\varphi^{-1}) A(V\circ\varphi^{-1})\,\mathrm{d}x = \frac{1}{4}\int_{\mathbb{S}} \frac{U_x V_x}{\varphi_x}\,\mathrm{d}x \tag{4.44}$$

for tangent vectors $U, V \in T_\varphi M^s \simeq E^s$ at $\varphi \in M^s$. Recall that the bilinear form $\langle\cdot,\cdot\rangle_{\mathrm{id}}$ at the identity, induced by the operator A defined in Lemma 3.5, is the $\dot{H}^1$-metric and that our definition of A ensures that $\langle\cdot,\cdot\rangle_{\mathrm{id}}$ is indeed a positive definite inner product[4]. Furthermore the metric (4.44) is compatible with the affine connection ∇ defined locally by

$$\nabla_X Y(\varphi) = DY(\varphi)\cdot X(\varphi) - \Gamma(\varphi; Y(\varphi), X(\varphi)),$$

where $\Gamma(\varphi;\cdot,\cdot) = R_\varphi \circ \Gamma(\mathrm{id},\cdot,\cdot)\circ R_\varphi^{-1}$ is the smooth Christoffel map for the HS equation with $\Gamma(\mathrm{id}, u, v) = -\frac{1}{2}A^{-1}(u_x v_x)_x$. As proved in [97], the geodesics of the $\dot{H}^1$ right-invariant metric are described by the HS equation: Let $J \subset \mathbb{R}$ be an open interval and let $\varphi\colon J \to H^s\mathrm{Diff}(\mathbb{S})$ be a smooth curve. Then the curve $u\colon J \to T_{\mathrm{id}}H^s\mathrm{Diff}(\mathbb{S})$ defined by $u\colon t \mapsto \varphi_t \circ \varphi^{-1}$ satisfies the HS equation (1.42) if and only if the curve $[\varphi]\colon J \to H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ given by $[\varphi]\colon t \mapsto [\varphi(t)]$ is a geodesic with respect to ∇. The geodesic in $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ can be found explicitly by the method of characteristics: For $u_0 \in T_{\mathrm{id}}M^s$ with $\langle u_0, u_0\rangle = 1$ the unique geodesic $\varphi\colon [0, T^*(u_0)) \to M^s$ with $\varphi(0) = \mathrm{id}$ and $\varphi_t(0) = u_0$ is given by

$$\varphi(t) = \mathrm{id} - \frac{1}{8}\left(A^{-1}\partial_x\left(u_{0x}^2\right)\right)(1-\cos 2t) + \frac{1}{2}u_0 \sin 2t,$$

where the maximal time of existence is

$$T^*(u_0) = \frac{\pi}{2} + \arctan\left(\frac{1}{2}\min_{x\in\mathbb{S}} u_{0x}(x)\right) < \pi/2.$$

Observe that the corresponding solution $u = \varphi_t\circ\varphi^{-1} \in C([0,T^*); E^s)\cap C^1([0,T^*); E^{s-1})$ of the HS is not unique; the set of solutions is

$$\{t \mapsto u(t, \cdot - c(t)) + c'(t)\} \subset C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S})),$$

where $T \leq T^*$ is the maximal time of existence, $c\colon [0,T) \to \mathbb{R}$ is an arbitrary C^1-function with $c(0) = c'(0) = 0$ and if $T < T^*$, then $|c(t)| \to \infty$ as $t \to T$ from below. Further geometric aspects of the HS equation are discussed in [95, 96].

4.4.2 The geodesic flow for the 2HS equation

We define $H^sG_0 = [H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})] \circledS E^{s-1}(\mathbb{S})$, $s \geq 3$; this definition is motivated by the results in Sect. 4.4.1, where we recalled that the group $H^s\mathrm{Diff}(\mathbb{S})$ modulo rotations is suitable for the one-component HS, and Sect. 4.2, where we saw for the 2CH that the

[4] The factor 1/4 is introduced to obtain that the sectional curvature for HS is identically equal to one. Here and in the sequel, we assume that A is an operator on $\{u \in H^s(\mathbb{S});\, u(0) = 0\}$ for $s \geq 3$.

product of the group and its tangent space at id (with regularity lowered by one) is a good candidate for the two-component generalization. Let us define a bilinear operator Γ on $E^s \times E^{s-1}$ by

$$\Gamma(X,Y) \coloneqq \begin{pmatrix} \Gamma^0_{\mathrm{id}}(X_1,Y_1) - \frac{1}{2}A^{-1}(X_2Y_2)_x \\ -\frac{1}{2}(X_{1x}Y_2 + Y_{1x}X_2) \end{pmatrix}; \tag{4.45}$$

A and Γ^0 are as in Lemma 3.5 and (4.43). As a map $\Gamma\colon (M^s \times E^{s-1}) \times (E^s \times E^{s-1})^2 \to E^s \times E^{s-1}$, Γ is defined by

$$\Gamma_{(\varphi,f)}(X,Y) = \Gamma((\varphi,f);X,Y) = \Gamma(X\circ\varphi^{-1}, Y\circ\varphi^{-1})\circ\varphi. \tag{4.46}$$

We next introduce the positive definite inner product $\langle u,v\rangle \coloneqq \langle u_1,v_1\rangle_{\dot{H}_1} + \langle u_2,v_2\rangle_{L_2}$, for $u,v \in E^s\times E^{s-1}$, and for $(\varphi,f)\in M^s\times E^{s-1}$ and $X,Y \in T_{(\varphi,f)}(M^s\times E^{s-1}) \simeq E^s\times E^{s-1}$ we define

$$\langle X,Y\rangle_{(\varphi,f)} \coloneqq \left\langle X\circ\varphi^{-1}, Y\circ\varphi^{-1}\right\rangle, \tag{4.47}$$

to obtain a right-invariant metric on H^sG_0. Furthermore, we let

$$\nabla_X Y(\varphi,f) = DY(\varphi,f)\cdot X(\varphi,f) - \Gamma_{(\varphi,f)}(Y(\varphi,f),X(\varphi,f)), \tag{4.48}$$

for vector fields X and Y on H^sG_0; here, $X,Y\colon M^s\times E^{s-1}\to E^s\times E^{s-1}$ are representatives of the vector fields X and Y in the global chart $M^s\times E^{s-1}$. As for the 2CH equation we establish that $\langle\cdot,\cdot\rangle_{(\varphi,f)}$ defines indeed a (weak) Riemannian metric which is compatible with the smooth connection ∇ to ensure the existence and uniqueness of geometric objects like geodesic flows or sectional curvatures.

Proposition 4.26. *Let $s\geq 3$. Let $H^sG_0 = [H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})]\circledS E^{s-1}(\mathbb{S})$ and let Γ be the Christoffel map defined in* (4.45) *and* (4.46). *Then Γ defines a smooth spray on H^sG_0, i.e., the map*

$$(\varphi,f)\mapsto\Gamma_{(\varphi,f)}\colon H^sG_0\to\mathcal{L}^2_{\mathrm{sym}}(E^s\times E^{s-1};E^s\times E^{s-1})$$

is smooth. Moreover, the metric $\langle\cdot,\cdot\rangle$ defined in (4.47) *is a smooth (weak) Riemannian metric on H^sG_0, i.e., the map*

$$([\varphi],f)\mapsto\langle\cdot,\cdot\rangle_{([\varphi],f)}:H^sG_0\to\mathcal{L}^2_{\mathrm{sym}}(T_{([\varphi],f)}H^sG_0;\mathbb{R})$$

is a smooth section of the bundle $\mathcal{L}^2_{\mathrm{sym}}(TH^sG_0;\mathbb{R})$. Finally, the connection ∇ in (4.48) *and the metric $\langle\cdot,\cdot\rangle$ are compatible in the sense that*

$$X\langle Y,Z\rangle = \langle\nabla_X Y,Z\rangle + \langle Y,\nabla_X Z\rangle$$

for all vector fields X,Y,Z on H^sG_0.

Proof. That the Christoffel map is smooth follows from the smoothness of $\varphi\mapsto\Gamma^0(\varphi,\cdot,\cdot)$ established in [97]. The other terms can be discussed as for the 2CH equation. That $([\varphi],f)\mapsto\langle\cdot,\cdot\rangle_{([\varphi],f)}$ is smooth follows from the smoothness of the $\dot{H}^1$ right-invariant metric on $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ and the fact that the second component makes the same contribution to the first component term as for the 2CH. Since the $\dot{H}^1$ right-invariant metric is compatible with the connection defined canonically by Γ^0 we are done since the terms including the second component can be discussed as in the proof of Proposition 4.1. □

By our definition of $\Gamma_{(\varphi,f)}$,

$$\begin{aligned}\begin{pmatrix}\varphi_{tt}\\ f_{tt}\end{pmatrix} &= \begin{pmatrix}(u_t+uu_x)\circ\varphi\\ (\rho_t+u\rho_x)\circ\varphi\end{pmatrix}\\ &= \begin{pmatrix}-[A^{-1}(u(Au)_x+2(Au)u_x-A(uu_x))]\circ\varphi-[A^{-1}(\rho\rho_x)]\circ\varphi\\ -(\rho u_x)\circ\varphi\end{pmatrix}\\ &= \begin{pmatrix}-[A^{-1}(u_xu_{xx})]\circ\varphi-[A^{-1}(\rho\rho_x)]\circ\varphi\\ -(\rho u_x)\circ\varphi\end{pmatrix}\\ &= \Gamma_{(\varphi,f)}((\varphi_t,f_t),(\varphi_t,f_t))\end{aligned} \tag{4.49}$$

so that the 2HS equation is in fact the geodesic equation for the connection ∇, i.e., $(u,\rho)=(\varphi_t,f_t)\circ\varphi^{-1}$ solves the 2HS if and only if (φ_t,f_t) is a solution of (4.49). Again, we conclude the following local well-posedness result from Proposition 4.26.

Theorem 4.27. *Let $s\geq 3$. Then there exists an open interval J centered at 0 and an open neighborhood U of $(0,0)\in E^s\times E^{s-1}$ such that for each $(u_0,\rho_0)\in U$ there exists a unique solution $(\varphi,f)\in C^\infty(J,H^sG_0)$ of (4.49) satisfying $(\varphi(0),f(0))=(\mathrm{id},0)$ and $(\varphi_t(0),f_t(0))=(u_0,\rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi\colon J\times U\to H^sG_0$ defined by $\Phi(t,u_0,\rho_0)=(\varphi(t;u_0,\rho_0),f(t;u_0,\rho_0))$ is a smooth map.*

It follows from Theorem 4.27 that 2HS is locally well-posed in $E^s\times E^{s-1}$ for $s\geq 3$. Recall that HS is not well-posed in $H^s(\mathbb{S})$; concerning unity of the second component solution we refer to Chap. 6.

Corollary 4.28. *Suppose $s\geq 3$. Then for any $(u_0,\rho_0)\in E^s\times E^{s-1}$ there exists an open interval J centered at 0 and a unique solution*

$$(u,\rho)\in C(J,E^s\times E^{s-1})\cap C^1(J,E^{s-1}\times E^{s-2})$$

of the Cauchy problem

$$\begin{cases} u_t+uu_x &= -\frac{1}{2}A^{-1}\left(u_x^2+\rho^2\right)_x,\\ \rho_t+u\rho_x &= -\rho u_x,\\ (u(0),\rho(0)) &= (u_0,\rho_0),\end{cases}$$

which depends continuously on the initial data (u_0,ρ_0).

Observe that we also have the following conservation laws.

Lemma 4.29. *For the 2HS equation, we have the conservation laws*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(m\circ\varphi)\varphi_x^2+(\rho\circ\varphi)f_x\varphi_x\right]=0 \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\rho\circ\varphi)\varphi_x]=0.$$

Proof. The proof is exactly the same as for Lemma 4.7. □

Concerning the question of local well-posedness of 2HS in the smooth category, we refer to the open problem chapter.

4.4.3 The sectional curvature for the 2HS equation

For the 2HS equation we obtain an expression for the curvature tensor which is in form similar to the equation in [95] for the curvature for HS. In particular we see that the sectional curvature for 2HS is identically equal to 1/4—as it has been established in [95] for the one-component HS. Note also that, in finite dimensions, any Riemannian manifold with constant positive curvature is locally isometric to a sphere, [91], and for the HS equation, an isometry from $H^s\mathrm{Diff}(\mathbb{S})/\mathrm{Rot}(\mathbb{S})$ to an open subset of an infinite-dimensional L_2-sphere is constructed in [95, 96]. Hence the geometric picture motivates to ask for extensions of solutions beyond their breaking time by extending the geodesic flow on a sphere. This problem has been considered by Lenells for the one-component HS in [95, 96].

Theorem 4.30. *The curvature tensor R for the 2HS equation on H^sG_0, $s \geq 3$, equipped with the right-invariant metric* (4.47)*, for vector fields X, Y, Z, is given by*

$$4R(X,Y)Z = X\langle Y,Z\rangle - Y\langle X,Z\rangle.$$

In particular, the sectional curvature for 2HS is constant and equal to 1/4.

Proof. We have the following local formula for R in terms of the Christoffel map (4.46):

$$R(X,Y)Z = D_1\Gamma_p(Z,X)Y - D_1\Gamma_p(Z,Y)X + \Gamma_p(\Gamma_p(Z,Y),X) - \Gamma_p(\Gamma_p(Z,X),Y),$$

for any vector fields X, Y, Z on H^sG_0. By right-invariance of Γ, i.e.,

$$\Gamma_p(X,Y)\circ\psi = \Gamma_{p\circ\psi}(X\circ\psi, Y\circ\psi),$$

it holds that $[R(X,Y)Z]\circ\varphi^{-1} = R(u,v)w$ if $X = u\circ\varphi$, $Y = v\circ\varphi$ and $Z = w\circ\varphi$. Therefore, it suffices to consider the curvature at $(\mathrm{id},0)$. We have

$$\begin{aligned} R(u,v)w &= D_1\Gamma(w,u)v - D_1\Gamma(w,v)u + \Gamma(\Gamma(w,v),u) - \Gamma(\Gamma(w,u),v),\\ &= -\Gamma(w_xv_1,u) - \Gamma(u_xv_1,w) + \Gamma(w,u)_xv_1\\ &\qquad +\Gamma(w_xu_1,v) + \Gamma(v_xu_1,w) - \Gamma(w,v)_xu_1\\ &\qquad +\Gamma(\Gamma(w,v),u) - \Gamma(\Gamma(w,u),v), \end{aligned}$$

using some of the results presented in the proof of Theorem 4.14. In the first component, we have the terms

$$\begin{aligned} &-\Gamma^0(w_{1x}v_1,u_1) + \frac{1}{2}A^{-1}(w_{2x}v_1u_2)_x - \Gamma^0(u_{1x}v_1,w_1) + \frac{1}{2}A^{-1}(u_{2x}v_1w_2)_x\\ &+\Gamma^0(w_1,u_1)_xv_1 - \frac{1}{2}[A^{-1}(w_2u_2)_x]_xv_1 + \Gamma^0(w_{1x}u_1,v_1) - \frac{1}{2}A^{-1}(w_{2x}u_1v_2)_x\\ &+\Gamma^0(v_{1x}u_1,w_1) - \frac{1}{2}A^{-1}(v_{2x}u_1w_2)_x - \Gamma^0(w_1,v_1)_xu_1 + \frac{1}{2}[A^{-1}(w_2v_2)_x]_xu_1\\ &+\Gamma^0\left(\Gamma^0(w_1,v_1) - \frac{1}{2}A^{-1}(w_2v_2)_x, u_1\right) + \frac{1}{4}A^{-1}(w_{1x}v_2u_2 + v_{1x}w_2u_2)_x\\ &-\Gamma^0\left(\Gamma^0(w_1,u_1) - \frac{1}{2}A^{-1}(w_2u_2)_x, v_1\right) - \frac{1}{4}A^{-1}(w_{1x}u_2v_2 + u_{1x}w_2v_2)_x. \end{aligned}$$

Using that $\partial_xA^{-1}\partial_x = \mu - 1$, cf. Remark 3.6, and the relation

$$\Gamma^0(\Gamma^0(w_1,v_1),u_1)-\Gamma^0(\Gamma^0(w_1,u_1),v_1)=-\frac{1}{4}u_1\mu(w_{1x}v_{1x})+\frac{1}{4}v_1\mu(w_{1x}u_{1x}),$$

cf. [95], we see that these terms equal

$$\begin{aligned}&\frac{1}{2}A^{-1}\partial_x\left[(w_{1x}v_1)_xu_{1x}+(u_{1x}v_1)_xw_{1x}-(w_{1x}u_1)_xv_{1x}-(v_{1x}u_1)_xw_{1x}\right]\\&-\frac{1}{2}(\mu-1)(w_{1x}u_{1x})v_1+\frac{1}{2}(\mu-1)(w_{1x}v_{1x})u_1\\&+\frac{1}{2}A^{-1}\left[(w_{2x}v_1u_2)_x+(u_{2x}v_1w_2)_x-(w_{2x}u_1v_2)_x-(v_{2x}u_1w_2)_x\right]\\&-\frac{1}{2}(\mu-1)(w_2u_2)v_1+\frac{1}{2}(\mu-1)(w_2v_2)u_1\\&-\frac{1}{4}u_1\mu(w_{1x}v_{1x})+\frac{1}{4}v_1\mu(w_{1x}u_{1x})-\frac{1}{2}A^{-1}\partial_x\left[\left(-\frac{1}{2}(\mu-1)(w_2v_2)\right)u_{1x}\right]\\&+\frac{1}{2}A^{-1}\partial_x\left[\left(-\frac{1}{2}(\mu-1)(w_2u_2)\right)v_{1x}\right]+\frac{1}{4}A^{-1}\partial_x(v_{1x}w_2u_2-u_{1x}w_2v_2).\end{aligned}\tag{4.50}$$

To see that the terms with $A^{-1}\partial_x$ cancel out, we use that $u_1(0)=v_1(0)=0$ so that, by Remark 3.6,

$$\begin{aligned}\frac{1}{2}w_{1x}u_{1x}v_1-\frac{1}{2}u_1w_{1x}v_{1x}&=-A^{-1}\partial_x^2\left(\frac{1}{2}w_{1x}u_{1x}v_1-\frac{1}{2}u_1w_{1x}v_{1x}\right)\\&=\frac{1}{2}A^{-1}\partial_x(w_{1xx}(u_1v_{1x}-u_{1x}v_1)\\&\qquad+w_{1x}u_1v_{1xx}-w_{1x}u_{1xx}v_1),\end{aligned}$$

which coincides up to sign with the first row terms in (4.50), and

$$\begin{aligned}\frac{1}{2}w_2u_2v_1-\frac{1}{2}w_2v_2u_1&=-A^{-1}\partial_x^2\left(\frac{1}{2}w_2u_2v_1-\frac{1}{2}w_2v_2u_1\right)\\&=\frac{1}{2}A^{-1}\partial_x(w_{2x}(v_2u_1-u_2v_1)+w_2(v_{2x}u_1-u_{2x}v_1)\\&\qquad+w_2(v_2u_{1x}-u_2v_{1x})).\end{aligned}$$

Using $A^{-1}\partial_x^2v_1=-v_1$ and $A^{-1}\partial_x^2u_1=-u_1$, the first component terms (4.50) thus reduce to

$$\frac{1}{4}u_1(\mu(w_{1x}v_{1x})+\mu(w_2v_2))-\frac{1}{4}v_1(\mu(w_{1x}u_{1x})+\mu(w_2u_2)),$$

which is the desired expression. The second component terms are

$$\begin{aligned}&\frac{1}{2}[(w_{1x}v_1)_xu_2+u_{1x}w_{2x}v_1]+\frac{1}{2}[(u_{1x}v_1)_xw_2+w_{1x}u_{2x}v_1]-\frac{1}{2}v_1[w_{1x}u_2+u_{1x}w_2]_x\\&-\frac{1}{2}[(w_{1x}u_1)_xv_2+v_{1x}w_{2x}u_1]-\frac{1}{2}[(v_{1x}u_1)_xw_2+w_{1x}v_{2x}u_1]+\frac{1}{2}u_1[w_{1x}v_2+v_{1x}w_2]_x\\&-\frac{1}{2}(\Gamma_1(w,v)_xu_2+u_{1x}\Gamma_2(w,v))+\frac{1}{2}(\Gamma_1(w,u)_xv_2+v_{1x}\Gamma_2(w,u))\end{aligned}\tag{4.51}$$

and with $\partial_xA^{-1}\partial_x=\mu-1$ we can simplify the last row terms

$$\Gamma_1(w,v)_x=\frac{1}{2}w_{1x}v_{1x}-\frac{1}{2}\mu(w_{1x}v_{1x})+\frac{1}{2}w_2v_2-\frac{1}{2}\mu(w_2v_2)$$

and

$$\Gamma_1(w,u)_x = \frac{1}{2}w_{1x}u_{1x} - \frac{1}{2}\mu(w_{1x}u_{1x}) + \frac{1}{2}w_2u_2 - \frac{1}{2}\mu(w_2u_2).$$

It is now easy to see that the terms in (4.51) reduce to

$$\frac{1}{4}u_2(\mu(w_{1x}v_{1x}) + \mu(w_2v_2)) - \frac{1}{4}v_2(\mu(w_{1x}u_{1x}) + \mu(w_2u_2))$$

so that we obtain

$$R(u,v)w = \frac{1}{4}u\left\langle v,w\right\rangle - \frac{1}{4}v\left\langle u,w\right\rangle.$$

By the definition of the sectional curvature, we have

$$S(u,v) = \frac{\left\langle R(u,v)v,u\right\rangle}{\left\langle u,u\right\rangle\left\langle v,v\right\rangle - \left\langle u,v\right\rangle^2} = \frac{1}{4}.$$

□

Remark 4.31. Since Lenells [95] uses a different scaling for the $\dot{H}^1$-metric, he comes to the result that the sectional curvature for the HS is identically equal to 1. Note carefully that we have only used that u_1 and v_1 vanish at zero; a corresponding assumption on the second component terms is not necessary in the above proof (cf. Chap. 6).

4.4.4 A two-component generalization of the μHS equation

As explained in Chap. 3, an interesting variant of the family (1.39) is obtained by setting $m = \mu(u) - u_{xx}$ where $\mu(u) = \int_{\mathbb{S}} u(t,x)\,\mathrm{d}x$. Equation (1.39) is then called μ-b-equation. Two-component generalizations of the μ-b-equation have not been studied so far. In this section, we are mainly interested in the case $b = 2$ and discuss the 2μHS equation which is obtained from (4.41) by setting $m = \mu(u) - u_{xx}$. First of all, we show the existence of a Lax pair for 2μHS in the following lemma.

Lemma 4.32. *Compatibility of the equations*

$$\psi_{xx} + (m\lambda + \rho^2\lambda^2)\psi = 0$$
$$\psi_t = -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}\psi u_x$$

together with isospectrality $\lambda_t = 0$ implies the 2μHS equation.

Proof. A straightforward computation, using that λ does not depend on time, shows that

$$\psi_{xxt} = -(m_t\lambda + 2\rho\rho_t\lambda^2)\psi + (m\lambda + \rho^2\lambda^2)\left(\frac{1}{2\lambda} + u\right)\psi_x - \frac{1}{2}u_x(m\lambda + \rho^2\lambda^2)\psi$$

and

$$\psi_{txx} = \frac{3}{2}u_x(m\lambda + \rho^2\lambda^2)\psi + \left(\frac{1}{2\lambda} + u\right)(m_x\lambda + 2\rho\rho_x\lambda^2)\psi + \left(\frac{1}{2\lambda} + u\right)(m\lambda + \rho^2\lambda^2)\psi_x + \frac{1}{2}u_{xxx}\psi.$$

Assuming $\psi_{txx} = \psi_{xxt}$ and using that $\psi \neq 0$, we get

$$\lambda^2(2u_x\rho^2 + 2\rho\rho_t + 2u\rho\rho_x) + \lambda(2mu_x + m_t + um_x + \rho\rho_x) + \frac{1}{2}(m_x + u_{xxx}) = 0.$$

Since $m_x = -u_{xxx}$, we obtain

$$m_t + um_x + 2mu_x + \rho\rho_x = 0 \quad \text{and} \quad 2\rho(\rho_t + (\rho u)_x) = 0$$

and hence the 2μHS equation. □

Remark 4.33. A two-component μDP equation has not been studied up to now; 2μDP could be the system

$$\begin{cases} m_t &= -um_x - 3mu_x + 2\rho\rho_x - \rho u_x, \\ \rho_t &= -2\rho u_x - \rho_x u, \end{cases} \qquad m = (\mu - \partial_x^2)u, \quad x \in \mathbb{S}, \quad t > 0, \tag{4.52}$$

or

$$\begin{cases} m_t &= -um_x - 3mu_x + k_3\rho\rho_x, \\ \rho_t &= -k_2\rho_x u - (k_1 + k_2)\rho u_x, \end{cases} \qquad m = (\mu - \partial_x^2)u, \quad x \in \mathbb{S}, \quad t > 0, \tag{4.53}$$

with $k_1 = k_2 = 1$ and k_3 arbitrary or $k_2 = 1$, $k_3 = 0$ and k_1 arbitrary. Observe that the replacement $\mu \to \mathrm{id}$ in Eq. (4.52) and Eq. (4.53) yields the two-component versions of the DP equation which are studied in [115]. Here, the author generalizes a Hamiltonian operator of the DP equation to an appropriate 2×2-matrix operator; nevertheless, the paper does not prove integrability of the 2DP equation and we are not aware of the integrability of Eqs. (4.52) and (4.53) either.

Let

$$\Gamma^0_{\mathrm{id}}(u, v) = -(\mu - \partial_x^2)^{-1}\left(\mu(u)v + \mu(v)u + \frac{1}{2}u_x v_x\right)_x$$

be the Christoffel operator for the μHS equation, as introduced in Sect. 3.5. To obtain the Christoffel operator for Eq. (4.41) with $m = \mu(u) - u_{xx}$, we set

$$\Gamma_{(\mathrm{id},0)}(X, Y) = \begin{pmatrix} \Gamma^0_{\mathrm{id}}(X_1, Y_1) - \frac{1}{2}(\mu - \partial_x^2)^{-1}(X_2Y_2)_x \\ -\frac{1}{2}(X_{1x}Y_2 + Y_{1x}X_2) \end{pmatrix}. \tag{4.54}$$

Again, we show that the connection

$$\nabla_X Y(\varphi, f) = DY(\varphi, f) \cdot X(\varphi, f) - \Gamma_{(\varphi,f)}(Y(\varphi, f), X(\varphi, f)) \tag{4.55}$$

with

$$\Gamma_{(\varphi,f)}(X, Y) = \Gamma_{(\mathrm{id},0)}(X(\varphi, f) \circ \varphi^{-1}, Y(\varphi, f) \circ \varphi^{-1}) \circ \varphi$$

and $\Gamma_{(\mathrm{id},0)}$ as in (4.54) is compatible with the right-invariant metric on H^sG given at the identity by

$$\langle X, Y\rangle = \mu(X_1)\mu(Y_1) + \langle X_{1x}, Y_{1x}\rangle_{L_2} + \langle X_2, Y_2\rangle_{L_2}. \tag{4.56}$$

The following proposition is quite similar to Proposition 4.1.

Proposition 4.34. *Let $s \geq 3$. Let $H^sG = H^s\mathrm{Diff}(\mathbb{S})Ⓢ H^{s-1}(\mathbb{S})$ and let Γ be the Christoffel map defined in (4.54). Then Γ defines a smooth spray on H^sG, i.e., the map*

$$(\varphi, f) \mapsto \Gamma_{(\varphi,f)} \colon H^sG \to \mathcal{L}^2_{\mathrm{sym}}(H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}))$$

is smooth. Moreover, the metric $\langle \cdot, \cdot \rangle$ defined by (4.56) is a smooth (weak) Riemannian metric on H^sG, i.e., the map

$$(\varphi, f) \mapsto \langle \cdot, \cdot \rangle_{(\varphi,f)} : H^sG \to \mathcal{L}^2_{\mathrm{sym}}\left(T_{(\varphi,f)}H^sG; \mathbb{R}\right)$$

is a smooth section of the bundle $\mathcal{L}^2_{\mathrm{sym}}(TH^sG; \mathbb{R})$. Finally, the connection ∇ is a Riemannian covariant derivative in the sense of Definition 1.24.

Proof. The smoothness of the Christoffel map is obtained as in the proof of Proposition 4.1; we simply have to replace the operator A by $\mu - \partial_x^2$. Clearly, $\langle \cdot, \cdot \rangle$ is a positive definite and symmetric bilinear from on $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$. It remains to check that the map sending (φ, f) to

$$g(\varphi, f)(X, Y) = \int_{\mathbb{S}} (X_1 \circ \varphi^{-1}) \left(\mu - \partial_x^2\right) (Y_1 \circ \varphi^{-1}) \,\mathrm{d}x + \int_{\mathbb{S}} (X_2 \circ \varphi^{-1})(Y_2 \circ \varphi^{-1}) \,\mathrm{d}x$$

is smooth for any $X, Y \in T_{(\varphi,f)}H^sG$. This follows from

$$g(\varphi, f)(X, Y) = \mu(X_1\varphi_x)\mu(Y_1\varphi_x) + \int_{\mathbb{S}} \frac{X_{1x}(y)Y_{1x}(y)}{\varphi_x(y)} \mathrm{d}y + \int_{\mathbb{S}} X_2(y)Y_2(y)\varphi_x(y)\mathrm{d}y.$$

The compatibility with ∇, in view of the preparatory work in Sect. 3.5, follows from the compatibility of Γ^0 with the right-invariant metric induced by the one-component inertia operator $\mu - \partial_x^2$; the remaining terms are of the same form as for the 2CH equation which we discussed in Proposition 4.1. □

We know that the 2μHS is a reexpression of the geodesic flow of the connection ∇ defined in (4.55) on the product H^sG. The geodesic equation reads as

$$(\varphi_{tt}, f_{tt}) = \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t)). \tag{4.57}$$

We have the following local well-posedness result.

Theorem 4.35. *Let $s \geq 3$. Then there exists an open interval J centered at 0 and an open neighborhood U of $(0, 0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, H^sG)$ of (4.57) satisfying $(\varphi(0), f(0)) = (\mathrm{id}, 0)$ and $(\varphi_t(0), f_t(0)) = (u_0, \rho_0)$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi\colon J \times U \to H^sG$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.*

We write the Cauchy problem for 2μHS in the form

$$\begin{cases} u_t + uu_x &= -(\mu - \partial_x^2)^{-1} \left(\frac{1}{2}u_x^2 + 2\mu(u)u + \frac{1}{2}\rho^2\right)_x, \\ \rho_t + u\rho_x &= -\rho u_x, \\ (u(0), \rho(0)) &= (u_0, \rho_0). \end{cases} \tag{4.58}$$

It follows from Theorem 4.35 that 2μHS is locally well-posed in $H^s \times H^{s-1}$ for $s \geq 3$.

Corollary 4.36. *Suppose $s \geq 3$. Then for any $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ there exists an open interval J centered at 0 and a unique solution*

$$(u, \rho) \in C(J, H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1(J, H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$$

of the Cauchy problem (4.58) *which depends continuously on the initial data (u_0, ρ_0).*

The previous results hold with the obvious changes also in the C^n-category. Assuming $n \geq 2$, the proofs are the same with H^sG replaced with C^nG. Observe that we also have the following conservation laws.

Lemma 4.37. *For the 2μHS equation, we have the conservation laws*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(m \circ \varphi)\varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x\right] = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\rho \circ \varphi)\varphi_x] = 0.$$

Proof. The proof is exactly the same as for Lemma 4.7. □

The discussion of the local well-posedness problem for 2μHS will be continued in Chap. 6. We are now concerned with the sectional curvature for the 2μHS.

Theorem 4.38. *The unnormalized sectional curvature $S(u, v) = S_2(u, v)$ for the 2μHS equation is given by*

$$S_2(u, v) = \langle \Gamma(u, v), \Gamma(u, v)\rangle - \langle \Gamma(u, u), \Gamma(v, v)\rangle - 3\mu(u_{1x}v_1)^2.$$

Proof. This is a straightforward calculation similar to the proof of Theorem 4.14; we simply have to replace the inertia operator by $\mu - \partial_x^2$. Observe that the additional term $-3\mu(u_{1x}v_1)^2$ comes from the sectional curvature formula for the μHS equation found in [79], see also Theorem 3.49; the unnormalized sectional curvature $S_1(u_1, v_1)$ for the μHS equation equals

$$\begin{aligned} S_1(u_1, v_1) = &\left\langle \Gamma^0(u_1, v_1), (\mu - \partial_x^2)\Gamma^0(u_1, v_1)\right\rangle_{L_2} \\ &- \left\langle \Gamma^0(u_1, u_1), (\mu - \partial_x^2)\Gamma^0(v_1, v_1)\right\rangle_{L_2} - 3\mu(u_{1x}v_1)^2. \end{aligned}$$

□

In [79], the authors establish positivity results for $S_1(u_1, v_1)$ by considering a decomposition of $T_{\mathrm{id}}H^s\mathrm{Diff}(\mathbb{S})$ according to the representation $u = \tilde{u} + \mu(u)$ with $\mu(\tilde{u}) = 0$, cf. Sect. 3.5. Since we have

$$S_2\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right) = S_1(u_1, v_1),$$

we see that the same positivity results are valid for H^sG in the $H^s\mathrm{Diff}(\mathbb{S})$-direction. To find a large subspace of positive sectional curvature for 2μHS with non-trival second component we compute $S_2(u, v)$ for

$$u = \begin{pmatrix} \cos k_1 x \\ \cos k_2 x \end{pmatrix}, \quad v = \begin{pmatrix} \cos l_1 x \\ \cos l_2 x \end{pmatrix},$$

where $k_i \neq l_i \in 2\pi\mathbb{N}$, $i = 1, 2$. Note that

$$
\begin{aligned}
S_2(u,v) &= S_1(u_1,v_1) + \frac{1}{4}\int_{\mathbb{S}} (u_2v_2)_x(\mu - \partial_x^2)^{-1}(u_2v_2)_x \,\mathrm{d}x - \int_{\mathbb{S}} \Gamma^0(u_1,v_1)(u_2v_2)_x \,\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\mathbb{S}} (u_{1x}v_2 + v_{1x}u_2)^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{S}} (u_2^2)_x(\mu - \partial_x^2)^{-1}(v_2^2)_x \,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(u_1,u_1)(v_2^2)_x \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(v_1,v_1)(u_2^2)_x \,\mathrm{d}x - \int_{\mathbb{S}} u_{1x}u_2v_{1x}v_2 \,\mathrm{d}x \\
&= S_1(u_1,v_1) + \sum_{j=1}^{4} I_j, \qquad (4.59)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{4}\int_{\mathbb{S}} (u_2v_2)_x(\mu - \partial_x^2)^{-1}(u_2v_2)_x \,\mathrm{d}x, \\
I_2 &= -\frac{1}{4}\int_{\mathbb{S}} (u_2^2)_x(\mu - \partial_x^2)^{-1}(v_2^2)_x \,\mathrm{d}x, \\
I_3 &= -\int_{\mathbb{S}} \Gamma^0(u_1,v_1)(u_2v_2)_x \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(u_1,u_1)(v_2^2)_x \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(v_1,v_1)(u_2^2)_x \,\mathrm{d}x, \\
I_4 &= \frac{1}{4}\int_{\mathbb{S}} (u_{1x}v_2 + v_{1x}u_2)^2 \,\mathrm{d}x - \int_{\mathbb{S}} u_{1x}u_2v_{1x}v_2 \,\mathrm{d}x.
\end{aligned}
$$

We write $A = \mu - \partial_x^2$ and apply the identity

$$
A^{-1}\partial_x^2 = \partial_x A^{-1}\partial_x = \partial_x^2 A^{-1} = \mu - 1,
$$

cf. Remark 3.3. Using integration by parts and the orthogonality relations (4.32) and (4.33) we find

$$
\begin{aligned}
S_1(u_1,v_1) &= \big\langle \Gamma^0(u_1,v_1), \Gamma^0(u_1,v_1)\big\rangle - \big\langle \Gamma^0(u_1,u_1), \Gamma^0(v_1,v_1)\big\rangle - 3\mu(u_{1x}v_1)^2 \\
&= -\frac{1}{2}\int_{\mathbb{S}} A^{-1}[\partial_x(u_{1x}v_{1x})]A\Gamma^0(u_1,v_1)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} A^{-1}[\partial_x(u_{1x}^2)]A\Gamma^0(v_1,v_1)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{S}} u_{1x}v_{1x}\Gamma^0(u_1,v_1)_x \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}} u_{1x}^2\Gamma^0(v_1,v_1)_x \,\mathrm{d}x \\
&= -\frac{1}{4}\int_{\mathbb{S}} u_{1x}v_{1x}(A^{-1}\partial_x^2)(u_{1x}v_{1x})\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{S}} u_{1x}^2(A^{-1}\partial_x^2)(v_{1x}^2)\,\mathrm{d}x \\
&= \frac{1}{4}\mu(u_{1x}^2)\mu(v_{1x}^2) \\
&= \frac{1}{16}k_1^2l_1^2. \qquad (4.60)
\end{aligned}
$$

Our choice of k_1 and l_1 implies that the one-component sectional curvature is strictly positive. All we have to show is that the second component terms do not contribute negative terms which make the total sectional curvature negative. Similar computations show that the terms I_1 and I_2 in (4.59) are

$$
I_1 = -\frac{1}{4}\int_{\mathbb{S}} u_2v_2\partial_x^2A^{-1}u_2v_2\,\mathrm{d}x = \frac{1}{4}\int_{\mathbb{S}} u_2v_2(1-\mu)(u_2v_2)\,\mathrm{d}x = \frac{1}{4}\int_{\mathbb{S}} u_2^2v_2^2\,\mathrm{d}x
$$

and

$$I_2 = \frac{1}{4}\int_{\mathbb{S}} u_2^2 \partial_x^2 A^{-1} v_2^2 \,\mathrm{d}x = \frac{1}{4}\int_{\mathbb{S}} u_2^2(\mu - 1) v_2^2 \,\mathrm{d}x = -\frac{1}{4}\int_{\mathbb{S}} u_2^2 v_2^2 \,\mathrm{d}x + \frac{1}{16}.$$

Since

$$\begin{aligned} -\int_{\mathbb{S}} \Gamma^0(u_1, v_1)(u_2 v_2)_x \,\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{S}} A^{-1}(u_{1x}v_{1x})_x (u_2 v_2)_x \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{S}} [(1-\mu)(u_{1x}v_{1x})] u_2 v_2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{S}} u_{1x} u_2 v_{1x} v_2 \,\mathrm{d}x \end{aligned}$$

we find that

$$\begin{aligned} I_3 + I_4 &= \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(u_1, u_1)(v_2^2)_x \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}} \Gamma^0(v_1, v_1)(u_2^2)_x \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{S}} (u_{1x}^2 v_2^2 + v_{1x}^2 u_2^2) \,\mathrm{d}x \\ &= \frac{1}{4}\mu(u_{1x}^2)\mu(v_2^2) + \frac{1}{4}\mu(v_{1x}^2)\mu(u_2^2) \\ &= \frac{1}{16}(k_1^2 + l_1^2). \end{aligned}$$

It follows from (4.59) and (4.60) that

$$S_2(u,v) = \frac{1}{16}\left(1 + k_1^2 + l_1^2 + k_1^2 l_1^2\right) > \frac{1}{16}.$$

The importance of this estimate lies in the fact that we cannot find a sequence $\{(u_n, v_n);\, n \in \mathbb{N}\}$ of elements in the subspace under consideration such that $S(u_n, v_n) \to 0$ as $n \to \infty$. Recall that, for the one-component μHS equation, we constructed a sequence of sine functions such that the sectional curvature of these functions with the constant function 1 tends to zero, cf. Sect. 3.5. This cannot happen within our subspace for the two-component extension of the μHS equation.

Our calculation also shows that the sectional curvature is equal to 1/16 in the direction of the second component since

$$S_2\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right) = I_1 + I_2 = \frac{1}{16}.$$

We have thus shown the following proposition.

Proposition 4.39. *Let $s \geq 3$. Let $S(u,v) := \langle R(u,v)v, u\rangle$ be the unnormalized sectional curvature on H^sG associated with the 2μHS equation. Then*

$$S(u,v) > \frac{1}{16}$$

for all vectors $u, v \in T_{(\mathrm{id},0)}H^sG$, of the form

$$u = \begin{pmatrix} \cos k_1 x \\ \cos k_2 x \end{pmatrix}, \quad v = \begin{pmatrix} \cos l_1 x \\ \cos l_2 x \end{pmatrix}, \qquad k_i \neq l_i \in \{2\pi, 4\pi, \dots\}.$$

Moreover, the normalized sectional curvature satisfies

$$\frac{S(u,v)}{|u \wedge v|^2} = \frac{1}{4}$$

for all vectors $u, v \in T_{(\mathrm{id},0)} H^s G$ *of the form*

$$u = \begin{pmatrix} 0 \\ \cos k_2 x \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ \cos l_2 x \end{pmatrix}, \qquad k_2 \neq l_2 \in \{2\pi, 4\pi, \dots\}.$$

Chapter 5
The non-periodic *b*-equation

Up to now we have only discussed periodic equations, i.e., we assumed that $x \in \mathbb{S} = \mathbb{R}/\mathbb{Z}$. In this section we will extend some of our results to the non-periodic case. Our aim is to discuss the b-equation on the real line, i.e., the family

$$m_t = -m_x u - b u_x m, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \quad t > 0. \tag{5.1}$$

In [48] the authors show that the Cauchy problem for the b-equation (5.1) is locally well-posed in the Sobolev spaces $H^s(\mathbb{R})$ for any $s > 3/2$. Furthermore, they explain the precise blow-up scenario and global well-posedness settings.

The goal of this chapter is to apply the theory in [41] to the non-periodic b-equation. Precisely, we will deal with the b-equation on the group of W_p^∞-diffeomorphisms on the real axis. Our key result is that the Eulerian velocity u in fact has a local flow on the diffeomorphism group under consideration, at least for a small time interval; this property is described by the notion *regularity* (in the sense of Milnor) in the analysis of infinite-dimensional Lie groups, cf. [108]. Once we have established regularity of our group, it will be straightforward to apply the theory explained in Sect. 3.3.1.

The main problem with the group $\mathrm{Diff}^\infty(\mathbb{R})$ of smooth and orientation-preserving diffeomorphisms $\mathbb{R} \to \mathbb{R}$ is that $\mathrm{Diff}^\infty(\mathbb{R})$ is *not* a regular Fréchet Lie group, and hence we cannot ensure the existence of local flows for our purposes. Strictly speaking not every element of the Lie algebra $\mathrm{Vect}^\infty(\mathbb{R})$ can be integrated into a one-parameter subgroup, cf. Appendix A. To overcome this, one suitable candidate was proposed in [105], namely the rapidly decreasing diffeomorphisms

$$\mathscr{S}\mathrm{Diff}(\mathbb{R}) \coloneqq \{\mathrm{id} + f;\ f \in \mathscr{S}(\mathbb{R}), f' > -1\}. \tag{5.2}$$

Here $\mathscr{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing functions. This group turned out to be a regular Fréchet Lie group. A simpler example is studied in [34]; it is the subgroup of $\mathrm{Diff}^\infty(\mathbb{R})$ defined by

$$H^\infty\mathrm{Diff}(\mathbb{R}) \coloneqq \{\mathrm{id} + f;\ f \in H^\infty(\mathbb{R}), f' > -1\} \tag{5.3}$$

where $H^\infty(\mathbb{R}) = \cap_{n=1}^\infty H^n(\mathbb{R})$ and $H^n(\mathbb{R})$ are the Sobolev spaces on the real line. In [34] the authors work out a proof that Eq. (5.1) for $b = 2$ is well-posed in $H^\infty(\mathbb{R})$. We extend this result to a larger class of function spaces and consider, for $1 < p < \infty$, the spaces

$$W_p^\infty(\mathbb{R}) := \bigcap_{k=1}^{\infty} W_p^k(\mathbb{R})$$

and the diffeomorphism groups

$$W_p^\infty \mathrm{Diff}(\mathbb{R}) := \{\mathrm{id} + f;\ f \in W_p^\infty(\mathbb{R}) \text{ and } f' > -1\} \tag{5.4}$$

and write the non-periodic b-equation as an evolution equation on $W_p^\infty\mathrm{Diff}(\mathbb{R})$. Note that the group $H^\infty\mathrm{Diff}(\mathbb{R})$ corresponds to the case $p = 2$ in our setting; hence our work is a generalization to arbitrary $1 < p < \infty$ and a general b. More precisely, we prove the following theorem in which the space $WC_p^3(\mathbb{R})$ stands for the intersection of the Sobolev space $W_p^3(\mathbb{R})$ with the space $C_b^3(\mathbb{R})$ consisting of $C^3(\mathbb{R})$-functions f such that f, f', f'' and f''' are bounded.

Theorem 5.1. *There is an open neighborhood $V \subset WC_p^3(\mathbb{R})$ of zero and a real number $\delta > 0$ such that for all $\xi \in V \cap W_p^\infty(\mathbb{R})$, the problem*

$$m_t = -(m_x u + b m u_x), \quad m = u - u_{xx}, \quad u(0) = \xi$$

has a unique solution $u \in C^\infty((-\delta,\delta), W_p^\infty(\mathbb{R}))$. Furthermore, the map

$$(-\delta,\delta) \times (V \cap W_p^\infty(\mathbb{R})) \to W_p^\infty(\mathbb{R}), \quad (t,\xi) \mapsto u(t)$$

is smooth.

For any regular Fréchet Lie group, the exponential map is well-defined (cf. Appendix A). For the b-equation on $\mathbb{R}$, we show that this map is a smooth local diffeomorphism.

Theorem 5.2. *The exponential map for the b-equation on $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is a smooth local diffeomorphism near $0 \in W_p^\infty(\mathbb{R})$ onto a neighborhood of $\mathrm{id} \in W_p^\infty\mathrm{Diff}(\mathbb{R})$.*

In a first step we establish that the diffeomorphism groups defined in (5.4) are regular Fréchet Lie groups. Let $k \in \mathbb{N}$ and $1 < p < \infty$ and recall that

$$W_p^k(\mathbb{R}) = \Big\{ f \in L_p(\mathbb{R});\ f^{(n)} \text{ exists in the weak sense and } f^{(n)} \in L_p(\mathbb{R}) \text{ for all } n = 1,\dots,k \Big\}$$

and

$$C_b^k(\mathbb{R}) = \left\{ f \in C^k(\mathbb{R});\ \sup_{x\in\mathbb{R}} \left| f^{(n)}(x) \right| < \infty \text{ for all } n = 0,\dots,k \right\},$$

endowed with the usual norms

$$\|f\|_{W_p^k(\mathbb{R})} := \left(\sum_{n=0}^{k} \left\| f^{(n)} \right\|_{L_p(\mathbb{R})}^p \right)^{1/p} \quad \text{and} \quad \|f\|_{C_b^k(\mathbb{R})} := \sum_{n=0}^{k} \left\| f^{(n)} \right\|_\infty .$$

Observe that since we do not fix $p = 2$, we have in general no inner product on the spaces $W_p^k(\mathbb{R})$. For $k = 0$ we define $W_p^0(\mathbb{R}) = L_p(\mathbb{R})$ and $C_b^0(\mathbb{R}) = C_b(\mathbb{R})$, the space of continuous and bounded functions on $\mathbb{R}$.

It is well-known that the spaces $(W_p^k(\mathbb{R}), \|\cdot\|_{W_p^k(\mathbb{R})})$ and $(C_b^k(\mathbb{R}), \|\cdot\|_{C_b^k(\mathbb{R})})$ are Banach spaces. Furthermore, according to Theorem 5.4 in [2], we have the imbedding

$$W_p^k(\mathbb{R}) \hookrightarrow C_b^{k-1}(\mathbb{R})$$

for $k \in \mathbb{N}$ and $1 < p < \infty$. We now define

$$WC_p^k(\mathbb{R}) := W_p^k(\mathbb{R}) \cap C_b^k(\mathbb{R}), \quad \|\cdot\|_{WC_p^k(\mathbb{R})} := \|\cdot\|_{W_p^k(\mathbb{R})} + \|\cdot\|_{C_b^k(\mathbb{R})}.$$

Lemma 5.3. *The spaces* $(WC_p^k(\mathbb{R}), \|\cdot\|_{WC_p^k(\mathbb{R})})$, $k \in \mathbb{N}$, $1 < p < \infty$, *are Banach algebras.*

Proof. Clearly, $(WC_p^k(\mathbb{R}), \|\cdot\|_{WC_p^k(\mathbb{R})})$ is a normed vector space. We check that $WC_p^k(\mathbb{R})$ is complete: Any Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset WC_p^k(\mathbb{R})$ is also a Cauchy sequence in the Banach spaces $(W_p^k(\mathbb{R}), \|\cdot\|_{W_p^k(\mathbb{R})})$ and $(C_b^k(\mathbb{R}), \|\cdot\|_{C_b^k(\mathbb{R})})$. Let us denote the limits by $f \in W_p^k(\mathbb{R})$ and $g \in C_b^k(\mathbb{R})$, i.e.,

$$f_n \to f \text{ in } W_p^k(\mathbb{R}), \quad f_n \to g \text{ in } C_b^k(\mathbb{R}).$$

It remains to check that $f = g$. This will follow from $\|f - g\|_{C_b^{k-1}(\mathbb{R})} = 0$. Note that $\|\cdot\|_{C_b^{k-1}(\mathbb{R})} \leq \|\cdot\|_{C_b^k(\mathbb{R})}$ implies that $f_n \to g$ in $C_b^{k-1}(\mathbb{R})$. Hence

$$\begin{aligned} \|f - g\|_{C_b^{k-1}(\mathbb{R})} &= \lim_{n\to\infty} \|f - f_n\|_{C_b^{k-1}(\mathbb{R})} \\ &\leq C \limsup_{n\to\infty} \|f - f_n\|_{W_p^k(\mathbb{R})} \\ &= 0. \end{aligned}$$

To see that $WC_p^k(\mathbb{R})$ is a Banach algebra[1] (where the product of two functions is defined by pointwise multiplication), we have to show that $fg \in WC_p^k(\mathbb{R})$ for all $f, g \in WC_p^k(\mathbb{R})$, with a suitable estimate. For this purpose, we apply the Leibniz rule

$$(fg)^{(j)} = \sum_{i=0}^{j} \binom{j}{i} f^{(i)} g^{(j-i)},$$

the estimate

$$\left\|(fg)^{(j)}\right\|_\infty \leq C \|f\|_{WC_p^k(\mathbb{R})} \|g\|_{WC_p^k(\mathbb{R})}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \left|(fg)^{(j)}(x)\right|^p \mathrm{d}x &\leq \int_{\mathbb{R}} \left(\sum_{i=0}^{j} \binom{j}{i} \left\|f^{(i)}\right\|_\infty \left|g^{(j-i)}(x)\right| \right)^p \mathrm{d}x \\ &\leq C^p \|f\|_{WC_p^k(\mathbb{R})}^p \int_{\mathbb{R}} \left(\sum_{i=0}^{k} \left|g^{(i)}(x)\right| \right)^p \mathrm{d}x \\ &\leq C^p \|f\|_{WC_p^k(\mathbb{R})}^p k^p \sum_{i=0}^{k} \left\|g^{(i)}\right\|_{L_p(\mathbb{R})}^p \\ &\leq (Ck)^p \|f\|_{WC_p^k(\mathbb{R})}^p \|g\|_{WC_p^k(\mathbb{R})}^p, \end{aligned}$$

for all $j \in \{0, \ldots, k\}$ and with $C = C(j)$. This achieves our proof. □

[1] That $W_p^k(\Omega)$ for $\Omega \subset \mathbb{R}^n$ having the cone property and $kp > n$ is a Banach algebra is proved in [2] where the author establishes the estimate $\|uv\|_{W_p^k(\Omega)} \leq K \|u\|_{W_p^k(\Omega)} \|v\|_{W_p^k(\Omega)}$ and redefines the W_p^k-norm by scaling with $K = K(n, k, p)$.

Next, we consider the infinite intersection

$$W_p^\infty(\mathbb{R}) = \bigcap_{k=1}^\infty W_p^k(\mathbb{R}).$$

Recall that we have the inclusions

$$W_p^{k+1}(\mathbb{R}) \subset WC_p^k(\mathbb{R}) \subset W_p^k(\mathbb{R}).$$

Hence

$$\bigcap_{k=1}^\infty W_p^k(\mathbb{R}) \subset \bigcap_{k=1}^\infty W_p^{k+1}(\mathbb{R}) \subset \bigcap_{k=1}^\infty WC_p^k(\mathbb{R}) \subset \bigcap_{k=1}^\infty W_p^k(\mathbb{R})$$

and we see that

$$W_p^\infty(\mathbb{R}) = \bigcap_{k=1}^\infty WC_p^k(\mathbb{R}).$$

In particular, the space $W_p^\infty(\mathbb{R})$ is approximated by Banach algebras in the sense of Definition 3.31.

Now we study the diffeomorphism groups (5.4) which we can approximate by the groups $WC_p^k\mathrm{Diff}(\mathbb{R}) := \left\{\mathrm{id} + f;\ f \in WC_p^k(\mathbb{R}) \text{ and } f' > -1\right\}$. Note that the derivative of any $\mathrm{id} + f$ is strictly positive whenever $f' > -1$ so that the diffeomorphisms considered here are orientation-preserving. The tangent space of $WC_p^k\mathrm{Diff}(\mathbb{R})$ at id is $WC_p^k(\mathbb{R})$. Note that we have no inner product on $WC_p^k(\mathbb{R})$, so that $WC_p^k\mathrm{Diff}(\mathbb{R})$ cannot be regarded as a Riemannian Banach manifold. In our next theorem, we show that $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is a regular Lie group with the Lie algebra $W_p^\infty(\mathbb{R})$, using techniques written down in [34, 105].

Proposition 5.4. *The diffeomorphism group* $W_p^\infty\mathrm{Diff}(\mathbb{R})$ *is a regular Fréchet Lie group.*

Proof. Our proof is subdivided into the following steps.

Step 1: We establish that $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is a group.
Let $f, g \in W_p^\infty(\mathbb{R})$ with $f', g' > -1$ be given. First, we show that $(\mathrm{id} + f) \circ (\mathrm{id} + g) \in W_p^\infty\mathrm{Diff}(\mathbb{R})$. To this end, we have to verify that $h(x) := g(x) + f(x + g(x))$ is in $W_p^\infty(\mathbb{R})$ with $h' > -1$. Clearly, $h \in C^\infty(\mathbb{R})$ and $f^{(j)}, g^{(j)}$ are in $L_p(\mathbb{R}) \cap C_b(\mathbb{R})$ for any $j \geq 0$. Furthermore, $1 + g'(x) \geq \varepsilon$ for some $\varepsilon > 0$ and hence

$$\int_\mathbb{R} \left| f^{(j)}(x + g(x)) \right|^p \mathrm{d}x = \int_\mathbb{R} \frac{\left| f^{(j)}(y) \right|^p}{1 + g'((\mathrm{id} + g)^{-1}(y))} \mathrm{d}y \leq \varepsilon^{-1} \left\| f^{(j)} \right\|_{L_p(\mathbb{R})}^p \tag{5.5}$$

for all $j \geq 0$. Since any derivative of $f \circ (\mathrm{id} + g)$ is a finite sum in which each term is a product of derivatives of $\mathrm{id}+g$ with some $f^{(j)} \circ (\mathrm{id}+g)$, the fact that $1+g'(x), g''(x), \ldots \in C_b(\mathbb{R})$ and (5.5) show that $h \in WC_p^k(\mathbb{R})$ for all $k \geq 0$. Finally,

$$\begin{aligned} h'(x) &= g'(x) + f'(x + g(x))(1 + g'(x)) \\ &= (1 + g'(x))[1 + f'(x + g(x))] - 1 \\ &> -1. \end{aligned}$$

Second, we must show the existence of $\tilde{f} \in W_p^\infty(\mathbb{R})$ with $\tilde{f}' > -1$ such that $(\mathrm{id} + f)^{-1} = \mathrm{id} + \tilde{f}$. Let us prove that

$$\tilde{f} = -f \circ (\mathrm{id} + f)^{-1}.$$

Of course, $(\mathrm{id}+\tilde{f})\circ(\mathrm{id}+f)=\mathrm{id}$. Clearly, $\tilde{f}$ is smooth and $\tilde{f}^{(j)}$ is bounded for any $j\geq 0$ since $f^{(j)}\in C_b(\mathbb{R})$ for all $j\geq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+f(x))^{-1}=\frac{1}{1+f'((\mathrm{id}+f)^{-1}(x))}\leq\varepsilon^{-1}<\infty$$

and similarly for higher order derivatives of $(\mathrm{id}+f)^{-1}$. The fact that all $f^{(j)}$ are $L_p(\mathbb{R})$-functions and the boundedness of the derivatives of $(\mathrm{id}+f)^{-1}$ immediately yield that $\tilde{f}^{(j)}\in L_p(\mathbb{R})$ for any $j\geq 0$. Finally, it follows from $\tilde{f}(x+f(x))=-f(x)$ with $y=x+f(x)$ that

$$\tilde{f}'(y)=-\frac{f'(x)}{1+f'(x)}=-1+\frac{1}{1+f'(x)}>-1.$$

Since $\mathrm{id}+f$ is a diffeomorphism, this holds for all $y\in\mathbb{R}$.
Clearly, id is the neutral element of $W_p^\infty\mathrm{Diff}(\mathbb{R})$ and $\circ$ is associative so that $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is a group.

Step 2: We show that $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is a Lie group, i.e., we show that the multiplication map and the inversion map are smooth as defined in Appendix A.

Let us assume that we are given smooth curves $t\mapsto\mathrm{id}+f(t,\cdot)$ and $t\mapsto\mathrm{id}+g(t,\cdot)$ in $W_p^\infty\mathrm{Diff}(\mathbb{R})$. By definition, we have to check that

$$t\mapsto(\mathrm{id}+f(t,\cdot))\circ(\mathrm{id}+g(t,\cdot))$$

and

$$t\mapsto[\mathrm{id}+f(t,\cdot)]^{-1}$$

are smooth. Obviously, $x+g(t,x)+f(t,x+g(t,x))$ depends smoothly on t and hence multiplication is smooth. Recall that we have $(\mathrm{id}+f(t,\cdot))^{-1}=\mathrm{id}+\tilde{f}(t,\cdot)$ with $f(t,x)=-\tilde{f}(t,(\mathrm{id}+\tilde{f}(t,\cdot))^{-1}(x))$. Thus

$$\begin{aligned}-\tilde{f}_t(t,x)&=\frac{\mathrm{d}}{\mathrm{d}t}f(t,x+\tilde{f}(t,x))\\&=f_t(t,x+\tilde{f}(t,x))+f_x(t,x+\tilde{f}(t,x))\tilde{f}_t(t,x)\end{aligned}$$

and hence

$$\tilde{f}_t(t,x)=-\frac{f_t(t,x+\tilde{f}(t,x))}{1+f_x(t,x+\tilde{f}(t,x))}.$$

By the successive computation of t-derivatives of the left-hand side, one proves inductively the smoothness of $t\mapsto(\mathrm{id}+f(t,\cdot))^{-1}$.

Step 3: We claim that $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is regular.

By definition, cf. Appendix A, we have to prove that for any $X\in C^\infty(\mathbb{R},W_p^\infty(\mathbb{R}))$ the problem

$$\begin{cases}\varphi(0,x)=x\\\dot{\varphi}(t,x)=X(t,\varphi(t,x))\end{cases}$$

has a solution $\varphi\in C^\infty(I_X,\mathrm{id}+W_p^\infty(\mathbb{R}))$, where $I_X\subset\mathbb{R}$ is some non-empty interval containing zero. First, there is a solution $\varphi\in C^\infty(I_X,WC_p^1\mathrm{Diff}(\mathbb{R}))$ satisfying

$$\varphi(t,x)=x+\int_0^t X(s,\varphi(s,x))\,\mathrm{d}s.$$

By the same arguments as in the proof of Theorem 2.7 in [34] one sees that φ is as desired. □

The proof of Proposition 5.4 also shows that $u \circ \varphi \in WC_p^k(\mathbb{R})$ for all $\varphi \in WC_p^k\mathrm{Diff}(\mathbb{R})$ and $u \in WC_p^k(\mathbb{R})$. Finally, we observe that the operator ∂_x^m maps the space $WC_p^k(\mathbb{R})$ continuously into $WC_p^{k-m}(\mathbb{R})$. In the proof of our main theorem, we apply the Cauchy-Lipschitz Theorem to obtain the existence and uniqueness of solutions in the geometric picture for the b-equation. As explained in Sect. 3.3.1, we write $P_\varphi(f) = R_\varphi \circ P \circ R_{\varphi^{-1}}$, where R_φ denotes the right translation given by φ, $P(u) = A^{-1}[3u_x u_{xx} + b(Au)u_x]$ and $A = 1 - \partial_x^2$, and set $Q = AP$.

Theorem 5.5. *There is an open neighborhood U of 0 in $WC_p^3(\mathbb{R})$ and a real number $\delta > 0$ so that for all $\xi \in U \cap W_p^\infty(\mathbb{R})$, there exists a unique $\varphi \in C^\infty((-\delta,\delta), W_p^\infty\mathrm{Diff}(\mathbb{R}))$ solving*

$$\begin{cases} \varphi_{tt}(t) = -P_\varphi(\varphi_t), \\ \varphi(0) \;\; = \mathrm{id}, \\ \varphi_t(0) = \xi. \end{cases} \tag{5.6}$$

Furthermore, the map

$$(-\delta,\delta) \times (U \cap W_p^\infty(\mathbb{R})) \to W_p^\infty\mathrm{Diff}(\mathbb{R}) \times W_p^\infty(\mathbb{R}), \quad (t,\xi) \mapsto (\varphi(t), \varphi_t(t))$$

is of class C^∞.

Proof. Since A, Q are polynomial differential operators with constant coefficients, it is easy to see that $R_\rho \circ A \circ R_{\rho^{-1}}$ and $R_\rho \circ Q \circ R_{\rho^{-1}}$ are again polynomial differential operators with coefficients being rational functions of derivatives of ρ (cf. [41]). Hence

$$\begin{aligned} WC_p^k\mathrm{Diff}(\mathbb{R}) \times WC_p^k(\mathbb{R}) &\to WC_p^{k-2}(\mathbb{R}), \\ (\rho, v) &\mapsto R_\rho \circ A \circ R_{\rho^{-1}} v \end{aligned}$$

and

$$\begin{aligned} WC_p^k\mathrm{Diff}(\mathbb{R}) \times WC_p^k(\mathbb{R}) &\to WC_p^{k-2}(\mathbb{R}), \\ (\rho, v) &\mapsto R_\rho \circ Q \circ R_{\rho^{-1}} v \end{aligned}$$

are smooth. Furthermore, the local inverse theorem shows that also

$$\begin{aligned} WC_p^k\mathrm{Diff}(\mathbb{R}) \times WC_p^{k-2}(\mathbb{R}) &\to WC_p^k(\mathbb{R}), \\ (\rho, v) &\mapsto R_\rho \circ A^{-1} \circ R_{\rho^{-1}} v \end{aligned}$$

is of class C^∞. Hence

$$\begin{aligned} WC_p^k\mathrm{Diff}(\mathbb{R}) \times WC_p^k(\mathbb{R}) &\to WC_p^k(\mathbb{R}), \\ (\rho, v) &\mapsto R_\rho \circ (A^{-1}Q) \circ R_{\rho^{-1}} v \end{aligned}$$

and

$$\begin{aligned} WC_p^k\mathrm{Diff}(\mathbb{R}) \times WC_p^k(\mathbb{R}) &\to WC_p^k(\mathbb{R}) \times WC_p^k(\mathbb{R}), \\ (\rho, v) &\mapsto (v, -P_\rho(v)) \end{aligned}$$

are smooth for all $k \geq 3$. From the Cauchy-Lipschitz Theorem we obtain the existence of an open neighborhood $U \subset WC_p^3(\mathbb{R})$ of zero and $\delta > 0$ so that (5.6) has a unique solution in $WC_p^3\mathrm{Diff}(\mathbb{R})$, defined on $(-\delta, \delta)$, and that for all $\xi \in U$ the map

$$(-\delta, \delta) \times U \to WC_p^3\mathrm{Diff}(\mathbb{R}) \times WC_p^3(\mathbb{R}), \quad (t, \xi) \mapsto (\varphi(t), \varphi_t(t))$$

is smooth. For $k \geq 3$, let

$$U_k := U \cap WC_p^k(\mathbb{R}).$$

Let us proof by induction that, assuming $\xi \in U_k$, $\varphi(t) \in WC_p^k\mathrm{Diff}(\mathbb{R})$ and $\varphi_t(t) \in WC_p^k(\mathbb{R})$ for any $t \in (-\delta, \delta)$ and all $k \geq 3$. The strategy of our proof can be found in [34]. For $k = 3$ there is nothing to show. Assume that $\varphi(t) \in WC_p^k\mathrm{Diff}(\mathbb{R})$ for some $k \geq 3$. By (3.19),

$$[(u - u_{xx}) \circ \varphi]\varphi_x^b = \xi - \xi_{xx},$$

with $u = \varphi_t \circ \varphi^{-1}$. Hence if $\xi \in U_{k+1}$, then $u(t) \in WC_p^{k+1}(\mathbb{R})$ for all $t \in (-\delta, \delta)$. A further application of the Cauchy-Lipschitz Theorem yields a solution $\tilde{\varphi} \in C^\infty(I_t, WC_p^{k+1}\mathrm{Diff}(\mathbb{R}))$ of $\varphi_{tt} = -P_\varphi(\varphi_t)$ with $\tilde{\varphi}(t) = \mathrm{id}$ and $\tilde{\varphi}_t(t) = u(t)$, where I_t is an open interval containing t. Unity of the solution implies that

$$\varphi(s) = \tilde{\varphi}(s) \circ \varphi(t) \tag{5.7}$$

for all $s \in I_t \cap (-\delta, \delta)$. That $\varphi(t) \in WC_p^{k+1}\mathrm{Diff}(\mathbb{R})$ for all $t \in (-\delta, \delta)$ follows from the fact that the set

$$\left\{t \in (-\delta, \delta);\ \varphi(t) \in WC_p^{k+1}\mathrm{Diff}(\mathbb{R})\right\}$$

is both open and closed in $(-\delta, \delta)$, which is a direct consequence of (5.7). □

Theorem 5.1 is an immediate consequence of the above result and of the fact that $W_p^\infty\mathrm{Diff}(\mathbb{R})$ is a regular Lie group. The proof of Theorem 5.2 is totally similar to the proof of Theorem 3.16 or Theorem 3.21.

Chapter 6
An outlook: open problems and further topics

In this final chapter, we present some open problems and further questions related to the issues of this thesis. The following list does not make any claims of being complete; it rather presents topics which might be worthwhile to study on account of our results. Maybe, some of the problems can be solved quite similarly to what we presented. Others might need some new and profound ideas.

1. A more general variant of the b-equation is discussed in [48] and reads as

$$u_t - \alpha^2 u_{txx} + c_0 u_x + (b+1) u u_x + \Gamma u_{xxx} = \alpha^2 (b u_x u_{xx} + u u_{xxx}), \qquad (6.1)$$

where α, b, Γ, c_0 are arbitrary real constants. Interestingly, Eq. (6.1) includes the KdV equation ($\alpha = 0$, $b = 2$) and reduces to (1.41) for $\Gamma = c_0 = 0$ and $\alpha = 1$. We call Eq. (6.1) the *dispersive b-equation* because of the terms proportional to u_x and u_{xxx}. For $b = 3$, additional results are presented in [57, 112]. In [48], the authors mainly establish local well-posedness of (6.1), and look for blow-up solutions and global strong and weak solutions. It is an interesting question whether Eq. (6.1) is suitable for a reformulation on the diffeomorphism group of the circle and to which novel results the geometric picture might lead. Perhaps, the geometric viewpoint results in a further interpretation of the effect of the dispersive terms. Similar questions can be asked for possible μ-variants and 2-component generalizations of Eq. (6.1).

2. In this thesis, we only discussed strong solutions, i.e., the functions under consideration inherited enough regularity to be plugged into our equations. In Chap. 1 we motivated, taking the example of peakons, that weak solutions also form a class of meaningful solutions. The CH and the DP can be written in the form

$$u_t + F(u)_x = 0, \quad t > 0,\ x \in \mathbb{S},$$

cf., e.g., [20, 43, 128]. Let u_0, the initial data, be in some function space $\mathcal{F}(\mathbb{S})$. A *weak solution* is a function $u\colon [0,\infty) \times \mathbb{S} \to \mathbb{R}$, if, given $T > 0$, $u \in L^\infty_{\text{loc}}([0,T); \mathcal{F}(\mathbb{S}))$ and the identity

$$\int_0^T \int_{\mathbb{S}} (u \varphi_t + F(u) \varphi_x) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{S}} u_0(x) \varphi(0,x) \,\mathrm{d}x = 0$$

is satisfied for all $\varphi \in C_c^\infty([0,T) \times \mathbb{S})$ where $\varphi \in C_c^\infty([0,T) \times \mathbb{S})$ if it is the restriction to $[0,T) \times \mathbb{S}$ of a function having continuous derivatives of arbitrary order on $\mathbb{R}^2$

with compact support contained in $(-T,T)\times\mathbb{R}$. If u is a weak solution on $[0,T)$ for every $T>0$ then it is called a *global* weak solution. A natural question which comes up is whether our geometric approach is appropriate to include weak solutions. Note that $s>3/2$ is the required assumption in order for $H^s\mathrm{Diff}(\mathbb{S})$ to be a topological group, but the interesting peaked solutions of the Camassa-Holm equation belong to $H^{3/2-\varepsilon}(\mathbb{S})$ for any $\varepsilon>0$ and *not* to $H^{3/2}(\mathbb{S})$. Lenells [94] remarks that peakons can at any rate not be captured rigorously by means of his approach.

3. In Chap. 3 we restricted ourselves to the case $b=3$ and discussed the μDP equation and its weakly dissipative variant. Presumably, the local well-posedness result presented in Sect. 3.3.2 can directly be generalized to arbitrary $b\in\mathbb{R}$. Then the formulas for the Christoffel map, the geodesic equation and the conservation law in Lemma 3.27 slightly change. It is a further task to find out whether our approach to the weakly dissipative μDP can be generalized. There might occur some difficulties when applying our arguments in Sect. 3.4 to the equation with general b since the weak form of the weakly dissipative μ-b-equation reads as
$$u_t+uu_x+\lambda u=-(\mu-\partial_x^2)^{-1}(bu_x\mu(u)-bu_xu_{xx}+3u_xu_{xx}),$$
and for $b\neq 3$ the last two parenthesis terms do not cancel out. What is also still open is whether the geometric theory applied in the proof of Theorem 3.38 yields a maximal existence time which is independent of s in the sense of Remark B.8 (which would be achieved from Kato's theory, cf. [125]). Are there strongly dissipative variants of μDP? How can dissipative effects be included in the two-component versions which we discussed?

4. By suggestions of a referee, it is worthwhile to study whether Theorem 3.26 can be improved by showing that the flow is analytic. It requires to improve Proposition 3.24 by showing that the vector field is actually analytic. The proof might become even somewhat shorter as it suffices to show that the vector field is complex differentiable. There are already results of this type in the literature.

5. The real line case: Most of our work yields results for periodic equations. A very general question is which changes have to be done to handle the non-periodic case. In our final chapter, we saw that we needed to exchange the manifold configuration space to extend our theory.

6. It would be nice to have some numerical results to reinforce our theorems. For instance, some computer-based calculations related to the blow-up result in Theorem 3.43 would be illustrative. In [60] the author applies a numerical scheme to compute solutions of the DP equation.

7. Up to today, only a few studies of two-component generalizations have been published so that our results, in particular the geometric aspects, are quite groundbreaking. There are several open problems for systems with two variables, e.g., derived from the DP equation. The question of integrability is not answered yet (although there are some conjectures, [115]). The construction of a Lax pair seems to be a hard exercise. Similarly: Is it possible to define an integrable two-component extension of μDP? In [12], the authors discuss a tree-component generalization of the Camassa-Holm equa-

tion, see also [50]. Further research projects might deal with three-component versions of our equations.

8. It would also be nice to have a local well-posedness result for the 2HS and the 2μHS in the smooth category. Note that we have established suitable conservation laws in Lemma 4.29 and Lemma 4.37. Maybe a problem is to generalize Proposition 4.9 to the 2HS and the 2μHS; recall that in the proof of Proposition 4.9 the condition (4.20) is crucial and it has been obtained from Kato's semi group approach. Thus a further task would be to find out whether Proposition 4.9 holds in a similar manner for the 2HS and the 2μHS. Using our approach, this would result in a well-posedness theorem as desired. It might also be possible to apply the techniques presented in [41], but with the second component we have the problem that integrating the equations in Lemma 4.29 gives
$$f(t) = \rho_0 \int_0^t \frac{1}{\varphi_x(s)} \,\mathrm{d}s$$
and
$$\varphi_{xx}(t) = \varphi_x(t) \left(\rho_0 \int_0^t \frac{f_x(s)}{\varphi_x(s)} \,\mathrm{d}s - m_0 \int_0^t \frac{1}{\varphi_x(s)} \,\mathrm{d}s \right);$$
hence φ_{xx} depends on f_x which itself depends on φ_{xx}. Thus it is not straightforward to apply the results of Escher and Kolev for the b-equation, cf. Sect. 3.3.1. The problem for 2μHS is similar.

9. We showed that 2HS is locally well-posed in $E^s \times E^{s-1}$, cf. Sect. 4.4.2. We do not expect that 2HS is well-posed in $H^s \times H^{s-1}$, since setting the second component ρ equal to zero we would obtain well-posedness of HS in H^s, which is not possible; recall Sect. 4.4.1 where we explained that the solution of HS is not unique. Our motivation for choosing the second component to be E^{s-1} is motivated by our experiences with the 2CH and the 2DP. Can we obtain local well-posedness for 2HS in $E^s \times H^{s-1}$? A closely related problem is to solve the geodesic equation for 2HS explicitly (as it has been done by Lenells [97] for the one-component equation). Integrating the first equation of 2HS gives
$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + \frac{1}{2}\rho^2 + c(t)$$
with some function c. Integrating this equation once again over $\mathbb{S}$ we obtain from $\int_{\mathbb{S}} u_{tx} \,\mathrm{d}x = 0$ that
$$c(t) = -\frac{1}{2} \int_{\mathbb{S}} (u_x^2 + \rho^2) \,\mathrm{d}x.$$
Moreover,
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}} u_x^2 \,\mathrm{d}x = \int_{\mathbb{S}} u_x u_{tx} \,\mathrm{d}x = \int_{\mathbb{S}} \left(-\frac{1}{2}u_x^3 - uu_x u_{xx} + \frac{1}{2}\rho^2 u_x \right) \mathrm{d}x = \frac{1}{2} \int_{\mathbb{S}} \rho^2 u_x \,\mathrm{d}x$$
and
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}} \rho^2 \,\mathrm{d}x = \int_{\mathbb{S}} \rho_t \rho \,\mathrm{d}x = -\int_{\mathbb{S}} (\rho u)_x \rho \,\mathrm{d}x = \int_{\mathbb{S}} u\rho\rho_x \,\mathrm{d}x = -\frac{1}{2} \int_{\mathbb{S}} u_x \rho^2 \,\mathrm{d}x$$

so that $c(t)$ is constant and assuming (u_0, ρ_0) is nontrivial, we may rescale to obtain $c = -2$. This shows that

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + \frac{1}{2}\rho^2 - 2.$$

Let φ be the geodesic flow associated with u, i.e., $\varphi_t = u \circ \varphi$ and $\varphi(0) = \mathrm{id}$. Since

$$(u_x \circ \varphi)_t = (u_{tx} + uu_{xx}) \circ \varphi$$

we conclude

$$(u_x \circ \varphi)_t = -\frac{1}{2}(u_x \circ \varphi)^2 + \frac{1}{2}(\rho \circ \varphi)^2 - 2$$

where

$$(\rho \circ \varphi)_t = (\rho_t + u\rho_x) \circ \varphi = -(u_x \circ \varphi)(\rho \circ \varphi).$$

In terms of the variables $z_1(t) = (u_x \circ \varphi)(t, x)$ and $z_2(t) = (\rho \circ \varphi)(t, x)$ we finally obtain the system

$$\begin{cases} \dot{z}_1(t) = -\frac{1}{2}(z_1(t)^2 - z_2(t)^2 + 4), \\ \dot{z}_2(t) = \qquad -z_1(t)z_2(t). \end{cases}$$

For $\rho = 0$, this system reduces to Eq. (5.4) in [97] and can be solved explicitly to obtain the solution formula presented in Sect. 4.4.1. Can we solve the above system for 2HS and what does the result tell us about the solution of 2HS? Finally, our curvature computation for 2HS raises the following issue: Is there a sphere interpretation for 2HS, similar to the one-component equation? Can we apply continuation arguments for the geodesic flow on this sphere? Is there a connection between the integrability of the 2HS equation and the fact that it describes geodesic motion on a sphere?

10. For 2CH and 2μCH, we computed the sectional curvature for pairs of vectors with a cosine function in each component. Similar computations could be carried out admitting the components to be equal to a sine function or 1. We did not concentrate on subspaces of negative sectional curvature.

11. In this thesis, we did not consider the *Virasoro group* $\mathrm{Diff}(\mathbb{S}) \times \mathbb{R}$. The Virasoro group and the *Virasoro algebra* $\mathrm{Vect}(\mathbb{S}) \times \mathbb{R}$ are one-dimensional extensions of the diffeomorphism group $\mathrm{Diff}(\mathbb{S})$ and the Lie algebra of vector fields. It could be shown that KdV, CH and HS can also be modelled on $\mathrm{Diff}(\mathbb{S}) \times \mathbb{R}$; precisely, these equations reexpress the geodesic flow with respect to different right-invariant metrics on an appropriate homogeneous space. In [80] the authors describe how Arnold's approach to the Euler equations as geodesic flows of one-sided invariant metrics extends from Lie groups to homogeneous spaces. For further reading, we recommend [24] where the authors consider geodesic exponential maps and prove that KdV on the Virasoro group does not allow for a local diffeomorphism near the origin.

Appendix A
Basic facts from Banach space analysis

In this appendix, we summarize some basic results from the analysis of Banach spaces and Fréchet spaces as explained in [85] (where the authors discuss even more general locally convex vector spaces) and [87].

A.1 Differential calculus in Banach spaces

Let E, F be Banach spaces, $U \subset E$ open and $x \in U$. A map $f\colon U \to F$ is called *differentiable* at x if there exists a continuous linear map $\lambda\colon E \to F$ and a map ψ defined for all sufficiently small h in E with values in F such that

$$f(x+h) = f(x) + \lambda(h) + |h|\psi(h), \quad \lim_{h\to 0} \psi(h) = 0.$$

The (unique) linear map λ is called the *derivative* of f at x and is also denoted by $f'(x)$ or $Df(x)$. If f is differentiable at all $x \in U$ we say f is differentiable in U. In that case, the derivative f' is a map

$$Df = f'\colon U \to \mathcal{L}(E, F)$$

from U to the Banach space of continuous linear maps $E \to F$, associating to each $x \in U$ the linear map $f'(x) \in \mathcal{L}(E, F)$. If f' is continuous, we say f is of class C^1. Since $\mathcal{L}(E, F)$ is a Banach space, we can define higher order derivatives inductively and we say that f is C^p if all derivatives $D^k f$ exist and are continuous for $1 \leq k \leq p$. Note that $D^p f(x) \in \mathcal{L}(E, \mathcal{L}(E, \ldots, \mathcal{L}(E, F)\ldots))$. The notion *smooth* is used for C^∞-maps. It is not hard to prove that many of the well-known results from calculus (e.g. the product rule, the chain rule etc.) are also true for maps $f\colon E \to F$ and the multi-variable analysis is also similar, cf. [87].

We also need a second notion of differentiability; recall Sect. 1.2.1. The following definition deals with Fréchet spaces, i.e., complete metrizable locally convex spaces. We will discuss Fréchet spaces in a subsequent section.

Definition A.1. Let E and F be Fréchet spaces, let $U \subset E$ be open and $f\colon U \to F$. We say that f is *Gateaux differentiable at* $x \in U$ if there exists a continuous linear map $Df(x)\colon E \to F$ such that

$$Df(x)v = \lim_{t\to 0} \frac{1}{t}(f(x+tv) - f(x))$$

for all $v \in E$. We call f *Gateaux differentiable in* U if f is Gateaux differentiable for all $x \in U$. We say that f is *Gateaux-*C^1 *in* U if f is Gateaux differentiable in U and the map

$$(x, v) \mapsto Df(x)v \colon U \times E \to F \tag{A.1}$$

is continuous (jointly on a subset of the product).

We say that f is Gateaux-C^2 if both f and the map in (A.1) are Gateaux-C^1. The notion of Gateaux-C^p for $p \geq 3$ is defined inductively. We refer to [58] for further details about the calculus for Gateaux differentiable functions in Fréchet spaces.

If f is a map between Banach spaces then both definitions of differentiability apply. If f is differentiable in the Fréchet sense then f is Gateaux differentiable and both derivatives coincide. Contrariwise, if f is Gateaux-C^1 then f is differentiable in the normal sense. Observe also that a Fréchet differentiable function is continuous and that this implication fails for Gateaux differentiability.

We know that the fact that a map f is C^p, $p \geq 1$, implies that f is Gateaux-C^p. In the converse direction we have the following result.

Proposition A.2. *Let E and F be Banach spaces, let $U \subset E$ be open and $f \colon U \to F$ be a continuous map. If f is Gateaux-C^{p+1} for some $p \geq 0$, then f is C^p. In particular for smooth maps between Banach spaces the two definitions coincide.*

A proof can be found in [77], p. 99 and p. 110. Note that our Gateaux-C^p maps correspond to the class C^p_c in [77]. We also have the following result.

Proposition A.3. *Let E, F and G be Banach spaces and let $U \subset E$ be open. Let $f \colon U \times F \to G$ be a C^p-mapping such that $f(x, u)$ is linear with respect to the second variable u. Set $h(x)u = f(x, u)$ and regard h has a mapping of U into $\mathcal{L}(F, G)$. Then h is a C^{p-1}-mapping.*

For a proof we refer to [113], Thm. 5.3. The way in which we use Proposition A.2 and Proposition A.3 is the following: Let E and F be Banach spaces, $U \subset E$ open and $(x, u, v) \mapsto P(x, u, v) \colon U \times E \times E \to F$ be a continuous mapping, linear in u and v. Assume that P is Gateaux-C^{p+1}. Then P is C^p by Proposition A.2. By Proposition A.3 the map

$$(x, u) \mapsto (v \mapsto P(x, u, v)) \colon U \times E \to \mathcal{L}(E, F)$$

is C^{p-1}. Since $\mathcal{L}(E, \mathcal{L}(E, F)) \simeq \mathcal{L}^2(E, F)$, where $\mathcal{L}^k(E, F)$ is the Banach space of continuous k-multilinear maps $E \to F$, another application of Proposition A.3 yields that

$$x \mapsto ((u, v) \mapsto P(x, u, v)) \colon U \to \mathcal{L}^2(E, F)$$

is C^{p-2}. We see that if P is Gateaux-smooth then $x \mapsto P_x = P(x, \cdot, \cdot)$ is a smooth map $U \to \mathcal{L}^2(E, F)$.

A.2 Inverse mappings and differential equations

Let $U \subset E$ be open, E a Banach space, and let $f \colon U \to F$ be a C^p-map for $p \geq 1$ into the Banach space F. Then f is called a *C^p-isomorphism* or *C^p-invertible on* U if the image $f(U) = V$ is open in F and there exists a C^p-map $g \colon V \to U$ such that $g \circ f$

and $f \circ g$ are the identity maps on U and V respectively. We say that f is a *local C^p-isomorphism* at a point $x \in U$ (or is *locally C^p-invertible* at x), if there exists an open set U_1, $x \in U_1 \subset U$, such that $f|_{U_1}$ is C^p-invertible on U_1. Clearly, the composite of two (local) C^p-isomorphisms is again a (local) C^p-isomorphism. The inverse mapping theorem provides a criterion for a map to be locally C^p-invertible in terms of its derivative.

Theorem A.4 (Inverse Mapping Theorem). *Let U be open in a Banach space E and let $f\colon U \to F$ be of class C^p. Let $x_0 \in U$ and assume that $f'(x_0)\colon E \to F$ is a topological isomorphism (i.e., invertible as a continuous linear map). Then f is a local C^p-isomorphism at x_0.*

An important corollary of the Inverse Mapping Theorem is the following.

Corollary A.5. *Let U and V be open subsets of Banach spaces and let $f\colon U \to V$ be a C^p-map which is also a C^1-diffeomorphism. Then f is a C^p-diffeomorphism.*

One of the most important theorems in the multivariable Banach space analysis is the Implicit Function Theorem.

Theorem A.6 (Implicit Function Theorem). *Let U, V be open sets in the Banach spaces E, F and let $f\colon U \times V \to G$ be a C^p-mapping. Let $(a, b) \in U \times V$ and assume that the second partial derivative $D_2 f(a, b)\colon F \to G$ is a topological isomorphism. Let $f(a, b) = 0$. Then there exists a continuous map $g\colon U_0 \to V$ defined on an open neighborhood U_0 of a such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in U_0$. If U_0 is taken to be a sufficiently small ball, then g is uniquely determined and is also of class C^p.*

By a *vector field* on $U \subset E$ we mean a mapping $f\colon U \to E$ which we interpret as assigning a vector to each point of U. Let $x_0 \in U$. An *integral curve* for f with initial condition x_0 is a mapping $\alpha\colon J \to U$, defined on some open interval J containing zero, such that $\alpha(0) = x_0$ and such that $\alpha'(t) = f(\alpha(t))$. An integral curve can also be viewed as a solution of the integral equation

$$\alpha(t) = x_0 + \int_0^t f(\alpha(s))\,\mathrm{d}s.$$

By a *local flow* of a vector field $f\colon U \to E$ at $x_0 \in U$ we mean a mapping $\alpha\colon J \times U_0 \to U$, where $0 \in J \subset \mathbb{R}$ and $U_0 \subset U$ are open and $x_0 \in U_0$, such that for each $x \in U_0$ the map

$$t \mapsto \alpha_x(t) = \alpha(t, x)$$

is an integral curve for f with initial condition $\alpha(0, x) = x$. We have the following existence and uniqueness result, see also [3].

Theorem A.7 (Cauchy-Lipschitz). *Let $f\colon U \to E$ be a vector field satisfying a Lipschitz condition*

$$\|f(x) - f(y)\| \le K\,\|x - y\|\,,$$

where $K > 0$. Let $x_0 \in U$. Let $0 < a < 1$, assume that the closed ball $\overline{B}_{2a}(x_0)$ is contained in U and that f is bounded by a constant $L > 0$ on this ball. If $b > 0$ satisfies $b < a/L$ and $b < 1/K$, then there exists a unique local flow

$$\alpha\colon (-b, b) \times B_a(x_0) \to U.$$

Note that any C^1-function is (locally) Lipschitz continuous. Concerning uniqueness, one can show that, if $f\colon U \to E$, $U \subset E$ open, is of class C^p, $p \geq 1$, and

$$\alpha_1\colon J_1 \to U, \quad \alpha_2\colon J_2 \to U$$

are two integral curves for f with the same initial condition x_0, then α_1 and α_2 are equal on $J_1 \cap J_2$. Concerning regularity of the flow, we have the following theorem.

Theorem A.8. *Let $1 \leq p \leq \infty$ and let $f\colon U \to E$ be a C^p-vector field. Then the flow of f is of class C^p on its domain of definition.*

If a C^p-vector field f has an additional dependence on time, i.e.,

$$f\colon J \times U \to E,$$

then, by setting

$$\overline{f}\colon J \times U \to \mathbb{R} \times E, \quad \overline{f}(t,x) = (1, f(t,x)),$$

one can regard $\overline{f}$ as a time-independent vector field on $J \times U$ and it is easy to see that the study of time-dependent vector fields reduces to the study of time-independent ones. The same is true if f depends on some additional parameters.

A.3 Fréchet spaces

The traditional differential calculus works well in finite-dimensional vector spaces and Banach spaces. Interestingly there are various differences to the analysis of general locally convex topological vector spaces. We will concentrate our attention to Fréchet spaces in this section and, of course, we will only mention the essential facts to develop an understanding of regular Fréchet Lie groups.

Recall that a Fréchet space is a locally convex topological vector space X such that the topology of X is generated by a countable family of semi-norms $(\rho_n)_{n\in\mathbb{N}}$ and such that (X,d) is a complete metric space where

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(x-y)}{1+\rho_n(x-y)}.$$

A major difference to Banach spaces is that there exists no Inverse Function Theorem for Fréchet spaces. Instead there is a theorem by John Forbes Nash and Jürgen Moser which can be regarded as a generalization of the Inverse Function Theorem on Banach spaces to the class of so called *tame* Fréchet spaces. In contrast to Banach spaces for which the invertibility of the derivative at a point is sufficient for a map to be locally invertible, the Nash-Moser theorem requires the derivative to be invertible in a whole neighbourhood. The theorem is widely used to prove local well-posedness for non-linear partial differential equations in spaces of smooth functions, cf. [58].

First of all, we need to have a notion of continuity and smoothness. Let E and F be locally convex topological spaces with families of semi-norms $(\rho_\alpha)_{\alpha\in A}$ and $(\eta_\beta)_{\beta\in B}$. We call a map $f\colon E \to F$ *continuous* if and only if for all $\beta \in B$ there exist $\alpha_1, \ldots, \alpha_n \in A$ and $C > 0$ with

$$\eta_\beta(f(x)) \leq C(\rho_{\alpha_1}(x) + \cdots + \rho_{\alpha_n}(x)).$$

If the family $(\rho_\alpha)_{\alpha\in A}$ is *directed*, i.e., for any $\alpha_1, \alpha_2 \in A$ there is $\alpha_3 \in A$ and a constant C such that $\rho_{\alpha_1}(x) + \rho_{\alpha_2}(x) \le C\rho_{\alpha_3}(x)$ for any $x \in E$, then f is continuous if and only if, for all $\beta \in B$,

$$\eta_\beta(f(x)) \le C\rho_\alpha(x)$$

for some $\alpha \in A$, cf. [117].

Let E and F be locally convex vector spaces. A curve c in E is called *smooth* or C^∞ if all derivatives exist and are continuous; Michor [105] calls this a concept without problems. The space of smooth curves in E is denoted by $C^\infty(\mathbb{R}, E)$. It turns out that this space does not depend on the locally convex topology of E, but on the associated *bornology* (which is the system of bounded sets), cf. [51]. A map $f\colon E \to F$ between locally convex vector spaces E and F is *smooth* if any smooth curve $c(t) \subset E$ is mapped to a smooth curve $(f \circ c)(t) \subset F$.

A *(Fréchet) Lie group* G is a smooth manifold modelled on open subsets of a Fréchet space and a group such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps. The *Lie algebra* $\mathfrak{g}$ of G is the tangent space at the neutral element e and consists of left-invariant vector fields on G. We say that G admits an *exponential mapping* if there exists a smooth mapping $\exp\colon \mathfrak{g} \to G$ such that $t \mapsto \exp(tX)$ is the (unique) one-parameter subgroup with tangent vector X at 0. Note that $\exp(0) = e$ and $D_0 \exp = \mathrm{id}$. If a suitable inverse function theorem is applicable, it follows that exp is a diffeomorphism from $0 \in \mathfrak{g}$ onto a neighborhood of $e \in G$. This holds true for smooth Banach Lie groups but in general not for diffeomorphism groups, cf. [105]. Lie groups in which an exponential mapping is defined, are also called *regular*. This notion goes back to Milnor, cf. [34, 106, 108].

Definition A.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For all $g \in G$, let $R_g\colon G \to G$ be the right translation $h \mapsto hg$. We define the *logarithmic derivative* $\delta_r\colon C^\infty(\mathbb{R}; G) \to C^\infty(\mathbb{R}; \mathfrak{g})$ by

$$(\delta_r\varphi)(t) = (D_{\varphi(t)}R_{\varphi^{-1}(t)})\varphi'(t), \quad t \in \mathbb{R}.$$

The Lie group G is called *regular*, if there is a smooth map

$$\mathrm{evol}_r\colon C^\infty(\mathbb{R}; \mathfrak{g}) \to C^\infty(\mathbb{R}, G),$$

called the *right evolution*, so that for all $X \in C^\infty(\mathbb{R}; \mathfrak{g})$ we have that

$$\delta_r \circ \mathrm{evol}_r(X) = X, \quad \mathrm{evol}_r(X)(0) = e.$$

Equivalently, a Lie group is regular, if any (smooth) left-invariant vector field has a local flow, cf. [1]. Up to now, all known Lie groups are regular, [105]. Finite-dimensional Lie groups and Banach Lie groups are regular. Note that for diffeomorphism groups, the evolution operator is just integration of the time-dependent vector fields with compact support. Each regular Lie group admits an exponential mapping, which is just the restriction of evol_r to the constant curves $\mathbb{R} \to \mathfrak{g}$, [85, 105].

Appendix B

Kato's theory for abstract quasi-linear evolution equations

Here our aim is to give a short introduction to Kato's theory proving the local well-posedness for a very general class of abstract evolution equations. For suitable non-linear equations Kato's semigroup method is fairly standard and our brief overview follows [19, 125]. To illustrate the theory we consider the periodic Camassa Holm equation (1.19). It is easy to generalize the results to the DP equation or the general b-equation as explained in [48, 125].

We use the short hand notation $\langle \cdot, \cdot \rangle_s$ for the H^s inner product, $s \geq 0$, and write $\|\cdot\|_s$ for the corresponding norm, cf. Sect. 3.1.4. Before we start let us recall the following definition, cf. [73, 75].

Definition B.1. Let $\mathcal{H}$ be a Hilbert space and let T be an operator on $\mathcal{H}$. We say that T is *accretive* if its numerical range is a subset of the right half-plane, i.e., $\operatorname{Re} \langle Tu, u \rangle \geq 0$ for all $u \in D(T)$. If, for $\operatorname{Re} \lambda > 0$, we have $(T + \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ with the estimate $\left\| (T + \lambda)^{-1} \right\| \leq 1/\operatorname{Re} \lambda$ we say that T is *m-accretive*. We call T *quasi-accretive* if $T + \alpha$ is accretive for some scalar α. Similarly, we say that T is *quasi-m-accretive* if $T + \alpha$ is m-accretive for some α.

Remark B.2. An m-accretive operator T is maximal accretive in the sense that T is accretive and has no proper accretive extension. An m-accretive operator is necessarily densely defined. That an operator is quasi-accretive means that its numerical range is contained in a half-plane of the form $\operatorname{Re} z \geq$ const. Like an m-accretive operator, a quasi-m-accretive operator is maximal quasi-accretive and densely defined.

Kato's famous theorem reads as follows.

Theorem B.3. *Consider the abstract quasi-linear evolution equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} v + A(v)v = f(v), \quad t \geq 0, \quad v(0) = v_0. \tag{B.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely injected into X and let $Q \colon Y \to X$ be a topological isomorphism. Furthermore, we assume:

1. The operator $A(y)$ is in $\mathcal{L}(Y, X)$ for any $y \in Y$ with

$$\|(A(y) - A(z))w\|_X \leq \mu_A \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y, \tag{B.2}$$

and $A(y)$ is quasi-m-accretive, uniformly on bounded sets in Y.

2. *For any $y \in Y$ there is a bounded operator $B(y) \in \mathcal{L}(X)$, uniformly on bounded sets in Y, such that $QA(y)Q^{-1} = A(y) + B(y)$ and*

$$\|(B(y) - B(z))w\|_X \leq \mu_B \|y - z\|_Y \|w\|_X, \quad y, z \in Y, \quad w \in X. \tag{B.3}$$

3. *The map $f\colon Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y and*

$$\|f(y) - f(z)\|_Y \leq \mu_1 \|y - z\|_Y, \quad y, z \in Y, \tag{B.4}$$

$$\|f(y) - f(z)\|_X \leq \mu_2 \|y - z\|_X, \quad y, z \in X. \tag{B.5}$$

Here μ_A, μ_B and μ_1 depend only on $\max\{\|y\|_Y, \|z\|_Y\}$ and the number μ_2 depends only on $\max\{\|y\|_X, \|z\|_X\}$. Then, given $v_0 \in Y$, there is a maximal time $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to Eq. (B.1) *such that*

$$v = v(\cdot, v_0) \in C([0,T); Y) \cap C^1([0,T); X).$$

Moreover the map $v_0 \mapsto v(\cdot, v_0)$ is continuous from Y to $C([0,T); Y) \cap C^1([0,T); X)$.

For a proof we refer to [73, 74, 76]. For the Camassa-Holm equation on the circle the natural choice is $X = L_2(\mathbb{S})$, $Y = H^1(\mathbb{S})$ and $Q = (1 - \partial_x^2)^{1/2}$. The momentum variable is denoted as $m = u - u_{xx}$ and we have

$$m_t + (Q^{-2}m)m_x = -2m(Q^{-2}m)_x, \quad m(0) = m_0$$

which is of type (B.1) if we set

$$A(y) = (Q^{-2}y)\partial_x, \quad f(y) = -2y(Q^{-2}y)_x, \quad y \in H^1(\mathbb{S}),$$

where $D(A(y)) = \{v \in L_2(\mathbb{S});\ (Q^{-2}y)v \in H^1(\mathbb{S})\}$. We now proceed in three steps to establish that the assumptions of Theorem B.3 are satisfied. In the first step, we check that the linear operator A is quasi-m-accretive. In [19] this is shown by considering operators D and D_0 with the common domain consisting of all $L_2(\mathbb{S})$-functions v such that $mv \in H^1(\mathbb{S})$ for some fixed $m \in H^2(\mathbb{S})$ and

$$Dv := (mv)_x - m_x v, \quad D_0 v := -(mv)_x.$$

For v in the domain of D we find that

$$Dv = (mv)_x - m_x v = m v_x \in L_2(\mathbb{S})$$

and the strategy is to prove that D and D_0 are both quasi-accretive in L_2. Since $D_0 = D^*$ it follows from the theory of semigroups that D is quasi-m-accretive, [118]. All we have to do is to give a proof of the following lemma. In the proof, we use that $C^\infty(\mathbb{S})$ is a core for the operator D, i.e., v belongs to the domain of D if and only if there is a sequence $(v_n)_{n\in\mathbb{N}}$ of smooth periodic functions such that $v_n \to v$ and $Dv_n \to Dv$ in $L_2(\mathbb{S})$. This follows from a standard mollification argument, [19].

Lemma B.4. *The operators D and D_0 are both quasi-accretive in $L_2(\mathbb{S})$ and $D_0 = D^*$.*

Proof. First we establish the existence of a positive constant K such that

$$\langle Dv, v\rangle_0 \le K\, \|v\|_0^2\,, \quad \langle D_0 v, v\rangle_0 \le K\, \|v\|_0^2$$

for all v in the domain of D. Therefore, we show that

$$\langle Dv, v\rangle_0 = \langle D_0 v, v\rangle_0 = -\frac{1}{2}\int_{\mathbb{S}} m_x v^2 \,\mathrm{d}x.$$

Approximating v by a sequence $(v_n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{S})$ such that $v_n \to v$ and $Dv_n \to Dv$ in $L_2(\mathbb{S})$ we come to the conclusion that

$$\begin{aligned}
\langle Dv, v\rangle_0 &= \int_{\mathbb{S}} (mv)_x v \,\mathrm{d}x - \int_{\mathbb{S}} m_x v^2 \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{S}} \left((mv_n)_x v_n - m_x v_n^2\right) \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{S}} m v_n (v_n)_x \,\mathrm{d}x \\
&= -\frac{1}{2}\lim_{n\to\infty} \int_{\mathbb{S}} m_x v_n^2 \,\mathrm{d}x \\
&= -\frac{1}{2}\int_{\mathbb{S}} m_x v^2 \,\mathrm{d}x.
\end{aligned}$$

Since $m_x \in C(\mathbb{S})$ we conclude that D is quasi-accretive. A similar computation for D_0 shows that D_0 is quasi-accretive. To prove that $D_0 = D^*$ we show that D_0 is an extension of D^* and vice versa. For fixed w belonging to the domain of D^* the map

$$F(\varphi) = \langle D\varphi, w\rangle_0 = \int_{\mathbb{S}} m\varphi_x w \,\mathrm{d}x = \langle \varphi, D^* w\rangle_0\,, \quad \varphi \in C^\infty(\mathbb{S}),$$

defines a continuous linear functional. Therefore $mw \in H^1(\mathbb{S})$ and

$$F(\varphi) = -\int_{\mathbb{S}} \varphi (mw)_x \,\mathrm{d}x, \quad \varphi \in C^\infty(\mathbb{S}).$$

Thus $w \in D(D_0)$ and $D_0 w = D^* w$ and hence $D^* \subset D_0$. Conversely, for $v \in D(D_0)$ the above approximation argument shows that $\langle Dz, v\rangle_0 = \langle z, w\rangle_0$ with $w = D_0 v$ and for every z in the domain of D. This proves that $D_0 \subset D^*$. □

In the next step we define the operator $B(y) = QA(y)Q^{-1} - A(y)$ where $y \in H^1(\mathbb{S})$ is fixed. Let $M(y)$ be multiplication with $Q^{-2}y$, i.e., $M(y)v = (Q^{-2}y)v$ for $v \in L_2(\mathbb{S})$. By direct computation

$$B(y) = [Q, M(y)]\partial_x Q^{-1} - M(y)[\partial_x, Q]Q^{-1}$$

on $C^\infty(\mathbb{S})$. Note that $[\partial_x, Q] = 0$ which follows from the representations

$$\partial_x f = \mathscr{F}^{-1}(2\pi i n \hat{f}_n), \quad Qf = \mathscr{F}^{-1}\left(\sqrt{1+4\pi^2 n^2}\,\hat{f}_n\right)$$

where $(\hat{f}_n)_{n\in\mathbb{Z}} \subset \mathscr{S}(\mathbb{Z})$ stands for the Fourier series of $f \in C^\infty(\mathbb{S})$ and the operator $\mathscr{F} \in \mathrm{Isom}(C^\infty(\mathbb{S}), \mathscr{S}(\mathbb{Z}))$ denotes the Fourier transform.

Lemma B.5. *Given $y \in H^1(\mathbb{S})$, the operator $B(y)$ extends to a map $B(y) \in \mathcal{L}(L_2(\mathbb{S}))$ and a map $B_1(y) \in \mathcal{L}(H^1(\mathbb{S}))$ that are uniformly bounded on bounded subsets of $H^1(\mathbb{S})$.*

Proof. Clearly, $\partial_x Q^{-1}$ extends to a bounded linear operator on $L_2(\mathbb{S})$ which is independent of $y \in H^1$. Since Q is a first-order pseudo-differential operator and $Q^{-2}y \in H^3(\mathbb{S})$ some standard results from harmonic analysis, cf. [19], show that $[Q, M(y)]$ extends to a bounded linear operator in $L_2(\mathbb{S})$ with norm less than or equal to $K \|y\|_1$. To complete the proof we estimate the norm of the operator $\partial_x[Q, M(y)]Q^{-1}$ in $\mathcal{L}(L_2(\mathbb{S}))$. Writing $M_x(y)$ for the multiplication operator induced by the function $\partial_x(Q^{-2}y) \in H^2$ we find that

$$\partial_x[Q, M(y)]Q^{-1} = QM_x(y)Q^{-1} + M_x(y) + [Q, M(y)]\partial_x Q^{-1}$$

and are done. □

Remark B.6. Since $B(y) - B(z) = B(y - z)$ the proof also establishes the estimate (B.3).

It remains to check that the estimates (B.2), (B.4) and (B.5) are valid. Obviously, $A(y) \in \mathcal{L}(H^1(\mathbb{S}), L_2(\mathbb{S}))$ for $y \in H^1(\mathbb{S})$ and by Lemma B.4 we know that $A(y)$ is quasi-m-accretive uniformly on bounded sets in $H^1(\mathbb{S})$. Let $y, z, w \in H^1(\mathbb{S})$. Then

$$\begin{aligned}
\|(A(y) - A(z))w\|_0^2 &= \int_{\mathbb{S}} \left[Q^{-2}(y - z)w_x\right]^2 \,\mathrm{d}x \\
&\le K \left\|Q^{-2}(y - z)\right\|_\infty^2 \|w\|_1^2 \\
&\le K \left\|Q^{-2}(y - z)\right\|_2^2 \|w\|_1^2 \\
&\le K \left\|Q^{-2}\right\|_{\mathcal{L}(L_2, H^2)}^2 \|y - z\|_0^2 \|w\|_1^2 .
\end{aligned}$$

Since $H^1(\mathbb{S})$ is a Banach algebra, f maps $H^1(\mathbb{S})$ into itself and is bounded on bounded sets in $H^1(\mathbb{S})$. Moreover, it extends to a map $L_2(\mathbb{S}) \to L_2(\mathbb{S})$ that satisfies the local Lipschitz properties (B.4) and (B.5), cf. [19].

Altogether, we have thus shown the following theorem.

Theorem B.7. *Given $m_0 \in H^1(\mathbb{S})$ there is a maximal time $T > 0$, depending only on $\|m_0\|_1$ and a unique solution m to the Camassa-Holm equation $m_t = -m_x u - 2u_x m$ satisfying $m(0) = m_0$ such that*

$$m = m(\cdot, m_0) \in C([0, T); H^1(\mathbb{S})) \cap C^1([0, T); L_2(\mathbb{S})).$$

Moreover the map $m_0 \to m(\cdot, m_0)$ is continuous from $H^1(\mathbb{S})$ to $C([0, T); H^1(\mathbb{S})) \cap C^1([0, T); L_2(\mathbb{S}))$.

Remark B.8. In [125], the author discusses the DP equation on the real line and works with the Sobolev spaces $H^s(\mathbb{R})$ for $s > 3/2$. Here the underlying spaces are $X = L_2(\mathbb{R})$ and $Y = H^s(\mathbb{R})$ and the isomorphism in this case is $(1 - \partial_x^2)^{s/2}$. A further nice result of Kato's approach is that the maximal existence time is independent of s in the following sense: If

$$v = v(\cdot, v_0) \in C([0, T), H^s(\mathbb{R})) \cap C^1([0, T), H^{s-1}(\mathbb{R}))$$

is a solution to (B.1) and if $v_0 \in H^{s'}(\mathbb{R})$ for some $s' \neq s$, $s' > 3/2$, then

$$v = v(\cdot, v_0) \in C([0, T), H^{s'}(\mathbb{R})) \cap C^1([0, T), H^{s'-1}(\mathbb{R})),$$

with the same value of T. In particular, we see that if $v_0 \in H^\infty(\mathbb{R}) = \cap_{s\geq 0} H^s(\mathbb{R})$, then $v \in C([0,T), H^\infty(\mathbb{R}))$.

Remark B.9. Since Kato's theorem requires Hilbert spaces it is clear why many authors model evolution equations like the CH on Sobolev spaces and not on C^n-spaces as we did in many of the previous considerations.

Appendix C
Integrable systems: Lax pairs and bi-Hamiltonian structures

Very often, the fact that a given equation is bi-Hamiltonian implies that one can find an infinite sequence of conservation laws. Rewriting a certain equation in Lax pair form, it might be integrated via the scattering approach. In this appendix we give a brief overview about the bi-Hamiltonian formalism and the method of inverse scattering and explain the corresponding theory for the Camassa-Holm equation (for which it works well). The main references are [11, 17, 18, 23, 62]. The integrable structure of the DP and HS is explained in [30, 66]. For the μ-variants of CH, DP and HS we refer to [79, 99].

C.1 The bi-Hamiltonian structure of the Camassa-Holm equation

Our first aim is to present the Hamiltonian structure of the Camassa-Holm equation (1.19). Since we do not deal with classical Hamiltonian systems, [6], we will start with a digression aimed at a more comprehensive picture of the Hamiltonian formalism we present. Our technical assumptions are kept deliberately vague in the sense that a function $f\colon \mathbb{R} \to \mathbb{R}$ will be either a smooth function vanishing rapidly at $\pm\infty$ (together with as many derivatives as necessary) or a smooth periodic function with period 1. We focus on what is happening rather than look for the sharpest technical conditions. This summary mainly presents the results of [17].

Let $F(f)$ be a functional defined on some underlying linear space of functions f. We call F *differentiable* if

$$\frac{\delta F}{\delta f}[g] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(f+\varepsilon g)\bigg|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(F(f+\varepsilon g) - F(f)\right)$$

exists for all f, g and is a linear functional of g. If this linear functional can be expressed as a scalar product (inherited from $L_2(\mathbb{R})$ or $L_2[0,1]$ accordingly to the considered case),

$$\frac{\delta F}{\delta f}[g] = \left\langle \frac{\partial F}{\partial f}, g \right\rangle,$$

we call $\frac{\partial F}{\partial f}$ the *gradient of* F *at* f. Note that $\frac{\partial F}{\partial f}$ is a function whereas $\frac{\delta F}{\delta f}$ is a functional. A linear operator $\mathcal{D}$ on the underlying space is called *Hamiltonian* if the bracket

$$[F, H](f) := \left\langle \frac{\partial F}{\partial f}, \mathcal{D}\frac{\partial H}{\partial f} \right\rangle$$

is *skew-symmetric*,

$$[F, H] = -[H, F],$$

and satisfies the *Jacobi identity*

$$[[F, G], H] + [[G, H], F] + [[H, F], G] = 0.$$

Clearly, $[\cdot, \cdot]$ is bilinear since $\frac{\partial(F+H)}{\partial f} = \frac{\partial F}{\partial f} + \frac{\partial H}{\partial f}$. We call $[\cdot, \cdot]$ the *Lie-Poisson bracket* defined by $\mathcal{D}$.

Let us consider evolution equations of the form

$$u_t = Au \tag{C.1}$$

where A is an operator, in general nonlinear, mapping the linear space to itself. We assume that the Cauchy problem for Eq. (C.1) is globally well posed, i.e., solutions are uniquely determined by their values at $t = 0$, the initial value can be prescribed arbitrarily (within the linear space) and solutions exist for all $t \geq 0$. The map sending initial data to the solution of (C.1) at time t can be thought of as a *flow*. We say that Eq. (C.1) can be written in *Hamiltonian* form if we can find a Hamiltonian operator $\mathcal{D}$ and a functional H such that the equation takes the form

$$u_t = \mathcal{D}\frac{\partial H}{\partial u}. \tag{C.2}$$

For any solution of Eq. (C.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F(u(t)) = \left\langle \frac{\partial F}{\partial u}, u_t \right\rangle = \left\langle \frac{\partial F}{\partial u}, \mathcal{D}\frac{\partial H}{\partial u} \right\rangle = [F, H]. \tag{C.3}$$

We say that F is a *conserved functional* if $F(u(t))$ is independent of t for all solutions of Eq. (C.2).

Theorem C.1. *1. F is a conserved functional if and only if $[F, H] = 0$.*
2. H is a conserved functional for Eq. (C.2)*.*
3. If F, G are conserved functionals for Eq. (C.2)*, then so is $[F, G]$.*

Proof. This follows from the definition of a conserved functional, Eq. (C.3) and the properties of the Lie bracket. □

The proof of our next proposition is a lengthy but straightforward computation which is written down in [17].

Proposition C.2.

$$\{F, H\}(m) = -\int \frac{\partial F}{\partial m}(\partial m + m\partial)\frac{\partial H}{\partial m}\,\mathrm{d}x$$

defines a Lie-Poisson bracket.

Define $H_1 = \frac{1}{2}\int(u^2 + u_x^2)\,\mathrm{d}x = \frac{1}{2}\int um\,\mathrm{d}x$ with the momentum $m = u - u_{xx}$. Then the Camassa-Holm equation can be written in the form

$$m_t = \{m, H_1\} = -(\partial m + m\partial)u$$

and we have thus shown that the Camassa-Holm equation is Hamiltonian.

Theorem C.3. *The Camassa-Holm equation* (1.19) *is Hamiltonian with the Hamilton operator* $-(m\partial + \partial m)$.

Observe that Eq. (1.19) is equivalent to

$$m_t = -\partial \left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} \right).$$

Let us try to rewrite the right-hand parenthesis as a variational derivative. Knowing that u or m include the same physical information the nature of this expression suggests that it would be easier to try to write it as $\frac{\partial H_2}{\partial u}$ with an appropriate H_2, i.e., we would like to find H_2 with

$$\int \frac{\partial H_2}{\partial u} f \, \mathrm{d}x = \frac{1}{2} \int (3u^2 - u_x^2 - 2uu_{xx}) f \, \mathrm{d}x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (H_2(u + \varepsilon f) - H_2(u)).$$

The presence of $3u^2$ suggests that under the integral in $H_2(u)$ there might be a term u^3. Since

$$\frac{\partial}{\partial u} \int u^3 \, \mathrm{d}x = 3u^2$$

we proceed with the next two terms that seem to suggest the presence of a term uu_x^2 in $H_2(u)$ under the integral sign. We have that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int [(u + \varepsilon f)(u_x^2 + 2u_x f_x \varepsilon + \varepsilon^2 f_x^2) - uu_x^2] \, \mathrm{d}x = \int (2uu_x f_x + u_x^2 f) \, \mathrm{d}x$$

and thus after integration by parts

$$\frac{\partial}{\partial u} \int uu_x^2 \, \mathrm{d}x = -u_x^2 - 2uu_{xx}$$

so that we come to the conclusion that

$$H_2 = \frac{1}{2} \int (u^3 + uu_x^2) \, \mathrm{d}x \tag{C.4}$$

is a good choice. Our next lemma establishes that the operator $1 - \partial_x^2$ does not only map the function u to the function m but also the derivative of functionals with respect to m to the corresponding derivative with respect to u.

Lemma C.4. *If F is a functional then*

$$\frac{\partial F}{\partial u} = (1 - \partial_x^2) \frac{\partial F}{\partial m}.$$

Proof. Let $g = f - f_{xx}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(m + \varepsilon g) \Big|_{\varepsilon=0} = \int \frac{\partial F}{\partial m} g \, \mathrm{d}x = \int \left(\frac{\partial F}{\partial m} - \left(\frac{\partial F}{\partial m} \right)'' \right) f \, \mathrm{d}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(m + \varepsilon g) \Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(u + \varepsilon f) \Big|_{\varepsilon=0} = \int \frac{\partial F}{\partial u} f \, \mathrm{d}x$$

and this achieves the proof. □

We thus can write the Camassa-Holm equation as

$$m_t = -(\partial - \partial^3)\frac{\partial H_2}{\partial m}.$$

If we could show that the operator $-\partial + \partial^3$ is Hamiltonian, we would obtain a second Hamiltonian structure. Before we prove that this is indeed so, let us explain what implications this has for the Camassa-Holm equation.

Let us assume that the equality

$$\mathcal{E}\frac{\partial F_0}{\partial m} = \mathcal{D}\frac{\partial F_1}{\partial m}$$

holds where F_0 and F_1 are functionals and $\mathcal{E}$ and $\mathcal{D}$ are Hamiltonian operators. We then say that F_0 raises to F_1 and F_1 lowers to F_0, in symbols $F_0 \uparrow F_1$ and $F_1 \downarrow F_0$. If we assume that $F_0 \uparrow F_1 \uparrow F_2 \uparrow F_3 \cdots$, i.e., the raising is unobstructed, then, for $i < j$,

$$[F_i, F_j]_{\mathcal{D}} = [F_i, F_{j-1}]_{\mathcal{E}} = -[F_{j-1}, F_i]_{\mathcal{E}} = -[F_{j-1}, F_{i+1}]_{\mathcal{D}} = [F_{i+1}, F_{j-1}]_{\mathcal{D}}$$

where $[\cdot,\cdot]_{\mathcal{D}}$ and $[\cdot,\cdot]_{\mathcal{E}}$ denote the Lie-Poisson brackets induced by $\mathcal{D}$ and $\mathcal{E}$ respectively. Hence if $k = \frac{i+j}{2}$ is an integer, then $[F_i, F_j]_{\mathcal{D}} = [F_k, F_k]_{\mathcal{D}} = 0$. Otherwise, $[F_i, F_j]_{\mathcal{D}} = [F_j, F_i]_{\mathcal{D}} = -[F_i, F_j]_{\mathcal{D}} = 0$. The most important consequence of this is that if the evolution equation $m_t = Au$ has the Hamiltonian form

$$m_t = \mathcal{E}\frac{\partial F_1}{\partial m}$$

then all the F_i, $i \geq 0$, are conserved functionals in view of Theorem C.1 since $[F_1, F_i]_{\mathcal{E}} = [F_1, F_{i+1}]_{\mathcal{D}}$ and the latter is zero as proved above. The only unpleasant thing in this construction is that the raising is unobstructed. To overcome this we introduce the notion of compatibility.

Definition C.5. We say that the Hamiltonian operators $\mathcal{D}$ and $\mathcal{E}$ are *compatible* if their sum $\mathcal{D} + \mathcal{E}$ is still a Hamiltonian operator. We say that an evolution equation is *bi-Hamiltonian* if it can be written in two different Hamiltonian forms with compatible Hamiltonians.

Remark C.6. In general the sum of two Lie-Poisson brackets would fail to satisfy the Jacobi identity.

A proof of the following lemma can be found in [9].

Lemma C.7. *Assume that $\mathcal{D}$ and $\mathcal{E}$ are compatible Hamiltonian operators. If the functions f_1, f_2, f_3 are such that*

$$\mathcal{E}f_1 = \mathcal{D}f_2, \quad \mathcal{E}f_2 = \mathcal{D}f_3$$

and there are functionals F_1, F_2 such that $f_1 = \frac{\partial F_1}{\partial m}$ and $f_2 = \frac{\partial F_2}{\partial m}$ then there is a functional F_3 such that $f_3 = \frac{\partial F_3}{\partial m}$.

Again, the following theorem follows from a lengthy computation written down in [17].

Theorem C.8. *The Camassa-Holm equation* (1.19) *is bi-Hamiltonian with the Hamiltonian operators* $\mathcal{E} = -(\partial m + m\partial)$ *and* $\mathcal{D} = -\partial + \partial^3$.

Proposition C.9. *For the compatible pair of Hamiltonian operators* $\mathcal{E} = -(\partial m + m\partial)$ *and* $\mathcal{D} = -\partial + \partial^3$, *if a functional* F *can be lowered, it can also be raised.*

Proof. Assume that F can be lowered to F_{-1},

$$\mathcal{E}\frac{\partial F_{-1}}{\partial m} = \mathcal{D}\frac{\partial F}{\partial m}.$$

Let f be the solution of the third-order linear differential equation

$$m'\frac{\partial F}{\partial m} + 2m\left(\frac{\partial F}{\partial m}\right)' = f' - f''',$$

i.e., $\mathcal{E}\frac{\partial F}{\partial m} = \mathcal{D}f$. By Lemma C.7 we deduce the existence of a functional F_1 such that $f = \frac{\partial F_1}{\partial m}$ and we can write

$$\mathcal{E}\frac{\partial F}{\partial m} = \mathcal{D}\frac{\partial F_1}{\partial m},$$

i.e., $F \uparrow F_1$. □

Now we obtain an infinite number of conservation laws for the Camassa-Holm equation. First,

$$H_0 = \int m\,\mathrm{d}x$$

is conserved for Eq. (1.19) since

$$\frac{\mathrm{d}}{\mathrm{d}t}H_0 = \int m_t\,\mathrm{d}x = -\int (m_x u + 2mu_x)\,\mathrm{d}x = -\int mu_x\,\mathrm{d}x = -\int (uu_x - u_x u_{xx})\,\mathrm{d}x = 0.$$

By construction we also have the conserved functional

$$H_1 = \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x$$

satisfying

$$\mathcal{E}\frac{\partial H_0}{\partial m} = \mathcal{D}\frac{\partial H_1}{\partial m}$$

and by Proposition C.9 we know that H_1 raises to some H_2, i.e.,

$$\mathcal{E}\frac{\partial H_1}{\partial m} = \mathcal{D}\frac{\partial H_2}{\partial m}.$$

We have already seen that we can choose H_2 as in (C.4). Again, by Proposition C.9, H_2 raises to some H_3 and so on. This procedure continues indefinitely because in the equality (expressing the fact that $H_n \uparrow H_{n+1}$)

$$(m\partial + \partial m)\frac{\partial H_n}{\partial m} = (\partial - \partial^3)\frac{\partial H_{n+1}}{\partial m}, \quad n \geq 0, \tag{C.5}$$

we see that $\frac{\partial H_{n+1}}{\partial m}$, the unknown, is differentiated three times, whereas $\frac{\partial H_n}{\partial m}$ is differentiated only once so that H_{n+1} is functionally independent of $H_0, \ldots, H_n$. To find H_{n+1}, we use Lemma C.4 to transform Eq. (C.5) to

$$(\partial m + m\partial)\frac{\partial H_n}{\partial m} = \partial \frac{\partial H_{n+1}}{\partial u}, \tag{C.6}$$

compute $\frac{\partial H_{n+1}}{\partial u}$ and finally H_{n+1}. Let

$$F_{n+1}(x) = \int_0^x \left(m' \frac{\partial H_n}{\partial m} + 2m \left(\frac{\partial H_n}{\partial m} \right)' \right) \mathrm{d}y,$$

i.e., for some $a \in \mathbb{R}$,

$$\frac{\partial H_{n+1}}{\partial u}(x) = F_{n+1}(x) + a.$$

If we can find a functional H^0_{n+1} such that $\frac{\partial H^0_{n+1}}{\partial u} = F_{n+1}(x)$ then the general solution of Eq. (C.6)—viewed as an equation in the unknown H_{n+1}—is

$$H_{n+1} = H^0_{n+1} + a \int u \, \mathrm{d}x + b \tag{C.7}$$

or equivalently $H_{n+1} = H^0_{n+1} + a \int m \, \mathrm{d}x + b$. To find H^0_{n+1} we "guess" as we did to find H_2 at the beginning. The fact that (C.7) yields all solutions is ensured by the following lemma.

Lemma C.10. *Let $H(f)$ be a functional. If $\frac{\partial H}{\partial f} \equiv 0$ then $H(f) = H(0)$.*

Proof. Clear. □

Hence if F_1 and F_2 are functionals such that $\frac{\partial F_1}{\partial f} = \frac{\partial F_2}{\partial f}$ then $F_1 = F_2 + c$ for some real c. The fact that (C.7) gives all solutions of (C.6) is now plain since

$$\frac{\partial}{\partial u} \int u \, \mathrm{d}x = 1.$$

From the general form (C.7) we have the liberty to choose H_{n+1} the neatest expression. An explicit calculation of H_3 is presented in [17].

Remark C.11. That $H_1 = \int (u^2 + u_x^2) \, \mathrm{d}x$ and $H_2 = \int (u^3 + uu_x^2) \, \mathrm{d}x$ are conserved for the CH equation (1.19) can also be verified by direct computation, cf. [19].

Remark C.12. Similarly, it is possible to produce conserved quantities by lowering $H_0 \downarrow H_{-1} \downarrow H_{-2} \downarrow \cdots$. Here, the assumption that m has no zeros is necessary to obtain functionals H_j for $j < 0$ from our recipe. It can be shown that for $m \in C^1(\mathbb{R})$ without zeros and for the compatible pair $\mathcal{E} = -(\partial m + m\partial)$ and $\mathcal{D} = -\partial + \partial^3$, if a functional F can be raised it can also be lowered.

The Camassa-Holm equation (1.19) possesses an infinite hierachy of independent conserved functionals obtained via the recursion formula (C.5).

C.2 The scattering approach for the Camassa-Holm equation

In Quantum Mechanics, the single particle motion under the influence of a potential $u(t,x)$ is described by the Schrödinger equation

$$\psi_{xx} + (\lambda - u)\psi = 0. \tag{C.8}$$

The Schrödinger equation can be seen as coming from the wave equation

$$\psi_{xx} - u\psi = \frac{1}{c^2}\psi_{tt} \tag{C.9}$$

since the plain wave ansatz

$$\psi(t,x) = \varphi(x)\exp(\pm \mathrm{i}\omega t)$$

gives Eq. (C.8), for the function φ, if $\lambda = (\omega/c)^2$. For $u = 0$ the wave equation (C.9) admits travelling waves of the form

$$\psi_\pm(t,x) = C_\pm \exp(-\mathrm{i}(kx \pm \omega t)), \quad C_\pm \in \mathbb{C}, \quad k,\omega > 0,$$

if $k = \omega/c$. More precisely, ψ_- describes right-propagating waves with velocity $c = \omega/k$ and ψ_+ describes left-propagating waves. For potentials belonging to the Schwartz class $\mathscr{S}(\mathbb{R})$, the Schrödinger equation has the asymptotic form

$$\psi_{xx} + \lambda\psi = 0, \quad |x| \to \infty,$$

and we expect that

$$\psi \to A\exp\left(\mathrm{i}\sqrt{\lambda}x\right) + B\exp\left(-\mathrm{i}\sqrt{\lambda}x\right).$$

Let us first assume that $\lambda < 0$. Let $\kappa := \sqrt{|\lambda|}$. In search of bounded solutions of (C.8), it turns out that, for certain values λ_n, there are solutions

$$\psi_n = A(\kappa,x)\exp(-\kappa x) + B(\kappa,x)\exp(\kappa x),$$

$A \to 0$ as $x \to -\infty$ and $B \to 0$ as $x \to +\infty$, called *bound states*. By Sturm-Liouville Theory the number of eigenvalues λ_n is finite. Indeed, if $\kappa^2 = |\lambda| > \max(-u)$ then

$$\frac{\psi_{xx}}{\psi} = u - \lambda = u + |\kappa|^2 > 0$$

and hence $\psi_{xx} > \psi$ if ψ is positive and this implies that ψ is unbounded. Hence

$$\min(u) < \lambda_1, \lambda_2, \ldots, \lambda_p < 0$$

for eigenvalues belonging to bound states. If $\lambda > 0$ one obtains oscillating eigenfunctions of the form

$$\psi \to \begin{cases} \exp(-\mathrm{i}\kappa x) + b(\kappa,u)\exp(\mathrm{i}\kappa x), & x \to \infty, \\ a(\kappa,u)\exp(-\mathrm{i}\kappa x), & x \to -\infty. \end{cases}$$

The physical interpretation of this solution is an incoming wave from the right which is reflected back to $+\infty$ and transmitted on to $-\infty$. We call $b(\kappa,u)$ the *reflection coefficient* and $a(\kappa,u)$ the *transmission coefficient*. In addition, the quantities

$$c_n(t) := \left(\int_{\mathbb{R}} \psi_n(t,x)^2 \, \mathrm{d}x \right)^{-1}, \quad \psi_n \text{ bound state,}$$

play a key role for the scattering problem for Eq. (C.8). We call $\{a_\kappa, b_\kappa, c_n\}$ the *scattering data* for the problem (C.8). Most importantly, knowing the scattering data at time zero we can calculate them for all positive t since they evolve according to linear ordinary differential equations.

Let us connect this concept with the well-known theory of finite-dimensional Hamiltonian systems: We describe a finite-dimensional Hamiltonian system in terms of a set of Hamiltonian functions

$$H_j \colon \mathbb{R}^{2n} = \{(p_i, q_i);\ i = 1, \ldots, n\} \to \mathbb{R}, \quad j \in \{1, \ldots, n\}.$$

We also assume that the differentials $\mathrm{d}H_j$ are linearly independent and that the Hamiltonians are in involution,

$$\{H_{j_1}, H_{j_2}\} = 0,$$

where $\{\cdot, \cdot\}$ is the canonical Poisson bracket on $\mathbb{R}^{2n}$. The equations $H_j = c_j$ with constants $c_1, \ldots, c_n$ define hyper surfaces in $\mathbb{R}^{2n}$ and their common level set has dimension n. Furthermore, the Arnold-Liouville Theorem (see [6]) says that the common level set[1] is diffeomorphic to an n-dimensional torus $\mathbb{T}$. Thus if we have n independent conservation laws, an integral curve $u(t)$ can be thought of a line winding around $\mathbb{T}$. Flattening out the torus and changing to a new set of variables (the so-called *action-angle variables*), one finally sees that the flow becomes linear. The inverse scattering approach generalizes this change of coordinates to infinite-dimensional Hamiltonian systems. While a flow $u(t)$ is described by the $2n$ coordinates $u_1(t), \ldots, u_{2n}(t)$ in the finite-dimensional case, we have a potential function $u(t,x)$ on the real axis[2] in the scattering problem for the Schrödinger operator $L = -\partial_x^2 + u$. The scattering data are also called action-angle variables and the crucial result is that the motion in these coordinates is linear.

For evolution equations $u_t = Au$ the idea behind the scattering transform is to find a suitable operator L and the corresponding scattering data (which requires information about the spectrum of L). In fact, the Schrödinger operator $L = -\partial_x^2 + u$ is the right candidate for the KdV equation and we now want to work out what we get for CH. Anyway, the upshot is:

For an integrable equation $u_t = Au$, suitable for the scattering approach, the scattering transform maps the initial problem to a sequence of separated ordinary differential equations for the action-angle variables which can be integrated trivially. Inverse scattering recovers the potential $u(t,x)$ from the scattering data (which is much harder from the mathematical point of view than vice versa).

We see that the inverse scattering approach is a method to integrate an equation of the form $u_t = Au$. Very often, inverse scattering is also called a *non-linear Fourier transform*: In the classical theory of ordinary differential equations the Fourier transform is used to solve certain classes of equations since it maps derivatives to polynomials.

[1] The common level set is assumed to be a smooth compact and connected manifold.

[2] Recall that bi-Hamiltonian equations like CH have an *infinite* hierarchy of conservation laws.

Solving the problem in terms of the co-variables (which is rather easy) and applying the inverse Fourier transform yields the solution of the initial problem.

Let us now explain the scattering approach for the CH equation (1.19). Therefore, we first introduce a spectral parameter λ. Recall the recursion relation

$$\mathcal{D}\frac{\partial H_n}{\partial m} = \mathcal{E}\frac{\partial H_{n-1}}{\partial m}, \quad \mathcal{D} = -(\partial - \partial^3), \quad \mathcal{E} = -(m\partial + \partial m).$$

Multiplying with λ^n and summing over n yields

$$\mathcal{D} \sum_{n=-\infty}^{\infty} \lambda^n \frac{\partial H_n}{\partial m} = \lambda\mathcal{E} \sum_{n=-\infty}^{\infty} \lambda^{n-1} \frac{\partial H_{n-1}}{\partial m}.$$

Let us introduce the squared eigenfunction

$$\psi^2(x,t;\lambda) \coloneqq \sum_{n=-\infty}^{\infty} \lambda^n \frac{\partial H_n}{\partial m}.$$

Then, formally,

$$\mathcal{D}\psi^2(x,t;\lambda) = \lambda\mathcal{E}\psi^2(x,t;\lambda). \tag{C.10}$$

Equation (C.10) is a third-order eigenvalue problem for the squared eigenfunction ψ^2 which is in fact equivalent to the following second order Sturm-Liouville problem for the function ψ.

Lemma C.13. *If ψ satisfies*

$$\lambda\left(\frac{1}{4} - \partial_x^2\right)\psi = \frac{1}{2}m\psi,$$

then ψ^2 is a solution of Eq. (C.10).

Proof. This is a straightforward computation, cf. [62]. □

Next, we assume that λ does not depend on time and that the time dependence of ψ is given by an evolution equation

$$\psi_t = a\psi_x + b\psi$$

with coefficients a and b so that the compatibility condition $\psi_{txx} = \psi_{xxt}$ implies the Camassa-Holm equation. Cross-differentiation shows that

$$b = -\frac{1}{2}a_x, \quad a = -(\lambda + u).$$

Consequently,

$$\psi_t = -(\lambda + u)\psi_x + \frac{1}{2}u_x\psi$$

is the desired evolution equation for the eigenfunction ψ.

Theorem C.14. *Equation* (1.19) *admits a Lax pair formulation: The eigenvalue problem*

$$\lambda\left(\frac{1}{4} - \partial_x^2\right)\psi = \frac{1}{2}m\psi$$

and the evolution equation

$$\psi_t = -(u+\lambda)\psi_x + \frac{1}{2}u_x\psi$$

imply (1.19) *if they are compatible,* $\psi_{xxt} = \psi_{txx}$*, and* λ *is constant in time (isospectrality).*

Proof. From the eigenvalue equation, we obtain

$$\begin{aligned}\lambda\psi_{xx} &= \left(\frac{\lambda}{4} - \frac{m}{2}\right)\psi,\\ \lambda\psi_{xxx} &= \left(\frac{\lambda}{4} - \frac{m}{2}\right)\psi_x - \frac{m_x}{2}\psi.\end{aligned}$$

Hence, differentiating the eigenvalue equation with respect to t and the evolution equation twice with respect to x, we obtain

$$\lambda\psi_{xxt} = \left(\frac{\lambda}{4} - \frac{m}{2}\right)\left(-(u+\lambda)\psi_x + \frac{1}{2}u_x\psi\right) - \frac{m_t}{2}\psi$$

and

$$\begin{aligned}\lambda\psi_{txx} = -(u+\lambda)&\left(\left(\frac{\lambda}{4} - \frac{m}{2}\right)\psi_x - \frac{m_x}{2}\psi\right) - 2u_x\left(\frac{\lambda}{4} - \frac{m}{2}\right)\psi\\ &+\frac{1}{2}\lambda u_{xxx}\psi + \frac{1}{2}u_x\left(\frac{\lambda}{4} - \frac{m}{2}\right)\psi\end{aligned}$$

and a careful examination shows that the compatibility condition implies that

$$(m_t + um_x + 2mu_x)\psi = 0.$$

Since ψ is an eigenfunction, it is nonzero and we obtain Eq. (1.19). □

The squared-eigenfunction approach leading to the isospectral problem for the CH equation goes back to [52]. We now explain the general Lax pair formalism, discovered by Peter Lax in 1968. Starting from the isospectral problem, we obtain operators L and B such that the equation under consideration is equivalent to

$$L\psi = \lambda\psi, \quad \psi_t = B\psi.$$

The operator L is linear and symmetric and B is the evolution operator for the eigenfunction ψ. Differentiating the first of these equations with respect to time, under the assumption that λ does not depend on time, we obtain

$$\begin{aligned}0 &= L_t\psi + L\psi_t - \lambda\psi_t\\ &= L_t\psi + LB\psi - \lambda B\psi\\ &= L_t\psi + LB\psi - B\lambda\psi\\ &= (L_t + LB - BL)\psi\end{aligned}$$

and hence

$$L_t = [B, L], \tag{C.11}$$

where $[B, L] = BL - LB$ denotes the usual commutator. Contrariwise, starting with Eq. (C.11) where L and B are spatial but time-dependent operators on some Hilbert space $\mathcal{H}$ and L is linear and symmetric, we consider the eigenvalue problem $L\psi = \lambda\psi$, $\psi \neq 0$. Differentiating the eigenvalue equation with respect to time we find that

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t$$

and hence

$$\begin{aligned}\lambda_t\psi &= (L-\lambda)\psi_t + [B, L]\psi \\ &= (L-\lambda)\psi_t + B\lambda\psi - LB\psi \\ &= (L-\lambda)\psi_t + (\lambda - L)B\psi \\ &= (L-\lambda)(\psi_t - B\psi). \end{aligned} \tag{C.12}$$

Let $\langle\cdot,\cdot\rangle$ denote the inner product in $\mathcal{H}$. Then

$$\begin{aligned}\lambda_t\langle\psi,\psi\rangle &= \langle\psi, (L-\lambda)(\psi_t - B\psi)\rangle \\ &= \langle(L-\lambda)\psi, (\psi_t - B\psi)\rangle \\ &= \langle 0, \psi_t - B\psi\rangle \\ &= 0.\end{aligned}$$

Since $\psi \neq 0$ it follows that $\lambda_t = 0$ and by (C.12),

$$(L-\lambda)(\psi_t - B\psi) = 0,$$

i.e., $\psi_t - B\psi$ is an eigenfunction of L with eigenvalue λ. Assuming that the eigenspace of λ has dimension one, we can find a function f only depending on time such that

$$\psi_t - B\psi = f(t)\psi.$$

Note that f commutes with the spatial operator L so that $\tilde{B} \coloneqq B + f$ satisfies both

$$\psi_t = \tilde{B}\psi, \quad L_t = \left[\tilde{B}, L\right].$$

Theorem C.15. *Let L be a symmetric linear spatial operator, B a spatial operator and suppose that $L\psi = \lambda\psi$ holds on some Hilbert space $\mathcal{H}$ and $\psi \neq 0$. Then:*

1. *If $\lambda_t = 0$ and $\psi_t = B\psi$ then $L_t = [B, L]$.*
2. *If $L_t = [B, L]$ then $\lambda_t = 0$ (and often one can redefine B to get $\psi_t = B\psi$).*

The pair L, B is called Lax pair *and $L_t = [B, L]$ is called* Lax equation.

Example C.16. For the Camassa-Holm equation (1.19) with potential $m = u - u_{xx}$ one finds the equivalent Lax pair representation

$$\psi_{xx} = \frac{1}{4}\psi + \lambda m\psi, \quad \psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi,$$

c.f., e.g., [18, 28, 92].

Example C.17. For the DP equation, we have the third order equation

$$\psi_x - \psi_{xxx} - \lambda m \psi = 0$$

in the corresponding scattering problem. The time evolution of the wave function is given by

$$\psi_t + \frac{1}{\lambda}\psi_{xx} + u\psi_x - \left(u_x + \frac{2}{3\lambda}\right)\psi = 0,$$

cf. [30]. Indeed, the compatibility of both equations implies the DP equation for the function u.

In this framework, forward scattering means determining the Lax pair for the given equation (so that the Lax equation recovers the original PDE). Then, for any fixed λ, the time evolution of ψ and the corresponding scattering data are determined; here we have to solve ordinary differential equations. Finally, the inverse scattering procedure yields the solution of the initial equation. Inverse scattering enables modern analytical approaches like the Riemann-Hilbert formalism and is a current area of research, cf. [98].

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Schulbildung

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- Stipendien aus Studiengebühren (2007, 2008), Stipendium der Studienstiftung des Deutschen Volkes (2008)
- Preis des Departments für Mathematik der Technischen Universität Braunschweig (2008, 2009)
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Wissenschaftliche Publikationen

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- Geometric aspects of the periodic μDP equation (with Joachim Escher and Boris Kolev, Birkhäuser, to appear in 2011)

  `arXiv:1004.0978v1 [math.AP]`

- The geometry of the two-component Camassa-Holm and Degasperis-Procesi equations (with Joachim Escher and Jonatan Lenells, J. Geom. Phys. **61** (2) 2011, 436–452)
- The periodic μ-b-equation and Euler equations on the circle (to appear in J. Nonlinear Math. Phys.)

  `arXiv:1010.1832v1 [math-ph]`

- Spectral properties of grain boundaries at small angles of rotation (with Rainer Hempel, to appear in Journal of Spectral Theory, EMS)

  `arXiv:1009.4039v1 [math-ph]`

- A variational approach to dislocation problems for periodic Schrödinger operators (with Rainer Hempel, submitted to J. Math. Anal. Appl.)

  `arXiv:1009.3581v1 [math-ph]`

- Global existence and blow-up for a weakly dissipative μDP equation (submitted to J. Nonlinear Sci.)

  `arXiv:1010.2355v1 [math-ph]`

- The curvature of semidirect product groups associated with 2HS and 2μHS (Preprint)

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